2008

A robust optimization approach for static portfolio management

Ban Kawas
Lehigh University

Kawas, Ban

A Robust Optimization Approach For Static Portfolio Management

September 2008

A Robust Optimization Approach For Static Portfolio Management

by

Ban Kawas

B.S. in Industrial Engineering (2005)

Presented to the Graduate and Research Committee

of Lehigh University

in Candidacy for the Degree of

Master of Science in Industrial Engineering

in the
Department of Industrial and Systems Engineering

Lehigh University

September 2008

Approved and recommended for acceptance as a dissertation in partial fulfillment of the requirements for the degree of Master of Science.

Date: 8/4/08

Certified by ....

Aurélie C. Thiele

P.C. Rossin Assistant Professor of Industrial and Systems Engineering

Thesis Supervisor

Accepted by ....

Gregory Tonkay

Chairperson of Department

Acknowledgements

This work was supported in part by National Science Foundation Grant CMMI-0757983.

Contents

List of Figures

List of Tables

A Robust Optimization Approach For Static Portfolio Management

by

Ban Kawas

Submitted to the Department of Industrial and Systems Engineering
on August 4, 2008, in partial fulfillment of the
requirements for the degree of
Master of Science in Industrial Engineering

Abstract

We present a robust optimization approach to portfolio management under uncertainty that builds upon insights gained from the well-known Lognormal model for stock prices, while addressing that model's limitations, in particular, the issue of fat tails being underestimated in the Gaussian framework and the active debate on the correct distribution to use. Our approach, which we call Log-robust in the spirit of the Lognormal model, does not require any probabilistic assumption, and incorporates the randomness on the continuously compounded rates of return by using range forecasts and a budget of uncertainty, thus capturing the decision-maker's degree of risk aversion through a single, intuitive parameter. Our objective is to maximize the worst-case portfolio value (over a set of allowable deviations of the random variables from their mean) at the end of the time horizon in a one-period setting; short sales are not allowed. We formulate the robust problem as a linear programming problem and derive theoretical insights into the worst-case uncertainty and the optimal allocation. We then compare in numerical experiments the Log-robust approach with the traditional robust approach, where range forecasts are applied directly to the stock returns. Our results indicate that the Log-robust approach significantly outperforms the benchmark with respect to 95% or 99% Value-at-Risk. This is because the traditional robust approach leads to portfolios that are far less diversified.

Chapter 1

Introduction

1.1 Literature Review

Portfolio management under uncertainty was pioneered in the 1950s by Markowitz (1959), who first articulated the investor's trade-off between risk and return. The optimal asset allocation, however, is very sensitive to parameter inputs, e.g., mean and covariance, so that small estimation errors can result in strategies that are far from optimal (Chopra and Ziemba (1993)). Goldfarb and Iyengar (2003) have proposed robust optimization approaches to minimize the worst-case variance and similar criteria over ellipsoidal uncertainty sets, which mitigates the impact of estimation errors for these performance measures; their work was later extended to active portfolio management with transaction costs in Erdogan et. al. (2004). (The reader is referred to Bertsimas and Thiele (2006) for a tutorial-level introduction to robust optimization, in essence, worst-case optimization over a bounded convex set of reasonable worst cases.) Recently, Value-at-Risk and Conditional Value-at-Risk have emerged as more pertinent risk measures in finance, and investors have become more concerned with maximizing the worst-case value of their portfolio than minimizing its standard deviation. This requires the development of new methodologies.

Ben-Tal and Nemirovski (1999) and Bertsimas and Sim (2004) have applied the robust optimization approach not on uncertain parameters but on random variables. In particular, they consider range forecasts at the level of the stock returns and their goal is to maximize the portfolio's worst-case return, where the worst case is computed over a set of allowable deviations of the stock returns from their mean to prevent over-conservatism. This approach is also at the core of the robust financial models developed in Bertsimas and Pachamanova (2008), Pachamanova (2006), Fabozzi et. al. (2007). Numerous studies of stock price behavior, however, (see Hull (2002) and the references therein) suggest that the true drivers of uncertainty are the continuously compounded rates of return, assumed to obey a Gaussian distribution in the famous Lognormal model developed by Black and Scholes (1973). This model gives rise to an elegant mathematical framework and closed-form formulas, for instance in pricing European options, but neglects the fact that the real distributions have fat tails (Jansen and deVries (1991), Cont (2001), Al Najjab and Thiele (2007)). In that sense, the Lognormal model leads the manager to take more risk than he is willing to accept. Furthermore, the empirical validity of that choice among possible distributions remains actively debated (Fama (1965), Blattberg and Gonedes (1974), Kon (1984), Jansen and deVries (1991), Richardson and Smith (1993), Cont (2001)). In particular, Jansen and deVries states: "Numerous articles have investigated the distribution of share prices, and find that the returns are fat-tailed. Nevertheless, there is still controversy about the amount of probability mass in the tails, and hence about the most appropriate distribution to use in modeling returns. This controversy has proven hard to resolve." Also, while risk aversion has long been incorporated to portfolio management through the use of utility functions, such functions are difficult to articulate in practice. Robust optimization, however, can capture risk aversion through a single parameter, called the budget of uncertainty, which determines the degree of protection against downside risk the manager requires for his investments.

The decision-maker seeking to protect his portfolio against downside risk needs to find an approach with the same ease of implementation as the Lognormal model, but that reflects the limited knowledge on the underlying distributions. The purpose of this paper is to provide such an approach for one-period portfolio management, based on robust optimization with polyhedral sets applied to the continuously compounded rates of return. To the best of our knowledge, this is the first time a robust optimization approach is applied to real-life models of stock price dynamics in portfolio management. We believe this approach gives more relevant results for finance practitioners than the traditional robust approach, while remaining theoretically insightful and numerically tractable.

1.2 Contributions

We make the following contributions to the literature:

- We provide a mathematical model that builds upon well-established features of stock prices behavior (specifically, the fact that the continuously compounded rates of return are i.i.d.), while addressing limitations of the Lognormal model, where the distribution is assumed to be Gaussian and tail events are underestimated.
- We reformulate the robust problem as a linear programming problem, which can be solved efficiently with commercial software, including in large-scale settings, and we provide insights into the optimal allocation and the worst-case deviations of the uncertain parameters.
- We compare the proposed approach with the traditional robust framework and show empirically that the latter leads to much less diversified portfolios, and hence much worse performance, in implementations with real financial data.

1.3 Outline

In Chapter 2 we describe and analyze the portfolio management problem with independent assets. We extend the formulation to the case of correlated stocks in Chapter 3. Chapter 4 contains our numerical experiments. We conclude in Chapter 5. All proofs are in the appendix.

Chapter 2

Portfolio Management With Independent Assets

2.1 Generalities

We will use the following notation throughout the paper.

n : the number of stocks,

T : the length of the time horizon,

$S_i(0)$: the initial (known) value of stock i,

$S_i(T)$: the (random) value of stock i at time T,

w_0 : the initial wealth of the investor,

μ_i : the drift of the Lévy process for stock i,

σ_i : the infinitesimal standard deviation of the Lévy process for stock i,

$\tilde{x}_i$: the number of shares invested in stock i,

x_i : the amount of money invested in stock i.

Short sales are not allowed. We start our analysis by assuming all stock prices are independent; this assumption is relaxed in Section 3. In the traditional Log-normal model (see Hull (2002) for an overview), the random stock price i at time T, $S_i(T)$, can be described as:

$$\ln \frac{S_i(T)}{S_i(0)} = \left(\mu_i - \frac{\sigma_i^2}{2}\right) T + \sigma_i \sqrt{T} Z_i. \tag{2.1}$$

where Z_i obeys a standard Gaussian distribution, i.e., $Z_i \sim N(0,1)$. The portfolio management problem, where the decision-maker seeks to maximize his expected wealth subject to a budget constraint and no short sales, is then formulated as:

$$\begin{aligned} \max_{\tilde{x}} \quad & \sum_{i=1}^{n} \tilde{x}_i \mathrm{E}[S_i(T)] \\ \text{s.t.} \quad & \sum_{i=1}^{n} \tilde{x}_i S_i(0) = w_0, \\ & \tilde{x}_i \geq 0, \quad \forall i. \end{aligned}$$

It is easy to see that the investor will allocate all his wealth to the stock with the highest ratio $\frac{E[S_i(T)]}{S_i(0)}$. To diversify the portfolio in the Lognormal framework, it is then necessary to introduce additional risk constraints, e.g., limiting portfolio variance.

Our goal in this paper is to investigate the optimal asset allocation when the stock price still satisfies Equation (2.1) but the distribution of the Z_i, $i = 1, \ldots, n$ is *not known*. (In particular, it might not be Gaussian at all.) Instead, we will model Z_i as uncertain parameters with nominal value of zero and known support, for all i. To avoid over-conservatism, we will follow the modeling practices in the robust optimization community and only consider a $100\, b_i\%$ confidence interval for Z_i, which is $[-c_i, c_i]$. (When the Z_i do obey a standard Gaussian distribution, we have $c_i = \Phi^{-1}\left(\frac{1+b_i}{2}\right)$.) For the purpose of clarity we will take the same b, and hence the same c, for all stocks; in experiments we use $b = 0.95$ and $c = 1.96$, which provide a high-probability confidence interval without making the approach too conservative. The choice of 95% confidence intervals is common practice in the robust optimization literature; researchers routinely pick the confidence intervals as deviating from the nominal value by two standard deviations on each side (left and right). See for instance Bertsimas and Thiele (2006) and the references therein for numerical experiments reflecting these choices. It appears that this choice strikes a good trade-off between covering a large number of the potential values taken by the random variables, while not being too conservative. Ben-Tal et. al. (2006) explores in depth the consequences of this modeling, in particular with respect to having realizations fall outside the uncertainty set.

In the remainder of the paper, we will describe the uncertain parameter Z_i, $i =$

$1, \ldots, n$, as:

$$Z_i = c\,\tilde{z}_i,$$

where $\tilde{z}_i \in [-1, 1]$ represents the *scaled deviation* of Z_i from its nominal value, which is zero. Furthermore, z_i will denote the *absolute value* of the scaled deviation, for all i.

2.2 Problem Formulation

To incorporate risk in the formulation, we adopt a worst-case approach where we seek to maximize the worst-case portfolio return over a set of feasible, "realistic" stock returns. The decision-maker has at his disposal the range forecasts $[-c, c]$ for the scaled uncertain parameters Z_i (it is possible to design different range forecasts for different assets; from a practical standpoint, however, there is little motivation to design a 95% confidence interval for one and a 99% confidence interval for the other). Furthermore, because the uncertain parameters are assumed *independent*, it is quite unrealistic that many of them turn out to be equal to their worst-case value; in practice, due to the assumption of independence, some will be higher than their nominal value and some will be lower, so that part of the uncertainty cancels itself out. This motivates the introduction of a budget-of-uncertainty constraint (first presented in Bertsimas and Sim (2004)), which bounds the total scaled deviation of the independent, uncertain parameters from their mean (here, zero) by a nonnegative budget denoted Γ:

$$\sum_{i=1}^{n} |\tilde{z}_i| \leq \Gamma, \;\; |\tilde{z}_i| \leq 1, \;\; \forall i.$$

If $\Gamma = 0$, all the uncertain parameters are equal to their mean (nominal value). If $\Gamma = n$, the budget-of-uncertainty constraint is redundant with $|\tilde{z}_i| \leq 1$ for all i and the decision-maker will protect the portfolio return against the worst possible value of each stock return. Selecting Γ between these two extremes allows the decision-maker to achieve a trade-off between not protecting the system against any uncertainty and being extremely conservative.

We hope that the simplicity of the uncertainty set, and the intuitive explanation behind it, will encourage financial practitioners to adopt the model. Other uncertainty sets are possible; in particular, ellipsoidal uncertainty sets have been advocated by Ben-Tal and Nemirovski (1999).

The robust portfolio management problem can then be formulated as:

$$\begin{aligned} \max_{\tilde{x}} \quad & \min_{\tilde{z}} \sum_{i=1}^{n} \tilde{x}_i S_i(0) \exp\left[(\mu_i - \frac{\sigma_i^2}{2})T + \sigma_i \sqrt{T} c \tilde{z}_i\right] \\ & \text{s.t.} \sum_{i=1}^{n} |\tilde{z}_i| \leq \Gamma, \\ & \quad |\tilde{z}_i| \leq 1 \ \forall i, \\ \text{s.t.} \quad & \sum_{i=1}^{n} \tilde{x}_i S_i(0) = w_0. \\ & \tilde{x}_i \geq 0 \ \forall i, \end{aligned}$$

or, using that the amount x_i of money invested in stock i at time 0 satisfies: $x_i = S_i(0)\,\tilde{x}_i$ for all i:

$$\begin{aligned} \max_{x} \quad & \min_{\tilde{z}} \sum_{i=1}^{n} x_i \exp\left[(\mu_i - \frac{\sigma_i^2}{2})T + \sigma_i \sqrt{T} c \tilde{z}_i\right] \\ & \text{s.t.} \sum_{i=1}^{n} |\tilde{z}_i| \leq \Gamma, \\ & \quad |\tilde{z}_i| \leq 1 \ \forall i, \\ \text{s.t.} \quad & \sum_{i=1}^{n} x_i = w_0. \\ & x_i \geq 0 \ \forall i. \end{aligned} \tag{2.2}$$

Robust optimization addresses the fat-tails issue by specifically planning against the rare (tail) events in the worst-case optimization framework, while these events are under-represented in the traditional Lognormal model, due to the mistakenly low probability estimates.

To ensure that Problem (2.2) can be solved efficiently using commercial software, our focus will be on rewriting the inner minimization problem as a maximization problem using duality arguments and studying the properties of the resulting formulation.

2.3 Worst-case deviations

We consider the inner minimization problem of Problem (2.2):

$$\begin{aligned} \min_{\tilde{z}} \quad & \sum_{i=1}^{n} x_i \exp\left[(\mu_i - \frac{\sigma_i^2}{2})T\right] \exp\left[\sigma_i \sqrt{T} c \tilde{z}_i\right] \\ \text{s.t.} \quad & \sum_{i=1}^{n} |\tilde{z}_i| \leq \Gamma, \\ & |\tilde{z}_i| \leq 1 \qquad \forall i. \end{aligned} \tag{2.3}$$

The following lemma allows us to discard the absolute values in Problem (2.3):

Lemma 2.3.1 *At optimality, $-1 \leq \tilde{z}_i \leq 0$ for all i and Problem (2.3) is equivalent to:*

$$\begin{aligned} \min_{z} \quad & \sum_{i=1}^{n} x_i \exp\left[(\mu_i - \frac{\sigma_i^2}{2})T\right] \exp\left[-\sigma_i \sqrt{T} c z_i\right] \\ s.t. \quad & \sum_{i=1}^{n} z_i \leq \Gamma, \\ & 0 \leq z_i \leq 1, \ \forall i. \end{aligned} \tag{2.4}$$

In the remainder of the paper, z_i will refer to the absolute value of the scaled deviation and the true worst-case scaled deviation $\tilde{z}_i$ will be negative. Problem (2.4) is convex; therefore, we study its optimal solution using a Lagrange relaxation approach (see Bertsekas (1999) for a review on nonlinear optimization). For notational convenience, we denote by k_i the constant $\exp\left[(\mu_i - \frac{\sigma_i^2}{2})T\right]$. We introduce the Lagrangian multipliers α, λ_i^0 and λ_i^1 for all i and obtain the unconstrained, convex Lagrange relaxation of Problem (2.4):

$$\min_{z} \quad \sum_{i=1}^{n} x_i \, k_i \exp(-\sigma_i \sqrt{T} c z_i) + \alpha \left(\sum_{i=1}^{n} z_i - \Gamma \right) - \sum_{i=1}^{n} \lambda_i^0 z_i + \sum_{i=1}^{n} \lambda_i^1 (z_i - 1) \tag{2.5}$$

Note that strong duality holds because the objective is convex and Slater's condition is satisfied.

Lemma 2.3.2 (Worst-case deviations) *(i) The optimal deviations z_i (for fixed val-*

ues of the Lagrange multipliers) are given by, for all i:

$$z_i = \frac{1}{\sigma_i \sqrt{T} c} \ln \left(\frac{x_i \, k_i \, \sigma_i \sqrt{T} c}{\alpha - \lambda_i^0 + \lambda_i^1} \right). \tag{2.6}$$

(If $x_i = 0$, we must have $\alpha - \lambda_i^0 + \lambda_i^1 = 0$.)

(ii) Specifically:

- *If $0 < z_i < 1$, then both $\lambda_i^0, \lambda_i^1 = 0$ and Equation (2.6) becomes:*

$$z_i = \frac{1}{\sigma_i \sqrt{T} c} \left[\ln \left(x_i \, k_i \, \sigma_i \sqrt{T} c \right) - \ln \alpha \right]. \tag{2.7}$$

- *If $z_i = 0$, then $\lambda_i^1 = 0$ and $\lambda_i^0 \geq 0$ is such that: $x_i \, k_i \, \sigma_i \sqrt{T} c = \alpha - \lambda_i^0$.*
- *If $z_i = 1$, then $\lambda_i^0 = 0$, and $\lambda_i^1 \geq 0$ is such that: $\ln \left(\frac{x_i \, k_i \, \sigma_i \sqrt{T} c}{\alpha + \lambda_i^1} \right) = \sigma_i \sqrt{T} c$, i.e., $\alpha + \lambda_i^1 = x_i \, k_i \, \sigma_i \sqrt{T} c \, \exp(-\sigma_i \sqrt{T} c)$.*

2.4 Robust convex counterpart and structure of the optimal solution

Theorem 2.4.1 shows the robust portfolio management problem (2.2) can be solved as a linear programming problem.

Theorem 2.4.1 (Optimal wealth and allocation)

(i) The optimal wealth in the robust portfolio management problem (2.2) is: $w_0 \exp(F(\Gamma))$, where F is the function defined by:

$$\begin{aligned} F(\Gamma) = \max_{\eta, \chi, \xi} \quad & \sum_{i=1}^{n} \chi_i \ln k_i - \eta \, \Gamma - \sum_{i=1}^{n} \xi_i \\ s.t. \quad & \eta + \xi_i - \sigma_i \sqrt{T} \, c \, \chi_i \geq 0, \ \forall i, \\ & \sum_{i=1}^{n} \chi_i = 1, \\ & \eta \geq 0, \ \chi_i, \ \xi_i \geq 0, \ \forall i. \end{aligned} \tag{2.8}$$

(ii) The optimal amount of money invested at time 0 in stock i is $\chi_i \, w_0$, for all i, where the χ_i are found by solving Problem (2.8).

Remarks:

- The optimal wealth is proportional to the initial amount of money invested, as expected.
- The worst-case scaled deviations become, for all i: $z_i = \dfrac{1}{\sigma_i \sqrt{T}\, c}\left[\ln k_i - F(\Gamma)\right].$
- It is easy to show by dualizing the coupling constraint $\sum_{i=1}^{n} \chi_i = 1$ in Problem (2.8) and setting η and ξ_i to their optimal values for all i, that at optimality, the decision-maker invests nothing in asset i if the drift falls below a threshold ($k_i < e^a$ with a the Lagrange multiplier) and invests an amount of money $\frac{\eta+\xi_i}{\sigma_i\sqrt{T}c} w_0$ in asset i if the drift exceeds that threshold ($k_i > e^a$). Note that this amount of money decreases with the standard deviation σ_i, everything else being equal.
- If the investor has additional requirements on the feasible allocation besides the budget constraint and non-negativity, we may still obtain a similar problem formulation using the specific structure of the constraint set. In practice, the easiest way to incorporate additional constraints on the asset allocation is to compute the optimal allocation using the proposed robust optimization approach, and then pick the feasible strategy that is "closest" (in a least-squares sense) to the theoretical one we have just obtained.

In the remainder of this section, we assume that the n assets are such that the terms $(\mu_i - \frac{\sigma_i^2}{2})T$ and $(\mu_i - \frac{\sigma_i^2}{2})T - \sigma_i\sqrt{T}c$ are all distinct. This is a reasonable assumption to make in practice because parameters are estimated using historical data, and more decimals can always be used to break ties. We use this assumption to derive further insights into the worst-case scaled deviations and the optimal asset allocation. First, we need the following lemma.

Lemma 2.4.2 *Assume it is known which scaled deviations hit their bounds (that is, equal 0 or 1) at optimality and let* $R = \sum_{i|0<z_i^*<1} \frac{1}{\sigma_i\sqrt{T}c}$. *The robust portfolio manage-*

ment problem is equivalent to the convex optimization problem:

$$\begin{aligned}
\max_x \quad & \left[\left(\exp\left[\sum_{i|z_i^*=1} 1 - \Gamma \right] \right)^{\frac{1}{R}} \cdot \left[\prod_{i|0<z_i^*<1} (x_i k_i \sigma_i \sqrt{T} c)^{\frac{1}{\sigma_i \sqrt{T} c}} \right]^{\frac{1}{R}} \right] \cdot R + \sum_{i|z_i^*=0} x_i k_i + \sum_{i|z_i^*=1} x_i k_i \exp(-\sigma_i \sqrt{T} c) \\
s.t. \quad & \sum_{i=1}^{n} x_i = w_0, \\
& x_i \geq 0, \ \forall i.
\end{aligned} \tag{2.9}$$

This allows us to derive the following theorem.

Theorem 2.4.3

(i) At most one scaled deviation can hit the bounds, i.e., be equal to either 0 or 1, among the stocks the manager invests in.

(ii) If no scaled deviation hits the bounds, then the optimal asset allocation is independent of the budget of uncertainty Γ *and is given by, for all* i*:*

$$x_i = \frac{w_0}{\sigma_i \cdot \left(\sum_{j=1}^{n} \frac{1}{\sigma_j} \right)}. \tag{2.10}$$

Remark: Equation (2.10) is identical to the optimal allocation in the Markowitz mean-variance model when the portfolio variance for independent assets is minimized and the expected return constraint is not binding, but the meaning of the σ_i is different. In the present paper, the σ_i are the standard deviations of the continuously compounded rates of return. In the Markowitz model, the σ_i traditionally denote the standard deviations of the rates of return over the one period considered. It is natural that the two approaches do not yield the same allocation, since the robust optimization framework does not rely on variance minimization, and instead on worst-case value maximization.

Chapter 3

Portfolio Management with Correlated Assets

3.1 Formulation

We now extend the approach described in Section 2 to the case with correlated assets. Equation (2.1), which characterizes the behavior of the stock prices, is replaced by:

$$\ln \frac{S_i(T)}{S_i(0)} = \left(\mu_i - \frac{\sigma_i^2}{2}\right) T + \sqrt{T} Z_i,$$

where the random vector Z is normally distributed with mean $\mathbf{0}$ and covariance matrix $\mathbf{Q}$. We define:

$$\mathbf{Y} = \mathbf{Q}^{-1/2}\mathbf{Z},$$

where $\mathbf{Y} \sim \mathcal{N}(\mathbf{0}, \mathbf{I})$ and $\mathbf{Q}^{1/2}$ is the square-root of the covariance matrix $\mathbf{Q}$, i.e., the unique symmetric positive definite matrix $\mathbf{S}$ such that $\mathbf{S}^2 = \mathbf{Q}$. In the robust optimization approach, the vector of scaled independent uncertainty drivers $\tilde{\mathbf{y}}$ is related to the vector of (here, non-scaled) deviations $\tilde{\mathbf{z}}$ as follows:

$$\tilde{z}_i = c \sum_{j=1}^{n} Q_{ij}^{1/2}\, \tilde{y}_i,$$

with each component $\tilde{y}_i$ belonging to $[-1, 1]$ so that $Y_i \in [-c, c]$, where c is the parameter corresponding to the two-sided $100b$%-confidence interval, as in Section 2.

The robust optimization model becomes:

$$
\begin{aligned}
\max_{x} \quad & \min_{\tilde{y}} \quad \sum_{i=1}^{n} x_i \exp\left[\left(\mu_i - \sigma_i^2/2\right) T + \sqrt{T} c \left(\sum_{j=1}^{n} Q_{ij}^{1/2}\, \tilde{y}_j\right)\right] \\
& \quad\ \text{s.t.} \quad \sum_{j=1}^{n} |\tilde{y}_j| \leq \Gamma, \\
& \qquad\qquad |\tilde{y}_j| \leq 1, \ \forall j, \\
\text{s.t.} \quad & \sum_{i=1}^{n} x_i = w_0, \\
& x_i \geq 0, \ \forall i.
\end{aligned}
\tag{3.1}
$$

We first need to reformulate the inner minimization problem:

$$
\begin{aligned}
\min_{\tilde{y}} \quad & \sum_{i=1}^{n} x_i \exp\left[\left(\mu_i - \sigma_i^2/2\right) T\right] \exp\left[\sqrt{T} c \left(\sum_{j=1}^{n} Q_{ij}^{1/2}\, \tilde{y}_j\right)\right] \\
\text{s.t.} \quad & \sum_{j=1}^{n} |\tilde{y}_j| \leq \Gamma, \\
& |\tilde{y}_j| \leq 1, \ \forall j.
\end{aligned}
\tag{3.2}
$$

as a maximization problem to keep the approach tractable.

Lemma 3.1.1 *Problem (3.2) is convex.*

Therefore, we can characterize the optimal solution using a Lagrangean relaxation approach.

3.2 Special Case

In the special case where the coefficients of the square root of the correlation matrix are all non-negative, we observe (using the same argument as in Lemma 2.3.1) that the minimum of the objective function in Problem (3.2) is achieved for $\tilde{y}_j \leq 0$. We define

$y_j = |\tilde{y}_j|$, so the minimization problem becomes:

$$\begin{aligned}
\min_{y} \quad & \sum_{i=1}^{n} x_i \exp\left[\left(\mu_i - \sigma_i^2/2\right) T\right] \exp\left[-\sqrt{T} c \left(\sum_{j=1}^{n} Q_{ij}^{1/2}\, y_j\right)\right] \\
\text{s.t.} \quad & \sum_{j=1}^{n} y_j \leq \Gamma, \\
& 0 \leq y_j \leq 1, \ \forall j.
\end{aligned} \tag{3.3}$$

Lemma 3.2.1 (Worst-case deviations) *The optimal solution to Problem (3.3) is given by:*

$$\mathbf{y} = \frac{1}{\sqrt{T}\, c}\, \mathbf{Q}^{-1/2} \ln(\sqrt{T}\, c\, \hat{\mathbf{x}}),$$

where:

$$\hat{x}_i = \frac{x_i \exp\left[(\mu_i - \sigma_i^2/2)\, T\right]}{[\mathbf{Q}^{-1/2}(\alpha\, \mathbf{e} - \lambda^0 + \lambda^1)]_i}, \ \forall i.$$

α, $\boldsymbol{\lambda}^0$ and $\boldsymbol{\lambda}^1$ are the Lagrange multipliers associated with the constraints of Problem (3.3).

Theorem 3.2.2 shows that the robust portfolio management problem with assets non-negatively correlated can be solved as a linear programming problem; hence, as in the non-correlated case (Theorem 2.4.1), the problem can be solved efficiently using commercial software.

Theorem 3.2.2 (Optimal wealth and allocation)

(i) The optimal wealth in the robust portfolio management problem (3.1) is: $w_0 \exp(F(\Gamma))$, where F is the function defined by:

$$\begin{aligned}
F(\Gamma) = \max_{\eta, \chi, \xi} \quad & \sum_{i=1}^{n} \chi_i \ln k_i - \eta\, \Gamma - \sum_{i=1}^{n} \xi_i \\
s.t. \quad & \eta + \xi_i - \sqrt{T}\, c \left(\sum_{j=1}^{n} Q_{ij}^{1/2}\, \chi_j\right) \geq 0, \ \forall i, \\
& \sum_{i=1}^{n} \chi_i = 1, \\
& \eta \geq 0, \ \chi_i, \ \xi_i \geq 0, \ \forall i.
\end{aligned} \tag{3.4}$$

(ii) The optimal amount of money invested at time 0 in stock i is $\chi_i\, w_0$, for all i, where

the χ_i are found by solving Problem (3.4).

3.3 General Correlated Case

We now address the general case in the presence of correlation, when the coefficients of the square root of the correlation matrix can be positive or negative.

Theorem 3.3.1 (Optimal wealth and allocation)

(i) The optimal wealth in the robust portfolio management problem (3.1) is: $w_0 \exp(F(\Gamma))$, *where F is the function defined by:*

$$
\begin{aligned}
F(\Gamma) = \max_{\eta, \chi, \xi} \quad & \sum_{i=1}^{n} \chi_i \ln k_i - \eta\,\Gamma - \sum_{i=1}^{n} \xi_i \\
\text{s.t.} \quad & \eta + \xi_i - \sqrt{T}\,c \left(\sum_{j=1}^{n} Q_{ij}^{1/2}\,\chi_j \right) \geq 0, \ \forall i, \\
& \eta + \xi_i + \sqrt{T}\,c \left(\sum_{j=1}^{n} Q_{ij}^{1/2}\,\chi_j \right) \geq 0, \ \forall i, \\
& \sum_{i=1}^{n} \chi_i = 1, \\
& \eta \geq 0, \ \chi_i, \ \xi_i \geq 0, \ \forall i.
\end{aligned}
\tag{3.5}
$$

(ii) The optimal amount of money invested at time 0 in stock i is $\chi_i w_0$, for all i, where the χ_i are found by solving the linear programming problem (3.5).

Chapter 4

Numerical Experiments

The purpose of this section is to compare the proposed Log-robust approach with the robust optimization approach that has been traditionally implemented in portfolio management. The traditional robust approach when the stock prices belong to polyhedral uncertainty sets is due to Bertsimas and Sim (2004). The presence of correlation in real-life data requires extending their formulation to incorporate this case; the mathematical details are straightforward and left to the reader. The traditional framework, using the notations introduced at the beginning of the paper, is:

$$
\begin{aligned}
\max_{x,p,q,r} \quad & \sum_{i=1}^{n} x_i \exp\left[\left(\mu_i - \frac{\sigma_i^2}{2}\right) T\right] E\left[\exp\left(\sum_{j=1}^{n} Q_{ij}^{1/2} Z_j\right)\right] - \Gamma p - \sum_{i=1}^{n} q_i \\
\text{s.t.} \quad & \sum_{i=1}^{n} x_i = w_0, \\
& p + q_i \geq c\, r_i, \ \forall i, \\
& -r_i \leq \sum_{k=1}^{n} M_{ki}^{1/2} x_k \leq r_i, \ \forall i, \\
& p,\, q_i,\, r_i,\, x_i \geq 0, \ \forall i,
\end{aligned}
$$

with $M^{1/2}$ the square root of the covariance matrix of $\exp\left[\left(\mu_i - \frac{\sigma_i^2}{2}\right) T + \sqrt{T}\left(\sum_{j=1}^{n} Q_{ij}^{1/2} Z_j\right)\right]$.

In both the traditional robust and the Log-robust models, we downloaded six months' worth of daily stock price data for 50 stocks from Yahoo! Finance, computed the drift parameters and covariance matrix Q based on the continuously compounded rates of return $\ln(S_t/S_{t-1})$ and generated 1,000 scenarios for the stock prices six months from now. In the traditional model, we then used these 1,000 scenarios to compute $M^{1/2}$.

Analysis of optimal solution.

Figure 4-1 studies the level of diversification achieved in both models by showing the number of stocks invested in as a function of Γ. A key observation we make is that,

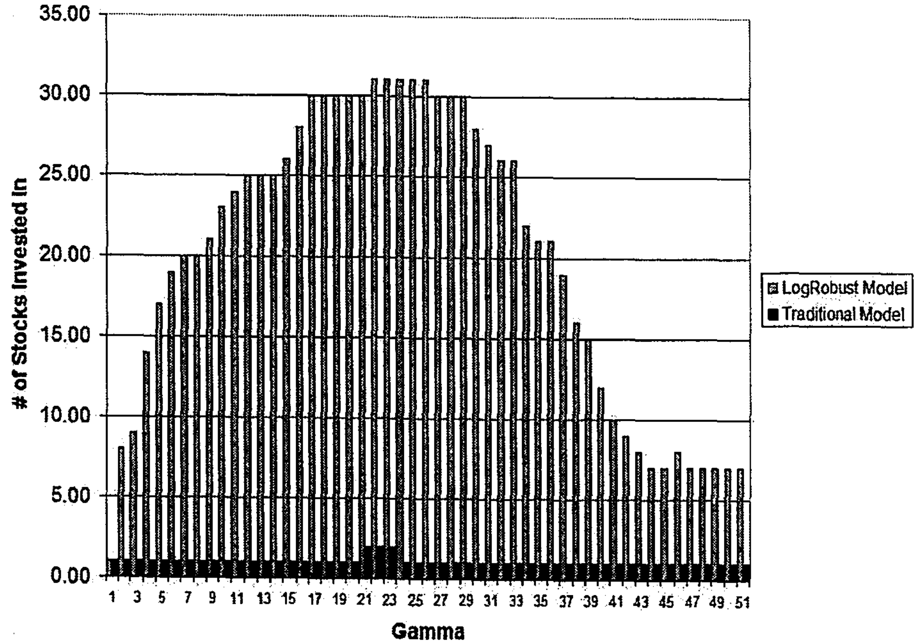

Figure 4-1: Number of stocks in optimal portfolio for Γ varying from 0 to 50, in the Traditional-Robust and Log-robust models.

while the numerical example in Bertsimas and Sim (2004) with artificial data (taken from Ben-Tal and Nemirovski (1999)) suggested that the robust approach would lead to a diversified portfolio for a wide range of budgets Γ, the results in Bertsimas and Sim (2004) appear to have been driven by the specific numerical values for the range forecasts of the stock returns (with tiny changes in mean and standard deviation from one stock to the next) and are not replicated with the real-life data we have considered. In our example, the traditional robust portfolio uses at most two stocks (and in general only one, as in the deterministic case). In contrast, the Log-robust model provides the manager with a diversified portfolio, with the number of stocks invested in increasing from 1 in the deterministic case ($\Gamma = 0$) to 31 for Γ between 21 and 25, and then decreasing steadily to 7 in the most conservative case of $\Gamma = 50$.

Figure 4-2 shows the number of shares bought in each of the stocks invested in either for $\Gamma = 10$ or $\Gamma = 20$, ranked in decreasing number of shares for $\Gamma = 10$. (Stocks that are not invested in in either case are not shown.) In the deterministic model and the Traditional-Robust model, the manager only invests in Air Products and Chemicals, Inc. (APD). Figure 4-2 indicates in particular that the number of shares bought in each stock is often quite substantial, so the diversification effect observed in Figure 4-1 is not due to the manager buying just one or two shares of more stocks; in other words, we achieve genuine diversification.

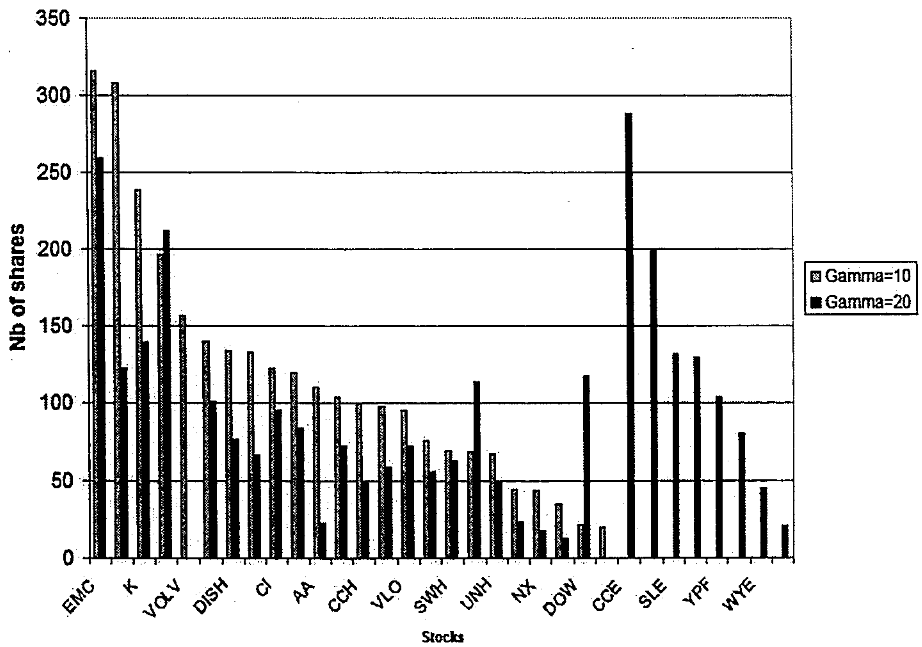

Figure 4-2: Number of shares in optimal portfolio for $\Gamma = 10$ and $\Gamma = 20$, for various stocks, in the Log-robust model.

Analysis of performance in simulations.

Since the goal of the proposed methodology is to protect against downside risk, we pay particular attention to the 99% and 95% Value-at-Risk of the portfolio in the Traditional-Robust and the Log-Robust models. We gather data on the other percentiles as well, to study under which circumstances the framework proposed here outperforms the traditional one. The simulations were performed using @Risk 4.5 from Palisade Corporation. We consider two cases: (i) the case where the random variables do obey a Normal distribution, and the only mistake made by the manager implementing the traditional robust approach is that he uses symmetric confidence intervals for the stock prices rather than for the true drivers of uncertainty, their continuously compounded rates of change, (ii) the case where the random variables have "fat tails", as has been observed in practice (see Hull (2002)), in which case the Lognormal model of stock price behavior underestimates rare events. This happens for instance when the scaled random variables Z obey a Logistic distribution. To calibrate the distribution we selected the same 95% confidence interval as that given by the Gaussian model with mean 0 and standard deviation 1, to keep the same range forecasts throughout. (The methodology was tested for other distributions as well and yielded similar results.)

Recall that the 99% and 95% Value-at-Risk are the 1% and the 5% percentiles of the portfolio wealth. For instance, the 99% VaR is the number such that there is only a 0.01 probability that the portfolio value will fall below that number. The decision-maker naturally wants these worst-case portfolio values to be as large as possible, so that investors remain wealthy even under adverse market conditions. In particular, 99% and 95% VaR are *risk-adjusted performance measures*, not risk measures, and should be maximized, not minimized.

Normal distribution.

Table 4.1 keeps track of the 99% Value-at-Risk for values of Γ varying from 0 to 50 in increments of 5, in the traditional and the Log-robust models; the last column shows the relative gain in 99% VaR when the manager implements the Log-robust approach. The values are obtained using 10,000 replications. We observe that 99% VaR decreases steadily as the level of conservatism (measured by the budget Γ) increases, so that the relative gain from using the Log-robust approach decreases from about 52% to about 33%. (Recall that for $\Gamma = 0$, both frameworks yield the deterministic model.) Because the stocks are correlated, the uncertain parameter z_i affects not only the stock price of asset i, but also the stock prices of the other assets. This is why, although the decision-maker invests in at most 31 stocks, the VaR keeps decreasing – instead of becoming

Γ	Traditional	Log-Robust	Relative Gain
5	70958.81	107828.94	51.96%
10	70958.81	104829.93	47.73%
15	70958.81	102502.79	44.45%
20	70958.81	101707.00	43.33%
25	70958.81	100905.96	42.40%
30	70958.81	101763.58	43.41%
35	70958.81	98445.23	38.74%
40	70958.81	96120.18	35.46%
45	70958.81	94253.62	32.83%
50	70958.81	94032.09	32.52%

Table 4.1: 99% VaR as a function of Γ for Gaussian distribution.

constant – for values of Γ greater than 31. Bertsimas and Sim (2004) have suggested selecting a value for the budget of uncertainty of the order of $\sqrt{n}$ (about 7 here) with n the number of uncertainty drivers, and Table 4.1 suggests that values of Γ in the 5-10 range are precisely those that maximize the benefit of using the Log-robust approach, at least for the 99% VaR. We investigate this point further in Figure 4-3, which shows the relative gain of the Log-robust approach for percentiles of the portfolio value between 5 and 95%, in increments of 5, and Γ between 0 and 50, in increments of 5. (Negative relative gains indicated that the traditional robust approach performs better.)

We observe that the relative performance of the Log-robust approach decreases as Γ increases and as the percentile increases; up to the 10th percentile (90% VaR), the Log-Robust model outperforms the traditional approach for any value of Γ. The Log-robust approach performs best for a risk-averse decision-maker (focusing on 99% or 95% VaR) and for moderate values of Γ (about 5 or 10).

Logistic distribution.

We now consider the more realistic case where the distribution has "fat tails," i.e., the Gaussian assumption underlying the Lognormal model underestimates the risk of extreme events. Table 4.2 keeps track of the 99% Value-at-Risk for values of Γ varying from 0 to 50 in increments of 5, in the traditional and the Log-robust models, when the scaled random variables Z obey Logistic distributions with 95% confidence intervals $[-c, c]$; as in Table 4.1, the last column shows the relative gain in 99% VaR when the

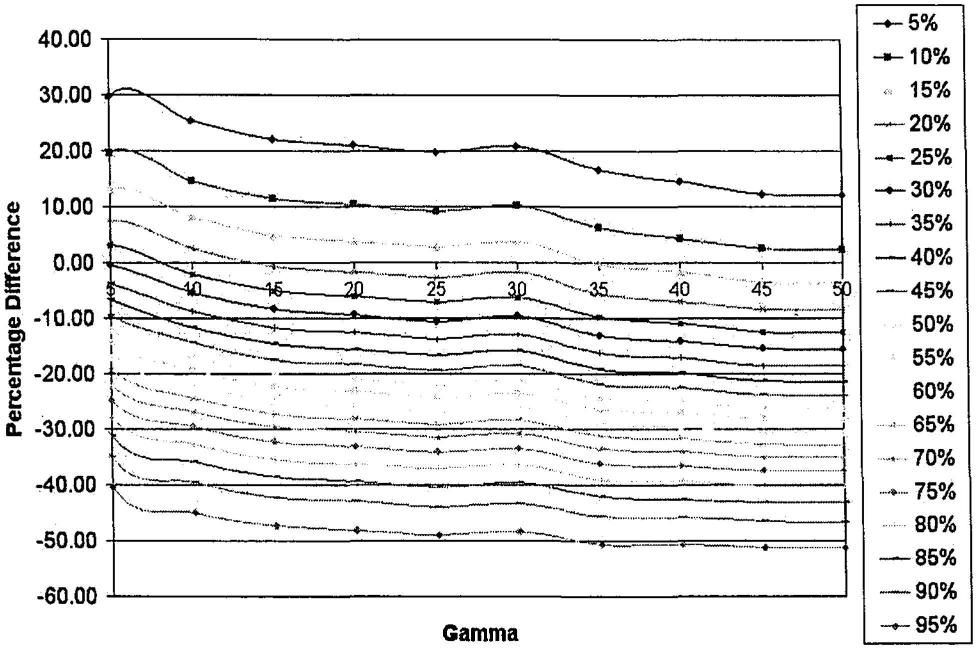

Figure 4-3: Relative gain of the Log-robust model compared to the Traditional robust model, for percentiles from 5% to 95% and Γ from 0 to 50, in the Gaussian case.

manager implements the Log-robust approach. As in the Gaussian case, the relative

Γ	Traditional	Log-Robust	Relative Gain
5	68415.97	108234.32	58.20%
10	68415.97	105146.66	53.69%
15	68415.97	102961.66	50.49%
20	68415.97	102124.75	49.27%
25	68415.97	101294.347	48.06%
30	68415.97	102206.73	49.39%
35	68415.97	98508.69	43.98%
40	68415.97	95940.01	40.23%
45	68415.97	93841.05	37.16%
50	68415.97	93562.59	36.76%

Table 4.2: 99% VaR as a function of Γ for Logistic distribution.

performance of the Log-robust approach decreases as Γ increases and as the percentile increases; up to the 10th percentile (90% VaR), the Log-Robust model outperforms the traditional approach for any value of Γ. The Log-robust approach performs best for a risk-averse decision-maker (focusing on 99% or 95% VaR) and for moderate values of Γ (about 5 or 10). This is shown in Figure 4-4. The changes compared to Figure 4-3 are minor; for instance, the relative gain for $\Gamma = 5$, considering the 20th percentile, has changed from 7.41% (Gaussian case) to 6.05% (Logistic case).

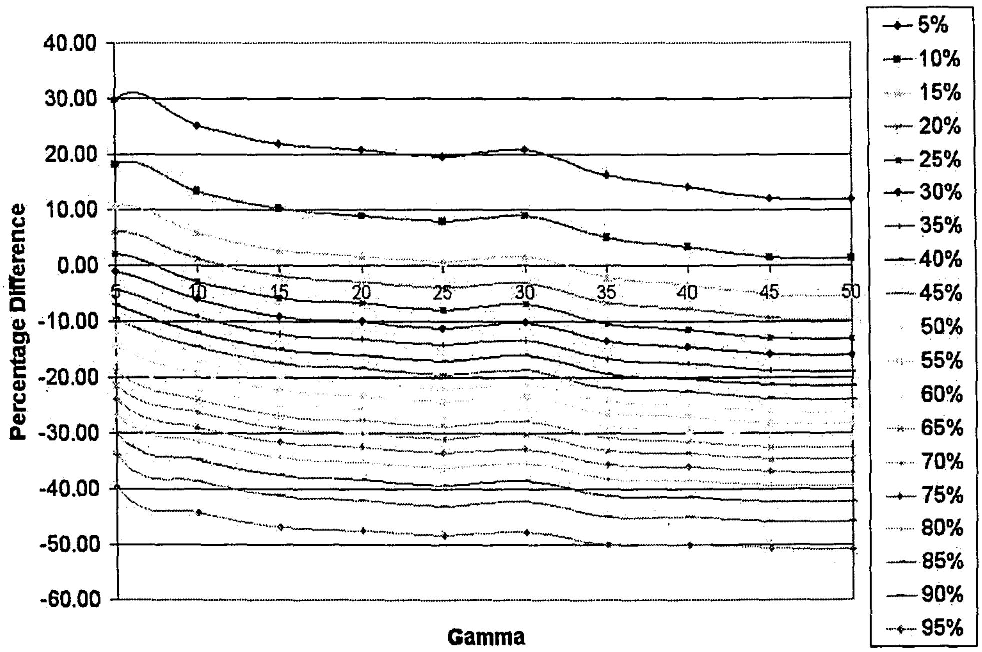

Figure 4-4: Relative gain of the Log-robust model compared to the Traditional robust model, for percentiles from 5% to 95% and Γ from 0 to 50, in the Logistic case.

Conclusions of Experiments.

Our numerical results indicate that incorporating robustness at the level of the true uncertainty driver, the continuously compounded rate of return, results in better performance for the risk-averse manager maximizing his 99% VaR (or 95% or 90% VaR). They also suggest that the budget of uncertainty should be of the order of the square root of the random variables to optimize the performance of the approach. This is in line with rules of thumb available in the literature.

Chapter 5

Conclusions and Future Work

5.1 Conclusions

We have proposed a robust optimization approach to portfolio management, where robustness is incorporated in the continuously compounded rates of return of the stock prices rather than in the prices themselves. This departure from the traditional robust framework aligns our model with the finance literature without requiring the mathematically convenient assumption of stock prices following a Lognormal process, which has been shown to underestimate extreme events in practice. We have obtained a robust formulation that is linear and thus can be solved efficiently, and have derived theoretical insights into the worst-case uncertainty and the optimal number of shares to buy of each stock. In numerical experiments when the decision-maker maximizes his 95% or 99% Value-at-Risk, the Log-robust approach outperforms the traditional robust optimization approach by double-digit margins, with an even more significant gain if the budget of uncertainty is well-chosen (about the square root of the number of stocks). This is because the traditional robust optimization approach does not achieve diversification for real-life financial data. Hence, we believe the Log-robust approach holds much potential in portfolio management under uncertainty.

5.2 Future Work

In future work, we plan to extend the model to the case with short sales. We also intend to consider the case with derivatives.

References

Al Najjab, M., A. Thiele. (2007) The Log-Logistic option pricing model. Technical Report, Lehigh University, Department of Industrial and Systems Engineering, Bethlehem, PA.

Ben-Tal, A., A. Nemirovski. (1999) Robust solutions to uncertain programs. *Oper. Res. Letters*, 25, 1-13.

Ben-Tal, A., S. Boyd, A. Nemirovski. (2006) Extending Scope of Robust Optimization: Comprehensive Robust Counterparts of Uncertain Problems. *Math. Prog. B,* 107, 63-89.

Bertsekas, D. (1999) *Nonlinear Programming* 2nd ed. Athena Scientific. Belmont, MA.

Bertsimas, D., D. Pachamanova. (2008) Robust Multiperiod Portfolio Management with Transaction Costs. *Computers and Operations Research*, 35(1), 3-17.

Bertsimas, D., M. Sim. (2004) The Price of Robustness. *Operations Research*, 52(1): 35-53.

Bertsimas, D., A. Thiele. (2006) Robust and data-driven optimization: modern decision-making under uncertainty, *INFORMS Tutorials in Operations Research,* Chapter 4.

Black, F., M. Scholes. (1973) The pricing of options and corporate liabilities. *Journal of Political Economy,* 81: 637-59.

Blattberg, R., N. Gonedes. (1974) A Comparison of the Stable and Student Distributions as Statistical Models for Stock Prices. *Journal of Business,* 47, 244-80.

Boyd, S., L. Vandenberghe. (2004) *Convex Optimization* Cambridge University Press. Cambridge, UK.

Chopra, V., W. Ziemba. (1993) The effect of errors in means, variances, and covariances on optimal portfolio choice. *Journal of Portfolio Management*, 19, 6-11.

Cont, R. (2001) Empirical Properties of Asset Returns – Stylized Facts and Statistical Issues. *Quantitative Finance,* 223-236.

Fabozzi, F., P. Kolm, D. Pachamanova, S. Focardi. (2007) *Robust Portfolio Optimization and Management.* Wiley.

Fama, E. (1965) The Behavior of Stock Prices. *Journal of Business,* 38, 34-105.

Erdogan, E., D. Goldfarb, G. Iyengar. (2004) Robust Portfolio Management. Technical Report, Department of Industrial Engineering and Operations Research, Columbia University, New York, NY.

Goldfarb, D., G. Iyengar. (2003) Robust Portfolio Selection Problems. *Mathematics of Operations Research,* 28, 1-38.

Hull, J. C. (2002) *Options, Futures and Other Derivatives.* 5th edition. Prentice Hall. Upper Saddle River, NJ.

Jansen, D., C. deVries. (1991) On the Frequency of Large Stock Returns – Putting Booms and Busts into Perspective. *The Review of Economics and Statistics,* 73(1), 18-24.

Kon, S. (1984) Models of Stock Returns – A Comparison. *Journal of Finance,* 39, 147-65.

Markowitz, H. M. (1959) *Portfolio selection: efficient diversification of investments.* Wiley, New York.

Pachamanova, D. (2006) Handling Parameter Uncertainty in Portfolio Risk Minimization. *Journal of Portfolio Management,* 32(4), 70-78.

Richardson, M., T. Smith. (1993) A Test for Multivariate Normality in Stock Returns. *Journal of Business,* 66, 295-321.

Appendix A

Proofs

A.1 Proofs of Chapter 2

A.1.1 Proof of Lemma 2.3.1

Because we do not allow short sales, the coefficient in front of the exponential is non-negative and the exponential is minimized for the smallest value of its argument. □

A.1.2 Proof of Lemma 2.3.2

(i) Problem (2.5) is an unconstrained convex optimization problem, and as such its optimal solution is found by setting the gradient of the objective to zero. (ii) follows from complementary slackness applied to Problem (2.4). □

A.1.3 Proof of Theorem 2.4.1

Injecting Lemma 2.3.2 into Equation (2.5), and using strong duality in convex programming with Slater's condition (see Bertsekas (1999)), we obtain that the robust portfolio management problem (2.2) is equivalent to:

$$\begin{aligned}
\max_{x,\alpha,\lambda^0,\lambda^1} \quad & \sum_{i=1}^{n} \left(\frac{\alpha - \lambda_i^0 + \lambda_i^1}{\sigma_i \sqrt{T}\, c} \right) \cdot \left[1 + \ln \left(\frac{x_i\, k_i\, \sigma_i \sqrt{T}\, c}{\alpha - \lambda_i^0 + \lambda_i^1} \right) \right] - \alpha\, \Gamma - \sum_{i=1}^{n} \lambda_i^1 \\
\text{s.t.} \quad & \sum_{i=1}^{n} x_i = w_0, \\
& \alpha \geq 0,\ \lambda_i^0,\ \lambda_i^1,\ x_i \geq 0,\ \forall i,
\end{aligned}$$

or alternatively, using the change of variable: $\beta_i = \dfrac{\alpha - \lambda_i^0 + \lambda_i^1}{\sigma_i \sqrt{T}\, c}$, which must be nonnegative due to the term in log:

$$\begin{aligned} \max_{x,\,\alpha,\,\beta,\,\lambda^1} \quad & \sum_{i=1}^{n} \beta_i \cdot \left[1 + \ln\left(\frac{x_i\, k_i}{\beta_i}\right)\right] - \alpha\,\Gamma - \sum_{i=1}^{n} \lambda_i^1 \\ \text{s.t.} \quad & \sum_{i=1}^{n} x_i = w_0, \\ & \alpha + \lambda_i^1 - \sigma_i \sqrt{T}\, c\, \beta_i \geq 0, \;\; \forall i, \\ & \alpha \geq 0, \; \beta_i, \; \lambda_i^1, \; x_i \geq 0, \;\; \forall i. \end{aligned} \tag{A.1}$$

We solve Problem (A.1) by first maximizing over the x_i and then over the remaining variables. The maximizing problem in the x_i can be formulated as:

$$\begin{aligned} \max_{x} \quad & \sum_{i=1}^{n} \beta_i \cdot \ln x_i \\ \text{s.t.} \quad & \sum_{i=1}^{n} x_i = w_0, \\ & x_i \geq 0, \;\; \forall i. \end{aligned} \tag{A.2}$$

Problem (A.2) is a convex optimization problem, which we solve using a Lagrange approach, obtaining $x_i = \frac{\beta_i\, w_0}{\sum_{j=1}^{n} \beta_j}$. (Note that this means that x_i and β_i are both zero or both positive, for each i.) We reinject the optimal asset allocation into Problem (A.1) and now have to solve:

$$\begin{aligned} \max_{\alpha,\,\beta,\,\lambda^1} \quad & \sum_{i=1}^{n} \beta_i \cdot \left[1 + \ln\left(\frac{w_0\, k_i}{\sum_{j=1}^{n} \beta_j}\right)\right] - \alpha\,\Gamma - \sum_{i=1}^{n} \lambda_i^1 \\ \text{s.t.} \quad & \alpha + \lambda_i^1 - \sigma_i \sqrt{T}\, c\, \beta_i \geq 0, \;\; \forall i, \\ & \alpha \geq 0, \; \beta_i, \; \lambda_i^1 \geq 0, \;\; \forall i. \end{aligned} \tag{A.3}$$

Because the right-hand side of the feasible set of Problem (A.3) is zero, we can parametrize over $\theta \geq 0$ where $\sum_{j=1}^{n} \beta_j = \theta$ (note that θ must be nonnegative for the logarithm to be defined) and scale the decision variables by $1/\theta$. Problem (A.3) becomes:

$$\begin{aligned}
\max_{\theta}\ \theta \cdot \max_{\eta,\chi,\xi}\ & \sum_{i=1}^{n} \chi_i \cdot \left[1+\ln\left(\frac{w_0\, k_i}{\theta}\right)\right] - \eta\,\Gamma - \sum_{i=1}^{n} \xi_i \\
\text{s.t.}\ & \eta + \xi_i - \sigma_i \sqrt{T}\, c\, \chi_i \geq 0,\ \forall i, \\
& \sum_{i=1}^{n} \chi_i = 1, \\
& \eta \geq 0,\ \chi_i,\ \xi_i \geq 0,\ \forall i.
\end{aligned}$$

(Note that, with these new notations, $x_i = \chi_i w_0$ for all i.) We then regroup the terms depending on θ and use $\sum_{i=1}^{n} \chi_i = 1$ to reformulate the robust optimization problem as:

$$\max_{\theta \geq 0}\ \left[1 + F(\Gamma) + \ln\left(\frac{w_0}{\theta}\right)\right] \theta, \tag{A.4}$$

where F is defined by Equation (2.8). The objective in Problem (A.4) is concave, as is easily checked by computing the second derivative, and the optimal value of θ follows by setting the first derivative to zero. This yields: $\theta = w_0 \exp(F(\Gamma))$. Reinjecting into the objective leads to an optimal wealth of $w_0 \exp(F(\Gamma))$. □

A.1.4 Proof of Lemma 2.4.2

Recall that, in this lemma, we assume that the optimal scaled deviations are already known to gain further insights into the optimal structure of the problem. We inject Equation (2.6) into Problem (2.2) and use the three cases identified in Lemma 2.3.2 (ii) to separate the objective function into three groups as follows:

$$\max_{x}\ \sum_{i|z_i^*=0} x_i\, k_i + \sum_{i|z_i^*=1} x_i\, k_i\, \exp(-\sigma_i \sqrt{T} c) + \sum_{i|0<z_i^*<1} \frac{\alpha}{\sigma_i \sqrt{T} c} \tag{A.5}$$

To find α we use that $\Gamma = \sum_{i=1}^{n} z_i^*$ and also separate the right-hand side of that equation into three groups:

$$\begin{aligned}
\Gamma &= \sum_{i=1}^{n} \frac{1}{\sigma_i \sqrt{T} c} \left[\ln\left(\frac{x_i\, k_i\, \sigma_i\, \sqrt{T} c}{\alpha - \lambda_i^0 + \lambda_i^1}\right)\right] \\
&= \sum_{i|z_i^*=0} 0 + \sum_{i|z_i^*=1} 1 + \sum_{i|0<z_i^*<1} \frac{1}{\sigma_i \sqrt{T} c} \left[\ln\left(\frac{x_i\, k_i\, \sigma_i \sqrt{T} c}{\alpha}\right)\right]
\end{aligned}$$

$$= \sum_{i|z_i^*=1} 1 + \sum_{i|0<z_i^*<1} \frac{1}{\sigma_i\sqrt{T}c} \ln(x_i\, k_i\, \sigma_i\sqrt{T}c) - \ln\alpha \cdot \sum_{i|0<z_i^*<1} \frac{1}{\sigma_i\sqrt{T}c}.$$

This yields:

$$\alpha = \exp\left[\frac{\left(\sum_{i|z_i^*=1} 1 - \Gamma\right)}{R}\right] \cdot \left[\prod_{i|0<z_i^*<1} (x_i\, k_i \sigma_i \sqrt{T}c)^{\frac{1}{\sigma_i\sqrt{T}c}}\right]^{\frac{1}{R}}, \tag{A.6}$$

where $R = \sum_{i|0<z_i^*<1} \frac{1}{\sigma_i\sqrt{T}c}$. We then inject Equation (A.6) into Problem (A.5) to obtain Problem (2.9), which is convex because the geometric mean is a concave function of its arguments. □

A.1.5 Proof of Theorem 2.4.3

To find the optimal value of the x_i's in Problem (2.9), we invoke the convexity of the problem and introduce the Lagrangian multipliers $\nu_i \geq 0$ and δ; the model becomes:

$$\max_x\ \alpha(x)R + \sum_{i|z_i^*=0} x_i\, k_i + \sum_{i|z_i^*=1} x_i k_i \exp(-\sigma_i\sqrt{T}c) - \delta\left(\sum_{i=1}^n x_i - w_0\right) + \sum_{i=1}^n \nu_i x_i.$$

We set the gradient of the objective to zero to find the optimum value of the x_i's, which depends on the z_i's as follows:

(a) For all i such that $z_i^* = 0$, we have: $k_i = \delta - \nu_i$.

(b) For all i such that $z_i^* = 1$, we have: $k_i \exp(-\sigma_i\sqrt{T}c) = \delta - \nu_i$.

(c) For all i such that $0 < z_i^* < 1$, we have:

$$\left(\exp\left[\sum_{j|z_j^*=1} 1 - \Gamma\right]\right)^{\frac{1}{R}} \cdot \left(\prod_{j|0<z_j^*<1} (x_j k_j \sigma_j \sqrt{T}c)^{\frac{1}{\sigma_j\sqrt{T}c}}\right)^{\frac{1}{R}} \cdot (x_i\, \sigma_i\, \sqrt{T}c)^{-1} = \delta - \nu_i. \tag{A.7}$$

By complementarity slackness, we know that if we have $x_i > 0$ for some i, then $\nu_i = 0$. Furthermore, it follows from (a) and (b) (injecting the fact that $k_i = \exp\left[(\mu_i - \frac{\sigma_i^2}{2})T\right]$ for all i) that $\delta = \exp\left[(\mu_i - \frac{\sigma_i^2}{2})T\right]$ for all i such that $x_i > 0$ and $z_i = 0$, and $\delta =$

$\exp\left[(\mu_i - \frac{\sigma_i^2}{2})T\right] \cdot \exp(-\sigma_i\sqrt{T}c)$ for all i such that $x_i > 0$ and $z_i = 1$. Because the $\exp\left[(\mu_i - \frac{\sigma_i^2}{2})T\right]$ and $\exp\left[(\mu_i - \frac{\sigma_i^2}{2})T\right] \cdot \exp(-\sigma_i\sqrt{T}c)$ are all distinct by assumption, δ can only be determined by at most one such equation and there can be at most one asset i the manager invests in ($x_i > 0$) for which the worst-case deviation hits the bounds. This proves (i).

If no z_i hits the bounds then Equation (A.7) can be written under the form, for $x_i > 0$:

$$x_i = \frac{K}{\delta\,\sigma_i\,\sqrt{T}\,c},$$

where K/δ is determined by the constraint: $\sum_{i=1}^{n} x_i = w_0$. This leads to Equation (2.10), proving (ii). □

A.2 Proofs of Chapter 3

A.2.1 Proof of Lemma 3.1.1

The feasible set is convex (all the constraints are less-than-or-equal to constraints with convex functions in the left-hand side) and the objective is the weighted sum with nonnegative coefficients of the composition of convex functions with affine functions of the decision variables (Boyd and Vandenberghe (2004)). □

A.2.2 Proof of Lemma 3.2.1

Follows immediately from solving the Lagrange relaxation of Problem (3.3) and invoking strong duality in convex optimization, since Slater's condition is satisfied. □

A.2.3 Proof of Theorem 3.2.2

Similar to that of Theorem 2.4.1. □

A.2.4 Proof of Theorem 3.3.1

The proof is similar to that of Theorem 2.4.1 and we only sketch the main ideas. We use the transformation: $\tilde{y}_j = y_j^+ - y_j^-$ (and hence, $|\tilde{y}_j| = y_j^+ + y_j^-$), which does not change the optimal objective because Problem (3.2) is convex. This yields, for the inner minimization problem:

$$\begin{aligned}
\min_{y^+, y^-} \quad & \sum_{i=1}^{n} x_i \, k_i \, \exp\left[\sqrt{T}c \sum_{j=1}^{n} Q_{ij}^{1/2} \, (y_j^+ - y_j^-)\right] \\
\text{s.t.} \quad & \sum_{j=1}^{n} (y_j^+ + y_j^-) \leq \Gamma, \\
& y_j^+ + y_j^- \leq 1, \; \forall j, \\
& y_j^-, y_j^+ \geq 0, \; \forall j.
\end{aligned} \tag{A.8}$$

We solve the convex optimization problem (A.8) using a Lagrange approach, with Lagrange multipliers α, λ_i^1, λ_i^{-0} and λ_i^{+0} for all i. Setting the gradient to zero yields:

$$2(\alpha + \lambda_j^1) - \lambda_j^{+0} - \lambda_j^{-0} = 0, \; \forall j,$$

and:

$$y_j^+ - y_j^- = \frac{1}{\sqrt{T}c} \sum_{i=1}^{n} Q_{ij}^{-1/2} \ln\left(\frac{[\mathbf{Q}^{1/2}(\alpha \mathbf{e} + \boldsymbol{\lambda}^1 - \boldsymbol{\lambda}^{-0})]_i}{\sqrt{T}c \, x_i \, k_i}\right), \; \forall j.$$

Using that $\lambda_j^{+0} = 2(\alpha + \lambda_j^1) - \lambda_j^{-0}$, introducing the change of variables: $\sqrt{T}c\boldsymbol{\beta} = \mathbf{Q}^{-1/2}(\alpha\mathbf{e} + \boldsymbol{\lambda}^1 - \boldsymbol{\lambda}^{-0})$, injecting the nonnegativity of λ_j^{-0} and λ_j^{+0} for all j and scaling by $\sum_{j=1}^{n} \beta_j$ yields the desired result, similarly to the proof of Theorem 2.4.1. □

Appendix B

Short Biography of the Candidate

Ban Kawas was born in 1983 in Amman, Jordan. She received her Bachelor of Science in Industrial Engineering from the University of Jordan in 2005. She worked as an engineer for a year before joining Lehigh University in 2006. She plans to continue her studies in the PhD program in Industrial Engineering at Lehigh University.

END OF TITLE