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A Continuous-Time Model for the Valuation of Mortgage-Backed Securities

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A Continuous-Time Model for the Valuation of Mortgage-Backed Securities

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Abstract

A mortgage model consists of three basic parts: the amortization model which examines the mortgage cash flows, the interest rate model which affects the mortgage price, and the prepayment model which measures the rates of mortgage termination when a property is sold, refinanced or foreclosed. A technique known as eigenfunction expansion has proven to be useful in pricing continuous-time mortgages.

The first part of this dissertation involves generalizing the existing mortgage model by analyzing the Cox-Ingersoll-Ross interest-rate model and including as an alternative the simpler Vasicek model and then comparing the results obtained by these methods. We also refine the relationship between interest rates and prepayments to reflect empirical data more accurately, particularly in low-interest rate scenarios by expanding the existing single-threshold prepayment model to include a secondary prepayment threshold.

The second problem expands the existing continuous prepayment model to include mortgage defaults. We use the default model to examine the price sensitivity of mortgages to loss severity and foreclosure rates. We also examine two practical applications of this model: accounting for wider spreads between mortgage yields and treasury yields during periods of economic stress, and estimating the value of the mortgage guarantee that government agencies such as Ginnie Mae provide to investors of mortgage-backed securities.

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Chapter 1. Introduction

In a recent paper (Kolbe and Zagst 2009) modeled prices of Ginnie Mae mortgage securities by modifying the prepayment model in two ways: Their endogenous prepayment model is a piecewise linear approximation to the S-curve relationship to interest rates, and their exogenous (baseline) prepayment model is a complex stochastic two-factor model; the first factor is a random noise component and the second factor is based on the GDP growth rate.

While exogenous prepayment rates are in reality stochastic, we believe the deterministic industry-standard models (PSA and CPR) are adequate to describe the exogenous prepayment rate.   Furthermore these industry-standard models are simpler. (Gorovoy and Linetsky 2007) use these standard models in both discrete and continuous forms in their eigenfunction expansion model; however, Kolbe and Zagst were unable to compare their model to that of Gorovoy and Linetsky due to numerical problems.   We were able to overcome these numerical problems and will demonstrate that applying an approach similar to that of Gorovoy and Linetsky actually produces superior results to Kolbe and Zagst over the period 1996-2006 without the additional complexity. Furthermore, we will show that the eigenfunction expansion approach is superior even if we select a different interest rate model.   We will also add our own endogenous piecewise linear prepayment model to further improve the results.   Finally, we will introduce a default model to see how various default assumptions could affect prices of mortgage-backed securities.   We will then demonstrate two applications of the default

model – one application is an alternative pricing model during economic crisis periods; the second application is to determine the value of loan guarantees.

Home ownership is synonymous with the American dream. Some of the most basic questions one asks when buying a house are: "How much house can I afford?", and "What is my monthly payment?" After one buys a house, a frequent question is: "What is the balance of my mortgage?" and "What is my house worth?" The answer to most of these questions requires that one knows what interest rate one is paying on the mortgage.

But how is that interest rate determined? Calculating the interest rate for a mortgage is more complex than for other fixed-income investments because the borrower not only pays down the balance over time, but he or she has the option to prepay the mortgage at any time. Furthermore the borrower also has the option to surrender the property to the bank or lender and walk away owing nothing. The lender usually absorbs a loss in this case, because the value of the underlying collateral minus foreclosure costs is usually less than the balance of the mortgage. (If this were not true, the borrower would simply sell the house at a profit rather than walk away and get nothing.)

Prepayments can occur for various reasons. There are three types of prepayments, exogenous prepayments, endogenous prepayments, and curtailments. Exogenous prepayments occur due to the sale of the house. Endogenous prepayments occur when the owner refinances to get a lower mortgage rate. Curtailments result when the borrower pays a little bit extra each month over and above the required mortgage payment. Prepayments can have a dramatic effect on the price of a mortgage because they affect the timing of the cash flows.

Another type of prepayment is a mortgage default. This occurs when the homeowner stops paying on the mortgage; the property goes into foreclosure until the bank can sell it, usually at a loss. In the past, defaults were ignored when mortgage were issued by quasi-government agencies such as Fannie Mae and Freddie Mac, because these agencies absorbed the losses and did not directly pass them on to investors. However, in 2008, Fannie Mae and Freddie Mac suffered huge losses due the mortgage crisis and required a large infusion of government cash. Thus modeling mortgage defaults has been shown to be extremely important for obtaining the fair mortgage rate.

Recent research has applied continuous-time models to mortgage valuation. There are three areas where these models are limited. First, they assume a particular interest rate model; second, the theoretical relationships between prepayments and current interest rates do not always agree with empirical data, and third, many models do not include defaults.

In this dissertation the eigenfunction expansion mortgage valuation model will be generalized to accept a different interest rate models such as Vasicek. This in turn will show what effects the choice of interest rate model has on predicted mortgage rates. The current model will also be improved by relating interest rates to prepayments to better reflect empirical data. This should permit us to model the effects of the very low interest rates that we are currently experiencing. Finally, the industry-standard default models will be embedded in the eigenfunction expansion model. This will help to model the recent effects of sub-prime mortgages and the recent drop in home prices by incorporating potential losses into price and yield calculations.

Chapter 2. Background and Motivation

The conventional 30-year mortgage with a 20% down payment became popular after the creation of the Federal Housing Administration by FDR in 1934.   The Federal National Mortgage Association (FNMA or Fannie Mae) was established in 1938 to provide government guarantees to lenders thus opening home ownership to the middle class.   Prior to that, most mortgages required a 50% down payment.

After World War II, American Soldiers returning from the War obtained VA loans through the GI Bill of Rights which allowed them to purchase homes with little or no down payment.   This gave leverage to middle-class Americans and allowed them to partake in the American Dream. Ginnie Mae (GNMA) was established in 1968 to handle government-backed mortgages (VA and FHA).   Fannie Mae (FNMA) and Freddie Mac (FHLMC) were created by the government, but were privately held.

In 1977, Lewis S. Ranieri created securities from a pool of mortgages. These securities freed up a bank's capital thus permitting the bank to issue new mortgages. The securities could then be traded on Wall Street. [*Business Week*, July 7, 2008]

2.1 Agency Mortgage Backed Securities

Mortgages which are guaranteed by Freddie Mac, Fannie Mae and Ginnie Mae are called Agency Mortgages.   The agencies issue mortgage-backed securities (MBS) to investors; these securities are protected against default risk. In spite of this protection, MBS present two other types of risks—prepayment risk and interest rate risk. Prepayments occur unexpectedly resulting in cash flows coming in sooner than expected.

If a certain prepayment rate is anticipated and does not materialize, extension risk occurs when payments come in later than expected.   When interest rates go up, the present value of a mortgage-backed security goes down; on the other hand when interest rates drop, there is no upside to the lender because many borrowers will refinance.

Huge losses at Fannie Mae and Freddie Mac leading up to 2008 resulted in a federal takeover of these institutions.   This may have been due to the unintended consequences of the Community Reinvestment Act of 1977 encouraging banks to lend to low-income people according to the Wall Street Journal Editorial "Not Everyone Should Own a Home" (Oct 6, 2008).   Many of these loans required no down payment or no income verification from the borrowers. When the housing bubble burst, these sub-prime borrowers simply walked away from the property leaving the banks holding the bag.

2.2 Collateral

The collateral consists of commercial and residential properties, but other assets such as vehicles may also be included. The two major types of loans are fixed-rate and adjustable rate mortgages (ARM's).   In addition, there are also interest-only mortgages, graduated payment (GPM) mortgages which have increasing payments but a constant interest rate, growing equity mortgages (GEM's) which are similar to GPM's but do not negatively amortize, and hybrid mortgages which combine several of these types of mortgages into one package.   (Fabozzi and Ramsey, Mortgages and Overview of Mortgage-Backed Securities 1997) describe these types of mortgages in detail.

The current sub-prime mortgage crisis illustrates the problem with borrowers defaulting on their mortgages.   Low or no down-payment mortgages are risky

investments because the property value can drop below the mortgage balance. This gives the borrower the option of walking away from the property and letting the bank assume possession. The bank takes a loss, but the borrower only loses the small down payment (if any) and the (typically small) amount of principal paid in thus far. This is equivalent to a put option on the property since its value can't drop below a certain level. The problem results because the price of the mortgage does not include this embedded put option which is based on the recent volatility in home prices.

2.3 Structuring

Originally banks held mortgages until they were paid off, collecting the payments every month. To free up capital to allow banks to issue new mortgages, they began selling mortgages to investors through mortgage companies. The individual mortgages were then pooled and the resulting cash flows were packaged into securities. These are called pass-through securities (whole-loan). The cash flows can also be carved up into more complex securities (derivatives) known as Collateralized Mortgage Obligations (CMO's) which have varying risks and maturities. These can be sold to investors who would not otherwise be attracted to mortgage investments. The process of carving up the cash flows into separate bond classes is known as *structuring*. (Hayre, Mohebbi and Zimmerman 1997) describe these various structures in detail. The danger here was that the original issuers of the loans no longer had to worry about default; they simply pushed the risk onto the investors.

The basic type of structuring is *Sequential Pay* where the cash flows are divided up among several bond classes (Ames 1997). For example, the Class A bond's principal

is paid down completely, while the remaining bond classes collect interest only. When the bond A class is retired, the B class receives the entire principal payments until it is retired. This process continues until the last bond class is retired. An accrual bond (Class Z) receives no interest until all other bonds are retired; it negatively amortizes until it is ready to receive cash flows. Mortgages which tend to be long-term investments can now be divided up into investments with shorter maturities and thus attract investors with differing investment horizons.

Optimal structuring was first proposed by (Zenios 1993) using a sequential pay structure with an accrual bond using a mixed-integer program with thousands of decision variables and constraints; the problem was not practical to solve for this reason. (Medina, Riano and Villarreal 2007) use a dynamic programming approach to create an optimal sequential pay structure using data from Colombia's Fannie Mae counterpart.

Treasury securities are often split into principal-only and interest-only strips. In a similar way, mortgage-backed securities may also be split into these classes known as PO's and IO's. PO's are immune from prepayment risk, but suffer from extension risk, while IO's are immune from extension risk, but are subject to prepayment risk. (Y. Goncharov 2006) shows how PO and IO strips can be priced by substituting the principal- only or interest only cash-flows into his pricing formula instead of the total cash flow. He also shows how this could be applied to sequential-pay CMO's.

Issuers often create a structure consisting of a protected investment-grade bond class along with a companion bond which absorbs the shock of the risk (e.g. interest rate, prepayment or default risk). The companion bond is usually held by the issuer; however, it may be sold as a junk bond with a suitably high interest rate.

Floaters protect against interest rate risk with a floating rate; and their companion classes are known as inverse floaters. Planned Amortization Classes (PACs) protect the bondholder from both prepayment risk and extension risk. (Huang, et al. 2007) propose an optimal CMO design for this type of structure. Targeted Amortization Classes (TACs) provide protection against either prepayment risk or extension risk but not both.

Finally, a senior/subordinated structure may be created to protect against default risk. The senior bonds will collect all available principal until they are retired, while the subordinated classes absorb all principal losses. This type of structure only applies to Non-Agency MBS (Lundy and Higgins 1977). As a rule this only applies to jumbo (non-conforming) and commercial mortgages. The rating agencies often gave favorable ratings to the senior bonds even though the underlying collateral was sub-prime. When the mortgage meltdown occurred, many investors were upset because they had assumed they were better protected from losses.

The behavior of the underlying collateral influences how the various bond structures behave. In reality many deals combine sequential pay with other types of structures creating exceedingly complex instruments. Understanding how prepayments and defaults affect the underlying collateral and the resulting bond structures is critical to managing these investments. We will use prepayments and defaults to understand the whole loan CMO's. This will leave open a huge area of future research to look at how various structures react to this type of activity.

2.4 Recent Mortgage Models

Since mortgages have a prepayment option, most models in the past have used Monte Carlo simulation to price the mortgage (Bhattercharjee and Hayre 2006). Recent research has shown that interest–rate option theory can be used to evaluate mortgages. Mortgage rates are based on an underlying index, such as 10-year treasury rates or the LIBOR, a spread, plus adjustments including those related to the property type, loan-to-value, and the credit score of the borrower.

2.4.1 Discrete Models

Recent research involves the calculation of mortgage rates from current short rates, given the borrowers' tendencies to refinance. (Pliska 2004) uses a simple discrete model which combines dynamic programming with game theory where the borrower's decision strategy is either to refinance or continue, and the lender's strategy is to choose the optimal mortgage rate. The goal is to find the equilibrium point where neither borrower nor lender would change his strategy. Unfortunately, the discrete approach suffers from the "curse of dimensionality" because the number of paths becomes exceedingly large in any realistic scenario. Pliska's example uses only 5 periods and four distinct interest rates; this illustrates the point but it is not realistic. (Kariya, Pliska and Ushiyama 2002) incorporate housing prices into the prepayment model, assuming that rising housing prices contribute to either sale of the property, or refinancing, to unlock additional equity.

2.4.2 Continuous Models

(Y. Goncharov 2006) proposed the first continuous-time approach using option-based models. The traditional option-based model described in the literature is optimal in the sense that the borrower is presumed to have enough financial sophistication to compare the expected present value of refinancing to the remaining balance of the current loan. The MRB (mortgage-rate based) model is sub-optimal in that the refinancing incentive is based only upon a crude comparison between the current mortgage rate and the contracted rate. Goncharov also demonstrates that calculating the refinancing incentive using the traditional option-based and MRB approaches can lead to contradictory results. While the existence and uniqueness of a fixed-point iterative solution to the endogenous mortgage rate can be proven using the traditional option-based approach; the same cannot be said for the MRB approach, as this remains an open problem.

(Goncharov, Okten and Shah 2007) use Randomized Quasi-Monte Carlo simulations to compute the endogenous mortgage rate. The equation to solve for the endogenous mortgage rate involves the balance of the mortgage as well as the prepayment rate, both of which are functions of the mortgage rate. Thus the equation must be solved iteratively. Using the lowest interest rate possible, one can assume no prepayments due to refinancing, so only the mortgage balance involves the mortgage rate. This can be solved iteratively. Higher mortgage rates are functions of all the lower rates, so it must be solved iteratively and convergence is assured by the fixed-point theorem.

Endogenous mortgage rates assume that all prepayments are due to interest rate fluctuations. In reality, prepayments occur for a variety of reasons other than refinancing

to obtain a lower interest rate. The sale of a house due to a job, divorce, marriage, or simply the decision to "trade up" or move into a smaller house after the children are grown (to prevent them from moving back in) also results in a prepayment and is independent of interest rates. (Gorovoy and Linetsky 2007) build upon Goncharov's model using an eigenfunction expansion approach and a prepayment rate that is a sum of stochastic and deterministic terms.

2.4.3 Other Models

Where the exogenous prepayment rate is deterministic in (Gorovoy and Linetsky 2007), (Kolbe and Zagst 2009) present a stochastic "baseline" prepayment rate which follows a two-factor Vasicek model. The first factor is to introduce uncertainty into the general turnover of real estate, while the second factor allows one to observe the influence of the GDP growth rate on home sales. When prepayment rates are plotted against the spread between the current mortgage rate and the contracted rate, (Kolbe and Zagst 2009) approximate an "S" curve with a constant rate of zero when the spread is negative, linearly increasing to a certain threshold when the spread is positive, and becoming again constant when the spread is large. The strength of this model is that the baseline (exogenous) prepayment in Kolbe-Zagst is stochastic; this can dependent on external factors such as the house price or GDP. However, this adds complexity to the model and makes it more difficult to do sensitivity analysis. Another weakness of this model is that when it reaches its maximum value for low interest rates it remains constant. Observed data show that while prepayments tend to level off somewhat when rates drop significantly, they still tend to increase albeit at a smaller rate.

Gorovoy and Linetsky's model also increases linearly when the spread becomes positive, but it does not become constant for large spreads. Empirical data from Freddie Mac between 1994 and 2003 clearly fit the "S" curve relationship better than the Gorovoy-Linetsky ramp; however, the data exhibit a slight upward trend rather than a constant when the spread is large. (See Figure 1.) This suggests enhancing the Gorovoy-Linetsky model to include a piecewise linear ramp with multiple breakpoints.

The Capponi heterogeneity model states that the prepayment rate depends upon the age of the loan as well as observable and hidden factors that vary from borrower to borrower. The hidden factors are each assumed to follow a probability distribution; these include the refinancing cost threshold and awareness of interest rates on the part of the borrower. The observable factors may include LTV (loan to value ratio) of property, or the borrower's credit score. Calibrating prepayments using this model involves taking derivatives of the Kullback-Leibler distance between the observed and theoretical prepayment probability distributions with respect to each of the parameters. The solution involves solving an unconstrained non-linear optimization problem. The problems with this model involve its complexity and the difficulty of obtaining and dealing with individual loan data, but the prepayment projections can be very accurate even with poor parameter estimates.

Two additional factors in Capponi's model are burnout and the media effect. Burnout is a decrease in the prepayment rate when most financially sophisticated borrowers who could have refinanced have already done so. The remainder of the pool refinances at a slower rate. When interest rates are very low, publicity about low rates,

as well as friends and neighbors refinancing tends to influence more people to refinance. The media effect thus increases the endogenous prepayment rate.

Capponi includes both the burnout and the media effects in his prepayment model. In his burnout model, he divides the pool into fast refinancers and slow refinancers. As the fast refinancers leave the pool, the prepayment rate tends toward that of the slow refinancers. In his media effect model, the prepayment rate is a function of a weighted average of past mortgage rates to the current rate. To incorporate these two effects into the prepayment rate Capponi generates an S-curve

$$S(x) = \begin{cases} A\left(1 - e^{-B(x-d)^2}\right) & x \geq d \\ 0 & x < d \end{cases}$$

where d is the prepayment threshold and A and B are influenced by the burnout and media effects.

The strength of Capponi's S-Curve is that it accounts for the burnout and media effects. Its weakness is that it is not piecewise linear. This would make it difficult to incorporate into the eigenfunction expansion model because the change of variable depends upon making the coefficient of the first-order term in the Sturm-Liouville equation a constant. That is impossible unless the prepayment function is piecewise linear.

The Citigroup Paper "Anatomy of a Prepayment" discusses the lock-in effect which is the disincentive to move because of increased interest rates. This means giving up a lower interest rate and taking on a higher one because of a move. This effect tends to lower exogenous prepayment rates. Certain VA and FHA loans are assumable which means that the new owner can assume the mortgage of the borrower, so this effect is

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mitigated in these cases.   Other effects include seasoning and housing inflation. During the "bubble", there were many cash-out refinancings and actual sales as homeowners could sell their existing properties at a significant profit and use the proceeds to trade up. Yet the bubble was influenced to a certain degree by interest rates. This shows there is some overlap between endogenous and exogenous prepayment rates. Since interest rates affect many other economic factors, such as GDP or housing prices, it is best to reduce the number of variables in order to avoid collinearity in estimating parameters.   This thesis takes the position that interest rates alone are the best way to predict prepayment rates.   Instead of trying to model the burnout/media effects specifically, we can simply construct a piecewise linear prepayment model from the actual prepayment and interest rate data and use the multiple threshold model to show the burnout effect indirectly.

Chapter 3. The Continuous-Time Mortgage Model

Mortgages involve complex mathematical calculations.   We often use continuous-time models to approximate the mortgage cash flows.   This simplifies the calculations to a certain degree, but we need to avoid loss of accuracy. Many continuous calculations can be very accurate; in fact we will show how the continuous amortization function is an extension of discrete amortization used in the industry in much the same way that the Gamma function is a continuous analogue of the factorial.   In this section we will review the mathematical background of continuous-time mortgages which will provide us with a framework upon which to build our new model.

Pricing of mortgages depends upon three things: the amortization schedule which is usually deterministic, the prepayments which are partly deterministic and partly stochastic, and the instantaneous risk-free rates which are stochastic. Mortgages are debt obligations which combine principal and interest payments into a cash flow stream. The debt is extinguished gradually over time unlike bonds which pay only interest until maturity.   Since the payments are made at regular intervals, a discrete model is the most accurate, but since payments occur frequently over a long time horizon, a continuous approximation can also be useful.

The present value of a series of cash flows is defined as the discounted sum of each of the cash flows. This is the price that one would pay to receive those cash flows. From the discrete model for the present value of an annuity

$$PV = \sum_{i=1}^{N} \frac{CF_i}{(1+r)^i} \tag{3.1}$$

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we generate the continuous model:

$$PV = \int_0^T ce^{-rt}dt \tag{3.2}$$

If we are dealing with a pool of mortgages with identical characteristics, i.e. their rates and terms are the same, the pool will amortize like a single mortgage before prepayments are introduced. Thus we can use (3.2) to represent a pool of mortgages as well. We next assume that homeowners prepay mortgages. The previously constant payment stream is now a function of time. The prepayments increase the amount of cash being returned, but this is offset by the reduced size of the pool which affects the scheduled principal and interest payments. Furthermore, since interest rates are constantly changing, we discount the cash flows by integrating the short rate over time. Letting both the continuous cash flow c and the short rate r be time-dependent, we arrive at:

$$PV = \int_0^T c_s e^{-\int_0^s r_u du}ds \tag{3.3}$$

3.1 Amortization Models

Amortization schedules can range from a standard 30-year or shorter term fixed-rate mortgage, to an ARM, (Adjustable-rate Mortgage), IO (Interest Only), GEM (Growing Equity Mortgage), GPM (Graduated Payment Mortgage) or Balloon Mortgage. Of these only the ARM is not deterministic.

The simplest mortgage is the fixed-rate amortization model. The borrower is contracted to pay off the balance of the loan at a specific rate for a specific term. At the end of the term, the balance is zero. Let B_0 be the original balance of the loan and

$B_T = 0$ is the value at maturity. The monthly payment consists of interest plus principal. We can express this by the following differential equation:

$$c = mB - \frac{dB}{dt} \qquad (3.4)$$

where B is the balance of the mortgage, m is the contracted interest rate and c is the payment. The interest equals the rate times the balance, while the principal paid is the change in the balance. Since balance always declines, this change is negative, so we subtract it to get a "positive" amount.

Thus using the boundary conditions, the amount borrowed, $B(0) = B_0$ respectively, we arrive at:

$$B(t) = B_0 e^{mt} - \frac{c}{m}\left(e^{mt} - 1\right), \qquad (3.5)$$

that is the balance at time t, $B(t)$ is equal to the future value of the original balance, minus the future value of all payments made to date. Using the boundary condition $B(T) = 0$ we get

$$0 = B_0 e^{mT} - \frac{c}{m}\left(e^{mT} - 1\right) \qquad (3.6)$$

Solving for the coupon rate gives:

$$c = \frac{B_0 m}{1 - e^{-mT}} \qquad (3.7)$$

We can substitute (3.7) into (3.5) and get the following expression for the continuous-time balance:

$$B(t) = B_0 \left(\frac{1 - e^{-m(T-t)}}{1 - e^{-mT}} \right) \qquad (3.8)$$

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Some may question the accuracy of continuous-time models such as (3.7) and (3.8) when in fact mortgage payments are made in discrete increments. However, by adjusting the mortgage rate from monthly compounding to continuous compounding, we can account for the difference. Thus letting m_d be the nominal mortgage rate, we can convert it to the continuous mortgage rate m :

$$m = 12\ln\left(1 + \frac{m_d}{12}\right). \tag{3.9}$$

Consider a numerical example of a loan with initial amount \$100,000, at 6% interest, and 30-year term. The balance after n months in the discrete model is determined by the following formula (Hayre, Mohebbi and Zimmerman 1997):

$$B = B_0 \frac{\left(1 + \frac{m_d}{12}\right)^N - \left(1 + \frac{m_d}{12}\right)^n}{\left(1 + \frac{m_d}{12}\right)^N - 1} \tag{3.10}$$

Substituting $B_0 = 100{,}000$, $m_d = .06$, $T = 30$, and $t = 10$ we obtain

$$B = 100000 \times \frac{1.005^{360} - 1.005^{120}}{1.005^{360} - 1} = \$83{,}685.73 \tag{3.11}$$

Now from (3.9) let $m = 12\ln(1.05) = 0.0598505$ which is close to the nominal rate of 6%. Substituting this value into (3.8) gives an identical result to that in (3.11):

$$B(t) = 100000\left(\frac{1 - e^{-.0598505 \times 20}}{1 - e^{-.0598505 \times 30}}\right) = \$83{,}685.73 \tag{3.12}$$

The discrete monthly payment is calculated from the formula

$$c_d = B_0 \frac{\left(\dfrac{m_d}{12}\right)\left(1+\dfrac{m_d}{12}\right)^N}{\left(1+\dfrac{m_d}{12}\right)^N - 1} = 100000 \times \frac{(.005)(1.005)^{360}}{(1.005)^{360} - 1} = \$599.55 \qquad (3.13)$$

In the continuous model, we have a cash flow rather than a discrete payment. The cash flow can be calculated from (3.7) using the continuous mortgage rate from (3.9)

$$c = \frac{100000 \times .05898505}{1 - e^{-.05898505 \times 30}} = \$7176.68 \quad (3.14)$$

This is an "annualized" rate which we can convert to monthly by integrating its continuous future value over the next month:

$$c_d = \int_0^{1/12} c e^{ms} ds = \frac{c}{m}\left(e^{m/12} - 1\right) \qquad (3.15)$$

Substituting into (3.15) we obtain an identical result to (3.13).

$$c_d = \frac{\$7176.68}{.05898505}\left(e^{.058598505/12} - 1\right) = \$599.55 \qquad (3.16)$$

The example above illustrates that the continuous amortization function is not an approximation, but rather an extension of the time domain of the amortization function from the positive integers to the positive real line.

3.2 Interest Rate Models

There are a variety of interest-rate models based on the short rate r_t. These short-rate models can be used to generate the yield curve. Let W_t, $t \geq 0$ be a Brownian motion and let κ, θ and σ be positive constants. Two particular short-rate models of interest which have mean-reverting characteristics are the Vasicek model (linear model):

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$$dr_t = \kappa(\theta - r_t)dt + \sigma dW_t \tag{3.17}$$

and the Cox-Ingersoll-Ross model (non-linear model):

$$dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}dW_t \tag{3.18}$$

Both models behave well most of the time, but the Vasicek model may become negative when the interest rate approaches zero. That is because the solution is a normal random variable. The solution of the CIR model is a positive random variable.

(Gorovoi and Linetsky 2004) address the Vasicek model problem when discussing very low Japanese Interest rates where they allow Vasicek to go negative (the shadow rate), but they create an "option" which prevents the interest rate from becoming negative. The strike price of the option is zero and the price of the option itself is added to the price of the bond.

The (Cox, Ingersoll and Ross 1985) Model (CIR) appears to be the most popular in mortgage finance literature. Both (Y. Goncharov 2006) and (Gorovoy and Linetsky 2007) use this particular model. (Kolbe and Zagst 2009) also use a one-factor CIR as the short-rate model.

The price of a zero coupon bond at time t with maturity T can be calculated using the affine model. See (Shreve 2004) pp. 272-274.

$$B(r_t;t,T) = \exp\left[C_1(t,T) - r_t C_2(t,T)\right] \tag{3.19}$$

where C_1 and C_2 are deterministic functions.

Solving for the Vasicek model, we obtain:

$$C_2(t,T) = \frac{1}{\kappa}\left[1 - e^{-\kappa(T-t)}\right] \tag{3.20}$$

$$C_1(t,T) = \theta(T-t) - \theta C_2(t,T) + \frac{\left[\sigma C_2(t,T)\right]^2}{4\kappa} \tag{3.21}$$

For the Cox-Ingersoll-Ross model, define $D(T) = (\rho + \kappa)(e^{\rho T} - 1) + 2\rho$ where $\rho = \sqrt{\kappa^2 + 2\sigma^2}$. Then substitute the following into (3.19):

$$C_1(t,T) = \frac{2\kappa\theta}{\sigma^2} \ln\left(\frac{2\rho e^{(\kappa+\rho)(T-t)/2}}{D(T-t)}\right) \qquad C_2(t,T) = \frac{2\left(e^{\rho(T-t)} - 1\right)}{D(T-t)} \tag{3.22}$$

To price a mortgage with no prepayments, we simply integrate the cash flows over time:

$$A(r;0,T) = \int_0^T c_s B(r_0;0,s)\,ds \tag{3.23}$$

For a fixed-rate mortgage, the cash flows are constant; thus we can treat it as a continuous annuity:

$$A(r_0;0,T) = c\int_0^T B(r_0;0,s)\,ds \tag{3.24}$$

While there is a closed form for $B(r;0,T)$ (3.24) cannot be solved analytically; however, using a method known as eigenfunction expansion as an alternative to numeric integration, it can be shown that the bond price can be expressed as an infinite sum:

$$B(r;0,T) = \sum_{n=0}^{\infty} c_n e^{-\lambda_n T} \varphi_n(r) \tag{3.25}$$

for suitable coefficients c_n, eigenvalues λ_n and eigenfunctions $\varphi_n(x)$. This will give the same result as (3.19); however, it can be easily integrated term by term:

$$A(r;0,T) = c\sum_{n=0}^{\infty} \frac{1 - e^{-\lambda_n T}}{\lambda_n} c_n \varphi_n(r) \tag{3.26}$$

The formulas for the eigenvalues, eigenfunctions and coefficients for Vasicek and CIR are known and easily calculated when no prepayments are involved. See (Gorovoi and Linetsky 2004) and Appendix C of (Gorovoy and Linetsky 2007),

3.3 Prepayment Models

Borrowers have the right to pay off their mortgages early. Thus while the total principal cash flow is certain, the timing of those cash flows is not. Also, interest rates tend to fluctuate, so a mortgage, along with most other bonds, suffers from interest rate risk. Most mortgages have embedded American call options which permit the borrower to pay the mortgage off early. It is this prepayment characteristic along with the gradual return of principal which differentiates mortgages from other fixed-income investments.

3.3.1 Exogenous Model

The mortgage industry uses two deterministic prepayment models, CPR (Constant Prepayment Rate) and PSA (Public Securities Administration). For simplicity we will start with the CPR model. When prepayments are involved, there are three cash flows: Scheduled Principal, Interest and Prepayments. The balance decreases each month by the scheduled principal plus interest. Taking the example in the previous section, let's see what happens if 8% of mortgages prepay each year. We convert CPR to single monthly mortality SMM which is:

$$SMM = 1 - (1 - CPR)^{1/12} \qquad (3.27)$$

Thus for 8% CPR the single month mortality is:

$$1 - (1 - .08)^{1/12} = .00692438 \qquad (3.28)$$

The discrete remaining balance can be described as:

$$\hat{B}_n = \hat{B}_{n-1} - \hat{P}_n - PR_n = B_n \left(1 - SMM\right)^n \tag{3.29}$$

Thus the balance including prepayments in the discrete case is:

$$\$83,685.73 \left(1 - .006924\right)^{120} = \$36,352.11 \tag{3.30}$$

The following chart shows all the conversions that are necessary in converting from the discrete to the continuous model:

$$
\begin{aligned}
t &= \frac{n}{12} \\
m &= 12\ln\left(1 + \frac{G}{1200}\right) \\
r &= 2\ln\left(1 + \tfrac{1}{200} BEY\right) = 12\ln\left(1 + \tfrac{1}{1200} CFY\right) \\
\gamma &= -\ln\left(1 - CPR\right) \\
c &= \frac{m \times PMT}{e^{m/12} - 1} = 14400 \times \left(\frac{PMT}{G}\right) \times \ln\left(1 + \frac{G}{1200}\right)
\end{aligned}
\tag{3.31}
$$

$$
\text{where}\ \begin{cases}
n = \text{Monthly Period} \\
G = \text{Nominal Interest Rate}\,(\%) \\
CPR = \text{Conditional Prepayment Rate(Decimal)} \\
PMT = \text{Monthly Payment} \\
CFY = \text{Cash Flow Yield or Mortgage Yield}(\%) \\
BEY = \text{Bond Equivalent Yield}(\%)
\end{cases}
$$

The continuous prepayment rate can be described as a hazard rate. To convert from the discrete rate to the hazard rate we use the following transformation:

$$\gamma = -\ln\left(1 - CPR\right) \tag{3.32}$$

For CPR = 8%, the hazard rate is:

$$\gamma = -\ln\left(1 - .08\right) = 0.08338160894 \tag{3.33}$$

The balance with prepayments after $t = 10$ years (120 months) is:

$$\hat{B}(t) = B(t)e^{-\gamma t} = \$83,685.73e^{-0.08338160894 \times 10} = \$36,352.11 \qquad (3.34)$$

With a no-prepayment fixed-rate mortgage the monthly cash flows are constant. However, with prepayments, the monthly cash flows change over time. There are three sets of cash flows: scheduled principal, interest and prepayments. Each of these has a discrete and continuous model. The discrete formulas are from (Hayre, Mohebbi and Zimmerman 1997). The nominal interest rate is expressed as percentage so it must be divided by 100; to convert it to a monthly effective rate; we further divide it by 12.

The scheduled principal for the next period (month 121) is:

$$\hat{P}_{n+1} = \frac{\hat{B}_n\left(\dfrac{G}{1200}\right)}{\left(1+\dfrac{G}{1200}\right)^{N-n}-1} = \frac{36352.11 \times 0.005}{1.005^{240}} = \$78.68 \qquad (3.35)$$

The interest for the next period is:

$$\hat{I}_{n+1} = \hat{B}_n\left(\frac{G}{1200}\right) = 36352.724 \times 0.005 = \$181.76 \qquad (3.36)$$

The expected prepayment for the next period is:

$$PR_{n+1} = SMM\left(\hat{B}_n - \hat{P}_{n+1}\right) SMM = 0.006924 \times (36352.11 - 78.12) = \$251.16 \quad (3.37)$$

The scheduled *payment* is the sum of scheduled principal plus interest.

The scheduled payment for the next period is:

$$\hat{M}_{n+1} = \hat{P}_{n+1} + \hat{I}_{n+1} = 78.68 + 181.76 = \$260.44 \qquad (3.38)$$

An alternative calculation is:

$$M_{n+1} = M\left(1 - SMM\right)^n = \$599.55\left(1 - .006924\right)^{120} = \$260.43 \qquad (3.39)$$

The difference between (3.38) and (3.39) is due to round off error.

Now let's look at the continuous equivalents. The continuous prepayment cash flow is:

$$PR(t) = \int_t^{t+1/12} \gamma B(s) e^{-\gamma s} ds \tag{3.40}$$

After performing the integration we obtain

$$PR(t) = \frac{B_0}{e^{-mT}-1}\left[e^{-\gamma t}\left(e^{\gamma/12}-1\right) + \frac{\gamma e^{-mT+(m-\gamma)t}\left(e^{(m-\gamma)/12}-1\right)}{m-\gamma} \right] \tag{3.41}$$

Applying this model to our data yields:

$$\mu - \gamma = 0.05985 - 0.08338 = -0.2353$$
$$mT = 0.05985 \times 30 = 1.7955 \tag{3.42}$$
$$PR(10) = \frac{100,000}{e^{-1.7955}-1}\left[e^{-0.8338}\left(e^{0.08338/12}-1\right) - \frac{\gamma e^{-1.7955-2.353}\left(e^{-0.08338/12}-1\right)}{0.2353} \right] = \$251.44$$

Notice that (3.42) is very close to (3.37).

Now let's look at the continuous scheduled payment stream. We integrate the continuous scheduled payment stream model for one month from t to t + 1/12:

$$M(t) = \int_t^{t+1/12} ce^{-\gamma s} e^{m(s-t)} ds = \frac{ce^{-\gamma t}\left(e^{(m-\gamma)/12}-1\right)}{m-\gamma} \tag{3.43}$$

$$M(t) = \frac{ce^{-\gamma t}\left(e^{(m-\gamma)/12}-1\right)}{m-\gamma} \tag{3.44}$$
$$= \frac{\$7176.68 \times e^{-0.08338 \times 10}\left(e^{-0.2353/12}-1\right)}{-0.2353} = \$259.53$$

Compare (3.44) to (3.39).

Prepayments can occur for one of three reasons: sale of the property, refinancing and curtailments (excess payments). The prepayment model for pricing a single

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mortgage uses the stopping-time τ which is a random variable indicating the time of prepayment, and the $\sigma -$ algebra $\mathcal{G}_t$, which contains all the information available to the investor at time t. The price of a mortgage M_t is the conditional expectation of the present value of the coupon payments up to τ plus the present value of the balance of the mortgage at that time.

Throughout this dissertation we will make use of the Iverson bracket notation, named for Kenneth E. Iverson, the Harvard mathematician who invented the computer language APL. Let P be any proposition. The expression $[P]$=1 if P is true and $[P]$=0 if P is false. Thus $[t{\leq}T]$ is equivalent to $\mathbf{1}_{\{t\leq T\}}$. Note that there is no ambiguity with other traditional uses of square brackets.

In continuous time the price can be expressed as:

$$M_t = E\left[\int_t^{\tau\wedge T} c_s e^{-\int_t^s r_u du} ds + [t < \tau \leq T] Z_\tau e^{-\int_t^\tau r_u du} \middle| \mathcal{G}_t\right]$$ (3.45)

where Z_τ is the value of a defaultable claim at the stopping time τ (Y. Goncharov 2006) .

3.3.2 Endogenous Model

We can restate (3.45) using the intensity process h_t and the reduced σ-algebra

$\mathcal{F}_t$ which represents all the information available to the investor prior to time t. See (Y. Goncharov 2006) and (Bielecki and Rutkowski 2002).

$$M_t = [\tau > t] E\left[\int_t^T (c_s + Z_s \gamma_s) e^{-\int_t^s r_u + h_u du} ds \middle| \mathcal{F}_t\right]$$ (3.46)

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The price of a mortgage in (3.46) can now be expressed as the conditional expectation of the scheduled payments plus prepayments. Note that the value of the mortgage is zero after the stopping time. This model can also be applied to a pool of identical mortgages if we remove the bracketed expression $[\tau > t]$ from (3.46). In the case of a mortgage pool, the coupon payments will decay at the prepayment rate as a proportion of mortgages are paid off each period. The total balance of the remaining mortgages will also decay at this rate. This results in the discounting of cash flows by both interest and prepayment rates.

Through integration by parts and some algebraic manipulation (see Appendix C or (Y. Goncharov 2006)) we can rewrite (3.46) as the sum of the mortgage balance plus the value of an interest-rate swap noting that the defaultable claim is the value of the mortgage at the time of prepayment.

$$M_t = B(t) + E\left[\int_t^T (m - r_s) B(s) e^{-\int_t^s h_u + r_u du} ds \Big| \mathcal{F}_t\right] \tag{3.47}$$

At time zero, the value of the mortgage should be exactly the same as the original balance to prevent arbitrage. Thus, the swap should be initially valued at zero:

$$E\left[\int_0^T (m - r_s) B(s) e^{-\int_0^s h_u + r_u du} ds \Big| r_0 = x\right] = 0 \tag{3.48}$$

We can solve (3.48) for the mortgage rate:

$$m = \frac{E\left[\int_0^T r_s B(s) e^{-\int_0^s h_u + r_u du} ds \Big| r_0 = x\right]}{E\left[\int_0^T B(s) e^{-\int_0^s h_u + r_u du} ds \Big| r_0 = x\right]} \tag{3.49}$$

Unfortunately, the mortgage rate is embedded in *B(t)* as defined in (3.8). It can also be embedded in the prepayment rate h_t. Goncharov uses an iterative approach in his earlier paper (Y. Goncharov 2006), starting with an initial guess for m_0 and iterating until the sequence converges. The problem is that this only works when the prepayment rate is independent of the mortgage rate which only exists for the lowest possible value for the interest rate. In this case there would be no reason to refinance.

(Goncharov 2009) finds a non-iterative approach. Instead of finding the mortgage rate for a given interest rate, he fixes a mortgage rate value and tries to find the interest rate to which it corresponds.

(Goncharov 2009) makes the assumption that prepayments due to sale of property is constant and that the difference between the current and contract rates influences the refinancing rate. However, this model can be solved in a reasonable amount of time only when the prepayment rate is constant whenever the refinancing incentive is above a certain threshold. Goncharov uses a very simple prepayment of the form:

$$h_t(m) = h_0 + (h_1 - h_0)\left[m > m^t + \delta \right] \tag{3.50}$$

where h_0 and h_1 are constants, m is the mortgage rate and δ represents the transaction costs. The difficulty with the Goncharov model is that m^t, the fixed mortgage rate for a loan originated at time *t*, has to be calculated iteratively. Also, the prepayment rate in this model does not vary with the magnitude of the difference between the current and contracted mortgages rates, it is only one of two specific values, h_0 or h_1 depending upon whether the new mortgage rate is above or below the refinancing threshold.

3.3.3 Combining Exogenous and Endogenous Prepayment Models

Gorovoy and Linetsky start with equation (3.49) and turn it inside out by moving the conditional expectation inside the integral and pulling out everything deterministic. Substitute $B(t)$ and then use the properties of conditional expectation to arrive at:

$$m = \frac{\int_0^T \left(1 - e^{-m(T-s)}\right) E\left[r_s\, e^{-\int_0^s h_u + r_u du} \,\middle|\, r_0 = x \right] ds}{\int_0^T \left(1 - e^{-m(T-s)}\right) E\left[e^{-\int_0^s h_u + r_u du} \,\middle|\, r_0 = x \right] ds} \tag{3.51}$$

(Gorovoy and Linetsky 2007) simplify Goncharov's model by using the fact that the short rate moves up and down with the mortgage rate, so they incorporate the short rate into the prepayment model instead of the mortgage rate. This eliminates calculating the mortgage rate iteratively, simplifying the model at a cost of slight inaccuracy. The Gororoy-Linetsky prepayment model has a deterministic part and a stochastic part:

$$h_t = h(t, r_t) = h_0(t) + \gamma (k - r_t)^+. \tag{3.52}$$

The parameter γ in (3.52) represents the sensitivity of the model to the spread, which was not present in the Goncharov model. The deterministic component $h_0(t)$ in (3.52) represents the exogenous prepayment rate which can be either CPR (constant prepayment rate) or the industry standard PSA (Public Securities Administration Model) (The Bond Market Association 1999) which increases linearly with time up to 2-1/2 years, becoming constant thereafter.

Gorovoy and Linetsky define the following functions:

$$Q(t, u) = E\left[e^{-\int_t^u (h_s + r_s) ds} \,\middle|\, r_t = x \right] = e^{-\int_t^u h_0(s) ds} E\left[e^{-\int_t^u r_s + \gamma (k - r_s)^+ ds} \,\middle|\, r_t = x \right] \tag{3.53}$$

$$R(t,u) = E\left[r_u e^{-\int_t^u h_s + r_s ds} \middle| r_t = x \right] = e^{-\int_t^u h_0(s) ds} E\left[r_u e^{-\int_t^u r_s + \gamma(k - r_s)^+ ds} \middle| r_t = x \right] \qquad (3.54)$$

This allows us to rewrite (3.51) as

$$m = \frac{\int_0^T \left(1 - e^{-m(T-u)}\right) R(0,u)\, du}{\int_0^T \left(1 - e^{-m(T-u)}\right) Q(0,u)\, du} \qquad (3.55)$$

reducing the problem to calculating (3.53) and (3.54) and solving for m implicitly.

3.3.4 Curtailments and Defaults

Curtailments are merely prepayments that occur before the mortgage is paid off completely. They fall under the category of endogenous prepayments since they do not involve the sale of the property. This means curtailments are sensitive to interest rate fluctuations. A borrower may opt to make an extra payment each month to pay off the mortgage faster. Curtailments represent a small portion of the total prepayments. According to Citigroup, they typically are less than 0.5% CPR which is about the same as the default rate. They tend to increase for loans with a small balances and remaining terms. Refinancings tend to decrease for mature loans because the closing costs would exceed the aggregate savings in interest payments. For example, no one would refinance a mortgage with less than a year to go because the closing costs would be higher than the savings in interest payments. Goncharov's model, using the refinancing incentive, would show that prepayments would drop off as the mortgage approaches maturity for this reason. But the Gorovoy-Linetsky model depends only upon the short rate, not on the maturity. A borrower might make extra payments each month to retire the mortgage earlier, particularly if interest rates have dropped. The increase in curtailments should

offset the decrease in refinancing. Borrowers can avoid closing costs by simply making extra payments each month. Curtailments are included in the calculation of the prepayment rate, so they are automatically included in the valuation of the mortgage.

Curtailments are partial prepayments and consequently do not result in termination of the mortgage. (Lin and Yang 2005) observe that while curtailments represent a small part of the prepayment rate particularly in the West, it is approximately equal to the default rate. (Lin, et al. 2005) examine the relationship between mortgage curtailment and default. Their research shows that cumulative curtailments are the most significant factor in predicting defaults in Asian countries.

Defaults can be considered prepayments which pay off a proportion of the remaining balance of the loan. When a mortgage defaults, the borrower stops paying and the property goes into foreclosure. After remaining in foreclosure for a period (usually less than a year), the property is sold, often at a loss. (Dunsky and Ho 2007) look at defaults in a discrete setting. (Kelly 2009) examines the effect of zero down payments on the default rate.

(Deng, Quigley and Order, Mortgage Terminations, Heterogeneity and the Exercise of Mortgage Options 2000) made some of the first attempts to study defaults and prepayments together with their correlation, using a competing-risks model. More recently, (Clapp, Deng and An 2006) look at an alternative approach to estimating the competing-risks model. The model in (Gorovoy and Linetsky 2007) looks only at prepayments, but it is easily adaptable to include defaults as we will demonstrate later.

Our model includes the loss premium based on the loss severity applied to the default rate. During the housing bubble, the loss severity would have been lower because

as housing prices increased, a foreclosed home would have brought a higher price when it was sold. The recent decline in housing prices increases the loss severity. Adding the loss premium to the mortgage rate would decrease the amount of money that could be lent to potential borrowers, forcing them to forego the "McMansions" and purchase more modestly priced homes or wait several years to save up for the down payment. By incorporating sensitivity analysis, our model allows one to observe how robust mortgage rates are to changes in loss severity.

Before 2008, the risk of default was smaller because housing prices increased rapidly according to the Case-Schiller Index. Agency Mortgages such as those issued by Fannie Mae and Freddie Mac are backed up by these quasi-government agencies--in the event of default, the loss is taken by Fannie Mae or Freddie Mac who pay the balance to the bondholders. Thus the investors in Agency Mortgage-Backed securities were previously concerned only with prepayment and interest rate risk, but did not worry about default risk. The solvency of Fannie and Freddie were not of concern because of their perceived government backing. As we know these agencies were not solvent; but the government did eventually bail them out by providing cash, thus sparing the bondholders. The Community Reinvestment Act of 1977 while well-intentioned had the unintended consequence of encouraging banks to issue sub-prime loans to poor and low-income people. When home prices dropped, many of these loans went into foreclosure.

Another problem is that lenders do not hold mortgages until maturity; rather they sell them to agencies such as Fannie and Freddie who package them into Mortgage-Backed Securities (MBS) and sell them to investors. Banks want to issue loans to make money; their only incentive not to do so is the fear of losing the principal. But since this

loss is absorbed by Fannie and Freddie (or the bond holders for non-Agency mortgages) the banks have no incentive not to lend. According to (Wallison 2011) in a Wall Street Journal editorial: "…thanks to rules adopted in 1995 under the Community Reinvestment Act, regulated banks as well as savings and loan associations had to make a certain number of loans to borrowers who were at or below 80% of the median income in the areas they served." In addition to the rate charged for a mortgage, the eligibility requirements need to be examined. Minimum down payments should be required so that borrowers have a stake. (Sowell 2009) states that creative financing gimmicks, such as no down payment, ARMS, or interest-only loans contributed to problems in the housing market. Borrowers suffer only a bad credit rating if they make no down payment. A higher interest rate would reduce the amount one can borrow, which may compel a borrower to come up with a down payment for a particular house; requiring a down payment would provide a cushion to the lender and thus reduce the severity rate. Thus, the probability of higher severity rates would be much lower in the sensitivity analysis of mortgages that had 20% down payments. (Kelly 2009) addresses this problem.

In manufacturing, quality engineers use control charts to determine whether to tinker with a process. This is discouraged unless the process falls outside pre-determined control limits because a change in the process may only be due to sampling error. In a similar fashion, the Federal Reserve observes economic data and sets overnight rates which indirectly influence mortgage rates. The Fed Funds rate dropped from over 5% in August 2007 down to 2% in August 2008 right before the economic meltdown. Should rates have been left alone?

According to (Wessel 2010) in the Wall Street Journal, Ben Bernanke claimed recently that the Fed's monetary policy of keeping interest rates low during the early part of the decade didn't have a significant effect on the housing bubble that followed based on the equations he used. But the article claims that the equations failed to predict the bubble. Keeping rates low forced investors to seek higher yields and MBS backed by subprime loans satisfied much of the demand. While the model presented here does not predict housing bubbles, its use of sensitivity analysis can show how robust mortgage rates are to differing economic conditions.

Chapter 4. Solutions to the Model

The uncertainty inherent in mortgage finance is primarily due to interest rate fluctuations. These affect both the discounting and the prepayment rates. In this section we will start with an arbitrary interest rate model and apply the eigenfunction expansion approach (Gorovoy and Linetsky 2007) to produce a general model. Next we will show how this model can be reduced to Cox Ingersoll Ross, and then use it to generate the Vasicek model. We also show another way to calculate the normalization constant using an integral rather than the differential approach used by Gorovoy and Linetsky.

4.1 The Basic Model

Option pricing theory states that the price of a payoff at some time t in the future which is contingent on the state of a process at time t can be seen as a risk-neutral expectation of a discounted payoff. Let us define the operator

$$\left(\mathcal{P}_t f\right)(x) = E\left[e^{-\int_0^t r(X_u)du} f\left(r_t\right) \middle| X_0 = x \right] \tag{4.1}$$

where X_t represents a Markov process and $f\left(X_t\right)$ is the value of the payoff at time t. Equation (4.1) is a general form for calculating (3.53) and (3.54) which we will need to solve(3.55). Since uncertainty is present in the interest rate as well as in the payoff, simulating X_t with a finite number of sample paths is the traditional method of calculating the price. As an alternative, (Linetsky 2004) uses the infinitesimal generator of the pricing semigroup. The definition is:

$$(\mathcal{P}_t f)(x) = E\left[\exp\left\{-\int_0^t \left(r_s + \gamma(k - r_s)^+\right)ds\right\} f(r_t)\middle| r_0 = x\right] \qquad (4.2)$$

We can show that $\{\mathcal{P}_t \mid t \geq 0\}$ is a semigroup according to the definition set forth in (Rudin 1973). The infinitesimal generator of a Markov process X_t is defined as

$$(\mathcal{G}f)(x) = \lim_{t \to 0} \frac{E\left[f(X_t)\middle| X_0 = x\right] - f(x)}{t} \qquad (4.3)$$

4.1.1 The Infinitesimal Generator

The Spectral Theorem for Self-Adjoint Compact Operators, Theorem 1.10.2 p. 190 (Debnath and Mukusinski 1999) proves there exists an orthonormal basis of a Hilbert space $\mathcal{H}$ consisting of the eigenvectors such that for any $f \in \mathcal{H}$

$$(\mathcal{G}f)(x) = \sum_{n=1}^{\infty} -\lambda_n \langle f, \varphi_n \rangle \varphi_n(x). \qquad (4.4)$$

where λ_n are eigenvalues and φ_n are eigenvectors.

Using the eigenvalues and eigenfunctions from the infinitesimal generator we can write:

$$(\mathcal{P}_t f)(x) = \sum_{n=1}^{\infty} e^{-\lambda_n t} c_n^f \varphi_n(x) \qquad (4.5)$$

where $c_n^f = \langle f, \varphi_n \rangle$. We now have a method to discount a claim $f(x)$ with respect to a stochastic short rate using an infinite sum.

We will start with a general Markov model for the interest rate:

$$dr_t = \mu(r_t, t)dt + \sigma(r_t, t)dW_t \qquad (4.6)$$

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From the Discounted Feynman-Kac Theorem (Theorem 6.4.3 p. 269-270 in (Shreve 2004), we can show that the price of a claim $f(t,x)$ satisfies the following partial differential equation:

$$f_t(t,x) + \mu(t,x) f_x(t,x) + \tfrac{1}{2}\sigma^2(t,x) f_{xx} = rf(t,x) \tag{4.7}$$

This can be decomposed into its time (t) and space (x) components; we can capture the space component (interest rate) using the infinitesimal generator. Appendix A shows how to derive the infinitesimal generator from the definition in (Rudin 1973).

$$(\mathcal{G}f)(x) = (\mathcal{D}f)(x) - f(x)\left[x + \gamma(k-x)^+\right] \tag{4.8}$$

Since r_t is stochastic, we must use Ito calculus to expand the first term of (4.8)

$$(\mathcal{G}f)(x) = \frac{1}{2}\sigma(x)^2 f''(x) + \mu(x) f'(x) - f(x)\left[x + \gamma(k-x)^+\right] \tag{4.9}$$

In the eigenfunction expansion method we need to find all f's and λ's which satisfy the following equation:

$$(\mathcal{G}f)(x) = -\lambda f(x) \tag{4.10}$$

(Gorovoy and Linetsky 2007) show that if the eigenvalues of $\mathcal{G}$ are $-\lambda_n$ then the eigenfunctions are the same and the corresponding eigenvalues of $\mathcal{P}_t$ are $e^{-\lambda_n t}$. Thus solving for the eigenfunctions and eigenvalues of the infinitesimal generator will also find them for the semigroup operators. Rewriting (4.10) as a Sturm-Liouville differential equation yields

$$\frac{1}{m(x)}\left(\frac{f'(x)}{s(x)}\right)' - \left(x + \gamma(k-x)^+ - \lambda\right) f(x) = 0 \tag{4.11}$$

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where the scale $\mathfrak{s}(x)$ and speed $\mathfrak{m}(x)$ densities are defined as follows:

$$\mathfrak{m}(x)\mathfrak{s}(x) = \exp\left[-\int^x \frac{2\mu(y)}{\sigma^2(y)}dy\right] \qquad \mathfrak{m}(x) = \frac{2}{\sigma^2(x)\mathfrak{s}(x)} \qquad (4.12)$$

Since the coefficient of $f(x)$ in (4.11) is piecewise linear, we can rewrite it in the following form:

$$\frac{1}{\mathfrak{m}(x)}\left(\frac{f'(x)}{\mathfrak{s}(x)}\right)' - \left(x(1-\gamma_k) + \gamma_k k - \lambda\right)f(x) = 0; \quad \gamma_k = \gamma[x < k] \qquad (4.13)$$

Thus when $x \geq k$ (4.13) simplifies to

$$\frac{1}{\mathfrak{m}(x)}\left(\frac{f'(x)}{\mathfrak{s}(x)}\right)' - (x - \lambda)f(x) = 0 \qquad (4.14)$$

It is known through Sturm-Liouville theory that the values of λ in (4.13) are positive and real and that the eigenfunctions are orthogonal. See (Al-Gwaiz 2008) . Two vectors are orthogonal when their inner product is zero; the inner product of two functions f and g in a Hilbert space $\mathcal{H}$ is defined as:

$$\langle f, g \rangle = \int_l^r f(x)g(x)\mathfrak{m}(x)dx \qquad (4.15)$$

where $m(x)$ is the speed density defined in(4.12). We will let $\{\varphi_i\}_{i=1}^\infty$ denote a set of orthonormal eigenfunctions in $\mathcal{H}$. The domain of the Sturm-Liouville operator $\mathcal{G}$ is

$\{f \in \mathcal{H} : f, f' \in AC_{\text{loc}}((e_1, e_2)), \mathcal{G}f \in \mathcal{H}, I = (e_1, e_2)\}$ where $AC_{\text{loc}}(I)$ is the space of functions absolutely continuous over compact subintervals of I. The boundary conditions at e_1 and e_2 are:

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$$\lim_{x \to e_1} \frac{f'(x)}{\mathfrak{s}(x)} = 0, \qquad \lim_{x \to e_2} \frac{f'(x)}{\mathfrak{s}(x)} = 0 \qquad\qquad (4.16)$$

See (Gorovoi and Linetsky 2004) p. 55 and (Gorovoy and Linetsky 2007) p. 553.

Equation (4.10) shows $\mathcal{G}$ is a non-positive operator in $\mathcal{H}$ and furthermore $\mathcal{G}$ is self-adjoint.

Its spectral representation is of interest because of its relation to the semigroup operators.

4.1.2 Change of Variable

To calculate the eigenfunctions of $\mathcal{G}$ we need to solve(4.13). Depending on the

interest rate model, we can perform a change of variable on both x and $f(x)$ to get (4.13)

into the form of a second-order differential equation with known solutions. The

independent variable x usually undergoes a linear transformation $z = \beta_0 + \beta_1 x$ for some

constants β_i; we change the dependent variable $f(x)$ with an exponential multiplier as

follows:

$$f(x) = e^{g(z)} u(z) \qquad\qquad (4.17)$$

Table 1 on page 41 summarizes finding the solution to(4.13). The Vasicek

infintesimal generator transforms to the stationary Schrodinger equation whose solutions

are the Parabolic Cylinder Functions while the CIR infinitesimal generator transforms to

the Confluent Hypergeometric equation whose solutions are the Confluent

Hypergeometric Functions. See Appendix B.

Model	**Vasicek**	**Cox-Ingersoll-Ross**
Equation	$dr_t = \sigma dW_t + \kappa(\theta - r_t)dt$	$dr_t = \sigma\sqrt{r_t}\,dW_t + \kappa(\theta - r_t)dt$
$\mu(x)$	$\kappa(\theta - x)$	$\kappa(\theta - x)$
$\sigma(x)$	σ	$\sigma\sqrt{x}$
$m(x)$	$\exp\left[\dfrac{\kappa(\theta - x)^2}{\sigma^2}\right]$	$\dfrac{2}{\sigma^2}x^{\beta-1}\exp\left(-\dfrac{2\kappa x}{\sigma^2}\right)$
$s(x)$	$\dfrac{2}{\sigma^2}\exp\left[-\dfrac{\kappa(\theta - x)^2}{\sigma^2}\right]$	$x^{-\beta}\exp\left(\dfrac{2\kappa x}{\sigma^2}\right)$
Independent Variable	$z = \dfrac{\sqrt{2\kappa}}{\sigma}(\theta - x)$	$z = \alpha_i x$
Dependent Variable	$f(x) = e^{z^2/4}u(z)$	$f(x) = \exp\left(\dfrac{(\kappa - \rho_i)z}{a_i\sigma^2}\right)u(z)$
Differential Equation	$u'' + \left(\tfrac{1}{2} + \mu_\gamma - \tfrac{1}{4}(\alpha_\gamma - z)^2\right)u = 0$ $\alpha_\gamma = \sigma(1-\gamma)\sqrt{2/k^3}$ $\mu_\gamma = \tfrac{1}{4}\alpha_\gamma^2 - \left[(1-\gamma)\theta + \gamma k - \lambda\right]/k$	$zu'' - (b - z)u' - au = 0$ $a = a_i - \dfrac{(\lambda - \gamma k)}{\rho_i} \quad b = \dfrac{2k\theta}{\sigma^2}$
Solutions	Parabolic Cylinder	Confluent Hypergeometric
$u_1(z)$	$D_\mu(z-a); \quad D_\mu(x) =$ $\dfrac{\sqrt{\pi}\,2^{\frac{1}{2}\mu}e^{-\frac{1}{4}x^2}}{\Gamma\left(\dfrac{1-\mu}{2}\right)}M\left(-\tfrac{1}{2}\mu, \tfrac{1}{2}, \tfrac{1}{2}x^2\right) -$ $\dfrac{\sqrt{\pi}\,2^{\frac{1}{2}(1+\mu)}xe^{-\frac{1}{4}x^2}}{\Gamma(-\mu/2)}M\left(\dfrac{1-\mu}{2}, \tfrac{3}{2}, \tfrac{1}{2}x^2\right)$ See (Abramowitz and Stegun 1972) 19.3.1 p. 687 and 19.12.3 p. 691	$M(a,b,z) = \sum\limits_{n=0}^{\infty}\dfrac{(a)_n}{(b)_n}\dfrac{z^n}{n!}$ See (Abramowitz and Stegun 1972) 13.1.2 p. 504
$u_2(z)$	$\dfrac{\Gamma(-\mu)}{\pi}\{D_\mu(z-a) -$ $D_\mu(a-z)\sin\left[\pi(\mu + \tfrac{1}{2})\right]\}$ See (Abramowitz and Stegun 1972) 19.3.8 p. 687	$U(a,b,z) = \dfrac{\Gamma(1-b)}{\Gamma(1+a-b)}M(a,b,z)$ $+\dfrac{\Gamma(b-1)}{\Gamma(a)}z^{1-b}M(1+a-b, 2-b, z)$ See (Slater 1960) p. 5

Table 1 – Eigenfunctions for Vasicek and Cox-Ingersoll Ross Interest Rate Models

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4.1.3 Continuity and Differentiability

In our model we have a breakpoint k above which there are no endogenous prepayments. We have to ensure that the eigenfunctions are both continuous and differentiable across the breakpoint. Thus if $\psi_\lambda(x)$ is a solution to (4.14) and $\phi_\lambda(x)$ is a solution to (4.13) for the eigenvalue λ then we must have $A_\lambda \psi_\lambda(k) - B_\lambda \phi_\lambda(k) = 0$ and $A\psi'_\lambda(k) - B\phi'_\lambda(k) = 0$. We can write this in matrix form:

$$\begin{bmatrix} \psi_\lambda(k) & -\phi_\lambda(k) \\ \psi'_\lambda(k) & -\phi'_\lambda(k) \end{bmatrix} \begin{bmatrix} A_\lambda \\ B_\lambda \end{bmatrix} = \mathbf{0} \tag{4.18}$$

If A_λ and B_λ are both zero, all $\varphi_n(x) = A_\lambda \psi_\lambda + B_\lambda \phi_\lambda = 0$, thus $\|\varphi\| = \langle \varphi, \varphi \rangle = \int_{e_1}^{e_2} |\varphi(x)|^2 \, \mathfrak{m}(x) \, dx = 0$ which cannot be true because the norm must be 1. Therefore the determinant of the matrix in (4.18) is zero. Using row operations, we have the matrix's echelon form:

$$\begin{vmatrix} 1 & -\phi_\lambda(k)/\psi_\lambda(k) \\ 0 & \psi_\lambda(k)\phi'_\lambda(k) - \phi_\lambda(k)\psi'_\lambda(k) \end{vmatrix} = 0 \tag{4.19}$$

That is

$$\psi_\lambda(k)\phi'_\lambda(k) - \phi_\lambda(k)\psi'_\lambda(k) = 0 \tag{4.20}$$

Finding the zeroes of (4.20) will determine the eigenvalues. The equation has countably many λ_n's. We now have all we need to calculate the solutions $\psi_\lambda(x)$ and $\phi_\lambda(x)$ for each λ_n.

In order for the continuity factor to hold, we must have $A_n = \dfrac{\phi_n(k)}{\psi_n(k)} B_n$.

Rewriting A_n and B_n in terms of a third parameter Δ_n yields:

$$A_n = \sqrt{\frac{\phi_n(k)}{\Delta_n \psi_n(k)}} \text{ and } B_n = \sqrt{\frac{\psi_n(k)}{\Delta_n \phi_n(k)}} \tag{4.21}$$

Now in order to normalize the eigenfunctions $\langle \varphi_n, \varphi_n \rangle = \|\varphi_n\|^2 = 1$:

$$1 = \langle \varphi_n, \varphi_n \rangle = \frac{\phi_n(k)}{\Delta_n \psi_n(k)} \int_0^k \psi_n^2(x) m(x) dx + \frac{\psi_n(k)}{\Delta_n \phi_n(k)} \int_k^\infty \phi_n^2(x) m(x) dx$$

Solving for Δ_n, we obtain the following formula for the normalization constant:

$$\Delta_n = \frac{\phi_n(k)}{\psi_n(k)} \int_0^k \psi_n^2(x) m(x) dx + \frac{\psi_n(k)}{\phi_n(k)} \int_k^\infty \phi_n^2(x) m(x) dx \tag{4.22}$$

Now we can defined the normalized eigenfunctions as:

$$\varphi_n(x) = \begin{cases} \sqrt{\dfrac{\phi_{\lambda_n}(k)}{\Delta_n \psi_{\lambda_n}(k)}} \psi_{\lambda_n}(x), & x \in (e_1, k] \\[4mm] \sqrt{\dfrac{\psi_{\lambda_n}(k)}{\Delta_n \phi_{\lambda_n}(k)}} \phi_{\lambda_n}(x), & x \in [k, e_2) \end{cases} \tag{4.23}$$

where Δ_n is defined in (4.22). Another solution for Δ_n is (Stackgold 1998) p. 441)

$$\Delta_n = \frac{d}{d\lambda} \left[\frac{\psi_\lambda(k) \phi_\lambda'(k) - \phi_\lambda(k) \psi_\lambda'(k)}{\mathfrak{s}(k)} \right]_{\lambda = \lambda_n} \tag{4.24}$$

Either (4.22) or (4.24) may be used to calculate Δ_n. Integrals may be easier to calculate numerically than derivatives; however, Mathematica gives the same results for both formulas.

Finally, we need to calculate the expansion coefficients on the interval $[e_1, e_2]$

$$c_n^f = \langle f, \varphi_n \rangle = \int_{e_1}^{e_2} f(x)\varphi(x)\mathfrak{m}(x)dx \qquad (4.25)$$

where $f(x)$ is the payoff function.

$$Q(0,s) = E\left[e^{-\int_0^s h_u + r_u du} \middle| r_0 = x \right] = \exp\left(-\int_0^s h_0(u)du\right)P_t f_Q(x) \qquad (4.26)$$

$$R(0,s) = E\left[r_s e^{-\int_0^s h_u + r_u du} \middle| r_0 = x \right] = \exp\left(-\int_0^s h_0(u)du\right)P_t f_R(x) \qquad (4.27)$$

In equation (4.26) the payoff function $f_Q(x) = 1$; in (4.27), the payoff function

$f_R(x) = x$.

If $\psi_{\lambda_n}(k)$ and $\phi_{\lambda_n}(k)$ have opposite signs, the function $\varphi_n(x)$ will not be

continuous at k. We can compensate for this by changing the sign of either function.

This may result in changing the sign of the normalized eigenfunction. Observe the

following:

$$\langle f, -\varphi_n \rangle = -\langle f, \varphi_n \rangle = -c_n^f \qquad (4.28)$$

Thus it doesn't matter whether we change the sign of the eigenfunctions because

this causes a corresponding change in the expansion coefficients. Since each term in

(4.5) involves the product of these two factors, the effect is cancelled.

Now to calculate the expansion coefficients for $Q\ (t,u)$ we can break (4.25) into

two pieces on either side of the threshold k:

$$c_n^Q = \sqrt{\frac{\phi_{\lambda_n}(k)}{\Delta_n \psi_{\lambda_n}(k)}} \int_{e_1}^{k} \psi_{\lambda_n}(x)\mathfrak{m}(x)dx + \sqrt{\frac{\psi_{\lambda_n}(k)}{\Delta_n \phi_{\lambda_n}(k)}} \int_{k}^{e_2} \phi_{\lambda_n}(x)\mathfrak{m}(x)dx \qquad (4.29)$$

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and similarly for $R(t,u)$:

$$c_n^R = \sqrt{\frac{\phi_{\lambda_n}(k)}{\Delta_n \psi_{\lambda_n}(k)}} \int_{e_1}^{k} x\psi_{\lambda_n}(x)\mathfrak{m}(x)dx + \sqrt{\frac{\psi_{\lambda_n}(k)}{\Delta_n \phi_{\lambda_n}(k)}} \int_{k}^{e_2} x\phi_{\lambda_n}(x)\mathfrak{m}(x)dx \qquad (4.30)$$

Thus using Gorovoy and Linetsky's eigenvalue expansion technique we calculate four sets of values for each eigenvalue problem determined by f: the eigenvalues themselves, the corresponding normalized eigenfunctions φ_n evaluated at r_0, the expansion coefficients c_n^Q and the expansion coefficients c_n^R. Although there are an infinite number of eigenvalues, the terms $e^{-\lambda_n t}$ approach zero rapidly as the eigenvalues increase. The finite sum in (4.5) is a good approximation.

4.1.4 Time Integrals

(Gorovoy and Linetsky 2007) make use of time integrals to solve for the mortgage rate (see Sections 5.2 and 5.3).

Substituting (4.26) and (4.27) into (3.55) and using (4.5) gives:

$$m = \frac{\int_0^T \left(1 - e^{-m(T-u)}\right) e^{-\int_0^u h(s)ds} \left(\sum_{n=1}^{\infty} e^{-\lambda_n u} c_n^R \varphi_n(x)\right) du}{\int_0^T \left(1 - e^{-m(T-u)}\right) e^{-\int_0^u h(s)ds} \left(\sum_{n=1}^{\infty} e^{-\lambda_n u} c_n^Q \varphi_n(x)\right) du} \qquad (4.31)$$

Define the following time integral:

$$L(t,T;\lambda,m) = \int_t^T \left(1 - e^{-m(T-u)}\right) \exp\left(-\int_t^u h(s)ds - \lambda u\right) du \qquad (4.32)$$

For a constant (exogenous) prepayment rate h, this can be written as:

$$L(t,T;\lambda,m) = \frac{m\left(1 - e^{-(h+\lambda)(T-t)}\right) - (h+\lambda)\left(1 - e^{-m(T-t)}\right)}{(h+\lambda)(m-h-\lambda)} \qquad (4.33)$$

A closed-form time integral is also available for a piecewise-linear exogenous prepayment rate. Using (4.32) and interchanging the infinite sums with the integrals we can reformulate (4.31) as:

$$m = \frac{\sum_{n=1}^{\infty} L(0,T;\lambda_n,m) c_n^R \varphi_n(x)}{\sum_{n=1}^{\infty} L(0,T;\lambda_n,m) c_n^Q \varphi_n(x)}. \tag{4.34}$$

Rearranging terms gives us a non-linear homogeneous equation:

$$\sum_{n=1}^{\infty} \varphi_n(r_0) \left[\left(m c_n^Q - c_n^R \right) L(0,T;\lambda_n,m) \right] = 0. \tag{4.35}$$

To obtain the mortgage rate, solve equation (4.35) iteratively by Newton's method using the closed-form solution for the time integral in (4.33).

Seasoned mortgages and pools can be priced by solving the following equation by (Gorovoy and Linetsky 2007):

$$M_t = P_t \left\{ 1 + \frac{\sum_{n=1}^{\infty} \varphi_n(r_0) \left[\left(m c_n^Q - c_n^R \right) L(t,T;\lambda_n,m) \right]}{1 - e^{-m(T-t)}} \right\} \tag{4.36}$$

The eigenvalues, eigenfunctions and expansion coefficients are available from the solution to the differential equation and the initial short rate r_0 and $(T\text{-}t)$ is equivalent to the WAM (Weighted Average Maturity). The net coupon will be assigned to the mortgage rate m, and P_t, the balance on the mortgage pool at time t will be given the par value of $100 as this is how mortgage pools are priced in the industry.

4.1.5 Eigenfunction Expansion Procedure

The entire procedure can be summarized as follows:

1.   Calibrate the interest rate parameters from 3-month Treasury data and find the prepayment parameters using least squares including the breakpoint k. (This is discussed further in Chapter 5).

2.   Obtain the scale and speed densities from Table 1

3.   Find the general solutions to the differential equation. These solutions are listed in Table 1 as $\phi_\lambda = u_2$ and $\psi_\lambda = u_1$.

4.   Create the Wronskian from the general solutions and solve iteratively to obtain eigenvalues using boundary conditions.

5.   Calculate the functions $\psi_\lambda(x), \phi_\lambda(x), \psi_\lambda(k), \phi_\lambda(k)$ for each eigenvalue

6.   Evaluate the derivative of the Wronskian with respect to λ for each eigenvalue or calculate the norms of the eigenfunctions using my integral method.

7.   Normalize the eigenfunctions by using the results of steps 5 and 6.

8.   Integrate the results of 7 with the speed density in step 2 to get the coefficients c_n^Q

9.   Integrate the results of 7 with the product of the rate and speed density to get the coefficients c_n^R

10.   Calculate the time integral $L(0,t;\lambda,m)$ for each eigenvalue.

11.   Using eigenvalues, eigenfunctions, expansion coefficients and time integrals, solve iteratively for the mortgage rate.

$\sum_{n=1}^{\infty} \varphi_n(r_0)\left[\left(mc_n^Q - c_n^R\right)L(\lambda_n,m)\right] = 0$ or using m as the Coupon rate and setting P

= 100, calculate the price from the following equation:

$$M_t = P_t \left\{ 1 + \frac{\sum_{n=1}^{\infty} \varphi_n(r_0)\left[\left(mc_n^Q - c_n^R\right)L(t,T;\lambda_n,m)\right]}{1-e^{-m(T-t)}} \right\}.$$

4.1.6 Mortgage Rate Example

Let us assume a Cox-Ingersoll-Ross interest rate model with $\theta = 0.06, \kappa = 0.25, \sigma = 0.1$, an exogenous prepayment rate $h_0 = 0.045, \gamma = 5, r_0 = 0.09, \; k = r_0$ and $T = 30$

Table 2 shows the first 18 eigenvalues, the associated normalized eigenfunctions, expansion coefficients and time integrals for the above.

n	λ_n	$\varphi_n(k)$	c_n^Q	c_n^R	$L(\lambda_n,m)$
1	0.20734	22.74321	0.054053	0.0038034	3.550047
2	0.47884	-7.81166	0.015381	-0.0009233	1.754071
3	0.74412	-15.69468	0.006317	0.0001217	1.171104
4	1.02106	-0.60631	0.001288	-0.0001377	0.869177
5	1.29713	12.54510	-0.000110	-0.0000781	0.691413
6	1.57718	12.25948	-0.000264	-0.0000349	0.572592
7	1.85936	2.56640	-0.000096	-0.0000037	0.488071
8	2.14140	-7.74337	0.000046	0.0000093	0.425315
9	2.42418	-12.25758	0.000093	0.0000103	0.376745
10	2.70820	-9.76229	0.000072	0.0000059	0.337979
11	2.99272	-2.71330	0.000028	0.0000010	0.306395
12	3.27722	4.98740	-0.000011	-0.0000024	0.280211
13	3.56187	9.98431	-0.000031	-0.0000036	0.258139
14	3.84697	10.90601	-0.000034	-0.0000032	0.239263
15	4.13246	7.82409	-0.000024	-0.0000018	0.222938
16	4.41810	2.33608	-0.000010	-0.0000004	0.208692
17	4.70737	-3.37958	-0.000222	-0.0000199	0.196007
18	4.98941	-8.09785	0.000121	0.0000014	0.185041

Table 2 Eigenvalues, Evaluated Eigenfunctions, Expansion Coefficients and Time Integrals

We use the secant method to obtain an iterative solution. Convergence is fairly quick; in this example it only takes 7 iterations to be accurate within 10 decimal places:

The results are as follows:

```
Iteration    Mortgage Rate
---------    ---------------
    0        0.09                  Initial Guess r(0)
    1        0.1
    2        0.07910587538
    3        0.07856263478
    4        0.07852811805
    5        0.07852804112
    6        0.07852804111
    7        0.07852804111        Solution m = 7.8528%
```

Table 3: Convergence of Mortgage Rate using Secant Method

4.2 The Piecewise-Linear Prepayment Model

Prepayment models usually exhibit two types of behavior; one type when interest rates are above a certain threshold and another type when they are below. The change of variables in both the Cox-Ingersoll-Ross and Vasicek models work best when prepayment rates are linear. However, when rates are below the refinancing threshold, the prepayment behavior in reality is not exactly linear as the data below show; the behavior changes when rates drop significantly. We propose a new piecewise linear model to account for this change in behavior.

An empirical view of prepayment data from 1995 to 2005 versus short term treasury rates is shown in Figure 1.
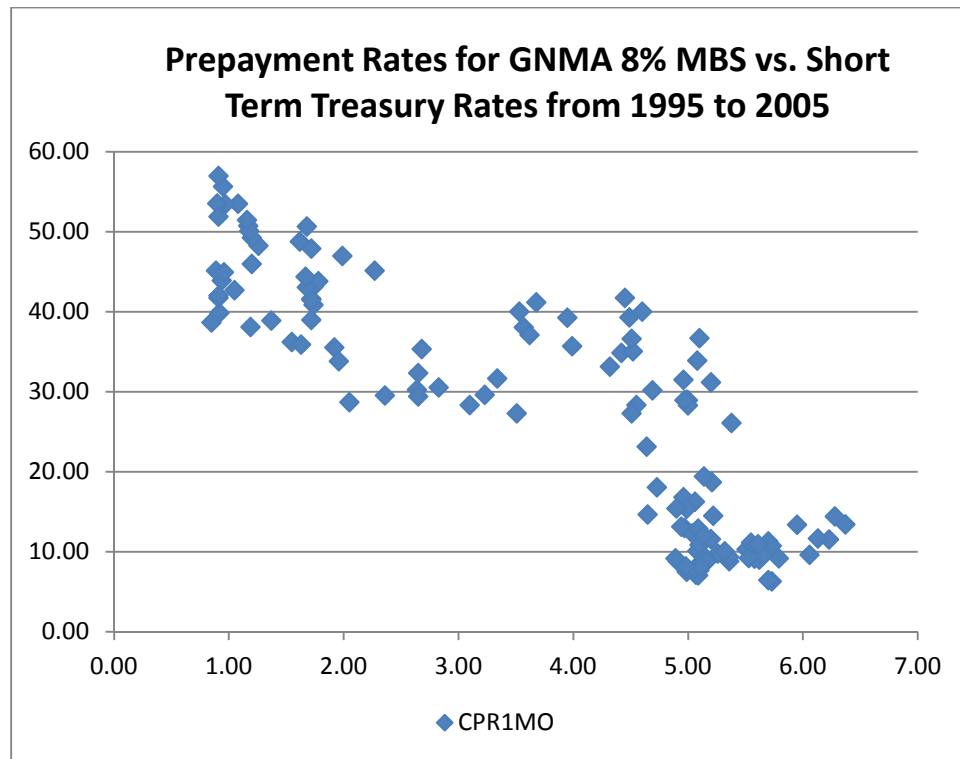
Figure 1 - Relationship between Mortgage Rates and Prepayment Rates

The model clearly indicates that prepayment rates increase significantly when the treasury rate drops below 5.25%. However, unlike (Gorovoy and Linetsky 2007), the slope of the prepayment function (the spread sensitivity) decreases significantly when the mortgage rate goes below 4.75%. In (Kolbe and Zagst 2009), the spread sensitivity drops to zero indicating a constant prepayment rate. The data in Figure 1 above suggest a piecewise linear model with two (or more) breakpoints instead of one.

4.2.1 Multiple Threshold Model

We propose a multiple threshold model. Let $k_1 > k_2 > \cdots > k_n > 0$ and let γ_i be the change in intensity at k_i where for all n, $\sum_{i=1}^{n} \gamma_i \geq 0$. We define the prepayment intensity as

$$h(r,t) = h_0(t) + \sum_{i=1}^{n} \gamma_i (k_i - r)^+ . \tag{4.37}$$

The infinitesimal generator for Cox-Ingersoll-Ross is:

$$(\mathcal{G}f)(x) = \frac{1}{2}\sigma^2 x f''(x) + \kappa(\theta - x) f'(x) - \left\{ r + \sum_{i=1}^{n} \gamma_i (k_i - x)^+ \right\} f(x) \tag{4.38}$$

Set $(\mathcal{G}f)(x) = -\lambda f(x)$ and perform the change of variable as in Appendix A of (Gorovoy and Linetsky 2007). When $x \geq k_1$, the problem is identical to that in Appendix A above where $k = k_1$, and for reasons that will become apparent later, we will set:

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$$k_1 > k_2 > \cdots > k_N > 0; \quad k_0 = \infty; \quad n = \max\{i : k_i > x\}$$

$$\Gamma_n = \sum_{i=1}^{n} \gamma_i = \text{Spread sensitivity immediately below } k_n; \; \Gamma_0 = 0$$

$$\Theta_n = \sum_{i=1}^{n} \gamma_i k_i = \text{Intensity/Threshhold dot product} \qquad (4.39)$$

$$\rho_n = \sqrt{\kappa^2 + 2\sigma^2(1-\Gamma_n)}; \quad a_n = \frac{\beta}{2} - \frac{\kappa^2\theta}{\sigma^2\rho_n}; \quad \alpha_n = \frac{2\rho_n}{\sigma^2}$$

See Appendix B for the change of variable procedure. It can be shown that the differential equation with multiple breakpoints is:

$$zu'' + \left(\frac{2\kappa\theta}{\sigma^2} - z\right)u' - \left(a_n + \frac{\Theta_n - \lambda}{\rho_n}\right)u = 0 \qquad (4.40)$$

The proof follows:

For any $x \in [k_{n+1}, k_n]$ the infinitesimal generator eigenvalue problem is:

$$\frac{1}{2}\sigma^2 x f''(x) + \kappa(\theta - x)f'(x) - \left(x + \sum_{i=1}^{n} \gamma_i(k_i - x)\right)f(x) = -\lambda f(x) \qquad (4.41)$$

This can be rewritten as:

$$\frac{1}{2}\sigma^2 x f''(x) + \kappa(\theta - x)f'(x) - \left((1-\Gamma_n)x + \Theta_n - \lambda\right)f(x) = 0 \qquad (4.42)$$

Performing the change of variable for Cox-Ingersoll-Ross using the substitutions for the dependent and independent variables from Table 1 on page 41, we gather the coefficients of u, u' and u'' separately. The coefficient of u is:

$$\frac{1}{2}\sigma^2 x \left(\frac{\kappa - \rho_n}{\sigma^2}\right)^2 + \kappa(\theta - x)\left(\frac{\kappa - \rho_n}{\sigma^2}\right) - \left[(1-\Gamma_n)x + \Theta_n - \lambda\right] \qquad (4.43)$$

Collecting terms involving the independent variable yields:

$$\left[\frac{(\kappa - \rho_n)^2}{2\sigma^2} + \frac{\rho_n\kappa - \kappa^2}{\sigma^2} - (1-\Gamma_n)\right]x + \left(\frac{\kappa^2\theta}{\sigma^2} - \frac{\rho_n\kappa\theta}{\sigma^2}\right) - \Theta_n + \lambda \qquad (4.44)$$

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The coefficient of x in (4.44) can be rearranged as follows:

$$\frac{\rho_n^2 - \kappa^2 - 2\sigma^2 (1-\Gamma_n)}{2\sigma^2} \tag{4.45}$$

But $\rho_n = \sqrt{\kappa^2 + 2\sigma^2 (1-\Gamma_n)}$ as defined in (4.39) above, reducing the expression

in (4.45) to zero. Thus (4.44) can be simplified using the definition of a_n in(4.39).

$$-a_n\rho_n - \Theta_n + \lambda \tag{4.46}$$

The coefficient of the first derivative u' is:

$$\frac{1}{2}\sigma^2 \left(\frac{z}{\alpha_n}\right) \left[2\alpha_n \left(\frac{\kappa - \rho_n}{\sigma^2}\right)\right] + \kappa \left(\theta - \frac{z}{\alpha_n}\right) \alpha_n \tag{4.47}$$

This can be simplified to

$$\rho_n \left(\frac{2\kappa\theta}{\sigma^2} - z\right) \tag{4.48}$$

And finally, the coefficient of the second derivative u'' is:

$$\frac{1}{2}\sigma^2 \left(\frac{z}{\alpha_n}\right)\alpha_n^2 = \frac{1}{2}\sigma^2 z\alpha_n = \frac{1}{2}\sigma^2 z\left(\frac{2\rho_n}{\sigma^2}\right) = \rho_n z \tag{4.49}$$

Putting together(4.46), (4.48) and (4.49) produces the differential equation:

$$\rho_n z u'' + \rho_n \left(\frac{2\kappa\theta}{\sigma^2} - z\right) u' - \left(a_n\rho_n + \Theta_n - \lambda\right)u = 0 \tag{4.50}$$

which after dividing by ρ_n gives equation(4.40) and therefore completing the proof. Any

solution to (4.40) is a linear combination of the confluent hypergeometric functions. See

(Abramowitz and Stegun 1972) and (Gorovoy and Linetsky 2007) Appendix B.

4.2.2 Example using Two Breakpoints with Cox-Ingersoll-Ross

Figure 2 - Continuity and Differentiability Across Two Breakpoints

Although our model allows for any number of breakpoints, we will examine the case where there are exactly two breakpoints k_1 and k_2. Let $N = 2$. Equation (4.40) becomes

$$zu'' + \left(\frac{2\kappa\theta}{\sigma^2} - z\right)u' - \left(a_2 + \frac{\gamma_1 k_1 + \gamma_2 k_2 - \lambda}{\rho_2}\right)u = 0, \quad x \in [0, k_2]$$

$$zu'' + \left(\frac{2\kappa\theta}{\sigma^2} - z\right)u' - \left(a_1 + \frac{\gamma_1 k_1 - \lambda}{\rho_1}\right)u = 0, \quad x \in [k_2, k_1] \qquad (4.51)$$

$$zu'' + \left(\frac{2\kappa\theta}{\sigma^2} - z\right)u' - \left(a_0 - \frac{\lambda}{\rho_0}\right)u = 0, \quad x \in [k_1, \infty)$$

The solutions to (4.51) where we used (4.16) boundary conditions on $[k_1, \infty)$ and $[0, k_2]$ are as follows:

$$\phi_1(x; \lambda) = \exp\left(\frac{(\kappa - \rho_0)x}{\sigma^2}\right) U\left(a_0 - \lambda/\rho_0, \beta, \alpha_0 x\right) \quad x \in [k_1, \infty)$$

$$\psi_1(x; \lambda) = \exp\left(\frac{(\kappa - \rho_1)x}{\sigma^2}\right) M\left(a_1 - (\lambda - \gamma_1 k_1)/\rho_1, \beta, \alpha_1 x\right) \quad x \in [k_2, k_1]$$

$$\phi_2(x; \lambda) = \exp\left(\frac{(\kappa - \rho_1)x}{\sigma^2}\right) U\left(a_1 - (\lambda - \gamma_1 k_1)/\rho_1, \beta, \alpha_1 x\right) \quad x \in [k_2, k_1]$$

$$\psi_2(x; \lambda) = \exp\left(\frac{(\kappa - \rho_2)x}{\sigma^2}\right) M\left(a_2 - (\lambda - \gamma_1 k_1 - \gamma_2 k_2)/\rho_2, \beta, \alpha_2 x\right) \quad x \in [0, k_2]$$

$$(4.52)$$

4.2.3 Finding the Coefficients

In order to solve for λ, we must assume that the solutions are continuous and differentiable across the breakpoints k_1 and k_2. Figure 2 shows the breakpoints where continuity and differentiability must be preserved for the eigenfunctions which are linear combinations of Ψ_1 and Φ_2 between the breakpoints and a multiple of Φ_1 above k_2 and a multiple of Ψ_2 below k_2. In order to accomplish this, we set up the following homogeneous equations:

$$A_1 \phi_1(k_1) - B_1 \psi_1(k_1) - A_2 \phi_2(k_1) = 0$$
$$A_1 \phi_1'(k_1) - B_1 \psi_1'(k_1) - A_2 \phi_2'(k_1) = 0$$
$$A_2 \phi_2(k_2) + B_1 \psi_1(k_2) - B_2 \psi_2(k_2) = 0$$
$$A_2 \phi_2'(k_2) + B_1 \psi_1'(k_2) - B_2 \psi_2'(k_2) = 0$$

We use the following shorthand notation: $\phi_i(k_j) = \phi_i(k_j; \lambda)$ and $\psi_i(k_j) = \psi_i(k_j; \lambda)$. Define the matrix

$$\mathbf{W}(\lambda) = \begin{bmatrix} \phi_1(k_1) & -\phi_2(k_1) & -\psi_1(k_1) & 0 \\ 0 & \phi_2(k_2) & \psi_1(k_2) & -\psi_2(k_2) \\ \phi_1'(k_1) & -\phi_2'(k_1) & -\psi_1'(k_1) & 0 \\ 0 & \phi_2'(k_2) & \psi_1'(k_2) & -\psi_2'(k_2) \end{bmatrix}$$

(4.53)

and let the vector $\mathbf{c}_\lambda = \begin{bmatrix} A_1 & A_2 & B_1 & B_2 \end{bmatrix}$

Since $\mathbf{c}_\lambda \neq \mathbf{0}$ and $\mathbf{W}\mathbf{c}_\lambda^{\mathbf{T}} = \mathbf{0}$ it follows that $|\mathbf{W}(\lambda)| = 0$. For the sake of simplification set:

$$W_{ij}(k_n) = \psi_i(k_n)\phi_j'(k_n) - \phi_j(k_n)\psi_i'(k_n) \quad i,j,n \in \{1,2\}$$
$$W_\phi(k_n) = \phi_1(k_n)\phi_2'(k_n) - \phi_2(k_n)\phi_1'(k_n)$$

(4.54)

Converting (4.53) to the algebraically equivalent echelon matrix using elementary operations yields:

$$\mathbf{W} \cong \begin{bmatrix} 1 & 0 & 0 & -W_{12}(k_1)Y \\ 0 & 1 & 0 & -W_{11}(k_1)Y \\ 0 & 0 & 1 & -W_\phi(k_1)Y \\ 0 & 0 & 0 & |\mathbf{W}| = 0 \end{bmatrix}$$

(4.55)

where

$$Y = \frac{\psi_2(k_2)}{\psi_1(k_2)W_\phi(k_1) + \phi_2(k_2)W_{11}(k_1)}$$

(4.56)

and

$$\begin{aligned}|\mathbf{W}| &= \phi_1(k_1)\phi_2(k_2)\psi_1'(k_1)\psi_2'(k_2) - \phi_1(k_1)\phi_2'(k_1)\psi_1(k_2)\psi_2'(k_2) \\ &+ \phi_1(k_1)\phi_2'(k_1)\psi_2(k_2)\psi_1'(k_2) - \phi_1(k_1)\phi_2'(k_2)\psi_2(k_2)\psi_1'(k_1) \\ &+ \phi_1'(k_1)\phi_2(k_1)\psi_1(k_2)\psi_2'(k_2) - \phi_1'(k_1)\phi_2(k_1)\psi_2(k_2)\psi_1'(k_2) \\ &+ \phi_1'(k_1)\phi_2'(k_2)\psi_1(k_2)\psi_2(k_2) - \phi_1'(k_1)\phi_2(k_2)\psi_1(k_1)\psi_2'(k_2)\end{aligned}$$

(4.57)

Thus the zeroes of (4.57) (with respect to λ) determine the eigenvalues. The coefficients can be expressed as follows:

$$
\begin{aligned}
A_1 &= W_{12}(k_1)YB_2 \\
A_2 &= W_{11}(k_1)YB_2 \\
B_1 &= W_\phi(k_1)YB_2
\end{aligned}
\tag{4.58}
$$

Set $\Delta = YB_2$. We can rewrite all coefficients in terms of the normalization constant Δ.

$$
\begin{aligned}
A_1 &= W_{12}(k_1)\Delta \\
A_2 &= W_{11}(k_1)\Delta \\
B_1 &= W_\phi(k_1)\Delta \\
B_2 &= \Delta/Y = \frac{\Delta}{\psi_2(k_2)}\left[\psi_1(k_2)W_\phi(k_1)+\phi_2(k_2)W_{11}(k_1)\right]
\end{aligned}
\tag{4.59}
$$

To normalize the eigenfunctions, we must have $\langle\varphi,\varphi\rangle=1$. Rewriting the norm as a piecewise integral:

$$
\int_0^{k_2}\left[B_2\psi_2(x)\right]^2 m(x)\,dx + \int_{k_2}^{k_1}\left[B_1\psi_1(x)+A_2\phi_2(x)\right]^2 m(x) \\
+ \int_{k_1}^{\infty}\left[A_1\phi_1(x)\right]^2 m(x)\,dx
\tag{4.60}
$$

and substituting (4.59) into (4.60) produces:

$$
\frac{\Delta^2}{Y^2}\int_0^{k_2}\psi_2^2(x)m(x)\,dx + \Delta^2\int_{k_2}^{k_1}\left[W_\phi(k_1)\psi_1(x)+W_{11}(k_1)\phi_2(x)\right]^2 m(x)\,dx \\
+ \Delta^2 W_{12}^2(k_1)\int_{k_1}^{\infty}\phi_1^2(x)m(x)\,dx = 1,
\tag{4.61}
$$

which gives:

$$\Delta = \left[\frac{1}{Y^2} \int_0^{k_2} \psi_2^2(x) m(x) dx + \int_{k_2}^{k_1} \left[W_\phi(k_1) \psi_1(x) + W_{11}(k_1) \phi_2(x) \right]^2 m(x) dx \right.$$
$$\left. + W_{12}^2(k_1) \int_{k_1}^\infty \phi_1^2(x) m(x) dx \right]^{-1/2}$$

(4.62)

This can be expanded to

$$\Delta = \left[\left[\frac{\psi_1(k_2) W_\phi(k_1) + \phi_2(k_2) W_{11}(k_1)}{\psi_2(k_2)} \right]^2 \int_0^{k_2} \psi_2^2(x) m(x) dx + \right.$$
$$W_\phi^2(k_1) \int_{k_2}^{k_1} \psi_1^2(x) m(x) dx + 2 W_\phi(k_1) W_{11}(k_1) \int_{k_2}^{k_1} \psi_1(x) \phi_2(x) m(x) dx$$

(4.63)

$$\left. + W_{11}^2(k_1) \int_{k_2}^{k_1} \phi_2^2(x) m(x) dx + W_{12}^2(k_1) \int_{k_1}^\infty \phi_1^2(x) m(x) dx \right]^{-1/2}$$

The normalization constant can be solved by numerical integration. Putting everything together gives us the following set of orthonormal eigenfunctions:

$$\varphi_n(x) = \begin{cases} \dfrac{\Delta_n \psi_2(x; \lambda_n)}{\psi_2(k_2; \lambda_n)} \left[\psi_1(k_2; \lambda_n) W_\phi(k_1; \lambda_n) + \phi_2(k_2; \lambda_n) W_{11}(k_1; \lambda_n) \right] & 0 \le x \le k_2 \\[2mm] \Delta_n \left[\psi_1(x; \lambda_n) W_\phi(k_1; \lambda_n) + \phi_2(x; \lambda_n) W_{11}(k_1; \lambda_n) \right] & k_2 \le x \le k_1 \\[2mm] \Delta_n W_{12}(k_1; \lambda_n) \phi_1(x; \lambda_n) & k_1 \le x < \infty \end{cases}$$

(4.64)

We have illustrated how to find the λ_n and the $\varphi_n(x)$. To calculate the expansion coefficients, we use the following piecewise integrals:

$$c_n^Q = \langle 1, \varphi_n \rangle = \Delta_n \left[\frac{\psi_1(k_2; \lambda_n) W_\phi(k_1; \lambda_n) + \phi_2(k_2; \lambda_n) W_{11}(k_1; \lambda_n)}{\psi_2(k_2; \lambda_n)} \int_0^{k_2} \psi_2(x; \lambda_n) m(x) dx \right.$$
$$+ W_\phi(k_1; \lambda_n) \int_{k_2}^{k_1} \psi_1(x; \lambda_n) m(x) dx + W_{11}(k_1; \lambda_n) \int_{k_2}^{k_1} \phi_2(x; \lambda_n) m(x) dx$$

(4.65)

$$\left. + W_{12}(k_1; \lambda_n) \int_{k_1}^\infty \phi_1(x; \lambda_n) m(x) dx \right]$$

and

$$c_n^R = \langle x, \varphi_n \rangle = \Delta_n \left[\frac{\psi_1(k_2;\lambda_n)W_\phi(k_1;\lambda_n) + \phi_2(k_2;\lambda_n)W_{11}(k_1;\lambda_n)}{\psi_2(k_2;\lambda_n)} \int_0^{k_2} x\psi_2(x;\lambda_n)m(x)dx \right.$$

$$+ W_\phi(k_1;\lambda_n)\int_{k_2}^{k_1} x\psi_1(x;\lambda_n)m(x)dx + W_{11}(k_1;\lambda_n)\int_{k_2}^{k_1} x\phi_2(x;\lambda_n)m(x)dx \qquad (4.66)$$

$$\left. + W_{12}(k_1;\lambda_n)\int_{k_1}^{\infty} x\phi_1(x;\lambda_n)m(x)dx \right]$$

To obtain the price and yield, it remains to complete the steps in Section 4.1.5.

4.2.4 Continuity and Differentiability of Eigenfunctions

The formulas for eigenfunctions differ on either side of each interest-rate threshold. At the threshold itself, the two formulas should not only produce the same value, their derivatives should also be equivalent. Thus we need to show:

$$\lim_{x \to k^-} \varphi_n(x) = \lim_{x \to k^+} \varphi_n(x) \text{ and } \lim_{x \to k^-} \varphi_n'(x) = \lim_{x \to k^+} \varphi_n'(x) \qquad (4.67)$$

For the single threshold model, it is fairly obvious that this is true because from (4.23) we arrive at:

$$\varphi_n(k) = \sqrt{\frac{\phi_n(k)\psi_n(k)}{\Delta_n}} \qquad (4.68)$$

For the double threshold model, we can derive the following eigenfunctions from (4.64).

$$\varphi(k_2) = \Delta_n \left[\psi_1(k_2;\lambda_n)W_\phi(k_1;\lambda_n) + \phi_2(k_2;\lambda_n)W_{11}(k_1;\lambda_n) \right] \qquad (4.69)$$

For the threshold at k_1 we must show that

$$\psi_1(k_1)W_\phi(k_1) + \phi_2(k_1)W_{11}(k_1) = W_{12}(k_1)\phi_1(k_1) \qquad (4.70)$$

This can be done by substituting (4.54) into (4.70) and the equality follows. We then can show that

$$\varphi\left(k_1\right)=\Delta_n W_{12}\left(k_1;\lambda_n\right)\phi_1\left(k_1;\lambda_n\right) \tag{4.71}$$

Continuity and differentiability follows from the theory of hypergeometric functions which states that Hypergeometric functions of the form $_1F_1(a,b;x)$ converge absolutely for all x. See (Andrews, Askey and Ranjan 1999) Theorem 2.1.1. Since the eigenfunctions are linear combinations of the Kummer and Triconmi confluent hypergeometric functions, it suffices to show that these functions and their derivatives converge at all meaningful points. For the Cox-Ingersoll-Ross model, the transformation for the change of variable is simply a constant, so we only need to show that convergence holds on the positive real line.

The Kummer confluent hypergeometric function is in the form $_1F_1(a,b;x)$ which is known to converge whenever $b \notin \mathbb{Z}^-$:

$$M\left(a,b,z\right)=\sum_{n=0}^{\infty}\frac{\left(a\right)_n z^n}{\left(b\right)_n n!} \tag{4.72}$$

The Tricomi confluent hypergeometric function, which is derived from the Kummer function, converges everywhere except at $z = 0$ when $b > 1$ which is required by the constraint $2k\theta > \sigma^2$:

$$U\left(a,b,z\right)=\frac{\Gamma\left(1-b\right)}{\Gamma\left(1+a-b\right)}M\left(a,b,z\right)+\frac{\Gamma\left(b-1\right)}{\Gamma\left(a\right)}z^{1-b}M\left(1+a-b,2-b,z\right) \tag{4.73}$$

The derivative of the Kummer function must also converge since it is simply a constant multiple of the shifted Kummer function:

$$\frac{d}{dz}M\left(a,b,z\right)=\frac{a}{b}M\left(a+1,b+1,z\right) \tag{4.74}$$

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The derivative of the Tricomi function also converges when $z > 0$ since it is likewise a multiple of the shifted Tricomi function:

$$\frac{d}{dz}U(a,b,z) = -aU(a+1,b+1,z) \tag{4.75}$$

Table 4 shows the values of the eigenfunctions and their derivatives at each of the breakpoints for each of the first ten eigenvalues for both the single- and double-threshold Cox-Ingersoll-Ross models.

Single Threshold Model			
n	λ	$\varphi(k)$	$\varphi'(k)$
1	0.1451	36.0216	212.53
2	0.4978	10.5354	-970.09
3	0.8225	-15.9138	-979.66
4	1.1496	-22.4293	78.35
5	1.4814	-11.8313	1128.88
6	1.8128	4.1439	1462.07
7	2.144	15.8601	957.32
8	2.4765	18.869	-37.6
9	2.8101	13.9053	-1027.3

Double Threshold Model				
n	λ	$\varphi(k_1)$	$\varphi(k_2)$	$\varphi'(k_1)$	$\varphi'(k_2)$
1	0.1601	33.8005	31.0856	355.795	386.75
2	0.5115	-17.7396	-23.0689	809.601	703.46
3	0.8233	-8.8276	0.1608	-1172.93	-1343.77
4	1.1474	21.6821	17.2892	356.855	871.16
5	1.4822	-17.9826	-21.2886	750.207	162.793
6	1.8139	5.3977	15.0088	-1474.62	-1179.05
7	2.1426	8.2843	-3.1843	1451.31	1710.51
8	2.4736	-16.6969	-8.2962	-755.271	-1562.92
9	2.8079	-17.9395	-15.6316	233.407	-888.589
10	3.1433	13.3155	17.8357	-1147.97	-50.5179

Table 4: Eigenvalues, Evaluated Eigenfunctions and Derivatives of Eigenfunctions for Single and Double Threshold Models

4.2.5 Discounted Claims

Table 5 shows the first ten eigenvalues, evaluated eigenfunctions at the breakpoint(s) and expansion coefficients for the single and double threshold CIR models. Since the pricing model is expressed in terms of the time t values of European style contingent claims maturing at time u with payoffs $f_Q(r_u)=1$ and $f_R(r_u)=r_u$, we also show the convergence of the maturity 0 and maturity 1 values of these claims at time 0. See (Gorovoy and Linetsky 2007) p. 552. The maturity-0 values should converge to 1 and r_0 respectively, while the maturity-1 values should be at a slight discount to 1 and the uncertain value r_1 respectively. Thus we can approximate the maturity-0 values by choosing a cutoff point, say N=10:

$$Q(0,0) = \sum_{n=1}^{N} c_n^Q \varphi_n(r_0) \approx 1 \text{ and } R(0,0) = \sum_{n=1}^{N} c_n^R \varphi_n(r_0) \approx r_0 \qquad (4.76)$$

We can also approximate the maturity-1 values:

$$Q(0,1) = e^{-h_0} \sum_{n=1}^{N} e^{-\lambda_n} c_n^Q \varphi_n(r_0) < 1 \text{ and } Q(0,1) = e^{-h_0} \sum_{n=1}^{N} e^{-\lambda_n} c_n^R \varphi_n(r_0) < r_1 \qquad (4.77)$$

For comparison, we have listed these Q(0,1) and R(0,1) which show the values discounted for one year. These values are more deeply discounted for the double threshold model than for the single threshold model.

| **April 2005 GNMA 8's – Single Threshold CIR** |
n	*λ*	*φ(x)*	*cQ*	*cR*	*Q(0,0)*	*R(0,0)*	*Q(0,1)*	*R(0,1)*
1	0.14512	27.0473	0.02890	0.00200	0.7817	0.05401	0.59804	0.04132
2	0.49780	33.5865	0.00389	-0.00076	0.9124	0.02856	0.66829	0.02764
3	0.82251	25.6948	0.00286	0.00010	0.9859	0.03124	0.69687	0.02868
4	1.14962	9.1200	0.00113	-0.00002	0.9963	0.03105	0.69977	0.02862
5	1.48138	-6.7726	0.00035	-0.00001	0.9939	0.03115	0.69930	0.02864
6	1.81280	-17.9582	0.00000	-0.00002	0.9938	0.03142	0.69929	0.02868
7	2.14401	-22.8867	-0.00010	-0.00001	0.9961	0.03167	0.69953	0.02871
8	2.47647	-21.9088	-0.00010	-0.00001	0.9983	0.03180	0.69969	0.02872
9	2.81010	-16.6552	-0.00006	0.00000	0.9992	0.03184	0.69974	0.02872
10	3.14406	-8.9543	-0.00002	0.00000	0.9994	0.03183	0.69974	0.02872
**April 2005 GNMA 8's – Double Threshold CIR**
n	*λ*	*φ(x)*	*cQ*	*cR*	*Q(0,0)*	*R(0,0)*	*Q(0,1)*	*R(0,1)*
1	0.16006	25.1738	0.0287	0.00202	0.7224	0.05097	0.54612	0.03853
2	0.51150	-33.7959	-0.00503	0.00068	0.8924	0.02807	0.63655	0.02635
3	0.82330	25.9557	0.00318	0.00011	0.9751	0.03091	0.66874	0.02746
4	1.14737	-7.99391	-0.00093	0.00003	0.9825	0.03070	0.67083	0.02740
5	1.48219	-7.91104	0.00003	-0.00002	0.9823	0.03088	0.67079	0.02744
6	1.81388	18.9625	0.00028	0.00002	0.9876	0.03132	0.67155	0.02750
7	2.14261	-23.549	-0.00029	-0.00002	0.9944	0.03170	0.67226	0.02754
8	2.47359	21.8266	0.00018	0.00001	0.9984	0.03187	0.67257	0.02755
9	2.80792	15.9947	0.00007	0.00000	0.9995	0.03189	0.67262	0.02755
10	3.14325	-8.11643	0.00002	0.00000	0.9993	0.03186	0.67245	0.02753

Table 5: April 2005 GNMA 8% MBS showing convergence of values of 0 and 1-year maturity claims

with payoffs $f_Q(r_u) = 1$ and $f_R(r_u) = r_u$

4.3 The Vasicek Model

The Vasicek model is a simpler mean-reverting model than Cox-Ingersoll-Ross because it is linear. The drawback is that it may occasionally become negative. For the Vasicek model we need to solve the following differential equation derived from the infinitesimal generator:

$$\frac{1}{2}\sigma^2 f''(x) + \kappa(\theta - x) f'(x) - \left[x + \gamma(k - x)^+ \right] f(x) = -\lambda f(x) \tag{4.78}$$

This can be divided into two sections:

$$\tfrac{1}{2}\sigma^2 f''(x) + \kappa(\theta - x) f'(x) - \left[(1 - \gamma) x + \gamma k \right] f(x) = -\lambda f(x), x < k \tag{4.79}$$

and

$$\tfrac{1}{2}\sigma^2 f''(x) + \kappa(\theta - x) f'(x) - x f(x) = -\lambda f(x), x \geq k \tag{4.80}$$

However we can simply use Equation (4.79) and set $\gamma = 0$ to obtain (4.80).

$$\frac{1}{2}\sigma^2 f''(x) + \kappa(\theta - x) f'(x) - \left[(1 - \gamma) x + \gamma k \right] f(x) = -\lambda f(x) \tag{4.81}$$

We make the following substitutions:

$$z = \frac{\sqrt{2\kappa}}{\sigma}(\theta - x) \text{ and } f(x) = e^{z^2/4} w(z) \tag{4.82}$$

From (4.82) we obtain the following derivatives:

$$f'(x) = -\frac{e^{z^2/4} \sqrt{\kappa/2}}{\sigma} \left[zw(z) + 2w'(z) \right] \tag{4.83}$$

$$f''(x) = \frac{2\kappa e^{z^2/4}}{\sigma^2} \left[w''(z) + zw'(z) + \left(\frac{z^2}{4} + \frac{1}{2} \right) w(z) \right] \tag{4.84}$$

Inserting (4.83) and (4.84) into (4.81) produces:

$$\left[w''(z) + zw'(z) + \left(\frac{z^2}{4} + \frac{1}{2} \right) w(z) \right] - \frac{z}{2} \left[zw(z) + 2w'(z) \right]$$
$$- \frac{\left[(1-\gamma)x + \gamma k - \lambda \right]}{\kappa} w(z) = 0 \tag{4.85}$$

Collecting like terms, we simplify(4.85):

$$w''(z) + (z - z)w'(z) + \left(\frac{z^2}{4} + \frac{1}{2} - \frac{z^2}{2} - \frac{\left[(1-\gamma)x + \gamma k - \lambda \right]}{\kappa} \right) w(z) = 0 \tag{4.86}$$

$$w''(z) + \left(\frac{1}{2} - \frac{z^2}{4} - \frac{\left[(1-\gamma)x + \gamma k - \lambda \right]}{\kappa} \right) w(z) = 0 \tag{4.87}$$

Now substitute $x = \theta - \frac{\sigma}{\sqrt{2\kappa}} z$:

$$\frac{\left[(1-\gamma)x + \gamma k - \lambda \right]}{\kappa} = \frac{\left[(1-\gamma)\left(\theta - \frac{\sigma}{\sqrt{2\kappa}} z \right) + \gamma k - \lambda \right]}{\kappa} = \frac{(1-\gamma)\theta + \gamma k - \lambda}{\kappa} - \frac{\sigma(1-\gamma)}{\sqrt{2\kappa^3}} z$$
(4.88)

Letting $\alpha_\gamma = \sigma \sqrt{\frac{2}{\kappa^3}}(1-\gamma)$ we can rewrite (4.88) as: $\frac{(1-\gamma)\theta + \gamma k - \lambda}{\kappa} - \frac{\alpha_\gamma}{2} z$

Now complete the square:

$$\frac{z^2}{4} + \frac{(1-\gamma)\theta + \gamma k - \lambda}{\kappa} - \frac{\alpha_\gamma z}{2} = \frac{z^2}{4} - \frac{\alpha_\gamma z}{2} + \frac{\alpha_\gamma^2}{4} - \frac{\alpha_\gamma^2}{4} + \frac{(1-\gamma)\theta + \gamma k - \lambda}{\kappa} =$$
$$\frac{1}{4}(z - \alpha_\gamma)^2 - \frac{\alpha_\gamma^2}{4} + \frac{(1-\gamma)\theta + \gamma k - \lambda}{\kappa}$$

Now letting $\mu = \frac{\alpha_\gamma^2}{4} - \frac{(1-\gamma)\theta + \gamma k - \lambda}{\kappa}$ and substituting into (4.87) gives:

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$$w''(z) + \left(\frac{1}{2} + \mu - \frac{1}{4}(z - \alpha_\gamma)^2 \right) w(z) = 0 \qquad (4.89)$$

which according to (Abramowitz and Stegun 1972), page 686, has the following solutions:

$$w_1(z) = D_\mu(z - \alpha_\gamma)$$
$$w_2(z) = \frac{1}{\pi} \Gamma(-\mu) \left[\sin\left(\pi(\mu + \tfrac{1}{2})\right) D_\mu(z - \alpha_\gamma) + D_\mu(\alpha_\gamma - z) \right] \qquad (4.90)$$

where $D_\mu(x)$ is the Parabolic Cylinder function.

Proposition 4.1 of (Gorovoi and Linetsky 2004) shows that $D_{\mu(\lambda)}(\alpha - z)$ is a solution to (4.89) . Another linearly independent solution to (4.89) is $D_{\mu(\lambda)}(z - \alpha)$ according to Chapter 19 (Parabolic Cylinder Functions) in (Abramowitz and Stegun 1972).

For $x \geq k$ we can set $\gamma = 0$. Thus $\mu_0(\lambda) = \dfrac{\sigma^2}{\kappa^3} + \dfrac{\lambda - \theta}{\kappa}$.

Since the upper boundary is the same as that described in (Gorovoi and Linetsky 2004) we can define

$$\phi_\lambda(x - k) = e^{z^2/4} D_\mu(\alpha - z); \, x \in [k, \infty) \qquad (4.91)$$

We can use the other linearly independent solution from (Abramowitz and Stegun 1972) (19.3.8 p. 687):

$$\psi_\lambda(x - k) = -e^{z^2/4} \frac{\Gamma(-\mu_\gamma(\lambda))}{\pi} \left[\sin\left(\pi\left[\mu_\gamma(\lambda) + \tfrac{1}{2}\right]\right) D_{\mu_\gamma(\lambda)}(\alpha - z) + D_{\mu_\gamma(\lambda)}(z - \alpha) \right]; \qquad (4.92)$$
$$x \in (-\infty, k]$$

4.3.1 Eigenfunctions and Boundary Conditions

Since the Vasicek model allows for negative interest rates, we will use Linetsky's reflecting boundary for the shadow rate (Linetsky 2004). We will divide the interest rate spectrum into three parts: greater than the refinancing threshold k, between 0 and the refinancing threshold, and the negative "shadow" rate. This is identical to the "triple-ramp" Cox-Ingersoll-Ross model except for two things: the choice of eigenfunctions $\phi_1, \psi_1, \phi_2, \psi_2$ and the interval (l,r) which is $(-\infty,\infty)$ instead of $[0, \infty)$.

For Vasicek, we will use the following transformations to define the eigenfunctions based on the solutions in (4.90):

$$\phi_1(x) = e^{z^2/4} D_\mu(\alpha - z); \quad k \le x < \infty$$

$$\left.\begin{array}{l} \phi_2(x) = e^{z^2/4} D_\mu(\alpha(1-\gamma) - z) \\[2mm] \psi_1(x) = e^{z^2/4} \dfrac{1}{\pi} \Gamma(-\mu)\left[\sin\left(\pi\left(\mu + \tfrac{1}{2}\right)\right) D_\mu(z - \alpha(1-\gamma)) + D_\mu(\alpha(1-\gamma) - z)\right] \end{array}\right\} 0 \le x < k$$

$$\psi_2(x) = e^{z^2/4} D_{\lambda/\kappa}(z); \quad -\infty < x < 0$$

where $z = \dfrac{\sqrt{2\kappa}}{\sigma}(\theta - x)$ and $\alpha = \sigma\sqrt{\dfrac{2}{\kappa^3}}$.

We will use the following boundary conditions:

$$\lim_{x \to -\infty} \frac{f'(x)}{\mathfrak{s}(x)} = 0 \text{ and } \lim_{x \to \infty} \frac{f'(x)}{\mathfrak{s}(x)} = 0$$

For Vasicek, $\mathfrak{s}(x) = \exp\left[\dfrac{\kappa(\theta - x)^2}{\sigma^2}\right] = e^{\frac{1}{2}z^2}$. The derivative of the Parabolic

Cylinder function $D'_\nu(z) = \nu D_{\nu-1}(z) - \dfrac{z}{2} D_\nu(z)$. Also note that $z'(x) = \dfrac{\sqrt{2\kappa}}{\sigma}$.

It can be shown that $\lim\limits_{z\to\pm\infty} e^{-\frac{z^2}{4}} D_\nu(z) = \lim\limits_{z\to\pm\infty} e^{-\frac{z^2}{2}} H_\nu(z) = 0$ because the Hermite

function is a polynomial of order ν. The Exponential functions grow much more rapidly

than the polynomial functions; hence the exponential function in the denominator forces

the limit to 0. From this we can conclude the following:

$$\phi_1'(x) = e^{\frac{z^2}{4}}\left[\frac{zz'}{2} D_\mu(\alpha - z) - D_\mu'(\alpha - z) z'\right]$$

$$\lim_{x\to\infty} \frac{\phi_1'(x)}{\mathit{s}(x)} = \lim_{z\to-\infty} \frac{\sqrt{2\kappa}}{\sigma} e^{-\frac{z^2}{4}}\left[D_\mu(\alpha - z) - \mu D_{\mu-1}(\alpha - z)\right] = 0$$

$$\psi_2'(x) = e^{\frac{z^2}{4}}\left[\frac{zz'}{2} D_\mu(z) + D_\nu'(z) z'\right]$$

$$\lim_{x\to\infty} \frac{\psi_2'(x)}{\mathit{s}(x)} = \lim_{z\to-\infty} \frac{\sqrt{2\kappa}}{\sigma} \nu e^{-\frac{z^2}{4}} D_{\nu-1}(z) = 0$$

$$\begin{aligned}
A_1\phi_1(k) - B_1\psi_1(k) - A_2\phi_2(k) &= 0 \\
A_1\phi_1'(k) - B_1\psi_1'(k) - A_2\phi_2'(k) &= 0 \\
A_2\phi_2(0) + B_1\psi_1(0) - B_2\psi_2(0) &= 0 \\
A_2\phi_2'(0) + B_1\psi_1'(0) - B_2\psi_2'(0) &= 0
\end{aligned} \qquad (4.93)$$

To find the eigenvalues we set the following determinant to zero:

$$\mathbf{W}(\lambda) = \begin{vmatrix}
\phi_1(k) & -\phi_2(k) & -\psi_1(k) & 0 \\
0 & \phi_2(0) & \psi_1(0) & -\psi_2(0) \\
\phi_1'(k) & -\phi_2'(k) & -\psi_1'(k) & 0 \\
0 & \phi_2'(0) & \psi_1'(0) & -\psi_2'(0)
\end{vmatrix} = 0 \qquad (4.94)$$

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4.3.2 Normalizing the constants

We can use the same process to normalize the eigenfunctions for the Vasicek model as with the two-breakpoint CIR model. The formula below is identical to (5.24) except for the limits of integration.

$$\left\|\varphi(x)\right\|^2 = \int_{-\infty}^{0}\left|B_2\psi_2(x)\right|^2 m(x)\,dx + \int_{0}^{k}\left|A_2\phi(x)+B_1\psi(x)\right|^2 m(x)\,dx + \int_{k}^{\infty}\left|A_1\phi_1(x)\right|^2 m(x)\,dx = 1$$

This allows us to represent the coefficients in terms of the normalization constant Δ:

$$A_1 = W_{12}(k)\Delta$$
$$A_2 = W_{11}(k)\Delta$$
$$B_1 = W_{\phi}(k)\Delta$$
$$B_2 = \Delta/Y = \frac{\Delta}{\psi_2(0)}\left[\psi_1(0)W_{\phi}(k)+\phi_2(0)W_{11}(k)\right]$$

$$\text{where } Y = \frac{\psi_2(0)}{\psi_1(0)W_{\phi}(k)+\phi_2(0)W_{11}(k)} \text{ and}$$

$$W_{ij}(k) = \psi_i(k)\phi_j'(k)-\phi_j(k)\psi_i'(k) \quad i,j\in\{1,2\}$$
$$W_{\phi}(k) = \phi_1(k)\phi_2'(k)-\phi_2(k)\phi_1'(k)$$

This allows us to solve for Δ:

$$\Delta = \left[\frac{1}{Y^2}\int_{-\infty}^{0}\psi_2^2(x)m(x)\,dx + \int_{0}^{k}\left[W_{\phi}(k)\psi_1(x)+W_{11}(k)\phi_2(x)\right]^2 m(x)\,dx\right.$$
$$\left. +W_{12}^2(k)\int_{k}^{\infty}\phi_1^2(x)m(x)\,dx\right]^{-1/2}$$

The completely normalized eigenfunction for the Vasicek model can be defined as follows:

$$\varphi_n(x) = \begin{cases} \dfrac{\Delta_n \psi_2(x;\lambda_n)}{\psi_2(k_2;\lambda_n)} \Big[\psi_1(0;\lambda_n)W_\phi(k;\lambda_n) + \phi_2(0;\lambda_n)W_{11}(k;\lambda_n) \Big] & -\infty \le x \le 0 \\[2mm] \Delta_n \Big[\psi_1(x;\lambda_n)W_\phi(k;\lambda_n) + \phi_2(x;\lambda_n)W_{11}(k;\lambda_n) \Big] & 0 \le x \le k \\[2mm] \Delta_n W_{12}(k;\lambda_n)\phi_1(x;\lambda_n) & k \le x < \infty \end{cases} \qquad (4.95)$$

The expansion coefficients can be written by integrating over the expanded interval $(-\infty,\infty)$ instead of the positive interval $[0,\infty)$ used by the Cox-Ingersoll-Ross model. In their complete form the Vasicek coefficients are:

$$c_n^Q = \langle x, \varphi_n \rangle =$$
$$\Delta_n \Bigg[\frac{\psi_1(0;\lambda_n)W_\phi(k;\lambda_n) + \phi_2(0;\lambda_n)W_{11}(k;\lambda_n)}{\psi_2(0;\lambda_n)} \int_{-\infty}^{0} \psi_2(x;\lambda_n)m(x)dx$$
$$+W_\phi(k;\lambda_n)\int_0^k \psi_1(x;\lambda_n)m(x)dx + W_{11}(k;\lambda_n)\int_0^k \phi_2(x;\lambda_n)m(x)dx$$
$$+W_{12}(k;\lambda_n)\int_k^\infty \phi_1(x;\lambda_n)m(x)dx \Bigg] \qquad (4.96)$$

and

$$c_n^R = \langle x, \varphi_n \rangle =$$
$$\Delta_n \Bigg[\frac{\psi_1(0;\lambda_n)W_\phi(k;\lambda_n) + \phi_2(0;\lambda_n)W_{11}(k;\lambda_n)}{\psi_2(0;\lambda_n)} \int_{-\infty}^{0} x\psi_2(x;\lambda_n)m(x)dx$$
$$+W_\phi(k;\lambda_n)\int_0^k x\psi_1(x;\lambda_n)m(x)dx + W_{11}(k;\lambda_n)\int_0^k x\phi_2(x;\lambda_n)m(x)dx$$
$$+W_{12}(k;\lambda_n)\int_k^\infty x\phi_1(x;\lambda_n)m(x)dx \Bigg] \qquad (4.97)$$

4.3.3 Applying The Vasicek Model

Applying the Vasicek model to the data, we must first calibrate the interest rate parameters to the Vasicek model and use the single threshold prepayment parameters. The Vasicek parameters are $\kappa = 0.40999$, $\sigma = 0.06062$ and $\theta = 0.05034$; the

prepayment parameters are the same as for the single-threshold CIR model in the previous section.

				April 2005 GNMA 8's – Vasicek Model				
n	**λ**	**φ(x)**	**cQ**	**cR**	**Q(0,0)**	**R(0,0)**	**Q(0,1)**	**R(0,1)**
1	0.14034	-0.08971	-10.4726	-0.63525	0.93950	0.05699	0.81649	0.04953
2	0.51259	0.02936	-0.05996	-0.57727	0.93774	0.04004	0.81543	0.03938
3	0.93183	0.06660	0.30990	-0.08401	0.95838	0.03444	0.82356	0.03717
4	1.35674	-0.03635	-0.32961	-0.01237	0.97036	0.03489	0.82665	0.03729
5	1.75768	-0.05214	-0.13477	-0.00376	0.97738	0.03509	0.82786	0.03732
6	2.18883	0.03943	0.19559	0.00186	0.98510	0.03516	0.82872	0.03733
7	2.59037	0.04211	0.04147	0.00214	0.98684	0.03525	0.82885	0.03734
8	3.01704	-0.04093	-0.12963	-0.00162	0.99215	0.03532	0.82911	0.03734
9	3.42386	-0.03455	-0.00483	-0.00143	0.99232	0.03537	0.82912	0.03734
10	3.84423	0.04166	0.08922	0.00136	0.99603	0.03543	0.82920	0.03734

Table 6 – Eigenvalues, evaluated eigenfunctions and expansion coefficients for the Vasicek Model

Applying the model results in a price of \$108.902, which slightly higher (21 basis points) than the single-threshold CIR model price of \$108.6901237 in the previous section. This example shows that the interest rate model (i.e. CIR or Vasicek) does not affect the results significantly. We will show later that this behavior is consistent over a ten-year period.

We can verify the continuity and differentiability at the breakpoints at k and 0 in a similar way that we did for the double threshold CIR model. The parabolic cylinder function is defined in (Abramowitz and Stegun 1972) as a linear combination of the Kummer function which converges everywhere on the real line:

$$D_\mu(x) = \frac{\sqrt{\pi}\,2^{\frac{1}{2}\mu}\,e^{-\frac{1}{4}x^2}}{\Gamma\left(\frac{1-\mu}{2}\right)} M\left(-\tfrac{1}{2}\mu, \tfrac{1}{2}, \tfrac{1}{2}x^2\right) - \frac{\sqrt{\pi}\,2^{\frac{1}{2}(1+\mu)}\,x\,e^{-\frac{1}{4}x^2}}{\Gamma(-\mu/2)} M\left(\frac{1-\mu}{2}, \tfrac{3}{2}, \tfrac{1}{2}x^2\right) \quad (4.98)$$

Since the argument *x* in (4.98) appears only in the numerator or as an exponent or as an argument to the Kummer function, we have convergence everywhere. Since the

change of variable for the Vasicek model is a linear function of *x*, we also have convergence on the real line for all possible values of the interest rate, positive and negative.

The derivative of the parabolic cylinder function can be expressed as a linear combination of itself and a shifted function which means that the derivative converges everywhere along the real line:

$$D_\nu'(z) = \nu D_{\nu-1}(z) - \tfrac{1}{2} z D_\nu(z) \tag{4.99}$$

The eigenvalues, evaluated eigenfunctions, expansion coefficients and convergence of *Q* and *R* are displayed in Table 6.

4.4 Default Models

We propose a new continuous default model which can be solved using the eigenfunction expansion approach. This model is flexible and can be adapted to a constant default rate (CDR) or to the time-dependent Standard Default Assumption (SDA) used in the industry.

Historically, defaults have represented a much smaller proportion of mortgage terminations than prepayments; however, in recent years, there has been a significant increase in the number of defaults especially in the subprime arena. Figure 3 shows how foreclosure rates have rapidly increased from January 2005 until June 2010. Although defaults are not exactly the same as foreclosures, since a borrower who is able to make payments on a house may choose not to do so if his mortgage is "under water", they are quite similar. According to (Bhattacharya, Berliner and Lieber 2006) "a default is defined as the event where the borrower loses title to the property." (The Bond Market

Association 1999) defines foreclosure as a "legal procedure for enforcing payment of a debt by seizing and selling the mortgaged property". While (Y. Goncharov 2006), (Gorovoy and Linetsky 2007) and (Kolbe and Zagst 2009) present continuous prepayment models, none of them have considered models that include defaults.
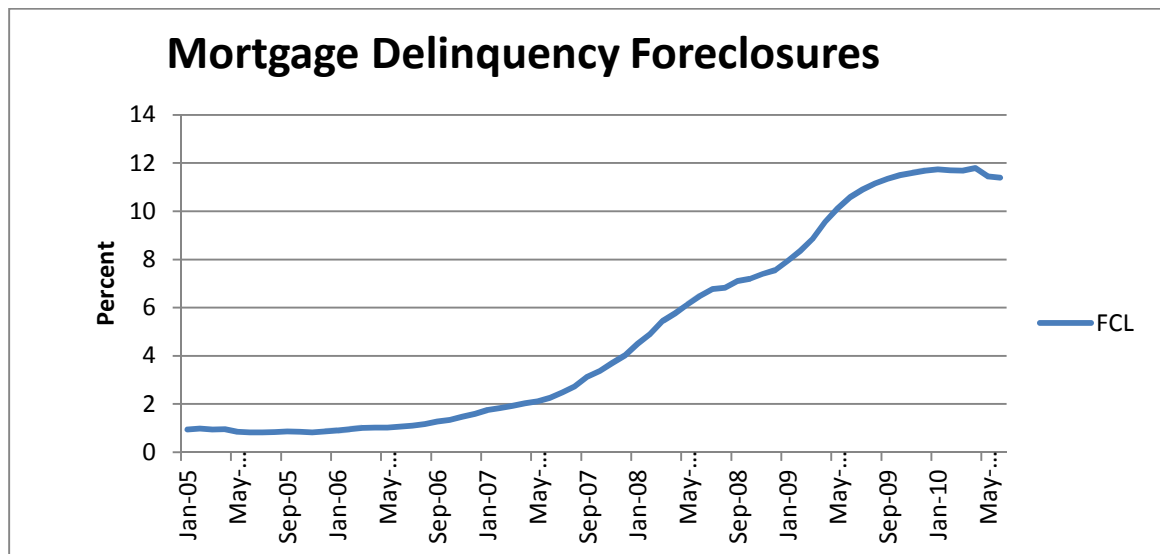
Figure 3: Foreclosure rates from Jan 2005 - June 2010 (Source: Bloomberg)

(Deng, Quigley and Van Order 2000) use a competing-risks model between prepayments and defaults where only the event that occurs first is observed and (Kelly 2009) has a complex default model with a high level of complexity. However, neither addresses the Standard Default Assumption. The proliferation of toxic assets in the mortgage industry has made defaults an essential part of the mortgage equation and has also shown that protecting investors from defaults tends to distort the market. Recently, Congress has been discussing the dissolution of Fannie Mae and Freddie Mac. This could result in an increase in private-label mortgages which are subject to default. Although many have shied away from defaults due to the complexity, rather than

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ignoring the problem, we introduce a default model with constant default rates and loss severities which can be expanded to become time-dependent.

(Goncharov 2009), (Gorovoy and Linetsky 2007), and (Kolbe and Zagst 2009) present continuous prepayment models, none of them have introduced a default model. (Deng, Quigley and Order, Mortgage Terminations, Heterogeneity and the Exercise of Mortgage Options 2000) and (Kelly 2009) have default models with a high level of complexity, but do not address the Standard Default Assumption. The proliferation of toxic assets in the mortgage industry has made this an essential part of the mortgage equation. Although many have shied away from defaults due to the complexity, rather than ignoring the problem, we introduce a rather simple default model which can be used as a platform for more complex models.

We assume a single mortgage will terminate at a stopping time $\tau \wedge T$ where in the simplest case τ has an exponential distribution. At that time either a prepayment will occur with probability $\dfrac{h}{h+\delta}$ or a default will occur with probability $\dfrac{\delta}{h+\delta}$. This differs from the competing risks model in (Deng, Quigley and Van Order 2000) with two stopping times, one for prepayments and one for defaults. We assume a homogeneous pool of mortgages, so that the likelihood of prepayment and default are the same across the pool. By using one stopping time, we can leverage the model of (Y. Goncharov 2006) without adding too much complexity. (See Figure 4) The only difference between the prepayments and defaults in our model is that at time $\tau \wedge T$, for a prepayment, the total balance will be returned, whereas with a default, only a proportion of the balance will be recovered, the remainder being lost. The time value of money lost due to the

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delay in receiving the proceeds while the property is in foreclosure is included in this loss

percentage.

Incorporating the Default Model (Single Stopping Time)

Mortgage selected from homogeneous pool.

Prepayment

$$p_\tau = \frac{h_\tau}{h_\tau + \delta(\tau)}$$

Payoff

$$B(\tau)$$

Default

$$q_\tau = \frac{\delta(\tau)}{h_\tau + \delta(\tau)}$$

$$(1 - S_\tau) B(\tau)$$

$$S_\tau = \text{Loss Severity}$$

Origination
$t = 0$

$\tau = $ Random
Stopping Time

$T = 30$ years
(Maturity)

t

3/30/2011
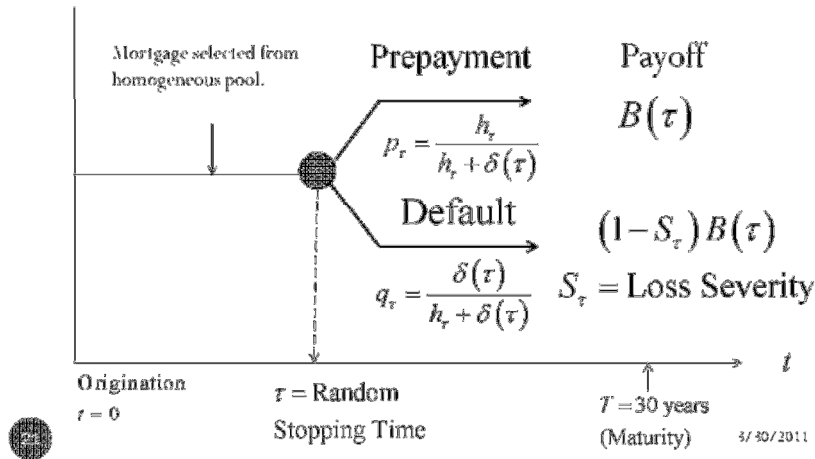
Figure 4 - Single Stopping-Time Model

According to the (The Bond Market Association 1999), the recursive equation for the discrete performing balance is:

$$\hat{B}_i = \hat{B}_{i-1} - D_i - PR_i - AM_i \qquad (4.100)$$

where $B_i =$ Scheduled Amortization assuming no prepayments

$\hat{B}_i =$ Performing Balance in month i

$D_i = \hat{B}_{i-1} MDR_i =$ New Defaults

$PR_i = \hat{B}_{i-1} \left(\dfrac{B_i}{B_{i-1}} \right) SMM_i =$ Voluntary Prepayments

$AM_i = \left(\hat{B}_{i-1} - D_i \right) \left(1 - \dfrac{B_i}{B_{i-1}} \right) =$ Actual Amortization

$SMM_i =$ Single Month Mortality

$MDR_i =$ Monthly Default Rate

Substituting into (4.100) we obtain

$$\hat{B}_i = \hat{B}_{i-1} \left(\frac{B_i}{B_{i-1}} \right) \left(1 - SMM_i - MDR_i \right). \qquad (4.101)$$

Since $\hat{B}_0 = B_0$ we can show that

$$\hat{B}_1 = B_1 \left(1 - SMM_1 - MDR_1 \right) \qquad (4.102)$$

and in general use recursion to show that the performing balance

$$\hat{B}_n = B_n \prod_{i=1}^{n} \left(1 - SMM_i - MDR_i \right). \qquad (4.103)$$

In the simple case, we use constant prepayment and default rates:

So (4.103) becomes $\hat{B}_n = B_n \left(1 - SMM - MDR \right)^n$ $\qquad (4.104)$

Now let $SMM = h/12$ and $MDR = \delta/12$ and define $t = n/12$:

$$\hat{B}(t) = B(t) \left(1 - \frac{h}{12} - \frac{\delta}{12} \right)^{12t}. \qquad (4.105)$$

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When the number of periods becomes large then:

$$\lim_{N \to \infty} \left(1 - \frac{h}{N} - \frac{\delta}{N}\right)^{Nt} = e^{-(h+\delta)t} \qquad (4.106)$$

Replacing (4.106) into (4.105) gives the continuous approximation for the performing balance:

$$\hat{B}(t) = B(t)e^{-(h+\delta)t}. \qquad (4.107)$$

In general, when the prepayment rates and default rates are not constant, (4.107) becomes:

$$\hat{B}(t) = B(t)\exp\left[-\int_0^t (h_s + \delta_s)ds\right] \qquad (4.108)$$

which is our definition for the Performing Balance.

The Basic Default Model consists of four cash flows, Scheduled Principal, Interest, Prepayments, and Principal Recoveries.

$$CF_t = P_t + I_t + PR_t + R_t \qquad (4.109)$$

Normally, defaults take up to a year to become liquidated. To simplify the model, we will assume that defaults are liquidated immediately. We will define the loss severity

$$S_t = 1 - \frac{L_t e^{-ml}}{B_t} \qquad (4.110)$$

where l is the liquidation period, L_t is the net amount received for the foreclosed property. This way we include the time value of money lost in the delay between foreclosure and liquidation in the loss severity.

The value of a mortgage with defaults now becomes the expected value of the discounted scheduled payment, prepayments and recoveries:

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$$M_t = E\left[\int_t^T \left[c_u + h_u B_u + (1 - S_u)\delta_u B_u\right]e^{-\int_t^u r_s + h_s + \delta_s ds}\,du\,\middle|\,\mathcal{F}_t\right] \qquad (4.111)$$

We will make a few simplifying assumptions about equation(4.111). Let us assume a fixed-rate, fixed-payment fully amortizing mortgage. Furthermore, let us assume that the exogenous prepayment rate, the default rate and the loss severity are deterministic. Only the short rate and endogenous prepayment rates are stochastic.

Note: This is a reasonable assumption if the only independent random factor in the model is the interest rate. Defaults are involuntary; they may be affected by interest rates, but for the sake of simplicity, we will assume they are constant or deterministic; exogenous prepayment rates are by definition not affected by interest rates. Endogenous prepayments are stochastic, but are correlated to interest rates.

We will use the following notation. A stochastic process is indicated by $(X_t)_{t\in[0,T]}$ while a deterministic function is indicated by $X(t)$.

Now we can rewrite (4.111) as follows:

$$M_t = E\left[\int_t^T \left\{c + h_u B(u) + \left[1 - S(u)\right]\delta(u) B(u)\right\}e^{-\int_t^u [r_s + h_s + \delta(s)]ds}\,du\,\middle|\,\mathcal{F}_t\right] \qquad (4.112)$$

Define

$$Q(t,u) = E\left[e^{-\int_t^u r_s + h_s + \delta_s ds}\,ds\,\middle|\,\mathcal{F}_t\right] \qquad (4.113)$$

$$H(t,u) = E\left[(h_u + \delta_u)e^{-\int_t^u r_s + h_s + \delta_s ds}\,ds\,\middle|\,\mathcal{F}_t\right] \qquad (4.114)$$

$$R(t,u) = E\left[r_u e^{-\int_t^u r_s + h_s + \delta_s ds}\,ds\,\middle|\,\mathcal{F}_t\right] \qquad (4.115)$$

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We can also rewrite (4.112) as:

$$M_t = \int_t^T \left[cQ(t,u) + B(u) H(t,u) - S(u)\delta(u) B(u) Q(t,u) \right] du . \qquad (4.116)$$

Using integration by parts, we derive (See Proof in Appendix C):

$$\int_t^T B_u \left(h_s + \delta_s \right) e^{-\int_t^u r_s + h_s + \delta_s ds} du = B_t + \int_t^T \left[\frac{dB_u}{du} - r_u B_u \right] e^{-\int_t^u r_s + h_s + \delta_s ds} du \qquad (4.117)$$

Substituting (4.117) into (4.116) produces

$$M(t) = B(t) + E \left[\int_t^T \left[\frac{dB}{du} - r_u B(u) \right] e^{-\int_t^u r_s + h_s + \delta_s ds} \bigg| \mathcal{F}_t \right] +$$

$$\int_t^T \left[c - S(u)\delta(u) B(u) \right] Q(t,u) du \qquad (4.118)$$

Rearranging terms gives

$$M(t) = B(t) - E \left[\int_t^T B(u) r_u e^{-\int_t^u r_s + h_s + \delta_s ds} \bigg| \mathcal{F}_t \right] du +$$

$$\int_t^T \left(c + \frac{dB(u)}{du} - S(u)\delta(u) B(u) \right) Q(t,u) du \qquad (4.119)$$

Using (4.115), recalling (3.4), and assuming a fixed-rate mortgage we can rewrite (4.119) using equation (3.4) as

$$M(t) = B(t) - \int_t^T B(u) R(t,u) du + \int_t^T \left[mB(u) - S(u)\delta(u) B(u) \right] Q(t,u) du . (4.120)$$

To prevent arbitrage, we need to price the mortgage at face value at time zero. Thus $M_0 = B_0$ and

$$\int_0^T B(u) R(0,u) du = \int_0^T \left(m - S(u)\delta(u) \right) B(u) Q(0,u) du . \qquad (4.121)$$

Solving for the mortgage rate, gives

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$$m = \frac{\int_0^T \left(1 - e^{m(T-u)}\right)\left[R(0,u) + S(u)\delta(u)Q(0,u)\right] du}{\int_0^T \left(1 - e^{-m(T-u)}\right)Q(0,u)\, du},$$ (4.122)

or broken into two pieces:

$$m = \frac{\int_0^T \left(1 - e^{-m(T-u)}\right)R(0,u)\, du}{\int_0^T \left(1 - e^{-m(T-u)}\right)Q(0,u)\, du} + \frac{\int_0^T \left(1 - e^{m(T-u)}\right)S(u)\delta(u)Q(0,u)\, du}{\int_0^T \left(1 - e^{-m(T-u)}\right)Q(0,u)\, du}$$ (4.123)

Thus, the mortgage rate equals the "average" short rate, the first term, plus the "average" loss percentage, the second term. We will refer to the second term in (4.123) as the loss premium.

4.4.1 Constant Default Rate and Loss Severity.

The simplest default model assumes a constant default rate. Furthermore assuming that the liquidation proceeds are a constant proportion of the future value of the balance of the mortgage at the time of default,

$$L_t = kB(t)e^{ml}.$$ (4.124)

Substituting (4.124) into (4.110) results in a constant severity rate. If we assume that the default and loss severity rates are both constant over time, (4.123) reduces to

$$m = S\delta + \frac{\int_0^T \left(1 - e^{m(T-u)}\right)R(0,u)\, du}{\int_0^T \left(1 - e^{-m(T-u)}\right)Q(0,u)\, du}$$ (4.125)

In this case, the loss premium is simply $S\delta$. Observe that if the severity rate is zero, the default rate is simply added to the prepayment rate. If the default rate is zero, (4.125) reduces to the standard prepayment model put forth by (Gorovoy and Linetsky

2007). The conditional expectation in (4.1) can be expanded to include both prepayments (as Gorovoy and Linetsky have done) and defaults (as proposed here):

$$E\left[\exp\left\{-\int_0^u \left(r_s + h_0(s) + \gamma(k-r)^+ + \delta_s\right)ds\right\} f(r_u)\Big| r_0 = r\right]$$ (4.126)

Factoring out the deterministic part from (4.126), which includes the default rate, we obtain:

$$\exp\left\{-\int_0^u h_0(s) + \delta(s)\right\} E\left[\exp\left\{-\int_0^u \left(r_s + \gamma(k-r)^+\right)ds\right\} f(r_u)\Big| r_0 = r\right]$$ (4.127)

The remaining conditional expectation is identical to that of Gorovoy and Linetsky; the eigenfunction expansion calculation does not need to be changed. This means that we can modify the deterministic (exogenous) prepayment and default rates and still use the same eigenvalues, eigenfunctions and expansion coefficients. Furthermore, the differential equation from the infinitesimal generator remains the same as in (Gorovoy and Linetsky 2007) if we use the Cox-Ingersoll-Ross interest rate model:

$$\mathcal{G} f(x) = \tfrac{1}{2}\sigma^2 x f''(x) + \kappa(\theta - x) f'(x) - \left(x + \gamma(k-x)^+\right) f(x)$$ (4.128)

With constant default rate and loss severity, using (4.125) we can solve for the mortgage rate from the following formula:

$$\sum_{n=1}^{\infty} \varphi_n(r)\left[(m - S\delta)c_n^Q - c_n^R\right] L_c(0,T;\lambda_n,m) = 0$$ (4.129)

Using the same parameters described in (Gorovoy and Linetsky 2007) , we find that the eigenvalues λ_n, eigenfunctions $\varphi_n(r)$ and expansion coefficients c_n^Q and c_n^R in (4.129) are not affected by the default rate, loss severity or exogenous prepayment rate. We can reuse the relatively difficult-to-compute eigenfunction expansion values and vary

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the exogenous parameters to obtain mortgage rates under various prepayment and default scenarios.

4.4.2 Mortgage Rate Sensitivity to Defaults

Using our model, we can observe how various default assumptions affect mortgage rates. The simplest model assumes a constant prepayment rate and a constant default rate over the life of a mortgage. Figure 5 shows how the price varies as we adjust the default rate and the loss severity assuming a constant prepayment rate. Table 7 shows mortgage rates under varying prepayment and default assumptions. Observe that if the CDR is 0%, the yield (mortgage rate) remains constant, but if the severity rate is 0%, the yield increases with the "default" rate because it is simply added to the prepayment rate and the payments are merely accelerated.

30-year Mortgage Rates Varying Default Parameters
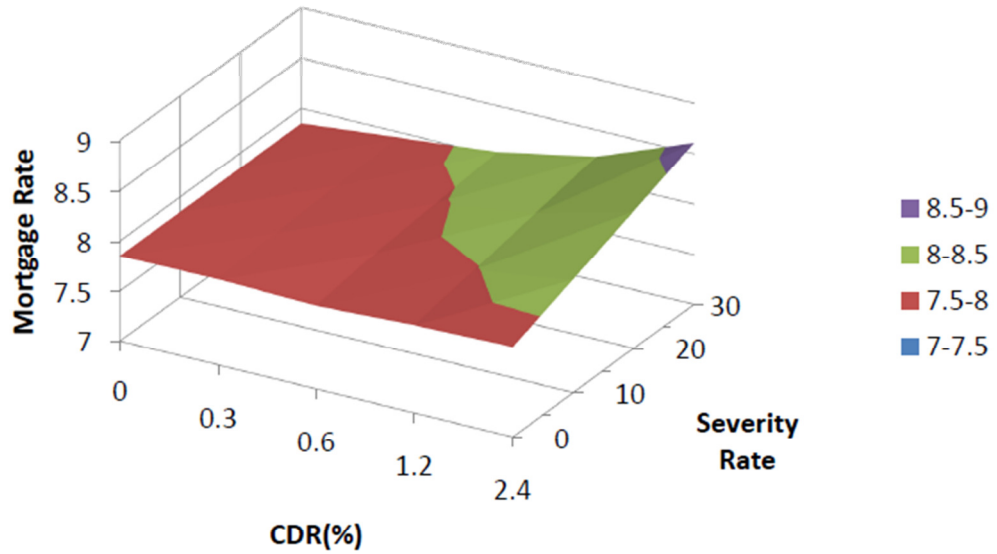
Figure 5 - How constant default parameters affect mortgage rates

15-Year Mortgage Rates under varying default and loss assumptions
holding exogenous prepayment rate constant at 4.5%.

Default Rates(%)	Loss Severity(%)			
	0.0000	10.0000	20.0000	30.0000
0.0000	7.9450	7.9450	7.9450	7.9450
0.3000	7.9488	7.9787	8.0086	8.0384
0.6000	7.9526	8.0124	8.0721	8.1318
1.2000	7.9601	8.0796	8.1991	8.3186
2.4000	7.9750	8.2139	8.4529	8.6919

30-Year Mortgage Rates under varying default and loss assumptions
holding exogenous prepayment rate constant at 4.5%.

Default Rates(%)	Loss Severity(%)			
	0.0000	10.0000	20.0000	30.0000
0.0000	7.8528	7.8528	7.8528	7.8528
0.3000	7.8579	7.8877	7.9176	7.9475
0.6000	7.8629	7.9227	7.9824	8.0422
1.2000	7.8728	7.9923	8.1119	8.2315
2.4000	7.8921	8.1313	8.3705	8.6098

15-Year Mortgage Rates under varying exogenous prepayment and
default rates holding severity rate constant at 20%

Prepay Rates(%)	Default Rates(%)				
	0.0000	0.3000	0.6000	1.2000	2.4000
0.0000	7.8857	7.9495	8.0132	8.1408	8.3956
4.0000	7.9386	8.0022	8.0658	8.1928	8.4467
4.5000	7.9450	8.0086	8.0721	8.1991	8.4529
5.0000	7.9514	8.0149	8.0784	8.2053	8.4590
5.5000	7.9576	8.0211	8.0846	8.2115	8.4650
6.0000	7.9639	8.0273	8.0908	8.2176	8.4710

30-Year Mortgage Rates under varying exogenous prepayment and
default rates holding severity rate constant at 20%

Prepay Rates(%)	Default Rates(%)				
	0.0000	0.3000	0.6000	1.2000	2.4000
0.0000	7.7720	7.8375	7.9029	8.0335	8.2945
4.0000	7.8443	7.9092	7.9741	8.1037	8.3625
4.5000	7.8528	7.9176	7.9824	8.1119	8.3705
5.0000	7.8612	7.9260	7.9907	8.1201	8.3784
5.5000	7.8695	7.9342	7.9989	8.1281	8.3862
6.0000	7.8776	7.9423	8.0069	8.1360	8.3939

Table 7- Mortgage Rates under varying default assumptions

4.4.3 Piecewise Linear Prepayment and Default Rates

If we allow the default rate to be time-dependent, but allow the severity rate to be constant, we can rewrite the formula in (4.122) as:

$$m = \frac{\int_0^T \left(1 - e^{m(T-u)}\right) R(0,u)\, du + S \int_0^T \left(1 - e^{-m(T-u)}\right) \delta(u) Q(0,u)\, du}{\int_0^T \left(1 - e^{-m(T-u)}\right) Q(0,u)\, du} \qquad (4.130)$$

We need to evaluate each of the integrals in (4.130) separately. Let us start with the denominator. First, by substituting the eigenfunction expansion in (4.5)

$$Q(0,u) = e^{-\int_0^u h_0(s)+\delta(s)\, ds} \sum_{n=1}^{\infty} e^{-\lambda_n u} c_n^Q \varphi_n(r_0) \qquad (4.131)$$

The denominator of (4.130) becomes (4.132) after we embed (4.131) and pull out the summation from the integral:

$$\sum_{n=1}^{\infty} c_n^Q \varphi_n(r_0) \int_0^T \left(1 - e^{-m(T-u)}\right) e^{-\int_0^u h_0(s)+\delta(s)\, ds - \lambda_n u}\, du \qquad (4.132)$$

Define the time integral

$$L(\lambda_n, m) = \int_0^T \left(1 - e^{-m(T-u)}\right) e^{-\int_0^u \left[h_0(s)+\delta(s)\right] ds - \lambda_n u}\, du, \qquad (4.133)$$

which is identical to Linetsky's definition except for the introduction of the default rate[1].

We now can write the denominator of (4.130) as

$$\int_0^T \left(1 - e^{-m(T-u)}\right) Q(0,u)\, du = \sum_{n=1}^{\infty} c_n^Q \varphi_n(r_0) L(\lambda_n, m). \qquad (4.134)$$

In a similar fashion

---

[1] Note that we use the abbreviated form $L(\lambda_n, m)$ to represent Linetsky's time integral $L(0,T;\lambda_n,m)$ for simplification $L(\lambda_n, m) = L(0,T;\lambda_n,m)$

$$\int_0^T \left(1 - e^{-m(T-u)}\right) R(0,u)\, du = \sum_{n=1}^{\infty} c_n^R \varphi_n(r_0) L(\lambda_n, m) \tag{4.135}$$

Now the remaining term, whose coefficient is the severity rate S, can be rewritten as:

$$\int_0^T \delta(u)\left(1 - e^{-m(T-u)}\right) e^{-\int_0^u h_0(s)+\delta(s)ds} \sum_{n=1}^{\infty} e^{-\lambda_n u} c_n^Q \varphi_n(r_0)\, du \tag{4.136}$$

Pulling out the summation from the integral gives

$$\sum_{n=1}^{\infty} c_n^Q \varphi_n(r_0) \int_0^T \delta(u)\left(1 - e^{-m(T-u)}\right) e^{-\int_0^u h_0(s)+\delta(s)ds} e^{-\lambda_n u}\, du \tag{4.137}$$

We define a new "default" time integral:

$$D(\lambda_n, m) = \int_0^T \delta(u)\left(1 - e^{-m(T-t)}\right) e^{-\int_t^u [h_0(s)+\delta(s)]ds - \lambda_n u}\, du\,, \tag{4.138}$$

so (4.137) can be written as:

$$\sum_{n=1}^{\infty} c_n^Q \varphi_n(r_0) D(\lambda_n, m)\,. \tag{4.139}$$

From (4.134), (4.135) and (4.139) we can rewrite equation (4.130) as

$$m = \frac{\displaystyle\sum_{n=1}^{\infty} c_n^R \varphi_n(r_0) L(\lambda_n, m) + S \sum_{n=1}^{\infty} c_n^Q \varphi_n(r_0) D(\lambda_n, m)}{\displaystyle\sum_{n=1}^{\infty} c_n^Q \varphi_n(r_0) L(\lambda_n, m)} \tag{4.140}$$

Rearranging terms of (4.140) gives:

$$\sum_{n=1}^{\infty} \varphi_n(r_0)\left[\left(mc_n^Q - c_n^R\right) L(\lambda_n, m) - S c_n^Q D(\lambda_n, m)\right] = 0 \tag{4.141}$$

Equation (4.141) can be solved for m using an iterative technique such as the Secant Method. First, however, we must show that the quantities $L(\lambda_n, m)$ and $D(\lambda_n, m)$ can be expressed in closed form.

4.4.4 Industry Prepayment and Default Models

The Public Securities Administration (PSA) and the Standard Default Assumption (SDA) models are piecewise linear deterministic prepayment and default models respectively, used by the (The Bond Market Association 1999) and discussed by (Fabozzi, Ramsey and Ramirez 1994). The integrals in the exponents in (4.133) and (4.138) involve the sum of prepayment and default rates. The sum $h_0(s) + \delta(s)$ is piecewise linear if both $h_0(s)$ and $\delta(s)$ are piecewise linear.

The standard prepayment rate using the PSA model increases linearly over the first 30 months to 6% and then remains constant thereafter. (Gorovoy and Linetsky 2007) extend this to the continuous model:

$$h_0(t) = b\left(at\left[t < T_1\right] + aT_1\left[t \geq T_1\right]\right) \qquad (4.142)$$

where $a = .024$ and $T_1 = 2.5$. The parameter b represents the prepayment speed; for 100% PSA, $b = 1$.

The Standard Default Assumption (SDA) established by the (The Bond Market Association 1999) applies to 30-year fixed-rate mortgages only. For 100% SDA the default rate starts at zero and rises linearly to 0.6% per annum over 30 months. During the next 30 months it remains constant at 0.6%, then over the next 60 months it drops to .03% where it remains constant until the end. 150% SDA would simply multiply the

default rate by 1.5.   The default rate using the SDA model can be expressed mathematically as:

$$\delta(t) = b_1 \left\{ a_1 t \left[t \le T_1 \right] + a_1 T_1 \left[t \ge T_1 \right] - a_2 \left(t - T_2 \right) \left[T_1 \le t \le T_3 \right] - a_2 \left(T_3 - T_2 \right) \left[t \ge T_3 \right] \right\}$$

where $b_1 = $ the default speed; e.g. for 150% SDA, $b_1 = 1.5$
$a_1 = 0.0024$
$a_2 = 0.00114$
$T_1 = 2.5$ years (30 months)
$T_2 = 5$ years $(T_1 + 30$ months)
$T_3 = 10$ $(T_2 + 60$ months)

(4.143)

It is perhaps easier to understand the SDA model by defining it in piecewise linear form:

$$\delta(t) = b_1 \begin{cases} a_1 t, & 0 = T_0 \le t \le T_1 \\ a_1 T_1, & T_1 < t \le T_2 \\ a_1 T_1 + a_2 T_2 - a_2 t, & T_2 < t \le T_3 \\ a_1 T_1 + a_2 T_2 - a_2 T_3, & T_3 < t \le T \end{cases}$$

(4.144)

Substituting numerical values we get the following:

$$\delta(t) = b_1 \begin{cases} .0024t, & 0 \le t \le 2.5 \\ 0.006, & 2.5 < t \le 5 \\ .0117 - .00114t, & 5 < t \le 10 \\ .0.003, & 10 < t \le 30 \end{cases}$$

Figure 6 shows the Constant Default Rate (CDR) versus the Standard Default Assumption (SDA) values in (4.143) at varying speeds as set by the Bond Market

Association. Figure 7 combines the PSA prepayment model with the Standard Default Assumption to show the overall prepayment rate.
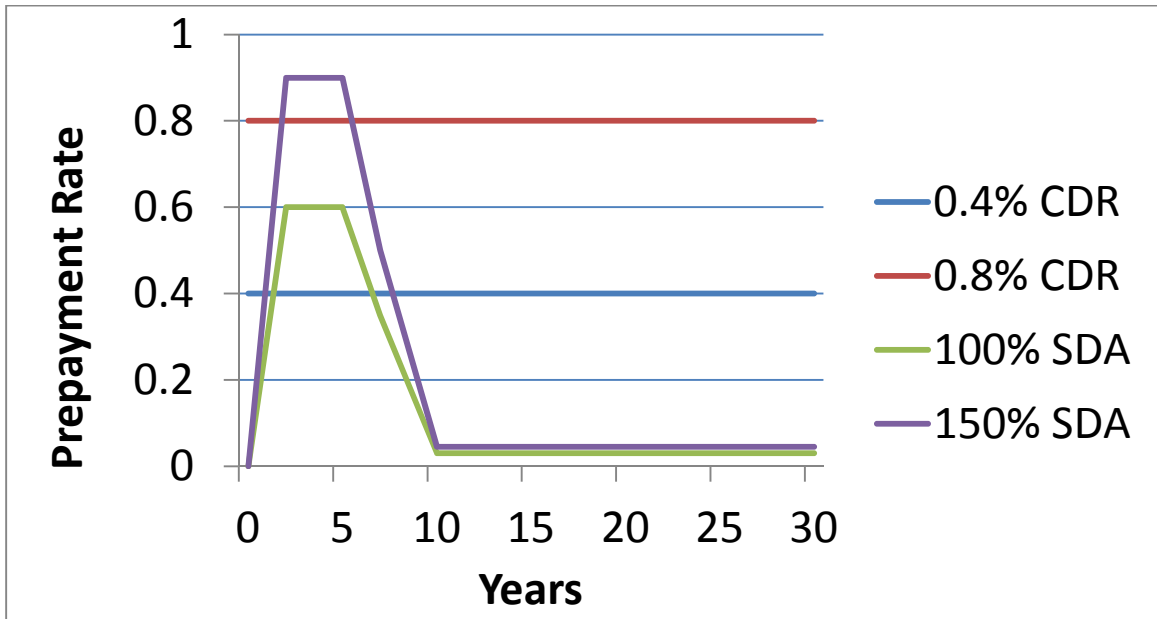
Figure 6 - Comparison of Constant Default Rate to Standard Default Assumption
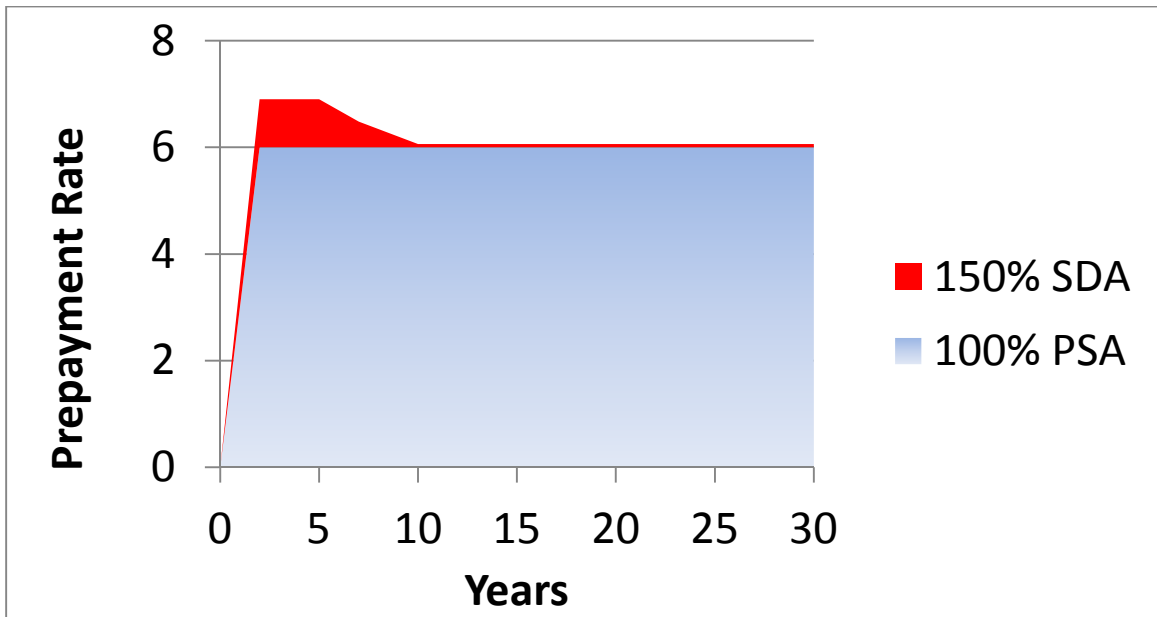
Figure 7 - Combining the PSA prepayment model with the Standard Default Assumption

The cumulative default rate using the SDA model is:

$$\int_0^u \delta(s)ds = b_1 \begin{cases} 0.0012u^2; & 0 \le u \le 2.5 \\ 0.006u - 0.0075; & 2.5 \le u \le 5 \\ -.00057u^2 + .0117u - 0.02175; & 5 \le u \le 10 \\ 0.0003u + .03525; & 10 \le u \le 29 \\ 0; & u \ge 29 \end{cases} \qquad (4.145)$$

The PSA rate can be expressed as: See (Gorovoy and Linetsky 2007).

$$\int_0^u h_0(s)ds = b \begin{cases} \frac{1}{2}au^2; u \le T_1 \\ -\frac{1}{2}aT_1^2 + aT_1u; u \ge T_1 \end{cases} \qquad (4.146)$$

Expressed numerically,

$$\int_0^u h_0(s)ds = b \begin{cases} 0.012u^2; u \le 2.5 \\ 0.06u - 0.075; 2.5 \le u \end{cases} \qquad (4.147)$$

Combining the default rate (SDA) in (4.145) and the PSA prepayment rate in (4.147) gives:

$$\int_0^u [h(s) + \delta(s)]ds = \begin{cases} (0.012b + 0.0012b_1)u^2; & 0 \le u \le 2.5 \\ 0.06bu + 0.006b_1u - 0.075b - 0.0075b_1u; & 2.5 \le u \le 5 \\ -.00057b_1u^2 + (.0117b_1 + 0.06b)u - (0.02175b_1 + 0.075b); 5 \le u \le 10 \\ (0.0003b_1 + 0.06b)u + .03525b_1 - 0.075b; & 10 \le u \le 30 \end{cases}$$

(4.148)

The general form is:

$$\int_0^t (h(s) + \delta(s))ds = \begin{cases} \frac{1}{2}Hu^2, u \le T_1 \\ -\frac{1}{2}H_1T_1^2 + H_1T_1u; & T_1 \le u \le T_2 \\ -\frac{1}{2}(H_1T_1^2 + H_2T_2^2) + (H_1T_1 + H_2T_2)u - \frac{1}{2}H_2u^2; & T_2 \le u \le T_3 \\ -\frac{1}{2}[HT_1^2 + H_2(T_2^2 - T_3^2)] + [H_1T_1 + H_2(T_2 - T_3)]u; T_3 \le u \le T \end{cases}$$

where $H_1 = ab + a_1b_1$ and $H_2 = a_2b_1$

(4.149)

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Using (4.149), we can convert (4.133) to closed form for the combined PSA, SDA model:

$$L_{PSA,SDA}\left(\lambda_n, m\right) =$$
$$\int_0^{T_1}\left(1 - e^{-m(T-u)}\right)e^{-\frac{1}{2}H_1 u^2 - \lambda_n u}\,du +$$
$$\int_{T_1}^{T_2}\left(1 - e^{-m(T-u)}\right)e^{\left(\frac{1}{2}H_1 T_1^2 - H_1 T_1 u - \lambda_n u\right)}\,du + \qquad (4.150)$$
$$\int_{T_2}^{T_3}\left(1 - e^{-m(T-u)}\right)e^{\frac{1}{2}\left(H_1 T_1^2 + H_2 T_2^2\right) - \left(H_1 T_1 + H_2 T_2\right)u + \frac{1}{2}H_2 u^2 - \lambda_n u}\,du +$$
$$\int_{T_3}^{T}\left(1 - e^{-m(T-u)}\right)e^{\frac{1}{2}\left[H_1 T_1^2 + H_2\left(T_2^2 - T_3^2\right)\right] - \left[H_1 T_1 + H_2\left(T_2 - T_3\right)\right]u - \lambda_n u}\,du .$$

The integral in the first term in (4.150) has already been solved by (Gorovoy and Linetsky 2007), the only difference being the constant coefficient of u^2. The remaining terms involve integrals with linear exponents of the form

$$\int e^{-(c_0 + c_1 u)}\,du = -\frac{e^{-(c_0 + c_1 u)}}{c_1} \qquad (4.151)$$

or with quadratic exponents of the form[2]:

$$\int e^{-\left(c_0 + c_1 u + c_2 u^2\right)}\,du = \frac{1}{2}\sqrt{\frac{\pi}{c_2}}\exp\left(-\frac{4c_0 c_2 - c_1^2}{4c_2}\right)\text{erf}\left(\frac{2c_2 u + c_1}{2\sqrt{c_2}}\right) \qquad (4.152)$$

where the error function is defined as:

$$\text{erf}\left(z\right) = \frac{2}{\sqrt{\pi}}\int_0^z e^{-t^2}\,dt . \qquad (4.153)$$

---

[2] Please note (4.151) and (4.152) are expressed as indefinite integrals without the constant of integration for clarity. The constants will cancel out when this is part of a definite integral.

Thus to calculate $L_{PSA,SDA}(\lambda_n, m)$ we simply need to obtain the coefficients c_i and the endpoints of each time interval and calculate the integral piecewise. Define the multivariate function

$$L_i(c_0, c_1, c_2) = \int_{T_i}^{T_{i+1}} e^{-(c_0 + c_1 u + c_2 u^2)} du .$$
(4.154)

Now we can define:

$$L_{CPR,MDA}(\lambda_n, m) = \sum_{i=0}^{3} L_i(c_{i0}, c_{i1}, c_{i2}) - L_i(c_{i0} + mT, c_{i1} - m, c_{i2}) .$$
(4.155)

From (4.149) we can obtain the coefficients c_{ij} for (4.155). They are listed in the matrix $\mathbf{C}$ below:

$$\mathbf{C} = \begin{bmatrix} 0 & \lambda_n & \frac{1}{2}H_1 \\ -\frac{1}{2}H_1 T_1^2 & \lambda_n + H_1 T_1 & 0 \\ -\frac{1}{2}(H_1 T_1^2 + H_2 T_2^2) & \lambda_n + H_1 T_1 + H_2 T_2 & -\frac{1}{2}H_2 \\ -\frac{1}{2}H_1 T_1^2 - \frac{1}{2}H_2(T_2^2 - T_3^2) & \lambda_n + H_1 T_1 + H_2(T_2 - T_3) & 0 \end{bmatrix}.$$
(4.156)

Observe that (4.152) involves complex numbers, since c_2 takes on negative values in (4.156). However, the result is real.

Let us rewrite (4.138) as:

$$\begin{aligned}
D_{PSA,SDA}(\lambda_n, m) = & \ b_1 \int_0^{T_1} a_1 u \left(1 - e^{-m(T-u)}\right) e^{-\frac{1}{2}H_1 u^2 - \lambda_n u} du + \\
& \ b_1 \int_{T_1}^{T_2} a_1 T_1 \left(1 - e^{-m(T-u)}\right) e^{-\frac{1}{2}H_1 T_1^2 + H_1 T_1 u - \lambda_n u} du + \\
& \ b_1 \int_{T_2}^{T_3} (a_1 T_1 + a_2 T_2 - a_2 u)\left(1 - e^{-m(T-u)}\right) e^{\frac{1}{2}(H_1 T_1^2 + H_2 T_2^2) - (H_1 T_1 + H_2 T_2)u + \frac{1}{2}H_2 u^2 - \lambda_n u} du + \\
& \ b_1 \int_{T_3}^{T} (a_1 T_1 + a_2 T_2 - a_2 T_3)\left(1 - e^{-m(T-u)}\right) e^{\frac{1}{2}[H_1 T_1^2 + H_2(T_2^2 - T_3^2)] - [H_1 T_1 + H_2(T_2 - T_3)]u - \lambda_n u} du
\end{aligned}$$
(4.157)

Each term of (4.157) can be written as a linear combination of the function (4.154) and (4.158) below:

$$D_i\left(c_0, c_1, c_2\right) = \int_{T_i}^{T_{i+1}} u e^{-\left(c_0 + c_1 u + c_2 u^2\right)} du .$$

(4.158)

Using the error function, we can write the integral in (4.158) as:

$$\int u e^{-\left(c_0 u + c_1 u + c_2 u^2\right)} du = -\frac{e^{-\left(c_0 + c_1 u + c_2 u^2\right)}}{2 c_2} - \frac{c_1 \sqrt{\pi} \exp\left(\frac{c_1^2}{4 c_2}\right) \mathrm{erf}\left(u \sqrt{c_2} + \frac{c_1}{2 \sqrt{c_2}}\right)}{4 c_2^{3/2}}$$

(4.159)

Now we have

$$
\begin{aligned}
D_{PSA,SDA}\left(\lambda_n, m\right) = {} & b_1 a_1 \left[D_0\left(c_{0,0}, c_{0,1}, c_{0,2}\right) - D_0\left(c_{0,0} + mT, c_{0,1} - m, c_{0,2}\right) \right] + \\
& b_1 a_1 T_1 \left[L_1\left(c_{1,0}, c_{1,1}, c_{1,2}\right) - L_1\left(c_{1,0} + mT, c_{1,1} - m, c_{1,2}\right) \right] + \\
& b_1 \left(a_1 T_1 + a_2 T_2\right) \left[L_2\left(c_{2,0}, c_{2,1}, c_{2,2}\right) - L_2\left(c_{2,0} + mT, c_{2,1} - m, c_{2,2}\right) \right] + \\
& b_1 a_2 \left[D_2\left(c_{2,0} + mT, c_{2,1} - m, c_{2,2}\right) - D_2\left(c_{2,0}, c_{2,1}, c_{2,2}\right) \right] + \\
& b_1 \left(a_1 T_1 + a_2 T_2 - a_2 T_3\right) \left[L_1\left(c_{3,0}, c_{3,1}, c_{3,2}\right) - L_1\left(c_{3,0} + mT, c_{3,1} - m, c_{3,2}\right) \right].
\end{aligned}
$$

(4.160)

We are set to solve the non-linear equation for the mortgage rate when the prepayment and default rates are piecewise linear over time, namely:

$$\sum_{n=1}^{\infty} \varphi_n\left(r_0\right) \left[\left(m c_n^Q - c_n^R\right) L_{PSA,SDA}\left(\lambda_n, m\right) - S c_n^Q D_{PSA,SDA}\left(\lambda_n, m\right) \right] = 0$$

(4.161)

where eigenvalue expansion parameters are the same as before, the severity rate S is a constant, and $L_{PSA,SDA}\left(\lambda_n, m\right)$ and $D_{PSA,SDA}\left(\lambda_n, m\right)$ are defined in (4.150) and (4.160) respectively. Note that when both the default and severity rates are zero, (4.161) simplifies to:

$$\sum_{n=1}^{\infty} \varphi_n(r_0) \left[\left(mc_n^Q - c_n^R \right) L_c(\lambda_n, m) \right] = 0. \tag{4.162}$$

This is identical to the non-linear equation to be solved in (Gorovoy and Linetsky 2007).

4.4.5 Comparing Discrete and Continuous Default Models

The default model is as follows starting with (4.120):

$$M(t) = B(t) - \int_t^T B(u) R(t,u) du + \int_t^T \left[mB(u) - S(u) \delta(u) B(u) \right] Q(t,u) du \tag{4.163}$$

For a fixed-rate mortgage

$$B(t) = \frac{1 - e^{-m(T-t)}}{1 - e^{-mT}} \tag{4.164}$$

Assuming a constant interest rate r

$$Q(t,u) = E\left[e^{-\int_t^u r_s + h_s + \delta_s ds} \middle| \mathcal{F}_t \right] = e^{-\int_t^u r_s + h_s + \delta_s ds} = e^{-(r+h+\delta)(u-t)} \tag{4.165}$$

And

$$R(t,u) = E\left[re^{-\int_t^u r_s + h_s + \delta_s ds} \middle| \mathcal{F}_t \right] = re^{-\int_t^u r_s + h_s + \delta_s ds} = re^{-(r+h+\delta)(u-t)} \tag{4.166}$$

Now applying (4.164) (4.165) and (4.166) to (4.163) we obtain:

$$M(t) = \frac{1 - e^{-m(T-t)} + (m - S\delta - r) \int_t^T \left(1 - e^{-m(T-u)} \right) e^{-(r+h+\delta)(u-t)} du}{1 - e^{-mT}} \tag{4.167}$$

Letting $\tilde{r} = r + h + \delta$, we have

$$\int_t^T \left(1 - e^{-m(T-u)} \right) e^{-\tilde{r}(u-t)} du = \frac{1 - e^{-\tilde{r}(T-t)}}{\tilde{r}} + \frac{e^{-\tilde{r}(T-t)} - e^{-m(T-t)}}{m - \tilde{r}} \tag{4.168}$$

Putting it all together gives the closed form solution:

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$$M(t) = \frac{1-e^{-m(T-t)} + (m-S\delta-r)\left[\dfrac{1-e^{-\tilde{r}(T-t)}}{\tilde{r}} + \dfrac{e^{-\tilde{r}(T-t)} - e^{-m(T-t)}}{m-\tilde{r}}\right]}{1-e^{-mT}}$$ (4.169)

We can rewrite this as:

$$M(t) = B(t)\left\{1 + \frac{(m-S\delta-r)\left[\dfrac{1-e^{-\tilde{r}(T-t)}}{\tilde{r}} + \dfrac{e^{-\tilde{r}(T-t)} - e^{-m(T-t)}}{m-\tilde{r}}\right]}{1-e^{-m(T-t)}}\right\}$$ (4.170)

The APL Function PriceDefaults is shown below:

```
     ∇PriceDefaults
[0]    PriceDefaults←{
[1]    Aα M T h d S
[2]    Aα m  Mortgage Rate
[3]        Aα T  Maturity in years
[4]        Aα h  Prepayment Rate
[5]        Aα d  Default Rate
[6]        Aα S  Severity Rate
[7]        Aω r  Yield
[8]        Aω t  Seasoning
[9]        A 0.08 30 0.12 0.12 0.2 PriceDefaults .08 0
[10]   (m T h d S)←α
[11]   r t←ω
[12]   R←r+h+d
[13]   z←(1-*-R×T-t)÷R
[14]   z←z+(-/*-R m×T-t)÷m-R
[15]   z←z×m-r+S×d
[16]   100×1+z÷1-*-m×T
[17]   }
     ∇
```

Consider an 8% MBS with a prepayment rate of 1% CPR and 1% MDR. The WAM is 30 years, loss severity is 20% and the time to liquidation after the loans stop paying is 12 months.

The cash flows calculated by the Bond Market Association produce an IRR or

yield of 5.037%. Now let us approximate this with a continuous model.

Convert the mortgage rate to continuous form:

```
    ⎕←m←12×⍟1+8÷1200
```

```
0.07973451262
```

Now annualize the prepayment and default rates:

```
    ⎕←CPR ADR←1-(1-.01)*12
0.1136151283
```

The adjusted loss severity S* is a function of discounted recovery amount L_t and the
current balance:

$$S_t{}^* = 1 - \frac{L_t e^{-ml}}{B_t} = 1 - \frac{(1-S_t)B_t e^{-ml}}{B_t} = 1 - (1-S_t)e^{-ml} \tag{4.171}$$

Letting $S_t = S = 0.20$, $l = 1.0$, $m = .0\ 7973$

Using the nominal rate of M = 8%, we can write

$$S^* = 1 - \frac{(1-s)}{\left(1+\dfrac{M}{12}\right)^{12}} \tag{4.172}$$

Thus $S^* = 1 - (1-0.2)e^{-0.7973} = 0.2613$

Or we can write: $S^* = 1 - (1-.2) / (1+.08/12)^{12} = 0.2613$

The original loss severity is S = 20%; the adjusted loss severity is S* = 26.13%.

The yield of the continuous model is 5.0024%

```
        f←{100-m 30 CPR ADR 0.2615 PriceDefaults ω 0}
        f SecAlg u,f¨u←0.045 0.055
0.05002415657
```

We define cumulative defaults as the total dollar amount of principal that is not paid due to default. This includes any amounts that may be recovered from foreclosure.

$$\text{CumulativeDefaults} = \int_0^T \delta(u) B(u) e^{-\int_0^u [h(s) + \delta(s)] ds} du \tag{4.173}$$

In the simple case with constant prepayments and defaults we have $\delta(u) = \delta$ and $h(u) = h$. Using the standard amortization $B(u) = \left(1 - e^{m(T-u)}\right) \Big/ \left(1 - e^{mT}\right)$, equation (4.173) reduces to:

$$\frac{\delta}{1 - e^{-mT}} \left[\frac{1 - e^{-(h+\delta)T}}{h + \delta} - e^{-mT} \frac{1 - e^{-(h+\delta-m)T}}{h + \delta - m} \right] \tag{4.174}$$

Now assume 1% MDR and 1% CPR with a coupon rate of 8.0% and a 30-year WAM. $CPR = 1 - (1 - .01)^{12} = .113615$ $h = -\ln(1\text{-}CPR) = 0.120604$ $m = 12$ ln(1+.08/12)=0.797345

The default rate is the same as the prepayment rate so $\delta = .120604$. Using these values we find cumulative defaults to be 47.53% of the total (over the lifetime of the pool). The discrete model gives 47.58%. Of course the constant rate of 1% per month is fairly large.

Now let's try the SDA (standard default assumption) model.

$$
\begin{aligned}
\text{CumDefaults} = {} & \frac{b_1}{1 - e^{-mT}} \left[a_1 \int_0^{T_1} u \left(1 - e^{-m(T-u)}\right) e^{-\frac{1}{2}H_1 u^2} du + \right. \\
& a_1 T_1 \int_{T_1}^{T_2} \left(1 - e^{-m(T-u)}\right) e^{\left(\frac{1}{2}H_1 T_1^2 - H_1 T_1 u\right)} du + \\
& \int_{T_2}^{T_3} \left(a_1 T_1 + a_2 T_2 - a_2 u\right)\left(1 - e^{-m(T-u)}\right) e^{\frac{1}{2}\left(H_1 T_1^2 + H_2 T_2^2\right) - (H_1 T_1 + H_2 T_2)u + \frac{1}{2}H_2 u^2} du + \\
& \left. \left(a_1 T_1 + a_2 T_2 - a_2 T_3\right) \int_{T_3}^{T} \left(1 - e^{-m(T-u)}\right) e^{\frac{1}{2}\left[H_1 T_1^2 + H_2\left(T_2^2 - T_3^2\right)\right] - \left[H_1 T_1 + H_2(T_2 - T_3)\right]u} du \right]
\end{aligned} \tag{4.175}
$$

Although (4.175) is a fairly complex integral, it can be written in closed form by using the error function.

The cumulative defaults for 150% PSA and 100% SDA are 2.78% for the discrete model. For the continuous model it is 2.79079%. For other combinations see Table 8 below. When prepayment rates and default rates are low, the values are within 1/100 of a percent. For higher prepayment/default rates, the values differ at most by about 1/3 of one percent.

```
Continuous Cumulative Defaults from Model

% PSA     50    100   150   200   250   300
        ------------------------------------
100     1.56  3.09  4.59  6.06  7.51  8.93
125     1.48  2.93  4.36  5.76  7.14  8.49
150     1.41  2.79  4.15  5.49  6.80  8.09
175     1.34  2.66  3.96  5.23  6.49  7.72
200     1.28  2.54  3.78  5.00  6.20  7.38
250     1.17  2.33  3.47  4.58  5.69  6.77
300     1.08  2.14  3.19  4.22  5.24  6.24
400     0.93  1.84  2.74  3.63  4.51  5.37
500     0.81  1.60  2.39  3.17  3.93  4.69


Discrete Cumulative Defaults
(from Standard Formulas - Bond Market Association)
                   % SDA
% PSA     50    100   150   200   250   300
        ------------------------------------
100     1.56  3.09  4.59  6.08  7.53  8.97
125     1.47  2.92  4.35  5.76  7.14  8.51
150     1.40  2.78  4.13  5.47  6.79  8.08
175     1.33  2.64  3.93  5.20  6.45  7.69
200     1.26  2.51  3.74  4.95  6.14  7.32
250     1.15  2.28  3.40  4.50  5.59  6.66
300     1.05  2.08  3.10  4.11  5.10  6.08
400     0.88  1.74  2.60  3.45  4.29  5.12
500     0.74  1.48  2.21  2.93  3.64  4.35
```

Table 8 - Cumulative Defaults

Chapter 5. Calibration and Parameter Estimation

In order to obtain parameter estimates for interest rate models, we first need to choose a theoretical forward rate model.   From data yields of varying maturities and we calculate empirical forward rates.   We then try to find the parameters of theoretical forward rates that most closely match the empirical forward rates using a least-squares approach.   In this chapter we will show some examples of calibration when the   yield curves exhibit typical upward-sloping behavior.   After that we will examine some of the problems that occur when the term structure of interest rates is flat, inverted or humped.

5.1 Short Rate Parameter Calibration

Interest rates r generally are comprised of six components:

$$r = r^* + IP + DP + MP + LP \pm OP \qquad (5.1)$$

where r^* is the base risk-free interest rate, IP is the Inflation Premium, DP is the Default premium, MP is the maturity premium, LP is the liquidity premium, and OP is the option premium. The Corporate "A" benchmark incorporates the inflation premium, the default premiums from the "A" rating, and the maturity premium, that is incorporated in the forward rate.   The liquidity premium may also be included in this benchmark.   The Treasury Rates also include the inflation and maturity premiums, but not the liquidity premium since they are easily traded.   The option premium is not present in either benchmark, but would be accounted for in the prepayment model. Since they are traded, MBS are more liquid than mortgage pools or individual mortgages. See (Kalotay, Yang and Fabozzi 2004). Thus, the liquidity premium would be the difference between an MBS and weighted average yield of the corresponding mortgage pool.

In the prepayment only model, the prepayment option is added to the discount rate which is comparable to the Corporate "A" benchmark with the same duration according to (Kalotay, Yang and Fabozzi 2004):

Mortgage Rate = Corporate "A" Rate + Prepayment Option

By including defaults, in addition to the prepayment option, we add a loss premium which can be calculated from the reduced cash flows. Thus we can use the risk-free rate with the same duration (Treasury rate) as a base; the prepayment option and the loss premium can be added to obtain the mortgage rate:

Mortgage Rate = Risk-Free Rate + Prepayment Option +   Loss Premium

Another way to achieve this is to estimate the spread between Corporate "A" rates and treasury rates. We then add this general spread to the treasury rates and calibrate from these augmented treasury rates the mortgage rate:

MortgageRate = Risk-Free Rate +Spread + Prepayment Option

Therefore, instead of calibrating the interest rate model to the Corporate A data, we can calibrate it directly to U.S. treasury rates. This is a distinct advantage since treasury issues are seen as a purer benchmark than corporate bonds and they are also more extensively archived.

5.1.1 Short Rate Models

The simplest short-rate model that can be calibrated to market data is the Ho-Lee model which consists of a deterministic, time-dependent part and a stochastic part. It is represented by the following stochastic differential equation:

$$dr_t = \theta_t dt + \sigma dW_t \tag{5.2}$$

where θ and σ are parameters, and W_t is a Brownian motion process with respect to risk-neutral measure.

The first mathematical model to describe the random fluctuations of interest rates using mean-reversion was the one introduced by (Vasicek 1977). It is a one-factor, short-rate model as it describes interest rate movements as being driven by only one source of market risk. The model can be used in the valuation of interest-rate derivatives. The main shortcoming of this model is that it allows the existence of negative interest rates. One can write a stochastic differential equation that describes the model as:

$$dr_t = \kappa(\theta - r_t)dt + \sigma dW_t \tag{5.3}$$

Under the risk-neutral measure, the parameter θ is the long-term mean interest rate, σ is the volatility term, and κ represents the rate of adjustment of the mean-reversion.

The model introduced by (Cox, Ingersoll and Ross 1985) also describes the evolution of interest rates. The stochastic differential equation that captures the essence of this model is given by:

$$dr_t = \kappa(\theta - r_t)dt + \sigma\sqrt{r_t}dW_t \tag{5.4}$$

The factor $\sqrt{r_t}$, introduced in the stochastic term, makes the model more complicated due to its non-linearity but also more realistic since if $2\kappa\theta \geq \sigma^2$, the interest rate cannot become negative.

In some variants of Vasicek and CIR models θ is a deterministic, time-dependent function. While this may be useful in pricing simple bonds such as treasuries, it adds too much complexity to the model to be useful in pricing mortgage-backed securities.

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Mortgages, unlike treasury bonds, contain embedded options and principal cash flows, so we prefer to use a simpler interest-rate model with a constant θ.

In order to find θ, κ and σ for the Cox-Ingersoll-Ross (CIR) and Vasicek models, we can obtain historical short rates or express the theoretical forward rates via the affine model for bond pricing. We can approximate short rates by looking at historical three-month treasuries for example. The problem with short rates is that we can only obtain them from the past. Forward rates, on the other hand, project future interest rates. Thus if we wish to predict the direction of interest rates it makes sense to examine forward rates. By transforming yields into forward rates and then applying the least squares method, we can fit the theoretical forward rate to that empirical forward rate data. The procedure boils down to a non-linear, non-convex minimization problem which can be addressed by an appropriate solver.

5.1.2 Theoretical Forward-Rate Models

To express the theoretical forward rate, recall that in the affine model, the bond price is described by

$$B(t,T,r_t) = e^{A(t,T)-C(t,T)r_t}.$$ (5.5)

The functions *A(t,T)* and *C(t,T)* are determined using the general affine model which is:

$$dr_t = \mu(t,r_t)dt + \sigma(t,r_t)dW_t$$ (5.6)

where $\mu(t,r_t) = \alpha(t)r_t + \beta(t)$ and $\sigma^2(t,r_t) = \gamma(t)r_t + \delta(t)$.

For both the Vasicek and CIR models, $\alpha(t) = -\kappa$ and $\beta(t) = \kappa\theta$. For Vasicek $\gamma(t) = 0$ and $\delta(t) = \sigma^2$; For CIR $\gamma(t) = \sigma^2$ and $\delta(t) = 0$.

The term structure differential equation for bonds is (See (Shreve 2004) p. 270):

$$rg = \frac{\partial g}{\partial r}\mu(t,r_t) + \frac{\partial g}{\partial t} + \tfrac{1}{2}\sigma^2(t,r_t)\frac{\partial^2 g}{\partial r^2} \tag{5.7}$$

Letting

$$g(t,r_t) = B(t,T) = \exp\left[A(t,T) - r_t C(t,T)\right] \tag{5.8}$$

$$\frac{\partial g}{\partial r} = -gC(t,T), \quad \frac{\partial g}{\partial t} = g\left(\frac{\partial A}{\partial t} - \frac{\partial C}{\partial t}r_t\right) \quad \text{and} \quad \frac{\partial^2 g}{\partial r^2} = gC^2.$$

We rewrite (5.7) as

$$rg = -gC(\alpha r + \beta) + g\left(\frac{\partial A}{\partial t} - \frac{\partial C}{\partial t}r\right) + \tfrac{1}{2}(\gamma r + \delta)gC^2 \tag{5.9}$$

Eliminating g and collecting like terms gives us:

$$A' - C\beta + \tfrac{1}{2}\delta C^2 = \left(1 + \alpha + C' - \tfrac{1}{2}\gamma C^2\right)r \tag{5.10}$$

Since r is stochastic, (5.10) will not hold unless the constant in front of r_t is 0. From this we can derive the system of differential equations for the general affine model (Shreve 2004)pp.272-277:

$$\alpha + C' - \tfrac{1}{2}\gamma C^2 = -1 \text{ and } A' = C\beta - \tfrac{1}{2}\delta C^2$$

5.1.3 Vasicek Model

For the Vasicek Model the differential equations for C and A can be written as:

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$$\frac{\partial C}{\partial t} - \kappa C = -1 \qquad \frac{\partial A}{\partial t} = \kappa \theta C - \tfrac{1}{2} C^2 \sigma^2$$

Solving the first equation and using the terminal condition C(T,T)=0 , we obtain

$$C(t,T) = \frac{1}{\kappa}\left(1 - e^{-\kappa(T-t)}\right) \tag{5.11}$$

Now substituting this into the second equation we get

$$\frac{\partial A}{\partial t} = \theta\left(1 - e^{-\kappa(T-t)}\right) - \frac{\sigma^2}{2\kappa^2}\left(1 - e^{-\kappa(T-t)}\right)^2 \tag{5.12}$$

Taking the derivative of (5.11) we obtain $\dfrac{\partial C}{\partial t} = -e^{-\kappa(T-t)}$ $\tag{5.13}$

The forward rate is defined as $f(t,T) = -\dfrac{\partial \ln B(t,T)}{\partial T} = \dfrac{\partial C}{\partial T} r_t - \dfrac{\partial A}{\partial T}$ $\tag{5.14}$

Substituting(5.12) and (5.13) into (5.14) we obtain (5.15) (See Baxter and Rennie p. 154).

$$f(0,T) = e^{-\kappa T} r_0 + \theta\left(1 - e^{-\kappa T}\right) - \frac{\sigma^2}{2\kappa^2}\left(1 - e^{-\kappa T}\right)^2 \tag{5.15}$$

For the Vasicek model $R(0,t)$ and $Q(0,t)$ can be obtained in closed form.

Observe that

$$\lim_{T \to \infty} f(0,T) = \theta - \frac{\sigma^2}{2\kappa^2} \tag{5.16}$$

demonstrates that in order to keep the theoretical forward rate from becoming negative,

$2\kappa^2\theta \geq \sigma^2$.

5.1.4 Cox-Ingersoll-Ross Forward Rate Model

For the CIR model the functions $A(t,T)$ and $C(t,t)$ can be found in (Shreve 2004) on pages 275-277 and they are:

$$A(t,T) = -\frac{2k\theta}{\sigma^2} \ln \left(\frac{\rho e^{\frac{1}{2}\kappa(T-t)}}{\rho\cosh\left[\frac{1}{2}\rho(T-t)\right] + \kappa\sinh\left[\frac{1}{2}\rho(T-t)\right]} \right) \qquad (5.17)$$

and

$$C(t,T) = \frac{2\sinh\left[\frac{1}{2}\rho(T-t)\right]}{\rho\cosh\left[\frac{1}{2}\rho(T-t)\right] + \kappa\sinh\left[\frac{1}{2}\rho(T-t)\right]} \qquad (5.18)$$

where $\rho = \sqrt{\kappa^2 + 2\sigma^2}$. Since the forward rate is:

$$f(t,T) = -\frac{\partial \ln B(t,T)}{\partial T} = -\frac{\partial A}{\partial T} + \frac{\partial C}{\partial T} r_t \qquad (5.19)$$

and we have

$$\frac{\partial A}{\partial T} = \frac{2\kappa\theta\sinh\left[\frac{1}{2}\rho(T-t)\right]}{\rho\cosh\left[\frac{1}{2}\rho(T-t)\right] + \kappa\sinh\left[\frac{1}{2}\rho(T-t)\right]} \qquad (5.20)$$

$$\frac{\partial C}{\partial T} = \frac{\rho^2}{\left(\rho\cosh\left[\frac{1}{2}\rho(T-t)\right] + \kappa\sinh\left[\frac{1}{2}\rho(T-t)\right]\right)^2} \qquad (5.21)$$

it follows that we can rewrite (5.19) as:

$$f(0,T) = -\frac{2\kappa\theta\sinh\left(\frac{1}{2}\rho T\right)}{G(T)} + \left[\frac{\rho}{G(T)}\right]^2 r_0 \qquad (5.22)$$

where $G(T) = \rho\cosh\left(\frac{1}{2}\rho T\right) + \kappa\sinh\left(\frac{1}{2}\rho T\right)$

We have now established theoretical forward rate models for Vasicek and Cox-Ingersoll-Ross in (5.15) and (5.22) respectively.

5.1.5 Empirical Forward Rates

Although mortgages are comparable to the credit risk of Corporate A Bonds, we prefer to use risk-free Treasuries. The main reason is that Corporate A interest rates for

various maturities are available from Bloomberg only as far back as September 2002. Also treasuries are the largest and most liquid bond market and are for all practical purposes considered "risk-free". (In light of S&P's recent downgrading of U.S. Debt, this may be called into question.) The ready availability of the data allows us to study its term structure in detail. We can easily approximate Corporate A rates by adding a spread to Treasury rates. This spread can be estimated by calculating the average difference between Corporate A rates and Treasury Rates over a specified period. Between September 2002 and October 2008, this average is 74 basis points.

Mortgage-backed securities are coupon bonds, so we need to look at the entire yield curve to capture the term structure of interest rates. The U.S. Treasury publishes daily yield curve rates on the web:

Because the one-month and 30-year rates are not available for the entire period of study, we will not include them in our estimation of forward rates. Also the one-month rates can be somewhat unstable. We will use the 3-month Treasury rates as an estimate for the short rate r_0.

The forward rates can be estimated from the spot rates in the following manner. See (Hull 2003) page 99):

$$R_F = \frac{R_2 T_2 - R_1 T_1}{T_2 - T_1} \tag{5.23}$$

where R_1 and R_2 are consecutive spot rates on the yield curve and T_1 and T_2 are their corresponding maturities.

A typical upward-sloping treasury yield curve is shown in Figure 8:
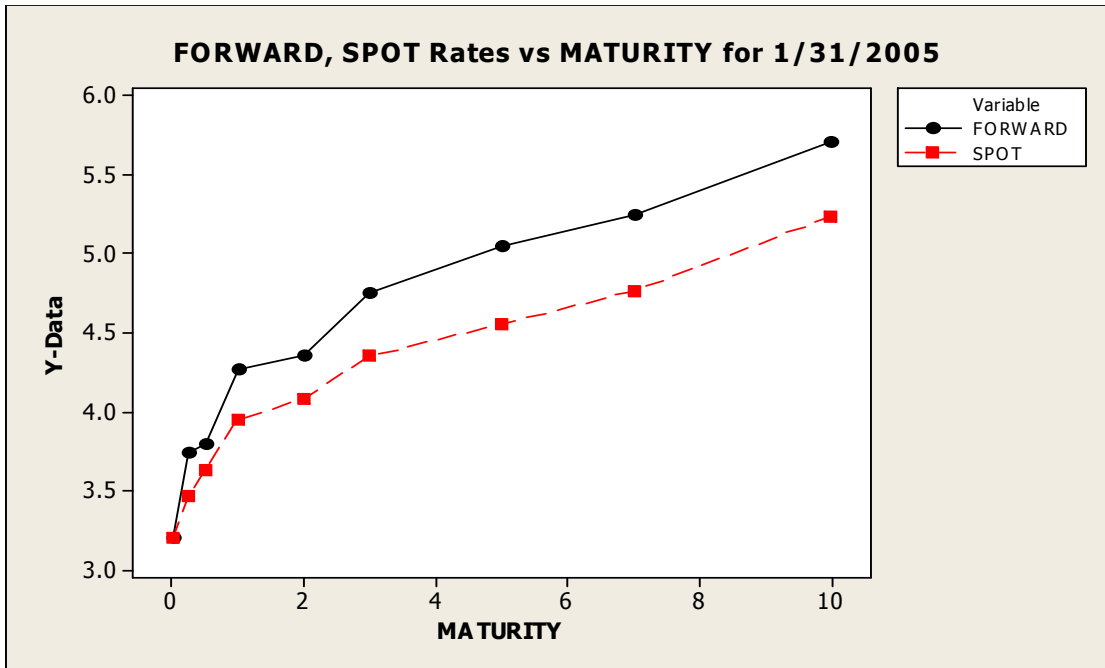
Figure 8 - Comparison of Spot Rates and Forward Rates
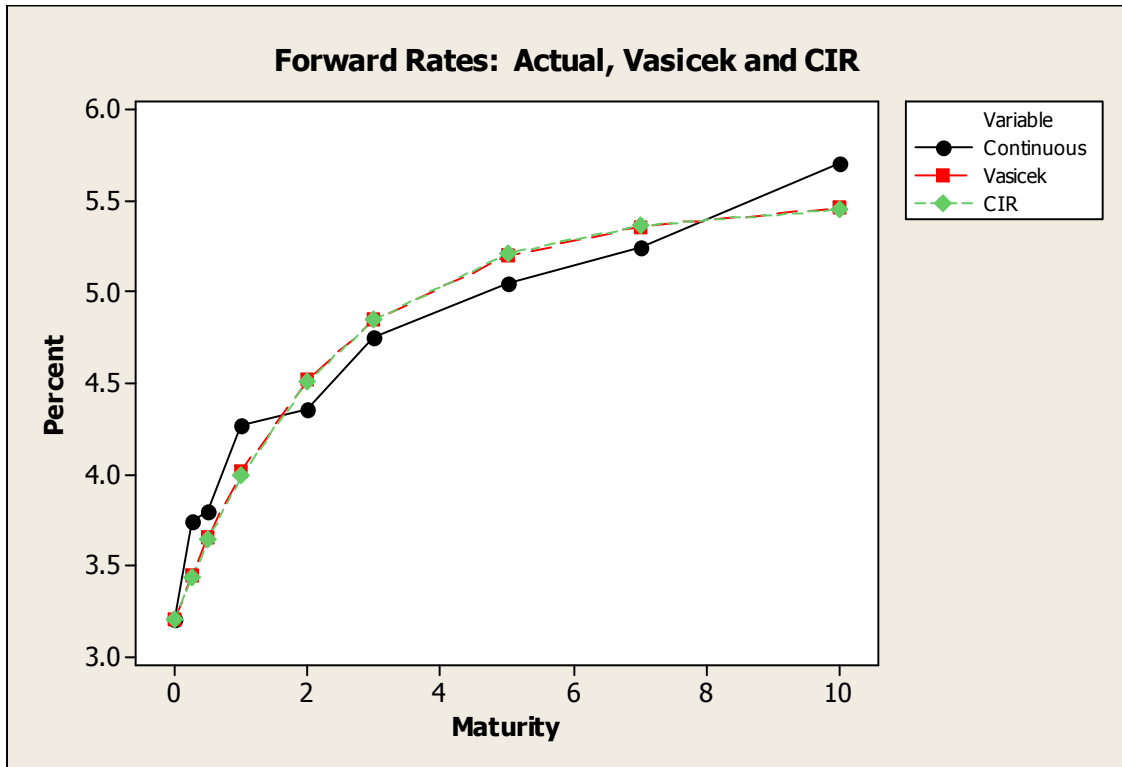
Figure 9 - Actual vs. Fitted Forward Rates

U.S. Treasury Spot Rates Corresponding to Figure 8									
Date	1 Mo	2 Mo	6 Mo	1 year	2 year	3 year	5 year	7 year	10 yr	20 yr
1/31/05	2.06%	2.51%	2.79%	2.96%	3.29%	3.43%	3.71%	3.92%	4.14%	4.64%

To calculate the one-year forward rate, we take the one-year spot rate and look for the nearest longer-maturity spot rate which is the two-year rate. We calculate the forward rate as follows:

$$\frac{3.29(2) - 2.96(1)}{2-1} = 3.62 \tag{5.24}$$

Since the forward rate calculation is a linear function of the spot rates, it does not matter when we add the Corporate A Spread. In other words adding a spread to the spot rate is equivalent to adding a spread to the forward rate:

$$\frac{(R_2 + S)T_2 - (R_1 + S)T_1}{T_2 - T_1} = \frac{R_2 T_2 - R_1 T_1 + S(T_2 - T_1)}{T_2 - T_1} = \frac{R_2 T_2 - R_1 T_1}{T_2 - T_1} + S \tag{5.25}$$

Figure 8 shows the relationship between spot rates and forward rates when the yield curve is upward-sloping.

5.1.6 Continuous-Time Forward Rates

The continuous model requires us to convert annualized rates to continuously compounded rates.

$$r_c = \ln\left[1 + \frac{R+S}{100}\right] \tag{5.26}$$

Applying a spread of 74 basis points to the 1-year forward rate on January 2005 obtained in (5.24), we arrive at the continuous rate:

$$\ln\left[1 + \frac{3.62 + 0.74}{100}\right] = 0.0427 \tag{5.27}$$

This continuous rate of 4.27% differs from the nominal rate of 3.62 + 0.74 = 4.34% by 7 basis points.

We can easily automate the calculation of a forward rate curve from a yield curve. Appendix D shows how to do this in both Mathematica and APL. Figure 8 shows the relationship between the adjusted continuous forward and spot rates for January 2005. Observe the upward-sloping of both the yield curve and the forward curve. As we will discover in the next section, it is fairly easy to calibrate the parameters when the forward rate exhibits this type of pattern.

5.1.7 Calibration of Vasicek and CIR Parameters

To calibrate parameters we need to take the empirical forward rates and find the best fit using the theoretical Vasicek and CIR Models. This can be defined as a non-linear, least-squares optimization problem. We define the objective function as:

$$\min \sum_{i=1}^{n} \left[\hat{f}(t,T_i) - f(t,T_i;\theta,\kappa,\sigma)\right]^2 \text{subject to} \quad \theta,\kappa,\sigma > 0 \tag{5.28}$$

where the $\hat{f}(t,T_i)$ are the empirical forward rates (calculated using (5.23) and the spot rates on the treasury web site), and the $f(t,T_i;\theta,\kappa,\sigma)$ are the theoretical forward rates expressed in terms of the model parameters $\boldsymbol{\theta}$, κ and σ that have to be estimated.

Mathematica provides several optimization functions: **Minimize, NMinimize, FindMinimum** and **FindFit**. **Minimize** performs symbolic optimization, while **NMinimize** does nonlinear constrained global optimization,

FindMinimum finds local constrained or unconstrained optimization and **FindFit** is restricted to a least squares (L_2) approach. Since there is no guarantee that the feasible region is convex, **NMinimize** appears to be the best choice.

Before calculating the parameters, we need to determine the short rate r_0. Our best estimate for this uses the 3-month Treasury rate because the one-month Treasury rate is not always available and is too volatile when the interest rates are very low. For January 31, 2005, the 3-month Treasury rate is 2.51%. To this we will add our spread of 74 basis points and we will convert to the continuous rate as follows:

$$\hat{r}_0 = \ln(1+r+s) = \ln(1+.0251+.0074) = \ln(1.0325) = 0.031983 \qquad (5.29)$$

To set up the Vasicek calibration, we obtain the empirical forward rates from treasury rates of varying maturities from http://www.treasury.gov:

```
FWD= {2.51, 3.07, 3.13, 3.62, 3.71, 4.13, 4.445, 4.65333, 5.14};
```

and their corresponding maturities:

```
T={1/4,1/2,1,2,3,5,7,10,20};
```

To ensure the parameter estimates are greater than zero, we create a dummy constraint which is slightly larger than zero:

```
dt=1/10^6;
```

We must include the spread to Corporate A bonds before the calibration:

```
Spread=0.74;
```

Now we set up the theoretical Vasicek forward rate equation from (5.15):

```
XX := Exp[-kappa*T]
FV[r0_]:=Drop[XX*r0+theta*(1-XX)-sigma^2/(2*kappa^2)*(1-XX)^2,-1]
```

We define the optimization model to estimate the Vasicek parameters as total least squares difference between observed and theoretical forward rates:

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```
PV[FWD_]:=NMinimize[{Total[(Drop[FWD,1]-FV[FWD[[1]]])^2],
        kappa>=dt,sigma>=dt,theta>=dt},{kappa,sigma,theta}]
```

Now we input the observed forward rates into the model

```
PV[Log[1+(FWD+Spread)/100]]
```

and obtain the following Vasicek parameter estimates:

$$\hat{\kappa}_V = 0.31695 \qquad \hat{\sigma}_V = 0.04332 \qquad \hat{\theta}_V = 0.06459 \tag{5.30}$$

Substituting the values in (5.30) into the Vasicek forward rate model (5.15) we can obtain the one-year theoretical forward rate by setting $T = 1$ and using the continuous 3-month Treasury rate plus spread:

$$1 - e^{-\hat{\kappa}_V} = 0.27163$$

$$\hat{f}_V(0,1) = e^{-\hat{\kappa}_V} r_0 + \hat{\theta}_V \left(1 - e^{-\hat{\kappa}_V}\right) - \frac{1}{2}\left[\frac{\hat{\sigma}_V}{\hat{\kappa}_V}\left(1 - e^{-\hat{\kappa}_V}\right)\right]^2$$

$$= 0.72837 \times 0.031983 + 0.06459 \times 0.27163 - 0.5\left(\frac{0.04332 \times .27163}{0.31695}\right)^2$$

$$= 0.040151$$

(5.31)

We compare this theoretical rate of 4.015% to the actual continuous one-year forward rate 4.26% which we obtained in(5.27). The difference is less than 25 basis points.

Now let's Calibrate the Cox-Ingersoll Ross model. Using the same empirical forward rates and maturities, we generate the theoretical Cox-Ingersoll-Ross forward rate model from(5.22):

```
rho := Sqrt[kappa^2+2*sigma^2]
A   := 2*kappa*theta*Sinh[0.5*rho*T]/G
G   := rho*Cosh[0.5*rho*T]+kappa*Sinh[0.5*rho*T]
CC  := rho/G
FC[r0_] := Drop[A+r0*CC^2,-1]
```

The Cox-Ingersoll-Ross parameter optimization model is also a least squares model and includes the constraint $2\kappa\theta > \sigma^2$ to prevent interest rates from going negative.

```
PC[FWD_]:=NMinimize[{Total[(Drop[FWD,1]-FC[FWD[[1]]])^2],
    2*kappa*theta-sigma^2>=dt,sigma>=0},{kappa,sigma,theta}]
```

We then apply the model to the observed forward rates

```
PC[Log[1+(FWD+Spread)/100]]
```

and obtain the Cox-Ingersoll-Ross forward rate parameters:

$$\hat{\kappa}_{CIR} = 0.32638 \qquad \hat{\sigma}_{CIR} = 0.17805 \qquad \hat{\theta}_{CIR} = 0.06210 \qquad (5.32)$$

Now we can calculate the 1-year theoretical CIR forward rate from (5.29) and (5.32)as follows:

$$\rho = \sqrt{\hat{\kappa}_{CIR}^2 + 2\hat{\sigma}_{CIR}^2} = 0.41223$$

$$G(1) = \rho\cosh\left(\tfrac{1}{2}\rho\right) + \hat{\kappa}_{CIR}\sinh\left(\tfrac{1}{2}\rho\right) = 0.41223\times1.0213 + .32638\times0.20758 = 0.48876$$

$$f(0,1) = \frac{2\hat{\kappa}_{CIR}\hat{\theta}_{CIR}\sinh\left(\tfrac{1}{2}\rho\right)}{G(1)} + \left[\frac{\rho}{G(1)}\right]^2 r_0 \qquad (5.33)$$

$$= \frac{2\times0.32638\times0.17805\times0.20758}{0.48876} + \left[\frac{0.41223}{0.48876}\right]^2 \times0.31983 = 0.039966$$

We compare this result of 3.9966% to the actual continuous forward rate of 4.267% which is less than 27 basis points away.

Table 9 and **Figure 9** show how well the theoretical model fits the actual data. What is remarkable is how close the Vasicek and CIR models are to each other even though the parameters and formulas are different. Vasicek and CIR are within 2 basis points of each other.

	Actual Forward Rate		Theoretical Forward Rate	
Maturity	Annualized	Continuous	Vasicek	CIR
0	3.25	3.1983	3.1983	3.1983
3 months	3.81	3.73921	3.4413	3.4313
6 months	3.87	3.79699	3.6561	3.6405
1 year	4.36	4.26763	4.0151	3.9966
2 year	4.45	4.35383	4.5233	4.5143
3 years	4.87	4.75513	4.8474	4.8503
5 years	5.185	5.05505	5.2002	5.2123
7 years	5.3933	5.25292	5.3625	5.3681
10 years	5.88	5.71362	5.4648	5.4535

Table 9:   Actual and Theoretical Forward Rates

5.1.8 Problems with Calibration – Inverted/Humped Yield Curve

When the yield curve is not in the familiar normal upward-sloping form, it may be difficult to calibrate the model. The interest rate models we are using assume that the long-term mean is constant. More complex models assume that the long-term mean is a deterministic function of time. The simpler parameterization is necessary to allow us to use the eigenfunction expansion method in our pricing model. The downside of this is the difficulty in handling inverted and humped yield curves. In these cases sometimes σ evaluates to zero or sometimes κ becomes extremely large. Most of the time θ is fairly stable, but occasionally even this parameter may become unusually large. When this occurs, we need to look at an alternative method of calibration.

The generalized p-norm measures the distance between the actual and theoretical forward rates:

$$\left\| f - \hat{f} \right\|_p = \sqrt[p]{\sum_i \left| f(t_i) - \hat{f}(t_i) \right|^p} \tag{5.34}$$

The standard least squares model we have been using is known as the Euclidean norm or L_2 norm and can be represented in (5.34) by setting $p = 2$.   This model is somewhat sensitive to outliers because these differences are squared.

A less-sensitive distance model involves summing the absolute value of each difference. This is accomplished by setting $p = 1$ in (5.34); in two dimensions this is known as the Manhattan norm or L_1 norm, because the shortest distance between any two points on a street grid is sum of the north-south distance and the east-west distance.

Many cases where the yield curve does not calibrate well are in times of economic crises. In these cases the spread between Corporate A Bonds and Treasuries tends to increase because of additional credit risk.   One example is in 2000 during the Internet bubble. The spread also went up between 2005 and the financial crisis of 2008. Since the Bloomberg Corporate A yields are not available before October 2005, we can estimate the crisis spread by looking at the period between 2005 and 2008. (After October 2008, the yields are too unstable.) Our estimate for the crisis spread is 95.4 basis points.

An isolated point is a single month where we do not get good calibration results. In this case, we simply use the calibrated parameters from the previous month. We use these parameters with the short rate and weighted average maturity for the new month in the price estimation.

Sometimes a whole year goes by when we cannot get good parameter estimates. We will define this as a crisis period. One example of this is the period from November 1999 until March 2001. This was when the internet bubble occurred. During crisis periods we cannot simply take the previous month's parameters.   There is more

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uncertainty in the market, so we will add the crisis spread to the Treasury forward rates instead of the normal spread. In addition, to avoid sensitivity to outliers we will use the L_1 norm as a distance measure instead of the usual L_2 (Euclidean) norm.

Figure 10 shows Treasury Rates for the period of study 1996-2006. The periods which caused the most problems in calibration are listed in Table 10. In all of these cases the interest rates for varying maturities were fairly close to each other. Another observation is that during two of these periods, the interest rates were on a steep downward trend. In general, most of the calibration problems occur when the yield curve is humped or decreasing. Since the eigenfunction expansion model requires that we use 3-parameters for both Cox-Ingersoll-Ross and Vasicek, we cannot use a time-varying but deterministic θ_t. We prefer to keep the model as simple as possible, but this comes at the expense of a poorer fit during crisis periods.

The yield curve for each month consists of 9 individual treasury yields of varying maturities. For each monthly yield curve we define an inversion as

$$y_{T_{i+1}} - y_{T_i} < 0 \tag{5.35}$$

The number of inversions is defined as:

$$\sum_{i=1}^{N-1} \left[y_{T_{i+1}} - y_{T_i} < 0 \right] \tag{5.36}$$

We also define the maximum inversion as:

$$\max \left| y_{T_i} - y_{T_{i+1}} \right|^{+} \tag{5.37}$$

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We define the functions Inversions and MaxInversion in APL and apply them to the yield curve for August 1998. This results in 5 inversions and a maximum inversion size of 0.08:

```
     Inversions←{0+.>2-/ω}
     MaxInversion←{⌈/0,2-/ω}
     YLDCV9808 ←4.96,5.03,4.95,4.91,4.85,4.91,5.03,5.05,5.45
     Inversions YLDCV9808
5
     MaxInversion YLDCV9808
0.08
```

Using the standard L_2 calibration procedure with the normal spread of 74 basis points we obtain the following Cox-Ingersoll-Ross parameters:

$$\kappa = 0.000327134, \ \sigma = 0.000242087, \text{ and } \theta = 1.77936$$

All three of these values are way outside the normal range. However, if we use L_1 calibration with the crisis spread of 94.5 basis points we obtain:

$$\kappa = 0.540341, \ \sigma = 0.078189, \text{ and } \theta = 0.0593712$$

If we constrain $\sigma >= 0.100$, the parameters are:

$$\kappa = 0.9698, \ \sigma = 0.14214, \text{ and } \theta = 0.05937$$

Two things are of interest here. Due to the non-linear non-convex model, the σ constraint is not binding. Also the parameter θ remains constant. The L_1 values seem much more reasonable. Now let's plot the theoretical forward rates with the actual. Notice that even though the forward rate is trending upward, it decreases for awhile and then increases. This produces a poor fit, but it's the best we can do with our model. Our price estimate for the GNMA 7.5% MBS for August 1998 uses the L_1 estimates with the $\sigma > 0.1$ constraint. The estimates are shown in Table 10.

GNMA 7.5% MBS For August 1998		
Model	**Estimate**	**Error (Basis Points)**
CIR1	$105.152	213.60
CIR2	$104.860	184.40
Kolbe-Zagst	$102.700	31.60
Bloomberg	$103.016	N/A

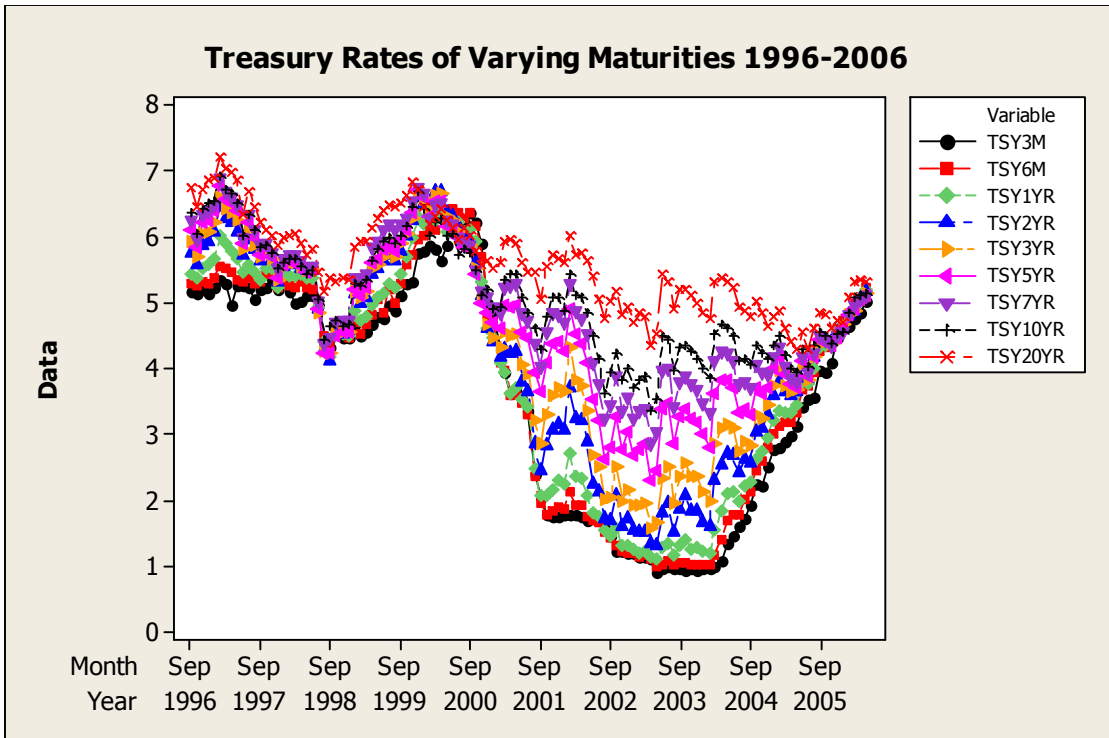| **Crisis Periods between 1996 and 2006**		
From	**To**	**Reason**
November 1997	January 1999	Unknown
November 1999	March 2001	Internet Bubble
December 2002	March 2004	Extremely Low Interest Rates (affects only Vasicek)
September 2005	June 2006	Immediately prior to Financial Meltdown of 2008

Table 10 – Pricing during Crisis Periods
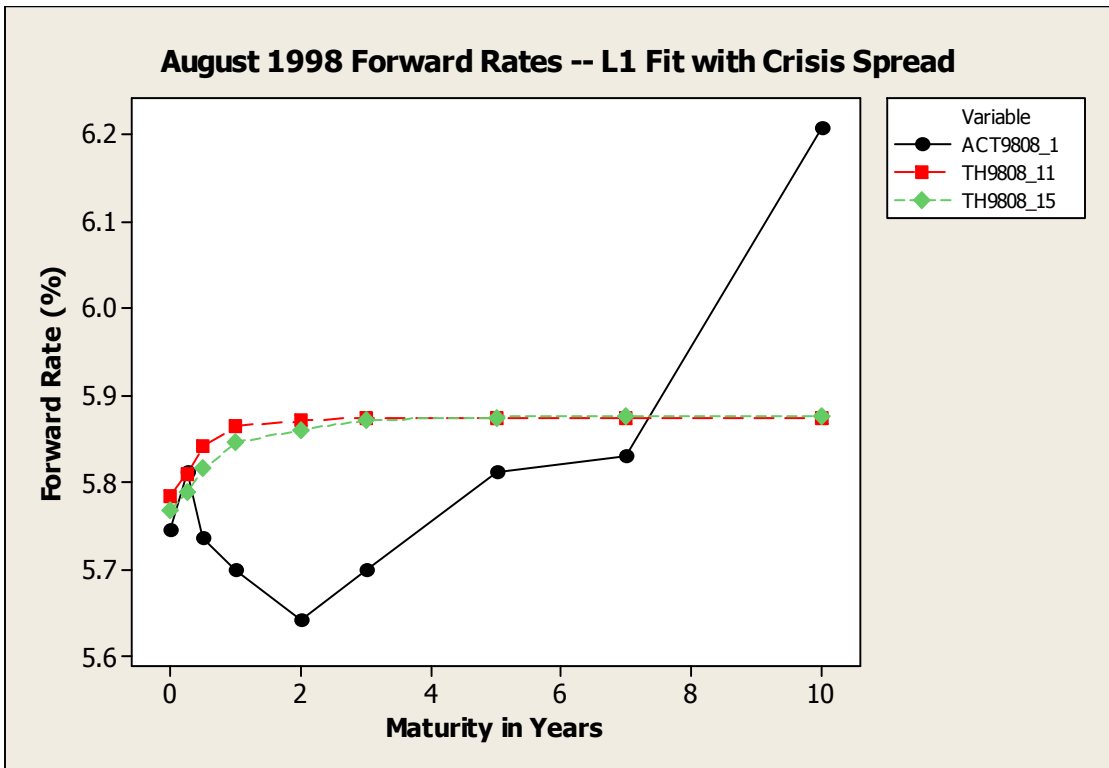
Figure 10 Treasury Rates from 1996-2006

Figure 11 - Fitting parameters during a Crisis Period

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CIR Parameter Estimates (Oct 1996 to June 2006)

DATE	KAPPA	SIGMA	THETA	NORM	DATE	KAPPA	SIGMA	THETA	NORM
199610	0.35205	0.17850	0.08170	2	200110	0.18354	0.17668	0.08504	2
199611	0.27790	0.12306	0.07727	2	200111	0.35264	0.23391	0.07758	2
199612	0.47628	0.25968	0.08199	2	200112	0.45902	0.26465	0.07629	2
199701	0.50872	0.28089	0.08347	2	200201	0.52007	0.27696	0.07375	2
199702	0.80105	0.35791	0.07996	2	200202	0.48818	0.26690	0.07296	2
199703	1.99990	0.54933	0.07958	1	200203	0.81636	0.34429	0.07260	2
199704	1.29400	0.45649	0.08060	1	200204	0.57948	0.29052	0.07283	2
199705	2.32025	0.59652	0.07714	1	200205	0.49973	0.27362	0.07492	2
199706	0.83765	0.36393	0.07908	2	200206	0.35344	0.23645	0.07911	2
199707	0.44499	0.19172	0.07416	2	200207	0.15307	0.17690	0.10223	2
199708	0.73640	0.33887	0.07797	2	200208	0.12005	0.15938	0.10580	2
199709	0.91007	0.36682	0.07393	2	200209	0.12005	0.15938	0.10580	2*
199710	0.35670	0.14021	0.07209	2	200210	0.05266	0.13779	0.18028	2
199711	1.42852	0.44510	0.06936	1	200211	0.22481	0.19499	0.08456	2
199712	0.83757	0.26164	0.06805	1	200212	0.10286	0.15307	0.11390	2
199801	0.72579	0.17986	0.06516	1	200301	0.16233	0.17332	0.09252	2
199802	0.70719	0.22711	0.06716	1	200302	0.09415	0.14797	0.11629	2
199803	2.70059	0.10856	0.06433	1	200303	0.11133	0.15650	0.11000	2
199804	4.67419	0.19522	0.06443	1	200304	0.13776	0.16426	0.09793	2
199805	3.76178	0.35417	0.06368	1	200305	0.04613	0.12676	0.17417	2
199806	3.14423	0.10534	0.06372	1	200306	0.10806	0.15140	0.10607	2
199807	2.26125	0.54474	0.06562	1	200307	0.22690	0.20292	0.09074	2
199808	0.96980	0.14240	0.05937	1	200308	0.27955	0.21449	0.08229	2
199809	0.41205	0.10000	0.05544	1	200309	0.15344	0.17216	0.09659	2
199810	0.17353	0.10000	0.06718	1	200310	0.23901	0.20162	0.08504	2
199811	0.75592	0.15443	0.05742	1	200311	0.30392	0.21765	0.07799	2
199812	0.81388	0.11422	0.05440	1	200312	0.25670	0.20439	0.08137	2
199901	0.62662	0.14630	0.05487	1	200401	0.26129	0.20329	0.07909	2
199902	0.20936	0.08643	0.07033	2	200402	0.20793	0.18597	0.08317	2
199903	0.19425	0.08813	0.07296	2	200403	0.16273	0.17197	0.09086	2
199904	0.27952	0.13294	0.07179	2	200404	0.39010	0.24080	0.07433	2
199905	0.59444	0.26359	0.07258	1	200405	0.46043	0.25812	0.07237	2
199906	0.60182	0.29964	0.07459	2	200406	0.43432	0.24994	0.07192	2
199907	0.81046	0.34583	0.07379	2	200407	0.37621	0.23367	0.07264	2
199908	0.69891	0.32503	0.07558	2	200408	0.21868	0.18652	0.07955	2
199909	0.54583	0.28339	0.07690	2	200409	0.23579	0.18962	0.07628	2
199910	0.69854	0.32516	0.07568	2	200410	0.15792	0.16441	0.08562	2
199911	3.20060	0.66913	0.07058	1	200411	0.23214	0.18915	0.07708	2
199912	4.91396	0.68375	0.07225	1	200412	0.24524	0.18900	0.07284	2
200001	3.32188	0.64965	0.07608	1	200501	0.32638	0.17805	0.06210	2
200002	2.64768	0.62993	0.07536	1	200502	0.44531	0.23452	0.06176	2
200003	0.52301	0.28528	0.07775	1	200503	0.78617	0.30313	0.05844	2
200004	0.52301	0.28528	0.07775	1	200504	0.40032	0.21908	0.05995	2
200005	0.52301	0.28528	0.07775	1	200505	0.30671	0.16600	0.05783	2
200006	1.61949	0.47216	0.07163	1	200506	0.25625	0.11900	0.05551	2
200007	0.66037	0.23037	0.07086	1	200507	1.84128	0.42523	0.05317	1
200008	1.49859	0.45349	0.06901	1	200508	0.43578	0.17475	0.05446	1
200009	1.30490	0.16786	0.06542	1	200509	1.54629	0.41056	0.05451	1
200010	3.33093	0.10084	0.06504	1	200510	1.52256	0.41248	0.05587	1
200011	2.46585	0.10000	0.06197	1	200511	2.50433	0.52007	0.05447	1
200012	3.81172	0.22763	0.05755	1	200512	1.99727	0.46494	0.05412	1
200101	0.26647	0.10000	0.06632	1	200601	2.46045	0.44314	0.05434	1
200102	1.69415	0.26621	0.05755	1	200602	0.52739	0.16962	0.05569	1
200103	0.19189	0.10605	0.07112	1	200603	0.63099	0.17379	0.05915	1
200104	0.15996	0.13193	0.08763	2	200604	0.60260	0.16955	0.06138	1
200105	0.18501	0.18708	0.09459	2	200605	0.78359	0.15869	0.06008	1
200106	0.18501	0.18708	0.09459	2*	200606	0.78052	0.16084	0.06034	1
200107	0.07503	0.13819	0.12725	2	200607	1.50321	0.10238	0.05778	1
200108	0.11966	0.14259	0.09679	2	200608	1.97883	0.00358	0.05361	2
200109	0.14293	0.16902	0.09994	2	200609	1.89065	0.45236	0.05412	2

*Used calibrated parameter estimates for the previous month.

Table 11 - CIR Calibrated Parameters 1996-2006

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```
      DATE    TSY3M    KAPPA    SIGMA    THETA
      ------  -------  -------  -------  -------
      200902  0.26000  0.20703  0.17083  0.07048
      200903  0.21000  0.15848  0.15195  0.07284
      200904  0.14000  0.20834  0.17504  0.07353
      200905  0.14000  0.25085  0.19110  0.07279
      200906  0.19000  0.31900  0.20583  0.06641
      200907  0.18000  0.30979  0.20345  0.06681
      200908  0.15000  0.29882  0.19822  0.06574
      200909  0.14000  0.29731  0.19531  0.06415
      200910  0.05000  0.28254  0.19613  0.06807
      200911  0.06000  0.19960  0.17381  0.07568
      200912  0.06000  0.32404  0.21499  0.07132
      201001  0.08000  0.24840  0.19170  0.07400
      201002  0.13000  0.22820  0.18680  0.07650
      201003  0.16000  0.27060  0.20050  0.07430
      201004  0.16000  0.26460  0.19510  0.07190
      201005  0.16000  0.21770  0.17650  0.07160
      201006  0.18000  0.16800  0.15750  0.07390
      201007  0.15000  0.12720  0.14800  0.08620
      201008  0.14000  0.11480  0.13400  0.07820
      201009  0.16000  0.07890  0.12790  0.10360
      201010  0.12000  0.03930  0.12260  0.19110
      201011  0.17000  0.07750  0.13510  0.11770
      201012  0.12000  0.17640  0.16820  0.08020
      201101  0.15000  0.13640  0.16200  0.09620
      201102  0.15000  0.19010  0.17390  0.07950
      201103  0.09000  0.22310  0.18280  0.07490
      201104  0.04000  0.17440  0.16870  0.08160
      201105  0.06000  0.13100  0.15290  0.08920
      201106  0.03000  0.13520  0.15760  0.09190
      201107  0.08000  0.09690  0.14450  0.10770
```

Table 12 - Recent CIR Parameters

5.1.10 Quadratic Variation

Sometimes sigma evaluates to zero. While this may be a decent fit to the forward rate, we cannot use sigma=0 in our model. We may be able to estimate sigma by using quadratic variation. Quadratic variation is defined as:

$$Q_\Pi(X_t) = \lim \sum_{i=1}^{n} \left(X_{t_i} - X_{t_{i-1}} \right)^2 \tag{5.38}$$

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The Vasicek formula is:

$$dr = \kappa(\theta - r)dt + \sigma dW_t \tag{5.39}$$

We can adapt the Vasicek formula to (5.38) by the following:

$$Q_\Pi(r_t) = \int_0^T (dr)^2 = \int_0^T \left[\kappa(\theta - r)dt + \sigma dW_t\right]^2 = \\ = \int_0^T \sigma^2 dt = \sigma^2 T \tag{5.40}$$

Thus

$$\sigma_V = \sqrt{\frac{Q_\Pi}{T}} \tag{5.41}$$

Taking one years' worth of 3-month Treasuries calculated on a daily basis, we can calculate the quadratic variation as follows:

$$\sum_{i=1}^n (r_i - r_{i-1})^2 \tag{5.42}$$

The Quadratic Variance for 3-month Treasuries in 1997 is 0.00005169.   Since T = 1 year,

$$\sigma_V = \sqrt{0.00005169} = 0.0071896 \tag{5.43}$$

For Cox-Ingersoll-Ross, the Quadratic Variance

$$Q_\Pi(r_t) = \int_0^T (dr)^2 = \int_0^T \left[\kappa(\theta - r)dt + \sigma\sqrt{r}dW_t\right]^2 = \\ = \int_0^T \sigma^2 r dt \approx \sigma^2 \sum_{i=1}^n r_i \Delta t \tag{5.44}$$

Thus

$$\sigma_{CIR} = \sqrt{\frac{Q_\Pi}{r_i \Delta t}} \tag{5.45}$$

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Using the 1997 data,

$$\sigma_{CIR} = \sqrt{\frac{0.00005169}{\frac{1}{251} \times 13.004}} = 0.031587 \tag{5.46}$$

We can fix σ=0.031587 and determine the other two parameters.

```
PC[FWD_]:=NMinimize[{PNorm[Drop[FWD,1]-
FC[FWD[[1]]]],2*kappa*theta-
sigma^2=dt,kappa>=0,theta>=0,sigma>=0.031587},{kappa,sigma,
theta}]
```

The problem is that in Table 11 the CIR parameter σ is about ten times larger than in (5.46). Experimentation has shown the CIR pricing model does not converge when σ < 0.05. Thus we are better off optimizing all three parameters simultaneously.

5.2 Theoretical Prepayment Rates

To calculate prepayment parameters, we need to observe the relationship between interest rates and prepayment rates. Since they are readily available, we will use treasury rates again.

The simplest prepayment model is the constant prepayment rate. In this case prepayments represent a constant proportion of the remaining balance. This means prepayments are independent of both time and interest rates. The second model is known as the PSA (Public Securities Administration) model. This model takes the position that prepayments do not occur as frequently during the first 2-1/2 years of a 30-year mortgage; they increase linearly up to that point, then remain constant thereafter. Both

the CPR and PSA models are deterministic. We will use the empirical 3-month CPR and 3-month PSA rates in our model.

5.2.1 Relationship between Interest Rates and Prepayments

To estimate the prepayment parameters, we obtain historical prepayment rates and the weighted average maturities for Ginnie Mae MBS from Bloomberg. The ticker is GNSF. For the 8.0% GNMA MBS, go to Bloomberg, then enter **GNSF 8 N <mtge> CPH <GO>**. We are interested in the following columns: Date, WAM and 3-Mo CPR. We will use the WAM (Weighted Average Maturity) for the maturity T, and we will perform a regression between the 3-month CPR and a suitable benchmark yield.

Although the Corporate A Benchmark index more closely tracks the credit risk of mortgage-backed securities, we prefer to use the 3-month treasuries as a baseline interest rate. The Corporate A index is only available after September 2002. To obtain a sufficient history, we must resort to Treasury rates which are readily available from a number of sources. We can obtain 3-month Treasury rates from the U.S. Treasury web site or from Bloomberg. To obtain these in Bloomberg select **C0793M <corp> HP <Go>.**

To account for interest rate fluctuations we introduce a stochastic component. Figure 12 shows the relationship between treasury rates and prepayment rates of five generic pools of Ginnie Mae Securities, each with a different coupon rate. There is a clear relationship between interest rates and prepayment rates. As interest rates decrease below a certain threshold, prepayment rates increase; however, if interest rates go above this threshold, they tend to remain constant. The 7.5% AND 8.0% GNMA's show a

piecewise linear relationship between interest rates and prepayments. The 6%, 6.5% and 7% show that the prepayment rates for 2004 and 2005 are somewhat higher than for 2001 and 2002 for the same interest rates. It appears that borrowers waited to refinance until rates were extremely low in 2003, then there was a rush to refinance when rates started moving back up again. With the 7.5% and 8.0% pools, the refinancing occurred much earlier, so that refinancing rates were much the same when interest rates came back up. With the 7.5% and 8.0% it is clear that there is a leveling off of prepayment rates after the initial rush to refinance, whereas with the lower coupon pools the model is more complex. We can model all pools with the single threshold model; however, the 7.5% and 8.0% can also be modeled with the double-threshold model.

Starting with the constant prepayment rate (CPR) model, we discuss three parameters: the exogenous prepayment rate h_o, the threshold k, and the slope γ. We also introduce the double threshold model which has five parameters—the exogenous prepayment rate as before, two thresholds k_1 and k_2 and two slopes γ_1 and γ_2. The exogenous prepayment rate can be either a constant or with a ramp.
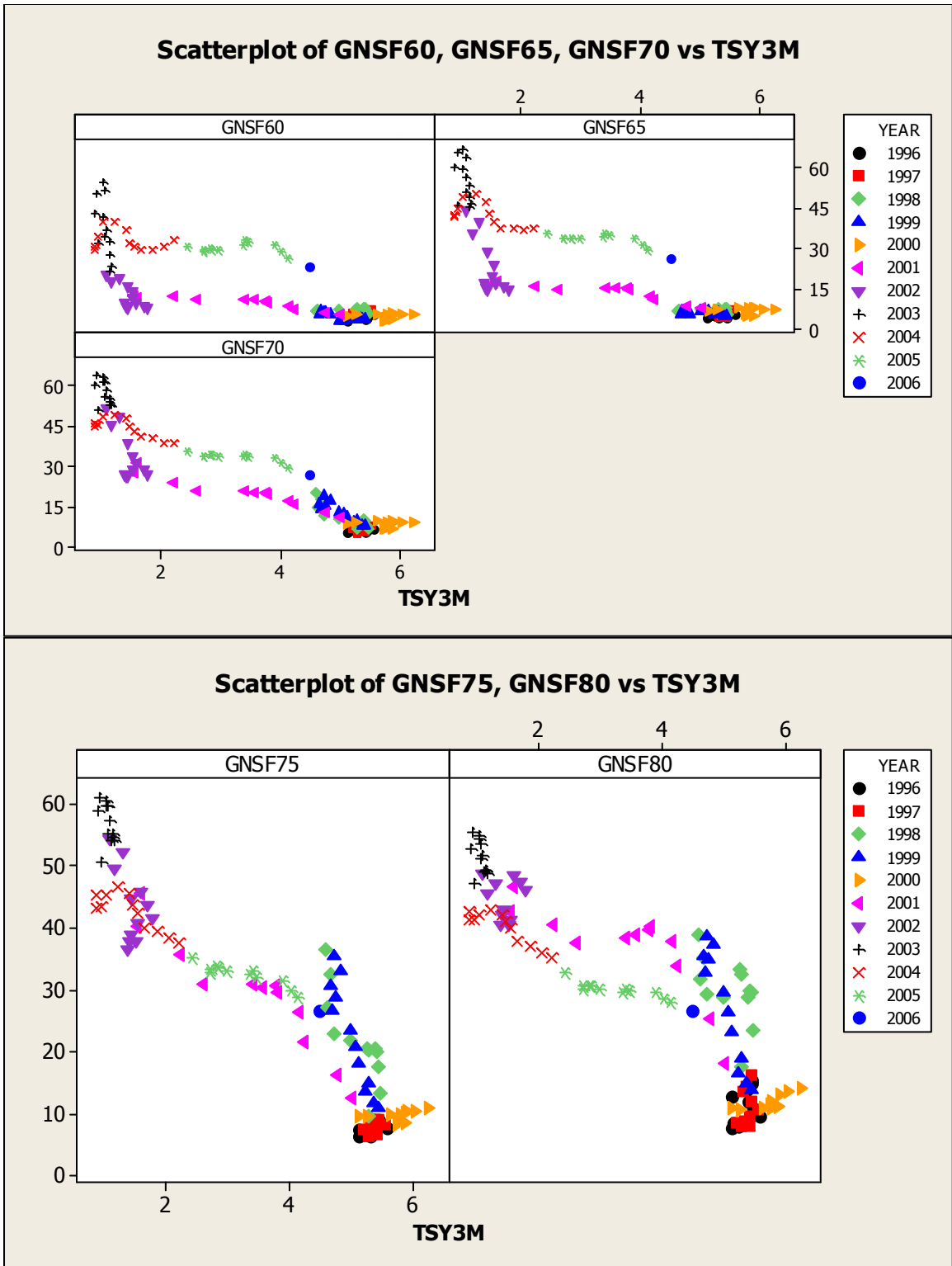
Figure 12: Relationships between Interest Rates and Prepayment Rates

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5.2.2 Single-Threshold Model

We will look at GNMA Mortgage-Backed Securities because they are backed by the full faith and credit of the U.S. Government unlike Fannie Mae and Freddie Mac which had an implicit guarantee which was exposed during the recent Mortgage crisis. We can obtain historical prepayment rates along with the Weighted Average Maturity for generic GNMA's with various coupons: 6.0%, 6.5%, 7.0%, 7.5% and 8.0%.

From Bloomberg, we obtain prepayment rates for Mortgage Backed Securities. We obtain 3-month Treasury data from Bloomberg. In Bloomberg, enter `C0793M <Govt> HP` to obtain historical prices and yields. To obtain historical GNMA prepayment rates enter `GNSF 8 N <Mtge> CPH <go>`. To obtain historical prices, enter `GNSF 8 N <Mtge>HP<go>`. We will use the 3-Month CPR and the WAM (Weighted Average Maturity

Calculating prepayment parameters is basically a linear regression problem with an additional twist of determining the breakpoint. The procedure involves selecting a breakpoint k, then performing a simple linear regression; the intercept h_0 is the exogenous prepayment rate, and the slope γ is the sensitivity of the prepayment rate to a drop in interest rates.

$$CPR_t = h_0 + \gamma \left[k - r_t \right]^+ + \varepsilon \qquad (5.47)$$

Observe that the prepayment rate remains constant when interest rates are above the threshold k. From historical data CPR_t and r_t are known for all t. Fixing k reduces (5.47) to a linear equation in the form

$$y_t = \beta_0 + \beta_1 x_t + \varepsilon_t \qquad (5.48)$$

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where $y_t = CPR_t$; $x_t = [k - r_t]^+$; $\beta_0 = h_0$;   and $\beta_1 = \gamma$. We can obtain estimates for β_0 and β_1 in (5.48) using simple regression (least squares) techniques.   This means we can also obtain estimates for h_0 and γ in (5.47) for each k we choose.   Furthermore we can measure the error sum-of-squares (SSE) once we have determined the prepayment parameters:

$$SSE(k) = \sum_i \left\{ CPR_i - h_0 - \gamma[k - r_i]^+ \right\}^2 \tag{5.49}$$

We choose a range of thresholds which we deem reasonable; in the worst case we can simply choose the range to include all values of r_t.   We then choose the value of k which minimizes *SSE(k)*.

We will use ten years' worth of monthly interest and prepayment rates for a total of 120 observations.   The range is from February 1996 through January 2006. Prepayment rates will differ for each generic GNMA pool because the coupon rates are different.

The prepayment parameter estimates are shown in Table 13:

5.2.3 Double-Threshold Model

The double-threshold model has five parameters.   These consist of the two thresholds k_1 and k_2, two slopes γ_1 and γ_2 and the exogenous prepayment rate h_0.   The model is

$$CPR_t = h_0 + \gamma_1 [k_1 - r_t]^+ + (\gamma_2 - \gamma_1)[k_2 - r_t]^+ + \varepsilon \tag{5.50}$$

Choosing values for k_1 and k_2 where $k_2 < k_1$ reduces equation (5.50) to a linear equation of the form

$$y_t = \beta_0 + \beta_1 x_{1,t} + \beta_2 x_{2,t} + \varepsilon_t \tag{5.51}$$

where $y_t = CPR_t$; $x_{1,t} = \left[k_1 - r_t\right]^+$; $x_{2,t} = \left[k_2 - r_t\right]^+$; $\beta_0 = h_0$; $\beta_1 = \gamma_1$; and $\beta_2 = \gamma_2 - \gamma_1$. To

calibrate this model, we must choose values for k_1 and k_2, $k_1 > k_2$, and then perform a

multiple regression. We choose the values of k_1 and k_2 which minimize the error.

$$SSE\left(k_1, k_2\right) = \sum_i \left\{ CPR_i - h_0 - \gamma_1 \left[k_1 - r_i\right]^+ - \left(\gamma_2 - \gamma_1\right)\left[k_2 - r_i\right]^+ \right\}^2 \tag{5.52}$$

Starting with $k_2 = k_1 - 0.01$ (one basis point), we perform a multiple regression

where the first predictor variable is the same as in the single threshold model and the

second predictor variable is the larger of 0 and the difference between k_2 and the current

3-month Treasury rate. We then find the y-intercept h_0 and the two slopes γ_1 and $(\gamma_2 - \gamma_1)$

as well as the error sum of squares (SSE). Now holding k_1 constant, we decrease k_2 by

one basis point and recalculate SSE. We keep decreasing k_2 until we hit the low end of

the range. The k_2 which minimizes the SSE within the range is then paired with k_1. We

now decrease k_1 by one basis point and repeat the procedure, finding its optimal k_2.

Eventually we have a set of pairs of (k_1, k_2) where the k_1's span the entire range of interest.

We now choose the pair (k_1, k_2) which minimizes(5.52). While this is a somewhat brute

force approach, it only takes about 96 seconds of CPU time. The double-threshold

parameters are also shown in Table 13.

We choose APL to do the parameter estimation because it can be done recursively

with arrays. The APL expression to accomplish this is:

```
Params←CPR⌹XX
```

The linear least-squares function, ⊟ (domino) also does matrix inversion. The left argument is a vector `CPR`, which represents the observed prepayment rates; the right argument `XX` is a matrix whose first column is 1's and whose second and third columns are the independent variables $x_{1,t} = [k_1 - r_t]^+$ and $x_{2,t} = [k_2 - r_t]^+$. The result `Params` is vector of parameter estimates—the intercept h_o, followed by the slopes γ_1 and $(\gamma_2 - \gamma_1)$

Single Threshold Model Parameters						
Coupon	**Error**	**h_o**	**γ**		**k**	
6.0%	8.456	5.100%	5.647		5.44%	
6.5%	9.074	6.136%	8.966		5.15%	
7.0%	6.495	7.908%	9.265		5.38%	
7.5%	5.920	9.805%	8.620		5.76%	
8.0%	6.951	13.792%	6.962		5.95%	
Double Threshold Model Parameters						
Coupon	**Error**	**h_o**	**γ_1**	**γ_2**	**k_1**	**k_2**
6.0%	7.912	4.802%	4.302	38.964	5.74%	1.32%
6.5%	7.886	6.232%	6.408	46.202	5.43%	1.44%
7.0%	5.684	8.204%	7.691	35.193	5..48%	1.44%
7.5%	5.057	10.367%	87.447	6.457	5.12%	4.97%
8.0%	5.801	14.319%	99.747	4.203	5.13%	4.98%

Table 13: Single and Double Threshold Model Parameters

Observe that the prepayment parameters for the double threshold model low coupon pools (6.0%, 6.5% and 7.0%) differ from the high coupon pools (7.5% and 8.0%). The higher coupon pools show that the slope increases below the first threshold k_1, but then it levels off below the second threshold k_2. For the lower coupon pools, the second threshold k_2 is much lower than the first threshold, which means that there is not much data below that threshold. Furthermore, the slope increases below this threshold. Thus we find that the double-threshold model is not useful for these lower coupon pools

because we do not have enough data to show the burnout effect which shows that

prepayments drop off when rates drop well below the refinancing threshold.

The APL source code for the dynamic function `FindPP` is shown below:

```
[0]    FindPP←{
[1]       ⍝⍺  Number of Threshold values: (1 or 2)
[2]       ⍝⍵  Short Rates,  Prepayment Rates(2xN Matrix)
[3]       ⍝←  Error,h0,Gamma1,[Gamma2],k1,[k2]
[4]       ⍝⍨  7535 12.23 7.49 5.79 ← 1 200209 200510 FindPP ⊃r CPR
[5]       ⍝⍨  5939 10.93 25.45 ¯21.13 5.26 4.49 ← 2 FindPP ⊃r CPR
[6]    n←↑⍺                       ⍝ Single or Double Threshold?
[7]    d←2↑1↓⍺,0 300000           ⍝ Start, End Dates
[8]    b←1≠×/×DATES∘.-d           ⍝ Date Selection
[9]    0∊b/⍵[0;]:'Error'          ⍝ Check for zero interest rates
[10]   FindK←{                    ⍝ Apply one or two breakpoints
[11]      0::1000000 0 0          ⍝
[12]      r c←↓⍵                  ⍝ Short Rates, Prepayment Rates
[13]      XX←1,0⌈-r∘.-,⍺          ⍝ Independent Variables
[14]      p←c⌹XX                  ⍝ Estimate Parameters
[15]      ch←XX+.×p               ⍝ Apply to model
[16]      e←(c-ch)+.*2            ⍝ Least Squares Error
[17]      e,p                     ⍝ Result is error, parameters
[18]   }
[19]   FindK2←{j←0.01×1↓⍳⌊100×⍺-1 ⍝ Find optimal k2 given k1
[20]      ob←⊃(⍺,¨⍺-j)FindK¨⊂⍵    ⍝
[21]      i←ob[;0]⍳⌊/ob[;0]       ⍝ Find optimal (minimum) value
[22]      ob[i;],⍺,⍺-j[i]         ⍝ Get parameters
[23]   }
[24]   k←2+0.01×⍳600             ⍝ Choose range of k values
[25]   ob←⊃n{⍺=1:k FindK¨⊂⍵      ⍝ If single threshold use FindK
[26]      k FindK2¨⊂⍵}b/⍵        ⍝   Else use FindK2
[27]   i←ob[;0]⍳⌊/ob[;0]         ⍝ Find optimal (minimum) value
[28]   z←ob[i;],(1=n)/k[i]       ⍝ Get objective, parameters
[29]   z[m]←+\z[m←2+⍳n]          ⍝ Cumulative slopes
[30]   z[0]←(z[0]÷¯1++/b)*÷2     ⍝ Standard Error
[31]   z
[32]   }


    1 199602 200601 FindPP ⊃C0793M GNSF8CPR
6.950662159 13.7922865 6.961978253 5.95
    2 199602 200601 FindPP ⊃C0793M GNSF8CPR
5.801102925 14.31851511 99.7472181 4.202734348 5.13 4.98
```

5.2.4 PSA Model

The PSA Model assumes that prepayments start at zero and increase linearly for two and a half years until they reach 6% annually. This is known as 100% PSA. A multiplier can be applied, i.e. 150% PSA reaches 9% linearly after 2.5 years. Gorovoy and Linetsky assume the PSA model applies to exogenous prepayment rates, thus

$$h_0(t) = ba \min(t, T^*) \tag{5.53}$$

We assume that T* is fixed at 2.5 and that $a=0.24$ to conform to the PSA standard. Thus we only need to estimate b.

We revise our prepayment model to

$$PSA_i = b + \frac{[k - r_i]^+}{a \min(t, T^*)} \gamma \tag{5.54}$$

for the single-threshold model. The double threshold PSA is:

$$PSA_i = b + \frac{\gamma_1 [k_1 - r_i]^+ + (\gamma_2 - \gamma_1)[k_2 - r_i]^+}{a \min(t, T^*)} \tag{5.55}$$

PSA values are available from Bloomberg. The seasoning factor, t is determined by subtracting the weighted average Maturity (WAM) from 30 years. Since a, t, T^* and r are known in (5.54), we only need to choose a value for k to have a simple linear regression model to estimate b and γ. For (5.55) we need to choose values for both k_1 and k_2 as we did in the previous section. In this case we have a multiple regression model in b, γ_1 and γ_2.

In our sample the WAM of the 7.5% and 8.0% GNMA's never exceeds 27.5 years, so the CPR model applies in all cases. Since these are the only two coupon rates

for which the double-threshold model is meaningful, we do not need to consider the PSA double-threshold model.

Taking PSA data from the 8% pools, we obtain the following results:

Pool	Start Date	End Date	b	γ	k
GNMA 8.0%	3/31/1996	3/31/2006	2.8612	6.529	6.05%

To convert from PSA to CPR, we observe that $h_o \approx abT^*$. Thus:

$$0.024 \times 2.8612 \times 2.5 = 0.171672 \qquad (5.56)$$

Observe that the GNMA 8.0% PSA value converts to approximately 17.16% CPR which is a little bit higher than the 13.792% CPR estimate in Table 13 The Dyalog APL source code for the PSA prepayment parameter estimates is shown below:

```
[0]    FindPSA←{
[1] ⍝α   Number of Threshold values: (1 or 2)
[2] ⍝ω   Short Rates,  Prepayment Rates(2xN Matrix)
[3] ⍝←   Error,h0,Gamma1,[Gamma2],k1,[k2]
[4] ⍝±  145.225 286.116 6.529 6.05  ←  199603 200603 FindPSA T6 PSA6 C0793M
[5]    b←1≠×/×DATES∘.-α           ⍝ Date Selection
[6]    (T PSA r)←ω                ⍝ Unpack right arg
[7]    M←0.024×2.5⌊30-b/T         ⍝ PSA to CPR conversion
[8]    D←⍝r PSA
[9]    0∈b/D[0;]:'Error'          ⍝ Check for zero interest rates
[10]   FindK←{                    ⍝ Apply one or two breakpoints
[11]      0::100000000 0 0        ⍝ Error condition
[12]      r c←↓ω                  ⍝ Short Rates, Prepayment Rates
[13]      XX←1,[0.5]0⌈(-r∘.-α)÷M  ⍝ Independent Variables
[14]      p←c⌹XX                  ⍝ Estimate Parameters
[15]      ch←XX+.×p               ⍝ Apply to model
[16]      e←(c-ch)+.*2            ⍝ Least Squares Error
[17]      e,p                     ⍝ Result is error, parameters
[18]   }
[19]   k←0.5+0.01×⍳700            ⍝ Choose range of k values
[20]   ob←⍝k FindK¨⊂b/D           ⍝ Apply regression for each k
[21]   i←ob[;0]⍳⌊/ob[;0]          ⍝ Find optimal (minimum) value
[22]   z←ob[i;],k[i]              ⍝ Get objective, parameters
[23]   z[0]←(z[0]÷¯1++/b)*÷2      ⍝ Standard Error
[24]   z
[25]   }
```

Chapter 6. Pricing MBS - A Practical Guide

Although the theory has been described in detail in the previous section, what matters for the practitioner is the means to price a mortgage or mortgage-backed security. This section describes in detail how to price a GNMA Mortgage-Backed Security using empirical data. All empirical data in this example were obtained from Bloomberg. All parameters are estimated from the data and then applied to theoretical formulas derived in the previous section. Although eigenfunction expansion consists of an infinite number of spectral values, we truncate these at a point where the convergence appears to a reasonable number of decimal places.

The following is general procedural approach to pricing MBS. It applies to either interest rate model (CIR or Vasicek) and to either prepayment models (single or double threshold):

1. Calculate forward rates from 3-month Treasury rates (Bloomberg or Treasury.gov)

2. Calibrate interest rate parameters from Forward rates

3. Estimate prepayment parameters from CPR and 3-month Treasury Rates (Bloomberg)

4. Get weighted average maturity (WAM) for generic GNMA securities (Bloomberg)

5. Solve differential equation to find eigenfunctions

6. Construct Wronskian and find eigenvalue solutions

7. Normalize the eigenfunctions obtained from step 3

8. Find Expansion coefficients using numerical integration

9. Apply the Time Integral to determine the price

10. Compare theoretical price to Bloomberg's empirical price.

In this section we will price a seasoned 8% GNMA mortgage-backed security on January 31, 2005. The first three steps were accomplished in Chapter 5. Historical 3-month Treasury rates are readily available on Bloomberg. The parameter estimates from the previous section are shown in Table 14.

Interest Rate Parameter Estimates: 1/31/2005

Model	κ	σ	θ
Vasicek	0.31695	0.04332	0.06459
CIR	0.32638	0.17805	0.06210

r_0 =2.51   Normal Spread = 0.74%   Crisis Spread = 0.954%

Bloomberg WAM FOR GNMA Securities 1/31/2005

GNMA Pool	6.0%	6.5%	7.0%	7.5%	8.0%
WAM	26.9167	24.4167	22.2500	20.4167	18.5833

Prepayment Parameter Estimates

Model	Single Threshold CPR			Double Threshold CPR				
Coupon	h_0	γ	κ	h_0	γ_1	γ_2	k_1	k_2
6.0%	5.100%	5.647	5.44%	4.802%	4.302	38.964	5.74%	1.32%
6.5%	6.136%	8.966	5.15%	6.232%	6.408	46.202	5.43%	1.44%
7.0%	7.908%	9.265	5.38%	8.204%	7.691	35.193	5..48%	1.44%
7.5%	9.805%	8.620	5.76%	10.367%	87.447	6.457	5.12%	4.97%
8.0%	13.792%	6.962	5.95%	14.319%	99.747	4.203	5.13%	4.98%

Table 14: Parameter Estimates for Various GNMA Securities on 1/31/2005

Figure 13 shows the relationships between the data and the various steps in pricing GNMA securities. In this section we will price five generic GNMA pools, each with a different coupon rate.   We will look at the standard Cox-Ingersoll Ross Model and then look at the double threshold model and the Vasicek Model.
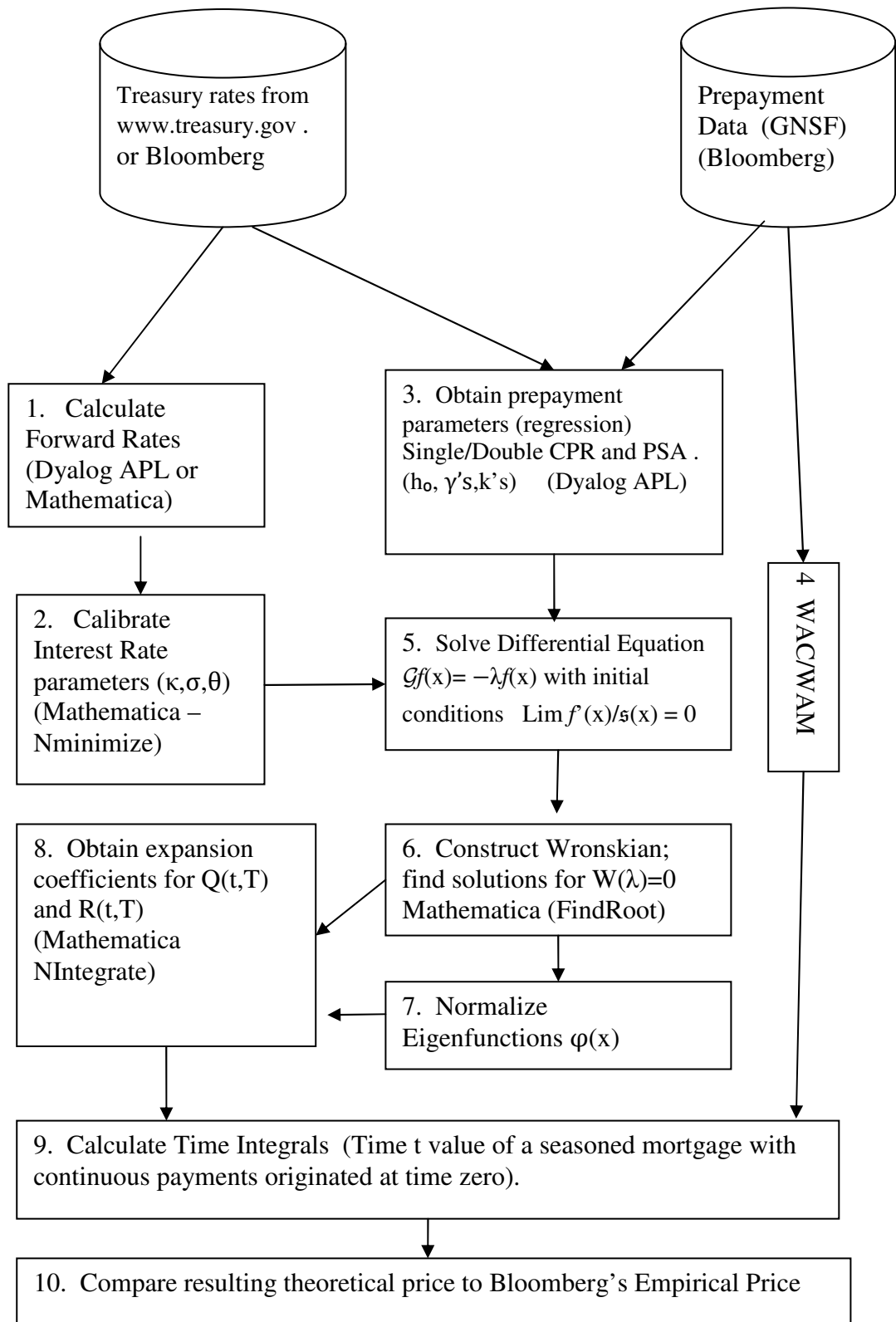
Figure 13 – General Flowchart showing procedure to price Mortgage-Backed Securities

After obtaining the solutions to the differential equation, we construct the Wronskian matrix and we find the eigenvalues, eigenfunctions and expansion coefficients. In this section we go into explicit detail for each of the three models: One-threshold CIR, two-threshold CIR and Vasicek. Code snippets from Mathematica can be cut and pasted into a Mathematica session and verified. We take the process all the way to pricing the security and comparing it to the empirical price obtained from Bloomberg. On the way we check intermediate values to verify continuity and differentiability of the solutions to the differential equation.

6.1 Single Threshold Cox-Ingersoll-Ross Model

The solutions to the Cox Ingersoll Ross Model are:

$$f_i(x) = \exp\left(\frac{(\kappa - \rho_i)z}{a_i\sigma^2}\right)u_i(z) \text{ where}$$

$$u_1(z) = M(a,b,z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n}\frac{z^n}{n!}$$ (6.1)

$$u_2(z) = U(a,b,z) = \frac{\Gamma(1-b)}{\Gamma(1+a-b)}M(a,b,z) + \frac{\Gamma(b-1)}{\Gamma(a)}z^{1-b}M(1+a-b,2-b,z)$$

This rather complex-looking function can easily be expressed using the Mathematica[3] functions **Hypergeometric1F1** and **HypergeometricU**.

```
psi[x_,lambda_]:=Exp[(kappa-rho1)*x/sigma^2]*
   Hypergeometric1F1[a1-(lambda-gam*k)/rho1,beta,alpha1*x]
phi[x_,lambda_]:=Exp[(kappa-rho2)*x/sigma^2]*
           HypergeometricU[a2-lambda/rho2,beta,alpha2*x]
```

---

[3]All Mathematica expressions in this section are fully executable and can be cut and pasted directly into a Mathematica 8.0 session for verification.

All variables can be defined in terms of the Cox, Ingersoll Ross parameters, $\{\kappa, \sigma, \theta\}$ or the prepayment parameters: $\{h_o, \gamma, k\}$:

```
rho1:=Sqrt[2*(1-gam)*sigma^2+kappa^2]
rho2:=Sqrt[2*sigma^2+kappa^2]
alpha1:=2*rho1/sigma^2
alpha2:=2*rho2/sigma^2
beta:=2*kappa*theta/sigma^2
a1:=beta/2-kappa^2*theta/(sigma^2*rho1)
a2:=beta/2-kappa^2*theta/(sigma^2*rho2)
```

We use the D operator in Mathematica to produce derivatives:

```
psid[k_,lambda_]:=D[psi[y,lambda],y]/.y->k
phid[k_,lambda_]:=D[phi[y,lambda],y]/.y->k
```

We next define the scale and speed densities:

```
s[x_]:=x^(-beta)*Exp[2*kappa*x/sigma^2]
m[x_]:=2/sigma^2*x^(beta-1)*Exp[-2*kappa*x/sigma^2]
```

And now we can construct the Wronskian:

```
w[lambda_]:=1/s[k]*(psi[k,lambda]*phid[k,lambda]-psid[k,lambda]
*phi[k,lambda])
```

We now set the Wronskian to zero and solve for it to obtain the eigenvalues:

```
FindLambda[e_]:=Re[x/.FindRoot[w[x],{x,e}]]
```

We must supply a guess to the function **FindLambda**. Different guesses will often produce different eigenvalues. We start with zero to produce the first eigenvalue. Then double the eigenvalue for the second guess. If this produces the same eigenvalue, then triple it. Keep increasing the value until there are two unique eigenvalues. Measure the interval between the two eigenvalues and use that as a step for the next guess. The concern here is that we do not skip over any eigenvalues, particularly early on as this will produce an incorrect result.

Now let us apply the model to the parameters obtained for January 31, 2005:

```
kappa=0.32638;sigma=0.17805;theta=0.06210;gam=6.962;
```

We adjust the threshold value by adding a spread and converting to the continuous rate:

```
Spread=0.74; k=Log[1+(5.95+Spread)/100]
```

```
0.0647572
```

Here is the first eigenvalue:

```
Lambda1=FindLambda[0]
```

```
0.195507
```

Now double the eigenvalue and use that for the next guess:

```
Lambda2=FindLambda[Lambda1*2]
```

```
0.584451
```

If the new guess had produced the same eigenvalue as before, we would triple the eigenvalue if necessary:

```
Lambda2=FindLambda[Lambda1*3]
```

Now estimate the step size between eigenvalues to produce the next guess and the next eigenvalue:

```
Step = Lambda2-Lambda1
```

```
0.388944
```

```
Lambda3 = FindLambda[Lambda2+Step]
```

```
0.962452
```

The Mathematica script below automates the process for the first 7 eigenvalues:

```
Lambda1:=FindLambda[0];
L2:=FindLambda[2*Lambda1];
Lambda2:=If[L2>Lambda1,L2,FindLambda[3*Lambda1]];
Lambda={Lambda1,Lambda2,0,0,0,0,0};
For[i=1,i<Length[Lambda]-1,i++,Lambda[[i+2]]=
     FindLambda[2*Lambda[[i+1]]-Lambda[[i]]]];

Lambda
```

```
{0.195507, 0.584451, 0.962452, 1.35902, 1.76459, 2.17169, 2.5789}
```

In the above example, we found the first 7 eigenvalues. Since there are an infinite number of eigenvalues, we must choose how many eigenvalues we wish to calculate. This depends upon how rapidly the series converges.

Now let's define the eigenfunctions in Mathematica. First we normalize them:

```
delta[x_]:=D[w[lambda],lambda]/.lambda->x
normal1[x_,lambda_]:=Sqrt[phi[k,lambda]/(delta[lambda]*psi[k,lambda])]*
        psi[x,lambda]
normal2[x_,lambda_]:=Sign[delta[lambda]]*Sqrt[psi[k,lambda]/
(delta[lambda]*phi[k,lambda])]*phi[x,lambda]
```

Now let's apply the normalized eigenfunctions to the initial short rate. Notice we have to adjust the short rate in the same way as the threshold value k:

```
r0=Log[1+(2.51+Spread)/100];

Ph1[lambda_]:=normal1[r0,lambda]
Ph2[lambda_]:=normal2[r0,lambda]
Ph[lambda_]:=Re[If[r0<k,Ph1[lambda],Ph2[lambda]]]
```

Now let's evaluate the eigenfunction corresponding to the principal eigenvalue:

```
Ph[Lambda1]
```

```
0.808661
```

We automate the process by using the Map operator with all of the calculated eigenvalues. Note that the eigenfunction values can be either positive or negative.

```
Phi=Map[Ph,Lambda]
```

```
{0.808661,0.685009,0.249787,-0.144693,-0.394074,-0.518877,-0.542557}
```

Now calculate the expansion coefficients c^Q and c^R. This requires the use of numerical integration. The expansion coefficients can also be either positive or negative.

```
cq1[lambda_]:=Re[NIntegrate[normal1[x,lambda]*m[x],{x,0,k}]]
cq2[lambda_]:=Re[NIntegrate[normal2[x,lambda]*m[x], {x,k,Infinity}]]
cQ=Map[cq1,Lambda]+Map[cq2,Lambda]
```

```
{1.06898,0.168328,0.122042,0.0437959,0.0168817,0.00419073,-0.00098589}
```

```
cr1[lambda_]:=Re[NIntegrate[x*normal1[x,lambda]*m[x],{x,0,k}]]
cr2[lambda_]:=Re[NIntegrate[x*normal2[x,lambda]*m[x], {x,k,Infinity}]]
```

```
cR=Map[cr1,Lambda]+Map[cr2,Lambda]
{.075692,-.048357,.0094047,-.002913,-.00041064,-.00066163,-.000463}
```

Now with the data we just calculated, we should like to examine and verify the

conditional expected value of a discounted claim:

$$(\mathcal{P}_t f)(x) = E_x \left[\exp \left\{ -\int_0^t \left(r_s + \gamma [k - r_s]^+ \right) ds \right\} f(r_t) \right]$$ (6.2)

According to Gorovoy and Linetsky , the payoffs $f_Q(r_T) = 1$ and $f_R(r_T) = r_T$ and

(6.2) is equivalent to:

$$\mathcal{P}_t f(x) = \sum_{n=1}^{\infty} e^{-\lambda_n t} c_n^f \varphi_n(x)$$ (6.3)

At maturity (6.3) reduces to:

$$\mathcal{P}_0 f(x) = f(x) = \sum_{n=1}^{\infty} c_n^f \varphi_n(x)$$ (6.4)

We can approximate (6.4) by a finite sum of the first 7 normalized eigenfunctions

weighted by their expansion coefficients:

$$r_0 = \ln(1 + r + s) = \ln(1 + .0251 + .0074) = 0.031983$$ (6.5)

$$\mathcal{P}_0 f_Q(r_0) = f_Q(r_0) \approx \sum_{n=1}^{7} c_Q^f \varphi_n(r_0) = 0.99560 \approx 1$$ (6.6)

$$\mathcal{P}_0 f_R(r_0) = f_R(r_0) \approx \sum_{n=1}^{7} c_Q^f \varphi_n(r_0) = 0.03161 \approx r_0$$ (6.7)

Let us now discount the payoffs by one year. Then

$$\mathcal{P}_1 f(x) \approx \sum_{n=1}^{N} e^{-\lambda} c_n^f \varphi_n(x)$$ (6.8)

We can approximate the one-year discounted claim

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$$\mathcal{P}_1 f_Q (r_0) \approx \sum_{n=1}^{7} e^{-\lambda_n} c_Q^f \varphi_n (r_0) = 0.809541 \tag{6.9}$$

$$\mathcal{P}_1 f_R (r_0) \approx \sum_{n=1}^{7} e^{-\lambda_n} c_R^f \varphi_n (r_0) = 0.0339127 \tag{6.10}$$

How can the discounting for $f_Q(x)$ be so large? That is because the "discounting" in this model involves the prepayment rate as well as the interest rate. How can $\mathcal{P}_1 f_R (r_0) > r_0$ if it is discounted for one year? Remember we are discounting the *expected* value of the short rate one year from now which is higher due to mean reversion. Thus

$$E[r_1] = r_0 + \kappa(\theta - r_0) = 0.031983 + 0.32638(.0621 - .031983) = 0.0418126 \tag{6.11}$$

Finally, we use time integrals to price the mortgage-backed security. Here we use the exogenous prepayment parameter h_0 in the case of CPR or b in the case of PSA. For the CPR model we use the formula:

$$L(t,T;\lambda,m) = \frac{m\left(1 - e^{-(h_0+\lambda)(T-t)}\right) - (h_0+\lambda)\left(1 - e^{-m(T-t)}\right)}{(h_0+\lambda)(m - h_0 - \lambda)} \tag{6.12}$$

Since we are dealing with seasoned mortgages, we use the following formula to value the mortgage:

$$M_t = P_t \left\{ 1 + \frac{\sum_{n=1}^{\infty} \varphi_n (r) \left(mc_n^Q - c_n^R\right) L_c (t,T;\lambda_n,m)}{1 - e^{-m(T-t)}} \right\} \tag{6.13}$$

Before we can evaluate the time integrals, we must know the exogenous prepayment rate and the coupon rate:

```
h0=13.792/100;T=18.5833; M=12*Log[1+8.0/1200]

0.0797345

ls:=h0+Lambda
Lc[m_]:=(m*(1-Exp[-ls*T])-ls*(1-Exp[-m*T]))/(ls*(m-ls))
Lc[M]

{2.1053,1.0307,0.68613,0.507684,0.400952,0.331064,0.281905}

Price=100*(1+Total[Phi*(M*cQ-cR)*Lc[M]]/(1-Exp[-M*T]))

107.628
```

We compare the theoretical price of $107.628 (based on the first 7 eigenvalues) to the Bloomberg price of $108.469 for a difference of 84 basis points. After 15 eigenvalues the theoretical price stabilizes at $107.626.

Now let's carry out the eigenfunction expansion a bit further. Figure xx shows the expansion up to $n=20$. Notice how much closer $Q(0,0)$ and $R(0,0)$ get to 1 and r_o respectively. Notice also that Q(0,1) and R(0,1) converge much faster than $Q(0,0)$ and R$(0,0)$. Since we integrate up to the WAM which in this case is over 18 years, the values $Q(0,t)$ and $R(0,t)$ for $t > 1$ will converge even faster. The last column shows the calculated price when the first n eigenvalues are used. After about 8 terms in the series the price stabilizes to within a tenth of a basis point.

```
 N    Lambda      Phi        cQ         cR       Q(0,0)     R(0,0)    Price
--  ---------  ---------  ---------  ---------  ---------  ---------  -------
 1    0.19551    0.80866    1.06898    0.07569    0.86444    0.06121  102.102
 2    0.58445    0.68501    0.16833   -0.04836    0.97975    0.02808  107.747
 3    0.96245    0.24979    0.12204    0.00940    1.01023    0.03043  107.754
 4    1.35902   -0.14469    0.04380   -0.00291    1.00389    0.03086  107.693
 5    1.76459   -0.39407    0.01688   -0.00041    0.99724    0.03102  107.657
 6    2.17169   -0.51888    0.00419   -0.00066    0.99507    0.03136  107.635
 7    2.57890   -0.54256   -0.00099   -0.00046    0.99560    0.03161  107.628
 8    2.98675   -0.48979   -0.00270   -0.00033    0.99693    0.03177  107.626
 9    3.39566   -0.38729   -0.00279   -0.00021    0.99801    0.03186  107.626
10    3.80554   -0.25849   -0.00223   -0.00012    0.99858    0.03189  107.626
11    4.21602   -0.12091   -0.00148   -0.00005    0.99876    0.03190  107.626
12    4.62675    0.01316   -0.00079    0.00000    0.99875    0.03190  107.626
13    5.03752    0.13496   -0.00023    0.00003    0.99872    0.03190  107.626
14    5.44831    0.23831    0.00017    0.00004    0.99876    0.03191  107.626
15    5.85914    0.31930    0.00042    0.00005    0.99889    0.03192  107.626
16    6.27009    0.37607    0.00054    0.00005    0.99910    0.03194  107.626
17    6.68120    0.40862    0.00056    0.00004    0.99933    0.03196  107.626
18    7.09249    0.41832    0.00052    0.00003    0.99955    0.03197  107.626
19    7.50393    0.40747    0.00044    0.00002    0.99973    0.03198  107.626
20    7.91547    0.37888    0.00034    0.00002    0.99986    0.03199  107.626
```

Table 15 Eigenfunction expansion for GNMA 8.0% on 1/31/2005

6.2 Double-Threshold CIR Model

There are two solutions at each threshold for the differential equation involving theTwo-Threshold CIR model . Thus we have a total of four solutions. These can be written in Mathematica along with their derivatives as:

```
phi1[x_,lambda_]:=Exp[(kappa-rho0)*x/sigma^2]*
    HypergeometricU[a0-lambda/rho0,beta,alpha0*x]
psi1[x_,lambda_]:=Exp[(kappa-rho1)*x/sigma^2]*
    Hypergeometric1F1[a1-(lambda-gam1*k1)/rho1,beta,alpha1*x]
phi2[x_,lambda_]:=Exp[(kappa-rho1)*x/sigma^2]*
    HypergeometricU[a1-(lambda-gam1*k1)/rho1,beta,alpha1*x]
psi2[x_,lambda_]:=Exp[(kappa-rho2)*x/sigma^2]*
    Hypergeometric1F1[a2-(lambda-gam1*k1-gam2*k2)/rho2,beta,alpha2*x]
psi1d[k_,lambda_]:=D[psi1[y,lambda],y]/.y->k
phi1d[k_,lambda_]:=D[phi1[y,lambda],y]/.y->k
psi2d[k_,lambda_]:=D[psi2[y,lambda],y]/.y->k
phi2d[k_,lambda_]:=D[phi2[y,lambda],y]/.y->k
```

All variables can be defined in terms of the CIR interest rate parameters, $\{\kappa,\sigma,\theta\}$ and the double-threshold prepayment parameters $\{h_0,\gamma_1,\gamma_2,k_1,k_2\}$.

```
rho0:=Sqrt[2*sigma^2 + kappa^2]
rho1:=Sqrt[2*(1-gam1)*sigma^2 + kappa^2]
rho2:=Sqrt[2*(1-gam1-gam2)*sigma^2 + kappa^2]
alpha0:=2*rho0/sigma^2
alpha1:=2*rho1/sigma^2
alpha2:=2*rho2/sigma^2
beta:=2*kappa*theta/sigma^2
a0:=beta/2-kappa^2*theta/(sigma^2*rho0)
a1:=beta/2-kappa^2*theta/(sigma^2*rho1)
a2:=beta/2-kappa^2*theta/(sigma^2*rho2)
```

Constructing the 4 x 4 "Wronskian" matrix is a bit more complex:

```
w11[k_,lam_]:=psi1[k,lam]*phi1d[k,lam]-phi1[k,lam]*psi1d[k,lam]
w12[k_,lam_]:=psi1[k,lam]*phi2d[k,lam]-phi2[k,lam]*psi1d[k,lam]
w22[k_,lam_]:=psi2[k,lam]*phi2d[k,lam]-phi2[k,lam]*psi2d[k,lam]
wp[k_,lam_]:=phi1[k,lam]*phi2d[k,lam]-phi1d[k,lam]*phi2[k,lam]
row1[k_,lam_]:={phi1[k,lam],-phi2[k,lam],-psi1[k,lam],0}
row2[k_,lam_]:={phi1d[k,lam],-phi2d[k,lam],-psi1d[k,lam],0}
row3[k_,lam_]:={0,phi2[k,lam],psi1[k,lam],-psi2[k,lam]}
row4[k_,lam_]:={0,phi2d[k,lam],psi1d[k,lam],-psi2d[k,lam]}
mat[lam_]:={row1[k1,lam],row2[k1,lam],row3[k2,lam],row4[k2,lam]}
w[lam_]:=Det[mat[lam]]
```

Before solving for the eigenvalues we supply the double-threshold parameters:

```
gam1=99.747;gam2=4.203-gam1;h0=14.319/100;
k1=Log[1+(5.13+Spread)/100];k2=Log[1+(4.98+Spread)/100];

Lambda1=FindLambda[0]

0.197216
```

The first seven eigenvalues are:

```
{0.197216, 0.575169, 0.951264, 1.3523, 1.76064, 2.16818, 2.57475}
```

The Eigenfunctions are normalized linear combinations of the solutions of the differential

equations:

```
(**** Normalizing Coefficients ****************************)
Y[lam_]:=psi2[k2,lam]/(psi1[k2,lam]*wp[k1,lam]+phi2[k2,lam]*w11[k1,lam]
);
D1[lam_]:=NIntegrate[psi2[x,lam]^2*m[x],{x,0,k2}]/Y[lam]^2;
D2[lam_]:=NIntegrate[(wp[k1,lam]*psi1[x,lam] +
                w11[k1,lam]*phi2[x,lam])^2*m[x],{x,k2,k1}];
D3[lam_]:=w12[k1,lam]^2*NIntegrate[phi1[x,lam]^2*m[x],{x,k1,Infinity}];
DD[lam_]:=1/Sqrt[D1[lam]+D2[lam]+D3[lam]];
A1[lam_]:=w12[k1,lam]*DD[lam];
A2[lam_]:=w11[k1,lam]*DD[lam];
B1[lam_]:=wp[k1,lam]*DD[lam];
```

```
B2[lam_]:=DD[lam]/Y[lam];
(*********** EigenFunctions *****************)
normal0[x_,lam_]:=A1[lam]*phi1[x,lam];
normal1[x_,lam_]:=A2[lam]*phi2[x,lam]+B1[lam]*psi1[x,lam];
normal2[x_,lam_]:=B2[lam]*psi2[x,lam];
Ph0[lam_]:=normal0[r0,lam];
Ph1[lam_]:=normal1[r0,lam];
Ph2[lam_]:=normal2[r0,lam];
Ph[lam_]:=Re[If[r0<k1,If[r0<k2,Ph2[lam],Ph1[lam]],Ph0[lam]]];
```

To obtain the first eigenfunction evaluated at the current short rate, we apply the defined

function **Ph** to the principal eigenvalue:

```
Ph[0.197216]
-0.80907
```

The expansion coefficients use numerical integration over three segments:

```
cq2[lambda_]:=NIntegrate[normal2[x,lambda]*m[x],{x,0,k2}]
cq1[lambda_]:=NIntegrate[normal1[x,lambda]*m[x],{x,k2,k1}]
cq0[lambda_]:=NIntegrate[normal0[x,lambda]*m[x],{x,k1,Infinity}]
cq[lambda_]:=cq0[lambda]+cq1[lambda]+cq2[lambda]
cr2[lambda_]:=NIntegrate[x*normal2[x,lambda]*m[x],{x,0,k2}]
cr1[lambda_]:=NIntegrate[x*normal1[x,lambda]*m[x],{x,k2,k1}]
cr0[lambda_]:=NIntegrate[x*normal0[x,lambda]*m[x],{x,k1,Infinity}]
cr[lambda_]:=cr0[lambda]+cr1[lambda]+cr2[lambda]
cQ=Map[cq,Lambda]
cR=Map[cr,Lambda]
```

The Double-Threshold CIR price convergence can be seen in Table 16. Observe

that the double-threshold price is closer to Bloomberg than the single threshold price of

$107.626.

```
N    Lambda      Phi        cQ         cR      Q(0,0)      R(0,0)    Price
--  ----------  ----------  ----------  ----------  ----------  ----------  -------
 1   0.19722   -0.80907   -1.07231   -0.07527    0.86757    0.06090  102.214
 2   0.57517    0.67071    0.16168   -0.04888    0.97601    0.02812  107.769
 3   0.95126   -0.20458   -0.10583   -0.01000    0.99767    0.03017  107.741
 4   1.35230   -0.17986    0.02932   -0.00308    0.99239    0.03072  107.676
 5   1.76064    0.41536   -0.00693    0.00052    0.98951    0.03094  107.653
 6   2.16818   -0.53169   -0.00240   -0.00075    0.99079    0.03133  107.641
 7   2.57475    0.54719    0.00508    0.00051    0.99357    0.03161  107.639
 8   2.98211   -0.48553   -0.00493   -0.00034    0.99596    0.03178  107.639
 9   3.39115    0.37610    0.00369    0.00019    0.99735    0.03185  107.640
10   3.80164   -0.24391   -0.00225   -0.00008    0.99790    0.03187  107.641
11   4.21285    0.10584    0.00098    0.00000    0.99800    0.03187  107.641
12   4.62414    0.02729   -0.00001    0.00004    0.99800    0.03188  107.641
13   5.03518   -0.14774   -0.00065   -0.00007    0.99810    0.03189  107.641
14   5.44594    0.24960    0.00103    0.00008    0.99836    0.03191  107.641
15   5.85657   -0.32870   -0.00119   -0.00008    0.99875    0.03193  107.641
16   6.26730    0.38299    0.00117    0.00007    0.99920    0.03196  107.641
17   6.67827   -0.41255   -0.00104   -0.00006    0.99962    0.03199  107.641
18   7.08957    0.41911    0.00083    0.00004    0.99997    0.03200  107.642
19   7.50115   -0.40543   -0.00060   -0.00003    1.00022    0.03201  107.642
20   7.91294    0.37461    0.00037    0.00001    1.00035    0.03202  107.642
```

Table 16 - Eigenfunction expansion for 8.0% GNMA on 1/31/2005 for Double Threshold Model

6.3 Vasicek Pricing Model

The Vasicek solutions are the Parabolic Cylinder Functions:

$$f_i(x) = e^{z^2/4} u_i(z), \quad z = \frac{2\kappa}{\sigma}(\theta - x)$$

$$u_1(z) = D_\mu(z - a) \tag{6.14}$$

$$u_2(z) = \frac{\Gamma(-\mu)}{\pi}\left\{D_\mu(z-a) - D_\mu(z-x)\sin\left[\pi\left(\mu + \tfrac{1}{2}\right)\right]\right\}$$

These solutions can be easily written in Mathematica script:

```
Dv[a_,x_]:=ParabolicCylinderD[a,x]
U[a_,x_]:=Dv[-a-0.5,x]
V[a_,x_]:=1/Pi*Gamma[a + 0.5]*(Sin[Pi*a]*Dv[-a-0.5,x]+Dv[-a-0.5,-x])
```

The independent variable can be written as:

```
z[x_]:=Sqrt[2*kappa]/sigma*(theta-x)
```

All variables can be defined in terms of the Vasicek Parameters, $\{\kappa, \sigma, \theta\}$ and the single-

threshold prepayment parameters $\{h_o, \gamma, k\}$:

```
mu[gam_,lam_]:=(sigma^2*(1-gam)^2)/(2*kappa^3)+
                          (lam-gam*k-theta*(1-gam))/kappa
v:=-mu[gam,lam]-.5
alpha[gam_]:=sigma*(1-gam)*Sqrt[2/kappa^3]
```

Although this is a single-threshold model, it is treated similar to a double threshold model

where the lower threshold $k_2=0$. We define four solutions to the differential equation:

```
phi1[x_,lam_]:=Exp[1/4*z[x]^2]*U[-mu[0,lam]-.5,alpha[0]-z[x]]
psi1[x_,lam_]:=Exp[1/4*z[x]^2]*V[-mu[gam,lam]-.5,alpha[gam]-z[x]]
phi2[x_,lam_]:=Exp[1/4*z[x]^2]*U[-mu[gam,lam]-.5,alpha[gam]-z[x]]
psi2[x_,lam_]:=Exp[1/4*z[x]^2]*U[-lam/kappa-.5,z[x]]
```

and their derivatives:

```
psi1d[k_,lam_]:=D[psi1[y,lam],y]/.y->k
phi1d[k_,lam_]:=D[phi1[y,lam],y]/.y->k
psi2d[k_,lam_]:=D[psi2[y,lam],y]/.y->k
phi2d[k_,lam_]:=D[phi2[y,lam],y]/.y->k
```

and from these construct a "Wronskian" matrix:

```
row1[k_,lam_]:={phi1[k,lam],-phi2[k,lam],-psi1[k,lam],0}
row2[k_,lam_]:={phi1d[k,lam],-phi2d[k,lam],-psi1d[k,lam],0}
row3[k_,lam_]:={0,phi2[k,lam],psi1[k,lam],-psi2[k,lam]}
row4[k_,lam_]:={0,phi2d[k,lam],psi1d[k,lam],-psi2d[k,lam]}
mat[lam_]:={row1[k,lam],row2[k,lam],row3[0,lam],row4[0,lam]}
w[lam_]:=Det[mat[lam]]
```

The Vasicek model has its own speed and scale densities:

```
s[x_]:=Exp[kappa*(theta-x)^2/sigma^2]
m[x_]:=2/sigma^2*Exp[-kappa*(theta-x)^2/sigma^2]
```

Now let's put in the Vasicek Interest Rate Parameters, along with the single-threshold
prepayment parameters:

```
kappa=0.31695; sigma=0.04332;theta=0.06459; gam=6.962;
k=Log[1+(5.95+Spread)/100]
```

We now can find the Eigenvalues. The first one is:

```
In[306]:= FindLambda[0]
Out[306]= 0.12756
```

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Here we define the normalizing coefficients as described in Section 4.2.3:

```
 (********Normalizing Coefficients ***************)
w11[k_,lam_]:=psi1[k,lam]*phi1d[k,lam]-phi1[k,lam]*psi1d[k,lam]
w12[k_,lam_]:=psi1[k,lam]*phi2d[k,lam]-phi2[k,lam]*psi1d[k,lam]
w22[k_,lam_]:=psi2[k,lam]*phi2d[k,lam]-phi2[k,lam]*psi2d[k,lam]
wp[k_,lam_]:=phi1[k,lam]*phi2d[k,lam]-phi1d[k,lam]*phi2[k,lam]
Y[lam_]:=psi2[0,lam]/(psi1[0,lam]*wp[k,lam]+  phi2[0,lam]*w11[k,lam])
D1[lam_]:=NIntegrate[psi2[x,lam]^2*m[x],{x,-Infinity,0}]/Y[lam]^2
D2[lam_]:=NIntegrate[(wp[k,lam]*psi1[x,lam] +
           w11[k,lam]*phi2[x,lam])^2*m[x],{x,0,k}]
D3[lam_]:=w12[k,lam]^2*NIntegrate[phi1[x,lam]^2*m[x],{x,k,Infinity}]
DD[lam_]:=1/Sqrt[D1[lam]+D2[lam]+D3[lam]]
A1[lam_]:=w12[k,lam]*DD[lam]
A2[lam_]:=w11[k,lam]*DD[lam]
B1[lam_]:=wp[k,lam]*DD[lam]
B2[lam_]:=DD[lam]/Y[lam]
```

Now we can normalize the eigenfunctions. The code is identical to the double-threshold

CIR model except that $k_2=0$.

```
(************ EigenFunctions ******************)
normal0[x_,lam_]:=A1[lam]*phi1[x,lam]
normal1[x_,lam_]:=A2[lam]*phi2[x,lam]+B1[lam]*psi1[x,lam]
normal2[x_,lam_]:=B2[lam]*psi2[x,lam]
Ph0[lam_]:=normal0[r0,lam]
Ph1[lam_]:=normal1[r0,lam]
Ph2[lam_]:=normal2[r0,lam]
Ph[lam_]:=Re[If[r0<k,If[r0<0,Ph2[lam],Ph1[lam]],Ph0[lam]]]
Phi=Map[Ph,Lambda]
```

First we find the principal eigenvalue:

```
Lambda1=FindLambda[0]
```

```
0.127556
```

We can check the Vasicek model for continuity at the breakpoint k:

```
In[52]:= normal0[k,Lambda1]
```

```
Out[52]= -0.0829837
```

```
In[53]:= normal1[k,Lambda1]
```

```
Out[53]= -0.0829837
```

We can also check it for differentiability:

```
In[58]:= D[normal0[x,Lambda1],x]/.{x->k}
```

```
Out[58]= -0.153603

In[59]:= D[normal1[x,Lambda1],x]/.{x->k}

Out[59]= -0.153603
```

Although the Vasicek model is technically a one-threshold model, there is an implied second threshold at r=0. This is because the Vasicek model can become negative. Thus we have to integrate between plus and minus infinity. Here we check for continuity and differentiability at zero:

```
In[54]:= normal1[0,Lambda1]

Out[54]= -0.0795716

In[55]:= normal2[0,Lambda1]

Out[55]= -0.0795716

In[60]:= D[normal1[x,Lambda1],x]/.{x->0}

Out[60]= 0.392158

In[61]:= D[normal2[x,Lambda1],x]/.{x->0}

Out[61]= 0.392158
```

The Vasicek expansion coefficients can be calculated using numerical integration in the same way that we did for the double-threshold CIR model. The only difference is the implied second threshold at r=0.

```
(****** Expansion Coefficients ***********************)
cq2[lam_]:=NIntegrate[normal2[x,lam]*m[x],{x,-Infinity,0}]
cq1[lam_]:=NIntegrate[normal1[x,lam]*m[x],{x,0,k}]
cq0[lam_]:=NIntegrate[normal0[x,lam]*m[x],{x,k,Infinity}]
cq[lam_]:=cq0[lam]+cq1[lam]+cq2[lam]
cr2[lam_]:=NIntegrate[x*normal2[x,lam]*m[x],{x,-Infinity,0}]
cr1[lam_]:=NIntegrate[x*normal1[x,lam]*m[x],{x,0,k}]
cr0[lam_]:=NIntegrate[x*normal0[x,lam]*m[x],{x,k,Infinity}]
cr[lam_]:=cr0[lam]+cr1[lam]+cr2[lam]
cQ=Map[cq,Lambda]
cR=Map[cr,Lambda]
```

Finally, we show the price convergence using the eigenvalue expansion for the first 20 terms in Table 17.

N	Lambda	Phi	cQ	cR	Q(0,0)	R(0,0)	Price
1	0.13947	-0.07579	-12.03120	-0.78442	0.91187	0.05945	104.160
2	0.42414	0.03540	0.02158	-0.63701	0.91264	0.03690	107.954
3	0.74490	0.05528	0.38886	-0.12319	0.93413	0.03009	108.886
4	1.08075	-0.04304	-0.50810	-0.01864	0.95600	0.03089	108.962
5	1.38500	-0.03922	-0.16920	-0.00613	0.96264	0.03113	108.980
6	1.72466	0.04560	0.29483	0.00284	0.97608	0.03126	109.030
7	2.03242	0.02741	0.03676	0.00294	0.97709	0.03134	109.030
8	2.36523	-0.04609	-0.19087	-0.00259	0.98588	0.03146	109.053
9	2.68025	-0.01838	0.01292	-0.00185	0.98565	0.03150	109.051
10	3.00549	0.04556	0.12749	0.00217	0.99145	0.03160	109.063
11	3.32654	0.01119	-0.03214	0.00114	0.99109	0.03161	109.061
12	3.64634	-0.04448	-0.08592	-0.00179	0.99492	0.03169	109.067
13	3.97097	-0.00529	0.03831	-0.00068	0.99471	0.03169	109.067
14	4.28789	0.04305	0.05749	0.00147	0.99719	0.03175	109.070
15	4.61379	0.00032	-0.03847	0.00039	0.99718	0.03175	109.070
16	4.92990	-0.04140	-0.03758	-0.00120	0.99873	0.03180	109.071
17	5.25543	0.00393	0.03582	-0.00020	0.99887	0.03180	109.071
18	5.57209	0.03960	0.02347	0.00098	0.99980	0.03184	109.072
19	5.89628	-0.00761	-0.03193	0.00008	1.00005	0.03184	109.072
20	6.21420	-0.03770	-0.01342	-0.00081	1.00055	0.03187	109.073

Table 17 - Eigenfunction Expansion for GNMA 8.0% - Vasicek Model

6.4 Performance

Each of the three models that we have proposed produces values that are reasonably close to Bloomberg Prices. The single-threshold Cox-Ingersoll-Ross model performs best at 114 seconds of CPU time. The double-threshold model takes about four times longer to compute mainly due to the additional complexity the second breakpoint resulting in solving for the determinant of a 4-by-4 matrix. The CPU time is 498 seconds. Surprisingly, the Vasicek model also involves a 4-by-4 matrix, but it requires only 10% more CPU time that the single-threshold CIR model; the Vasicek CPU time is 127 seconds. This may be due to the fact that the parabolic cylinder function solutions to Vasicek require less time to compute than the Hyperbolic geometric solutions to the CIR model.

Chapter 7. Comparison of Model to Empirical Data

In order to test the model we need to look at prices of mortgage-backed securities over a range of interest rates. One reason we choose the10-year period from September 1996 until June 2006 is because it precedes the financial crisis of 2008 by two years; forward rates began to deviate from their usual upward-sloping tendency possibly due to events leading up to the mortgage crisis. We also use this period because we can compare our results to those of an existing model (Kolbe and Zagst 2009) which studied GNMA securities over the same period. This appeared to be the best model currently available because it used a piecewise linear prepayment S-curve similar to our double-threshold model. Furthermore Kolbe and Zagst reported that they were unable to compare the accuracy of the (Gorovoy and Linetsky 2007) approach to their data due to numerical problems. In this section we will do exactly that. In addition to the Gorovoy and Linetsky model, we will also compare Kolbe and Zagst's results to both the double-threshold and Vasicek models that we proposed and developed. We will also take a look at more recent data in 2010 and 2011 where parameter calibration is feasible.

Figure 14 shows U.S. Treasury rates from 1996 to 2011 at varying maturities. What one notices is that interest rates are on a distinct downward trend. The short rate shows much bigger fluctuations than longer term rates. When short-term rates drop radically, the long-term rates (20-year) are not much affected. In general, the CIR calibration works well when the yield curve is upward sloping (the typical situation). When the rates are close to each other, the yield curve may be flat or humped. This occurred during 1998, 2000 and 2006. When the interest rates are very low, the Vasicek

calibration tends to break down because the model allows for negative interest rates. These low interest rate periods occurred during 2002 and after 2008.
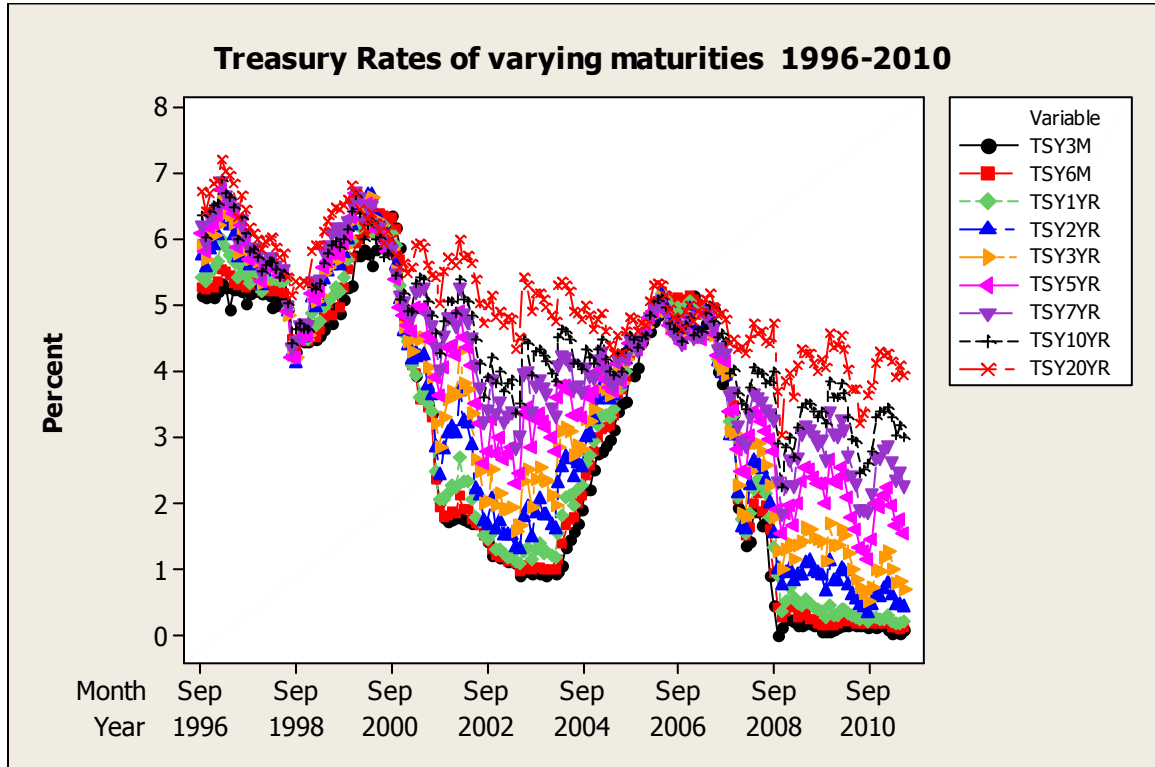

Figure 14: Treasury Rates with varying maturities.

7.1 Historical data

For the historical period from 1996-2006, we calibrate the interest rate models each month. Unfortunately one method of calibration does not work over the entire period of interest. Where possible, we will use the least squares approach proposed in Section 5. This distance measure is the L_2 norm

$$\min \left\| f - \hat{f} \right\|_2 = \sqrt{\sum_i \left(\hat{f}_{T_i} - f\left(T_i; k, \sigma, \theta\right) \right)^2} \qquad (7.1)$$

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During extended periods when this calibration method fails, there is usually some external event such as the dot-com bubble of 2000 or the pre-financial crisis period from 2006-2008 which we will denote a "crisis period". We will use the L_1 norm and a spread based on the difference between single-A corporate rates and treasuries during crisis periods. (According to (Kalotay, Yang and Fabozzi 2004) single-A corporate rates approximate the credit risk of mortgages.) This "crisis" spread is approximately 20 basis points above the normal spread between the single-A corporate rates and treasuries. The L_1 norm is defined as:

$$\min \left\| f - \hat{f} \right\|_1 = \sum_i \left| \hat{f}_{T_i} - f\left(T_i; k, \sigma, \theta\right) \right| \tag{7.2}$$

The L_1 norm is less sensitive to outliers that the L_2 norm, so when we have one or two points which cause the yield to deviate from its typical upward shape those points will not be weighted as heavily. The combination of using the L_1 norm with the crisis spread will allow us to calibrate the parameters in most cases when the yield curve is humped or inverted.

Simply using the L_1 norm without the crisis spread did not yield reasonable parameters in many cases. Using the crisis spread with the L_2 norm did not solve the calibration problem either because adding a spread does not change the shape of the yield curve.

In rare cases even the L_1 norm with a crisis spread failed to successfully calibrate. When this occurred for interest-rate model parameters for an isolated month or two (and not an extended period of time), we simply used the previous month's calibrated

parameters along with the current short rate r_0 and the weighted average maturity (WAM) for the current month.

The Kolbe-Zagst model performed the best using the 7.5% GNMA MBS compared to other coupon rates. The performance measure we use is the average absolute pricing error between the model prices and empirical prices obtained from Bloomberg. According to (Kolbe and Zagst 2009), the average absolute pricing error over this 10-year period was 116 basis points. The individual Kolbe-Zagst price estimates were unavailable, but the time series plots of the price estimates were published in (Kolbe and Zagst 2009). Communications with Kolbe and Zagst did not yield the actual data from their paper. However, an old drafting trick with an engineer's scale allows a visual price to be estimated for each month during the period 1996-2006 from grid lines drawn on a blown-up chart of the time series. These estimates for the Kolbe-Zagst model resulted in an average absolute pricing error of about 114 basis points, which is reasonably close to the published estimate considering the crudeness of this method.

When we exclude the crisis periods, our model performs extremely well compared to Kolbe-Zagst. For the 7.5% GNMA securities, we were able to calibrate the CIR parameters using the L_2 norm for 79 out of 117 months. For these 79 months the single-threshold CIR model had an average absolute deviation was 87 basis points. For the double-threshold CIR model it was only 83 basis points. For the same 79 monthly periods, the Kolbe-Zagst deviation was 110 basis points.

Now let's include all of the data. The Kolbe-Zagst difference is 116 basis points by their own estimate. The single-threshold CIR model for the entire period now averages 107 basis points. The double-threshold model average is 102 basis points.

For the Vasicek model, the L_2 calibrations work for most points. For some isolated points, we simply use the previous month's parameters as we did in the CIR model. The main problem with the Vasicek model occurs during periods of low interest rates. There were 24 monthly periods out of 117 where interest rates were too low to successfully calibrate the Vasicek parameters. This includes the period from July 2002 until March 2004. For the remaining 93 months the average absolute difference between our estimates and Bloomberg's was 109 basis points compared to the Kolbe-Zagst average for the same periods of 130 basis points.

Thus we have seen that all of our models compare favorably to Kolbe-Zagst when applied to the 7.5% GNMA pools. We next examine five generic GNMA pools ranging between 6% and 8%. Prices of all 5 pools are plotted against the observed (Bloomberg) data and the Kolbe-Zagst estimates. We shall also examine the mean absolute difference for each pool.

The absolute pricing error for the single-threshold CIR model was consistently between 107 and 120 basis points away from Bloomberg. We contrast this to Kolbe-Zagst whose errors tended to increase as coupon rates decreased. For the GNMA 6% pool the Kolbe-Zagst average absolute pricing error exceeded 250 basis points.

When there was significant prepayment data for interest rates much lower than the coupon rate, the double-ramp CIR model performed better than the single-ramp CIR model. This worked for the GNMA 7.5% and GNMA 8.0% pools. The lower coupon

pools did not have enough prepayment data to show any advantage to user the double-threshold over the single-threshold model.

When we exclude the low interest rate periods, we can compare Vasicek to Kolbe-Zagst. For comparison, we also include the CIR1 and CIR2 pricing errors when the low-interest rate periods are excluded. The Vasicek model produces a significantly closer price estimate than Cox-Ingersoll-Ross for the GNMA 6.0%, 6.5% and 7.0% pools. For the 7.5% and 8.0%, there was no sign cant difference in the price estimate.

For each of the Ginnie Mae securities, we show the Bloomberg market prices and Kolbe-Zagst prices followed by our model prices using the Vasicek interest-rate model, CIR interest-rate model with one prepayment threshold and CIR interest-rate model with two prepayment thresholds over the period October 1996 through June 2006. See Figure 15, Figure 16 and **Figure 17**. Our Cox-Ingersoll-Ross and Vasicek models beat the Kolbe-Zagst model using the mean absolute deviation measurement.

Next we examine the residual errors. We disregard residual patterns because prices are highly autocorrelated; however, we would like to see that our price estimates are unbiased. We created box-plots to observe not only the size of the errors, but also how well they are centered about zero. We also would like to observe the shape of the residuals—are they normal/symmetric or are they skewed left or right. Figure 18 shows that the Vasicek model has much smaller errors for the 6.0% and 6.5% GNMA securities. The errors in the CIR model are slightly larger than those in the Vasicek model, but they are much better centered about zero, which indicates that the CIR model, while less accurate the Vasicek, is also less biased. Both the CIR and Vasicek model outperform Kolbe-Zagst both in magnitude and estimation bias. Both Kolbe-Zagst and Vasicek based

models tended to overestimate the price. The GNMA 6.5% errors appear to be more symmetric than the GNMA 6.0% errors.

For the GNMA 7.0% securities, the Kolbe-Zagst errors appear to be fairly well centered on zero, but they are right-skewed. The CIR errors seem to be fairly well-centered on zero and are less skewed, while the Vasicek errors are smaller and symmetric but are not centered on zero.

For the higher coupons, we have both the single and double threshold CIR models. In these models, Figure 18 shows the Vasicek errors are again smaller but slightly off-center. Kolbe-Zagst is also off-center, but not as much as Vasicek, and both CIR models' errors are fairly well-centered, and slightly smaller than Kolbe-Zagst. All errors seem to be somewhat symmetric.

To test for significance, we performed a matched sample t-test for the difference between the absolute errors of Kolbe-Zagst and the absolute errors of our models. The differences in absolute errors appeared to follow an AR(1) time series model; the partial autocorrelation function showed a single spike at lag 1 for all coupon rates. Due to the autocorrelation, we had to adjust the variance of the mean difference by the following factor:

$$\mathrm{var}\left[\bar{X}\right] = \frac{\sigma^2}{n}\left[1 + 2\sum_{k=1}^{n}\left(1 - \frac{k}{n}\right)\rho_k\right] \tag{7.3}$$

where each ρ_k is the autocorrelation at lag k. Most of the autocorrelations completely decay by lag 10 so we truncated the autocorrelations and assumed a value of zero after the appropriate number of lags. For the CIR models we had a fairly large sample size

(n=117). For the Vasicek model we had a minimum sample size of n=84.   We obtain the correction factor from (7.3):

$$\text{Correction Factor} = \sqrt{1 + \sum_{k=1}^{n}\left(1 - \frac{k}{n}\right)\rho_k} \tag{7.4}$$

The standard error of the mean of correlated paired differences is calculated by multiplying the correction factor by the sample standard deviation divided by the square root of the sample size.

From Table 18 our model shows a significant improvement at the 5% level over Kolbe-Zagst for each of the lower coupons (6.0% and 6.5%). Our model performed better over Kolbe-Zagst for each of the other coupons (7.0%, 7.5% and 8.0%); although all the individual results were not all significant, taken together they indicate a marked improvement over Kolbe-Zagst.   For the Vasicek model, Table 19 shows a significant difference for the 6.0%, 6.5% and 7.0% coupons; the higher coupons were also closer than Kolbe-Zagst.

In conclusion, it appears that our model based on the eigenfunction expansion method proposed by Gorovoy and Linetsky is superior to Kolbe and Zagst's more complex model on GNMA securities.   This appears to be true regardless of coupon rate and it does not depend upon the underlying interest rate model since both Vasicek and CIR provide more accurate results. Moreover these two models are produces results that are close to each other. The double-ramp model represents a slight improvement over the single-ramp model for the higher-coupon models. The Vasicek model performs well during normal interest-rate periods; however, there are two concerns: (1) the model

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performs poorly when interest rates are low, and (2) while the errors are smaller, Vasicek

tends to overestimate the price since the residuals tend to be positive.

Sample Period 10/1996-6/2006	Model	GNMA Coupon				
		6.0%	6.5%	7.0%	7.5%	8.0%
n=117 (Entire Period)	Kolbe-Zagst	265	185	139	114	122
	CIR1	119	120	111	106	109
	CIR2	119*	120*	111*	101	107
n = 89 (Excluding Low Interest Rate Periods)	Kolbe-Zagst	272	199	161	130	129
	Vasicek	91	82	84	109	120
	CIR1	120	127	116	103	103
	CIR2	120*	127*	116*	99	100
	* CIR1 model used because second threshold not useful					

Table 18 - Absolute differences in Basis Points between Market and Model GNMA Security Prices

Model	GNMA Coupon	Mean Difference	Standard Deviation	n	Correction Factor	Standard Error	t	p-value
CIR1	6.0%	145.90	175.90	117	2.6614	43.2790	3.3711	0.0005
	6.5%	64.57	159.20	117	2.5796	37.9671	1.7007	0.0458
	7.0%	27.60	138.90	117	2.2541	28.9460	0.9535	0.1712
	7.5%	8.35	122.50	117	1.3715	15.5325	0.5376	0.2959
	8.0%	12.60	135.21	117	2.2248	27.8103	0.4531	0.3257
CIR2	7.5%	13.42	118.00	117	1.2173	13.2793	1.0106	0.1571
	8.0%	14.37	130.30	117	2.1340	25.7071	0.5590	0.2886
Vasicek	6.0%	180.47	184.80	89	2.0021	39.2180	4.6017	0.0000
	6.5%	119.90	146.60	84	1.9375	30.9903	3.8690	0.0001
	7.0%	79.10	121.60	84	1.9359	25.6845	3.0797	0.0014
	7.5%	19.80	127.30	92	2.0169	26.7688	0.7397	0.2305
	8.0%	14.35	126.10	90	2.5414	33.7810	0.4248	0.3359

Table 19 – Differences between Kolbe Zagst mean absolute error and Model mean absolute error

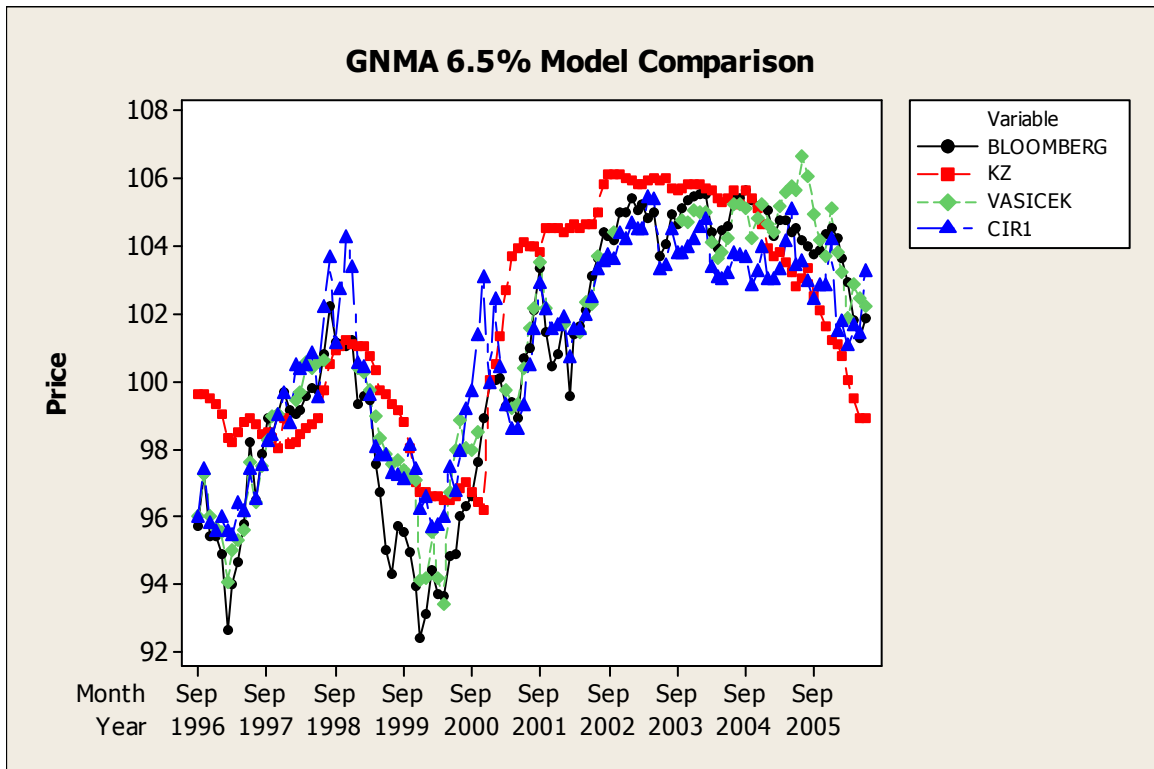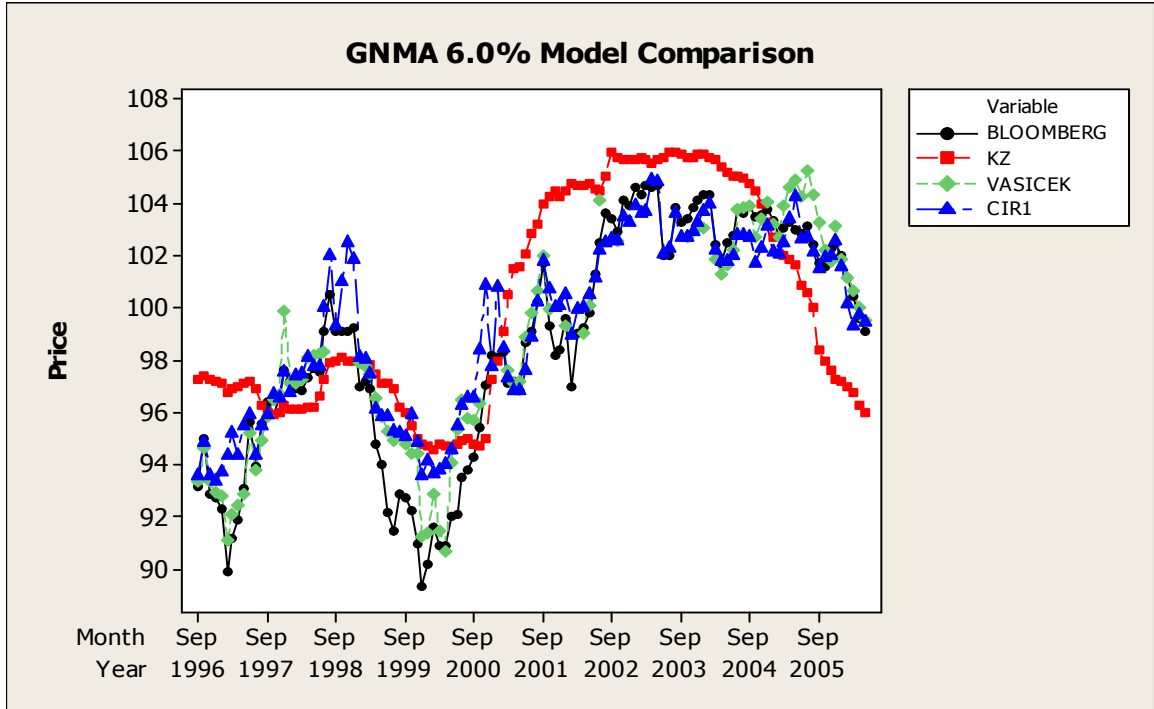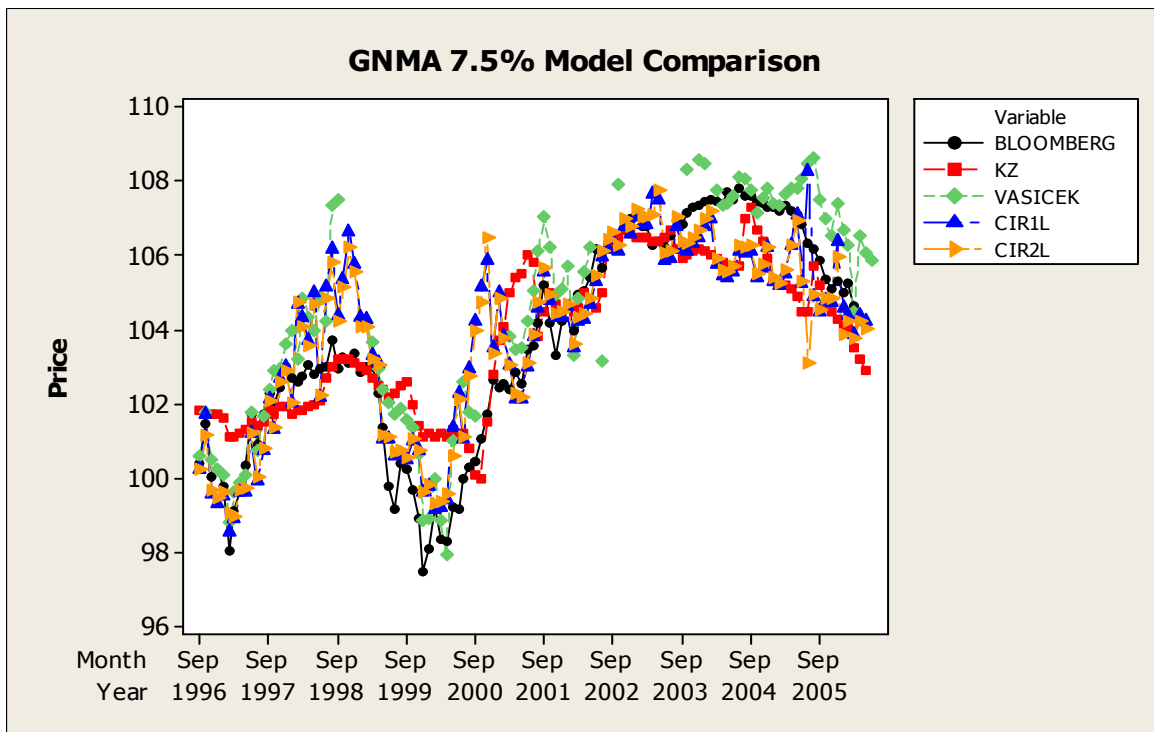**Figure 15 - Market and model prices for various GNMA passthru securities (I)**

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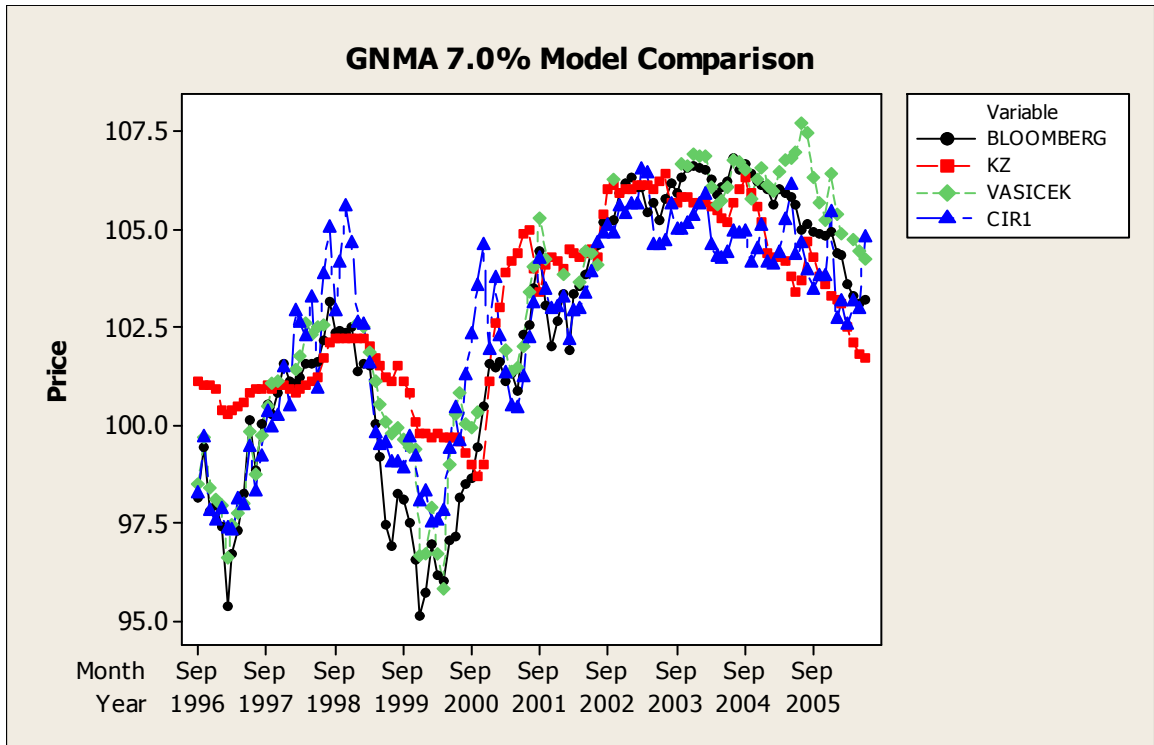
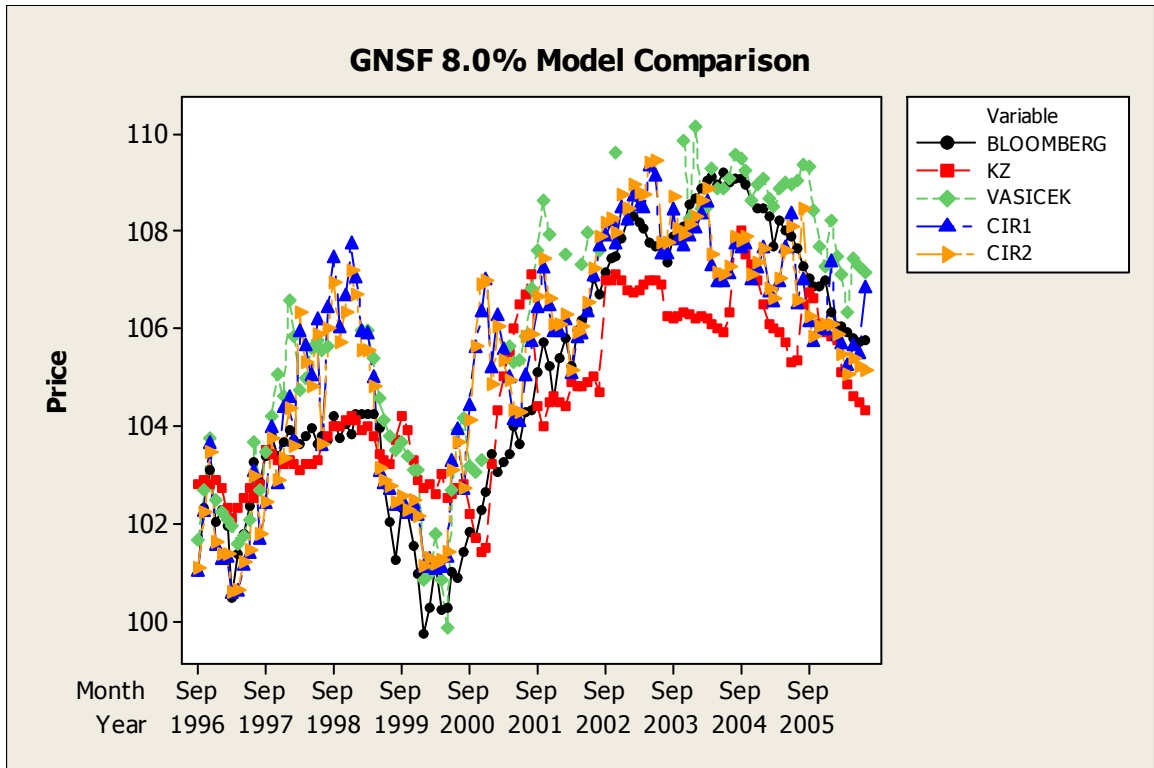**Figure 16 - Market and model prices for various GNMA passthru securities (II)**

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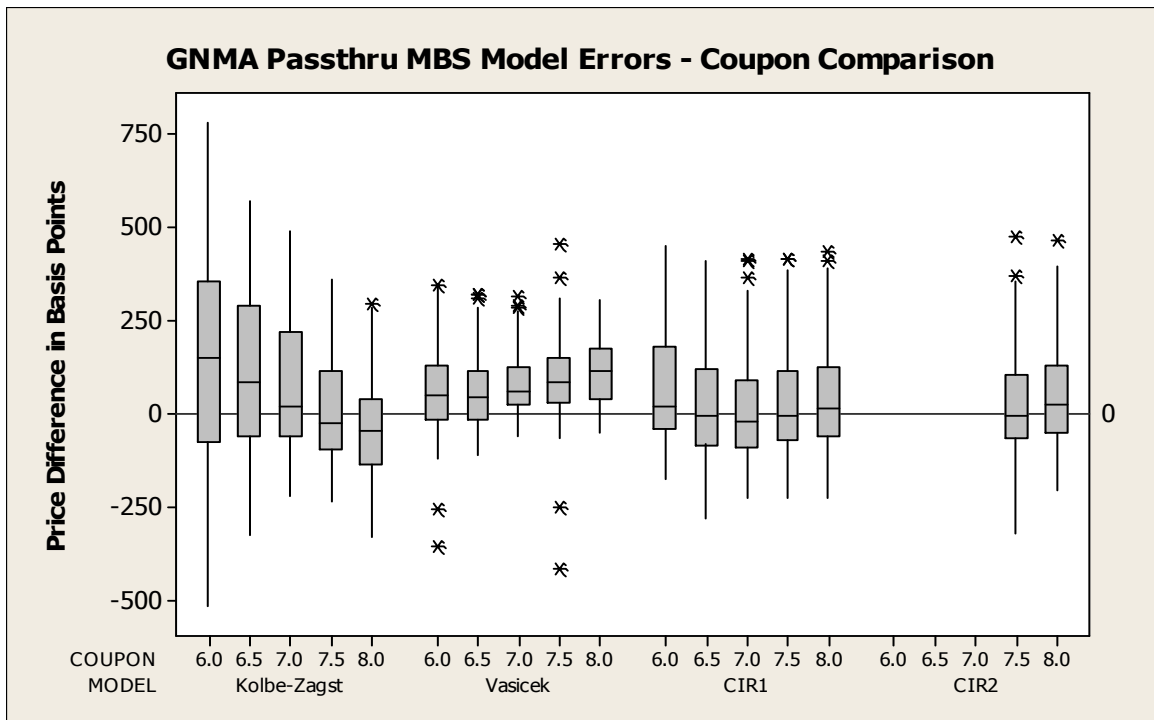**Figure 17 - Market and model prices for various GNMA passthru securities (III)**

Figure 18 - GNMA Passthru Model Errors

7.2 Recent Price Estimates

Although the mortgage crisis of 2008 has made it difficult to calibrate interest-rate models during that period, recently the yield curve has started to exhibit the more typical upward-sloping form. Prior to January 2009, in the months leading up to the 2008 financial crisis and shortly thereafter, the yield curve was largely unstable and our calibration produced strange results, in many cases producing a volatility of zero. Because recent short-term rates are extremely low, longer-term rates must necessarily be higher. Although the low interest rates make it difficult to use the Vasicek model because of the possibility of negative interest rates, we are able to calibrate the Cox-Ingersoll-Ross parameters.

Since interest rates are much lower in the recent period, it makes sense to will look at GNMA pools with lower coupon rates, i.e. 5.0% and 5.5% GNMA pools from the period of January 2009 until July 2011. The results of the least squares CPR parameter estimation for the recent period are based on the piecewise linear relationship between prepayment rates and treasury rates over this recent period. Thus we use a single set of parameters for each coupon rate:

Pool	h_o	γ	k
GNMA 5.0%	5.92190	8.89	1.11
GNMA 5.5%	7.01946	15.58	1.26

The recent spread between 3-month Corp A and Treasuries is about 64 basis points. During the crisis period between October 2008 and December 2009 it was much higher. Using the recent spread, we obtain the price estimates for the 5.0% and 5.5% GNMA's. This is because the pool size of the 6.0% - 8.0% GNMA's was very small in 2009-2011 because most of those loans have since refinanced.

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The Treasury yield curve for March 2010 is:

3 month	6 month	1 year	2 year	3 year	5 year	7 year	10 year	20 year
0.16	0.24	0.41	1.02	1.60	2.55	3.28	3.84	4.55

Adding the 64 basis points to the treasury yields, we are able to estimate the CIR parameters: $\kappa = 0.27$, $\sigma = 0.20$, $\theta = 0.074$. The parameters for the other months are listed in Table 20.

Plots of the 5.0% and 5.5% GNMA prices are shown in Figure 19. The model prices seem to be reasonably close to the market prices. In fact the mean absolute difference between the model and Bloomberg is 141.1 basis points for GNMA 5.0% MBS and 128.5 basis points for GNMA 5.5% MBS. After the third quarter 2010, the model prices appear to diverge from market prices. In fact, the mean absolute difference for the 5.0% MBS is only 101.4 basis points before November 2010; from November 2010 until July 2011 our model under predicts market prices consistently by an average of 198.5 basis points. For the 5.5% MBS, the mean absolute difference is 80.1 before November 2010; the model under predicts the market by 198.4 basis points from November 2010 until July 2011. The model is much more accurate before November 2010 than afterwards. We note that the second round of Quantitative Easing known as QE2 occurred at approximately the same time, when the Fed indicated it would purchase $600 billion worth of Treasury Securities. According to (Swanson 2011) of the Federal Reserve Bank of San Francisco:

> The QE2 program has been controversial, with detractors conjecturing that the risks or costs of the policy are large while the benefits are small. For example, an open letter to Federal Reserve Chairman Bernanke signed by several prominent economists and published in full-page ads in *The Wall Street Journal* and *The New York Times* asserted that the purchases "risk currency debasement and inflation" and could "distort financial markets".

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The market prices after November 2010 are much higher than those predicted by the model, possibly due to the artificial demand created by the Federal Reserve. This appears to bear out the market distortion predicted by the economists.

DATE	TSY3M	WAM		CIR Parameters		
		5.0%	5.5%	κ	σ	θ
200910	0.05	28.1667	26.6667	0.28254	0.19613	0.06807
200911	0.06	28.1667	26.5833	0.19960	0.17381	0.07568
200912	0.06	28.1667	26.4167	0.32404	0.21499	0.07132
201001	0.08	28.0833	26.3333	0.24840	0.19170	0.07400
201002	0.13	28.0833	26.2500	0.22820	0.18680	0.07650
201003	0.16	28.0000	26.1667	0.27060	0.20050	0.07430
201004	0.16	27.9167	26.0833	0.26460	0.19510	0.07190
201005	0.16	27.9167	26.0000	0.21770	0.17650	0.07160
201006	0.18	27.8333	25.9167	0.16800	0.15750	0.07390
201007	0.15	27.7500	25.7500	0.12720	0.14800	0.08620
201008	0.14	27.6667	25.6667	0.11480	0.13400	0.07820
201009	0.16	27.5833	25.5833	0.07890	0.12790	0.10360
201010	0.12	27.5000	25.5000	0.03930	0.12260	0.19110
201011	0.17	27.4167	25.3333	0.07750	0.13510	0.11770
201012	0.12	27.3333	25.2500	0.17640	0.16820	0.08020
201101	0.15	27.2500	25.1667	0.13640	0.16200	0.09620
201102	0.15	27.1667	25.0833	0.19010	0.17390	0.07950
201103	0.09	27.0833	25.0000	0.22310	0.18280	0.07490
201104	0.04	27.0000	24.9167	0.17440	0.16870	0.08160
201105	0.06	26.9167	24.8333	0.13100	0.15290	0.08920
201106	0.03	26.8333	24.7500	0.13520	0.15760	0.09190
201107	0.08	26.7500	24.6667	0.09690	0.14450	0.10770

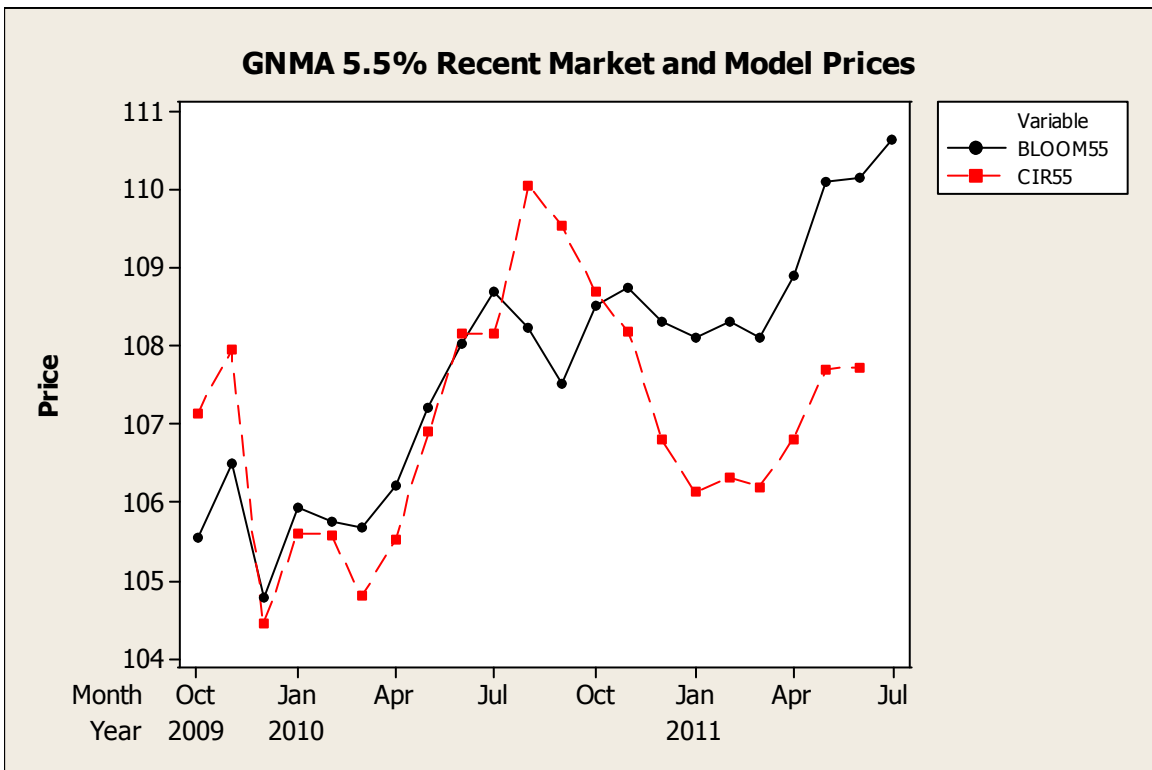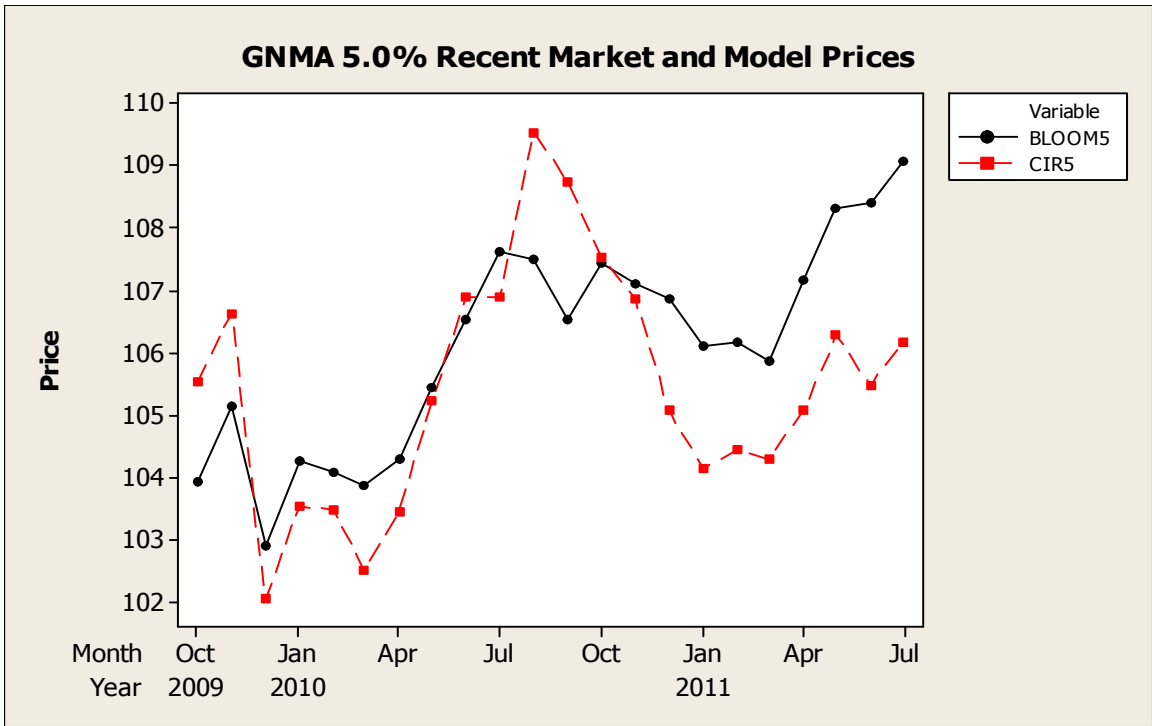**Table 20: Recent CIR Parameters**

Figure 19 - Recent MBS Prices Showing Possible Effect of Quantitative Easing in 4th Qtr 2010

7.3 Applications of the Default Model

Although agency MBS are not subject to defaults because they have corporate or government backing, the default model proposed in section 4.4 can be useful in several ways. We examine two specific ways to study the pricing of GNMA MBS by comparing prices and yields of similar instruments both with and without defaults. During periods of economic upheaval the spread to treasuries tends to increase for MBS; however, by pricing a theoretical GNMA MBS with "defaults" using the typical spread, we are able to approximate the same price as we did using the crisis spread. Since agency mortgages are guaranteed against defaults, we can use the default model to estimate the value of this guarantee by comparing the yield of an MBS with no defaults with that of a theoretical MBS with defaults. The difference in basis points can then be compared to the guarantee fee charged by Ginnie Mae, for example.

7.3.1 Using Defaults in Lieu of a Crisis Spread

During the 10-year period we examined between 1996 and 2006 we observed several periods where we introduced a "crisis spread" of 95.4 basis points. Instead of using a crisis spread, we can use the normal spread of 74 basis points alongside a constant default rate. Unfortunately, private mortgage companies do not release default data and Ginnie Mae, Fannie Mae and Freddie Mac cover defaults with guarantee fees. However, we can use the foreclosure rates available in Bloomberg as a proxy for default rates. Bloomberg's foreclosure rates are only available as far back as January 2005, so we will examine the period from September 2005 until June 2006.

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Instead of using a crisis spread, we can use the normal spread of 74 basis points along with the default model. Since we were unable to calibrate Treasury rates plus the normal spread during the crisis period, we will use the parameters from the calibration of the crisis spread of 95.4 basis points. However, since the difference between the crisis spread and the normal spread is 21.4 basis points, we will adjust the long term rate parameter by this amount and use the adjusted parameter in our default model:

$$\tilde{\theta} = \theta - 0.0214$$

We now apply the default model to GNMA 6.0%, 6.5%, 7.0%, 7.5% and 8.0% MBS. Figures 22, 23 and 24 show the default model prices versus the crisis spread along with Bloomberg and Kolbe Zagst. Observe that the Default Models and Crisis Spread Models are very close to each other over this period. There is a small spread for the 8.0% GNMA's, but both models are better than Kolbe-Zagst.
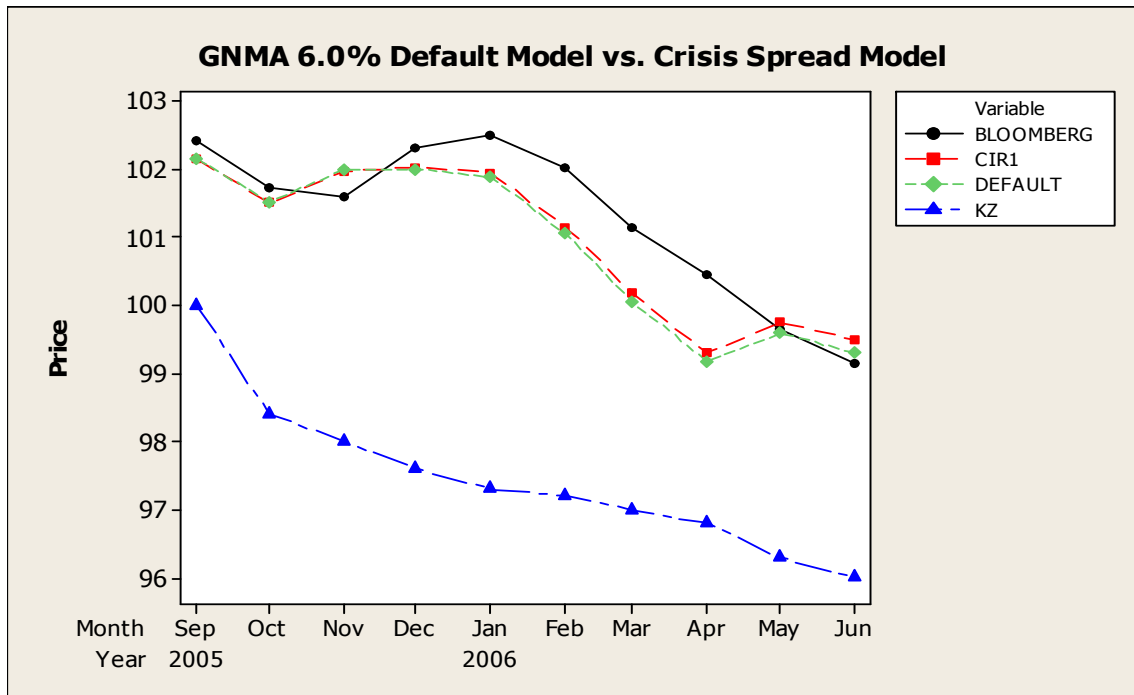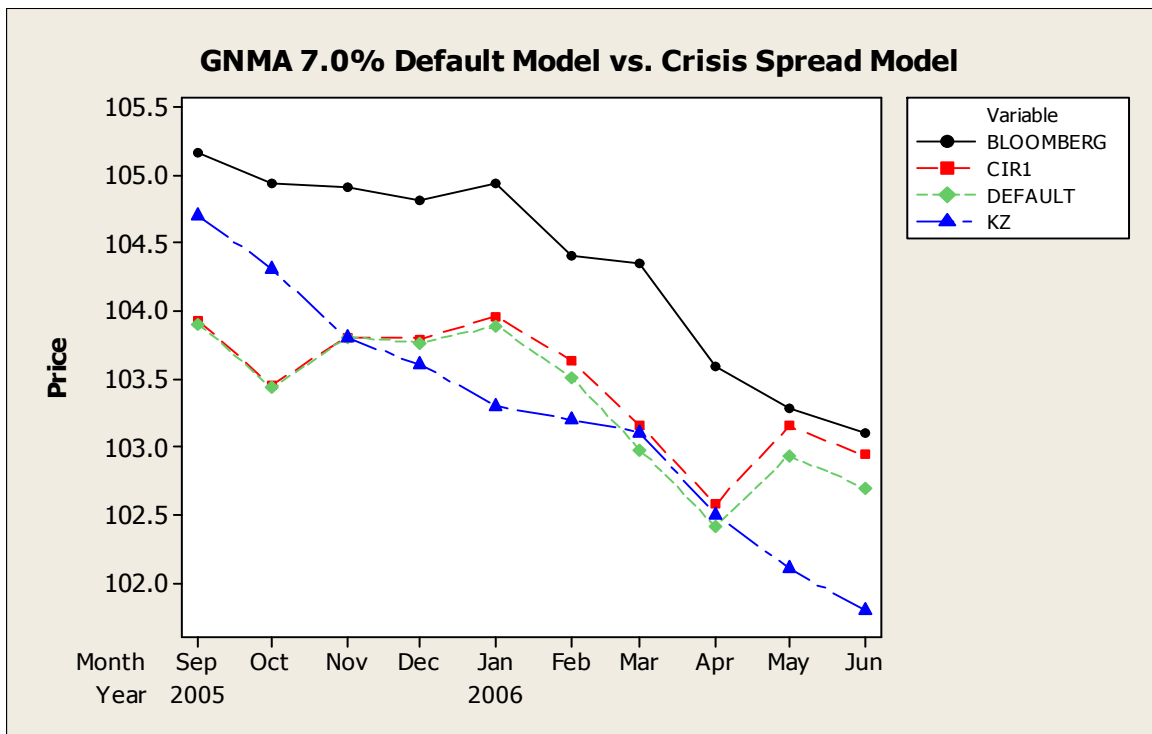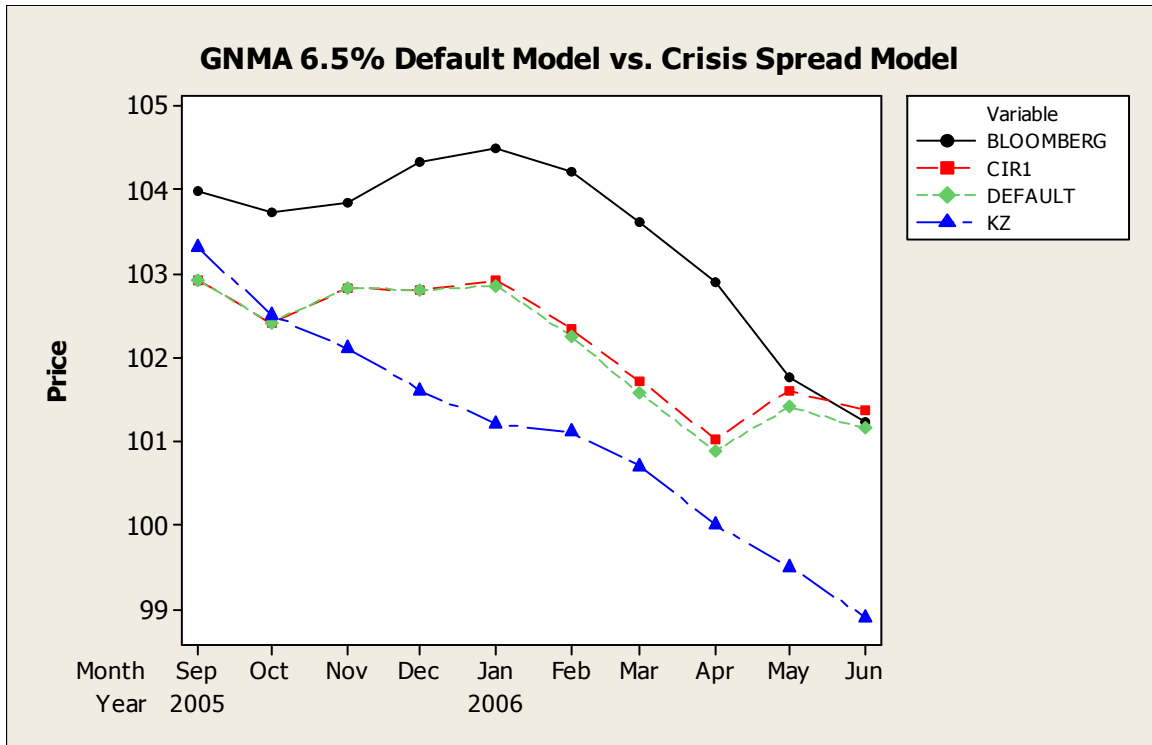

Figure 20 - Default vs. Crisis Spread Model (I)

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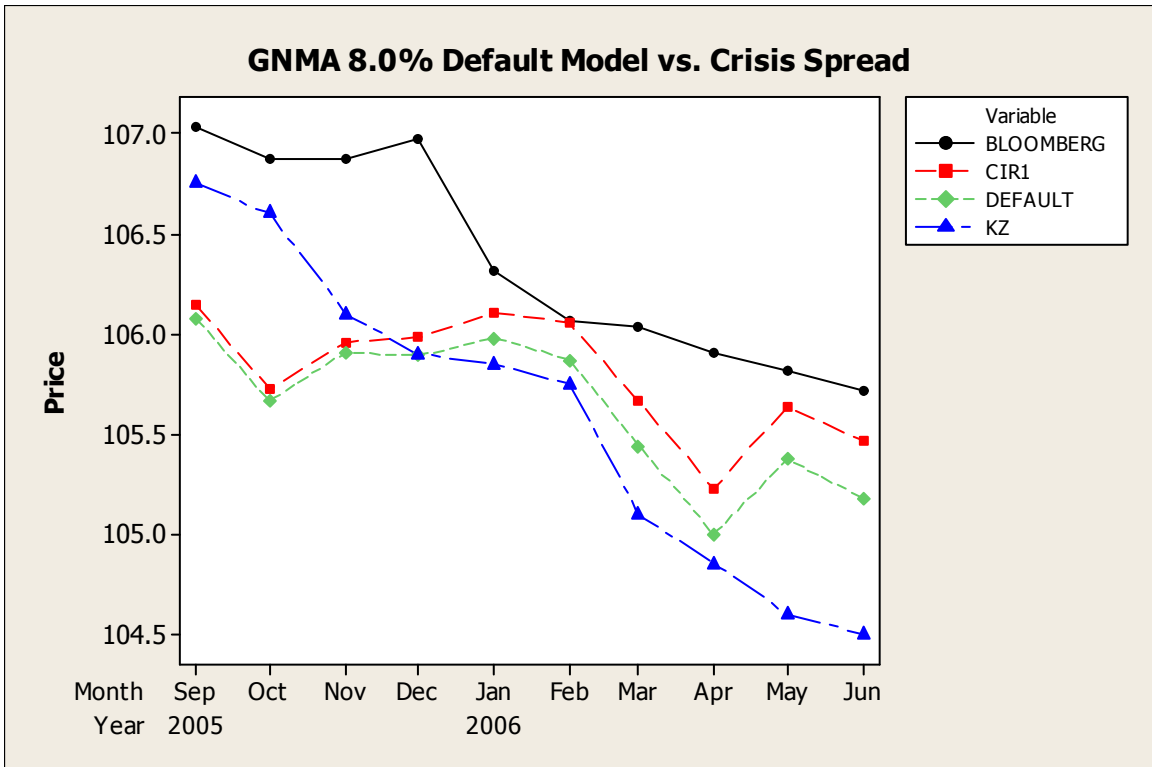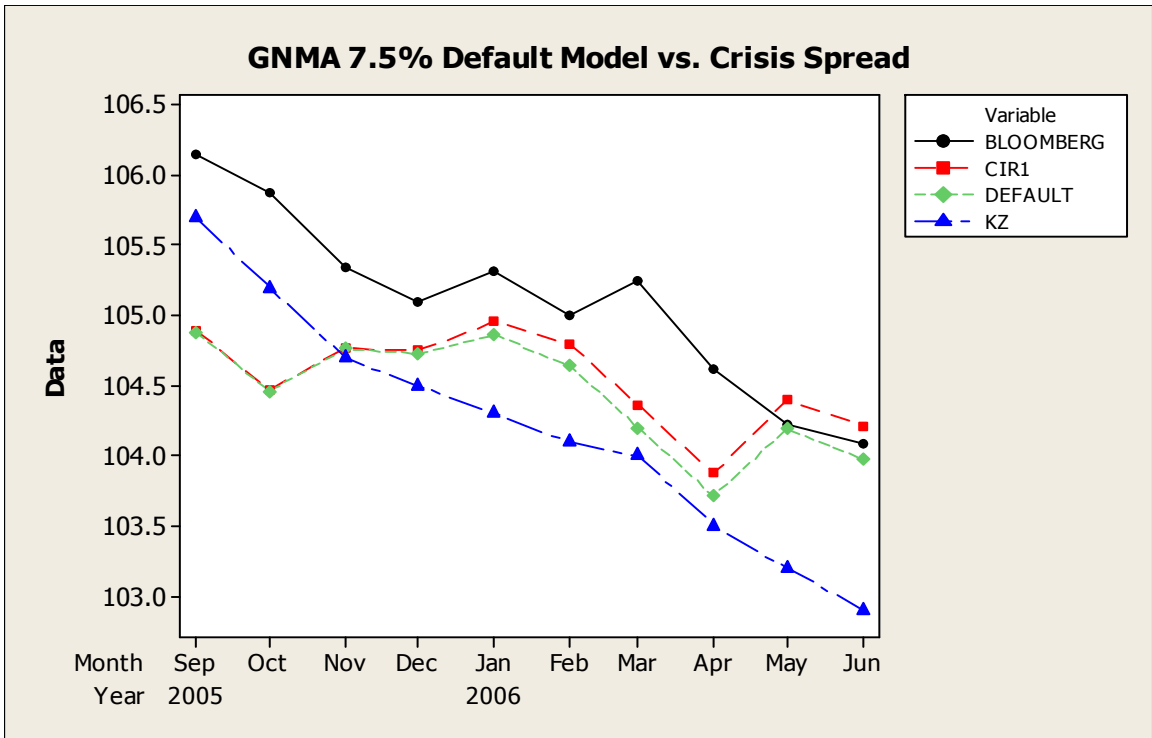**Figure 21 - Default vs. Crisis Spread Model (II)**

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Figure 22 - Default vs. Crisis Spread Model (III)

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7.3.2 Determining the Value of the GNMA Guarantee Fee

For a six-basis point guarantee fee on the balance of the securities, GNMA pays the issuer the lost principal and interest from defaults each month. Although defaulting mortgages take up to a year to be settled through the foreclosure process, we treat a default as an immediate prepayment except that only a fraction of the total balance due is received at that time. The proportion of the total balance due is based on three factors-- the recovery amount, the current mortgage rate and the time to foreclosure. The recovery amount is the market value of the house, less foreclosure expenses incurred by the bank. Since many mortgages are currently underwater, this recovery amount should be substantially less than it was prior to 2008. To account for the delay in receiving the recovery amount, we must also discount it by the mortgage rate for the period from first default to foreclosure.

What is the value of the guarantee provided by Ginnie MAE? To determine this, we must first determine the yield of an MBS with prepayments and zero defaults. We then determine the yield of an MBS with prepayments, a specified default rate and a loss severity rate. The difference between the yields determines the minimum number of basis points necessary for the guarantee fee to cover the losses associated with default.

(Gorovoy and Linetsky 2007) present a theoretical example where the short rate is $r_0 = 9.0\%$, $\theta = 0.06$, $\sigma = 0.10$, $\kappa = 0.25$. The refinancing threshold $k = 0.09$, and refinancing intensity $\gamma = 5.0$. Using the constant prepayment and constant default model we assume 4.5% CPR (Constant Prepayment Rate) and 0.6% ADR (Annual Default Rate) respectively. First we ignore the default rate and estimate the mortgage rate for a 30-year mortgage which is 7.85272%. This is the same value obtained in (Gorovoy and Linetsky

2007). Now let us apply a constant default rate (CDR)of 0.6% and loss severity of 20%. This results in a mortgage rate of 8.01942%. The difference between the two rates is 16.65 basis points. This assumes however, that the default rate of 0.6% remains constant throughout the life of the pool.

A slightly more realistic approach is to use the PSA prepayment model with the standard default assumption (SDA) described in Section 4.4.4 where the prepayment and default rates depend upon the seasoning of the pool. For illustrative purposes, we take another example using CIR parameters from previously calibrated by (Gorovoy and Linetsky 2007): θ =0.0702, κ=0.147, σ=0.095, r_o=0.0395, b=0.545, γ=15.5, s=0.00375. For a zero default rate, the interest rate is 5.61223%. Table 21 shows what happens to the mortgage rate by varying the default rate as a multiple of SDA and varying the loss severity from 0% to 50%. The table also shows the difference between the theoretical mortgage rate and the no-default benchmark in basis points. Observe that the loss premium is negative for very low (1%) loss-severity rates. This is because the yield curve predicts that interest rates will rise and the losses are too small to offset this advantage. To eliminate this "negative" loss premium, we can compare the yields to the same default rate with a loss severity of 0. These differences are also listed in Table 21. Here the loss premiums up to 100% SDA are small enough for the 6 basis-point guarantee fee to be effective. However, in recent years, foreclosure rates and hence default rates have skyrocketed. Furthermore, an increase in loss severity due to the bursting of the recent housing bubble shows that Ginnie Mae's guarantee fee is insufficient to cover losses due to foreclosures.

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Now let us apply this method to actual data from Ginnie Mae. First we look at a 6.0% GNMA sold on January 31, 2005. The price according to Bloomberg on that date was $P_t = 103.797$. The WAM (weighted average maturity) was 26.5 years. We use the calibrated CIR parameters $\kappa = 0.32368$, $\sigma = 0.17805$ and $\theta = 0.06210$, the short rate 2.51% and our prepayment estimates of $\gamma = 5.647$, $h_o = 0.051$, and prepayment threshold of 5.44%. Solving (4.170) numerically with $\delta = 0$ gives us a baseline mortgage rate of 6.14778%. (This is approximately the same as the effective annual rate of a nominal 6.0% mortgage compounded monthly: 6.168%.) Now let us use the foreclosure rate of 0.95% for January 2005 as a proxy for the default rate. Since this is a constant rate, we use the simpler CPR model with a constant default rate. Let us assume a loss severity of 20% which is typical for the industry. Using the function $S*(m) = 1 - (1-0.2)e^{-m}$ from above we find that the adjusted loss severity S* = 0.246476. From this we find that the mortgage rate $m = 6.38851\%$. The difference between the mortgage rate with defaults and the baseline mortgage rate is 24.07 basis points which is close to the loss premium

$$S\delta = 0.246476 \times 0.95 = 0.234152 \tag{7.5}$$

It is also close to the difference between the normal (74 BP) and crisis (95 BP) spreads we examined in the previous section.

Although a loss severity of 20% was often assumed in the mortgage industry, in recent years due to the housing bubble, this amount is likely to be much higher. The S&P/Case-Schiller Home Price Index, Seasonally Adjusted U.S.National Index Levels has declined 32.4% in the five year period ending in the first quarter 2011. Including the

costs of foreclosure would increase this figure. So it is useful to observe the sensitivity of MBS yields to loss severity. Holding everything else constant, we vary the loss severity from 0% to 50% and observe the difference in basis points. Table 22 shows the implied value of the Ginnie Mae Loan guarantee by comparing implied yields of a GNMA 6.0% security at the Bloomberg price of $103.797 on January 31, 2005 with various loss severities to the yield of a no-default GNMA 6.0% MBS and the yield of a GNMA 6.0% with an implied default rate of 0.95% (reflecting the foreclosure rate in January 2005) with no loss severity. The latter yield basically adds the default rate to the prepayment rate. Since the prepayment rate is imbedded in the MBS yield, this implied value may be more accurate.

We define the loss premium as the value of the Ginnie Mae Guarantee. Observe that even in 2005 the loss premium exceeds the six basis points charged by Ginnie Mae. This indicates that the guaranty fee charged by Ginnie Mae is a good deal for most banks and is, in fact, subsidized.

For a more recent case, we look at a GNMA 5.5% MBS sold on July 31st, 2010. The market price from Bloomberg is $108.703; assuming no defaults and solving for the yield, we obtain 5.59669%. The foreclosure rate for July 2010 was extremely high at 11.1%; however, at this rate nearly two-thirds of all mortgages will have defaulted after 10 years. Instead we will average the foreclosure rates over the previous four years to approximate the annual default rate (ADR); this results in a default rate of 6.64%. The implied value of the loan guarantee at this default rate for the July 2010 MBS is also listed in Table 22. We also include loss severities of 40% and 50%, since housing prices

fell substantially after 2008. It is clear from this table that 6 basis points are inadequate to cover the losses from defaults if we assume a typical 20% loss severity.

Fannie and Freddie MBS were obviously overpriced because investors did not perceive the risk involved, assuming government backing. But maybe they were right— after all the government did bail them out. As we have seen, even Ginnie Mae—which did not have the problems of Fannie Mae and Freddie Mac—cannot be sustained by charging only six basis points to guarantee the payments of defaulting mortgages.

Sensitivity of Mortgage Rates to Standard Default Assumption and Loss Severity

% SDA	0%	1%	10%	20%	30%	40%	50%
0%	5.61223	5.61223	5.61223	5.61223	5.61223	5.61223	5.61223
50%	5.60988	5.61066	5.61759	5.62530	5.63300	5.64070	5.64840
100%	5.60752	5.60906	5.62292	5.63831	5.65370	5.66908	5.68445
200%	5.60274	5.60582	5.63348	5.66419	5.69488	5.72555	5.75620
500%	5.58804	5.59569	5.66443	5.74067	5.81677	5.89274	5.96856
1000%	5.56253	5.57766	5.71359	5.86412	6.01413	6.16362	6.31258

Basis Point difference to 0% Loss Severity and 0% SDA

% SDA	0%	1%	10%	20%	30%	40%	50%
0%	0.000	0.000	0.000	0.000	0.000	0.000	0.000
50%	-0.235	-0.157	0.536	1.307	2.077	2.847	3.617
100%	-0.471	-0.317	1.069	2.608	4.147	5.685	7.222
200%	-0.949	-0.641	2.125	5.196	8.265	11.332	14.397
500%	-2.419	-1.654	5.220	12.844	20.454	28.051	35.633
1000%	-4.970	-3.457	10.136	25.189	40.190	55.139	70.035

Basis Point difference to 0% Loss Severity with Same Default Rate

% SDA	0%	1%	10%	20%	30%	40%	50%
0%	0.000	0.000	0.000	0.000	0.000	0.000	0.000
50%	0.000	0.078	0.771	1.542	2.312	3.082	3.852
100%	0.000	0.154	1.540	3.079	4.618	6.156	7.693
200%	0.000	0.308	3.074	6.145	9.214	12.281	15.346
500%	0.000	0.765	7.639	15.263	22.873	30.470	38.052
1000%	0.000	1.513	15.106	30.159	45.160	60.109	75.005

Table 21 - Effect of Defaults on Mortgage Rates (SDA Model)

MBS (Seasoned)		Loss Severity	CPR/CDR Required Yield(%)	Implied Value of Loan Guarantee (BP)	
				$\delta=0, S=0$	S = 0
		0%	6.20881%	6.10	0.00
GNMA	6.00%	5%	6.25379%	10.60	4.50
Date	Jan-05	10%	6.29874%	15.10	8.99
Price	$ 103.797	15%	6.34364%	19.59	13.48
Yield	6.14778%	20%	6.38851%	24.07	17.97
Default Rate	0.95%	25%	6.43333%	28.56	22.45
WAM (yrs)	26.5	30%	6.65686%	50.91	44.81
		0%	5.87759%	56.60	0.00
GNMA	5.50%	5%	6.20719%	89.56	32.96
Date	Jul-10	10%	6.53472%	122.31	65.71
Price	$ 108.703	15%	6.86022%	154.86	98.26
Yield	5.31163%	20%	7.18370%	187.21	130.61
Default Rate	6.64%	25%	7.50519%	219.36	162.76
WAM(yrs)	27.75	30%	7.82470%	251.31	194.71
		40%	8.45790%	314.63	258.03
		50%	9.08348%	377.19	320.59

Table 22- Sensitivity of Implied Loan Guarantee Values to Loss Severity

Chapter 8. Conclusion and Future Research

The results show that the eigenfunction expansion method not only works to calculate mortgage rates as presented by Gorovoy and Linetsky, but it can also be used to price mortgage-backed securities. Kolbe and Zagst were unable to apply Gorovoy and Linetsky's methodology to GNMA securities, claiming to have encountered numerical problems. We were able to overcome this hurdle during periods when the forward rates were well-behaved. During crisis periods we used a larger spread to the treasury rates and used the L_1 norm as a measure of distance instead of the usual Euclidean norm (L_2) because the former was less sensitive to outliers. As an alternative, we applied our default model with a normal spread during crisis periods and received very similar results. This gave us a means to circumvent the numerical problems Kolbe and Zagst may have encountered during calibration. Furthermore, our model depends only upon the interest rate, not on external factors such as GDP.

While the model works reasonably well, there are some things that are not explained by the model. First, interest rates are not truly random because the Federal Reserve deliberately manipulates short-term interest rates. Secondly, the home mortgage tax deduction distorts home buying and refinancing decisions.

The continuous model offers us an opportunity to study the behavior of mortgages under various economic conditions. We have shown that the eigenfunction expansion method is insensitive to the choice of interest rate model.   We obtained similar results using the Vasicek model. Using short term U.S. Treasury yields over a set period of

time, we will be able to calibrate the short rate to both the Cox-Ingersoll-Ross and the Vasicek model parameters.

We have also modeled the relationship between interest rates and the prepayment rate using two prepayment thresholds, the refinancing threshold and the low-interest rate threshold, to reflect empirical data more closely. We used prepayment data from Ginnie Mae over the same period as the U.S. Treasury data in order to determine the thresholds and prepayment sensitivities.

We presented a continuous mortgage default model with a deterministic piecewise- linear default rate. This model works with the eigenfunction expansion approach of (Gorovoy and Linetsky 2007) and uses both the constant default rate (CDR) and the Standard Default Assumption (SDA) model endorsed by the Public Securities Administration. Treating endogenous and exogenous prepayment rates, defaults, and severity rates separately allows us to generate different scenarios using the same eigenvalues. This reduces the amount of computations necessary in order to observe the effects of differing economic conditions on a particular mortgage-backed security, and the reuse allows analysts to run many default scenarios to test for robustness without a lot of additional overhead. The model can be used to assign a value to mortgage guarantees such as those provided by Ginnie Mae by finding the difference between prices and/or yields estimated using this model with and without losses and defaults.

On the technical side, we have shown a simpler way to calculate the normalizing constant for the single and double breakpoint prepayment models using an integral approach rather than the differential approach of (Gorovoy and Linetsky 2007) Using a

differential approach can lead to problems both computationally and conceptually; the integral approach is easier to grasp and is less prone to round-off error.

The continuous model can be further expanded to include the industry measures of investment life of mortgage-backed securities described in (Hayre, Mohebbi and Zimmerman 1997). These include weighted average life (WAL) and Macaulay duration. As an example, we have developed a WAL formula for a continuous mortgage-backed security with a constant prepayment rate which closely approximates the discrete industry model (See Appendix E). The mortgage industry often prefers *effective duration* and *effective convexity*, both of which use the relationship between interest rates and prepayments that we have presented, making them more accurate measures of price sensitivity. Formulating these measures for continuous models should lead to new insights into mortgage finance.

Recent problems in the mortgage industry due to the housing bubble have made it necessary to pay more attention to the problems of default. Government-sponsored entities such as Ginnie Mae can no longer ignore the risk of default without increasing the deficit.

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Appendices

Appendix A: Infinitesimal Generator of the Pricing Semigroup

Appendix B: Applying the Change of Variable to the Infinitesimal Generator

Appendix C: Integration by Parts

Appendix D: Calculating Forward Rates—A comparison of Mathematical and APL

Appendix E: Weighted Average Life—Continuous Model

Appendix A: Infinitesimal Generator of the Pricing Semigroup

To show that an operator family $Q(t)$ defined on a Hilbert space $\mathcal{H}$ forms a semigroup (Rudin 1973), we must show three things. Let $f \in \mathcal{H}$:

(a) $Q(0)f = If$; (b) $Q(s)Q(t)f = Q(s+t)f$; and (c) $\lim_{t \to 0} \|Q(t)f - f\| = 0$

In our case: $Q(t)f_s = E\left[\exp\left\{-\int_0^{t+s} r_u + \gamma(k - r_u)^+ \, du\right\} f(r_{t+s})\Big| r_t\right]$

and

$$\left(Q(t)f\right)(x) = E\left[\exp\left\{-\int_0^t r_u + \gamma(k - r_u)^+ \, du\right\} f(r_t)\Big| x = r_0\right]$$

Let us check (a) – (c).

To prove (a) is easy:

$$\left(Q(0)f\right)(x) = E\left[\exp\left\{-\int_0^0 r_u + \gamma(k - r_u)^+ \, du\right\} f(r_0)\Big| x = r_0\right]$$
$$= E\left[f(r_0)\Big| x = r_0\right] = f(x) = (If)(x) \Rightarrow Q(0) = I$$

Here is the proof for part (b):

We apply the definition of $\left(Q(s)Q(t)f\right)(x)$:

$$E\left[\exp\left\{-\int_0^s r_u + \gamma(k - r_u)^+ \, du\right\} E\left[\exp\left\{-\int_0^t r_u + \gamma(k - r_u)^+ \, du\right\} f(r_t)\Big| r_0\right]\Big| x = r_0\right]$$

The theory of semigroups is very well established on Markov processes. (Wong 1964) studied stochastic differential equations of the form

$$dX_t = \rho_1(X_t)\, dt + \rho_2\left(|X_t|^{1/2}\right) dW_t \ .$$

where ρ_1 and ρ_2 are polynomials. Let ρ_1 be a linear function with a negative slope to account for mean-reversion and let ρ_2 be a polynomial up to degree 2. When ρ_2 is a positive constant, this can be used to model the Vasicek interest-rate process. When ρ_2 is a linear function with positive slope, we have a Feller square root process which is a feature of the Cox-Ingersoll-Ross interest-rate model. (Wong 1964) established that both of these are stationary processes. See also (Ethier and Kurtz 1986). Now using the established time-homogeneity of the CIR and Vasicek models we can shift both the function argument and the limits of integration in $Q(t)$:

$$E\left[\exp\left\{-\int_0^s r_u + \gamma(k - r_u)^+ du\right\} E\left[\exp\left\{-\int_s^{s+t} r_u + \gamma(k - r_u)^+ du\right\} f\left(r_{s+t}\right)\bigg| r_s\right]\bigg| x = r_0\right]$$

Since conditional expectation is linear we can move everything inside:

$$E\left[E\left[\exp\left\{-\int_0^{s+t} r_u + \gamma(k - r_u)^+ ds\right\} f\left(r_{s+t}\right)\bigg| r_s\right]\bigg| x = r_0\right]$$

And now using the tower law we have:

$$\left(Q(s)Q(t)f\right)(x) = E\left[\exp\left\{-\int_0^{s+t} r_u + \gamma(k - r_u)^+ ds\right\} f\left(r_{s+t}\right)\bigg| x = r_0\right] = \left(Q(s+t)f\right)(x)$$

(c) To prove $\lim_{t\to 0}\|Q(t)f - f\| = 0$ we compute:

$$\lim_{t\to 0}\left\|E\left[\exp\left\{-\int_0^t r_u + \gamma(k - r_u)^+ du\right\} f(r_t) - f(r_0)\bigg| x = r_0\right]\right\|$$

$$= \left\|\lim_{t\to 0} E\left[\left\{1 - \int_0^t r_u + \gamma(k - r_u)^+ du + \frac{1}{2}\left(\int_0^t r_u + \gamma(k - r_u)^+ du\right)^2 - \cdots\right\} f(r_t) - f(r_0)\bigg| x = r_0\right]\right\|$$

$$= \left\|E\left[\lim_{t\to 0} f(r_t) - f(r_0) + \lim_{t\to 0} f(r_t)\left[\int_0^t r_u + \gamma(k - r_u)^+ du + \frac{1}{2}\left(\int_0^t r_u + \gamma(k - r_u)^+ du\right)^2\right]\bigg| x = r_0\right]\right\|$$

$$= \left\| E\left[f(r_0) - f(r_0) + f(r_0)\left[\int_0^0 r_u + \gamma(k - r_u)^+ \, du + \frac{1}{2}\left(\int_0^0 r_u + \gamma(k - r_u)^+ \, du \right)^2 \right] \middle| x = r_0 \right] \right\| = 0$$

To derive the infinitesimal generator of $(Q(t))$,

$$(Q(t)f)(x) = (\mathcal{P}_t f)(x) = E\left[\exp\left\{ -\int_0^t r_s + \gamma(k - r_s)^+ \, ds \right\} f(r_t) \middle| r_0 = x \right],$$

we note that

$$(\mathcal{G}_\varepsilon f)(x) = \frac{1}{\varepsilon}\left[(Q(\varepsilon)f)(x) - x \right] = \frac{1}{\varepsilon}\left[\mathcal{P}_\varepsilon f(x) - f(x) \right]$$

$$= \frac{1}{\varepsilon}\left\{ E\left[\exp\left\{ -\int_0^\varepsilon r_s + \gamma(k - r_s)^+ \, ds \right\} f(r_\varepsilon) \middle| r_0 = x \right] - f(r_0) \right\}$$

$$= \frac{1}{\varepsilon}\left\{ E\left[\left(1 - \int_0^\varepsilon r_s + \gamma(k - r_s)^+ \, ds + \frac{1}{2}\left[\int_0^t r_s + \gamma(k - r_s)^+ \, ds \right]^2 - \cdots \right) f(r_\varepsilon) \middle| r_0 = x \right] - f(r_0) \right\}$$

$$= E\left[\frac{f(r_\varepsilon) - f(r_0)}{\varepsilon} - f(r_\varepsilon)\frac{\int_0^\varepsilon r_s + \gamma(k - r_s)^+ \, ds}{\varepsilon}\left(\sum_{n=2}^\infty \frac{\left[-\int_0^\varepsilon r_s + \gamma(k - r_s)^+ \, ds \right]^{n-1}}{n!} \right) \middle| r_0 = x \right]$$

Using the following limits:

$$\lim_{\varepsilon \to 0} \frac{\int_0^\varepsilon g(s)\, ds}{\varepsilon} = \lim_{\varepsilon \to 0} \frac{G(\varepsilon) - G(0)}{\varepsilon} = G'(0) = g(0)$$

$$\lim_{\varepsilon \to 0} \frac{\left(\int_0^\varepsilon g(s)\, ds \right)^{n-1}}{n!} = \lim_{\varepsilon \to 0} \frac{\left[G(\varepsilon) - G(0) \right]^{n-1}}{n!} = \begin{cases} 1 \text{ for } n = 1 \\ 0 \text{ for } n > 1 \end{cases}$$

we can remove all but the first term in the summation below:

$$\lim_{\varepsilon \to 0} f(r_\varepsilon)\frac{\int_0^\varepsilon r_s + \gamma(k - r_s)^+ \, ds}{\varepsilon}\left(\sum_{n=2}^\infty \frac{\left[-\int_0^\varepsilon r_s + \gamma(k - r_s)^+ \, ds \right]^{n-1}}{n!} \right) = f(r_0)\left[r_0 + \gamma(k - r_0)^+ \right].$$

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$$(\mathcal{D}f)(r_0) = \lim_{\varepsilon \to 0} E\left[\left.\frac{f(r_\varepsilon) - f(r_0)}{\varepsilon}\right| r_0\right]$$

$$df(r_t) = f'(r_t)dr_t + \frac{1}{2}f''(r_t)(dr_t)^2$$

$$(dr_t)^2 = \left[\sigma(r_t)dW_t + \mu(r_t)dt\right]^2 = \sigma^2(r_t)dt$$

$$dr = \kappa(\theta - r_t)dt$$

$$df(r_t) = f'(r_t)dr + \frac{1}{2}\sigma^2(r_t)f''(r_t)dt$$

And so $\mathcal{G}f(r_0) = f'(r_0)\mu(r_0) + \frac{1}{2}\sigma^2(r_0)f''(r_0)$

$$(\mathcal{G}f)(x) =$$

$$\lim_{\varepsilon \to 0} E\left[\left.\frac{f(r_\varepsilon) - f(r_0)}{\varepsilon} - f(r_\varepsilon)\frac{\int_0^\varepsilon r_s + \gamma(k - r_s)^+ ds}{\varepsilon}\left(\sum_{n=2}^\infty \frac{\left[-\int_0^\varepsilon r_s + \gamma(k - r_s)^+ ds\right]^{n-1}}{n!}\right)\right| r_0 = x\right]$$

$$E_x\left[\lim_{\varepsilon \to 0}\frac{f(r_\varepsilon) - f(r_0)}{\varepsilon}\right] - E_x\left[f(r_\varepsilon)\frac{\int_0^\varepsilon r_s + \gamma(k - r_s)^+ ds}{\varepsilon}\left(\sum_{n=2}^\infty \frac{\left[-\int_0^\varepsilon r_s + \gamma(k - r_s)^+ ds\right]^{n-1}}{n!}\right)\right]$$

$$(\mathcal{G}f)(x) = (\mathcal{D}f)(r_0) - f(x)\left[r_0 + \gamma(k - x)^+\right]$$

To find $(\mathcal{D}f)(r_0)$ we must go to the original stochastic differential equations:

$dr = \mu(t)dt + \sigma(t)dW_t$. Differentiating implicitly, we must include the Ito term:

$$df(r_t) = \mu(x)f'(x)dt + \frac{1}{2}\left[\sigma(x)dW_t\right]^2 f''(x) = \mu(x)f'(x)dt + \frac{1}{2}\left[\sigma(x)\right]^2 f''(x)dt$$

The infinitesimal generator is

$$(\mathcal{G}_t f)(x) = \frac{1}{2}\sigma(x)^2 f''(x) + f'(x)\mu(x) - f(x)\left[x + \gamma(k - x)^+\right]$$

For the Vasicek interest rate model, the infinitesimal generator is:

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$$(\mathcal{G}f)(x) = f'(x)\kappa(\theta - r_t) + \frac{1}{2}\sigma^2 f''(x) - f(x)\left[x + \gamma(k - x)^+\right],$$

and for the Cox-Ingersoll-Ross interest rate model, the infinitesimal generator is:

$$(\mathcal{G}f)(x) = f'(x)\kappa(\theta - r_t) + \frac{1}{2}\sigma^2 x f''(x) - f(x)\left[x + \gamma(k - x)^+\right].$$

Appendix B: Applying the Change of Variable to the Infinitesimal Generator

Let us start with the infinitesimal generator.

$$(\mathcal{G}f)(x) = \frac{1}{2}\sigma(x)^2 f''(x) + \mu(x) f'(x) - \left[x + \gamma(k-x)^+ - \lambda\right] f(x) \tag{9.1}$$

and let $f(x) = e^{g(z)} u(z)$ and $z = \alpha x + \beta$

Now taking the first and second derivatives, we obtain:

$$f'(x) = \alpha e^{g(z)} \left[u' + g'(z) u\right] \tag{9.2}$$

and

$$f''(x) = \alpha^2 e^{g(z)} \left\{u'' + 2g'(z) u' + \left[g'(z)^2 + g''(z)\right] u\right\} \tag{9.3}$$

Substituting (9.2) and (9.3) into (9.1) yields:

$$\frac{1}{2}\sigma(x)^2 \alpha^2 e^{g(z)} \left\{u'' + 2g'(z) u' + \left[g'(z)^2 + g''(z)\right] u\right\} + $$
$$\mu(x) \alpha e^{g(z)} \left[u' + g'(z) u\right] - \left[x + \gamma(k-x)^+ - \lambda\right] e^{g(z)} u = 0 \tag{9.4}$$

Factoring out the exponential and grouping by derivatives of *u* gives:

$$\tfrac{1}{2}\sigma^2(x)\alpha^2 u'' + \left[\alpha^2\sigma^2(x) g'(z) + \alpha\mu(x)\right] u' + $$
$$\left[\tfrac{1}{2}\sigma^2(x)\alpha^2 \left(g'(z)^2 + g''(z)\right) + \alpha\mu(x) g'(z) + \lambda + x(\gamma-1) - \gamma k\right] u = 0 \tag{9.5}$$

CIR and Confluent Hypergeometric Equations

In the CIR model $\sigma(x) = \sigma_0 \sqrt{x}$, $\mu(x) = \kappa(\theta - x)$ $\alpha = \dfrac{2\rho}{\sigma^2}, \beta = 0$ and $g'(z) = \dfrac{(\kappa-\rho)}{\alpha\sigma_0^2}$.

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Substituting the above into (9.5) gives

$$\tfrac{1}{2}x\sigma^2\alpha^2 u'' + \left[(\kappa-\rho)x + \alpha\kappa(\theta-x)\right]u' +$$

$$\left[\frac{x(\kappa-\rho)^2}{2\sigma^2} + \frac{\kappa(\theta-x)(\kappa-\rho)}{\sigma^2} + \lambda + x(\gamma-1) - \gamma k\right]u = 0 \qquad (9.6)$$

Set $\rho = \sqrt{2(1-\gamma)\sigma^2 + \kappa^2}$ and the coefficient of u in (9.6) simplifies to:

$$-\frac{2\rho^2}{\sigma^2} - \gamma k + \lambda \qquad (9.7)$$

Notice that the variable x disappears in (9.7), making the coefficient of u a constant.

The coefficient of u' in (9.6) simplifies to

$$\rho\left(\frac{2\kappa\theta}{\sigma^2} - z\right) \qquad (9.8)$$

and the coefficient of u'' becomes

$$\rho z \qquad (9.9)$$

Putting (9.7), (9.8) and (9.9) into (9.6) and dividing by ρ gives :

$$zu'' + \left(\frac{2\kappa\theta}{\sigma^2} - z\right)u' - \left(\frac{2\rho}{\sigma^2} + \frac{\gamma k - \lambda}{\rho}\right)u = 0 \qquad (9.10)$$

This is in the same form as the confluent hypergeometric differential equation

$$zu'' + (b-z)u' - au = 0, \qquad (9.11)$$

whose known solutions are the Kummer and Tricomi functions. (Abramowitz and Stegun 1972) p. 504 Chapter 13 and (Slater 1960) p. 5).

The Kummer function is described in (Abramowitz and Stegun 1972) as:

$$M(a,b,z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!} \qquad (9.12)$$

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The Tricomi function is described in (Slater 1960) as:

$$U(a,b,z) = \frac{\Gamma(1-b)}{\Gamma(1+a-b)} M(a,b,z) + \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b} M(1+a-b, 2-b, z) \qquad (9.13)$$

An alternative formulation of the Tricomi function found in (Slater 1960) and (Abramowitz and Stegun 1972) is:

$$U(a,b,z) = \frac{\pi}{\sin \pi b} \left\{ \frac{M(a,b,z)}{\Gamma(1+a-b)\Gamma(b)} - z^{1-b} \frac{M(1+a-b, 2-b, z)}{\Gamma(a)\Gamma(2-b)} \right\}. \qquad (9.14)$$

Vasicek and the Parabolic Cylinder Functions

For the Vasicek model $\sigma(x)$ is a constant and the parameterization of $\mu(x)$ is the same as in CIR above, $\alpha = -\frac{\sqrt{2\kappa}}{\sigma}$, $\beta = \frac{\theta\sqrt{2\kappa}}{\sigma}$ and $g(z) = \frac{z^2}{4}$. Substituting those expressions into (9.5) yields

$$\kappa u'' + \left[\kappa z - \frac{\sqrt{2\kappa}}{\sigma} \kappa(\theta - x) \right] u' +$$
$$\left[\kappa \left(\frac{z^2}{4} + \frac{1}{2} \right) - \frac{\sqrt{2\kappa}}{\sigma} \kappa(\theta - x) \frac{z}{2} + \lambda + x(\gamma - 1) - \gamma k \right] u = 0 \qquad (9.15)$$

Now dividing (9.15) by κ and making the following substitution:

$$x = \theta - \frac{\sigma z}{\sqrt{2\kappa}} \qquad (9.16)$$

we arrive at:

$$u'' + \left[\left(\frac{z^2}{4} + \frac{1}{2} \right) - \frac{z^2}{2} + \left(\frac{\theta}{\kappa} - \frac{\sigma z}{\kappa\sqrt{2\kappa}} \right)(\gamma - 1) + \frac{\lambda - \gamma k}{\kappa} \right] u = 0 \qquad (9.17)$$

Note that the coefficient of u' in (9.17) disappears. Rearranging terms we get

$$u'' + \left[\frac{1}{2} - \frac{z^2}{4} + \left(\frac{\sigma z}{\kappa \sqrt{2\kappa}} \right)(1-\gamma) + \frac{\lambda - \gamma k - \theta(1-\gamma)}{\kappa} \right] u = 0 \qquad (9.18)$$

Completing the square in (9.18) gives

$$u'' + \left[\frac{1}{2} + \frac{\sigma^2 (\gamma-1)^2}{2k^3} + \frac{\lambda - \gamma k - \theta(1-\gamma)}{\kappa} - \left(\frac{\sigma(\gamma-1)}{\kappa\sqrt{2\kappa}} + \frac{z}{2} \right)^2 \right] u = 0 \qquad (9.19)$$

Set $v = \dfrac{\sigma^2 (1-\gamma)^2}{2\kappa^3} + \dfrac{\lambda - \gamma k - \theta(1-\gamma)}{\kappa}$ and rearrange terms to arrive at:

$$u'' + \left[\frac{1}{2} + v - \frac{1}{4} \left(\sigma(1-\gamma)\sqrt{\frac{2}{\kappa^3}} - z \right)^2 \right] u = 0 \qquad (9.20)$$

For $x > k$, $\gamma = 0$. This reduces (9.20) to

$$u'' + \left[\frac{1}{2} + \frac{1}{\kappa} \left(\frac{\sigma^2}{2k^2} + \lambda - \theta \right) - \frac{1}{4} \left(\sigma\sqrt{\frac{2}{\kappa^3}} - z \right)^2 \right] u = 0 \qquad (9.21)$$

Both (9.20) and (9.21) follow the general pattern in the differential equation (9.22) known as the one-dimensional Schrodinger equation.

$$u'' - \left(\frac{1}{4}v^2 + a \right)u = 0 \qquad (9.22)$$

The equation (9.22) has known solutions $U(a,x)$ and $V(a,x)$ which are the parabolic cylinder functions (See Chapter 19, (Abramowitz and Stegun 1972) p. 687 Equations 19.3.7 and 19.3.8), where:

$$U(v,x) = D_{-v-\frac{1}{2}}(x)$$

$$U(a,x) = \frac{\sqrt{\pi}\,2^{-\frac{1}{4}-\frac{1}{2}a}\,e^{-\frac{1}{4}x^2}}{\Gamma\left(\frac{3}{4}+\frac{1}{2}a\right)} M\left(\tfrac{1}{2}a+\tfrac{1}{4},\tfrac{1}{2},\tfrac{1}{2}x^2\right) - \frac{\sqrt{\pi}\,2^{\frac{1}{4}-\frac{1}{2}a}\,xe^{-\frac{1}{4}x^2}}{\Gamma\left(\frac{1}{4}+\frac{1}{2}a\right)} M\left(\tfrac{1}{2}a+\tfrac{3}{4},\tfrac{3}{2},\tfrac{1}{2}x^2\right), \qquad (9.23)$$

$$V(\nu, x) = \frac{\Gamma\left(\frac{1}{2} + \nu\right)}{\pi} \left\{ \sin(\pi\nu) U(\nu, x) + U(\nu, -x) \right\}. \tag{9.24}$$

Appendix C: Integration by Parts

We want to prove that

$$\int_t^T B(u)(h_s + \delta_s) e^{-\int_t^u r_s + h_s + \delta_s ds} du = B(t) + \int_t^T \left[\frac{dB_u}{du} - r_u B(u)\right] e^{-\int_t^u r_s + h_s + \delta_s ds} du \quad (10.1)$$

Let $f(u) = B(u) e^{-\int_t^u r_s ds}$ and $g'(u) = (h_u + d_u) e^{-\int_t^u (h_s + \delta_s) ds}$. So

$$\int_t^T B(u)(h_s + \delta_s) e^{-\int_t^u r_s + h_s + \delta_s ds} du = \int_t^T f(u) g'(u) du \quad (10.2)$$

Differentiating $f(u)$ and integrating $g'(u)$ gives:

$$f'(u) = \frac{d}{du}\left[B(u) e^{-\int_t^u r_s ds}\right] = e^{-\int_t^u r_s ds}\left[\frac{dB}{du} - B(u) r_u\right] \quad (10.3)$$

$$g(u) = \int (h_u + \delta_u) e^{-\int_t^u (h_s + \delta_s) ds} du = -e^{-\int_t^u (h_s + \delta_s) ds} \quad (10.4)$$

Now integration by parts yields:

$$f(u) g(u) - \int f'(u) g(u) du = \int \left[\frac{dB}{du} - B(u) r_u\right] e^{-\int_t^u (r_s + h_s + \delta_s) ds} du - B(u) e^{-\int_t^u (r_s + h_s + \delta_s) ds} ,(10.5)$$

or

$$\int_t^T f(u) g'(u) du = \int_t^T \left[\frac{dB}{du} - B(u) r_u\right] du + B(t) - B(T) e^{-\int_t^T (r_s + h_s + \delta_s) ds} \quad (10.6)$$

Note that $B(T) = 0$ so (10.2) reduces to

$$\int_t^T f(u) g'(u) du = \int_t^T \left[\frac{dB}{du} - B(u) r_u\right] du + B(t) \quad (10.7)$$

Q.E.D.

Appendix D: Calculating Forward Rates: A Comparison of Mathematica and APL

We can use Mathematica to calculate the forward rates by creating vectors for the maturities and corresponding U.S. Treasury rates:

```
T={0.25,0.5,1,2,3,5,7,10,20}

R={2.51,2.79,2.96,3.29,3.43,3.71,3.92,4.14,4.64}
```

and coding (5.23) inside a loop:

```
For[i=1,i<Length[T],i++,FWD[[i+1]]=(R[[i+1]]*T[[i+1]]-
R[[i]]*T[[i]])/(T[[i+1]]-T[[i]])];
```

Then we can transform the result to the continuous forward rates:

```
Spread = 0.74

100*Log[1+(FWD+Spread)/100]

{3.1983,3.7392,3.797,4.2676,4.3538,4.7551,5.0551,5.2529}
```

Although Mathematica is very efficient, sometimes the notation is rather clumsy. It is too easy to get lost in the brackets and parentheses in the Mathematica loop above. Let's see how to approach the same problem in APL.

We first create a vector of maturities expressed in years:

```
T←.25 .5 1 2 3 5 7 10 20
```

Next we create a vector of Treasury spot rates obtained from the U.S. Treasury web site corresponding to those maturities:

```
R←2.51 2.79 2.96 3.29 3.43 3.71 3.92 4.14 4.64
```

Now we assemble a two-row matrix from the vectors using the APL function disclose

(⊃):

```
⊃T R
0.25 0.5  1    2    3    5    7    10   20
2.51 2.79 2.96 3.29 3.43 3.71 3.92 4.14 4.64
```

The APL operator scan (\) applies its left operand × (multiplication) cumulatively down the columns of the matrix. This leaves the first row as the maturities (T) and converts the second row to total returns--the product of the rates and maturities (T×R):

```
×\⊃T R

0.25   0.5   1    2    3     5     7     10   20
0.6275 1.395 2.96 6.58 10.29 18.55 27.44 41.4 92.8
```

Observe that we are constructing and modifying an array by applying functions to the left of the current expression. The function ⊖ reverses the rows:

```
⊖×\⊃T R
0.6275 1.395 2.96 6.58 10.29 18.55 27.44 41.4 92.8
0.25   0.5   1    2    3     5     7     10   20
```

The pairwise reduction operator (2-/) applies its operand (-) between pairs of columns. This gives us the differences between returns $T_i R_i - T_{i+1} R_{i+1}$ as well as the differences between maturities $T_i - T_{i-1}$. Notice that pairwise reduction reduces the number of columns by 1.

```
2-/⊖×\⊃T R
¯0.7675 ¯1.565 ¯3.62 ¯3.71 ¯8.26 ¯8.89 ¯13.96 ¯51.4
¯0.25   ¯0.5   ¯1    ¯1    ¯2    ¯2    ¯3     ¯10
```

And finally, the simple reduction operator (/) applied to the division (÷) lets us divide the first row by the second row producing the forward rates. Notice that since both numerator and denominator are negative, the result is positive.

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```
÷≠2-/⊖×\⊃T R
```
3.07 3.13 3.62 3.71 4.13 4.445 4.653 5.14

We now append the current 3-month spot rate as the current forward rate.

```
    R[0],÷≠2-/⊖×\⊃T R
```
2.51 3.07 3.13 3.62 3.71 4.13 4.445 4.653 5.14

We can embed the entire process in a dynamic function where α represents the left argument (maturity) and ω represent the right argument (spot rates):

```
    Fwd←{ω[0],÷≠2-/⊖×\⊃α ω}
```

Compare this streamlined function to the original Mathematica loop function **For** which is reproduced below for comparison:

```
For[i=1,i<Length[T],i++,FWD[[i+1]]=(R[[i+1]]*T[[i+1]]-
R[[i]]*T[[i]])/(T[[i+1]]-T[[i]])];
```

We can apply the dynamic function we just defined as follows:

```
    T Fwd R
```
2.51 3.07 3.13 3.62 3.71 4.13 4.445 4.6533 5.14

The spread between Corp A and Treasuries over the period 1996 to 2006 is 74 basis points:

```
    Spread←0.74
```

We can then add the spread to the forward rate

```
 Spread+T Fwd R
```

3.81 3.87 4.36 4.45 4.87 5.185 5.3933 5.88

and use the natural log (APL symbol ⊛) to convert to the continuous forward rate.

```
    100×⊛1+.01×Spread+T Fwd R
```

3.1983 3.7392 3.797 4.2676 4.3538 4.7551 5.0551 5.2529 5.7136

Typical programmers in other languages measure productivity in thousands of lines of code (KLOC), whereas APL programmers measure productivity in token count. A token is a name such as Fwd or a symbol such as ⊖. In APL the forward rate calculations require a token count of 16. Excluding appending the current forward rate reduces the token count to 11. The best APL programmer is the usually the one who can minimize the number of tokens for a given program.

Appendix E: Weighted Average Life Continuous Model

Weighted Average Life (WAL) measures the investment life of a mortgage-backed security (MBS). The advantage to WAL is that it is simple and can be compared across the industry. The disadvantage is that it does not account for interest payments.

The definition of WAL in the discrete model is:

$$WAL = \sum_{n=1}^{N} nP_n \bigg/ \sum_{n=1}^{N} P_n = \sum_{n=1}^{N} B_n \bigg/ B_0 \tag{11.1}$$

where

$$N = \text{Number of periods}$$
$$P_n = \text{Principal paid in period } n$$
$$B_n = \text{Balance at beginning of period } n$$

and in the continuous model:

$$WAL = \int_0^T tP(t)\,dt \bigg/ \int_0^T P(t)\,dt = \int_0^T B(t)\,dt \bigg/ B(0). \tag{11.2}$$

Performing the change of variable we obtain the identity:

$$\int_0^T tP(t)\,dt = \int_0^T t\left(-\frac{dB}{dt}\right)dt = -\int_{B_0}^{B_T} t\,dB = \int_0^{B_0} t\,dB \tag{11.3}$$

Substituting the continuous amortization formula we obtain a closed-form definition of WAL:

$$WAL = \int_0^T \frac{B(t)}{B(0)}\,dt = \int_0^T \left[\frac{1-e^{-m(T-t)}}{1-e^{-mT}}\right]dt = \frac{T}{1-e^{-mT}} - \frac{1}{m} \tag{11.4}$$

For a 30-year mortgage at 10% with no prepayments, the WAL is approximately 21.6 years:

$$WAL = \frac{30}{1-e^{-0.1\times30}} - \frac{1}{0.1} = 21.57 \tag{11.5}$$

Prepayments shorten the weighted average life.   We introduce the following notation:

$$\{\gamma_t\} = \text{Prepayment Process}$$

$$\Gamma_t = \int_0^t \gamma_s ds = \text{Cumulative Prepayment Rate}$$

$$B_t e^{-\Gamma_t} = \text{Balance at time } t \text{ given } \{\gamma_t\}$$

$$e^{-\Gamma_t}(c_t + \gamma_t B_t) = \text{Cash Flow at time } t \text{ given } \{\gamma_t\}$$

The WAL with a constant prepayment rate is:

$$\int_0^T \frac{B(t)e^{-ht}}{B(0)} dt = \int_0^T \left[\frac{1 - e^{-m(T-t)}}{1 - e^{-mT}} \right] e^{-ht} dt = \frac{1}{1 - e^{-mT}} \left[\frac{1 - e^{-hT}}{h} + \frac{e^{-\gamma T} - e^{-mT}}{h - m} \right] \quad (11.6)$$

Thus $\lim_{h \to 0} WAL(h) = \frac{T}{1 - e^{-mT}} - \frac{1}{m}$ which is the same as (11.4).

Using the continuous model, we can take the derivative of (11.6) with respect to the prepayment rate to observe the sensitivity of the Weighted Average Life. Given $h=0.6$; $T = 30$ and $m = 0.1$, the weighted average life is 11.60 years as opposed to 21.57 with no prepayments. If prepayments increase by 1%, average life decreases by 1.0254 years:

$$\frac{\partial WAL}{\partial h} = \frac{1}{1 - e^{-mT}} \left[\frac{(hT + 1)e^{-\gamma T} - 1}{h^2} + \frac{(mT - hT - 1)e^{-hT} + e^{-mT}}{(h - m)^2} \right] \quad (11.7)$$

$$WAL = 11.60; \quad \frac{\partial WAL}{\partial h} = -102.54$$

Vita

Stephen M. Mansour was born on October 3, 1952 in Washington, D.C., the son of Farris and Annie Laurie Mansour (both deceased). He has three children--Darlene, George, and Lisa and is currently engaged to Regina Okrasinksi of Scranton, Pennsylvania. He is a veteran of the United States Coast Guard, honorably discharged in 1980. He received a Bachelor's Degree in Mathematics with Distinction in General Scholarship from the University of California at Berkeley in 1981 where he graduated Phi Beta Kappa. He worked at IBM East Fishkill, New York from 1982 to 1994 as an APL programmer. While at IBM he received a Division Award for "Outstanding Team Leadership During the Development of the ALORS2 Data Base". During his tenure at IBM, he also received a Master's Degree in Operations Research and Applied Statistics from Union College in 1992. He worked at Check-Free Corporation in Jersey City, New Jersey from 1994-1996 where he developed a billing system for portfolio managers, and at The Carlisle Group in Scranton, Pennsylvania from 1996-2008 where he developed an APL-based pricing and portfolio optimization system for mortgage companies. He is currently teaching statistics at the University of Scranton and at Penn State Worthington Scranton.