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Abstract

Risk defined as the chance that the outcome of an uncertain event is different than expected. In practice, the risk reveals itself in different ways in various applications such as unexpected stock movements in the area of portfolio management and unforeseen demand in the field of new product development. In this dissertation, we present four essays on data-driven risk management to address the uncertainty in portfolio management and capacity expansion problems via stochastic and robust optimization techniques.

The third chapter of the dissertation (Portfolio Management with Quantile Constraints) introduces an iterative, data-driven approximation to a problem where the investor seeks to maximize the expected return of his/her portfolio subject to a quantile constraint, given historical realizations of the stock returns. Our approach involves solving a series of linear programming problems (thus) quickly solves the large scale problems. We compare its performance to that of methods commonly used in finance literature, such as fitting a Gaussian distribution to the returns. We also analyze the resulting efficient frontier and extend our approach to the case where portfolio risk is measured by the inter-quartile range of its return. Furthermore, we extend our modeling framework so that the solution calculates the corresponding conditional value at risk (CVaR) value for the given quantile level.

The fourth chapter (Portfolio Management with Moment Matching Approach) focuses on the problem where a manager, given a set of stocks to invest in, aims to minimize the probability of his/her portfolio return falling below a threshold while keeping the expected portfolio return no worse than a target, when the stock returns are assumed to be Log-Normally distributed. This assumption, common in finance literature, creates computational difficulties. Because the

portfolio return itself is difficult to estimate precisely. We thus approximate the portfolio return distribution with a single Log-Normal random variable by the Fenton-Wilkinson method and investigate an iterative, data-driven approximation to the problem. We propose a two-stage solution approach, where the first stage requires solving a classic mean-variance optimization model, and the second step involves solving an unconstrained nonlinear problem with a smooth objective function. We test the performance of this approximation method and suggest an iterative calibration method to improve its accuracy. In addition, we compare the performance of the proposed method to that obtained by approximating the tail empirical distribution function to a Generalized Pareto Distribution, and extend our results to the design of basket options.

The fifth chapter (New Product Launching Decisions with Robust Optimization) addresses the uncertainty that an innovative firm faces when a set of innovative products are planned to be launched a national market by help of a partner company for each innovative product. The innovative company investigates the optimal period to launch each product in the presence of the demand and partner offer response function uncertainties. The demand for the new product is modeled with the Bass Diffusion Model and the partner companies' offer response functions are modeled with the logit choice model. The uncertainty on the parameters of the Bass Diffusion Model and the logic choice model are handled by robust optimization. We provide a tractable robust optimization framework to the problem which includes integer variables. In addition, we provide an extension of the proposed approach where the innovative company has an option to reduce the size of the contract signed by the innovative firm and the partner firm for each product.

In the sixth chapter (Log-Robust Portfolio Management with Factor Model), we investigate robust optimization models that address uncertainty for asset pricing and portfolio management. We use factor model to predict asset returns and treat randomness by a budget of uncertainty. We obtain a tractable robust model to maximize the wealth and gain theoretical insights into the optimal investment strategies.

Chapter 1

Literature Review

This literature review examines on robust optimization, risk measures and portfolio risk management, portfolio management with log-Normal sum approximation methods, and real options.

1.1 Decision Making Under Uncertainty and Robust Optimization

This section summarizes traditional approaches for decision making under uncertainty and provides definitions for robust optimization. In addition, recent studies in robust optimization with financial engineering applications are mentioned.

1.1.1 Stochastic Programing

Incomplete information is one of the major and most common challenges faced in real life applications. Therefore, the optimization models applied in real life problems must handle the issue of incomplete information.

Stochastic Programming (SP) is introduced as the pioneer in the field of decision making under uncertainty. SP depends on the assumption that uncertainty could be explained by probability distributions. Dantzig [65] explains uncertain parameters as random variables obeying a known discrete distribution, and he optimizes the expected value of the function of interest over possible scenarios generated based on this distribution. The fact that information is disclosed in stages in real life application is reflected in modeling techniques in SP. For instance, two-stage problems are widely used in SP literature. Two-stage SP formulation suggests that the first-stage decisions are made without complete information. The second-stage problem, which is called "recourse problem" is formulated assuming the first stage decision variables are given.The reader is referred to Birge and Louveaux [34], Kall and Wallace [109], Prékopa [173], and Ruszczyński and Shapiro [182] for further information.

According to Dyer and Stougie [71], two-stage SP problems are NP hard when the stochastic parameters are independently distributed. In addition, number scenarios increase exponentially as the number of uncertain parameter increases. Furthermore, it is hard to obtain an accurate estimation for probability distribution. Moreover, if the first or the second-stage problem contains integer decision variables, the complexity significantly increases. When SP is formulated as a multiple-stage problem, the drawbacks mentioned above are intensified. We refer the reader to Shapiro and Nemirovski [194] for a more detailed discussion about the complexity of the SP problems.

1.1.2 Robust Optimization

Robust optimization (RO) is a relatively modern/recent approach for decision making under uncertainty. RO depends on the assumption that uncertainty can be modeled by bounded sets. In other words, any realization of the uncertainty belongs to the defined uncertainty set. Sosyter [205] is one of the pioneers of RO. The author defines an interval for each uncertain parameter and formulates a model which optimizes the worst case objective function value. However, since this work requires each uncertain parameter to take its worst case value, it is found to be very conservative for practical implementations. However, this work provides a valuable base for the later studies. Especially, in 1990s robust optimization literature was significantly extended.

Ben-Tal and Nemirovski [17], [18], [19], El-Ghaoui and Lebret [72] and El-Ghaoui et al. [73]

use ellipsoidal uncertainty sets to define any possible realization of the uncertain parameters in the uncertainty constraints of the mathematical programing problem. They formulate robust optimization problems so that the worst case objective function value is maximized considering this deterministic uncertainty set. They interpret the robust counterpart of the nominal deterministic problem by tractable second-order cone problems. In addition, the degree of conservatism is adjusted by the size of the radius of the ellipsoid. In addition, Ben-Tal and Nemirovski [20] provide robust optimization applications to conic quadratic and semidefinite programming problems with uncertain parameters. However, the complexity of the nominal problem increases when robust optimization with ellipsoidal uncertainty set is applied to it.

Bertsimas and Sim [30, 31] and Bertsimas et al. [29] use polyhedral uncertainty sets to define uncertainty. The uncertainty set is implied by the ranges defined for all uncertain parameters. The range of an uncertain parameter is an interval which covers its possible realizations. The constraint called "budget of uncertainty constraint" manages the number of parameters which can possibly take their worst case value. In this way, the degree of conservatism can be managed by the decision maker. In addition, the robust counterpart of the linear nominal problem is also nominal; however, additional decision variables and constraints exist in the robust counterpart. We refer the reader to the survey written by Bertsimas and Thiele [32] for robust optimization literature until 2006.

The classical robust optimization approach with polyhedral uncertainty set can be described as below. Let c be the uncertain objective coefficient vector of size n. The general model is:

$$\begin{aligned} \max \quad & \mathbf{c}'\mathbf{x} \\ \text{s.t.} \quad & \mathbf{x} \in \mathcal{X}, \end{aligned} \tag{1.1}$$

where X defines the feasible region which could be formed by the investment budget constraint and non-negativity constraints for an investment problem. According to traditional robust optimization approach of Bertsimas and Sim [29], [31], each uncertain parameter $c_i,\ i \in$

$\{1, .., n\}$ can be modeled as an uncertain parameter belonging to the interval $[\bar{c}_i - \hat{c}_i, \bar{c}_i + \hat{c}_i]$, where $\bar{c}_i$ is the nominal value of the parameter, and range is defined by $\hat{c}_i$. Since each decision variable is non-negative, in the worst case model, none of them will be higher than its nominal value. In other words, in this specific setting, RO is interested in the left side of the interval. Therefore, each uncertain parameter can be represented as $c_i = \bar{c}_i + \hat{c}_i y_i$, for all i, where y_i is the scaled deviation such that $y_i \in [-1, 0]$, for all i. The polyhedral uncertainty set is defined as:

$$\mathcal{P} = \{y | \sum_{i=1}^{n} |y_i| \leq \Gamma, \quad |y_i| \leq 1, \, \forall i\}.$$

Parameter $\Gamma \in [0, n]$ is named as "uncertainty budget" and is used to control the risk conservatism.

- If $\Gamma = 0$, each uncertain parameter takes its nominal value $(\bar{c}_i)$; therefore, the robust counterpart and the deterministic problem are equal.
- If $\Gamma = n$, each uncertain parameter takes its worst case value $([\bar{c}_i - \hat{c}_i)$, as in Sosyter's study [205].
- If $0 < \Gamma < n$, the decision maker reflects his/her risk conservatism to the uncertainty budget Γ in order to find the balance between the protection level that he/she desired and the price of robustness (loss in the objective function) that he/she can pay for this level of protection.

The robust problem becomes:

$$\begin{aligned} \max \quad & \min \quad && \sum_{i=1}^{n} (\bar{c}_i + \hat{c}_i \, y_i) \, x_i \\ & \text{s.t.} && \mathbf{y} \in \mathcal{P} \\ \text{s.t.} \quad & \mathbf{x} \in \mathcal{X}. \end{aligned} \tag{1.2}$$

Theorem 1.1 (Bertsimas and Sim [31]) *The robust counterpart of Problem (1.1) is:*

$$
\begin{aligned}
\max \quad & \sum_{i=1}^{n} \bar{c}_i\, x_i \;-\Gamma z_0 - \sum_{i=1}^{n} z_i \\
\textit{s.t.} \quad & \mathbf{x} \in \mathcal{X} \\
& z_i + z_0 \geq \hat{c}_i\, x_i, \qquad \forall i, \\
& z_i, z_0 \geq 0 \qquad \forall i.
\end{aligned}
\tag{1.3}
$$

Proof. This is a direct application of Bertsimas and Sim [31] to Problem (1.2) with a special case where $y_i \leq 0$ and the decision $x_i \geq 0$ for all i.

More recent researchers have extended these works in both theory and application. Bertsimas and Brown [27] use coherent risk measures to express the decision maker's risk preferences and construct the uncertainty set accordingly. Goh and Sim [90] provide tractable approximations to distributionally robust optimization by incorporating piece-wise linear decision rules.

Robust optimization has been applied in finance where decision makers face high level of uncertainty. Goldfarb and Iyengar [91] apply robust optimization techniques to minimize the worst case variance in the portfolio selection framework. Erdogan et. al [75] extends this work taking transaction costs into consideration. Bertsimas and Pachamanova [28] apply robust optimization to multi-period portfolio optimization with transaction costs. Kawas and Thiele [111] address the uncertainty in the continuously compounded rate of return in the log-normal asset pricing of Hull [101] and use robust optimization techniques to the portfolio management problem with and without short sales.

1.2 Risk Measures and Portfolio Risk Management

In this section, the most commonly used risk measures are explained, and portfolio management problems with some of these risk measures are described.

In the finance world, risk is basically defined as the chance that the return on the investment will be different than expected. Olsen [160] stresses the importance of a general definition of investment risk shared by managers and their clients. Olsen defines perceived investment risk by considering four attributes: the potential for a large loss, the potential for a below target return, the feeling of control, and the perceived level of knowledge.

According to Harlow [96], in modern portfolio optimization framework, the investment decision process requires first evaluating capital market information and quantifying ex-ante measures of both risk and expected return for the appropriate set of assets, and then selecting the combinations of assets that are most efficient in the sense of the providing the lowest level of risk for a desired level of return. The final step is choosing a combination which is consistent with the risk tolerance of the investor. Harlow [96] believes that the most obscure and crucial task in this decision making process is defining the risk. As Cornuejols and Tutuncu [59] claim, managing risk requires a good understanding of quantitative risk measures. In the next section, we briefly introduce the common risk measures used in risk management.

1.2.1 Quantitative Risk Measures and Their Properties

Szegö [208] explains measuring risk as establishing a map ρ between the space of random variables Y and a nonnegative real number, i.e. $\rho : Y \to R$. For instance, in a portfolio management case, Y can be the returns of a specified set of investments. Scaler measures of risk let decision makers compare the investment alternatives. However, scalar risk measures might lead to inconsistencies, unless they have some specific properties. Szegö [208] explains these properties by three conditions. Let $\rho : Y \to R$ be a function; in order to define ρ as a possible risk measure (not a precise measure), the distance between two points in the space Y must satisfy the following conditions:

- the distance between a point and itself is zero,
- inverting the two points does not change the distance,

- for a specified set of three points and three different pairs of point combinations, the sum of distances between two pairs is always greater than (or equal to) the distance between the other pair.

  If a function satisfies these conditions, it is accepted to be a distance. However, these conditions do not define a precise or a proper risk measure, but only a class of possible measures.

Artzer, Delbaen, Eber and Health [6] name every proper risk measure as a "coherent" measure if it satisfies transitional invariance, sub-additivity, positive homogeneity, and monotonicity conditions, where

- Transitional invariance: $\rho(y + \alpha r_0) = \rho(y) - \alpha$ for all random variables y, real numbers α, and risk-free rates r_0,
- Sub-additivity: $\rho(y_1 + y_2) \leq \rho(y_1) + \rho(y_2)$ for all random variables y_1 and y_2,
- Positive homogeneity: $\rho(\beta y) = \beta \rho(y)$ for all random variables y, real numbers β,
- Monotonicity: $y_1 \leq y_2$ implies $\rho(y_1) \leq \rho(y_2)$ for all random variables y_1 and y_2.

Szegö [208], provides economic interpretations of these conditions. Transitional invariance implies that the value of risk measure ρ decreases by including some risk-free return αr_0 to a random return y. Moreover, sub-additivity suggests that the risk involved in a diversified portfolio is less than or equal to the sum of the risk of each single risky component of the portfolio. In addition, positive homogeneity means that enlarging the size of the portfolio by a real number β results in a new risk measure which is β times the risk of the original portfolio. Finally, monotonicity means that if one risk source always leads to higher losses than another risk source, then this risk measure of the former one should always be greater than that of the other one.

Some of the risk measures that exist in finance literature and applied in practice are described in the rest of the section.

Variance

Markovitz [139] introduces variance as a risk measure in the sense of dispersion of an asset's return from the mean of its return distribution. In the case of a portfolio, covariance is used to account for the risk arising from the interaction between different assets' returns. They are formulated as:

$$Cov[R_i, R_j] = E[R_i, R_j] - E[R_i]E[R_j]$$

$$Var[R_i] = E[R_i^2] - E[R_i]^2$$

where R_i and R_j are random variables.

According to Szegö [208] the most significant contribution of Markovitz is to measure the risk of the portfolio via joint distribution of returns of all assets. However, Markovitz's model depends on an appropriate investor utility function which determines the efficiency of the assets and their combinations. Szegö claims that Markovitz's model works only when the random returns are generated from elliptic distributions, such as normal or $t-$ distributions with finite variances. Moreover, determining a utility function for each investor is an obscure task. Roy [179] expresses this drawback as: " *A man who seeks advice about his actions will not be grateful for the suggestion that he maximizes expected utility.*", besides, the variance also considers the excess return, which investors seek for, as a part of risk.

Semi-Variance

Markovitz [140] introduces "semi-variance" to address the drawbacks of the variance. Markovitz provides two alternative semi-variance calculations: below-mean semi-variance (SV_m) and below-target semi-variance (SV_t). Then, the semi-variances of a given portfolio are written as follows:

$$SV_m = \frac{1}{n}\sum_{i=1}^{n}(\max\{0, -R_i + E[R_p]\})^2$$

$$SV_t = \frac{1}{n}\sum_{i=1}^{n}(\max\{0, -R_i + R_t\})^2$$

where R_i, $E[R_p]$, R_t, and n are respectively the return rate of asset i, the expected portfolio return rate, the target for portfolio rate of return, and the number of assets considered in the portfolio.

According to Markovitz [140], the variance is superior to the semi-variance with respect to the computational cost and the convenience. For instance, an analysis based on the variance needs only the variance and covariance information. On the other hand, for an analysis based on SV_m or SV_t, the entire joint distribution is required [140]. However, the semi-variance does not punish excess returns.

Safety-First Ratio

Roy [179] defines "disaster" as facing a net loss as a result of some investment, if the income is less than what it would almost certainly be in some other occupation. Roy believes that such a disaster idea exists for many investors, and these investors will seek to reduce the chance of such a disaster. The author shows that for a given asset i minimizing the probability of the disaster is equivalent to maximizing the asset's safety-first ratio, SFR, which is written as:

$$SFR_i = \frac{E[R_i] - R_{min}}{\sqrt{Var[R_i]}}$$

where $E[R_i]$ is the expected return, R_{min} is the minimum acceptable return, and $\sqrt{Var[R_i]}$ is the standard deviation of the return for asset i. A portfolio's safety-first ratio is formulated as:

$$SFR_p = \frac{E[R_p] - R_{pmin}}{\sqrt{Var[R_p]}}$$

where $E[R_p]$ is the expected portfolio return, R_{pmin} is the minimum acceptable portfolio return, and $\sqrt{Var[R_p]}$ is the standard deviation of the portfolio return.

Nawrocki [151] mentions that Roy's safety-first criterion inspired researchers to develop more downside risk measures.

Sharpe Ratio

The sharpe ratio, S, is actually a risk-adjusted return measurement. Sharpe [196] defines this reward-to-variability ratio so that the numerator shows the difference between the asset's return rate and the benchmark rate (reward), while the denominator is equal to the standard deviation of the asset's return rate (variability). Sharpe [198] considers both ex-ante or ex-post Sharpe ratios.

Ex-Ante Sharpe Ratio

Let R_i and R_b be the rate of return on asset i, and that on the benchmark security or portfolio, if asset returns are not known in advance, the reward is:

$$\tilde{d} = \tilde{R_i} - \tilde{R_b}.$$

Then, the ex-ante Sharpe ratio (S) is:

$$S = \frac{\bar{d}}{\sigma_d},$$

where $\bar{d}$ and σ_d are the expected value and the predicted standard deviation of the reward, respectively.

Ex-Post Sharpe Ratio

Let $R_{i,t}$, $R_{b,t}$, and d_t be the rate of return on asset i, that on benchmark portfolio, and the reward in period t. Then,

$$d_t = R_{i,t} - R_{b,t},$$

$$\bar{d} = \frac{1}{T}\sum_{t=1}^{T} d_t,$$

and

$$\sigma_d = \sqrt{\frac{\sum_{t=1}^{T}(d_t - \bar{d})^2}{T-1}}.$$

Then, ex-post historic Sharpe Ratio is calculated as:

$$S_h = \frac{\bar{d}}{\sigma_d}.$$

Portfolio β -Market Neutrality

β_i measures the linear dependence between the rate of return on asset i (R_i) and that on market portfolio (R_m). It is formulated as:

$$\beta_i = \frac{Cov[R_i, R_m]}{Var[R_m]},$$

while β of a portfolio is calculated as:

$$\beta_p = \sum_{i=1}^{n} x_i \beta_i,$$

where x_i is the portion of the portfolio invested in asset i.

Szegö [208] states that the motivation behind the introduction of β as a risk measure in 1960s was the computational cost of the Markovitz model. In addition, portfolio β requires a smaller data set than the Markovitz model does. However, with the computational power of current computers, these problems are not valid anymore. Moreover, the idea of using β as a risk measure led to the development of one of the most commonly used pricing models, CAPM [208].

Lower Partial Moment

Bawa [11] and Fishburn [81] introduce and improve lower partial moment (LPM) method. Nawrocki [151] believes that LPM provides more freedom on selecting the utility function than the variance and semi-variance since it resents a large number of Von Neumann-Morgenstern utility functions, whereas the variance and semi-variance use only quadratic utility functions. Moreover, risk seeking, risk neutral, and risk aversion behaviors can be represented by LPM approach. Given a risk tolerance value a and return target R_t, LPM is calculated as:

$$LPM(a, R_t) = \frac{1}{n}\sum_{i=1}^{n} \max[0, (R_t - R_i)^a],$$

where R_i, R_t and n are respectively the rate of return on asset i, the target for portfolio rate of return, and the number of assets considered in the portfolio [151]. Moreover, Bawa [11] proves the connection between LPM and the stochastic dominance when the risk tolerance value is 0, 1 and 2. Moreover, Fishburn [81] shows the equivalence of LPM to stochastic dominance for all risk tolerance values higher than zero. In addition, Fishburn demonstrates that the cases $a < 1$, $a = 1$, and $a > 1$ respectively represent risk seeking, risk neutral, and risk averse behaviors.

Sortino Ratio

Sortino ratio [204], which is a modification of Sharpe ratio, measures the risk adjusted return. Unlike Sharpe ratio, Sortino ratio penalizes only the downside risk. In other words, the denominator of the ratio is the standard deviation of returns below the benchmark. Actually, the downside standard deviation is not different than the lower partial moment (degree 2) of the asset's rate of return distribution.

Mean Absolute Deviation

Mean absolute deviation (MAD) is also used as a risk measure, which is suitable for concave quadratic utility functions like variance. It is formulated as:

$$MAD = \frac{1}{n}\sum_{i=1}^{n} |-R_i + E[R_p]|,$$

where R_i, $E[R_p]$ and n are the rate of return on asset i, the expected portfolio rate of return, and the number of assets considered in the portfolio.

Sharpe [197] considers MAD in portfolio analysis through an algorithm for a regression problem, which relates the portfolio's rate of return to that of a market portfolio. Konno & Yamazaki [119] propose a portfolio optimization model using the mean absolute deviation as a response to computational difficulties of the Markovitz model. Using the mean absolute deviation as a risk measure instead of the variance transforms the quadratic Markovitz model into a linear model. This transformation decreases the computational cost, especially in large scale problems. The authors show that the performance of the optimal portfolio suggested by the MAD model is quite similar to that of the Markovitz model.

In fact, efforts to handle the portfolio management problem by a linear model goes back to 1970s. Sharpe [195] approximates the quadratic variance function to edge-to-edge collection of several piecewise linear functions and solves the portfolio optimization problem as a linear problem.

Downside Mean Semi-Deviation

Downside mean semi-deviation was introduced by Speranze [206] and formulated as:

$$E[|\min\{0, \sum_{j=1}^{n} R_j x_j - E[(\sum_{j=1}^{n} R_j x_j)]\}|],$$

where R_j, R_t and n are the rate of return on asset j, the specified rate of return level, and the number of assets considered in the portfolio.

Gini's Mean Difference

Yitzhaki [225] introduced a new portfolio optimization model based on Gini's mean difference as a risk measure. The author claims that the new model is almost as simple as the Markovitz model. Also, it enables constructing stochastic dominance efficient portfolios, and has a simple geometric representation.

Corrado Gini introduced the Gini Index as a measure of the inequality among values of a frequency distribution, such as the welfare distribution of a society. Gini's mean difference is defined as the average of the absolute value of the difference between two independent values belonging to the same probability distribution.

In a general case, where no assumption is made on the asset return distribution, while $R_{i,j}$ is the value of the rate of return on asset i in observation j, and x_i is the share of asset i in the portfolio, let

$$y_j = \sum_{i=1}^{n} R_{i,j} x_i,$$

$$\delta_{i,j,k} = R_{i,j} - R_{i,k}, \text{ and}$$

$$y_j - y_k = \sum_{i=1}^{n} \delta_{i,j,k} x_i.$$

Then, Gini's mean difference (GMD) for a portfolio is:

$$GMD = \frac{1}{J^2} \sum_{j=1}^{J} \sum_{k=1}^{J} |y_j - y_k|.$$

Yitzhaki [226] provides the similarities and differences between the GMD and variance, then shows that GMD is a better risk measure than the variance in terms of stochastic dominance, exchangeability, and stratification. Both the variance and GMD can be defined without reference to a location parameter, are sensitive to all observation, can be represented graphically as the difference between the first moment distribution and cumulative distribution, can be represented as the weighted sum of adjacent observations, and can be calculated as the weighted sum of order statistics. On the other hand, the difference function of GMD is L1, whereas; that of the variance is L2. Moreover, GMD can be used to build necessary conditions for the second-degree stochastic dominance, whereas the variance cannot. Also, GMD obtains two correlation coefficients, which improve the comparison when the base of the comparison affects the direction of the results. For instance, while measuring the changes over a period, the result might change depending on the direction either from past to future or vice-versa. Besides, when subpopulations exist in the overall distribution, GMD is sensitive to this stratification [226].

Expected Regret

Expected regret is a risk measure which is the expected value of the loss distribution beyond a benchmark portfolio or a threshold ζ, i.e. Harlow [96] used the term "low partial moment" for naming the expected regret.

Let $f(\mathbf{x}, \mathbf{y})$ be a loss function with a decision vector of asset allocation $\mathbf{x}$ and a random vector $\mathbf{y}$ for asset returns, and let $p(\mathbf{y})$ be the joint density function where $\mathbf{y}$ is drawn. Then, the expected regret for a given threshold ζ is formulated as:

$$G_\zeta = \int_{\mathbf{y} \in \mathbf{R^m}} [f(\mathbf{x}, \mathbf{y}) - \zeta]^+ \mathbf{p}(\mathbf{y}) \, \mathrm{d}\mathbf{y}.$$

In the case of scenario approach, the expected regret can be used as a risk measure in a linear portfolio management problem as follows:

$$\min_{\mathbf{x}} \mathbf{p^T}[\mathbf{y} - \zeta]^+$$

where $\mathbf{p}$ is the vector keeps the probability of each scenario.

Maximum Loss-Minimax

Young [227] introduced "minimax" as a new data driven portfolio selection principle. The author describes the optimal portfolio selection as the combination of assets that minimizes the maximum loss while satisfying the target for the portfolio rate of return over the historical period. Let

n : the number of decision variables, e.g., assets,

T : the number of observations, e.g. time periods in historical data set,

$y_{i,t}$: the rate of return on one dollar invested in asset i in time period t,

$\bar{y_i}$: the average rate of return on asset i,

x_i : the portfolio allocation on asset i,

$y_{p,t}$: the return on portfolio in time period t,

$\bar{y_p}$: the average return on portfolio,

M_p : the minimum return on portfolio,

then, the portfolio optimization problem is formulated as:

$$\begin{aligned} \max_{M_p,x} \quad & M_p \\ \text{s.t.} \quad & \textstyle\sum_{i=1}^{n} x_i y_{i,t} - M_p \geq 0, t = 1, ..., T, \\ & \textstyle\sum_{i=1}^{n} x_i \bar{y_i} - G \geq 0, \\ & \textstyle\sum_{i=1}^{n} x_i - W \geq 0, \\ & x_i \geq 0, i = 1, ..., n, \end{aligned}$$

where G and W are the target for the mean of the portfolio return and initial investment budget.

Young [227] shows that when assets' return rates are normally distributed, the solution of the linear minimax problem is similar to the quadratic minimum variance problem with the same portfolio rate of the return target. Moreover, the author proves that under some certain distributions, such as the log-Normal distribution, the minimax is a better risk measure than the variance. However, the minimax rule is very sensitive to individual outliers in the historical data. In addition, Krokhmal, Uryasev and Zrazhevsky [121] suggest that if the returns are drawn from a continuous function, the minimax risk measure might be infinite unless the distribution is truncated.

Value at Risk (VaR)

Linsmeier and Pearson [129] explain the need for a measure like VaR by referring to the enormous volatility in exchange rates, interest rates, and commodity prices. Some major financial firms started employing VaR in late 1980s. VaR has been widely used since JP Morgan's attempt to standardize the risk measurement though out the market in 1994. For instance, Basle Committee on Banking Supervision (1996) lets banks calculate their capital requirements for the market risk according to their own VaR models, and the U.S. Securities and Exchange Commission (1997) suggests VaR as one of the three possible disclosure methods [129].

Rockafeller and Uryasev [174] define the β-VaR of a portfolio with a given probability β as the lowest value for α such that the loss will not exceed α with the probability of β. Let $\mathbf{x}$ be the decision vector selected from a certain subset $\mathbf{X}$ in R^n, vector $\mathbf{y}$ be the random return vector in R^m, and $f(\mathbf{x}, \mathbf{y})$ be the loss function associated with $\mathbf{x}$ and $\mathbf{y}$. In addition, the underlying distribution of $\mathbf{y}$ is assumed to have a density $p(\mathbf{y})$ just for convenience. Then, the probability of $f(\mathbf{x}, \mathbf{y})$ not exceeding the threshold level α is formulated as [174]:

$$\psi(\mathbf{x}, \alpha) = \int_{f(\mathbf{x},\mathbf{y}) \leq \alpha} p(\mathbf{y}) \, \mathrm{d}\mathbf{y},$$

and

$$\beta - VaR(x) = \alpha_\beta = \min\{\alpha \in R : \psi(\mathbf{x}, \alpha) \geq \beta\}.$$

VaR measures the downside risk, and it is applicable to nonlinear instruments such as options. However, it does not give any information about risks exceeding VaR [174]. In other words, it cannot tell if the losses that are worse than VaR are slightly worse or devastating. In addition, when the losses are not normally distributed, VaR is not stable. Moreover, VaR is not a coherent measure according to the consistency rules determined by Artzner, Delbaen, Eber and Health [6]. Actually, it is coherent only when it is calculated based on the standard deviation of normally distributed random numbers. Furthermore, VaR is non-sub-additive and non-convex.

That is, as the portfolio becomes more diversified, the overall risk might seem to be increasing. In addition, optimizing VaR requires long computation time when it is calculated according to scenarios.

In spite of the drawbacks mentioned above, VaR is still commonly used in practice.

Conditional Value at Risk -CVaR

Even though VaR is a very popular and simple measure, because of the drawbacks mentioned above, researchers sought an alternative method: conditional value at risk-CVaR. Embrechts, Küppelberg and Mikosch [74] introduced k-expected shortfall. Artzner, Delbaen, Eber and Helath [6] used the term conditional tail expectation. For continuous distributions CVaR is also known as the tail-VaR or tail conditional expectation or coherent-tail Var and it is calculated as the weighted average of losses exceeding VaR. However,, Rockafeller and Uryasev [174] calculates CVaR as the weighted average of VaR and the losses greater than VaR for general distributions, including discrete distributions.

Pflug [169] shows that CVaR is a coherent risk measure which is transition-equivariant, positive homogeneous, convex and monotonic with respect to the stochastic dominance of order 1. Rockafeller and Uryasev [174] propose a tractable method of optimizing the CVaR over a set of scenario and calculating VaR in a linear problem form. This work provides a convenient way of calculating and optimizing CVaR of the portfolios including linear and nonlinear derivatives; evaluating market, credit, and operational risks.

We focus on the quantile (VaR) management in the third chapter. In addition, we provide an extension to our study which calculates CVaR while eliminating the quantile value falling below a specified target and maximizing the expected portfolio return.

1.2.2 Portfolio Management with Quantile-Based Risk Measures

Rodriguez [175] provides an overview of optimization models with quantile-based functions. The author categorizes these techniques into three main groups, namely general, non-gradient-based, and gradient-based optimization methods.

General Quantile-Based Optimization ([175])

Rodriguez [175] formulates random function $\tilde{W}(x)$ as $\tilde{W}(x) = x'\tilde{b}$ where x is the investment allocation vector, and $\tilde{b}$ is the random parameter vector. The author suggests optimizing a general function, namely $Q(x)$, which involves linear combinations of quantile functions:

$$Q(x) = \sum_{i=1}^{k} \lambda_i q(\alpha_i, x) + H(x)$$

where $Q(x)$ is the weighted sum of k quantiles for different values of α_i and positive weights $\lambda_i, i = 1, ..k$, and $H(x)$ is an arbitrary concave function. $Q(x)$ is a general function which can represent different quantile-based risk measures such as VaR. If each quantile function $q(\alpha_i, x)$ is concave, then the function $Q(x)$ is also concave [175]. The gradient of Q is formulated as:

$$\Delta_x Q(x) = \sum_{i=1}^{k} \lambda_i \Delta_x q(\alpha_i, x) + \Delta_x H(x).$$

The author considers two cases :

- **Case 1:** The optimization of $Q(x)$ over a convex set C defined by a finite number of equalities or inequalities, such as:

  $$\max\ Q(x)\ s.t.\ x \in C.$$

  Then the problem can be solved by classical constrained nonlinear optimization techniques such as penalty-based constrained optimization.

- **Case 2:** The optimization of a convex function $G(x)$ over the intersection of the convex set C and convex set $Q(x) \geq L$ where L is a predefined constant, such as:

$$\max\ G(x)\ s.t.\ x \in C,\ x \in \{Q(x) \leq L\}.$$

These stochastic optimization problems need to be approximated. Simulation is a classical way of approximating. The random parameters can be replaced by artificially generated random variables or can be bootstrapped from a set of historical samples. Another simulation method is **the non-recursive method**, in which a sequence of random variables are generated such that the empirical measure

$$r\hat{h}o_n = \frac{1}{n}\sum_{i=1}^{n}\delta(\zeta - \zeta_i)$$

converges weakly to ρ_x, where $\delta(\zeta - \zeta_0)$ represents the point mass at the point ζ_0. Alternatively, **recursive simulation methods** involve one random sequence of approximate solutions (x_k), where the next sequence of approximate solutions (x_{k+1}) depends on x_k and the random sequence generated at step k. In addition, if the distribution function for the uncertain parameter is known to be a certain parametric form, we can reach the quantile function. Especially elliptic distributions provide convenience to approximate the quantile function to a closed form formulation ([175]).

Non-Gradient-Based Optimization Methods ([175])

Rodriguez [175] describes how the brute force method, a mixed integer programming, and the greedy linear programming methods solve the linear case for the quantile optimization problem where $\tilde{W}(x) = x'\tilde{b}$, and $E[\tilde{W}(x)]$ is to be maximized using both quantile constraints and the constraints forcing x to belong the polyhedral P.

Brute force method

Let us say Q_p is a predefined target for the quantile function which can be approximated to $\hat{q}_{\alpha,n}$ for a given probability level α and n possible random return scenarios, where $\hat{q}_{\alpha,n}$ is the $k-$th order of the statistic $x'\tilde{b}$ (for $k = \alpha n$). That is, quantile constraint assures that given n samples of the optimal vector x^*, $k-1$ samples will be less than a predefined value for Q_p. The problem is formulated as:

$$\begin{aligned} W_A = \max\nolimits_x \quad & E[\tilde{W}(x)] \\ \text{s.t.} \quad & Y_A \geq Q_p \mathbf{1}, \\ & x \in P, \end{aligned}$$

where the complement of set A, namely A', is a subset with $k-1$ samples from the set of n samples; **1** is a vector composed of ones, and

$$Y_A = (b_i'),\ \forall i \in A.([175])$$

The Brute force method requires solving this sub-problem for all $\binom{n}{k-1}$ subsets A. This method assures global optimal solution; however, it runs in exponential time. Therefore, this approach is applicable for small values of n and k. [175]

Mixed Integer Programming

The problem above can be modeled as a MIP problem as:

$$\begin{aligned} \max\nolimits_x \quad & E[\tilde{W}(x)] \\ \text{s.t.} \quad & Y_U x + c{\cdot}p \geq Q_p \mathbf{1}, \\ & x \in P, \\ & p'\mathbf{1} = k-1, \\ & p_i = \{0,1\}\ \forall i \in \{1,..,n\}, \end{aligned}$$

where U is the complete set of samples [175].

This method runs in exponential time and it becomes computationally very intense when n is large [175].

Greedy Linear Programming

According to Rodriguez [175], greedy heuristic is a very fast method to obtain approximate results. Let B_k be a sequence of sets each of which satisfies the quantile target enforced. Actually, the quantile target is planned to be satisfied by at least $k^* = \alpha n$ samples. The heuristic is an iterative algorithm which requires solving the following linear problem iteratively.

$$\begin{aligned} W_{B_k} = \max\nolimits_x \quad & E[\tilde{W}(x)] \\ \text{s.t.} \quad & Y_{b_x} \geq Q_p \mathbf{1}, \\ & x \in P, \end{aligned}$$

where $Y_{B_k} = (b'_k),\ \forall k \in A$. The algorithm is as follows:

- Step1: Set the iteration index to zero, $k = 0$ and solve the linear problem above for B_k. Note that B_0 includes all the available samples.

- Step2: Mark the sample which has the most negative dual variable, remove this sample from the B_k, and update the iteration index, $k = k + 1$.

- Step3: Continue with Step1 and Step2 as long as $k \leq k^* - 1$.

Since the greedy algorithm solves a linear problem at each iteration, it runs in polynomial time [175].

Gradient-Based Optimization Methods

Rodriguez [175] formulates the general stochastic quantile optimization problem as:

$$\max Q(x) \text{ subject to } x \in P$$

If a non-recursive approach is used to find an approximate solution to the problem, all the n samples are employed to obtain the estimators for $Q(x)$ and $\Delta_x Q(x)$. In this case, the estimator $\Delta_x \hat{Q}(x)$ will add a constant error, which is equal to the sum of the gradient estimator bias and zero mean random error due to the finite sampling since no new samples are introduced.

Therefore, if the non-recursive converges to a solution x^*, then it is actually the solution of

$$\max Q(x) + v'x \text{ subject to } \ x \in P. \text{ ([175])}$$

Thus, a reasonable strategy to improve the approximated solution could be keeping the error v as small as possible. In addition, validity of the gradient-based optimization method depends on whether the quantile functions are "well behaved" or not, i.e. if they are concave or not. ([175])

The CVaR calculation and optimization approach proposed by Rockafeller and Uryasev [174] is as fallows:

Let $\mathbf{x}$ be the decision vector selected from a certain subset $\mathbf{X}$ in R^n, vector $\mathbf{y}$ be the random return vector in R^m, and $f(\mathbf{x}, \mathbf{y})$ be the loss function associated with $\mathbf{x}$ and $\mathbf{y}$. In addition, the underlying distribution of $\mathbf{y}$ is assumed to have a density $p(\mathbf{y})$ just for convenience. Then, the probability of $f(\mathbf{x}, \mathbf{y})$ not exceeding a threshold level α is formulated as:

$$\psi(\mathbf{x}, \alpha) = \int_{f(\mathbf{x},\mathbf{y}) \leq \alpha} p(\mathbf{y}) \, \mathrm{d}\mathbf{y},$$

and for a specified probability level β, $\beta - VaR$ and $\beta - CVaR$, which are denoted by α_β and ϕ_β respectively, are calculated as:

$$\alpha_\beta(\mathbf{x}) = \min\{\alpha \in \mathbf{R} : \psi(\mathbf{x}, \alpha) \geq \beta\},$$

$$\phi_\beta(\mathbf{x}) = (1-\beta)^{-}1 \int_{f(\mathbf{x},\mathbf{y}) \geq \alpha_\beta(x)} f(\mathbf{x}, \mathbf{y}) p(\mathbf{y}) \, \mathrm{d}\mathbf{y}.$$

Rockafeller and Uryasev [174] define a function $F_\beta(\mathbf{x}, \alpha)$ on $\mathbf{X}$x$\mathbb{R}$ such that

$$F_\beta(\mathbf{x}, \alpha) = \alpha + (1-\beta)^{-1} \int_{y \in R^m} [f(\mathbf{x}, \mathbf{y}) - \alpha]^+ p(\mathbf{y}) \, \mathrm{d}\mathbf{y}, \text{ where } [\mathbf{t}]^+ = \max\{\mathbf{0}, \mathbf{t}\}.$$

$F_\beta(\mathbf{x}, \alpha)$ is convex function of $\mathbf{x}$; therefore, for the minimization problem whose objective function is $F_\beta(\mathbf{x}, \alpha)$ and the decision vector is $\mathbf{x}$, the only local minimum corresponds to the global minimum. Rockafeller and Uryasev [174] show that $\beta - CVaR$ of the loss associated with $x \in X$ is obtained by the formula:

$$\phi_\beta(x) = \min_{\alpha \in \mathbb{R}} \; F_\beta(\mathbf{x}, \alpha).$$

This problem leads to a set of consisting values of α for which the minimum is obtained, such as:

$$A_\beta(\mathbf{x}) = \text{argmin}_{\alpha \in \mathbb{R}} F_\beta(\mathbf{x}, \alpha).$$

Set A is a nonempty closed bounded interval and $\beta - VaR$ of the loss is

$$\alpha_\beta(x) = \text{left endpoint of} A_\beta(\mathbf{x}).$$

In particular, the authors reach that

$$\alpha_\beta(\mathbf{x}) \in \text{argmin}_{\alpha \in \mathbb{R}} F_\beta(\mathbf{x}, \alpha) \text{ and } \phi_\beta(\mathbf{x}) = F_\beta(\mathbf{x}, \alpha_\beta(\mathbf{x})).\text{[174]}$$

Moreover, the integral in the formulation of $F_\beta(\mathbf{x}, \alpha)$ can be approximated as:

$$\tilde{F}_\beta(\mathbf{x}, \alpha) = \alpha + \frac{1}{q(1-\beta)} \sum_{k=1}^{q} [f(\mathbf{x}, \mathbf{y_k}) - \alpha]^+.$$

The function $\tilde{F}_\beta(\mathbf{x}, \alpha)$ is convex and piecewise linear with respect to α. This approximation is not differentiable; however, it can be minimized by either a line search method or by a linear problem representation [174].

Therefore, Rockafeller and Uryasev [174] formulate the portfolio management problem which minimizes $\beta - CVaR$ of losses for a specified probability β as:

$$\begin{aligned} \min_{x,u} \quad & \alpha + \frac{1}{q(1-\beta)} \sum_{k=1}^{q} \mathbf{u_k} \\ \text{s.t.} \quad & \mathbf{x}^T \mathbf{y_k} + \alpha + \mathbf{u_k} \geq 0, k = 1, ..., r, \\ & \mathbf{u_k} \geq 0, \\ & \mathbf{x} \in X, \end{aligned} \tag{1.4}$$

where

- q : the total number of scenarios,
- $\mathbf{y_k}$: the return vector on one dollar invested in specified set of assets in observation k,
- $\mathbf{u_k}$: the auxiliary variable for observation k,
- $\mathbf{x}$: the investment decision vector,
- r : $\lfloor q * \beta \rfloor$, the number of loss scenarios falling above $(100^*\beta)^{th}$ percentile.

This optimization problem calculates $\beta - VaR$, which is equal to α, and optimizes $\beta - CVaR$ over a $\mathbf{x} \in \mathbf{X}$ at the same time. Even though it does not provide the optimal $\beta - VaR$ value, the authors claim that the portfolios with low CVaR necessarily have low VaR as well.

The most important contribution of Rockafeller and Uryasev [174] is optimizing the CVaR of a portfolio with a linear problem formulation. In addition, the authors show that, under the assumption of normally distributed asset returns, $\beta - CVaR$ and $\beta - VaR$ of loss can be expressed in terms of the mean ($\mu(\mathbf{x})$) and variance ($\sigma(\mathbf{x})$) by:

$$\alpha_\beta(\mathbf{x}) = \mu(\mathbf{x}) + \sqrt(2)\text{erf}^{-1}(2\beta - 1)\sigma(\mathbf{x}),$$

$$\phi_\beta(\mathbf{x}) = \mu(\mathbf{x}) + [\sqrt{2\pi} e^{(\text{erf}^{-1}(2\beta-1))^2}(1-\beta)]^{-1}$$

where $\text{erf}^{-1}(z) = \frac{2}{\sqrt{\pi}} \int_o^z e^{-t^2} \text{d}t$.

The authors claim that evidently when the constraint for the portfolio expected return target is active and the returns are normally distributed, minimizing either $\alpha_\beta(\mathbf{x})$ or $\phi_\beta(\mathbf{x})$ is equivalent to minimizing the variance ($\sigma(\mathbf{x})$) over $\mathbf{x} \in \mathbf{X}$.

A data-driven iterative VaR optimization algorithm introduced by Larsen, Mausser, and Uryasev [124] (is referred as Algorithm-A1) provides an approximated solution to the quantile optimization problem by iteratively solving a linear optimization problem which maximizes the CVaR of the portfolio return that was introduced by Rockafeller and Uryasev [174].

Konno, Waki and Yuuki [118] conclude that the linear CVaR optimization model of Rockafeller and Uryasev [174] can control the downside risk when the distribution of returns on instruments are not normal nor symmetric. Krokhmal, Palmquist and Uryasev [120] extend the linear formulation of Rockafeller and Uryasev [174] to the formulation with CVaR constraints and weighted return-CVaR performance function by considering transaction costs.

Bardou, Frikha and Pagès [8] focus on estimating VaR and CVaR using the stochastic approximation based on Rockafeller-Uryasev's identity for the CVaR and VaR. The authors believe that the main disadvantage of the Rocakfeller-Uryasev method of optimizing CVaR and calculating VaR is the fact that the number of quantile constraints of the linear program is equal to the number of scenarios. They suggest using Robbins-Monro approximation method to estimate both VaR and CVaR and apply a recursive and adaptive importance sampling to increase the convergence rate. The authors claim that a significant contribution of this approach is that only the quantiles of interest are predicted, not the whole inverse of the distribution function.

A. Balbàs and R. Balbàs and S. Mayoral [7] propose a general approach which minimizes several risk measures. First, a general risk minimization problem is transferred to a minimax problem. Then, this minimization problem is transformed to a couple of linear problems each of which is dual of the other one between infinite-dimensional Banach spaces of continuous functions and inner regular σ-additive measures. The authors provide necessary and sufficient optimality conditions for problems which do not use subgradients of the risk measure. When a portfolio choice problem with finite number of instruments is considered, the dual linear problem is usually a semi-infinite problem. Next, the extreme points of the dual problem of the main portfolio choice problem are determined. Then, a simplex-like algorithm is generated to solve it. This simplex-like algorithm leads to primal and dual optimal solutions. The authors provide application of the approach with different risk measures such as the standard deviation, usual dispersions, conditional value-at-risk, value-at-risk and distortion functions. Besides, the authors claim that the convergence rate of the algorithm is really fast.[7]

Shaw [199] points the difficulty of computing VaR and CVaR based on continuous distributions and proposes a method which performs this job by evaluating the functions of the moments for Student-t return distributions. The author's starting point is the existence of Student-t characteristics in daily log-returns of major indices. The author computes the inverse CDF of the Student-t distribution via inverse beta function representation, then formulates VaR and CVaR as a function of the mean, standard deviation and the inverse CDF function. The author represents the CDF of a general positive real v in terms of regularized β-functions as follows:

$$F_v(x) = \frac{1}{2}\left(1 + \mathrm{sgn}(x)(1 - I(\frac{v}{x^2+v})\left(\frac{v}{2},\frac{1}{2}\right)\right),$$

while regularized β-function $I_x(a,b)$ is given by:

$$I_x(a,b) = \frac{B_x(a,b)}{B(a,b)},$$

$B(a,b)$ is ordinary β-function, and $B_x(a,b)$ is

$$B_x(a,b) = \int_0^x t^{(a-1)}(1-t)^{(b-1)}\,\mathrm{d}t.$$

Then, the quantile function $Q(u,v)$ is formulated as:

$$Q(u,v) = \operatorname{sgn}\left(u - \frac{1}{2}\right)\sqrt{v\left(\frac{1}{I^{-1}_{\text{If}[u\leq\frac{1}{2},2u,2(1-u)]}\left(\frac{v}{2},\frac{1}{2}\right)} - 1\right)}.$$

VaR and CVaR for a specific probability level u are represented as:

$$VaR(u) = -\mu - \sigma\sqrt{\frac{v-2}{v}}Q(u,v), \text{ and}$$

$$CVaR(u) = -\mu + \sigma\sqrt{\frac{v-2}{v}}k(Q(u,v),u)\frac{1}{u}, \text{ where}$$

μ is the mean, σ is the standard deviation and $k(t,v) = \frac{v^{\frac{v}{2}}\Gamma\left(\frac{v-1}{2}\right)(v+t^2)^{\frac{1}{2}-\frac{v}{2}}}{2\sqrt{\pi}\Gamma\left(\frac{v}{2}\right)}$ [199].

According to Shaw [199], this method lets the analysts to examine the risk properties of the portfolio carefully based on the risk function, the return distribution, and the event frequency. In addition, with this new calculation approach, the portfolio optimization problem minimizing CVaR becomes a trivial moment-based optimization problem [199].

Zhu and Fukushima [230] deal with portfolio optimization when only partial information on the underlying loss distribution is available. The authors focus on minimizing the worst-case CVaR under the mixture distribution uncertainty, box uncertainty, and ellipsoidal uncertainty by robust optimization. The deterministic problem is formulated as Problem (1.4) of Rockafeller and Uryasev [174]. In addition, the authors provide the formulation of the robust portfolio optimization problem which maximizes the worst case portfolio return while satisfying the constraint on the worst case CVaR. According to the authors, their approach provides more flexibility in the portfolio decision analysis and leads to more robust solutions. However, they emphasize the importance of determining the uncertainty set for a successful practical application.

Clemente and Romano [56] use CVaR to measure the portfolio credit risk considering the non-normality of the credit loss distribution and multiple default event for credit assets. A Monte Carlo simulation is applied to construct the loss distribution of the loan portfolio with two states:

default and not default. Monte Carlo scenarios for time to default for each obligor are created according to the t-copula approach. Specifically, these scenarios are generated from a multi-variate distribution by assuming the dependence structure of the credit defaults are driven by Student's t-copula and exponentially distributed margins. Next, CVaR of the loan portfolio is minimized according to Rockafeller-Uryasev's [174] scenario based the linear programming approach.

Similarly, Deng, Ma and Yang [66] point out that the CVaR is easily affected by the tail distribution of the risk factors; therefore, they apply the extreme value theory (EVM) to model tails of the return more accurately. However, since the return series might not be independently and identically distributed, first a GARCH-based model is used to fit the return series, then EVT is applied to the innovations. In addition, the authors apply a copula approach in order to capture the nonlinear dependencies between tails of the asset returns. Then, the authors solve the linear CVar optimization Problem (problem 1.4) of Rockafeller and Uryasev [174] with the scenarios generated according to the proposed Copula-GARCH-EVT model.

Wozabal [223] formulates VaR as the difference between two CVaR. The author solves the portfolio optimization problem with a VaR constraint by the difference of convex (DCA) algorithm.

1.3 Log-Normal Sum Approximation in Portfolio Optimization

In this section, some log-Normal sum approximation methods are explained. The motivation results from the fact that empirical single stock return distributions are close to the log-Normal distribution. In the portfolio management framework, the portfolio return is not different from a linear combination of log-Normally distributed random variables under the assumption that single stock returns are log-Normally distributed.

1.3.1 Log-Normal Sum Approximation Methods

Signal shadowing in wireless communications and stock returns in finance are well modeled by the log-Normal distribution. Economics, reliability, biology, atmospheric sciences, geology and actuarial science are some of the other disciplines that use the log-Normal distribution. Since the log-Normal sum distribution is not known in closed form and is difficult to compute numerically, several approximation methods to log-Normal sum distributions have been developed.

Fenton [79] approximated the sum of log-Normal random variables with a log-Normally distributed random variable based on a moment matching approach. This approach is later called the Fenton-Wilkinson approximation since it was built upon Wilkinson's [222] idea of log-Normal sum approximation. The Fenton-Wilkinson method approximates the sum of N log-Normally distributed random variables L_i with a single log-Normally distributed variable Z by matching the first two moments of each. Pirinen [171] expresses these approximation as:

$$L = \sum_{i=1}^{N} L_i = e^{\sum_{i=1} Y_i} \cong e^{Z}, \tag{1.5}$$

where Y_i and Z are Gaussian random variables. In addition, the correlation coefficient of Y_i and Y_j can be written as:

$$\rho_{i,j} = \frac{E[(Y_i - m_{y_i})(Y_j - m_{y_j})]}{\sigma_{y_i}\sigma_{y_j}},$$

where m and σ correspond to the mean and the standard deviation of the indexed random variables.

Matching the first and second moments (u_1 and u_2, respectively) of Z and L enables us to obtain closed form expressions for the mean (m_z) and the standard deviation of Z (σ_z) [171].

Considering equation 1.5, the first moment is calculated as:

$$u_1 = E[L] = E[e^{Z}] = \sum_{i=1}^{N} e^{m_{y_i} + \frac{\sigma_{y_i}^2}{2}}. \tag{1.6}$$

The second moment is expressed as:

$$u_2 = E[L^2] = E[e^{2Z}] = \sum_{i=1}^{N} e^{2m_{y_i}+2\sigma_{y_i}^2} + 2\sum_{i=1}^{N-1}\sum_{j=i+1}^{N}\left(e^{m_{y_i}+m_{y_j}} e^{\frac{1}{2}(\sigma_{y_i}^2+\sigma_{y_j}^2+2\rho_{i,j}\sigma_{y_i}\sigma_{y_j})}\right). \quad (1.7)$$

Solving (1.6) and (1.7) leads to following expressions for m_z and σ_z:

$$m_z = 2ln\ u_1 - \tfrac{1}{2}ln\ u_2, \quad (1.8)$$

$$\sigma_z^2 = ln\ u_2 - 2ln\ u_1. \quad (1.9)$$

Given m_z and σ_z formulation above, the cumulative distribution function (CDF) can be written as:

$$Pr(L \geq \gamma) = Pr(e^Z \geq \gamma) = Pr(Z \geq ln\ \gamma) = \phi\left(\tfrac{ln\gamma - m_z}{\sigma_z}\right), \quad (1.10)$$

where $\phi(\cdot)$ is CDF of a zero mean, unit variance Gaussian random variable [158].

According to Beaulieu [13], the Fenton-Wilkinson approximation is valid only for a limited range of small values of σ_{y_i}. However, Beaulieu, Abu-Dayya, and McLane [158] mention that this does not imply that the Fenton-Wilkinson approximation to CDF is poor. In fact, the authors conclude that according to the results of the numerical studies, the Fenton-Wilkinson may provide a good estimate of CDF.

The Schwarts-Yeh (SY) approximation [189] is another method based on the assumption that power sum is log-Normally distributed. However, the first and the second moments of random variable Z are not calculated according the same assumption. The exact expression for the sum of two log-Normal random variables is calculated. The first two moments of the sum of more than two log-Normally distributed variables are calculated by a recursive algorithm assuming that the sum of two log-Normal random variables is also log-Normally distributed [158]. The calculations for the mean and standard deviation of the sum of two log-Normal random variables are as follows [171]:

$$
\begin{aligned}
m_z &= m_{y_1} + G_1, \\
\sigma_z^2 &= \sigma_{y_1}^2 - G_1^2 - 2\sigma_{y_1}^2(I_2 + I_0) + G_2, \\
G_1 &= E[ln(1+e^w)] = A_0 + I_1, \\
G_2 &= E[ln^2(1+e^w)] = I_3 + 2I_4 + \sigma_w^2 I_0 + m_w A_0, \\
G_3 &= E[(w-m_w)ln(1+e^w)] = \sigma_w^2(I_0 + I_2), \\
I_4 &= \sigma_w^2[f_w(0)ln2 - I_5] + m_w I_6, \\
A_0 &= \tfrac{\sigma_w}{\sqrt{2\pi}} e^{\frac{-m_w^2}{2\sigma_w^2}} + m_w I_0, \\
I_i &= \textstyle\int_0^1 h_i(v)v^{-1}dv, \\
h_0 &= \tfrac{1}{\sqrt{2\pi}} e^{-\frac{\left(ln\ v + \frac{m_w}{\sigma_w}\right)^2}{2}}, \\
h_1 &= [f_w(ln\ v) + f_w(-ln\ v)](ln\ (1+v)), \\
h_2 &= [f_w(ln\ v) - f_w(-ln\ v)](1+v^{-1})^{-1}, \\
h_3 &= [f_w(ln\ v) + f_w(-ln\ v)](ln^2\ (1+v)), \\
h_4 &= -f_w(-ln\ v)ln\ v\ ln(1+v), \\
h_5 &= f_w(-ln\ w)(1+v^{-1})^{-1}, \\
h_6 &= f_w(-ln\ v)ln(1+v), \\
f_w(w) &= \tfrac{1}{\sqrt{2\pi\sigma_w^2}} e^{\left(-\frac{(w-m_w)^2}{2\sigma_w^2}\right)}.
\end{aligned}
\tag{1.11}
$$

Applying the original SY approximation could be cumbersome because some complex calculations need to be done in order to reach the desired accuracy in digits [171]. Ho [99] provides a modified and simpler version of the SY method. Pirinen [171] mentions that, according to the literature, the SY method might underestimate the variance of the sum of log-Normally distributed random variables when the summation components are identically distributed. In addition, as more components are involved in the summation, this approximation error gets larger.

Beaulieu, Abu-Dayya, and McLane [158] observe that when the components in the log-Normal sum are uncorrelated, the cumulative distribution function derived by Ho's SY approximation seems to be close to that obtained by Monte-Carlo simulation, which represents the actual distribution. On the contrary, in the correlated components case, the Fenton-Wilkinson approximation seems to be capturing the cumulative distribution function of simulated points better than Ho's SY approximation method.

Schleher [186] assumes that the cumulative distribution function of the sum of log-Normal random variables is nearly log-Normal. The author uses a cumulants matching approach to approximate the cumulative distribution function of the sum of log-Normal random variables. The author shows that accurate approximation of the log-Normal sum CDF can be obtained for a large range ($10^{-1} - 0.9$). The approach suggests dividing the whole range into three or fewer sub-regions and finding the parameters of the new log-Normal distribution which are optimized on a local basis.

Farley's approximation is also used for approximating the cumulative distribution function of a log-Normal sum [158]. For t independent and identically distributed Gaussian random variables $(Y_1, ..., Y_t)$ each with the mean m_y and standard deviation σ_y, Farley's approximation is formulated as:

$$Pr(L \geq \gamma) \cong 1 - \left(1 - \phi(\frac{(ln\,\gamma - m_y)}{\sigma_y})\right)^t.$$

Beaulieu, Abu-Dayya, and McLane [158] compare the performance of the Fenton-Wilkinson's approximation, the Schwarts and Yeh's method, the cumulants matching approach of Schleher, and the Farley's method in terms of accuracy of their CDF approximation of independent log-Normally distributed random variables. According to the numerical experiments, the CDF approximation according to the Farley's method is closest to the simulated data for each considered

CDF value. The other three methods' relative performances change according to the σ value and CDF level.

Beaulieu and Xie [13] use a linearizing transform on the log-Normal sum distribution and apply linear minimax approximation in the transformed domain to obtain a log-Normal approximation to the log-Normal sum distribution. The authors aim to construct a log-Normal probability paper and obtain an optimal straight line approximation to sum distributions on the log-Normal probability paper on which the cumulative distribution function of the log-Normal distribution is represented as a linear line. Therefore, the process of finding an optimal straight line approximation on log-Normal paper, actually, minimizes the maximum error [13]. The authors, transform the CDF, $F(x)$, of a log-Normal sum distribution according to:

$$f(x) = \phi^{-1}(F(x)) = \phi^{-1}(F(e^t)), \tag{1.12}$$

where $\phi^{-1}(x)$ is the inverse function of standard normal CDF and $t = ln\ x$. Then,

$$f(t) = \phi^{-1}(F(e^t)) = \tfrac{1}{\sigma}t - \tfrac{m}{\sigma}, \tag{1.13}$$

where $f(t)$ is a linear function of t. The authors plot $t, f(t)$ pairs on the log-Normal paper and determine the cumulative distribution function of the log-Normal sum, $F(x)$, numerically according to 1.12. The straight line approximation, $\tilde{f}(t)$, on the log-Normal paper is represented as $\tilde{f}(t) = c_0 + c_1 t$. Then, the following problem is solved to obtain the optimal c_1 and c_0 values:

$$\min\nolimits_{c_0,c_1} \max\nolimits_{t\in[a,b]} \quad |f(t) - (c_0 + c_1 t)| \tag{1.14}$$

where $F(e^a) = 10^{-6}$ and $F(e^b) = 1 - 10^{-6}$. The existence of the solution is proved by Cheney [50]. The authors mention that the CDF of the log-Normal sum on log-Normal probability paper is concave with $f''(t) < 0$, and c_0 and c_1 are calculated as:

$$\begin{aligned} c_1 &= \tfrac{f(b)-f(a)}{b-a}, \\ c_0 &= \tfrac{1}{2}[f(a)+f(t_0)] - c_1\tfrac{a+t_0}{2}, \end{aligned} \tag{1.15}$$

where t_0 is the unique solution of $c_1 \cong f'(t_0)$. Therefore, the mean, m_z^* and the standard deviation σ_z^* of the optimal log-Normal approximation are represented as:

$$\begin{aligned} m_z &= -\tfrac{c_0}{c_1}, \\ \sigma_z &= \tfrac{1}{c_1}. \end{aligned} \tag{1.16}$$

The authors observe that neither the Schwarts-Yeh method nor the Fenton-Wilkinson method performs better than the minimax method, which provides a close approximation to the log-Normal sum distribution over the whole range.

The Schwarts-Yeh method, the Fenton-Wilkinson method, the Farley's approximation, and the minimax approximation to the CDF of the log-Normal sum proposed by Beaulieu and Xie [13] depend on the assumption that the sum of independent log-Normal random variables can be approximated by a single log-Normally distributed random variable. Beaulieu and Rajwani [12] propose a new method based on the representation of the distribution on the log-Normal probability paper. The authors observe that the proposed method, which provides a simple closed form for log-Normal sum distributions, is highly accurate. Let L_i be a log-Normal random variable. To show the classic approach, the authors define the Gaussian random variable X_i such that $X_i = 10log_{10}L_i$ where X_i has decibel units (dB), and has the PDF

$$f_{X_i}(x) = \tfrac{1}{\sqrt{2\pi}\sigma_{x_i}} exp\left(-\tfrac{(x-m_{x_i})^2}{2\sigma_{x_i}^2}\right). \tag{1.17}$$

Also, Y_i in equation 1.5 is defined as:

$$Y_i = ln\ L_i = \lambda X_i \tag{1.18}$$

where $\lambda = \frac{ln\,10}{10}$. The mean and the standard deviation of Y_i are written as follows:

$$m_{y_i} = \lambda m_{x_i}, \sigma_{y_i} = \lambda \sigma_{x_i}. \tag{1.19}$$

From (1.17) and (1.19), the PDF of L_i is given by

$$f_{L_i}(l) = \frac{1}{l\sigma_{y_i}\sqrt{2\pi}} exp\left(\frac{-(ln\ l - m_{y_i})^2}{2\sigma_{y_i}^2}\right) \tag{1.20}$$

and the CDF is represented as:

$$F_{L_i}(l) = \int_0^1 \frac{1}{t\sigma_{y_i}\sqrt{2\pi}} exp\left(\frac{-(ln\ t - m_{y_i})^2}{2\sigma_{y_i}^2}\right) dt \tag{1.21}$$

which is equivalent to:

$$\phi\left(\frac{lnl - \lambda m_{x_i}}{\lambda \sigma_{x_i}}\right) \tag{1.22}$$

Beaulieu and Rajwani [12] observe that the CDF of log-Normal sums generally does not lead to a straight line CDF representation on the log-Normal paper. This implies that this CDF of log-Normal sum is not really log-Normal. This observation denies the validity of the classic approach described above and the others which are developed based on this assumption such as the Schwarts-Yeh method, the Fenton-Wilkinson method, the Farley's approximation, and the minimax approximation to the CDF of the log-Normal sum proposed by Beaulieu and Xie [13]. However, Beaulieu and Rajwani [12] realize that the sum distributions are not log-Normal (straight lines on log-Normal paper); however, they are smooth and convex curves which become increasingly convex as the number of components in the sum, N, increases. Therefore, the authors approximate this convex curve (on the log-Normal paper), to the function represented as:

$$f(\gamma) = a_0 - a_1 e^{-a_2\gamma}, \tag{1.23}$$

where $a_0,\ a_1,$ and a_2 are parameters to be determined. Therefore, the CDF according to approximation is written as:

$$Pr(Z \leq \gamma) = \phi(f(\gamma)) = \phi(a_0 - a_1 e^{-a_2\gamma}) = \phi\left(f\left(\frac{ln\ z}{\lambda}\right)\right) = \phi(a_0 - a_1 z^{\frac{-a_2}{\lambda}}), \tag{1.24}$$

In this way, a closed form representation is obtained. In addition, the results of the numerical experiments support that this approximation method is highly accurate when values of $a_0,\ a_1,$ and a_2 are appropriately selected. The authors use the least squares method to determine these parameters' values. The numerical tests run, when the parameters are determined according to the least squares method, imply that the approximation method is highly accurate.

Almhana, Wang, and McGorman [2] state that numerical results show that (1.24) approximates log-Normal distributions with very high accuracy; however, if $a_2 \geq 0$, the k^{th} moment of (1.24) exists only if $k \leq a_2$. The authors include that the fitted values of parameter a_2 are usually in the interval (0,1) because of practical interests. Therefore, the approximations based on (1.24) do not have finite mean or higher order moments. As an alternative method, the authors suggest a two-component mixture log-Normal model to approximate log-Normal sum distributions. Recall that, if $x > 0$, then the PDF log-Normal function is written as:

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma_x} exp\left(-\frac{(x-m_x)^2}{2\sigma_x^2}\right). \tag{1.25}$$

Then, a two-component mixture log-Normal model can be written as:

$$f_X(x) = \alpha_1 f_{X_1}(x) + (1-\alpha) f_{X_2}(x),\ \alpha \in (0,1), \tag{1.26}$$

where

$$f_{X_i}(x) = \frac{1}{\sqrt{2\pi}\sigma_{x_i}} exp\left(-\frac{(x-m_{x_i})^2}{2\sigma_{x_i}^2}\right),\ i \in \{1,2\}. \tag{1.27}$$

Similarly, the cumulative distribution function is represented as:

$$F_X(x) = \alpha_1 F_{X_1}(x) + (1-\alpha)F_{X_2}(x),\ \alpha \in (0,1), \tag{1.28}$$

where $F_{X_i}(x) = \phi\left(\frac{(ln\ x - \lambda m_{x_i})}{\lambda\sigma_{x_i}}\right)$.

The authors use (1.28) to approximate the cumulative distribution function of the log-Normal sum and call the CDF to be approximated as $G(x)$. Next, in order to capture the tails of $G(x)$, they solve the following non-linear optimization problem:

$$min_{\alpha,\theta}\left\{sup_x\left(\phi^{-1}(G(x)) - \phi^{-1}(F_X(x))\right)^2\right\}. \tag{1.29}$$

The CDF $G(x)$ does not have a closed for representation, so it needs to be solved numerically. Instead of matching the whole CDF region, the authors prefer to match only K quantile points since it enhances simplicity and improves the approximation accuracy at these quantile points. In addition, in practice, quantiles represent the probability levels that is critical. The authors formulate the problem as:

$$min_{\alpha\in(0,1),\theta}\ U(\alpha,\theta), \tag{1.30}$$

where

$$U(\alpha,\theta) = max_{1\le k\le K}\left\{\left(\phi^{-1}(G(x_k)) - \phi^{-1}(F_X(x_k))\right)^2\right\}. \tag{1.31}$$

The authors mention that, numerical experiments show that (1.30) is sensitive to γ, so they develop and use an algorithm which is a combination of the Nelder-Mead Algorithm [153] and one dimensional search. The algorithm is stated by the authors [2] as: “

Algorithm 1.2

Step 1 *Initialization:* $\alpha_0 = 0, \bar{\alpha} = 1, m = 10, \epsilon = 0.001$.

Step 2 *Set* $\alpha_t = \alpha_0 + t(\bar{\alpha} - \alpha_0) \setminus m, t = 0, ..., m$. *For each t, find* $\theta_t = argmin_\theta\{U_{\alpha_t,\theta}\}$ *using the Nelder-Mead algorithm, and let* $(\alpha_{\tilde{t}}, \theta_{\tilde{t}}) = argmin\{U(\alpha_t, \theta_t), 0 \leq t \leq m\}$. *Let* $[\alpha_0, \bar{\alpha}] = [\alpha_0, \alpha_1]$ *if* $\tilde{t} = 0$; $[\alpha_0, \bar{\alpha}] = [\alpha_{m-1}, \alpha_m]$ *if* $\tilde{t} = m$; *otherwise,* $[\alpha_0, \bar{\alpha}] = [\alpha_{\tilde{t}-1}, \bar{\alpha}_{\tilde{t}+1}]$.

Step 3 *If* $|\bar{\alpha} - \alpha_0| \leq \epsilon$, *then stop; otherwise, go to Step 2. "*

According to the simulation results, the mixture log-Normal model proposed by Almhana, Wang, and McGorman [2] provides a very accurate approximation to log-Normal sum distributions.

Pratesi, Santucci and Graziosi [172] focus on the moment matching approximation method for the sum of log-Normal random variables since it provides a simple and closed form expression for the parameters of the approximated log-Normal random variable. The authors propose an approximation method which generalizes the Fenton-Wilkinson method so that the first n, $(n \geq 2)$ moments are matched during the approximation process. As it was shown earlier in (1.8) and (1.9), according to the Fenton-Wilkinson method, the mean m_z and the standard deviation σ_z of the approximated log-Normal random variable (Z) are represented as :

$$\begin{aligned} m_z &= 2ln\ u_1 - \tfrac{1}{2}ln\ u_2, \\ \sigma_z^2 &= ln\ u_2 - 2ln\ u_1. \end{aligned} \tag{1.32}$$

where u_1 and u_2 are the first and second moments. Pratesi, Santucci and Graziosi [172] provide extensions of the Fenton-Wilkinson method with higher order moments. For instance, when the second and the third orders are matched for the approximation, then the mean and the standard deviation of Z are represented as:

$$\begin{aligned} m_z &= \tfrac{3}{2} ln\ u_2 - \tfrac{2}{3} ln\ u_3, \\ \sigma_z &= \sqrt{-ln\ u_2 + \tfrac{2}{3} ln\ u_3}. \end{aligned} \tag{1.33}$$

Alternatively, if the third and fourth moments are matched, then

$$\begin{aligned} m_z &= \tfrac{4}{3} ln\ u_3 - \tfrac{3}{4} ln\ u_4, \\ \sigma_z &= \sqrt{-\tfrac{2}{3} ln\ u_3 + \tfrac{1}{2} ln\ u_4}. \end{aligned} \tag{1.34}$$

The authors mention that according to the numerical results, the approximation method is quite accurate.

Berggren [23] provides a closed form formulation for the error bound ($\Delta(\gamma)$) of the distribution function for the Fenton-Wilkinson method such as:

$$\Delta(\gamma) = \left(\sum_{k=0}^{2} a_k \gamma^{-k} \right)^{-1}, \tag{1.35}$$

where

$$\begin{aligned} a_0 &= u_2 \setminus (u_2 - u_1^2), \\ a_1 &= -2u_1 \setminus (u_2 - u_1^2), \\ a_2 &= -1 \setminus (u_2 - u_1^2). \end{aligned} \tag{1.36}$$

The error bound is a function of the parameters $\gamma, \sigma,$ and N since, in the tail, the error bound $\Delta(\gamma)$ approaches to a_0^{-1}, ($\lim_{\gamma \to \infty} \Delta(\gamma) = a_0^{-1}$), which decreases as N increases. Therefore, the error bound gets tighter as N increases in the tail [23].

Nie and Chen [157] focus on approximating the log-Normal sum with type-IV Pearson distribution by matching mean, variance, skewness and kurtosis. Later, Nie, Chen and Ayers-Glassey [183] propose a more accurate method which is a variant of the previous approach.

Zhao and Ding [229] apply the least squares linear and the least squares quadratic approximation approaches to model the sum of log-Normal random variables as a single log-Normal random variable. Li, Wu, Chakravarthy and Wu [133] suggest a low complexity approximation method (the log skew Normal approximation) to approximate the log-Normal sum distribution. The moment matching technique was used to determine the parameters of the approximation.

1.3.2 Portfolio Management with Moment Matching Approach

The log-Normal approximation methods have also been studied with financial engineering applications. Hakala and Wystup [94] use the Fenton-Wilkinson method in valuation of Foreign Exchange Basket Options. Basket options are European options based on a common base currency and a set of foreign currencies. The option value is determined by the difference between the basket value and the strike price at the expiration day. Each single correlated component of the basket is modeled as a log-Normal process. Therefore, the basket spot is not different than the sum of log-Normally distributed random variables. The mean and the standard deviation of the basket spot is determined by matching the first two moments. Next, well known the Black-Scholes-Merton method is applied to price the basket option. The authors extend their approach by introducing one more term, which is calculated by matching the third moment of the market spot and the model spot within the Ito-Taylor expansion of the basket spot [94].

Henriksen [98] also applies the Fenton-Wilkinson method to calculate the price of a basket option composed of two correlated components. Similar to Hakala and Wystup [94], Henriksen [98] approximates the sum of two coupled log-Normal variables by matching the mean and the variance with one dimensional log-Normal variable. The author mentions that the moment matching approach provides an accurate approximation as long as the correlation between log-returns of the two log-Normal variables is non-negative.

1.4 Revenue Management and Customized Pricing

1.4.1 Revenue Management and Pricing

Netessine and Shumsky [156] define yield management as a set of tools used in many service industries to describe techniques to allocate limited sources among a variety of customers such as the airplane seat allocation to the business and leisure classes. The firm can optimize the total revenue or yield on investment in capacity by adjusting this allocation by using the help of pricing. Phillips [170] classifies pricing and revenue optimization as a tactical function. The main target of pricing and revenue management is providing a guidance on how prices should change in a dynamic environment by analytical techniques derived from the management science.

The history of revenue management with analytical tools goes back to 1980s. Passenger airlines and hotel chains are the first business groups implemented revenue management. According to Boyd [38], revenue management techniques build up an annual revenue increase of $500 million and $300 million for American Airlines and Delta Airlines respectively. According to Phillips [170], development of e-commerce and availability of customer data through customer relationship management (CRM) let revenue management techniques expand into other industries such as automotive, retail, telecommunications, financial services, and manufacturing. Geraghty and Johnson [87] describe the importance of the revenue management through the success story of National Car Rental. National Car Rental, which faced a liquidation in 1993, implemented a revenue management program to manage the capacity, pricing and reservation in 1993, and improved its revenues by $56 million in 1994.

Netessine and Shumsky [156] mention cases where firms employ yield management as follows:

- Storing excess resource is expensive or impossible: Tonight's room for use cannot be stored for tomorrow night's customers.
- Commitments need to be made while the future demand is uncertain: Some seats must be

set aside business customers without knowing the future demand for business class seats.

- Customers are divided into segments and each segment has different demand characteristics. The demand curve of leisure customers is very sensitive to price while that of business customers is not.

- The same unit of capacity can be used to deliver many products or services: Rooms are the same; however, they can be used by both leisure and business travelers.

- Producers are profit-oriented and have a broad freedom of action: A hotel can withhold rooms from current customers for future profit; however, such practices are illegal and immoral in emergency wards.

According to Phillips [170], the problem of pricing had not been observed until the modern market economies arose in the West in 17th and 18th centuries since the prices used to be set by custom or by law or by imperial fiat until industrialization started. However, a debate on the theory of value, or price, was begun by Aristotle who developed an input-based theory of value. Aristotle claims that all things ought to be valued by land and labor. Sir William Petty addresses the discrepancy between the market price and the natural price. Market prices are subject to effects of several dynamics and are difficult to be theorized whereas natural prices can be theorized by some market fundamentals. The determinants of natural price change within the Classical School. For instance, Adam Smith claims that the natural value is determined by labor, profit, and rent. However, Ricardo believes that the rent should be price determined instead of price determining.

Phillips [170] claims that the most important insight of classical economists was that the price of a good in a capital economy is determined by not only any intrinsic value but also by the interaction between the supply and the demand. Indeed, the price of a good is determined by the interaction between people willing to sell the good with the willingness of other people to buy the good. Of course, the value of the good for possible buyers and the value (cost) of the good for the sellers are also effective. [170]

The classical economics could explain the origin of the pricing; however, they left some questions unanswered such as the price stability without an anchor and how an economy with unregulated prices can work. According to Phillips [170], one of the most important achievements of 20^{th} century neoclassical economics is showing how an unregulated market work under the assumption of the perfect market competition. In such a market, there is no pricing decisions since the prices are determined by the iron law of market. For instance, if a supplier sells a good at a higher price than the market price, customers abandon him/her and do not return until he/she decreases the price level to market (equilibrium) price. Alternatively, if he sells a lower price than an equilibrium price, then arbitragers buy at this price and sell at a higher price until the price of the good reaches the market price.

However, the perfectly competitive market structure assumption holds if the following conditions are satisfied:

- Many firms are active in the market and each firm has an insubstantial share in the market.
- Each firm produces identical products (with the same quality) through the same processes.
- Each firm possesses perfect information.
- Firms can enter the market for free if the other firms in the market obtain higher profits than normal profits.
- Firms are price takers and they can sell as much as they can produce.

Actually it is impossible to satisfy all these conditions in real markets. In addition, the tools that are employed for pricing are more likely to be selected from statistics and operations research rather than economics. These tools compromise quantitative analysis of marketing initiatives such as predicting market response, forecasting sales, product planning, pricing, promotions, sales, and marketing strategy. However, due to the fact that pricing decisions have become more tactical and operational in nature, the gap between this managerial science theory and its applications in practice has been growing. Companies need to update their pricing decisions so rapidly in order to respond to competitive actions or changes in market dynamics. This necessity points out the importance of revenue management and pricing. The success of revenue management in several industries, developments in enterprise resource planning (ERP) and consumer relationship management (CRM) software systems, and the rise of e-commerce and improvements in analytic supply chain management software systems encourage companies to be involved in the pricing and revenue management activities. [170]

According to Marn and Rosiello [141], pricing is the most effective and fastest way to maximize the profit of a company. In addition, according to Phillips [170], pricing is often the area which could be improved the most with the least investment.

1.4.2 Traditional Pricing Approaches

Cost-Plus Pricing

The cost-plus pricing method determines the price level as summation of cost of each product and a profit margin which ensures that the firm obtains a target rate of return. This way of determining prices seems fair and applicable. However, it doesn't account for the market competition and customers' preferences. In addition, it doesn't consider the differences among firms' cost structures. In addition, Phillips [170] states that cost determination for an item compromises many subjective judgements. Therefore, it might yield to highly distorted prices. Dolan and Simon [67] claim that the cost-plus pricing is not an acceptable method. According to Drury [68], the cost-plus pricing method has some advantages such as being easy to implement and encouraging price stability by enabling firms to predict the competitors' prices. Actually, this approach is useful in public-utility pricing. In addition, according to the survey conducted by Drury and Tayles [69] on 187 UK organizations, 60 % of the survey participants use cost-plus pricing.

Market-Based Pricing

The market-based pricing method is also known as the competition-based pricing strategy. In this pricing strategy, firms determine their price levels according to the prices of similar products in the market. For instance, a local car producer, can adjust its price according to the price of another car with similar properties. According to Phillips [170], the market-based pricing might be an effective strategy for a low-cost supplier intending to enter a new market. This pricing method is applicable to commodity markets where each firm in the market is a price taker. However, this approach keeps the firms away from considering the changing value perceptions of the customers. [170]

Value-Based Pricing

The value-based pricing method suggests that the price level of a service or a good should be determined based on the value that the customer assigns to it. According to Phillips [170], the value-based pricing is used as synonym for 'personalized' or 'one-on-one' pricing in which each customer is offered a different price based on his/her value for the product. The value of a customer for a product is determined by customer surveys, focus groups and conjoint analysis. However, determining the customer values appropriately is difficult. In addition, charging different prices for the same product introduces the arbitrage risk. Moreover, the market competition will force suppliers to set lower prices than those that they actually assume.

1.4.3 Stated Preference Models and Methods

According to Lauviere, Hensher and Swait [125] individuals' choices are determined by some factors, such as habit, inertia, experience, advertising, peer pressure, environmental constraints, accumulated opinion, household, and family constraints, etc. Lancaster [123] defines consumption as an activity in which goods are considered as inputs, and the output is a set of characteristics. Furthermore, the author suggests that the consumer makes decision based on the utility which ranks the goods according to the characteristics that they possess. Lancaster's study stands for a remarkable divergence from the traditional consumer behavior theory where goods are the main sources of the utility. Rosen [178] diverges from Lancaster on the assumption of divisibility, and develops a model for the indivisible (discrete) goods. Lauviere, Hensher and Swait [125] suggests a "modified Lancester-Rosen" model by assuming that individuals consume commodities as services provided by the commodities. That is:

$$u = U(s_1, s_2, s_3, ..., s_K)$$

where s_k is the kth consumption service amount that is obtained by consumption of the commodity k, where $k \in \{1, 2, ..., K\}$.

If the uncertainty in the service supplied by commodities is considered, then the utility function that the customer use to make decisions can be represented as:

$$u = U(se_1, se_2, se_3, ..., se_K)$$

where se_k is the expected amount of kth consumption service that the consumer enjoys by the consumption of the commodity k, where $k \in \{1, 2, ..., K\}$.

Actually, the analyst is not able to reach the same information level as the individuals while they make decisions. Therefore, the utility function from the point of view of the analyst is written as:

$$u = U((se_o + se_{uo})_1, (se_o + se_{uo})_2, (se_o + se_{uo})_3, ..., (se_o + se_{uo})_K)$$

where subscripts o and $u0$ represents observed and unobserved portion of the utility by the analyst [125]. Therefore, a good consumer choice model should capture not only the observed terms in the utility function but also the structure of the unobserved terms.

Discrete choice methods have been widely used by researchers to examine the consumer choices. Logit, multi-nominal logit, nested logit, probit, and mixed logit are the major discrete choice models. In the following section, each of these models will be briefly described.

Logit Model

According to Train, [114] the logit model is the easiest and the most commonly used discrete choice model. It was first introduced by Luce [132] in 1959.

As mentioned earlier, the utility obtained from consuming a product is composed of two terms: observed and unobserved components. Let us call the utility that the customer n obtains from the product j $U_{n,j}$. It is decomposed into the observed utility ($V_{n,j}$) and the unobserved utility ($\epsilon_{n,j}$), that is

$$U_{n,j} = V_{n,j} + \epsilon_{n,j}. \tag{1.37}$$

The logit model assumes that each $\epsilon_{n,j}$ is an independently and identically distributed extreme value. The independence assumption could be interpreted as that each unobserved utility portion $\epsilon_{n,j}$ is not related to the unobserved utility portions obtained from any other consumption alternatives [114]. The probability that the customer n selects the alternative j is represented as:

$$P_{n,j} = Prob(V_{n,j} + \epsilon_{n,j} \geq V_{n,k} + \epsilon_{n,k},\ \forall k \neq j) \tag{1.38}$$

After some calculations and algebraic manipulations, this probability is written as [114]:

$$P_{n,j} = \frac{e^{V_{n,j}}}{\sum_k e^{V_{n,k}}} \tag{1.39}$$

The deterministic utility portion ($V_{n,j}$) is usually represented as a linear function of the observed variables ($x_{n,j}$ such that:

$$V_{n,j} = \beta' x_{n,j} \tag{1.40}$$

where β could be estimated trough logistic regression models.

Train [114] mentions some other desirable properties of the logit model:

- Each choice probability $(P_{n,j})$ is necessarily between zero and one.
- The sum of each choice probability $(P_{n,j})$ is equal to one.
- The relation of the choice probability $(P_{n,j})$ to the observed utility $(V_{n,j})$ is sigmoid, or S-Shaped. That is, the effect of change in observed utility has little impact on the probability if the utility level is very low or very high. Change in the observed utility has the highest effect on the probability level when it is in moderate levels, or the probability level is around 0.5.
- The logit model captures the systematic taste variations that are directly related to the observed characteristics of the consumer. However, it cannot explain the random taste variations which result from unobserved characteristics.
- The logit model depends on the assumption that the unobserved utility portions are independently and identically distributed. Therefore, it cannot perform well if the factors affecting unobserved utility portion are correlated.

Nested Logit Model

The Generalized Extreme Value (GEV) models assume that unobserved utility portions $(\epsilon_{n,j})$ are jointly distributed as a generalized extreme value. Therefore, the situations where the factors affecting the unobserved utility portion are correlated can be handled by the GEV models. The most popular GEV model is the nested logit model [114].

According to Train [114], a nested logit model should be used if the decision alternatives can be partitioned into two subsets (nests), where

- Within a nest, the ratio probabilities of two alternatives is independent of the attributes or the existence of all the other alternatives.

- For any two alternatives belonging to different nests, the ratio of the of the probabilities might depend on the attributes of the other alternatives in these two nests.

These assumptions could be interpreted as the unobserved utility portion $\epsilon_{n,j}$ is correlated with $\epsilon_{n,m}$ if the alternatives m and j are in the same nest B_k; otherwise, they are uncorrelated. The parameter $\lambda_{n,k}$ is introduced to reflect the degree of independence in the unobserved utility components among the alternatives in the nest B_k. A higher value of λ_k is interpreted as less correlation and higher independence within the nest B_k.

Since the nested logit models assume that each unobserved $\epsilon_{n,j}$ is a univariate extreme value, the cumulative distribution for $\epsilon_{n,k}$ is formulated as:

$$exp\left(-\sum_{k=1}^{K}\left(\sum_{j\in B_k} e^{\frac{\epsilon_{n,j}}{\lambda_{n,k}}}\right)^{\lambda_{n,k}}\right). \tag{1.41}$$

After the necessary calculations are applied, the probability of the customer n's selecting the alternative j belonging to nest B_k is written as:

$$P_{n,i} = \frac{e^{V_{n,j}\backslash\lambda_k}\left(\sum\limits_{i\in B_k} e^{V_{n,i}\backslash\lambda_k}\right)^{\lambda_k - 1}}{\sum\limits_{l=1}^{K}\left(\sum\limits_{i\in B_l} e^{V_{n,i}\backslash\lambda_l}\right)^{\lambda_l}}. \tag{1.42}$$

Train [114] summarizes the nested logit model as a generalization of the logit model where the correlation among the unobserved utility portions are handled in a particular pattern.

Probit Model

The nested logit model considers only the correlation among the unobserved utility terms in a particular manner; however, it still cannot explain the cases where the unobserved factors are correlated over time and express the random taste variation. The probit model is introduced to satisfy these needs.

The probit model depends on the assumption that all the unobserved utility components are distributed according to a normal distribution [114].

The choice probability is represented as:

$$\begin{aligned} P_{n,j} = \quad & Prob(V_{n,j} + \epsilon_{n,j} > V_{n,k} + \epsilon_{n,k} \, \forall k \neq j) \\ & \int I((V_{n,k} + \epsilon_{n,k} > V_{n,k} + \epsilon_{n,k} \, \forall k \neq j)\phi(\epsilon_n) d_{\epsilon_n} \end{aligned}$$

where $\epsilon_n \sim N(0, \omega)$.

Since the integral in Equation (1.43) does not have a closed form expression, a probit model does not provide a closed form expression for the choice probability.

Mixed Logit Model

According to Train, [114] the mixed logit model is developed to handle the three issues that the standard logit model cannot, which are

- Random taste variations,
- Unrestricted substitution patterns,
- The correlation among the unobserved factors over time.

Even though the probit model could explain these issues, it is restricted by the jointly normal distribution assumption. However, the mixed logit model allows the analyst to use any distribution for the customer taste parameters (β) while the unobserved utility components are identically and independently distributed random variables.

A mixed logit choice probability is calculated based on the standard logit choice probability formulation and the distribution of parameters. In fact, it is the integral of its formulation according to the standard logit model over a density of the unobserved utility components [114]. That is, given that the standard logit probability choice formulation is as:

$$P_{n,j}(\beta) = \frac{e^{V_{n,j}(\beta)}}{\sum_{i=1}^{I} e^{V_{n,i}(\beta)}}$$

and the density function for the consumer preference parameters , (β), is represented as $f(\beta)$, the mixed logit choice probability ($\tilde{P_{n,j}}$) is represented as [114]:

$$\tilde{P_{n,j}}(\beta) = \int \frac{e^{V_{n,j}(\beta)}}{\sum_{i=1}^{I} e^{V_{n,i}(\beta)}} f(\beta) d(\beta). \tag{1.43}$$

Random taste modeling, unrestricted substitution patterns, and correlation in unobserved utility components can be expressed by the stochastic customer preferences parameters (β) with a probability density function $f(\beta)$.

1.4.4 Customized Pricing with Discrete Choice Models

Hanson and Martin [95] address the concavity of the revenue function when the probability of purchase is determined by a multi-nominal logit function and difficulty of obtaining a closed-form optimal solution. The authors claim that if the demand curve is sufficiently elastic in price, the second order conditions for concavity fail. Therefore, the authors propose a path following approach which relies on perturbing not-concave profit function to a concave one by artificially making the choices less responsive to the product attributes and prices. Agrawal and Ferguson [1] address the need for price optimization methods in a business-to-business setting where a customer requests bids from a group of firms and makes his/her decision. The authors use the logit model to determine the probability of winning a bid opportunity (bid-response function) considering the price related and non-price related factors. The authors suggest methods to extend the logit bid-response function modeling approach to the cases where market segmentation and market competition are also considered. Broder and Rusmevichientong [44] also use the logit function to predict the bid-response function. The authors formulate the problem from a monopolist's point of view which offers price to the customers whose decisions depend on the price while the parameters of the logit model is unknown. The authors minimize the regret, which is the difference between the revenue obtained under perfect information on the logit model parameters and the revenue obtained when the logit parameters are uncertain.

Talluri and Ryzin [209] propose a dynamic programming model for a network revenue management problem where demand and capacity are modeled as continuous random variables, and buyers' choice behavior is represented by the multi-nominal logit model. The proposed method requires determining and ranking efficient sets which are composed of possible fare products to offer. Rusmevichientong, Shmoys, and Topaloglu [180] propose an assortment optimization problem where multiple customer classes are considered, and demand of each class is modeled by the multi-nominal logit model. Rusmevichientong and Topaloglu [181] apply robust optimization to handle the uncertainty in the parameters of the multi-nominal logit model of the demand. The authors apply their price optimization approach in both static and dynamic setting.

1.5 Real Options

1.5.1 Introduction

Real options have been the focus of significant research interest in the financial economics literature since they were first introduced by Myers [149]. Myers addresses the gap in the finance theory with respect to the corporate debt policy and introduces the analogy between call options and corporate investment opportunities. Trigeorgis [216] combines his own contributions to the flexibility in capital budgeting decisions with those of previous researchers such as Louis Bachelier, Samuelson Fisherback, Myron Scholes, Robert Merton, and Steward Myers. Trigeorgis [216] discusses a wide range of real option types with their applications in corporate finance. Copeland and Antikarov [58] and Rogers [177] describe various methods of pricing and implementing real options on a more practical level for corporate finance managers. Later researchers have extended the application areas of real options and improved real options valuation techniques.

Traditional Capital Budgeting

In this section, we explain some traditional capital budgeting methods.

Discounted Cash Flow Analysis

Capital budgeting focuses on the allocation of limited resources among possible investment projects. The main target is maximizing the firm's market value which is a function of the profitability of the firm's projects. Net present value (NPV), payback period, accounting rate of return, and internal rate of return are some of the traditional measures which use discounted cash flow (DCF) analysis to evaluate the project's profitability. According to Trigeorgis [216], the net present value analysis is widely regarded as being the most accurate one among these measurements.

According to DCF, the NPV of a project with a discount factor r and a stochastic cash flow C_t, whose expected value is $\mathrm{E}[C_t]$, is calculated as:

$$NPV = \sum_{t=0}^{N} \frac{\mathrm{E}[C_t]}{(1+r)^t},$$

where t and N stand for the discrete time and life time of the project, respectively. Let us consider an investment project whose expected cash flow is - \$1000, \$200, \$650 and \$500 in the first, second, third and fourth year, respectively, while the life time of the project is four years and the discount rate is 5%. Then, the NPV of the project is calculated as:

$$-\$209.481 = \frac{-1000}{1.05} + \frac{200}{(1.05^2)} + \frac{650}{(1.05)^3} + \frac{500}{(1.05)^4}.$$

The project is rejected because its NPV is negative.

The DCF methods disregard the effect of managerial control during the lifetime of the project. They assume that managers do not revise their decisions regarding the project. In fact, the market is dynamic and subject to multiple uncertainty sources. Therefore, as Trigeorgis [216] points out, managers do update their decisions according to information revealed up to that point so that they can defer, extend, or abandon the project during its life time.

In addition, the NPV approach calculates the project value based on an "expected cash flow" scenario by assuming that the cash flow structure of the project is static. Trigeorgis [216] asserts that this assumption may lead to an unrealistic project valuation especially when the probability distribution of the project return is asymmetric. Because of the skewness, absolute value of a loss realization can be much higher than a profit realization. According to Neely and Neufville [152], this problem can be addressed by reforming the analysis according to different cash flow scenarios. The authors include that this approach might not be practical.

Uncertainty, which results from the sources such as effective tax rate, inflation rate, and the project's time life can be captured by defining a risk adjusted discount factor as in the capital asset pricing model (CAPM). Then, the value of the project equals to the sum of the expected value of the future net cash flows discounted by the risk adjusted rate, as Fama suggests [76]. According to CAPM, risk adjusted risk factor r' is defined as:

$$r' = r_f + \beta(r_m - r_f),$$

where r_f is the risk free interest rate, r_m is the expected market return, and the beta of the project is formulated as

$$\beta = \text{Cov}(r', r_m)/\text{Var}(r_m).$$

The NPV approach assumes that the beta of the project stays the same during the project's life time. However, in practice the beta of the project can change over the time. Moreover, it considers neither the market competition nor the interaction between different projects. For instance, the rival firms' reaction to an R&D project implementation might affect not only the R&D project's but also the other ongoing projects' cash flow structure.

Sensitivity Analysis

Sensitivity analysis is another traditional capital budgeting technique. As mentioned above, the present value of a project depends on the estimated values of several factors such as the project's life time, the cash flow structure, the risk free rate, the market rate etc. In the words of Trigeorgis [216]:

> "Sensitivity analysis is the process of delving into these forecasts to identify the key primary variables and determining the impact upon NPV of a given variation in each key variable at a time, with other variables held constant; it's sometimes called "what if " analysis, since it addresses questions of the form "What is the consequence or effect on the investment decision (NPV) if there is an error or mis-estimation of the variable x by a certain amount, assuming the other variables are estimated correctly?". When the sensitivity analysis determines the critical variables, more time and effort can be spent to improve the accuracy of these variables' estimations. However, if these variables are interdependent, then sensitivity analysis may not give realistic insights."

Traditional simulation techniques, including Monte Carlo simulation, are applied to determine the probability distribution of the NPV of the project by repeatedly sampling randomly from the probability distributions of the crucial variables leading to the calculation of the net cash flows for each period according to previously embedded set of mathematical equations. However, according to Trigeorgis [216], reflecting interdependencies between primary variables through their probability distributions with high accuracy is a complex task. In addition, the simulation gives the risk profile of the NPV without an exact discount factor value. Therefore, conclusions obtained based on traditional simulation analysis are questionable.

Decision-Tree Analysis

The decision-tree analysis (DTA) has been employed to value a project in the presence of uncertainty and the possibility of decision deferral. DTA represents a project as a sequence of decisions and possible realizations of chance events with real probabilities in a tree structure during the lifetime of the project. It expresses the interdependency between decisions given at different time points and the effect of different realizations of chance events on the cash flow structure of the project. This eases the management's task to visualize the project's inherent options and price them into the NPV of the project. However, according to Trigeorgis [216], as the number of decisions or chance event realizations increase, the number of possible paths in the decision tree increases geometrically, which converts the decision-tree analysis to the decision-bush analysis. In addition, DTA uses the same discount factor during the life time of the project and neglects the dynamic nature of the riskiness of the project [216]. According to Schulmerich [188], updating the discount factor based on available information at each time period could help overcome this problem; however, this idea is hard to implement in practice.

As an example, let us consider an R&D project for an electronic device, which requires an initial investment of $10,000. The project is expected to be successful with 30% probability at the end of the first year. Production set-up costs are expected to be $20,000 in the second year. In the third year, the firm expects to get a high, or medium, or low level demand which leads to $40,000, $25,000, and $10,000 of total profit, respectively. The discount factor is assumed to be 5%. The case is represented as a decision tree as follows:

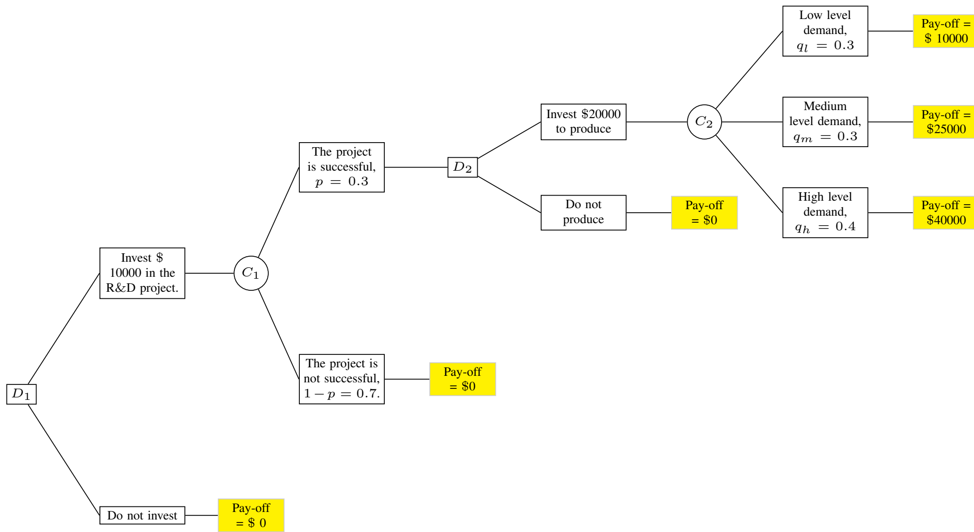


Figure 1.1: Analyzing an R&D project via the decision tree framework

$$\text{the NPV at } \ C_2 = 25,238 = \frac{0.4{*}40,000 + 0.3{*}25,000 + 0.3{*}10,000}{1.05},$$

$$\text{the NPV at } D_2 = 4,030 = max(0, \frac{25,238}{1.05} - 20,000),$$

$$\text{the NPV at } D_1 = 0 = max(0, \frac{4,030{*}0.3}{1.05} - 10,000).$$

So, the optimal decision is not to invest according to the decision tree analysis.

Contingent-Claim Analysis

The contingent-claim analysis (CCA) was first introduced by Trigeorgis [216]. The motivation for CCA is described in words of Trigeorgis:

> "The fundamental problem with the traditional approaches to capital budgeting lies in the valuation of investment opportunities whose claims are not symmetrical or proportional. The asymmetry resulting from operating flexibility options and other strategic aspects of various projects can nevertheless be properly analyzed by thinking of discretionary investment opportunities as options on real assets (or as real

options) through the technique of contingent claim analysis."

Contingent-claim analysis views real investment opportunities as a collection of options on real assets. Trigeorgis [216] overcomes the discount rate problem of DTA by financial options valuation approach on the basis of no-arbitrage equilibrium. CCA constructs an analogy between financial options and operating options. For instance, the holder of an American option on an asset owns the right but not the obligation to buy an asset at a predetermined price (exercise price) on or before a predetermined day (exercise day). Similarly, the owner of a discretionary investment opportunity has the right to gain the gross present value of expected cash flows by making an investment defrayal on or before the date until the investment opportunity is predicted to be available. The analogy between a call option on a stock and a real option on a project is summarized in Table 1.1 as:

Table 1.1: The analogy between a call option on a stock and a real option on a project [216]

Call Option on stock	Real option on project
Current Value of stock	Gross PV of expected cash flows
Exercise Price	Investment Cost
Time to expiration	Time until opportunity vanishes
Stock value uncertainty	Project Value Uncertainty
Risk free interest rate	Risk free interest rate

Trigeorgis constructs a twin portfolio of traded securities by issuing bonds to replicate the payoff of options and to calculate risk neutral probabilities from real probabilities. By this way, a constant risk free discount rate can be used in the decision tree [216]. Since a real option's underlying asset is not usually tradable, a twin portfolio which imitates the risk and payoff structure of the investment with the real option is constructed from tradable securities and issued bonds. Then, the risk adjusted discounted rate is calculated as in the financial options pricing approach.

Schulmerich [188] provides a detailed explanation of the CCA through an example. The following notation will be used in the example describing the project valuation by the CCA approach.

V : Project's overall value,

S : Price of the security which is perfectly correlated with the project,

n : Number of twin security in the twin portfolio,

B : Amount of bond issued to construct the twin portfolio,

P : Twin portfolio which replicates the payoff structure of the project,

k : Return of the twin security,

r : Risk-free interest rate,

ρ : Risk-neutral probability for upward movements of V and S in each period,

q : Real probability for upward movements of V and S in each period,

u : Return of V and S in the case of a upward movement in each period,

d : Return of V and S in the case of a downward movement in each period.

First, the twin portfolio has to be constructed by buying n twin securities at the price S and issuing B amounts of bond. Let us assume

$V =$ 100,

$S =$ 40,

$r =$ 5%,

$q =$ 0.6,

$u =$ 1.2,

$d =$ 0.8.

Then, two possible realizations of the twin portfolio are shown as:

Following equations make the twin portfolio generate the same payoff structure with the project:

$$V = nS - B,$$

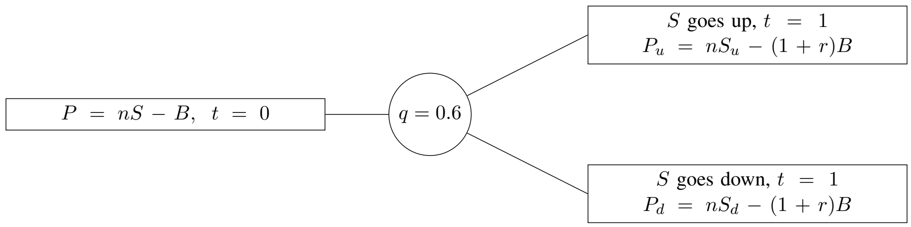


Figure 1.2: Possible values of the twin portfolio in the next period

$$Vu = nSu - (1+r)B = P\_u,$$

$$Vd = nSd - (1+r)B = P\_d.$$

Solution of the system is as follows:

$$n = \frac{P\_u - P\_d}{Su - Sd},$$

$$B = \frac{SdP\_u - SuP\_d}{(Su - Sd)(1+r)},$$

$$P = \frac{\rho P\_u + (1-\rho)P\_d}{1+r}$$

$$\rho = \frac{(1+r) - d}{u - d},$$

$$k = \frac{qSu + (1-q)Sd}{S} - 1.$$

These formulations with the parameters specific to this example yield to:

$$\rho = 0.625,$$

$$V = 100,$$

$$B = 0,$$

$$n = 2.5.$$

Once the risk-neutral probability ρ is obtained, the risk-free interest rate r can be used as a discount rate in the CCA framework, where chance events are defined in terms of risk-neural probabilities.

If we assume that the project has an investment cost of \$90, the investment cost of the deferred project will be \$94.5 ($= 1.05^{*}90$) in the following year. The CCA framework of the project is represented as:

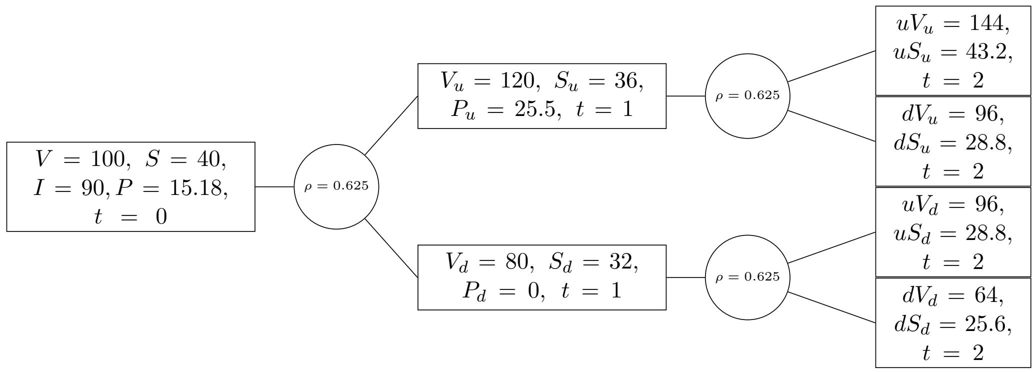


Figure 1.3: Analyzing an investment project with an option to defer within the CCA framework

Since the investor has an option to defer the investment, project value at the beginning of time horizon is calculated as:

$$P\_u = max\{120 - 94.5, 0\} = 25.5,$$

$$P\_d = max\{80 - 94.5, 0\} = 0,$$

$$P = \frac{\rho P\_u + (1-\rho) P\_d}{(1+r)} = \frac{0.625 * 25.5 + 0}{1.05} = 15.178.$$

The value of the project with option to defer is determined to be \$15.178 according to the CCA approach.

Trigeorgis [214] applies the CCA to the valuation of lease contracts with a variety of embedded operating options such as options to buy, sell, and renew. His study also examines a case with multiple interacting real options.

The CCA can be regarded as an improved version of the DTA in terms of accounting for strategic options; however, it is not free of limitations. Trigeorgis [216] lists the most important weaknesses of the analogy between real options and call options as:

- Exclusiveness of ownership and competition interaction,
- Non-tradability and preemption,

- Across-time interdependencies and option compoundness.

1.5.2 Real Options

According to Insead and Levinthal [105] the real options framework is built upon the realization that future investment opportunities are functions of their prior investment decisions. However, Schulmerich [188] points out that the real options approach is not applicable in every investment situation. Amram and Kulatilaka [3] present a list of situations where the real options approach is applicable as follows (in their words):

- *When there is a contingent investment decision and no other approach can correctly value this type of opportunity,*
- *When uncertainty is large enough that is sensible to wait for more information, avoiding regret for irreversible investment,*
- *When the value seems to be captured in possibilities for future growth options rather than current cash flow,*
- *When uncertainty is large enough to make flexibility a consideration. Only the real options approach can correctly value the investment in flexibility,*
- *When there will be project updates and mid-course strategy corrections.*

They also classify investments from the real options point of view. Irreversible investments, flexibility investments, insurance investments, modular investments, platform investments, and learning investments are some of these investment types. These investment categories are consistent with Trigeorgis's [213] real options classification:

- **Option to defer** provides the flexibility of delaying the initiation of an investment according to available information. For instance, managers can postpone opening a new production plant if a financial crisis, which shrinks the demand, occurs.

- **Time-to-build option** can be regarded as a combination of several options to abandon having consecutive exercise times. At each single decision point, the management has an opportunity to quit the project based on market conditions or the investor's interest. For instance, opening a tobacco production plant decision can be reviewed several times even after a series of outlays. If, at some point, the government applies high tax on cigarette or forbids smoking in public areas, continuing to build a new tobacco production plant might not be profitable, then the management quits the project.

- **Option to alter operating scale** lets managers adapt the scale of production according to the changes in factors affecting the profitability of the project. For instance, managers of the tobacco firm operating in the country, where the tax rate on cigarette has risen, can decrease the production amount as a response to decreased demand.

- **Option to abandon** gives the opportunity to abandon the project permanently if the market conditions deteriorate severely, e.g. managers quit the new tobacco production plant project because of high tax rate.

- **Option to switch** enables the management to modify the output mix of facility (product flexibility) when the price or demand changes. Alternatively, the same outputs can be produced using different types of inputs (process flexibility). For instance, because of increased gasoline prices, demand for big cars might reduce. Therefore, managers of a car production plant might decide on shifting the production from high gasoline consuming cars to low gasoline consuming ones.

- **Growth options** can be considered as inter-project compound options. An early investment or outlay can be regarded as prerequisite for the following investment opportunities. If the market conditions are promising, then later projects can be implemented. For example, let us consider an Italian pasta producer which enters the Turkish market by selling only pasta. If the demand for pasta is high enough, then the managers might consider introducing canned pasta sauce for Turkish consumers.

- **Multiple interaction options** is a combination of upward potential-enhancing and downward protection options. Generally the combined value of multiple interacting options is different from the sum of their separate values. For instance, the Italian pasta producer might have an option to built a production plant in Turkey in addition to an option to switch the production scale between Italy and Turkey.

Insead and Levinthal [105] explore the boundary between real options and sequential stream of path dependent investment decisions based on the flexibility of the market and technical agenda. Authors claim that as the flexibility of markets and technical agenda increases, decision activities turn into path dependent "probe and learn" activities.

1.5.3 Real Options Valuation Methods

Schulmerich [188] provides a detailed review of real options valuation methods. The author classifies these methods into analytical and numerical methods, then summarizes contributions of pioneer researchers as below:

Analytical Models

Trigeorgis [216] states that several researchers have applied financial option pricing methods in order to obtain analytic models for valuation of real options in recent years.

Option to defer: McDonald and Siegel [145], and Paddock, Siegel and Smith [163] focus on option to defer. McDonald and Siegel model the gross project value $(V_t)_{t\geq 0}$ by a diffusion process given via SDE

$$dV_t = \alpha V_t dt + \sigma V_t dB_t,\ t \geq 0,\, \alpha \in R^+, \sigma \in R^+,$$

where α is the instantaneous expected return on the project and σ is the instantaneous standard deviation. Paddock, Siegel and Smith value the option to defer the project, which has a payout rate D, with the SDE

$$dV_t = (\alpha - D)V_t dt + \sigma V_t dB_t,\ t \geq 0,\ \alpha \in R^+, \sigma \in R^+$$

Option to abandon: McDonald and Siegel [144] model the unit output price's diffusion process $(P_t)_{t\geq 0}$ as follows:

$$dP_t = \alpha P_t dt + \sigma P_t dB_t,\ t \geq 0,\ \alpha \in R^+, \sigma \in R^+.$$

While Myers and Majd [150] apply the following process:

$$dP_t = \alpha(D - P_t)dt + \sigma P_t dB_t,\ t \geq 0,\ \alpha \in R^+, \sigma \in R^+,$$

where D is the instantaneous cash payout or dividend.

Option to switch: Margrabe [138] values an option to exchange one risky asset for another with the same diffusion process for each asset's price, V and S respectively, but with different coefficients.

$$dV_t = \alpha_1 V_t dt + \sigma_1 V_t dB_t,\ t \geq 0,\ \alpha_1 \in R^+, \sigma_1 \in R^+,$$

$$dS_t = \alpha_2 S_t dt + \sigma_2 S_t dB_t,\ t \geq 0,\ \alpha_2 \in R^+, \sigma_2 \in R^+.$$

According to Schulmerich [188], analytical methods can value a single real option; however, they cannot account for the interaction between several real options properly. Therefore, analytical methods are not capable of valuing complex real options. In addition, analytical methods depend on the assumption that describing partial differential equations can be written with the underlying stochastic process. However, this assumption is not always valid in practice.

Numerical Models

Approximation of the partial differential equations

Finite difference and numerical integration methods are considered as approximation of

PDEs. Trigeorgis [216] provides a detailed review of partial differential approximation approach. Parkinson [166] works on numerical integration. Brennan [41], and Brennan and Schwartz [42] explore implicit and explicit finite difference schemes. Borone-Adesi and Whaley [37] apply quadratic approximation.

Approximation of the underlying stochastic process

The Monte Carlo simulation of Boyle [39], in addition to several lattice approaches like binomial approximations of Cox, Ross and Rubinstein [62], Hull and White [102], and Trigeorgis [212] are considered in this group. The Monte Carlo simulation and lattice approaches can be used to price both American and European type options.

Trigeorgis [212] develops the log-transformed lattice approach with constant risk free interest rate. This method is claimed to be a consistent, stable, and efficient binomial tree method which can value complex investments with interacting real options. According to Trigeorgis, lattice approaches are superior to Monte Carlo simulation in terms of simplicity and flexibility in handling different stochastic processes, options payoffs, early exercise of the other intermediate decisions (interaction), etc. In addition, they can handle real option packages and compounding real options. However, Schulmerich [188] points out that the lattice approaches value the option for only one underlying start value at each time and this requires running all steps several times with various starting points, which is time consuming.

Schulmerich [188] modifies the binomial tree approach of Cox, Ross, and Rubinstein [62] and the log-transformed binomial tree approach of Trigeorgis [212] in order to be able to value real options under stochastic interest rates.

Real Options Valuation Methods with Different Applications

Investment decisions are subject to multiple uncertainty sources such as project life time, interest rate, currency rate, market share, oil prices, etc. Managerial control during the lifetime of the project is a frequently applied tool to employ the new information in favor of the shareholders. Busby and Pitts [49] interview with several finance officers about the occurrence of different types of flexibility in their capital expenditure projects and summarize their results as:

Table 1.2: Frequency of occurrence of types of flexibility in capital investments [49]

Frequency (%)	Postponement	Abandonment	Rescaling	Growth	Technical Change
0-20	21	49	30	14	43
21-40	16	28	23	21	29
41-60	16	9	16	12	12
61-80	16	9	16	28	10
81-100	30	5	14	26	7

Real options have been the focus of significant research interest since they fill the gap between traditional capital budgeting techniques and the presence of managerial control in practice. R&D projects, mergers and acquisitions, product development, strategic investment, supply chain management, revenue management and pricing, and commercial lease contracts are some of the areas that real options approach is employed as a decision making tool.

Lease Contracts and Real Options

Trigeorgis [215] evaluates lease contracts with operational options such as option to buy, cancel, and renew, by contingent claim analysis. The author suggests a CCA-based numerical analysis for leasing contracts with multiple interacting options. Grenadier [92] develops an endogenous process for rent, supply and asset values by considering fundamental economic uncertainty and the market competition. These processes determine the entire term structure of lease rates. The model is flexible to determine the equilibrium rate for leases of different structures such as forward leases, adjustable rate leases, leases with options to cancel or renew and leases with payments contingent on the intensity of the asset's usage. Buetow and Albert [46] model the market price of a real-estate and its rental rate via both Geometric Brownian Motion and mean reverting processes. Using no-arbitrage assumption and a variant of riskless-hedge portfolio, authors obtain the system of stochastic differential equations, whose solution is approximated by FDM. Finally, values of the option to renew the lease at a rent indexed to CPI and the option to purchase the leased space at a price indexed to CPI are determined.

Market Competition and Real Options

Recently, researchers have begun to combine market competition and real options approach. Ferreira, Kar and Trigeorgis [80] provide a toolkit for the strategic investment decisions in a competitive environment. Grenadier [93] presents a continuous time model which prices real estate leases with competitive interactions. Schwartz and Torous [191] test implications of Grenadier's real estate lease pricing model. Results of the study suggest that competitive nature of the local real estate market affects the number of new building starts. Cunningham [63] provides robust empirical evidences claiming that existence of real options delay investments increases land prices in King County, Washington. Bulan, Mayer, and Somerville [47] provide the first study that differentiates impacts of market and idiosyncratic risk on real options and investment decisions. Authors define the competition as number of potential competitors in the market and show empirical results claiming that the competition has an insignificant effect unless it interacts with the volatility. In addition, authors show that the effect of idiosyncratic risk on development decreases as the competition increases, which means that real option exercise decisions become more robust when the market is more competitive. Also, the study asserts that increase in both idiosyncratic and market risk encourage investors to postpone real estate developments.

The real options approach with competitive interactions has been studied in other application areas as well. Folta and Miller[82] focus on real options to strategy in buyouts and equity purchases of partner biotechnology firms. The authors consider two different types of buyout events: capturing the majority stake and capturing an additional stake while the firm also has previously acquired stakes. For each case, hazard rate, which is a measure for the effect of option exercise decision, is modeled separately. The study shows that when uncertainty level is low, the number of equity partners increases likelihood of partial buyouts decreases. On the other hand, in the presence of high uncertainty, a higher number of equity partners leads to an increase in the rate of additional acquisition.

Kim and Sanders [115] provide a low technical level framework of strategic actions considering competitors' reaction based on real options approach. Savva and Scholtes [185] combine the cooperative game theory and the real options theory. In the presence of uncertainty and complete market assumption, the effect of the European type cooperative options, which are exercised in the interest of partnership, and the non-cooperative options, which are exercised by a single party based on its own interest, on the partnership synergies is analyzed by using a dynamic programming model. Smit and Trigeorgis [203] integrate the real options approach with the game theory principles to evaluate corporate investment opportunities under uncertainty. The authors compare two strategies: competition and strategic alliance, through an example in consumer electronics market and provide a number of insights. Thijssen [210], and Kong and Kwork [117] focus on two players real options game with cases player-specific uncertainty, and asymmetric sunk cost and revenue flows, respectively.

R&D Projects and Real Options

R&D projects have been the most common application area for real options. Benaroch and Kauffman [21] analyze a case for evaluating information technology project investments by using the real options approach. Authors apply the Black-Sholes option pricing method for the exercise time of an deployment option of POS debit services by Yankee 24. Panayi and Trigeorgis [165] consider R&D projects with multi-stage decisions as compound options which are combinations of sequential investment options. Each of these sequential call options is valid at one of the three main stages: research, technical construction-development, and implementation-commercialization. Two examples from the real life: information technology infrastructure project of a telecommunication firm and an international expansion project of a bank are evaluated by decision tree analysis with the real options approach.

The book Real R&D Options edited by Paxson [167] provides seventeen articles written by various researchers on real R&D options from a wide perspective including real R&D options with learning and real R&D options under incomplete information. Neely and Neufville [152] propose a hybrid method to evaluate product development projects. The authors assume that there are two main sources of uncertainty: market dynamics, which could be reflected by financial options theory, and technical difficulties, which could be expressed by decision tree analyses. Therefore, the hybrid method suggests inserting chance events reflecting market conditions into decision tree which already includes chance events expressing technical difficulties and investment decision nodes. Next, the project value is calculated by discounting project values at each node, which is determined based on option exercising decisions, by risk adjusted discount rate. The discount rate is calculated by the financial options approach. Finally, the effects of assumptions and parameters on the project value are checked by the sensitivity analysis.

Tsui [219] applies real options to value an innovative R&D project in the automotive industry. First, uncertain demand is predicted by the Monte Carlo simulation. Then, a linear optimization model is solved to obtain the optimal product portfolio for cases with and without the innovative product at each decision node. The difference between the profit amounts promised by each case determines the decision to exercise the option. Optimal exercise time is obtained by the backward recursion method. Bekkum, Pennings, and Smit [15] analyze portfolios of R&D projects by the real options approach. The authors show that if the projects are positively correlated, diversification is an effective tool for reducing the risk. On the other hand, strategies such as synergies and spill overs should be considered rather than diversification under negative correlation. In addition, they observe that if high-risk projects are considered in the portfolio, then the overall portfolio risk is less sensitive to the correlation.

Real Options with Other Applications

Brosch [45] formulates the real options portfolio selection problem as a stochastic mixed integer problem with dynamic budget and path dependency constraints while the objective function is the expected value of the optimal real options exercise policy. The model accounts for managerial flexibility, inter-project and intra-project options interactions. The model is solved according to a simultaneous forward and backward looking procedure which introduces path dependency and backward recursion. However, due to the complexity of the problem, a closed form solution cannot be obtained.

Chow and Regan [54] propose a model to determine the value of a network design deferral option (NIDO). The basic approach assumes that the solution of the network design problem as an investment, whereas NIDO is a deferral option on this investment. However, problems with different assumptions are also considered. The optimal option exercising time is obtained by solving a dynamic program with the network design subproblem according to the least squares-Monte Carlo simulation algorithm. The model handles network investment under uncertainty; however, it requires to network design subproblem to be solved at each time period. In addition, accurate estimations of necessary parameters in the stochastic demand process might be difficult.

Graf and Kimms [135] employ an option-based procedure for the capacity control problem for the strategic alliance of two airlines while the main decision is the number of seats allocated to booking classes of each airline in the alliance. Miller and Bertus [147] argue the applicability of the real options approach to license valuation in the aerospace maintenance, repair, and overhaul industry.

Madlener and Stoverink [136] value a coal-fired power plant investment project by the real options theory considering the market liberalization. Cheng, Lo and Lin [51] use compound real options for cleaner energy development projects considering the lead time for power plant investments and demand uncertainty.

Childs, Riddiough and Triantis [53] examine the effect of mixed uses and redevelopment options on the property value. The model assumes that one of the two possible uses is active at a moment. The instantaneous cash flow is represented as a function of the net instantaneous cash flow per unit of improved properties and land usages. Property valuation is obtained by the finite difference method. The authors provide mixed use and redevelopment options examples on undeveloped and re-developable property. They observe that the contribution of the flexibility to the property value is higher when the correlation between cash flows of asset usage types and/or the re-development cost are lower.

Healthcare is another application area where the real options approach is becoming more popular. The survey by Hartmann and Hassan [97] supports this idea. Özgül, Karsak and Ethem [162] build a model to value a real world hospital information system (HIS) project with compound options. The authors define HIS as a customized and upgraded enterprise resource planning (ERP) system which supports strategic service offering, resource and supply chain planning, collaborative care support, patient management, enterprise management, and support capabilities. The binomial lattice model is applied to real options pricing. The sensitivity analysis supports that the method is robust against the uncertainty in parameters and interaction between options. In addition, Palmer and Smith [164] use the real options approach to evaluate an irreversible healthcare technology investment decision with options to defer. The authors also address the applicability of the real option approach at the microlevel of the individual patient treatment in which uncertainty and reversibility is observed.

New research partnership models between pharmaceutical companies and universities where pharmaceutical companies collaborate with the universities in new product development processes have become more popular in the pharmaceutical industry. Kinase Consortium at University of Dundee, Scotland, which is funded by an industry consortium on a five-yearly basis; Centre for Drug Research and Development, Canada, which is funded by The Province of British Columbia, some charitable foundations, and Pfizer Research; Imperial College Drug Discovery Centre, UK; and Broad Institute, Cambridge,MA, USA, are some of the examples that Chesbrough and Schwartz [52] mentioned. In addition, Washington University and Pfizer signed a five-year $22.5 agreement in 2010 which brings scientist from both institutions to conduct research jointly on a wide range of disease areas where the university has a significant scientific expertise including Alzheimer's disease, cancer, diabetes, asthma, and chronic obstructive pulmonary disease in order to develop new drugs. Moreover, AstraZeneca and Vanderbilt University are involved in a similar partnership with the purpose of developing new treatments for major brain disorders in January, 2013.

Traditional business models for R&D projects suggest developing a product from internal technology and handling the production, marketing and selling processes using the internal sources ([52]). The innovative research partnership models provide the university research labs with the research funding, proximity to the real data, and expertise through a collaboration with the pharmaceutical companies. Similarly, the pharmaceutical companies benefit from sharing the high risk in R&D projects with the research labs and reallocating their research staff to some other research projects.

In the traditional R&D project management models, the pharmaceutical company bears all of the costs and risks from the first stage. However, in the innovative research partnership models the pharmaceutical company (generally) makes some upfront payments to the university research lab to finance the research activities, milestone payments when the drug completes a stage, or a phase in the clinical trials stage; and royalty payments when the product is successfully commercialized and being sold in the market. The university research lab conducts the research and discusses its results with the pharmaceutical company. The pharmaceutical company has the right to quit the partnership and stop funding the research if it thinks that the results obtained are not promising (or because of any other reason). If the pharmaceutical company dissolves the partnership, then the research lab can search for another partner to conduct the research together.

The fact that the new innovative research partnership models allow the pharmaceutical companies to dissolve the partnership at certain points during the project life time provides them with managerial flexibility. Vanhaverbeke et al. [220] mention that the pharmaceutical companies benefit from delayed financial commitment, early exits reducing the downward losses, and delayed exit in case it spins off a venture if they are involved in R&D projects in the open innovation framework. In addition, Vanhaverbeke et al. [220] claim that this type of research partnerships thus can be considered as a series of real options where the firms have the option to terminate the partnership at each state. A real option is “the right, but not the obligation, to take an action in the future” ([4]).

Real Options with Parameter Uncertainty

Uncertainty involving real life systems reveals itself in various stages of the decision making process such as valuation of input parameters and determining possible outcomes of decision alternatives. In this section of the survey, we focus on the real options literature addressing parameter uncertainty.

Parameter Uncertainty in Revenue Management

The dynamic nature of the customer demand, row material and commodity prices, and exchange rates and shifts in market competition in supply chain systems result in the need for the ability to adapt to changes and the flexibility in decision making process. Burnetas and Rithcken [48] investigate the effect of contract options on the wholesale and retail prices of a product supplied by a monopolist. The authors consider reordering contracts, which are call options providing the retailer with the right to purchase additional products at a specified time for a pre-determined price, and return contracts, which are put options providing the retailer with the right to return unsold products for a previously determined salvage price. The manufacturer determines the terms of the contracts and prices of the contract options in addition to the wholesale price. The authors formulate the demand as a linear function of the wholesale price and a stochastic parameter (α) such that the inverse demand function is formulated as:

$$S_1 = \alpha - \delta Q,$$

where $\delta > 0$ and α/δ is the maximum size of the market. Uncertainty in the demand curve is represented using a Bernoulli process for the two possible cases (high and low demand values) with two possible realizations of the stochastic parameter α. the high and low values of the parameter α are formulated using the mean and the standard deviation of intercept of the demand curve under the risk-neutral measure. The authors formulate problem for determining the optimal wholesale price and contract options’ price as a standard Stackelberg game with complete information and the manufacturer being the leader. They conclude that the existence of the contract options change the equilibrium prices in favor of the manufacturer. On the other hand, it affects the retailer positively (negatively) when the volatility of the demand curve is high (low), which is measured by the standard deviation of the intercept of the demand curve.

Nembhard, Shi and Aktan [154] point out the fact that there is a time gap between the time when the real option is decided to be exercised and the time that the decision is implemented. The authors investigate the effect of this time lag on the on the outcome of the switching (supplier, production plant, etc.) decisions in a supply chain under exchange rate uncertainty. The exchange rate, e_i between the home country currency and that of foreign country i is assumed to follow geometric Brownian motion as:

$$\frac{de_{i,t}}{e_{i,t}} = \mu_i dt + \sigma_i dz_i,$$

where μ_i is the drift of the exchange rate changes in the unit time, σ_i is the volatility of the exchange rate, and dz is a standard Wiener disturbance term. The authors formulate the problem which values the switching option under exchange rate uncertainty as a stochastic dynamic problem where at each stage the recursive value function is optimized in order to maximize the profit by selecting an option for the given state variable (exchange rate) value and the option selected in the previous stage. The option valuation process is handled via modeling exchange rate movements by two alternative approaches, namely, a multi-nominal lattice approach and a Monte-Carlo simulation. They observe that the option value decreases as time lag increases. In addition, the proposed Monte Carlo simulation method provides closer approximations to the true option value than the proposed lattice approach. Moreover, the proposed Monte Carlo simulation handles the valuation process for the cases with large number of variables more efficiently than the lattice approach does.

Fujita [84] re-formulates the international trade model considering the stochastic exchange rates using the real option approach and measure the effect of foreign exchange rates on the exporting country. The world price is is formulated by a stochastic equation as:

$$\frac{dp_t}{p_t} = \alpha dt + \sigma dZ,$$

where α, σ, and dZ are drift, variance and an increment of a Wiener process, respectively. The profit of a firm is calculated as the difference between the revenue cost of labor input and discounted with a constant factor. In addition, utility value of a household is calculated as sum of the wage revenue, disutility of labor, and dividend obtained from firms. Equilibrium conditions for labor are obtained by the first order condition for the utility maximization and equilibrium wage is formulated accordingly. Next, the critical cut-off price (the foreign exchange rate value of a firm's country where the firm exports if the world price is higher than this value) is calculated using the standard real options theory and assuming that each firm determines its time to export based on the equilibrium wages that it will encounter. The author observes that a higher uncertainty on foreign exchange rates leads to a higher growth rate and a variance of the welfare of the exporting country.

Berling and Rosling [24] consider the systematic risk of the stochastic demand and purchase price and analyze their effect on the inventory policies in a real options framework. The stochastic Wiener process is used to model the stochastic factors such as demand and price, and two inventory models (a single-period newsboy model and an infinite horizon model with a fixed set-up cost) are employed. The authors aim to maximize the market value of a firm which is calculated according to *Consumption-Capital Asset Pricing Model* (the reader is referred to Breeden [40] for further information) and the firm's inventory policies. They observe that the systematic purchase-price risk has a notable effect on the inventory policies (re-order point and order quantity) whereas that of the systematic risk of stochastic demand is negligible.

Bengtsson and Olgaher [22] use the real options approach to value the product-mix flexibility considering the uncertain demand, correlation between products, and relative demand distribution within the product-mix. The authors formulate the problem which maximizes the total contribution margin of the production subject to production capacity and demand constraints as an optimization problem with a piecewise linear objective function and linear constraints, where the decision variables are production amounts of all product types in different product lines in a multi-period time horizon. The objective function is represented as the payoff function of the real option which gives the right to produce a pre-determined product in a pre-determined line with a given set-up cost serving as the strike price of the option in the option valuation process. The authors formulate the demand for each single product with a mean reverting stochastic differential equation. The demand equations include the correlated Wiener process terms in order to reflect the correlation between the demand values for each product type. The authors use a Monte Carlo simulation method, where the optimal product-mix is determined by solving the optimization problem maximizing the total contribution margin the for given simulated demand values at each iteration. They repeat this process several times and estimate the value of the option using the pay-off values over all simulation runs. In addition, the authors address the need for using an equilibrium model such as in-temporal capital asset pricing method (ICAPM) while using the traditional option pricing method.

Bollen [36] criticizes the usage of stochastic differential equations with constant expected growth rates for demand and price, and the methods underestimating the product life cycle models, especially in high-technology goods market. The author values a capacity extension option using a regime switching stochastic process for a product type. The demand for the product at time t, Q_t^D, is formulated as a linear function of price, P_t, and a stochastic demand parameter θ_t as:

$$Q_t^D = \theta_t - \lambda P_t.$$

The stochastic parameter θ is assumed to be normally distributed with parameters changing across a growth and a decay regime. The author uses a dynamic programming approach assuming that the product cycle starts with the growth regime, the decay regime follows the growth regime and the probability of switching over is constant. At each stage, the capacity update decision is made according to the maximum project net present value criterion. The value of the real option is calculated using the optimized pay-off values over all simulation iterations.

Parameter Uncertainty in Energy Markets

Tseng and Lin [218] consider the real option to commit or decommit a generating unit at a power plant. The decision to exercise the real option depends on the fuel and electricity prices because the power plant consumes a particular fuel for fuel generation necessary for the electricity production. The authors assume that the fuel and electricity prices follow correlated geometric mean reverting processes and propose a lattice framework to represent the price movements and convergence property of the joint distribution. The option valuation problem is formulated as a stochastic dynamic programming.

Thompson, Davison and Rasmussen [211] propose an algorithm to value hydroelectric and thermal power generation plants and to determine the optimal operating strategies in deregulated electricity markets. The electricity price is subject to uncertainty because of the competition in the deregulated market and dynamic nature of the demand and production cost. The authors model the price and cost via mean reverting stochastic differential equations with jumps. The electricity price P is formulated as:

$$dP = \mu_1(P,t)dt + \sigma_1(P,t)dX_1 + \sum_{k=1}^{N} \gamma_k(P,t,J_k)dq_k,$$

where μ, σ, and γ_ks can represent any function of price and/or time, and the J_ks follow some other distributions $Q_k(J)$. dX_1 is the standard increment of Brownian motion; however, dq_ks are Poisson processes with two possible values (0 and 1) defining the price jumps. In addition, the authors consider other sources of uncertainty including the water inflow, power function, cost of fuel, lead time in power generation, control response time lags, and output rates. The authors solve the equations using current complex numerical methods and determine the optimal operational strategies along with the expected cash flow.

Tsai and Hung [217] address the demand uncertainty in Internet retailing and propose a dynamic pricing method integrating the real options (RO) approach with goal programming (GO) and the analytic hierarchy process (AHP) for the revenue management problem of Internet auctions. The RO is used to determine upper and lower bounds of the value of each auction commodity; the AHP is employed to calculate the increment and decrement volumes for each commodity based on some criteria such as demand growth, market share, life cycle, competitive power, and long term return/volatility ratio. Timely quota increment and decrement values are calculated based on AHP weights and updated as new information is obtained. A goal programming approach is used to minimize the penalties resulting from the under and over achievements of the targeted goals while satisfying the available budget, limited capacity, and AHP process-related constraints where quotas of the auction commodities, increment and decrement of the initial quota, deviation variables denoting under and over achievement of the targeted goals on the revenue are the decision variables. The authors observe that a firm can increase the profitability of its Internet auction practices by following the inferences obtained from the proposed method since it incorporates the risk information.

Parameter Uncertainty in Strategic Investment Decisions

Dangl [64] investigates the real options approach for a strategic investment problem of a firm where the optimal timing and capacity of an irreversible investment have to be determined under demand uncertainty. The author considers an option to invest a production plant whose maximum capacity is given (m) and the inverse demand function is formulated as :

$$P = \theta(t) - \delta q),\ P \geq 0,$$

where q is the output of the firm, P is the price, and δ is the effect of unit change in quantity to the price. The parameter θ is the demand shift parameter and follows a multiplicative geometric Brownian motion as:

$$d\theta = \alpha\theta dt + \sigma\theta dz,$$

where dz denotes a Wiener process, α is the expected relative drift of θ per unit time, and σ^2 is the relative variance per unit time. The problem of determining the timing and size of the capacity extension decision is solved using a stochastic dynamic programming approach based on maximum net present value criterion.

Cortazar, Schwartz, and Salinas [61] consider a firm, a copper production plant, which has to obey an environmental regulation schedule limiting the disposal amount. The environmental impact of the production facilities can be lessened by investing in R&D projects and new technologies, which are assumed to be irreversible investments and to increase the operational costs; otherwise, the production amount should be kept in low levels to match the regulations. The authors use the real options approach to determine the optimal output price level at which the investment option on environmental technologies is exercised. The Geometric Brownian motion is used to formulate the output (copper) and input (copper concentrate) prices. The authors propose a model which lets continuous environmental investments at each point of time where the environmental investment schedule and the plant production levels are the decision variables. The original problem does not have an analytical solution; however, it can be solved by numerical methods. If the input (concentrate) price is assumed to be a fraction of the output (copper) price, then only one uncertainty source of price remains in the model and the problem can be solved analytically. The authors conclude that the environmental regulations might cause production plants under emission restrictions to decrease their output levels instead of investing in environmental technologies when the output price volatility is high.

Cortazar, Gravet, and Urzua [60] point out the fact that the real options valuation is more cumbersome than the financial options valuation by addressing the longer time to maturity and the higher risk exposure during this longer time period, and real investments' nature leading to a more complex set of interactive American options. They investigate a computer-simulation based least squares estimation method (LSM) incorporating a three factor stochastic process to model commodity prices to efficiently and effectively value the American type real options on coal mine investments with initiate, temporarily chase, and completely stop the production alternatives. The authors conclude that the simulation based real options valuation methods are promising tools which provide a higher degree of freedom to use rigorous models than the classical methods do without the concern of obtaining analytical solutions.

Schwartz and Smith [190] use a two-factor stochastic commodity price model that reflects the mean-reversion in the short-run prices and the uncertainty in the equilibrium price level to which prices converge in the long-run. The changes in the equilibrium price level is formulated according to the Geometric Brownian motion with drift expressing the expectations of the consumption of the existing supply, improvements in production technologies, new commodity reserve discoveries, inflation, and political and regulatory effects. The Mean-reverting Ornstein-Uhlenbeck process is used to model the short term deviations (the difference between the spot and the equilibrium prices) which revert to zero. These deviations result from some short-term changes in demand, supply, or price dynamics. Kalman Filtering, an iterative procedure for estimating unobserved state variables based on observations whose values are affected by these state variables, is employed. The authors use the proposed stochastic commodity pricing approach in a real options valuation problem where the decision maker has a right to build an oil production plant which starts producing oil after a determined time lag. The problem to determine the value of the investment and the optimal exercise strategy is solved by a discrete-time, infinite-horizon dynamic programming where at each period the decision maker either exercises the option to develop the production plant or postpones the decision till next period. They observe that the proposed method provides closer commodity price estimations and, therefore, real options valuations than the benchmark models.

Parameter Uncertainty with Jump-Diffusion Process and Fuzzy Uncertainty Sets

Martzoukos and Trigeorgis [142] propose an asset valuation approach where the underlying asset follows a mixed jump-diffusion process with multiple jumps each of which is assumed to be independent of each other and to have a log-normally distributed jump-size and a Poisson-distributed inter-arrival time as:

$$\frac{dS}{S} = \mu dt + \sigma dZ + \sum_{i=1}^{N}(k_i dq_i),$$

where S, μ, σ, dZ are the stochastic asset price, drift and instantaneous standard deviation (without jumps' effect), and an increment to a standard Wiener process, respectively. The third term in the stochastic differential equation represents the total effect of the rare events each of which has an annual frequency λ_i and a jump counter dq_i. dq_i becomes 1 with probability $\lambda_1 dt$ or 0 with probability $(1 - \lambda_1)dt$. The authors provide a general valuation framework and analytical solution for the European type real options and a Markov-chain solution approach for valuing both the American and the European type real options incorporating the proposed asset pricing model. The authors think that the proposed asset valuation with multiple jumps method leads to more realistic option values than the prevailing methods for both financial and real options since it is more capable of capturing the price dynamics.

Secomandi [193] investigates the research question: *"What is the structure of the optimal inventory-trading policy, both in terms of in-terms of inventory availability and prevailing commodity price, when the storage asset features both space and capacity constraints?"* The author considers the exogenous Markov process to model the commodity spot price evolution. The decision maker has control over both operational and inventory trading decisions which corresponds to capacity injection/withdrawal. According to the author, decoupling these two types of decisions is generally tough. The author proposes an optimal trading policy at each iteration. The operational decisions and capacity injection/withdrawal decisions depend on both the spot price and the inventory level. In other words, the author links these two decisions considering the inter-dependence structure and shows the value of such an interface using real data from natural gas industry.

Parameter Uncertainty in R&D Projects

Huchzermeier and Loch [100] consider five types of operational uncertainty, namely, the market pay-off, budget, performance, market requirement, and schedule uncertainties. They evaluate the impact of these operational uncertainties on the value of managerial flexibility in R&D projects using the real options approach incorporating the stochastic dynamic programming method. The authors observe that uncertainty may decrease the probability of real options being exercised; therefore, decrease the value of the flexibility. In addition, the value of the flexibility increases with the uncertainty level if the decision is made after uncertainty is cleared up and before costs and revenue augment.

Santiago and Vakili [184] discuss whether the value of a R&D project increases or not as uncertainty increases. The authors consider real options with three decision alternatives, namely, continue, improve, and abandon. They formulate the performance state of the project at the end of the stage t as:

$$X_{t+1} = \begin{cases} X_t + k(u_t) + w_t & \text{if } u_t = \text{continue or improve,} \\ stopped & \text{if } u_t = \text{abandon} \end{cases}$$

where w_t is the uncertainty of the development process during stage t, and u_t is the decision made at the beginning of the stage t and k is a function with binary outcomes such that $k(continue) = 0$ and $k(improve) = 1$. The development uncertainty parameter is a random variable such that $w_t = i/2$ with probability p/N and $w_t = -i/2$ with probability $(1-p)/N$ for $i = 1, ..., N$, where the parameter N can be counted as the development uncertainty measure. The authors formulate the problem of determining the optimal series managerial decisions under project development uncertainty as a dynamic programming model. The authors conclude that no general statements about the effect of increasing uncertainty on the R&D project value can be made when the source of uncertainty is the project development uncertainty.

Wang and Hwang [221] mention that the traditional financial analysis approaches underestimate the R&D project value because they ignore the fact that long lead times of R&D projects decrease the credibility of the original data collected in order to determine the optimal portfolio of R&D projects. The authors name this type of information corruption"R&D uncertainty" and suggest that a fuzzy integer portfolio selection model can overcome this deficiency. The authors combine compound options pricing model introduced by Geske [89] with the fuzzy set theory in order to calculate the R&D projects' value under R&D uncertainty. Next, they transform the fuzzy integer programming problem into a crisp mathematical model using a qualitative possibility theory and the new model can be solved by an optimization technique.

Parameter Uncertainty and Game-Theory

Kogut [116] points out the difficulties in developing a marking position and competitive power for a single firm and the collaborations in the joint-ventures form to overcome such difficulties in practice due to risk sharing and reduction in overall investment costs. The ownership structure of the joint-venture might change with assignments of right to buy and sell equity in the venture. In other words, one party might have the right to buy the ownership interest of the other party. The value of these options depends on the value of the venture. The author formulates the venture value as the sum of the value of its current assets and that of the embedded options. Therefore, the valuation of the venture and determining the timing of the acquisition require the valuation of embedded options and the asset values over time while the acquisition options itself is considered as a real option to expand. The author use product market signal which proxy the venture's valuation and determines the optimal time to exercise the acquisition option when the estimated venture value increases the base venture forecast value.

Pennings and Lint [168] use the real options approach to find the optimal timing and region to roll-out a new product with known unit cost and stochastic profit margin and demand following the correlated geometric Brownian motion considering the market competition. Therefore, the cash flow value at each time unit is represented by a stochastic equation and financial options valuation method is used to calculate the value of the real option which gives the right to roll-out a new product. The authors provide a case study on Philips Electronics introducing CD-1to the market and conclude that the market and technology uncertainty impact the value of phased roll-out strategy. Lin and Wu [128] consider an export-oriented manufacturer planning to the transfer production location from a domestic country to a foreign country. The exchange rate is assumed to follow a geometric Brownian motion. The problem of determining the optimal labor and row material allocation decisions along with production shift decision (American type options) is formulated as a stochastic control problem and dynamic programming and the Lagrange multipliers approaches are used to obtain the productive value of the exporter's productive value. Schwartz and Zozaya-Gorostiza [192] investigate the ways of evaluating the IT development and acquisition projects considering the technical and input cost and cash flow uncertainties in the real options framework. The authors propose more sophisticated and capable contingent claim models to value the IT development and acquisition projects than the models in the existing literature.

Murto, Näsäkkälä and Keppo [148] focus on the valuation of the investment projects (with the purpose of adjusting production cost and capacity) in a oligopoly market for a homogeneous commodity. The authors aim to determine the optimal timing of the granular investment project considering the oligopolistic competition and price uncertainty. The price uncertainty arises from the exogenous uncertainty and the new capacity investments' impact. Market demand evolves stochastically and the firms move sequentially. The authors first obtain a unique Markov-perfect Nash equilibrium, then a Monte Carlo simulation is run to generate demand realizations over time which will be used to determine the values of the firms as a result of their investment decisions.

Kong and Kwok [117] propose a modeling framework to analyze the game between two firms competing for the optimal entry in a project. The sunk cost and cash flows of the investment are asymmetric and stochastic for both firms. The authors' target is determining the value of the real investment options and optimal entrance time. The cash flows for each firm is assumed to be a unique multiple of a the base cash flow evolving according to a Geometric Brownian motion process. In addition, the sunk cost of the investment for each firm is adjusted (and become asymmetric) based on its cash flow. The authors follow a typical dynamic game backward solution approach; that is, they first solve the problem for the follower firm, then the leader's problem is solved.

Clark and Easaw [55] study the problem determining the optimal access price to enter the natural monopolistic networks under cash flow uncertainty. The price of the commodity and the demand evolve according to a Geometric Brownian motion. The entrant firm has an option to postpone entrance where entrance to the network corresponds to undertaking the entire investment, since the network is a natural monopoly. The value of the option to invest is calculated in the real options framework.

Siddiqui and Takashima [201] study games of lumpy capacity expansion projects under output price uncertainty with different settings including monopolistic and duopolistic markets. Sequential decision making for capacity expansion offers managerial right to defer the exercising investment option until it is in their best interest to do so based on market competition and the output price. The industry output price (P_t) is formulated as:

$$P_t = x_t D(K_t),$$

where K_t, $D(.)$ and x_t are the installed capacity, demand (as a function of installed capacity), and the exogenous shock to demand, respectively. In addition, the exogenous shock to demand is assumed to evolve according to a Geometric Brownian motion. The sequential capacity expansion decisions are determined by a dynamic programming problem in the case of the monopolistic market. In the duopolistic market case, a dynamic sequential game approach is used to determine the optimal timing for the capacity expansion decision. The authors provide insights about the effect of the uncertainty on the value of flexibility for both cases.

Lukas, Reuer, and Welling [134] study mergers and acquisition deals with involving contingent earn-outs in a game-theoretic real options approach. The authors consider a buyer and a target firm both of which are risk-neutral. The target firm's cash flows are assumed to follow a Geometric Brownian motion. Sunk transaction costs occur in the acquisition process; however, the buyer firm enjoys the possible synergies and future cash flows of the target firm later. Possible synergies are modeled as a positive, monotonously increasing, and a concave function. The problem of determining the optimal earnout and initial payment conditions and the timing of the acquisition is solved by means of dynamic programming.

Martzoukos and Zacharias [143] develop a real options framework to study a research joint venture where two firms have to decide on both the optimal level of coordination in R&D activities and the optimal level of effort and money spent on information acquisition activities considering the spillover effects. In other words, each firm holds an investment option and aims to maximize the profit potential, though information acquisition or investing in R&D projects to improve the potential for cost reduction and revenue increase. The authors propose a game theoretic approach allowing firms to coordinate their R&D activities due to the spillover effect between the firms' R&D actions. A two-stage closed-loop stochastic game is proposed to determine the optimal set of decisions for the firms and the values of the embedded real options.

Parameter Uncertainty with Information Asymmetry

Shibata [200] considers the effect of uncertainty on real options valuation by using a model extended version of the model that Bernardo and Chowdhry [25] employ. While the standard real options pricing models consider only profit uncertainty, Shibata's model accounts for three uncertainty sources: profit, information, and estimation uncertainty. Information and estimation uncertainty results from incomplete information. The main motivation for the paper is the fact that the cumulative profit of the initial action at time t can be observed; however the current realized value of underlying (state variable) is not determined certainly. The author provides the effect of the three uncertainty sources on real options value.

Bellalah [16] provides a valuation method for lease contracts in a real options framework under incomplete information. The author presents the term structure of lease rates under incomplete information and a framework for the equilibrium lease rate. The incomplete information modeling is inspired from Grenadier [92] where two sources of uncertainty (demand shocks and construction costs shocks) are considered. Bellalah computes the equilibrium rents on leases with options to renew and options to cancel. Löffler, Pfeiffer, and Schneider [131] examine the vendor selection process with several key variables including the timing of the contracting, transfer payments, and set-up, switching and abandonment decisions in an asymmetric information setting when a new supplier enters the market. The information asymmetry arises from the fact that new entrant has imperfect information about its costs whereas the incumbent supplier has perfect information about its own costs. The buyer selects one of these two suppliers to form a supply chain. The authors focus on the impact of the asymmetric information on the timing of contracting with the new entrant firm and that of the current supplier on the buyer's set of actions.

Oh and Özer [159] study time in forecast information sharing and decision making under uncertainty with multiple decision makers having asymmetric information. Specifically, the authors focus on the problem of a supplier extracting credible forecast information from a manufacturer to plan its capacity investment decision. The supplier has an option to defer the capacity investment decision and obtain more information from the manufacturer which will decrease the degree of uncertainty that the investment decision is subject to. On the other hand, waiting for further information leads to tighter deadline for the capacity expansion project which increases the cost of the project. Specifically, the supplier decides on timing the capacity expansion, whether to corporate with the manufacturer for information sharing (at a cost), and size of the capacity expansion. The authors represent the degree of demand uncertainty and that of information asymmetry by parameters whose value changes by time and propose a model for the dynamic evolutions of asymmetric forecasts. In addition, the value of the option to refer the capacity expansion is valued based on proposed forecasting approach.

1.6 The Diffusion of Innovations and the Bass Model

1.6.1 Diffusion of Innovations

According to Rogers [176], the diffusion of innovation is achieved by propagation of the innovation through certain communication channels by time in a social system. Geroski [88] mentions that the adaptation of new technologies by time usually follows an S-curve. The author classifies the models explaining the adaptation rate into four main groups: epidemic models, probit models, models of density dependence, and models of information cascades. Epidemic models generally assume that the diffusion of innovation occurs by means of direct contact with the previous adopters or by imitating them. In addition, it depends on the premise that the potential adopters form a homogeneous population in terms of their needs and willingness to adopt. However, the probit models address that different potential adopters have heterogeneous preferences and abilities to adopt the new technologies at different times. The models of density dependence explain adoption of new technology and finally limits its benefits by mentioning the balance of the impacts of legitimation and competition. Finally, the main idea behind the models relying on information cascade is that the adopters make sequential decisions rationally based on the information that they have. In addition, the subsequent speed of the diffusion of the new technology depends on the initial choice of the adopter.

Mahajan, Muller, and Bass [137] mention that these communication channels are both mass media and interpersonal communications. The diffusion of innovation was first introduced to the marketing science in 1960s by researchers including Arndt [5]; Bass [9]; and Frank, Massy, and Morrison [83]. Among these studies, a new product development model suggested by Bass [9] and its revised versions used in estimating diffusion of innovation in various markets including retail service, pharmaceutical industry, consumer durables market, and industrial technology

([137]). We will focus on the Bass model for a new product development.

The Bass model of diffusion considers two types of potential adopters, namely innovators and imitators. According to Tidd [104], the diffusion process occurs in the epidemic form for imitators; however, the innovators are not subject to social emulation. Therefore, the adaptation of the innovators in early periods is followed by that of the imitators in later periods. This leads to a skewed S-curve for the adaptation rate for the whole population.

1.6.2 The Bass Diffusion Model

The Bass model of diffusion considers two types of potential adopters, namely innovators and imitators and assumes that the two communication channels used to influence the potential adopters are the mass media and word of mouth. According to Tidd [104], the diffusion process occurs in the epidemic form for imitators; however, the innovators are not subject to social emulation. In other words, the innovators are impacted only by the external influence (mass media), whereas the imitators are impacted by the internal influence (the word-of-mouth). Therefore, the adoption of the innovators in early periods is followed by that of the imitators in later periods. This leads to a skewed S-curve for the adaptation rate for the whole population.

According to Lilien, Rangaswamy, and Bruyn [127], the Bass model can estimate the long terms sales patterns of new technologies for the following two cases:

- The new product has already been introduced to the market and first few periods' sales amounts have been observed,
- The product has not been introduced to the market; however, an existing product's diffusion process can be used as a proxy for the product of interest.

The basic Bass model also assumes that a member of the population can adapt the product only once and the probability of an adoption at time t can be modeled as a hazard rate. Let us denote the density function of time to adoption as $f(t)$ and the cumulative fraction of adopters at time t as $F(t)$, then the hazard function leads to the following equality:

$$\frac{f(t)}{(1 - F(t))} = p + qF(t),$$

where the parameter p stands for the external influence and the parameter q reflects the internal influence resulting from earlier adopters. The function $F(t)$ is assumed to be a non-decreasing function and approaches to 1 as t gets larger. In addition, it assumes that the process starts with no initial adopters, in other words $F(0) = 0$ and $f(0) = 0$.

If q is a zero, $f(t)$ follows the negative exponential distribution ([137]). Lilien, Rangaswamy, and Bruyn [127] interpret that if $q \geq p$, then the innovation influence is dominated by the imitation influence and the plot of $f(t)$ versus time has an inverted U shape. Otherwise, the innovation influence prevails the imitation influence and the highest amount of sales are observed at the introduction and the rate of adoption decreases as time passes. In addition, a decrease in p leads to a longer time period to realize the sales growth for the innovation. Furthermore, if both of p and q are large, the adaptation rate takes off rapidly and falls off quickly after reaching its peak point ([137]).

If the parameter m stands for the potential number of ultimate adopters, the number of adopters at time t, $S(t)$, and the cumulative number of adopters at time t, $C(t)$, are represented as:

$$S(t) = mf(t), \text{ and}$$

$$C(t) = mF(t).$$

Next, we can express the relationship between $f(t)$ and $F(t)$ as:

$$S(t) = \frac{dC(t)}{dt} = p[m - C(t)] + \frac{q}{p}C(t)[m - C(t)].$$

The second term represents the number of new imitators, whereas the second term represent the number of new innovators at time t. After some basic mathematical operations $f(t),\ F(t),\ S(t),$ and $C(t)$ are expressed as:

$$\begin{aligned} f(t) &= \frac{(m+q)^2}{p}\frac{e^{-(p+q)t}}{\left(1+\frac{q}{p}e^{-(p+q)t}\right)^2}, \\ S(t) &= m\frac{(m+q)^2}{p}\frac{e^{-(p+q)t}}{\left(1+\frac{q}{p}e^{-(p+q)t}\right)^2}, \\ F(t) &= \frac{1-e^{-(p+q)t}}{1+\frac{q}{p}e^{-(p+q)t}}, \\ C(t) &= m\frac{1-e^{-(p+q)t}}{1+\frac{q}{p}e^{-(p+q)t}}. \end{aligned}$$

In addition, the period when the sales amount peaks (T^*), the cumulative and marginal amounts of sales at the peak time are derived by further differentiations as follows:

$$T^* = -\frac{1}{p+q}ln\left(\frac{p}{q}\right) \text{ and } S(T^*) = m\left(\frac{1}{2}-\frac{p}{2q}\right)$$

1.6.3 Parameter Estimations for the Bass Model

The usage of the basic Bass model requires estimating three parameters: $m,\ p,$ and q. As it is mentioned in Section 1.6, the Bass model can estimate the sales of an innovative product under two conditions: the product has already been introduced to the market and some sales observations are available or the product has not been launched yet; however, another product which has some similarities with the original product is in the market and is used as a proxy.

In the first case, historical data sets are used for estimating the parameters. Time invariant estimation procedures such as the ordinary least squares (OLS) ([228]) and maximum likelihood estimation procedures ([187]) are the main methods used by researchers. However, Hyman [103] points that the estimates for these parameters depend on the number of data points used in the estimation procedure. Moreover, Srinivasan and Mason [202] show that reliable estimations for the parameters can be obtained when the available data set is large enough to cover the peak of the rate of adoption curve. Therefore, time-varying estimation procedures have been proposed lately including Bayesian estimating procedures and adaptive-filtering methods. Sultan, Farley, and Lehmann [78] update the initial estimates of the parameters p and q after obtaining new estimates of them by taking weighted sum of these two estimates. Bretschneider and Mahajan [43] propose a time-varying parameter estimation method based on a feedback filter.

In the second case, where there is no data available, parameters can be estimated by expert judgments or using historical observations of the diffusion process of an analogous product.

1.6.4 Extensions of the Basic Bass Model

Kalish and Lilien [108] address the impacts of perceived product quality and information level in the market place (advertisement) in a period on the number of new adopters in that period. Bass, Krishnan, and Jain [10] reformulate the relationship between $f(t)$ and $F(t)$ so that the new formulation incorporates pricing and advertising decisions as:

$$\frac{f(t)}{(1-F(t))} = (p + qF(t))x(t),$$

where $x(t)$ is a function of price $(P(t))$ and advertisement expenditure $(A(t))$ at time t and formulated as:

$$x(t) = 1 + \alpha\frac{[P(t) - P(t-1)]}{P(t-1)} + \beta \max\left(0, \frac{[A(t) - A(t-1)]}{A(t-1)}\right)$$

α : the parameter reflecting the diffusion process's sensitivity to the fractional chance in the product price,

$P(t)$: the price of the product in period t

β : the parameter reflecting the diffusion process's sensitivity to the fractional chance in the money spent on advertisement,

$A(t)$: the amount of money spent on advertisement in period t

Kamrad, Lele, and Siddique [110] propose a stochastic model of innovation diffusion and determine the optimal advertisement and pricing policies using a stochastic dynamic programming approach.

Kumar and Krishan [122] reformulate the adaptation rate formulation so that one country's diffusion process impacts the other. The authors consider the lag-lead, lead-lag, and lag-lag (simultaneous) impacts of inter-country interactions on the diffusion processes. The diffusion rate of country i is formulated as:

$$\frac{f_i(t)}{(1-F_i(t))} = (p_i + q_i F_i(t))x_i(t),\ i \in \{1,2\} \text{ where}$$

$$x_1(t) = 1 + b_{21}\frac{dF_2(t)}{dt} \text{ and } x_2(t) = 1 + b_{12}\frac{dF_1(t)}{dt}.$$

Thus, the cumulative fractions of adopters in both countries are formulated as:

$$F_1(t) = \frac{1 + exp(-(p_1+q_1)(t + b_{21}F_2(t))}{1 + \frac{q_1}{p_1}exp(-(p_1+q_1)(t + b_{21}F_2(t))},$$

$$F_2(t) = \frac{1 + exp(-(p_2+q_2)(t + b_{12}F_1(t))}{1 + \frac{q_2}{p_2}exp(-(p_2+q_2)(t + b_{12}F_1(t))}.$$

Chapter 2

Positioning and Contributions

2.1 Positioning

This dissertation deals with risk management especially in portfolio management and revenue management problems. In Chapter 3 which is named “Portfolio Management with Quantile Constraints”, we are interested in a portfolio management problem where the risk is defined as the negative of the quantile function for a given probability level. We propose an iterative, data-driven approximation to the problem which maximizes the expected return and keeps risk below a specified target. This work could be applied to revenue management problems as well, such as airlines admission problems. In Chapter 4, which is labeled as“Portfolio Management with Moment Matching Approach”, we use a moment matching approach, which is widely used in communication systems technologies, to approximate portfolio return and define risk as the probability of the portfolio return’s being less than a specified level. We provide a tractable mathematical formulation to the problem minimizing the probability of having the portfolio return value less than a specific target while keeping the expected portfolio return value not less than a determined value. We transform the mathematical model of problem which has a non-linear objective function of the decision variables into a two stage problem which reaches the optimal solution by solving two convex nonlinear problems. In addition, we provide an algorithm to improve the accuracy of the log-Normal sum approximation on the left tail of the portfolio return distribution. In Chapter 5, which we name "New Product Launching Decisions with Robust Optimization", we address the uncertainty involved in the introduction of innovative products to a regional market considering an innovative company which seeks the optimal product launching schedule under periodic investment budget limitations. In addition, the innovative company seeks a partner company for each product in order to establish the infrastructure for the product. We use the new product growth model introduced by Bass [9] to formulate the demand for each product. Moreover, each potential partner companies' willingness to accept the collaboration offer is modeled by the logit choice model. We handle the uncertainty involved in the model parameters by robust optimization techniques. Furthermore, we extend our approach to the case where the innovative company has an option to update the size of the contract signed by a partner company for an innovative product. In Chapter 6, which is called "Log-Robust Portfolio Management with Factor Model", we incorporate robust portfolio optimization with asset pricing with factor models. The portfolio risk is defined as the uncertainty resulting from factor modeling used for asset pricing and stochastic processes used for pricing the factors. We aim to maximize the worst case portfolio return given a budget of uncertainty.

Stock returns are known to be highly volatile and have heavy left tailed distributions as Jansen and Vries [106] mentioned. Traditional risk measures such as variance and semi-variance might not be so productive against occasional market crashes. Quantile is a risk-adjusted return measure, and relatively stable. However, representing the quantile function in an optimization problem in a tractable manner is a difficult task. As Rodriguez [175] mentioned, the Ranking and Selection, Gradient-based procedures, and integer programing-based models are some of the methods applied in the literature. However, these methods might require too much time or computational effort. Therefore, we investigated a data driven approximation method which converges to a solution close enough to the optimal in few iterations.

The Fenton-Wilkinson [79] method, which is well known in communication systems technologies literature, provides a closed form formulation for the log-Normal sum distribution. In other words, if it is applied to log-Normally distributed stock returns, the portfolio return is represented as a single log-Normally distributed random variable which lets us formulate tractable optimization models. In the finance literature, this approach is applied to basket options, which is a collection of several European style call options, by researchers such as Henriksen [98], and Hakala and Wystup [94]. Our study covers both equities in the portfolio optimization problem and the basket options in the option design problem.

Bass [9] models the diffusion of innovation considering two types of diffusion channels: mass media and word-of-mouth. The new product development model proposed by Bass (the Bass diffusion model), defines two types of potential adopters, namely innovators and imitators. The innovators are assumed to be influenced by the external sources such as mass media; whereas, the imitators are assumed to be affected by the internal sources such as the customers who have already adopted the innovation. The Bass model forecasts the rate of adoption and periodic sales by using estimations for three parameters, namely the coefficient of innovation, that of imitation, and the potential number of ultimate adopters. These parameters are estimated before the diffusion process starts; therefore, their estimations are subject to uncertainty.

According to Lauviere, Hensher and Swait [125] individuals' choices are determined by some factors, such as habit, inertia, experience, advertising, peer pressure, environmental constraints, accumulated opinion, household and family constraints, etc. Discrete choice methods have been widely used by researchers to examine consumer choices and forecast customer demand in revenue management problems. Another application area for the discrete choice models is customized-pricing in business-to-business environments where one side of the contract does not know the offer response function of the other side of the contract but has some observations on its previous offers to it and the results of these offers. The parameters of these discrete choice models are estimated based on available data or managerial judgments. Therefore, the parameters of the discrete choice model of a firm in this type of business-to-business settings are subject to uncertainty.

As Trigeorgis [216] explains, the notion of real option arises from the managerial flexibility that allows the managers to update their decisions at a certain time period according to the information revealed up to that point so that they can defer, extend, or abandon a project during its life-time. According to Amram and Kulatilaka [3], the real options approach is the most applicable when the first course of decisions are given in the presence of large uncertainty and there will be information updates, and mid-course strategy correction opportunities. This definition suits well to the case where a company plans to launch an innovative product and determines the size of the infrastructure-outsourcing contracts related to the product by estimating demand-related parameters before the market meets the product. The company will obtain some observations and information on the actual demand as time progress and will be able to update the size of the outsourcing contracts for the product's infrastructure.

In Chapter 5, we use the Bass diffusion model forecast the diffusion pattern of an innovative product, and the logit choice model to estimate the potential partner companies' response to a partnership offer. We address the uncertainty on the parameters of both of the models with robust optimization techniques. Furthermore, we use the real options approach to value the managerial flexibility of updating the contract size with the partners after obtaining information on the actual demand for the products.

Factor modeling is a common asset pricing method. Famous Fama-French three factor model [77] is an example for equity pricing with factor models. Other equities' returns, stock market indexes, currency rates, commodity prices could be used as factors while pricing an equity. Even though factor loadings (coefficients of factors) are known, future prices of factors are subject to uncertainty. In both literature and practice, stochastic processes such as Geometric Brownian Motion (GBM) and OrnsteinUhlenbeck Processes (OUP) [85] are used to model and forecast future levels of factors. GBM is mostly used for equity pricing and it is not mean reverting. Kawas and Thiele [112] use the traditional log-Normal model (see Hull [101]) which depends on GBM. OUP is used to model mean reverting random variables such as interest rate and currency rate in finance. Considering the fact that both mean reverting and non-mean reverting factors could be effective in asset pricing, we use both GBM and OUP to forecast future factor levels and construct factor models for each single asset price via linear regression. We consider the stochasticity resulting from each factor's pricing formulation and the residual in the each linear regression model for each asset as uncertainty sources. We handle this uncertainty via robust optimization techniques. Our work could be counted as an extension to Kawas and Thiele's joint work [112]. Different from that work, we use factor models for asset pricing, provide more flexibility for asset pricing by the choice of GBM or OUP, and handle more uncertainty sources in a single portfolio management problem setting.

2.2 Contributions

Our contributions to the literature is as follows:

Portfolio Management with Quantile Constraints

- We approximate the quantile function without any assumption on the return distribution, but based on available scenarios.
- We model the problem which maximizes the expected return not falling below a threshold given percent of the time.
- Our method, which involves solving a series of linear problems, can be quickly solved for large scale problems.
- Numerical studies imply that our algorithm provides more robust investment decisions against adverse realizations of stock returns than classical Gaussian approximation models (Normal and log-Normal approximation models).

- Numerical studies imply that our algorithm leads to solutions which are close to optimal solutions.
- We extend our approach to the portfolio management problem where risk is measured by inter-quantile range of portfolio return.
- We provide another version of our approach which calculates Tail Conditional Expectation (TCE) at the same time.

Portfolio Management with Moment Matching Approach

- We use the Fenton-Wilkinson log-Normal sum approximation method to approximate the random portfolio return as a log-Normal sum to a single log-Normally distributed random variable.
- We formulate the model which minimizes the probability of obtaining a portfolio return less than a specified threshold level while keeping expected portfolio return above a specified target.
- We provide an approach which divides the overall problem into two sub-problems and solves the risk management problem as an unconstrained nonlinear programing problem with a smooth objective function based on the other sub-problem's solution.
- We suggest an algorithm which improves the accuracy of the log-Normal sum approximation method.
- We extend our work to the basket options design problem.

New Product Launching Decisions with Robust Optimization

- We use the Bass diffusion model to forecast the demand for an innovative product.

- We employ the logit choice model to formulate a potential partner firm's willingness to accept the partnership offer made by the innovative company while the only variable of interest is the unit price per product/service.

- We suggest a partner selection model which determines the potential partner whose worst-case probability to accept the partnership offer is higher than a specified level with the minimum customized price.

- We combine the robust optimization model for the partner selection decisions with that for the product launching decisions which maximizes the worst-case profit obtained from the innovative products' and determines the optimal launching time for each product.

- We handle the uncertainty involved in the parameters of the Bass diffusion model and the logit choice model with robust optimization techniques.

- We propose an iterative approach to transform the in-tractable robust optimization formulation for the product launching problem into a tractable model.

- We handle the managerial flexibility of updating the size of the contract between a partner company and the innovative company by incorporating the robust product launching problem formulation with the real options approach.

Log-Robust Portfolio Management with Factor Models

- We treat randomness on asset pricing by a budget of uncertainty.

- We maximize the worst-case portfolio return at the end of the time horizon in a one-period setting.

- We gain insights into the worst-case scaled deviations and the structure of the optimal strategies.

- We drive a tractable robust formulation, specifically a linear optimization model.

Chapter 3

Portfolio Management with Quantile Constraints

3.1 Portfolio Management with Quantile Constraints

3.1.1 Problem Setup

Suppose that the investment portfolio is re-adjusted once per period in the presence of transaction fees γ_i^- and γ_i^+ proportional to the amount asset i sold and bought, respectively. We aim to maximize the expected value of a random objective bilinear in the decision variables and the random variables, while guaranteeing that the random objective achieves a target with a given probability based on discrete scenarios. Specifically, we aim to maximize the expected portfolio return with quantile constraints. We will use the following notation:

n : the number of assets,

x_i^+ : the dollar amount transacted into asset i,

x_i^- : the dollar amount transacted out asset i,

x_i^0 : the current holding in asset i,

W : the current wealth i,

lb_i : the lower bound for the holding in asset i,

ub_i : the upper bound for the holding in asset i,

$I_{i,l}$: the binary indicator which takes value of 1 if asset i belongs to the sector l,

β_l : the maximum amount that can be invested in sector l,

μ_i : the sample mean of the i-th random coefficient, i.e., the mean of the , return rate of a stock i,

$\tilde{\mu}_i$: the sample mean of the natural logarithm of the i-th random coefficient,

$\tilde{\sigma}_i$: the standard deviation of the natural logarithm of the i-th random coefficient,

$\tilde{\rho}_{i,j}$: the correlation coefficient between the natural logarithm of the i-th and the j-th random coefficients,

τ : the target expected portfolio return,

T : the number of observations, e.g., time periods in historical data set,

r_{ti} : the t-th observation of random variable i, e.g., the return of stock i on day t,

α : the specified quantile level, $\alpha \in (0, 1)$,

m : the index of the observation that corresponds to the $100\alpha^{th}$ quantile, i.e., $m = \lceil \alpha \cdot T \rceil$,

q_m : the desired value for the $100\alpha^{th}$-quantile (m-th smallest observation),

$y_{(k)}$: the k-th smallest value in the set $(y_1, \ldots, y_n)$ for $k = 1, \ldots, n$,

X : the feasible set for the decision variables formulated considering sector limits, change in the amount of asset i invested, and limits on x_i.

The feasible set X is defined by the following set of constraints (the budget constraint, the constraints for the lower and upper bound on money invested in each asset, and the constraint on amount invested in each sector):

$$\begin{aligned}
&\sum_{i=1}^{n}(x_i^+ - x_i^-) + \sum_{i=1}^{n}(\gamma_i^+ x_i^+ + \gamma_i^- x_i^-) \leq 0,\\
&x_i^0 + x_i^+ - x_i^- \geq lb_i, \forall i,\\
&x_i^0 + x_i^+ - x_i^- \leq ub_i, \forall i,\\
&\sum_{i=1}^{n} I_{i,l}(x_i^0 + x_i^+ - x_i^-) \leq \beta_l,\ \forall l,\\
&x_i^+, x_i^- \geq 0, \forall i.
\end{aligned} \tag{3.1}$$

The portfolio management problem with a quantile constraint can be formulated as:

$$\begin{aligned}
\max \quad & \frac{1}{W}\sum_{i=1}^{n}\mu_i\,(x_i^0 + x_i^+ - x_i^-)\\
\text{s.t.} \quad & \frac{1}{W}\left(\sum_{i=1}^{n} r_{\cdot i}\,(x_i^0 + x_i^+ - x_i^-)\right)_{(m)} \geq q_m,\\
& x_i^+,\, x_i^- \in X,
\end{aligned} \tag{3.2}$$

where $\frac{1}{W}\left(\sum_{i=1}^{n} r_{\cdot i}\,(x_i^0 + x_i^+ - x_i^-)\right)_{(m)}$ refers to the m-th lowest value of the portfolio returns $\frac{1}{W}\sum_{i=1}^{n}\mu_i\,(x_i^0 + x_i^+ - x_i^-), t = 1, \ldots, T$.

If $m = 1$, Problem (3.2) can be linearized easily, because, in that case, portfolio return for all realizations must be at least the threshold. However, if $m > 1$, Problem (3.2) is hard to solve because it involves ranking the objective (portfolio return) values for every candidate solution. Our goal in Section 3.1.2 is to investigate an efficient approximation approach to solve Problem (3.2) for the cases when $m > 1$.

3.1.2 Solution Approach

The specific method we analyze involves solving a linear problem iteratively where set of constraints used to express the quantile constraint changes at each iteration. At each iteration a set of portfolio return scenarios for which the objective must exceed the threshold is determined. Next, the linear problem maximizes the expected portfolio return with the constraint corresponding to these scenarios. Specifically, $m-1$ scenarios for which the objective does not need to exceed the threshold is identified by simply ranking the portfolio return scenarios in ascending order identifying the first $m-1$ of them by the vector $z \in R^T$.

When $z_j = 1$ for a scenario j, the portfolio return calculated in that scenario j does not need to exceed the threshold level q_m since only the m^{th} greatest portfolio return or higher should exceed the threshold q_m. The master problem to obtain the decision allocation is then formulated as follows:

$$
\begin{aligned}
\max \quad & \frac{1}{W}\sum_{i=1}^{n} \mu_i (x_i^0 + x_i^+ - x_i^-) \\
\text{s.t.} \quad & (1-z_t)\frac{1}{W}\sum_{i=1}^{n} r_{t,i}(x_i^0 + x_i^+ - x_i^-) \geq q_m (1-z_t), \; \forall t \in \{1,\ldots,T\}, \\
& x^+, x^- \in X.
\end{aligned}
\tag{3.3}
$$

We repeat solving the Problem (3.3) for a scenario identification vector z and, then, update the vector z based on the latest portfolio allocation decision X iteratively until the algorithm converges to an approximate solution of the original problem. We provide our algorithm in more details below.

Algorithm 3.1

Step 1 *Start with a feasible solution $x \in X$ to serve as a candidate solution $\bar{x}$ and set the iteration number, $s = 1$.*

Step 2 *Obtain a new active scenario selection decision z, namely z^s for the candidate solution $\bar{x}$. If there are more than one scenarios leading to the same portfolio return value and they are both candidates to be the $(m-1)^{th}$ scenario based on the current investment decision, then select the one with the smallest index.*

Step 3 *Solve the linear problem for z^s identified in Step 2. Obtain a new candidate solution, x^s, and set $s = s + 1$ and $\bar{x} = x^s$.*

Step 4 *Repeat Steps 2 and 3 until the algorithm generates the same set of active scenarios or the same candidate solution $x \in X$ in two consecutive steps, whichever happens sooner.*

In addition, the feasible region of the master problem is a closed polyhedron, therefore in the case of multiple solutions at, an interior point method algorithm terminates at the analytic center of the optimal face (see Colombo [57]). Namely, if the master problem is solved by an interior point algorithm, the algorithm will have a unique solution at each iteration s for a given passive scenario set $A^s\{i : 1 \leq i \leq T \cap z_i^s = 1\}$.

The algorithm will terminate if the sets A^s and A^{s+1} are identical, since the constraint sets for the Problem (3.3) will be identical for the iterations s and $s + 1$ which lead to the same solution under the assumption that it is solved by interior point method.

If there is one scenario, scenario-t, belonging to set A^{s+1} but not the set A^s, then the portfolio return value obtained by scenario-t is less than or equal to that obtained by the scenario-t' which is in the set A^s but not in the set A^{s+1}. If these two scenarios lead to the same portfolio return value for the given investment decision x^s, the current investment decision x_s will be a feasible decision with the set of constraints $r_j'x \geq q_m, \ \forall j \in A^{s+1}$, therefore the objective function value at iteration $s + 1$ will improve with a new investment decision or stay the same with the same investment decision and the algorithm will terminate. If the scenario $t \in A^{s+1}$ leads to a lower portfolio return value than the scenario-t' in A^s for the decision x^s, then this implies that the scenario-t' in set A^s leads to a higher portfolio return value than the target q_m, since at iteration s the master problem is solved while $z_t = 0$ and the solution x^s satisfies the inequality $r_t' x^s \geq qm$. Considering that the portfolio return obtained by the scenario-t' is greater than the portfolio return value obtained by the scenario-t' which is greater than or equal to q_m, we can conclude that the quantile target is satisfied for a smaller probability level at iteration s. Therefore, transforming the active set from A^s to A^{s+1} enlarges the feasible region for the master problem and it provides an improved objective function value. For the cases where there are more than one scenarios belonging to the set A^{s+1} but not to the set A^s, the same argument is also valid. Therefore, the proposed algorithm improves at each iteration until convergence.

3.1.3 Numerical Results

This section tests the performance of Algorithm 3.1 in terms of solution time and return-risk efficiency. As comparison benchmarks, we use portfolio optimization models with quantile constraints where the asset returns are assumed to be Normally and Log-Normally distributed as in the previous studies such as [113], [130], and [174]. We refer to these benchmark models as the "Normal Approximation" and the "Log-Normal Approximation" methods. In particular, we compare our solution with the optimal solution in the Normal Approximation method which assumes that the portfolio return is a Normally distributed random variable and that in the Log-Normal Approximation method which is built by approximating the portfolio return by a Log-Normally distributed random variable based on a moment matching approach. In addition, a data-driven iterative VaR optimization algorithm introduced by Larsen, Mausser, and Uryasev [124] (will be referred as Algorithm-A1) is used an another benchmark model. Algorithm A1 provides an approximated solution to the quantile optimization problem by iteratively solving a linear optimization problem which maximizes the CVaR of the portfolio return that was introduced by Rockafeller and Uryasev [174]. We will observe that:

- The iterative algorithm that we propose converges in a small number of iterations. Total solution time in terms of CPU seconds is close to that obtained with the benchmark methods. Indeed, in some experiments with relatively small observations, the Linear Approximation methods terminates earlier than the benchmark models, especially the Log-Normal Approximation method.

- For a given data set, the number iterations and time to reach a solution change for the proposed Linear Approximation method in a set of experiments, however those for Algorithm-A1 stay relatively consistent within the same data set. The upper bounds of the ranges of the observed number of iterations and time to convergence for the Linear Approximation problem in different numerical experiments are closer to the number of iterations and time to convergence for Algorithm-A1 than the lower bounds of the ranges.

- The proposed method generally outperforms the benchmark methods in terms of return-risk efficiency in both in-sample and out-of-sample performance tests. That is, for a given quantile target q_m, the proposed algorithm generally leads to a portfolio allocation decision providing higher expected portfolio return with both the training and testing data set.

- Portfolios generated by the proposed algorithm are generally more robust against unexpected stock return realizations in the out-of-sample data sets than the ones generated by the benchmark methods.

- The risk-return efficiencies of Algorithm-A1 and the linear approximation are very close to each other. Because both of them are data-driven approximation methods and the way that the quantile function is approximated is similar in both approaches.

Setup

The traditional approach for the portfolio management problem with quantile constraints assumes that the asset returns follow a jointly Gaussian distribution. In other words, the quantile constraint, which is hard to formulate, is in general approximated by the quantile function of a Normal distribution. Another approach, which is known as the Fenton-Wilkinson Method [79], calculates an approximation to the Log-Normal sum distribution based on a moment matching method. In contrast with the Gaussian case, a linear combination of Log-Normal random variables is not Log-Normal, therefore this is an approximation even if each single stock return series obeys a Log-Normal distribution. The Fenton-Wilkinson method approximates the Log-Normal sum by a single Log-Normal random variable by matching the first and the second moments. Therefore, we will refer to these models as the Normal and the Log-Normal Approximation methods, respectively. Our proposed algorithm will be referred as the Linear Approximation method since it involves solving a series of linear problems.

The portfolio management problem according to the Normal Approximation method for a given α probability level is formulated as:

$$\begin{aligned} \max \quad & \tfrac{1}{W}\mu^T x \\ \text{s.t.} \quad & \mu^T x + \phi^{-1}(\alpha)\sqrt{x^T Q x} \geq q_m W, \\ & x_i = (x_i^0 + x_i^+ - x_i^-)\ \forall i \ \in \{1, ..., n\}, \\ & x^+, x^- \in X, \end{aligned} \tag{3.4}$$

where ϕ is the CDF of a standard Gaussian random variable.

The portfolio management problem according to the Log-Normal Approximation (Fenton-Wilkinson) method for a given α probability level is written as follows:

$$
\begin{aligned}
\max \quad & \tfrac{1}{W} b^T x \\
\text{s.t.} \quad & 2\ln(b^T x) - \frac{1}{2}\ln(x^T A x) + \phi^{-1}(\alpha)\sqrt{\ln(b^T x) - 2\ln(x^T A x)} \geq \ln(W q_m), \\
& x_i = (x_i^0 + x_i^+ - x_i^-)\ \forall i \ \in \{1, ..., n\}, \\
& x^+, x^- \in X,
\end{aligned}
\tag{3.5}
$$

where the vector $b \in \mathcal{R}^n$ is such that

$$b[i] = e^{\left(\tilde{\mu_i} T + \frac{\tilde{\sigma_i}^2 T}{2}\right)}\ \forall i \in \{1, ..., n\},$$

and matrix A $\in \mathcal{R}^{nxn}$ such that

$$A[i,j] = e^{\left((\tilde{\mu_i} + \tilde{\mu_j})T + \frac{T}{2}(\tilde{\sigma_i}^2 + \tilde{\sigma_j}^2 + 2\rho_{i,j}\tilde{\sigma_i}\tilde{\sigma_j})\right)}\ \forall i \in \{1, ..., n\}\ ,\ \forall j \in \{1, ..., n\},\ \text{and}\ i \neq j$$

$$A[i,i]] = e^{2T\tilde{\mu_i} + 2T\tilde{\sigma_i}^2}\ \forall i \in \{1, ..., n\}.$$

The explanation of the Fenton-Wilkinson method and derivation of the Log-Normal approximation problem are provided in Appendix 1.

Note that the Log-Normal Approximation method approximates the linear combination of Log-Normally distributed stock returns by a single Log-Normally distributed random variable. In addition, the objective function formulation according to this approach is different than that of the Linear Approximation method. In order to have a fair comparison, we update this benchmark model so that the objective function is the same as that of the Linear Approximation method (namely, the sample average of return rates), while the quantile function is approximated according to Fenton-Wilkinson method. This hybrid benchmark model is formulated as:

$$\begin{aligned}
\max \quad & \tfrac{1}{W}\mu^T x \\
\text{s.t.} \quad & 2\ln(b^T x) - \frac{1}{2}\ln(x^T A x) + \phi^{-1}(\alpha)\sqrt{\ln(b^T x) - 2\ln(x^T A x)} \geq \ln(W q_m), \\
& x_i = (x_i^0 + x_i^+ - x_i^-)\ \forall i\ \in \{1, ..., n\}, \\
& x^+, x^- \in X.
\end{aligned} \tag{3.6}$$

Algorithm-A1 provides an approximated solution to the quantile maximization problem by iteratively maximizing the tail conditional expectation of the portfolio return for updated probability levels so that at the next iteration the new tail conditional expectation, which will be maximized (by using the linear problem suggested by Rockafeller and Uryasev [174]) is a closer lower bound to the original quantile level in interest. The linear tail conditional expectation optimization problem and the algorithm introduced by Larsen, Mausser, and Uryasev [124] is adjusted to our problem setting and represented below:

Algorithm 3.2

Step 1 *Assign a lower bound on the expected portfolio return, the probability level parameter for the tail conditional expectation, and a value for the algorithm constant* ζ, $0 \leq \zeta \leq 1$.

Step 2 *Set* $\alpha_0 = \alpha$ *and* $s = 0$.

Step 3 *Solve the tail conditional expectation maximization problem:*

$$\begin{aligned}
\max_{x^+,x^-,x,\kappa} \quad & \frac{1}{W}\sum_{t=1}^{t_s}\left(\sum_{i=1}^{n} r_{.i}x_i\right)_t \\
\text{s.t.} \quad & \mu^T x \geq \eta, \\
& \sum_{i=1}^{n} r_{t,i}x_i \geq \kappa \; \forall t \geq t_s, \\
& \sum_{i=1}^{n} r_{t,i}x_i \leq \kappa \; \forall t < t_s, \\
& x_i = (x_i^0 + x_i^+ - x_i^-) \; \forall i \; \in \{1, ..., n\}, \\
& x \in X.
\end{aligned} \tag{3.7}$$

Step 4 *Sort the scenarios according to their return values* $\frac{1}{W}\sum_{i=1}^{n} r_{t,i}x_i^s$ *based on the solution of the Problem (3.7) at iteration* s.

Step 5 *Set* $s = s + 1$, $b_s = \alpha + (1-\alpha)(1-\zeta)^s$, $t_s = \lfloor T(1-b_s) \rfloor$, *and* $\alpha_s = 1 - \frac{1-\alpha}{b_s}$.

Step 6 *If* $t_s \leq \lfloor [T\alpha] \rfloor$ *repeat Step 3,4, and 5, otherwise exit.*

During the numerical experiments, the constant ζ is set to 0.5. In this section, we will compare the risk-return efficiency of the suggested algorithm with respect to these three methods as benchmark models.

Time and Number of Iterations to Convergence

We compare the CPU seconds used by the solver calls (by the variable $\_solve\_time$) for each approach using different quantile targets over different data sets with various sample sizes and number of assets. The Mosek solver is used through AMPL modeling language on a 2.10 GHz Pentium(R) machine. The results are provided in Table 3.1. The number of decision variables increases by the number of stocks considered (with the same rate) in all of the approximation methods. However, as number of scenario increases, the number of constraints of the Linear Approximation method and Algorithm-A1 increase. Therefore, total time spent by solvers for these methods are more vulnerable to the data set size than those in the Normal and LogNormal Approximation methods. In addition, generally the Linear Approximation method requires fewer iterations and less time to terminate than Algorithm-A1 does.

Table 3.1: Total Solution Time in CPU Seconds

Data Set			Linear	Approx.	Normal	Approx.	LogNormal	Approx.		Algorithm.	A1
Sample Size	Asset Number	Iteration Range	Min. Solution Time	Max. Solution Time	Min. Solution Time	Max. Solution Time	Min. Solution Time	Max. Solution Time	Iteration Range	Min. Solution Time	Max. Solution Time
100	30	[2,4]	0.0920	0.1440	0.0960	0.1480	0.1680	0.2000	4	0.1480	0.1760
1000	30	[4,9]	0.4920	1.4081	0.0960	0.1600	0.1520	0.1760	7	1.1041	1.1481
2000	30	[4,15]	1.0161	5.1283	0.0800	0.1480	0.1400	0.1640	8	2.5242	2.6962
5000	30	[3,15]	0.7000	5.0763	0.0960	0.1560	0.1400	0.1680	8	2.5242	2.6642
100	50	[2,4]	0.1480	0.2440	0.1880	0.5080	0.7120	0.7961	4	0.2680	0.2920
1000	50	[2,11]	0.9721	3.4482	0.2120	0.5400	0.8201	0.8441	7	1.9881	2.0681
2000	50	[4,11]	2.0521	7.1324	0.4640	0.5320	0.7561	0.8481	8	4.9883	5.2763
5000	50	[3,11]	5.7444	20.3210	0.4600	0.5080	0.7320	0.8521	9	15.5410	16.5650
100	100	[3,5]	0.4720	1.0481	1.4161	3.2122	9.7846	11.2530	[4,5]	0.8081	1.4161
1000	100	[2,9]	2.7602	8.0485	2.4322	3.1122	11.2050	12.7370	7	5.2363	5.3403
2000	100	[5,11]	7.7685	21.0010	2.5482	3.2962	10.9010	12.9330	8	12.4770	13.9930
5000	100	[4,8]	16.4010	38.2580	2.8202	3.4042	9.7006	12.3130	9	40.6790	43.7510
100	200	[3,15]	0.6880	5.1283	0.1000	0.1520	0.1480	0.1640	[4,5]	1.1921	1.6401
1000	200	[5,11]	54.2030	116.0000	12.0210	13.2890	101.2400	135.9100	[7,9]	63.5200	88.9060
2000	200	[6,11]	25.6100	56.3640	9.8606	14.2730	98.4500	104.8500	8	31.8100	34.5500
5000	200	[4,11]	40.6490	253.1600	2.8802	21.6890	74.2130	116.6800	9	69.8720	132.3400

Each row in Table 3.1 summarizes a set of experiments conducted with different quantile targets (between 0.90 and 1.09) over the same training data identified by the number of assets and sample size. For each set of experiments, the minimum and maximum values of the observed solution time values for each approximation method are recorded in CPU seconds. In addition, the minimum and maximum values of the observed number of iterations to converge for the iterative methods are also presented.

Risk-Return Efficiency and Robustness

In this section, we analyze the performance of approximation methods based on two different types of data sets, namely training and testing data sets. The allocation decision is determined based on the training data set for each approach (Linear, Normal, and Log-Normal Approximation methods, and Algorithm-A1). Efficient frontiers of all approaches are calculated by using the training data set. Linear, Normal, and Log-Normal Approximation methods are run for given quantile targets to obtain highest expected portfolio return. Then, Algorithm-A1 is run (for each expected return-quantile target pair of the efficient frontier of the Linear Approximation method) to maximize the approximated quantile function treating each of given expected return value obtained by the Linear Approximation method as a target for the expected portfolio return and the investment decisions according to Algorithm-A1 are obtained. Furthermore, we compare the portfolio return rates based on these allocation decisions associated with the risk-return pairs and stock return realizations over the testing data sets, which are used as out-of-sample data sets for testing purposes. In addition, we use a performance measure (ω) which is the ratio between the portfolio return realization of the Linear Approximation method and that of the benchmark method with the training and testing data set. We provide 95% one-sided confidence interval (CI) of ω for each case. Please note that, 95% CI of ω while Algorithm A1 is used as a benchmark model is considered only for the testing (out-of-sample) data, since the expected portfolio return generated by the suggested algorithm and Algorithm-A1 are the same on the training data because of the efficient frontier generation method explained above.

We follow the same approach for four different numerical experiments set in each of which a different scenario generation method is used. Also, in each set of numerical experiments three different cases (daily, weekly, and monthly stock return scenarios) are considered. The target is to test the performance of the proposed data-driven algorithm over several data sets with different structures. For instance, 30, 50, and 100 stocks listed in the New York Stock Exchange (NYSE) are considered in all of the four sets of numerical experiments.

The goal of the proposed approach is to manage the downside risk. Therefore, we are particularly interested in the low quantile values such as the 5^{th} one. The quantile constraints will enforce that approximated quantile function values do not fall below the pre-specified q_m level 95% of the time (hence $\phi^{-1}(\alpha) = 1.645$). the linear transaction cost coefficients γ^+ and γ^- are assumed to be 0.02. Also, the lower bound and upper bounds on the holding in a single asset are selected to be 30% and 0% of the overall wealth. In addition, limit on sector holdings is assumed to be 50%. Simulations are performed using MATLAB R2012a and R Statistical Software.

Numerical Experiments Set 1

In this set of numerical experiments, three training data sets are composed of 100 daily, weekly, and monthly rate of return (ROR) observations of 30, 50, and 100 stocks listed in the New York Stock Exchange (NYSE) so that the last observation is of as of January 30, 2012. For each case, a testing data set is a random data set of 100 scenarios generated by Monte-Carlo simulation assuming that the stock returns follow a multivariate t-distribution. The parameters of the multivariate distribution are extracted from the historical data. In addition, for each stock a set of additional noisy data is generated from its left tail distribution (5^th percentile and lower). The additional noisy data are included in each testing data set in order to assure that the empirical probability distribution of the return rate has heavier left tail. In other words, we seek to compare the dependence between random stock returns, and the fat tailed nature of stock returns by the $t-copula$ and additional adverse return realizations. This way, we compare the robustness of the approximation methods against unexpected return rate movements within a similar (perturbed by additional noisy data) interdependence structure of the stock market.

In other words, the losses (rate of return values less than 1) are more likely to occur in testing data sets than in corresponding training data sets. This lets us compare the robustness of the approximation methods against undesired realizations of the stock returns. In other words, if the Linear Approximation approach performs better than benchmark models in terms of return-risk efficiency, then it can be inferred that the Linear Approximation method is more robust against undesired return movements within a similar (perturbed by additional noisy data) interdependence structure of the stock market.

The portfolio return realization according to both the testing (out-of-sample) and training (in-sample) data sets are calculated based on the allocation decision obtained over the training data for all of the approximation methods. Next, 95% confidence intervals for ω are calculated in order to compare the risk-return performance of the linear approximation method with that of the benchmark models.

Table 3.2 provides the 95% CI of ω for all benchmark models over both the testing and training data sets.The relative performance of the Linear Approximation method with respect to Algorithm-A1 over the training data set is not provided, since the expected portfolio return values of these approaches over the training data sets are the same because of the efficient frontier generation method explained earlier. Table 3.2 suggests that generally Linear Approximation method's investment decisions perform better than the those of of the benchmark models in both the testing and training data sets (except Algorithm-A1) with more than 95% confidence.

Numerical Experiments Set 2

In this section, we test the performance of the Linear Approximation method in terms of risk/return efficiency while daily, weekly and monthly training data sets are generated according to Monte Carlo simulation with Geometric Brownian Motion (GBM) and corresponding testing data sets are historical stock return observations. That is, 100 daily, weekly, and monthly historical observations of 30, 50 and, 100 stocks (Listed in NYSE) are used to forecast stock return realizations for the following 100 days, 100 weeks, and 100 months, respectively. These daily, weekly and, monthly stock return forecasts are used as training data sets and the actual stock return values during the same period are used as testing data sets. Investment decisions according to all of the quantile management approaches are determined based on these training data sets. Portfolio return realizations of these investment decisions with the actual stock returns (testing data) are compared.

95% one-sided confidence interval (CI) of ω for each case is constructed in order to compare the portfolio return realizations with both the testing and training data sets. Table 3.2 summarizes the results. The results suggest that the Linear Approximation method and the benchmark models' performances are similar when the data frequency is a day. However, the Normal Approximation benchmark method provides better portfolio return realizations in testing data for some observations. On the other hand, the Linear Approximation method method outperforms the benchmark methods with 95% confidence when the data frequency is a month and a week in both testing and training data sets.

Numerical Experiments Set 3

In this section, we generate three training data sets (daily, weekly, and monthly) using the Fama-French three-factor model. Daily, weekly and monthly series of factors obtained from Prof. Kenneth R. French's website
(http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/). That is, 100 daily, 100 weekly, and 100 monthly historical factor values are used to construct the Fama-French three-factor model for each of 30, 50, and 100 stocks (listed in NYSE). Next, stock return values of each stock for the following 100 days, 100 weeks, and 100 months respectively are forecast according to the corresponding three-factors model. The actual stock return values during the same periods are used as testing data sets. Investment decisions according to each method are determined based on training data sets. Portfolio return realizations over both actual stock return observations (testing data) and training data are compared.

Portfolios generated by the Linear Approximation method usually lead to higher return values than those generated by benchmark methods with both training and testing data sets according to Table 3.2.The risk-return efficiencies of Algorithm-A1 and the linear approximation are very close to each other.

Numerical Experiments Set 4

In this section we generate daily, weekly and monthly training data sets by using a multi-factor model with macro factors. We follow the forecasting approach presented in a working paper of the International Monetary Fund prepared by Oyama [161]. First, effective macro factors are selected among 10 macro factors by principal component analysis (PCA) using 100 daily, weekly and, monthly observations of each factor. The macro factors are West Texas Intermediate (WTI) Crude Oil Spot Price, Dow Jones Industrial Average Index (DJI), Aruoba-Diebold-Scotti (ADS) Business Conditions Index, US Dollar to Japanese Yen Exchange Rate, EURO to US Dollar Exchange Rate, Chicago Board of Options Exchange (CBOE) Volatility Index (VIX), BofA Merrill

Lynch US Corp AA Total Return Index, BofA Merrill Lynch US Corp BBB Total Return Index, 1-Year Treasury Constant Maturity Rate and 3-Month Treasury Constant Maturity Rate. In addition, we analyze the 30,50, and 100 stocks considered in the NYSE. We obtain each stock's exposure to macro factors by regressing its returns on the series of daily, weekly and monthly changes in daily, weekly and monthly growth rates of macro factor over the estimation period.

During principal component analysis both the Kaiser criterion and the value of the cumulative proportion of variance explained by the components are considered. That is, the components whose corresponding eigenvalues are greater than 1 are accepted. If the cumulative proportion of variance explained by the components is less than 80%, an additional component with the next highest eigenvalue is accepted as well. Effective factors are selected by associating each component with a factor by the VARIMAX rotation method in Principle Component Analysis (PCA). According to our study, when the data frequency is a day, a week, and a month, the number of effective macro factors are five, five, and four respectively.

Oyama [161] uses the residual of each regression model as an index representing the information explained by the market but not by other variables and uses this index as another factor. We follow the same approach and regress each individual stock's returns on effective macro factors and on this residual index in order to obtain the factor loadings for each stock.

We treat each factor as a stationary time series (according to Augmented Dickey-Fuller (ADF) test results) and fit a suitable Autoregressive Moving Average (ARMA) model to it by considering the autocorrelation function, the partial autocorrelation function and the maximum likelihood function value. In addition, the quality of fit for each time series is controlled via residual analysis.

We generate 100 daily, weekly, and monthly future scenarios for each factor based on its corresponding time series model. Next, future return scenarios for each individual stock are calculated according to the corresponding multi-factor model. These forecast scenarios stand for the training data set and investment decisions are made based on this training data set. Actual stock return realizations over the same period form the testing data set.

According to Table 3.2, when the data frequency is a month the Normal Approximation method provides higher expected portfolio return values than the Linear Approximation method over testing data sets for given quantile targets.

Table 3.2: Relative Risk-Return Efficiency of the Linear Approximation Method with respect to the Benchmark Models

			– Benchmark: Normal Approx.				–Benchmark: LogNormal Approx.				–Benchmark: Algorithm A1	
Data Set	Data Freq.	Asset Size	Testing Set Mean	Testing Set L.Bound	Training Set Mean	Training Set L.Bound	Testing Set Mean	Testing Set L.Bound	Training Set Mean	Training Set L.Bound	Testing Data Mean	Testing Data L.Bound
Set-1	Day	30	1.00081	1.00065	1.00200	1.00133	1.00794	1.00776	1.00741	1.00723	1.00013	1.00012
Set-1	Week	30	1.00136	1.00072	1.00633	1.00616	1.01205	1.01184	1.01156	1.01135	1.00012	1.00011
Set-1	Month	30	1.01456	1.00899	1.01108	1.01083	1.02333	1.02285	1.02357	1.02261	1.00005	1.00002
Set-1	Day	50	1.00094	1.00029	1.00075	1.00021	1.00830	1.00814	1.00746	1.00730	1.00011	1.00011
Set-1	Week	50	1.00098	1.00039	1.00367	1.00361	1.00883	1.00870	1.00922	1.00830	1.00012	1.00011
Set-1	Month	50	1.02058	1.01118	1.01113	1.01098	1.01894	1.01801	1.01875	1.01782	1.00015	1.00013
Set-1	Day	100	1.00114	1.00049	1.00066	1.00008	1.00906	1.00858	1.00878	1.00830	1.00012	1.00011
Set-1	Week	100	1.00115	1.00056	1.00202	1.00173	1.01113	1.01053	1.01108	1.01048	1.00012	1.00011
Set-1	Month	100	1.01472	1.01332	1.01115	1.01096	1.01256	1.01248	1.01166	1.01157	1.00015	1.00012
Set-2	Day	30	1.00012	0.99970	1.01171	1.01075	1.00213	1.00192	1.00166	1.00145	1.00012	0.99999
Set-2	Week	30	1.00923	1.00600	1.01214	1.01018	1.00929	1.00737	1.00969	1.00697	1.00011	1.00011
Set-2	Month	30	1.01238	1.01020	1.02036	1.01960	1.01206	1.01101	1.01158	1.01053	1.00011	0.99999
Set-2	Day	50	1.00016	0.99962	1.00994	1.00909	1.00125	1.00102	1.00095	1.00072	1.00011	1.00011
Set-2	Week	50	1.00671	1.00337	1.01670	1.01535	1.00639	1.00477	1.00725	1.00391	1.00011	1.00011
Set-2	Month	50	1.01235	1.00744	1.02279	1.02271	1.01203	1.00945	1.01134	1.00876	1.00011	1.00011
Set-2	Day	100	1.00010	0.99970	1.00916	1.00795	1.00130	1.00106	1.00057	1.00033	1.00002	1.00002
Set-2	Week	100	1.00940	1.00613	1.01684	1.01632	1.00934	1.00754	1.00839	1.00659	1.00012	1.00012
Set-2	Month	100	1.01487	1.01233	1.02385	1.02317	1.01516	1.01354	1.01508	1.01332	1.00012	1.00012
Set-3	Day	30	0.99999	0.99980	1.00116	1.00040	1.00190	1.00176	1.00120	1.00105	1.00010	1.00004
Set-3	Week	30	1.01331	1.00910	1.02303	1.02294	1.01333	1.01277	1.01287	1.01231	1.00003	0.99999
Set-3	Month	30	1.00143	1.00132	1.03006	1.02983	1.01543	1.01419	1.01465	1.01341	1.00010	1.00009
Set-3	Day	50	1.00115	1.00070	1.00582	1.00511	1.00114	0.99979	1.00183	1.00048	1.00010	1.00009
Set-3	Week	50	0.99993	0.99992	1.02561	1.02518	1.02310	1.02171	1.02258	1.02119	1.00004	0.99999
Set-3	Month	50	1.00096	1.00051	1.03666	1.03568	1.02910	1.02875	1.02876	1.02841	1.00002	0.99999
Set-3	Day	100	1.00052	1.00034	1.01907	1.01873	1.00252	1.00053	1.00343	1.00145	1.00011	1.00009
Set-3	Week	100	1.01460	1.00999	1.02776	1.02722	1.02436	1.02398	1.02383	1.02346	1.00010	1.00009
Set-3	Month	100	1.00184	1.00173	1.03989	1.03920	1.02847	1.02758	1.02821	1.02731	1.00010	1.00009
Set-4	Day	30	1.00151	1.00139	1.00048	1.00045	1.00150	1.00132	1.00184	1.00166	1.00010	1.00009
Set-4	Week	30	1.00126	1.00091	1.00343	1.00324	1.00126	1.00085	1.00109	1.00068	1.00010	1.00009
Set-4	Month	30	1.00265	1.00243	1.01224	1.01207	1.00256	1.00242	1.00188	1.00174	1.00010	1.00009
Set-4	Day	50	1.00066	1.00041	1.00165	1.00156	1.00066	1.00040	1.00092	1.00066	1.00010	1.00008
Set-4	Week	50	1.03039	1.02715	1.00888	1.00871	1.03024	1.02674	1.02834	1.02485	1.00000	0.99998
Set-4	Month	50	0.99819	0.99551	1.01386	1.01379	1.02013	1.01780	1.01947	1.01714	1.00010	1.00008
Set-4	Day	100	1.01098	1.00777	1.00269	1.00257	1.01098	1.00749	1.01064	1.00715	1.00010	1.00009
Set-4	Week	100	1.00630	1.00426	1.01174	1.01168	1.00626	1.00386	1.00571	1.00331	1.00010	1.00009
Set-4	Month	100	1.02177	1.01837	1.01976	1.01971	1.02079	1.01736	1.02081	1.01738	1.00010	1.00008

Closeness to Optimality

In this section, we measure the closeness of the solutions proposed by the Linear Approximation method to optimality. We assume that the optimal solution is obtained by the Normal (Log-Normal) Approximation method when the data set is composed of Normally (Log-Normally) distributed stock return scenarios since the Normal (Log-Normal) Approximation method assumes that the stock returns are Normally (Log-Normally) distributed.

Multi-variate Normally and Log-Normally distributed rate of return (ROR) scenarios are generated according to the sample mean and standard deviation of historical data set composed of 100 observations of 30 stocks listed in NYSE. Sixty different cases are considered, namely, the cases where the data sets are composed of 500, 1000 and 2000 scenarios for Normally and Log-Normally distributed RORs of 10, 20, 30,..,90, and 100 assets. Figure 3.1, Figure 3.2, and Figure 3.3 represent the empirical probability distribution of averages of Normally and Log-Normally distributed 1000 ROR scenarios of 10, 50 and 100 stocks, respectively.

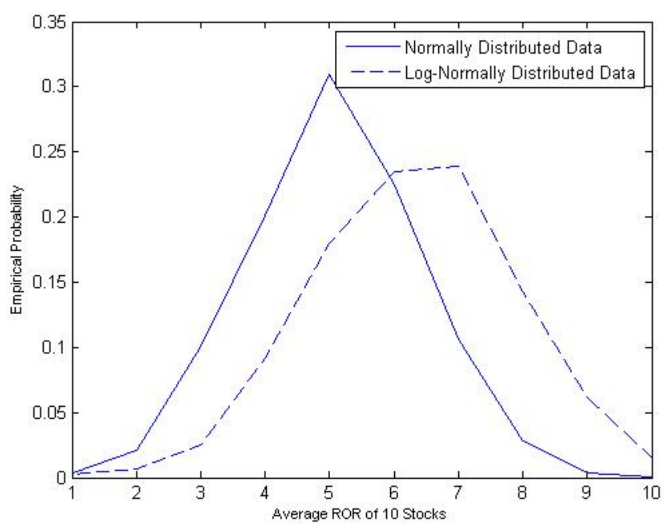


Figure 3.1: Empirical Distribution of Average ROR of 10 Stocks

We define the measure for closeness to optimality, θ, as the relative difference between the

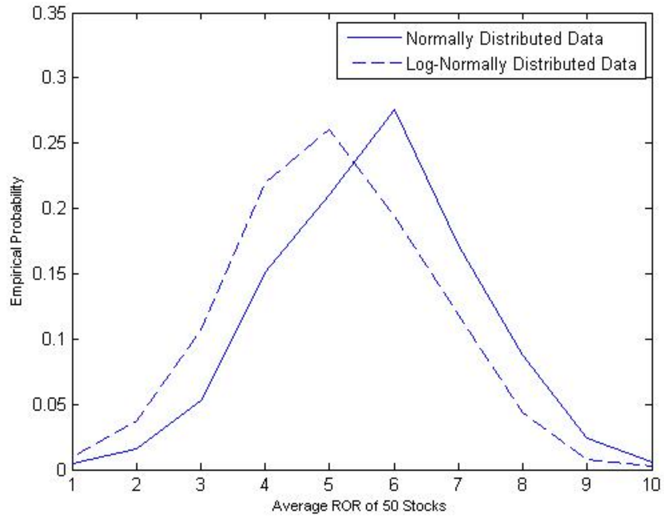


Figure 3.2: Empirical Distribution of Average ROR of 50 Stocks

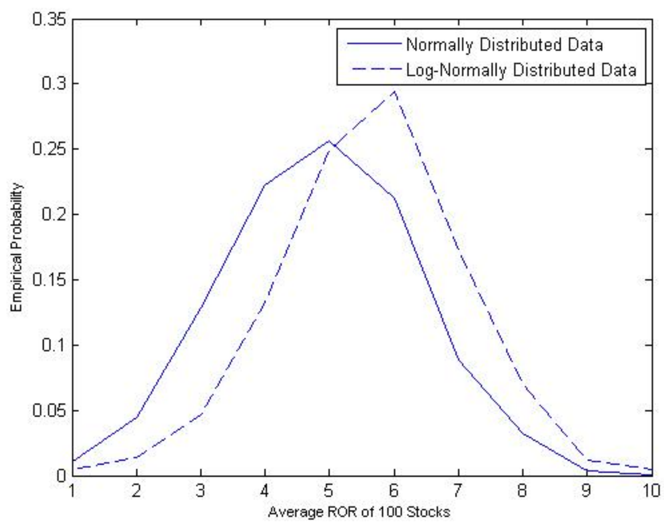


Figure 3.3: Empirical Distribution of Average ROR of 100 Stocks

objective function values of the Linear Approximation and Normal (LogNormal) Approximation methods, where

$$\theta = \frac{|Obj^{*} - Obj_{App}|}{|Obj^{*}|}.$$

Table 3.3 and Table 3.4 provide the 95^{th} quantile value for θ obtained by comparing the proposed Linear Approximation method and the benchmark approximation methods (the Normal approximation and the log-Normal approximation) over samples of observations. For each data set, the Linear Approximation method and the corresponding benchmark model are run several times with different quantile targets (q_m), then θ values are obtained for each single run. Next, the 95^{th} quantile value for θ is calculated from the sample of θ specific to the corresponding data set. Table 3.3 and Table 3.4 suggest that solutions suggested provided by the Linear Approximation approach are close to optimality.

We use also Brute Force method as another benchmark to measure the closeness of the Linear Approximation method to optimality. The Brute Force method consists in enumerating all possible candidates for the solution and selecting the one which satisfies the constraints and provides the best objective function. Therefore, it leads to the optimal solution [175].

In this study, 2 assets and 500 observations of both are considered. Both the Linear Approximation method and the Brute Force method are run for the same q_m targets and the 95^{th} quantile value is calculated for 10 different data sets. The results are summarized in Table 3.5.

Table 3.3: 95^{th} Quantile for θ, Normally Distributed Data

Number of Scenarios	Number of Assets	95^{th} Quantile for θ
500	10	0.000101
500	20	0.000017
500	30	0.000302
500	40	0.000333
500	50	0.000103
500	60	0.000272
500	70	0.000178
500	80	0.000112
500	90	0.000133
500	100	0.000099
1000	10	0.000089
1000	20	0.000156
1000	30	0.000083
1000	40	0.000193
1000	50	0.000188
1000	60	0.000385
1000	70	0.000371
1000	80	0.000032
1000	90	0.000107
1000	100	0.000061
2000	10	0.000033
2000	20	0.000077
2000	30	0.000115
2000	40	0.000343
2000	50	0.000071
2000	60	0.000063
2000	70	0.000102
2000	80	0.000143
2000	90	0.000126
2000	100	0.000184

Table 3.4: 95^{th} Quantile for θ, Log-Normally Distributed Data

Number of Scenarios	Number of Assets	95^{th} Quantile for θ
500	10	0.000175
500	20	0.000403
500	30	0.000136
500	40	0.000426
500	50	0.000269
500	60	0.000279
500	70	0.000214
500	80	0.000220
500	90	0.000212
500	100	0.000131
1000	10	0.000049
1000	20	0.000003
1000	30	0.000148
1000	40	0.001165
1000	50	0.000088
1000	60	0.000216
1000	70	0.000282
1000	80	0.000074
1000	90	0.000528
1000	100	0.000529
2000	10	0.000116
2000	20	0.000158
2000	30	0.000050
2000	40	0.000110
2000	50	0.000180
2000	60	0.000136
2000	70	0.000140
2000	80	0.000052
2000	90	0.000056
2000	100	0.000118

Table 3.5: 95^{th} Quantile for θ, Benchmark: Brute Force Method

Data Set Index	95% CI for θ
1	0.000383
2	0.000045
3	0.000104
4	0.000181
5	0.000025
6	0.000002
7	0.000047
8	0.000032
9	0.000063
10	0.000006

3.2 Extension to Inter-quartile Range Management

3.2.1 Problem Setup

In this section, we describe how our approach can be applied to risk management problems where risk (to be minimized) is defined as the inter-quantile range of a random variable, i.e., the 75^{th} percentile minus the 25^{th} percentile of a variable such as a portfolio return. This measure is commonly used in financial management to quantify risk but has not been used so far in the context of portfolio optimization due to the difficulty in optimizing quantiles. Our approach can be extended to the difference of any quantiles of the random objective. For instance, a decision maker interested in downside risk may prefer to minimize the difference between the median of the objective and its 25^{th} percentile.

In what follows, we will refer to a, respectively b, as the rank of the observation corresponding to the lower, respectively higher, quantile. The problem, using similar notation in Problem (3.2) is as follows:

$$\begin{aligned}
\min \quad & \frac{1}{W}\left[\left(\sum_{i=1}^{n} r_{t,i}x_i\right)_{(b)} - \left(\sum_{i=1}^{n} r_{t,i}x_i\right)_{(a)}\right] \\
\text{s.t.} \quad & \sum_{i=1}^{n} \mu_i x_i \geq \tau W, \\
& x_i = x_i^0 + x_i^+ - x_i^-, \ \forall i, \\
& x_i^+, x_i^- \in X.
\end{aligned} \tag{3.8}$$

Problem (3.8) can be written as:

$$
\begin{aligned}
\min \quad & q_b - q_a \\
\text{s.t.} \quad & q_a \leq \tfrac{1}{W} \left(\sum_{i=1}^{n} r_{t,i} x_i \right)_{(a)}, \\
& q_b \geq \tfrac{1}{W} \left(\sum_{i=1}^{n} r_{t,i} x_i \right)_{(b)}, \\
& \sum_{i=1}^{n} \mu_i x_i \geq W\tau, \\
& x_i = x_i^0 + x_i^+ - x_i^-, \ \forall i, \\
& x^+, x^- \in X.
\end{aligned}
$$

We need to rank the scenarios in order to determine the worst $a-1$ and b scenarios while the investment decision is given. For specific ranks a and b, the vectors z^a and z^b are the active-scenario identification vectors such that we have $z_t^a = 1$, if scenario t is among those that achieve the $a-1$ smallest returns and $z_t^b = 1$, if scenario t among those that achieve b smallest returns. The vectors z^a and z^b will have $a - 1 + T - b$ values in common ($a-1$ "ones" and $T-b$ "zeros").

The problem to obtain a decision allocation for given ranking vectors z_t^a and z_t^b is formulated as follows:

$$
\begin{aligned}
\min_{q_m, q_a, x} \quad & q_b - q_a \\
\text{s.t.} \quad & (1 - z_t^a) q_a \leq \frac{1}{W} \sum_{i=1}^{n} r_{t,i} x_i (1 - z_t^a), \ \forall t, \\
& z_t^b q_b \geq \frac{1}{W} \sum_{i=1}^{n} x_i r_{t,i} z_t^b, \ \forall t, \\
& \sum_{i=1}^{n} \mu_i x_i \geq W\tau, \\
& x_i = x_i^0 + x_i^+ - x_i^-, \ \forall i, \\
& x^+, x^- \in X.
\end{aligned}
\tag{3.9}
$$

The algorithm for the interquantile range management problem is as provided below is nearly the same as the algorithm 3.1.The only difference is that we need to identify $a - 1$ and b worst case scenarios instead of $m - 1$ scenarios as in the algorithm 3.1.

3.2.2 Numerical Results

Numerical results related to CPU time and number of iteration to convergence, and a representative efficient frontier for the inter-quartile range management algorithm are provided in this section. Each row in Table 3.6 represents a set of experiments where the interquartile range management problem is solved with several expected portfolio return targets over the data set with the same number of observations and assets. The numerical experiments are repeated with different data sets having various number of scenarios and assets.

From Table 3.6, we see that the number of iterations and time to convergence for the inter-quartile range management algorithm are slightly higher than those for the quantile management algorithm presented in the previous section. As number of scenarios increases, the number of iteration to convergence also increases, because determining the worst case scenarios leading to the min inter-quartile range value becomes harder as more scenarios are considered.

Figure 3.4 points out the trade-off between the inter-quartile range (the difference between the third and the first quartile values of the return) and the target or the expected portfolio ROR on the data set which is composed of 100 observations of 50 stocks. As the target for expected portfolio return increases, the risk, which is defined by the difference between two pre-specified percentile levels, also increases.

Table 3.6: Amount of Time (CPU Seconds) and Number of Iterations to Convergence

Sample Size	Asset Number	Iteration Range	Min. Solution Time	Max. Solution Time
100	30	[2,6]	0.0600	0.3240
1000	30	[3,7]	0.3120	0.8401
2000	30	[2,18]	0.3200	6.4924
5000	30	[2,16]	1.0121	7.6165
100	50	[3,4]	0.1720	0.2440
1000	50	[2,13]	0.3120	4.4723
2000	50	[5,16]	3.1002	10.4726
5000	50	[4,17]	7.6085	24.4694
100	100	[5,7]	1.0521	1.2841
1000	100	[5,10]	3.4723	8.8246
2000	100	[4,12]	7.6565	24.1252
5000	100	[3,6]	10.1850	26.4140
100	200	[2,5]	0.5120	1.9081
1000	200	[3,12]	31.3380	133.0380
2000	200	[2,15]	9.9006	63.2880
5000	200	[3,18]	40.0430	293.5320

3.3 Extension to Portfolio Management with TCE

The specific methodology we propose in this extension involves representing the observation of a given rank using the difference between sum of sorted values, specifically, if $y_{(m)}$ is the m-th lowest observation among $y_1, ..., y_T$, we have:

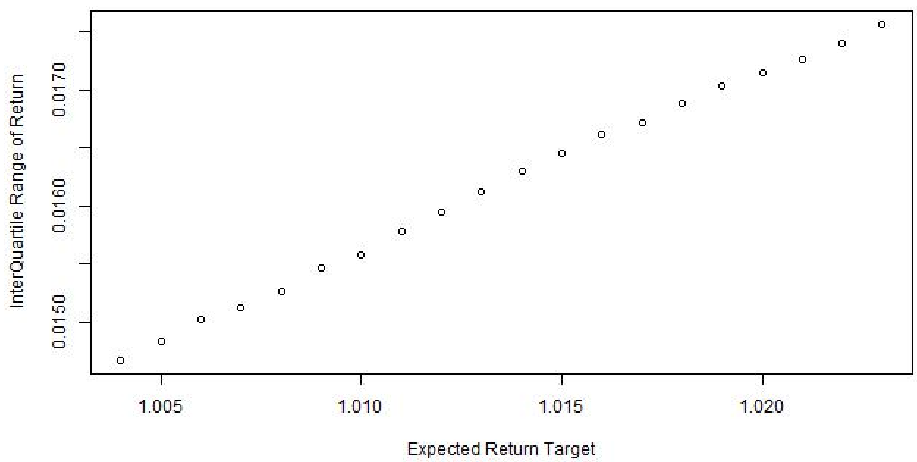


Figure 3.4: Efficient Frontier, Daily Return Data, $\alpha = 0.05$

$$y_{(m)} = \sum_{j=1}^{m} y_{(j)} - \sum_{k=1}^{m-1} y_{(k)}.$$

The following lemma reminds the reader that the value of a sum of sorted values can be computed by solving a linear programming problem.

Lemma 3.3 *Let $y_1, \ldots, y_T$ be a given series of numbers. $\sum_{j=1}^{r} y_{(j)}$ is the optimal objective of:*

$$\begin{aligned} \min \quad & \sum_{j=1}^{T} y_j z_j \\ \textit{s.t.} \quad & \sum_{j=1}^{T} z_j = w, \\ & 0 \leq z_j \leq 1, \ \forall j, \end{aligned} \tag{3.10}$$

or equivalently of:

$$\begin{aligned} \max \quad & w \cdot \theta_w + \sum_{j=1}^{T} \beta_{wj} \\ \text{s.t.} \quad & \theta_w + \beta_{wj} \leq y_j, \ \forall j, \\ & \beta_{wj} \leq 0, \ \forall j. \end{aligned} \tag{3.11}$$

Proof: The first formulation is proved for instance in [33]. The second formulation follows from applying strong duality to the first formulation, which holds since the feasible set of the primal is non-empty and bounded. □

This leads us to the following (not yet tractable) reformulation of Problem (3.2):

Theorem 3.4 (Formulation with many problems, $m = \lfloor (1-\alpha) * T \rfloor$**)** *Let S_{m-1} be the number of extreme points of the set $\mathcal{Z} = \{\sum_{j=1}^{T} z_j = m-1, \ 0 \leq z_j \leq 1, \ \forall j\}$ and let z^s_{m-1}, $s = 1, \ldots, S_{m-1}$ be the corresponding extreme points. The optimal objective and solution of Problem (3.2) are obtained by solving Subproblem s, $s = 1, \ldots, S_{m-1}$:*

$$\begin{aligned} \max \quad & \frac{1}{W} \sum_{i=1}^{n} \mu_i \, x_i \\ \text{s.t.} \quad & m \cdot \theta_m + \sum_{t=1}^{T} \beta_{mt} - \sum_{t=1}^{T} \left(\sum_{i=1}^{n} r_{ti} \, x_i \right) z^s_{t,m-1} \geq W q_m, \\ & \theta_m + \beta_{mt} \leq \sum_{i=1}^{n} r_{ti} \, x_i, \ \forall t, \\ & \beta_{mt} \leq 0, \ \forall t, \\ & \\ & x_i = x_i^0 + x_i^+ - x_i^-, \ \forall i, \\ & x^+, x^- \in X. \end{aligned} \tag{3.12}$$

and keeping the optimal objective and solution of the subproblem with the highest objective value.

We refer the reader to Section 3.4.2 for the proof of Theorem 3.4.

Our implementation of the approach described in Theorem 3.4 is based on the observation that we actually do not need to generate all the subproblems, since our goal is really to find $\min_{z \in \mathcal{Z}_{m-1}} \sum_{t=1}^{T} \left(\sum_{i=1}^{n} r_{ti} \, x_i \right) z_t$. So we will implement a delayed problem generation approach, where we only generate subproblems corresponding to corner points z_{m-1}^s that have been previously found to achieve the minimum over $\mathcal{Z}_{m-1}$ of $\sum_{t=1}^{T} \left(\sum_{i=1}^{n} r_{ti} \, x_i \right) z_t$ for a candidate solution x. Other (non-optimal) z_{m-1}^s do not need to be considered since they would lead to a no-smaller (than what is already available) value of the right-hand side of the first constraint in Problem (3.12) and therefore a no-larger feasible set and a no-larger objective.

We provide our algorithm in more details below.

Algorithm 3.5 (Delayed Problem Generation)

Step 1 *Start with a feasible solution $x \in X$ to serve as a candidate solution $\bar{x}$.*

Step 2 *Solve Problem (3.10) for the candidate solution $\bar{x}$ and $r = m - 1$, and obtain an optimal corner point z_{m-1}^s for some $s = 1, \ldots, S_{m-1}$.*

Step 3 *Solve the subproblem s defined by Problem (3.12) for the s and z_{m-1}^s identified in Step 2. Obtain a new candidate solution.*

Step 4 *Repeat Steps 2 and 3 until the algorithm generates the same corner point of $\mathcal{Z}_{m-1}$ or the same candidate solution x in two consecutive steps, whichever happens sooner.*

Note that Step 4 checks both consecutive z and consecutive x because there may be multiple optimal solutions.

Figure 3.6 and Figure 3.5 shows how portfolio Return and TCE changes when α is 5%.

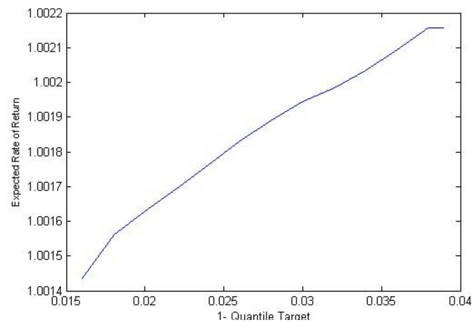


Figure 3.5: Efficient Frontier, Daily Return Data, $\alpha = 0.05$

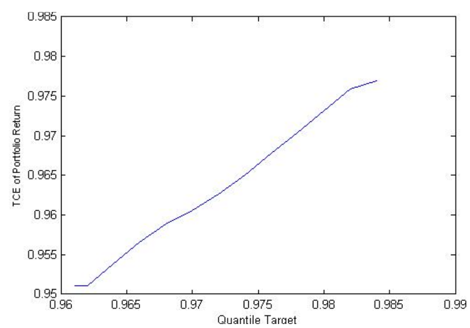


Figure 3.6: TCE vs Risk, Daily Return Data, $\alpha = 0.05$

3.4 Conclusions

In this chapter, we investigated an approximation method to solve the portfolio management problem with quantile constraints. The algorithm that we proposed is tractable, since it requires solving a series of linear problems iteratively. The numerical experiments we performed suggest that our method leads to robust portfolio decisions against adverse realizations of the returns. In addition, our method usually leads to more efficient portfolio allocation decisions than well known Gaussian approximation methods ([113], [130], and [174]) and an iterative data-driven approximation algorithm ([124]). We also extended our work to the inter-quantile range risk management problem.

Acknowledgments

We would like to thank to the audience of our talk at the 21st International Symposium on Mathematical Programming (ISMP 2012) in Berlin, Germany for their insightful comments.

Appendix

3.4.1 Log-Normal Sum Approximation with Moment Matching Approach

Denote $e^{R_i^t}$ the return of stock i during time period t. Then return of stock i from time 1 to time T is $e^{\sum_{t=1}^T R_i^t}$. Therefore, the portfolio return over T period can be formulated as:

$$W = \sum_{i=1}^{n} x_i e^{\sum_{t=1}^T R_i^t}.$$

Then, the first and the second moments of the portfolio return are calculated as:

$$\begin{aligned} E[W] &= \sum_{i=1}^{T} x_i E[e^{\sum_{t=1}^T R_i^t}] \\ &= \sum_{i=1}^{n} e^{(\tilde{\mu_i} T + \frac{\tilde{\sigma_i}^2 T}{2})} \end{aligned} \quad (3.13)$$

$$\begin{aligned} E[W^2] &= E\left[\left(\sum_{i=1}^{T} x_i e^{\sum_{t=1}^T R_i^t}\right)^2\right] \\ &= \sum_{i=1}^{n}\left(x_i^2 e^{2T\tilde{\mu_i}+2T\tilde{\sigma_i}^2} + \sum_{j=1, j\neq i}^{n} x_i x_j e^{\left((\tilde{\mu_i}+\tilde{\mu_j})T + \frac{T}{2}(\tilde{\sigma_i}^2+\tilde{\sigma_j}^2+2\rho_{i,j}\tilde{\sigma_i}\tilde{\sigma_j})\right)}\right) \end{aligned} \quad (3.14)$$

We define the vector $b \in \mathcal{R}^n$ such that

$$b[i] = e^{(\tilde{\mu_i} T + \frac{\tilde{\sigma_i}^2 T}{2})} \; \forall i \in \{1, ..., n\},$$

and the matrix A $\in \mathcal{R}^{nxn}$ such that

$$A[i,j] = e^{\left((\tilde{\mu_i}+\tilde{\mu_j})T+\frac{T}{2}(\tilde{\sigma_i}^2+\tilde{\sigma_j}^2+2\rho_{i,j}\tilde{\sigma_i}\tilde{\sigma_j})\right)} \; \forall i \in \{1,...,n\}\,,\; \forall j \in \{1,...,n\}, \text{ and } i \neq j$$

$$A[i,i]] = e^{2T\tilde{\mu_i}+2T\tilde{\sigma_i}^2} \; \forall i \in \{1,...,n\}.$$

The Log-Normal approximation of portfolio return is represented as e^Y where $Y \sim N(\mu^*, \sigma^*)$. Then the following equations hold:

$$\begin{aligned} E[W] &= b'x = E[e^Y] = e^{\mu^*+\frac{\sigma^{*2}}{2}} \\ E[W^2] &= x'Ax = e^{2\mu^*+2\sigma^{*2}} \end{aligned} \tag{3.15}$$

The solution of this system of equations is as follows:

$$\begin{aligned} \mu^* &= 2ln(b'x) - \frac{1}{2}ln(x'Ax) \\ \sigma^{*2} &= ln(x'Ax) - 2ln(b'x) \end{aligned} \tag{3.16}$$

Then, the expected return maximization problem with quantile constraint is written as:

$$\begin{aligned} \max \quad & \left(e^{\mu^*+\frac{\sigma^{*2}}{2}}\right) \\ \text{s.t.} \quad & \mu^* + \phi^{-1}(\alpha)\sigma^* \geq ln(q_m), \\ & x \in X, \end{aligned}$$

which is equivalent to

$$
\begin{aligned}
\max \quad & b^T x \\
\text{s.t.} \quad & 2\ln(b^T x) - \frac{1}{2}\ln(x^T A x) + \phi^{-1}(\alpha)\sqrt{\ln(b^T x) - 2\ln(x^T A x)} \geq \ln(q_m), \\
& x \in X.
\end{aligned} \tag{3.17}
$$

3.4.2 Proof of Theorem 3.12

Proof: Follows from using Problem (3.11) to represent $\sum_{k=1}^{m} \left(\sum_{i=1}^{n} r_{\cdot i}\, x_i\right)_{(k)}$ with $w = m$ and Problem (3.10) to represent $\sum_{k=1}^{m-1} \left(\sum_{i=1}^{n} r_{\cdot i}\, x_i\right)_{(k)}$ with $w = m - 1$. (The two sums of sorted values must be treated differently since they are on different sides of the inequality.)

Looking at the part corresponding to Problem (3.11) only, we have the optimal objective of a maximization problem that must be equal to or greater than a threshold, so it is necessary and sufficient to find a feasible solution whose objective is equal to or greater than that threshold. (In other words, we can remove the maximization operator.)

Looking at the part corresponding to Problem (3.10) only, we have the optimal objective of a minimization problem that must be equal to or smaller than a threshold, but since it is a linear problem and the feasible set is non-empty and bounded there will be an optimal solution at a corner point of the feasible set, so it is necessary and sufficient to have one corner point (which we enumerate, leading to S_{m-1} subproblems) whose objective is equal to or greater than that threshold.

Chapter 4

Portfolio Management with Moment Matching Approach

4.1 Problem Setup

We will use the following notation throughout the chapter.

T : the length of the investment horizon,

W : the portfolio return,

x_i : the fraction of the portfolio allocated to asset i,

R_i^t : the natural logarithm of the return on asset i at time t,

μ_i : the location parameter of the Log-Normal distribution of the return on asset i,

σ_i : the scale parameter of the Log-Normal distribution of the return on asset i,

$\rho_{i,j}$: the correlation coefficient between the natural logarithms of two Log-Normally distributed random variables with indexes i and j,

μ : the location parameter of the Log-Normal distribution of the portfolio return,

σ : the scale parameter of the Log-Normal distribution of the portfolio return,

w : the left tail parameter for the portfolio return distribution,

w_f : the target for the expected portfolio return,

n : the total number of stocks,

Y : the random variable representing the natural logarithm of the portfolio return.

We assume that $w < w_f$ throughout the chapter and define the matrix A $\in \mathcal{R}^{n\times n}$ such that:

$$A_{i,j} = e^{(\mu_i+\mu_j)T+\frac{T}{2}(\sigma_i^2+\sigma_j^2+2\rho_{i,j}\sigma_i\sigma_j)} \; \forall i, j$$

and the vector $b \in \mathcal{R}^n$ such that:

$$b_i = e^{\mu_i T+\frac{\sigma_i^2 T}{2}} \; \forall i \in \{1, ..., n\}.$$

Let Φ be the cumulative distribution function of a standard Gaussian random variable.

Lemma 4.1 (Probability in the Moment Matching Approximation) *For a given allocation x, the cumulative probability value of the portfolio return at w is approximated by:*

$$P(W \leq w) \;\; \approx \;\; \Phi\left(\frac{\ln(w) - 2\ln(b'x) + \frac{1}{2}\ln(x'Ax)}{\sqrt{\ln(x'Ax) - 2\ln(b'x)}}\right). \tag{4.1}$$

Proof. Denote $e^{R_i^t}$ the return on stock i during time period t. Then the return on stock i from time 1 to time T is $e^{\sum_{t=1}^T R_i^t}$ and the portfolio return over T periods is given by:

$$W = \sum_{i=1}^n x_i \, e^{\sum_{t=1}^T R_i^t},$$

with $\sum_{t=1}^T R_i^t$ obeying a Normal distribution with mean $T\mu_i$ and variance $T\sigma_i$. Therefore the return on stock i from time 1 to time T, as Log-Normal random variable, has mean $e^{\mu_i T+\frac{\sigma_i^2 T}{2}}$

and variance $\left(e^{\sigma_i^2 T}-1\right)\cdot e^{2\mu_i T+\sigma_i^2 T}$ with the first moment:

$$\begin{aligned} E[W] &= \sum_{i=1}^{n} x_i\, e^{\mu_i T+\frac{\sigma_i^2 T}{2}} \\ &= b'x, \end{aligned}$$

and the second moment:

$$\begin{aligned} E[W^2] &= E\left[\left(\sum_{i=1}^{n} x_i e^{\sum_{t=1}^{T} R_i^t}\right)^2\right] \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j E\left[e^{\sum_{t=1}^{T} R_i^t} e^{\sum_{t=1}^{T} R_j^t}\right] \\ &= \sum_{i=1}^{n}\sum_{j=1}^{n} x_i x_j E\left[e^{\sum_{t=1}^{T}(R_i^t+R_j^t)}\right] \\ &= x'Ax, \end{aligned}$$

where we have used that $\sum_{t=1}^{T}(R_i^t+R_j^t)$ obeys a Normal distribution with mean $(\mu_i+\mu_j)T$ and variance $(\sigma_i^2+\sigma_j^2+2\,\rho_{i,j}\sigma_i\sigma_j)T$. Note that because $Var(W)=E(W^2)-E(W)^2$ is always non-negative, we always have $x'Ax\geq(b'x)^2$.

The Log-Normal approximation of the portfolio return is denoted e^Y where $Y\sim N(\mu,\sigma^2)$. Matching the first two moments of the approximation and the true distribution yields:

$$\begin{aligned} E[W] &= b'x = E[e^Y] = e^{\mu+\frac{\sigma^2}{2}} \\ E[W^2] &= x'Ax = e^{2\mu+2\sigma^2}. \end{aligned}$$

This leads to:

$$\begin{aligned} \mu &= 2\ln(b'x)-\frac{1}{2}\ln(x'Ax) \\ \sigma &= \sqrt{\ln(x'Ax)-2\ln(b'x)}. \end{aligned}$$

Eq.(5.3) follows by computing the probability of Y, which obeys a Gaussian distribution, falling below $\ln(w)$.

4.2 Portfolio Optimization Approach

Our goal here is to minimize the risk (measured by the probability of the portfolio return's falling below w) while keeping the expected return on the portfolio at a specified level w_f or above, using the approximate formulations obtained via the Fenton-Wilkinson method. Let X be the feasible set for the allocation x, assumed to be a polyhedron. (In its simplest form, $X = \{x | e'x = 1\}$, indicating that fractions invested must sum to 1. If short sales are not allowed, there will also be non-negativity constraints on the decision variables.)

Theorem 4.2 (Solving the approximated problem) *Let F be the function defined on $[w_f, \infty)$ by the quadratic programming problem:*

$$\begin{aligned} F(v) = \min \quad & x'Ax \\ \textit{s.t.} \quad & b'x = v \\ & x \in X. \end{aligned} \tag{4.2}$$

The optimal allocation x of the portfolio problem approximated via moment matching is the optimal solution of Problem 4.2 with v the optimal solution of:

$$\min_{v \geq w_f} \quad \frac{\ln\left(\frac{w\sqrt{F(v)}}{v^2}\right)}{\sqrt{\ln\left(\frac{F(v)}{v^2}\right)}}. \tag{4.3}$$

Proof. Using Lemma 4.1, the portfolio management problem is formulated as:

$$\begin{aligned} \min_{x} \quad & \frac{\ln(w) - 2\ln(b'x) + \frac{1}{2}\ln(x'Ax)}{\sqrt{\ln(x'Ax) - 2\ln(b'x)}} \\ \text{s.t.} \quad & b'x \geq w_f, \\ & x \in X. \end{aligned}$$

or equivalently, by conditioning on $b'x$:

$$\min_{v \geq w_f, x \in X, b'x = v} \quad \frac{\ln(w) - 2\ln(v) + \frac{1}{2}\ln(x'Ax)}{\sqrt{\ln(x'Ax) - 2\ln(v)}}.$$

Let denote y^* the smallest value of $\ln(x'Ax)$ achievable over $\{x \in X,\, b'x = v\}$ for a given v. (We drop the dependence of y^* in v for the sake of clarity.) We check by taking the first derivative that the slope of the function f defined over $[y^*, \infty)$ by $f(y) = \dfrac{\alpha + \frac{1}{2}y}{\sqrt{y - \beta}}$ (with $\alpha = \ln(w) - 2\ln(v)$ and $\beta = 2\ln(v)$) is first negative then positive with a sign change at $y = 2(\alpha + \beta) = 2\ln(w)$. Therefore, its minimum is achieved at $2\ln(w)$ if $2\ln(w) \geq y^*$ and y^* otherwise; however, we have seen earlier that $x'Ax > (b'x)^2$ which, with the present notations, translates into $y > 2\ln(v)$, and we have stated at the beginning of the chapter that we assume $w < w_f$ throughout the study due to the meaning of w (left-tail parameter) and w_f (expected ROR target), hence $w < v$ (because $w_f \leq v$). It follows that we must have $2\ln(w) < 2\ln(v) < y$. This eliminates $2\ln(w)$ as the candidate minimum. As a result, the minimum of our problem is achieved for $y = y^*$, or equivalently, $x'Ax$ as small as possible over the set of feasible solutions: $\{x \in X,\, b'x = v\}$ at v given.

Remark: Because $E[W] = b'x$ is set, minimizing $x'Ax = E[W^2]$ is equivalent to minimizing the variance of the portfolio. Hence, the feasible allocations for the problem are reduced to those on the efficient frontier, in the Markowitz (mean-variance) sense, and the manager only has to decide which expected return to request, or equivalently, where on the efficient frontier to place himself, which is achieved by solving Problem (4.3).

4.3 Portfolio Management Algorithm

As shown in the numerical experiments in the next section, the Fenton-Wilkinson approximation seems to capture well the central tendency of the portfolio return; however, the dispersion is not caught as well as the central tendency. This is an issue for the application considered, since we use the moment-matching approximation to estimate a left-tail probability with the goal of protecting the decision-maker against adverse events. Therefore, we present here a method to update the matrix A (in a way made precise below) to yield more precise estimates of the probability of interest; a change in A is directly reflected to the function $\ln(x'Ax)$, which is the natural logarithm of the second moment of the portfolio return when the allocation x is given, and thus is related to the dispersion or volatility of the portfolio return.

In order to test and control the accuracy of the Log-Normal sum approximation at this point, we define a performance measure, θ, which is the difference between the *empirical value* of the cumulative probability of the Log-Normal sum and the *actual value* of the Log-Normal sum approximation at w for a given portfolio allocation decision x. Let denote the cumulative probability for the Log-Normal sum approximation as π, calculated according to the random sample generated. Then,

$$\theta = P(W \leq w) - \pi.$$

Ideally, we would want θ to be zero. Our goal is to adjust A so that $P(W \leq w)$, computed with the moment-matching approximation for a new A, is equal to π. Since A appears in Eq.(5.3) through $\ln(x'Ax)$, we define z such that:

$$\Phi\left(\frac{ln(w^*) - 2ln(b'x) + \frac{1}{2}z}{\sqrt{z - 2ln(b'x)}}\right) = \pi$$

We consider an update for A of the type:

$$A := e^{\epsilon}A.$$

It is easy to check that we will have $z = \ln(x'Ax)$ with the new A for ϵ given by:

$$\epsilon = z - ln(x'Ax).$$

This leads to the following algorithm, for given stopping parameters $\epsilon_1 > 0$ and $\epsilon_2 > 0$:

Algorithm 4.3 (Iterative algorithm)

Step 1 *Start with a feasible solution $x \in X$ to serve as a candidate solution and set the iteration number, $s = 0$. Apply the Fenton-Wilkinson approximation and obtain the matrix A and the vector b.*

Step 2 *Apply the portfolio optimization approach to obtain the optimal investment fractions x^s. If $|x^s - x^{s-1}| \leq \epsilon_1$, then STOP; else go to Step 3.*

Step 3 *Calculate θ^s,*
If $|\theta^s| \leq \epsilon_2$ or $|\theta^s| > |\theta^{s-1}|$, then STOP; else go to Step 4.

Step 4 *Repeat Step 2 and Step 3 until one of the conditions to STOP is satisfied.*

4.4 Numerical Experiments

In this section, we measure the approximation accuracy of the proposed portfolio management approach. We provide cumulative histogram plots of the actual Log-Normal sum and those of the proposed Log-Normal sum approximation over a sample data set with three different frequencies (day, week, and month) to measure the accuracy of the suggested method. In addition, we compare the performance of the suggested Log-Normal sum approximation method to that of the approximation to the tail empirical distribution function with the Generalized Pareto Distribution. We also estimate the risk (the probability of the portfolio return's being not higher than the specified target w) according to the two approximation methods and compare these estimations with the empirical cumulative probability of the actual Log-Normal sum calculated over a sample generated by Monte-Carlo simulation.

We will see that:

- Our proposed approximation and calibration algorithm improve the accuracy of the Fenton-Wilkonson approximation, so that it generally measures the risk more accurately than the commonly used tail empirical distribution function approximation by a Generalized Pareto Distribution (GPD). For instance, the suggested method outperforms the benchmark approximation method in 63% of the numerical tests summarized in Table 4.1.

4.4.1 Setup

We consider historical observations over four consecutive time periods between Jan 01, 2007 and March 08, 2013. The first (Sep 1, 2007 - Sep 30, 2008), second (Oct 1, 2008- March 31, 2010), third (April 1, 2010 - June 30, 2011), and the fourth (July 1, 2011- March 8, 2013) time periods are determined to capture the timeline of 2008 financial crises by considering the beginning date of the crises and the dates of the first three quantitative-easing decisions. We analyze separately daily, weekly and monthly historical observations of three sets of stocks – 30, 50 and 100 stocks listed in New York Stock Exchange (NYSE) – over the four consecutive periods.

As described in the previous section, the Fenton-Wilkinson approximation method is applied to forecast the location μ and scale σ parameters of the approximated Log-Normal distribution for the overall portfolio ROR. Next, we apply the proposed portfolio management approach to obtain the optimal asset allocation. Then, given the scale and the location parameters, and the portfolio allocation decision, we generate a sample of random portfolio return realizations for both the Log-Normal sum and the Log-Normal sum approximation. In addition, the left and the right tails of the portfolio return distribution are approximated by a Generalized Pareto Distribution (GPD). The 25^{th} percentile values of the empirical distribution of the portfolio return and the portfolio rate of loss are used as thresholds for approximating the left and right tail distributions by GPD, respectively.

The Log-Normal sum realization at time t, W_t, is calculated using i.i.d. standard Gaussian random variables r_t^i for all t and i such that:

$$W_t = \sum_{i=1}^{n} e^{\mu_i + \sigma_i r_t^i} x_i$$

while the portfolio ROR approximation for time t is calculated using a standard Gaussian random variable r_t' for all t such that:

$$W_t' = e^{\mu + \sigma r'}.$$

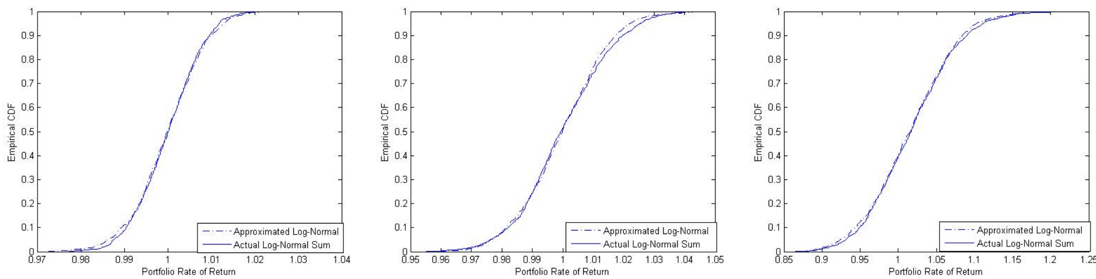
The cumulative returns are then computed as $W = \prod_{t=1}^{T} W_t$ and $W' = \prod_{t=1}^{T} W_t'$. The performance measure Ω is defined to assess the quality of the cumulative probability forecast at w:

$$\Omega = 100 \cdot \frac{|P(W \leq w) - P(W' \leq w)|}{P(W \leq w)}.$$

Here our goal is to observe the accuracy of the approximation method in estimating the specified risk (the probability of obtaining a portfolio return less than or equal to a specified level w). In addition, we use the approximation suggested by GPD as the benchmark. The cumulative probability value at w based on the proposed approach with the Fenton-Wilkinson Approximation (FWA) and the one based on the GPD approximation are denoted $P_{FWA}(W' \leq w^*)$ and $P_{GPD}(W' \leq w^*)$, respectively.

4.4.2 Measuring the Accuracy of the Approximated Distribution

In this section we measure the accuracy of the proposed approximation method by comparing the cumulative histogram plots of the portfolio return (Log-Normal sum) and the proposed approximation. The set of stocks considered is 30 stocks of the NYSE during the first period, while the data frequency is a day, a week, or a month and w is set to 0.99.



(a) Daily Data (b) Weekly Data (c) Monthly Data

Figure 4.1: Cumulative Histogram Plot

Figure 4.1 shows that the cumulative histogram plots of the Log-Normal sum (portfolio return) and those of the Log-Normal sum approximation (proposed approach) nearly coincide, which indicates the accuracy of the proposed approximation method.

4.4.3 Accuracy in Estimating the Risk

We calculate and compare Ω values for both the proposed approach (Ω_{FWA}) and the approximation with Generalized Pareto Distribution (Ω_{GPD}), over each data set with a unique data frequency. Each single row in Table 4.1 summarizes the numerical experiments with the same data frequency and same number of assets over the four different periods (namely, Period 1, 2, 3, and, 4). We provide the average, min, and max value of Ω_{FWA} and Ω_{GPD} for each set of experiments summarized in a row in Table 4.1. In addition, the number of the experiments (out of 4) in which the proposed method provides higher accuracy than the GPD approximation is presented in the last column of Table 4.1.

Table 4.1: Analysis for the data set consisting 30, 50 and 100 stocks listed in NYSE with three different data frequencies

Asset	Freq.	T	w	Iteration Number (Avg.)	$P(W \leq w)$ (Actual)	$P(W' \leq w)$ (FWA)	$P(W' \leq w)$ (GPD)	Ω_{FWA} (Avg.)	Ω_{GPD} (Avg.)	Ω_{FWA} (Min.)	Ω_{FWA} (Max.)	Ω_{GPD} (Min.)	Ω_{GPD} (Max.)	# of Obs.
30	Day	1	0.990	3.50	0.0808	0.0809	0.0808	0.0190	0.0934	0.0138	0.0120	0.0301	0.1902	3
30	Day	5	0.990	3.75	0.0581	0.0595	0.0579	0.0668	0.0492	0.0000	0.0208	0.0952	0.0892	1
30	Day	10	0.990	2.75	0.0729	0.0622	0.0714	0.7482	0.1286	0.0455	0.0237	1.0000	0.3028	1
50	Day	1	0.988	4.00	0.0309	0.0294	0.0313	0.0794	0.1496	0.0000	0.0122	0.2000	0.3333	3
50	Day	5	0.988	4.00	0.0310	0.0309	0.0322	0.0568	0.1462	0.0067	0.0232	0.1111	0.4553	3
50	Day	10	0.988	3.25	0.0663	0.0543	0.0675	0.4384	0.1785	0.0467	0.0069	0.9981	0.4002	2
100	Day	1	0.988	3.00	0.0612	0.0683	0.0609	0.3021	0.1259	0.0293	0.0086	0.9543	0.2618	2
100	Day	5	0.988	2.25	0.2145	0.2221	0.1951	0.2720	0.0716	0.0106	0.0153	0.6497	0.1942	1
100	Day	10	0.988	4.00	0.0389	0.0393	0.0389	0.0520	0.0631	0.0114	0.0177	0.1538	0.1244	2
30	Week	1	0.980	4.00	0.1056	0.1071	0.1029	0.0264	0.0469	0.0062	0.0180	0.0461	0.0918	3
30	Week	2	0.980	3.75	0.1055	0.1071	0.1041	0.0382	0.0583	0.0013	0.0001	0.1111	0.1784	2
30	Week	3	0.980	3.75	0.1095	0.1066	0.1053	0.0298	0.0495	0.0057	0.0305	0.0476	0.0706	4
50	Week	1	0.980	4.00	0.0736	0.0736	0.0733	0.0176	0.0560	0.0000	0.0115	0.0530	0.1197	4
50	Week	2	0.990	3.00	0.1655	0.1680	0.1612	0.0162	0.0280	0.0090	0.0134	0.0281	0.0597	4
50	Week	3	0.983	3.25	0.0736	0.0747	0.0734	0.0284	0.0361	0.0092	0.0161	0.0462	0.0556	3
100	Week	1	0.990	3.50	0.1363	0.1370	0.1337	0.0345	0.0190	0.0106	0.0067	0.0975	0.0377	2
100	Week	2	0.990	3.25	0.1412	0.1438	0.1377	0.0149	0.0219	0.0054	0.0006	0.0367	0.0396	2
100	Week	3	0.980	4.00	0.0534	0.0543	0.0546	0.0498	0.0386	0.0019	0.0039	0.0838	0.1056	2
30	Month	1	0.980	2.75	0.1016	0.0986	0.1011	0.1021	0.0170	0.0089	0.0095	0.3660	0.0340	3
30	Month	2	0.980	1.50	0.0997	0.1017	0.0989	0.0241	0.0354	0.0100	0.0184	0.0392	0.0774	3
30	Month	3	0.980	2.50	0.0904	0.0866	0.0915	0.0324	0.0278	0.0039	0.0003	0.0553	0.0674	3
50	Month	1	0.980	3.00	0.0593	0.0590	0.0591	0.0420	0.0436	0.0106	0.0160	0.1140	0.0772	3
50	Month	2	0.980	3.00	0.0662	0.0655	0.0654	0.0195	0.0280	0.0080	0.0127	0.0500	0.0499	3
50	Month	3	0.980	3.00	0.0696	0.0700	0.0685	0.0268	0.0205	0.0018	0.0081	0.0947	0.0414	3
100	Month	1	0.990	4.00	0.1278	0.1280	0.1256	0.0059	0.0185	0.0015	0.0100	0.0083	0.0381	4
100	Month	2	0.990	4.00	0.1309	0.1279	0.1293	0.0275	0.0113	0.0064	0.0027	0.0508	0.0218	1
100	Month	3	0.980	3.25	0.0679	0.0688	0.0672	0.0407	0.0457	0.0040	0.0274	0.1344	0.0692	3

Ω_{FWA} is less than Ω_{GPD} in 68 of 108 observations in Tables 4.1. This can be interpreted as that the proposed approximation approach outperforms the GPD approximation 63% of the time. In addition, the suggested approximation method outperforms the approximation with GPD the most often when the frequency of the data set is one month and one week.

4.5 Extension to the Design of Basket Options

In this section we describe how our approach can be applied to basket option designing problems. We will minimize the price of the basket call option where the strike price, which is a function of weights of the stocks in the basket, is determined to assure that the probability of not exercising the basket option is no worse than a specified probability level.

The price of a basket option is difficult to calculate exactly because of the multiple random variables that define the underlying assets. Various approximation methods have been proposed in the literature for basket option pricing including conditional expectation techniques [14], an approximation using the geometric average [86], a Log-Normal moment matching method [126], a Taylor expansion [107], and a reciprocal Gamma approximation [146]. In our work we will incorporate Levy's moment-matching approach for basket option pricing, which is the work most relevant to the present chapter, and the iterative approximation approach that is proposed in the previous section. We will use the following notation:

T : the time to expiration,

y_i : the fraction of the basket that is allocated to stock i,

$S_i(t)$: the price of stock i at time t,

$F_i(t)$: the forward price of stock i at time t,

$B(t)$: the value of the basket at time t,

μ_i : the location parameter of the Log-Normal distribution of ROR on stock i,

σ_i : the scale parameter of the Log-Normal distribution of ROR on stock i,

$\rho_{i,j}$: the correlation coefficient between the natural logarithms of two Log-Normally distributed random variables with indexes i and j,

μ_b : the location parameter of the Log-Normal distribution of the basket ROR,

σ_b : the scale parameter of the Log-Normal distribution of the basket ROR,

K : the strike price of the basket call option,

n : total number of stocks,

r : the risk-free interest rate.

The basket call-option pricing method proposed by [126] approximates the distribution of the value of the basket by a Log-Normal distribution obtained by matching the first two moments of this distribution and those of the original Log-Normal sum of the stock prices.

Let A and b the matrix and vector defined by:

$$A_{ij} = F_i(T)F_j(T)e^{\sigma_i\sigma_j\rho_{i,j}T}\ \forall i,j, \text{ and } b_i = F_i(T)\ \forall i,$$

so that:

$$E[B(T)] = b'y \text{ and } V = E[B(T)^2] = y'Ay.$$

The basket value at time t is calculated as: $B(t) = \sum_{i=1}^{n} y_i S_i(t)$ and the price at time 0 of a European call with strike K on the basket $B(T)$ at maturity is given by:

$$C(0) = e^{-rT} E^Q[(B(T) - K)^+],$$

where Q is the risk-neutral measure as in the famous Black-Scholes option pricing model (we refer the reader to [35] for more information). [126] approximates the price of the basket option by:

$$C(0) = e^{-rT} \left[E(B(T))\Phi(d_1) - K\Phi(d_2) \right],$$

with

$$d_1 = \frac{\mu_b - ln(K) + \sigma_b^2}{\sigma_b}, \;\; d_2 = d_1 - \sigma_b,$$

where

$$\mu_b = 2\ln(b'y) - \frac{1}{2}\ln(y'Ay),$$

$$\sigma_b^2 = \ln(y'Ay) - 2\ln(b'y),$$

Lemma 4.4 (Basket Options) *The probability of not exercising the basket call option, when the basket value is approximated as a Log-Normally distributed random variable with parameters μ_b and σ_b, is given by:*

$$P(B(T) \leq K) \approx \Phi\left(\frac{\ln(K) - 2\ln(b'y) + \frac{1}{2}\ln(y'Ay)}{\sqrt{\ln(y'Ay) - 2\ln(b'y)}} \right).$$

Proof. As for Lemma 4.1, the result follows directly from $\ln(B(T))$ being approximated by a Normal distribution with mean μ_b and standard deviation σ_b.

The problem of minimizing the price of the basket call option subject to the probability of not exercising the option being less than a specified threshold value p is formulated as:

$$\begin{aligned} \min \quad & e^{-rT}\left[b'y\,\Phi\left(\frac{\frac{1}{2}\ln(y'Ay)-\ln(K)}{\sqrt{\ln(y'Ay)-2\ln(b'y)}}\right)-K\Phi\left(\frac{2\ln(b'y)-\frac{1}{2}\ln(y'Ay)-\ln(K)}{\sqrt{\ln(y'Ay)-2\ln(b'y)}}\right)\right] \\ \text{s.t.} \quad & \Phi\left(\frac{\ln(K)-2\ln(b'y)+\frac{1}{2}\ln(y'Ay)}{\sqrt{\ln(y'Ay)-2\ln(b'y)}}\right)\leq p, \\ & y\in Y. \end{aligned}$$

The following theorem explains how to solve this problem.

Theorem 4.5 (Solving the basket options designing problem) *Let F be the function defined on $[w_f,\infty)$ by the quadratic programming problem:*

$$\begin{aligned} F(v)=\min \quad & y'Ay \\ \textit{s.t.} \quad & b'y=v \\ & y\in Y. \end{aligned} \tag{4.4}$$

The optimal decision y of the basket options designing problem is the optimal solution of Problem (4.4) with v the optimal solution of

$$\min_{v\geq w_f} e^{-rT}\left(v\Phi\left(-\Phi^{-1}(p)+\sqrt{F(v)-2ln(v)}\right)-(1-p)e^{2ln(v)-0.5ln(F(v))+\Phi^{-1}(p)\sqrt{ln(F(v))-2ln(v)}}\right) \tag{4.5}$$

Proof: The chance constraint in the basket options designing problem is tight at optimality; because, the cumulative probability value p decreases only if the strike price K increases, while the basket allocation decision y is given, which, at the same time, increases the price of the basket option. Therefore, the chance constraint will be forced to be tight in order to minimize the price of the basket option. Thus, the strike price, K, is formulated as:

$$K = \frac{b'y^2}{\sqrt{y'Ay}} exp\left(\Phi^{-1}(p)\sqrt{ln(y'Ay) - 2ln(b'y)}\right)$$

Injecting the strike price formulation into the basket options price function leads to the following formulation for the options designing problem:

$$\begin{aligned} \min_{v \geq w_f} \quad & e^{-rT}\left(v\Phi\left(-\Phi^{-1}(p) + \sqrt{ln(y'Ay) - 2ln(v)}\right)\right) - (1-p)\frac{b'y^2}{\sqrt{ln(y'Ay)}} exp\left(\Phi^{-1}(p)\sqrt{ln(y'Ay) - 2ln(b'y)}\right) \\ & y \in Y. \end{aligned} \tag{4.6}$$

Note that the objective function is a nondecreasing function of $y'Ay$ (assuming α is less than or equal to 0.5); in other words, while $b'y = w_f$ is given and $y \in Y$, the minimum price value is obtained when $y'Ay$ takes the minimum value satisfying $b'y = v$. As a result, the optimal decision y of the basket options designing problem is the optimal solution of Problem (4.4) with v the optimal solution of Problem (4.5).

In addition, we use the approximation accuracy measure θ (with $\pi = p$) and the accuracy improvement approach which are introduced in the previous chapter.

Our algorithm is given as follows, for stopping parameters ϵ_1 and ϵ_2:

Algorithm 4.6

Step 1 *Start with a feasible solution $y \in Y$ to serve as a candidate solution and set the iteration number, $s = 0$. Apply Fenton-Wilkinson approximation and obtain the matrix A and the vector b.*

Step 2 *Apply the basket option designing approach explained above by solving Problem (4.4) and Problem(4.5) to obtain the basket allocation fractions (y^s). If $|y^s - y^{s-1}| \leq \epsilon_1$, then STOP; else go to Step 3.*

Step 3 *Calculate θ^s,*

If $|\theta^s| \leq \epsilon_2$ or $|\theta^s| > |\theta^{s-1}|$, then STOP; else go to Step 4.

Step 4 *Repeat Step 2 and Step 3 until one of one of the conditions to STOP is satisfied.*

The proposed approach lets us design a European basket call option whose strike price is set in order to keep the probability of not exercising the option at a specified level. However, the basket call option designed according to the suggested approach may not be traded in the market. Even though some big market players have enough power to persuade financial institutions to trade the basket call option that they designed, this might not be the case for other market players. Therefore, we will refer to the previous studies in the literature for the strategies which replicate the payoff structure of the designed call basket option by individual European call options.

A Static Super-Replicating Strategy

[207] proposed this method to obtain a super-replicating portfolio whose final payoff is never worse than the payoff of the underlying basket call option. The author uses Jensen's inequality for the final payoff of the replication,

$$C(0) = \left(\sum_{i=1}^{n} y_i S_i(T) - K\right)^{+} = \left(\sum_{i=1}^{n} y_i \left(S_i(T) - \frac{q_i}{y_i}K\right)\right)^{+} \leq \sum_{i=1}^{n} y_i \left(S_i(T) - \frac{q_i}{y_i}K\right)^{+},$$

where $\sum_{i=1}^{n} q_i = 1$ and y_i is the basket fraction allocated to stock i.

Therefore, the payoff of the portfolio consisting of n plain vanilla call options (each of which has a strike price equal to $\frac{q_i}{y_i}K,\ i \in \{1, .., n\}$) is not worse than that of basket call option. Considering the no-arbitrage argument, the cost the super-replicating portfolio must be greater than or equal to that of the basket call option, in other words the following inequality holds:

$$\sum_{i=1}^{n} e^{-rT} E^{\mathbb{Q}} \left[(y_i S_i(T) - K)^{+}\right] \leq \sum_{i=1}^{n} y_i e^{-rT} E^{\mathbb{Q}} \left[\left(S_i(T) - \frac{q_i}{y_i}K\right)^{+}\right] \tag{4.7}$$

where $\mathbb{Q}$ is the risk-neutral measure as in the famous Black-Scholes option pricing model. Therefore, one approach to design the replicating portfolio might be minimizing the cost of super-replicating portfolio as follows:

$$
\begin{aligned}
\min_{q} \quad & \sum_{i=1}^{n} y_i e^{-rT} E^{\mathbb{Q}} \left[\left(S_i(T) - \frac{q_i}{y_i} K \right)^{+} \right] \\
\text{s.t.} \quad & e'q = 1.
\end{aligned}
\tag{4.8}
$$

[207] provides the solution of Problem(4.8), i.e., the optimal sequence of weights q_i^* by the following proposition.

Chapter 5

New Product Launching Decisions with Robust Optimization

5.1 Introduction

In this chapter, we consider the introduction of a set of innovative products or services to a national market. We assume that the company which has innovated the products plans to launch them in a national market in a limited time period and is interested in determining the optimal time to launch each product. The company is assumed to have a limited marketing budget at each period. The products or services under consideration can be durable goods with different purposes or drugs with different treatments so that the substitution effect among the products is neglected.

We use the new product growth model suggested by the Bass [9] to estimate the adoption rate of the customers for each product. The Bass diffusion model and its revised versions have been used for forecasting the diffusion of innovation in various areas including durable goods, pharmaceutical, and industrial technology markets. As it is mentioned in chapter 1.6 the Bass model of diffusion considers two types of potential adopters, namely innovators and imitators.

It assumes that two communication channels are used to influence the potential adopters: mass media and word of mouth. The innovators are affected by the external influence (mass media), whereas the imitators' motivation to adopt the innovation comes from the internal influence of the customers who have already adopted the innovation. According to Lilien, Rangaswamy, and Bruyn [127], the Bass model can estimate the long term sales patterns of an innovative product for the following two cases:

- The new product has already been introduced to the market and the first few periods' sales amounts have been observed,
- The new product has not been introduced to the market; however, an existing product's diffusion process can be used as a proxy for the product of interest.

The usage of the basic Bass model requires estimating three parameters: $m,\ p,$ and q for each product. The parameters m, p, and q stand for the potential number of ultimate adopters in the market, the coefficient of the external influence, and that of the internal influence, respectively. In addition, the Bass model formulates the number of new adopters of the product i in period t ($S_i(t)$) as:

$$S_i(t) = m_i \frac{(m_i + q_i)^2}{p_i} \frac{e^{-(p_i+q_i)t}}{\left(1 + \frac{q_i}{p_i} e^{-(p_i+q_i)t}\right)^2}.$$

Srinivasan and Mason [202] show that reliable estimations for the parameters can be obtained when the available data set is large enough to cover the peak of the rate of the adaption curve for the product under consideration. Therefore, the estimations for the parameters and that for the number of new adopters in each period made before the original diffusion process starts are subject to uncertainty and might depend on time. On the other hand, the parameter m_i (the ultimate number of adopters of product i) is not expected to be time-dependent. However, it is subject to estimation errors, and it can be forecast more accurately after the first few periods' sales amounts are revealed and analyzed.

For each product, the innovative company seeks a partner whose willingness to accept the innovator company's partnership offer depends on the proposed unit payments for the service that it will provide. The partner might help the innovative company establish the infrastructure for the new adopters or provide some services such as shipping, installment, customer service, etc. For instance, a bank introduces a new service such as accepting and evaluating loan applications through text messages. A new account is opened for each new adopter when he/she uses the system for the first time and the same account is reserved for the same user for future loan applications. The account keeps some information regarding the user such as his/her social security number, highest education level, current job, income level, and current credit rating. Therefore, an additional adopter requires an extension and improvement in the information technology system (ITS) of the bank. Under the assumption that the bank outsources the ITS projects, it will look for a partner to satisfy the need for the technology infrastructure improvement arising from the new adopters starting using the product.

A potential partner's probability to accept an offer is modeled as a logit model where the payment amount per unit is the main variable in the logit model. In other words, the innovator company offers the partner company a specific amount of payment per new adopter for its collaboration starting from the period when the product is launched until the time period that the diffusion process of the product terminates. For a given specific probability level, α, the innovator company selects a partner whose inverse probability function corresponds to the minimum payment value per unit. Please note that a given amount of product or service minimum unit payment corresponds to the minimum net present value (NPV) of the total payments. For given the logit model parameters a_j and b_j, the probability of the potential partner j agreeing to collaborate with the innovative company when the payment per unit offered by the company is R_j is represented as:

$$P_j(\text{Yes}) = \frac{e^{(a_j+b_jR)}}{1+e^{(a_j+b_jR_j)}}$$

The parameters a_j and b_j of the logit functions are also estimated based on the available data or managerial judgments. Therefore, they are subject to uncertainty as well.

Our main target in this chapter is to propose a tractable mathematical framework which handles the parameter uncertainty and answers the following questions of the innovative company:

- How should it schedule launching of each innovative product/service?
- Which partner should it select in each region so that the total profit expected to be obtained from the innovative product throughout a specific time horizon is maximized?

We assume that the innovative company is a monopoly for these products due to the innovative nature of the products. In addition, the diffusion of innovation of a product does not affect the other products' diffusion processes. Moreover, we assume that the current unit price of the product is determined. Therefore, we calculate the present value of the revenue obtained from new products' sales by discounting the number of adopters in each period by a constant discount factor.

5.1.1 Motivations for Robust Optimization Applications with the Bass Model

In this section, we show how some key measures of the diffusion of innovation (for the innovative product) and the logit choice model (for a potential partner's response to an offer) are affected by some changes in the parameters of the corresponding models. Specifically, we show how the adoption rate $(f_i(t))$, cumulative adoption rate $(F_i(t))$, and total discounted number of adopters $(CS_i'(t))$ diverse by different values of the parameters p_i and q_i of the product i. Furthermore, we provide a similar analysis for the logit response function of a representative potential partner with changing values of the parameters a_j and b_j.

The Bass Model Parameters

As it is explained in the chapter 1.6, according to the basic Bass model, the adoption rate $(f_i(t))$, cumulative adoption rate $(F_i(t))$, number adopters $(S_i(t))$, and the cumulative number of adopters $(CS_i(t))$ at period t for the innovative product i are formulated as:

$$f_i(t) = \frac{(m_i + q_i)^2}{p_i} \frac{e^{-(p_i+q_i)t}}{\left(1 + \frac{q_i}{p_i} e^{-(p_i+q_i)t}\right)^2},$$

$$F_i(t) = \frac{1 - e^{-(p_i+q_i)t}}{\left(1 + \frac{q_i}{p_i} e^{-(p_i+q_i)t}\right)},$$

$$S_i(t) = m_i \frac{(m_i + q_i)^2}{p_i} \frac{e^{-(p_i+q_i)t}}{\left(1 + \frac{q_i}{p_i} e^{-(p_i+q_i)t}\right)^2},$$

$$CS_i(t) = m_i \frac{1 - e^{-(p_i+q_i)t}}{\left(1 + \frac{q_i}{p_i} e^{-(p_i+q_i)t}\right)},$$

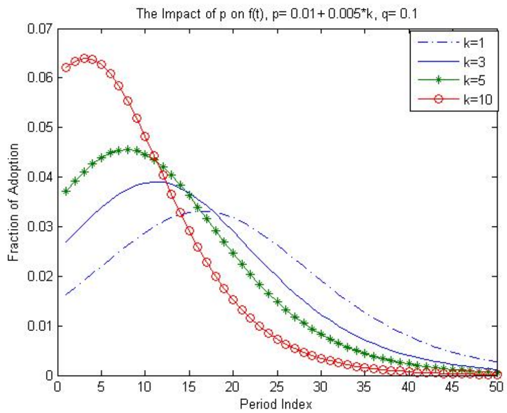
where the parameters $q_i,\ p_i$, and m_i are assumed to be estimated for the product i under consideration. Then, under the assumption that the discount factor is constant and represented by r, the total discounted number of adopters until the period T becomes:

$$CS_i'(t) = \sum_{t=1}^{T} \frac{1}{(1+r)^t} \left(m_i \frac{(m_i + q_i)^2}{p_i} \frac{e^{-(p_i+q_i)t}}{\left(1 + \frac{q_i}{p_i} e^{-(p_i+q_i)t}\right)^2} \right).$$

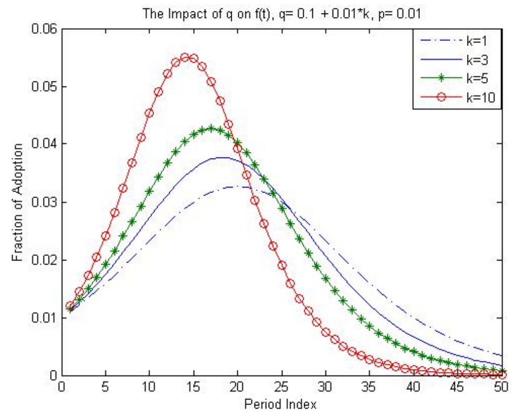
Please note that if the period unit is small enough, then summation of the number of adopters at each period is a close approximation for the cumulative number of adopters $(CS_i(t))$.

Figure 5.1 shows that as p_i or q_i increases, the peak marginal adoption rate increases and the time to reach the peak adoption rate decreases. According to Figure 5.2, an increase in the parameter p_i or the parameter q_i results in a higher cumulative adoption rate at a given time period t. In other words, time to reach a given cumulative adoption rate decreases as p_i or q_i increases. If we consider the time value of adoption, the impact of increase in p_i or q_i on the total number of adopted customers becomes more significant as it can be observed in Figure 5.3.

The main idea inferred from Figures 5.1, 5.2, and 5.3 is that the estimations for the total

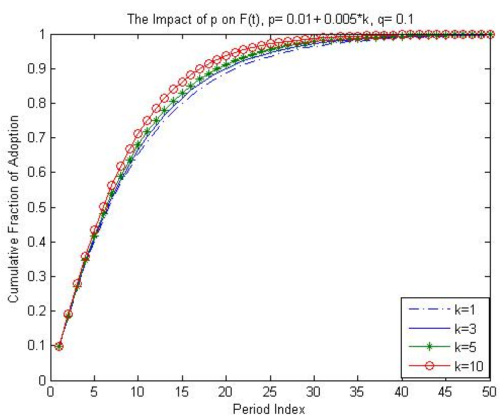


(a) Varying p, constant q

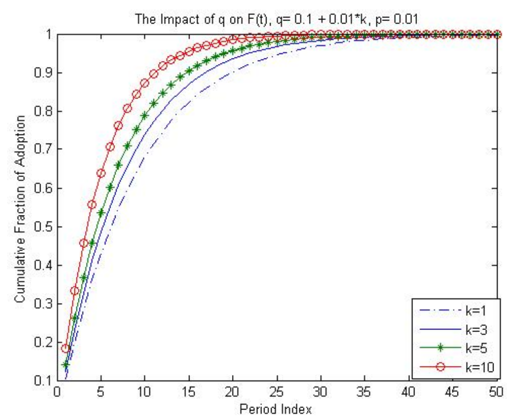


(b) Varying q, constant p

Figure 5.1: The impact of q and p on marginal adoption rates



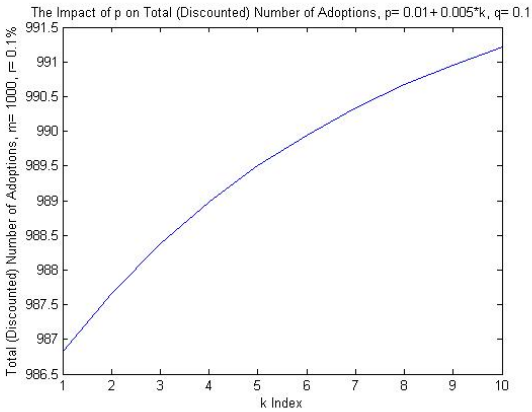
(a) Varying p_i, constant q_i



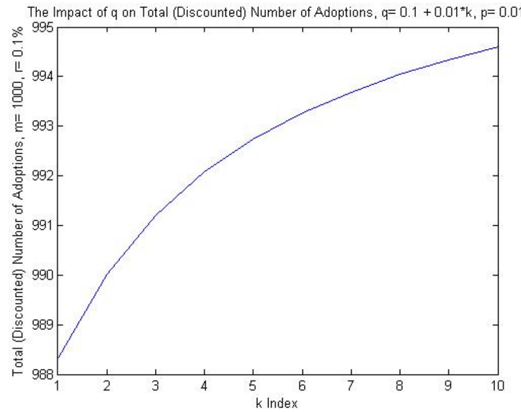
(b) Varying q_i, constant p_i

Figure 5.2: The impact of q_i and p_i on cumulative adoption rates

number of adopters during a period and the number of new adopters in a period depend on the parameters p_i and q_i. In addition, in the worst case the parameter m_i takes its smallest possible value independent from p_i's and q_i's values (both of p_i and q_i impact the new adoption rate and only one among several possible adoption rate combinations is selected; however, m_i is used to calculate the number of new adopters at a period by being multiplied by the new adoption rate in this period). Therefore, we address the uncertainty involved in the parameters p_i and q_i of



(a) Varying p_i, constant q_i



(b) Varying q_i, constant p_i

Figure 5.3: The impact of q_i and p_i on total discounted number of adopters while $m = 1000,\ r = 0.001$

each product i by robust optimization techniques and assume that the parameter m_i (the ultimate number potential adopters) of each product is constant in Chapter 5.2.2. However, in Chapter 5.2.4, we combine the real options approach to handle the uncertainty involved in the parameter m_i of each product with the robust optimization model in Chapter 5.2.2, which takes uncertainty in the parameters q_i and p_i of each product i into consideration.

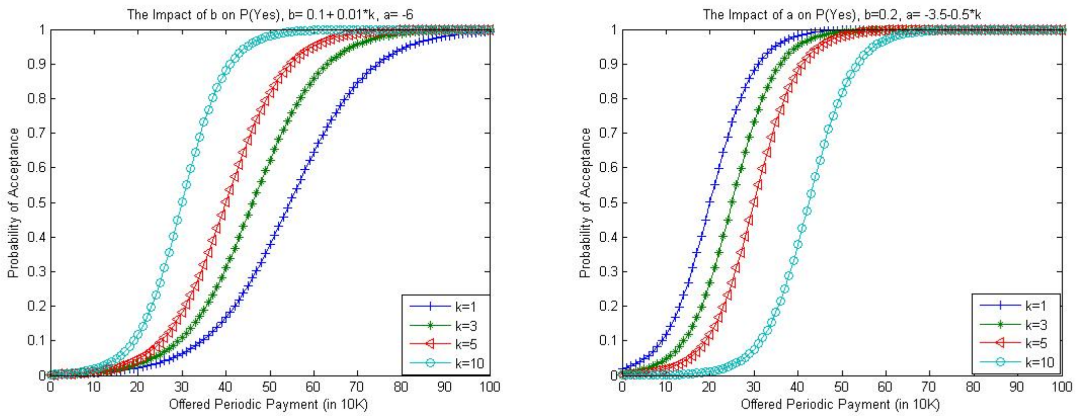
The Logit Choice Model Parameters

As it is explained in Chapter 1.4 the probability of a potential partner j accepting an offer when the offered unit price (the only variable in the logit model) is R is formulated as:

$$P_j(\text{Yes}) = 1 - \frac{e^{-(a_j+b_jR_j)}}{1+e^{-(a_j+b_jR_j)}}$$

It is expected that the potential partner's willingness to accept the offer to collaborate with the innovative company increases as the offered payment per unit R_j increases. Figure 5.4 summarizes an imaginary potential partner's probability of accepting an offer where the offer is defined as the periodic unit payment. In addition, Figure 5.4 shows that as the parameters b_j or a_j increases, the probability of acceptance for a given offered payment per unit R_j increases.

Therefore, we can conclude that a potential partner's response to an offer depends on the estimations for the parameters a_j and b_j, which are estimated by analyzing the historical data or by managerial judgment. As a result, these parameters and the offer acceptance probability for a potential partner are subject to uncertainty.



(a) Varying b_j, constant a_j

(b) Varying a_j, constant b_j

Figure 5.4: The impact of a_j and b_j on the probability of acceptance

We have shown that both the total discounted number of adopters of an innovative product and the probability of a potential partner accepting an offer depend on the estimations for the parameters used in the corresponding models. In other words, the uncertainty involved in parameters p_i and q_i of the Bass model of the product i and in parameters a_j and b_j of the logit model of the potential partner j impact the product launching decisions and the offered price to a potential partner aiming to obtain a certain chance of acceptance.

For instance, let us consider a case where the innovative firm plans to launch five innovative products in a region so that only one product is introduced at each period during consecutive five periods. Considering that there are five products, each of which will be launched exactly at one of the five periods, one can order the products to launch in one hundred and twenty different ways. In other words, there are one hundred and twenty different strategies, and we name each strategy the index of itself. Furthermore, let us use the estimated Bass model parameter values provided in Table 5.1 along with the assumption that the discount rate is 0.001. For this small example, we assume that each product is identical in terms of potential partners and their choice models in order to show the impact of the uncertainty involved in the Bass model parameters on the optimal strategy.

Table 5.1: The nominal values of the estimations for the Bass model parameters for each product

p	q	m
0.046	0.45	1000
0.045	0.44	1000
0.044	0.42	1000
0.047	0.4	1000
0.043	0.43	1000
0.042	0.44	1000

Table 5.2 provides the optimal strategies for the cases where only one product's q parameter takes a value which is equal to the multiplication of its nominal value by the specified coefficient located in the first column while the rest of the parameters take their nominal values. Table 5.3 summarizes the outcomes of the same analysis repeated for p parameters. Tables 5.2 and 5.3 show the sensitivity of the optimal strategy to the parameters p and q of each product, respectively.

In summary, we have observed that:

- The total discounted number of adopters, the cumulative rate of adoption, and the new adoption rate for a product are significantly affected by changes in the values of the parameters of the Bass model.
- The optimal sequence of the products to be launch changes at least once when only one parameter changes within the range of $\mp 40\%$ of its nominal value for the small example mentioned above.
- The probability of a potential customer accepting a given offer is sensitive to the logit model parameters.

Therefore, we believe that robust optimization techniques can be used to handle the uncertainty involved in the parameters (a and b) of the logit choice model and those (p and q) of the Bass model. The parameters of the logit choice model and the Bass model are estimated through different processes, and they are subject to different sources of uncertainty. For instance, the uncertainty affecting the logit choice parameters mostly results from the difficulty of decoding the potential partners' utility and offer response functions. On the other hand, the uncertainty involved in p and q parameters of the Bass diffusion model is generally caused by the potential customers' willingness to adopt the innovative product and the communication channels, i.e. mass media or word-of-mouth. Therefore, we define a different uncertainty budget for each model's parameters in the robust optimization model that we develop.

Intervals for the Parameters of the Bass and Logit Choice Models

We define the worst case as the lowest adoption rate for the Bass Model. We assume that initial estimations of the product-specific Bass models are subject to interval uncertainty. Specifically, we assume that each parameter q_i and p_i $(i \in \{1, 2, ..., N\})$ can take values in an interval $[q_i - \hat{q}_i, q_i + \hat{q}_i]]$ and $[p_i - \hat{p}_i, p_i + \hat{p}_i]]$ at each time period upon the diffusion process starts for the product i. As the two graphs in Figure 5.1 suggest, the diffusion process with the smallest p parameter value results in the lowest adoption rates from the beginning of the process until the 14^{th} period. On the other hand, the process with the highest p parameter value leads to the smallest adoption rates after the 14^{th} period. Similarly, the diffusion process with the smallest q parameter value provides the smallest adoption rates until the 21^{st} period; however, the one with the largest q parameter value leads to the smallest adoption rates thereafter. Therefore, the parameters p_i and q_i can take their smallest or highest values at a period depending on the time passed since the diffusion process starts in the worst case. We define the uncertainty budget for the Bass diffusion parameters as the maximum number of parameters taking its smallest or highest value in a period until either the diffusion process terminates. However, we do not impose another budget of uncertainty for the deviations within the ranges. Therefore, in the worst case, the parameter values are expected to take the edge values of their intervals. Thus, t periods after the diffusion process of the product i starts, the new adoption rate takes one of the following nine values:

- $f_i^0(t) = f(t, \bar{q}_i, \bar{p}_i)$
- $f_i^1(t) = f(t, \bar{q}_i - \hat{q}_i, \bar{p}_i)$
- $f_i^2(t)= f(t, \bar{q}_i + \hat{q}_i, \bar{p}_i)$
- $f_i^3(t)= f(t, \bar{q}_i, \bar{p}_i - \hat{p}_i)$
- $f_i^4(t)= f(t, \bar{q}_i, \bar{p}_i + \hat{p}_i)$
- $f_i^5(t)= f(t, \bar{q}_i - \hat{q}_i, \bar{p}_i + \hat{p}_i)$
- $f_i^6(t)= f(t, \bar{q}_i + \hat{q}_i, \bar{p}_i + \hat{p}_i)$
- $f_i^7(t)= f(t, \bar{q}_i - \hat{q}_i, \bar{p}_i - \hat{p}_i)$
- $f_i^8(t)= f(t, \bar{q}_i + \hat{q}_i, \bar{p}_i - \hat{p}_i)$

where $f(t, q, p)$ stands for the rate of adoption of the product i at a period which is t periods after the diffusion starts with the corresponding p and q parameters. Please note that if one the last four cases happen, both p_i and q_i take their smallest or highest values, which means that 2 out of the overall uncertainty budget for the Bass model is taken for this case.

According to the logit choice model, the probability of acceptance is an increasing function of the offered unit payment R, as shown in Figure 5.4. In addition, the probability of acceptance increases as parameters a or b increases for a given R. Furthermore, from the innovator company's point of view, the probability of a potential customer accepting an offer is lower in the worst case. Therefore, we will be interested in the intervals $[\bar{a}_{ij} - \hat{a}_{ij}, \bar{a}_{ij}]$ and $[\bar{b}_{ij} - \hat{b}_{ij}, \bar{b}_{ij}]$ for the response function of the potential partner j for the product i. Therefore, the probability of the potential partner j accepting the collaboration offer when the offered periodic payment is R_j can be one of the following four values:

- $P^0_{ij}(R_j) \;=\; P(\bar{a}_{ij}, \bar{b}_{ij}, R_j)$
- $P^1_{ij}(R_j) \;=\; P(\bar{a}_{ij} - \hat{a}_{ij}, \bar{b}_{ij}, R_j)$
- $P^2_{ij}(R_j) \;=\; P(\bar{a}_{ij}, \bar{b}_{ij} - \hat{b}_{ij}, R_j)$
- $P^3_{ij}(R_j) \;=\; P(\bar{a}_{ij} - \hat{a}_{ij}, \bar{b}_{ij} - \hat{b}_{ij}, R_j)$

Please note that when P^3_{ij} makes both $a_{i,j}$ and $b_{i,j}$ take their lowest values. Given that the innovator company seeks for a partner whose probability to accept the collaboration offer exceeds a specified probability level, α, with the minimum periodic payment offer R_j, and the probability of accepting an offer is an increasing function of the offered unit payment R_j, the innovative company actually looks for the potential customer whose worst-case inverse logit probability function value is the minimum.

As it is mentioned earlier, the ultimate number of adopters of product i, m_i, is estimated before the diffusion process of the product i starts and is subject to the risk of having an estimation error. However, the innovative company observes the sales amounts and updates its estimates on the parameter m_i while the diffusion process continues. Therefore, we will handle the uncertainty on the estimation for m_i using the real options approach. The option will give the innovative company the right to decrease the size of the contract (in terms of unit) a specific number of periods (η) later to a specific fraction (κ) of it. The innovative company is assumed to have some-scenario based estimations for the value of the updated estimation for m_i with their corresponding probabilities, η periods later the start of the diffusion process starts. However, the innovative company avoids returning customers (new adopters) and applies a conservative strategy so that the maximum of the nine possible periodic adoption rates ($f^+_i(t) = \max\{f^k_i(t)\},\ k \in \{0, .., 8\}$) is used as the estimated new adoption rate and $m_i f^+_i(t)$ units of capacity is reserved at the partner company. However, η periods after the process starts the real option can be exercised and the new periodic reserved capacity can be updated to $m_i' f_i^+(t)$ with a new estimation for the ultimate number of adopters , m_i'.

5.2 New Product Launching Decisions with Robust Optimization

5.2.1 Problem Setup

In this section, we will provide a tractable robust optimization formulation for the problem of an innovative company which:

- maximizes the global profit obtained from the set of innovative products considering
  - the worst case discounted total number of adopters of each product,
  - the minimum foreseen partnership payments per unit ensuring the worst case probability to accept the offer is no less than a specific target,
  - unit present value of each product,
  - product-specific sets of potential partners with different choice model parameters, and
  - product specific set-up cost and available investment budget limitation per time period,
- decides on:
  - sequence of the products to be launched,
  - the product-specific potential partner whose worst case probability to accept the collaboration offer for the product exceeds a specified probability level, α, with the minimum periodic payment offer R,
- by addressing the uncertainty structure mentioned in the Chapter 5.1.1 under

- the uncertainty budget for the Bass model parameters,
- the uncertainty budget for the Logit choice model parameters, and
- the estimation errors on estimations for the ultimate market size for each product.

We will use the following notation:

General Parameters

N : the total number of products,

K_i : the maximum number of periods (considering all possible values of q_i and p_i) between the time period when the diffusion process of the product i starts and the time period when its cumulative rate of adoption reaches 1,

T : the end of the time horizon considered,

S : the latest time period until which all of the products are launched,

r : the discount rate,

μ_i : the current price of the innovative product i where $i \in \{1, .., N\}$

m_i : the estimation for the number of ultimate adopters of the product i,

A^i : the set of potential partners for the product i,

α : the specified probability level for the partner selection process,

B_t : the available investment budget for the time period t,

D_i : the set up cost of launching the product i,

η_i : the number of periods between the time period that the product i is launched and the time period that the real option on the product i can be exercised,

κ_i : the fraction to which the original total size of the service or product requested from the partner can be decreased if the real option on the product i is exercised η_i periods later than the product i's diffusion process starts,

The Bass Model and Logit Choice Model Parameters

$f_i^k(t)$: the adoption rate of the product i, t periods after the diffusion process starts when the parameters of the Bass model belongs to case $k,\ k \in \{1,..,8\}$ i.e., $k = 0$ when both parameters are at their nominal value

$f_i^+(t)$: the maximum possible adoption rate of the product i, t periods after the diffusion process starts,

$P_{ij}^k(R)$: the probability of the potential partner j for the product i accepting the unit payment offer R when the parameters of the logit choice model belongs to case $k,\ k \in \{1,..,3\}$,

$Q_{ij}^k(\alpha)$: the inverse logit probability function for the potential partner j for the product i and the probability level α when parameters of the logit choice model belongs to case k,

$\tilde{m}_i^l$: the updated estimation for the ultimate number of adopters of the product i according to the scenario l when η_i periods have passed since the product i is launched,

π_i^l : the probability that the updated estimation for the ultimate number of adopters for the product i is $\tilde{m}_i^l$

Robust Optimization Parameters and Decision Variables

Γ_B : the uncertainty budget for the Bass model parameters restricting the number of parameters whose value deviate from the nominal value,

Γ_L : the uncertainty budget for the logit choice model parameters restricting the number of parameters whose value deviate from the nominal value,

$x_{i\tau}$: the binary variable which becomes 1 if the product i is launched at time τ, where $i \in \{1, .., N\},$ and $\tau \in \{1, .., S\}$,

y_{ij}: the binary variable which becomes 1 if the potential partner j is selected for the product i, where $i \in \{1, .., N\},$ and $j \in A^i$,

v_{is}^k: the binary variable which becomes 1 if the adoption rate of the product i is $f_i^k(s)$ s years after the diffusion process starts where $s \in \{1, .., K_i\},\ k \in \{1, .., 8\},$ and $i \in \{1, .., N\}$,

w_{ij}^k: the binary variable which becomes 1 if the probability of acceptance by the potential partner j for the product i is P_{ij}^k, where $k \in \{1, .., 3\},\ i \in \{1, .., N\},$ and $j \in A^i$.

The deterministic product launching problem where each product can be launched at most once is formulated as:

$$
\begin{aligned}
\max_x \quad & \sum_{i=1}^{N}\sum_{\tau=1}^{S} \frac{x_{i\tau}}{(1+r)^{\tau-1}} \left[\sum_{s=1}^{K_i} \left(\frac{m_i}{(1+r)^s}[\mu_i f_i^0(s) - f_i^+(s)\Phi_i^*(\alpha)]\right)\right] \\
s.t. \quad & \sum_{\tau=1}^{S} x_{i\tau} \leq 1, \ \ \forall i, \\
& \sum_{i=1}^{N} x_{i\tau} D_i (1+r)^{(\tau-1)} \leq B_\tau, \ \ \forall \tau, \\
& x_{i\tau} \in \{0,1\}, \ \ \forall i, \ \forall \tau,
\end{aligned}
\tag{5.1}
$$

where $\Phi_i(\alpha)$ is the partner selection problem for the product i formulated as:

$$\begin{aligned} \Phi_i(\alpha) = \quad & \min_{y_i} \sum_{j \in A^i} y_{ij} \min_{R_j} \left\{ \ R_{i,j} : \ P^0_{ij}(R_{ij}) \geq \alpha . \ \right\} \\ & s.t. \sum_{j \in A^i} y_{ij} = 1, \ \ \forall i, \\ & y_{ij} \in \{0,1\}, \ \forall j. \end{aligned} \tag{5.2}$$

Lemma 5.1 (Inverse Logit Probability Function) *For a given probability level α the optimal solution of the inner minimization problem in Problem (5.2) corresponds to the inverse probability formulation of the logit probability model with given logit choice parameters.*

$$R^*_{ij} = \frac{1}{\bar{b}_{ij}} \left(-\bar{a}_{ij} + ln \left(\frac{\alpha}{1-\alpha} \right) \right) = Q^0_{ij}(\alpha). \tag{5.3}$$

Proof. The logit probability function is a continuous and increasing function of the variable R_{ij}. The inner minimization problem:

$$\min_{R_j} \left\{ \ R_{i,j} : \ \ \left(P^0_{ij}(R_{ij}) \geq \alpha, \ \ R_{ij} \geq 0. \ \right) \right\}$$

is reformulated by rearranging the constraint using the inverse probability function as:

$$\min_{R_j} \left\{ \ R_{i,j} : \left(\frac{1}{\bar{b}_{ij}} \left(-\bar{a}_{ij} + ln \left(\frac{\alpha}{1-\alpha} \right) \right) \right) \leq R_{ij} \right\}$$

and the optimal solution occurs at $R^*_{ij} = \frac{1}{\bar{b}_{ij}} \left(-\bar{a}_{ij} + ln \left(\frac{\alpha}{1-\alpha} \right) \right)$ which is the inverse logit probability function value with the nominal values of the parameters, $Q^0_{ij}(\alpha)$.

Therefore, the Problem $\Phi_i(\alpha)$ is reformulated as:

$$\begin{aligned} \Phi_i(\alpha) = \quad & \min_{y_i} \sum_{j \in A^i} y_{ij} Q^0_{ij}(\alpha) \\ & s.t. \sum_{j \in A^i} y_{ij} = 1, \ \forall i, \\ & y_{ij} \in \{0, 1\}, \ \forall j. \end{aligned} \tag{5.4}$$

which is equivalent to:

$$\Phi_i(\alpha) = \inf\{Q^0_{ij}(\alpha), i \in A^i\}, \ \forall i.$$

Therefore, the deterministic product launching problem can be solved by first determining $\Phi_i(\alpha)$ for each product i and then solving the Problem (5.1).

5.2.2 Robust Product Launching

As Chapter 5.1.1 explains, the uncertainty involved in the Bass model's parameters leads to 9 possible adoption rates of a product at a time period under the uncertainty budget restricting the total number of parameters depicting from their nominal value throughout the time horizon (but without an uncertainty budget limiting the deviation within the uncertainty intervals). Similarly, the uncertainty involved in the logit choice parameters results in 4 possible acceptance probability values (and 4 possible inverse probability function values for a probability level α) for each potential partner and for each product with a given periodic payment offer under the uncertainty budget restricting the total number of logit choice parameters taking their worst case values.

The robust product launching problem is formulated as:

$$
\begin{aligned}
\max_x \quad & \min_v \sum_{i=1}^{N}\sum_{\tau=1}^{S}\frac{x_{i\tau}}{(1+r)^{\tau-1}}\left[\sum_{s=1}^{K_i}\left(\frac{m_i}{(1+r)^s}\left[\mu_i\left(f_i^0(s)+\sum_{k=1}^{8}v_{is}^k[f_i^k(s)-f_i^0(s)]\right)-f_i^+(s)\Theta_i^*(\alpha,\Gamma_L)\right]\right)\right] \\
& s.t. \ \sum_{k=1}^{8} v_{is}^k \le 1, \forall i, \ \forall s, \\
& \quad \sum_{i=1}^{N}\sum_{s=1}^{K_i}\left(\sum_{k=1}^{4} v_{is}^k + 2\sum_{k=5}^{8} v_{is}^k\right) \le \Gamma_B, \\
& \quad v_{is}^k \in \{0,1\}, \ \forall i, \ \forall s, \ \forall k, \\
s.t. \quad & \sum_{\tau=1}^{S} x_{i\tau} \le 1, \ \forall i, \\
& \sum_{i=1}^{N} x_{i\tau} D_i (1+r)^{(\tau-1)} \le B_\tau, \ \forall \tau, \\
& x_{i\tau} \in \{0,1\}, \ \forall i, \ \forall \tau.
\end{aligned} \tag{5.5}
$$

where the $\Theta_i^*(\alpha, \Gamma_L)$ stands for the minimum of the worst case inverse logit probability function values of the potential partners of the product i for a given probability level α under the uncertainty budget Γ_L. The robust optimization problem for the potential partner selection decision is as follows:

$$
\begin{aligned}
\Theta(\alpha, \Gamma_L) = \quad \min_y \qquad & \max_w \sum_{i=1}^{N}\sum_{j\in A^i} y_{ij}\left(Q_{ij}^0 + \sum_{k=1}^{3}\left(w_{ij}^k[Q_{ij}^k - Q_{ij}^0]\right)\right) \\
& s.t. \ \sum_{k=1}^{3} w_{ij}^k \le 1, \forall i, \ \forall j, \\
& \quad \sum_{i=1}^{N}\sum_{j\in A^i}\left(\sum_{k=1}^{2} w_{ij}^k + 2w_{ij}^3\right) \le \Gamma_L, \\
& \quad w_{ij}^k \in \{0,1\}, \ \forall i, \ \forall j, \ \forall k, \\
s.t. \ & \sum_{j\in A^i} y_{ij} = 1, \ \forall i, \\
& y_{ij} \in \{0,1\}, \ \forall i, \ \forall j.
\end{aligned} \tag{5.6}
$$

The tractability of Problem (5.5) (Problem (5.6)) above depends on the decision maker's ability to transfer the inner minimization (maximization) problem into a maximization (minimization) problem so that the outer maximization (minimization) problem incorporates the inner maximization (minimization) problem over a tractable feasible region. Bertsimas and Sim [31] recall the strong duality theorem to obtain a tractable formulation for the robust optimization model when the inner problem is a linear program. However, the inner problems of Problem (5.5) and Problem (5.6) have integer variables which makes expressing the inner minimization problem by a maximization problem difficult. Therefore, we seek for the ways of expressing the inner problems as linear programming models and using strong duality. Specifically, we will investigate totally unimodularity of the constraint matrices of the inner problems.

5.2.3 Problem Solution Approach

For a given feasible x decision, the inner minimization the Problem (5.5) handling the uncertainty affecting the Bass model parameters is formulated as:

$$\begin{aligned}
\min_{v} \sum_{i=1}^{N}\sum_{\tau=1}^{S} \frac{x_{i\tau}}{(1+r)^{\tau-1}} & \left[\sum_{s=1}^{K_i}\left(\frac{m_i}{(1+r)^s}\left[\mu_i\left(f_i^0(s)+\sum_{k=1}^{8} v_{is}^k[f_i^k(s)-f_i^0(s)]\right)-f_i^+(s)\Theta_i^*(\alpha,\Gamma_L)\right]\right)\right] \\
s.t. \ & \sum_{k=1}^{8} v_{is}^k \leq 1, \forall i,\ \forall s, \\
& \sum_{i=1}^{N}\sum_{s=1}^{K_i}\left(\sum_{k=1}^{4} v_{is}^k + 2\sum_{k=5}^{8} v_{is}^k\right) \leq \Gamma_B, \\
& v_{is}^k \in \{0,1\},\ \forall i,\ \forall s,\ \forall k,
\end{aligned} \tag{5.7}$$

For a given feasible decision vector y, the inner maximization problem of the Problem (5.6) is represented as:

$$\begin{aligned}\Theta_i^*(\alpha,\Gamma_L) = \quad & \max_w \sum_{i=1}^{N}\sum_{j\in A^i} y_{ij}\left(Q_{ij}^0 + \sum_{k=1}^{3}\left(w_{ij}^k[Q_{ij}^k - Q_{ij}^0]\right)\right)\\ & s.t. \ \sum_{k=1}^{3} w_{ij}^k \le 1, \forall i,\ \forall j,\\ & \sum_{i=1}^{N}\sum_{j\in A^i}\left(\sum_{k=1}^{2} w_{ij}^k + 2w_{ij}^3\right) \le \Gamma_L,\\ & w_{ij}^k \in \{0,1\},\ \forall i,\ \forall j,\ \forall k.\end{aligned} \tag{5.8}$$

Neither Problem (5.7) nor Problem (5.8) has a totally unimodular constraint matrix. However, they have a similar structure allowing reformulating the uncertainty budget constraint by introducing two new integer parameters Γ'_L ($\Gamma'_L \le \lfloor 0.5\Gamma_L \rfloor$) and Γ'_B ($\Gamma_B \le \lfloor 0.5\Gamma'_B \rfloor$) and decomposing the original uncertainty budget constraints so that the both problems' constraint matrices become totally unimodular. Problem (5.7) and Problem (5.8) are reformulated as:

$$\begin{aligned}\min_v \ & \sum_{i=1}^{N}\sum_{\tau=1}^{S}\frac{x_{i\tau}}{(1+r)^{\tau-1}}\left[\sum_{s=1}^{K_i}\left(\frac{m_i}{(1+r)^s}\left[\mu_i\left(f_i^0(s) + \sum_{k=1}^{8} v_{is}^k[f_i^k(s) - f_i^0(s)]\right) - f_i^+(s)\Theta_i^*(\alpha,\Gamma_L)\right]\right)\right]\\ s.t. \ & \sum_{k=1}^{8} v_{is}^k \le 1, \forall i,\ \forall s,\\ & \sum_{i=1}^{N}\sum_{s=1}^{K_i}\left(\sum_{k=1}^{4} v_{is}^k + 2\sum_{k=5}^{8} v_{is}^k\right) \le \Gamma_B - 2\Gamma'_B,\\ & \sum_{k=5}^{8} v_{is}^k \le \Gamma'_B,\\ & v_{is}^k \in \{0,1\},\ \forall i,\ \forall s,\ \forall k.\end{aligned} \tag{5.9}$$

and,

$$
\begin{aligned}
&\max_{w} \sum_{i=1}^{N} \sum_{j \in A^i} y_{ij} \left(Q_{ij}^0 + \sum_{k=1}^{3} \left(w_{ij}^k [Q_{ij}^k - Q_{ij}^0] \right) \right) \\
&s.t. \ \sum_{k=1}^{3} w_{ij}^k \leq 1, \forall i, \forall j, \\
&\quad \sum_{i=1}^{N} \sum_{j \in A^i} \sum_{k=1}^{2} w_{ij}^k \leq \Gamma_L - 2\Gamma_L', \\
&\quad \sum_{i=1}^{N} w_{ij}^3 \leq \Gamma_L', \\
&\quad w_{ij}^k \in \{0,1\}, \ \forall i, \ \forall j, \ \forall k.
\end{aligned} \tag{5.10}
$$

Assuming that newly introduced parameters h_1 and h_2 hold the following equalities $h_1 = \sum_{i=1}^{N} K_i$ and $h_2 = \sum_{i=1}^{N} \left| A^i \right|$, constraint matrices of Problem (5.9) and Problem (5.10) have the following structures, respectively,

$$
P'^1 = \begin{pmatrix}
I^{h_1Xh_1} & I^{h_1Xh_1} & I^{h_1Xh_1} & I^{h_1Xh_1} & I^{h_1Xh_1} & I^{h_1Xh_1} & I^{h_1Xh_1} & I^{h_1Xh_1} \\
1^{1Xh_1} & 1^{1Xh_1} & 1^{1Xh_1} & 1^{1Xh_1} & 0^{1Xh_1} & 0^{1Xh_1} & 0^{1Xh_1} & 0^{1Xh_1} \\
0^{1Xh_1} & 0^{1Xh_1} & 0^{1Xh_1} & 0^{1Xh_1} & 1^{1Xh_1} & 1^{1Xh_1} & 1^{1Xh_1} & 1^{1Xh_1}
\end{pmatrix}
$$

$$
P'^2 = \begin{pmatrix}
I^{h_2Xh_2} & I^{h_2Xh_2} & I^{h_2Xh_2} \\
1^{1Xh_2} & 1^{1Xh_2} & 0^{1Xh_2} \\
0^{1Xh_2} & 0^{1Xh_2} & 1^{1Xh_2}
\end{pmatrix}
$$

Lemma 5.2 *The constraint matrices of Problem (5.9) and Problem (5.10), namely P'^1 and P'^2, are totally unimodular.*

Proof. A totally unimodular matrix stays totally unimodular after multiplying a row by -1 (see Nemhauser and Wolsey [155]). Therefore, we multiply the $(h_1+1)^{th}$ row of P'^1 and the $(h_2+1)^{th}$ row of P'^2 by -1 and obtain:

$$P''^1 = \begin{pmatrix} I^{h_1 X h_1} & I^{h_1 X h_1} & I^{h_1 X h_1} & I^{h_1 X h_1} & I^{h_1 X h_1} & I^{h_1 X h_1} & I^{h_1 X h_1} & I^{h_1 X h_1} \\ -1^{1 X h_1} & -1^{1 X h_1} & -1^{1 X h_1} & -1^{1 X h_1} & 0^{1 X h_1} & 0^{1 X h_1} & 0^{1 X h_1} & 0^{1 X h_1} \\ 0^{1 X h_1} & 0^{1 X h_1} & 0^{1 X h_1} & 0^{1 X h_1} & 1^{1 X h_1} & 1^{1 X h_1} & 1^{1 X h_1} & 1^{1 X h_1} \end{pmatrix}$$

$$P''^2 = \begin{pmatrix} I^{h_2 X h_2} & I^{h_2 X h_2} & I^{h_2 X h_2} \\ -1^{1 X h_2} & -1^{1 X h_2} & 0^{1 X h_2} \\ 0^{1 X h_2} & 0^{1 X h_2} & 1^{1 X h_2} \end{pmatrix}$$

Lemma 5.3 (Nemhauser and Wolsey [155], p.544) *Let A be a $(0, -1, 1)$ matrix with no more than two nonzero elements in each column. Then, A is totally unimodular if and only if the rows of A can be partitioned into two subsets Q_1 and Q_2 such that if a column contains two nonzero elements, the following statements are true:*

- *If both nonzero elements have the same sign, then one is in a row contained in Q_1 and the other is in a row contained in Q_2.*
- *If the two nonzero elements have opposite sign, then both are in rows contained in the same subset.*

The matrices P''^1 and P''^2 satisfy these conditions.

Theorem 5.4 *The robust optimization problem 5.5 is equivalent to the following mixed integer programing problem:*

$$
\begin{aligned}
\max_{x,\sigma,\gamma,\delta,\epsilon,\nu} \quad & \sum_{i=1}^{N}\sum_{\tau=1}^{S}\frac{x_{i\tau}}{(1+r)^{\tau-1}}\left[\sum_{s=1}^{K_i}\left(\frac{m_i}{(1+r)^s}\left(\mu_i f_i^0(s) - f_i^+(s)\Theta_i^*(\alpha,\Gamma_L)\right)\right)\right] \\
& - \left(\sum_{i=1}^{N}\sum_{s=1}^{K_i}\epsilon_{is} + \sigma\Gamma_B + \nu\lfloor 0.5\Gamma_B\rfloor + \sum_{i=1}^{N}\sum_{s=1}^{K_i}\sum_{k=1}^{8}\delta_{is}^k\right) \\
s.t. \quad & \epsilon_{is} + \sigma + \delta_{is}^k \geq \sum_{\tau=1}^{S}\frac{x_{i\tau}\mu_i m_i[f_{is}^0 - f_{is}^k]}{(1+r)^{(s+\tau-1)}},\ \forall i,\ \forall s\in\{1,...,K_i\},\ \forall k\in\{1,..,4\}, \\
& \epsilon_{is} + \gamma + \delta_{is}^k \geq \sum_{\tau=1}^{S}\frac{x_{i\tau}\mu_i m_i[f_{is}^0 - f_{is}^k]}{(1+r)^{(s+\tau-1)}},\ \forall i,\ \forall s\in\{1,...,K_i\},\ \forall k\in\{5,..,8\}, \\
& \nu \geq \gamma - 2\sigma, \\
& \sum_{\tau=1}^{S} x_{i\tau} \leq 1,\ \forall i, \\
& \sum_{i=1}^{N} x_{i\tau} D_i \leq \frac{B_\tau}{(1+r)^{\tau-1}},\ \forall \tau, \\
& x_{i,\tau}\in\{0,1\},\ \forall i,\ \forall\tau \\
& \epsilon,\gamma,\delta,\sigma,\nu \geq 0,
\end{aligned} \tag{5.11}
$$

where $\Theta^*(\alpha, \Gamma_L)$ is represented as:

$$\begin{aligned}
\min_{\theta,\beta,\chi,\zeta,z,y} \quad & \sum_{i=1}^{N}\sum_{j\in A^i} y_{ij}O_{ij}^0 + \sum_{k=1}^{3}\sum_{i=1}^{N}\sum_{j\in A^i}\beta_{ij} + \chi\Gamma_L + \zeta\lfloor 0.5\Gamma_L\rfloor + \sum_{i=1}^{N}\sum_{j\in A^i}\sum_{k=1}^{3} z_{ij}^k \\
s.t. \quad & \beta_{ij} + \chi + z_{ij}^k - y_{ij}[O_{ij}^k - O_{ij}^0] \geq 0, \forall i,\ \forall j\in A^i,\ \forall k\in\{1,2\}, \\
& \beta_{ij} + \theta + z_{ij}^k - y_{ij}[O_{ij}^3 - O_{ij}^0] \geq 0, \forall i,\ \forall j \in A^i, \\
& \zeta \geq \theta - 2\chi, \\
& \sum_{j\in A^i} y_{ij} = 1,\ \forall i, \\
& y_{ij}\in\{0,1\},\ \forall i,j, \\
& \theta, \beta_{ij}, \chi, \zeta, z_{ij}^k \geq 0.
\end{aligned} \tag{5.12}$$

Proof. The inner minimization problem (5.9) has a totally unimodular constraint matrix and the right hand side coefficients are integers; therefore, the solution of the linear relaxation of the problem the same as that of the original problem with binary variables. This enables us to invoke strong duality in the robust optimization formulation. This approach was introduced by [70]. Then, the dual of the problem (5.9) is formulated as

$$\begin{aligned}
\max_{\sigma,\gamma,\delta,\epsilon} \quad & -\left(\sum_{i=1}^{N}\sum_{s=1}^{K_i}\epsilon_{is} + \sigma(\Gamma_B - 2\Gamma_B') + \gamma\Gamma_B' + \sum_{i=1}^{N}\sum_{s=1}^{K_i}\sum_{k=1}^{8}\delta_{is}^k\right) \\
s.t. \quad & \epsilon_{is} + \sigma + \delta_{is}^k \geq \sum_{\tau=1}^{S}\frac{x_{i\tau}\mu_i m_i[f_{is}^0 - f_{is}^k]}{(1+r)^{(s+\tau-1)}},\ \forall i,\ \forall s\in\{1,...,K_i\},\ \forall k\in\{1,..,4\}, \\
& \epsilon_{is} + \gamma + \delta_{is}^k \geq \sum_{\tau=1}^{S}\frac{x_{i\tau}\mu_i m_i[f_{is}^0 - f_{is}^k]}{(1+r)^{(s+\tau-1)}},\ \forall i,\ \forall s\in\{1,...,K_i\},\ \forall k\in\{5,..,8\}, \\
& \epsilon, \gamma, \delta, \sigma \geq 0.
\end{aligned} \tag{5.13}$$

The overall robust NPV maximization problem requires, first, solving the minimization (with nonnegative objective function) problem (5.13) over Γ'_B, $0 \leq \Gamma'_B \leq \lfloor 0.5\Gamma_B \rfloor$, then maximizing over x. That is, the suggested robust optimization approach is composed of, from the innermost to the outermost, a maximization over the auxiliary dual variables $\epsilon, \gamma, \delta, \sigma$, a minimization problem whose decision variable is Γ_B, and a maximization over x. The innermost minimization problem's objective function is bilinear in Γ'_B and γ, σ, and linear in all other variables. The feasible set for Γ'_B is closed and bounded below by 0 and above by $\lfloor 0.5\Gamma_B \rfloor$. We relax the integrality constraint on Γ'_B, which will lead to an integer solution. Each of the decision variables $\epsilon, \gamma, \delta, \sigma$ in the problem (5.13) is less than or equal to $\max_i \ \max_{s \in \{1,\ldots,K_i\}} \sum_{\tau=1}^{S} \frac{x_{i\tau}\mu_i m_i max_k |f_{is}^0 - f_{is}^k|}{(1+r)^{(s+\tau-1)}}$, which can be observed from the constraints, therefore the decision variables $\epsilon, \gamma, \delta, \sigma$ are bounded from below by 0 and from above by $\max_i \ \max_{s \in \{1,\ldots,K_i\}} \sum_{\tau=1}^{S} \frac{x_{i\tau}\mu_i m_i max_k |f_{is}^0 - f_{is}^k|}{(1+r)^{(s+\tau-1)}}$. Therefore, Proposition 5.4.4 p.532 in [26] applies and we can switch the order of the minimization over Γ'_B and the innermost maximization. The coefficient in front of Γ'_B is $-\gamma + 2\sigma$, therefore, the minimization over Γ'_B will bring $\min(0, -\gamma + 2\sigma)\lfloor 0.5\Gamma_B \rfloor$ into the overall objective function. We observed that Γ'_B is an integer and equal to either $\lfloor 0.5\Gamma_B \rfloor$ or zero at optimality; however, it can take other integer values when the equality $\sigma = \gamma$ holds at optimality. Therefore, while solving the maximization problem over x, we linearize the piece-wise linear formulation $\max(0, \gamma - 2\sigma)$ by introducing a new variable v and reach the formulation of the Problem (5.11).

A very similar argument proves modeling approach for the Problem (5.12).

5.2.4 Product Launching with a Real Option: Option to Update the Contract Size

In this chapter, we address the error involved in the estimation for the parameter m_i, the ultimate number of adopters of the product i. The parameter m_i does not change during the diffusion process in contrast with the other parameters of the Bass model: p_i and q_i for the product i. It is estimated before the process starts by using some analog products which have been in the market or based on managerial judgment. However, the innovative company can have a more educated estimation for the parameter m_i by observing the first few periods' sales amounts after the product is launched. The real option considered in this chapter provides the innovative company with the right to reduce the size of the contract with the partners by a given fraction at a certain time period. The reader is reminded that in the absence of the real option, the innovative company sets the size of the contract according to the highest possible new adoption rates considering the parameters $\bar{p}_i,\ \bar{q}_i,\ \hat{p}_i,$ and, $\hat{q}_i$ and the estimation for m_i so that for the period which is s periods later than the process starts $f_i^+(s)m_i$ unit of capacity is reserved at the partner company. Although this strategy provides safety against the case of underestimated m_i, it might result in extra payments made to the selected partners in the case of overestimated m_i. Therefore, the option to update the contract size allow the innovative company to rearrange the size of the contract (reserved service and committed payment) based on the current estimate for m_i. Therefore, we extend our approach to the case where the innovative company selects its prospective partners in the presence of the option to update contract size at a certain period. This real option can be considered a European type option to shrink the size of the contract while keeping the payment per unit constant.

The partner firm (the writer of the option) has to honor the innovative company's (the buyer of the option) wish to decrease the contract size by the fraction $((1-\kappa_i))$ stated in the contract at the specified date (the option expiration date which is η periods after the product is launched) specified in the contract because of the definition of the option. Therefore, the innovative firm pays an option premium (Ω_i) to the partner to make it willing to commit to the requirements brought by the real option for the product i. If the innovative company exercises the real option for the product i, the selected partner for this product has to give up the profit that it could have gained by conducting business for $(1-\kappa)$ units of products.

We assume the partners are rational decision makers, and they use the NPV of the net profit they obtain as the measurement to compare the two alternatives: the business contract with the innovative company in the absence of the real option and in the presence of the real option. However, the innovative company does not know the unit cost that the partner company encounters for the product i. Therefore, we use the industry profit margin average (ψ_i) as an estimation for the profit margin of each of the potential partners for the product i (We use S&P 500 Sectors and Industries Profit Margins Report by Yardeni and Abbott [224] as a reference.). The potential partner, which is foreseen to have a business with the innovative company, is expected to be paid by $m_i f_i^+(s)$ amount of money at the s^{th} period of their partnership where s changes between 1 and K_i. Therefore, the NPV of the total payments foreseen to be made to the partner j for the product i, which is introduced to the market at time period τ when the real option is not available, is formulated as:

$$NPV(TotalPayment) = \frac{1}{(1+r)^{\tau-1}} \sum_{s=1}^{K_i} \frac{f_i^+(s) m_i \Theta_i^*(\alpha, \Gamma_L)}{(1+r)^s}.$$

Then, using the industry profit margin for the partner for the product i, the NPV of the profit that the partner obtains from this business is formulated as:

$$NPV(TotalProfit) = \frac{\psi_i}{(1+r)^{\tau-1}} \sum_{s=1}^{K_i} \frac{f_i^+(s) m_i \Theta_i^*(\alpha, \Gamma_L)}{(1+r)^s}.$$

If the innovative company has purchased the real option and exercises it, the partner will be paid by:

$$NPV(TotalPayment)' = \frac{1}{(1+r)^{\tau-1}} \left(\Omega_i + \sum_{s=1}^{\eta_i} \frac{f_i^+(s) m_i \Theta_i^*(\alpha, \Gamma_L)}{(1+r)^s} + \sum_{s=\eta_i+1}^{K_i} \frac{f_i^+(s) \kappa_i m_i \Theta_i^*(\alpha, \Gamma_L)}{(1+r)^s} \right)$$

and the total net profit of the partner will be:

$$NPV(TotalProfit)' = \frac{1}{(1+r)^{\tau-1}} \left(\Omega_i + \psi_i \sum_{s=1}^{\eta_i} \frac{f_i^+(s) m_i \Theta_i^*(\alpha, \Gamma_L)}{(1+r)^s} + \psi_i \sum_{s=\eta_i+1}^{K_i} \frac{f_i^+(s) \kappa_i m_i \Theta_i^*(\alpha, \Gamma_L)}{(1+r)^s} \right)$$

The partner company needs to have at least the same profit when the innovative company exercises it in order to accept the contract with the real option. Therefore, the option premium should be at least equal to the NPV of the profit that could have been obtained by the decreased portion of the contract. In other words, $NPV(TotalProfit) <= NPV(TotalProfit)'$ should be satisfied so that the partner company accepts the requirements of the real option. This results in the following option premium formulation:

$$\Omega_i^* = \psi_i \sum_{s=\eta_i+1}^{K_i} \frac{f_i^+(s)(1-\kappa_i)m_i\Theta_i^*(\alpha,\Gamma_L)}{(1+r)^s}.$$

Please note that since all of the potential partners for a given product are assumed to have the same profit margin (market profit margin), and the NPV calculation of the profit is linear function of the unit payments $\Theta_i^*(\alpha,\Gamma_L)$, this way of calculating the option premium does not contradict the logit choice models of the potential partners and the partner selection strategy of the innovative firm in Problem (5.2).

In the presence of the real option, the innovative company exercises the real option if the new estimation for m_i is less than the amount specified in the option ($\kappa_i m_i$). It is assumed that the innovative company has a set of scenarios for the possible future estimations for the ultimate number of adopters ($\tilde{m}_i^l$) and their corresponding probabilities (π_i^l). Let us define a set N^i such that $N^i = \{l : \tilde{m}_i^l < \kappa_i m_i\}$. Then, the probability that the innovative firm exercises the option (ρ_i) for the product i is calculated as:

$$\rho_i = \sum_{l \in N^i} \pi_i^l.$$

The expected value of the NPV of the payments foreseen to be made by the innovative company to the prospective partner is equal to:

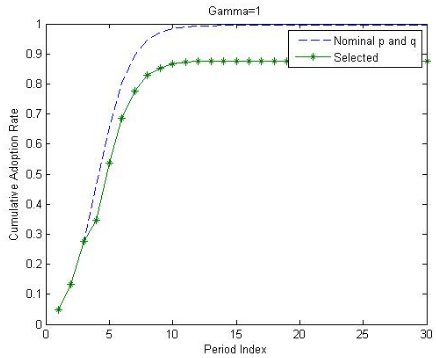
$$\frac{1}{(1+r)^{\tau-1}}\left(\Omega_i^* + \sum_{s=1}^{\eta_i}\frac{f_i^+(s)m_i\Theta_i^*(\alpha,\Gamma_L)}{(1+r)^s} + \rho_i \sum_{s=\eta_i+1}^{K_i}\frac{f_i^+(s)\kappa_i m_i\Theta_i^*(\alpha,\Gamma_L)}{(1+r)^s} + (1-\rho_i)\sum_{s=\eta_i+1}^{K_i}\frac{f_i^+(s)m_i\Theta_i^*(\alpha,\Gamma_L)}{(1+r)^s}\right).$$

Therefore, in the presence of the real options, Problem (5.11) becomes:

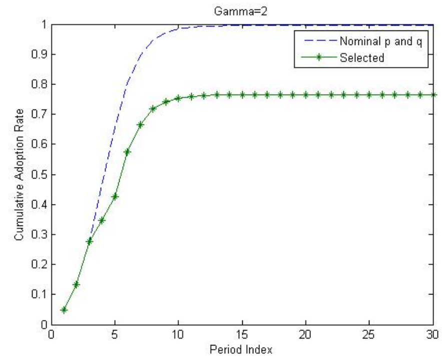
$$
\begin{aligned}
\max_{x,\sigma,\gamma,\delta,\epsilon,\nu} \quad & \sum_{i=1}^{N}\sum_{\tau=1}^{S}\left(\frac{x_{i\tau}}{(1+r)^{\tau-1}}\left[\sum_{s=1}^{\eta_i}\left(\frac{m_i}{(1+r)^s}\left(\mu_i f_i^0(s) - f_i^+(s)\Theta_i^*(\alpha,\Gamma_L,\Gamma_L')\right)\right)\right]\right) \\
& \sum_{i=1}^{N}\sum_{l\in N^i}\pi_i^l\sum_{\tau=1}^{S}\left(\frac{x_{i\tau}}{(1+r)^{\tau-1}}\left[\sum_{s=\eta_i+1}^{K_i}\left(\frac{1}{(1+r)^s}\left(m_i^l\mu_i f_i^0(s) - \kappa_i m_i f_i^+(s)\Theta_i^*(\alpha,\Gamma_L,\Gamma_L')\right)\right)\right]\right) \\
& \sum_{i=1}^{N}\sum_{l\in N^{i'}}\pi_i^l\sum_{\tau=1}^{S}\left(\frac{x_{i\tau}}{(1+r)^{\tau-1}}\left[\sum_{s=\eta_i+1}^{K_i}\left(\frac{1}{(1+r)^s}\left(\min(m_i^l,m_i)\mu_i f_i^0(s) - m_i f_i^+(s)\Theta_i^*(\alpha,\Gamma_L,\Gamma_L')\right)\right)\right]\right) \\
& -\left(\sum_{i=1}^{N}\sum_{s=1}^{K_i}\epsilon_{is} + \sigma\Gamma_B + \nu\lfloor 0.5\Gamma_B\rfloor + \sum_{i=1}^{N}\sum_{s=1}^{K_i}\sum_{k=1}^{8}\delta_{is}^k\right) \\
s.t. \quad & \epsilon_{is} + \sigma + \delta_{is}^k \geq \sum_{\tau=1}^{S}\frac{x_{i\tau}\mu_i\left(\sum_{l\in N^i} m_i^l\pi_i^l + \sum_{l\in N^{i'}}\pi_i^l\min(m_i,m_i^l)\right)[f_{is}^0 - f_{is}^k]}{(1+r)^{(s+\tau-1)}},\ \forall i,\ \forall s\in\{\eta_i+1,...,K_i\}\ \forall k=\{1,..,4\}, \\
& \epsilon_{is} + \sigma + \delta_{is}^k \geq \sum_{\tau=1}^{S}\frac{x_{i\tau}\mu_i m_i[f_{is}^0 - f_{is}^k]}{(1+r)^{(s+\tau-1)}},\ \forall i,\ \forall s\in\{1,...,\eta_i\},\ \forall k=\{1,..,4\}, \\
& \epsilon_{is} + \gamma + \delta_{is}^k \geq \sum_{\tau=1}^{S}\frac{x_{i\tau}\mu_i m_i[f_{is}^0 - f_{is}^k]}{(1+r)^{(s+\tau-1)}},\ \forall i,\ \forall s\in\{1,...,\eta_i\},\ \forall k=\{5,..,8\}, \\
& \epsilon_{is} + \gamma + \delta_{is}^k \geq \sum_{\tau=1}^{S}\frac{x_{i\tau}\mu_i\left(\sum_{l\in N^i} m_i^l\pi_i^l + \sum_{l\in N^{i'}}\pi_i^l\min(m_i,m_i^l)\right)[f_{is}^0 - f_{is}^k]}{(1+r)^{(s+\tau-1)}},\ \forall i,\ \forall s\in\{\eta_i+1,...,K_i\}\ \forall k=\{5,..,8\}, \\
& \nu \geq \gamma - 2\sigma, \\
& \sum_{\tau=1}^{S} x_{i\tau} \leq 1,\ \forall i, \\
& \sum_{i=1}^{N} x_{i\tau}(D_i(1+r)^{(\tau-1)} + \Omega_i^*(\alpha,\Gamma_L,\Gamma_L')) \leq B_\tau,\ \forall\tau, \\
& x_{i,\tau}\in\{0,1\},\ \forall i,\ \forall\tau, \\
& \epsilon,\gamma,\delta,\sigma,\nu \geq 0.
\end{aligned}
\tag{5.14}
$$

5.2.5 Numerical Experiments

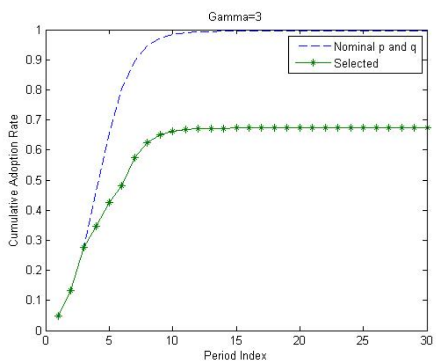
In this section, we analyze how the adoption rate selection decision of the inner robust optimization model, Problem (5.7), is affected by the increase in the parameter Γ_B value for a product. We will see that as uncertainty budget increases, the adoption rate proposed by robust optimization disperses from the adoption rate obtained when the parameter values are at their nominal values. Figures 5.5, 5.6, 5.7, and 5.8 provide the adoption rates determined by robust optimization for various uncertainty budget values.



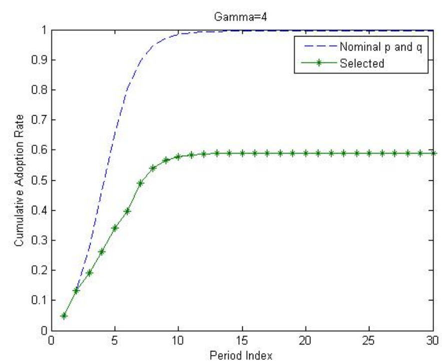
(a) $\Gamma_B = 1$



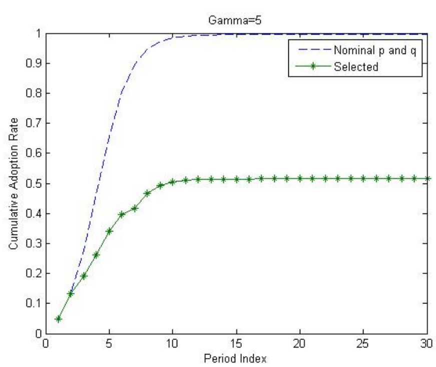
(b) $\Gamma_B = 2$



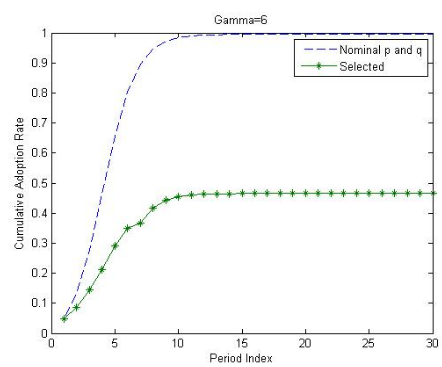
(c) $\Gamma_B = 3$



(d) $\Gamma_B = 4$

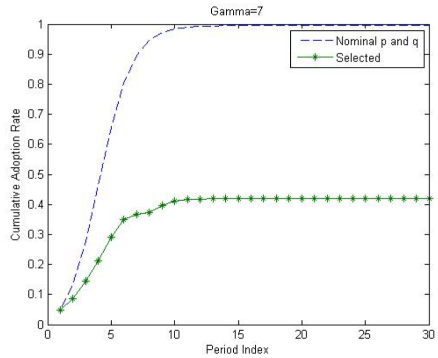


(e) $\Gamma_B = 5$

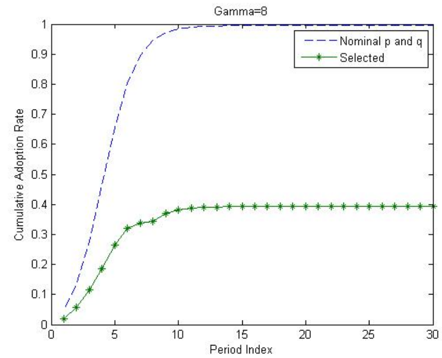


(f) $\Gamma_B = 6$

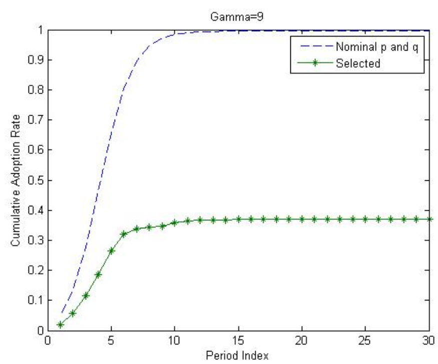
Figure 5.5: The Impact of the Uncertainty Budget Parameter on the Robust Cumulative Adoption Rates ($1 \leq \Gamma_B \leq 6$)



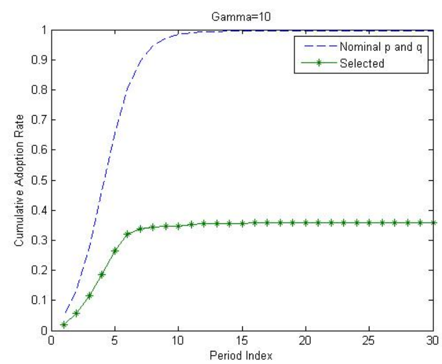
(a) $\Gamma_B = 7$



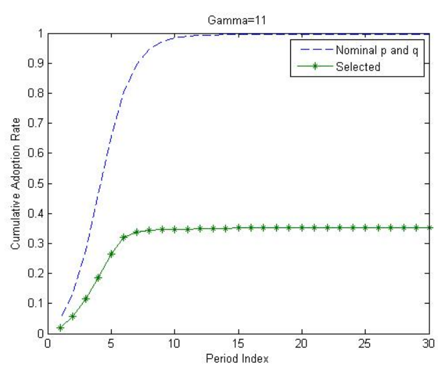
(b) $\Gamma_B = 8$



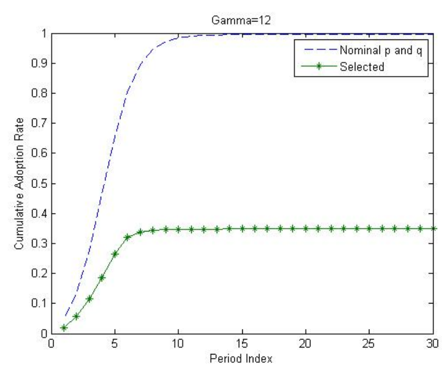
(c) $\Gamma_B = 9$



(d) $\Gamma_B = 10$

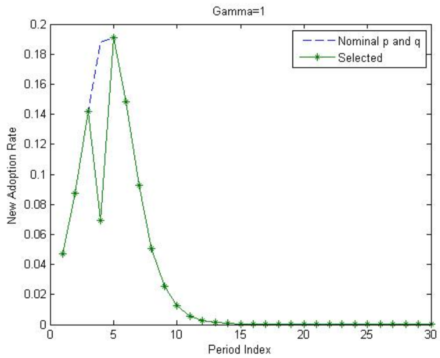


(e) $\Gamma_B = 11$

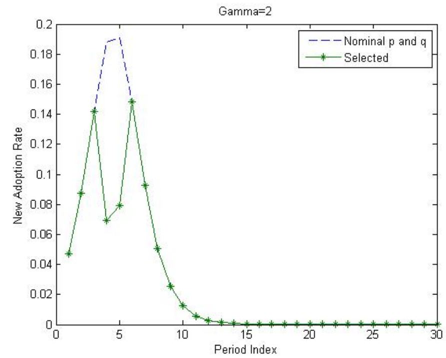


(f) $\Gamma_B = 12$

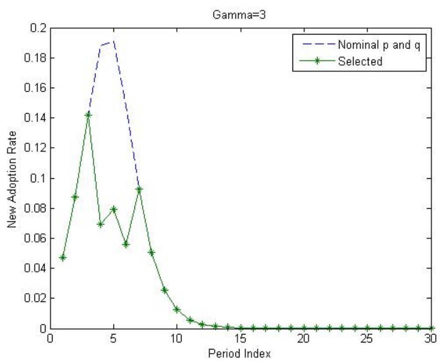
Figure 5.6: The Impact of the Uncertainty Budget Parameter on the Robust Cumulative Adoption Rates ($7 \leq \Gamma_B \leq 12$)



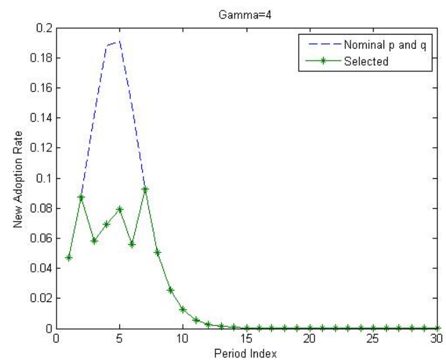
(a) $\Gamma_B = 1$



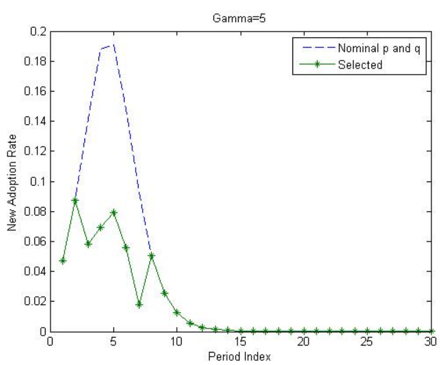
(b) $\Gamma_B = 2$



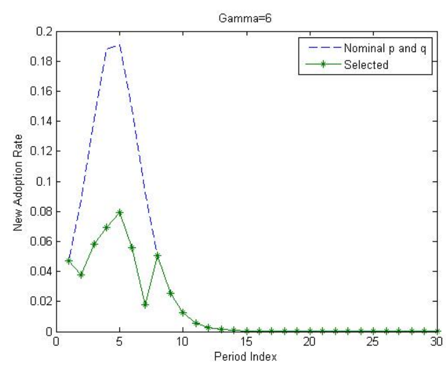
(c) $\Gamma_B = 3$



(d) $\Gamma_B = 4$

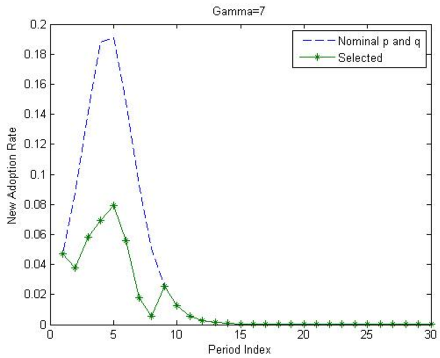


(e) $\Gamma_B = 5$

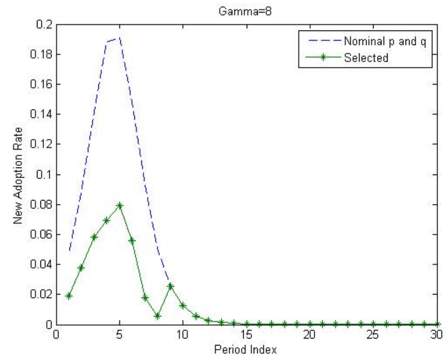


(f) $\Gamma_B = 6$

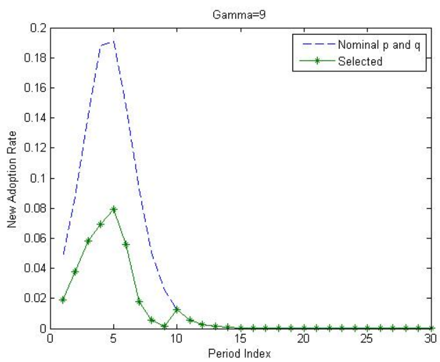
Figure 5.7: The Impact of the Uncertainty Budget Parameter on the Robust New Adoption Rates $(1 \leq \Gamma_B \leq 6)$



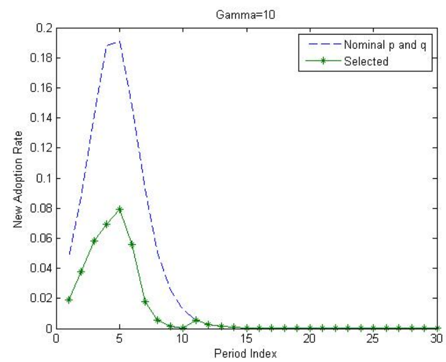
(a) $\Gamma_B = 7$



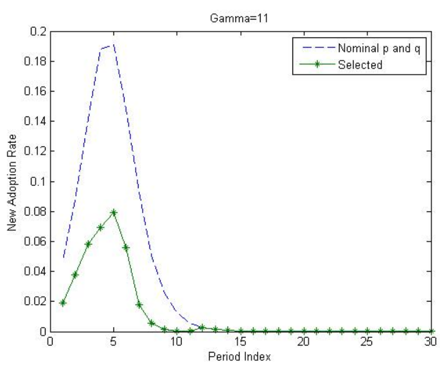
(b) $\Gamma_B = 8$



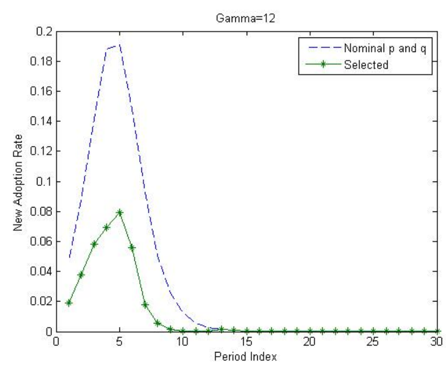
(c) $\Gamma_B = 9$



(d) $\Gamma_B = 10$



(e) $\Gamma_B = 11$



(f) $\Gamma_B = 12$

Figure 5.8: The Impact of the Uncertainty Budget Parameter on the Robust New Adoption Rates ($7 \leq \Gamma_B \leq 12$)

Table 5.2: Sensitivity of the optimal strategy to the parameter q_i of the region i

Coefficient of q_i	only q_1 changes	only q_2 changes	only q_3 changes	only q_4 changes	only q_5 changes
0.6	87	9	3	7	88
0.64	87	9	3	7	88
0.68	87	9	3	7	88
0.72	87	9	3	7	88
0.76	85	7	3	9	90
0.8	85	7	3	9	90
0.84	85	7	1	9	90
0.88	85	1	7	9	90
0.92	85	1	7	9	90
0.96	87	7	7	9	90
1	7	7	7	7	7
1.04	112	33	7	111	7
1.08	112	113	3	111	7
1.12	112	113	107	111	7
1.16	112	113	107	111	7
1.2	112	113	107	111	7
1.24	112	113	107	111	7
1.28	112	113	107	111	7
1.32	112	113	107	111	7
1.36	112	113	107	111	7
1.4	112	113	107	111	7

Table 5.3: Sensitivity of the optimal strategy to the parameter p_i of the region i

Coefficient of p_i	only p_1 changes	only p_2 changes	only p_3 changes	only p_4 changes	only p_5 changes
0.6	112	7	1	9	7
0.64	112	7	1	9	7
0.68	112	7	1	9	7
0.72	112	7	1	9	7
0.76	112	7	1	9	7
0.8	112	7	1	9	7
0.84	112	7	1	9	7
0.88	112	7	7	9	7
0.92	112	7	7	9	7
0.96	112	7	7	7	7
1	7	7	7	7	7
1.04	7	7	7	7	112
1.08	7	9	7	7	112
1.12	7	9	7	7	112
1.16	7	9	7	7	112
1.2	7	9	7	87	112
1.24	7	9	7	87	112
1.28	7	9	7	87	112
1.32	7	33	7	87	112
1.36	7	33	7	87	112
1.4	7	33	7	87	112

Chapter 6

Log-Robust Portfolio Management with Factor Model

6.1 Log-Robust Portfolio Management with Factor Model

6.1.1 Generalities for Asset Pricing

In this section, we present our asset pricing approach. We employ a factor-model to formulate asset prices. We use either the Geometric Brownian Motion or the Ornstein-Uhlenbeck Process to model each factor considered in the asset pricing model.

Asset Pricing via Multi-Factor Model Throughout the paper, we use generic term "asset" since we believe that our asset pricing approach is applicable to several asset types including foreign currencies and publicly traded stocks since it allows including both mean-reverting and non-mean-reverting factors into the asset pricing model.

We analyzed historical data sets of several asset types such as monthly direct cross currency quotes between 12/2001 and 11/2011 while the USA is the home country and the US Dollar is the primal currency. The correlation coefficients between the natural logarithms of some exchange rate quotes are represented in Figure 6.1 as an example.

	BRL	CAD	CNY	DKK	HKD	INR	JPY	MYR	MXN	NOK	ZAR	SGD	KRW	LKR	SEK	CHF
BRL	1															
CAD	0.89	1														
CNY	0.83	0.76	1													
DKK	0.7	0.89	0.68	1												
HKD	0.24	0.29	0.34	0.26	1											
INR	0.37	0.52	-0	0.52	-0.3	1										
JPY	0.6	0.66	0.85	0.61	0.42	-0.1	1									
MYR	0.87	0.83	0.89	0.67	0.08	0.29	0.76	1								
MXN	-0.4	-0.5	-0.7	-0.6	-0.6	0.23	-0.8	-0.45	1							
NOK	0.74	0.91	0.66	0.94	0.12	0.61	0.57	0.76	-0.4	1						
ZAR	0.15	0.49	-0.1	0.52	0.11	0.58	0.19	0.11	-0.1	0.52	1					
SGD	0.88	0.89	0.94	0.79	0.24	0.24	0.85	0.96	-0.6	0.82	0.2	1				
KRW	0.32	0.49	-0.1	0.42	-0.2	0.82	-0.2	0.19	0.27	0.56	0.65	0.18	1			
LKR	-0.8	-0.8	-0.8	-0.8	-0.4	-0.2	-0.7	-0.73	0.76	-0.7	-0.1	-0.8	-0.1	1		
SEK	0.64	0.87	0.53	0.93	0.06	0.67	0.53	0.67	-0.4	0.95	0.66	0.74	0.6	-0.6	1	
CHF	0.75	0.87	0.86	0.86	0.28	0.24	0.89	0.86	-0.7	0.85	0.39	0.95	0.19	-0.77	0.82	1

Figure 6.1: Correlation Coefficients Between Natural-Logarithm of the Currency Rates.

The correlation coefficients between the natural logarithms of the exchange rates are too significant to be neglected. However, if the correlation values are directly used in the portfolio optimization problem in which n currencies are considered to invest, n standard deviations and $\frac{n(n-1)}{2}$ correlation coefficients have to be predicted. This argument is valid for other assets too.

Usually, a historical data set is the main source for this estimation process. However, this process is subject to estimation errors and the precision of the estimation for the overall portfolio risk and the accuracy of the investment decisions are negatively affected by these errors. In addition, determining the size of the historical data sample is another matter. For instance, a financial or political crisis in a country may affect the performance of a sector far more than that of other sectors in a certain period. In that case, this extreme event has a significant effect on the covariance values in this period. Therefore, an investor who has an interest in several assets should find a historical data set which is large enough to make accurate estimations and free from impacts of extreme events, and find a way of reflecting the consequences of these extreme events. Therefore, estimating the correlation coefficients explicitly is cumbersome.

Therefore, we decided to express the correlation between the asset prices by a multi-factor model in order to keep our approach practical and accurate. The overall impact of using the same factors with different loadings in each asset's pricing formula reflects the correlation between the asset prices. In addition, while estimating n assets' prices by the m-factors model, mn factor loadings and n error terms must be predicted. Also, at most three input parameters for the stochastic process of each factor and a volatility parameter for the error term of the factor model need to be forecast. That is, at most $mn + n + 3m$ parameters need to be predicted according to our approach. Given that m (number of factors) is usually smaller than n (number of assets), the m-factors model requires fewer parameters to be estimated than forming a covariance matrix.

Throughout the chapter, the following notation will be used.

n : the number of assets,

m : the number of factors,

T : the length of the time horizon,

$S_j(0)$: the initial (known) value of factor j,

$S_j(T)$: the (random) value of factor j at time T,

$R_i(T)$: the (random) value of asset i at time T,

w_0 : the amount of money to invest at the beginning of the time period,

μ_j : the drift of the stochastic process for factor j,

σ_j : the volatility term of the stochastic process for factor j,

$\tilde{\sigma_j}$: the volatility term of the stochastic process for the error term (ϵ_i) in the factor model for asset i,

θ_j : the rate by which the variable reverts towards to μ_j in the stochastic process for factor j,

$a_{i,j}$: the factor loading coefficient of factor j for asset i,

Γ : the uncertainty budget for the portfolio,

$\epsilon_i(T)$: the error term of the factor model for asset i at time T,

Z_i : the uncertain parameter with nominal value of zero and known support $[-c_i, c_i]$ for all i,

$\tilde{z}_i$: the *scaled deviation* of Z_i from its mean, which is 0, such that $Z_i = c_i \, \tilde{z}_i$ and $\tilde{z}_i \in [-1, 1]$,

Y_j : the uncertain parameter with nominal value of zero and known support $[-d_j, d_j]$ for all j,

$\tilde{y}_j$: the *scaled deviation* of Y_j from its mean, which is 0, such that $Y_j = d_j \, \tilde{y}_j$ and $\tilde{y}_j \in [-1, 1]$,

x_i : the amount of money invested in asset i,

$\tilde{x}_i$: the fraction of the portfolio invested in asset i.

According to our factor model, the price of asset i, $R_i(T)$, is as follows:

$$\ln R_i(T) = \sum_{j=1}^{m}(a_{i,j} \ln S_j(T)) + \epsilon_i(T) \quad \forall i \in \{1, 2.., n\}. \tag{6.1}$$

$\ln S_j(T)$ represents the value of the factor j at time T and $a_{i,j}$ stands for the sensitivity of the natural logarithm of the price of the asset i ($\ln R_i(T)$) to the factor $\ln S_j(T)$ while $\epsilon_i(T)$ is the idiosyncratic component of $\ln R_i(T)$.

We assume that each idiosyncratic component ($\epsilon_i(T)$ where $i \in \{1, 2, .., n\}$) is independent of other idiosyncratic components and factors ($\ln S_j(T)$ where $j \in \{1, 2, .., m\}$). In addition, the correlation between factors is neglected. In other words, the only sources of correlation among $\ln R_i(T)$ values are the factors and their loadings in the asset pricing formula.

Each idiosyncratic component $\epsilon_i(T)$ is assumed to follow a Geometric Brownian Motion with zero drift such that:

$$\frac{\mathrm{d}\epsilon_i(t)}{\epsilon_i(t)} = \tilde{\sigma_i}\mathrm{dW}_t,$$

where the parameter $\tilde{\sigma_i}$, $\tilde{\sigma_i}>0$, which is estimated from historical data, represents the volatility and dW_t stands for a Wiener process. After applying the Weiner integration, the equation turns into the following form:

$$\epsilon_i(T) = \tilde{\sigma_i}\sqrt{T}Z_i,$$

where Z_i obeys a Gaussian distribution, i.e., $Z_i \sim N(0, 1)$. Then, the equation (6.1) can be written as:

$$\ln R_i(T) = \sum_{j=1}^{m}(a_{i,j} \ln S_j(T)) + \tilde{\sigma_i}\sqrt{T}Z_i \quad \forall i \in \{1, 2.., n\}. \tag{6.2}$$

Each factor loading ($a_{i,j}$) is estimated as a regression coefficient through a linear regression model where the dependent variable is $\ln R_i$, the regressors are $\ln S_1$, $\ln S_2, ...,$ $\ln S_m$, and the error term is ϵ_i.

The Ornstein-Uhlenbeck Model for Mean-Reverting Factors

Commodity prices, interest rates, and exchange rates are observed to be mean-reverting. Any mean reverting measurement used as a factor in the asset price modeling is assumed to follow an Ornstein-Uhlenbeck process:

$$\mathrm{d}s_j = \theta_j(\mu_j - s_j)\mathrm{d}_t + \sigma_j \mathrm{dW}_t,$$

where s_j stands for $\ln S_j$, $\theta_j{>}0$, μ_j and $\sigma_j{>}0$ are the parameters estimated from historical data, and dW_t stands for a Wiener process.

After applying Itō's-Doeblin's formula and integrating the both sides of the equation from 0 to ∞, we obtain:

$$s_j(t) = s_j(0)e^{-\theta_j t} \, + \, \mu_j(1 - e^{-\theta_j t}) \, + \, \int_0^t \sigma_j e^{\theta_j(v-t)} \mathrm{dW}_v.$$

After integrating the last term and rearranging, we reach:

$$S_j(T) = S_j(0)e^{\left(e^{-\theta_j T} + \mu_j\left(1 - e^{-\theta_j T}\right) + \sigma_j e^{-\theta_j T}\sqrt{\frac{e^{2\theta_j T}-1}{2\theta_j}} Y_j\right)}, \tag{6.3}$$

where Y_j obeys a Gaussian distribution, i.e. $Y_j \sim N(0, 1)$.

The Geometric Brownian Motion for Non-Mean-Reverting Factors

Some factors such as stock price indexes are not mean reverting. Non-mean-reverting factors in the asset price modeling are assumed to follow a Geometric Brownian Motion with log-normal property and satisfy the following equation:

$$\mathrm{d}S_j = \mu_j S_j \mathrm{d}t + \sigma_j S_j \mathrm{dW}_t,$$

where $S_j(t)$ is the value of the factor j at time t, μ_j and $\sigma_j>0$ are the drift and volatility parameters estimated from historical data, and dW_t is a Wiener process.

After applying Itō's-Doeblin's formula and integrating both sides of the equation from 0 to ∞, we obtain:

$$S_j(T) = S_j(0)e^{\left(\mu_j - \frac{\sigma_j^2}{2}\right)T + \sigma_j\sqrt{T}Y_j}, \tag{6.4}$$

where Y_j obeys a Gaussian distribution, i.e. $Y_j \sim N(0,1)$.

The Asset Pricing Model as an Input for the Portfolio Optimization Problem

We assume that the factors' values at time $t = 0$, $(\ln S_1(0),\ \ln S_2(0), .., \ln S_m(0))$, are known. The riskiness of the asset i results from the random variables Z_i of asset i and $Y_1,\ Y_2, .., Y_j, .., Y_m$ of factors $1, 2, .., j, .., m$ which are used in the factor model for the asset i in equations (6.2), (6.3) and (6.4).

Y_j is a random variable whose nominal value is zero and which belongs to the interval $[-d_j, d_j]$. This is valid for every j, where $j \in \{1, 2, .., m\}$ and d_j is a positive constant used for defining an interval for possible Y_j values. For each j, we define $\tilde{y}_j$ as the *scaled deviation* of Y_j from its mean, which is 0, such that $Y_j = d_j\,\tilde{y}_j$ and $\tilde{y}_j \in [-1, 1]$.

Similarly, Z_i is a random variable whose nominal value is zero and which belongs to the interval $[-c_i, c_i]$. This is valid for every i, where $i \in \{1, 2, .., n\}$ and c_i is a positive constant used for defining an interval for possible c_i values. For each i, we define $\tilde{z}_i$ as the *scaled deviation* of Z_i from its mean, which is 0, such that $Z_i = c_i\,\tilde{z}_i$ and $\tilde{z}_i \in [-1, 1]$.

For notational convenience, the model for exponential of the factor j in equations 6.3 and 6.4 can be generalized as:

$$S_j(T) = S_j(0)e^{p_j + q_j Y_j},$$

where p_j and q_j are as follows:

$$p_j = \begin{cases} \left(\mu_j - \frac{\sigma_j^2}{2}\right) T, & \text{if the factor } j \text{ follows a Geometric Brownian Motion,} \\ \left(e^{-\theta_j T} + \mu_j \left(1 - e^{-\theta_j T}\right)\right), & \text{if the factor } j \text{ follows an Ornstein-Uhlenbeck Process;} \end{cases}$$

$$q_j = \begin{cases} \left(\sigma_j \sqrt{T}\right), & \text{if the factor } j \text{ follows a Geometric Brownian Motion,} \\ \left(\sigma_j e^{-\theta_j T} \sqrt{\frac{e^{2\theta_j T} - 1}{2\theta_j}},\right), & \text{if the factor } j \text{ follows an Ornstein-Uhlenbeck Process.} \end{cases}$$

Next, we can formulate the value of the asset i, $R_i(T)$, as follows:

$$R_i(T) = \left(\prod_{j=1}^{m} (S_j(0)^{a_{i,j}}\right) \exp\left[\sum_{j=1}^{m} (a_{i,j}(p_j + q_j Y_j)) + \tilde{\sigma}_i \sqrt{T} Z_i\right] \quad \forall i.$$

$S_j(0)$ and T are known. Whereas $a_{i,j}, \mu_j$, σ_j and θ_j are estimated. Considering that all of these parameters are deterministic. For a more convenient notation, we can define a parameter k_i as follows:

$$k_i = \left(\prod_{j=1}^{m} (S_j(0)^{a_{i,j}}\right) \exp\left[\sum_{j=1}^{m} (a_{i,j} p_j)\right] \quad \forall i.$$

This leads to the following formulation:

$$R_i(T) = k_i \exp\left[\sum_{j=1}^{m} (a_{i,j} q_j Y_j) + \tilde{\sigma}_i \sqrt{T} Z_i\right] \quad \forall i.$$

Since $Y_j = d_j \tilde{y_j}$, and $Z_i = c_i \tilde{z}_i$, the value of asset i could be formulated as follows:

$$R_i(T) = k_i \exp\left[\sum_{j=1}^{m} (a_{i,j} q_j d_j \tilde{y_j}) + \tilde{\sigma}_i \sqrt{T} c_i \tilde{z}_i\right] \quad \forall i.$$

where $-1 \leq \tilde{z}_i \leq 1$ and $\tilde{z}_i \sim N(0,1)\ \forall i$, and $-1 \leq \tilde{y_j} \leq 1$ and $\tilde{y_j} \sim N(0,1)\ \forall j$.

Then, the portfolio wealth is formulated as follows:

$$\sum_{i=1}^{n} \left(x_i k_i \exp\left[\sum_{j=1}^{m} (a_{i,j} q_j d_j \tilde{y_j}) + \tilde{\sigma}_i \sqrt{T} c_i \tilde{z}_i\right]\right).$$

The investment portfolio is assumed to be formed by several assets such as the foreign currencies in a given domestic country, publicly traded stocks, etc. The exchange rates are expressed according to a direct quotation. That is, a foreign exchange rate is defined as the domestic currency per unit of the foreign currency whereas the other assets are assumed to be traded in domestic currency.

6.1.2 General Guidance to Log-Robust Portfolio Management Problem

The ultimate target is to maximize the worst-case wealth of the portfolio at time T using robust optimization techniques. The investment decisions are restricted by an initial budget of w_0, and short sales are not allowed.

Since robust optimization does not require any underlying distribution for the uncertain parameters, in our problem, $Z_1, Z_2, .., Z_n$ and $Y_1, Y_2, Y_3, .., Y_m$ are not necessarily Gaussian. They have the same mean, which is zero, and they belong to the intervals $[-c_1, c_1], [-c_2, c_2], .., [-c_n, c_n]$ and $[-d_1, d_1], [-d_2, d_2], .., [-d_m, d_m]$, respectively. Similarly, the scaled deviations $\tilde{z_1}, ..., \tilde{z_i}, .., \tilde{z_n}$ and $\tilde{y_1}, \tilde{y_2}, \tilde{y_3}, ..., \tilde{y_j}, .., \tilde{y_m}$ are not necessarily Gaussian either. They have the same mean, which is zero, and belong to the same interval, which is $[-1, 1]$.

The decision maker adjusts his/her conservatism by considering the trade off between the protection against uncertainty and the cost of robustness, which implies less portfolio wealth. Because, as the degree of conservatism increases, the scaled deviations take smaller negative values (higher absolute values) and this leads to a reduction in the final welfare of the investor. Γ is a mathematical expression of his/her conservatism such that sum of the absolute values of all the scaled deviations cannot be higher than Γ. That is

$$\sum_{i=1}^{n} |\tilde{z_i}| + \sum_{j=1}^{m} |\tilde{y_i}| \leq \Gamma.$$

represents the “uncertainty budget constraint” of our robust portfolio optimization problem.

Decision variables $\tilde{z_1}, \tilde{z_2}, \tilde{z_3}, ..., \tilde{z_i}, .., \tilde{z_n}$ and $\tilde{y_1}, \tilde{y_2}, \tilde{y_3}, ..., \tilde{y_j}, .., \tilde{y_m}$ determine the worst-case asset and factor levels according to the decision maker's judgement on Γ. For instance, if $\Gamma = 0$, then each uncertain parameter takes its nominal value, namely 0. This corresponds to the case in which there is no uncertainty. Alternatively, if $\Gamma = n + m$, the most conservative case occurs and each uncertain parameter takes its worst-case value, namely -1.

The decision variable x_i stands for the amount of money invested in the asset i at the beginning of the time horizon. We aim to obtain a tractable formulation for the robust problem, which can be solved easily, and to obtain theoretical insights for the structure of the optimal solution.

6.1.3 Log-Robust Portfolio Optimization

The robust portfolio management problem in max-min framework is modeled as fallows:

$$
\begin{aligned}
\max_{x} \quad & \min_{\tilde{z},\tilde{y}} \sum_{i=1}^{n} x_i k_i \exp\left[\sum_{j=1}^{m} (a_{i,j} q_j d_j \tilde{y_j}) + \tilde{\sigma}_i \sqrt{T} c_i \tilde{z}_i\right] \\
& \text{s.t.} \sum_{i=1}^{n} |\tilde{z_i}| + \sum_{j=1}^{m} |\tilde{y_i}| \leq \Gamma, \\
& \quad |\tilde{z_i}| \leq 1 \; \forall i, \\
& \quad |\tilde{y_j}| \leq 1 \; \forall j, \\
\text{s.t.} \quad & \sum_{i=1}^{n} x_i = w_0, \\
& x_i \geq 0 \; \forall i.
\end{aligned}
\tag{6.5}
$$

Lemma 6.1 *At optimality, $-1 \leq \tilde{z_i} \leq 0$ for all i and $-1 \leq \tilde{y_j} \leq 0$ for all j. Therefore, Problem (6.5) is equivalent to:*

$$
\begin{aligned}
\max_{x} \quad & \min_{\tilde{z},\tilde{y}} \sum_{i=1}^{n} x_i k_i \; exp \left[- \sum_{j=1}^{m} (|a_{i,j}| q_j d_j \tilde{y_j}) - \tilde{\sigma_i} \sqrt{T} c_i \tilde{z_i} \right] \\
& \textit{s.t.} \; \sum_{i=1}^{n} \tilde{z_i} + \sum_{j=1}^{m} \tilde{y_i} \leq \Gamma, \\
& \quad 0 \leq \tilde{z_i} \leq 1 \; \forall i, \\
& \quad 0 \leq \tilde{y_j} \leq 1 \; \forall j, \\
\textit{s.t.} \quad & \sum_{i=1}^{n} x_i = w_0, \\
& x_i \geq 0 \; \forall i.
\end{aligned}
\tag{6.6}
$$

It can be realized that the objective function of Problem (6.6) is a linear function of investment decisions (x_i). In addition, it is a nonlinear but convex function of scaled deviations ($\tilde{z_i}, \tilde{y_j}$). Therefore, we solve the inner minimization problem by invoking convexity, unconstrained optimization by Lagrange relaxation approach, and strong duality. Next, we insert its solution into the outer maximization problem to obtain a single maximization problem.

Lemma 6.2 *Problem (6.6), which is a robust optimization problem in max-min framework, can be converted to the following maximization problem:*

$$
\begin{aligned}
\max_{Q,\,\alpha,\,\lambda,\,\beta,\,x} \quad & \sum_{i=1}^{n} Q_i \left(1 + \ln\left(\frac{x_i k_i}{Q_i}\right)\right) - \alpha\Gamma - \sum_{i=1}^{n} \lambda_i - \sum_{j=1}^{m} \beta_j \\
\textit{s.t.} \quad & -q_j d_j \sum_{i=1}^{n} |a_{i,j}| Q_i + \beta_j + \alpha \geq 0 \; \forall j, \\
& -Q_i c_i \sqrt{T} \sigma_i + \lambda_i + \alpha \; \forall i, \\
& \sum_{i=1}^{n} x_i = w_0, \\
& \alpha \geq 0, \\
& \beta_j \geq 0 \; \forall j, \\
& \lambda_i, Q_i \geq 0 \; \forall i.
\end{aligned}
\tag{6.7}
$$

where $\lambda,\ \lambda^0\beta,\ \beta^0$and α are the lagrangian multipliers arising while solving inner minimization problem by Lagrangian relaxation, and $Q_i = \frac{\lambda_i - \lambda_i^0 + \alpha}{\sigma_i \sqrt{T} c_i}$.

Theorem 6.3 (Optimal Worst-Case Wealth and Investment Decisions)

(i) The optimal portfolio value for the log-robust portfolio optimization problem (6.5) equals to $w_0 \exp(F(\Gamma))$, where F is the linear problem formulated as:

$$
\begin{aligned}
F(\Gamma) = \max_{\gamma,\,\delta,\,\zeta,\,\chi} \quad & \sum_{i=1}^{n} \chi_i \ln k_i - \zeta\Gamma - \sum_{i=1}^{n} \gamma_i - \sum_{j=1}^{m} \delta_j \\
\textit{s.t.} \quad & -q_j d_j \textstyle\sum_{i=1}^{n} |a_{i,j}| \chi_i + \delta_j + \zeta \geq 0 \; \forall j, \\
& -c_i \sqrt{T} \chi_i \sigma_i + \gamma_i + \zeta \geq 0 \; \forall i, \\
& \sum_{i=1}^{n} \chi_i = 1, \\
& \zeta \geq 0, \\
& \gamma_i,\ \chi_i \geq 0 \; \forall i, \\
& \delta_j \geq 0 \; \forall j.
\end{aligned}
\tag{6.8}
$$

(ii) For each i, χ_i is the fraction of the budget (w_0) invested in asset i. Therefore, the total amount of money invested in asset i is $w_o \chi_i$.

6.1.4 Numerical Experiments

This section presents the impact of the uncertainty budget Γ value on the objective function value, and the optimal portfolio diversification. In addition, we explore the structure of the optimal solution through the model parameters and their impact on the dynamics among decision variables. We will see that:

- As Γ increases the optimal objective function value decreases.
- As Γ increases the diversity, the number of assets invested, until some point, then drops.
- The theoretical insights regarding the optimal solution structure are justified through numerical experiments.

Setup

The natural logarithm of a 10-year treasury bond yield rate, Dow Jones Index (DJI), and crude oil spot price are used as factors throughout the numerical experiments. In addition, 25 stocks listed in NYSE are considered as assets. 100 monthly historical observations are analyzed to extract the factor parameters (μ, σ, θ) and linear regression coefficients used in the asset pricing $(a_{i,j})$ by MATLAB R2012a. In addition, all of the linear programming calculations are carried out by CPLEX on a 2.10 GHz Pentium(R) machine.

Figure 6.2 implies that as uncertainty budget increases, the investment decisions become more protective, the risky assets become less preferable and the optimal objective function value (the worst case rate of return) decreases. In addition, Figure 6.3 shows that as the uncertainty budget increases the investor becomes more risk-averse. Therefore, the model prefers a higher diversity in the portfolio to be protected against uncertainty. However, if the uncertainty budget is too high or the investor is very risk-averse, then the model invests in the safest asset only.

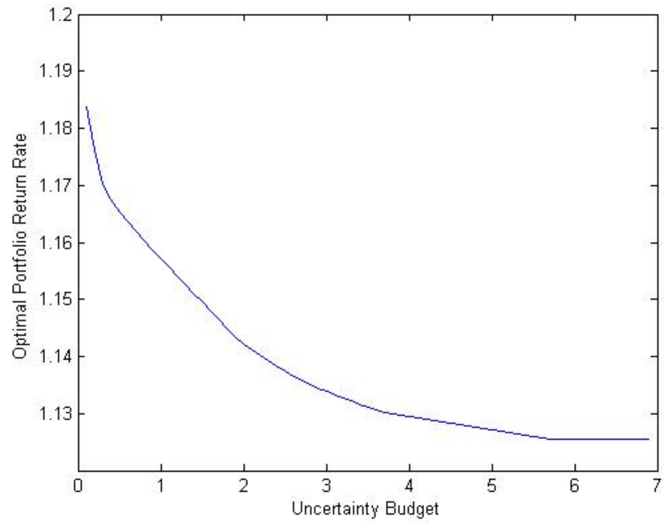


Figure 6.2: Portfolio Rate of Return vs Uncertainty Budget

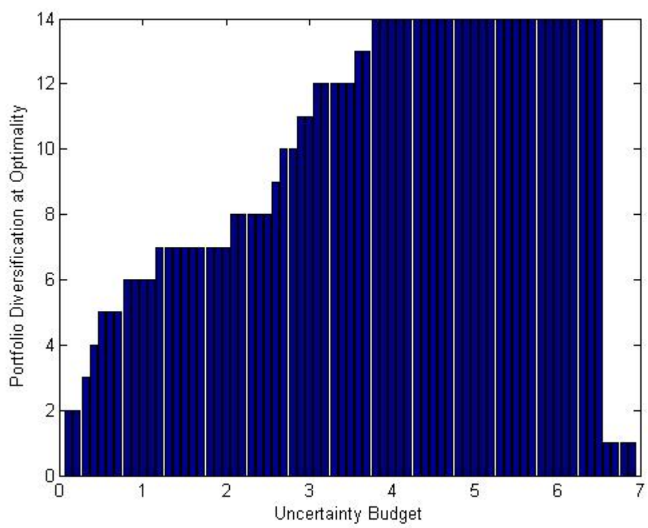


Figure 6.3: Diversification vs Uncertainty Budget

Exploring Structure of the Optimal Solution

Dual of Problem (6.8) is as follows:

$$
\begin{aligned}
\min_{v,u,\theta} \quad & \theta \\
\text{s.t.} \quad & \sum_{j=1}^{m} |a_{i,j}| v_j d_j q_j + \sigma_i \sqrt{T} c_i u_i + \theta \geq \ln k_i \ \forall i, \\
& 0 \leq u_i \leq 1 \ \forall i, \\
& 0 \leq v_j \leq 1 \ \forall j, \\
& \sum_{i=1}^{n} u_i + \sum_{j=1}^{m} v_j \leq \Gamma,
\end{aligned}
\tag{6.9}
$$

By strong duality, the following equalities hold:

- $(u_i - 1)\,\gamma_i = 0$
- $(v_j - 1)\,\delta_j = 0$
- $\sum_{j=1}^{m} \left(|a_{i,j}| v_j d_j q_j + \sigma_i \sqrt{T} c_i u_i + \theta - \ln k_i \right) \chi_i = 0$
- $\left(\sum_{i=1}^{n} u_i + \sum_{j=1}^{m} v_j - \Gamma \right) \zeta = 0$
- $\left(-q_j d_j \sum_{i=1}^{n} |a_{i,j}| \chi_i + \delta_j + \zeta \right) v_j = 0$
- $\left(-c_i \sqrt{T} \chi_i \sigma_i + \gamma_i + \zeta \right) u_i = 0$

Special Cases

- if $(v_1 = v_2 =, .., = v_m = 0)$,

  The decision maker selects the assets by ranking them according to their nominal return, i.e. $k_1 < k_2 < k3 < ... < k_n$, as in Kawas and Thiele's study [112]. Assuming that it is strictly sub-optimal to invest in only one stock and there exists an index h, at optimality, the decision maker invests only in assets whose ranking is not greater than h with the allocation:

$$\chi_i = \frac{1}{\sigma_i\left(\sum_{l=1}^{j} \frac{1}{\sigma_l}\right)}. \tag{6.10}$$

If all v_j variables are zero, then the constraints regarding the factor uncertainty could be disregarded, then the Problem (6.8) becomes (as it is in Kawas and Thiele's study [112]:

$$\begin{aligned} F(\Gamma) = \max_{\gamma,\,\zeta,\,\chi} \quad & \sum_{i=1}^{n} \chi_i \ln k_i - \zeta\Gamma - \sum_{i=1}^{n} \gamma_i \\ \text{s.t.} \quad & -c_i\sqrt{T}\chi_i\sigma_i + \gamma_i + \zeta \geq 0 \; \forall i, \\ & \sum_{i=1}^{n} \chi_i = 1, \\ & \zeta \geq 0, \\ & \gamma_i,\, \chi_i \geq 0 \; \forall i. \end{aligned} \tag{6.11}$$

Solution of Problem 6.11 leads to the allocation

$$\chi_i = \frac{1}{\sigma_i\left(\sum_{l=1}^{j} \frac{1}{\sigma_l}\right)}$$

where h is the index for raking the assets according to their nominal return, i.e. $k_1 < k_2 < k3 < ... < k_n$.

The results summarized in Table 6.1 support that the assets selected according to their ranking with respect to their nominal returns and the portfolio allocation to each asset is inversely proportional to its volatility σ_i.

Table 6.1: Optimal Asset Allocation

Asset	σ_i	ln k_i	$x_i(\Gamma = 0.1)$	$x_i(\Gamma = 0.3)$	$x_i(\Gamma = 0.4)$	$x_i(\Gamma = 0.5)$	$x_i(\Gamma = 0.8)$	$x_i\sigma_i(\Gamma = 0.1)$	$x_i\sigma_i(\Gamma = 0.3)$	$x_i\sigma_i(\Gamma = 0.4)$	$x_i\sigma_i(\Gamma = 0.5)$	$x_i\sigma_i(\Gamma = 0.8)$
x_4	0.0492	0.1758	0.4250	0.2203	0.1311	0.1005	0.0916	0.0209	0.0108	0.0065	0.0049	0.0045
x_{18}	0.0364	0.1743	0.5750	0.2980	0.1773	0.1360	0.1240	0.0209	0.0108	0.0065	0.0049	0.0045
x_{22}	0.0225	0.1581	0	0.4817	0.2867	0.2198	0.2004	0	0.0108	0.0065	0.0049	0.0045
x_{14}	0.0159	0.1565	0	0	0.4049	0.3105	0.2830	0	0	0.00645	0.0049	0.0045
x_{16}	0.0212	0.1532	0	0	0	0.2333	0.2127	0	0	0	0.0049	0.0045
x_1	0.0511	0.1497	0	0	0	0	0.0883	0	0	0	0	0.0045
x_{23}	0.0249	0.1440	0	0	0	0	0	0	0	0	0	0
x_8	0.0549	0.1433	0	0	0	0	0	0	0	0	0	0
x_{20}	0.0231	0.1433	0	0	0	0	0	0	0	0	0	0
x_6	0.1099	0.1347	0	0	0	0	0	0	0	0	0	0
x_{21}	0.0093	0.1337	0	0	0	0	0	0	0	0	0	0
x_{25}	0.0161	0.1322	0	0	0	0	0	0	0	0	0	0
x_{19}	0.0106	0.1318	0	0	0	0	0	0	0	0	0	0
x_{15}	0.0200	0.1294	0	0	0	0	0	0	0	0	0	0
x_{24}	0.0118	0.1277	0	0	0	0	0	0	0	0	0	0
x_{17}	0.0225	0.1263	0	0	0	0	0	0	0	0	0	0
x_{12}	0.0203	0.1260	0	0	0	0	0	0	0	0	0	0
x_9	0.0350	0.1259	0	0	0	0	0	0	0	0	0	0
x_5	0.1222	0.1257	0	0	0	0	0	0	0	0	0	0
x_{13}	0.0014	0.1225	0	0	0	0	0	0	0	0	0	0
x_7	0.0642	0.1221	0	0	0	0	0	0	0	0	0	0
x_3	0.1981	0.1212	0	0	0	0	0	0	0	0	0	0
x_2	0.0419	0.1207	0	0	0	0	0	0	0	0	0	0
x_{11}	0.0037	0.0634	0	0	0	0	0	0	0	0	0	0
x_{10}	0.0181	0.0629	0	0	0	0	0	0	0	0	0	0

- if $(v_1 = v_2 =, .., = v_m = 1)$,

  The decision maker selects the assets by ranking them according to their nominal return, i.e. $ln\ k_1 - \sum_{j=1}^{m} a_{1,j} q_j d_j < ... < k_n - \sum_{j=1}^{m} a_{n,j} q_j d_j$, as in Kawas and Thile's study [112]. Assuming that it is strictly sub-optimal to invest in only one stock and there exists an index h, at optimality, the decision maker invests only in assets whose ranking is not greater than h with the allocation:

$$\chi_i = \frac{1}{\sigma_i \left(\sum_{l=1}^{j} \frac{1}{\sigma_l} \right)}. \tag{6.12}$$

  The results summarized in Table 6.2 support that the assets selected according to their ranking with respect to their nominal returns and the portfolio allocation to each asset is inversely proportional to its volatility σ_i.

Table 6.2: Optimal Asset Allocation

Asset	$\ln\ k_i - \sum_{j=1}^{3} q_j d_j a_{i,j}$	σ_i	$x_i(\Gamma = 7.3)$	$x_i(\Gamma = 7.6)$	$x_i(\Gamma = 8.3)$	$x_i\sigma_i(\Gamma = 7.3)$	$x_i\sigma_i(\Gamma = 7.6)$	$x_i\sigma_i(\Gamma = 8.3)$
x_1	0.1420	0.0511	0.0387	0.0359	0.0330	0.0020	0.0018	0.0017
x_4	0.1407	0.0492	0.0402	0.0372	0.0343	0.0020	0.0018	0.0017
x_{21}	0.1274	0.0929	0.0213	0.0197	0.0182	0.0020	0.0018	0.0017
x_{14}	0.1269	0.0159	0.1241	0.1150	0.1059	0.0020	0.0018	0.0017
x_{25}	0.1147	0.0161	0.1228	0.1137	0.1047	0.0020	0.0018	0.0017
x_{24}	0.1058	0.0118	0.1679	0.1556	0.1432	0.0020	0.0018	0.0017
x_{18}	0.1037	0.0364	0.0544	0.0504	0.0464	0.0020	0.0018	0.0017
x_{19}	0.1004	0.0106	0.1858	0.1721	0.1585	0.0020	0.0018	0.0017
x_{15}	0.1000	0.0200	0.0988	0.0915	0.0842	0.0020	0.0018	0.0017
x_{13}	0.0987	0.0135	0.1460	0.1353	0.1245	0.0020	0.0018	0.0017
x_{23}	0.0828	0.0249	0	0.0736	0.0678	0	0.0018	0.0017
x_{16}	0.0819	0.0212	0	0	0.0795	0	0	0.0017
x_{20}	0.0763	0.0231	0	0	0	0	0	0
x_{22}	0.0670	0.0225	0	0	0	0	0	0
x_{11}	0.0589	0.0369	0	0	0	0	0	0
x_8	0.0570	0.0549	0	0	0	0	0	0
x_9	0.0566	0.0350	0	0	0	0	0	0
x_2	0.0564	0.0419	0	0	0	0	0	0
x_{17}	0.0551	0.0225	0	0	0	0	0	0
x_{12}	0.0471	0.0203	0	0	0	0	0	0
x_7	0.0414	0.0642	0	0	0	0	0	0
x_3	0.0315	0.0198	0	0	0	0	0	0
x_{10}	0.0139	0.0181	0	0	0	0	0	0
x_5	-0.0332	0.0122	0	0	0	0	0	0
x_6	-0.0628	0.0110	0	0	0	0	0	0

6.2 Conclusions

In this chapter, we integrated the factor models with the GBM and/or the OUP to have reliable forecasts for the future asset price levels with the flexibility of being able to modify the model according to the factors' nature. We handle the uncertainty in the asset pricing model by robust optimization and obtain a tractable model, $F(\Gamma)$. Any additional factor brings 1 new decision variable and 1 new constraint; however, the problem $F(\Gamma_1, \Gamma_2)$ remains linear. We obtain insights on the optimal solution structure for two special cases.

6.3 Proofs

6.3.1 Proof of Lemma (6.2)

The inner minimization problem in Problem (6.6) is:

$$\begin{aligned}
\min_{\tilde{z},\tilde{y}} \quad & \sum_{i=1}^{n} x_i k_i \text{exp}\left[-\sum_{j=1}^{m}(|a_{i,j}|q_j d_j \tilde{y_j}) - \tilde{\sigma_i}\sqrt{T}c_i\tilde{z_i}\right] \\
\text{s.t.} \quad & \textstyle\sum_{i=1}^{n}\tilde{z_i} + \sum_{j=1}^{m}\tilde{y_i} \leq \Gamma, \\
& 0 \leq \tilde{z_i} \leq 1 \;\forall i, \\
& 0 \leq \tilde{y_j} \leq 1 \;\forall j.
\end{aligned} \tag{6.13}$$

This is a convex problem of $\tilde{y}$ and $\tilde{z}$. Therefore, the optimal solution of Problem (6.13) can be found by Lagrangian relaxation approach.

We define α; $\beta_j$and β_j^0 $\forall j$; λ_i and λ_i^0 $\forall i$, as Lagrangian multipliers and obtain the following unconstrained nonlinear problem:

$\min(\sum_{i=1}^{n} x_i k_i e^{\left[-\sum_{j=1}^{m}(a_{i,j}q_j d_j \tilde{y_j}) - \tilde{\sigma_i}\sqrt{T}c_i\tilde{z_i}\right]} + \alpha(\sum_{i=1}^{n}\tilde{z_i} + \sum_{j=1}^{m}\tilde{y_j} - \Gamma) - \sum_{i=1}^{n}\lambda_i^0\tilde{z_i} + \sum_{i=1}^{n}\lambda_i(\tilde{z_i}-1) - \sum_{j=1}^{m}\beta_j^0\tilde{y_j} + \sum_{j=1}^{m}\beta_j(\tilde{y_j}-1))$

The solution of the system created by KKT conditions for this problem supplies the following equations:

$$\text{exp}\left[-\sum_{j=1}^{m}(a_{i,j}q_j d_j \tilde{y_j}) - \tilde{\sigma_i}\sqrt{T}c_i\tilde{z_i}\right] = \frac{\lambda_i - \lambda_i^0 + \alpha}{x_i k_i \sigma_i \sqrt{T}c_i}\;\forall i, \tag{6.14}$$

$$\sum_{i=1}^{n} -x_i k_i q_j a_{i,j} d_j \text{exp}\left[-\sum_{j=1}^{m}(a_{i,j}q_j d_j \tilde{y_j}) - \tilde{\sigma_i}\sqrt{T}c_i\tilde{z_i}\right] + \beta_j - \beta_j^0 + \alpha = 0\;\forall j, \tag{6.15}$$

$$-\sum_{i=1}^{n}\left(\frac{q_j d_j a_{i,j}}{\sigma_i\sqrt{T}c_i}(\lambda_i - \lambda_i^0 + \alpha)\right) + \beta_j - \beta_j^0 + \alpha = 0\;\forall j, \tag{6.16}$$

Employing these equalities in the Lagrangian function leads to the following representation of the Lagrangian function:

$$\min\left(\sum_{i=1}^{n}\left[\left(\frac{\lambda_i-\lambda_i^0+\alpha}{\sigma_i\sqrt{T}c_i}\right)\left(1+\ln\left(\frac{x_ik_i\sigma_i\sqrt{T}c_i}{\lambda_i-\lambda_i^0+\alpha}\right)\right)\right]-\alpha\Gamma-\sum_{i=1}^{n}\lambda_i-\sum_{j=1}^{m}\beta_j\right). \quad (6.17)$$

Since the Problem (6.13) is convex and Slater's condition is satisfied, strong duality holds. Then, by strong duality (see Bertekas 1999), we embed the inner minimization problem, Problem (6.13), into the outer max problem. Then, we obtain an equivalent formulation for log-robust portfolio management problem, Problem (6.6).

$$\begin{aligned}
\max_{\alpha,\lambda,\lambda^0,\beta,\beta^0,x} \quad & \left(\sum_{i=1}^{n}\left[\left(\frac{\lambda_i-\lambda_i^0+\alpha}{\sigma_i\sqrt{T}c_i}\right)\left(1+\ln\left(\frac{x_ik_i\sigma_i\sqrt{T}c_i}{\lambda_i-\lambda_i^0+\alpha}\right)\right)\right]-\alpha\Gamma-\sum_{i=1}^{n}\lambda_i-\sum_{j=1}^{m}\beta_j\right)\\
\text{s.t.} \quad & \sum_{i=1}^{n}x_i=w_0,\\
& \alpha\geq 0,\\
& \beta_j,\beta_j^0\geq 0\ \forall j,\\
& \lambda_i,\lambda_i^0,x_i\geq 0\ \forall i.
\end{aligned} \quad (6.18)$$

We define a new decision variable, Q_i, such that $Q_i = \left(\frac{\lambda_i-\lambda_i^0+\alpha}{\sigma_i\sqrt{T}c_i}\right)$. Also, each λ_i^0 and β_j^0 can be defined through Q_i . In addition, due to the natural logarithm term, every Q_i has to be nonnegative. Then, the Problem (6.18) turns into:

$$
\begin{aligned}
\max_{Q,\,\alpha,\,\lambda,\,\beta,\,x} \quad & \sum_{i=1}^{n} Q_i\left(1+\ln\left(\frac{x_i k_i}{Q_i}\right)\right) - \alpha\Gamma - \sum_{i=1}^{n}\lambda_i - \sum_{j=1}^{m}\beta_j \\
\text{s.t.} \quad & -q_j d_j \sum_{i=1}^{n} a_{i,j} Q_i + \beta_j + \alpha \geq 0 \;\forall j, \\
& -Q_i c_i \sqrt{T}\sigma_i + \lambda_i + \alpha \;\forall i, \\
& \sum_{i=1}^{n} x_i = w_0, \\
& \alpha \geq 0, \\
& \beta_j \geq 0 \;\forall j, \\
& \lambda_i, Q_i, x_i \geq 0 \;\forall i.
\end{aligned}
\tag{6.19}
$$

6.3.2 Proof of Theorem (6.3)

We firstly solve Problem (6.19) for x, then for the remaining variables. The maximization problem over x is written as:

$$
\begin{aligned}
\max_{x} \quad & \sum_{i=1}^{n} Q_i \ln x_i \\
& \sum_{i=1}^{n} x_i = w_0, \\
& x_i \geq 0 \;\forall i.
\end{aligned}
\tag{6.20}
$$

Since Problem (6.20) is a convex problem, it could be solved by the Lagrangian relaxation approach which leads to:

$$
x_i = \frac{Q_i w_0}{\sum_{i=1}^{n} Q_i} \;\forall i.
$$

If we replace each x_i by $\frac{Q_i w_0}{\sum_{i=1}^{n} Q_i}$ in the log-robust portfolio problem, Problem (6.19), it becomes:

$$
\begin{aligned}
\max_{Q,\,\alpha,\,\lambda,\,\beta,\,x} \quad & \sum_{i=1}^{n} Q_i \left(1 + \ln\left(\frac{w_0 k_i}{\sum_{i=1}^{n} Q_i}\right)\right) - \alpha\Gamma - \sum_{i=1}^{n} \lambda_i - \sum_{j=1}^{m} \beta_j \\
\text{s.t.} \quad & -q_j d_j \sum_{i=1}^{n} a_{i,j} Q_i + \beta_j + \alpha \geq 0 \; \forall j, \\
& -Q_i c_i \sqrt{T} \sigma_i + \lambda_i + \alpha \; \forall i, \\
& \alpha \geq 0, \\
& \beta_j \geq 0 \; \forall j, \\
& \lambda_i, Q_i \geq 0 \; \forall i.
\end{aligned}
\tag{6.21}
$$

Since the right-hand side of the constraints in Problem (6.21) are zero, it can be parametrized. We define a scale factor $\frac{1}{\omega}$ such that $\omega = \sum_{i=1}^{n} Q_i$ to parametrize the problem by ω and scale the decision variables of Problem (6.21) by $\frac{1}{\omega}$ so that :

$$\gamma_i = \frac{\lambda_i}{\omega} \; \forall i,$$

$$\delta_j = \frac{\beta_j}{\omega} \; \forall j,$$

$$\zeta = \frac{\alpha}{\omega} \text{ and}$$

$$\chi_i = \frac{Q_i}{\omega} = \frac{x_i}{w_0}.$$

The Problem (6.21) is equivalent to:

$$
\begin{aligned}
\max_{\omega} \omega \quad & \max_{\gamma,\delta,\zeta,\chi} \sum_{i=1}^{n} \chi_i \ln k_i - \zeta\Gamma - \sum_{i=1}^{n} \gamma_i - \sum_{j=1}^{m} \delta_j \\
& \text{s.t.} - q_j d_j \sum_{i=1}^{n} a_{i,j}\chi_i + \delta_j + \zeta \geq 0 \,\forall j, \\
& \quad -c_i\sqrt{T}\chi_i\sigma_i + \gamma_i + \zeta \geq 0 \,\forall i, \\
& \quad \sum_{i=1}^{n} \chi_i = 1, \\
& \quad \zeta \geq 0, \\
& \quad \gamma_i \chi_i \geq 0 \,\forall i, \\
& \quad \delta_j \geq 0 \,\forall j.
\end{aligned}
\tag{6.22}
$$

Problem (6.22) can be written as:

$$\max_{\omega} \omega \left(F(\Gamma) + 1 + \ln \frac{w_0}{\omega} \right) \tag{6.23}$$

where

$$
\begin{aligned}
F(\Gamma) = \max_{\gamma,\delta,\zeta,\chi} \quad & \sum_{i=1}^{n} \chi_i \ln k_i - \zeta\Gamma - \sum_{i=1}^{n} \gamma_i - \sum_{j=1}^{m} \delta_j \\
\text{s.t.} \quad & -q_j d_j \textstyle\sum_{i=1}^{n} a_{i,j}\chi_i + \delta_j + \zeta \geq 0 \,\forall j, \\
& -c_i\sqrt{T}\chi_i\sigma_i + \gamma_i + \zeta \geq 0 \,\forall i, \\
& \sum_{i=1}^{n} \chi_i = 1, \\
& \zeta \geq 0, \\
& \gamma_i,\, \chi_i \geq 0 \,\forall i, \\
& \delta_j \geq 0 \,\forall j.
\end{aligned}
\tag{6.24}
$$

The Problem (6.23) has a concave objective function with a single decision variable. For a given $F(\Gamma)$, the optimal solution of Problem (6.23) can be found by setting the first derivative of objective function (with respect to ω) to zero. Therefore, optimal ω is equal to $w_0 \exp F(\Gamma)$ and optimal objective function value of Problem (6.23) is calculated as $w_0 \exp F(\Gamma)$. In other words, optimal portfolio wealth at time T is $w_0 \exp F(\Gamma)$.

Chapter 7

Conclusions and Future Research

We present four essays on risk management with financial investment and revenue management applications. In the Chapter 3 (Portfolio Management with Quantile Constraints), the quantile function is used as risk measure. In Chapter 4 (Portfolio Management with Moment Matching Approach) the risk is defined as the probability of the portfolio return falling below a threshold; whereas, in Chapter 6 (Log-Robust Portfolio Management with Factor Model) the implied risk is the adverse stock return realizations resulting from volatility terms in the stochastic processes which factors used in the asset pricing model follow. In addition, in Chapter 5 (New Product Launching Decisions with Robust Optimization), the risk reveals itself as unexpected adoption rate of an innovative product in a market and unexpected response of a potential partner to a partnership offer. We manage the risk using robust optimization and stochastic optimization techniques.

In Chapter 3 (Portfolio Management with Quantile Constraints), we approximate the quantile function of the portfolio return distribution by using available scenarios or historical data sets. We solve the problem which maximizes the expected return while keeping the approximated quantile function value (for a given probability level) at least equal to a threshold value. The proposed method requires solving a series of linear problems. Therefore, it can be quickly solved for large scale problems. In our numerical experiments, we measure the performance of the algorithm in terms of the closeness the optimality and time to reach a solution. In addition, we compare its performance in terms of the risk/return efficiency with three benchmark models. Numerical studies imply that our algorithm is a fast and tractable approximation method which leads to close-to-optimal solutions to the problem which is cumbersome to be optimized due to the difficulty of expressing the quantile function in an optimization problem. The proposed approach is a data-driven approach; therefore, the data set used in the optimization model should reflect the actual distributions of the asset returns. In addition, as the data set enlarges, the time to reach to a solution by the algorithm increases. Therefore, investigating the optimal size of the data set and the best sampling method to prepare the data set used as an input for this problem can be an interesting future research direction.

In Chapter 4 (Portfolio Management with Moment Matching Approach), we approximate the random portfolio return as a log-Normal sum to a single log-Normally distributed random variable with the Fenton-Wilkinson log-Normal sum approximation method. We formulate a stochastic optimization model which minimizes the probability of the portfolio return falling below a target while keeping expected portfolio return above a specified target. Next, we decompose the problem into two sub-problems so that the risk management problem (a sub-problem) is solved as an unconstrained nonlinear programing problem with a smooth objective function based on the other sub-problem's solution which is formulated as a quadratic programming model. In addition, we extend our work to the basket options design problem. In our study, we consider only the first two moments of the portfolio return. The future research on the portfolio management incorporating with the third and/or fourth moments should improve the accuracy of the approximation and result in valuable contributions.

In Chapter 5 (New Product Launching Decisions with Robust Optimization), we use the new product development model suggested Bass [9] and a customized pricing model based on a logit-choice model in an optimization model which determines the optimal timing of the introduction of a set of innovative products to a regional market. Uncertainty on the model parameters are handled with the robust optimization techniques. An iterative approach is proposed to obtain a tractable robust optimization formulation. We include the real options which allow the innovative company to reduce the size of the contract in the robust product launching problem model. In our study, we only consider the option to reduce the contract size and assume that the unit price per product is locked by contracts. However, another problem setting, where the real options to update the contract price based on revealed demand are considered, is more realistic and interesting for future research.

Chapter 6 (Log-Robust Portfolio Management), we use factor models to formulate asset prices and treat randomness on asset pricing by a budget of uncertainty. We present a tractable robust optimization formulation which maximizes the worst-case portfolio return. In our study, we neglect the correlation between the factor values used in asset pricing. This project can be extended to a case where the factors are correlated.

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