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Analysis & Synthesis of Distributed Control Systems with Sparse Interconnection Topologies

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ANALYSIS & SYNTHESIS OF DISTRIBUTED CONTROL SYSTEMS WITH SPARSE INTERCONNECTION TOPOLOGIES

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To my teachers,
my friends,
and my loving family

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Abstract

This dissertation is about control, identification, and analysis of systems with sparse interconnection topologies. We address two main research objectives relating to sparsity in control systems and networks. The first problem is optimal sparse controller synthesis, and the second one is the identification of sparse network. The first part of this dissertation starts with the chapter focusing on developing theoretical frameworks for the synthesis of optimal sparse output feedback controllers under pre-specified structural constraints. This is achieved by establishing a balance between the stability of the controller and the systems quadratic performance. Our approach is mainly based on converting the problem into rank constrained optimizations.

We then propose a new approach in the syntheses of sparse controllers by employing the concept of $\mathcal{H}_p$ approximations. Considering the trade-off between the controller sparsity and the performance deterioration due to the sparsification process, we propose solving methodologies in order to obtain robust sparse controllers when the system is subject to parametric uncertainties.

Next, we pivot our attention to a less-studied notion of sparsity, namely row sparsity, in our optimal controller design. Combining the concepts from the majorization theory and our proposed rank constrained formulation, we propose an exact reformulation of the optimal state feedback controllers with strict row sparsity constraint,

which can be sub-optimally solved by our proposed iterative optimization techniques.

The second part of this dissertation focuses on developing a theoretical framework and algorithms to derive linear ordinary differential equation models of gene regulatory networks using literature curated data and micro-array data. We propose several algorithms to derive stable sparse network matrices. A thorough comparison of our algorithms with the existing methods are also presented by applying them to both synthetic and experimental data-sets.

Chapter 1

Introduction

The growth of large-scale dynamical systems such as power grids, transportation systems, and wireless data networks, and the impotence of traditional control/identification schemes have caused the problem of sparse/structured controller design/system identification to receive increasing attention over the past few years. In modeling and control of such dynamical system, it is crucial to consider the network underlying topology, as it dictates the pattern of information flow among the sub-systems.

In conventional control, it is usually assumed that all measurements are accessible to a centralized controller, while in large scale interconnected systems this assumption is not practical, since the subsystem level information is not globally accessible throughout the network in medium to large-scale systems. Furthermore, in large network of dynamical systems, it is desirable that subsystems only communicate with a few neighboring components due to the high cost, security concerns, or infeasibility of communication links. Needless to say that incorporation of the inherent structure of the system in the identification processes is paramount in order to obtain a precise model of the system. Therefore, the need to exploit particular structures, obtained based on the layout of the system network, seems undeniable.

One common desired structure in the control of network of dynamical systems is the sparsity of the network, which could correspond to a simpler controller topology, fewer sensors/actuators, and minimization of long distance communications. However, fewer measurement/communication links leads to performance deterioration and sometimes even instability of the overall system. Therefore, there exists a trade off between the stability and performance of the system and minimizing the number of non-zero entries of the feedback gain matrices. As for the problem of large-scale network identification, the assumption of sparsity is reasonable, as nodes rarely directly interact with the majority of other nodes.

In general, the problems of controller design and system identification subject to additional constraints, such as sparsity, are challenging problems. The complexity of the problems originates from two main sources. One is the non-convexity of the stability conditions, and the other is the combinatorial behaviour of the sparsity measuring function. In recent years, numerous attempts have been made to provide distributed controller synthesis approaches for different classes of systems [1, 2, 3, 4, 5]. Bamieh *et al*, in [6, 7], investigated the distributed control of spatially invariant systems, then the work in [8] has proved that the solution of Riccati and Lyapunov equations for systems consisting of Spatially Decaying (SD) operators has SD property, which lends credibility to the search for controllers that have access only to local measurements. The design of optimal state feedback gain in the presence of an *a priori* specified structure, usually in the form of sparsity patterns, is considered in [4]. In their recent papers, Lavaei *et al.* [9, 10, 11] cast the problem of optimal decentralized control for discrete time systems as a rank constrained optimization problem, developed results on the possible rank of the resulting feasible set, and introduced several rank-reducing heuristics as well. Wang *et al.* studied the problem of localized LQR/LQG control and presented a synthesis algorithm for large-scale localizable systems [12, 13]. Frequency domain approaches to design optimal decentralized controllers are also presented in [14, 15, 16].

Regarding the issues caused by the sparsity requirements, it suffices to say that the problem of minimizing the number of nonzero elements of a vector/matrix subject to a set of convex constraints, which arises in many fields such as Compressive Sensing [17, 18, 19], in inherently NP-hard. To alleviate the issues caused by the combinatorial nature of cardinality functions, several convex/non-convex functions have been proposed as surrogates for the cardinality functions in optimization problems. For example, in cases where the optimization constraint is affine, ℓ_1-norm , as a convex relaxation of ℓ_0-norm, has proved to work reliably under certain conditions, namely Restricted Isometry Property (RIP) [20, 21, 22]. Thus, ℓ_1-norm and its weighted versions have been extensively used in signal processing and control applications [23, 24, 25, 26, 2, 27, 23, 28, 29, 30]. Non-convex relaxations of the cardinality function, such as ℓ_q-quasi-norm $(0 < q < 1)$, have also received considerable attention recently [31, 32]. In [33, 34, 35], it is shown that, for a large class of SD systems, the quadratically-optimal feedback controllers inherit spatial decay property from the dynamics of the underlying system. Moreover, the authors have proposed a method, based on new notions of q-Banach algebras, by which sparsity and spatial localization features of the same class can be studied when q is chosen sufficiently small.

In this dissertation, we study the problems of optimal sparse/row-sparse controller synthesis, and the sparse network identification with application to Gene Regulatory

Networks. Our approach is mainly based on converting the problems into rank constrained optimizations, which can be solved efficiently though our proposed solving algorithms.

1.1 Contributions

The contributions of this dissertation can be summarized as the following.

Optimal Sparse Output Feedback Controller Design. We consider the problem of optimal sparse output feedback controller synthesis for continuous linear time invariant systems when the feedback gain is static and subject to specified structural constraints. Introducing an additional term penalizing the number of nonzero entries of the feedback gain into the optimization cost function, we show that this inherently non-convex problem can be equivalently cast as a rank constrained optimization, hence, it is an NP-hard problem. We then obtain upper/lower bounds for the optimal cost of the sparse output feedback control problem by proposing a convex optimization problem that conservatively solves our main problem. Moreover, we show that our problem reformulation allows us to incorporate additional implementation constraints, such as norm bounds on the control inputs or system output, by assimilating them into the rank constraint. We further exploit our rank constrained approach to define a structured output feedback control feasibility problem with global convergence property, and subsequently propose to utilize a version of the Alternating Direction Method of Multipliers (ADMM) as an efficient method to sub-optimally solve the equivalent rank constrained problem. As a special case, we study the problem of designing the sparsest stabilizing output feedback controller, and show that it is, in fact, a structured matrix recovery problem where the matrix of interest is simultaneously sparse and low rank. Furthermore, we show that this matrix recovery problem can be equivalently cast in the form of a canonical and well-studied rank minimization problem. We finally illustrate performance of our proposed methodology using numerical examples.

Output Feedback Controller Sparsification. The problem of optimal sparse output feedback control design for continuous linear time invariant systems is considered. This work adopts the concept of $\mathcal{H}_p$-approximation to develop an optimization algorithm capable of synthesizing a structured sparse static controller gain for which the overall closed loop system exhibits empirical frequency characteristics resembling that of the system controlled with a pre-designed centralized controller. We, moreover, modify our optimization problem so that the control signal generated by the sparse controller falls into the vicinity of the centralized control input, in the sense of L_2^2 norm. Furthermore, we show that our optimization problem can be equivalently reformulated into a rank constrained problem for which we propose to use a tailored version of Alternating Direction Method of Multipliers (ADMM) as a computationally efficient algorithm to sub-optimally solve it. Finally, we illuminate the effectiveness of our proposed method by testing it on randomly generated sample network models.

Controller Sparsification Under Parametric Uncertainties. We consider the problem of output feedback controller sparsification for systems with parametric uncertainties. We develop an optimization scheme that minimizes the performance deterioration from that of a well-performing pre-designed centralized controller, while enhancing sparsity pattern of the feedback gain. In order to improve temporal proximity of the pre-designed control system and its sparsified counterpart, we also incorporate an additional constraint into the problem formulation such that the output of the controlled system is enforced to stay in the vicinity of the output of the pre-designed system. It is shown that the resulting nonconvex optimization problem can be equivalently reformulated into a rank constrained problem. We then study the effect of the magnitude of the parametric uncertainties on the controller sparsification process by means of running a series of simulations. Overall, with the growth of the uncertainties magnitude, a decreasing trend in the sparsification performance is observed.

Optimal State Feedback Controllers with Strict Row Sparsity Constraints. The problem of optimal row sparse state feedback controller design for LTI systems, where the controller is assumed to be static with pre-specified structural constraint, is considered. Incongruous to the existing literature on the sparsity promoting control synthesis, we do not employ convex relaxation of the sparsity representing terms, such as ℓ_0-norm of the controller gain, in our proposed framework. Borrowing the results from the theory of majorization, we develop an exact rank constrained reformulation of the s-sparse vector recovery from a convex set, and, then, utilized it to cast our row sparse control problem into a an optimization problem where all constraints are convex, except a single rank constraint. Furthermore, we propose a necessary and sufficient condition for the feasibility of a stabilizing row s-sparse controller, and exploited it to propose a bi-linear minimization problem, subject to convex constraints, which solve the derived equivalent rank constrained problem to deliver an optimal row sparse state feedback controller. The benefits of approach are demonstrated though several numerical simulations.

Gene Regulatory Network Modeling Building on the linear matrix inequality formulation developed recently by Zavlanos *et al* (2011), we present a theoretical framework and algorithms to derive a class of ordinary differential equation models of gene regulatory networks using literature curated data and micro-array data. The solution proposed by Zavlanos *et al* (2011) requires that the micro-array data be obtained as the outcome of a series of controlled experiments in which the network is perturbed by over-expressing one gene at a time. We note that this constraint may be relaxed for some applications and, in addition, demonstrate how the conservatism in these algorithms may be reduced by using the Perron-Frobenius diagonal dominance conditions as the stability constraints. Due to the LMI formulation, it follows that the bounded real lemma may easily be used to make use of additional information. We present case studies that illustrate how these algorithms can be used on data sets to derive ODE models of the underlying regulatory networks.

1.1.1 Dissertation Overview

This dissertation consists of three main parts. Part I is devoted sparsity promoting control, and the approach is mainly based on converting the problem into equivalent rank constrained optimization problems, then, finding their sub-optimal solutions. At the end of each chapter, we summarize the main contributions and discuss future research directions.

In Chapter 2, we consider the problem of sparsity promoting control with quadratic performance measure. We provide an equivalent rank constrained optimization optimization. We describe how the input/output constraints can be Incorporated into the design without affecting the structure of the equivalent optimization program. Then, we demonstrate that the modified alternating direction method of multipliers is well-suited to solve the rank constrained problem.

In chapter 3, we take a look at the sparsity promoting control problem from a different point of view. While the previous chapter construct the sparse controller by minimizing a cost function, here our approach is to find a sparse approximation of a pre-designed well-performing controller by minimizing a weighted sum of the performance deviation and the density of the controller gain. By utilizing the concepts from mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem, we find the rank constrained reformulation of the problem which can be solved using the ADMM algorithm. Chapter 4 extends the results of chapter two by considering the systems with parametric uncertainties. Here we show that, on average, a decreasing trend in sparsification performance can be observed as with enlarge the uncertainties magnitude.

In Chapter 5, we consider the row sparsity of the controller instead of the matrix sparsity. We show that, using the results from majorization theory, it is possible to rewrite the problem of finding s-sparse stabilizing controller as a rank con trained optimization. Then, we propose a necessary and sufficient condition for the feasibility of a stabilizing row s-sparse controller. We, also, demonstrate that the optimal row sparse state feedback controller can be derived by iteratively solving a bi-linear optimization subject to convex constraints.

Part II of this thesis addresses the problem of sparse network modeling with application to Gene Regulatory Networks (GRNs). In Chapter 6, we first formulate the GRN identification problem as an optimization program with bi-linear matrix inequality constraints. In addition to proposing several algorithms to sub-optimally solve the problem, we also describe the rank constrained reformulation of the problem which can be solved using the algorithms developed in part I. We conclude this part by applying our algorithms on experimental data sets.

Part III is dedicated to summarizing the work presented in this dissertation and present future research directions.

Part I

Sparsity Promoting Optimal Feedback Controller Synthesis

Chapter 2

Optimal Sparse Output Feedback Controller Design: A Rank Constrained Optimization Approach

2.1 Introduction

The problem of optimal linear quadratic controller design has been extensively studied for several decades. In conventional control, it is usually assumed that all measurements are accessible to a centralized controller, while in large scale interconnected systems this assumption is not practical, since it is often desirable that subsystems only communicate with a few neighboring components due to the high cost, security concerns, or infeasibility of communication links. Therefore, the need to exploit a particular controller structure, obtained based on the layout of the system network, seems undeniable. Furthermore, the traditional controller synthesis methods, which are closely related to solving the Algebraic Riccati Equation, no longer work when additional constraints are imposed on the structure of the controller.

In general, the problem of designing constant gain feedback controllers subject to additional constraints is NP-hard [36]. In recent years, numerous attempts have been made to provide distributed controller synthesis approaches for different classes of systems [1, 2, 3, 4, 5]. Bamieh *et al*, in [6, 7], investigated the distributed control of spatially invariant systems, then the work in [8] has proved that the solution of Riccati and Lyapunov equations for systems consisting of Spatially Decaying (SD) operators has SD property, which lends credibility to the search for controllers that have access only to local measurements. The design of optimal state feedback gain in the presence of an *a priori* specified structure, usually in the form of sparsity patterns, is considered in [4]. In their recent papers, Lavaei *et al.* [9, 10, 11] cast the problem of optimal decentralized control for discrete time systems as a rank constrained optimization problem, developed results on the possible rank of the resulting feasible set, and introduced several rank-reducing heuristics as well. Wang *et al.* studied the problem of localized LQR/LQG control and presented a synthesis algorithm for large-scale localizable systems [12, 13]. Frequency domain approaches to design optimal decentralized controllers are also presented in [14, 15, 16].

In the design of linear feedback controllers for interconnected systems, a common desired structure is the sparsity of the controller matrices, which could correspond to a simpler controller topology, fewer sensors/actuators, and minimization of long distance communications. However, fewer measurement/communication links leads to performance deterioration and sometimes even instability of the overall system. Therefore, there exists a trade off between the stability and performance of the system and minimizing the number of non-zero entries of the feedback gain matrices. On the other hand, the problem of minimizing the number of nonzero elements of a vector/matrix subject to a set of constraints, which arises in many fields such as Compressive Sensing [17, 18, 19], in inherently NP-hard.

To alleviate the issues caused by the combinatorial nature of cardinality functions, several convex/non-convex functions have been proposed as surrogates for the cardinality functions in optimization problems. For example, in cases where the optimization constraint is affine, ℓ_1-norm , as a convex relaxation of ℓ_0-norm, has proved to work reliably under certain conditions, namely Restricted Isometry Property (RIP) [20, 21, 22]. Thus, ℓ_1-norm and its weighted versions have been extensively used in signal processing and control applications [23, 24, 25]. Non-convex relaxations of the cardinality function, such as ℓ_q-quasi-norm ($0 < q < 1$), have also received considerable attention recently [31, 32]. In [33, 34, 35], it is shown that, for a large class of SD systems, the quadratically-optimal feedback controllers inherit spatial decay property from the dynamics of the underlying system. Moreover, the authors have proposed a method, based on new notions of q-Banach algebras, by which sparsity and spatial localization features of the same class can be studied when q is chosen sufficiently small.

In this chapter, we consider the problem of optimal sparse feedback controller synthesis for linear time invariant system, in which convex constraints are imposed on the structure of the controller feedback gain. The main contribution of our chapter is to propose a novel approach which allows us to equivalently represent the intrinsically nonlinear constraints, such as closed loop stability condition and enforcement of controller structure, with a *single* rank constraint in an otherwise convex optimization program. Having all non-linearities encapsulated in only one rank constraint allows us to employ one of several existing algorithms to efficiently solve the resulting problem.

Our results are distinct from those reported in [23], as we present an alternative formulation which not only solves the regular sparse controller design problem, but also enables us to solve the output feedback control problem. Furthermore, integrating various types of nonlinear system constraints, such as constraints on the controller matrix and its norms, into the existing rank constraint can be effortlessly implemented in our approach. It should also be noted that the rank constraint emerging in our approach originates from the positive definiteness of the Lyapunov matrix and the properties of fixed rank matrices, thus the ratio of matrix dimension and its rank does not grow with the size of system. In contrast, the rank one constraint appears in [11] results from utilizing the auxiliary variable introduced by self multiplying the vector formed by augmenting the states, inputs, and outputs, hence there exist a linear growth of the ratio of the dimension of the matrix to its rank as the number of variable increases, which is a computational drawback in controller synthesis for large scale systems.

We start by augmenting the ℓ_0-norm of the feedback gain matrix to the quadratic cost function of our optimization problem. This additional term penalizes the extra communication links in the feedback pathway. We then reformulate it into an equivalent optimization problem where the non-convex constraints are lumped into a rank constraint. Based on the notions of holdable ellipsoid, we propose a reformulation of the problem to incorporate norm bounds on the control inputs and outputs of the system, which usually appear in controller implementations. Employing a convex relaxation of the added cardinality term, based on the weighted ℓ_1-norm, we argue that Alternating Direction Method of Multipliers (ADMM) is well-suited to solve our problem, since our search is to obtain a solution with an a priori known rank. ADMM iteratively solves the rank-unconstrained problem and projects the solution into the space of the matrices with the desired rank until the convergence criteria are met. We further investigate the special case of designing the sparsest stabilizing controller, and show that this problem can be rewritten as a rank minimization problem. Rank minimization problems have received considerable attention in recent years [37, 38]. In [39], it is shown that if a certain Restricted Isometry Property holds for the linear transformation defining the constraints, the minimum rank solution can be recovered by solving the minimization of the nuclear norm over the feasible space. Therefore, the nuclear norm may be used as a proxy for the rank minimization in our problem.

The remainder of this chapter is organized as follows. In Section 2.2, the general optimal sparse output feedback control problem setup is defined. Section 2.3, we reformulate the optimal sparse output feedback control problem as a rank constrained problem, and develop several results based on the proposed reformulation. In Section 2.4, we study the convex relaxation of this problem, and discuss the application of ADMM in solving the problem. The special case where the sparsity penalizing factor dominates the quadratic terms in the cost function is described in Section 2.5. Numerical examples illustrating the proposed methods are provided in 2.6. Finally, Section 2.7 concludes the chapter.

Notations: Throughout the chapter, the following notations are adopted. The space of n by m matrices with real entries is indicated by $\mathbb{R}^{n\times m}$. The n by n identity matrix is denoted I_n. Operators $\mathbf{Tr}(.)$ and $\mathbf{rank}(.)$ denote the trace and rank of the matrix operands. The transpose and vectorization operators are denoted by $(.)^{\mathrm{T}}$ and $\mathbf{vec}(.)$, respectively. The Hadamard product is represented by $\circ$. A matrix is said to be Hurwitz if all its eigenvalues lie in the open left half of the complex plane. $\|.\|_0$ represents the cardinality of a vector/matrix, while $\|.\|_1$ and $\|.\|_F$ denote ℓ_1 and Frobenius norm operators.Also, the norm $\|.\|_{L^q_\infty(\mathbb{R}^n)}$ is defined by

$$\|x\|_{L^q_\infty(\mathbb{R}^n)} \triangleq \sup_{t\geq 0} \|x(t)\|_q$$

A real symmetric matrix is said to be positive definite (semi-definite) if all its eigenvalues are positive (non-negative). $\mathbb{S}^n_{++}$ ($\mathbb{S}^n_{+}$) denotes the space of positive definite (positive semi-definite) real symmetric matrices, and the notation $X \succeq Y$ ($X \succ Y$) means $X - Y \in \mathbb{S}^n_+$ ($X - Y \in \mathbb{S}^n_{++}$).

2.2 Problem Formulation

2.2.1 Structurally Constrained Sparse Output Feedback Controllers

Let a linear time invariant system be given by its state space realization

$$\begin{cases} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{cases} \tag{2.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $y(t) \in \mathbb{R}^p$ is the output of the system, $u(t) \in \mathbb{R}^m$ is the control input, and matrices A, B and C have appropriate dimensions. We consider designing a constant gain output feedback stabilizing controller

$$u(t) = Ky(t), \quad K \in \mathcal{K} \tag{2.2}$$

with the minimum number of non-zero entries that minimizes a quadratic objective function. We further assume that the set of all acceptable *a priori* specified structures for feedback gains, denoted by $\mathcal{K}$, is a convex set. The reason behind this call is that such an assumption not only reduces the complexity of the problem, but also convex constraints on controller constraints have broad real-world applications. For example, there exist numerous applications in which establishing a link between two particular nodes is impractical either due to physical constraints or extremely high costs; such limitations can be incorporated into the design process by imposing the convex constraints that the corresponding entry of the controller gain should be zero. Also, other regularly occurring limitations such as upper bounds on the entries of the controller matrix can be also be implemented by convex constraints on matrix K.

The search for such a controller can be formulated as an optimization problem, in which the sparsity of the feedback gain is incorporated by adding the ℓ_0-norm of the gain matrix to the objective function. The ℓ_0-norm denotes the cardinality of the feedback gain, hence, it penalizes the number of non-zero entries of the matrix. Therefore, we have the following optimization problem

$$\begin{aligned} \min_{K,x,u} \; & J = \int_0^\infty [x(t)^{\mathrm{T}} Q x(t) + u(t)^{\mathrm{T}} R u(t)] dt + \lambda \|K\|_0 \qquad (\mathbf{P1}) \\ \text{s.t.} \; & \dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = x_0 \\ & u(t) = KCx(t), \quad K \in \mathcal{K}, \end{aligned}$$

where $Q \in \mathbb{R}_+^n$ and $R \in \mathbb{R}_{++}^m$ are performance weight matrices, x_0 is the initial state, and $\lambda \in \mathbb{R}_+$ is the regularization parameter. In this paper, we first address the general problem of sparse output feedback control; then, we study the problem of finding the sparsest stabilizing controller, which can be considered as a special case of our main problem when the matrices R and Q are set to zero.

2.2.2 Equivalent Formulations

It is possible to rewrite our main optimization problem as follows

$$\begin{aligned} J &= \int_0^\infty \mathbf{Tr}[x(t)^{\mathrm{T}}(Q + K^{\mathrm{T}}RK)x(t)]dt + \lambda\|K\|_0 \\ &= \mathbf{Tr}[(Q + K^{\mathrm{T}}RK)\int_0^\infty (x(t)x(t)^{\mathrm{T}})dt] + \lambda\|K\|_0 \\ &= \mathbf{Tr}[(Q + K^{\mathrm{T}}RK)\int_0^\infty (e^{(A+BK)t}x_0)x_0^{\mathrm{T}}e^{(A+BK)^{\mathrm{T}}t})dt] + \lambda\|K\|_0 \end{aligned}$$

Assuming the asymptotic stability of the closed loop system under the state feedback K is guaranteed, there exist a symmetric matrix X_{11} satisfying the following equation [40, p. 11]

$$(A + BK)X_{11} + X_{11}(A + BK)^{\mathrm{T}} = -x_0 x_0^{\mathrm{T}} \tag{2.3}$$

Plugging the left hand side of equation (2.3) into our cost function, the integrand can be easily integrated as follows

$$\begin{aligned} J = \mathbf{Tr}\Big[-(Q + K^{\mathrm{T}}RK)\int_0^\infty (e^{(A+BK)t}\left[(A + BK)X_{11} + X_{11}(A + BK)^{\mathrm{T}}\right]e^{(A+BK)^{\mathrm{T}}t})dt\Big] \\ + \lambda\|K\|_0 \end{aligned}$$

$$
\begin{aligned}
&= \mathbf{Tr}\left[(Q+K^{\mathrm{T}}RK)\int_0^\infty \frac{-d}{dt}(e^{(A+BK)t}X_{11}e^{(A+BK)^{\mathrm{T}}t})dt\right] + \lambda\|K\|_0 \\
&= \mathbf{Tr}\left[-(Q+K^{\mathrm{T}}RK)e^{(A+BK)t}X_{11}e^{(A+BK)^{\mathrm{T}}t}\right]_0^\infty + \lambda\|K\|_0 \\
&= \mathbf{Tr}\left[(Q+K^{\mathrm{T}}RK)X_{11}\right] + \lambda\|K\|_0
\end{aligned}
$$

The last equality holds, since the controller is assumed to stabilize the system, i.e. $e^{(A+BK)t}$ vanishes as t tends to infinity. Hence, the following minimization is equivalent to our main optimization problem.

$$
\begin{aligned}
\min_{X_{11},K} \quad & \mathbf{Tr}[QX_{11}] + \mathbf{Tr}[RKCX_{11}C^{\mathrm{T}}K^{\mathrm{T}}] + \lambda\|K\|_0 \\
\text{s.t.} \quad & (A+BKC)X_{11} + X_{11}(A+BKC)^{\mathrm{T}} + x_0x_0^{\mathrm{T}} = 0, \\
& (A+BKC) \text{ Hurwitz}, \\
& K \in \mathcal{K}.
\end{aligned} \tag{2.4}
$$

The feedback gain matrix K derived from solving the above optimization problem depends on the value of the initial state x_0. To avoid re-solving the minimization problem for every value of x_0, we design a state feedback controller which minimizes the expected value of the cost function assuming that the entries of x_0 are independent Gaussian random variables with zero mean and covariance matrix equal to the positive definite matrix N, i.e. $x_0 \in \mathcal{N}(0,N)$. Using Lyapunov stability theorem, it can be easily checked that the global asymptotic stability of the closed loop system is guaranteed if and only if the matrix X_{11} is positive definite, thus we can rewrite the optimization problem as follows

$$\min_{\substack{X_{11},X_{12}\\X_{22},K}} \mathbf{Tr}[QX_{11}] + \mathbf{Tr}[RX_{22}] + \lambda\|K\|_0 \tag{2.5a}$$

$$\text{s.t.}\quad AX_{11} + X_{11}A^{\mathrm{T}} + BX_{12}^{\mathrm{T}} + X_{12}B^{\mathrm{T}} + N = 0, \tag{2.5b}$$

$$X_{11} \succ 0, \tag{2.5c}$$

$$K \in \mathcal{K}, \tag{2.5d}$$

$$X_{22} = (KC)X_{11}(KC)^{\mathrm{T}},\ X_{12}^{\mathrm{T}} = KCX_{11}, \tag{2.5e}$$

where $X_{11} \in \mathbb{R}^{n\times n}$, $X_{12} \in \mathbb{R}^{n\times m}$, and $X_{22} \in \mathbb{R}^{m\times m}$. In optimization problem (2.5), the constraints (2.5b-2.5d) are convex, nevertheless, the constraints (2.5e) are nonlinear.

2.3 Equivalent Rank Constrained Formulation

In traditional LQR problems, the nonlinear constraints can be replaced by a linear matrix inequality to form an equivalent convex problem. However, the addition of the sparsity penalizing term to the cost function, the existence of structural constraints on the feedback gain matrix, and incorporation of input/output bounds differentiate our problem from the conventional LQR problem, making the conventional approach inapplicable. Here, we propose a controller synthesis approach based on the idea that the non-convex constraints can be replaced by a rank constraint. Before proceeding, lets state the following lemma.

Lemma 2.3.1. *Let $U \in \mathbb{R}^{n\times n}$, $V \in \mathbb{R}^{n\times m}$, $W \in \mathbb{R}^{m\times m}$, and $Y \in \mathbb{R}^{m\times n}$, with $U \succ 0$.*

Then, $\mathbf{rank}(M) = n$ *if and only if* $W = YUY^T$ *and* $V^T = YU$*, where*

$$M = \begin{bmatrix} U & V \\ V^T & W \\ I_n & Y^T \end{bmatrix}$$

Proof. Since $\mathbf{rank}(U) = n$, its inverse exists and the matrix M can be decomposed as

$$M = \begin{bmatrix} I_n & 0 \\ \begin{bmatrix} V^{\mathrm{T}} \\ I_n \end{bmatrix} U^{-1} & I_{m+n} \end{bmatrix} \bar{M} \begin{bmatrix} I_n & U^{-1}V \\ 0 & I_m \end{bmatrix},$$

where

$$\bar{M} = \begin{bmatrix} U & 0 \\ 0 & \begin{bmatrix} W \\ Y^{\mathrm{T}} \end{bmatrix} - \begin{bmatrix} V^{\mathrm{T}} \\ I_n \end{bmatrix} U^{-1}V \end{bmatrix}.$$

Since the matrices pre/post-multiplied by the matrix $\bar{M}$ are full rank, the matrix M is rank n if and only if the rank of the matrix $\bar{M}$ is n, which is equivalent to

$$\begin{bmatrix} W \\ Y^{\mathrm{T}} \end{bmatrix} - \begin{bmatrix} V^{\mathrm{T}} \\ I_n \end{bmatrix} U^{-1}V = 0_{2n+m}.$$

This completes the proof of the lemma. □

The following corollary is now immediate.

Corollary 2.3.2. *Assuming* $X_{11} \succ 0$*, the constraint*

$$\mathbf{rank} \begin{bmatrix} X_{11} & X_{12} & I_n \\ X_{12}^T & X_{22} & (KC) \\ I_n & (KC)^T & Z \end{bmatrix} = n$$

is equivalent to

$$\begin{cases} \mathbf{rank} \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \\ I_n & (KC)^T \end{bmatrix} = n, \\ Z = X_{11}^{-1} \end{cases}$$

For legibility purposes, we first develop the equivalent formulation for the case with no constraint imposed on the control inputs/outputs; then, we incorporate the bounds on the input/output of the closed loop system.

2.3.1 Rank Constrained Formulation

Assuming that no upper bound is defined for the input/output of the controlled system, the next proposition states that the nonlinear Semidefinite Program (2.5) can be cast as an optimization problem, where all constraints are convex except one, which is a rank constraint.

Proposition 2.3.3. *The optimization program (2.5a-2.5e) is equivalent to the following rank constrained problem*

$$\begin{aligned} \min_{\substack{X_{11},X_{12}\\X_{22},K}} \ & \mathbf{Tr}[QX_{11}] + \mathbf{Tr}[RX_{22}] + \lambda\|K\|_0 \qquad \textbf{(P2)} \\ s.t. \ & AX_{11} + X_{11}A^T + BX_{12}^T + X_{12}B^T + N = 0, \\ & X_{11} \succ 0, \\ & K \in \mathcal{K}, \\ & \mathbf{rank}(X) = n, \end{aligned}$$

where

$$X = \begin{bmatrix} X_{11} & X_{12} & I_n \\ X_{12}^T & X_{22} & (KC) \\ I_n & (KC)^T & Z \end{bmatrix}.$$

Proof. Applying Lemma 2.3.1 to the constraints $X_{22} = (KC)X_{11}(KC)^{\mathrm{T}}$ and $X_{12}^{\mathrm{T}} = (KC)X_{11}$, they can be equivalently replaced by the rank constraint

$$\mathbf{rank} \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^{\mathrm{T}} & X_{22} \\ I_n & (KC)^{\mathrm{T}} \end{bmatrix} = n,$$

since X_{11} is constrained to be positive definite. Introducing the auxiliary matrix variable Z, we can employ Corollary 2.3.2 to rewrite the above rank constraint as a rank constraint on a symmetric matrix, i.e. $\mathbf{rank}(X) = n$. □

It should be noted that augmenting the matrix $\left[\begin{smallmatrix} I_n & KC & Z \end{smallmatrix}\right]^{\mathrm{T}}$ to the original rank constrained matrix only adds some redundant constraints along with an extra variable. Although we increase the number of variables by introducing the new *n-by-n* variable Z, having a symmetric rank constrained matrix has proved to be helpful, as we aim to use a positive semidefinite relaxation of the rank constraint later in this chapter, thus, it is crucial to associate the rank constraint to a symmetric matrix.

The following corollary is an immediate result of Lemma 2.3.1.

Corollary 2.3.4. *The optimal value of Z in problem (**P2**) is the inverse of the optimal X_{11}, i. e. $Z^* = {X_{11}^*}^{-1}$.*

Next, we exploit the stated rank constrained formulation (**P2**) to investigate the bounds on the optimal cost of the optimization problem (**P2**). Assuming feasibility, the lower bound for the optimal cost can be evidently achieved by relaxing the rank constraint $\mathbf{rank}(X) = n$ by the positive semidefinite constraint $X \succeq 0$, since the PSD constraint defines a super-set for the set determined by the rank constraint. As a result, the feasible set of the rank constrained optimization **P2** is a subset of the feasible set of the relaxed problem, hence, the optimal cost of the relaxed optimization provides us with a lower bound for original problem. A more detailed discussion is provided in Section 2.4.2.

As for the upper bound, the results of the theorem 2.3.5 can be utilized to obtain a conservative solution to the problem (**P2**) when either there is no pre-defined structure on the controller gain or the set of the acceptable controller structures, i.e. $\mathcal{K}$ is assumed to be invariant with respect to positive scaling. This assumption covers the highly applicatory structural constraint, where the feasibility/infeasibility of feedback paths are *a priori* specified generally through a directed graph representation.

In such cases the feedback link can be established only if its corresponding edge of the graph, i.e. the pair $(\mathcal{V}, \mathcal{E})$ of vertices and edges respectively, is existent, as shown in equation (2.6).

$$\mathcal{K} = \{K \mid K_{ij} = 0 \text{ if } (v_i, v_j) \notin \mathcal{E}\} \tag{2.6}$$

Theorem 2.3.5. *Assuming the set $\mathcal{K}$ is invariant under positive scaling, the optimization problem (2.7) sub-optimally solves (**P2**).*

$$\begin{aligned} \min_{X} \;& \mathbf{Tr}[RX_{22}] + \mathbf{Tr}[QX_{11}] + \lambda\|\tilde{K}\|_0 \\ s.t. \;& AX_{11} + X_{11}A^T + BX_{12}^T + X_{12}B^T + N = 0, \\ & X_{11} \succ 0, \quad \tilde{K} \in \mathcal{K}, \quad \alpha > 0, \\ & X \succeq 0, \end{aligned} \tag{2.7}$$

where

$$X = \begin{bmatrix} X_{11} & X_{12} & \alpha I_n \\ X_{12}^T & X_{22} & (\tilde{K}C) \\ \alpha I_n & (\tilde{K}C)^T & 2\alpha I_n - X_{11} \end{bmatrix}.$$

*Also, in the case of feedback controller synthesis, i.e. $C = I_n$, the problem (**P2**) can be sub-optimally solved by the following optimization problem*

$$\begin{aligned} \min_{X} \;& \mathbf{Tr}[RX_{22}] + \mathbf{Tr}[QX_{11}] + \lambda\|\tilde{K}\|_0 \\ s.t. \;& AX_{11} + X_{11}A^T + BX_{12}^T + X_{12}B^T + N = 0, \end{aligned} \tag{2.8}$$

$$X_{11} \succ 0, \quad \tilde{K} \in \mathcal{K},$$

$$\Gamma = \mathbf{diag}(\alpha_1, \cdots, \alpha_n) \succ 0,$$

$$X \succeq 0,$$

where

$$X = \begin{bmatrix} X_{11} & X_{12} & \Gamma \\ X_{12}^T & X_{22} & \tilde{K} \\ \Gamma & \tilde{K}^T & 2\Gamma - X_{11} \end{bmatrix}.$$

Furthermore, the suboptimal stabilizing controller gain can be achieved from the optimal values of $\tilde{K}$ *and* Γ *utilizing* $K^* = \tilde{K}^* \Gamma^{*-1}$.

Proof. Here, we present the proof for the state feedback problem, as the proof for the output feedback controller synthesis case is pretty similar; hence, omitted. To prove the second part of the theorem, we assume the problem (2.8) is feasible and its optimum is X^*. For the positive definite matrix X_{11}^* and the positive scaler Γ^*, we have the matrix identity $\Gamma^{*-\frac{1}{2}} X_{11}^* \Gamma^{*-\frac{1}{2}} + \Gamma^{*\frac{1}{2}} {X_{11}^*}^{-1} \Gamma^{*\frac{1}{2}} \succeq 2I$. As a result, we can write

$$\Gamma^* {X_{11}^*}^{-1} \Gamma^* \succeq 2\Gamma^* - X_{11}^*$$

Therefore, the constraint $X^* \succeq 0$ implies $\bar{X}^* \succeq 0$, where

$$\bar{X}^* = \begin{bmatrix} X_{11}^* & X_{12}^* & \Gamma^* \\ {X_{12}^*}^{\mathrm{T}} & X_{22}^* & \tilde{K}^* \\ \Gamma^* & \tilde{K}^{*\mathrm{T}} & \Gamma^* {X_{11}^*}^{-1} \Gamma^* \end{bmatrix}.$$

Due to the positive definiteness of X_{11}^*, the matrix inequality $\bar{X}^* \succeq 0$ is equivalent to the positive definiteness of its Schur complement, that is

$$\begin{bmatrix} X_{22}^* & \tilde{K}^* \\ \tilde{K}^{*\mathrm{T}} & \Gamma^* {X_{11}^*}^{-1} \Gamma^* \end{bmatrix} - \begin{bmatrix} {X_{12}^*}^{\mathrm{T}} \\ \Gamma^* \end{bmatrix} {X_{11}^*}^{-1} \begin{bmatrix} X_{12}^* & \Gamma^* \end{bmatrix} \succeq 0$$
$$\begin{bmatrix} X_{22}^* - {X_{12}^*}^{\mathrm{T}} {X_{11}^*}^{-1} X_{12}^* & \tilde{K}^* - {X_{12}^*}^{\mathrm{T}} {X_{11}^*}^{-1} \Gamma^* \\ \tilde{K}^{*\mathrm{T}} - \Gamma^* {X_{11}^*}^{-1} X_{12}^* & 0 \end{bmatrix} \succeq 0$$

which holds if and only if $\tilde{K}^* = {X_{12}^*}^{\mathrm{T}} {X_{11}^*}^{-1} \Gamma^*$, and $X_{22} = X_{12}^{\mathrm{T}} X_{11}^{-1} X_{12} + M$, where $M \succeq 0$. Therefore, the matrix K^*, defined by $K^* = \tilde{K}^* \Gamma^{*-1}$, satisfies $K^* = {X_{12}^*}^{\mathrm{T}} {X_{11}^*}^{-1}$; thus, $A + BK^*$ satisfies the stability condition with the positive definite Lyapunov matrix X_{11}^*.

Furthermore, The ℓ_0-norm is invariant under positive scaling, hence $\|K\Gamma\|_0 = \|K\|_0$. Also, Since the set $\mathcal{K}$ is assumed to be invariant under positive scaling, the constraint $K \in \mathcal{K}$ is identical to the matrix $K\Gamma$ belonging to the set of admissible controller structures. Therefore, the problem (2.8) provides a suboptimal solution to (**P2**). □

2.3.2 Rank Constrained Formulation in Presence of Input/ Output Constraints

Next, we present how an upper bound on the norm of the control input/output can be incorporated into our rank constrained formulation. It is known that for the positive scaler γ satisfying $x_0^{\mathrm{T}} X_{11}^{-1} x_0 \leq \gamma^{-1}$, where x_0 is the initial state of the system and X_{11} is the solution to the Lyapunov stability condition, the set

$$\mathcal{M} = \{x \in \mathbb{R}^n \mid x^{\mathrm{T}} X_{11}^{-1} x \leq \gamma^{-1}\} \tag{2.9}$$

is an invariant set for the closed loop system. Employing the concept of invariant sets for linear systems, we can develop the rank constrained formulation of control system with bounded input norms. The details for two choices of norms utilize to bound the control input in given in the sequel.

- **System Norm:** The next theorem describes how the upper bound on the system norm of the control input can be incorporated into the controller synthesis problem using our proposed rank constrained formulation.

  Theorem 2.3.6. *The optimization problem (***P2***) can be modified to conservatively incorporate an upper bound on the system norm of the control input, i.e.* $\|u\|_{L^2_\infty(\mathbb{R}^m)} \leq u_{max}$*, as follows.*

$$\begin{aligned} \min_{X} \quad & \mathbf{Tr}[RX_{22}] + \mathbf{Tr}[QX_{11}] + \lambda \|K\|_0 && (\mathbf{P3}) \\ s.t. \quad & AX_{11} + X_{11}A^T + BX_{12}^T + X_{12}B^T + N = 0, \\ & X_{11} \succ 0, \end{aligned}$$

$$K \in \mathcal{K},$$

$$\begin{bmatrix} W & (KC)^T \\ (KC) & u_{max}^2 I_m \end{bmatrix} \succeq 0,$$

$$x_0^T W x_0 \leq 1,$$

$$\mathbf{rank}(X) = n,$$

where x_0 denotes the initial state and

$$X = \begin{bmatrix} X_{11} & X_{12} & I_n \\ X_{12}^T & X_{22} & (KC) \\ I_n & (KC)^T & Z \\ \gamma I_n & Y & W \end{bmatrix}.$$

Proof. Based on the lines in [41, p. 103], we have

$$\begin{aligned} \|u\|_{L^2_\infty(\mathbb{R}^m)} &= \sup_{t\geq 0} \|u(t)\|_2 = \sup_{t\geq 0} \|KCx(t)\|_2 \\ &\leq \sup_{x\in\mathcal{M}} \|KCx\|_2 \\ &= \sup_{x\in\mathcal{M}} \|KCX_{11}^{1/2}X_{11}^{-1/2}x\|_2 \\ &= \sqrt{\lambda_{\max}(X_{11}^{1/2}(KC)^{\mathrm{T}}(KC)X_{11}^{1/2})\gamma^{-1}} \end{aligned}$$

Thus, the input constraint $\|u\|_{L^2_\infty(\mathbb{R}^m)} \leq u_{\max}$ holds for all $t \geq 0$ if

$$\begin{bmatrix} \gamma X_{11}^{-1} & (KC)^{\mathrm{T}} \\ (KC) & u_{\max}^2 I_m \end{bmatrix} \succeq 0,$$

$$x_0^{\mathrm{T}} \gamma X_{11}^{-1} x_0 \leq 1.$$

The existence of the term γX_{11}^{-1} in the above matrix inequality makes it nonlinear, however, Utilizing Lemma 2.3.1, it can be verified that the rank constraint $\mathbf{rank}(X) = n$, applied on the modified matrix X, is equivalent to introducing the variables $W = \gamma X_{11}^{-1}$. The rest of the proof is straightforward. □

- **Infinity Norm:** If the constraint on the system output is in the form of $\|y(t)\|_{L_\infty^\infty(\mathbb{R}^m)} \leq y_{\max}$, it can be represented using the following matrix inequalities [41, p. 104].

$$\begin{bmatrix} V & C \\ (C)^{\mathrm{T}} & \gamma X_{11}^{-1} \end{bmatrix} \succeq 0,$$

$$V_{ii} \leq u_{\max}^2$$

$$x_0^{\mathrm{T}} \gamma X_{11}^{-1} x_0 \leq 1.$$

Therefore, this problem can also be posed as a rank constrained problem through the next theorem.

Theorem 2.3.7. *The optimization problem (**P2**) can be modified to conservatively incorporate an upper bound on the infinity norm of the control input, i.e.* $\|y\|_{L_\infty^\infty(\mathbb{R}^m)} \leq y_{max}$, *as follows.*

$$\min_{X} \ \mathbf{Tr}[RX_{22}] + \mathbf{Tr}[QX_{11}] + \lambda \|K\|_0 \qquad \textbf{(P3')}$$

$$s.t. \ \ AX_{11} + X_{11}A^T + BX_{12}^T + X_{12}B^T + N = 0,$$

$$X_{11} \succ 0,$$

$$K \in \mathcal{K},$$

$$\begin{bmatrix} V & C \\ (C)^T & W \end{bmatrix} \succeq 0,$$

$$V_{ii} \leq y_{max}^2,$$

$$x_0^T W x_0 \leq 1,$$

$$\mathbf{rank}(X) = n,$$

where x_0 denotes the initial state and

$$X = \begin{bmatrix} X_{11} & X_{12} & I_n \\ X_{12}^T & X_{22} & (KC) \\ I_n & (KC)^T & Z \\ \gamma I_n & Y & W \end{bmatrix}.$$

Remark 2.3.8. *Other norms such as element-wise bound on the control input or the norm bounds on the system outputs can also be assimilated into the rank constraint using similar techniques. The details are omitted with the purpose of improving the readability of the paper.*

All of the optimization problems posed so far are NP-hard due to the existence of the ℓ_0-norm in the cost function and the rank constraint. Therefore, no polynomial time algorithm capable of solving them in the general form, exists. In the next two sections, we propose a method to sub-optimally solve the problem, then, discuss a special case of the problem where only the sparsity of the controller is of importance.

2.4 Convex Relaxation of the Optimal Control Problem

In this section, we study the general problem of designing a optimal sparse output feedback controller. Although the results presented in the sequel are applicable to the controller design problem with the input/output norm bounds, to enhance the legibility of the chapter, we choose to state them in the absence of the constraints on the control inputs and system outputs. Hence, we consider the problem (**P2**), which is a combinatorial problem, due to the existence of the ℓ_0-norm, in fact a quasi-norm, in the cost and the rank constraint.

To reduce the complexity of the problem, we first adopt the weighted ℓ_1 relaxation of the ℓ_0 based on the notion that weighted ℓ_1 minimization problem is a reliable heuristic for cardinality minimization [17, 42, 22]. Substituting the cardinality penalizing term with the weighted ℓ_1-norm of the controller gain matrix, we obtain the following relaxed optimization problem

$$\begin{aligned}
&\min_{X} \mathbf{Tr}[QX_{11}] + \mathbf{Tr}[RX_{22}] + \lambda \|W \circ K\|_1 && \textbf{(C1)}\\
&\text{s.t.} \quad AX_{11} + X_{11}A^{\mathrm{T}} + BX_{12}^{\mathrm{T}} + X_{12}B^{\mathrm{T}} + N = 0,\\
&\qquad X_{11} \succ 0,\\
&\qquad K \in \mathcal{K},\\
&\qquad \mathbf{rank}(X) = n.
\end{aligned}$$

where the weight matrix W is a positive matrix with appropriate dimensions and

$$X = \begin{bmatrix} X_{11} & X_{12} & I_n \\ X_{12}^{\mathrm{T}} & X_{22} & (KC) \\ I_n & (KC)^{\mathrm{T}} & Z \end{bmatrix}. \tag{2.10}$$

Combinatorial nature of rank constrained optimization problems make the search for the optimal point computationally intractable. Therefore, a systematic solution to general rank constrained optimization problem has remained open [43, 44]. Nonetheless, attempts have been made to solve specific rank constrained problems, and algorithms proposed to locally solve such problems [45, 46]. Here, we propose to use a particular form of Alternating Direction Method of Multipliers (ADMM) to solve our rank constrained problem.

2.4.1 Feasibility of the Output Feedback Control Problem

Before proceeding with describing the ADMM algorithm to solve the relaxed problem, we discuss how our proposed rank constrained reformulation can be utilized in investigating the feasibility of the output feedback control problem under constraints such as controller pre-defined structure and input/output constraint. The next theorem introduces a feasibility test for the existence of a stabilizing output feedback controller with predefined structure.

Theorem 2.4.1. *The linear time invariant system (2.1) can be stabilized using the output feedback controller described in (2.2), with the optimal cost less than or equal to J^*, if and only if the optimal cost of the following optimization problem is equal to* *zero.*

$$\begin{aligned} \min_{X,Y} \quad & \mathbf{Tr}(Y^T X) \\ s.t. \quad & AX_{11} + X_{11}A^T + BX_{12}^T + X_{12}B^T + N = 0, \\ & \mathbf{Tr}[QX_{11}] + \mathbf{Tr}[RX_{22}] + \lambda \|W \circ K\|_1 \leq J^*, \\ & X_{11} \succ 0, \\ & K \in \mathcal{K}, \\ & X \succeq 0, \\ & 0 \preceq Y \preceq I_{2n+m}, \\ & \mathbf{Tr}(Y) = n + m, \end{aligned} \tag{2.11}$$

where

$$X = \begin{bmatrix} X_{11} & X_{12} & I_n \\ X_{12}^T & X_{22} & (KC) \\ I_n & (KC)^T & Z \end{bmatrix}.$$

Proof. Applying the results from [47, p.266], if the matrix X is positive semidefinite, i.e. $X \in \mathbb{S}_+^{2n+m}$, we have

$$\begin{aligned} \sum_{i=n+1}^{2n+m} \lambda_i(X) = \min_{Y \in \mathbb{R}^{2n+m}} \quad & \mathbf{Tr}(Y^{\mathrm{T}} X) \\ \text{s.t.} \quad & 0 \preceq Y \preceq I_{2n+m}, \\ & \mathbf{Tr}(Y) = (2n + m) - n, \end{aligned}$$

where $\lambda_1(X) \geq \cdots \geq \lambda_{2n+m}(X)$ are the eigenvalues of X. Due to the positive semidefiniteness of X, the optimal cost of (2.11) is lower bounded by zero. Now, using our rank constraint formulation, it can be verified that such an output feedback controller, satisfying the predefined structure, stabilizes the LTI system (2.1) if and only if the feasible set of (2.11) contains at least a matrix X with rank n for which the sum of $n+m$ smaller eigenvalues is equal to zero, i.e. $\sum_{i=n+1}^{2n+m} \lambda_i(X) = 0$. □

The optimization problem (2.11) is non-convex due to the existence of the the bi-linear term in its cost function. However, it can be solved utilizing an optimization algorithm, which iteratively solves the problem for X and Y till it reaches the convergence [45, 48].

2.4.2 ADMM for Solving the Relaxed Problem

ADMM was originally developed in 1970s [49, 50], and has been used for optimization purposes since. Boyd *et al.*, in [51], argued that this method can be efficiently applied to large-scale optimization problems. For non-convex problems, the convergence of ADMM is not guaranteed, also, it may not reach the global optimum when it converges, thus, the convergence point should be considered as a local optimum.

For the optimization problem (**C1**), one way to perform convex relaxation is replacing the rank constraint on matrix X with a positive semi-definite constraint, i.e. $X \succeq 0$. Since X_{11} is positive definite, using lemma 2.3.1, it can be seen that the rank constraint in (**C1**) is equivalent to

$$\begin{bmatrix} X_{22} & (KC) \\ (KC)^{\mathrm{T}} & Z \end{bmatrix} - \begin{bmatrix} X_{12}^{\mathrm{T}} \\ I_n \end{bmatrix} X_{11}^{-1} \begin{bmatrix} X_{12} & I_n \end{bmatrix} = 0, \tag{2.12}$$

which implies that the Schur complement of the matrix X should be equal to zero, while $X \succeq 0$ is the same as positive semi-definiteness of its Schur complement. Therefore, the set defined by the PSD constraint is a super-set for the one defined by the rank constraint. Now, if we define the convex set

$$\mathcal{C} = \{X \,| AX_{11} + X_{11}A^{\mathrm{T}} + BX_{12}^{\mathrm{T}} + X_{12}B^{\mathrm{T}} + N = 0,$$
$$X_{11} \succ 0, \quad K \in \mathcal{K}, \quad X \succeq 0\}$$

where the structure of X is given in (2.10), and $\mathcal{S}$ denotes the set of $(2n+m)\times(2n+m)$ symmetric matrices with rank equal to n, the minimization (**C1**) can be represented as

$$\begin{aligned} \min_{X} \quad & f(X) \\ s.t \quad & X \in \mathcal{C} \cap \mathcal{S} \end{aligned} \tag{2.13}$$

where

$$f(X) = \mathbf{Tr}[RX_{22}] + \mathbf{Tr}[QX_{11}] + \lambda \|W \circ K\|_1$$

and the weight matrix W is a positive real matrix with appropriate dimensions. Considering the above formulation, the ADMM algorithm can be carried out by repeatedly performing the steps stated in the sequel till certain convergence criteria is satisfied [51, p. 74].

$$X^{(k+1)} = \arg\min_{X \in \mathcal{C}} \quad f(X) + (\rho/2)\|X - V^{(k)} + Y^{(k)}\|_F^2 \tag{2.14a}$$

$$V^{(k+1)} = \Pi_{\mathcal{S}}(X^{(k+1)} + Y^{(k)}) \tag{2.14b}$$

$$Y^{(k+1)} = Y^{(k)} + X^{(k+1)} - V^{(k+1)} \tag{2.14c}$$

$$w_{ij}^{(k+1)} = \frac{1}{|k_{ij}^{(k)}| + \delta} \tag{2.14d}$$

where w_{ij} and k_{ij} denote the (i,j) entries of the matrices W and K, respectively. The convexity of the cost function and the constraints makes (2.14a) a convex problem, hence, it can be solved by various computationally efficient methods. The operator $\Pi_{\mathcal{S}}(.)$, in (2.14b), denotes projection onto the set $\mathcal{S}$. Although the projection on a non-convex set is generally not an easy task, it can be carried out exactly in the case of projecting on the set of matrices with pre-defined rank. In our case, the set $\mathcal{S}$ is the set of matrices with rank n, thus, $\Pi_{\mathcal{S}}(.)$ can be determined by carrying out Singular Value Decomposition (SVD) and keeping the top dyads, i.e.

$$\Pi_{\mathcal{S}}(X) \triangleq \sum_{i=1}^{n} \sigma_i u_i v_i^{\mathrm{T}} \tag{2.15}$$

where σ_i, $i = 1, \cdots, n$ are the n largest singular values of matrix x, and the vectors $u_i \in \mathbb{R}^{(2n+m)}$ and $v_i \in \mathbb{R}^{(2n+m)}$ are their corresponding left and right singular vectors. The step (2.14c) in the algorithm is a simple matrix manipulation to update the auxiliary variable u, which is exploited in the next iteration.

The last step of the heuristic (2.14) is to update the weight on the entries of the controller matrix approximately inversely proportional to the value of the corresponding matrix entry recovered from the previous iteration. Hence, the next iteration optimization will be forced to concentrate on the entries with smaller magnitudes, which results in promoting the controller sparsity. It should also be noted the relatively small constant δ is added to the denominator of the update law (2.14d) to avoid instability of the algorithm, especially when a recovered controller entry turns out to be zero in the previous iteration [22].

Initializing with the stabilizing LQR controller along with its corresponding Lyapunov matrix, a sub-optimal minimizer to the problem (**C1**) can be obtained by iterating the steps (2.14a-2.14c) until the convergence is achieved. The algorithm's stopping criteria is either reaching the maximum number of iterations or $\varepsilon < \varepsilon^*$, where ε update is performed using the following equation.

$$\varepsilon^{(k+1)} \triangleq \mathbf{max}(\|X^{(k+1)} - V^{(k+1)}\|_F, \|V^{(k+1)} - V^{(k)}\|_F) \tag{2.16}$$

The small enough entries of the generated controller gain can then be truncated to yield a sparse controller matrix, namely $\bar{K}$, while considering the extent of its adverse effect on the stability and performance of the closed loop system. The step-by-step procedure is described in Algorithm 1. As said before, the truncation step in the algorithm should be performed with the necessary precautions, since not only does it deteriorate the obtained optimal performance but it also may destabilize the closed loop system. The following proposition provides the sufficient condition under which the truncation process does not have cause instability in the closed loop system.

Proposition 2.4.2. *The truncated controller, denoted by $\bar{K}$, stabilizes the system if the truncation threshold ξ is bounded by*

$$\xi < \frac{\sigma_{min}(N)}{\sum_{ij} \|BE_{ij}CX_{11} + X_{11}(BE_{ij}C)^T\|_2} \tag{2.17}$$

Algorithm 1: Solution to **C1**

Inputs: A, B, C, Q, R, λ, $\mathcal{K}$, ρ, δ and ε^*

1: *Initialization:*
   Find $X^{(0)}$ by solving (2.14a) for $\lambda = 0, \rho = 0$ (LQR),
   Set $V^{(0)} = X^{(0)}$, $Y^{(0)} = 0 \times I_{(2n+m)}$, and $n = 0$,
2: **While** $\varepsilon^{(n)} \leq \varepsilon^*$ **do**
3: Update $X^{(n+1)}$ by solving (2.14a),
4: Update $V^{(n+1)}$ using Eq. (2.14b),
5: Update $Y^{(n+1)}$ using Eq. (2.14c),
6: Update $W^{(n+1)}$ using Eq. (2.14d),
7: Update $\varepsilon^{(n+1)}$ using Eq. (2.16),
8: $n \leftarrow n + 1$,
9: **end while**
10: Truncate K,

Output: $\bar{K}$

where $\sigma_{min}(N)$ denotes the smallest singular value of the matrix N, which is the positive definite matrix satisfying

$$(A + BKC)X_{11} + X_{11}(A + BKC)^T + N = 0,$$

and $E_{ij} \in \mathbb{R}^{m \times p}$ is the matrix whose only nonzero entry, equal to 1, is its (i, j)-entry.

Proof. Defining the matrix of the truncated entries of the controller as $K_\xi = K - \bar{K}$, we will have

$$\begin{aligned} 0 =&(A + B(\bar{K} + K_\xi)C)X_{11} \\ &+ X_{11}(A + B(\bar{K} + K_\xi)C)^{\mathrm{T}} + N, \\ =&(A + B\bar{K}C)X_{11} + X_{11}(A + B\bar{K}C)^{\mathrm{T}} \\ &+ BK_\xi CX_{11} + X_{11}(BK_\xi C)^{\mathrm{T}} + N. \end{aligned}$$

Hence, the truncated controller stabilized the system if

$$BK_\xi CX_{11} + X_{11}(BK_\xi C)^{\mathrm{T}} + N \succ 0,$$

which is equivalent to the following inequality, for any nonzero vector x with appropriate dimension,

$$x^{\mathrm{T}}(BK_\xi CX_{11} + X_{11}(BK_\xi C)^{\mathrm{T}} + N)x > 0.$$

The previous inequality holds if we have

$$|x^{\mathrm{T}}(BK_\xi CX_{11} + X_{11}(BK_\xi C)^{\mathrm{T}})x| < \sigma_{min}(N)x^{\mathrm{T}}x.$$

Noting that $K_\xi = \sum_{(i,j)\in\mathcal{D}} k_{ij}E_{ij}$, where $\mathcal{D} = \{(i,j)|\ |k_{ij}| < \xi\}$, we rewrite the above inequality as

$$|x^{\mathrm{T}}(\sum_{(i,j)\in\mathcal{D}} k_{ij}[BE_{ij}CX_{11} + X_{11}(BE_{ij}C)^{\mathrm{T}}])x| < \sigma_{min}(N)x^{\mathrm{T}}x.$$

which is true if

$$\sum_{(i,j)\in\mathcal{D}} |k_{ij}|\|BE_{ij}CX_{11} + X_{11}(BE_{ij}C)^{\mathrm{T}}\|_2 < \sigma_{min}(N).$$

Since $|k_{ij}| < \xi$ for all $(i,j) \in \mathcal{D}$, we can conservatively replace the above inequality with

$$\xi \sum_{\forall(i,j)} \|BE_{ij}CX_{11} + X_{11}(BE_{ij}C)^{\mathrm{T}}\|_2 < \sigma_{min}(N),$$

which completes our proof. □

Remark 2.4.3. *For the problem of optimal sparse state feedback control design, i.e* $C = I_n$*, if there exists no* a priori *defined controller structure or the constraint on the controller matrix is in the form of sparsity pattern, one way to perform the truncation is to solve the minimization problem, assuming that all of the variables have already converged to their optimal values except the controller matrix. Thus, we will have*

$$\begin{aligned} \min_{K} \quad & \lambda\|K\| + (\rho/2)\|K - (K^{(V^*)} - K^{(Y^*)})\|_F^2 \\ s.t. \quad & K \in \mathcal{K}. \end{aligned} \tag{2.18}$$

where $K^{(V^*)}$ *and* $K^{(Y^*)}$ *are the sub-blocks of the optimal values of* V^* *and* Y^**, respectively, which correspond to the controller gain matrix, and* $\|.\|$ *can be chosen as either* ℓ_1 *or* ℓ_0*-norm. Moreover, in such problems, the problem (2.18) has a unique solution that can be obtained analytically as follows [23, 51]. For example, if the norm used in (2.18) is* ℓ_0*-norm, the optimal values of the elements, not constrained to zero, can be obtained through the following element-wise truncation operator*

$$K^*_{ij} = \begin{cases} K^{(V^*)}_{ij} - K^{(Y^*)}_{ij}, & |K^{(V^*)}_{ij} - K^{(Y^*)}_{ij}| > \sqrt{2\lambda/\rho} \\ 0, & otherwise. \end{cases} \tag{2.19}$$

2.5 Sparsest Stabilizing Output Feedback Controller Design

Next, we study the special case in which obtaining a stabilizing constant gain feedback controller with the sparsest feasible structure, i.e. considering the constraints, is desirable. To this end, we eliminate the terms which penalize the system performance from the cost, i.e. both R and Q are zero. One of the applications that can be addressed using this problem setup is the problem of stabilizing controller synthesis for networks/systems where establishing communication links between nodes are so costly that the control effort and error cost are almost negligible. Having $R = 0$, it can be seen the variable X_{22} is irrelevant in this case, so its corresponding constraints can be removed from the optimization program. Therefore, we will have

$$\begin{aligned} \min_{X_{11},X_{12},K,N} & \|K\|_0 \\ \text{s.t.} \quad & AX_{11} + X_{11}A^{\mathrm{T}} + BX_{12}^{\mathrm{T}} + X_{12}B^{\mathrm{T}} + N = 0, \\ & X_{11} \succ 0, \ N \succ 0 \\ & K \in \mathcal{K}, \\ & \mathbf{rank} \begin{bmatrix} X_{11} & X_{12} \\ I_n & (KC)^{\mathrm{T}} \end{bmatrix} = n, \end{aligned} \tag{\textbf{P4}}$$

The following lemma helps us convert rank constrained cardinality minimization problem (**P4**) into an affine rank minimization problem.

Lemma 2.5.1. *Consider the following rank constrained cardinality minimization problem*

$$\begin{aligned} \min_{Y} \ & \|W_1 Y W_2\|_0 \\ s.t. \ & \mathcal{L}_1(Y) = \mu, \\ & \mathcal{L}_2(Y) \succeq 0, \\ & \mathbf{rank}(Y) = \mathbf{rank}(Y_{11}) = n, \end{aligned} \tag{2.20}$$

where Y is partitioned as $Y = \begin{bmatrix} Y_{11} & Y_{12} \\ Y_{21} & Y_{22} \end{bmatrix} \in \mathbb{R}^{p\times q}$, $W_1 \in \mathbb{R}^{a\times p}$ and $W_2 \in \mathbb{R}^{q\times b}$ are weight matrices, $\mathcal{L}_1$ and $\mathcal{L}_2$ are two arbitrary maps, and $Y_{11} \in \mathbb{R}^{n\times n}$ is a full rank square matrix ($n < min\{p, q\}$). If the optimization problem (2.20) is feasible, it can be equivalently formulated as

$$\begin{aligned} \min_{Y} \quad & \|W_1 Y W_2\|_0 + \nu\mathbf{rank}(Y) \\ s.t. \quad & \mathcal{L}_1(Y) = \mu, \\ & \mathcal{L}_2(Y) \succeq 0, \\ & \mathbf{rank}(Y_{11}) = n, \end{aligned} \tag{2.21}$$

for any $\nu > ab$.

Proof. Let Y^* be the optimum of (2.20), then $\mathbf{rank}(Y^*_{11}) = n$ and it satisfies both equality and inequality constraints. Therefore, it belongs to the feasible set of (2.21). Furthermore, for every point Y in the feasible set of (2.21) with the rank greater than n, we have

$$\begin{aligned} J - J^* &= \|W_1 Y W_2\|_0 + \nu\mathbf{rank}(Y) \\ &\quad - (\|W_1 Y^* W_2\|_0 + \nu\mathbf{rank}(Y^*)) \\ &= (\|W_1 Y W_2\|_0 - \|W_1 Y^* W_2\|_0) \\ &\quad + \nu(\mathbf{rank}(Y) - \mathbf{rank}(Y^*)) \\ &\geq -\|W_1 Y^* W_2\|_0 + \nu(\mathbf{rank}(Y) - \mathbf{rank}(Y^*)) \end{aligned}$$

Since $W_1 Y^* W_2 \in \mathbb{R}^{a\times b}$, it is safe to bound the cardinality as $\|W_1 Y^* W_2\|_0 \leq ab$. Using

$\mathbf{rank}(Y) - \mathbf{rank}(Y^*) \geq 1$, we can write

$$J - J^* > -ab + \nu$$

Hence, the cost for all Y, with rank greater than n, is higher than the cost of Y^*, if $\nu > ab$. This means the optimum of (2.21) should be of rank n. Knowing that Y^* has the minimum cardinality among the matrices with rank equal to n, we conclude that Y^* is also the optimum for (2.21).

Conversely, let $\bar{Y}$ be the optimal point for (2.21). As it is shown in the first part of the proof, the cost generated by matrices, with the rank higher than n is greater than that of rank n matrices, for $\nu > ab$. Thus, the rank of $\bar{Y}$ must be n, unless no point with the rank equal to n exists in the feasible set of (2.21). However, this implies that (2.20) is infeasible, which contradicts the lemma's assumption. Therefore, $\bar{Y}$ is the minimizer of the cardinality term of the cost function among all rank n matrices in the feasible set of (2.21), i.e. $\bar{Y}$ is the minimizer of (2.20). □

Remark 2.5.2. *In the optimization problem (2.20), if the cost which is to be minimized is the rank of the matrix $W_1 Y W_2$, instead of its cardinality, lemma 2.5.1 can still be applied to the problem for any $\nu > min\{a, b\}$.*

Applying lemma 2.5.1 to (**P4**), we can equivalently write it as

$$\min_{X_{11}, X_{12}, K, N} \|K\|_0 + \nu \mathbf{rank} \begin{bmatrix} X_{11} & X_{12} \\ I_n & (KC)^{\mathrm{T}} \end{bmatrix} \tag{2.22}$$

$$\text{s.t.} \quad AX_{11} + X_{11}A^{\mathrm{T}} + BX_{12}^{\mathrm{T}} + X_{12}B^{\mathrm{T}} + N = 0,$$

$$\mathbf{diag}(X_{11}, N) \succ 0,$$

$$K \in \mathcal{K},$$

with $\nu > mn$. Note that the matrix X_{11} is full rank due to its positive definiteness, therefore, all of the requirements of lemma 2.5.1 are satisfied.

Remark 2.5.3. *The solution to equation (2.22) falls into the category of the problem of recovery of simultaneously structured models where the matrix of interest is both sparse and low-rank [52, 53]. Oymak* et al.*, in their recent paper, have shown that minimizing a combination of the known norm penalties corresponding to each structure (for example, ℓ_1-norm for sparsity and nuclear norm for matrix rank) will not yield better results than an optimization exploiting only one of the structures. They have concluded that an entirely new convex relaxation is required in order to fully utilize both structures [52].*

Without loss of generality, the following theorem is stated assuming $m < n$.

Theorem 2.5.4. *The optimization problem (**P4**), if feasible, is equivalent to*

$$\min_{\substack{X_{11}, X_{12}, \\ C, K, N, \varepsilon}} \mathbf{rank}(\mathbf{diag}[\mathbf{vec}(K), \Psi_1, \cdots, \Psi_\nu, \Phi_1, \cdots, \Phi_\rho]) \tag{2.23}$$

$$\begin{aligned} s.t.\quad & AX_{11} + X_{11}A^T + BX_{12}^T + X_{12}B^T + N = 0, \\ & K \in \mathcal{K}, \\ & \varepsilon > 0, \end{aligned}$$

where

$$\Psi_i = \left[\begin{array}{cc|c} X_{11} & X_{12} & \\ I_n & (KC)^T & 0_{(2n\times(n-m))} \end{array}\right] \quad i = 1, \cdots, \nu$$

$$\Phi_i = \begin{bmatrix} I_{2n} & D \\ D^T & \mathbf{diag}(X_{11}, N) - \varepsilon I_{2n} \end{bmatrix} \quad i = 1, \cdots, \rho$$

and the parameters ν *and* ρ *are integers satisfying*

$$\rho > mn + \nu.\mathbf{max}\{2n, (n+m)\}$$

$$\nu > mn$$

Proof. For a function that maps matrices into $q \times q$ symmetric matrices, positive semi-definiteness can be equivalently expressed as a rank constraint, i.e. $f(X) \succeq 0$ is equivalent to

$$\mathbf{rank} \begin{bmatrix} I_q & U \\ U^{\mathrm{T}} & f(X) \end{bmatrix} \leq q \tag{2.24}$$

for some $U \in \mathbb{R}^q$ [39]. Since $\mathbf{diag}(X_{11}, N) \succ 0$ is equivalent to $\mathbf{diag}(X_{11}, N) \succeq \varepsilon I_{2n}$ for some $\epsilon > 0$, it can be written as the following rank constraint

$$\mathbf{rank} \begin{bmatrix} I_{2n} & D \\ D^{\mathrm{T}} & \mathbf{diag}(X_{11}, N) - \varepsilon I_{2n} \end{bmatrix} = 2n$$

Noting that the cost function in (2.22) is bounded by $mn + \nu.\mathbf{max}\{2n, (n+m)\}$, we can use an argument similar to the one used in the proof of lemma 2.5.1 to to show that (**P4**), if feasible, can be equivalently cast in the following form

$$\begin{aligned}\min_{X_{11},X_{12},C,K,N} \|K\|_0 &+ \nu \mathbf{rank} \begin{bmatrix} X_{11} & X_{12} \\ I_n & (KC)^{\mathrm{T}} \end{bmatrix} \\ &+ \rho \mathbf{rank} \begin{bmatrix} I_{2n} & D \\ D^{\mathrm{T}} & M - \varepsilon I_{2n} \end{bmatrix} \\ \text{s.t.} \quad & AX_{11} + X_{11}A^{\mathrm{T}} + BX_{12}^{\mathrm{T}} + X_{12}B^{\mathrm{T}} + N = 0, \\ & K \in \mathcal{K}, \\ & \varepsilon > 0,\end{aligned} \tag{2.25}$$

where

$$\nu > mn$$
$$\rho > mn + \nu.\mathbf{max}\{2n, (n+m)\}.$$

Next, we are going to show that the cost function of (2.25) is equal to the cost function of (2.23) for ρ and ν chosen to be integers satisfying the conditions. It can be easily verified that $\|K\|_0 = \mathbf{rank}(\mathbf{diag}(\mathbf{vec}(K)))$, also, the ranks of the square matrices Ψ_i's are equal to the rank of $\begin{bmatrix} X_{11} & X_{12} \\ I_n & (KC)^{\mathrm{T}} \end{bmatrix}$.

If the parameters ρ and ν are integers, we can construct a block diagonal matrix in the following form

$$\mathbf{diag}[\mathbf{vec}(K), \Psi_1, \cdots, \Psi_\nu, \Phi_1, \cdots, \Phi_\rho]$$

Thus, the rank of such matrix is equal to the sum of the rank of its constructing block matrices. Therefore, it is equal to the cost function of the optimization problem (2.23), which completes our proof. □

Assuming the structural constraints on the controller gain to be in the form of equality constraints, the above formulation is in the form of *Affine Rank Minimization Problem* (ARMP), which consists of minimizing the rank of a matrix subject to affine/convex constraints with the general form

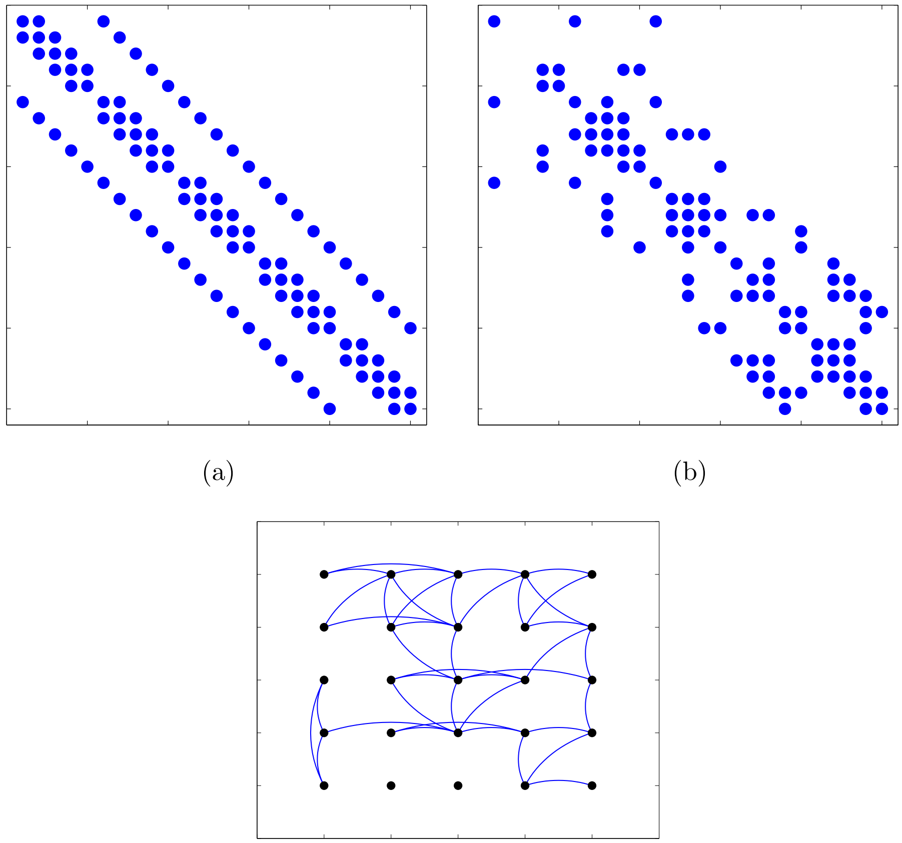
$$\begin{aligned} &\min_{X} \ \mathbf{rank}(X) \\ &\text{s.t.} \ \ \mathcal{A}(X) = b \end{aligned}$$

for a fixed infinitesimal $\varepsilon > 0$. ARMP has been investigated thoroughly in the past decade and several heuristics have been proposed to solve it. For example, Recht *et al.* in [39] showed that nuclear norm relaxation of rank can recover the minimum rank solution if certain property, namely Restricted Isometry Property (RIP), holds for the linear mapping. A family of Iterative Re-weighted Least Squares algorithms which minimize Schatten-p norm, i.e. $\|X\|_{S_p} = \mathbf{Tr}(X^{\mathrm{T}}X + \gamma I)^{p/2}$, of the matrix as a surrogate for its rank is also introduced in [54]. Singular Value Projection (SVP) algorithm is also guaranteed to recover the low rank solution for affine constraints which satisfy RIP [55].

Remark 2.5.5. *The discrete-time counterpart of the optimization problem (***P4***) can be formulated as*

$$\min_{\substack{X_{11},X_{12},\\ K,N}} \|K\|_0 \tag{P5}$$

$$s.t. \ \ A^T X_{11} A + A^T X_{12} + X_{12}^T A + X_{22} - X_{11} + N = 0,$$



(a) (b)

(c)

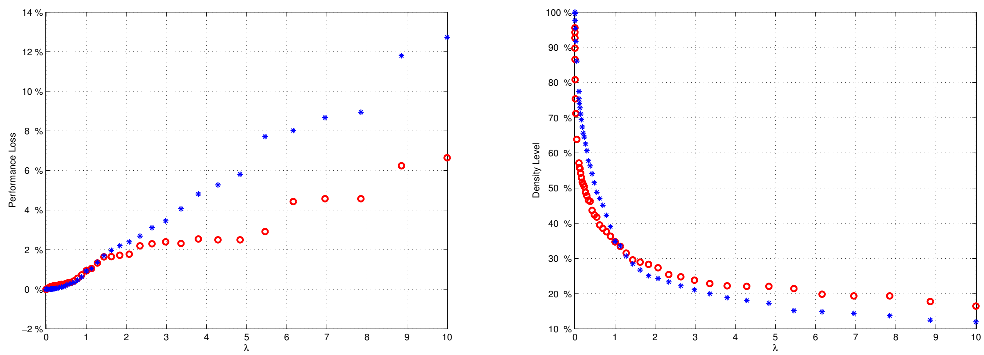
Figure 2.1: Sparsity pattern of (a) the network system (b) the optimal sparse feedback controller $\{\lambda = 10,\, \rho = 100\}$. (c) representation of the underlying graph of the sparse controller.

$$X_{11} \succ 0, \;\; N \succ 0,$$

$$Y^T = BKC,$$

$$K \in \mathcal{K},$$

$$\textbf{rank} \begin{bmatrix} X_{11} & X_{12} \\ X_{12}^T & X_{22} \\ I_n & Y^T \end{bmatrix} = n.$$



(a) (b)

Figure 2.2: (a) Percentage of optimal quadratic cost degradation relative to the LQR optimal cost and (b) Density level of the controller gain for different values of λ, and for the two controller design approaches: SPOFC ($*$) and our proposed method ($\circ$)

Hence, the results, developed in this section, are applicable to the problem of identifying the sparsest stabilizing controller for discrete-time linear time invariant systems.

2.6 Simulation Results

In this section, we use several examples to demonstrate how our proposed rank constrained optimization approach can be exploited to solve the optimal sparse output feedback controller design problem considering the input/output constraints.

2.6.1 Unstable Lattice Network System

Here, we illustrate an example in which we design an optimal sparse state feedback controller for an unstable networked system with 25 states defined on a 5×5 lattice. The entries of its corresponding system matrix are randomly generated scalars drawn from the standard uniform distribution on the open interval $(-1, 1)$, and it is assumed the state performance matrix Q to be an identity matrix, while the control performance weight $R = 10I$. Here, we used the traditional LQR controller as the benchmark to measure the performance of our proposed algorithm. Performing standard LQR design method, our results show that the optimal cost, for the case of LQR control design, is $J^* = 211.173$.

Next, we applied Algorithm 1 to design an optimal sparse controller with the parameters values $\lambda = 10$ and $\rho = 100$, while keeping the performance weights unchanged. It can be observed that the optimal controller cost function increases to $J^* = 230.6989$, which is about 9.2% higher, comparing to that of the LQR design. On the other hand, the number of non-zero entries of the controller gain drops to 97, i.e. $\|K\|_0 = 97$. This means a major decrease in the number of non-zero entries of the controller gain. Figures 2.1a and 2.1b show the sparsity structure of the system network and the obtained sparse controller. The figures basically visualize the controller matrix by using solid blue circles to represent the non-zero entries of the matrix and leaving the zero entries as blanks. In Figure 2.1c the graph representation of the generated sparse controller is depicted.

Additionally, we present a brief case study that compares our approach with the Sparsity Promoting Optimal Feedback Control (SPOFC) method, proposed in [27, 23]. The SPOFC method essentially solves a different control problem, since it solves the $\mathcal{H}_2$ problem, modified by adding a sparsity promoting penalty function to its cost function and obtain a sub-optimal sparse state feedback controller, while our proposed approach is built upon adjusting the LQR problem to achieve a sparse *output* feedback controller. Moreover, the approach in SPOFC algorithm fails to directly incorporate the norms bounds on the inputs/outputs and the controller predefined structure.

Nonetheless, for comparison purposes and demonstrating the comparable performance of our method, we have obtained the MATLAB source code for SPOFC from the website *www.ece.umn.edu/mihailo/sofware/lqrsp*, and applied both our method and SPOFC to design sparse state feedback controllers for the randomly generated system. Fig. 2.2 depicts the results of the simulations performed using both controller design methods. As predicted, the quadratic cost of the closed loop system increases, as the the parameter λ becomes larger. Moreover, increasing the value of this parameters on the system promotes the sparsity level of feedback gain matrix. Figure 2.2 depicts the effect of the parameter λ on the performance of the closed loop system and the number of non-zero entries of the controller gain. In Fig. 2.2a the Y-axis represents percentage of the performance loss, which is defined as $(J^* - J^*_{\text{LQR}})/J^*_{\text{LQR}}$. The density level percentage of the controller gain is also shown in Fig. 2.2b when the parameter λ varies from 10^{-3} to 10.

The simulation results, demonstrated in figure 2.2, show that the SPOFC approach compromises the performance for a sparser controller in comparison to the our method. Our proposed method assures less performance loss by obtaining denser feedback controller. The disagreement between the optimal solutions of the two algorithms is mainly due to convergence to different local optima. It should also be noted the optimization parameter ρ plays an important, but different, role in adjusting the convergence properties in both of the methods. Hence, setting the parameter ρ to the same value in both optimizations may not be the most accurate choice for the comparison purposes. Moreover, the choice of the system also affects the design performance of both methods. Overall, our extensive simulation results suggest the comparable performance of both approaches. Considering the fact that our problem

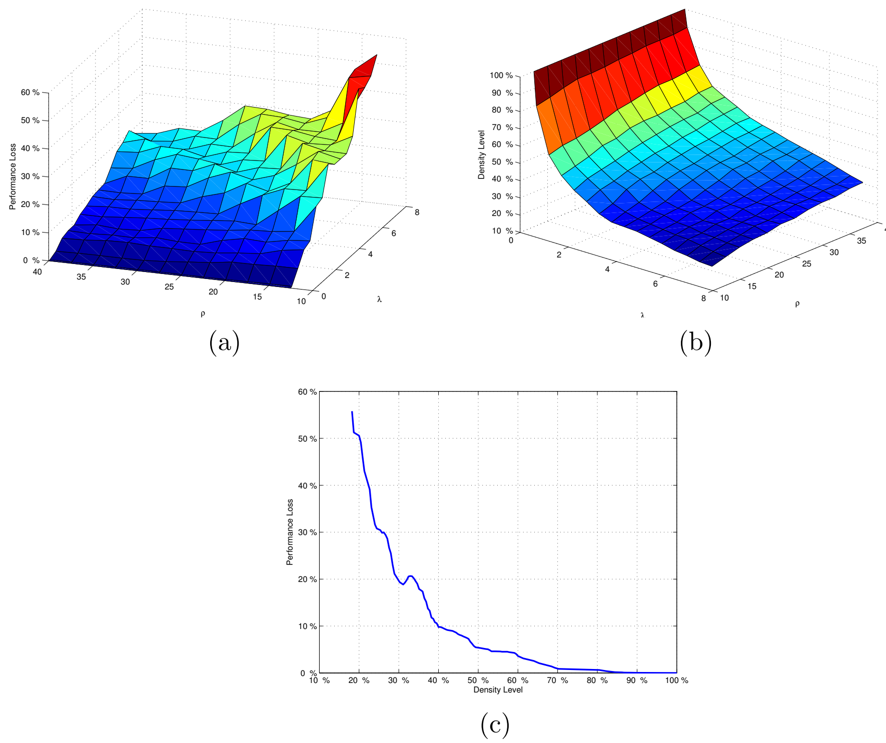


Figure 2.3: The characteristics of the sparse controller designed for a randomly generated spatially decaying system with parameters values $\{C_A = 10,\ C_B = 2,\ \alpha_A = 1,\ \alpha_B = 0.4,\ \beta_A = 3$, and $\beta_B = 0.9\}$. (a) Performance loss vs. ρ and λ, (b) Density level vs. ρ and λ (c) Density level vs. controller performance degradation

formulation and solving procedure, which is completely different from the preceding method, generates roughly the same sparse controller, it can be concluded that the derived sparse controller is likely to be the best we can obtain.

2.6.2 Sub-exponentially Spatially Decaying System

To study the effects of parameters λ and ρ on the performance of our proposed method, we have run extensive simulations on a randomly generated sub-exponentially spatially decaying system [34]. In such systems, it is assumed the entries of the system matrices decay as they get further from the diagonal, thus we define the matrices

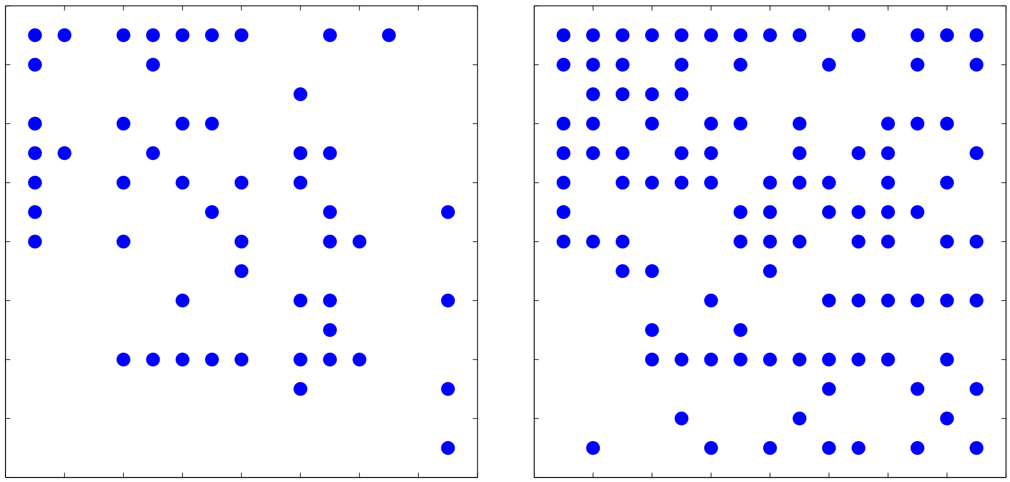
$A = [a_{ij}]$ and $B = [b_{ij}]$ as

$$\begin{cases} a_{ij} = C_A \mathfrak{a} \; e^{-\alpha_A |i-j|^{\beta_A}} \\ b_{ij} = C_B \mathfrak{b} \; e^{-\alpha_B |i-j|^{\beta_B}} \end{cases}$$

where $\mathfrak{a}$ and $\mathfrak{b}$ are uniformly distributed random variables on the open interval $(-1, 1)$. By employing Algorithm 1 till the rank constraint is satisfied, we have depicted the performance degradation and density level of the generated controllers in figure 2.3 for different values of ρ and λ. Although the proposed algorithm has converged for all choices of parameters in this simulation, It seems that the choice of the optimization parameter ρ is needed to be at least one order of magnitude larger than the parameter λ in order to guarantee the convergence to a proper sub-optimal minimum. In addition, since the main objective in designing a sparse controller is to obtain a controller with minimum number of nonzero entries and lowest performance decline, we have also presented the plot of the lowest performance loss obtained for particular values of density level in figure 2.3c. As expected, it can be observed the performance loss grows as the sprasity level of the controller increases.

2.6.3 Optimal Sparse Controller with Upper Bound Imposed on the Control Input Norm

In this example, we illustrate the effect of bounding the norm of the control input on the sparsity of the controller matrix. Considering a randomly generated 16×16 sub-exponential spatially decaying system, with the same parameter values used in section 2.6.2, we first designed a sparse controller with no constraint on the control input. Our results show that the controllers number of nonzero entries and its performance



(a) (b)

Figure 2.4: Sparsity pattern of (a) optimal sparse feedback control with no bound on the control input (b) the optimal sparse feedback controller with upper bound imposed on the system norm of the control input. Design Parameters for both figures are $Q = I$, $R = 10I$, $\lambda = 10$ and $\rho = 100$.

loss, with respect to the cost of the LQR controller which is 639.1912, are 55 and 9.3% respectively. It is also observed that for the generated controller, we have $\|u\|_{L^2_\infty(\mathbb{R}^m)} = 228.66$.

We then redesigned the controller, using the *re-weighted ℓ_1 minimization* method, by containing its control input norm in the interval $[0, 200]$, and obtained controller has the following characteristics: $\|K\|_0 = 105$ and $J = 737.16$. Although we bounded the control input norm to an approximately 10% lower value, the obtained controller demonstrates 50% less sparse pattern and 6% higher performance loss. The simulations results, depicted in figure 2.4, not only verifies the capability of our method to incorporate bounds on the control input, as well as the system output, but also reveals the adverse impact of sparsifying the controller matrix on the control input norm.

2.7 Conclusions

In this chapter, We have proposed a new framework for optimal sparse output feedback control design, which is capable of incorporating structural constraints on the feedback gain matrix as well as norm bounds on the inputs/outputs of the system. We have shown that problem can be converted to a rank constrained optimization problem with no other non-convex constraints. Using the proposed formulation, we have presented an optimization problem which yields an upper bound for the optimal value of the optimal sparse state feedback control problem. Exploiting the relaxation the ℓ_0-norm with the ℓ_1-norm, We have also expressed that local optimum of the relaxed optimization problem, in its general form, can be obtained by performing ADMM algorithm, which is, in essence, iteratively solving the relaxed problem and projecting its solution to the space of matrices with rank n. For the special case, where the objective is merely sparsity pattern recognition of the controller gain, we have demonstrated that the problem can be reduced to an Affine Rank Minimization. The simulation results are also provided to illustrate the utility and performance of our proposed approach. As compared to the results of [23], our results show that while our proposed method has the advantage of performing the output feedback control design restricted by various forms of nonlinear constraints, the performance of our approach is on a par with theirs when applied to the regular sparse state feedback controller design problem.

Chapter 3

Output Feedback Controller Sparsification via $\mathcal{H}_2$-Approximation

3.1 Introduction

The growth of large-scale dynamical systems such as power networks and transportation systems, and the impotence of traditional centralized controllers in controlling these systems have caused the problem of sparse/structured controller design to receive increasing attention over the past few years. In such control paradigms the effort is focused on synthesizing robust controllers performing as well or even better than the centralized ones, while limiting the communications to the neighboring subsystems or imposing particular structure on the underlying network.

Although numerous works have been done in the area of distributed controller design [3, 6, 1, 2, 4] a systematic approach capable of efficiently solving the general problem is yet to be developed. For some classes of systems such as spatially invariant systems and spatially decaying systems useful results on the structure of the solution space have been derived [7, 8, 34, 35]. Furthermore, several other design frameworks, each with their specific shortfalls, have also been proposed to design sparse/structured controllers for the continuous/discrete time linear time invariant systems both in time and frequency domain [28, 14, 12, 13, 9, 23, 9] The common controller design approach in the area of synthesis of distributed controllers is to minimize a cost function, defined based on the specific needs of the application and the desired performance, while stabilizing the overall system. Unlike these methods, our proposed framework in this chapter is based on the assumption that there already exists a well performing controller, like a centralized controller synthesized using conventional robust control methods [56]. We focus our attempts to find a sparse/structured controller which is capable of manifesting near-optimal performance characteristics to that of the available pre-designed controller.

We adopt concepts from mixed $\mathcal{H}_2/\mathcal{H}_\infty$ control [57, 58] to not only achieve minimum gap in the frequency characteristics of the closed loop transfer functions, but also consider the difference between the characteristics of the control signals generated by both controllers in our design framework. We show that our presented method can be reformulated into a rank constrained optimization where all non-convexities are collected into the rank constraints. Choosing the ℓ_0-measure as the measure of the controller sparsity, we have shown useful results which help transforming the problem into a conventional rank minimization problem under convex constraints. We further employ and modify the ADMM algorithm as a tool for our specific purpose of searching for the sub-optimal solution of our problem.

This chapter is structured as follows. In Section 3.2, we formally state the problem we wish to solve. In Section 3.3, we elaborate how our problem can be equivalently reformulated into an optimization problem constrained to several linear matrix inequalities and a rank constraint. Section 3.4 provides some insights into our choice of sparsity measure and the algorithm we opt for in solving out rank constrained optimization problem. The numerical results validating our theoretical results are presented in Section 3.5. Finally, Section 3.6 concludes our chapter.

Notations: Throughout the chapter, the following notations are adopted. The space of n by m matrices with real entries is indicated by $\mathbb{R}^{n\times m}$. The n by n identity matrix is denoted I_n. Operators $\mathbf{Tr}(.)$ and $\mathbf{rank}(.)$ denote the trace and rank of the matrix operands. The transpose and vectorization operators are denoted by $(.)^{\mathrm{T}}$ and $\mathbf{vec}(.)$, respectively. The Hadamard product is represented by $\circ$. A matrix is said to be Hurwitz if all its eigenvalues lie in the open left half of the complex plane. $\|.\|_0$ represents the cardinality of a vector/matrix, while $\|.\|_1$ and $\|.\|_F$ denote ℓ_1 and Frobenius norm operators. Also, the norm $\|.\|_{L_2^q(\mathbb{R}^n)}$ is defined by

$$\|x\|^2_{L_2^q(\mathbb{R}^n)} \triangleq \int_0^\infty \|x(t)\|_2^q \; dt$$

A real symmetric matrix is said to be positive definite (semi-definite) if all its eigenvalues are positive (non-negative). $\mathbb{S}^n_{++}$ ($\mathbb{S}^n_{+}$) denotes the space of positive definite (positive semi-definite) real symmetric matrices, and the notation $X \succ Y$ ($X \succeq Y$) means $X - Y \in \mathbb{S}^n_{++}$ ($X - Y \in \mathbb{S}^n_{+}$).

3.2 Problem Formulation

Let a linear time invariant (LTI) continuous-time system be given by its state space realization

$$\begin{cases} \dot{x}(t) = Ax(t) + B_1u(t) + B_2d(t) \\ y(t) = Cx(t) \end{cases}, \tag{3.1}$$

where $A \in \mathbb{R}^{n\times n}$, $B_1 \in \mathbb{R}^{n\times m_1}$, $B_2 \in \mathbb{R}^{n\times m_2}$ and $C \in \mathbb{R}^{p\times n}$. It is assumed that the pair (A, B_1) is controllable. Our goal is to design a constant gain output feedback controller

$$u(t) = Ky(t), \quad K \in \mathcal{K} \tag{3.2}$$

which achieves minimum performance difference comparing to a reference well-performing pre-designed state controller, namely $\hat{K}$, while minimizing the number of non-zero entries of the controller matrix. We, also, desire that controller to be contained in a set of admissible feedback gains with previously specified structure, denoted by $\mathcal{K}$. In the current chapter, we only consider the case where the set $\mathcal{K}$ is convex, since it reduces the complexity of the problem, and, more importantly, it covers a wide range of practical constraints on the controller that should be considered in the synthesis of the controller. For example, in some applications, it is practically infeasible to establish a feedback link between particular nodes; such limitations translate to the convex constraints that the corresponding entry of the controller gain should be zero. Other practical limitations such as upper bounds on the entries of the controller matrix, imposed by technological shortcomings, can be also be addressed by convex constraint on matrix K.

In addition, we prefer the energy level of the input/output signals, generated by the synthesized sparse controller, to be in the vicinity of that of the input/output, produced by the original controller when an input signal $d(t)$ with bounded energy is fed to the closed loop plant. Representing the closed loop systems controlled by the controllers K and $\hat{K}$ by the state space realizations $\mathcal{S}$ and $\hat{\mathcal{S}}$, respectively, we can formulate the search for controller K as the following optimization problem

$$\min_{K} \quad \|\mathcal{S} - \hat{\mathcal{S}}\|^2 + \lambda \|K\|_s \tag{3.3a}$$

$$\text{s.t.} \quad K \in \mathcal{K}, \tag{3.3b}$$

$$A + B_1 K C \text{ Hurwitz}, \tag{3.3c}$$

$$\|y_{\mathcal{S}} - y_{\hat{\mathcal{S}}}\|_{L_2^2(\mathbb{R}^p)} \leq \varepsilon_y \|d\|_{L_2^2(\mathbb{R}^{m_2})}, \tag{3.3d}$$

where $\|.\|$ is the norm defined in accordance with the objectives of the problem, $\|K\|_s$ represents a norm measuring the sparsity level of the matrix K, and λ is the regularization parameter. Moreover, the constant ε_y is chosen to be a positive real number.

The common choices for the norm used in the cost function of the optimization problem (3.3) are $\mathcal{H}_2/\mathcal{H}_\infty$ norms. However, $\mathcal{H}_\infty$ minimization problem suffers from non-uniqueness of the solution. Hence, in this chapter, we restrict our attention to the case where the similarity in frequency characteristics of the networks are measured by $\mathcal{H}_2$ norm of the difference between the systems empirical transfer functions. Before proceeding, it is helpful to note that the first term in the cost function of the optimization problem (3.3) can be simplified into the $\mathcal{H}_2$ norm of an augmented system, namely $\bar{\mathcal{S}}$, constructed by the following state space realization matrices

$$\begin{aligned}\bar{A} &= \begin{bmatrix} A + B_1 K C & 0 \\ 0 & A + B_1 \hat{K} C \end{bmatrix} \in \mathbb{R}^{2n\times 2n}, \\ \bar{B} &= \begin{bmatrix} B_2^{\mathrm{T}} & B_2^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^{2n\times m_2}, \\ \bar{C} &= \begin{bmatrix} C & -C \end{bmatrix} \in \mathbb{R}^{p\times 2n}.\end{aligned} \tag{3.4}$$

Furthermore, the constraints (3.3b-3.3d) can be equivalently cast by imposing bounds on the $\mathcal{H}_\infty$ norm of the closed loop transfer functions from $d(t)$ to $y(t)$ and $u(t)$. Therefore, we can re-formulate our problem into the $\mathcal{H}_2$ norm minimization of the augmented system subject to the performance constraints as follows

$$\begin{aligned} \min_{K} \quad & \left\|\bar{C}(sI - \bar{A})^{-1}\bar{B}\right\|_{\mathcal{H}_2}^2 + \lambda\|K\|_s \\ \text{s.t.} \quad & K \in \mathcal{K}, \\ & A + B_1 K C \text{ Hurwitz}, \\ & \|\bar{C}(sI - \bar{A})^{-1}\bar{B}\|_{\mathcal{H}_\infty} < \varepsilon_y, \end{aligned} \tag{3.5}$$

In problem (3.5), the first term in the objective function is formulated such that it captures the gap between the frequency response of the systems in the sense of $\mathcal{H}_2$ norm. Hence, it makes it possible to identify another stable network with sparser communication structure and approximately the same frequency characteristics. In contrast to the design of sparse LQR controllers, introduced in [23, 28], administering such approaches in the design of the controllers with sparse structures has the advantages of enabling us to exploit the deliverable merits in various controller synthesis strategies.

Next, we are going to describe how to formulate our problem as an optimization program with linear/bilinear matrix inequality/equality constraints. Then, we elaborate how to combine all nonlinear constraints into a rank constrained problem.

3.3 Fixed Rank Optimization Reformulation

In this section, we state some lemmas which can help us cast the constraints of the optimization problem as rank constrained linear matrices inequalities.

Lemma 3.3.1. *Assuming $\mathcal{P}$ is a Linear Time Invariant System with realization matrices (A, B, C), where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$, and the pair (A, B) is controllable; then, $\|\mathcal{P}\|^2_{\mathcal{H}_2} \leq \gamma$ if and only if there exists a positive definite matrix $X \succ 0$ such that*

$$\begin{aligned}\mathbf{Tr}(CXC^T) &< \gamma,\\ Y + Y^T + BB^T &\preceq 0,\\ \mathbf{Rank}\begin{bmatrix} X & Y \\ I_n & A^T \end{bmatrix} &= n.\end{aligned}$$

The previous lemma helps cast the $\mathcal{H}_2$-optimal sparsification problem as a rank constrained optimization problem where all nonlinear constraints are lumped into a single rank constraint. Several solving algorithms have been proposed to efficiently solve rank constrained optimization problems [59, 60, 43, 61]. Hence, we aim to make such algorithms applicable in solving our problem by collecting various forms of non-convex/combinatorial constraints into a single rank constraint.

Similar to the rank constrained reformulation of the $\mathcal{H}_2$ problem, the $\mathcal{H}_\infty$ constraints of the problem (3.5) can also be proved to be equivalent to a certain set of rank constrained LMI's. Next lemma helps us in our attempt to accommodate the $\mathcal{H}_\infty$ constraints in the framework of rank constrained optimizations.

Lemma 3.3.2. *Given $\mathcal{P}$ is a Linear Time Invariant System with realization matrices (A, B, C), where $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, and $C \in \mathbb{R}^{p\times n}$, the matrix A is Hurwitz and $\|\mathcal{P}\|_{\mathcal{H}_\infty} < \gamma$ if and only if there exists a positive definite matrix $X \succ 0$ such that*

$$\begin{bmatrix} Y + Y^T + C^TC & XB \\ B^TX & -\gamma^2 I_n \end{bmatrix} \prec 0,$$
$$\textbf{Rank} \begin{bmatrix} X & Y \\ I_n & A \end{bmatrix} = n.$$

Consequently, we can reformulate the problem (3.5) into a rank constrained problem, as described in the sequel.

Theorem 3.3.3. *The mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem (3.5) is equivalent to the following rank constrained optimization program*

$$\begin{aligned} \min_{K,\beta,\Phi} \quad & \beta + \lambda\|K\|_s \\ s.t. \quad & K \in \mathcal{K}, \\ & X_i \succ 0, \quad i = 1, 2, \\ & \begin{bmatrix} Y_1 + Y_1^T & X_1\bar{C}^T & \bar{B} \\ \bar{C}X_1 & -I_p & 0 \\ \bar{B}^T & 0 & -\varepsilon_y^2 I_{m_2} \end{bmatrix} \prec 0, \end{aligned} \tag{3.6}$$

$$
\begin{aligned}
&Y_2 + Y_2^T + \bar{B}\bar{B}^T \preceq 0,\\
&\mathbf{Tr}(\bar{C}X_2\bar{C}^T) < \beta,\\
&\mathbf{rank}(\Phi) = 2n.
\end{aligned}
$$

where

$$
\Phi = \begin{bmatrix} I & \bar{A}^T \\ X_1 & Y_1 \\ X_2 & Y_2 \end{bmatrix}
$$

Proof. Applying the results from corollary (3.3.2) and (3.3.1) to the $\mathcal{H}_2/\mathcal{H}_\infty$ norms existent in the optimization problem (3.5) yields the desired results. □

3.3.1 Bounding $\mathcal{H}_\infty$ Norm of the Control Signal

In the process of designing controllers for practical purposes such as industrial automation or guided vehicles, several implementation considerations are usually taken into account. For example, it is common to impose some constraints on the control signal generated by the controller unit. In this section, we study if it is possible to incorporate some types of constraints into our optimization problem so that the signal generated by the synthesized sparse controller is forced to stay in the vicinity of the output of the pre-designed controller. Hence, we avoid the violation of the implementation constraints, considered in the initial control design, as we sparsify the controller matrix. A sensible choice, that we plan to pursue here, is to bound the L_2^2 norm of the difference between the control signals generated by the pre-desinged and the sparse controllers for a bounded energy input signal, i.e.

$$\|u_{\mathcal{S}} - u_{\hat{\mathcal{S}}}\|_{L_2^2(\mathbb{R}^{m_1})} < \varepsilon_u^2, \tag{3.7}$$

for input signals satisfying $\|d\|_{L_2^2(\mathbb{R}^{m_2})} \leq 1$. A discussion similar to what we had for bounding the output difference can be applied to verify that the above constraint is equivalent to the following norm constraint

$$\|\bar{C}_u(sI - \bar{A}_0 - \bar{B}_K K \bar{C}_K)^{-1}\bar{B}\|_{\mathcal{H}_\infty} < \varepsilon_u^2 \tag{3.8}$$

where

$$\bar{A}_0 = \begin{bmatrix} A & 0 \\ 0 & A + B_1\hat{K}C \end{bmatrix} \in \mathbb{R}^{2n\times 2n},$$
$$\bar{B}_K = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \in \mathbb{R}^{2n\times m_1},$$
$$\bar{C}_K = \begin{bmatrix} C & 0 \end{bmatrix} \in \mathbb{R}^{p\times 2n},$$
$$\bar{C}_u = \begin{bmatrix} KC & -\hat{K}C \end{bmatrix} \in \mathbb{R}^{m_1\times 2n}.$$

The following theorem establishes an equivalent formulation of our optimal sparsification problem where all non-convex constraints are combined into a fixed rank constraint.

Theorem 3.3.4. *The mixed $\mathcal{H}_2/\mathcal{H}_\infty$ problem (3.5) with the additional constraint*

(3.8) is equivalent to the following rank constrained optimization program

$$\min_{K,\beta,\Phi} \quad \beta + \lambda \|K\|_s \tag{3.9a}$$

$$s.t. \quad K \in \mathcal{K}, \tag{3.9b}$$

$$X_i \succ 0, \quad i = 1, 2, 3, \tag{3.9c}$$

$$\begin{bmatrix} M_1 & X_1 \bar{C}^T & \bar{B} \\ \bar{C} X_1 & -\varepsilon_y^2 I_p & 0 \\ \bar{B}^T & 0 & -I_{m_2} \end{bmatrix} \prec 0, \tag{3.9d}$$

$$\begin{bmatrix} M_2 & N & \bar{B} \\ N^T & -\varepsilon_u^2 I_p & 0 \\ \bar{B}^T & 0 & -I_{m_2} \end{bmatrix} \prec 0, \tag{3.9e}$$

$$M_3 + \bar{B}\bar{B}^T \preceq 0, \tag{3.9f}$$

$$\mathbf{Tr}(\bar{C} X_3 \bar{C}^T) \leq \beta, \tag{3.9g}$$

$$\mathbf{rank}(\Phi) = 2n. \tag{3.9h}$$

where

$$M_i = X_i \bar{A}_0^T + Y_i \bar{B}_K^T + \bar{A}_0 X_i + \bar{B}_K Y_i^T, \quad i = 1, 2, 3,$$

$$N = Y_2 - X_{2(2)} (\hat{K} C)^T,$$

$$\Phi = \begin{bmatrix} I & (K \bar{C}_K)^T \\ X_1 & Y_1 \\ X_2 & Y_2 \\ X_3 & Y_3 \end{bmatrix},$$

and the positive definite matrix X_2 is partitioned into two $2n \times n$ blocks as

$$X_2 = \begin{bmatrix} X_{2(1)} & X_{2(2)} \end{bmatrix}.$$

3.3.2 Controller Sparsification with Bounded $\mathcal{H}_2$ Gap

In problem (3.5), the aim is to find a sparse approximation of the a pre-designed controller for which the closed loop system demonstrates characteristics comparable to those achieved by the original controller. As it can be seen, it is possible to regularize the norm of the gap between the systems $\mathcal{S}$ and $\hat{\mathcal{S}}$ by adjusting the value of the parameter λ. For example, as the parameter λ gets smaller, the optimization program yields less sparse controllers and, as a result, closed loop systems with more similar frequency characteristics. Due to some practical purposes, it is sometimes required that a gap larger than a certain value, measured in the sense of $\mathcal{H}_2$ norm, is not tolerated. This is equivalent to adding the inequality $\beta \leq \beta_{max}$ into the set of constraints in the optimizations (3.6) or (3.9).

Although bounding the value of β is a limiting constraint and may lead to the synthesis of a non-sparse controller, it can be advantageous in simplifying our problem formulation as described in the next theorem. The theorem essentially states that, in the presence of such a constraint, the rank constraint (3.9h) can be moved to the objective function; hence, our problem can be equivalently cast as an optimization program on a convex feasible set.

Theorem 3.3.5. *Assuming the optimization problem (3.9), with the upper bound on β and the ℓ_0 norm as the measure of sparsity, is feasible, it can be equivalently cast* *as*

$$\min_{K,\beta,\Phi} \quad \beta + \lambda\|K\|_0 + \nu\mathbf{rank}(\Phi) \tag{3.10}$$

$$s.t. \quad (3.9b) - (3.9g),$$

$$\beta \leq \beta_{max},$$

for any $\nu > \beta_{max} + \lambda m_1 p$*, where*

$$\Phi = \begin{bmatrix} I & (K\bar{C}_K)^T \\ X_1 & Y_1 \\ X_2 & Y_2 \\ X_3 & Y_3 \end{bmatrix}$$

Remark 3.3.6. *In some practical applications, minimizing the value of* β *is not of interest and we only intend to keep the* $\mathcal{H}_2$ *norm of the gap in the desired range. In such cases,* β *is eliminated from the objective function of the optimization problem, thus, only the terms representing the sparsity level of the controller and the rank of the matrix* Φ *remain to be minimized. Since* ℓ_0 *norm of the matrix* K *is identical to the rank of the diagonal matrix* $\mathbf{diag}(K)$*, the problem can be considered as a rank minimization problem subject to convex constraints. More detailed discussion can be found in [28].*

3.4 The Choice of the Sparsity Measure and a Sub-optimal Design Protocol

There are quite a number of sparsity measures of readily used in diverse areas of science. Among the functions used to measure the sparsity of matrices, ℓ_1 norm and its weighted versions, as convex relaxations of the ℓ_0 norm, [17, 22] and the references within, are definitely the most common ones and have been employed in various applications [23, 25]. Non-convex surrogates for the cardinality function, such as ℓ_q qusi-norm for $0 < q < 1$, have also received growing attention in the literature recently [34, 35, 33] However, since employing weighted ℓ_1 norm in optimization programs does not cause numerical issues, which typically emerge in ℓ_q and ℓ_0 norm minimization problems due to their non-convex and combinatorial natures, respectively, we choose to utilize weighted ℓ_1, as the measure of the sparsity of the controller gain matrix in the current chapter.

The choice of weighted ℓ_1 norm significantly reduces the complexity of our problem, since the norm is a convex function and, as a result, the only non-convexity arises in problem (3.9) becomes the rank constraint (3.9h). However, the presence of the rank constraint still makes our optimization problem computationally demanding. Although an efficient methodical algorithm to solve rank constrained problem has yet to be developed [43], there exists a number of optimization protocols which are capable of solving particular types of rank constrained problems by achieving sub-optimal solutions. In our previous paper [28], we proposed to put the Alternating Direction Method of Multipliers (ADMM), originally developed in 1970, to use to solve a rank constrained problem. The method has been proved to be useful in determining the optimal solution of large-scale optimization problem [51]; however, its convergence has not been proved for non-convex problems.

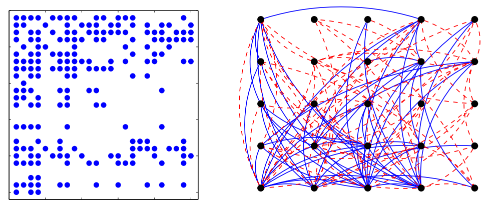
Before presenting the ADMM algorithm, it is important to note that the rank constraint in the optimization problem (3.9), can be equivalently replaced by $\mathbf{rank}(\Psi) = 2n$, where Ψ is a symmetric square matrix constructed as

$$\Psi = \begin{bmatrix} X_1 & * & * & * & * \\ Y_1^{\mathrm{T}} & - & * & * & * \\ X_2 & Y_2 & - & * & * \\ X_3 & Y_3 & - & - & * \\ I & (K\bar{C}_K)^{\mathrm{T}} & - & - & - \end{bmatrix} \tag{3.11}$$

where the entries with no particular importance are shown by "-". As discussed in [28], the rank constraint on the matrix Ψ can be relaxed by replacing it with a positive semi-definite constraint, i.e. $\Psi \succeq 0$, due to the assumption $X_1 \in \mathbb{S}^{2n}_{++}$. Denoting the feasible set of the convex optimization, obtained by relaxing the the rank constraint (3.9h), by $\mathcal{C}$, and the set of $(8n + m_1) \times (8n + m_1)$ matrices with rank equal to $2n$ by $\mathcal{S}$, the minimization problem (3.9) becomes

$$\begin{aligned} \min_{x} \quad & f(x) \\ s.t. \quad & x \in \mathcal{C} \cap \mathcal{S} \end{aligned}$$

where $f(x) = \beta + \lambda \|W \circ K\|_1$ and the weight matrix $W = [w_{ij}] \in \mathbb{R}^{m_1 \times p}$ is positive. Therefore, the program (3.9) can be carried out by repetitively performing the following steps (3.12-3.15) until either the stopping criteria is met or the maximum



(a) (b)

Figure 3.1: (a) Sparsity pattern of the optimal sparse controller (b) Graph representation of the optimal sparse controller. The Solid (dashed) lines represent Bi-directional (one-directional) links. ($\lambda = 2.5$, $\rho = 50$, $\varepsilon_y = 1.5$)

number of iterations is reached.

$$x^{k+1} = \arg\min_{x \in \mathcal{C}} \quad f(x) + (\rho/2)\|x - z^k + u^k\|_F^2 \tag{3.12}$$

$$z^{k+1} = \Pi_{\mathcal{S}}(x^{k+1} + u^k) \tag{3.13}$$

$$u^{k+1} = u^k + x^{k+1} - z^{k+1} \tag{3.14}$$

$$w_{ij}^{(k+1)} = \frac{1}{|K_{ij}^{(k)}| + \delta} \tag{3.15}$$

where $\Pi_{\mathcal{S}}$ is projection onto $\mathcal{S}$. Taking into account that the cost function and constraints of the optimization (3.12) are convex, this step can be performed numerically efficient is convex. Also, the projection on the non-convex set $\mathcal{S}$, denoted by $\Pi_{\mathcal{S}}(.)$, can be determined by the computationally effective method of carrying out Singular Value Decomposition and keeping the top $2n$ dyads.

It should be noted that the constant $\delta > 0$, chosen to be a relatively small constant, is introduced into the denominator of the update law (3.15) to guarantee the stability of the algorithm, especially when $K_{ij}^{(n)}$ turns out to be zero in the previous iteration [22]. The step (3.14) is designed to update the variable u, which is to be utilized in the next iteration. Moreover, the termination criterion is defined by $\varepsilon < \varepsilon^*$, where ε^*

is the desired precision, with the update law

$$\mathbf{max}(\|x^{k+1} - z^{k+1}\|_F, \|z^{k+1} - z^k\|_F) < \varepsilon. \tag{3.16}$$

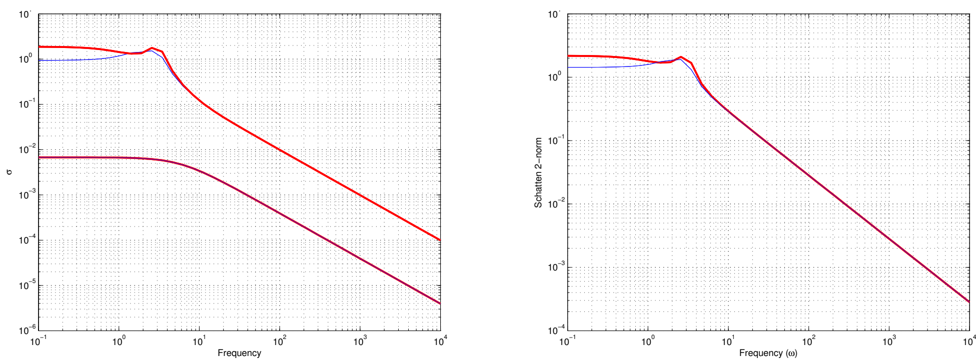
3.5 Simulation Results

Next, we apply our proposed method to a sub-exponentially decaying (SD) system to near-optimal design a sparse feedback controller in the vicinity of the corresponding LQR controller. We considered a 5×5 grid model governed by randomly generated SD system defined by the system matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ as

$$\begin{cases} a_{ij} = \mathfrak{X}\ e^{-\alpha_A d_{ij}^{\beta_A}} \\ b_{ij} = \mathfrak{X}\ e^{-\alpha_B d_{ij}^{\beta_B}} \end{cases}$$

where $\alpha_A = 1$, $\alpha_B = 0.7$, $\beta_A = 0.4$, $\beta_B = 0.7$, and $\mathfrak{X}$ is chosen to be a uniformly distributed random variable on the interval $(-20, 20)$. Also, the matrix C is assumed to be the identity matrix and d_{ij} denotes the distance between nodes i and j.

The SD model is mainly of interest since it captures the decay in the coupling weight caused by increase of the distance between nodes, which is a common phenomenon in networks such as power grids. Having synthesized the LQR controller for the weight values $Q = 10I$ and $R = I$, we set the parameter values $\lambda = 2.5$ and $\rho = 50$ and employed our algorithm, for $\varepsilon_y = 1.5$ and no bound on ε_u, to obtain an stabilizing sparse controller. It is observed that the $\mathcal{H}_2$ norm difference caused by the sparsification process is 0.89, which is around 34% of the $\mathcal{H}_2$ norm corresponding to the LQR controller. Furthermore, after truncating the controller entries smaller than



(a) (b)

Figure 3.2: Frequency characteristics of the closed loop systems controlled by the LQR (blue) and the sparse controller (red) (a) Maximum and Minimum singular values (b) Schatten 2-norm.

10^{-10}, we noticed that the number of non-zero entries of the sparse controller had dropped to $\|K\|_0 = 383$, while the same value for the LQR controller is 625; hence, there is approximately 39% decrease in the controller cardinality number.

We have depicted the sparsity pattern of the sparse controller designed by our proposed algorithm in Figure 3.1a. Fig. 3.1a is basically a visual representation of the controller gain matrix where the nonzero terms are shown by solid blue circles and the zero entries are left blank. Also, Fig. 3.1b is used to illustrate the underlying graph of the obtained controller, in which the blue lines show bi-directional communication between the nods and the dashed red lines represent one way links. It should also be noted that the obtained gap between the $\mathcal{H}_\infty$ norm of the sparse controller and the linear quadratic regulator is 1.35, which satisfies the optimization constraint.

To further illustrate the similar frequency behaviour of the systems, we have reproduced two additional plots shown in Fig. 3.2. Representing the closed loop systems controlled by the sparse and the LQR controllers as $\mathcal{S}$ and $\hat{\mathcal{S}}$, respectively, the largest and smallest singular values of both $\mathcal{S}$ and $\hat{\mathcal{S}}$ are depicted in Fig. 3.2a.

It can be seen that the smallest singular values of both system match for almost the whole frequency range and large singular values obtain the same values for higher frequencies. Also, the plot of Schatten 2-norm of the systems $\mathcal{S}$ and $\hat{\mathcal{S}}$, shown in Figure 3.2b, visually establishes the proximity of the $\mathcal{H}_2$ norm of both systems. Interestingly, it seems that the sparsification of the controller does not have any effect on the higher frequency content of the the closed loop system.

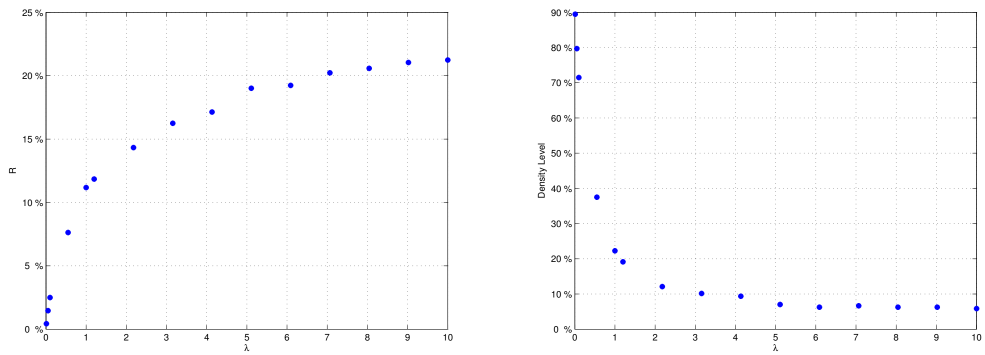
Next, we ran a series of simulations for different values of λ, ranging from 5×10^{-2} to 10, on a 4×4 lattice with the same parameters values, and depicted the results in Fig. 3.3. As predicted, the outcome of our optimization method converges to the LQR controller as λ goes to zero if the optimization heuristic is initialized at a proper point in the feasible domain. Fig. 3.3a represents the ratio of the $\mathcal{H}_2$ norm of the systems difference to the $\mathcal{H}_2$ norm of the system controlled by the LQR controller, i.e.

$$R = \frac{\|\mathcal{S} - \hat{\mathcal{S}}\|_{\mathcal{H}_2}}{\|\hat{\mathcal{S}}\|_{\mathcal{H}_2}}$$

for different values of λ. As expected, the value of R increases with the growth of the parameter λ. On the other hand, this increase in the value of parameter λ promotes the sparsity level of feedback gain as shown in Fig. 3.3b.

3.6 Conclusions

We have proposed a new approach for the design of optimal sparse controllers. Our method, basically, attempts to alter an available pre-designed controller towards a sparse controller, while heeding the performance deterioration caused by the process



(a) (b)

Figure 3.3: (a) Percentage of the $\mathcal{H}_2$ norm deviation, caused by the sparsification process, relative to the $\mathcal{H}_2$ norm of system controlled by the LQR controller (b) Density level of the controller gain obtained for various values of λ.

sparsification. By equivalently reformulating the problem into a fixed rank optimization, we could achieve a controller synthesis method, by which a sparse structured controller capable of exhibiting similar frequency and time characteristics of the pre-designed controller, in the sense of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms. Our method can also be modified to incorporate constraints on the control signal. Furthermore, we proposed the Alternating Direction Method of Multipliers (ADMM), modified to include weighted ℓ_1 norm minimization, as computationally tractable algorithm to sub-optimally solve our problem. The simulation results are also provided to illustrate the effectiveness of our proposed framework.

Chapter 4

Controller Sparsification Under Parametric Uncertainties via $\mathcal{H}_p$ Approximations

4.1 Introduction

In control theory, there has always been a desire to achieve the best possible performance, while taking into account the feasibility and cost of the communication between the subsystems. The reason behind such yearning is mostly rooted in the fact that unlike small-scale dynamical systems, where centralized control methodologies can be efficiently applied due to the availability of information from all subsystems, the subsystem level information is not globally accessible throughout the network in medium to large-scale systems. With the emergence and growth of ultra large-scale interconnected systems, e.g. power grids, transportation systems, and wireless data networks, exploiting the particulars from the underlying structure of the system in the control synthesis has become inescapable. Hence, the concept of distributed and decentralized controllers has received increasing attention in recent years [3, 6, 2, 4, 9].

It has been known that optimal controller design under controller structural constraints is a hard problem, and the structured controller synthesis is still an open problem. Nonetheless, numerous works have been done to either propose controller synthesis frameworks or reveal inherent structural properties of controllers for special classes of systems [14, 12, 13, 7, 8, 34]

Another concern in the design of control systems for large-scale systems is the number of communication links among the subsystems, which poses major issues especially when establishing links between nodes are highly costly. The synthesis of controller gains with minimum number of non-zero entries can mitigate the communication overflow issues emergent in large interconnected systems, since sparsifying the controller gain leads to fewer information pathways as well as fewer controller sensors and actuators. In the sparsity promoting control problem, the ultimate objective is to minimize the number of feedback links without losing the control performance. This is achieved by incorporating additional functions into the optimization cost function to penalize the number of communication links. The problem has been addressed by a number of researchers , who opted for various techniques to tackle the inherently non-convex problem [26, 2, 27, 23, 28, 29, 30] . For example, in [27, 23] the Alternating Direction Method of Multipliers is utilized to handle the non-convex terms in the problem formulation. In [28], the authors proposed a novel framework in which all non-convexities are lumped into a rank constraint, which enables it to address output feedback problems with norm constraints on the input/output signals.

In this chapter the proposed sparse controller synthesis framework is founded on the ground we have a pre-designed well performing controller available; then, we pivot our effort to finding a sparse/structured controller with performance characteristics resembling that of the pre-designed controller. We first formally define the problem of controller sparsfication, where the ultimate goal is to obtain a sparse feedback controller approximating the attributes and qualities of the original well-preforming controller. By providing some insights on our choice of the proximity measures used in assessing the controller approximation performance, we utilize the results from $\mathcal{H}_2$ and $\mathcal{H}_\infty$ control [62, 63, 64] to reformulate our problem into a rank constrained optimization where all non-convexities are collected into the rank constraints. Choosing the weighted ℓ_1-norm as the measure for the controller sparsity, we employ a modified ADMM algorithm to reach the sub-optimal solution of the rank constrained optimization problem. We, then, use the proposed procedure to study the trade-off between the controller sparsification rate and the system uncertainty level.

This chapter is structured as follows. In Section 4.2, we formally state the problem we aim to solve. In Sections 4.3 and 4.4, we elaborate how our problem can be equivalently reformulated into an optimization problem constrained to several linear matrix inequalities and a single rank constraint. Section 4.5 provides justifications on the choice weighted ℓ_1 norm as the sparsity measure; then, describe the ADMM algorithm used in solving our rank constrained optimization problem. The simulation results studying the effect of system uncertainties on the feedback sparsification process are presented in Section 4.6. Finally, we end with concluding remarks in Section 4.7.

Notations: Throughout this chapter, matrices are customarily named with capital letters, and the entries are named using the corresponding lower-case letters, but with subscripts. The vectors, on the other hand, are symbolized by lower-case letters, with components denoted by the same letter, but subscripted. Also, the following notations are adopted. The set of real numbers is denoted by $\mathbb{R}$. The space of n by m matrices with real entries is indicated by $\mathbb{R}^{n\times m}$. The set of real matrices with non-negative (positive) entries is represented by $\mathbb{R}_{+}^{n\times m}$ ($\mathbb{R}_{++}^{n\times m}$). The n by n identity matrix is denoted I_n.

The number of nonzero elements of a matrix is denoted by $\|.\|_0$, while $\|.\|_1$ and $\|.\|_F$ denote ℓ_1 and Frobenius norm. Also, the norm $\|.\|_{L_2^q(\mathbb{R}^n)}$ is defined by

$$\|x\|_{L_2^q(\mathbb{R}^n)}^2 \triangleq \int_0^\infty \|x(t)\|_2^q \, dt$$

Tr(.) and **rank**(.) denote the trace and rank of the operands, which are matrices. The vectorization operator is denoted by **vec**(.). The entry wise product of two matrices, i.e. Hadamard product, is represented by $\circ$. A matrix is said to be Hurwitz if all its eigenvalues lie in the open left half of the complex plane.

A real symmetric matrix is said to be positive definite (semi-definite) if all its eigenvalues are positive (non-negative). $\mathbb{S}_{++}^n$ ($\mathbb{S}_{+}^n$) denotes the space of positive definite (positive semi-definite) real symmetric matrices, and the notation $X \succeq Y$ ($X \succ Y$) means $X - Y \in \mathbb{S}_{+}^n$ ($X - Y \in \mathbb{S}_{++}^n$).

Remark 4.1.1. *For simplicity of our notations, we will use a new notation in statements of theorems, where we use symbol "*" to represent the upper triangular sub-blocks of symmetric matrices. Moreover, in the occasions when the optimal solutions of the optimization problems in these theorems do not depend on some of the sub-blocks of matrices, we use symbol "-" to represent such sub-blocks with no apparent utilization in the problem.*

4.2 Problem Formulation

4.2.1 LTI Systems with Parametric Uncertainties

Consider an uncertain linear time-invariant (LTI) continuous-time system defined by the state space realization

$$\begin{aligned} \dot{x}(t) &= [A + \Delta_A]x(t) + [B_1 + \Delta_{B_1}]u(t) + B_2 d(t) \\ y(t) &= Cx(t), \end{aligned} \tag{4.1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^{m_1}$ is the control input, and $d(t) \in \mathbb{R}^{m_2}$ represents the exogenous disturbance input. We assume that the matrices $A \in \mathbb{R}^{n\times n}$, $B_1 \in \mathbb{R}^{n\times m_1}$, $B_2 \in \mathbb{R}^{n\times m_2}$, and $C \in \mathbb{R}^{m_3\times n}$ are constant real matrices describing the dynamics of the nominal system, whereas Δ_A and Δ_{B_1} represent the parameter uncertainties of the matrices A and B_1, respectively. In this chapter, we consider a special uncertainty structure expressed by

$$\begin{bmatrix} \Delta_A & \Delta_{B_1} \end{bmatrix} = D\Delta \begin{bmatrix} E_A & E_{B_1} \end{bmatrix} \tag{4.2}$$

where D, E_A and E_{B_1} are pre-known constant real matrices with appropriate dimensions, which characterize the structure of the uncertainties, while Δ is an unknown i by j real matrix which is constrained by

$$\Delta^{\mathrm{T}}\Delta \preceq \rho^2 I_j. \tag{4.3}$$

This class of uncertain linear systems are initially reported by Petersen in papers [65, 66] and later thoroughly addressed in the paper by Khargonekar *et* al. [67].

Assumption 1. *We Assume that the pair* (A, B_1) *is controllable and* (A, C) *is detectable.*

4.2.2 Controller Sparsification via $\mathcal{H}_p$ Approximations

Suppose that a pre-designed well-performing controller, namely $\hat{K}$, is readily available and the nominal system controlled by such a controller, represented by $\hat{\mathcal{S}}$, has all the desired characteristics. The objective is to synthesize a constant gain output feedback controller of the form

$$u(t) = Ky(t), \quad K \in \mathcal{K} \tag{4.4}$$

with minimum number of non-zero entries, while minimizing the performance deterioration from that of the closed-loop system $\hat{\mathcal{S}}$. In (4.4), $\mathcal{K}$ denotes a set of admissible feedback gains which holds desirable properties such as pre-defined communication layout.

Assumption 2. *It is assumed that the set* $\mathcal{K}$ *is convex.*

This assumption not only reduces the complexity of the problem, but also it covers a wide range of real-world constraints on controllers. There exist numerous applications associated with such convexly constrained controller synthesis. For example, in power grids or multi-UAV systems, it is sometimes practically infeasible to establish a communication link between particular nodes due to the nodes distant locations or security/privacy issues in networks. There are also cases where the attenuation/amplification in certain feedback paths are upper bounded, due to technological shortcomings. Such restrictions are addressed by forcing the corresponding controller entries to be contained in a convex set.

Representing the closed loop systems controlled by the controller K by the state space realizations $\mathcal{S}$, we can formulate the search for controller K as the following optimization problem

$$\min_{K,\varepsilon_y,\varepsilon_{\mathcal{S}}} \quad \varepsilon_{\mathcal{S}} + \lambda_1\varepsilon_y + \lambda_2\|K\|_0 \tag{4.5a}$$

$$\text{s.t.} \quad K \in \mathcal{K}, \tag{4.5b}$$

$$\mathcal{S} \text{ Stable}, \tag{4.5c}$$

$$\|y_{\mathcal{S}} - y_{\hat{\mathcal{S}}}\|_{L_2^2(\mathbb{R}^{m_3})} < \varepsilon_y\|d\|_{L_2^2(\mathbb{R}^{m_2})}, \tag{4.5d}$$

$$\|\mathcal{S} - \hat{\mathcal{S}}\|^2 < \varepsilon_{\mathcal{S}}, \tag{4.5e}$$

where $\|.\|$ is the norm defined in accordance with the objectives of the problem. In this chapter, we consider the frequency response of the system $\mathcal{S} - \hat{\mathcal{S}}$ as the measure of the proximity between the systems controlled by the sparse and the pre-designed controllers; hence, we opt for the $\mathcal{H}_2$ norm to replace the norm in (4.5e). The term $\|K\|_0$, which is a quasi-norm measuring the sparsity level of the matrix K, is added to the cost function to penalize the density of the controller gain, and λ_1 and λ_2 are the regularization parameters.

In the optimization problem (4.5), the constraint (4.5e) is to ensure that the closed-loop nominal system $\hat{S}$ is well approximated by the system controlled by the sparse controller K. To enhance our approximation, we also incorporate another criterion into our design scheme, which is represented by the constraint (4.5d). This criterion enforces the output signal energy level of $\mathcal{S}$ to be in the vicinity of that of the system $\hat{\mathcal{S}}$ when a disturbance input bounded energy is fed to the closed loop plants.

As said before, the aim of this chapter is to utilize the previously defined sparse controller approximation in order to thoroughly study the effect of system uncertainties on the sparsification of stabilizing controllers. However, finding the optimal solution of the problem (4.5) is not an easy task. The next sections are dedicated to discuss the equivalent problem reformulation exploited in numerically solving our optimization problem.

4.3 Equivalent Reformulation

The first two terms in the cost function of the optimization problem (4.5) can be simplified into the $\mathcal{H}_2/\mathcal{H}_\infty$ norms of an augmented system, namely $\bar{\mathcal{S}}$, constructed by the following state space realization matrices

$$\begin{aligned}\bar{A} &= \begin{bmatrix} \bar{A}_{11} & 0 \\ 0 & A + B_1\hat{K}C \end{bmatrix}, \\ \bar{B} &= \begin{bmatrix} B_2^{\mathrm{T}} & B_2^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}, \\ \bar{C} &= \begin{bmatrix} C & -C \end{bmatrix}.\end{aligned} \tag{4.6}$$

where $\bar{A}_{11} = [A+\Delta_A]+[B_1+\Delta_{B_1}]KC$. As it can be seen, the system $\bar{\mathcal{S}}$ represents the difference between the nominal system controlled by the pre-designed controller and the uncertain system, stabilized by closing its feedback loop using a sparse controller.

Hence, we can re-formulate our problem into the $\mathcal{H}_2/\mathcal{H}_\infty$ norm minimization of the augmented system as follows

$$\begin{aligned}
\min_{K,\varepsilon_y,\varepsilon_\mathcal{S}} \max_{\Delta_A,\Delta_{B_1}} \quad & \varepsilon_\mathcal{S} + \lambda_1\varepsilon_y + \lambda_2\|K\|_0 \\
\text{s.t.} \quad & K \in \mathcal{K}, \\
& \bar{A}_{11} \text{ Hurwitz}, \\
& \|\bar{C}(sI - \bar{A})^{-1}\bar{B}\|_{\mathcal{H}_\infty} < \varepsilon_y, \\
& \left\|\bar{C}(sI - \bar{A})^{-1}\bar{B}\right\|_{\mathcal{H}_2}^2 < \varepsilon_\mathcal{S}.
\end{aligned} \tag{4.7}$$

In problem (4.7), the attempt is to minimize the worst case gap between the frequency response of the systems in the sense of a weighted sum of the $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms. Therefore, unlike the design schemes introduced in [23, 28], the approach proposed in this chapter allows us to exploit the advantages offered by other controller synthesis schemes in the sparse controller synthesis.

In the next section, we show that the optimization problem (4.7) includes bilinear matrix inequality constraints mainly due to the existence of the Lyapunov stability conditions. Here, we intend to employ the idea of lumping all nonlinear constraints into a rank constrained problem, proposed in [28], to rewrite problem as a rank constrained optimization. Based on the obtained reformulation, it is possible to either develop heuristics to sub-optimally solve the problem or provide necessary and sufficient conditions for the feasibility of the points with particular desired costs.

4.4 Fixed Rank Optimization Reformulation

The approach adopted in this chapter is based on solving the problem of sparse controller approximation via rank constrained optimization. Hence, we start by stating the main lemma which helps us cast the constraints of the optimization problem as rank constrained linear matrix inequalities.

Lemma 4.4.1 ([28]). *Let $\mathcal{U} \in \mathbb{R}^{n\times n}$, $\mathcal{V} \in \mathbb{R}^{n\times m}$, $\mathcal{W} \in \mathbb{R}^{m\times m}$, and $\mathcal{Y} \in \mathbb{R}^{m\times n}$, with $\mathcal{U} \succ 0$. Then, $\mathbf{rank}(\mathcal{M}) = n$ if and only if $\mathcal{W} = \mathcal{Y}\mathcal{U}\mathcal{Y}^T$ and $\mathcal{V}^T = \mathcal{Y}\mathcal{U}$, where*

$$\mathcal{M} = \begin{bmatrix} \mathcal{U} & \mathcal{V} \\ \mathcal{V}^T & \mathcal{W} \\ I_n & \mathcal{Y}^T \end{bmatrix}.$$

The above lemma can be utilized to collect almost all non-convex terms of the optimization problems in one and only one constraint in the form of a rank constraint. Since there are a number of algorithms proposed to efficiently solve rank constrained optimization problems [60, 59, 43, 61], we, in this chapter, target to make such algorithms applicable in solving our inherently nonlinear controller sparsification problem by collecting various forms of non-convex/combinatorial constraints into a single rank constraint.

As the first step, we show how the $\mathcal{H}_2$ norm of an uncertain system can be formulated by rank constrained linear matrix inequalities.

Lemma 4.4.2. *Given a strictly proper uncertain linear system $\mathcal{P}$ with state space realization $(\mathcal{A} + \Delta_{\mathcal{A}}, \mathcal{B}, \mathcal{C})$, where $\mathcal{A} \in \mathbb{R}^{n\times n}$, $\mathcal{B} \in \mathbb{R}^{n\times m}$, $\mathcal{C} \in \mathbb{R}^{r\times n}$, $\Delta_{\mathcal{A}} = \mathcal{D}\Delta\mathcal{E}$ and $\Delta^T\Delta \preceq \rho^2 I_j$, then $\mathcal{P}$ is stable and $\|\mathcal{P}\|^2_{\mathcal{H}_2} < \gamma$ if and only if there exists a positive* *definite matrix $\mathcal{X} \succ 0$ and a positive scalar ε such that*

$$\begin{aligned}
\mathbf{Tr}(\mathcal{C}\mathcal{X}\mathcal{C}^T) &< \gamma, \\
\begin{bmatrix} \mathcal{Y}_1 + \mathcal{Y}_1^T + \mathcal{B}\mathcal{B}^T + \varepsilon\rho\mathcal{D}\mathcal{D}^T & \sqrt{\rho}\mathcal{Y}_2 \\ \sqrt{\rho}\mathcal{Y}_2^T & -\varepsilon I_j \end{bmatrix} &\prec 0, \\
\mathbf{rank}\begin{bmatrix} \mathcal{X} & * & * & * \\ \mathcal{Y}_1^T & - & * & * \\ \mathcal{Y}_2^T & - & - & * \\ I_n & \mathcal{A}^T & \mathcal{E}^T & - \end{bmatrix} &= n.
\end{aligned}$$

Proof. The system $\mathcal{P}$ is stable with $\mathcal{H}_2$ norm less than γ if and only if there exists a positive definite matrix $\mathcal{X}$ such that [68, p. 210]

$$\begin{aligned}
\mathbf{Tr}(\mathcal{C}\mathcal{X}\mathcal{C}^{\mathrm{T}}) &< \gamma, \\
(\mathcal{A} + \Delta_{\mathcal{A}})\mathcal{X} + \mathcal{X}(\mathcal{A} + \Delta_{\mathcal{A}}) + \mathcal{B}\mathcal{B}^{\mathrm{T}} &\prec 0.
\end{aligned}$$

Substituting $\Delta_{\mathcal{A}} = \mathcal{D}\Delta\mathcal{E}$ into the second equation, we have

$$\mathcal{A}\mathcal{X} + \mathcal{X}\mathcal{A}^{\mathrm{T}} + \mathcal{B}\mathcal{B}^{\mathrm{T}} + \mathcal{D}\Delta\mathcal{E}\mathcal{X} + \mathcal{X}(\mathcal{D}\Delta\mathcal{E})^{\mathrm{T}} \prec 0.$$

Since the term $\mathcal{A}\mathcal{X} + \mathcal{X}\mathcal{A}^{\mathrm{T}} + \mathcal{B}\mathcal{B}^{\mathrm{T}}$ is symmetrical and $\Delta^{\mathrm{T}}\Delta \preceq \rho^2 I_j$, the above linear matrix inequality holds if and only if there exist a positive scalar $\varepsilon > 0$ such that [63]

$$\mathcal{A}\mathcal{X} + \mathcal{X}\mathcal{A}^{\mathrm{T}} + \mathcal{B}\mathcal{B}^{\mathrm{T}} + \varepsilon\rho\mathcal{D}\mathcal{D}^{\mathrm{T}} + \varepsilon^{-1}\rho\mathcal{X}\mathcal{E}^{\mathrm{T}}\mathcal{E}\mathcal{X} \prec 0.$$

This is equivalent to having

$$\begin{bmatrix} \mathcal{A}\mathcal{X} + \mathcal{X}\mathcal{A}^{\mathrm{T}} + \mathcal{B}\mathcal{B}^{\mathrm{T}} + \varepsilon\rho\mathcal{D}\mathcal{D}^{\mathrm{T}} & \sqrt{\rho}(\mathcal{E}\mathcal{X})^{\mathrm{T}} \\ \sqrt{\rho}(\mathcal{E}\mathcal{X}) & -\varepsilon I_j \end{bmatrix} \prec 0.$$

Applying Lemma 4.4.1, the last LMI can be equivalently rewritten as shown below.

$$\begin{bmatrix} \mathcal{Y}_1 + \mathcal{Y}_1^{\mathrm{T}} + \mathcal{B}\mathcal{B}^{\mathrm{T}} + \varepsilon\rho\mathcal{D}\mathcal{D}^{\mathrm{T}} & \sqrt{\rho}\mathcal{Y}_2 \\ \sqrt{\rho}\mathcal{Y}_2^{\mathrm{T}} & -\varepsilon I_j \end{bmatrix} \prec 0,$$

$$\mathbf{rank} \begin{bmatrix} \mathcal{X} & \mathcal{Y}_1 & \mathcal{Y}_2 \\ I_n & \mathcal{A}^{\mathrm{T}} & \mathcal{E}^{\mathrm{T}} \end{bmatrix} = n.$$

Augmenting proper rows and columns to the rank constrained matrix to make it symmetric completes our proof. □

Similar to Lemma 4.4.2, which paves the way in casting the $\mathcal{H}_2$ norm term in our optimal controller sparsification problem, as a rank constrained optimization problem, the $\mathcal{H}_\infty$ norm term of the problem (4.7) can also be equivalently represented with a set of rank constrained linear matrix inequalities. In the next lemma, we prove such equivalence, which later helps in accommodating the whole problem of controller sparsification into the framework of rank constrained optimization.

Lemma 4.4.3. *Suppose a strictly proper uncertain LTI plant $\mathcal{P}$, represented in the state space triplet $(\mathcal{A}+\Delta_{\mathcal{A}}, \mathcal{B}, \mathcal{C})$, where $\mathcal{A} \in \mathbb{R}^{n\times n}$, $\mathcal{B} \in \mathbb{R}^{n\times m}$, $\mathcal{C} \in \mathbb{R}^{r\times n}$, $\Delta_{\mathcal{A}} = \mathcal{D}\Delta\mathcal{E}$ and $\Delta^T\Delta \preceq \rho^2 I_j$, then the system is stable with $\mathcal{H}_\infty$ norm less than γ if and only if* *there exists a positive definite matrix* $\mathcal{X} \succ 0$ *and a positive scalar* $\varepsilon > 0$ *satisfying*

$$\begin{bmatrix} \mathcal{Y}_1 + \mathcal{Y}_1^T + \varepsilon\rho\mathcal{D}\mathcal{D}^T & * & * & * \\ \mathcal{B}^T & -\gamma^2 & * & * \\ (\mathcal{C}\mathcal{X}) & 0 & -I_r & * \\ \sqrt{\rho}\mathcal{Y}_2^T & 0 & 0 & -\varepsilon I_j \end{bmatrix} \prec 0.$$

$$\textbf{rank} \begin{bmatrix} \mathcal{X} & * & * & * \\ \mathcal{Y}_1^T & - & * & * \\ \mathcal{Y}_2^T & - & - & * \\ I_n & \mathcal{A}^T & \mathcal{E}^T & - \end{bmatrix} = n.$$

Proof. Employing Lemma 7.4 in [68, p. 221], the plant $\mathcal{P}$ is stable with $\|\mathcal{P}\|_\infty < \gamma$ if and only if there exists a positive definite matrix $\mathcal{Z}$ such that

$$\mathcal{Z}(\mathcal{A} + \Delta_\mathcal{A}) + (\mathcal{A} + \Delta_\mathcal{A})^\mathrm{T}\mathcal{Z} + \mathcal{C}^\mathrm{T}\mathcal{C} + \gamma^{-2}\mathcal{Z}\mathcal{B}\mathcal{B}^\mathrm{T}\mathcal{Z} \prec 0.$$

Pre and Post multiplying the above LMI by the inverse of $\mathcal{Z}$, namely $\mathcal{X}$, we have

$$(\mathcal{A} + \Delta_\mathcal{A})\mathcal{X} + \mathcal{X}(\mathcal{A} + \Delta_\mathcal{A})^\mathrm{T} + \mathcal{X}\mathcal{C}^\mathrm{T}\mathcal{C}\mathcal{X} + \gamma^{-2}\mathcal{B}\mathcal{B}^\mathrm{T} \prec 0,$$
$$\mathcal{A}\mathcal{X} + \mathcal{X}\mathcal{A}^\mathrm{T} + \gamma^{-2}\mathcal{B}\mathcal{B}^\mathrm{T} + \mathcal{X}\mathcal{C}^\mathrm{T}\mathcal{C}\mathcal{X} + \Delta_\mathcal{A}\mathcal{X} + \mathcal{X}\Delta_\mathcal{A}^\mathrm{T} \prec 0.$$

Plugging $\Delta_\mathcal{A} = \mathcal{D}\Delta\mathcal{E}$ into the previous inequality, we get

$$\begin{aligned} \mathcal{A}\mathcal{X} + \mathcal{X}\mathcal{A}^\mathrm{T} + &\gamma^{-2}\mathcal{B}\mathcal{B}^\mathrm{T} + \mathcal{X}\mathcal{C}^\mathrm{T}\mathcal{C}\mathcal{X} \\ &+ (\mathcal{D}\Delta\mathcal{E})\mathcal{X} + \mathcal{X}(\mathcal{D}\Delta\mathcal{E})^\mathrm{T} \prec 0, \end{aligned}$$

$$\mathcal{A}\mathcal{X} + \mathcal{X}\mathcal{A}^{\mathrm{T}} + \gamma^{-2}\mathcal{B}\mathcal{B}^{\mathrm{T}} + \mathcal{X}\mathcal{C}^{\mathrm{T}}\mathcal{C}\mathcal{X} \\ + \rho\mathcal{D}(\Delta/\rho)\mathcal{E}\mathcal{X} + \rho\mathcal{X}\mathcal{E}^{\mathrm{T}}(\Delta^{\mathrm{T}}/\rho)\mathcal{D}^{\mathrm{T}} \prec 0.$$

Having $(\Delta^{\mathrm{T}}/\rho)(\Delta/\rho) \leq I_j$, the above inequality is valid for all acceptable values of Δ if and only if there exists a positive $\varepsilon > 0$ for which the following holds.

$$\mathcal{A}\mathcal{X} + \mathcal{X}\mathcal{A}^{\mathrm{T}} + \gamma^{-2}\mathcal{B}\mathcal{B}^{\mathrm{T}} + \mathcal{X}\mathcal{C}^{\mathrm{T}}\mathcal{C}\mathcal{X} \\ + \varepsilon\rho\mathcal{D}\mathcal{D}^{\mathrm{T}} + \varepsilon^{-1}\rho\mathcal{X}\mathcal{E}^{\mathrm{T}}\mathcal{E}\mathcal{X} \prec 0.$$

It can be observed that the last LMI is the Schur complement of the negative definite constraint, shown below.

$$\begin{bmatrix} \mathcal{A}\mathcal{X} + \mathcal{X}\mathcal{A}^{\mathrm{T}} + \varepsilon\rho\mathcal{D}\mathcal{D}^{\mathrm{T}} & * & * & * \\ \mathcal{B}^{\mathrm{T}} & -\gamma^2 & * & * \\ (\mathcal{C}\mathcal{X}) & 0 & -I_r & * \\ \sqrt{\rho}\mathcal{E}\mathcal{X} & 0 & 0 & -\varepsilon I_j \end{bmatrix} \prec 0.$$

The rest of the proof is straightforward; hence, omitted. □

Consequently, we can reformulate the problem (4.7) into a rank constrained problem, as described in the sequel.

Theorem 4.4.4. *The optimization problem (4.7) is equivalent to the following rank constrained optimization program*

$$\min_{K,\varepsilon_y,\varepsilon_{\mathcal{S}}} \varepsilon_{\mathcal{S}} + \lambda_1\varepsilon_y + \lambda_2\|K\|_0 \tag{4.8}$$

$$s.t. \quad K \in \mathcal{K},$$

$$X_p \succ 0, \quad p = 1, 2,$$

$$\varepsilon_p > 0, \quad p = 1, 2,$$

$$\mathbf{Tr}(\bar{C} X_1 \bar{C}^T) < \varepsilon_{\mathcal{S}},$$

$$\begin{bmatrix} P_1 + \bar{B}\bar{B}^T & * \\ \sqrt{\rho} Y_2^T & -\varepsilon_1 I_j \end{bmatrix} \prec 0,$$

$$\begin{bmatrix} P_2 & * & * & * \\ \bar{B}^T & -\varepsilon_y^2 I_{m_2} & * & * \\ (\bar{C} X_2) & 0 & -I_{m_3} & * \\ \sqrt{\rho} Y_4^T & 0 & 0 & -\varepsilon_2 I_j \end{bmatrix} \prec 0,$$

$$\mathbf{rank}(M_1) = 2n,$$

where

$$P_p = Y_{2p-1} + Y_{2p-1}^T + \varepsilon_p \rho \bar{D}\bar{D}^T, \qquad p = 1, 2,$$

$$M_1 = \begin{bmatrix} X_1 & * & * & * & * \\ Y_1^T & - & * & * & * \\ Y_2^T & - & - & * & * \\ X_2 & Y_3 & Y_4 & - & * \\ I_{2n} & A_{cl}^T & E_{cl}^T & - & - \end{bmatrix},$$

$$A_{cl} = \begin{bmatrix} A + B_1 K C & 0 \\ 0 & A + B_1 \hat{K} C \end{bmatrix} \in \mathbb{R}^{2n \times 2n},$$

$$\bar{D} = \begin{bmatrix} D \\ 0 \end{bmatrix} \in \mathbb{R}^{2n \times j},$$

$$E_{cl} = \begin{bmatrix} E_A + E_{B_1}KC & 0 \end{bmatrix} \in \mathbb{R}^{j\times 2n}.$$

Proof. It can be observed that the closed-loop system can be represented using the state representation $(A_{cl} + \bar{\Delta}_A, \bar{B}, \bar{C}, 0)$, where $\bar{B}$ and $\bar{C}$ are defined in (4.6) and $\bar{\Delta}_A = \bar{D}\Delta E_{cl}$. Therefore, applying the results from lemmas 4.4.2 and 4.4.3 yields the desired result. □

Now, the next corollary is immediate.

Corollary 4.4.5. *The optimization problem (4.7) can be equivalently cast as the following rank constrained optimization program*

$$\begin{aligned}
&\min_{K,\varepsilon_y,\varepsilon_{\mathcal{S}}} \varepsilon_{\mathcal{S}} + \lambda_1\varepsilon_y + \lambda_2\|K\|_0 && (4.9)\\
&s.t. \quad K \in \mathcal{K},\\
&\qquad X_p \succ 0, \quad p = 1, 2,\\
&\qquad \varepsilon_p > 0, \quad p = 1, 2,\\
&\qquad \mathbf{Tr}(\bar{C}X_1\bar{C}^T) < \varepsilon_{\mathcal{S}},\\
&\qquad \begin{bmatrix} Q_1 + \bar{B}\bar{B}^T + \varepsilon_1\rho\bar{D}\bar{D}^T & * \\ \sqrt{\rho}R_1 & -\varepsilon_1 I_j \end{bmatrix} \prec 0,\\
&\qquad \begin{bmatrix} Q_2 + \varepsilon_2\rho\bar{D}\bar{D}^T & * & * & * \\ \bar{B}^T & -\varepsilon_y^2 I_{m_2} & * & * \\ (\bar{C}X_2) & 0 & -I_{m_3} & * \\ \sqrt{\rho}R_2 & 0 & 0 & -\varepsilon_2 I_j \end{bmatrix} \prec 0,\\
&\qquad \mathbf{rank}(M_2) = 2n,
\end{aligned}$$

where

$$
\begin{aligned}
Q_p &= X_p A_o^T + A_o X_p + Y_p B_K^T + B_K^T Y_p^T, \quad p = 1, 2 \\
R_p &= E_o X_p + E_{B_1} Y_p^T, \quad p = 1, 2 \\
M_2 &= \begin{bmatrix} X_1 & * & * & * \\ Y_1^T & - & * & * \\ X_2 & Y_2 & - & * \\ I_{2n} & (KC_K)^T & - & - \end{bmatrix}, \\
A_o &= \begin{bmatrix} A & 0 \\ 0 & A + B_1 \hat{K} C \end{bmatrix} \in \mathbb{R}^{2n \times 2n}, \\
\bar{D} &= \begin{bmatrix} D \\ 0 \end{bmatrix} \in \mathbb{R}^{2n \times j}, \\
E_o &= \begin{bmatrix} E_A & 0 \end{bmatrix} \in \mathbb{R}^{j \times 2n}, \\
C_K &= \begin{bmatrix} C & 0 \end{bmatrix} \in \mathbb{R}^{m_3 \times 2n}, \\
B_K &= \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \in \mathbb{R}^{2n \times m_1}.
\end{aligned}
$$

4.5 Convex Relaxations and Numerical Algorithms

For the simulation purposes, we plan to solve the problem formulated in (4.9). The terms in our optimization problem are all convex except the density penalizing term in the cost function and the rank constraint. This section is dedicated to shed light on our approach in dealing with the non-convex and combinatorial terms. It breaks into two parts: (i) Choice of Sparsity Measure (ii) An ADMM Computational Algorithm.

4.5.1 Choice of Sparsity Measure

Since the ℓ_0 quasi-norm is an integer function, utilizing it in our formulation introduces the complications of combinatorial optimization. In order to reduce the complexity sparse vector/matrix recovery problems, quite a number of sparsity measures have been proposed and commonly used in various areas of science. The ℓ_1 norm and its weighted versions, as convex surrogates of the ℓ_0 quasi-norm, are among the most common functions used to measure the sparsity, and have been utilized in diverse applications [23, 25]. Recently, non-convex substitutes for the cardinality function, such as ℓ_q quasi-norm for $0 < q < 1$, have also received increasing consideration in the literature [34, 35]. However, since utilization of weighted ℓ_1 norm in optimization programs does not cause numerical issues, occurring in the cases of ℓ_q and ℓ_0 quasi-norms, we choose to employ weighted ℓ_1 norm to penalize the density of the controller gain in this chapter.

The choice of weighted ℓ_1 norm notably reduces the complexity of our problem; consequently, the only apparent non-convexity in the problem (4.9) becomes the rank constraint. However, the presence of the rank constraint still makes solving our optimization problem computationally challenging. Although no efficient algorithm to solve the general rank constrained problem has been developed yet [43], there exist a number of optimization protocols which enable the solving of particular types of rank constrained problems by obtaining sub-optimal solutions [46, 45].

4.5.2 An ADMM Computational Algorithm

In paper [28], it is proposed to make use of Alternating Direction Method of Multipliers (ADMM) in order to solve the rank constrained problems, and the effectiveness of such a method has been demonstrated through simulations. As discussed in the paper, the rank constraint on the matrix M_2 can be relaxed by replacing it with a positive semi-definite constraint, i.e. $M_2 \succeq 0$, due to the assumption $X_1 \in \mathbb{S}^{2n}_{++}$. Denoting the feasible set of the convex optimization, obtained by relaxing the the arisen rank constraint, by $\mathcal{N}$, and the set of $(6n+m_1) \times (6n+m_1)$ matrices with rank equal to $2n$ by $\mathcal{M}$, the minimization problem (4.9) becomes

$$\begin{aligned} \min_{\Theta} \quad & f(\Theta) \\ s.t. \quad & \Theta \in \mathcal{N} \cap \mathcal{M}, \end{aligned}$$

where Θ represents the collection of corresponding optimization variables,

$$f(\Theta) = \varepsilon_{\mathcal{S}} + \lambda_1 \varepsilon_y + \lambda_2 \|W \circ K\|_1,$$

and the weight matrix $W = [w_{pq}] \in \mathbb{R}^{m_1 \times m_3}$ is positive. Hence, the program (4.9) can be implemented by repetitively executing the following steps (4.10-4.13) until either the stopping criteria is satisfied or the maximum number of iterations is reached.

$$\Theta^{(k+1)} = \arg\min_{\Theta \in \mathcal{N}} \quad f(\Theta) + \frac{\lambda}{2} \|\Theta - \Gamma^{(k)} + \Lambda^{(k)}\|_F^2. \tag{4.10}$$

$$\Gamma^{(k+1)} = \Pi_{\mathcal{M}}(\Theta^{(k+1)} + \Lambda^{(k)}). \tag{4.11}$$

$$\Lambda^{(k+1)} = \Lambda^{(k)} + \Theta^{(k+1)} - \Gamma^{(k+1)}. \tag{4.12}$$

$$w_{pq}^{(k+1)} = \frac{1}{|k_{pq}^{(k)}| + \delta}, \quad \forall p, q. \tag{4.13}$$

Taking into account that the cost function and constraints of the optimization (4.10) are convex, this step can be performed numerically efficient. Also, the projection on

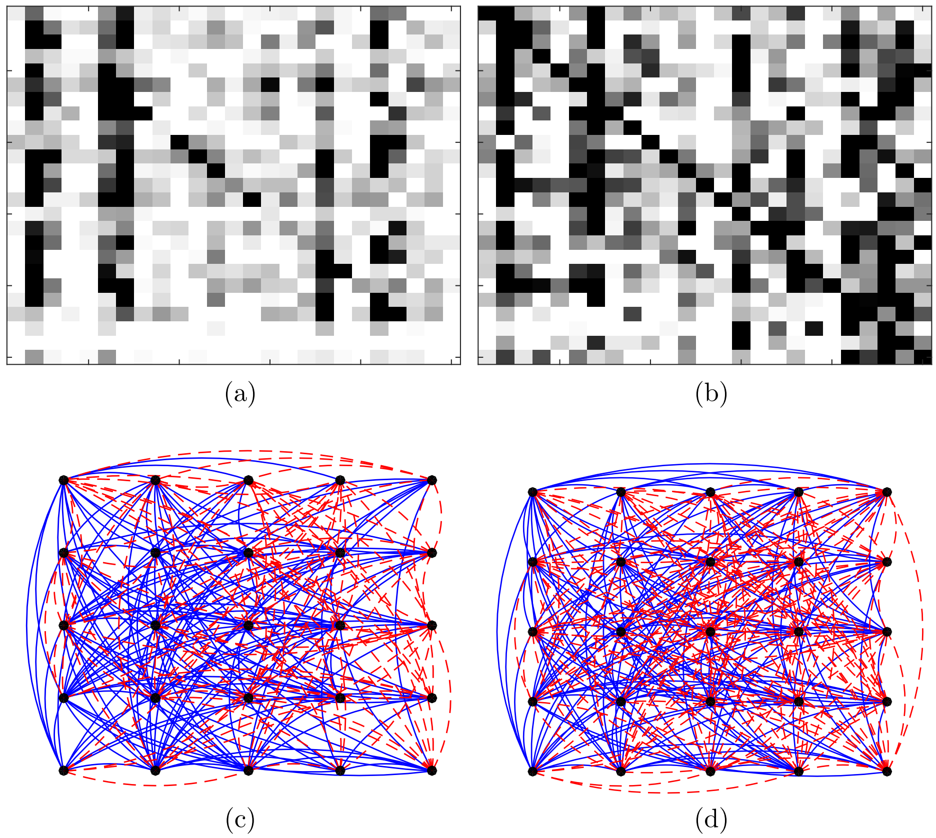


Figure 4.1: Sparsity patterns and graph representations of the synthesized sparse controllers: The left and right sub-figures correspond to the cases of $\rho = 0$ and $\rho = 5$, respectively.

the non-convex set $\mathcal{M}$, denoted by $\Pi_{\mathcal{M}}(.)$, can be determined by the computationally efficient method of executing Singular Value Decomposition and keeping the top $2n$ dyads.

It is worth to note that the constant $\delta > 0$ which is chosen as a relatively small constant, is augmented to the denominator of the update law (4.13) to guarantee the stability of the algorithm, especially, when $k_{pq}^{(k)}$ turns out to be zero in the previous iteration [22]. The step (4.12) is designed to update the variable Λ, which is to be utilized in the next iteration. Moreover, the stopping criteria is defined by $\varepsilon^{(k+1)} < \varepsilon^{*}$,

Algorithm 1: Solution to problem (4.8)

Inputs: A, B, C, Q, R, λ_2, $\mathcal{K}$, λ, δ and ε^*

1: *Initialization:*
Find $\Theta^{(0)}$ by solving (4.10) for $\lambda_2 = 0, \lambda = 0$ (LQR),
Set $\Gamma^{(0)} = \Theta^{(0)}$, $\Lambda^{(0)} = 0$, $\varepsilon^{(0)} > \varepsilon^*$, and $k = 0$,
2: **While** $\varepsilon^{(k)} \leq \varepsilon^*$ **do**
3: Update $\Theta^{(k+1)}$ by solving (4.10),
4: Update $\Gamma^{(k+1)}$ using Eq. (4.11),
5: Update $\Lambda^{(k+1)}$ using Eq. (4.12),
6: Update $W^{(k+1)}$ using Eq. (4.13),
7: Update $\varepsilon^{(k+1)}$ using Eq. (4.14),
8: $k \leftarrow k + 1$,
9: **end while**
10: Truncate K,
Output: K

where ε^* is the desired precision, with the update law

$$\varepsilon^{(k+1)} = \mathbf{max}\left(\frac{\|\Theta^{(k+1)} - \Gamma^{(k+1)}\|_F}{\|\Gamma^{(k+1)}\|_F}, \frac{\|\Gamma^{(k+1)} - \Gamma^{(k)}\|_F}{\|\Gamma^{(k+1)}\|_F}\right). \tag{4.14}$$

The step-by-step procedure is described in Algorithm 1.

4.6 Simulation Results

In this section, we aim to study the impact of the choice of the parameters λ_1 and λ_2 on the controller sparsification process. To this end, we consider sub-exponentially decaying systems by defining the system matrices $A = [a_{pq}]$ and $B = [b_{pq}]$ as follows

$$\begin{cases} a_{pq} = \mathfrak{X}_A \; exp(-\alpha_A d_{pq}^{\beta_A}) \\ \\ b_{pq} = \mathfrak{X}_B \; exp(-\alpha_B d_{pq}^{\beta_B}) \end{cases}$$

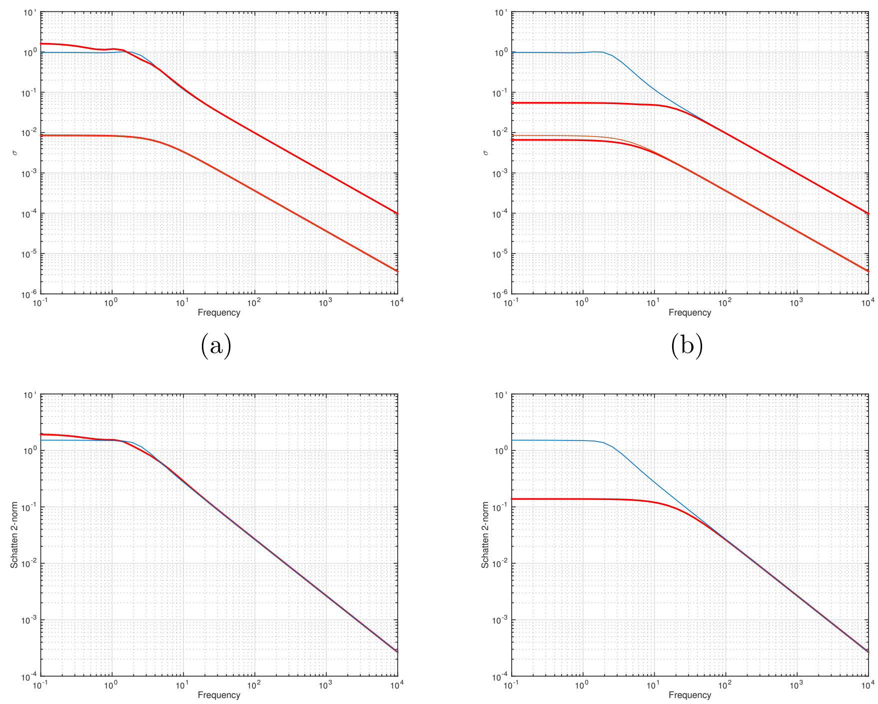
where $\mathfrak{X}_A$ and $\mathfrak{X}_B$ are randomly generated matrices with entries from the uniform distributions $[-a_{max}, a_{max}]$ and $[-b_{max}, b_{max}]$, respectively. Also, the parameters d_{pq}'s denote the distance between nodes p and q, i.e.

$$d_{pq} = \sqrt{(x_p - x_q)^2 + (y_p - y_q)^2}.$$

The SD model is mainly of interest. Because, it captures the decay in the coupling weight caused by increase of the distance between nodes, which is a common fact in networks such as power grids.

First, we consider a 5×5 grid model governed by randomly generated SD system, where $\alpha_A = 1$, $\alpha_B = 4$, $\beta_A = 0.4$, $\beta_B = 0.7$, $a_{max} = 10$, and $b_{max} = 2$. Utilizing our ADMM algorithm, we design two near optimal sparse state feedback controllers in the vicinity of the corresponding LQR controller, one for the case of system with no uncertainties, i.e. $\rho = 0$, and one for the uncertain system with $\rho = 5$ (around 59% for a norm of the system's A matrix). It should be noted that the reference LQR controller was synthesized for the weights $Q = 10I$ and $R = I$, and the controller designs were conducted by setting the parameters values $\lambda_1 = 0.25$, $\lambda_2 = 10$, and $\lambda = 50$.

It is observed that for the case of system with no uncertainties the $\mathcal{H}_2$ norm difference caused by the sparsification process is 0.74, which is about 45% of the $\mathcal{H}_2$ norm corresponding to the LQR controller. In addition, truncating the controller entries smaller than 10^{-7} reduces the number of non-zero entries of the controller to $\|K\|_0 = 407$, while the corresponding value for the LQR one is equal to 625; hence, there is approximately 34.88% decrease in the controller cardinality number. As for the case of the uncertain system with $\rho = 5$, the controller density is around



(a) (b) (c) (d)

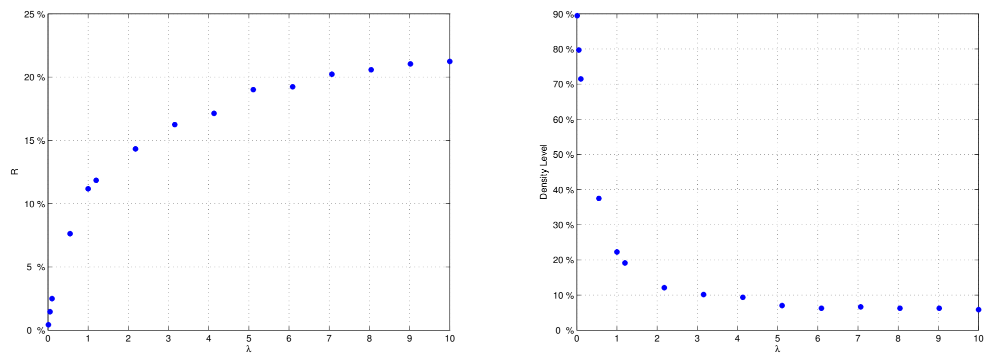
Figure 4.2: Frequency characteristics of the closed loop systems controlled by the LQR (blue) and the sparse controller (red). The upper left and right sub-figures depict maximum and minimum singular values for the cases of $\rho = 0$ and $\rho = 5$, respectively. Lower sub-figures exhibit the Schatten 2-norm of the system ((c) nominal case (d) uncertain case)

72.16%, which shows a reduction of 7.04% in the controller sparsity. Also, the $\mathcal{H}_2$ norm deterioration is raised to approximately 92.32%. These results confirms our prediction that the growth of uncertainties magnitudes in the system has negative effects on the sparsification process.

To further illustrate our results, we exhibit the sparsity pattern of the synthesized sparse controllers in figures 4.1a and 4.1b. The figures are essentially the visualizations of the controller matrices where the brightness of each square is inversely proportional to the absolute value of its corresponding entry. In addition, we plot the underlying graph of the obtained controllers, in which the blue lines show bi-directional communication links between the nodes and the dashed red lines illustrate one way links.

Furthermore, additional plots are presented in Fig. 4.2 to show the similarity of the frequency behaviour of the sparse systems to that of the LQR control system. The upper left sub-figure, i.e. Fig. 4.2a, depicts the largest and smallest singular values of $\mathcal{S}$ and $\hat{\mathcal{S}}$ for the case of $\rho = 0$. It can be seen that the smallest singular values of the systems matches for almost the whole frequency range and large singular values achieve the same values for higher frequencies. Similar plots for the case of uncertain system with $\rho = 5$ are shown in Fig. 4.2a. The plots show that the deviation of the maximum eigenvalue, cause by increasing the magnitude of the uncertainties, is much larger in comparison with the deviation of the minimum eigenvalue. Also, the plots of Schatten 2-norm of the systems $\mathcal{S}$ and $\hat{\mathcal{S}}$, are presented in lower sub-figures of 4.2 for both nominal and uncertain cases. Interestingly, in neither of the cases, the sparsification process does not seem to affect the higher frequency content of the closed loop systems.

Moreover, we run a series of simulations to study the effect of the magnitude of the parameters λ_2 and ρ on our sparsisification method. First, we consider a randomly generated 4×4 lattice with no uncertainty. Then, we apply our controller synthesis algorithm for different values of λ_2, ranging from 5×10^{-2} to 10, while keeping ε_y less than the unit. The results are shown in Fig. 4.3. As predicted, the outcome of our controller synthesis method converges to the LQR controller as λ_2 goes to zero if the initialization of optimization heuristic is done appropriately at a point in the feasible region. Fig. 4.3a demonstrates the ratio of the $\mathcal{H}_2$ norm of the systems difference to



(a) (b)

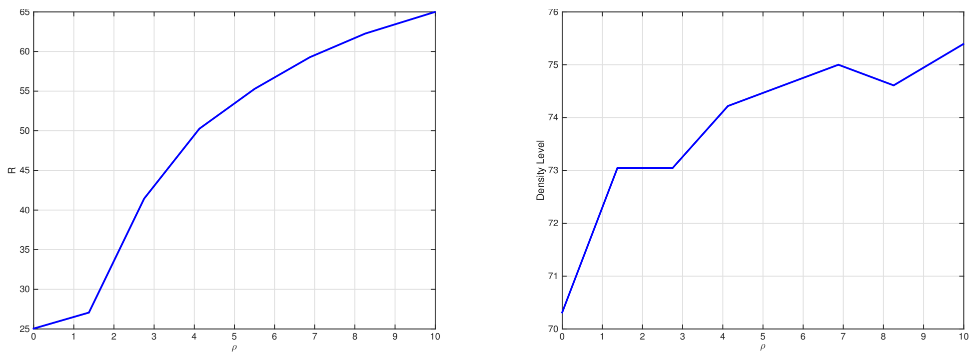
Figure 4.3: (a) Percentage of the $\mathcal{H}_2$ norm deviation, caused by the sparsification process, relative to the $\mathcal{H}_2$ norm of system controlled by the LQR controller (b) Density level of the controller gain obtained for different values of λ_2.

the $\mathcal{H}_2$ norm of the system controlled by the LQR controller, i.e.

$$R = \frac{\|\mathcal{S} - \hat{\mathcal{S}}\|_{\mathcal{H}_2}}{\|\hat{\mathcal{S}}\|_{\mathcal{H}_2}}$$

for different values of λ_2. It is not surprising that the value of R increases with the growth of the parameter λ_2. On the other hand, this increase in the value of parameter λ_2 boosts the sparsity level of feedback gain matrix as depicted in Fig. 4.3b.

Next, we fix the parameter λ_2 and sweep the uncertainty magnitude ρ from zero to 10% of the norm of the system matrix A. The simulation parameters are chosen as $E_A = I$, $E_{B_1} = 0$, $\lambda_1 = 0.25$, $\lambda_2 = 10$, and $\lambda = 100$. Similar to the previous simulation setups, we consider a 4×4 grid with dynamics defined by a randomly generated SD system with parameters exactly the same as the one used before, i.e. $\alpha_A = 1$, $\alpha_B = 4$, $\beta_A = 0.4$, $\beta_B = 0.7$, $a_{max} = 10$, and $b_{max} = 2$. The results, shown in Fig. 4.4, corroborate our initial conjecture that larger upper bounds on the



(a) (b)

Figure 4.4: (a) Ratio of the $\mathcal{H}_2$ norm deviation to the $\mathcal{H}_2$ norm of $\hat{\mathcal{S}}$ (b) Density level of the controller gain ($\lambda_1 = 0.25$, $\lambda_2 = 10$, and $\rho \in [0, 0.1] \times \|A\|_2$.)

norm of the additive uncertainty leads to denser stabilizing feedback controllers. This is mainly due to the fact that for a fixed nominal system, the controllers designed for larger uncertainties belong to the set of controllers which stabilize the systems with smaller uncertainty level. Therefore, for large enough λ_2's, the optimal sparse controller designed for the smaller uncertainty are sparser than the ones synthesized for uncertainties with larger magnitudes.

4.7 Conclusions

We have proposed a new approach for the design of optimal sparse controllers. This method is developed based on altering an available pre-designed controller towards a sparse controller, while heeding the performance deterioration caused by the process sparsification as well as the parameter uncertainties in the system. We start with formulating an optimization problem which seeks a sparse structured controller capable of exhibiting similar frequency and time characteristics of the pre-designed controller, in the sense of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms. By equivalently reformulating the problem into a fixed rank optimization, we propose to utilize the Alternating Direction Method of Multipliers (ADMM), modified to include weighted ℓ_1 norm minimization, as a computationally tractable algorithm to sub-optimally solve our problem. The simulation results are also provided to demonstrate the effect of various parameters values in performance of our proposed method. On average, the results reveal that while the increase of the weight on the density penalizing term in the cost function enhances the sparsity promoting properties of our method, the growth of the uncertainty level has adverse effects on the sparsity level of the synthesized controller.

Chapter 5

Optimal State Feedback Controllers with Strict Row Sparsity Constraints

5.1 Introduction

In recent years, there has been an increasing interest in sparse and decentralized controller synthesis motivated by the advent of large scale systems and challenges in implementation of centralized controllers for such systems. In control systems, consisting of large number of sub-systems, the underlying structure of the controller is usually restricted due to implementation-related concerns and issues such as actuators/sensors limitations or communication costs. Since the optimal controller structure is not always available, one approach is to sparsify the controller gain in order to obtain the best viable control system by minimizing the number of communication links between the subsystems, while improving the overall closed loop performance.

However, the sparsification of the controller matrix does not always deliver the desired structure/performance, as a matrix can be sparse while having a row or column with no zero entry which translates into having a node connected to all other nodes. Therefore, it is also of importance to consider the row/columns sparsity of the controller gain.

In general, the problem of designing state feedback controllers with entries constrained to lie in a pre-specified set is NP-hard.[36]. Recently, a number of endeavors has been made to identify the structural properties of the stabilizing controllers in particular systems. Bamieh et al., in [6], tackled the control problem for the class of spatially invariant systems with quadratic performance criteria and showed that the problem of optimal controller for spatially invariant systems, with funnel causality properties, can be cast as convex problem [7]. In [8], Motee et al. attempted to derive optimal sparse controller by using spatial truncation techniques; they also introduced novel sparsity measures for a broad class of spatially distributed systems and categorizes the largest class of spatially distributed systems for which their corresponding quadratically optimal controllers inherent spatial decay property. Localized LQR optimal control problem is formulated in [12, 13] where the authors drive its analytic solution for a distributed system with non-scalar subsystems. Furthermore, [4] studies design of the optimal state feedback gains subject to structural constraints on the distributed controller. As to the problem of sparse controller syntheis, a framework for sparsity promoting and design sparse and block sparse feedback gains that minimize the quadratic cost of distributed systems were developed in [23], in which Alternating Direction Method of Multipliers (ADMM) is exploited to provide a solving algorithm to the inherently non-convex problem. Arastoo et al. [28, 64] offered an alternate formulation for the problem of sparse output feedback control for a LTI system where all non-linear constraints where encapsulated in one constraint on the rank. Regarding optimal decentralized control for discrete time systems with norm constraints on the input and output, the paper [9] and [11] have proposed a novel approach to address the problem, where authors cast the problem as rank one optimization problem and put forward heuristics to solve it.

In this chapter, we mainly consider the problem of optimal row sparse state feedback controller synthesis, where the rows of the desired controller are all s-sparse. We start by adopting the results from the majorization theorem presented in [69] to convert the problem of s-sparse vector recovery into an optimization problem, which is convex except for a constraint on the rank of a matrix variable. We, then, exploit our results to propose a novel, and more importantly exact, reformulation of the optimal row s-sparse controller synthesis problem. Besides, we develop an extension to our reformulation method in order to cover the problem of optimal row sparse controller design. Subsequently, we put forward a bi-linear optimization problems, which provides a necessary and sufficient condition for the existence of a row s-sparse stabilizing controller for a given system. We, then, provide an algorithm capable of solving the row sparsity problems, and, further, state some results on the optimality of the solutions yielded by our proposed algorithm.

Our results differ from that of [23, 2, 70], for we present a disparate approach to solve the optimal sparse state feedback control problem in the sense that we do not employ convex surrogates for the ℓ_0 norm of the controller matrix, which are typically utilized to simplify the inherently combinatorial problem. In addition, the notion of row sparsity in [2, 70] where the sparsity of state feedback is intended; however, this chapter investigates the problem of improving the sparsity of the rows of feedback gain matrix.

The rest of this chapter is organized as follows. In section 5.3, we define the row s-sparse feedback synthesis optimization problem. Section 5.5 is devoted to studying the problem of recovering a s-sparse vector from a set determined by convex constraints. In section 5.6, we reformulate our row sparse control problem as a rank constrained problem. In section 5.7, we develop an algorithm to solve our rank constrained optimization problem. Our numerical results are presented in section 5.8. Section 5.9 summarizes our results.

5.2 Preliminaries and Notations

Throughout this chapter, matrices are customarily named with capital letters, and the entries are named using the corresponding lower-case letters, but with subscripts. The vectors, on the other hand, are symbolized by lower-case letters, with components denoted by the same letter, but subscripted. Also, the following notations are adopted. The set of real numbers is denoted by $\mathbb{R}$. The space of n by m matrices with real entries is indicated by $\mathbb{R}^{n\times m}$. The set of real matrices with non-negative (positive) entries is represented by $\mathbb{R}_{+}^{n\times m}$ ($\mathbb{R}_{++}^{n\times m}$). The n by n identity matrix is denoted I_n.

Definition 5.2.1. *An $n \times 1$ vector x is s-sparse, if at most $s < n$ elements of X are non-zero.*

The s-sparse vector ξ^s is defined by

$$\xi^s \triangleq [\overbrace{0, \cdots, 0}^{n-s}, \overbrace{1, \cdots, 1}^{s}]^{\mathrm{T}}$$

The number of nonzero elements of matrix X is shown by $\|X\|_0$. The ℓ_1 norm of a vector $x \in \mathbb{R}^n$ is defined by

$$\|X\|_1 := \sum_{i=1}^{n} |x_i|.$$

Definition 5.2.2. *The row sparsity measure of the m by n matrix X is defined by*

$$\|X\|_{r-0} = \max_{1 \leq i \leq m} \sum_{j=1}^{n} |X_{ij}|^0$$

The trace and rank of matrix X is represented by $\mathbf{Tr}(X)$ and $\mathbf{rank}(X)$.

A matrix is said to be Hurwitz if all its eigenvalues lie in the open left half of the complex plane. A real symmetric matrix is said to be positive definite (semi-definite) if all its eigenvalues are positive (non-negative). $\mathbb{S}^n_{++}$ ($\mathbb{S}^n_{+}$) denotes the space of positive definite (positive semi-definite) real symmetric matrices, and the notation $X \succeq Y$ ($X \succ Y$) means $X - Y \in \mathbb{S}^n_{+}$ ($X - Y \in \mathbb{S}^n_{++}$). For simplicity of our notations, we will use a new notation in statements of Theorems, 4, 6, 7, and 8. In these occasions, we use symbol "*" to represent the upper triangular sub-blocks of symmetric matrices M_i for $i = 1, ..., 4$. Moreover, the optimal solutions of the optimization problems in these theorems do not depend on some of the sub-blocks of matrices M_i for $i = 1, ..., 4$. In such occasions, we use symbol "-" to represent such sub-blocks with no apparent utilization in the problem.

Definition 5.2.3. *For any column vector $x = [x_1, x_2, \cdots, x_n]^T \in \mathbb{R}^n$, we denote the* *increasing rearrangement of the vector* x *by* $x_{\uparrow}$*, that is, if*

$$x_{[1]} \leq x_{[2]} \leq \cdots \leq x_{[n]}$$

denote the components of x *in increasing order, then*

$$x_{\uparrow} = [x_{[1]}, x_{[2]}, \cdots, x_{[n]}]^T.$$

5.3 Problem Formulation

We consider the following class of linear time-invariant systems

$$\dot{x}(t) = Ax(t) + Bu(t), \tag{5.1}$$

where $x(t) \in \mathbb{R}^{n\times 1}$ is the state vector, $u(t) \in \mathbb{R}^{m\times 1}$ is the control input, and matrices A and B have appropriate dimensions. It is assumed that initial condition of the system $x(0) = x_0$ is a random variable with standard normal distribution. We aim to design an optimal stabilizing constant gain state feedback controller

$$u(t) = Kx(t), \quad K \in \mathcal{K}, \tag{5.2}$$

where $K \in \mathbb{R}^{m\times n}$ with rows satisfying the s-sparse condition, i.e. $\|K\|_{r-0} \leq s$. We further assume the set of all acceptable feedback gains with predefined structure, denoted by $\mathcal{K}$, is convex. The convexity assumption on $\mathcal{K}$ not only reduces the complexity of the problem, but it is also capable of addressing a wide span of real-world constraints on controllers. There exist numerous applications associated with such convexly constrained controller synthesis. For example, in power grids or multi-UAV systems, it is sometimes practically infeasible to establish a communication link between particular nodes due to the nodes distant locations or technological/security/privacy issues in networks. There are also cases where the attenuation/amplification in certain feedback paths are upper bounded, due to technological shortcomings. Such restrictions is addressed by forcing the corresponding controller entries to be contained in a convex set. The s-sparsity of the controller gain rows is of importance in cases where there is a hard limit on the number of channels providing feedback to the states. One example is a platoon of cars where each car can at most communicate with s neighbors due to wireless communication limitations [71]. To obtain the desired controller, we formulate the following optimization problem

$$\begin{aligned} \min_{x,K} \quad & J = \mathbb{E}\{\int_0^\infty [x(t)^{\mathrm{T}} Q x(t) + u(t)^{\mathrm{T}} R u(t)] dt\} && \textbf{(P1)} \\ \text{s.t.} \quad & \dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0, \\ & u(t) = Kx(t), \qquad K \in \mathcal{K}, \\ & \|K\|_{r-0} \leq s, \end{aligned}$$

where $Q \in \mathbb{S}_+^n$ and $R \in \mathbb{S}_{++}^m$ are performance weight matrices, and x_0 is the initial state.

5.4 Equivalent Formulation

Assuming that the components of the initial state x_0 are independent Gaussian random variables with zero mean and positive definite covariance matrix N, i.e. $x_0 \in \mathcal{N}(0, N)$, the well-known techniques in control systems theory can be exploited to reveal that the optimal controller, which minimizes the expected value of the cost function, can be obtained by solving the following optimization problem [72, 28].

$$\min_{\substack{X_{11}, X_{12}, \\ X_{22}, K}} \mathbf{Tr}[QX_{11}] + \mathbf{Tr}[RX_{22}] \tag{5.4a}$$

$$\text{s.t.} \quad AX_{11} + X_{11}A^{\mathrm{T}} + BX_{12}^{\mathrm{T}} + X_{12}B^{\mathrm{T}} + N = 0, \tag{5.4b}$$

$$X_{11} \succ 0, \tag{5.4c}$$

$$K \in \mathcal{K}, \tag{5.4d}$$

$$X_{22} = KX_{11}K^{\mathrm{T}}, \tag{5.4e}$$

$$X_{12}^{\mathrm{T}} = KX_{11}, \tag{5.4f}$$

$$\|K\|_{r-0} \leq s, \tag{5.4g}$$

where $X_{11} \in \mathbb{S}^{n}_{++}$, $X_{12} \in \mathbb{R}^{n \times m}$, and $X_{22} \in \mathbb{S}^{m}_{+}$. As it can be observed, unlike the constraints (5.4b)-(5.4d), which are convex, the constraints (5.4e)-(5.4g) are either nonlinear or combinatorial; thus, the problem is non-convex. Applying the results from Lemma III.1 in [28], we can equivalently replace the nonlinear constraints (5.4e)-(5.4f) with the rank constraint

$$\mathbf{rank}\begin{bmatrix} X_{11} & X_{12} \\ X_{12}^{\mathrm{T}} & X_{22} \\ I_n & K^{\mathrm{T}} \end{bmatrix} = n.$$

However, having two disparate non-convex constraints in an optimization problem can cause serious convergence issue; therefore, it is preferable to boil down our controller design problem into an optimization program constrained to only one non-convex constraint. In the following sections, we develop a novel equivalent rank constraint representation of the row sparsity condition, which enables us to incorporate the combinatorial constraint (5.4g) into the rank constraint. Hence, we can rewrite our problem into an optimization problem, where the constraints are all convex except a single rank constraint.

5.5 s-Sparse Vector Recovery

Before proceeding to the rank constrained representation of the optimal row sparse control problem, we study the problem of recovering a s-sparse vector from a convex set, stated in (5.5), whose results are directly applicable to our controller design problem. Let us consider the following feasibility problem

$$\begin{aligned} \text{find} \quad & x \\ \text{s.t.} \quad & \|x\|_0 \leq s, \end{aligned} \tag{5.5}$$

$$Ax = b,$$
$$Ex < f.$$

where $A \in \mathbb{R}^{m\times n}$, $E \in \mathbb{R}^{p\times n}$, and the vectors x, b, f have appropriate dimensions. We prove that this inherently NP-hard problem can be equivalently characterized as a rank constrained program.

Definition 5.5.1. *We say the vector x weakly super-majorize y, written as $x \overset{W}{\succeq} y$, if and only if*

$$\sum_{1}^{k} x_{[i]} \geq \sum_{1}^{k} y_{[i]} \quad \textit{for} \;\; k = 1, \cdots, n.$$

To establish a link between weak super-majorization and s-sparsity of a vector, we state the following lemma which provides a necessary and sufficient condition for the s-sparisty of a non-negative vector.

Lemma 5.5.2. *For any non-negative vector x, $\|x\|_0 \leq s$ if and only if there exist $\alpha > 0$ such that the vector ξ^s satisfies $\xi^s \overset{W}{\succeq} \alpha x$*

Proof. ($\Leftarrow$) For the non-negative vector x, $\xi^s \overset{\mathrm{W}}{\succeq} \alpha y$ results

$$0 \leq \sum_{i=1}^{q} \alpha x_{[i]} \leq 0, \qquad q = 1, \cdots, n-s$$

Hence, $x_{[i]} = 0,\ 1, \cdots, n-s$, i.e. $\|x\|_0 \leq s$.

($\Rightarrow$) The vectors x and αx have the same number of non-zero components. Thus, it suffices to set the parameter α to a value less than or equal to $\|x\|_\infty^{-1}$ in order to satisfy $\xi^s \overset{\mathrm{W}}{\succeq} \alpha x$. □

Definition 5.5.3. *A real valued function* $\phi : \mathbb{R} \to \mathbb{R}_+$ *is positive definite if* $\phi(0) = 0$ *and for any nonzero* $x \in \mathbb{R}$, $\phi(x) > 0$.

Definition 5.5.4. *A square matrix* $P = [p_{ij}]$ *is said to be* doubly α-super-stochastic *if there exist a doubly stochastic matrix* D *and a non-negative matrix* Q *such that*

$$P = \alpha D + Q.$$

Lemma 5.5.2 is only valid for non-negative vectors. However, its applicability domain can be expanded to include all real valued vectors using the following theorem.

Theorem 5.5.5. *A real valued vector* x *satisfies* $\|x\|_0 \leq s$ *if and only if there exist* $\alpha > 0$ *such that*

$$\xi^s \overset{W}{\succeq} \alpha \left[\phi_1(x_1), \phi_2(x_2), \cdots, \phi_n(x_n)\right]^T \tag{5.6}$$

where ϕ_i*'s are positive definite functions. Furthermore, the weak super-majorization (5.6) can be equivalently cast as*

$$\xi^s = P\left[\phi_1(x_1), \phi_1(x_2), \cdots, \phi_n(x_n)\right]^T, \tag{5.7}$$

where P *is a doubly* α*-super-stochastic matrix.*

Proof. The weak majorization relation (5.6) is equivalent to the s-sparsity of the vector $\alpha\left[\phi_1(x_1), \phi_2(x_2), \cdots, \phi_n(x_n)\right]^{\mathrm{T}}$. Since the function ϕ_i's are assumed to be positive definite, at least $n - s$ components of vector x are zeros, which implies $\|x\|_0 \leq s$.

The second part of the lemma follows from the theorem stating that for non-negative vectors x and y, the $x \overset{\mathrm{W}}{\succeq} y$ is equivalent to $x = \bar{P}y$ where $\bar{P} = \bar{D} + \bar{Q}$

for some doubly stochastic matrix $\bar{D}$ and a non-negative matrix $\bar{Q}$ [69]. Hence, the relation (5.6) is identical to

$$\xi^s = P\left[\phi_1(x_1), \phi_1(x_2), \cdots, \phi_n(x_n)\right]^{\mathrm{T}},$$

with $P = \alpha(D + \tilde{Q})$, where D is a doubly stochastic matrix and $\tilde{Q} \in \mathbb{R}^n_+$. Since α is positive, introducing $Q = \alpha\tilde{Q} \in \mathbb{R}^n_+$ reveals that P is a doubly α-super-stochastic matrix. This concludes our proof. □

Remark 5.5.6. *The choices of ϕ's should not be necessarily the same, and can be arbitrarily chosen. Among the positive definite functions, a preferable choice can be $\phi_i(x) = x^2$, $i = 1, \cdots, n$, since it is positive definite and both continuous and differentiable.*

Remark 5.5.7. *The class of doubly α-super-stochastic matrices is convex, since such matrices can be characterized by the following linear equalities.*

$$\begin{aligned} &P = D + Q, \quad D, Q \in \mathbb{R}^{n\times n}_+, \\ &\textstyle\sum_i d_{ij} = \alpha, \quad i = 1, \cdots, n, \\ &\textstyle\sum_j d_{ij} = \alpha, \quad j = 1, \cdots, n, \end{aligned}$$

Using lemma 5.5.5, we are now able to assert the following theorem, which helps transform problem (5.5) into a nonlinear optimization problem. The next lemma can be utilized to reformulate the non-linear constraints into a single rank constraint through the following lemma.

Lemma 5.5.8 ([28]). *Let $U \in \mathbb{R}^{n\times n}$, $V \in \mathbb{R}^{n\times m}$, $W \in \mathbb{R}^{m\times m}$, and $Y \in \mathbb{R}^{m\times n}$,* *where U is a full rank matrix. Then,* $\mathbf{rank}(M) = n$ *if and only if* $W = YUY^T$ *and* $V^T = YU$*, where*

$$M = \begin{bmatrix} U & V \\ V^T & W \\ I_n & Y^T \end{bmatrix}.$$

Proof. Since $\mathbf{rank}(U) = n$, its inverse exists and the matrix M can be decomposed as

$$M = \begin{bmatrix} I_n & 0 \\ \begin{bmatrix} V^{\mathrm{T}} \\ I_n \end{bmatrix} U^{-1} & I_{m+n} \end{bmatrix} \bar{M} \begin{bmatrix} I_n & U^{-1}V \\ 0 & I_m \end{bmatrix},$$

where

$$\bar{M} = \begin{bmatrix} U & 0 \\ 0 & \begin{bmatrix} W \\ Y^{\mathrm{T}} \end{bmatrix} - \begin{bmatrix} V^{\mathrm{T}} \\ I_n \end{bmatrix} U^{-1}V \end{bmatrix}.$$

Since the matrices pre/post-multiplied by the matrix $\bar{M}$ are full rank, the matrix M is rank n if and only if the rank of the matrix $\bar{M}$ is n, which is equivalent to

$$\begin{bmatrix} W \\ Y^{\mathrm{T}} \end{bmatrix} - \begin{bmatrix} V^{\mathrm{T}} \\ I_n \end{bmatrix} U^{-1}V = 0_{2n+m}.$$

This completes the proof of the lemma. □

Applying the above lemma, we can state the following theorem.

Theorem 5.5.9. *The program (5.5) is equivalent to the following rank constraint problem.*

$$\begin{aligned} \textit{find} \quad & x \\ \textit{s.t.} \quad & \xi^s = \mathbf{vec}(\textstyle\sum_j s_{ij}) \\ & P = D + Q, \quad D, Q \in \mathbb{R}_+^{n\times n}, \\ & \textstyle\sum_i d_{ij} = \alpha, \quad i = 1, \cdots, n, \\ & \textstyle\sum_j d_{ij} = \alpha, \quad j = 1, \cdots, n, \\ & Ax = b, \quad Ex < f, \quad \alpha > 0, \\ & \mathbf{rank}(M_1) = n, \end{aligned} \tag{5.8}$$

where

$$M_1 = \begin{bmatrix} I_n & * & * & * \\ \mathbf{diag}(x) & R & * & * \\ P & - & - & * \\ R & - & S^T & - \end{bmatrix}$$

is symmetric.

Proof. Applying lemma 5.5.8 to the rank constraint $\mathbf{rank}(M) = n$, we obtain $R = \mathbf{diag}(x)\mathbf{diag}(x)$ and $S = PR$. Thus, we have

$$\xi^s = \mathbf{vec}(\textstyle\sum_j s_{ij}) = P\left[x_1^2, x_2^2, \cdots, x_n^2\right]^{\mathrm{T}}.$$

which is equivalent to $\|x\|_0 \leq s$ according to the equation (5.7) of Lemma 5.5.5. This concludes our proof. □

The next result is essentially a corollary of Lemma 5.5.5, and is stated without proof

Corollary 5.5.10. *The real valued vector x satisfies $\|x\|_0 \leq s$ if and only if there exist $\alpha > 0$, $\beta > 0$, non-negative z_i's, $i = 1, \cdots, n$, and a doubly α-super-stochastic matrix P such that*

$$\begin{cases} \xi^s = P\left[z_1, z_2, \cdots, z_n\right]^T \\ \beta x_i^2 \leq z_i, \quad i = 1, \cdots, n \end{cases},$$

The previously presented corollary helps us reduce the dimension of the rank constrained matrix by providing an equivalent quadratically constrained reformulation of (5.8), in which all constraints are convex except the rank constraint on the matrix of the size $2n$ by $n + 1$.

Theorem 5.5.11. *The program (5.8) is equivalent to the following rank constraint problem.*

$$\begin{aligned} \textit{find} \quad & x && && (5.9) \\ \textit{s.t.} \quad & P = D + Q, && D, Q \in \mathbb{R}_+^{n\times n}, \\ & \sum\nolimits_i d_{ij} = \alpha, && i = 1, \cdots, n, \\ & \sum\nolimits_j d_{ij} = \alpha, && j = 1, \cdots, n, \\ & Ax = b, \quad Ex < f, \quad \alpha > 0, \\ & \beta x_i^2 \leq z_i, && i = 1, \cdots, n, \end{aligned}$$

$$\mathbf{rank}(M_2) = n,$$

where β is a positive real number, $z \in \mathbb{R}^{n\times 1}$, and

$$M_2 = \begin{bmatrix} I_n & * & * \\ z^T & - & * \\ P & \xi^s & - \end{bmatrix}$$

Proof. Lemma 5.5.8 asserts that the rank constraint $\mathbf{rank}(M) = n$ is equivalent to $\xi^s = Pz$. From corollary 5.5.10, it can be deduced that $\xi^s = P\left[z_1, z_2, \cdots, z_n\right]^{\mathrm{T}}$, along with $\beta x_i^2 \leq z_i,\ i = 1, \cdots, n$, is equivalent to $\|x\|_0 \leq s$, since P is a doubly α-super-stochastic matrix. Hence, the programs (5.9) and (5.5) are identical. □

In summary, we have shown that the problem of s-sparse vector recovery from a convex set, characterized by linear constraints, can be alternatively formulated by a rank constrained feasibility problem. Rank constrained optimizations are still NP-hard, however, several solving algorithms have been proposed to efficiently solve such problems [46, 43, 61, 59, 60]. Therefore, by lumping various forms of non-convex/combinatorial constraints into a single rank constraint, one of such algorithms come in handy in providing a solution to such inherently non-convex problems. In the next section, we describe how the results obtained so far can be applied to solve the problem of designing an optimal s-sparse feedback controller for linear time invariant systems.

5.6 Optimal Row s-Sparse Controller Design

In this section, we apply the results, obtained for the recovery of s-sparse vectors, to reformulate our row sparse optimal control problem as a rank constrained optimization. Unlike the existing methods for development of sparse/row-sparse controllers, where convex surrogates for the sparsity penalizing terms are often employed, our proposed reformulation, stated in the next theorem, is exact.

Theorem 5.6.1. *The optimization problem (***P1***) can be equivalently cast as follows*

$$\min_{M,\alpha} \ \mathbf{Tr}[QX_{11}] + \mathbf{Tr}[RX_{22}] \tag{5.10a}$$

$$s.t. \quad AX_{11} + X_{11}^T A^T + BX_{12}^T + X_{12}B^T + N = 0, \tag{5.10b}$$

$$X_{11} \succ 0, \tag{5.10c}$$

$$K \in \mathcal{K}, \tag{5.10d}$$

$$\beta k_{ij}^2 \leq z_{ji}, \qquad \forall i,j, \tag{5.10e}$$

$$T_{(ci)}^i = \xi^s, \qquad i = 1, \cdots, m, \tag{5.10f}$$

$$P^i \ \textit{doubly } \alpha\textit{-super-stochastic}, \quad i = 1, \cdots, m, \tag{5.10g}$$

$$\mathbf{rank}(M_3) = n, \tag{5.10h}$$

where

$$M_3 = \begin{bmatrix} X_{11} & * & * & * & * & \cdots & * \\ X_{12}^T & X_{22} & * & * & * & \cdots & * \\ I_n & K^T & - & * & * & \cdots & * \\ - & - & Z^T & - & * & \cdots & * \\ P^1 & - & - & T^1 & - & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ P^m & - & - & T^m & - & \cdots & - \end{bmatrix} \tag{5.11}$$

is a symmetric matrix, $T^i_{(ci)}$ *denotes the i-th column of the matrix* T^i*, and* $\beta \in \mathbb{R}_{++}$.

Proof. From lemma 5.5.8, it is straightforward to verify that the rank constraint in the optimization program (5.13) is identical to having $T^i = P^i Z$ for $i = 1, \cdots, m$. Hence, the equality constraints $T^i_{(ci)} = \xi^s$ are the same as

$$\xi^s = P^i \left[z_{1i}, z_{2i}, \cdots, z_{ni}\right]^{\mathrm{T}},$$

with P^i's are doubly α-super-stochastic; Considering the quadratic constraints of problem (5.13), we can write

$$\begin{cases} \xi^s = P^i \left[z_{1i}, z_{2i}, \cdots, z_{ni}\right]^{\mathrm{T}} \\ \beta k_{ij}^2 \leq z_{ji}, \qquad j = 1, \cdots, n \end{cases},$$

which, in view of corollary (5.5.10), is equivalent to the constraint $\|K_{(ri)}\|_0 \leq s$. The rest of the proof is straightforward. $\square$

5.6.1 Extension to Row Sparse Controller Synthesis Problem

Next, we generalize problem (**P1**) by adding a term representing the maximum row cardinality of the controller matrix. The proposed modification makes the problem distinct from the one introduced previously in the sense that the former problem formulation is devised to accept the controllers maximum row sparsity measure as a design parameter, while the later minimizes it by penalizing the non-zero elements in each row of the controller gain. Hence, we have the following optimization problem

$$\begin{aligned}
\min_{x,s,K} \quad & J = \int_0^\infty [x(t)^{\mathrm{T}} Q x(t) + u(t)^{\mathrm{T}} R u(t)] dt + \lambda s && (\mathbf{P2}) \\
\text{s.t.} \quad & \dot{x}(t) = Ax(t) + Bu(t), \qquad x(0) = x_0, \\
& u(t) = Kx(t), \qquad K \in \mathcal{K}, \\
& \|K\|_{r-0} \leq s,
\end{aligned}$$

where Q and R have appropriate dimensions, and $\lambda > 0$ is the regularization parameter. Having the parameter s as an optimization variable brings more complexity to the problem since it only admits positive integer values less than or equal to the number of columns of the controller matrix. Furthermore, exploiting the results derived in the previous section is not straightforward, since not only ξ^s is not a constant vector, but also its entries belong to the set $\{0, 1\}$, which makes the corresponding equality constraint combinatorial. Nonetheless, as stated by the next theorem, this problem can be cast as a rank constrained problem likewise.

Theorem 5.6.2. *The optimization problem (*$\mathbf{P2}$*) is equivalent to the following rank* *constrained program*

$$
\begin{aligned}
\min_{G,s,\alpha} \quad & \mathbf{Tr}[QX_{11}] + \mathbf{Tr}[RX_{22}] + \lambda s && && (5.13) \\
s.t. \quad & AX_{11} + X_{11}^T A^T + BX_{12}^T + X_{12}B^T + N = 0, \\
& X_{11} \succ 0, \\
& K \in \mathcal{K}, \\
& \sum_{i=1}^{n} d_{ii} = n, \\
& \sum_{i=1}^{n} e_i = s, \\
& \beta k_{ij}^2 \leq z_{ji}, && \forall i, j, \\
& e_i = (c_{ii}+1)/2, \quad |c_{ii}| \leq 1 && i = 1, \cdots, n, \\
& T^i_{(ci)} = e, && i = 1, \cdots, m, \\
& P^i \textit{ doubly } \alpha\textit{-super-stochastic}, && i = 1, \cdots, m, \\
& \mathbf{rank}(M_4) = n,
\end{aligned}
$$

where

$$M_4 = \begin{bmatrix} X_{11} & * & * & * & * & * & * & \cdots & * \\ X_{12}^T & X_{22} & * & * & * & * & * & \cdots & * \\ I_n & K^T & - & * & * & * & * & \cdots & * \\ - & - & Z^T & - & * & * & * & \cdots & * \\ - & - & C & - & - & * & * & \cdots & * \\ C & - & - & - & - & D & * & \cdots & * \\ P^1 & - & - & T^1 & - & - & - & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ P^m & - & - & T^m & - & - & - & \cdots & - \end{bmatrix}$$

is a symmetric matrix, the matrices C and D are diagonal, $\beta > 0$, and $T^i_{(ci)}$ denotes the i-th column of the matrix T^i.

Proof. The proof follows from the fact that ξ^s can be represented by the following equations.

$$\begin{aligned} &\sum_{i=1}^{n} q_i^2 = n, \\ &|q_i| \leq 1, \quad i = 1, \cdots, n, \\ &\xi_i^s = \frac{q_i + 1}{2}, \\ &\sum_{i=1}^{n} \xi_i^s = s. \end{aligned}$$

There, implementing the above equations into the constraints and the rank constraint yields the desired result. □

5.7 Algorithm for Optimal s-Sparse Controller Synthesis

In this section, we present our algorithm to solve the rank constraint optimization problem. Our approach is based on relaxing the rank constraint by a positive semi-definite constraint, and, then, introducing an additional term penalizing the rank of the matrix into the optimization cost function. The obtained optimization problem is all convex except for the additional bi-linear term added to compensate the rank constraint relaxation, hence, several optimization methods that can be found in the literature can be utilized to solve it. The next theorem states an optimization problem which provides a tool to check the feasibility of a stabilizing s-sparse state feedback controller for a given system.

Theorem 5.7.1. *The row s-sparse state feedback controller (5.2) stabilizes the linear time invariant system (5.1), with the optimal cost less than or equal to J^* if and only if the optimal value of the objective of the following bi-linear optimization is equal to zero.*

$$\begin{aligned} \min_{M,Y,\alpha} \quad & \mathbf{Tr}(Y^T M) \qquad (5.14) \\ s.t. \quad & (5.10b) - (5.10g), \\ & \mathbf{Tr}[QX_{11}] + \mathbf{Tr}[RX_{22}] \leq J^*, \\ & 0 \preceq Y \preceq I_{mn+2(n+m)}, \\ & \mathbf{Tr}(Y) = mn + 2m + n, \\ & M \succeq 0, \end{aligned}$$

where M is given by (5.11).

Proof. Since the matrix M is positive semi-definite, we have

$$\begin{aligned}\sum_{i=1}^{mn+2m+n} \lambda_{[i]}(M) = \min_{Y \in \mathbb{R}^{mn+2(m+n)}} \quad & \mathbf{Tr}(Y^{\mathrm{T}} M) \\ \text{s.t.} \quad & 0 \preceq Y \preceq I_{mn+2(m+n)}, \\ & \mathbf{Tr}(Y) = mn + 2m + n,\end{aligned}$$

where $\lambda(.)$ denotes the vector of eigenvalues of a given matrix [47, p.266]. We have proved that the row s-sparse stabilizing controller exists if and only if there is a matrix M with rank n in the feasible set of our optimization problem. This is equivalent to having a matrix M satisfying $\sum_{i=1}^{mn+2m+n} \lambda_{[i]}(M) = 0$. Therefore, the existence of such a controller is identical to having at least one point in the feasible set of our optimization with the corresponding cost equal zero. Such a point is also the optimal cost of (5.14), since the positive semi-definiteness of the matrices M and Y guarantees the non-negativity of the the cost function. This completes our proof. □

Although the bi-linear term in the cost function of (5.14) makes it a non-convex problem, it optimum can be achieved by solving the problem for Y and $\{M, \alpha\}$, until convergence criteria is met [45, 48]. Next, we employ the above result to propose a routine for solving the problems (5.10) and (5.13). Here, we describe the method with a focus on (5.10) problem; however, it can be effortlessly modified to become applicable to the rank constrained problem (5.13) as well.

Theorem 5.7.2. *Consider the optimization problem*

$$\begin{aligned}
\min_{M,Y,\alpha} \quad & \mathbf{Tr}[QX_{11}] + \mathbf{Tr}[RX_{22}] + \nu \mathbf{Tr}(Y^T M) \\
s.t. \quad & (5.10b) - (5.10g), \\
& 0 \preceq Y \preceq I_{mn+2(n+m)}, \\
& \mathbf{Tr}(Y) = mn + 2m + n, \\
& M \succeq 0,
\end{aligned} \tag{5.15}$$

where M is given by (5.11), and also let $M^*(\nu)$ and $Y^*(\nu)$ denote the optimal solution to (5.15) for different values of ν. Assuming the the problem (5.10) is feasible; then, one of the following hold.

- *There exist a positive real number ν^* such that for all $\nu \geq \nu^*$ the optimization problem (5.15) solves the rank constrained program (5.10).*
- *There exists a constant, namely η, such that the optimal values of $\mathbf{Tr}(Y^T M)$ is bounded by $\eta\nu^{-1}$, i.e.*

$$\mathbf{Tr}({Y^*}^T(\nu) M^*(\nu)) < \eta\nu^{-1}$$

Proof. If there is a $\nu^* > 0$ for which the solution of (5.15) gives the optimum of (5.10), then, $M^*(\nu^*)$ is a rank n matrix, hence, $\mathbf{Tr}({Y^*}^{\mathrm{T}}(\nu^*) M^*(\nu^*)) = 0$. Now assume that for $\nu > \nu^*$ the rank of $M^*(\nu)$ is not n; thus, we have

$$\begin{aligned}
&\mathbf{Tr}[QX_{11}^*(\nu^*)] + \mathbf{Tr}[RX_{22}^*(\nu^*)] \\
\leq &\mathbf{Tr}[QX_{11}^*(\nu)] + \mathbf{Tr}[RX_{22}^*(\nu)] + \nu^* \mathbf{Tr}({Y^*}^{\mathrm{T}}(\nu) M^*(\nu)).
\end{aligned}$$

Since $\mathbf{Tr}(Y^{*\mathrm{T}}(\nu)M^*(\nu)) > 0$, we can write

$$\begin{aligned}&\mathbf{Tr}[QX_{11}^*(\nu^*)] + \mathbf{Tr}[RX_{22}^*(\nu^*)]\\<&\mathbf{Tr}[QX_{11}^*(\nu)] + \mathbf{Tr}[RX_{22}^*(\nu)] + \nu\mathbf{Tr}(Y^{*\mathrm{T}}(\nu)M^*(\nu)),\end{aligned}$$

which is a contradicts the fact that $X_{11}^*(\nu)$, $X_{22}^*(\nu)$, $Y^*(\nu)$, and $M^*(\nu)$ are the optimum of (5.15) for the particular value of ν.

Next, we consider the case where no such ν^* exists. Then, the following holds for all values of ν.

$$\begin{aligned}&\mathbf{Tr}[QX_{11}^*] + \mathbf{Tr}[RX_{22}^*]\\\geq&\mathbf{Tr}[QX_{11}^*(\nu)] + \mathbf{Tr}[RX_{22}^*(\nu)] + \nu\mathbf{Tr}(Y^{*\mathrm{T}}(\nu)M^*(\nu)),\end{aligned}$$

where X_{11}^* and X_{22}^* are the optimal point of problem (5.10). Knowing that $\mathbf{Tr}[QX_{11}^*(\nu)] + \mathbf{Tr}[RX_{22}^*(\nu)] \geq 0$, the following is true.

$$\mathbf{Tr}(Y^{*\mathrm{T}}(\nu)M^*(\nu)) \leq (\mathbf{Tr}[QX_{11}^*] + \mathbf{Tr}[RX_{22}^*])\nu^{-1}$$

Therefore, there exist a constant η for which $\eta\nu^{-1}$ bounds the $\mathbf{Tr}(Y^{*\mathrm{T}}(\nu)M^*(\nu))$ from above. Hence, the proof is complete. □

The previous theorem basically express that it is possible to decrease the optimal value of the additional term in (5.15) arbitrarily by enlarging the parameter ν. As this value represent the sum of the smallest $mn + 2n + n$ eigenvalues of the matrix M, diminishing this term forces the rank of M to obtain the desired rank, i. e. n.

In the bi-linear program (5.15), the optimization variable pair $\{M, \alpha\}$ is constrained to the convex set $\mathcal{C}$, defined by the constraints (5.10b)-(5.10g) along with $M \succeq 0$, and the convex set constructed by the linear matrix inequalities $0 \preceq Y \preceq I_{mn+2(n+m)}$ and the linear equality $\mathbf{Tr}(Y) = mn + 2m + n$, namely $\bar{\mathcal{C}}$, declares the set of feasible Y's. Exploiting the new notations, the optimization (5.15) can be represented as

$$\begin{aligned} \min_{M,\alpha,Y} \quad & f(M, \alpha, Y) \\ \text{s.t.} \quad & \{M, \alpha\} \in \mathcal{C}, \quad Y \in \bar{\mathcal{C}}, \end{aligned} \tag{5.16}$$

where

$$f(\{M, \alpha\}, Y) = \mathbf{Tr}[QX_{11}] + \mathbf{Tr}[RX_{22}] + \nu \mathbf{Tr}(Y^{\mathrm{T}} M),$$

and the structure of M is given by (5.11). Considering the above formulation, the problem (5.15) can be solved using the algorithms for constrained bi-convex optimization [73, 48]. The approach we are utilizing here is adopted from [48] which is based on alternating minimization between the optimization variables $\{M, \alpha\}$ and Y until the convergence criteria is satisfied, as shown in the sequel.

$$\{M, \alpha\}^{(k+1)} = \underset{\{M,\alpha\} \in \mathcal{C}}{\operatorname{argmin}} f(M, \alpha, Y^{(k)}) \tag{5.17}$$

$$Y^{(k+1)} = \underset{Y \in \bar{\mathcal{C}}}{\operatorname{argmin}} f(\{M, \alpha\}^{(k+1)}, Y) \tag{5.18}$$

Remark 5.7.3. *It should be noted that the problem (5.15) can be relaxed into a convex problem by setting Y to an identity matrix. As a result, the term $\mathbf{Tr}(Y^T M)$ reduces* *to the nuclear norm of M, i.e. $\|M\|_*$, which a well known convex surrogate for the rank operator [39]. This special form of our proposed approach can be interpreted as penalizing the nuclear norm of the matrix in order to force its to the smallest feasible value.*

5.8 Simulation Results

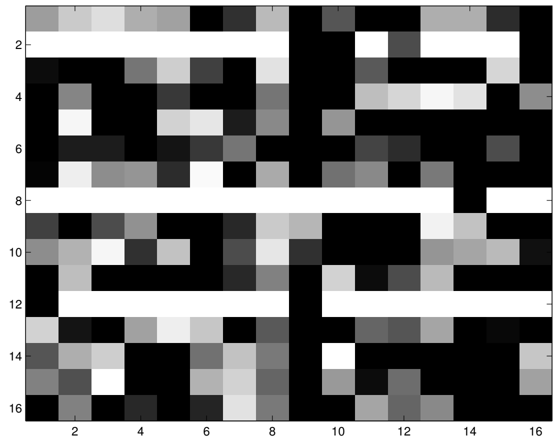
In this section, we demonstrate the validity of our results by applying our proposed algorithm on two randomly generated sub-exponentially decaying (SD) systems defined on grids by the system matrices $A = [a_{ij}]$ as

$$a_{ij} = \mathfrak{X}\ e^{-\alpha d_{ij}^{\beta}}$$

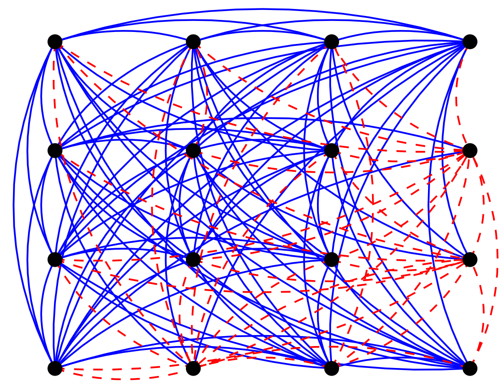
where d_{ij} denotes the distance between nodes i and j, and $\mathfrak{X}$ is chosen to be a uniformly distributed random variable on the interval $(-1, 1)$. Also, the matrix B is taken to be a randomly generated square matrix. This class of system is of interest to us, since it is capable of capturing the decaying effect in the coupling weights caused by increase of the distance between nodes, which is a common phenomenon in networks such as power grids.

5.8.1 Case A

In this case, we consider a 4×4 grid model generated by setting the SD parameters $\alpha = 1$, $\beta = 0.5$. We also choose the quadratic performance weights as $Q = 10I$ and $R = I$, and the regularization parameters ν are set to 10^4 and 10, respectively. The objective in this case study is to enforce 4-sparsity on the rows 2, 8, and 12. The figure



(a)

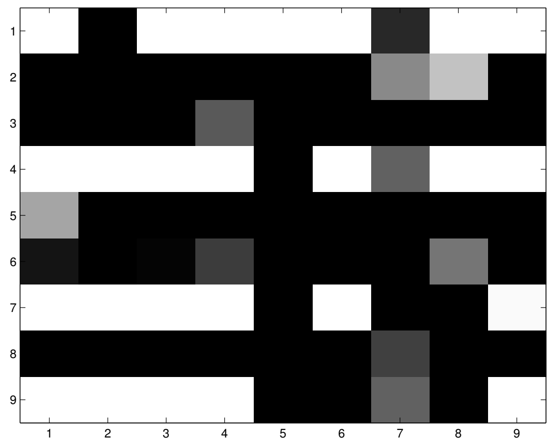


(b)

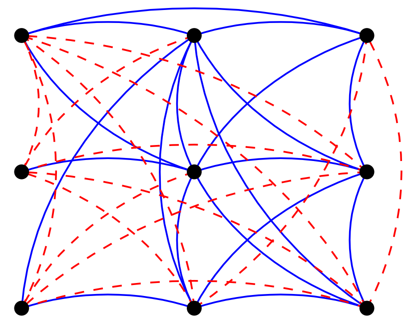
Figure 5.1: (a) Sparsity pattern of the controller (b) Graph representation of the synthesized controller.

5.1a visually represent the synthesized controller matrix truncated by the threshold 10^{-3}. The graph representation of the controller, showing the spatial of the nodes in the grid, is also depicted in 5.1b. In this figure, the blue lines show bi-directional communication between the nods and the dashed red lines represent one way links.

According to our data, the controller quadratic performance has deteriorated 71% and the number of nonzero entries of the controller is 219, which is around 85% of the total controller entries. It should be noted that the above examples not only reveal the effectiveness of our approach in the design of s-sparse controller, it also shows that our approach enables the user to enforce the s-sparsity constraint on arbitrarily chosen rows.



(a)



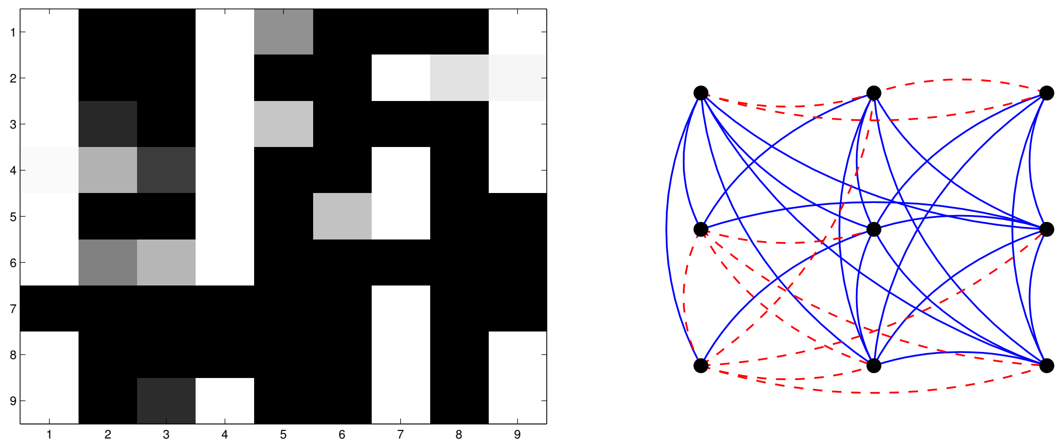
(b)

Figure 5.2: (a) Sparsity pattern of the controller with sparsity constraints on the rows 1, 4, 7, and 9 (b) Graph representation of the controller.

5.8.2 Case B

The purpose of this section is to demonstrate the versatility of our design method by presenting that it is capable of synthesizing row sparse, column sparse, column/row sparse, and iso-structured row/column sparse controllers. We consider a randomly generated 3×3 grid model with SD properties. The system parameters, performance weights, and the regularization parameters used here are all the same as the ones used in 5.8.1.

We start by utilizing our method to synthesize a controller, with 4-sparsity constraints on the rows 1, 4, 7, and 9, for the randomly generated system. The visual representation of the sparsity pattern of the controller is depicted in Figure 5.2a. Also, Fig. 5.2b describes the underlying graph of the obtained controller. It should also be noted that the gap between the quadratic performance the sparse controller and the linear quadratic regulator is about 82.14%.



(a) (b)

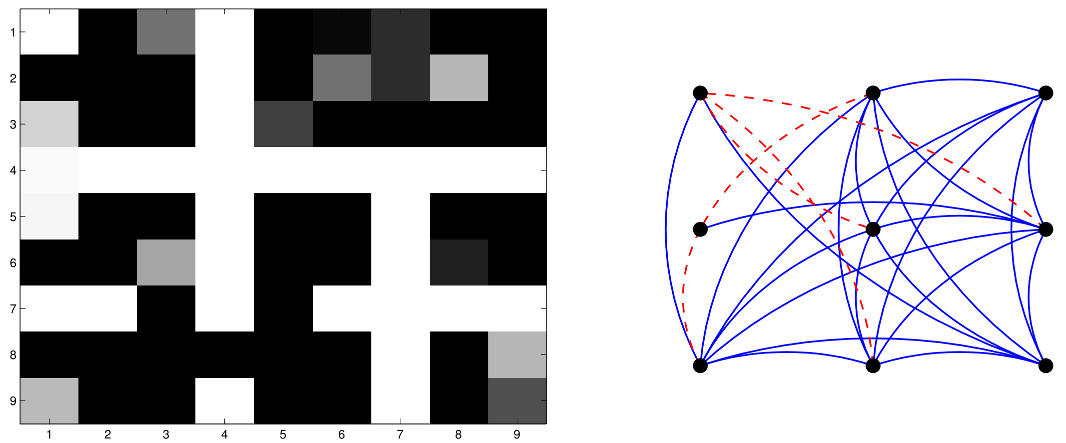
Figure 5.3: (a) Sparsity pattern of the controller with sparsity constraints on the columns 1, 4, 7, and 9 (b) Graph representation of the controller.

Next, we show how our method can be employed to synthesize row s-sparse controller. By simply modifying the constraint 5.10e with the constraint

$$\beta k_{ij}^2 \leq z_{ij}, \qquad\qquad \forall i, j,$$

the optimization problem yields column s-sparse controllers. Applying this modification, we synthesize a controller, with 4-sparsity constraints on the columns 1, 4, 7, and 9, for the same randomly generated system. The visual representation of the sparsity pattern of the controller is shown in Figure 5.3a. Also, Fig. 5.3b illustrates the underlying graph of the obtained controller. In this case, the quadratic performance deterioration, comparing to that of the LQR controller, is about 60.13%.

There are also applications in which we desire to seclude certain nodes from all other nodes as much as possible. This is viable by enforcing the sparsity constraint on both rows and column associated with the nodes. Interestingly, this cases is also



(a) (b)

Figure 5.4: (a) Sparsity pattern of the controller with sparsity constraints on the rows/columns 4 and 7 (b) Graph representation of the controller.

implementable in our appraoch by selectively choosing the constraints 5.10e or its modified versions. In Figure 5.4, we present the controller synthesized by imposing such limitations on the rows 4 and 7 as well as the columns 4 and 7. The performance deterioration in this cases is about 67.75%

Furthermore, there are cases where the sparsity structure of the columns and rows associated with particular nodes are aimed to be similar. This means that the nodes receive information form the exact same nodes they send information to; hence, the minimum number of transmission lines can be minimized substantially. Such "iso-structure sparsity" patterns can be implemented in our approach by simply associating the particular rows/columns to a certain column of the matrix Z. The results presented in Figure 5.5 shows the synthesized controller with the iso-structure s-sparsity constraint the rows/columns 4 and 7. It can be see that the symmetry between the structure of the rows and columns is plain. Also, the performance deterioration associate with case is around 71%, which is slightly larger that the previous

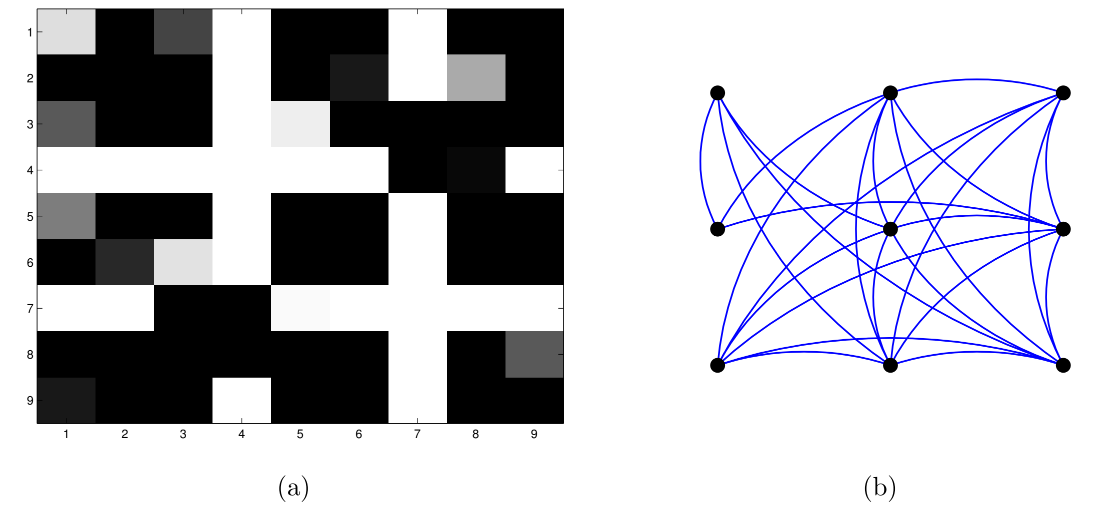


Figure 5.5: (a) Sparsity pattern of the controller with iso-structure row/column sparsity constraints on the rows/columns 4 and 7 (b) Graph representation of the controller.

case.

In our simulations, performed using the optimization software CVX in MATLAB [74], we have noticed that the optimization problem may become unbounded or produce unacceptable results. Our inspections revealed that the issue is not always due to the shortcoming of the optimization algorithm, as it sometimes caused by the insufficient precision of the convex optimization algorithms used by CVX. For example, the parameter α should not be allowed to dwindle significantly, since the matrix D may become negligible, for the software precision, when being added to Q. Hence, the resulted P does not satisfy the doubly α-super-stochasticity condition. On the other hand, significantly large Q reproduces the same issue, so suitable modification should be implemented into the optimization algorithm. It is expected that other numerical issues may associated with the algorithm, which is beyond the scope of this chapter.

5.9 Conclusions

A method is developed for the synthesis of controller with strict row sparsity constraints. We have shown that the recovery of a s-sparse vector can be formulated using a rank constraint formulation. Then, extend our results to propose an equivalent reformulation where all non-convex and combinatorial constraints are lumped into a single fixed rank constraint. Unlike, the common methods which uses relaxation for the ℓ_0-norm of the vectors/matrices, we have not employ any form of relaxation; thus, our reformulation is exact. We further show how our formulation can be extended to accommodate the row sparse control design paradigm. In addition, we propose that the equivalent rank constrained optimization problem can be solved using a bi-linear optimization with convex constraints. We have also provided numerical examples in order to exhibit the effectiveness of our proposed methodology.

Part II

Identification of Sparse Stable Networks

Chapter 6

Gene Regulatory Network Modeling Using Literature Curated and High Throughput Data

The phenotypic expression of a genome, including the response to external stimuli, is a complex process involving multiple levels of regulation. This regulation includes controls over the transcription of *messenger RNA* (mRNA) and translation of mRNA into protein via *gene regulatory networks* (GRNs). Advances in microarray and assay technologies are facilitating increasingly large amounts of laboratory data for analysis of these networks. If the network is operating sufficiently close to a steady-state, Gardner *et al* [75] have shown that multiple linear regressions can be applied to this data to derive a linear *ordinary differential equation* (ODE) model of the form $\dot{x} = Ax + u$, where x is the vector of gene expression values and u is the exciting input (see [75] and [76]). Now, in addition to this data, information on the interactions between genes, proteins, and metabolites is available through published literature.

Observing that this information can be included as a constraint in the optimization problem solved in [75], Zavlanos *et al* [77] have performed convex relaxations on the modified optimization problem and have given a *linear matrix inequality* (LMI) based solution to derive linear ODE models of gene regulatory networks. In particular, [77] re-formulates the approach of [75] using LMI's and includes sufficient conditions for asymptotic stability, given by the Lyapunov stability theorem (see [78], [79], and [80]), as the additional constraints to ensure that the linear ODE model is stable. In [77] , the problem formulation and its solution is presented in a highly lucid manner and its choice of LMI formulation is likely to lead to a number of LMI-based solutions for such network modeling problems.

The chapter is organized as follows. After stating our modeling assumptions, we present the network modeling algorithms of [77] and our extensions of those algorithms. We then show that our algorithms perform at least as well as those algorithms when presented with a synthetic dataset that is generated using the procedure given in [77]. We then show how these results can be used to derive a protein regulatory network of malaria infected patients.

6.1 Linear ODE Models of Gene Regulatory Networks

The problem of how the gene expression data should be used to obtain linear ODE models of the underlying gene regulatory networks has been well researched (see for example [75], [76], [81], [82], [83], [84], and references therein). We shall focus on deterministic models. The ODE model is of the form $\dot{x} = Ax + Bu$, where A

Symbol	Meaning
$(\mathbb{R}^+)$ $\mathbb{R}$	Set of all (nonnegative) real numbers
$\mathbb{R}^n$	n-dimensional ($n \times m$) real-valued vector (matrix)
$\mathbb{R}^{n \times m}$	$n \times m$ real-valued matrix
$\mathbb{C}$	Set of all complex numbers
$\mathbb{Z}$	Set of all integers
$(\cdot)^{\mathrm{T}}$	Transpose of a vector or a matrix $(\cdot)$
Herm$(\cdot)$	$\frac{1}{2}((\cdot) + (\cdot)^{\mathrm{T}})$ ... (Hermitian of $(\cdot)$)
$A \succ 0$ $(A \prec 0)$	A is positive semidefinite (negative definite).
$\lVert z \rVert_1$	$= \sum_i \lvert z_i \rvert$ if z is a vector $(= \sum_{i,j} \lvert z_{i,j} \rvert$ if z is a matrix)
card(A)	Number of nonzero elements of A ... (cardinality)
$\lambda_i(A)$	i-th eigenvalue of the matrix A
diag(a_i)	Diagonal matrix with a_i as its diagonal elements
$\dot{x}$	$= dx/dt$ (derivative of x with respect to time)

and B are real-valued matrices of suitable sizes, x is the vector of gene expression values, and u is the vector (or matrix) of exciting inputs. Laboratory data on the gene expression values for varying inputs furnishes the datasets X and U, where the matrix X comprises the vectors of gene expression values and the matrix U comprises the vectors of corresponding excitations. Now, the objective is to solve for A and B such that some performance metric is optimized. Assuming the availability of time-series data for the gene expression values, such models are derived in [83] and [84] whereas this requirement is relaxed in [75], [76], and [77]. All of these approaches rest on the assumption that the network is operating sufficiently close to a stable equilibrium point. Under this assumption, solving the ODE $\dot{x} = Ax + Bu$ for A and B effectively reduces to solving the equation $0 = Ax + Bu$ for A and B. In addition, it is assumed in [75], and therefore in [77], that the inputs u can be controlled to selectively over-express precisely one gene at a time. This reduces the matrix B to an identity matrix and, as a result, only the matrix A needs to be solved for. However, in practice, such controlled excitation is rarely performed, at least as of today. Instead, most pharmaceutical companies and cosmetic firms have large repositories of snapshots of the gene expression values for the *control* cases, i.e., for normal subjects, and for the *treatment* cases, i.e., for the cases in which the subject is either abnormal or exposed to an excitation or a treatment (such as a radiation or a drug dose). Here, it rarely holds that the excitation input u selectively over-expresses (or suppresses) precisely one gene at a time. We shall show that the approach of [77] is applicable even when its overly restrictive constraint $B = I$ is relaxed.

6.2 Method

6.2.1 Assumptions

Our main assumptions are as follows.

- The network can be modeled as $\dot{x} = f(x, u)$ for some function f.
- The network has a stable equilibrium point, x_{eq}, in the neighborhood of which $\dot{x} = f(x, u)$ can be approximated as $\dot{\tilde{x}} = A\tilde{x} + Bu$, where $\tilde{x} \doteq x - x_{eq}$, for some matrices A and B.
- The operating point of the network is sufficiently close to the stable equilibrium.
- The matrix A is invariant across all treatments and all subjects.
- The matrix A is sparse (see [85, 86]).
- The input u is to be computed as follows. The exogenous excitation is a transcription perturbation in which individual genes are over-expressed using an

episomal expression plasmid. After the perturbation, these cells are allowed to grow under constant physiological conditions to a steady-state and the difference in the mRNA concentrations of these cells and that of normal cells, i.e., those having reporter genes as opposed to the over-expressed genes is to be noted down (see [87]). In general, a perturbation will affect $p \leq n$ genes in the n-gene network.

- Specific genes encode the *transcription factors* (TFs) — proteins that can bind DNA (either independently or as part of a complex), usually in the upstream regions of target genes (promoter regions), and so regulate their transcription. Since the targets of a TF can include genes encoding for other TFs, as well as those encoding for proteins of other function, interactions between transcriptional and translational levels of the system take place. In addition, post-translational and epigenetic effects also influence the network. We assume these can be accounted for indirectly in the gene regulatory network.

6.2.2 Background Results

Let us now note the main results of [77]. To begin with, let us denote the i-th element of a vector v as v_i and the (i, j)-th element of a matrix A as either $a_{i,j}$ or a_{ij}. Let m be the number of available transcription perturbations. Let n denote the number of genes. Let $U \doteq [u_1 \ u_2 \ \ldots \ u_m] \in \mathbb{R}^{p\times m}$ and $\tilde{X} \doteq [\tilde{x}_1 \ \tilde{x}_2 \ \ldots \ \tilde{x}_m] \in \mathbb{R}^{n\times m}$ be the matrices containing transcriptional perturbation values and their associated mRNA expression values, respectively, for the m experiments. Then, if the network modeled as $\dot{x} = Ax + Bu$ is at the stable equilibrium, then it holds that $A\tilde{X} + BU = 0$. In general, the measured deviation in x can be different from the deviation predicted by the linear ODE model. Therefore, let $X \doteq \tilde{X} + \Delta X$, where X comprises the measured values and ΔX is the mismatch due to nonlinearities, measurement noise, etc. Then, $AX + BU = A\tilde{X} + BU + \eta$, where $\eta \doteq A\Delta X$. The network modeling problem can now be stated as follows: Given X and U, determine a sparse stable matrix A that minimizes η subject to the constraint that it satisfies the constraints laid down by *a priori* information.

The *a priori* information is often in the form of sign pattern S that captures the interaction between the nodes i and j. The convention is that s_{ij} is (i) '+' if the node j activates the node i, (ii) '-' if the node j inhibits the node i, (iii) zero if the nodes i and j do not interact, and (iv) '?' if no *a priori* information is available on how the node j affects the node i. Then,

$$A \in S \Leftrightarrow \begin{cases} a_{ij} \geq 0 & \text{if } s_{ij} = +; \\ a_{ij} \leq 0 & \text{if } s_{ij} = -; \\ a_{ij} = 0 & \text{if } s_{ij} = 0; \\ a_{ij} \in \mathbb{R} & \text{if } s_{ij} = ?. \end{cases} \tag{6.1}$$

The stability constraint is satisfied if every eigenvalue of A has a negative-valued real component. Since minimizing card$(\cdot)$ might have an adverse effect on η and vice versa, a convex combination of card$(\cdot)$ and η is minimized in [77] — specifically, the Problem 1 is first re-cast as the following optimization problem **P1:**

$$\begin{aligned} &\text{minimize} && t\ \text{card}(A) + (1-t)\epsilon \\ &\text{subject to} && \|AX + BU\|_1 \leq \epsilon,\ \epsilon > 0, \quad A \in S, \end{aligned}$$

Algorithm Z: Solution to **P1** (see [77, Algorithm 1])
Input: t, δ, S, X, and U
1: *Initialization:* Set $w_{ij} = 1$ for all $i, j = 1, \cdots, n$
2: **for** iteration = 1 to J **do**
3: Solve **P1** for A and ϵ,
4: Update the weights w_{ij} using Eq. (2),
5: Update the weights v_{ij} using (6.3),
6: **end for**
Output: A

where $t \in [0, 1]$ is a user defined parameter. Now, card$(\cdot)$ is a non-convex function. Hence, it is relaxed in [77] to a convex function, namely, a weighted ℓ_1-norm $\sum_{i,j=1}^{n} w_{ij}|a_{ij}|$, where the weights w_{ij} are defined as

$$w_{ij} = \frac{\delta}{\delta + \mid a_{ij} \mid}, \quad i, j = 1, \cdots, n, \tag{6.2}$$

where $\delta > 0$. If δ is chosen sufficiently small then the value of $w_{ij}|a_{ij}| \approx 1$ if $a_{ij} \neq 0$ and is zero otherwise. The following algorithm, viz., [77, Algorithm 1], solves this optimization problem.

To ensure that the system is stable, the eigenvalues of A must be constrained to have negative valued real part so that **P1** is modified into the following optimization problem **P2:**

$$\begin{aligned} \text{minimize} \quad & t \sum_{i,j} w_{ij}|a_{ij}| \ + (1-t)\epsilon \\ \text{subject to} \quad & \|AX + BU\|_1 \leq \epsilon, \ \epsilon > 0 \\ & \text{real}(\lambda_i(A)) < 0 \ \ \forall i, \quad A \in S, \end{aligned} .$$

where $t \in [0, 1]$ is a user defined parameter. In [77], a solution to **P2** is obtained by using the Gershgorin's circle theorem as follows (see Algorithm 2 of [77]).

Theorem 6.2.1. [Gershgorin's Circle Theorem (see [88])]
Let $A \in \mathbb{R}^{n\times n}$. For all $i \in \{1, \cdots, n\}$, define the deleted absolute row sums of A as $R_i(A) \doteq \sum_{j\neq i} |a_{ij}|$. Then, all eigenvalues of A lie within the union $G(A)$ of n discs that is defined as

$$G(A) \doteq \bigcup_{i=1}^{n} \{z \in \mathbb{C} |\ |z - a_{ii}| \leq R_i(A)\}.$$

Furthermore, if a union of k of these n discs forms a connected region that is disjoint from every other disc then that region contains precisely k eigenvalues of A. □

From Theorem 6.2.1, it follows that the matrix A is stable if $a_{ii} \leq -\sum_{i\neq j} |a_{ij}| \quad \forall i$, which holds if A is diagonally dominant with non-positive diagonal entries. To relax this restrictive requirement, a similarity transformation V can be applied to A since the eigenvalues of $V^{-1}AV$ are the same as those of A. An easy choice for V is $V = \text{diag}(v_i)$ with $v_i > 0$. Then, using 6.2.1, it follows that the matrix $V^{-1}AV$ is stable if $a_{ii} \leq -\frac{1}{v_i}\sum_{j\neq i} v_j|a_{ij}| \quad \forall i$. Therefore, it follows (see [77]) that the solution A of **P2** is guaranteed to be stable if it is obtained by solving the following modified optimization problem **P3:**

$$\begin{aligned} \text{minimize} \quad & t \sum_{i,j} w_{ij}|a_{ij}| \ + (1-t)\epsilon \\ \text{subject to} \quad & \|AX + BU\|_1 \leq \epsilon,\ \epsilon > 0 \\ & a_{ii} \leq -\tfrac{1}{v_i}\textstyle\sum_{i\neq j} v_j|a_{ij}| \ \ \forall i, \quad v_i > 0\ \forall i, \quad A \in S, \end{aligned}$$

where $t \in [0, 1]$ is a user defined parameter. The matrices V and W can be chosen as follows (see [77]). Initialize $V = I$ where I is the identity matrix of suitable size and set $w_{ij} = 1 \; \forall i, j$. Then, repeatedly solve **P3**, updating w_{ij} using Eq. (2) and v_{ii} using

$$v_{ii} \doteq \begin{cases} 1 + \frac{|a_{ii}| - R_i(A) - \beta}{\delta + (|a_{ii}| - R_i(A) - \beta)} & \text{if } \mid a_{ii} \mid - R_i(A) > \beta; \\ \frac{\delta}{\delta - (|a_{ii}| - R_i(A) - \beta)} & \text{if } \mid a_{ii} \mid - R_i(A) \leq \beta, \end{cases} \tag{6.3}$$

where $\beta \doteq \sum_{i=1}^{n} (|a_{i,i}| - R_i(A))/n$.

Remark 6.2.2. In [77], it is claimed that this procedure, described in [77, Algorithm 2], usually requires no more than $J = 20$ iterations but may yield periodic solutions for certain ill-condition problems. □

Remark 6.2.3. [77, Algorithm 2] is somewhat ad-hoc since the parameter δ is left undefined in it. □

Remark 6.2.4. In [77], another solution to **P2** is obtained by using the Lyapunov stability theorem to ensure the stability (see [77, Algorithm 3]). □

6.2.3 Main Results

The values of v_{ii} in the above algorithm can be updated at the end of each iteration using a number of known results. For example, it is shown in [89] that the optimal diagonal postcompensator V to render the matrix VA row dominant can be obtained by computing the left Perron eigenvectors of the $\mathbb{R}^{n \times n}$ nonnegative matrix T having $|a_{ij}|$ as its elements, provided it is a *primitive* matrix. Also, it is known that the Perron eigenvalue and its corresponding eigenvector can be easily computed using the following iterative method: select an arbitrary unit vector x_0, then iterate it as

Algorithm 1: (Solution to **P3**)

Input: t, δ, Δ, S, X, and U

Initialization: $V = I$ and $w_{ij} = 1$ for all $i, j = 1, \cdots, n$

for iteration = 1 to J **do**

  Solve **P3** for A and ϵ,

  while $\|\bar{x}_{k+1} - \bar{x}_k\| > \Delta$ **do**

    Update $\bar{x}_k$ and $\bar{x}_{k+1}$ using Eq. (6),

  end while

  Update the weights v_{ii} using Eq. (7),

  Update the weights w_{ij} using Eq. (2),

end for

Output: A

follows:

$$\bar{x}_{k+1} = T\bar{x}_k / \|T\bar{x}_k\| \tag{6.4}$$

until $\|\bar{x}_{k+1} - \bar{x}_k\| < \Delta$, where $\Delta > 0$ is arbitrarily small. Now, $\bar{x}_{k+1}$ is a reasonable approximation of the right perron eigenvector of T, and its corresponding eigenvalue r can be obtained by solving $T\bar{x}_{k+1} \simeq r\bar{x}_{k+1}$ (see [89]). If the column-dominance of A is to be optimized then the same procedure should be applied to A^{T} and then the result should be transposed. Therefore, Perron eigenvector of T seems to be a good choice for the construction of the scaling matrix V, where

$$V \doteq \mathrm{diag}(\bar{x}_{k+1}). \tag{6.5}$$

Hence, Algorithm 1, an improvement over [77, Algorithm 2], can be stated as follows.

Another approach to modify Algorithm Z so that its output A is a stable matrix is as follows (see [77]). If the output A is unstable, perturb it by a *small enough* perturbation D such that the perturbed matrix $\widetilde{A} \doteq A+D$ is stable and, furthermore, an element of S. By Lyapunov stability theorem, $\widetilde{A}$ is stable if there exists a $P = P^{\mathrm{T}} \succ 0$ such that $\mathrm{Herm}(\widetilde{A}^{\mathrm{T}}P) \prec 0$, i.e., if

$$\mathrm{Herm}(A^{\mathrm{T}}P + L) \prec 0, \tag{6.6}$$

where $L \doteq PD$. Now, (6.6) is an LMI that can be efficiently solved by solving the following semidefinite program **P4**:

$$\begin{array}{ll} \text{minimize} & \|LX\|_2 \\ \text{subject to} & \mathrm{Herm}(A^T P + L) \prec 0, \quad P \succ 0, \end{array} .$$

the solution of which gives the perturbation as $D = P^{-1}L$ (see [90]). However, while this perturbation ensures the stability of $\widetilde{A} \doteq A + D$, it does not ensure $\widetilde{A} \in S$. In [77], this difficulty is resolved by using the Lyapunov matrix P, obtained as a solution of **P4**, in solving the following optimization problem **P5**:

$$\begin{array}{ll} \text{minimize} & t \displaystyle\sum_{i,j=1}^{n} w_{ij} \mid a_{ij} \mid + (1-t)\epsilon \\ \text{subject to} & \|AX + BU\|_1 \le \epsilon, \ \ \epsilon > 0, \\ & \mathrm{Herm}(A^{\mathrm{T}}P) \prec 0, \quad A \in S. \end{array} .$$

A solution to this problem is given by [77, Algorithm 3].

If the network is sufficiently damped then $\|Gu\|_2/\|u\|_2$ can be approximated by $\|y_{ss}\|_2/\|u\|_2$ where G is the transfer function of the linearized system, and y_{ss} is the steady-state response of the system, which is the same as state vector if $C = I_n$. Therefore, if sufficient amount of the steady-state data is available then $\|G(s)\|_\infty$ can be approximated as:

$$\sup_i \|y_{ss}^i\|_2 / \|u^i\|_2 \simeq \|G(s)\|_\infty \simeq \gamma, \tag{6.7}$$

where the maximization is performed over the experiment trials. Now, the well-known *bounded real lemma* (BRL) can be used to derive a more powerful network modeling algorithm.

Theorem 6.2.5. [Bounded Real Lemma [91]]
Let the system $G(s)$ be given in the state-space form as

$$\dot{x} = Ax + Bu,$$
$$y = Cx + Du.$$

Then, A is stable and $\|G(s)\|_\infty < \gamma$ if and only if the system of LMI's:

$$\begin{bmatrix} AP + PA^T & B & PC^T \\ B^T & -\gamma I & D^T \\ CP & D & -\gamma I \end{bmatrix} \prec 0, \quad P \succ 0$$

has a symmetrix matrix P as its solution. □

Therefore, we can identify out network model by solving the following optimization

Algorithm 2: (Solution to **P6**)
Input: t, δ, S, X and U
1: Apply Algorithm Z to obtain A
2: Approximate γ using (6.7)
3: **while** A is unstable or $\|G(s)\| > \gamma$ **then**
4: Solve **P4** for a Lyapunov matrix P,
5: Initialize $w_{ij} = 1$ for all $i, j = 1, \cdots, n$,
6: **for** iteration = 1 to J **do**
7: Solve **P6** for A and ϵ,
8: Update the weights w_{ij} using Eq. (2),
9: **end for**
10: **end while**
Output: A

problem **P6**:

$$
\begin{aligned}
\text{minimize} \quad & t \sum_{i,j=1}^{n} w_{ij} \mid a_{ij} \mid + (1-t)\epsilon \\
\text{subject to} \quad & \|AX + BU\|_1 \leq \epsilon, \;\; \epsilon > 0, \\
& \begin{bmatrix} AP + PA^{\mathrm{T}} & B & P \\ B^{\mathrm{T}} & -\gamma I & 0 \\ P & 0 & -\gamma I \end{bmatrix} \prec 0 \\
& P \succ 0, \quad A \in S.
\end{aligned}
$$

A solution to this problem is obtained by using Algorithm 2. In all algorithms considered thus far, the matrix B is assumed to be known. However, as observed earlier, such is rarely the case in practice. If A and B both need to be estimated then more *a priori* information on A is required since, otherwise, $A = 0$ and $B = 0$ is a trivial solution to $0 = Ax + Bu$. Such a meaningless solution can be readily ruled out by stipulating $a_{ii} < \sigma_i \; \forall i$ for some σ_i as a constraint in the optimization problem. This constraint is valid in reality since every gene and protein down-regulates its own production through self-degradation. Using Gershgorin's circle theorem to guarantee the stability, the estimation of A and B can be obtained from the solution of the following optimization problem **P7**:

$$
\begin{aligned}
\text{minimize} \quad & t \sum_{i,j=1}^{n} w_{ij} \mid a_{ij} \mid + (1-t)\epsilon \\
\text{subject to} \quad & \|AX + BU\|_1 \le \epsilon, \;\; \epsilon > 0, \\
& \text{Herm}(A^{\mathrm{T}} P) \prec 0, \quad a_{ii} < -\sigma_i \; \forall i, \quad A \in S.
\end{aligned}
$$

where $\Sigma \doteq \text{diag}(\sigma_i) \in \mathbb{R}^{n \times n}$ is a diagonal matrix that has the self-degeneration rates as its diagonal elements. The estimation of B introduces a scaling difficulty: if (A^*, B^*) is a solution of our optimization problem, then $(\alpha A^*, \alpha B^*)$ is also a valid solution for every scalar α that satisfies $|\alpha| < 1$. In fact, scaling by such an α facilitates smaller modeling errors. This difficulty can be resolved by scaling A and B by a suitable positive number, say $\kappa(A, B)$, so that the absolute value of the largest element of A becomes equal to 1. Depending on its sign, one can then set the elements having absolute value less than an arbitrary small value such as, say, $\nu = 10^{-4}$: we refer to these matrices as $\tilde{A}$ and $\tilde{B}$ (see Algorithm 3). The elements of $\tilde{A}$ and $\tilde{B}$ are defined as

$$
\begin{aligned}
\tilde{a}_{ij} &= \begin{cases} a_{ij} & \text{if } |a_{ij}| \ge \nu; \\ 0 & \text{if } |a_{ij}| < \nu; \end{cases} \\
\tilde{b}_{ij} &= \begin{cases} b_{ij} & \text{if } |b_{ij}| \ge \nu; \\ 0 & \text{if } |b_{ij}| < \nu. \end{cases}
\end{aligned} \tag{6.8}
$$

In **P4**, we solve an optimization problem to find a small perturbation that makes

Algorithm 3: (Solution to **P7**)
Input: t, ν, Σ, S, X, and U
1: Apply Algorithm Z to obtain A and B
2: **while** A is unstable **then**
3: Solve **P4** for a Lyapunov matrix P,
4: Initialize $w_{ij} = 1$ for all $i, j = 1, \cdots, n$,
5: **for** iteration = 1 to J **do**
6: Solve **P7** for A, B and ϵ,
7: Update the weights w_{ij} using Eq. (2),
8: **end for**
9: **end if**
10: Scale A and B by $\kappa(A, B)$
11: Define $\tilde{A}$ and $\tilde{B}$ as per Eq. (10),
Output: A, B, $\tilde{A}$, and $\tilde{B}$

matrix A stable, while minimizing an upper bound of the 2-norm of the difference between $AX + BU$ and $\tilde{A}X + \tilde{B}U$ (see [77]). If the eigenvectors of A can be estimated well enough then A can be stabilized by perturbing its eigenvalues while keeping its eigenvectors fixed. Hence, a revised optimization problem **P8** is as follows:

$$\begin{aligned} &\text{minimize} \quad h\|D^{-1}(\lambda_A + \lambda)DX + BU\|_1 + (1-h)\sum_{i=1}^{n} \lambda_i^2 \\ &\text{subject to} \quad \lambda_A + \lambda > 0, \quad \lambda \in \Lambda_A, \end{aligned}.$$

where Λ_A is the set of matrices having the canonical structure of the Jordan normal form of A. Now, P can be obtained by solving

$$(A + D^{-1}\lambda D)^{\mathrm{T}} P + P(A + D^{-1}\lambda D) \prec 0. \tag{6.9}$$

Then, A and B can be computed by solving **P7** iteratively.

Now, suppose our experimental data can be partitioned into q separate sets of data, X_i's, and each set contains the response of our network to the same input

Algorithm 4: (Solution to **P9**)

Input: t, h, δ, ν, Σ, S, X, and U

Apply Algorithm Z to obtain A and B
if A is unstable **then**
  Decompose A to its Jordan normal from,
  Solve **P8** for a λ,
  Find P using (6.9),
  Initialize $w_{ij} = 1$ for all $i, j = 1, \cdots, n$,
  for iteration = 1 to J **do**
    Solve **P7** for A, B and ϵ,
    Update the weights w_{ij} using Eq. (2),
  end for
end if
Scale A and B by $\kappa(A, B)$
Define $\tilde{A}$ and $\tilde{B}$ as per Eq. (10),

Output: A, B, $\tilde{A}$, and $\tilde{B}$

value. Therefore, we have

$$\|AX_i + BU_i\| \simeq 0 \quad i = 1, \cdots, q, \quad X_i \in \mathbb{R}^{n \times m_i}, \quad U_i \in \mathbb{R}^{p \times m_i}, \tag{6.10}$$

where $m_i > 0$ is the number of data columns in each set, $\sum_{i=1}^{q} m_i = m$, and all columns of U_i's are the same. Now, if we construct matrix $X_{i0} \in \mathbb{R}^{n \times m_i}$ with columns equal to one arbitrarily column chosen from X_i, it holds that

$$\begin{aligned} \|A(X_i - X_{i0})\| &= \|(AX_i + BU_i) - (AX_{i0} + BU_i)\| \\ &< \|(AX_i + BU_i)\| + \|(AX_{i0} + BU_i)\| \simeq 0 \ \ \forall i. \end{aligned}$$

Therefore, we can claim that $X' = \cup_{i=1}^{q}(X_i - X_{i0})$ approximately spans the subspace corresponding to the eigenvectors corresponding to the small eigenvalues of A. As a result, Algorithm Z estimates the eigenvectors of matrix A regardless of its stability. Assuming that the eigenvectors can be estimated well enough, A can be stabilized by perturbing its eigenvalues while keeping its eigenvectors fixed. This gives rise to a revised optimization problem **P9** presented below:

$$\begin{aligned} \text{minimize} \quad & h\|D^{-1}(\lambda_A + \lambda)DX + BU\|_1 + (1-h)\sum_{i=1}^{n} \lambda_i^2 \\ \text{subject to} \quad & \lambda_A + \lambda > 0, \quad \lambda \in \Lambda_A, \end{aligned} \tag{6.11}$$

where Λ_A is the set of matrices having the canonical structure of the Jordan normal form of A. Now, we can derive the positive definite Lyapanov matrix P by solving equation (6.9) and then compute A and B by solving **P7** iteratively. This solution is implemented in Algorithm 4.

6.2.4 GRN Modeling As a Rank Constrained Problem

Using lemma 2.3.1, the minimization problem, corresponding to the GRN identification, can be equivalently cast as a rank constrained semi-definite program

$$\begin{aligned} \underset{A}{min} \ & t\|A\|_0 + (1-t)\epsilon \\ \text{s.t.} \ & \|AX + BU\|_1 \leq \epsilon, \quad \epsilon > 0, \\ & Y + Y^T \prec 0, \\ & P > 0, \quad A \in S, \\ & rank(\begin{bmatrix} P & Y \\ I & A^T \end{bmatrix}) = n. \end{aligned} \tag{6.12}$$

Therefore, the ADMM method is applicable in this case too. The user-defined design parameter t in (6.12) weighs the trad-off between the sparsity of the network matrix and the model fit error ϵ, however, in scenarios where the error ϵ is upper bounded by a pre-defined value, denoted by ϵ_u, the gene regulatory problem can be reformulated with cardinality of matrix A as the cost function, as shown below

$$\begin{aligned}
\min_{A} \quad & \|A\|_0 \\
\text{s.t.} \quad & \|AX + BU\|_1 \leq \epsilon_u, \\
& Y + Y^T \prec 0, \\
& P > 0, \quad A \in S, \\
& rank\left(\begin{bmatrix} P & Y \\ I & A^T \end{bmatrix}\right) = n.
\end{aligned} \tag{6.13}$$

Taking a closer look at the structure of minimization program 6.13 unveils its similarity to *s*parsity promoting feedback controller design problems; hence, it can be reformulated as a rank minimization problem and solved accordingly.

6.3 Results and Discussion

6.3.1 Comparison of Our Algorithms With the Algorithms Derived in [77]

We now present a brief case-study that compares the performance of our algorithms with that of the algorithms presented in [77] for the same synthetic dataset. For this comparison, a wide range of the parameter t is chosen. To provide results consistent with the ones given in [77], the *receiver operating characteristic* (ROC) curves are used as the performance measures. Following [77], we define *sensitivity* and *specificity* as follows:

$$\text{Sensitivity} = \frac{\text{The Number of Correctly Identified Non-Zero Elements}}{\text{The Number of Non-Zero Elements}},$$

$$\text{Specificity} = \frac{\text{The Number of Correctly Identified Zero Elements}}{\text{The Number of Zero Elements}}.$$

Clearly, an identification with 100% sensitivity and specificity is the best possible result. We used the method described in Section 5 of [77] to generate the 20×20 random sparse matrix A, and its associated dataset X as $X = -A^{-1}BU + \nu N$ where $BU \in \mathbb{R}^{n\times m}$ and $N \in \mathbb{R}^{n\times m}$ are zero mean and unit variance normally distributed random matrices. Then, we identified the system from both full datasets and partial datasets for several values of t. For the case of full dataset, the number of samples are equal to the dimension of the system matrix, i.e., $m = n$, the noise coefficient is $\nu = 10\%$, and a priori knowledge is available for 30% of the matrix entries. For the case of partial dataset, no a priori knowledge is available, the noise coefficient is $\nu = 50\%$, and the number of samples is roughly one third of the dimension of matrix A. The results are shown in Fig. 1 and Fig. 2. The simulation results show that our algorithms perform at least as well as the ones derived in [77]: the improvement is not surprising since besides reducing the conservatism in the stability constraint used in [77], we have not altered the structure of the algorithms [77] by a great extent.

6.3.2 Illustrative Example: GRN for Malaria Patients

Malaria is a mosquito-borne infectious disease caused in humans and other animals by eukaryotic protists of the genus *Plasmodium*. Five species of *Plasmodium* can

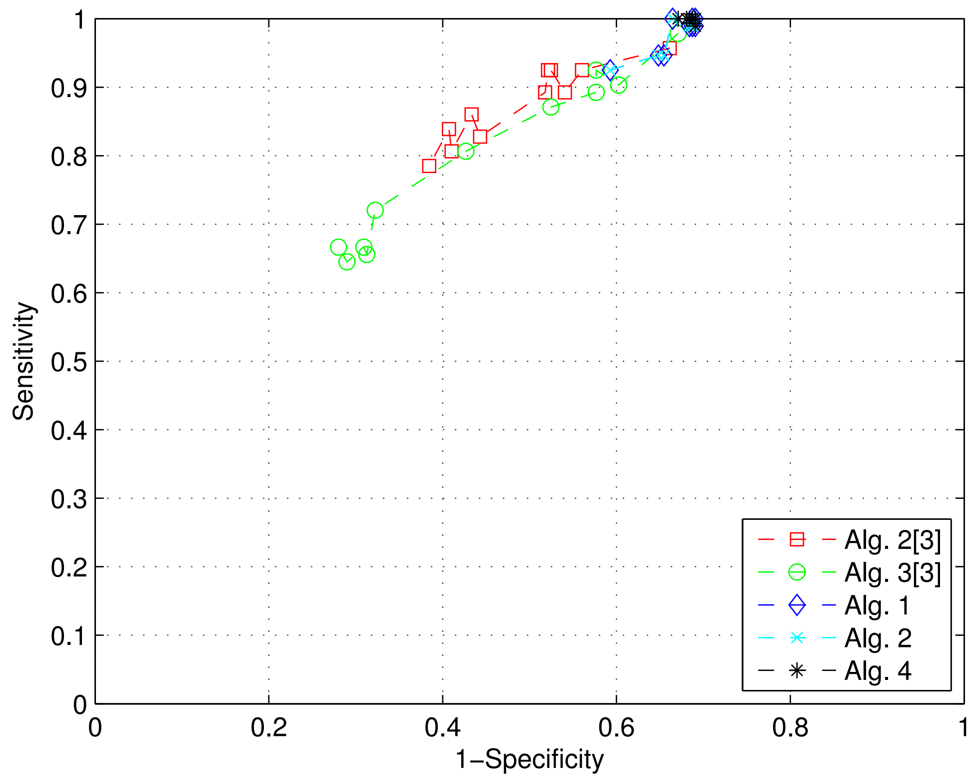


Figure 6.1: ROC plots of different algorithms for a network of size $n = 20$ and connectivity $c = 20\%$ using full data ($m = n$, $\sigma = 30\%$ and $\nu = 10\%$)

infect humans with this disease. Among these, the infection from *Plasmodium falciparum* can be fatal. The infection caused by others, including *Plasmodium vivax*, is rarely fatal. We now reconstruct the gene-protein regulatory network using two sets of expression data on 30 proteins collected from patients suffering from malaria. GeneSpring version 11.5.1 was used to perform the pathway analysis. GeneSpring has its own pathway database wherein the relations in the database were mainly derived from published literature abstracts using a proprietary Natural Language Processing (NLP) algorithm. Additional interactions from experimental data available in public repositories like IntAct were also included in the pathway database of GeneSpring.

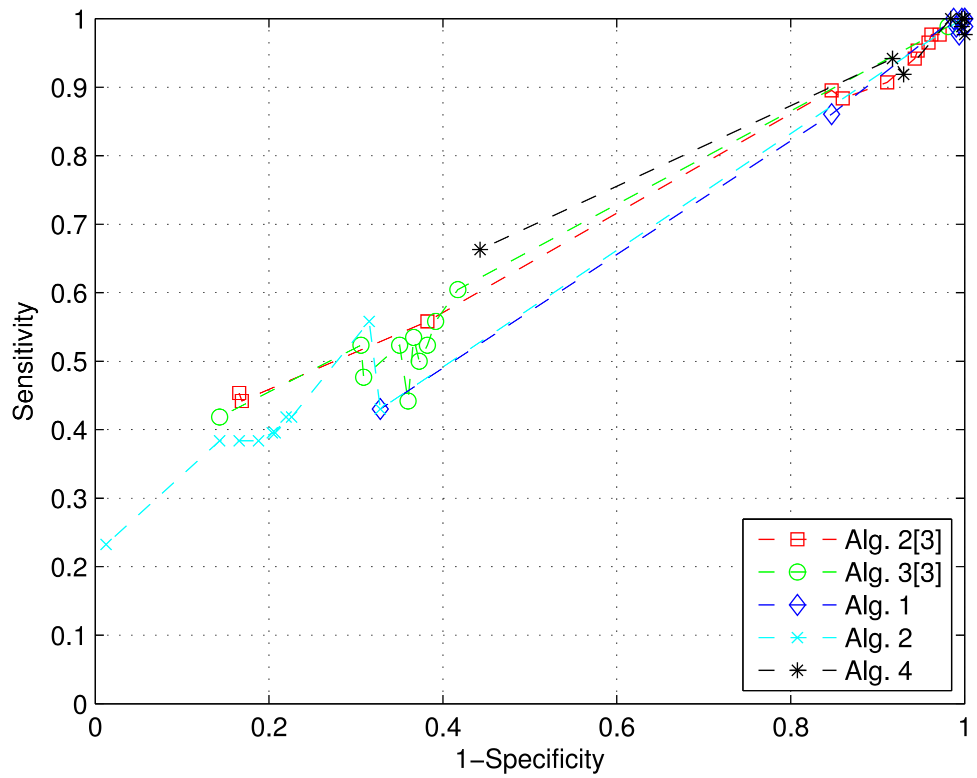


Figure 6.2: ROC plots of different algorithms for a network of size $n = 20$ and connectivity $c = 20\%$ using partial data ($m = \lceil \frac{n}{3} \rceil$, $\sigma = 0\%$ and $\nu = 50\%$)

The list of Entrez IDs corresponding to the proteins was used to find the key interactions involved in Malaria. The data collected from patients infected by *Plasmodium falciparum* is tagged FM whereas the data was collected from patients infected by *Plasmodium vivax* is tagged VM. In addition, we collected the expression data for healthy control samples as well. This data is tagged HC. In all, there are 8 sets of data for HC and a combined 8 sets of data for FM and VM.

$$X_1 = \begin{bmatrix} HC_{11} & VM_1 \\ HC_{12} & VM_2 \end{bmatrix}, \quad X_2 = \begin{bmatrix} HC_{21} & FM_1 \\ HC_{22} & FM_2 \end{bmatrix},$$

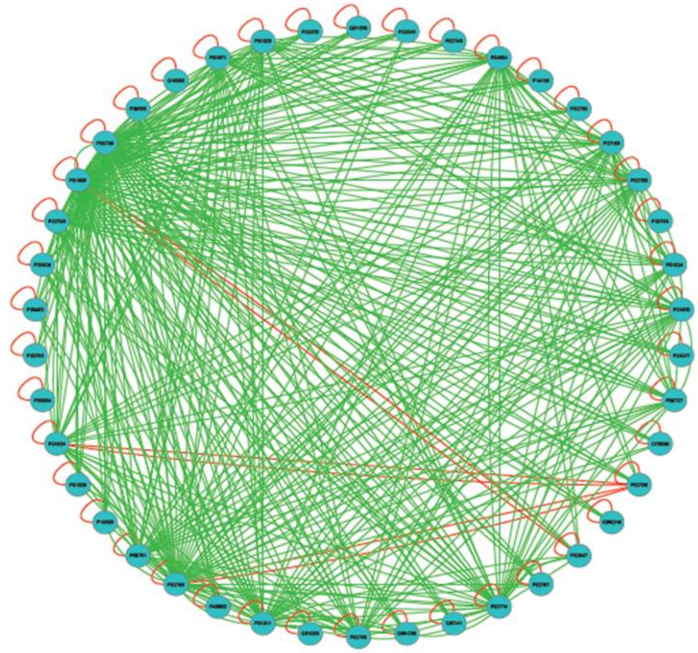


Figure 6.3: The gene-protein regulatory network in malaria affected patients. The network has 30 nodes. GeneSpring version 11.5.1 was used to perform the pathway analysis in data collected from hospital patients. Then, our algorithms to obtain linear ODE models of the form $\dot{x} = Ax + Bu$ were run on the data. This diagram illustrates the network interconnection, determined by the matrix A, and is created using Cytoscape. Green edges represent activation whereas red edges represent inhibition.

where $HC_{11} \in \mathbb{R}^{18\times 8}$, $VM_1 \in \mathbb{R}^{18\times 8}$, $HC_{12} \in \mathbb{R}^{12\times 8}$, $VM_2 \in \mathbb{R}^{12\times 8}$, $HC_{21} \in \mathbb{R}^{18\times 8}$, $FM_1 \in \mathbb{R}^{18\times 8}$, $HC_{22} \in \mathbb{R}^{12\times 8}$, and $FM_2 \in \mathbb{R}^{12\times 8}$. As can be seen, we partitioned the data rows into two parts (one with 18 rows and one with 12 rows). The reason is that among the proteins with available differential expression, only 18 are common in the two data sets, therefore, there are 12 proteins in each data set that expressed in only one type of Malaria. Since our objective was to derive a unified network model, we needed a method to somehow integrate these sets of data together. Hence, we used the average expression values of healthy control samples in one data set to replace the expression value data that are not exhibited in another data set. The reason behind what we did is that if a particular protein, for example $P00751$, is specific for Falciparum Malaria, it indicates there is no change in expression level in vivax malaria for that specific protein, hence, we can take the same value that is exhibited by healthy controls. Thus, our matrix $X \in \mathbb{R}^{42\times 32}$ is:

$$X = \begin{bmatrix} HC_{11} & HC_{21} & FM_1 & VM_1 \\ HC_{12} & \overline{HC}_{12} & \overline{HC}_{12} & VM_2 \\ \overline{HC}_{22} & HC_{22} & FM_2 & \overline{HC}_{22} \end{bmatrix},$$

where $\overline{M}$ represents a matrix with entries equal to the average of elements in the same row of matrix M. Taking each type of Malaria as an independent input to the system, i.e. $U_{FM} = [1\ 0]^{\mathrm{T}}$ and $U_{VM} = [0\ 1]^{\mathrm{T}}$, the input matrix $U \in \mathbb{R}^{2\times 30}$ corresponding to our dataset X is $U = [M_1\ M_2\ M_3]$, where $M_1 \in \mathbb{R}^{2\times 16}$ is an all-zero matrix, and $M_2, M_3 \in \mathbb{R}^{2\times 8}$ are given as

$$M_2 = \begin{bmatrix} 1\,1\ \dots\ 1 \\ 0\,0\ \dots\ 0 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 0\,0\ \dots\ 0 \\ 1\,1\ \dots\ 1 \end{bmatrix}.$$

Now, we can model the system as $\dot{X} = AX + BU$. Using the first 29 columns of X, we trained our network model using Algorithm Z and [77, Algorithm 2]. Verification of our results using the remaining columns of our data showed that [77, Algorithm 2] is not working in this case, and generates a very large error which may be caused by the very conservative stability condition laid down by Gershgorin's Circle Theorem. However, Algorithm [77, Algorithm 3] works properly with a fairly low error of $\|AX + BU\|_1 \simeq 0.01$. We used Cytoscape (see [92]) to visualize the matrix as a network of interactions. Interactions between all proteins in the matrix were specified in the *Simple Interaction File* (sif) format and were given to Cytoscape as the input. The SIF file lists each interaction using a source node, a relationship type (or edge type), and the target node. For example, for proteins P1 and P2, the structure **P1 1 P2** represents the relationship *P1 activates P2* and the structure **P1 -1 P2** represents the relationship *P1 inhibits P2*. The edges in the resulting network are colored by their interaction - a green edge represents activation and a red edge represents inhibitory interaction between the proteins. A representative network diagram is shown in Fig. 3.

6.4 Conclusion

We have presented a theoretical framework, and associated algorithms, to obtain a class of nonlinear *ordinary differential equation* (ODE) models of gene regulatory networks assuming the availability of literature curated data and microarray data. We build on a *linear matrix inequality* (LMI) based formulation developed recently by Zavlanos *et al* [77] to obtain linear ODE models of such networks. However, whereas the solution proposed in [77] requires that the microarray data be obtained as the outcome of a series of controlled experiments in which the network is perturbed by over-expressing one gene at a time, this requirement is not necessary to implement our approach. We have shown how the algorithms derived in [77] can be easily extended to derive the required stable linear ODE model. In addition, we have built on these algorithms by using new stability constraints that ensure the diagonal dominance of a given matrix: our case study on a synthetic dataset shows that our algorithms perform at least as well as those given in [77]. We have then presented a case-study of how these algorithms can be applied to derive a protein regulatory network model of malaria-infected patients. Our approach to network reconstruction differs from that of [93] in that [93] needs a large number of data samples that are in either a cue-response form or in a time-series form. Our approach to network reconstruction differs from that of [94] in that [94] mandates that the data samples should be the outcomes of independent perturbations to the so-called modules of the network. We have implemented our algorithms in MATLAB to successfully reconstruct a sparse 35-node network in which the maximum number of nodes adjacent to a node is 9.

6.5 Acknowledgments

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Part III

Summary and Directions for Future Work

Chapter 7

Summary and Directions for Future Work

7.1 Summary

With the recent growth of large scale systems and the failure of traditional control and identification methods in addressing their related issues, the problems of sparse/structured controller synthesis and system identification have received increasing attention in recent years. The main theme of this dissertation involves proposing methods to incorporate desired structures in controller design and system modeling problems. We proposed several approaches capable of addressing a wide range of issues emergent in the so called area, and proved that rank constrained optimization problem can be used as a unifying platform in solving many of the problems in this field.

We started with proposing a new framework for optimal sparse output feedback control design, which is capable of incorporating structural constraints on the feedback gain matrix as well as norm bounds on the inputs/outputs of the system. Then, we showed that problem can be converted to a rank constrained optimization problem with no other non-convex constraints. Using the proposed formulation, we presented an optimization problem which yields an upper bound for the optimal value of the optimal sparse state feedback control problem. Exploiting the relaxation of the ℓ_0-norm with the weighted ℓ_1-norm, we have also expressed that local optimum of the relaxed optimization problem, in its general form, can be obtained by performing ADMM algorithm. In chapter 2, a novel approach for the design of optimal sparse controllers is proposed. The new method is based on constructing a sparse controller by altering an available pre-designed controller towards a sparse controller, while heeding the performance deterioration caused by the process sparsification. We, again, showed that this problem can be equivalently reformulated into a fixed rank optimization problem. In the next chapter, we extend the results from chapter 2 so that they can be applied to the linear time invariant systems with parametric uncertainties. We started with formulating an optimization problem which seeks a sparse structured controller capable of exhibiting similar frequency and time characteristics of the predesigned controller, in the sense of $\mathcal{H}_2$ and $\mathcal{H}_\infty$ norms. By equivalently reformulating the problem into a fixed rank optimization, we again proposed to utilize the Alternating Direction Method of Multipliers (ADMM), modified to include weighted ℓ_1 norm minimization, as a computationally tractable algorithm to sub-optimally solve our problem.

In chapter 4, we adopted a totally different approach in sparse controller synthesis, as we utilized row sparsity as the measure for the sparsity of the controller gain. Without using any relaxation for the sparsity measuring function, a method is developed for the synthesis of controller with strict row sparsity constraints. We have shown that the recovery of a s-sparse vector can be formulated using a rank constraint formulation. Then, extended this results to propose an equivalent reformulation where all non-convex and combinatorial constraints are lumped into a single fixed rank constraint. Unlike, the common methods which use relaxation for the ℓ_0-norm of the vectors/matrices, we have not employ any form of relaxation; thus, our reformulation is exact. We further showed how our formulation can be extended to accommodate the row sparse control design paradigm. In addition, we proposed that the equivalent rank constrained optimization problem can be solved using a bi-linear optimization with convex constraints.

Part II of this dissertation, is dedicated to presenting a theoretical framework, and associated algorithms, to obtain a class of nonlinear *ordinary differential equation* (ODE) models of gene regulatory networks assuming the availability of literature curated data and microarray data. In addition to proposing several algorithms to obtain linear ODE models of such networks, we showed that this problem can also be cast as a rank constrained optimization problem. Therefore, a majority of the results derived in Part I of this dissertation are applicable to this problem as well.

7.2 Future Works

The area of sparse/structure control is still in its infancy; hence, there are still a large number of problems ready to be explored. We believe that the framework proposed in this dissertation can pave the way in solving many of the unsolved issues in this area. However, further exploration is still needed to make the method of rank constrained optimization, proposed here, perfect and applicable to large scale systems.

One area of study is to develop new optimization methods that can solve the rank constrained optimization problem in a timely manner. Although our proposed methods can theoretically address a wide range of controller sparsification problems, there are still challenges to overcome. One may be interested in studying the convergence properties of our proposed ADMM method, as it still lacks a convergence proof. As for the iterative solution of the bi-linear optimization, the convergence is theoretically guaranteed; however, we sometimes witness numerical anomalies in our simulations. In our opinion, such numerical issues originate from the bugs in optimization softwares, such as CVX, used for the simulation purposes. Therefore, a proper continuation of this work can be developing a dedicated optimization software such that not only the numerical issues are resolved, but also larger size problems can be solved in a timely manner.

As said before, the main focus of this dissertation is developing methods to sparsify the controller gains. Based on the results provided here, a nice research direction is to explore sparsity of the control signal vector. The important of such a study lies in the fact that sparsifying the control signal may lead to lower energy consumption in the control systems. Another interesting research direction in this area would be studying the sparsity of the control signal in time domain, i. e. reducing the frequency of control command transmission. This is of great importance, especially in battery operated wireless control systems, as in such systems it is desired to maintain the performance of the system with the minimum packet transmission rate due the limited power of the batteries.

Despite being mentioned at the end, extending our results to the more general case of dynamic feedback design problems, where controllers consist of dynamic subsystems connected through sparse communication networks, would be a great avenue for future researches.

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