ABSTRACT

Title of dissertation: MATHEMATICAL TOPICS IN FLUID-PARTICLE INTERACTION

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Models for particles interacting with compressible fluids are useful to several areas of science. This dissertation considers some of the mathematical issues of the Navier-Stokes-Smoluchowski and Euler-Smoluchowski models for compressible fluids. First, well-posedness for the NSS system is investigated. Among the results are the existence of weakly dissipative solutions obeying a relative entropy inequality. An approximating scheme using an artificial pressure and vanishing viscosity is employed to this end. The existence of these weakly dissipative solutions is used to show a weak-strong uniqueness result, using a Gronwall's argument on the relative entropy inequality. The existence of smooth solutions for finite time to the NSS system under certain compatibility conditions is shown using an iterative approximation.

Next, two scaled regimes for the NSS system are considered. It is shown that for these low Mach number regimes, the solutions of the compressible system can be approximated by solutions of simpler models. In particular, the solutions to the model in a low stratification regime can be approximated by solutions to a model for

incompressible flows with a Boussinesq relation. Solutions to the model in a strong stratification regime can be approximated by solutions to a model for anelastic flows. Much of the analysis for these limits relies on a Helmholtz free energy inequality, which bounds many of the quantities needed for the analysis.

Lastly, the Euler-Smoluchowski model for inviscid, compressible fluids is considered. Finite-time existence of smooth solutions is shown using an iterative approximation and the results of Friedrichs and Majda for existence of smooth solutions for symmetric hyperbolic systems.

MATHEMATICAL TOPICS IN FLUID-PARTICLE INTERACTION

by

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Dedication

To my family and friends, without whose support my successes would not be possible.

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Undoubtedly, I have omitted some here who have given me support, and I apologize to those I have inadvertently left out.

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List of Abbreviations

NSS	Navier-Stokes-Smoluchowski
ES	Euler-Smoluchowski
$C(\Omega; X)$	$:= \{f : \Omega \mapsto X$ such that f is continuous$\}$
$C(\Omega)$	$:= C(\Omega; \mathbb{R})$
$C_c(\Omega; X)$	Set of continuous functions $f : \Omega \mapsto X$ with compact support
$C_c(\Omega)$	$:= C_c(\Omega; \mathbb{R})$
$\mathcal{D}$	Signifies the set of test functions
$C^{k,\alpha}(\Omega; X)$	$:= \{f : \Omega \mapsto X$ such that $D^\beta f$ is continous for all multi-indices $\|\beta\| \leq k$ and $\sup_{x,y\in\Omega, x\neq y} \frac{\|D^\beta f(x) - D^\beta f(y)\|}{\|x-y\|^\alpha} < \infty$ for multi-indices $\|\beta\| = k\}$
$C^{k,\alpha}(\Omega)$	$:= C^{k,\alpha}(\Omega; \mathbb{R})$
$L^p(\Omega; X)$	$:= \{f : \Omega \mapsto X$ such that $\int_\Omega \|f\|^p \,\mathrm{d}x < \infty\}$
$L^p(\Omega)$	$:= L^p(\Omega; \mathbb{R})$
$L^p_+(\Omega)$	$:= \{f \in L^p(\Omega)$ such that $f(x) \geq 0 \;\forall x \in \Omega\}$
$L^p(0, T; X)$	Set of functions f such that $\int_0^T \\|f\\|_X^p \,\mathrm{d}t < \infty$
$W^{k,p}(\Omega; X)$	$:= \{f : \Omega \mapsto X$ such that $D^\alpha f \in L^p(\Omega; X)$ for all multi-indices α such that $\|\alpha\| \leq k\}$
$W^{k,p}_0(\Omega; X)$	Set of functions in $W^{k,p}(\Omega; X)$ with zero trace
$W^{k,p}(\Omega)$	$:= W^{k,p}(\Omega; \mathbb{R})$
$W^{k,p}_0(\Omega)$	$:= W^{k,p}_0(\Omega; \mathbb{R})$
ϱ	Fluid mass density
$\mathbf{u}$	Fluid velocity field
η	Particle density
$\mathbf{n}$	Outward unit normal vector
∇_x	Spatial gradient operator
∇_x^2	Hessian operator
div_x	Spatial divergence operator
Δ_x	$\mathrm{div}_x \nabla_x$
D	Particle dispersion coefficient
ζ	Particle-fluid drag coefficient
μ	Shear viscosity coefficient
λ	Bulk viscosity coefficient
$\mathbb{S}$	Stress tensor
Φ	External potential
$\in_b$	Contained and bounded in

Chapter 1

Introduction

Fluid-particle systems encountered in many scientific and engineering applications pose significant modeling and analytical challenges, and are of great significance in sedimentation analysis of disperse suspensions of particles in fluids. One of the challenges in this context is the separation of the solid grains from the fluid by external forces such as settling processes due to gravitation or such as centrifugal forces. These procedures have applications in fields such as biotechnology, medicine, waste-water recycling and mineral processing, as well as in combustion theory, when modeling Diesel engines or rocket propulsors.

In what follows, the focus is on the macroscopic description of the dispersed phase obtained by taking averages with respect to the microscopic variable ξ of the probability distribution function $f(t,x,\xi)$, with $f(t,x,\xi)\mathrm{d}\xi\ \mathrm{d}x$ denoting the number of particles enclosed at time $t \geq 0$ in the infinitesimal domain on the phase space centered on $(x,\xi) \in \mathbb{R}^3 \times \mathbb{R}^3$, with volume $\mathrm{d}\xi\ \mathrm{d}x$.

It is assumed throughout the dissertation that $\Omega \subset \mathbb{R}^3$ is a $C^{2,\nu}$ spatial domain for some $\nu > 0$ and that $t \in (0,T)$ for some $0 < T \leq \infty$. In the macroscopic description, the density of the particles $\eta(t,x)$ is related to the probability distribution function $f(t,x,\xi)$ through the relation

$$\eta(t,x) = \int_{\mathbb{R}^3} f(t,x,\xi)\,\mathrm{d}\xi,$$

the fluid mass density is denoted by $\varrho(t,x)$, and the fluid velocity field is given by $\mathbf{u}(t,x)$.

1.1 Navier-Stokes-Smoluchowski System

In this context, the primitive conservation equations governing fluid-particle flows in the bubbling regime express the conservation of mass, the balance of momentum, and the balance of particle densities often referred as the Smoluchowski equation and are given as follows:

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0 \tag{1.1}$$

$$\partial_t(\varrho\mathbf{u}) + \operatorname{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) + \nabla_x(p_F(\varrho)+\eta) - \mu\Delta_x\mathbf{u} - \lambda\nabla_x \operatorname{div}_x \mathbf{u} = -(\beta\varrho+\eta)\nabla_x\Phi \tag{1.2}$$

$$\partial_t \eta + \operatorname{div}_x(\eta\mathbf{u} - \eta\nabla_x\Phi) - \Delta\eta = 0. \tag{1.3}$$

Constitutive relations between certain quantities are given below.

- The fluid pressure p_F is taken to be

$$p_F(\varrho) := a\varrho^\gamma, \text{ with } a > 0 \text{ and } \gamma > \frac{3}{2}.$$

- The total pressure $P = P(\varrho,\eta)$ in the mixture depends on the density of the particles and the density of the fluid and is given by

$$P(\varrho,\eta) = p_F(\varrho) + \eta.$$

- The viscous stress tensor $\mathbb{S} = \mathbb{S}(\nabla_x \mathbf{u})$ is assumed to satisfy *Newton's Law for Viscosity* which requires that

$$\mathbb{S} = \mu(\nabla \mathbf{u} + \nabla \mathbf{u}^T) + \lambda \operatorname{div}_x \mathbf{u}\, \mathbb{I},$$

where μ and λ are constant viscosity coefficients satisfying

$$\mu > 0,\ \ \lambda + \frac{2}{3}\mu \geq 0.$$

Thus,

$$\operatorname{div}_x \mathbb{S}(\nabla_x \mathbf{u}) = \mu \Delta_x \mathbf{u} + \lambda \nabla_x \operatorname{div}_x \mathbf{u}.$$

The external potential

$$\Phi : \Omega \to \mathbb{R}^+$$

represents the effects of gravity and buoyancy and β in (1.2) is a constant reflecting the differences in how the external force affects the fluid and the particles.

The no-slip boundary condition is imposed for the velocity vector leading to a no-flux condition for the fluid density through the boundaries and the no-flux condition for the particle density leading to the boundary conditions

$$\mathbf{u}|_{\partial\Omega} = \nabla_x \eta \cdot \mathbf{n} + \eta \nabla_x \Phi \cdot \mathbf{n} = 0 \ \text{ on } \ (0,T) \times \partial\Omega, \tag{1.4}$$

with $\mathbf{n}$ denoting the outer normal vector to the boundary $\partial\Omega$. The problem is supplemented with the initial data $\{\varrho_0, \mathbf{m}_0, \eta_0\}$ such that

$$\begin{aligned} \varrho(0,x) &= \varrho_0 \in L^\gamma(\Omega) \cap L^1_+(\Omega), \\ (\varrho \mathbf{u})(0,x) &= \mathbf{m}_0 \in L^{\frac{6}{5}}(\Omega) \cap L^1(\Omega), \\ \eta(0,x) &= \eta_0 \in L^2(\Omega) \cap L^1_+(\Omega). \end{aligned} \tag{1.5}$$

The total energy of the system is given by

$$\mathcal{F}(\eta,\varrho,\mathbf{u})(t) := \int_\Omega \left[\frac{1}{2}\varrho(t)|\mathbf{u}(t)|^2 + \frac{a}{\gamma-1}\varrho^\gamma(t) + (\eta\ln\eta)(t) + (\beta\varrho+\eta)\Phi\right]\,\mathrm{d}x(t) \tag{1.6}$$

At the formal level, the total energy can be viewed as a Lyapunov function satisfying the energy inequality

$$\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}t} + \int_\Omega \left[\mu|\nabla_x\mathbf{u}|^2 + \lambda|\operatorname{div}_x\mathbf{u}|^2 + |2\nabla_x\sqrt{\eta} + \sqrt{\eta}\nabla_x\Phi|^2\right]\,\mathrm{d}x \le 0. \tag{1.7}$$

System (1.1)-(1.3) is derived by formal asymptotics from a mesoscopic description in similar to the inviscid model investigated in [13], which is expanded to a viscous fluid by an argument in [15]. This is based on a kinetic equation for the particle distribution of Fokker-Planck type coupled to fluid equations. In this scaling limit, particles are assumed to have a negligible density with respect to the fluid, and due to buoyancy effects, they typically move upwards in a system under gravity. For that reason this scaling regime is known as the bubbling regime. This limit for an analogous flowing regime problem is derived rigorously by Mellet and Vasseur in [41].

The coupling between the kinetic and the fluid equations is obtained through the friction forces that the fluid and the particles exert mutually. The friction force is assumed to follow Stokes' law and thus is proportional to the relative velocity vector, namely

$$F_\varepsilon = \int_{\mathbb{R}^3}\left(\frac{\xi}{\sqrt{\varepsilon}} - u_\varepsilon(t,x)\right) f(t,x,\xi)\,\mathrm{d}\xi.$$

This forcing term affects the momentum equation in the Navier-Stokes system which is now enhanced by an additional forcing term taking into account the action of the cloud of particles on the fluid. The cloud of particles is described by its distribution function $f_\varepsilon(t,x,\xi)$ on phase space, which is the solution to the dimensionless Vlasov-Fokker-Planck equation

$$\partial f_\varepsilon + \frac{1}{\sqrt{\varepsilon}}\Big(\xi\cdot\nabla_x f_\varepsilon - \nabla_x\Phi\cdot\nabla_\xi f_\varepsilon\Big) = \frac{1}{\varepsilon}\mathrm{div}_\xi\Big(\big(\xi-\sqrt{\varepsilon}u_\varepsilon\big)f+\nabla_\xi f_\varepsilon\Big). \tag{1.8}$$

Here, $\varepsilon > 0$ is a dimensionless parameter and the drag force is independent of the fluid density ϱ_ε, but proportional to the relative velocity of the fluid and the particles.

1.2 Confinement Hypotheses

Part of this work considers weak solutions to the two-phase flow problem (1.1)-(1.3) in two different geometrical constraints of interest in the applications: for bounded domains and for unbounded domains under confinement conditions due to the external potential. The assumptions concerning the geometry Ω and the external potential Φ are collected under the generic name of *confinement hypotheses*. The external potential Φ is always defined up to a constant; therefore, for external potentials Φ which are bounded below, it is assumed without loss of generality by adding a suitable constant that

$$\inf_{x\in\Omega}\Phi(x) = 0. \tag{1.9}$$

Definition 1.2.1. Given a domain $\Omega \in C^{2,\nu}$, $\nu > 0$, $\Omega\subset\mathbb{R}^3$, and given a bounded-below external potential $\Phi : \Omega \longrightarrow \mathbb{R}_0^+$ satisfying (1.9), (Ω,Φ) verifies the confinement hypotheses **(HC)** for the two-phase flow system (1.1)-(1.3) coupled with no-flux boundary conditions (1.4) whenever:

(HC-Bounded) If Ω is bounded, Φ is bounded and Lipschitz continuous in $\overline{\Omega}$ and the sub-level sets $[\Phi < k]$ are connected in Ω for any $k > 0$.

(HC-Unbounded) If Ω is unbounded, $\Phi \in W^{1,\infty}_{\text{loc}}(\Omega)$, $\beta > 0$, the sub-level sets $[\Phi < k]$ are connected in Ω for any $k > 0$,

$$e^{-\Phi/2} \in L^1(\Omega),$$

and

$$|\Delta\Phi(x)| \leq c_1 |\nabla_x \Phi(x)| \leq c_2 \Phi(x), \;\; |x| > R, \tag{1.10}$$

for some large $R > 0$.

Remark 1.2.1. The condition on the connectedness of the sublevel sets is needed to show long-time behavior toward a steady-state solution in [15]. It is not needed for the work presented here, but is mentioned for the sake of completeness.

The confinement assumption **(HC)** has physical relevance in the setting under consideration as it is verified for several domains Ω with Φ being the gravitational potential. For instance,

1. when $\Omega = \{x \in \mathbb{R}^3 \,|\, (x_1, x_2) \in [a,b]^2, \, x_3 \in [0, H]\}$ and $\Phi(x) = g x_3$, where $\beta = 1 - \frac{\varrho_F}{\varrho_P}$.

2. when $\Omega = \{x \in \mathbb{R}^3 \,|\, (x_1, x_2) \in [a,b]^2, \, x_3 > 0\}$ and $\Phi(x) = g x_3$, where $\beta = 1 - \frac{\varrho_F}{\varrho_P}$ and $\varrho_F < \varrho_P$.

3. when $\Omega = \mathbb{R}^3 \setminus \overline{B(0,R)}$ and $\Phi(x) = g|x|$, where $B(0,R)$ is the ball centered at the origin with radius R and $\beta > 0$.

Here, ϱ_F and ϱ_P are the typical mass density of fluid and particles, respectively. Note that Example 1 corresponds to the standard bubbling case (see [13]) in which particles move upwards due to buoyancy.

1.3 Euler-Smoluchowski System

In addition to the Navier-Stokes-Smoluchowski system for viscous fluids, this dissertation also considers the Euler-Smoluchowski system for inviscid, compressible fluids. This work is inspired by work done in [48] on the Euler equations for compressible fluids. In this paper, Sideris *et al.* examine long-time behavior for the Euler system with a damping forcing term. They note that finite-time existence of smooth solutions follows from the work of Friedrichs in [35] and [36] and of Majda in [40] and transforming the Euler system into a symmetric hyperbolic system.

The Euler-Smoluchowski model considered in this dissertation is as follows.

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0 \tag{1.11}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x(p_F(\varrho) + \eta) = -(\beta\varrho + \eta)\nabla_x \Phi \tag{1.12}$$

$$\partial_t \eta + \mathrm{div}_x(\eta \mathbf{u} - \eta \nabla_x \Phi) - \Delta_x \eta = 0 \tag{1.13}$$

In addition, the spatial domain is taken to be $\mathbb{R}^3$. With the addtion of (1.13), the system loses hyperbolicity. However, considering the observation that (1.11)-(1.12) form a hyperbolic system if η is known, an iterative approximation scheme detailed in Chapter 4 is used to establish finite-time existence of solutions to (1.11)-(1.13). Then like in [48], an energy inequality is used with a physical dissipative condition on Φ to determine long-time behavior of solutions.

1.4 Outline of Dissertation

The rest of this dissertation discusses various mathematical results for the NSS and ES models. The results described herein are outlined below.

1. Chapter 2 presents existence and regularity results for the NSS model. More specifically, a previous result from [15] is presented establishing the existence of *renormalized weak solutions* to the model. Then, the existence of a new class of solutions, *weakly dissipative solutions*, which obey a relative entropy inequality, is proven. This relative entropy inequality is then used to establish a *weak-strong uniqueness* result, which states that if a solution of a certain regularity class exists, the weakly dissipative solution coincides with the solution of the proposed regularity class. This is the focus of the candidate's work in [7]. Finally, the existence of such strong solutions is tackled, establishing *compatibility conditions* for which smooth solutions of the NSS system will exist, at least for finite time. This is also discussed in the candidate's work in [3].

2. Chapter 3 explores certain scaling regimes for which the NSS system can be approximated by simpler models. In particular, situations in which the speed of the fluid is small compared to the speed of sound in the fluid are considered.

In the low stratification case, it is shown that the model can be approximated by a model for incompressible fluids supplemented with a Boussinesq relation. A strong stratification case is also explored and an approximation with an anelastic condition is considered. These results also are explored in the candidate's work in [5] and [4].

3. Chapter 4 considers an inviscid model, the Euler-Smoluchowski system. Here, the viscosity coefficients μ and λ are taken to be zero. The existence of smooth solutions for finite time is shown for appropriate initial data. This is accomplished using an iterative approximation similar to that used for the finite-time existence of smooth solutions to the NSS system.

4. Chapter 5 provides a summary of the results in this dissertation. In addition, directions for future research are suggested.

Chapter 2

Existence and Uniqueness of Solutions

2.1 Existence of Weak Solutions

The existence of weak solutions to (1.1)-(1.7) in the sense of the following definiton was proven using in [15]. In this paper, the authors use a time discretization approximation and show the convergence of these approximate solutions to solutions of the NSS system in the following sense.

Definition 2.1.1. Assume that (Ω, Φ) satisfy the confinement hypotheses **(HC)**. $\{\varrho, \mathbf{u}, \eta\}$ is a free-energy solution of problem (1.1)-(1.3) supplemented with boundary data satisfying (1.4) and initial data $\{\varrho_0, \mathbf{m}_0, \eta_0\}$ satisfying (1.5) provided that the following hold:

- $\varrho \geq 0$ in $L^\infty(0,T; L^\gamma(\Omega))$ represents a renormalized solution of (1.1) on $(0,\infty)\times\Omega$, i.e., for any test function $\phi \in \mathcal{D}([0,T)\times\overline{\Omega}), T > 0$ and any b, B such that

$$b \in L^\infty([0,\infty)) \cap C([0,\infty)),\ B(\varrho) := B(1) + \int_1^\varrho \frac{b(z)}{z^2} dz,$$

the renormalized continuity equation

$$\int_0^T \int_\Omega B(\varrho)\partial_t \phi + B(\varrho)\mathbf{u}\cdot\nabla_x\phi - b(\varrho)\phi \operatorname{div}_x \mathbf{u} \,\mathrm{d}x\,\mathrm{d}t = -\int_\Omega B(\varrho_0)\phi(0,\cdot)\,\mathrm{d}x \tag{2.1}$$

holds.

- The balance of momentum holds in the sense of distributions, i.e., for any $\mathbf{w} \in \mathcal{D}([0,T);\mathcal{D}(\overline{\Omega};\mathbb{R}^3))$,

$$\int_0^T \int_\Omega \varrho \mathbf{u} \cdot \partial_t \mathbf{w} + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \mathbf{w} + (p_F(\varrho) + \eta) \operatorname{div}_x \mathbf{w} \, \mathrm{d}x \, \mathrm{d}t$$
$$= \int_0^\infty \int_\Omega \mu \nabla_x \mathbf{u} \nabla_x \mathbf{w} + \lambda \operatorname{div}_x \mathbf{u} \operatorname{div}_x \mathbf{w} - (\beta \varrho + \eta) \nabla_x \Phi \cdot \mathbf{w} \, \mathrm{d}x \, \mathrm{d}t - \int_\Omega \mathbf{m}_0 \cdot \mathbf{w}(0,\cdot) \, \mathrm{d}x \tag{2.2}$$

  All quantities are required to be integrable, so in particular, $\mathbf{u} \in L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3))$, thus the velocity field can be required to vanish on $\partial\Omega$ in the sense of traces.

- $\eta \geq 0$ is a weak solution of (1.3), i.e.,

$$\int_0^T \int_\Omega \eta \partial_t \phi + \eta \mathbf{u} \cdot \nabla_x \phi - \eta \nabla_x \Phi \cdot \nabla_x \phi - \nabla_x \eta \cdot \nabla_x \phi \, \mathrm{d}x \, \mathrm{d}t = -\int_\Omega \eta_0 \phi(0,\cdot) \, \mathrm{d}x \tag{2.3}$$

  Again, terms in this equation must be integrable on $(0,T) \times \Omega$, so in particular $\eta \in L^2(0,T;L^3(\Omega)) \cap L^1(0,T;W^{1,\frac{3}{2}}(\Omega))$.

- The energy of the system

$$\mathcal{F}(\varrho,\mathbf{u},\eta)(\tau) := \int_\Omega \frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{a}{\gamma-1}\varrho^\gamma + \eta \ln \eta + (\beta\varrho + \eta)\Phi \, \mathrm{d}x(\tau)$$

  is finite and bounded by the initial energy. Also

$$\int_0^T \int_\Omega \mu|\nabla_x \mathbf{u}|^2 + \lambda|\operatorname{div}_x \mathbf{u}|^2 + |2\nabla_x \sqrt{\eta} + \sqrt{\eta}\nabla_x \Phi|^2 \, \mathrm{d}x \, \mathrm{d}t \leq \mathcal{F}(\varrho_0,\mathbf{u}_0,\eta_0)$$

2.2 Existence of Weakly Dissipative Solutions and Weak-Strong Uniqueness

Motivated by the stability arguments in [13], the numerical investigation presented in [14], a number of studies on numerical experiments and scale analysis on the NSS model (see [8]), as well as the analytical results in [15], this section of the dissertation investigates the issue of *weak-strong uniqueness*, presenting a new class of weak solutions with additional regularity properties. There are many results, mostly devoted to the incompressible Navier-Stokes system, concerning conditional regularity of the weak solutions. Roughly speaking, these results are that the weak solutions are regular as soon as they belong to a critical regularity class. Results in this direction are presented in Prodi [46], Serrin [47], or more recently, Neustupa *et al.* [43], [44]. In the context of compressible fluids related results are presented by Feireisl, Jin, and Novotný in [31], by Feireisl, Novotný, and Sun in [34], and by Mellet and Vasseur in [41]. The present work establishes the existence of weakly dissipative solutions obeying a relative entropy inequality. The results and ingredients of the approach can be formulated as follows:

- An inherent definition of *weakly dissipative solutions* to the Navier-Stokes Smoluchowski system (1.1) -(1.3) is introduced satisfying a relative entropy inequality with respect to any hypothetical strong solution to the problem. The analysis is motivated by the pioneering work of Dafermos [21] and DiPerna [23], the results of Germain [37], the analysis of Mellet and Vassuer [41] as well as the approach of Feireisl *et al.* [34]. The global existence of weakly-dissipative solutions is established by the construction of an approximating scheme with approximate solutions obeying an approximate relative entropy inequality. Convergence arguments are employed to show that the solutions to the actual system obey the given relative entropy inequality.

- By monitoring the evolution of a relative entropy functional and by employing an argument using Gronwall's Lemma, a weak-strong uniqueness result is established stating that a weakly dissipative solution agrees with a classical solution with the same initial data when such a classical solution exists.

- Physically grounded hypotheses are imposed on the domain Ω and the external potential Φ (confinement hypotheses **(HC)**). The analysis herein treats both the case of a bounded physical domain Ω as well as the case of an unbounded domain. The confinement hypotheses **(HC)** on (Ω, Φ) plays a crucial role in providing control of the negative contribution of the physical entropy $\eta \ln \eta$ in the free-energy bounds for unbounded domains.

2.2.1 Relative Entropy

In the spirit of Dafermos [21], given an entropy $\mathcal{E}(U)$ the relative entropy is defined as

$$\mathcal{H}(U|\overline{U}) := \mathcal{E}(U) - \mathcal{E}(\overline{U}) - D\mathcal{E}(\overline{U}) \cdot (U - \overline{U}) \tag{2.4}$$

where D stands for the total differentiation operator with respect to $\varrho, \mathbf{m}, \text{and}\, \eta$. In the present context,

$$U = \begin{bmatrix} \varrho \\ \mathbf{m} := \varrho \mathbf{u} \\ \eta \end{bmatrix}, \quad \overline{U} = \begin{bmatrix} r \\ \overline{\mathbf{m}} := r\mathbf{U} \\ s \end{bmatrix}$$

where $\overline{U}$ can be considered to be a smooth solution and

$$\mathcal{E}(U) := \frac{|\mathbf{m}|^2}{2\varrho} + \frac{a}{\gamma - 1}\varrho^\gamma + \eta \ln \eta + (\beta \varrho + \eta)\Phi. \tag{2.5}$$

Thus, from the definition, the relative entropy is

$$\begin{aligned}
\mathcal{H}(U|\overline{U}) &= \frac{|\mathbf{m}|^2}{2\varrho} + \frac{a}{\gamma - 1}\varrho^\gamma + \eta \ln \eta + (\beta \varrho + \eta)\Phi \\
&- \frac{|\overline{\mathbf{m}}|^2}{2r} - \frac{a}{\gamma - 1} r^\gamma - s \ln s - (\beta r + s)\Phi \\
&- \begin{bmatrix} -\frac{|\mathbf{U}|^2}{2} + \frac{a\gamma}{\gamma - 1} r^{\gamma - 1} + \beta \Phi \\ \mathbf{U} \\ \ln s + 1 + \Phi \end{bmatrix} \cdot \begin{bmatrix} \varrho - r \\ \varrho \mathbf{u} - r\mathbf{U} \\ \eta - s \end{bmatrix} \\
&= \frac{\varrho |\mathbf{u}|^2}{2} + \frac{a}{\gamma - 1}\varrho^\gamma + \eta \ln \eta + \beta \varrho \Phi + \eta \Phi \\
&- \frac{r|\mathbf{U}|^2}{2} - \frac{a}{\gamma - 1} r^\gamma - s \ln s - \beta r \Phi - s\Phi \\
&+ \frac{\varrho |\mathbf{U}|^2}{2} - \frac{r|\mathbf{U}|^2}{2} - \frac{a\gamma}{\gamma - 1} r^{\gamma - 1}\varrho + \frac{a\gamma}{\gamma - 1} r^\gamma - \beta \varrho \Phi + \beta r \Phi \\
&- \varrho \mathbf{u} \cdot \mathbf{U} + r|\mathbf{U}|^2 - \eta \ln s + s \ln s - \eta + s - \eta \Phi + s\Phi
\end{aligned} \tag{2.6}$$

After some basic calculations, the relative entropy is calculated to be

$$\mathcal{H}(U|\overline{U}) = \frac{\varrho}{2}|\mathbf{u} - \mathbf{U}|^2 + \frac{a}{\gamma - 1}\left(\varrho^\gamma - r^\gamma\right) - \frac{a\gamma}{\gamma - 1} r^{\gamma - 1}(\varrho - r)$$

$$+\eta \ln \eta - s \ln s - (\ln s + 1)(\eta - s), \tag{2.7}$$

or equivalently,

$$\mathcal{H}(U|\overline{U}) = \frac{\varrho}{2}|\mathbf{u} - \mathbf{U}|^2 + E_F(\varrho, r) + E_P(\eta, s),$$

where

$$H_F(\varrho) := \frac{a}{\gamma - 1}\varrho^\gamma$$

$$P_F(\varrho) := H_F'(\varrho) = \frac{a\gamma}{\gamma - 1}\varrho^{\gamma - 1}$$

$$E_F(\varrho, r) := H_F(\varrho) - H_F'(r)(\varrho - r) - H_F(r)$$

$$H_P(\eta) := \eta \ln \eta$$

$$P_P(\eta) := H_P'(\eta) = \ln \eta + 1$$

$$E_P(\eta, s) := H_P(\eta) - H_P'(s)(\eta - s) - H_P(s)$$

Remark 2.2.1. The integrals of the quantities H_F and H_P over Ω represent the physical quantities of the entropy of the fluid and the entropy of the particles, respectively.

Note that the relative entropy does not contain any information regarding the external potential Φ. This is expected since one of the motivations of the relative entropy functional is to reflect information about quadratic terms, but not linear terms. Next, *weakly dissipative solutions* are defined using the ideas of relative entropy. The key addition to the definition of weak solutions is the *relative entropy inequality.* Letting

$$r = r(t, x),\ \ \mathbf{U} = \mathbf{U}(t, x),\ \ s = s(t, x)$$

be smooth functions on $[0,T]\times\overline{\Omega}$ with $r,s>0$ on $[0,T]\times\overline{\Omega}$ and

$$\mathbf{U}|_{\partial\Omega}=0,$$

it is shown in Section 2.2.8 that for $\{\varrho,\mathbf{u},\eta\}$,

$$\begin{aligned}
&\int_\Omega \frac{1}{2}\varrho|\mathbf{u}-\mathbf{U}|^2+E_F(\varrho,r)+E_P(\eta,s)\ \mathrm{d}x(\tau)\\
&+\int_0^\tau\int_\Omega[\mathbb{S}(\nabla_x\mathbf{u})-\mathbb{S}(\nabla_x\mathbf{U})]:\nabla_x(\mathbf{u}-\mathbf{U})\ \mathrm{d}x\ \mathrm{d}t\\
&\leq\int_\Omega\frac{1}{2}\varrho_0|\mathbf{u}_0-\mathbf{U}_0|^2+E_F(\varrho_0,r_0)+E_P(\eta_0,s_0)\ \mathrm{d}x+\int_0^\tau\mathcal{R}(\varrho,\mathbf{u},\eta,r,\mathbf{U},s)\ \mathrm{d}t
\end{aligned}\tag{2.8}$$

where

$$\begin{aligned}
&\mathcal{R}(\varrho,\mathbf{u},\eta,r,\mathbf{U},s)\\
&\quad:=\int_\Omega \mathrm{div}_x(\mathbb{S}(\nabla_x\mathbf{U}))\cdot(\mathbf{U}-\mathbf{u})\ \mathrm{d}x\\
&\quad-\int_\Omega\varrho(\partial_t\mathbf{U}+\mathbf{u}\cdot\nabla_x\mathbf{U})\cdot(\mathbf{u}-\mathbf{U})\ \mathrm{d}x\\
&\quad-\int_\Omega\partial_tP_F(r)(\varrho-r)+\nabla_xP_F(r)\cdot(\varrho\mathbf{u}-r\mathbf{U})\ \mathrm{d}x\\
&\quad-\int_\Omega[\varrho(P_F(\varrho)-P_F(r))-E_F(\varrho,r)]\,\mathrm{div}_x\,\mathbf{U}\ \mathrm{d}x\\
&\quad-\int_\Omega\partial_tP_P(s)(\eta-s)+\nabla_xP_P(s)\cdot(\eta\mathbf{u}-s\mathbf{U})\ \mathrm{d}x\\
&\quad-\int_\Omega[\eta(P_P(\eta)-P_P(s))-E_P(\eta,s)]\,\mathrm{div}_x\,\mathbf{U}\ \mathrm{d}x\\
&\quad-\int_\Omega\nabla_x(P_P(\eta)-P_P(s))\cdot(\nabla_x\eta+\eta\nabla_x\Phi)\ \mathrm{d}x\\
&\quad-\int_\Omega(\beta\varrho+\eta)\nabla_x\Phi\cdot(\mathbf{u}-\mathbf{U})\ \mathrm{d}x-\int_\Omega\frac{\eta\nabla_xs}{s}\cdot(\mathbf{u}-\mathbf{U})\ \mathrm{d}x.
\end{aligned}\tag{2.9}$$

Definition 2.2.1. $\{\varrho,\mathbf{u},\eta\}$ is a *weakly dissipative solution* of (1.1)-(1.7) with initial data $\{\varrho_0,\mathbf{u}_0,\eta_0\}$ if and only if

- $\{\varrho, \mathbf{u}, \eta\}$ is a weak solution in the sense of Definition 2.1.1, except that the time interval is taken to be $(0, T)$ for some $T > 0$ instead of $(0, \infty)$ and that the energy inequality becomes for $0 < \tau \leq T$

$$
\begin{aligned}
&\int_\Omega \frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{a}{\gamma - 1}\varrho^\gamma + \eta \ln \eta + \eta\Phi \, \mathrm{d}x(\tau) \\
&+ \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} + |2\nabla_x\sqrt{\eta} + \sqrt{\eta}\nabla_x\Phi|^2 \, \mathrm{d}x \, \mathrm{d}t \\
&\leq \int_\Omega \frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + \frac{a}{\gamma - 1}\varrho_0^\gamma + \eta_0 \ln \eta_0 + \eta_0\Phi \, \mathrm{d}x \\
&- \beta \int_0^\tau \int_\Omega \varrho\mathbf{u} \cdot \nabla_x\Phi \, \mathrm{d}x \, \mathrm{d}t. \qquad (2.10)
\end{aligned}
$$

- $\{\varrho, \mathbf{u}, \eta\}$ obeys inequality (2.8) for any suitably smooth functions $\{r, \mathbf{U}, s\}$.

The main result of this section of the dissertation is as follows:

Theorem 2.2.1 (Suitable weak solutions)**.** *Assume that* (Ω, Φ) *satisfy the confinement hypotheses* **(HC)** *with* $\Omega \subset \mathbb{R}^3$ *a bounded domain of class* $C^{2,\nu}, \nu > 0$. *Suppose the initial data* $\{\varrho_0, \boldsymbol{u}_0, \eta_0\}$ *satisfy*

$$\varrho_0 \textit{ not identically zero, } \varrho_0|\boldsymbol{u}_0|^2 \in L^1(\Omega), \textit{ and } \eta_0 \ln \eta_0 \in L^1(\Omega)$$

in addition to the conditions on the initial data specified in (1.5). *Then the Navier-Stokes-Smoluchowski system in* (1.1)-(1.6) *has a weakly dissipative solution in the sense of Definition* 2.2.1.

Section 2.2 is outlined as follows:

1. In Section 2.2.2 a suitable three level approximation scheme to the Navier-Stokes-Smoluchowski system in the spirit of [30] is introduced. The reader

should contrast the approximating procedure presented here with the time discretization approximation scheme in [15].

2. In Sections 2.2.3-2.2.7 the convergence of the approximate solutions to a weak solution in the sense of Definition 2.1.1 is shown.

3. In Section 2.2.8 the approximate relative entropy inequality is established, and with the aid of the convergence results in Section 2.2.3-2.2.7, it is shown that the weak solutions to the Navier-Stokes-Smoluchowski system satisfy a relative entropy inequality, proving Theorem 2.2.1.

2.2.2 Approximation Scheme

This section of the work uses the typical method: find a suitable approximation scheme which has solutions and then show the approximate solutions converege to solutions of the original problem. However, showing that the approximate solutions converge in function spaces satisfying *a priori* estimates is the main task, as stated by Evans in [27]. The weakly dissipative solutions here are constructed using a three-level approximation scheme in the spirit of [30]. First, an artificial pressure in terms of some small $\delta > 0$ and then a vanishing viscosity in terms of some small $\varepsilon > 0$ are introduced. Finally, a family of finite dimensional spaces X_n for $n \in \mathbb{N}$ consisting of smooth vector-valued functions on $\overline{\Omega}$ vanishing on $\partial\Omega$ is considered. The ε-regularizations are included to guarantee that certain *a priori* estimates hold true while the energy inequality remains valid at each level of the approximation. The δ-regularization serves to introduce the artificial pressure term. Thus, the task becomes to consider the approximate system:

$$\partial_t \varrho_n + \mathrm{div}_x(\varrho_n \mathbf{u}_n) = \varepsilon \Delta_x \varrho_n \tag{2.11}$$

$$\partial_t \eta_n + \mathrm{div}_x(\eta_n \mathbf{u}_n - \eta_n \nabla_x \Phi) = \Delta_x \eta_n \tag{2.12}$$

$$\begin{aligned} \int_\Omega \partial_t(\varrho_n \mathbf{u}_n) \cdot \mathbf{w} \, \mathrm{d}x &= \int_\Omega \varrho_n \mathbf{u}_n \otimes \mathbf{u}_n : \nabla_x \mathbf{w} + (a\varrho_n^\gamma + \eta_n + \delta \varrho_n^\alpha) \, \mathrm{div}_x \mathbf{w} \, \mathrm{d}x \\ - \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}_n) : \nabla_x \mathbf{w} &+ \varepsilon \nabla_x \varrho_n \cdot \nabla_x \mathbf{u}_n \cdot \mathbf{w} \, \mathrm{d}x - \int_\Omega (\beta \varrho_n + \eta_n) \nabla_x \Phi \cdot \mathbf{w} \, \mathrm{d}x \end{aligned} \tag{2.13}$$

for any $\mathbf{w} \in X_n$. Also, the boundary conditions

$$\nabla_x \varrho_n \cdot \mathbf{n} = 0 \text{ on } (0,T) \times \partial\Omega \text{ and } \mathbf{u}_n = \nabla_x \eta_n \cdot \mathbf{n} + \eta_n \nabla_x \Phi \cdot \mathbf{n} = 0 \text{ on } (0,T) \times \partial\Omega$$

are imposed. For notational simplicity, $\{\varrho_n, \mathbf{u}_n, \eta_n\}$ will denote $\{\varrho_{n,\varepsilon,\delta}, \mathbf{u}_{n,\varepsilon,\delta}, \eta_{n,\varepsilon,\delta}\}$ and $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \eta_\varepsilon\}$ will denote $\{\varrho_{\varepsilon,\delta}, \mathbf{u}_{\varepsilon,\delta}, \eta_{\varepsilon,\delta}\}$. Here, α is an appropriate constant. The approximation scheme is also supplemented by the approximate initial data $\{\varrho_{0,\delta}, \mathbf{m}_{0,\delta}, \eta_{0,\delta}\}$. The approximate initial data are modifications of the original initial data in that

- $0 < \delta \leq \varrho_{0,\delta} \leq \delta^{-1/2\alpha}$ for all $x \in \Omega$, $\varrho_{0,\delta} \to \varrho_0$ in $L^\gamma(\Omega)$,

  and $|\{x \in \Omega | \varrho_{0,\delta}(x) < \varrho_0(x)\}| \to 0$ as $\delta \to 0$

- $\mathbf{m}_{0,\delta}(x)$ is the same as $\mathbf{m}_0(x)$ unless $\varrho_{0,\delta}(x) < \varrho_0(x)$, in which case $\mathbf{m}_{0,\delta}(x) = 0$.

- $0 < \delta \leq \eta_{0,\delta} \leq \delta^{-1/2\alpha}$ for all $x \in \Omega$, $\eta_{0,\delta} \to \eta_0$ in $L^2(\Omega)$,

  and $|\{x \in \Omega | \eta_{0,\delta}(x) < \eta_0(x)\}| \to 0$ as $\delta \to 0$

The theory of parabolic equations gives the existence of a unique regular solution $\{\varrho_n, \mathbf{u}_n, \eta_n\}$ for each fixed $n \in \mathbb{N}$ for each fixed $\varepsilon, \delta > 0$. Specifically, equations

(2.11) and (2.12) are parabolic, so $|\mathbf{u}_n|$, ϱ_n, and η_n are smooth and ϱ_n and η_n are bounded above and below away from zero for all $t \in [0,T]$. Given $\mathbf{u}_n$, ϱ_n and η_n are obtained using fixed point arguments in the spirit of Ladyzhenskaya (see also Chapter 7 of [30]). Next, the existence of $\{\mathbf{u}_n\}$ is obtained by employing the Faedo-Galerkin approximation and using an iteration argument in the spirit of [30]. Indeed, the following bounds are obtained:

$$\mathbf{u}_n \in C^1([0,T]; X_n), \text{ and}$$

the quantities

$$\varrho_n,\ \partial_t\varrho_n,\ \nabla_x\varrho_n,\ \nabla_x^2\varrho_n,\ \eta_n,\ \partial_t\eta_n,\ \nabla_x\eta_n,\ \nabla_x^2\eta_n$$

are Hölder continuous on $(0,T] \times \overline{\Omega}$.

The interested reader is referred to Chapter 7 of [30] for more details.

Note that by integrating (2.11) and (2.12) over Ω and applying the boundary conditions, it can be shown that the total fluid mass $M_F = \int_\Omega \varrho_n \,\mathrm{d}x$ and the total particle mass $M_P = \int_\Omega \eta_n \,\mathrm{d}x$ are constant for each δ, ε, and n, and so for all times the total masses are the initial total masses.

2.2.3 Convergence of the Approximate Solutions

Now, the goal is to show that the approximate solutions $\{\varrho, \mathbf{u}, \eta\}_{n,\varepsilon,\delta}$ converge to a solution $\{\varrho, \mathbf{u}, \eta\}$ in the sense of Definition 2.1.1. The limits are taken as follows.

- take $n \to \infty$ to obtain $\varrho_n \to \varrho_\varepsilon$, $\mathbf{u}_n \to \mathbf{u}_\varepsilon$, and $\eta_n \to \eta_\varepsilon$ in the Faedo-Galerkin approximations.

- take $\varepsilon \to 0$ to obtain $\varrho_\varepsilon \to \varrho_\delta$, $\mathbf{u}_\varepsilon \to \mathbf{u}_\delta$, and $\eta_\varepsilon \to \eta_\delta$.

- take $\delta \to 0$ for $\varrho_\delta \to \varrho$, $\mathbf{u}_\delta \to \mathbf{u}$, and $\eta_\delta \to \eta$.

2.2.4 Uniform Bounds

In order to provide bounds on the various quantities, an approximate energy balance is derived by using $\mathbf{u}_n$ as a test function in (2.13).

Thus, (2.13) becomes

$$\int_\Omega \partial_t(\varrho_n \mathbf{u}_n) \cdot \mathbf{u}_n \, \mathrm{d}x = \int_\Omega [\varrho_n \mathbf{u}_n \otimes \mathbf{u}_n - \mathbb{S}(\nabla_x \mathbf{u}_n)] : \nabla_x \mathbf{u}_n \, \mathrm{d}x$$

$$+ \int_\Omega (a\varrho_n^\gamma + \delta\varrho_n^\alpha + \eta_n) \operatorname{div}_x \mathbf{u}_n \, \mathrm{d}x - \int_\Omega (\beta\varrho_n + \eta_n)\nabla_x \Phi \cdot \mathbf{u}_n \, \mathrm{d}x$$

$$-\varepsilon \int_\Omega \nabla_x \varrho_n \nabla_x \mathbf{u}_n \cdot \mathbf{u}_n \, \mathrm{d}x.$$

However, noting that

$$\int_\Omega \partial_t(\varrho_n \mathbf{u}_n) \cdot \mathbf{u}_n - (\varrho_n \mathbf{u}_n \otimes \mathbf{u}_n) : \nabla_x \mathbf{u}_n \, \mathrm{d}x$$

$$= \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \frac{1}{2} \varrho_n |\mathbf{u}_n|^2 \, \mathrm{d}x + \frac{1}{2} \int_\Omega [\partial_t \varrho_n + \operatorname{div}_x(\varrho_n \mathbf{u}_n)] |\mathbf{u}_n|^2 \, \mathrm{d}x$$

$$= \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \frac{1}{2} \varrho_n |\mathbf{u}_n|^2 \, \mathrm{d}x + \frac{1}{2} \varepsilon \int_\Omega \Delta_x \varrho_n |\mathbf{u}_n|^2 \, \mathrm{d}x,$$

and by (2.11) noting that the fluid mass is constant over time for the approximate solutions,

$$\int_\Omega a\varrho_n^\gamma \operatorname{div}_x \mathbf{u}_n \, \mathrm{d}x = -\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \varrho_n \frac{a}{\gamma - 1} \varrho_n^{\gamma-1} \, \mathrm{d}x - \int_\Omega \operatorname{div}_x \left(\varrho_n \frac{a}{\gamma - 1} \varrho_n^{\gamma-1} \mathbf{u}_n \right) \, \mathrm{d}x$$

$$-\varepsilon \int_\Omega \gamma \varrho_n^{\gamma-2} |\nabla_x \varrho_n|^2 \, \mathrm{d}x.$$

Multiplying (2.12) by $(\eta_n \ln \eta_n)'$ and integrating over Ω, it can be shown that

$$\int_\Omega \eta_n \operatorname{div}_x \mathbf{u}_n \ \mathrm{d}x = -\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \eta_n \ln \eta_n \ \mathrm{d}x - \int_\Omega \operatorname{div}_x(\eta_n \ln \eta_n(\mathbf{u}_n - \nabla_x \Phi)) \ \mathrm{d}x$$
$$- \int_\Omega (\ln \eta_n + 1)\Delta_x \eta_n + \eta_n \Delta_x \Phi \ \mathrm{d}x.$$

Using the above relations and the boundary conditions, a preliminary approximate energy inequality is obtained:

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}\varrho_n|\mathbf{u}_n|^2 + \frac{a}{\gamma-1}\varrho_n^\gamma + \frac{\delta}{\alpha-1}\varrho_n^\alpha + \eta_n \ln \eta_n \ \mathrm{d}x \\
&+ \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}_n) : \nabla_x \mathbf{u}_n + |2\nabla_x \sqrt{\eta_n} + \sqrt{\eta_n}\nabla_x \Phi|^2 \ \mathrm{d}x \\
&+\varepsilon \int_\Omega |\nabla_x \varrho_n|^2 (a\gamma \varrho_n^{\gamma-2} + \delta a \varrho_n^{\alpha-2}) \ \mathrm{d}x \\
\leq& - \int_\Omega (\beta \varrho_n + \eta_n)\nabla_x \Phi \cdot \mathbf{u}_n \ \mathrm{d}x + \int_\Omega \nabla_x \eta_n \cdot \nabla_x \Phi + \eta_n |\nabla_x \Phi|^2 \ \mathrm{d}x.
\end{aligned}$$

Now, noting that by (2.12) and integration by parts

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \eta_n \Phi \ \mathrm{d}x = -\int_\Omega \nabla_x \eta_n \cdot \nabla_x \Phi \ \mathrm{d}x + \int_\Omega \eta_n \mathbf{u}_n \cdot \nabla_x \Phi \ \mathrm{d}x - \int_\Omega \eta_n |\nabla_x \Phi|^2 \ \mathrm{d}x,$$

the desired approximate energy balance is obtained:

$$\begin{aligned}
&\int_\Omega \frac{1}{2}\varrho_n|\mathbf{u}_n|^2 + \frac{a}{\gamma-1}\varrho_n^\gamma + \frac{\delta}{\alpha-1}\varrho_n^\alpha + \eta_n \ln \eta_n + \eta_n \Phi \ \mathrm{d}x(\tau) \\
&+ \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}_n) : \nabla_x \mathbf{u}_n + |2\nabla_x \sqrt{\eta_n} + \sqrt{\eta_n}\nabla_x \Phi|^2 \ \mathrm{d}x \ \mathrm{d}t \\
&+ \varepsilon \int_0^\tau \int_\Omega |\nabla_x \varrho_n|^2 (a\gamma \varrho_n^{\gamma-2} + \delta a \varrho_n^{\alpha-2}) \ \mathrm{d}x \ \mathrm{d}t \\
&= \int_\Omega \frac{1}{2}\varrho_{0,\delta}|\mathbf{u}_{0,\delta}|^2 + \frac{a}{\gamma-1}\varrho_{0,\delta}^\gamma + \frac{\delta}{\alpha-1}\varrho_{0,\delta}^\alpha + \eta_{0,\delta}\ln \eta_{0,\delta} + \eta_{0,\delta}\Phi \ \mathrm{d}x \\
&- \beta \int_0^\tau \int_\Omega \varrho_n \mathbf{u}_n \cdot \nabla_x \Phi \ \mathrm{d}x \ \mathrm{d}t.
\end{aligned} \tag{2.14}$$

Using a Gronwall's argument in the spirit of [30] and [32] on the last right-hand side term in (2.14), it is apparent that $\mathbf{u}_n$ is controlled in $L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3))$.

Now, (2.14) with the convergence of the initial data will be used to obtain bounds on the approximate quantities. The following bounds are evident from a quick inspection of (2.14):

- $\{\sqrt{\varrho}\mathbf{u}\}_{n,\varepsilon,\delta} \in_b L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))$
- $\{\varrho\}_{n,\varepsilon,\delta} \in_b L^\infty(0,T;L^\gamma(\Omega))$
- $\{\eta \ln \eta\}_{n,\varepsilon,\delta} \in_b L^\infty(0,T;L^1(\Omega))$
- $\{\mathbf{u}\}_{n,\varepsilon,\delta} \in_b L^2(0,T;W_0^{1,2}(\Omega))$
- $\{\nabla_x \sqrt{\eta}\}_{n,\varepsilon,\delta} \in_b L^2(0,T;L^2(\Omega))$

Using the embedding of $W^{1,2}(\Omega)$ in $L^6(\Omega)$ on the last bound listed above, it is clear that $\{\eta\}_{n,\varepsilon,\delta} \in_b L^2(0,T;L^3(\Omega))$. Using this result, and that

$$\nabla_x \sqrt{\eta} = \frac{\nabla_x \eta}{2\sqrt{\eta}},$$

it is also clear that

$$\{\eta\}_{n,\varepsilon,\delta} \in_b L^2(0,T;W^{1,\frac{3}{2}}(\Omega)).$$

2.2.5 Faedo-Galerkin Limit: $n \to \infty$

The first step in the approximating procedure is to take the Faedo-Galerkin limit, that is to take $n \to \infty$. Much of this work has been performed by Feireisl in [30]. However, work has to be done to perform the limit in the approximate Smoluchowski equation. Starting first with the approximate continuity equation, since

$$\{\varrho\}_n \in_b L^\infty(0,T;L^\alpha(\Omega)),$$

$$\{\mathbf{u}\}_n \in_b L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)), \text{ and}$$

$$\{\eta\}_n \in_b L^2(0,T;L^3(\Omega)),$$

the quantity

$$\left|\int_0^T \int_\Omega (a\varrho_n^\gamma + \eta_n)\operatorname{div}_x \mathbf{u}_n \ \mathrm{d}x \ \mathrm{d}t\right| \le c\int_0^T \int_\Omega (\varrho_n^\gamma + \eta_n)|\operatorname{div}_x \mathbf{u}_n| \ \mathrm{d}x \ \mathrm{d}t$$

is bounded independently of n provided α is large enough. Also, by the approximate energy balance above, the quantity

$$\varepsilon\delta \int_0^T \int_\Omega |\nabla_x \varrho_n|^2 \varrho_n^{\alpha-2} \ \mathrm{d}x \ \mathrm{d}t$$

is bounded independently of n, thus by Poincaré's inequality, the following uniform bound is obtained

$$\{\varrho\}_n \in_b L^2(0,T;W^{1,2}(\Omega)).$$

By this bound and the bound of the approximate velocities, it is clear that

$$\nabla_x \varrho_n \cdot \mathbf{u}_n \in_b L^1(0,T;L^{3/2}(\Omega)).$$

However, this quantity is only just integrable with respect to time. To get around this and obtain uniform bounds on $\partial_t \varrho_n$ and $\Delta_x \varrho_n$, the approximate continuity equation is multiplied by $G'(\varrho_n)$ to obtain

$$\partial_t \int_\Omega G(\varrho_n) \ \mathrm{d}x + \varepsilon \int_\Omega G''(\varrho_n)|\nabla_x \varrho_n|^2 \ \mathrm{d}x = \int_\Omega (G(\varrho_n) - G'(\varrho_n)\varrho_n)\operatorname{div}_x \mathbf{u}_n \ \mathrm{d}x,$$

which can be considered as a parabolic version of the renormalized continuity equation. Taking $G(\varrho_n) = \varrho_n \ln \varrho_n$ and using the bounds on $\varrho_n \operatorname{div}_x \mathbf{u}_n$ and ϱ_n, the quantity

$$\varepsilon \int_0^T \int_\Omega \frac{|\nabla_x \varrho_n|^2}{\varrho_n} \ \mathrm{d}x \ \mathrm{d}t$$

is bounded independently of n. Since

$$\|\nabla_x \varrho_n \cdot \mathbf{u}_n\|_{L^1(\Omega)} \leq \left\| \frac{\nabla_x \varrho_n}{\sqrt{\varrho_n}} \right\|_{L^2(\Omega;\mathbb{R}^3)} \|\sqrt{\varrho_n}\mathbf{u}_n\|_{L^2(\Omega;\mathbb{R}^3)},$$

by interpolation,

$$\{\nabla_x \varrho \cdot \mathbf{u}\}_n \in_b L^q(0,T;L^p(\Omega))$$

for $p \in \left(1, \frac{3}{2}\right)$ and q depending on p in $(1,2)$. By $L^p - L^q$ theory, the sequences $\{\partial_t \varrho_n\}_n$ and $\{\partial_{x_i}\partial_{x_j}\varrho_n\}_n$ are bounded in $L^q(0,T;L^p(\Omega))$. Thus, the limits $\varrho_\varepsilon, \mathbf{u}_\varepsilon$ obey the equation

$$\partial_t \varrho_\varepsilon + \operatorname{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon) = \varepsilon \Delta_x \varrho_\varepsilon.$$

Strong convergence of the gradients $\nabla_x \varrho_n \to \nabla_x \varrho_\varepsilon$ in $L^2(0,T;L^2(\Omega;\mathbb{R}^3))$ follows from the renormalized parabolic approximate continuity equation with $G(z) = z^2$.

The next step is to obtain convergence for the Smoluchowski equation. The approach is similar as the one used for the continuity equation. First, it is noted that

$$[\partial_t - \Delta_x]\eta_n = -\nabla_x \eta_n \cdot \mathbf{u}_n - \eta_n \operatorname{div}_x \mathbf{u}_n + \nabla_x \eta_n \cdot \nabla_x \Phi + \eta_n \Delta_x \Phi.$$

As with the continuity equation, previously mentioned bounds control all the terms on the right side of this equation except for $-\nabla_x \eta_n \cdot \mathbf{u}_n$. However, by Hölder's inequality

$$\|\nabla_x \eta_n \cdot \mathbf{u}_n\|_{L^{6/5}(\Omega)} \leq \|\nabla_x \eta_n\|_{L^{3/2}(\Omega;\mathbb{R}^3)} \|\mathbf{u}_n\|_{L^6(\Omega)}$$

where the right side of the inequality is bounded in $L^2(0,T)$. Thus, it is seen that the limit functions $\eta, \mathbf{u}$ obey

$$\partial_t \eta_\varepsilon + \operatorname{div}_x(\eta_\varepsilon(\mathbf{u}_\varepsilon - \nabla_x \Phi)) = \Delta_x \eta_\varepsilon,$$

using the interpolation arguments and $L^p - L^q$ theory used for showing the convergence of the derivatives of ϱ_n.

In accordance with the above convergences and bounds, the convergence of most of the terms of the momentum equation follow directly; the only issues arise with the convective term $\varrho_\varepsilon \mathbf{u}_n \otimes \mathbf{u}_n$. By the bounds on $\varrho_n |\mathbf{u}_n|^2$ and $\mathbf{u}_n$, the convective term converges weakly to $\overline{(\varrho \mathbf{u} \otimes \mathbf{u})}_\varepsilon$ in $L^q((0,T) \times \Omega; \mathbb{R}^3)$ for some $q > 1$. Also, by the bounds on ϱ_n, $\varrho_n \mathbf{u}_n$ converges weakly-$*$ to $\varrho_\varepsilon \mathbf{u}_\varepsilon$ in $L^\infty(0,T; L^{5/4}(\Omega; \mathbb{R}^3))$

However, a quick inspection of the momentum equation shows that the functions $t \mapsto \int_\Omega \varrho_n \mathbf{u}_n \cdot \phi \, \mathrm{d}x$ are well-defined and bounded in $C[0,T]$. Thus, by Arzelà-Ascoli,

$$\varrho_n \mathbf{u}_n \to \varrho_\varepsilon \mathbf{u}_\varepsilon \text{ in } C_{\text{weak}}([0,T]; L^{5/4}(\Omega; \mathbb{R}^3))$$

which is compactly embedded in $C_{\text{weak}}([0,T]; W^{-1,2}(\Omega; \mathbb{R}^3))$. Thus,

$$\varrho_n \mathbf{u}_n \to \varrho_\varepsilon \mathbf{u}_\varepsilon$$

strongly in $C_{\text{weak}}([0,T]; W^{-1,2}(\Omega; \mathbb{R}^3))$. Thus, with the bounds on $\mathbf{u}_n$,

$$(\overline{\varrho \mathbf{u} \otimes \mathbf{u}})_\varepsilon = \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon.$$

Also, the following lemma cited by Simon in [49] is used for the convergence of the approximate particle density.

Lemma 2.2.1. *: Let $X \subset B \subset Y$ be Banach spaces with $X \subset B$ compactly. Then, for $1 \leq p < \infty$, $\{v : v \in L^p(0,T; X), v_t \in L^1(0,T; Y)\}$ is compactly embedded in $L^p(0,T; B)$.*

Thus,

$$\{\eta\}_{n,\varepsilon} \to \eta_\delta \text{ in } L^2(0,T; L^3(\Omega)).$$

by applying Lemma 2.2.1 with $p=2$, $X=W^{1,\frac{3}{2}}(\Omega)$, $B=L^3(\Omega)$, and $Y=L^1(\Omega)$.

2.2.6 Artificial Diffusion Limit: $\varepsilon \to 0$

After taking the Faedo-Galerkin limit in Section 2.2.5, δ is now fixed, and for each $\varepsilon > 0$, there exist $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \eta_\varepsilon\}$ satisfying

$$\partial_t \varrho_\varepsilon + \operatorname{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon) = \varepsilon \Delta_x \varrho_\varepsilon \tag{2.15}$$

$$\partial_t \eta_\varepsilon + \operatorname{div}_x(\eta_\varepsilon \mathbf{u}_\varepsilon - \eta_\varepsilon \nabla_x \Phi) = \Delta_x \eta_\varepsilon \tag{2.16}$$

$$\begin{aligned}\int_\Omega \partial_t(\varrho_\varepsilon \mathbf{u}_\varepsilon)\cdot \mathbf{w}\ \mathrm{d}x &= \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla_x \mathbf{w} + (a\varrho_\varepsilon^\gamma + \eta_\varepsilon + \delta \varrho_\varepsilon^\alpha)\operatorname{div}_x \mathbf{w}\ \mathrm{d}x \\ -\int_\Omega \mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) : \nabla_x \mathbf{w} + \varepsilon \nabla_x \varrho_\varepsilon \cdot \nabla_x \mathbf{u}_\varepsilon \cdot \mathbf{w}\ \mathrm{d}x &- \int_\Omega (\beta \varrho_\varepsilon + \eta_\varepsilon)\nabla_x \Phi \cdot \mathbf{w}\ \mathrm{d}x\end{aligned} \tag{2.17}$$

for any test function $\mathbf{w}$,

$$\nabla_x \varrho_\varepsilon \cdot \mathbf{n} = 0 \tag{2.18}$$

on $(0,T)\times\partial\Omega$

$$\mathbf{u}_\varepsilon = \nabla_x \eta_\varepsilon \cdot \mathbf{n} + \eta_\varepsilon \nabla_x \Phi \cdot \mathbf{n} = 0 \tag{2.19}$$

on $(0,T)\times\partial\Omega$, and

$$\begin{aligned}&\int_\Omega \frac{1}{2}\varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 + \frac{a}{\gamma-1}\varrho_\varepsilon^\gamma + \frac{\delta}{\alpha-1}\varrho_\varepsilon^\alpha + \eta_\varepsilon \ln \eta_\varepsilon + \eta_\varepsilon \Phi\ \mathrm{d}x(\tau) \\ &+ \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) : \nabla_x \mathbf{u}_\varepsilon + |2\nabla_x \sqrt{\eta_\varepsilon} + \sqrt{\eta_\varepsilon}\nabla_x \Phi|^2\ \mathrm{d}x\ \mathrm{d}t \\ &+ \varepsilon \int_0^\tau \int_\Omega |\nabla_x \varrho_\varepsilon|^2 (a\gamma \varrho_\varepsilon^{\gamma-2} + \delta a \varrho_\varepsilon^{\alpha-2}) + \beta \nabla_x \varrho_\varepsilon \cdot \nabla_x \Phi\ \mathrm{d}x\ \mathrm{d}t \\ &\le \int_\Omega \frac{1}{2}\varrho_{0,\delta}|\mathbf{u}_{0,\delta}|^2 + \frac{a}{\gamma-1}\varrho_{0,\delta}^\gamma + \frac{\delta}{\alpha-1}\varrho_{0,\delta}^\alpha + \eta_{0,\delta}\ln \eta_{0,\delta} + \eta_{0,\delta}\Phi\ \mathrm{d}x \\ &- \int_0^\tau \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x \Phi\ \mathrm{d}x\ \mathrm{d}t.\end{aligned} \tag{2.20}$$

First, noting the uniform bounds from the last section,

$$\varrho_\varepsilon \to \varrho_\delta \text{ weakly-}* \text{ in } L^\infty(0,T;L^\alpha(\Omega))$$

$$\mathbf{u}_\varepsilon \to \mathbf{u}_\delta \text{ weakly in } L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3))$$

$$\eta_\varepsilon \to \eta_\delta \text{ weakly in } L^2(0,T;L^3(\Omega)\cap W^{1,3/2}(\Omega)).$$

for some $\{\varrho_\delta, \mathbf{u}_\delta, \eta_\delta\}$.

It is noted that the last term on the right side of (2.20) can be controled in a method similar to its analog in the Faedo-Galerkin approximation.

The next step in taking the limit as ε goes to zero is to show that $\varrho_\delta, \mathbf{u}_\delta$ solve the equation of continuity in the sense of distributions. By (2.15),

$$\varrho_\varepsilon \partial_t \varrho_\varepsilon + \varrho_\varepsilon \operatorname{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon) = \varepsilon \varrho_\varepsilon \Delta_x \varrho_\varepsilon,$$

so by integration by parts,

$$\int_\Omega \frac{1}{2}\varrho_\varepsilon^2 \, \mathrm{d}x(\tau) + \varepsilon \int_0^\tau \int_\Omega |\nabla_x \varrho_\varepsilon|^2 \, \mathrm{d}x \, \mathrm{d}t = \int_\Omega \frac{1}{2}\varrho_{0,\delta}^2 \, \mathrm{d}x - \frac{1}{2}\int_0^\tau \int_\Omega \varrho_\varepsilon^2 \operatorname{div}_x \mathbf{u}_\varepsilon \, \mathrm{d}x \, \mathrm{d}t$$

for any $\tau \in [0,T]$. Thus, if α is large enough,

$$\{\sqrt{\varepsilon}\nabla_x \varrho_\varepsilon\} \in_b L^2(0,T;L^2(\Omega;\mathbb{R}^3)),$$

so

$$\varepsilon \nabla_x \varrho_\varepsilon \to 0 \text{ in } L^2(0,T;L^2(\Omega;\mathbb{R}^3)).$$

As with the Faedo-Galerkin limit,

$$\varrho_\varepsilon \to \varrho_\delta \text{ in } C_{\text{weak}}([0,T];L^\alpha(\Omega))$$

by Arzelà-Ascoli. Thus it is clear that

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \to \varrho_\delta \mathbf{u}_\delta \text{ weakly-}* \text{ in } L^\infty(0,T;L^{2\alpha/\alpha+1}(\Omega;\mathbb{R}^3)).$$

and it can be concluded that for any test function ϕ,

$$\int_0^T \int_\Omega \varrho_\delta \partial_t \phi + \varrho_\delta \mathbf{u}_\delta \cdot \nabla_x \phi \,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \varrho_{0,\delta}\phi(0,\cdot)\,\mathrm{d}x = 0.$$

The next step is to obtain a convergence result for the Smoluchowski equation. Multiplying (2.16) by η_ε and integrating by parts:

$$\int_\Omega \eta_\varepsilon^2 \,\mathrm{d}x(\tau) + \int_0^\tau \int_\Omega |\nabla_x \eta_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t = \int_\Omega \eta_{0,\delta}\,\mathrm{d}x + \int_0^\tau \int_\Omega \frac{1}{2}(\eta_\varepsilon^2 \operatorname{div}_x \mathbf{u}_\varepsilon - \nabla_x \eta_\varepsilon^2 \cdot \nabla_x \Phi)\,\mathrm{d}x\,\mathrm{d}t,$$

where the right hand side is bounded, so

$$\{\eta_\varepsilon\} \in_b L^\infty(0,T;L^2(\Omega)),$$

$$\{\nabla_x \eta_\varepsilon\} \in_b L^2(0,T;L^2(\Omega;\mathbb{R}^3)).$$

Thus, with similar arguments as with the convergence of the continuity equation:

$$\int_0^T \int_\Omega \eta_\delta \partial_t \phi + (\eta_\delta \mathbf{u}_\delta - \eta_\delta \nabla_x \Phi - \nabla_x \eta_\delta) \cdot \nabla_x \phi \,\mathrm{d}x\,\mathrm{d}t = -\int_\Omega \eta_{0,\delta}\phi(0,\cdot)\,\mathrm{d}x.$$

The next step is to show the convergence of the approximate momentum equation. First, noting that $\varepsilon \nabla_x \varrho_\varepsilon \to 0$ in $L^2(0,T;L^2(\Omega;\mathbb{R}^3))$ and that $\{\mathbf{u}_\varepsilon\}$ is bounded in $L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3))$, it is clear that $\varepsilon \nabla_x \varrho_\varepsilon \nabla_x \mathbf{u}_\varepsilon \to 0$ in $L^1((0,T)\times\Omega;\mathbb{R}^3)$. Thus,

$$\varepsilon \int_0^\tau \int_\Omega |\nabla_x \varrho_\varepsilon|^2 (a\gamma \varrho_\varepsilon^{\gamma-2} + a\delta \varrho_\varepsilon^{\alpha-2}) + \beta \nabla_x \varrho_\varepsilon \cdot \nabla_x \Phi \,\mathrm{d}x\,\mathrm{d}t \to 0.$$

Next, since $\mathbf{u}_\varepsilon \to \mathbf{u}_\delta$ weakly in $L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3))$

$\mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) \to \mathbb{S}(\nabla_x \mathbf{u}_\delta)$ weakly in $L^p((0,T)\times\Omega)$ for some $p > 1$.

Thus, the only terms to consider in the momentum and energy balances are the pressure-related terms. First, using the Bogovskii operator $\mathcal{B}$, analogous to the inverse of the divergence (see [30] and [32]), the test function $\mathbf{w} := \psi\varphi$ where $\psi \in C_c^\infty(0,T)$ and $\varphi := \mathcal{B}[\varrho_\varepsilon - \overline{\varrho}]$ where $\overline{\varrho} := \frac{1}{|\Omega|}\int_\Omega \varrho_\varepsilon \,\mathrm{d}x$, in the approximate momentum equation and performing some analysis yields

$$
\begin{aligned}
&\int_0^T \psi \int_\Omega (a\varrho_\varepsilon^\gamma + \eta_\varepsilon + \delta\varrho_\varepsilon^\alpha)\varrho_\varepsilon \,\mathrm{d}x \,\mathrm{d}t \\
&= \int_0^T \psi\overline{\varrho} \int_\Omega a\varrho_\varepsilon^\gamma + \eta_\varepsilon + \delta\varrho_\varepsilon^\alpha \,\mathrm{d}x \,\mathrm{d}t \\
&- \int_0^T \psi \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \partial_t \varphi \,\mathrm{d}x \,\mathrm{d}t \\
&- \int_0^T \psi \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla_x \varphi \,\mathrm{d}x \,\mathrm{d}t \\
&+ \int_0^T \psi \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) : \nabla_x \varphi \,\mathrm{d}x \,\mathrm{d}t \\
&+ \int_0^T \psi \int_\Omega (\beta\varrho_\varepsilon + \eta_\varepsilon)\nabla_x \Phi \cdot \varphi \,\mathrm{d}x \,\mathrm{d}t \\
&- \int_0^T \psi' \int_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \varphi \,\mathrm{d}x \,\mathrm{d}t \\
&+ \varepsilon \int_0^T \psi \int_\Omega \nabla_x \varrho_\varepsilon \nabla_x \mathbf{u}_\varepsilon \cdot \varphi \,\mathrm{d}x \,\mathrm{d}t.
\end{aligned}
$$

However, all the terms on the right-hand side are bounded. Thus, $a\varrho_\varepsilon^\gamma + \eta_\varepsilon + \delta\varrho_\varepsilon^\alpha$ has a weak limit as $\varepsilon \to 0$. Note that the form of the last integral above follows from the choice of test function, and is the only reason for separating the limits for ε and δ (see [32]).

The next goal is to show that the weak limit of the pressure term is $a\varrho_\delta^\gamma + \eta_\delta + \delta\varrho_\delta^\alpha$. To do this, the strong (pointwise) convergence of the densities must be shown. The strong convergence of $\{\eta_\varepsilon\}$ follows from Lemma 2.2.1. To show the strong convergence of the fluid density, the test function

$$\psi(t)\zeta(x)\varphi_1(x)$$

is used in the approximate (level-ε) momentum equation where

$\psi \in C_c^\infty(0,T), \zeta \in C_c^\infty(\Omega)$, and $\varphi_1 := \nabla_x \Delta_x^{-1}(\mathbf{1}_\Omega \varrho_\varepsilon)$. Since $\varrho_\varepsilon \mathbf{u}_\varepsilon$ and $\nabla_x \varrho_\varepsilon$ have zero normal traces, the approximate continuity equation can be extended to all of $\mathbb{R}^3$ to obtain

$$\partial_t(\mathbf{1}_\Omega \varrho_\varepsilon) + \operatorname{div}_x(\mathbf{1}_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon) - \varepsilon \operatorname{div}_x(\mathbf{1}_\Omega \nabla_x \varrho_\varepsilon) = 0.$$

Thus, for α large, some straight-forward analysis gives

$$\begin{aligned}
&\int_0^T \int_\Omega \psi\zeta((a\varrho_\varepsilon^\gamma + \eta_\varepsilon + \delta\varrho_\varepsilon^\alpha)\varrho_\varepsilon - \mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) : \mathcal{RT}(\mathbf{1}_\Omega \varrho_\varepsilon)) \, \mathrm{d}x \, \mathrm{d}t \\
&= \int_0^T \int_\Omega \psi\zeta(\varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathcal{RT}(\mathbf{1}_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon) - (\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) : \mathcal{RT}(\mathbf{1}_\Omega \varrho_\varepsilon)) \, \mathrm{d}x \, \mathrm{d}t \\
&- \varepsilon \int_0^T \int_\Omega \psi\zeta\varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x \Delta_x^{-1}(\operatorname{div}_x(\mathbf{1}_\Omega \nabla_x \varrho_\varepsilon)) \, \mathrm{d}x \, \mathrm{d}t \\
&+ \int_0^T \int_\Omega \psi\zeta(\beta\varrho_\varepsilon + \eta_\varepsilon)\nabla_x \Phi \cdot \nabla_x \Delta_x^{-1}(\mathbf{1}_\Omega \varrho_\varepsilon) \, \mathrm{d}x \, \mathrm{d}t \\
&- \int_0^T \int_\Omega \psi(a\varrho_\varepsilon^\gamma + \eta_\varepsilon + \delta\varrho_\varepsilon^\alpha)\nabla_x \zeta \cdot \nabla_x \Delta_x^{-1}(\mathbf{1}_\Omega \varrho_\varepsilon) \, \mathrm{d}x \, \mathrm{d}t \\
&+ \int_0^T \int_\Omega \psi\mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) : \nabla_x \zeta \otimes \nabla_x \Delta_x^{-1}(\mathbf{1}_\Omega \varrho_\varepsilon) \, \mathrm{d}x \, \mathrm{d}t \\
&- \int_0^T \int_\Omega \psi(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla_x \zeta \otimes \nabla_x \Delta_x^{-1}(\mathbf{1}_\Omega \varrho_\varepsilon)) \, \mathrm{d}x \, \mathrm{d}t \\
&- \int_0^T \int_\Omega \zeta\varrho_\varepsilon \mathbf{u}_\varepsilon \partial_t \psi \cdot \nabla_x \Delta_x^{-1}(\mathbf{1}_\Omega \varrho_\varepsilon) \, \mathrm{d}x \, \mathrm{d}t \\
&+ \varepsilon \int_0^T \int_\Omega \psi\zeta\nabla_x \varrho_\varepsilon \nabla_x \mathbf{u}_\varepsilon \cdot \nabla_x \Delta_x^{-1}(\mathbf{1}_\Omega \varrho_\varepsilon) \, \mathrm{d}x \, \mathrm{d}t
\end{aligned}$$

where $\mathcal{RT}$ is the double Riesz transform defined componentwise as

$$\mathcal{RT}_{i,j} := \partial x_i \Delta_x^{-1} \partial_{x_j}.$$

Similarly, the test function $\psi\zeta\varphi_2$ is used in the weak limit of the approximate (level-ε) momentum equation, where $\varphi_2 := \nabla_x \Delta_x^{-1}(\mathbf{1}_\Omega \varrho)$ to obtain

$$
\begin{aligned}
&\int_0^T \int_\Omega \psi\zeta((\overline{a\varrho^\gamma + \eta + \delta\varrho^\alpha})_\delta \varrho_\delta - \mathbb{S}(\nabla_x \mathbf{u}_\delta) : \mathcal{R}\mathcal{T}(\mathbf{1}_\Omega \varrho_\delta)) \,\mathrm{d}x \,\mathrm{d}t \\
&= \int_0^T \int_\Omega \psi\zeta(\varrho_\delta \mathbf{u}_\delta \cdot \mathcal{R}\mathcal{T}(\mathbf{1}_\Omega \varrho_\delta \mathbf{u}_\delta) - (\varrho_\delta \mathbf{u}_\delta \otimes \mathbf{u}_\delta) : \mathcal{R}\mathcal{T}(\mathbf{1}_\Omega \varrho_\delta)) \,\mathrm{d}x \,\mathrm{d}t \\
&+ \int_0^T \int_\Omega \psi\zeta(\beta\varrho_\delta + \eta_\delta)\nabla_x \Phi \cdot \nabla_x \Delta_x^{-1}(\mathbf{1}_\Omega \varrho_\delta) \,\mathrm{d}x \,\mathrm{d}t \\
&- \int_0^T \int_\Omega \psi(\overline{a\varrho^\gamma + \eta + \delta\varrho^\alpha})_\delta \nabla_x \zeta \cdot \nabla_x \Delta_x^{-1}(\mathbf{1}_\Omega \varrho_\delta) \,\mathrm{d}x \,\mathrm{d}t \\
&+ \int_0^T \int_\Omega \psi \mathbb{S}(\nabla_x \mathbf{u}_\delta) : \nabla_x \zeta \otimes \nabla_x \Delta_x^{-1}(\mathbf{1}_\Omega \varrho_\delta) \,\mathrm{d}x \,\mathrm{d}t \\
&- \int_0^T \int_\Omega \psi(\varrho_\delta \mathbf{u}_\delta \otimes \mathbf{u}_\delta : \nabla_x \zeta \otimes \nabla_x \Delta_x^{-1}(\mathbf{1}_\Omega \varrho_\delta)) \,\mathrm{d}x \,\mathrm{d}t \\
&- \int_0^T \int_\Omega \zeta \varrho_\delta \mathbf{u}_\delta \partial_t \psi \cdot \nabla_x \Delta_x^{-1}(\mathbf{1}_\Omega \varrho_\delta) \,\mathrm{d}x \,\mathrm{d}t.
\end{aligned}
$$

From the convergence results stated earlier in this subsection and the fact that from the theory of elliptic problems (see [32]), the operator $\nabla_x \Delta_x^{-1}$ gains a spatial derivative, i.e., due to the embedding $W^{1,\alpha}(\Omega) \hookrightarrow C(\overline{\Omega})$,

$$
\nabla_x \Delta_x^{-1}(\mathbf{1}_\Omega \varrho_\varepsilon) \to \nabla_x \Delta_x^{-1}(\mathbf{1}_\Omega \varrho_\delta)
$$

in $C([0,T] \times \overline{\Omega}; \mathbb{R}^3)$. Thus, taking the limit as $\varepsilon \to 0$ in the previous two equations, it follows that

$$
\begin{aligned}
&\lim_{\varepsilon \to 0} \int_0^T \int_\Omega \psi\zeta((a\varrho_\varepsilon^\gamma + \eta_\varepsilon + \delta\varrho_\varepsilon^\alpha)\varrho_\varepsilon - \mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) : \mathcal{R}\mathcal{T}(\mathbf{1}_\Omega \varrho_\varepsilon)) \,\mathrm{d}x \,\mathrm{d}t \\
&= \int_0^T \int_\Omega \psi\zeta((\overline{a\varrho^\gamma + \eta + \delta\varrho^\alpha})_\delta \varrho_\delta - \mathbb{S}(\nabla_x \mathbf{u}_\delta) : \mathcal{R}\mathcal{T}(\mathbf{1}_\Omega \varrho_\delta)) \,\mathrm{d}x \,\mathrm{d}t \\
&+ \lim_{\varepsilon \to 0} \int_0^T \int_\Omega \psi\zeta(\varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \mathcal{R}\mathcal{T}(\mathbf{1}_\Omega \varrho_\varepsilon \mathbf{u}_\varepsilon) - (\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) : \mathcal{R}\mathcal{T}(\mathbf{1}_\Omega \varrho_\varepsilon)) \,\mathrm{d}x \,\mathrm{d}t \\
&- \int_0^T \int_\Omega \psi\zeta(\varrho_\delta \mathbf{u}_\delta \cdot \mathcal{R}\mathcal{T}(\mathbf{1}_\Omega \varrho_\delta \mathbf{u}_\delta) - (\varrho_\delta \mathbf{u}_\delta \otimes \mathbf{u}_\delta) : \mathcal{R}\mathcal{T}(\mathbf{1}_\Omega \varrho_\delta)) \,\mathrm{d}x \,\mathrm{d}t.
\end{aligned}
$$

The goal now is to show that the difference of the last two integrals above vanishes when the limit for ε is taken. This follows from the following lemma which follows from the Div-Curl Lemma (see [30]).

Lemma 2.2.2. *Let $\boldsymbol{V}_\varepsilon \to \boldsymbol{V}$ weakly in $L^p(\mathbb{R}^3;\mathbb{R}^3)$ and $r_\varepsilon \to r$ weakly in $L^q(\mathbb{R}^3)$. Define s such that*

$$\frac{1}{p}+\frac{1}{q}=\frac{1}{s}<1.$$

Then $r_\varepsilon \mathcal{RT}(\boldsymbol{V}_\varepsilon)-\mathcal{RT}(r_\varepsilon)\boldsymbol{V}_\varepsilon \to r\mathcal{RT}(\boldsymbol{V})-\mathcal{RT}(r)\boldsymbol{V}$ weakly in $L^s(\mathbb{R}^3;\mathbb{R}^3)$

Using the Commutator Lemma in Section 3.6.5 in [32] and some analysis, the weak compactness identity for the pressure is derived:

$$[\overline{(a\varrho^\gamma+\eta+\delta\varrho^\alpha)\varrho}]_\delta-\left(\frac{4}{3}\mu+\lambda\right)(\overline{\varrho\operatorname{div}_x\mathbf{u}})_\delta$$

$$=(\overline{a\varrho^\gamma+\eta+\delta\varrho^\alpha})_\delta\varrho_\delta-\left(\frac{4}{3}\mu+\lambda\right)\varrho_\delta\operatorname{div}_x\mathbf{u}_\delta.$$

By multiplying the approximate continuity equation (2.15) by $G'(\varrho_\varepsilon)=\varrho_\varepsilon\ln\varrho_\varepsilon$ noting that G is a smooth convex function, integrating by parts, and taking the weak limit, the following equation is obtained.

$$\int_\Omega(\overline{\varrho\ln\varrho})_\delta\ \mathrm{d}x(\tau)+\int_0^\tau\int_\Omega(\overline{\varrho\operatorname{div}_x\mathbf{u}})_\delta\ \mathrm{d}x\ \mathrm{d}t=\int_\Omega\varrho_{0,\delta}\ln\varrho_{0,\delta}\ \mathrm{d}x$$

Also, since $\varrho_\delta,\mathbf{u}_\delta$ solve the level-δ approximation of the equation of continuity, they represent a renormalized solution of the equation of continuity of the actual Naiver-Stokes-Smoluchowski system (c.f. Lemma 3.7 in [32]). Thus,

$$\int_\Omega \varrho_\delta \ln \varrho_\delta \, \mathrm{d}x(\tau) + \int_0^\tau \int_\Omega \varrho_\delta \operatorname{div}_x \mathbf{u}_\delta \, \mathrm{d}x \, \mathrm{d}t \leq \int_\Omega \varrho_{0,\delta} \ln \varrho_{0,\delta} \, \mathrm{d}x.$$

After some analysis, it can be shown that

$$(\overline{\varrho \ln \varrho})_\delta = \varrho_\delta \ln \varrho_\delta$$

which since $z \mapsto z \ln z$ is strictly convex, implies that $\varrho_\varepsilon \to \varrho_\delta$ almost everywhere on $(0, T) \times \Omega$.

2.2.7 Vanishing Artificial Pressure Limit: $\delta \to 0$

After taking the limit of $\varepsilon \to 0$, the approximate Navier-Stokes-Smoluchowski system reduces to

$$\int_0^T \int_\Omega \varrho_\delta B(\varrho_\delta)(\partial_t \phi + \mathbf{u}_\delta \cdot \nabla_x \phi) \, \mathrm{d}x \, \mathrm{d}t + \int_\Omega \varrho_{0,\delta} B(\varrho_{0,\delta}) \phi(0, \cdot) \, \mathrm{d}x$$
$$= \int_0^T \int_\Omega b(\varrho_\delta) \operatorname{div}_x \mathbf{u}_\delta \phi \, \mathrm{d}x \, \mathrm{d}t \tag{2.21}$$

where $b \in L^\infty[0, \infty) \cap C[0, \infty)$, and $B(\varrho) := B(1) + \int_1^\varrho \frac{b(z)}{z^2} dz$

$$\int_0^T \int_\Omega \eta_\delta \partial_t \phi + (\eta_\delta \mathbf{u}_\delta - \eta_\delta \nabla_x \Phi - \nabla_x \eta_\delta) \cdot \nabla_x \phi \, \mathrm{d}x \, \mathrm{d}t = -\int_\Omega \eta_{0,\delta} \phi(0, \cdot) \, \mathrm{d}x, \tag{2.22}$$

$$\int_\Omega \partial_t (\varrho_\delta \mathbf{u}_\delta) \cdot \mathbf{w} \, \mathrm{d}x = \int_\Omega \varrho_\delta \mathbf{u}_\delta \otimes \mathbf{u}_\delta : \nabla_x \mathbf{w} + (a \varrho_\delta^\gamma + \eta_\delta + \delta \varrho_\delta^\alpha) \operatorname{div}_x \mathbf{w} \, \mathrm{d}x$$
$$- \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}_\delta) : \nabla_x \mathbf{w} \, \mathrm{d}x - \int_\Omega (\beta \varrho_\delta + \eta_\delta) \nabla_x \Phi \cdot \mathbf{w} \, \mathrm{d}x \tag{2.23}$$

for any test functions ϕ and $\mathbf{w}$ and

$$\mathbf{u}_\delta = \nabla_x \eta_\delta \cdot \mathbf{n} + \eta_\delta \nabla_x \Phi \cdot \mathbf{n} = 0 \text{ on } (0,T) \times \partial\Omega \tag{2.24}$$

$$\begin{aligned}
&\int_\Omega \frac{1}{2}\varrho_\delta |\mathbf{u}_\delta|^2 + \frac{a}{\gamma-1}\varrho_\delta^\gamma + \frac{\delta}{\alpha-1}\varrho_\delta^\alpha + \eta_\delta \ln \eta_\delta + \eta_\delta \Phi \, \mathrm{d}x(\tau) \\
&+ \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}_\delta) : \nabla_x \mathbf{u}_\delta + |2\nabla_x \sqrt{\eta_\delta} + \sqrt{\eta_\delta}\nabla_x \Phi|^2 \, \mathrm{d}x \, \mathrm{d}t \\
&\leq \int_\Omega \frac{1}{2}\varrho_{0,\delta}|\mathbf{u}_{0,\delta}|^2 + \frac{a}{\gamma-1}\varrho_{0,\delta}^\gamma + \frac{\delta}{\alpha-1}\varrho_{0,\delta}^\alpha + \eta_{0,\delta}\ln \eta_{0,\delta} + \eta_{0,\delta}\Phi \, \mathrm{d}x \\
&- \beta \int_0^\tau \int_\Omega \varrho_\delta \mathbf{u}_\delta \cdot \nabla_x \Phi \, \mathrm{d}x \, \mathrm{d}t.
\end{aligned} \tag{2.25}$$

The first step in performing the limit $\delta \to 0$ is to find uniform bounds on the various quantities. Since the masses of the fluid and particles are constant and since $\varrho_{0,\delta} \to \varrho_0$ in $L^\gamma(\Omega)$ and $\eta_{0,\delta} \to \eta_0$ in $L^2(\Omega)$, the bounds on $\int_\Omega \varrho_\delta \, \mathrm{d}x$ and $\int_\Omega \eta_\delta \, \mathrm{d}x$ are uniform. So by the convexity properties of

$$H_F(\varrho) + H_P(\eta) := \frac{a}{\gamma-1}\varrho^\gamma + \eta \ln \eta,$$

the approximate energy balance can be used to bound certain terms. From the approximate energy balance, the following bounds can be obtained:

$$\{\mathbf{u}_\delta\} \in_b L^2(0,T; W_0^{1,2}(\Omega;\mathbb{R}^3))$$

$$\{\eta_\delta\} \in_b L^2(0,T; L^3(\Omega) \cap W^{1,3/2}(\Omega))$$

$$\{\varrho_\delta\} \in_b L^\infty(0,T; L^\gamma(\Omega))$$

$$\{\sqrt{\varrho_\delta}\mathbf{u}_\delta\} \in_b L^\infty(0,T; L^2(\Omega;\mathbb{R}^3))$$

Note that from the first two bounds, weak limits $\eta, \mathbf{u}$ are obtained, and from the bound on ϱ_δ, a weak-$*$ limit ϱ is obtained.

Much of the work in showing convergence to the weak formulation in Definition 2.1.1 is similar to that in Section 2.2.6 (for more details, see Section 3.7 in [32]). However, the main difference is in showing the pointwise convergence of $\{\varrho_\delta\}$. Here the family of cutoff functions $T_k(z)$ are used and are defined by

$$T_k(z) := kT\left(\frac{z}{k}\right)$$

where $T \in C^\infty[0,\infty)$ is concave and defined by

$$T(z) := \begin{cases} z, & z \in [0,1], \\ 2, & z \in [3,\infty). \end{cases}$$

Here, the calculations are similar to those in section 2.2.6 changing the definitions of φ_1, φ_2 to

$$\varphi_1 := \nabla_x \Delta_x^{-1}(\mathbf{1}_\Omega T_k(\varrho_\delta))$$

$$\varphi_2 := \nabla_x \Delta_x^{-1}(\mathbf{1}_\Omega T_k(\varrho))$$

After some analysis, the details of which are similar to those in the previous subsection and are carried out in detail in Section 3.7.4 in [32], it becomes clear that strong pointwise convergence of the fluid density will follow if the following two things can be shown:

- For

  $$L_k(\varrho) := \int_1^\varrho \frac{T_k(z)}{z^2} dz,$$

  the relation

  $$\int_0^T \int_\Omega \varrho L_k(\varrho)\partial_t\phi + \varrho L_k(\varrho)\mathbf{u}\cdot\nabla_x\phi - T_k(\varrho)\,\mathrm{div}_x\,\mathbf{u}\phi \;\mathrm{d}x\;\mathrm{d}t = -\int_\Omega \varrho_0 L_k(\varrho_0)\phi(0,\cdot)\;\mathrm{d}x$$

- $\int_0^\tau \int_\Omega T_k(\varrho) \operatorname{div}_x \mathbf{u} - \overline{T_k(\varrho)} \operatorname{div}_x \mathbf{u} \, \mathrm{d}x \, \mathrm{d}t \to 0$ as $k \to \infty$.

To prove these statements, the oscillation defect measure defined below must be controlled.

Definition 2.2.2. : Let $Q \subset \Omega$ and $q \geq 1$. Then

$$\operatorname{osc}_q[\varrho_\delta - \varrho](Q) := \sup_{k \geq 1} \left(\limsup_{\delta \to 0+} \int_Q |T_k(\varrho_\delta) - T_k(\varrho)|^q \, \mathrm{d}x \right).$$

It is clear that since $T_k(\varrho), \overline{T_k(\varrho)} \to \varrho$ as $k \to \infty$ in $L^1((0,T) \times \Omega$ and that $\operatorname{div}_x \mathbf{u} \in L^2((0,T) \times \Omega)$, if $\operatorname{osc}_q[\varrho_\delta - \varrho]((0,T) \times \Omega) < \infty$ for $q > 2$, then

$$\int_0^\tau \int_\Omega T_k(\varrho) \operatorname{div}_x \mathbf{u} - \overline{T_k(\varrho)} \operatorname{div}_x \mathbf{u} \, \mathrm{d}x \, \mathrm{d}t \to 0$$

as $k \to \infty$.

Also, the following will prove that

$$\int_0^T \int_\Omega \varrho L_k(\varrho) \partial_t \phi + \varrho L_k(\varrho) \mathbf{u} \cdot \nabla_x \phi - T_k(\varrho) \operatorname{div}_x \mathbf{u} \phi \, \mathrm{d}x \, \mathrm{d}t = - \int_\Omega \varrho_0 L_k(\varrho_0) \phi(0, \cdot) \, \mathrm{d}x.$$

Lemma 2.2.3. *Let $Q \subset \mathbb{R}^4$ be open and assume that*

$$\varrho_\delta \to \varrho \text{ weakly in } L^1(Q)$$

$$\boldsymbol{u}_\delta \to \boldsymbol{u} \text{ weakly in } L^2(Q; \mathbb{R}^3)$$

$$\nabla_x \boldsymbol{u}_\delta \to \nabla_x \boldsymbol{u} \text{ weakly in } L^2(Q; \mathbb{R}^3)$$

$$\operatorname{osc}_q[\varrho_\delta - \varrho](Q) < \infty \text{ for } q > 2.$$

Then the limit functions $\varrho, \boldsymbol{u}$ solve the renormalized continuity equation (2.1).

A proof of this lemma is given in Section 3.7.5 in [32]. Thus, once $\mathrm{osc}_q[\varrho_\delta - \varrho]$ is controlled, the strong convergence of the fluid density will have been shown. However, as argued in [32], this bound follows from the concavity of T_k and the convexity of the fluid pressure $\varrho \mapsto a\varrho^\gamma$.

Therefore, a solution in the sense of Definition 2.1.1 has been constructed from the Faedo-Galerkin with artificial diffusion and pressure approximation.

2.2.8 Approximate Relative Entropy Inequality

This section of the work follows in spirit the approach used in [31] and [34]. The approximate difference $\mathbf{u}_n - \mathbf{U}_m$ is used as a test function in the approximate momentum equation (2.13). This difference and its quadratic form are employed in the construction of the approximate relative entropy functional. Monitoring the evolution in time of this functional leads first to the approximate relative inequality (2.31), and subsequently, by passing to the limit, to the existence of weakly dissipative solutions.

Now, an approximation for (2.8) for each fixed $\varepsilon, \delta,$ and n is derived. First, smooth functions $\mathbf{U}_m \in C^1([0,T]; X_m)$, r_m and s_m on $[0,T] \times \overline{\Omega}$ with $r_m, s_m > 0$ on $[0,T] \times \overline{\Omega}$ and $\mathbf{U}_m|_{\partial\Omega} = 0$ are considered. Thus, $\mathbf{u}_n - \mathbf{U}_m$ can be taken as a suitable test function and substituted for $\mathbf{w}$ in (2.13) and performing some straight-forward calculations, it can be shown that

$$
\begin{aligned}
&\int_\Omega \varrho_n(\partial_t(\mathbf{u}_n - \mathbf{U}_m) + \mathbf{u}_n \cdot \nabla_x(\mathbf{u}_n - \mathbf{U}_m)) \cdot (\mathbf{u}_n - \mathbf{U}_m) \, \mathrm{d}x \\
&+ \int_\Omega \nabla_x(p(\varrho_n) + \delta\varrho_n^\alpha - p(r_m) + \eta_n - s_m) \cdot (\mathbf{u}_n - \mathbf{U}_m) \, \mathrm{d}x \\
&= - \int_\Omega [\mathbb{S}(\nabla_x \mathbf{u}_n) - \mathbb{S}(\nabla_x \mathbf{U}_m)] : \nabla_x(\mathbf{u}_n - \mathbf{U}_m) \, \mathrm{d}x \\
&- \int_\Omega r_m \left(\partial_t \mathbf{U}_m + \mathbf{U}_m \cdot \nabla_x \mathbf{U}_m + \nabla_x P_F(r_m) - \frac{a}{r_m} \operatorname{div}_x(\mathbb{S}(\nabla_x \mathbf{U}_m)) \right) \cdot (\mathbf{u}_n - \mathbf{U}_m) \, \mathrm{d}x \\
&+ \int_\Omega (r_m - \varrho_n)(\partial_t \mathbf{U}_m + \mathbf{U}_m \cdot \nabla_x \mathbf{U}_m) \cdot (\mathbf{u}_n - \mathbf{U}_m) \, \mathrm{d}x \\
&+ \int_\Omega \varrho_n(\mathbf{U}_m - \mathbf{u}_n) \cdot \nabla_x \mathbf{U}_m \cdot (\mathbf{u}_n - \mathbf{U}_m) \, \mathrm{d}x \\
&- \varepsilon \int_\Omega \Delta_x \varrho_n \mathbf{u}_n \cdot (\mathbf{u}_n - \mathbf{U}_m) + \nabla_x \varrho_n \nabla_x \mathbf{u}_n \cdot (\mathbf{u}_n - \mathbf{U}_m) \, \mathrm{d}x \\
&- \int_\Omega [(\beta\varrho_n + \eta_n)\nabla_x \Phi + \nabla_x s_m] \cdot (\mathbf{u}_n - \mathbf{U}_m) \, \mathrm{d}x. \qquad (2.26)
\end{aligned}
$$

By multiplying (2.11) by $\frac{1}{2}|\mathbf{u}_n - \mathbf{U}_m|^2$, it can be deduced that

$$
\begin{aligned}
&\partial_t \left(\frac{1}{2}\varrho_n |\mathbf{u}_n - \mathbf{U}_m|^2 \right) + \operatorname{div}_x \left(\frac{1}{2}\varrho_n \mathbf{u}_n |\mathbf{u}_n - \mathbf{U}_m|^2 \right) - \frac{\varepsilon}{2} \Delta_x \varrho_n |\mathbf{u}_n - \mathbf{U}_m|^2 \\
&= \varrho_n(\mathbf{u}_n - \mathbf{U}_m) \cdot \partial_t(\mathbf{u}_n - \mathbf{U}_m) + \varrho_n \mathbf{u}_n \cdot (\mathbf{u}_n - \mathbf{U}_m) \cdot \nabla_x(\mathbf{u}_n - \mathbf{U}_m). \qquad (2.27)
\end{aligned}
$$

Thus by (2.26)-(2.27) and some straight-forward calculations

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{1}{2}\varrho_n|\mathbf{u}_n-\mathbf{U}_m|^2\ \mathrm{d}x+\int_\Omega [\mathbb{S}(\nabla_x\mathbf{u}_n)-\mathbb{S}(\nabla_x\mathbf{U}_m)]:\nabla_x(\mathbf{u}_n-\mathbf{U}_m)\ \mathrm{d}x\\
&+\int_\Omega \varrho_n\left(\nabla_x P_F(\varrho_n)-\nabla_x P_F(r_m)\right)\cdot(\mathbf{u}_n-\mathbf{U}_m)\ \mathrm{d}x+\delta\int_\Omega \nabla_x\varrho_n^\alpha\cdot(\mathbf{u}_n-\mathbf{U}_m)\ \mathrm{d}x\\
&+\int_\Omega \eta_n\nabla_x\left(\ln\eta_n-\ln s_m\right)\cdot(\mathbf{u}_n-\mathbf{U}_m)\ \mathrm{d}x\\
&=\int_\Omega\left(\frac{r_m-\varrho_n}{r_m}\right)\mathrm{div}_x(\mathbb{S}(\nabla_x\mathbf{U}_m))\cdot(\mathbf{u}_n-\mathbf{U}_m)\ \mathrm{d}x\\
&-\int_\Omega\varrho_n\left(\partial_t\mathbf{U}_m+\mathbf{u}_n\cdot\nabla_x\mathbf{U}_m+\nabla_x P_F(r_m)-\frac{1}{r_m}\mathrm{div}_x(\mathbb{S}(\nabla_x\mathbf{U}_m))\right)\cdot(\mathbf{u}_n-\mathbf{U}_m)\ \mathrm{d}x\\
&+\int_\Omega\varrho_n(\mathbf{U}_m-\mathbf{u}_n)\cdot\nabla_x\mathbf{U}_m\cdot(\mathbf{u}_n-\mathbf{U}_m)\ \mathrm{d}x+\varepsilon\int_\Omega\nabla_x\varrho_n\cdot\mathbf{U}_m\cdot\nabla_x(\mathbf{u}_n-\mathbf{U}_m)\ \mathrm{d}x\\
&-\int_\Omega(\beta\varrho_n+\eta_n)\nabla_x\Phi\cdot(\mathbf{u}_n-\mathbf{U}_m)\ \mathrm{d}x-\int_\Omega\frac{\eta_n\nabla_x s_m}{s_m}\cdot(\mathbf{u}_n-\mathbf{U}_m)\ \mathrm{d}x. \qquad (2.28)
\end{aligned}
$$

Following the techniques from [37]

$$E_F(\varrho,r)=H_F(v+r)-H_F'(r)v-H_F(r)\text{ where }v:=\varrho-r,$$

$$E_P(\eta,s)=H_F(w+s)-H_F'(s)w-H_F(s)\text{ where }w:=\eta-s.$$

Thus,

$$
\begin{aligned}
&\frac{\partial E_F(\varrho,r)}{\partial v}=P_F(\varrho)-P_F(r),\ \frac{\partial E_F(\varrho,r)}{\partial r}=P_F(\varrho)-P_F(r)-P_F'(r)(\varrho-r)\\
&\frac{\partial E_P(\eta,s)}{\partial w}=P_P(\eta)-P_P(s),\ \frac{\partial E_P(\eta,s)}{\partial s}=P_P(\eta)-P_P(s)-P_P'(s)(\eta-s).
\end{aligned}
$$

Multiplying (2.11) by $P_F(\varrho_n)-P_F(r_m)$ yields

$$
\begin{aligned}
&\int_\Omega\varrho_n\left(\nabla_x P_F(\varrho_n)-\nabla_x P_F(r_m)\right)\cdot(\mathbf{u}_n-\mathbf{U}_m)\ \mathrm{d}x\\
&=\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega E_F(\varrho_n,r_m)\ \mathrm{d}x+\int_\Omega P_F'(r_m)(\varrho_n-r_m)(\partial_t r_m+\mathrm{div}_x(r_m\mathbf{u}_n))\ \mathrm{d}x\\
&+\int_\Omega\left(\frac{\partial E_F}{\partial v}(\varrho_n,r_m)(\varrho_n-r_m)+\frac{\partial E_F}{\partial r}(\varrho_n,r_m)r_m-E_F(\varrho_n,r_m)\right)\mathrm{div}_x\,\mathbf{U}_m\ \mathrm{d}x\\
&+\varepsilon\int_\Omega(P_F(\varrho_n)-P_F(r_m))\Delta_x\varrho_n\ \mathrm{d}x \qquad (2.29)
\end{aligned}
$$

and multiplying (2.12) yields

$$
\begin{aligned}
&\int_\Omega \eta_n \left(\nabla_x P_P(\eta_n) - \nabla_x P_P(s_m)\right) \cdot (\mathbf{u}_n - \mathbf{U}_m) \ \mathrm{d}x \\
&= \int_\Omega P_P'(s_m)(\eta_n - s_m)(\partial_t s_m + \operatorname{div}_x(s_m \mathbf{U}_m)) \ \mathrm{d}x \\
&+ \int_\Omega \left(\frac{\partial E_P}{\partial w}(\eta_n, s_m)(\eta_n - s_m) + \frac{\partial E_P}{\partial s} s_m - E_P(\eta_n, s_m) \right) \operatorname{div}_x \mathbf{U}_m \ \mathrm{d}x \\
&- \int_\Omega (P_P(\eta_n) - P_P(s_m))(\Delta_x \eta_n + \operatorname{div}_x(\eta_n \nabla_x \Phi)) \ \mathrm{d}x. \qquad (2.30)
\end{aligned}
$$

Thus, combining (2.26)-(2.30) and using (2.14), the approximate relative entropy equation is obtained:

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \frac{1}{2} \varrho_n |\mathbf{u}_n - \mathbf{U}_m|^2 + E_F(\varrho_n, r_m) + E_P(\eta_n, s_m) \ \mathrm{d}x \\
&+ \int_\Omega [\mathbb{S}(\nabla_x \mathbf{u}_n) - \mathbb{S}(\nabla_x \mathbf{U}_m)] : \nabla_x (\mathbf{u}_n - \mathbf{U}_m) \ \mathrm{d}x + \frac{\delta}{\alpha - 1} \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \varrho_n^\alpha \ \mathrm{d}x \\
&\leq \int_\Omega \operatorname{div}_x(\mathbb{S}(\nabla_x \mathbf{U}_m)) \cdot (\mathbf{u}_n - \mathbf{U}_m) \ \mathrm{d}x \\
&- \int_\Omega \varrho_n (\partial_t \mathbf{U}_m + \mathbf{u}_n \cdot \nabla_x \mathbf{U}_m) \cdot (\mathbf{u}_n - \mathbf{U}_m) \ \mathrm{d}x \\
&- \int_\Omega \partial_t P_F(r_m)(\varrho_n - r_m) + \nabla_x P_F(r_m) \cdot (\varrho_n \mathbf{u}_n - r_m \mathbf{U}_m) \ \mathrm{d}x \\
&- \int_\Omega [\varrho_n (P_F(\varrho_n) - P_F(r_m)) - E_F(\varrho_n, r_m)] \operatorname{div}_x \mathbf{U}_m \ \mathrm{d}x \\
&- \int_\Omega \partial_t P_P(s_m)(\eta_n - s_m) + \nabla_x P_P(s_m) \cdot (\eta_n \mathbf{u}_n - s_m \mathbf{U}_m) \ \mathrm{d}x \\
&- \int_\Omega [\eta_n (P_P(\eta_n) - P_P(s_m)) - E_P(\eta_n, s_m)] \operatorname{div}_x \mathbf{U}_m \ \mathrm{d}x \\
&- \int_\Omega \nabla_x (P_P(\eta_n) - P_P(s_m)) \cdot (\nabla_x \eta_n + \eta_n \nabla_x \Phi) \ \mathrm{d}x \\
&- \int_\Omega (\beta \varrho_n + \eta_n) \nabla_x \Phi \cdot (\mathbf{u}_n - \mathbf{U}_m) \ \mathrm{d}x - \int_\Omega \frac{\eta_n \nabla_x s_m}{s_m} \cdot (\mathbf{u}_n - \mathbf{U}_m) \ \mathrm{d}x \\
&+ \varepsilon \int_\Omega \nabla_x \varrho_n \cdot \mathbf{U}_m \cdot \nabla_x (\mathbf{u}_n - \mathbf{U}_m) - \nabla_x (P_F(\varrho_n) - P_F(r_m)) \cdot \nabla_x \varrho_n \ \mathrm{d}x \\
&- \delta \int_\Omega \varrho_n^\alpha \operatorname{div}_x \mathbf{U}_m \ \mathrm{d}x. \qquad (2.31)
\end{aligned}
$$

By taking the limits $n \to \infty, \varepsilon \to 0, \delta \to 0$ as worked earlier in the section and replacing $\{r_m, \mathbf{U}_m, s_m\}$ with $\{r, \mathbf{U}, s\}$ by means of a density argument, inequality (2.8) is obtained, proving Theorem 2.2.1.

2.2.9 Regularity Required for Smooth Solutions

First the required regularity for $\{r, \mathbf{U}, s\}$ is determined such that $\mathcal{R}(\varrho, \mathbf{u}, \eta, r, \mathbf{U}, s)$ is well-defined. A quick inspection shows that the following are required to ensure all the integrals in (2.8) and the remainder term are defined:

$$r \in C_{\text{weak}}([0,T]; L^{\gamma}(\Omega))$$

$$\mathbf{U} \in C_{\text{weak}}([0,T]; L^{2\gamma/\gamma-1}(\Omega; \mathbb{R}^3))$$

$$\nabla_x \mathbf{U} \in L^2(0,T; L^2(\Omega; \mathbb{R}^{3\times 3})),\ \mathbf{U}|_{\partial\Omega} = 0$$

$$s \in C_{\text{weak}}([0,T]; L^1(\Omega)) \cap L^1(0,T; L^{6\gamma/\gamma-3}(\Omega))$$

$$\partial_t \mathbf{U} \in L^1(0,T; L^{2\gamma/\gamma-1}(\Omega; \mathbb{R}^3)) \cap L^2(0,T; L^{6\gamma/5\gamma-6}(\Omega; \mathbb{R}^3))$$

$$\nabla_x^2 \mathbf{U} \in L^1(0,T; L^{2\gamma/\gamma+1}(\Omega; \mathbb{R}^{3\times 3\times 3})) \cap L^2(0,T; L^{6\gamma/5\gamma-6}(\Omega; \mathbb{R}^{3\times 3\times 3}))$$

$$\partial_t P_F(r) \in L^1(0,T; L^{\gamma/\gamma-1}(\Omega))$$

$$\nabla_x P_F(r) \in L^1(0,T; L^{2\gamma/\gamma-1}(\Omega; \mathbb{R}^3)) \cap L^2(0,T; L^{6\gamma/5\gamma-6}(\Omega; \mathbb{R}^3))$$

$$\partial_t P_P(s) \in L^1(0,T; L^{\infty}(\Omega)) \cap L^{\infty}(0,T; L^{3/2}(\Omega))$$

$$\nabla_x P_P(s) \in L^{\infty}(0,T; L^3(\Omega; \mathbb{R}^3))$$

$$\nabla_x s \in L^{\infty}(0,T; L^2(\Omega; \mathbb{R}^3)) \cap L^2(0,T; L^{6\gamma/5\gamma+3}(\Omega; \mathbb{R}^3)).$$

2.2.10 The Weak-Strong Uniqueness Result

The theorem that is the aim of this section of the dissertation can now be stated. First $\{r, \mathbf{U}, s\}$ is taken to be a solution of (1.1)-(1.7) with the regularity stated above and r and s are taken to be bounded above and bounded below by some positive constant. Also, $\mathbf{U}$ is taken to be bounded above in magnitude and the following conditions are imposed:

$$\nabla_x r \in L^2(0,T; L^q(\Omega; \mathbb{R}^3)) \tag{2.32}$$

$$\nabla_x^2 \mathbf{U} \in L^2(0,T; L^q(\Omega; \mathbb{R}^{3\times 3\times 3})) \tag{2.33}$$

$$\alpha := \nabla_x s + s\nabla_x \Phi \in L^2(0,T; L^q(\Omega; \mathbb{R}^3)) \tag{2.34}$$

where

$$q > \max\left\{3, \frac{3}{\gamma - 1}\right\}.$$

Thus, by embeddings, $\mathbf{U} \in L^2(0,T; W^{1,\infty}(\Omega;\mathbb{R}^3))$ since $\gamma > \frac{3}{2}$, $q > \frac{6\gamma}{5\gamma-6}$. Now the weak-strong uniqueness result is stated:

Theorem 2.2.2 (Weak-Strong Uniqueness)**.** *Assume* $\{\varrho, \boldsymbol{u}, \eta\}$ *is a weakly dissipative solution of* (1.1)-(1.7) *in the sense of Definiton* 2.2.1. *Assume that* $\{r, \boldsymbol{U}, s\}$ *is a smooth solution of* (1.1)-(1.7) *with the regularity stated in Section* 2.2.9 *and obeying the hypotheses* (2.32)-(2.34). *Then* $\{\varrho, \boldsymbol{u}, \eta\}$ *is identically* $\{r, \boldsymbol{U}, s\}$.

Proof. To begin with, some simple algebra yields the following alternative expression for $\mathcal{R}(\varrho, \mathbf{u}, \eta, r, \mathbf{U}, s)$:

$$
\begin{aligned}
&\mathcal{R}(\varrho, \mathbf{u}, \eta, r, \mathbf{U}, s) \\
&\quad := \int_\Omega \frac{1}{r} \operatorname{div}_x(\mathbb{S}(\nabla_x \mathbf{U})) \cdot (\mathbf{U} - \mathbf{u}) \, \mathrm{d}x \\
&\quad - \int_\Omega \varrho(\partial_t \mathbf{U} + \mathbf{u} \cdot \mathbf{U}) \cdot (\mathbf{u} - \mathbf{U}) \, \mathrm{d}x \\
&\quad - \int_\Omega \partial_t P_F(r)(\varrho - r) + \nabla_x P_F(r) \cdot (\varrho \mathbf{u} - r\mathbf{U}) \, \mathrm{d}x \\
&\quad - \int_\Omega [\varrho(P_F(\varrho) - P_F(r)) - E_F(\varrho, r)] \operatorname{div}_x \mathbf{U} \, \mathrm{d}x \\
&\quad - \int_\Omega \partial_t P_P(s)(\eta - s) + \nabla_x P_P(s) \cdot (\eta \mathbf{u} - s\mathbf{U}) \, \mathrm{d}x \\
&\quad - \int_\Omega [\eta(P_P(\eta) - P_P(s)) - E_P(\eta, s)] \operatorname{div}_x \mathbf{U} \, \mathrm{d}x \\
&\quad - \int_\Omega \nabla_x (P_P(\eta) - P_P(s)) \cdot (\nabla_x \eta + \eta \nabla_x \Phi) \, \mathrm{d}x \\
&\quad - \int_\Omega (\beta\varrho + \eta) \nabla_x \Phi \cdot (\mathbf{u} - \mathbf{U}) \, \mathrm{d}x. \qquad (2.35)
\end{aligned}
$$

By the conditions on the stress tensor,

$$
[\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U})] : \nabla_x(\mathbf{u} - \mathbf{U}) = \mu |\nabla_x(\mathbf{u} - \mathbf{U})|^2 + \lambda |\operatorname{div}_x(\mathbf{u} - \mathbf{U})|^2 \geq 0.
$$

A straight-forward manipulation of (2.35) using $\{r, \mathbf{U}, s\}$ as a solution for (1.2) yields

$$
\begin{aligned}
&\mathcal{R}(\varrho, \mathbf{u}, \eta, r, \mathbf{U}, s) \\
&\quad = \int_\Omega \frac{1}{r}(\varrho - r)\,\mathrm{div}_x(\mathbb{S}(\nabla_x \mathbf{U})) \cdot (\mathbf{U} - \mathbf{u})\ \mathrm{d}x \\
&\quad + \int_\Omega \varrho(\mathbf{u} - \mathbf{U}) \cdot \nabla_x \mathbf{U} \cdot (\mathbf{U} - \mathbf{u})\ \mathrm{d}x \\
&\quad + \int_\Omega [E_F(\varrho, r) - \varrho(P_F(\varrho) - P_F(r)) + rP_F'(r)(\varrho - r)]\,\mathrm{div}_x\,\mathbf{U}\ \mathrm{d}x \\
&\quad + \int_\Omega [E_P(\eta, s) - \eta(P_P(\eta) - P_P(s)) + sP_F'(s)(\eta - s)]\,\mathrm{div}_x\,\mathbf{U}\ \mathrm{d}x \\
&\quad + \int_\Omega \left(\frac{\varrho}{r}\nabla_x s + \frac{\varrho s}{r}\nabla_x \Phi - \eta \nabla_x \Phi - \frac{\eta}{s}\nabla_x s\right) \cdot (\mathbf{u} - \mathbf{U})\ \mathrm{d}x \\
&\quad - \int_\Omega (\nabla_x P_P(\eta) - \nabla_x P_P(s)) \cdot (\nabla_x \eta + \eta \nabla_x \Phi)\ \mathrm{d}x \\
&\quad + \int_\Omega \Delta_x s + \mathrm{div}_x(s\nabla_x \Phi) - \frac{\eta}{s}(\Delta_x s + \mathrm{div}_x(s \nabla_x \Phi))\ \mathrm{d}x
\end{aligned} \tag{2.36}
$$

Similarly to [34] and [37],

$$
\left| \int_\Omega \varrho(\mathbf{u} - \mathbf{U}) \cdot \nabla_x \mathbf{U} \cdot (\mathbf{U} - \mathbf{u})\ \mathrm{d}x \right| \le h(\tau) \int_\Omega \frac{1}{2}\varrho |\mathbf{u} - \mathbf{U}|^2\ \mathrm{d}x \tag{2.37}
$$

and

$$
\begin{aligned}
&\left| \int_\Omega [E_F(\varrho, r) - \varrho(P_F(\varrho) - P_F(r)) + rP_F'(r)(\varrho - r)]\,\mathrm{div}_x\,\mathbf{U}\ \mathrm{d}x \right| \\
&+ \left| \int_\Omega [E_P(\eta, s) - \eta(P_P(\eta) - P_P(s)) + sP_F'(s)(\eta - s)]\,\mathrm{div}_x\,\mathbf{U}\ \mathrm{d}x \right| \\
&\le h(\tau) \int_\Omega E_F(\varrho, r) + E_P(\eta, s)\ \mathrm{d}x
\end{aligned} \tag{2.38}
$$

for some $h \in L^2(0, T)$.

Also, by the embedding of $W^{1,2}(\Omega)$ in $L^6(\Omega)$ and Korn's inequality,

$$
\int_\Omega [\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U})] : \nabla_x(\mathbf{u} - \mathbf{U})\ \mathrm{d}x \ge \Lambda \|\mathbf{u} - \mathbf{U}\|^2_{W_0^{1,2}(\Omega;\mathbb{R}^3)}.
$$

So, for $Q >> 1$

$$\int_{\varrho \leq Q} \left| \frac{1}{r}(\varrho - r) \operatorname{div}_x(\mathbb{S}(\nabla_x \mathbf{U})) \cdot (\mathbf{U} - \mathbf{u}) \right| \, \mathrm{d}x$$

$$\leq \frac{\Lambda}{2} \|\mathbf{u} - \mathbf{U}\|^2_{W_0^{1,2}(\Omega;\mathbb{R}^3)} + c(\Lambda, Q) \|\mathbf{U}\|^2_{W^{2,3}(\Omega;\mathbb{R}^3)} \int_\Omega E_F(\varrho, r) \, \mathrm{d}x \tag{2.39}$$

and for $Q << 1$

$$\int_{\varrho > Q} \left| \frac{1}{r}(\varrho - r) \operatorname{div}_x(\mathbb{S}(\nabla_x \mathbf{U})) \cdot (\mathbf{U} - \mathbf{u}) \right| \, \mathrm{d}x$$

$$\leq c(Q, r) \|\mathbf{u} - \mathbf{U}\|^2_{W_0^{1,2}(\Omega;\mathbb{R}^3)} \| \operatorname{div}_x \mathbb{S}(\nabla_x \mathbf{U})\|_{L^q(\Omega;\mathbb{R}^3)} \int_\Omega E_F(\varrho, r) \, \mathrm{d}x \tag{2.40}$$

as argued in [34].

The fifth integral in (2.36) is rewritten as

$$\int_\Omega \frac{1}{r}(\varrho - r)\alpha \cdot (\mathbf{u} - \mathbf{U}) \, \mathrm{d}x - \int_\Omega \frac{1}{s}(\eta - s)\alpha \cdot (\mathbf{u} - \mathbf{U}) \, \mathrm{d}x. \tag{2.41}$$

Thus, using a technique similar to obtaining the bounds in (2.39) and (2.40) for $Q >> 1$

$$\int_{\varrho \leq Q} \left| \frac{1}{r}(\varrho - r)\alpha \cdot (\mathbf{u} - \mathbf{U}) \right| \, \mathrm{d}x$$

$$\leq \frac{\Lambda}{2} \|\mathbf{u} - \mathbf{U}\|^2_{W_0^{1,2}(\Omega;\mathbb{R}^3)} + c(\Lambda, Q) \|\alpha\|^2_{L^3(\Omega;\mathbb{R}^3)} \int_\Omega E_F(\varrho, r) \, \mathrm{d}x \tag{2.42}$$

and for $Q << 1$

$$\int_{\varrho > Q} \left| \frac{1}{r}(\varrho - r)\alpha \cdot (\mathbf{U} - \mathbf{u}) \right| \, \mathrm{d}x$$

$$\leq c(Q, r) \|\mathbf{u} - \mathbf{U}\|^2_{W_0^{1,2}(\Omega;\mathbb{R}^3)} \|\alpha\|_{L^q(\Omega;\mathbb{R}^3)} \int_\Omega E_F(\varrho, r) \, \mathrm{d}x \tag{2.43}$$

and also

$$\int_{\eta \leq Q} \left| \frac{1}{s}(\eta - s)\alpha \cdot (\mathbf{u} - \mathbf{U}) \right| \, \mathrm{d}x$$

$$\leq \frac{\Lambda}{2} \|\mathbf{u} - \mathbf{U}\|^2_{W_0^{1,2}(\Omega;\mathbb{R}^3)} + c(\Lambda, Q) \|\alpha\|^2_{L^3(\Omega;\mathbb{R}^3)} \int_\Omega E_P(\eta, s) \, \mathrm{d}x \tag{2.44}$$

and for $Q << 1$

$$\int_{\eta > Q} \left| \frac{1}{s}(\eta - s)\alpha \cdot (\mathbf{U} - \mathbf{u}) \right| \, \mathrm{d}x$$
$$\leq c(Q,s) \|\mathbf{u} - \mathbf{U}\|^2_{W_0^{1,2}(\Omega;\mathbb{R}^3)} \|\alpha\|_{L^q(\Omega;\mathbb{R}^3)} \int_\Omega E_P(\eta, s) \, \mathrm{d}x. \tag{2.45}$$

A simple calculation using that $(\nabla_x s + s\nabla_x \Phi) \cdot \mathbf{n} = 0$ on $\partial\Omega$ yields

$$\begin{aligned} &- \int_\Omega (\nabla_x P_P(\eta) - \nabla_x P_P(s)) \cdot (\nabla_x \eta + \eta \nabla_x \Phi) \, \mathrm{d}x \\ &+ \int_\Omega \Delta_x s + \mathrm{div}_x(s\nabla_x \Phi) - \frac{\eta}{s}(\Delta_x s + \mathrm{div}_x(s\nabla_x\Phi)) \, \mathrm{d}x \\ &= - \int_\Omega \frac{1}{s} \left| \sqrt{\frac{\eta}{s}} \nabla_x s - \sqrt{\frac{s}{\eta}} \nabla_x \eta \right|^2 \, \mathrm{d}x \leq 0. \end{aligned} \tag{2.46}$$

Thus, by applying (2.37)-(2.46) and Gronwall's inequality,

$$\begin{aligned} &\int_\Omega \frac{1}{2}\varrho|\mathbf{u} - \mathbf{U}|^2 + E_F(\varrho, r) + E_P(\eta, s) \, \mathrm{d}x(\tau) \\ &+ \int_0^\tau \int_\Omega [\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U})] : \nabla_x(\mathbf{u} - \mathbf{U}) \, \mathrm{d}x \, \mathrm{d}t \\ &\leq c(T) \int_\Omega \frac{1}{2}\varrho_0|\mathbf{u}_0 - \mathbf{U}_0|^2 + E_F(\varrho_0, r_0) + E_P(\eta_0, s_0) \, \mathrm{d}x \end{aligned} \tag{2.47}$$

for *a.e.* $\tau \in (0,T)$ which proves the result. □

2.2.11 Ω Unbounded Domain in $\mathbb{R}^3$

Given an unbounded domain Ω and an external potential Φ satisfying the assumptions **(HC)**, an increasing sequence of domains Ω_r, with $r > 0$ can be constructed such that each Ω_r is bounded and (Ω_r, Φ) satisfies **(HC)**. The domains Ω_r approximate Ω in the sense that $\bigcup_{r>0} \Omega_r = \Omega$. Using the previous subsection, for any $r > 0$, there is a solution on Ω_r. In this subsection, it is shown that the limit $r \to \infty$ can be taken to obtain a solution on Ω.

One of the key issues in this problem for unbounded domains Ω is providing a control for the negative contribution of the physical entropy $\eta \ln \eta$ in the free-energy bounds, noted $\eta \ln_- \eta$. Here, the confinement conditions **(HC)** on (Ω, Φ) are crucial. Most of these lemmas can be seen in [25] and [15] but are included here for the sake of completeness.

Lemma 2.2.4. *Assume that* (Ω, Φ) *satisfy the hypotheses* **(HC)**. *For any density* $\eta \in L^1_+(\Omega)$,

$$\int_\Omega \eta(x) \ln_- \eta(x)\ dx \le \frac{1}{2}\int_\Omega \Phi(x)\eta(x)\ dx + \frac{1}{e}\int_\Omega e^{-\Phi(x)/2}\ dx\ .$$

Proof. Let $\overline{\eta} := \eta\,\chi_{\{\eta\le 1\}}$ and $\overline{M} = \int_\Omega \overline{\eta}(x)\ \mathrm{d}x \le \int_\Omega \eta(x)\ \mathrm{d}x = M$. Then

$$\int_\Omega \overline{\eta}(x)\left(\ln \overline{\eta}(x) + \frac{1}{2}\Phi(x)\right)\ \mathrm{d}x = \int_\Omega [Y(x)\ln Y(x)]\mu\ \mathrm{d}x - \overline{M}\ln Z$$

where $Y := \overline{\eta}/\mu$, $\mu(x) = e^{-\Phi(x)/2}/Z$ with $Z = \int_\Omega e^{-\Phi(x)/2}\ \mathrm{d}x$. The Jensen inequality yields

$$\int_\Omega [Y(x)\ \ln Y(x)]\mu\ \mathrm{d}x \ge \left(\int_\Omega Y(x)\mu\ \mathrm{d}x\right)\ \ln\left(\int_\Omega Y(x)\mu\ \mathrm{d}x\right) = \overline{M}\ \ln \overline{M}$$

and

$$\begin{aligned} -\int_\Omega \eta(x)\ \ln_- \eta(x)\ \mathrm{d}x = \int_\Omega \overline{\eta}(x)\ \ln \overline{\eta}(x)\ \mathrm{d}x &\ge \overline{M}\ln\overline{M} - \overline{M}\ln Z - \frac{1}{2}\int_\Omega \Phi(x)\,\overline{\eta}(x)\ \mathrm{d}x \\ &\ge -\frac{Z}{e} - \frac{1}{2}\int_\Omega \Phi(x)\,\eta(x)\ \mathrm{d}x\ , \end{aligned}$$

from which the desired claim follows. □

This previous lemma leads immediately to the following consequence.

Corollary 2.2.1. *Assume that* (Ω, Φ) *satisfy the hypotheses* **(HC)**. *For any density* $\eta \in L^1_+(\Omega)$, *if*

$$\int_\Omega \eta(x)\, \ln \eta(x)\ dx + \int_\Omega \Phi(x)\eta(x)\ dx \le C\,,$$

then $\eta \ln \eta \in L^1(\Omega)$ *and there exists* $D > 0$ *depending on* C *and* Φ *such that*

$$\int_\Omega \eta(x)\, \ln_+ \eta(x)\ dx \le D \qquad \text{and} \qquad \int_\Omega \Phi(x)\eta(x)\ dx \le D\,.$$

Finally, the above estimates can be used to control the mass of the densities η outside a large ball to avoid loss of mass at infinity.

Lemma 2.2.5. *Given any domain* Ω *such that* $e^{-\Phi} \in L^1_+(\Omega)$ *and any density* $\eta \in L^1_+(\Omega)$, *then*

$$\int_\Omega \eta(x)\, \ln \eta(x)\ dx + \int_\Omega \Phi(x)\eta(x)\ dx \ge \int_\Omega \eta(x)\ dx\, \ln\left(\frac{\int_\Omega \eta(x)\ dx}{\int_\Omega e^{-\Phi(x)}\ dx}\right).$$

As a consequence, if $e^{-\Phi} \in L^1_+(\Omega)$ *and*

$$\int_\Omega \eta(x)\, \ln \eta(x)\ dx + \int_\Omega \Phi(x)\eta(x)\ dx \le C\,,$$

then, for any $\epsilon > 0$ *there exists* $R > 0$ *depending on* C *and* Φ *only such that*

$$\int_{\Omega\cap(\mathbb{R}^3 - B(0,R))} \eta(x)\ dx < \epsilon\,.$$

Proof. A direct use of Jensen's inequality shows the first inequality by using the convexity of $x \mapsto x \ln x$. Application of the first inequality to the domain $\Omega^c_R := \Omega \cap (\mathbb{R}^3/B(0,R))$ starts to show the second claim.

$$\int_{\Omega^c_R} \eta(x)\ \mathrm{d}x\ \ln\left(\frac{\int_{\Omega^c_R} \eta(x)\ \mathrm{d}x}{\int_{\Omega^c_R} e^{-\Phi(x)}\ \mathrm{d}x}\right) \le D \tag{2.48}$$

for some $D > 0$, where Lemma 2.2.4 and Corollary 2.2.1 were used. Now, arguing by contradiction, if the second claim were not true,

$$\exists \epsilon_0 > 0 \; \forall R_0 > 0 \; \exists R > R_0 \text{ such that } \int_{\Omega_R^c} \eta(x) \, \mathrm{d}x \geq \epsilon_0.$$

Since $e^{-\Phi} \in L^1_+(\Omega)$, R_0 can be assumed to be large such that

$$\int_{\Omega_R^c} e^{-\Phi(x)} \, \mathrm{d}x \leq \int_{\Omega_{R_0}^c} e^{-\Phi(x)} \, \mathrm{d}x < \epsilon_0 \leq \int_{\Omega_R^c} \eta(x) \, \mathrm{d}x$$

and thus due to (2.48),

$$\int_{\Omega_R^c} \eta(x) \, \mathrm{d}x \leq \int_{\Omega_R^c} e^{-\Phi(x)} \, \mathrm{d}x \, e^{D/\epsilon_0} \leq \int_{\Omega_{R_0}^c} e^{-\Phi(x)} \, \mathrm{d}x \, e^{D/\epsilon_0} \, .$$

This leads to a contradiction since the right-hand side can be made arbitrarily small by taking R_0 large enough. □

What follows are sketches of proofs for the existence of weakly dissipative solutions and the uniqueness result in the case for unbounded Ω. The main idea, as in [15] is to construct solutions on an increasing sequence of bounded subsets Ω_r of Ω such that $\bigcup_{r>0} \Omega_r = \Omega$.

Theorem 2.2.3 (Suitable weak solutions)**.** *Assume that* (Ω, Φ) *satisfies the confinement hypotheses* **(HC)** *with* $\Omega \subset \mathbb{R}^3$ *an unbounded domain of class* $C^{2+\nu}, \nu > 0$. *Suppose the initial data* $\{\varrho_0, \boldsymbol{u}_0, \eta_0\}$ *satisfy*

$$0 < \varrho_0 \in L^\gamma(\Omega), \; \varrho_0 |\boldsymbol{u}_0|^2 \in L^1(\Omega), \; \eta_0 \ln \eta_0 \in L^1(\Omega)$$

in addition to the conditions on the initial data specified in Section 2.2.9. *Then the Navier-Stokes-Smoluchowski system in* (1.1)-(1.6) *has a weakly dissipative solution in the sense of Definition* 2.2.1.

Sketch of Proof. As stated before, for each bounded subset Ω_r of Ω, there is a weakly dissipative solution $\{\varrho_r, \mathbf{u}_r, \eta_r\}$. The key point in showing a solution $\{\varrho, \mathbf{u}, \eta\}$ on Ω is that

$$\int_{\Omega_r} \left|2\nabla_x\sqrt{\eta_r} + \sqrt{\eta_r}\nabla_x\Phi\right|^2 \, \mathrm{d}x$$

is bounded by some constant C which is independent of r. Thus

$$\int_{\Omega_r} 4|\nabla_x\sqrt{\eta_r}|^2 + 2\nabla_x\eta_r \cdot \nabla_x\Phi + \eta_r|\nabla_x\Phi|^2 \, \mathrm{d}x = \int_{\Omega_r} \left|2\nabla_x\sqrt{\eta_r} + \sqrt{\eta_r}\nabla_x\Phi\right|^2 \, \mathrm{d}x \leq C.$$

Thus, by a reordering of terms and an integration by parts

$$\int_{\Omega_r} 4|\nabla_x\sqrt{\eta_r}|^2 + \eta_r|\nabla_x\Phi|^2 \, \mathrm{d}x \leq C + \int_{\Omega_r} \eta_r|\Delta_x\Phi| \, \mathrm{d}x.$$

From this (c.f. [15]) and using the confinement hypotheses and their consequences stated above,

$$\begin{aligned}
\int_{\Omega_r} \eta_r|\Delta_x\Phi| \, \mathrm{d}x &= \int_{\Omega_R} \eta_r|\Delta_x\Phi| \, \mathrm{d}x + \int_{\Omega_r-\Omega_R} \eta_r|\Phi| \, \mathrm{d}x \\
&\leq \|\Delta_x\Phi\|_{L^\infty(\Omega_R)} \int_{\Omega_R} \eta_r \, \mathrm{d}x + C\int_{\Omega_r-\Omega_R} \eta_r\Phi \, \mathrm{d}x \leq C.
\end{aligned}$$

From this, $\{\varrho_r, \mathbf{u}_r, \eta_r\}$ have the necessary bounds to obtain the necessary convergence to $\{\varrho, \mathbf{u}, \eta\}$.

Next is stated the uniqueness result for unbounded domains. Here, the key point is that $\alpha \in L^3(\Omega;\mathbb{R}^3)$, as in the unbounded case, $L^3(\Omega;\mathbb{R}^3)$ need not be embedded in $L^q(\Omega;\mathbb{R}^3)$. Due to the additional hypotheses, the proof of the theorem below differs only slightly from the proof of Theorem 2.2.2 and is omitted here.

Theorem 2.2.4 (Uniqueness on Unbounded Ω)**.** *Assume that* (Ω, Φ) *satisfy the confinement hypotheses* **(HC)** *with* $\Omega \subset \mathbb{R}^3$ *an unbounded domain of class* $C^{2+\nu}, \nu >$

0. *Suppose the initial data* $\{\varrho_0, \boldsymbol{u}_0, \eta_0\}$ *satisfy*

$$0 < \varrho_0 \in L^\gamma(\Omega),\ \varrho_0|\boldsymbol{u}_0|^2 \in L^1(\Omega),\ \eta_0 \ln \eta_0 \in L^1(\Omega)$$

in addition to the conditions on the initial data specified in Section 2.2.9. *Assume that* $\{\varrho, \boldsymbol{u}, \eta\}$ *is a weakly dissipative solution of the system and that* $\{r, \boldsymbol{U}, s\}$ *is a solution of the Navier-Stokes-Smoluchowksi system with the same initial data as* $\{\varrho, \boldsymbol{u}, \eta\}$ *enjoying higher regularity* (2.32)-(2.34). *Also, assume that*

$$\nabla_x r \in L^2(0, T; L^p(\Omega; \mathbb{R}^3))$$

$$\nabla_x^2 \boldsymbol{U} \in L^2(0, T; L^p(\Omega; \mathbb{R}^{3\times3\times3}))$$

$$\alpha := \nabla_x s + s\nabla_x \Phi \in L^2(0, T; L^p(\Omega; \mathbb{R}^3))$$

where

$$p < \min\left\{3, \frac{3}{\gamma - 1}\right\}.$$

Then $\{\varrho, \boldsymbol{u}, \eta\}$ *is identically* $\{r, \boldsymbol{U}, s\}$.

2.3 Existence of Smooth Solutions

The attentive reader will notice that the previous section on weak-strong uniqueness makes no claim about the existence of suitably smooth solutions $\{r, \mathbf{U}, s\}$. A key difficulty in proving the existence of smooth solutions is the existence of possible vacuum states in the fluid density. In that case, the momentum equation loses its parabolicity. However, work done on other models of compressible fluid flows has shown that there are conditions on the initial data that if imposed, will guarantee existence of smooth solutions for finite time even with initial fluid density with vacuum states (see [20] for the compressible Navier-Stokes model and [19] for compressible heat-conducting flows). The local result can then be combined with blow-up conditions along the lines of [28] and [29] that if satisfied, allow the local regularity result to be extended to a global result. This section proves compatibility conditions on the initial data that will guarantee the existence of smooth solutions for finite time and follows the spirit of the work in [19].

If only the continuity equation (1.1) is considered and $\mathbf{u}$ is taken to be given with reasonable bounds on $\operatorname{div}_x \mathbf{u}$, the equation becomes a basic linear first order transport condition, and the representation of the solution, as shown in [30], shows that if the initial density is bounded below by some positive constant, then the density will remain positive at all times. However, the initial data under consideration for this work does allow for vacuum states. Thus, in order to preserve the parabolicity of the momentum equation, a compatibility condition must be imposed on the initial data. Formally, the proposed compatibility condition is derived by considering the momentum equation and taking the limit as $t \to 0$. As such, the following condition arises: there is a vector field $\mathbf{h}$ such that

$$\varrho_0 \mathbf{h} = \nabla_x (a\varrho_0^\gamma + \eta_0) - \operatorname{div}_x \mathbb{S}(\nabla_x \mathbf{u}) + \eta_0 \nabla_x \Phi.$$

However, if the initial data have high enough regularity, this condition can be weakened to the existence of a vector field $\mathbf{h} \in L^2(\Omega; \mathbb{R}^3)$ such that

$$\sqrt{\varrho_0}\mathbf{h} = \nabla_x (a\varrho_0^\gamma + \eta_0) - \operatorname{div}_x \mathbb{S}(\nabla_x \mathbf{u}) + \eta_0 \nabla_x \Phi, \tag{2.49}$$

and the following result still holds.

Theorem 2.3.1 (Local Existence of Smooth Solutions)**.** *Consider the NSS system* (1.1)-(1.3) *with boundary condition* (1.4) *on a bounded* $C^{2,\nu}$ *domain* Ω*. Assume the stress tensor* $\mathbb{S}$ *satisfies Newton's Law for Viscosity and that in addition to the confinement hypotheses,* $\Phi \in W^{2,2}(\Omega)$*. Also assume that in addition to the initial conditions* (1.5)*, the initial data satisfy*

$$\begin{aligned} &\varrho_0 \in W^{1,q}(\Omega) \\ &\boldsymbol{u}_0 \in W_0^{1,2}(\Omega;\mathbb{R}^3) \cap W^{2,2}(\Omega;\mathbb{R}^3) \\ &\eta_0 \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega) \end{aligned} \tag{2.50}$$

where $q \in (3, 6]$*. Then there is some time* $T > 0$ *such that there is a unique solution* $\{\varrho, \boldsymbol{u}, \eta\}$ *to* (1.1)-(1.3) *on* $[0, T] \times \Omega$ *such that*

$$\begin{aligned} &\varrho \in C([0,T]; W^{1,q}(\Omega)) \\ &\varrho_t \in C([0,T]; L^q(\Omega)) \\ &\boldsymbol{u} \in C([0,T]; W_0^{1,2}(\Omega;\mathbb{R}^3) \cap W^{2,2}(\Omega;\mathbb{R}^3)) \cap L^2(0,T; W^{2,q}(\Omega;\mathbb{R}^3)) \\ &\boldsymbol{u}_t \in L^2(0,T; W_0^{1,2}(\Omega;\mathbb{R}^3)) \\ &\eta \in C([0,T]; W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)) \cap L^2(0,T; W^{2,q}(\Omega)) \\ &\eta_t \in L^2(0,T; W_0^{1,2}(\Omega)). \end{aligned}$$

2.3.1 Linear Approximation

For the analysis to begin the proof of Theorem 2.3.1, the linear problem below is considered

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{v}) = 0 \tag{2.51}$$

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{v} \otimes \mathbf{u}) + \nabla_x(a\varrho^\gamma + \eta) = \mu \Delta_x \mathbf{u} + \lambda \nabla_x \operatorname{div}_x \mathbf{u} - (\beta\varrho + \eta)\nabla_x \Phi \tag{2.52}$$

$$\partial_t \eta + \operatorname{div}_x(\eta \mathbf{v} - \eta \nabla_x \Phi) - \Delta_x \eta = 0 \tag{2.53}$$

on $(0,T) \times \Omega$ where T is some value greater than zero. Here, $\mathbf{v} : (0,T) \times \Omega \mapsto \mathbb{R}^3$ is given with the regularity

$$\mathbf{v} \in C([0,T]; W_0^{1,2}(\Omega;\mathbb{R}^3) \cap W^{2.2}(\Omega;\mathbb{R}^3)) \cap L^2(0,T; W^{2,q}(\Omega;\mathbb{R}^3))$$

and

$$\mathbf{v}_t \in L^2(0,T; W_0^{1,2}(\Omega;\mathbb{R}^3))$$

where $q \in (3,6]$. It follows from classical Sobolev theorems that $\mathbf{v} \in C([0,T]; C^{0,\frac{1}{2}}(\Omega;\mathbb{R}^3))$. The initial data have the regularity

$$\varrho_0 \in W^{1,q}(\Omega) \cap C_c(\Omega)$$

$$\mathbf{u}_0 \in W_0^{1,2}(\Omega;\mathbb{R}^3) \cap W^{2,2}(\Omega;\mathbb{R}^3)$$

$$\eta_0 \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$$

and the boundary condition

$$\mathbf{u}|_{\partial\Omega} = (\eta \nabla_x \Phi + \nabla_x \eta) \cdot \mathbf{n}|_{\partial\Omega} = 0$$

completely analogous to the usual boundary conditions for the NSS model is imposed.

A second level is added to this approximation by bounding the initial density below by some $\delta > 0$. The program is to find solutions for a fixed $\mathbf{v}$ for each δ, and then take the limit as $\delta \to 0$. Thus, the first part of the analysis after showing the existence of solutions for each δ is to find bounds on these solutions independent of δ.

In light of (2.54), if it is also assumed that for all $x \in \Omega$, $\varrho_0(x) \geq \delta > 0$, then the fluid density is positive for all times $t \in [0, T]$ everywhere in the spatial domain. Considering (2.53), it is clear that if the initial particle density is positive anywhere, then the particle density is positive everywhere as argued in [15]. At this point, the proposed approximation scheme becomes clear.

1. Approximate the NSS system with the linear system (2.51)-(2.53) for some fixed $\mathbf{v}$ with the regularity mentioned above.

2. Approximate the initial fluid density with a fluid density bounded below by $\delta > 0$.

Since (2.51) and (2.53) have only ϱ and η as unknowns, respectively, they can be used to solve for these values and then with (2.52) used to solve for $\mathbf{u}$. After finding approximate solutions $\{\varrho_\delta, \mathbf{u}_\delta, \eta_\delta\}$, the limit $\delta \to 0$ is taken. Then, using an iteration argument, $\mathbf{v}$ can be taken to the unknown $\mathbf{u}$ as done in [19].

However, first the existence of approximate solutions is shown. Using the methods of characteristics, the solution for ϱ is given by

$$\varrho(t,x) = \varrho_0(U(0,t,x)) \exp\left[-\int_0^t \operatorname{div}_x \mathbf{v}(s, U(s,t,x))\ \mathrm{d}s\right] \tag{2.54}$$

where $U \in C([0,T] \times \overline{\Omega} \times [0,T])$ solves

$$\begin{cases} \frac{\partial}{\partial t} U(t,x,s) = \mathbf{v}(t, U(t,x,s)) \\ U(s,x,s) = x \end{cases}$$

Thus, using the method of characteristics, the fluid density has the regularity

$$\varrho \in C([0,T]; W^{1,q}(\Omega)),\ \varrho_t \in C([0,T]; L^q(\Omega)).$$

By Sobolev embedding theorems, the fluid density enjoys the regularity $\varrho \in C([0,T]; C^{0,1-\frac{3}{q}}(\Omega))$. Note that this regularity does not depend upon the initial density being uniformly positive. However, since $\varrho_0 \geq \delta > 0$, $\varrho \geq \overline{\delta}$ for some $\overline{\delta} > 0$.

Rewriting (2.53) as

$$\partial_t \eta + (\mathbf{v} - \nabla_x \Phi) \cdot \nabla_x \eta + \eta \operatorname{div}_x(\mathbf{v} - \nabla_x \Phi) - \Delta_x \eta = 0$$

shows it to be a classic linear parabolic equation, with η as the only unknown, and no quantities dependent upon δ. Thus, using classic parabolic results (see [19] and [26]), there exists a unique solution η to (2.53) such that

$$\eta \in C([0,T]; W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)) \cap L^2(0,T; W^{2,q}(\Omega))$$

$$\eta_t \in C([0,T]; L^2(\Omega)) \cap L^2(0,T; W_0^{1,2}(\Omega))$$

$$\eta_{tt} \in L^2(0,T; W^{-1,2}(\Omega)).$$

Similarly, rewriting (2.52) as

$$\partial_t \mathbf{u} + \mathbf{v} \cdot \nabla_x \mathbf{u} - \frac{1}{\varrho} \operatorname{div}_x \mathbb{S}(\nabla_x \mathbf{u}) = -\frac{1}{\varrho}[\nabla_x(a\varrho^\gamma) + \nabla_x \eta + \eta \nabla_x \Phi] - \beta \nabla_x \Phi$$

suggests another linear parabolic problem in one unknown (since ϱ and η are already determined) with a unique solution $\mathbf{u}$ such that

$$\mathbf{u} \in C([0,T]; W_0^{1,2}(\Omega;\mathbb{R}^3) \cap W^{2,2}(\Omega;\mathbb{R}^3)) \cap L^2(0,T; W^{2,q}(\Omega;\mathbb{R}^3))$$

$$\mathbf{u}_t \in C([0,T]; L^2(\Omega;\mathbb{R}^3)) \cap L^2(0,T; W_0^{1,2}(\Omega;\mathbb{R}^3))$$

$$\mathbf{u}_{tt} \in L^2(0,T; W^{-1,2}(\Omega;\mathbb{R}^3)).$$

Note that since (2.52) depends on ϱ, estimates on $\mathbf{u}$ will depend upon δ. Thus to be able to pass through the limit of δ, bounds independent of δ must be calculated for ϱ and $\mathbf{u}$. This is the focus of the next subsection.

2.3.2 Bounds Independent of δ for the Linear NSS System

In order to find estimates on ϱ independent of δ, the constants

$$c_0 \geq 1 + \|\varrho_0\|_{W^{1,q}} + \|\mathbf{u}_0\|_{W^{2,2}} + \|\eta_0\|_{W^{2,2}} + \|\mathbf{h}\|_{L^2},$$

$$c_1 \geq \sup_{t\in[0,T]} \left(\|\mathbf{v}(t)\|_{W_0^{1,2}} + \kappa^{-1}\|\mathbf{v}(t)\|_{W^{2,2}} \right) + \int_0^T \left(\|\mathbf{v}_t(t)\|^2_{W_0^{1,2}} + \|\mathbf{v}(t)\|^2_{W^{2,q}} \right) \, \mathrm{d}t,$$

and

$$c_2 = \kappa c_1 > c_1$$

are defined. Using the representation of the solution ϱ,

$$\|\varrho(t)\|_{W^{1,q}} \leq C c_0 \exp\left(C \int_0^t \|\nabla_x \mathbf{v}\|_{W^{1,q}} \, \mathrm{d}s \right)$$

which by application of Hölder's and Young's inequalities on the integral above implies

$$\|\varrho(t)\|_{W^{1,q}} \leq C c_0$$

and in conjunction with (2.51)

$$\|\varrho_t(t)\|_{L^q} \leq Cc_2$$

for $t \in [0, \min(T, T_1)]$ where $T_1 = c_2^{-1}$, and C is a constant depending only upon μ, λ, γ, T, and q.

From these bounds and the representation of ϱ, it follows that

$$C^{-1}\delta \leq \varrho(t, x) \leq Cc_0$$

on $\overline{\Omega} \times [0, \min(T, T_1)]$.

The next step is to find estimates on the pressure $p_F(\varrho)+\eta$. Then the estimates on the pressure will be used to find estimates on the velocity field $\mathbf{u}$ from (2.52). Unlike for the Navier-Stokes-Fourier system, the pressure for the NSS system is defined explicitly, not through the internal energy defined through its own equation. Since the pressure term contains two parts, the fluid pressure dependent only upon ϱ and the contribution of the particles through the η term. As such, estimates for the pressure will be divided into two parts: estimates on the fluid pressure based on the estimates on ϱ from the continuity equation, and estimates on η arising from the Smoluchowski equation.

Since $p_F(\varrho) = a\varrho^\gamma$, and ϱ is continuous, p_F is a continuous function. Since $\nabla_x p_F = a\gamma\varrho^{\gamma-1}\nabla_x\varrho$ and

$$\|\nabla_x\varrho(t)\|_{L^q(\Omega;\mathbb{R}^3)} \leq \|\varrho\|_{W^{1,q}(\Omega)} \leq Cc_0,$$

it is clear that

$$\|\nabla_x p_F(t)\|_{L^q(\Omega;\mathbb{R}^3)} \leq Cc_0.$$

Similarly,

$$\|\partial_t p_F(t)\|_{L^q(\Omega)} \leq Cc_2.$$

Since η has the regularity

$$\eta \in C([0,T]; W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega))$$

$$\eta_t \in C([0,T]; L^2(\Omega))$$

the Sobolev embedding theorems give that $\eta \in C\left([0,T]; C^{0,\frac{1}{2}}(\Omega)\right)$, and $\nabla_x \eta \in C\left([0,T]; L^6(\Omega; \mathbb{R}^3)\right)$. However, since (2.53) has no dependence on ϱ or $\mathbf{u}$, the norms of η and its derivatives do not involve the lower bound of the fluid density δ. As such, the following estimates for the pressure term are obtained.

$$P(\varrho, \eta)(t) \text{ is continuous on } \Omega \tag{2.55}$$

$$\|\nabla_x P(\varrho, \eta)(t)\|_{L^q(\Omega;\mathbb{R}^3)} \leq Cc_0 + c_g \tag{2.56}$$

$$\|\partial_t P(\varrho, \eta)(t)\|_{L^2(\Omega)} \leq Cc_2 + c_g \tag{2.57}$$

where c_g is a constant depending on $\mathbf{v}$, Ω, q, T, and Φ.

These pressure estimates will become important in the following analysis of (2.52) to obtain bounds on $\mathbf{u}$ independent of δ.

To obtain δ-independent bounds on $\mathbf{u}$, (2.52) is differentiated with respect to time, multiplied by $\mathbf{u}_t$, and integrated over Ω to obtain

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varrho|\mathbf{u}_t|^2 \,\mathrm{d}x + \int_\Omega \mu|\nabla_x \mathbf{u}_t|^2 + \lambda(\operatorname{div}_x \mathbf{u}_t)^2 \,\mathrm{d}x \\ &= -\int_\Omega \varrho_t(\mathbf{v}\cdot\nabla_x\mathbf{u})\cdot\mathbf{u}_t + \varrho(\mathbf{v}_t\cdot\nabla_x\mathbf{u})\cdot\mathbf{u}_t + 2\varrho(\mathbf{v}\cdot\nabla_x\mathbf{u}_t)\cdot\mathbf{u}_t \\ &- \int_\Omega \nabla_x P_t\cdot\mathbf{u}_t + (\beta\varrho_t + \eta_t)\nabla_x\Phi\cdot\mathbf{u}_t \,\mathrm{d}x. \end{aligned} \tag{2.58}$$

First, noting that

$$-\int_\Omega \nabla_x P_t \cdot \mathbf{u}_t \, \mathrm{d}x = \int_\Omega P_t \operatorname{div}_x \mathbf{u}_t \, \mathrm{d}x,$$

this term is bounded by

$$\frac{1}{2}\|P_t\|^2_{L^2(\Omega)} + \frac{1}{2}\|\operatorname{div}_x \mathbf{u}_t\|^2_{L^2(\Omega)} \tag{2.59}$$

using Young's inequality. Note that the second term can be placed on the left side of (2.58). Next to be considered is the term

$$\int_\Omega \beta \varrho_t \nabla_x \Phi \cdot \mathbf{u}_t + \eta_t \nabla_x \Phi \cdot \mathbf{u}_t \, \mathrm{d}x.$$

This is bounded by

$$C\|\nabla_x \Phi\|_{L^2(\Omega;\mathbb{R}^3)} \|\varrho_t\|_{L^3(\Omega)} \|\mathbf{u}_t\|_{L^6(\Omega;\mathbb{R}^3)} \tag{2.60}$$

$$\leq C\left(\|\nabla_x \Phi\|^2_{L^2(\Omega;\mathbb{R}^3)} \|\varrho_t\|^2_{L^3(\Omega)} + \|\nabla_x \mathbf{u}_t\|^2_{L^2(\Omega;\mathbb{R}^{3\times 3})}\right), \tag{2.61}$$

the second inequality using Young's inequality and one of the Sobolev inequalities.

It can also be shown that

$$\left|\int_\Omega \varrho_t(\mathbf{v} \cdot \nabla_x \mathbf{u}) \cdot \mathbf{u}_t \, \mathrm{d}x\right|$$
$$\leq C\left(\|\varrho_t\|^2_{L^3(\Omega)} \|\mathbf{v}\|^2_{L^\infty(\Omega;\mathbb{R}^3)} \|\nabla_x \mathbf{u}\|^2_{L^2(\Omega;\mathbb{R}^{3\times 3})} + \|\nabla_x \mathbf{u}_t\|^2_{L^2(\Omega;\mathbb{R}^{3\times 3})}\right) \tag{2.62}$$

and

$$\left|\int_\Omega 2\varrho(\mathbf{v} \cdot \nabla_x \mathbf{u}_t) \cdot \mathbf{u}_t \, \mathrm{d}x\right|$$
$$\leq C\left(\|\varrho\|_{L^\infty(\Omega)} \|\mathbf{v}\|^2_{L^\infty(\Omega;\mathbb{R}^3)} \|\sqrt{\varrho}\mathbf{u}_t\|^2_{L^2(\Omega;\mathbb{R}^3)} + \|\nabla_x \mathbf{u}_t\|^2_{L^2(\Omega;\mathbb{R}^{3\times 3})}\right) \tag{2.63}$$

The next term is handled by a variant of Young's inequality and the Sobolev inequalities:

$$\left|\int_\Omega \varrho(\mathbf{v}_t \cdot \nabla_x \mathbf{u}) \cdot \mathbf{u}_t \ \mathrm{d}x\right| \leq C\varepsilon \|\nabla_x \mathbf{v}_t\|^2_{L^2(\Omega;\mathbb{R}^{3\times 3})} \|\sqrt{\varrho}\mathbf{u}_t\|^2_{L^2(\Omega;\mathbb{R}^3)}$$

$$+ C\varepsilon^{-1} \|\varrho\|_{L^\infty(\Omega)} \|\nabla_x \mathbf{u}\|_{L^2(\Omega;\mathbb{R}^{3\times 3})} \|\nabla_x \mathbf{u}\|_{W^{1,2}(\Omega;\mathbb{R}^{3\times 3})}. \tag{2.64}$$

The final term on the right of (2.58) is handled as follows

$$\left|\int_\Omega \eta_t \nabla_x \Phi \cdot \mathbf{u}_t \ \mathrm{d}x\right|$$

$$\leq C\|\eta_t\|^2_{L^2(\Omega)} \|\nabla_x \Phi\|^2_{L^\infty(\Omega)} + C\|\nabla_x \mathbf{u}_t\|^2_{L^2(\Omega;\mathbb{R}^{3\times 3})}. \tag{2.65}$$

Combining (2.58)-(2.65) gives the following estimate.

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\varrho}\mathbf{u}_t\|^2_{L^2(\Omega;\mathbb{R}^3)} + \mu\|\nabla_x \mathbf{u}_t\|^2_{L^2(\Omega;\mathbb{R}^{3\times 3})}$$

$$\leq \frac{1}{2}\|P_t\|^2_{L^2(\Omega)} + C(\|\nabla_x \Phi\|^2_{L^\infty(\Omega;\mathbb{R}^3)} \|\varrho_t\|^2_{L^3(\Omega)}$$

$$+ \|\varrho_t\|^2_{L^3(\Omega)} \|\mathbf{v}\|^2_{L^\infty(\Omega;\mathbb{R}^3)} \|\nabla_x \mathbf{u}\|^2_{L^2(\Omega;\mathbb{R}^{3\times 3})} + \|\eta_t\|^2_{L^2(\Omega)} \|\nabla_x \Phi\|^2_{L^\infty(\Omega;\mathbb{R}^3)})$$

$$+ C\varepsilon\|\nabla_x \mathbf{v}_t\|^2_{L^2(\Omega;\mathbb{R}^{3\times 3})} \|\sqrt{\varrho}\mathbf{u}_t\|^2_{L^2(\Omega;\mathbb{R}^3)}$$

$$+ C\varepsilon^{-1}\|\varrho\|_{L^\infty(\Omega)} \|\nabla_x \mathbf{u}\|_{L^2(\Omega;\mathbb{R}^{3\times 3})} \|\nabla_x \mathbf{u}\|_{W^{1,2}(\Omega;\mathbb{R}^{3\times 3})}$$

$$+ C\|\nabla_x \mathbf{u}_t\|^2_{L^2(\Omega;\mathbb{R}^{3\times 3})} \tag{2.66}$$

Using the definitions of the constants, (2.66) is transformed to the inequality (with the aid of Young's inequality)

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\sqrt{\varrho}\mathbf{u}_t\|^2_{L^2(\Omega;\mathbb{R}^3)} + \mu\|\nabla_x \mathbf{u}_t\|^2_{L^2(\Omega;\mathbb{R}^{3\times 3})}$$

$$C\left(c_2^2 + M^2 c_2^2 + +c_g^2 M^2 + c_1^2 c_2^2 \|\nabla_x \mathbf{u}\|^2_{L^2(\Omega;\mathbb{R}^{3\times 3})}\right)$$

$$C\varepsilon c_1^2 \|\sqrt{\varrho}\mathbf{u}_t\|^2_{L^2(\Omega;\mathbb{R}^3)} + C\|\nabla_x \mathbf{u}_t\|^2_{L^2(\Omega;\mathbb{R}^{3\times 3})}$$

$$+ C\varepsilon^{-1}\left(c_0^4 \|\nabla_x \mathbf{u}\|^2_{L^2(\Omega;\mathbb{R}^{3\times 3})} + \|\nabla_x \mathbf{u}\|^2_{W^{1,2}(\Omega;\mathbb{R}^{3\times 3})}\right) \tag{2.67}$$

where M is the $W^{1,\infty}$ bound on Φ. Thus, in order to apply Gronwall's inequality on (2.67), estimates on $\|\nabla_x \mathbf{u}\|_{L^2(\Omega;\mathbb{R}^{3\times 3})}$ and $\|\nabla_x \mathbf{u}\|_{W^{1,2}(\Omega;\mathbb{R}^{3\times 3})}$ are needed.

To obtain the bounds on $\|\nabla_x \mathbf{u}\|_{W^{1,2}(\Omega;\mathbb{R}^{3\times 3})}$, the following lemma from [18] is used.

Lemma 2.3.1 (Elliptic Regularity)**.** *Assume $\boldsymbol{u} \in W_0^{1,2}(\Omega;\mathbb{R}^3) \cap W^{1,r}(\Omega;\mathbb{R}^3)$ solves the problem*

$$L\boldsymbol{u} = G$$

on Ω where $G \in L^r(\Omega;\mathbb{R}^3)$ where $r \in (1,\infty)$. Then $\boldsymbol{u} \in W^{2,r}(\Omega;\mathbb{R}^3)$ and

$$\|\boldsymbol{u}\|_{W^{2,r}(\Omega;\mathbb{R}^3)} \leq C\left(\|G\|_{L^r(\Omega;\mathbb{R}^3)} + \|\boldsymbol{u}\|_{W^{1,r}(\Omega;\mathbb{R}^3)}\right).$$

Thus, using $r = 2$ and

$$G = -\left(\partial_t(\varrho\mathbf{u}) + \mathrm{div}_x(\varrho\mathbf{v}\otimes\mathbf{u}) + \nabla_x P + (\beta\varrho + \eta)\nabla_x\Phi\right),$$

$$\begin{aligned}
&\|\nabla_x \mathbf{u}\|_{W^{1,2}(\Omega;\mathbb{R}^{3\times 3})} \leq C(\|\sqrt{\varrho}\|_{L^\infty(\Omega)}\|\sqrt{\varrho}\mathbf{u}_t\|_{L^2(\Omega;\mathbb{R}^3)} + \|\nabla_x P\|_{L^2(\Omega;\mathbb{R}^3)} \\
&+ \|\varrho\mathbf{v}\|_{L^\infty(\Omega;\mathbb{R}^3)}\|\nabla_x \mathbf{u}\|_{L^2(\Omega;\mathbb{R}^{3\times 3})} + |\beta|\|\nabla_x\Phi\|_{L^\infty(\Omega;\mathbb{R}^3)}\|\varrho\|_{L^2(\Omega)} \\
&+ \|\nabla_x\Phi\|_{L^\infty(\Omega;\mathbb{R}^3)}\|\eta\|_{L^2(\Omega)} + \|\nabla_x\mathbf{u}\|_{L^2(\Omega;\mathbb{R}^{3\times 3})}) \\
&+ C(c_0 M + c_0\|\sqrt{\varrho}\mathbf{u}_t\|_{L^2(\Omega;\mathbb{R}^3)} + c_0 c_1\|\nabla_x\mathbf{u}\|^{1/2}_{L^2(\Omega;\mathbb{R}^{3\times 3})}\|\nabla_x\mathbf{u}\|^{1/2}_{W^{1,2}(\Omega;\mathbb{R}^{3\times 3})} \\
&+ c_2 + c_g M + \|\nabla_x\mathbf{u}\|_{L^2(\Omega;\mathbb{R}^{3\times 3})})
\end{aligned}$$

Thus, using Gronwall's lemma, for $t \in [0,T]$ for some finite time T (not necessarily the T from before), the following bound holds:

$$\begin{aligned}
&\|\mathbf{u}(t)\|_{W_0^{1,2}(\Omega;\mathbb{R}^3)} + c_v\|\mathbf{u}(t)\|_{W^{2,2}(\Omega;\mathbb{R}^3)} + \|\sqrt{\varrho}\mathbf{u}_t(t)\|_{L^2(\Omega;\mathbb{R}^3)} \\
&+ \int_0^t \|\mathbf{u}_t(s)\|^2_{W_0^{1,2}(\Omega;\mathbb{R}^3)} + \|\mathbf{u}(s)\|^2_{W^{2,q}(\Omega;\mathbb{R}^3)}\,\mathrm{d}s \leq Cc_* \qquad (2.68)
\end{aligned}$$

Here, c_v and c_* depend only on c_0, but just as with C, they have no dependence on δ.

2.3.3 Existence for Linear Vacuum System

The next step in the analysis is to take δ to zero, allowing for a vacuum state in the initial density ϱ_0. The following conditions on the initial data are imposed, similar to the case of non-zero initial density.

$$
\begin{aligned}
&0 \leq \varrho_0 \in W^{1,q}(\Omega) \\
&\mathbf{u}_0 \in W_0^{1,2}(\Omega;\mathbb{R}^3) \cap W^{2,2}(\Omega;\mathbb{R}^3) \\
&\eta_0 \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)
\end{aligned} \tag{2.69}
$$

The same conditions on Φ and the same boundary conditions are imposed.

The compatibility condition requires a $\mathbf{h} \in L^2(\Omega;\mathbb{R}^3)$ such that

$$
\varrho_0^{1/2}\mathbf{h} = \nabla_x(a\varrho_0^\gamma + \eta_0) - \operatorname{div}_x \mathbb{S}(\nabla_x \mathbf{u}_0) + \eta_0 \nabla_x \Phi. \tag{2.70}
$$

The following conditions are also placed on $\mathbf{v}$.

$$
\begin{aligned}
&\mathbf{v} \in C([0,T]; W_0^{1,2}(\Omega;\mathbb{R}^3) \cap W^{2,2}(\Omega;\mathbb{R}^3)) \cap L^2(0,T; W^{2,q}(\Omega;\mathbb{R}^3)) \\
&\mathbf{v}_t \in L^2(0,T; W_0^{1,2}(\Omega;\mathbb{R}^3)) \\
&\sup_{0\leq t\leq T} \left(\|\mathbf{v}(t)\|_{W_0^{1,2}(\Omega;\mathbb{R}^3)} + \kappa^{-1}\|\mathbf{v}(t)\|_{W^{2,2}(\Omega;\mathbb{R}^3)} \right) \\
&+ \int_0^T \|\mathbf{v}_t(t)\|^2_{W_0^{1,2}(\Omega;\mathbb{R}^3)} + \|\mathbf{v}(t)\|^2_{W^{2,q}(\Omega;\mathbb{R}^3)} \, \mathrm{d}t \leq c_3(c_0)
\end{aligned} \tag{2.71}
$$

where

$$
c_0 = 2 + \|\varrho_0\|_{W^{1,q}(\Omega)} + \|\eta_0\|_{W^{2,2}(\Omega)} + \|\mathbf{u}_0\|_{W^{2,2}(\Omega;\mathbb{R}^3)} + \|\mathbf{h}\|^2_{L^2(\Omega;\mathbb{R}^3)}.
$$

Then using the fact that the vacuum-free case has strong solutions, it can be shown that linear NSS system with conditions (2.69)-(2.71) has a strong solution for some time T (not relabeled) such that

$$\begin{aligned}
&\varrho \in C([0,T]; W^{1,q}(\Omega)),\ \varrho_t \in C([0,T]; L^q(\Omega)) \\
&\eta \in C([0,T]; W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)) \cap L^2(0,T; W^{2,q}(\Omega)) \\
&\eta_t \in L^2(0,T; W_0^{1,2}(\Omega)) \\
&\mathbf{u} \in C([0,T]; W_0^{1,2}(\Omega;\mathbb{R}^3) \cap W^{2,2}(\Omega;\mathbb{R}^3)) \cap L^2(0,T; W^{2,q}(\Omega;\mathbb{R}^3)) \\
&\mathbf{u}_t \in L^2(0,T; W_0^{1,2}(\Omega;\mathbb{R}^3)) \\
&\sqrt{\varrho}\mathbf{u}_t \in L^\infty(0,T; L^2(\Omega;\mathbb{R}^3)).
\end{aligned} \tag{2.72}$$

To this end, initial densities $\varrho_0^\delta := \varrho_0 + \delta$ are defined for $\delta > 0$. Then, $\mathbf{h}^\delta$ is defined such that

$$\varrho_0^\delta \mathbf{h}^\delta = \nabla_x(a\varrho_0^\delta + \eta_0) - \mathrm{div}_x\, \mathbb{S}(\nabla_x \mathbf{u}_0) + \eta_0 \nabla_x \Phi.$$

Therefore, for small enough δ

$$c_0 \geq 1 + \delta + \|\varrho_0^\delta - \delta\|_{W^{1,q}(\Omega)} + \|\eta_0\|_{W^{2,2}(\Omega)} + \|\mathbf{u}_0\|_{W^{2,2}(\Omega;\mathbb{R}^3)} + \|\mathbf{h}^\delta\|^2_{L^2(\Omega;\mathbb{R}^3)}.$$

Thus, the result for vacuum-less initial data can be used to arrive at solutions $\{\varrho^\delta, \mathbf{u}^\delta, \eta^\delta\}$ for each δ. By the bounds produced in the previous subsection, limits $\{\varrho, \mathbf{u}, \eta\}$ exist and smoothly solve the linear NSS system. Uniqueness of $\{\varrho, \mathbf{u}, \eta\}$ follows from results from linear parabolic equations, the representation of ϱ, and an argument similar to that in [19]. Assuming that there are two solutions to the linear problem $\{\varrho_1, \mathbf{u}_1, \eta_1\}$ and $\{\varrho_2, \mathbf{u}_2, \eta_2\}$, it is clear that η^δ is the same for each $\delta > 0$, so $\eta_1 = \eta_2$. The uniqueness of ϱ follows from a uniqueness result of DiPerna and Lions in [24].

To handle the uniqueness of $\mathbf{u}$, define $\overline{\mathbf{u}} := \mathbf{u}_1 - \mathbf{u}_2$. Then using the fact that $\varrho_1 = \varrho_2$ and $\eta_1 = \eta_2$, subtracting the linear momentum equations for $\{\varrho_1, \mathbf{u}_1, \eta_1\}$ and $\{\varrho_2, \mathbf{u}_2, \eta_2\}$ yields

$$\partial_t(\varrho_1 \overline{\mathbf{u}}) + \mathrm{div}_x(\varrho_1 \mathbf{v} \otimes \overline{\mathbf{u}}) - \mathrm{div}_x\, \mathbb{S}(\nabla_x \overline{\mathbf{u}}) = 0. \tag{2.73}$$

By multiplying (2.73) by $\overline{\mathbf{u}}$ and using a Gronwall's argument, it is clear that $\overline{\mathbf{u}}$ is zero on $[0,T] \times \Omega$.

The time-continuity of ϱ follows from the fact that $L^\infty(0,T;L^q(\Omega))$ solutions of the (2.51) are unique and the time-continuity of η follows from the fact that η in the linear, vacuumless approximation has no dependence on δ. The time-continuity of $\mathbf{u}$ follows from the spaces the velocities are in and the elliptic regularity result from Lemma 2.3.1 (see [19]).

Thus, the linear NSS system for nonnegative ϱ_0 has a unique solution for finite time that has the regularity given by (2.72).

2.3.4 Existence for Nonlinear System

Now that the existence and uniqueness result for the linear NSS system has been established, the task now is to extend the result to the nonlinear system. To this end, a sequence $\{\mathbf{v}^k\}$ is defined inductively. First, $\mathbf{u}^0$ is defined as the solution to the parabolic problem

$$\partial_t \mathbf{u} - \Delta_x \mathbf{u} = 0 \text{ in } (0,\infty)\times\Omega$$

$$\mathbf{u}(0,\cdot) = \mathbf{u}_0.$$

Because of the hypothesized regularity of $\mathbf{u}_0$, the solution $\mathbf{u}^0$ will have the regularity required of the given vector field $\mathbf{v}$ in the linear NSS model. Assuming $\mathbf{u}^k$ is defined, the quantities $\{\varrho^{k+1}, \mathbf{u}^{k+1}, \eta^{k+1}\}$ are defined by solving the linear NSS system using $\mathbf{u}^k$ in place of $\mathbf{v}$. The goal now is to show that the sequence $\{\varrho^k, \mathbf{u}^k, \eta^k\}$ converges to a solution $\{\varrho, \mathbf{u}, \eta\}$ of the NSS system.

Recalling that estimates on the solutions depend on the initial data, which are identical for each $k \in \mathbb{N}$, the following estimate holds for some $\overline{C} > 1$.

$$\begin{aligned}
&\sup_{t\in[0,T]} \left(\|\varrho^k(t)\|_{W^{1,q}(\Omega)} + \|\varrho^k_t(t)\|_{L^q(\Omega)} + \|\eta^k(t)\|_{W^{1,2}(\Omega)\cap W^{2,2}(\Omega)}\right)\\
&+ \sup_{t\in[0,T]} \|\mathbf{u}^k(t)\|_{W^{1,2}(\Omega;\mathbb{R}^3)\cap W^{2,2}(\Omega;\mathbb{R}^3)} + \text{ess}\sup_{t\in[0,T]} \|\sqrt{\varrho^k}\mathbf{u}^k_t\|_{L^2(\Omega;\mathbb{R}^3)}\\
&+ \int_0^T \|\eta^k_t(t)\|^2_{W^{1,2}_0(\Omega)} + \|\mathbf{u}^k_t(t)\|^2_{W^{1,2}_0(\Omega;\mathbb{R}^3)}\ \mathrm{d}t\\
&+ \int_0^T \|\eta^k(t)\|^2_{W^{2,q}(\Omega)} + \|\mathbf{u}^k(t)\|^2_{W^{2,q}(\Omega;\mathbb{R}^3)}\ \mathrm{d}t \le \overline{C}
\end{aligned} \tag{2.74}$$

Defining the differences as

$$\overline{\varrho}^{k+1} := \varrho^{k+1} - \varrho^k \quad \overline{\mathbf{u}}^{k+1} := \mathbf{u}^{k+1} - \mathbf{u}^k \quad \overline{\eta}^{k+1} := \eta^{k+1} - \eta^k$$

the equations for the differences become

$$\partial_t \overline{\varrho}^{k+1} + \mathrm{div}_x \left(\overline{\varrho}^{k+1} \mathbf{u}^k \right) + \mathrm{div}_x (\rho^k \overline{\mathbf{u}}^k) = 0 \tag{2.75}$$

$$\begin{aligned} &\varrho^{k+1} \partial_t \overline{\mathbf{u}}^{k+1} + \varrho^{k+1} \mathbf{u}^k \cdot \nabla_x \overline{\mathbf{u}}^{k+1} - \mathrm{div}_x \mathbb{S}(\nabla_x \overline{\mathbf{u}}^{k+1}) \\ &= -\overline{\varrho}^{k+1} (\partial_t \mathbf{u}^k + \mathbf{u}^{k-1} \cdot \nabla_x \mathbf{u}^k) - \varrho^{k+1} \overline{\mathbf{u}}^k \cdot \nabla_x \mathbf{u}^k \\ &- \nabla_x (p_F(\varrho^{k+1}) - p_F(\varrho^k) + \overline{\eta}^{k+1}) - (\beta \overline{\varrho}^{k+1} + \overline{\eta}^{k+1}) \nabla_x \Phi \end{aligned} \tag{2.76}$$

$$\partial_t \overline{\eta}^{k+1} + \mathrm{div}_x (\overline{\eta}^{k+1} \mathbf{u}^k - \overline{\eta}^{k+1} \nabla_x \Phi) + \mathrm{div}_x (\eta^k \overline{\mathbf{u}}^k) = \Delta_x \overline{\eta}^{k+1}. \tag{2.77}$$

Estimates for $\overline{\varrho}^{k+1}$ follow from using (2.75) which is the same as in [19]. Multiplying (2.75) by $\mathrm{sgn}(\overline{\varrho}^{k+1}) |\overline{\varrho}^{k+1}|^{1/2}$ and integrating over Ω yields

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega |\overline{\varrho}^{k+1}|^{3/2} \, \mathrm{d}x \\ &\leq C \int_\Omega |\nabla_x \mathbf{u}^k| |\overline{\varrho}^{k+1}|^{3/2} + \left(|\nabla_x \varrho^k| |\overline{\mathbf{u}}^k| + \varrho^k |\nabla_x \overline{\mathbf{u}}^k| \right) |\overline{\varrho}^{k+1}|^{1/2} \, \mathrm{d}x \\ &\leq C \|\nabla_x \mathbf{u}^k\|_{W^{1,q}(\Omega; \mathbb{R}^{3\times 3})} \|\overline{\varrho}^{k+1}\|^{3/2}_{L^{3/2}(\Omega)} \\ &+ C \|\varrho^k\|_{W^{1,2}(\Omega)} \|\nabla_x \overline{\mathbf{u}}^k\|_{L^2(\Omega; \mathbb{R}^{3\times 3})} \|\overline{\varrho}^{k+1}\|^{1/2}_{L^{3/2}(\Omega)}. \end{aligned}$$

Multiplying this result by $\|\overline{\varrho}^{k+1}\|^{1/2}_{L^{3/2}(\Omega)}$ gives

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\overline{\varrho}^{k+1}\|^2_{L^2(\Omega)} \leq A^k_\varepsilon(t) \|\overline{\varrho}^{k+1}\|^2_{L^{3/2}(\Omega)} + \varepsilon \|\nabla_x \overline{\mathbf{u}}^k\|^2_{L^2(\Omega; \mathbb{R}^{3\times 3})} \tag{2.78}$$

where $A^k_\varepsilon(t) := C \|\nabla_x \mathbf{u}^k\|_{W^{1,q}(\Omega; \mathbb{R}^{3\times 3})} + \varepsilon^{-1} C \|\varrho^k(t)\|^2_{W^{1,q}(\Omega)}$.

From (2.74), it is clear that

$$\int_0^t A^k_\varepsilon(s) \, \mathrm{d}s \leq \overline{C} + \overline{C}_\varepsilon t$$

for any $k \in \mathbb{N}$ and $t \in [0, T]$, where $\overline{C}_\varepsilon$ is a constant with the same dependence as $\overline{C}$ but also with a dependence on ε. Similar techniques show that

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\overline{\varrho}^{k+1}\|^2_{L^2(\Omega)} \leq B^k_\varepsilon(t) \|\overline{\varrho}^{k+1}\|^2_{L^2(\Omega)} + \varepsilon \|\nabla_x \overline{\mathbf{u}}^k\|^2_{L^2(\Omega; \mathbb{R}^{3\times 3})} \tag{2.79}$$

for some $B_\varepsilon^k \in L^1(0,T)$ such that

$$\int_0^t B_\varepsilon^k(s)\ \mathrm{d}s \leq \overline{C} + \overline{C}_\varepsilon t.$$

The key difference in the approach used in [19] and the problem here is handling the bounding of the η^k terms. However, this can be handled by multiplying (2.77) by η^{k+1} and integrating over Ω to obtain

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega (\overline{\eta}^{k+1})^2\ \mathrm{d}x + \int_\Omega |\nabla_x \overline{\eta}^{k+1}|^2\ \mathrm{d}x \\
&= \int_\Omega \overline{\eta}^{k+1}\nabla_x\overline{\eta}^{k+1}\cdot\mathbf{u}^k - \overline{\eta}^{k+1}\nabla_x\overline{\eta}^{k+1}\cdot\nabla_x\Phi + \eta^k\nabla_x\overline{\eta}^{k+1}\cdot\overline{\mathbf{u}}^k\ \mathrm{d}x
\end{aligned} \tag{2.80}$$

noting that $(\overline{\eta}^{k+1}\nabla_x\Phi + \nabla_x\overline{\eta}^{k+1})\cdot\mathbf{n} = 0$ on $\partial\Omega$. From (2.80), the bounds for η^k can be obtained as for the quantities just above, and considering the bounds from

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\sqrt{\varrho^{k+1}}\overline{\mathbf{u}}^{k+1}|^2\ \mathrm{d}x + \mu\int_\Omega |\nabla_x\overline{\mathbf{u}}^{k+1}|^2\ \mathrm{d}x \\
&= -\int_\Omega \overline{\varrho}^{k+1}\overline{\mathbf{u}}^{k+1}\cdot(\partial_t\mathbf{u}^k + \mathbf{u}^{k-1}\cdot\nabla_x\mathbf{u}^k) + \varrho^{k+1}(\overline{\mathbf{u}}^k\cdot\nabla_x\mathbf{u}^k)\cdot\overline{\mathbf{u}}^{k+1}\ \mathrm{d}x \\
&- \int_\Omega \nabla_x[p_F(\varrho^{k+1}) - p_F(\varrho^k) + \overline{\eta}^{k+1}]\cdot\overline{\mathbf{u}}^{k+1} + (\beta\overline{\varrho}^{k+1} + \overline{\eta}^{k+1})\nabla_x\Phi\cdot\overline{\mathbf{u}}^{k+1}\ \mathrm{d}x
\end{aligned} \tag{2.81}$$

which arises from (2.76), the analysis proceeds as it does for [19] yielding the convergence of

$$\varrho^k \to \varrho$$

$$\mathbf{u}^k \to \mathbf{u}$$

$$\eta^k \to \eta$$

where $\{\varrho, \mathbf{u}, \eta\}$ are in the spaces given by Theorem 2.3.1 and solve the NSS system. This proves the existence part of Theorem 2.3.1. The uniqueness follows from the weak-strong uniqueness result in Section 2.2.

Chapter 3

Hydrodynamic Limits

While Chapter 2 shows the existence of solutions to the NSS system, expressing these solutions requires numerical methods. However, these methods often are computationally expensive for compressible models. But in certain scaling regimes, the solutions to the compressible NSS system can be approximated by solutions to simpler problems. In this chapter, two scalings low Mach number scalings, a low stratification scaling and a strong stratification scaling, are considered and shown to be approximated by solutions systems which are less computationally expensive to solve numerically.

Before scaling the system (1.1)-(1.3), the values D, describing the dispersion of the particles in the fluid, and ζ, a drag coefficient, must be added to ensure consistency of the physical units in the equations. Specifically, the pressure term in the momentum equation becomes

$$\nabla_x \left(a\varrho^\gamma + \frac{D}{\zeta}\eta \right),$$

the Smoluchowski equation becomes

$$\partial_t \eta + \operatorname{div}_x(\eta \mathbf{u}) - \operatorname{div}_x(\zeta \eta \Phi) - D\Delta_x \eta = 0,$$

and the energy inequality becomes

$$\int_\Omega \frac{1}{2}\varrho|\mathbf{u}|^2 + \frac{a}{\gamma-1}\varrho^\gamma + \frac{D}{\zeta}\eta \ln \eta + (\beta\varrho + \eta)\Phi \; \mathrm{d}x(\tau)$$

$$+\int_0^\infty \int_\Omega \mu|\nabla_x \mathbf{u}|^2 + \lambda|\operatorname{div}_x \mathbf{u}|^2 + \left|2\nabla_x \frac{D}{\sqrt{\zeta}}\sqrt{\eta} + \sqrt{\zeta\eta}\nabla_x \Phi\right|^2 \,\mathrm{d}x\,\mathrm{d}t$$
$$\leq \int_\Omega \frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + \frac{a}{\gamma-1}\varrho_0^\gamma + \frac{D}{\zeta}\eta_0 \ln \eta_0 + (\beta\varrho_0 + \eta_0)\Phi \,\mathrm{d}x.$$

To begin the scaling of the Navier-Stokes-Smoluchowski model, the quantities

$$\varrho, \mathbf{u}, \eta, \zeta, D, p_F, p_P, \Phi, \mu, \text{ and } \lambda,$$

where $p_P(\eta) := \frac{D}{\zeta}\eta$, as well as the time and length scales, must be made non-dimensional. This is done by defining for each quantity A a reference value A_{ref} which also reflects the physical unit of measurement for that quantity, such as meter, second, meter per second, and so on. Then, the dimensionless value A' is defined as

$$A' := \frac{A}{A_{ref}}.$$

After some application of the chain rule and some straight-forward algebra, the formal dimensionless Navier-Stokes-Smoluchowski system becomes (omitting the primes for the sake of notational simplicity)

$$\mathrm{Sr}\partial_t \varrho + \operatorname{div}_x(\varrho\mathbf{u}) = 0 \tag{3.1}$$

$$\mathrm{Sr}\partial_t(\varrho\mathbf{u}) + \operatorname{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) + \frac{1}{\mathrm{Ma}^2}\nabla_x\left(a\varrho^\gamma + \mathrm{Pc}\frac{D}{\zeta}\eta\right) = \frac{1}{\mathrm{Re}}(\mu\Delta_x\mathbf{u} + \lambda\nabla_x \operatorname{div}_x \mathbf{u})$$
$$-\frac{1}{\mathrm{Fr}^2}(\beta\varrho + \mathrm{Dc}\eta)\nabla_x\Phi \tag{3.2}$$

$$\mathrm{Sr}\partial_t\eta + \operatorname{div}_x(\eta\mathbf{u}) - \mathrm{Za}\operatorname{div}_x(\zeta\eta\nabla_x\Phi) - \mathrm{Da}D\Delta_x\eta = 0 \tag{3.3}$$

$$
\begin{array}{|cccccc|}
\hline
\text{Sr:=} & \dfrac{L_{ref}}{\mathbf{u}_{ref}t_{ref}} & \text{Ma:=} & \dfrac{\mathbf{u}_{ref}}{\sqrt{p_{F_{ref}}/\varrho_{ref}}} & \text{Re:=} & \dfrac{\varrho_{ref}\mathbf{u}_{ref}L_{ref}}{\mu_{ref}} \\
\text{Fr:=} & \dfrac{\mathbf{u}_{ref}}{\sqrt{L_{ref}f_{ref}}} & \text{Za:=} & \dfrac{\zeta_{ref}f_{ref}}{\mathbf{u}_{ref}} & \text{Da:=} & \dfrac{D_{ref}}{L_{ref}\mathbf{u}_{ref}} \\
\text{Pc:=} & \dfrac{p_{P_{ref}}}{p_{F_{ref}}} & \text{Dc:=} & \dfrac{\eta_{ref}}{\varrho_{ref}}. & & \\
\hline
\end{array}
$$

Table 3.1: Definitions of the Dimensionless Parameters

The total energy inequality for the scaled system takes the form.

$$
\begin{aligned}
&\mathrm{Sr}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \frac{\mathrm{Ma}^2}{2}\varrho|\mathbf{u}|^2 + \frac{a}{\gamma-1}\varrho^\gamma + \mathrm{Pc}\frac{D\eta}{\zeta}\ln\eta + \frac{\mathrm{Ma}^2}{\mathrm{Fr}^2}(\beta\varrho + \mathrm{Dc}\eta)\Phi \ \mathrm{d}x \\
&+ \int_\Omega \frac{\mathrm{Ma}^2}{\mathrm{Re}}\mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{u} \ \mathrm{d}x \\
&+ \int_\Omega \mathrm{PcDa}D^2\frac{|\nabla_x\eta|^2}{\zeta\eta} + 2\mathrm{Za}D\nabla_x\eta\cdot\nabla_x\Phi + \frac{\mathrm{Za}^2}{\mathrm{Da}}\zeta\eta|\nabla_x\Phi|^2 \ \mathrm{d}x \le 0. \qquad (3.4)
\end{aligned}
$$

The non-dimensional parameters used in (3.1)-(3.4) are defined in Table 3, where the quantities

$$
L_{ref}, \mathbf{u}_{ref}, t_{ref}, p_{F_{ref}}, \varrho_{ref}, \mu_{ref}, f_{ref}, \zeta_{ref}, D_{ref}, p_{P_{ref}}, \text{ and } \eta_{ref}
$$

represent the reference values for the length, velocity, time, fluid pressure, fluid density, viscosity coefficient, force (equal to $\nabla_x\Phi$), drag coefficient, diffusivity coefficient, particle pressure, and particle density, respectively. Taking $e_{F_{ref}}$ and $e_{P_{ref}}$ to the reference internal energies of the fluid and particles, respectively, the compatibility conditions $\mu_{ref} = \lambda_{ref}$ and $p_{F_{ref}} = \varrho_{ref}e_{F_{ref}}$, $p_{P_{ref}} = \eta_{ref}e_{P_{ref}}$ are also imposed to obtained the scaling, the second and third of which follow naturally from Maxwell's relation. Note also that Ma represents the Mach number, Sr the Strouhal number, Re the Reynolds number, and Fr the Froude number used in other works on singular limits (see [32]). Since existence of solutions to the scaled system follows from [7] and [15] for any choices of positive values of the dimensionless parameters, various singular limits can be explored.

3.1 Low Stratification Limit

The current section considers a low-Mach-number limit, with Ma taken to be a small parameter ε, Za scaled as Ma, and Fr=$\sqrt{\varepsilon}$.

$$\partial_t \varrho_\varepsilon + \operatorname{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon) = 0 \tag{3.5}$$

$$\varepsilon^2[\partial_t(\varrho_\varepsilon \mathbf{u}_\varepsilon) + \operatorname{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon)] + \nabla_x \left(a\varrho_\varepsilon^\gamma + \frac{D}{\zeta}\eta_\varepsilon \right)$$

$$= \varepsilon^2(\mu \Delta_x \mathbf{u}_\varepsilon + \lambda \nabla_x \operatorname{div}_x \mathbf{u}_\varepsilon) - \varepsilon(\beta\varrho_\varepsilon + \eta_\varepsilon)\nabla_x \Phi \tag{3.6}$$

$$\partial_t \eta_\varepsilon + \operatorname{div}_x(\eta_\varepsilon \mathbf{u}_\varepsilon) - \varepsilon \operatorname{div}_x(\zeta \eta_\varepsilon \nabla_x \Phi) - D\Delta_x \eta_\varepsilon = 0 \tag{3.7}$$

$$\begin{aligned}
&\int_\Omega \frac{\varepsilon^2}{2}\varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 + \frac{a}{\gamma-1}\varrho_\varepsilon^\gamma + \frac{D\eta_\varepsilon}{\zeta}\ln \eta_\varepsilon + \varepsilon(\beta\varrho_\varepsilon + \eta_\varepsilon)\Phi \, \mathrm{d}x(T) \\
&\quad + \int_0^T \int_\Omega \varepsilon^2(\mu|\nabla_x \mathbf{u}_\varepsilon|^2 + \lambda|\operatorname{div}_x \mathbf{u}_\varepsilon|^2 \, \mathrm{d}x \, \mathrm{d}t \\
&\quad + \int_0^T \int_\Omega \left| 2\frac{D}{\sqrt{\zeta}}\nabla_x \sqrt{\eta_\varepsilon} + \varepsilon\sqrt{\zeta\eta_\varepsilon}\nabla_x \Phi \right|^2 \, \mathrm{d}x \, \mathrm{d}t \\
&\le \int_\Omega \frac{\varepsilon^2}{2}\varrho_0|\mathbf{u}_0|^2 + \frac{a}{\gamma-1}\varrho_0^\gamma + \frac{D\eta_0}{\zeta}\ln \eta_0 + \varepsilon(\beta\varrho_0 + \eta_0)\Phi \, \mathrm{d}x
\end{aligned} \tag{3.8}$$

3.1.1 Formal Calculations

The formal technique is to expand $\varrho_\varepsilon, \mathbf{u}_\varepsilon,$ and η_ε as

$$\varrho_\varepsilon = \overline{\varrho} + \sum_{i=1}^{\infty} \varepsilon^i \varrho^{(i)}$$

$$\mathbf{u}_\varepsilon = \overline{\mathbf{u}} + \sum_{i=1}^{\infty} \varepsilon^i \mathbf{u}^{(i)}$$

$$\eta_\varepsilon = \overline{\eta} + \sum_{i=1}^{\infty} \varepsilon^i \eta^{(i)}$$

plug these expansions into (3.5)-(3.8), and equate terms of equal orders of ε. In doing so, it becomes clear that since the right side of (3.8) is bounded uniformly in ε, as it is just the initial energy, it must be true that

$$\nabla_x \sqrt{\overline{\eta}} = 0.$$

Thus, $\overline{\eta}$ is constant on Ω for each time t and

$$\overline{\eta} = \frac{1}{|\Omega|} \int_\Omega \eta_0 \ \mathrm{d}x$$

in the formal limit. Moving to the momentum equation (3.6) and equating terms of order one, the formal equation becomes

$$\nabla_x \left(a\overline{\varrho}^\gamma + \frac{D}{\zeta}\overline{\eta} \right) = 0.$$

Since $\overline{\eta}$ is constant, it follows formally that

$$\overline{\varrho} = \frac{1}{|\Omega|} \int_\Omega \varrho_0 \ \mathrm{d}x.$$

Using this fact in the continuity equation (3.5) and equating terms of order one yields the incompressibility condition for the limit velocity

$$\mathrm{div}_x \, \overline{\mathbf{u}} = 0.$$

Returning to (3.8) and equating terms of order ε^2, it is easy to show formally that

$$\overline{\varrho}[\partial_t \overline{\mathbf{u}} + \mathrm{div}_x(\overline{\mathbf{u}} \otimes \overline{\mathbf{u}})] + \nabla_x \Pi = \mu \Delta_x \overline{\mathbf{u}} - (\beta r + \theta) \nabla_x \Phi$$

where r, θ are related to the limit quantities by

$$\nabla_x \left(a r^\gamma + \frac{D}{\zeta}\theta \right) = -(\beta\overline{\varrho} + \overline{\eta})\nabla_x \Phi,$$

which is found by equating terms of order ε in (3.6) and relabeling $\varrho^{(1)}$ and $\eta^{(1)}$. Thus, the formal low stratification low Mach number limit for the Navier-Stokes-Smoluchowski system becomes

$$\overline{\eta} = \frac{1}{|\Omega|}\int_\Omega \eta_0(x)\ \mathrm{d}x \tag{3.9}$$

$$\overline{\varrho} = \frac{1}{|\Omega|}\int_\Omega \varrho_0(x)\ \mathrm{d}x \tag{3.10}$$

$$\mathrm{div}_x\, \overline{\mathbf{u}} = 0 \tag{3.11}$$

$$\overline{\varrho}[\partial_t \overline{\mathbf{u}} + \mathrm{div}_x(\overline{\mathbf{u}} \otimes \overline{\mathbf{u}})] + \nabla_x \Pi = \mu \Delta_x \overline{\mathbf{u}} - (\beta r + \theta)\nabla_x \Phi \tag{3.12}$$

where r, θ satisfy

$$\nabla_x \left(a r^\gamma + \frac{D}{\zeta}\theta \right) = -(\beta\overline{\varrho} + \overline{\eta})\nabla_x \Phi \tag{3.13}$$

and Π is a function incorporating the terms for which a gradient is taken.

3.1.2 Rigorous Derivation of the Low Stratification Limit

In this section, the formal limit derived in Subsection 3.1 is rigorously proven. First is introduced the notion of solution for the scaled system (3.5)-(3.8).

3.1.2.1 Free energy solutions

Definition 3.1.1. Assume that (Ω, Φ) satisfy the confinement hypotheses **(HC)** with $\Omega \subset \mathbb{R}^3$ a domain of class $C^{2+\nu}, \nu > 0$. Also, assume that μ, λ, ζ, and D are positive constants. Then $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \eta_\varepsilon\}$ represent a weak solution of the low stratification system with Mach number ε if and only if

- $\varrho_\varepsilon \geq 0$ represents a renormalized solution of the continuity equation on $(0,\infty)\times\Omega$, i.e., for any test function $\phi \in \mathcal{D}([0,T)\times\overline{\Omega})$, $T>0$ and any b, B such that

  $$b \in L^\infty([0,\infty)) \cap C([0,\infty)), B(\varrho) := B(1) + \int_1^\varrho \frac{b(z)}{z^2} \mathrm{d}z,$$

  the renormalized continuity equation

  $$\begin{aligned}&\int_0^T \int_\Omega B(\varrho_\varepsilon)\partial_t\phi + B(\varrho_\varepsilon)\mathbf{u}_\varepsilon \cdot \nabla_x \phi - b(\varrho_\varepsilon)\phi \operatorname{div}_x \mathbf{u}_\varepsilon \ \mathrm{d}x \ \mathrm{d}t \\ &= -\int_\Omega B(\varrho_0)\phi(0,\cdot) \ \mathrm{d}x \end{aligned} \tag{3.14}$$

  holds.

- The balance of momentum holds in the sense of distributions, i.e., for any $\mathbf{w} \in \mathcal{D}([0,T); \mathcal{D}(\overline{\Omega}; \mathbb{R}^3))$,

  $$\begin{aligned}&\int_0^T \int_\Omega \varepsilon^2(\varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \partial_t \mathbf{w} + \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla_x \mathbf{w}) + \left(p_F(\varrho_\varepsilon) + \frac{D}{\zeta}\eta_\varepsilon\right) \operatorname{div}_x \mathbf{w} \ \mathrm{d}x \ \mathrm{d}t \\ &= \int_0^T \int_\Omega \varepsilon^2(\mu \nabla_x \mathbf{u}_\varepsilon \nabla_x \mathbf{w} + \lambda \operatorname{div}_x \mathbf{u}_\varepsilon \operatorname{div}_x \mathbf{w}) - \varepsilon(\beta \varrho_\varepsilon + \eta_\varepsilon)\nabla_x \Phi \cdot \mathbf{w} \ \mathrm{d}x \ \mathrm{d}t \\ &-\varepsilon^2 \int_\Omega \mathbf{m}_0 \cdot \mathbf{w}(0,\cdot) \ \mathrm{d}x \end{aligned} \tag{3.15}$$

- $\eta_\varepsilon \geq 0$ is a weak solution of (3), i.e.,

  $$\begin{aligned}&\int_0^T \int_\Omega \eta_\varepsilon \partial_t \phi + \eta_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x \phi - \varepsilon \zeta \eta_\varepsilon \nabla_x \Phi \cdot \nabla_x \phi - D \nabla_x \eta_\varepsilon \cdot \nabla_x \phi \ \mathrm{d}x \ \mathrm{d}t \\ &= -\int_\Omega \eta_0 \phi(0,\cdot) \ \mathrm{d}x \end{aligned} \tag{3.16}$$

  for any test function $\phi \in \mathcal{D}([0,T)\times\overline{\Omega})$

- The energy inequality below holds

$$\int_\Omega \frac{\varepsilon^2}{2}\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2 + \frac{a}{\gamma-1}\varrho_\varepsilon^\gamma + \frac{D\eta_\varepsilon}{\zeta}\ln\eta_\varepsilon + \varepsilon(\beta\varrho_\varepsilon+\eta_\varepsilon)\Phi \,\mathrm{d}x(T) +$$
$$\int_0^T\int_\Omega \varepsilon^2(\mu|\nabla_x\mathbf{u}_\varepsilon|^2 + \lambda|\operatorname{div}_x\mathbf{u}_\varepsilon|^2 \,\mathrm{d}x\,\mathrm{d}t +$$
$$\int_0^T\int_\Omega \left|2\frac{D}{\sqrt{\zeta}}\nabla_x\sqrt{\eta_\varepsilon} + \varepsilon\sqrt{\zeta\eta_\varepsilon}\nabla_x\Phi\right|^2 \,\mathrm{d}x\,\mathrm{d}t$$
$$\leq \int_\Omega \frac{\varepsilon^2}{2}\varrho_0|\mathbf{u}_0|^2 + \frac{a}{\gamma-1}\varrho_0^\gamma + \frac{D\eta_0}{\zeta}\ln\eta_0 + \varepsilon(\beta\varrho_0+\eta_0)\Phi \,\mathrm{d}x \tag{3.17}$$

By the existence results in [7] and [15], and from Chapter 2, it is clear that such $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \eta_\varepsilon\}$ exist for each $\varepsilon > 0$. Next is introduced the notion of weak solutions of the target system (3.9)-(3.13) called the Oberbeck-Boussinesq approximation.

Definition 3.1.2. $\{\overline{\mathbf{u}}, \varrho^{(1)}, \eta^{(1)}\}$ is a variational solution of the target system (3.9)-(3.13) supplemented with the boundary conditions

$$\overline{\mathbf{u}} = 0 \text{ on } \partial\Omega \tag{3.18}$$

and the initial conditions

$$\overline{\mathbf{u}}(0,\cdot) = \mathbf{u}_0, \tag{3.19}$$

if the following conditions hold

- $\overline{\mathbf{u}} \in L^\infty(0,T;L^2(\Omega;\mathbb{R}^3)) \cap L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3))$,

- The incompressibility condition

$$\mathrm{div}_x\overline{\mathbf{u}} = 0 \ a.e. \text{ on } (0,T)\times\Omega.$$

  holds,

- The integral identity

$$\int_0^T \int_\Omega \Big(\overline{\varrho}\overline{\mathbf{u}} \cdot \partial_t \varphi + \overline{\varrho}(\overline{\mathbf{u}} \otimes \overline{\mathbf{u}}) : \nabla_x \varphi \Big) \, \mathrm{d}x \, \mathrm{d}t$$
$$= \int_0^T \int_\Omega \Big(\mu \nabla_x \overline{\mathbf{u}} - (\beta \varrho^{(1)} + \eta^{(1)}) \nabla_x \Phi \Big) \cdot \varphi \, \mathrm{d}x \, \mathrm{d}t - \int_\Omega \overline{\varrho}\overline{\mathbf{u}} \cdot \varphi(0, \cdot) \, \mathrm{d}x \tag{3.20}$$

  holds for any test function

$$\varphi \in \mathcal{D}((0,T) \times \Omega; \mathbb{R}^3), \ \mathrm{div}_x \varphi = 0 \text{ in } \Omega, \ \varphi \cdot \mathbf{n}|_{\partial\Omega} = 0.$$

- The quantities $\varrho^{(1)}$ and $\eta^{(1)}$ are interrelated via the so-called Boussinesq relation:

$$\varrho^{(1)} = -\frac{1}{a\gamma\overline{\varrho}^{\gamma-1}} \left[(\beta\overline{\varrho} + \overline{\eta})\Phi + \frac{D}{\zeta}\eta^{(1)} \right],$$

  which holds weakly.

Next, a geometric condition on Ω is introduced which plays a crucial role in the study of propagation of the acoustic waves. Considering the problem

$$-\Delta\phi = \lambda\phi \ \text{ in } \ \Omega, \ \ \frac{\partial\phi}{\partial\mathbf{n}} = 0 \ \text{ on } \ \partial\Omega, \tag{3.21}$$

where ϕ is constant on $\partial\Omega$, a solution of the problem (3.21) is called trivial if $\lambda = 0$ and ϕ is constant. Also, Ω is said to verify assumption (H) if all solutions of the problem (3.21) are trivial. Notice that Schiffer's conjecture shows that every Ω satisfies (H) except the ball and Feireisl, Novotny, Petzeltova [33] gives an example of domain Ω which is trivial. In two dimensional space, it is proven that every bounded, simply connected open domain $\Omega \subset \mathbb{R}^2$ whose boundary is Lipschitz but not real analytic satisfies (H).

Theorem 3.1.1 (Low stratification limit). *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with a boundary of class $C^{2+\nu}$, $\nu > 0$ verifying the suitable assumption (H) for* 3.21. *Let* (Ω, Φ) *satisfy the confinement hypothesis* **(HC)** *and assume* $Za = \mathrm{Ma} = \varepsilon, \mathrm{Fr} = \sqrt{\varepsilon}$ *and* $\{\varrho_\varepsilon, \boldsymbol{u}_\varepsilon, \eta_\varepsilon\}_{\varepsilon>0}$ *is a family of free energy solutions to the scaled Navier-Stokes Smoluchowski system in the sense of Definition* 3.1.1 *with the boundary conditions*

$$\boldsymbol{u}|_{\partial\Omega} = (\varepsilon\eta_\varepsilon\nabla_x\Phi + \nabla_x\eta_\varepsilon)\cdot\boldsymbol{n}|_{\partial\Omega} = 0.$$

Assume the initial condition as follows.

$$\varrho_\varepsilon(0,\cdot) = \varrho_{\varepsilon,0} = \bar{\varrho} + \varepsilon\varrho_{\varepsilon,0}^{(1)},\ \mathbf{u}_\varepsilon(0,\cdot) = \mathbf{u}_{\varepsilon,0},\ \eta(0,\cdot) = \bar{\eta} + \varepsilon\eta_{\varepsilon,0}^{(1)} \tag{3.22}$$

where

$$\bar{\varrho} = \frac{1}{|\Omega|}\int_\Omega \varrho_{\varepsilon,0}\ dx,\ \bar{\eta} = \frac{1}{|\Omega|}\int_\Omega \eta_{\varepsilon,0}\ dx, \tag{3.23}$$

and

$$\varrho_{\varepsilon,0} \rightharpoonup \varrho_0^{(1)},\ \mathbf{u}_{\varepsilon,0} \rightharpoonup \mathbf{u}_0^{(1)},\ \eta_{\varepsilon,0} \rightharpoonup \eta_0^{(1)}, \tag{3.24}$$

as ε tends to 0 *using weak$-*$ convergence in $L^\infty(\Omega)$. Then, up to subsequences,*

$$\begin{cases} \varrho_\varepsilon \to \bar{\varrho} \ \text{ in } \ C([0,T];L^1(\Omega)) \cap L^\infty(0,T;L^{\frac{5}{3}}(\Omega)), \\ \\ \eta_\varepsilon \to \bar{\eta} \ \text{ in } \ L^2(0,T;W^{1,2}(\Omega)), \\ \\ \boldsymbol{u}_\varepsilon \to \bar{\mathbf{u}} \ \text{ strongly in } L^2(0,T;L^2(\Omega;\mathbb{R}^3)), \end{cases} \tag{3.25}$$

and

$$\begin{cases} \varrho_\varepsilon^{(1)} = \dfrac{\varrho_\varepsilon - \bar{\varrho}}{\varepsilon} \to \varrho^{(1)} \text{ weakly} - * \text{ in } L^\infty(0,T;L^q(\Omega)),, q = \min\{2,\gamma\} \\ \\ \eta_\varepsilon^{(1)} = \dfrac{\eta_\varepsilon - \bar{\eta}}{\varepsilon} \to \eta^{(1)} \text{ weakly in } L^2(0,T;L^2(\Omega)), \end{cases} \tag{3.26}$$

where $\{\overline{\mathbf{u}}, \varrho^{(1)}, \eta^{(1)}\}$, *solve the target system in the sense of Definition* 3.1.2 *with the boundary condition* $\mathbf{u}|_{\partial\Omega} = 0$ *and the initial data*

$$\mathbf{u}(0) = \mathbf{H}[\mathbf{u}_0], \tag{3.27}$$

where the Helmholtz's projection $\mathbf{H}$ *is defined by*

$$\mathbf{H} = \mathbf{I} - \mathbf{H}^{\perp},\ \ \mathbf{H}^{\perp} = \nabla_x \Delta_x^{-1} \operatorname{div}_x . \tag{3.28}$$

3.1.2.2 Free Energy Inequality and Uniform Bounds

The first step in rigorously deriving the convergence stated in Theorem 3.1.1 is to obtain bounds uniform in ε which will yield the weak limits. To do this, analogs of the Helmholtz free energy function defined below are utilized:

$$E_F(\varrho) := \frac{a}{\gamma-1}\varrho^\gamma - (\varrho - \overline{\varrho})\frac{a\gamma}{\gamma-1}\overline{\varrho}^{\gamma-1} - \frac{a}{\gamma-1}\overline{\varrho}^\gamma$$

and

$$E_P(\eta) := \frac{D}{\zeta}\eta\ln\eta - \frac{D}{\zeta}(\eta - \overline{\eta})(\ln\overline{\eta} + 1) - \frac{D}{\zeta}\overline{\eta}\ln\overline{\eta}.$$

Basic calculations show that E_F and E_P have global minima at $\overline{\varrho}$ and $\overline{\eta}$ respectively, and are both convex, facts that will be used later in the proof. Thus after some analysis, the energy inequality given by (3.17) can be rewritten as

$$\begin{aligned}
&\int_\Omega \frac{1}{2}\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2 + \frac{1}{\varepsilon^2}(E_F(\varrho_\varepsilon) + E_P(\eta_\varepsilon)) + \frac{1}{\varepsilon}(\beta\varrho_\varepsilon + \eta_\varepsilon)\Phi\ \mathrm{d}x(T)\\
&+ \int_0^T\int_\Omega \mu|\nabla_x\mathbf{u}_\varepsilon|^2 + \lambda|\operatorname{div}_x\mathbf{u}_\varepsilon|^2 + \frac{1}{\varepsilon^2}\left|\frac{2D\nabla_x\sqrt{\eta_\varepsilon}}{\sqrt{\zeta}} + \varepsilon\sqrt{\zeta\eta_\varepsilon}\nabla_x\Phi\right|^2\ \mathrm{d}x\ \mathrm{d}t\\
&\leq \int_\Omega \frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + \frac{1}{\varepsilon^2}(E_F(\varrho_0) + E_P(\eta_0)) + \frac{1}{\varepsilon}(\beta\varrho_0 + \eta_0)\Phi\ \mathrm{d}x
\end{aligned} \tag{3.29}$$

By the hypotheses on the initial data, the right side of this equation is bounded by a constant (c.f. Chapter 5.1 in [32]). Thus, the following uniform in ε bounds are obtained:

$$\{\mathbf{u}_\varepsilon\}_{\varepsilon>0} \in_b L^2(0,T;W^{1,2}(\Omega))$$

$$\{\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon\}_{\varepsilon>0} \in_b L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))$$

$$\left\{\frac{1}{\varepsilon}\left(\frac{2D\nabla_x\sqrt{\eta_\varepsilon}}{\sqrt{\zeta}} + \varepsilon\sqrt{\zeta\eta_\varepsilon}\nabla_x\Phi\right)\right\}_{\varepsilon>0} \in_b L^2(0,T;L^2(\Omega;\mathbb{R}^3))$$

Next, the following sets are defined:

$$\mathcal{O}_{\text{ess}} := \{(\varrho,\eta) \in \mathbb{R}^2 | \overline{\varrho}/2 \leq \varrho, \eta \leq 2\overline{\eta}\}$$

$$\mathcal{M}^\varepsilon_{\text{ess}} := \{(t,x) \in (0,T)\times\Omega | (\varrho_\varepsilon(t,x), \eta_\varepsilon(t,x)) \in \mathcal{O}_{\text{ess}}\}$$

$$\mathcal{M}^\varepsilon_{\text{res}} := ((0,T)\times\Omega) - \mathcal{M}^\varepsilon_{\text{ess}}$$

Since ϱ^γ and $\eta\ln\eta$ are clearly strongly convex on $\mathcal{M}^\varepsilon_{\text{ess}}$,

$$\mathcal{H}(\rho_\varepsilon,\eta_\varepsilon) := E_F(\varrho_\varepsilon) + E_P(\eta_\varepsilon) \geq C(|\varrho-\overline{\varrho}|^2 + |\eta-\overline{\eta}|^2) \text{ on } \mathcal{M}^\varepsilon_{\text{ess}}.$$

And by the properties of E_F, E_P mentioned above,

$$E_F(\varrho) \geq E_F(\overline{\varrho}/2) > 0 \text{ for } \varrho < \overline{\varrho}/2 \text{ and } E_P(\eta) \geq E_P(2\overline{\eta}) > 0 \text{ for } \eta > 2\overline{\eta}.$$

Thus, on $\mathcal{M}^\varepsilon_{\text{res}}$, $\mathcal{H}(\varrho,\eta) \geq c > 0$ for some constant c. It also becomes clear that the right hand side of (3.29) is uniformly bounded by some finite, positive constant.

Using the coercivity of E_F and E_P and the boundedness of (3.29), it can be shown that the measures of the residual sets $\mathcal{M}^{\varepsilon}_{\text{res}}[t] := \{x \in \Omega | (t,x) \in \mathcal{M}^{\varepsilon}_{\text{res}}\}$ go as ε^2. Indeed, using the set $\{\rho_\varepsilon(t) \leq \overline{\rho}/2\}$ as an example,

$$|\{\rho_\varepsilon(t) \leq \overline{\rho}/2\}|$$

$$\leq c_1 \int_\Omega 1_{\{\rho_\varepsilon(t)\leq\overline{\rho}/2\}} \mathcal{H}(\rho_\varepsilon, \eta_\varepsilon) dx \leq \int_\Omega \mathcal{H}(\rho_\varepsilon, \eta_\varepsilon) dx \leq \varepsilon^2 c_2.$$

Thus, using the coercivity of $\mathcal{H}$ and (3.29), the following bounds can be obtained:

$$\operatorname*{ess\,sup}_{t\in(0,T)} |\mathcal{M}^{\varepsilon}_{\text{res}}[t]| \leq \varepsilon^2 c \tag{3.30}$$

$$\left\{ \left[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \right]_{\text{ess}} \right\} \in_b L^\infty(0,T; L^2(\Omega)) \tag{3.31}$$

$$\left\{ \left[\frac{\eta_\varepsilon - \overline{\eta}}{\varepsilon} \right]_{\text{ess}} \right\} \in_b L^\infty(0,T; L^2(\Omega)) \tag{3.32}$$

$$\{\mathbf{u}_\varepsilon\} \in_b L^2(0,T; W^{1,2}(\Omega)) \tag{3.33}$$

$$\{\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon\} \in_b L^\infty(0,T; L^2(\Omega; \mathbb{R}^3)) \tag{3.34}$$

$$\left\{ \frac{1}{\varepsilon} \left(\frac{2D\nabla_x \sqrt{\eta}}{\sqrt{\zeta}} + \varepsilon\sqrt{\zeta\eta}\nabla_x \Phi \right) \right\} \in_b L^2(0,T; L^2(\Omega; \mathbb{R}^3)) \tag{3.35}$$

$$\{[\varrho_\varepsilon]_{\text{res}}\} \in_b L^\infty(0,T; L^\gamma(\Omega)). \tag{3.36}$$

3.1.2.3 Convergence

From the uniform bounds in (3.30)-(3.36), the following convergences are easily obtained:

- There exists $\varrho^{(1)}$ such that

$$\left[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \right]_{\text{ess}} \to \varrho^{(1)}$$

weakly in $L^2(0,T;L^2(\Omega))$.

- Also,

$$\left[\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}\right]_{\mathrm{res}} \to 0$$

weakly-$*$ in $L^\infty(0,T;L^\gamma(\Omega))$.

- There exists $\eta^{(1)}$ such that

$$\left[\frac{\eta_\varepsilon - \eta}{\varepsilon}\right]_{\mathrm{ess}} \to \eta^{(1)}$$

weakly in $L^2(0,T;L^2(\Omega))$.

- There exists $\overline{\mathbf{u}}$ such that $\mathbf{u}_\varepsilon \to \overline{\mathbf{u}}$ weakly in $L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3))$.

By (3.35),

$$\left[\frac{\eta_\varepsilon - \overline{\eta}}{\varepsilon}\right]_{\mathrm{res}} \to 0$$

weakly in $L^2(0,T;L^2(\Omega))$. Therefore, letting $q := \min\{2,\gamma\}$

$$\varrho_\varepsilon \to \overline{\varrho} \qquad \text{weakly in} \qquad L^2(0,T;L^q(\Omega)) \tag{3.37}$$

$$\eta_\varepsilon \to \overline{\eta} \qquad \text{weakly in} \qquad L^2(0,T;L^2(\Omega)) \tag{3.38}$$

$$\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \to \varrho^{(1)} \qquad \text{weakly in} \qquad L^2(0,T;L^q(\Omega)) \tag{3.39}$$

$$\frac{\eta_\varepsilon - \overline{\eta}}{\varepsilon} \to \eta^{(1)} \qquad \text{weakly in} \qquad L^2(0,T;L^2(\Omega)). \tag{3.40}$$

Using these convergence results and taking $b(z) = 0$ and $B(1) = 1$ in the renormalized continuity equation, ε can be taken to zero to yield that

$$\int_0^T \int_\Omega \overline{\mathbf{u}} \cdot \nabla_x \phi dx = 0, \tag{3.41}$$

that is, $\overline{\mathbf{u}}$ is weakly divergence-free.

To complete the proof of Theorem 3.1.1, the convergence of the momentum equation must be shown. The first thing to note is that by using the uniform bounds and the compact embedding of $W^{1,2}(\Omega; \mathbb{R}^3)$ into $L^6(\Omega; \mathbb{R}^3)$,

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \to \overline{\varrho \mathbf{u}} \tag{3.42}$$

weakly in $L^2(0, T; L^{6q/q+6}(\Omega; \mathbb{R}^3))$ and weakly-$*$ in $L^\infty(0, T; L^{2q/q+1}(\Omega; \mathbb{R}^3))$. Thus

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon \to \overline{\varrho \mathbf{u} \otimes \mathbf{u}}$$

weakly in $L^2(0, T; L^{6q/4q+3}(\Omega; \mathbb{R}^{3\times 3}))$. So the taking the limit as $\varepsilon \to 0$ in the momentum equation yields

$$\begin{aligned} &\int_0^T \int_\Omega \overline{\varrho \mathbf{u}} \cdot \partial_t \mathbf{v} + \overline{\varrho \mathbf{u} \otimes \mathbf{u}} : \nabla_x \mathbf{v} \; \mathrm{d}x \; \mathrm{d}t \\ &\int_0^T \int_\Omega \mu \nabla_x \overline{\mathbf{u}} : \nabla_x \mathbf{v} - (\beta \varrho^{(1)} + \eta^{(1)}) \nabla_x \Phi \cdot \mathbf{v} \; \mathrm{d}x \; \mathrm{d}t - \int_\Omega \overline{\varrho \mathbf{u}_0} \cdot \mathbf{v} \; \mathrm{d}x \end{aligned} \tag{3.43}$$

for all divergence-free $\mathbf{v} \in C_c^\infty([0, T) \times \Omega; \mathbb{R}^3)$.

At this point, the original momentum equation 3.15 can be multiplied by ε and taking $\varepsilon \to 0$, with the aid of the uniform estimates, a relation for the quantities $\varrho^{(1)}$ and $\eta^{(1)}$ can be obtained as

$$\int_0^T \int_\Omega \left(a\gamma \overline{\varrho}^{\gamma-1} \varrho^{(1)} + \frac{D}{\zeta} \eta^{(1)} \right) \mathrm{div}_x \, \mathbf{w} \; \mathrm{d}x \; \mathrm{d}t = -\int_0^T \int_\Omega (\beta \overline{\varrho} + \overline{\eta}) \nabla_x \Phi \cdot \mathbf{w} \; \mathrm{d}x \tag{3.44}$$

for any test function (not necessarily divergence-free) $\mathbf{w}$. Thus, at least weakly,

$$\varrho^{(1)} = \frac{1}{a\gamma\overline{\varrho}^{\gamma-1}}\left[(\beta\overline{\varrho}+\overline{\eta})\Phi - \frac{D}{\zeta}\eta^{(1)}\right].$$

3.1.2.4 Convective Term

All that is left to do to prove Theorem 3.1.1 is to show that the divergence of $\overline{\varrho}\overline{\mathbf{u}}\otimes\overline{\mathbf{u}} - \overline{\varrho\mathbf{u}\otimes\mathbf{u}}$ converges weakly to a gradient. To do this, the standard Helmholtz decomposition is employed to decompose the quantity into a divergence-free and a gradient part. Here, $\mathbf{H}[\mathbf{v}]$ will denote the divergence-free (solenoidal) part and $\mathbf{H}^{\perp}[\mathbf{v}]$ will denote the gradient part of the vector $\mathbf{v}$. Thus, the convective term can be rewritten as

$$\varrho_\varepsilon\mathbf{u}_\varepsilon\otimes\mathbf{u}_\varepsilon = \mathbf{H}[\varrho_\varepsilon\mathbf{u}_\varepsilon]\otimes\mathbf{u}_\varepsilon + \mathbf{H}^{\perp}[\varrho_\varepsilon\mathbf{u}_\varepsilon]\otimes\mathbf{H}\mathbf{u}_\varepsilon + \mathbf{H}^{\perp}[\varrho_\varepsilon\mathbf{u}_\varepsilon]\otimes\mathbf{H}^{\perp}[\mathbf{u}_\varepsilon].$$

By the convergence results and the continuity of the Helmholtz decomposition

$$\mathbf{H}[\varrho_\varepsilon\mathbf{u}_\varepsilon] \to \mathbf{H}[\overline{\varrho}\overline{\mathbf{u}}] = \overline{\varrho}\overline{\mathbf{u}}$$

in $C_{\text{weak}}([0,T];L^{2q/q+1}(\Omega;\mathbb{R}^3))$. Since

$$\overline{\varrho}\mathbf{H}[\mathbf{u}_\varepsilon]\cdot\mathbf{u}_\varepsilon = \left(\varepsilon\mathbf{H}\left[\frac{\overline{\varrho}-\varrho_\varepsilon}{\varepsilon}\mathbf{u}_\varepsilon\right] + \mathbf{H}[\varrho_\varepsilon\mathbf{u}_\varepsilon]\right)\cdot\mathbf{u}_\varepsilon \to \overline{\varrho}|\overline{\mathbf{u}}|^2$$

weakly in $L^1(\Omega)$, it follows that $\mathbf{H}[\mathbf{u}_\varepsilon] \to \overline{\mathbf{u}}$ in $L^2(0,T;L^2(\Omega;\mathbb{R}^3))$. Therefore,

$$\mathbf{H}[\varrho_\varepsilon\mathbf{u}_\varepsilon]\otimes\mathbf{u}_\varepsilon \to \overline{\varrho}\overline{\mathbf{u}}\otimes\overline{\mathbf{u}} \tag{3.45}$$

$$\mathbf{H}^{\perp}[\varrho_\varepsilon\mathbf{u}_\varepsilon]\otimes\mathbf{H}[\mathbf{u}_\varepsilon] \to 0 \tag{3.46}$$

weakly in $L^2(0,T;L^{6q/4q+3}(\Omega;\mathbb{R}^{3\times3}))$. Thus, it remains to show that the singular term $\mathbf{H}^{\perp}[\varrho_\varepsilon\mathbf{u}_\varepsilon]\otimes\mathbf{H}^{\perp}[\mathbf{u}_\varepsilon]$ converges weakly to a gradient so that it can be absorbed into the term Π in the limit.

Noting that $\overline{\varrho}$ and $\overline{\eta}$ are constant, the scaled weak formulation of the Navier-Stokes-Smoluchowski system can be rewritten as

$$\int_0^T \int_\Omega \varepsilon \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \partial_t \phi + \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla_x \phi \, \mathrm{d}x \, \mathrm{d}t = 0 \tag{3.47}$$

$$\begin{aligned} &\int_0^T \int_\Omega \varepsilon \varrho_\varepsilon \mathbf{u}_\varepsilon \cdot \partial_t \mathbf{v} + \left[\frac{[p(\varrho_\varepsilon, \eta_\varepsilon)]_{ess} - p(\overline{\varrho}, \overline{\eta})}{\varepsilon} + (\beta \overline{\varrho} + \overline{\eta}) \Phi \right] \mathrm{div}_x \, \mathbf{v} \, \mathrm{d}x \, \mathrm{d}t \\ &= \int_0^T \int_\Omega [\beta(\overline{\varrho} - \varrho_\varepsilon) + (\overline{\eta} - \eta)] \nabla_x \Phi \cdot \mathbf{v} \, \mathrm{d}x \, \mathrm{d}t \\ &+ \int_0^T \int_\Omega \left[\varepsilon \mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) - \varepsilon \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon - \frac{[p(\varrho_\varepsilon, \eta_\varepsilon)]_{res}}{\varepsilon} \mathbb{I} \right] : \nabla_x \mathbf{v} \, \mathrm{d}x \, \mathrm{d}t \end{aligned} \tag{3.48}$$

$$\int_0^T \int_\Omega \varepsilon \frac{\eta_\varepsilon - \overline{\eta}}{\varepsilon} \partial_t \phi + [\eta \mathbf{u}_\varepsilon - \varepsilon \zeta \eta_\varepsilon \nabla_x \Phi - D \nabla_x \eta_\varepsilon] \cdot \phi \, \mathrm{d}x \, \mathrm{d}t = 0 \tag{3.49}$$

for test functions $\phi \in C_c^\infty((0,T) \times \Omega), \mathbf{v} \in C_c^\infty((0,T) \times \Omega; \mathbb{R}^3)$. Thus, defining the following quantities

$$\begin{aligned} \varrho_\varepsilon^{(1)} &:= \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \\ \eta_\varepsilon^{(1)} &:= \frac{\eta_\varepsilon - \overline{\eta}}{\varepsilon} \\ r_\varepsilon &:= \varrho_\varepsilon^{(1)} + \frac{D}{a\gamma \overline{\varrho}^{\gamma-1} \zeta} \eta_\varepsilon^{(1)} + \frac{(\beta \overline{\varrho} + \overline{\eta})\Phi}{a\gamma \overline{\varrho}^{\gamma-1}} \\ \omega &:= a\gamma \overline{\varrho}^{\gamma-1} \\ \mathbf{V}_\varepsilon &:= \varrho_\varepsilon \mathbf{u}_\varepsilon \\ h_\varepsilon^1 &:= \varepsilon \mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) - \varepsilon \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon - \frac{[p(\varrho_\varepsilon, \eta_\varepsilon)]\mathrm{res}}{\varepsilon} \mathbb{I} \\ \mathbf{h}_\varepsilon^2 &:= \frac{1}{\omega} \left[\varepsilon D \eta_\varepsilon \nabla_x \Phi + \frac{D^2}{\zeta} \nabla_x \eta_\varepsilon - \frac{D}{\zeta} \eta_\varepsilon \mathbf{u}_\varepsilon \right] \\ h_\varepsilon^3 &:= \frac{[p(\varrho_\varepsilon, \eta_\varepsilon)]\mathrm{ess} - p(\overline{\varrho}, \overline{\eta})}{\varepsilon} - p_F'(\overline{\varrho}) \varrho_\varepsilon^{(1)} + p_P'(\overline{\eta}) \eta_\varepsilon^{(1)} \end{aligned}$$

the system (3.47)-(3.49) becomes after some algebra

$$\int_0^T \int_\Omega \varepsilon \varrho_\varepsilon^{(1)} \partial_t \phi + \mathbf{V}_\varepsilon \cdot \nabla_x \phi \, \mathrm{d}x \, \mathrm{d}t = \int_0^T \int_\Omega \mathbf{h}_\varepsilon^2 \cdot \nabla_x \phi \, \mathrm{d}x \, \mathrm{d}t \tag{3.50}$$

$$\int_0^T \int_\Omega \varepsilon \mathbf{V}_\varepsilon \cdot \partial_t \mathbf{v} + \omega \varrho_\varepsilon^{(1)} \operatorname{div}_x \mathbf{v} \, \mathrm{d}x \, \mathrm{d}t$$
$$= \int_0^T \int_\Omega [\beta(\overline{\varrho} - \varrho_\varepsilon) + (\overline{\eta} - \eta_\varepsilon)] \nabla_x \Phi \cdot \mathbf{v} + h_\varepsilon^1 : \nabla_x \mathbf{v} - h_\varepsilon^3 \operatorname{div}_x \mathbf{v} \, \mathrm{d}x \, \mathrm{d}t \tag{3.51}$$

for test functions $\phi \in C_c^\infty((0,T) \times \Omega), \mathbf{v} \in C_c^\infty((0,T) \times \Omega; \mathbb{R}^3)$. By the uniform bounds and convergence results, it is clear that

$$\|h_\varepsilon^1\|_{L^s(0,T;L^1(\Omega;\mathbb{R}^{3\times 3}))} \leq \varepsilon c$$

for some $s > 1$ and

$$\mathbf{h}_\varepsilon^2 \to 0$$

weakly in the appropriate Lebesgue space. By the following lemma (adapted from Proposition 5.2 in [32]), $h_\varepsilon^3 \to 0$ weakly-$*$ in $L^\infty(0,T;L^1(\Omega))$:

Lemma 3.1.1. *Let $\{\varrho_\varepsilon\}_{\varepsilon>0}, \{\eta_\varepsilon\}_{\varepsilon>0}$ be sequences of non-negative measurable functions such that $[\varrho_\varepsilon^{(1)}]_{ess} \to \varrho^{(1)}, [\eta_\varepsilon^{(1)}]_{ess} \to \eta^{(1)}$ weakly-$*$ in $L^\infty(0,T;L^2(\Omega))$. Suppose*

$$\operatorname*{ess\,sup}_{t \in (0,T)} |\mathcal{M}_{res}^\varepsilon[t]| \leq \varepsilon^2 c$$

and that $p \in C^2(\overline{\mathcal{O}_{ess}})$. Then defining h_ε^3 as above,

$$h_\varepsilon^3 \to 0 \text{ weakly} - * \text{ in } L^\infty(0,T;L^2(\Omega)).$$

Also, from the section on convergence and the properties of Φ, it is clear that

$$[\beta(\overline{\varrho} - \varrho_\varepsilon) + (\overline{\eta} - \eta_\varepsilon)] \nabla_x \Phi$$

converges to zero weakly in the appropriate Lebesgue space. Thus (3.50) and (3.51) represent a system of wave equations for which the right sides converge to zero. Now the associated eigenvalue problem for the left sides of (3.50) and (3.51) are considered:

$$\operatorname{div}_x \mathbf{w} = \lambda q$$

$$\omega \nabla_x q = \lambda \mathbf{w}$$

$$\mathbf{w} \cdot \mathbf{n}|_{\partial\Omega} = 0,$$

which can easily be reformulated as

$$-\Delta_x q = \Lambda q \tag{3.52}$$

$$\nabla_x q \cdot \mathbf{n}|_{\partial\Omega} = 0$$

$$-\Lambda = \frac{\lambda^2}{\omega}$$

(note that λ here is unrelated to the λ from the stress tensor). As is well known (c.f. [32]), the system in (3.52) admits a countable system of eigenvalues

$$0 = \Lambda_0 < \Lambda_1 \leq \Lambda_2 \leq \Lambda_3 \leq ...$$

with associated eigenfunctions $\{q_n\}_{n=0}^{\infty}$ which form an orthogonal basis of $L^2(\Omega)$. Then, corresponding eigenfunctions $\mathbf{w}_{\pm n}$ are defined as

$$\mathbf{w}_{\pm n} := \pm i \sqrt{\frac{\omega}{\Lambda_n}} \nabla_x q_n$$

for each positive n. Also, the space $L^2(\Omega; \mathbb{R}^3)$ can be composed orthogonally into

$$L^2(\Omega; \mathbb{R}^3) = L^2_\sigma(\Omega; \mathbb{R}^3) \oplus L^2_g(\Omega; \mathbb{R}^3)$$

where

$$L^2_g(\Omega;\mathbb{R}^3) := \mathrm{closure}_{L^2}\left\{\mathrm{span}\left\{\frac{-i}{\omega}\mathbf{w}_n\right\}_{n=1}^{\infty}\right\}$$

represents the closure of the gradient functions and

$$L^2_\sigma(\Omega;\mathbb{R}^3) := \mathrm{closure}_{L^2}\{\mathbf{v}\in C_c^\infty(\Omega;\mathbb{R}^3)|\,\mathrm{div}_x\,\mathbf{v}=0\}$$

represents the space of divergence-free functions.

With these spaces defined, the following projection can be defined:

$$\mathbf{P}_M : L^2(\Omega;\mathbb{R}^3) \to \mathrm{span}\left\{\frac{-i}{\sqrt{\omega}}\right\}_{n\leq M}$$

for each $M\in\mathbb{N}$. Noting that $\mathbf{P}_M$ and $\mathbf{H}^\perp$ commute, for the sake of notational simplicity, the operator $\mathbf{H}^\perp_M$ will be defined by

$$\mathbf{H}^\perp_M[\mathbf{v}] := \mathbf{P}_M\mathbf{H}^\perp[\mathbf{v}] = \mathbf{H}^\perp[\mathbf{P}_M\mathbf{v}].$$

Returning to the singular term, it is noted that

$$\begin{aligned}&\int_0^T\int_\Omega \mathbf{H}^\perp[\varrho_\varepsilon\mathbf{u}_\varepsilon]\otimes\mathbf{H}^\perp[\mathbf{u}_\varepsilon] : \nabla_x\mathbf{v}\ \mathrm{d}x\ \mathrm{d}t\\ &=\int_0^T\int_\Omega \mathbf{H}^\perp[\varrho_\varepsilon\mathbf{u}_\varepsilon]\otimes\mathbf{H}^\perp_M[\mathbf{u}_\varepsilon] : \nabla_x\mathbf{v}\ \mathrm{d}x\ \mathrm{d}t\\ &+\int_0^T\int_\Omega \mathbf{H}^\perp[\varrho_\varepsilon\mathbf{u}_\varepsilon]\otimes(\mathbf{H}^\perp[\mathbf{u}_\varepsilon]-\mathbf{H}^\perp_M[\mathbf{u}_\varepsilon]) : \nabla_x\mathbf{v}\ \mathrm{d}x\ \mathrm{d}t\end{aligned} \tag{3.53}$$

and by estimates shown in Section 5.4.6 of [32] and since $\varrho_\varepsilon\mathbf{u}_\varepsilon \to \overline{\varrho\mathbf{u}}$ weakly-$*$ in $L^\infty(0,T;L^{2q/q+1}(\Omega;\mathbb{R}^3))$,

$$\left|\int_0^T\int_\Omega \mathbf{H}^\perp[\varrho_\varepsilon\mathbf{u}_\varepsilon]\otimes(\mathbf{H}^\perp[\mathbf{u}_\varepsilon]-\mathbf{H}^\perp_M[\mathbf{u}_\varepsilon]) : \nabla_x\mathbf{v}\ \mathrm{d}x\ \mathrm{d}t\right|\to 0$$

uniformly in ε as $M\to\infty$. Also, since for fixed $\mathbf{v}\in[W^{1,2}(\Omega;\mathbb{R}^3)]^*$ defined by the standard Riesz formula

$$\|\mathbf{H}^\perp[\mathbf{v}]-\mathbf{H}^\perp_M[\mathbf{v}]\|^2_{[W^{1,2}(\Omega;\mathbb{R}^3)]^*}\to 0$$

uniformly in ε as $M \to \infty$, the problem of showing the weak convergence of $\mathbf{H}^{\perp}[\varrho_\varepsilon \mathbf{u}_\varepsilon] \otimes \mathbf{H}^{\perp}[\mathbf{u}_\varepsilon]$ to a gradient reduces to showing that for any fixed $M \in \mathbb{N}$

$$\int_0^T \int_\Omega \mathbf{H}_M^{\perp}[\varrho_\varepsilon \mathbf{u}_\varepsilon] \otimes \mathbf{H}_M^{\perp}[\mathbf{u}_\varepsilon] : \nabla_x \mathbf{v} \; \mathrm{d}x \; \mathrm{d}t \to 0$$

or by (3.37)

$$\int_0^T \int_\Omega \mathbf{H}_M^{\perp}[\varrho_\varepsilon \mathbf{u}_\varepsilon] \otimes \mathbf{H}_M^{\perp}[\varrho_\varepsilon \mathbf{u}_\varepsilon] : \nabla_x \mathbf{v} \; \mathrm{d}x \; \mathrm{d}t \to 0$$

as $\varepsilon \to 0$ for any divergence-free $\mathbf{v} \in C_c^\infty((0,T) \times \Omega; \mathbb{R}^3)$ with $\mathbf{v} \cdot \mathbf{n}|_{\partial\Omega} = 0$.

In order to handle this term, the test functions ϕ and $\mathbf{v}$ are defined as

$$\phi(t,x) = \psi(t) q_n(x)$$

and

$$\mathbf{v}(t,x) = \psi(t) \frac{1}{\sqrt{\Lambda_n}} \nabla_x q_n(x)$$

where $\psi \in C_c^\infty(0,T)$ and q_n is the corresponding eigenfunction from above. After some basic manipulations, the system (3.50) and (3.51) becomes

$$\varepsilon \partial_t b_n[\varrho_\varepsilon^{(1)}] - \sqrt{\Lambda_n} a_n[\mathbf{V}_\varepsilon] = \chi_{\varepsilon,n}^1 \tag{3.54}$$

$$\varepsilon \partial_t a_n[\mathbf{V}_\varepsilon] + \omega \sqrt{\Lambda_n} b_n[\varrho_\varepsilon^{(1)}] = \chi_{\varepsilon,n}^2 \tag{3.55}$$

where

$$a_n[\mathbf{V}_\varepsilon] := \frac{1}{\Lambda_n} \int_\Omega \mathbf{V}_\varepsilon \cdot \nabla_x q_n dx$$
$$b_n[\varrho_\varepsilon^{(1)}] := \int_\Omega \varrho_\varepsilon^{(1)} q_n dx$$

are the appropriate Fourier coefficients, and $\chi_{\varepsilon,n}^1$, $\chi_{\varepsilon,n}^2$ are defined appropriately. It is easily seen that for each n, $\chi_{\varepsilon,n}^1$ and $\chi_{\varepsilon,n}^2$ converge to zero in $L^1(\Omega)$ from the bounds on the remainder terms h_ε^i. Rewriting (3.54)-(3.55) in terms of the Helmholtz projectors, the system becomes

$$\varepsilon \partial_t [\varrho_\varepsilon^{(1)}]_M + \mathrm{div}_x (\mathbf{H}_M^\perp [\varrho_\varepsilon \mathbf{u}_\varepsilon]) = \chi_{\varepsilon,M}^3 \tag{3.56}$$

$$\varepsilon \partial_t \mathbf{H}_M^\perp [\varrho_\varepsilon \mathbf{u}_\varepsilon] + \omega \nabla_x [\varrho_\varepsilon^{(1)}]_M = \chi_{\varepsilon,M}^4 \tag{3.57}$$

where

$$[\varrho_\varepsilon^{(1)}]_M = \sum_{n=1}^{M} b_n [\varrho_\varepsilon^{(1)}] q_n$$

and $\chi_{\varepsilon,M}^3$ and $\chi_{\varepsilon,M}^4$ both converge to zero in $L^1(\Omega)$. Note also that $[\varrho_\varepsilon^{(1)}]_M$ and $\mathbf{H}_M^\perp[\varrho_\varepsilon \mathbf{u}_\varepsilon]$ are both twice spatially differentiable and absolutely continuous in time. Thus the system (3.56)-(3.57) is defined and the potential $\Psi_{\varepsilon,M}$ can be defined such that

$$\Psi_{\varepsilon,M} = \mathbf{H}_M^\perp [\varrho_\varepsilon \mathbf{u}_\varepsilon], \int_\Omega \Psi_{\varepsilon,M} \, \mathrm{d}x = 0.$$

Thus,

$$\int_0^T \int_\Omega \mathbf{H}_M^\perp [\varrho_\varepsilon \mathbf{u}_\varepsilon] \otimes \mathbf{H}_M^\perp [\varrho_\varepsilon \mathbf{u}_\varepsilon] : \nabla_x \mathbf{v} \, \mathrm{d}x \, \mathrm{d}t = - \int_0^T \int_\Omega \Delta_x \Psi_{\varepsilon,M} \nabla_x \Psi_{\varepsilon,M} \cdot \mathbf{v} \, \mathrm{d}x \, \mathrm{d}t$$

for any test function $\mathbf{v}$ which has zero normal trace and is divergence free. Rewriting the right side of this equation as

$$\begin{aligned} &\int_0^T \int_\Omega \Delta_x \Psi_{\varepsilon,M} \nabla_x \Psi_{\varepsilon,M} \cdot \mathbf{v} \, \mathrm{d}x \, \mathrm{d}t \\ &= \varepsilon \int_0^T \int_\Omega [\varrho_\varepsilon^{(1)}]_M \nabla_x \Psi_{\varepsilon,M} \cdot \partial_t \mathbf{v} \, \mathrm{d}x \, \mathrm{d}t \\ &+ \int_0^T \int_\Omega \chi_{\varepsilon,M}^3 \mathbf{H}_M^\perp [\varrho_\varepsilon \mathbf{u}_\varepsilon] \cdot \mathbf{v} + [\varrho_\varepsilon^{(1)}]_M \chi_{\varepsilon,M}^4 \cdot \mathbf{v} \, \mathrm{d}x \, \mathrm{d}t \end{aligned} \tag{3.58}$$

by using (3.56) and (3.57), it is clear from the convergences of $\chi_{\varepsilon,M}^3$ and $\chi_{\varepsilon,M}^4$ that the right side of (3.58) converges to zero for any fixed zero normal trace, divergence free $\mathbf{v}$ as ε goes to zero. Thus, it has been shown that $\mathbf{H}^{\perp}[\varrho_\varepsilon \mathbf{u}_\varepsilon] \otimes \mathbf{H}^{\perp}[\mathbf{u}_\varepsilon]$ converges weakly to a gradient, completing the proof of Theorem 3.1.1.

Remark 3.1.1. The interested reader will notice that while the work in this section follows the outline of the work of Feireisl and Novotný, no time lifting is performed. This is due to the fact that the Navier-Stokes-Fourier system investigated in [32] contains an entropy production term that behaves as a measure instead of an integral over the time domain. This complication does not arise in the Navier-Stokes-Smoluchowski system investigated here.

3.2 Strong Stratification Limit

The next limit considered here is a strong stratification limit. In this case, the Froude number is scaled the same as the Mach number limit, and the values are scaled as stated below:

- Ma is taken to be a small parameter $\varepsilon > 0$.
- Za, Da are taken to be ε^{-1}.
- Fr is taken to be ε.
- Other parameters are taken to be of order 1.
- The external potential takes the form $\Phi = g x_3$ where g is a constant (gravity/buoyancy).

Thus, the scaled NSS system becomes

$$\partial_t \varrho_\varepsilon + \operatorname{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon) = 0 \tag{3.59}$$

$$\varepsilon^2[\partial_t(\varrho_\varepsilon \mathbf{u}_\varepsilon) + \operatorname{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon)] + \nabla_x \left(a\varrho_\varepsilon^\gamma + \frac{D}{\zeta}\eta_\varepsilon \right)$$

$$= \varepsilon^2(\mu \Delta_x \mathbf{u}_\varepsilon + \lambda \nabla_x \operatorname{div}_x \mathbf{u}_\varepsilon) - (\beta\varrho_\varepsilon + \eta_\varepsilon)\nabla_x \Phi \tag{3.60}$$

$$\varepsilon\left[\partial_t \eta_\varepsilon + \operatorname{div}_x(\eta_\varepsilon \mathbf{u}_\varepsilon)\right] - \operatorname{div}_x(\zeta \eta_\varepsilon \nabla_x \Phi) - D\Delta_x \eta_\varepsilon = 0 \tag{3.61}$$

$$\varepsilon \frac{\mathrm{d}}{\mathrm{d}t} \int_\Omega \frac{\varepsilon^2}{2}\varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 + \frac{a}{\gamma - 1}\varrho_\varepsilon^\gamma + \frac{D\eta_\varepsilon}{\zeta} \ln \eta_\varepsilon + (\beta\varrho_\varepsilon + \eta_\varepsilon)\Phi \ \mathrm{d}x$$

$$+ \varepsilon \int_\Omega \varepsilon^2 \mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) : \nabla_x \mathbf{u}_\varepsilon \ \mathrm{d}x + \int_\Omega \left| D\frac{\nabla_x \eta_\varepsilon}{\sqrt{\zeta\eta_\varepsilon}} + \sqrt{\zeta\eta_\varepsilon}\nabla_x \Phi \right|^2 \ \mathrm{d}x \leq 0. \tag{3.62}$$

Now, assuming $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \eta_\varepsilon\}$ have the following expansions

$$\varrho_\varepsilon = \tilde{\varrho} + \sum_{i=1}^\infty \varepsilon^i \varrho_\varepsilon^{(i)}$$

$$\eta_\varepsilon = \tilde{\eta} + \sum_{i=1}^\infty \varepsilon^i \eta_\varepsilon^{(i)}$$

$$\mathbf{u}_\varepsilon = \tilde{\mathbf{u}} + \sum_{i=1}^\infty \varepsilon^i \mathbf{u}_\varepsilon^{(i)}$$

substition into (3.59)-(3.62) formally yields the target system

$$g\tilde{\eta} = -\frac{D}{\zeta}\frac{\mathrm{d}\tilde{\eta}}{\mathrm{d}x_3}$$

$$\frac{\mathrm{d}}{\mathrm{d}x_3}\left[a\tilde{\varrho}^\gamma\right] = -\beta g \tilde{\varrho}$$

$$\operatorname{div}_x(\tilde{\varrho}\tilde{\mathbf{u}}) = 0$$

$$\tilde{\varrho}\partial_t \tilde{\mathbf{u}} + \operatorname{div}_x(\tilde{\varrho}\tilde{\mathbf{u}} \otimes \tilde{\mathbf{u}}) + \nabla_x \Pi = \mu\Delta_x \tilde{\mathbf{u}} + \lambda \nabla_x \operatorname{div}_x \tilde{\mathbf{u}} - \left(\beta\varrho^{(2)} + \eta^{(2)}\right)\nabla_x \Phi.$$

3.2.1 Rigorous Justification of the Strong Stratification Limit

For the strong stratification scaling, the weak formulation follows:

Definition 3.2.1. $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon, \eta_\varepsilon\}$ form a *weak solution to the scaled strong stratification NSS system* if and only if

- $\varrho_\varepsilon \geq 0$ and $\mathbf{u}_\varepsilon$ form a renormalized solution of the scaled continuity equation, i.e.,

$$\int_0^T \int_\Omega B(\varrho_\varepsilon)\partial_t\varphi + B(\varrho_\varepsilon)\mathbf{u}_\varepsilon \cdot \nabla_x\varphi - b(\varrho_\varepsilon)\operatorname{div}_x \mathbf{u}_\varepsilon\varphi \; \mathrm{d}x \; \mathrm{d}t$$
$$= -\int_\Omega B(\varrho_0)\varphi(0,\cdot) \; \mathrm{d}x \tag{3.63}$$

where $b \in L^\infty([0,\infty)) \cap C([0,\infty)),\ B(\varrho) := B(1) + \int_1^\varrho \frac{b(z)}{z^2}\mathrm{d}z$.

- The scaled momentum balance holds in the sense of distributions:

$$\int_0^T \int_\Omega \varepsilon^2 \left(\varrho_\varepsilon\mathbf{u}_\varepsilon \cdot \partial_t\mathbf{w} + \varrho_\varepsilon\mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon : \nabla_x\mathbf{w}\right) + \left(p_F(\varrho_\varepsilon) + \frac{D}{\zeta}\eta_\varepsilon\right)\operatorname{div}_x \mathbf{w} \; \mathrm{d}x \; \mathrm{d}t$$
$$= \int_0^T \int_\Omega \varepsilon^2 \left(\mu\nabla_x\mathbf{u}_\varepsilon\nabla_x\mathbf{w} + \lambda \operatorname{div}_x \mathbf{u}_\varepsilon \operatorname{div}_x \mathbf{w}\right) - (\beta\varrho_\varepsilon + \eta_\varepsilon)\nabla_x\Phi \cdot \mathbf{w} \; \mathrm{d}x \; \mathrm{d}t$$
$$- \varepsilon^2 \int_\Omega \mathbf{m}_0 \cdot \mathbf{w}(0,\cdot) \; \mathrm{d}x. \tag{3.64}$$

- $\eta_\varepsilon \geq 0$ is a weak solution of the scaled Smoluchowski equation:

$$\int_0^T \int_\Omega \varepsilon \left[\eta_\varepsilon\partial_t\varphi + \eta_\varepsilon\mathbf{u}_\varepsilon \cdot \nabla_x\varphi\right] - \zeta\eta_\varepsilon\nabla_x\Phi \cdot \nabla_x\varphi - D\nabla_x\eta_\varepsilon \cdot \nabla_x\varphi \; \mathrm{d}x \; \mathrm{d}t$$
$$= -\int_\Omega \eta_0\varphi(0,\cdot) \; \mathrm{d}x. \tag{3.65}$$

- The scaled energy inequality is satisfied:

$$\int_\Omega \frac{\varepsilon^2}{2}\varrho_\varepsilon|\mathbf{u}_\varepsilon|^2 + \frac{a}{\gamma-1}\varrho_\varepsilon^\gamma + \frac{D}{\zeta}\eta_\varepsilon \ln\eta_\varepsilon + (\beta\varrho_\varepsilon + \eta_\varepsilon)\Phi \; \mathrm{d}x(T)$$
$$+ \int_0^T \int_\Omega \varepsilon^2(\mu|\nabla_x\mathbf{u}_\varepsilon|^2 + \lambda|\operatorname{div}_x \mathbf{u}_\varepsilon|^2) \; \mathrm{d}x \; \mathrm{d}t$$
$$+ \frac{1}{\varepsilon}\int_0^T \int_\Omega \left|2\frac{D}{\sqrt{\zeta}}\nabla_x\sqrt{\eta_\varepsilon} + \sqrt{\zeta\eta_\varepsilon}\nabla_x\Phi\right|^2 \; \mathrm{d}x \; \mathrm{d}t$$
$$\leq \int_\Omega \frac{\varepsilon^2}{2}\varrho_0|\mathbf{u}_0|^2 + \frac{a}{\gamma-1}\varrho_0^\gamma + \frac{D}{\zeta}\eta_0 \ln\eta_0 + (\beta\varrho_0 + \eta_0)\Phi \; \mathrm{d}x. \tag{3.66}$$

Note that for this scaling, Φ takes the form $\Phi = gx_3$, where x_3 is the vertical coordinate, and g is a constant greater than zero. Also defined is the target system.

Definition 3.2.2. $\{\tilde{\varrho}, \tilde{\mathbf{u}}, \tilde{\eta}, \varrho^{(2)}, \eta^{(2)}\}$ solve the *strong stratification target system* if and only if:

- $$\int_0^T \int_\Omega \tilde{\varrho}\tilde{\mathbf{u}} \cdot \nabla_x \phi \, \mathrm{d}x \, \mathrm{d}t = 0 \tag{3.67}$$

  for all $\phi \in C_C^\infty((0,T) \times \Omega)$,

- $$g\tilde{\eta} = -\frac{D}{\zeta}\frac{\mathrm{d}\tilde{\eta}}{\mathrm{d}x_3} \tag{3.68}$$

  $$\frac{\mathrm{d}}{\mathrm{d}x_3}\left[a\tilde{\varrho}^\gamma\right] = -\beta g \tilde{\varrho} \tag{3.69}$$

  with the conditions

  $$\int_\Omega \tilde{\varrho} \, \mathrm{d}x = \int_\Omega \varrho_0 \, \mathrm{d}x$$

  $$\int_\Omega \tilde{\eta} \, \mathrm{d}x = \int_\Omega \eta_0 \, \mathrm{d}x,$$

- $$\int_0^T \int_\Omega \tilde{\varrho}\tilde{\mathbf{u}} \cdot \mathbf{w} + \tilde{\varrho}\tilde{\mathbf{u}} \otimes \tilde{\mathbf{u}} : \nabla_x \mathbf{w} \, \mathrm{d}x \, \mathrm{d}t$$
  $$= \int_0^T \int_\Omega \mu \nabla_x \tilde{\mathbf{u}} \nabla_x \mathbf{w} - \left(\beta \varrho^{(2)} + \eta^{(2)}\right) \nabla_x \Phi \cdot \mathbf{w} \, \mathrm{d}x \, \mathrm{d}t \tag{3.70}$$

  for all $\mathbf{w} \in C_C^\infty((0,T) \times \Omega; \mathbb{R}^3)$ such that $\operatorname{div}_x \mathbf{w} = 0$.

Much like for the low stratification limit, many of the bounds and convergences used in the analysis arise from the free energies defined as

$$E_F(\varrho, \tilde{\varrho}) := \frac{a}{\gamma - 1}\varrho^\gamma - (\varrho - \tilde{\varrho})\frac{a\gamma}{\gamma - 1}\tilde{\varrho}^{\gamma-1} - \frac{a}{\gamma - 1}\tilde{\varrho}^\gamma$$

$$E_P(\eta, \tilde{\eta}) := \frac{D}{\zeta}\eta \ln \eta - \frac{D}{\zeta}(\eta - \tilde{\eta})(\ln \tilde{\eta} + 1) - \frac{D}{\zeta}\tilde{\eta} \ln \tilde{\eta},$$

and the resulting inequality formed from these and the energy inequality:

$$\begin{aligned}
&\int_\Omega \frac{1}{2}\varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 + \frac{1}{\varepsilon^2}\left[E_F(\varrho_\varepsilon, \tilde{\varrho}) + E_P(\eta_\varepsilon, \tilde{\eta})\right]\ \mathrm{d}x(T) \\
&\int_0^T \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) : \nabla_x \mathbf{u}_\varepsilon \ \mathrm{d}x \ \mathrm{d}t + \frac{1}{\varepsilon^3}\int_0^T \int_\Omega \left| \frac{D\nabla_x \eta_\varepsilon}{\sqrt{\zeta \eta_\varepsilon}} + \sqrt{\zeta \eta_\varepsilon}\nabla_x \Phi \right|^2 \ \mathrm{d}x \ \mathrm{d}t \\
&\leq \int_\Omega \frac{1}{2}\varrho_0 |\mathbf{u}_0|^2 + \frac{1}{\varepsilon^2}[E_F(\varrho_0, \tilde{\varrho}) + E_P(\eta_0, \tilde{\eta})]\ \mathrm{d}x.
\end{aligned} \tag{3.71}$$

Next, the essential and residual sets are defined similarly to those in section 3.1:

$$\mathcal{O}_{\text{ess}} := \{(\varrho, \eta) \in \mathbb{R}^2 | \tilde{\varrho}/2 \leq \varrho \leq 2\tilde{\varrho}, \tilde{\eta}/2 \leq \eta \leq 2\tilde{\eta}\}$$

$$\mathcal{M}^\varepsilon_{\text{ess}} := \{(t, x) \in (0, T) \times \Omega | (\varrho_\varepsilon(t, x), \eta_\varepsilon(t, x)) \in \mathcal{O}_{\text{ess}}\}$$

$$\mathcal{M}^\varepsilon_{\text{res}} := ((0, T) \times \Omega) - \mathcal{M}^\varepsilon_{\text{ess}}$$

Thus, by using (3.71), assuming appropriate bounds on the initial data,

$$\{\sqrt{\varrho_\varepsilon}\mathbf{u}_\varepsilon\}_{\varepsilon>0} \in_b L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))$$

$$\|[\varrho_\varepsilon - \tilde{\varrho}]\text{ess}\|_{L^\infty(0,T;L^2(\Omega))} \leq \varepsilon^2 c$$

$$\|[\eta_\varepsilon - \tilde{\eta}]\text{ess}\|_{L^\infty(0,T;L^2(\Omega))} \leq \varepsilon^2 c$$

$$\{\mathbf{u}_\varepsilon\}_{\varepsilon>0} \in_b L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3))$$

$$\left\| \frac{D\nabla_x \eta_\varepsilon}{\sqrt{\zeta\eta_\varepsilon}} + \sqrt{\zeta\eta_\varepsilon}\nabla_x\Phi \right\|_{L^2(0,T;L^2(\Omega;\mathbb{R}^3))} \leq \varepsilon^3 c$$

$$\left\{ \left[\frac{\varrho_\varepsilon - \tilde{\varrho}}{\varepsilon} \right]_{\text{ess}} \right\}_{\varepsilon>0} \in_b L^\infty(0,T;L^2(\Omega))$$

$$\left\{ \left[\frac{\eta_\varepsilon - \tilde{\eta}}{\varepsilon} \right]_{\text{ess}} \right\}_{\varepsilon>0} \in_b L^\infty(0,T;L^2(\Omega))$$

and since the measure of the residual set goes as ε^2 for each fixed t,

$$\|[\varrho_\varepsilon]\text{ess}\|_{L^\infty(0,T;L^\gamma(\Omega))} \leq \varepsilon^2 c$$

$$\{\varrho_\varepsilon\mathbf{u}_\varepsilon\}_{\varepsilon>0} \in_b L^\infty(0,T;L^{2q/q+1}(\Omega;\mathbb{R}^3)) \cap L^{6q/q+6}(\Omega;\mathbb{R}^3))$$

where $q := \min\{2, q\}$. Thus, $\varrho^{(1)}, \eta^{(1)} \in L^\infty(0,T;L^2(\Omega))$ and $\tilde{\mathbf{u}} \in L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3))$ exist such that up to subsequences

$$\varrho_\varepsilon \to \tilde{\varrho} \text{ strongly in } L^\infty(0,T;L^q(\Omega))$$

$$\eta_\varepsilon \to \tilde{\eta} \text{ strongly in } L^\infty(0,T;L^2(\Omega))$$

$$\mathbf{u}_\varepsilon \rightharpoonup \tilde{\mathbf{u}} \text{ weakly in } L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3))$$

$$\frac{\varrho_\varepsilon - \tilde{\varrho}}{\varepsilon} \rightharpoonup \varrho^{(1)} \text{ weakly-}* \text{ in } L^\infty(0,T;L^q(\Omega))$$

$$\frac{\eta_\varepsilon - \tilde{\eta}}{\varepsilon} \rightharpoonup \varrho^{(1)} \text{ weakly-}* \text{ in } L^\infty(0,T;L^2(\Omega)).$$

These bounds along with work similar to that for the low stratification limit and in [32] suggest the following theorem.

Theorem 3.2.1 (Strong stratification limit)**.** *Let* (Ω, Φ) *satisfy the confinement hypothesis and for each* $\varepsilon > 0$*,* $\{\varrho_\varepsilon, \boldsymbol{u}_\varepsilon, \eta_\varepsilon\}$ *solves* (3.59)-(3.62) *in the sense of the definition of the scaled strong stratification system. Assume the initial data can be expressed as follows:*

$$\varrho_\varepsilon(0,\cdot) = \varrho_{\varepsilon,0} = \tilde{\varrho} + \varepsilon \varrho^{(1)}_{\varepsilon,0},\ \boldsymbol{u}_\varepsilon(0,\cdot) = \boldsymbol{u}_{\varepsilon,0},\ \textit{and}\ \eta_\varepsilon(0,\cdot) = \eta_{\varepsilon,0} = \tilde{\eta} + \varepsilon \eta^{(1)}_{\varepsilon,0}.$$

where $\tilde{\varrho}, \tilde{\eta}$ *are the densities defined by* (3.69)-(3.68)*. Assume also that as* $\varepsilon \to 0$*,*

$$\varrho^{(1)}_{\varepsilon,0} \rightharpoonup \varrho^{(1)}_0,\ \boldsymbol{u}_{\varepsilon,0} \rightharpoonup \overline{\boldsymbol{u}}_0,\ \eta^{(1)}_{\varepsilon,0} \rightharpoonup \eta^{(1)}_0$$

weakly-$*$ in $L^\infty(\Omega)$ *or* $L^\infty(\Omega;\mathbb{R}^3)$ *as the case may be. Then up to a subsequence and letting* $q := \min\{\gamma, 2\}$*,*

$$\varrho_\varepsilon \to \tilde{\varrho}\ \textit{in}\ C([0,T];L^1(\Omega)) \cap L^\infty(0,T;L^q(\Omega))$$

$$\eta_\varepsilon \to \tilde{\eta}\ \textit{in}\ L^2(0,T;L^2(\Omega))$$

$$\boldsymbol{u}_\varepsilon \to \tilde{\mathbf{u}}\ \textit{weakly in}\ L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3))$$

where $\{\tilde{\varrho}, \tilde{\mathbf{u}}, \tilde{\eta}\}$ *solve the target system* (3.67)-(3.70)*.*

Chapter 4

Inviscid Model

Thus far, this dissertation has analyzed the NSS model for viscous compressible flows for the bubbling regime. The attentive reader will note the existence of the stress tensor $\mathbb{S}$. However, this stress tensor is non-zero only because the viscosity coefficients μ and λ are non-zero. If the fluid is assumed to be inviscid, that is, the viscosity coefficients are zero, and follow the Euler equations for fluid flow, then the following model can be considered for the bubbling regime.

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0 \tag{4.1}$$

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x(a\varrho^\gamma + \eta) = -(\beta\varrho + \eta)\nabla_x \Phi \tag{4.2}$$

$$\partial_t \eta + \operatorname{div}_x(\eta \mathbf{u} - \eta \nabla_x \Phi) = \Delta_x \eta \tag{4.3}$$

This model is derived from the mesoscopic description of the particles in a fluid obeying the Euler equations and a Vlasov-Fokker-Planck equation in [13].

Here, the boundary conditions become

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = (\eta \nabla_x \Phi + \nabla_x \eta) \cdot \mathbf{n}|_{\partial\Omega} = 0 \tag{4.4}$$

the first condition being the typical boundary condition on $\mathbf{u}$ for inviscid flows (see [45]). However, for the rest of the analysis, the Cauchy problem on $(0, T) \times \mathbb{R}^3$ will be considered, making the boundary condition (4.4) moot.

4.1 Local-In-Time Existence

The ES system for compressible fluids can be written in matrix form as

$$\mathbb{A}_0(U)\partial_t U + \sum_{i=1}^{3} \mathbb{A}_i(U)\partial_{x_i} U + \mathbb{A}_4 = 0 \tag{4.5}$$

where

$$U := \begin{bmatrix} \varrho \\ \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \\ \eta \end{bmatrix},$$

$$\mathbb{A}_0 := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & \varrho & 0 & 0 & 0 \\ 0 & 0 & \varrho & 0 & 0 \\ 0 & 0 & 0 & \varrho & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbb{A}_1 := \begin{bmatrix} \mathbf{u}_1 & \varrho & 0 & 0 & 0 \\ p_F'(\varrho) & \varrho\mathbf{u}_1 & 0 & 0 & 1 \\ 0 & 0 & \varrho\mathbf{u}_1 & 0 & 0 \\ 0 & 0 & 0 & \varrho\mathbf{u}_1 & 0 \\ 0 & \eta & 0 & 0 & \mathbf{u}_1 - \partial_{x_1}\Phi \end{bmatrix},$$

$$\mathbb{A}_2 := \begin{bmatrix} \mathbf{u}_2 & 0 & \varrho & 0 & 0 \\ 0 & \varrho\mathbf{u}_2 & 0 & 0 & 0 \\ p_F'(\varrho) & 0 & \varrho\mathbf{u}_2 & 0 & 1 \\ 0 & 0 & 0 & \varrho\mathbf{u}_2 & 0 \\ 0 & 0 & \eta & 0 & \mathbf{u}_2 - \partial_{x_2}\Phi \end{bmatrix},$$

$$\mathbb{A}_3 := \begin{bmatrix} \mathbf{u}_3 & 0 & 0 & \varrho & 0 \\ 0 & \varrho\mathbf{u}_3 & 0 & 0 & 0 \\ 0 & 0 & \varrho\mathbf{u}_3 & 0 & 0 \\ p_F'(\varrho) & 0 & 0 & \varrho\mathbf{u}_3 & 1 \\ 0 & 0 & 0 & \eta & \mathbf{u}_3 - \partial_{x_3}\Phi \end{bmatrix},$$

and

$$\mathbb{A}_4 := \begin{bmatrix} 0 \\ (\beta\varrho + \eta)\partial_{x_1}\Phi \\ (\beta\varrho + \eta)\partial_{x_2}\Phi \\ (\beta\varrho + \eta)\partial_{x_3}\Phi \\ -\Delta_x\eta - \eta\Delta_x\Phi \end{bmatrix}.$$

Notice that the matrices $\mathbb{A}_i$, $i = 1, 2, 3$, are not symmetric matrices. Thus, the goal is to find transformations for ϱ and η that will yield a symmetric system in the spirit of [48]. This will enable the use of the existence results for symmetric hyperbolic systems of Majda and others. Clearly, the transformation

$$w := \frac{2}{\gamma - 1}\left(\sqrt{p_F'(\varrho)} - \overline{\sigma}\right) \tag{4.6}$$

where $\overline{\sigma}$ represents the sound speed where the fluid has some background density $\overline{vr}$ will resolve the $p'_F(\varrho)/\varrho$ asymmetry. However, there will need to be a transformation for η as well to resolve the $\eta/1$ asymmetry.

Indeed, using (4.6) yields the system

$$\partial_t V + \sum_{i=1}^{3} \mathbb{B}_i(V)\partial_{x_i} V + \mathbb{B}_4 = 0 \tag{4.7}$$

where

$$V := \begin{bmatrix} w \\ \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \\ \eta \end{bmatrix},$$

$$\mathbb{B}_1 := \begin{bmatrix} \mathbf{u}_1 & \overline{\sigma} + \frac{\gamma-1}{2}w & 0 & 0 & 0 \\ \overline{\sigma} + \frac{\gamma-1}{2} & \mathbf{u}_1 & 0 & 0 & f(w) \\ 0 & 0 & \mathbf{u}_1 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{u}_1 & 0 \\ 0 & \eta & 0 & 0 & \mathbf{u}_1 - \partial_{x_1}\Phi \end{bmatrix},$$

$$\mathbb{B}_2 := \begin{bmatrix} \mathbf{u}_2 & 0 & \overline{\sigma} + \frac{\gamma-1}{2}w & 0 & 0 \\ 0 & \mathbf{u}_2 & 0 & 0 & 0 \\ \overline{\sigma} + \frac{\gamma-1}{2}w & 0 & \mathbf{u}_2 & 0 & f(w) \\ 0 & 0 & 0 & \mathbf{u}_2 & 0 \\ 0 & 0 & \eta & 0 & \mathbf{u}_2 - \partial_{x_2}\Phi \end{bmatrix},$$

$$
\mathbb{B}_3 := \begin{bmatrix} \mathbf{u}_3 & 0 & 0 & \overline{\sigma} + \frac{\gamma-1}{2}w & 0 \\ 0 & \mathbf{u}_3 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{u}_3 & 0 & 0 \\ \overline{\sigma} + \frac{\gamma-1}{2}w & 0 & 0 & \mathbf{u}_3 & f(w) \\ 0 & 0 & 0 & \eta & \mathbf{u}_3 - \partial_{x_3}\Phi \end{bmatrix},
$$

and

$$
\mathbb{B}_4 := \begin{bmatrix} 0 \\ (\beta + f(w)\eta)\partial_{x_1}\Phi \\ (\beta + f(w)\eta)\partial_{x_2}\Phi \\ (\beta + f(w)\eta)\partial_{x_3}\Phi \\ -\Delta_x\eta - \eta\Delta_x\Phi \end{bmatrix}.
$$

Now, the asymmetry is between η and $f(w)$, which represents ϱ^{-1}.

Rewriting the system (4.1)-(4.2) and considering η to be given and not unknown, the matrix representation is

$$
\partial_t V + \sum_{i=1}^{3} \mathbb{B}_i \partial_{x_i} V + \mathbb{B}_4 \tag{4.8}
$$

where

$$
V := \begin{bmatrix} w \\ \mathbf{u}_1 \\ \mathbf{u}_2 \\ \mathbf{u}_3 \end{bmatrix},
$$

$$\mathbb{B}_1 := \begin{bmatrix} \mathbf{u}_1 & \overline{\sigma} + \frac{\gamma-1}{2}w & 0 & 0 \\ \overline{\sigma} + \frac{\gamma-1}{2}w & \mathbf{u}_1 & 0 & 0 \\ 0 & 0 & \mathbf{u}_1 & 0 \\ 0 & 0 & 0 & \mathbf{u}_1 \end{bmatrix},$$

$$\mathbb{B}_2 := \begin{bmatrix} \mathbf{u}_2 & 0 & \overline{\sigma} + \frac{\gamma-1}{2}w & 0 \\ 0 & \mathbf{u}_2 & 0 & 0 \\ \overline{\sigma} + \frac{\gamma-1}{2}w & 0 & \mathbf{u}_2 & 0 \\ 0 & 0 & 0 & \mathbf{u}_2 \end{bmatrix},$$

$$\mathbb{B}_3 := \begin{bmatrix} \mathbf{u}_3 & 0 & 0 & \overline{\sigma} + \frac{\gamma-1}{2}w \\ 0 & \mathbf{u}_3 & 0 & 0 \\ 0 & 0 & \mathbf{u}_3 & 0 \\ \overline{\sigma} + \frac{\gamma-1}{2}w & 0 & 0 & \mathbf{u}_3 \end{bmatrix},$$

and

$$\mathbb{B}_4 := \begin{bmatrix} 0 \\ (\beta + f(w)\eta)\partial_{x_1}\Phi + f(w)\partial_{x_1}\eta \\ (\beta + f(w)\eta)\partial_{x_2}\Phi + f(w)\partial_{x_2}\eta \\ (\beta + f(w)\eta)\partial_{x_3}\Phi + f(w)\partial_{x_3}\eta \end{bmatrix}.$$

Thus, if η is known and has high enough regularity, (4.8) is a symmetric hyperbolic system.

Consider the pseudo-symmetrized ES system

$$\partial_t w + \overline{\sigma}\,\mathrm{div}_x\,\mathbf{u} = -\mathbf{u}\cdot\nabla_x w - \frac{\gamma-1}{2} w\,\mathrm{div}_x\,\mathbf{u} \tag{4.9}$$

$$\partial_t \mathbf{u} + \overline{\sigma}\nabla_x w + f(w)\nabla_x\eta + [\beta + f(w)\eta]\nabla_x\Phi = -(\mathbf{u}\cdot\nabla_x)\mathbf{u} - \frac{\gamma-1}{2} w\nabla_x w \tag{4.10}$$

$$\partial_t\eta + \eta\,\mathrm{div}_x(\mathbf{u} - \nabla_x\Phi) + (\mathbf{u} - \nabla_x\Phi)\cdot\nabla_x\eta - \Delta_x\eta = 0. \tag{4.11}$$

To begin with, the following results follow from the calculations of the transformation above and some basic calculations.

Lemma 4.1.1. *For any $T > 0$, if ϱ, $\eta \in C^1([0,T]\times\mathbb{R}^3)$ and $\boldsymbol{u} \in C^1([0,T]\times\mathbb{R}^3;\mathbb{R}^3)$ solve* (4.1)-(4.3) *with $\varrho > 0$, then w, $\eta \in C^1([0,T]\times\mathbb{R}^3])$ and $\boldsymbol{u} \in C^1([0,T]\times\mathbb{R}^3;\mathbb{R}^3)$ solve* (4.9)-(4.11) *with $f(w) > 0$. Conversely, if w, $\eta \in C^1([0,T]\times\mathbb{R}^3)$ and $\boldsymbol{u} \in C^1([0,T]\times\mathbb{R}^3;\mathbb{R}^3)$ solve* (4.9)-(4.11) *with $f(w) > 0$, then ϱ, $\eta \in C^1([0,T]\times\mathbb{R}^3)$ and $\boldsymbol{u} \in C^1([0,T]\times\mathbb{R}^3;\mathbb{R}^3)$ solve* (4.1)-(4.3) *with $\varrho > 0$.*

And using the method of characteristics, the following lemma shows that if the initial fluid density is positive, then the fluid density remains positive provided the solution is uniformly bounded.

Lemma 4.1.2. *If ϱ, $\eta \in C^1([0,T]\times\mathbb{R}^3)$ and $\boldsymbol{u} \in C^1([0,T]\times\mathbb{R}^3;\mathbb{R}^3)$ solve* (4.1)-(4.3) *and are uniformly bounded, then $\varrho > 0$ on $[0,T]\times\mathbb{R}^3$ provided $\varrho_0 > 0$ on $\mathbb{R}^3$. Additionally, if w, $\eta \in C^1([0,T]\times\mathbb{R}^3)$ and $\boldsymbol{u} \in C^1([0,T]\times\mathbb{R}^3;\mathbb{R}^3)$ solve* (4.9)-(4.11) *and are uniformly bounded with $f(w_0) > 0$ on $\mathbb{R}^3$, then $f(w) > 0$ on $[0,T]\times\mathbb{R}^3$.*

As stated previously, (4.9)-(4.10) form a symmetric hyperbolic system in the unknowns w and $\mathbf{u}$.

Next, a sequence $\{w^k, \mathbf{u}^k, \eta^k\}$ of approximate solutions to the ES system is constructed. An approximate solution η^1 is found by using a solution to a heat equation. Then, η^1 will be substituted into (4.9)-(4.10) to obtain $\mathbf{u}^1$ and w^1. Then, $\mathbf{u}^1$ will be substituted into (4.11) to solve for η^2, which will be plugged into (4.9)-(4.10) to obtain w^2 and $\mathbf{u}^2$, and so on, continuing inductively. To begin, consider the Cauchy problem

$$\partial_t \mathbf{v} - \Delta_x \mathbf{v} = 0 \tag{4.12}$$

$$\mathbf{v}(x, 0) = \mathbf{u}_0.$$

In order to be able to use the theorem for local existence of the symmetric hyperbolic system (4.9)-(4.10), assume the following regularity on the initial data w_0 and $\mathbf{u}_0$:

$$w_0 \in W^{3,2}(\mathbb{R}^3)$$

$$\mathbf{u}_0 \in W^{3,2}(\mathbb{R}^3; \mathbb{R}^3).$$

Assume also that the support of w_0 and $\mathbf{u}_0$ is within a compact subset of $\mathbb{R}^3$. By the Sobolev embedding theorems, it is clear that

$$w_0 \in C^{1,1/2}(\mathbb{R}^3)$$

$$\mathbf{u}_0 \in C^{1,1/2}(\mathbb{R}^3; \mathbb{R}^3).$$

Because of this, if $\mathbf{u}^0$ solves (4.12), $\mathbf{u}^0 \in C^\infty(\mathbb{R}^3; \mathbb{R}^3)$ from basic properties of the heat equation with Neumann boundary conditions (see, for example, Chapter 2.3 in [26]). If $\Phi \in C^2(\mathbb{R}^3)$, which will be assumed throughout the rest of this chapter because this makes the coefficients in (4.11) continuous (using $\mathbf{u}^0$ for $\mathbf{u}$), using results in Chapter 7.1 of [26] yields a solution

$$\eta^1 \in C^1([0,T]; C^2(\mathbb{R}^3)). \tag{4.13}$$

Consider the approximation of the system (4.9)-(4.11):

$$\partial_t w^k + \overline{\sigma}\,\mathrm{div}_x\,\mathbf{u}^k = -\mathbf{u}^k \cdot \nabla_x w^k - \frac{\gamma-1}{2} w^k\,\mathrm{div}_x\,\mathbf{u}^k \tag{4.14}$$

$$\partial_t \mathbf{u}^k + \overline{\sigma}\nabla_x w^k + [\beta + f(w^k)\eta^k]\nabla_x \Phi = -(\mathbf{u}^k \cdot \nabla_x)\mathbf{u}^k - \frac{\gamma-1}{2} w^k \nabla_x w^k \tag{4.15}$$

$$\partial_t \eta^k + \eta^k\,\mathrm{div}_x(\mathbf{u}^{k-1} - \nabla_x\Phi) + (\mathbf{u}^{k-1} - \nabla_x\Phi)\cdot\nabla_x\eta^k - \Delta_x\eta^k = 0. \tag{4.16}$$

If η^k is known, adapting Theorem 2.1 from [40] used also in [38] and [48] on the existence of smooth solutions for local time of symmetric hyperbolic systems to this problem gives the following theorem.

Theorem 4.1.1 (Solutions for Symmetric Hyperbolic Systems)**.** *Let $w_0 \in W^{3,2}(\mathbb{R}^3)$ and $\boldsymbol{u}_0 \in W^{3,2}(\mathbb{R}^3;\mathbb{R}^3)$ with the support of w_0 and $\boldsymbol{u}_0$ contained in some compact subset K of $\mathbb{R}^3$. Assume also that $\eta^k \in C^1([0,T]; C^2(\mathbb{R}^3))$. Then there is a time interval $[0,T]$ with $T > 0$ such that there is a unique classical solution*

$$\begin{aligned} w^k &\in C([0,T]; W^{3,2}(\mathbb{R}^3)) \cap C^1([0,T]; W^{2,2}(\mathbb{R}^3)) \\ \boldsymbol{u}^k &\in C([0,T]; W^{3,2}(\mathbb{R}^3;\mathbb{R}^3)) \cap C^1([0,T]; W^{2,2}(\mathbb{R}^3;\mathbb{R}^3)). \end{aligned} \tag{4.17}$$

Further, T depends only on w_0, $\boldsymbol{u}_0$ and K.

Remark 4.1.1. Due to the Sobolev embedding theorems,

$$w^k \in C^1([0,T]; C^{0,1/2}(\mathbb{R}^3)) \cap C([0,T]; C^{1,1/2}(\mathbb{R}^3))$$

$$\mathbf{u}^k \in C^1([0,T]; C^{0,1/2}(\mathbb{R}^3;\mathbb{R}^3)) \cap C([0,T]; C^{1,1/2}(\mathbb{R}^3;\mathbb{R}^3))$$

Note that Theorem 4.1.1 implies that the maximal time of existence will be the same positive number T for each k in the sequence. Thus when taking the limit, there is no worry about the limiting maximal time of existence being zero. Second, due to the regularity of $\mathbf{u}^k$, it can be used in (4.16) to obtain $\eta^{k+1} \in C^1([0,T]; C^2(\mathbb{R}^3))$, which is then used with Theorem 4.1.1 to obtain w^{k+1} and $\mathbf{u}^{k+1}$, leading to the following theorem:

Theorem 4.1.2 (Existence of Approximate Smooth Solutions)**.** *Let*

$$w_0 \in W^{3,2}(\mathbb{R}^3)$$

$$\boldsymbol{u}_0 \in W^{3,2}(\mathbb{R}^3; \mathbb{R}^3)$$

$$\eta_0 \in W^{3,2}(\mathbb{R}^3)$$

all with support contained in some compact subset K of $\mathbb{R}^3$. Let

$$\boldsymbol{u}^0 \in C^1([0,T]; C^2(\mathbb{R}^3))$$

solve (4.12)*. Then there exists some $T > 0$ such that for all $k \in \mathbb{N}$, there exist solutions $\{w^k, \boldsymbol{u}^k, \eta^k\}$ of* (4.14)-(4.16) *such that*

$$w^k \in C([0,T]; W^{3,2}(\mathbb{R}^3)) \cap C^1([0,T]; W^{2,2}(\mathbb{R}^3))$$

$$\boldsymbol{u}^k \in C([0,T]; W^{3,2}(\mathbb{R}^3; \mathbb{R}^3)) \cap C^1([0,T]; W^{2,2}(\mathbb{R}^3; \mathbb{R}^3))$$

$$\eta^k \in C^1([0,T]; C^2(\mathbb{R}^3)).$$

Further, T depends only on w_0, $\boldsymbol{u}_0$ and K.

Proof. The proof follows from induction on k. Consider the case where $k = 1$. Defining $\mathbf{u}^0$ as the solution of the Cauchy problem of the heat equation with initial data $\mathbf{u}_0$, the existence of $\eta^1 \in C^1([0,T];C^2(\mathbb{R}^3))$ follows from the classical theory of parabolic equations. With the existence of η^1, (4.9)-(4.11) is a symmetric hyperbolic system in the unknowns w^1 and $\mathbf{u}^1$. The existence of

$$w^1 \in C([0,T];W^{3,2}(\mathbb{R}^3)) \cap C^1([0,T];W^{2,2}(\mathbb{R}^3))$$

and

$$\mathbf{u}^1 \in C([0,T];W^{3,2}(\mathbb{R}^3;\mathbb{R}^3)) \cap C^1([0,T];W^{2,2}(\mathbb{R}^3;\mathbb{R}^3))$$

follows from Theorem 4.1.1.

The argument to show the existence of w^{k+1}, $\mathbf{u}^{k+1}$, and η^{k+1} given the existence of w^k, $\mathbf{u}^k$, and η^k is identical to the argument for the $k = 1$ case because of the regularity of $\mathbf{u}^k$.

Also by Theorem 4.1.1, T depends only on the initial data and is therefore independent of $k \in \mathbb{N}$. $\square$

In order to pass through to the limit $k \to \infty$, a similar set of calculations used for the existence of local smooth solutions in Section 2.3.4 is employed, yielding the following theorem.

Theorem 4.1.3. *Let w_0, $\eta_0 \in W^{3,2}(\mathbb{R}^3)$ and $\boldsymbol{u}_0 \in W^{3,2}(\mathbb{R}^3;\mathbb{R}^3)$ all with support in some compact subset K of $\mathbb{R}^3$. Then there is some $T > 0$ such that there exists a solution $\{w, \boldsymbol{u}, \eta\}$ to the symmetrized ES system such that*

$$w \in C([0,T];W^{3,2}(\mathbb{R}^3)) \cap C^1([0,T];W^{2,2}(\mathbb{R}^3))$$

$$\boldsymbol{u} \in C([0,T];W^{3,2}(\mathbb{R}^3;\mathbb{R}^3)) \cap C^1([0,T];W^{2,2}(\mathbb{R}^3;\mathbb{R}^3))$$

$$\eta \in C([0,T];W^{3,2}(\mathbb{R}^3)) \cap C^1([0,T];W^{2,2}(\mathbb{R}^3)).$$

Remark 4.1.2. The reader will note that the result of Theorem 4.1.3 requires initial data with compact support. However, by using a background density, $\overline{\varrho}$, alluded to with the transformation to w, along with appropriate decay conditions on $\mathbf{u}$ and η, the result is able to be passed through to initial data that is positive on $\mathbb{R}^3$ by considering $\varrho - \overline{\varrho}$.

Chapter 5

Summary and Future Work

5.1 Summary

The work covered in this dissertation focuses on the analysis of two systems of partial differential equations modeling the interaction of a compressible fluid with particles in the bubbling regime. In particular, well-posedness results are presented for both the viscous case and the inviscid case. In addition, certain approximations to solutions in the viscous case are considered and their properties investigated.

5.1.1 Viscous Case–Well-Posedness

The well-posedness results for the viscous case can be summarized by three results. First, in Theorem 2.2.1, the existence of weakly-dissipative solutions is shown. These are weak solutions which obey a relative entropy inequality. This relative entropy inequality is then used for the second result, the weak-strong existence result of Theorem 2.2.2. This result states that if a suitably smooth solution exists for the given initial data, that it is unique among the weakly-dissipative solutions. These results are expanded to unbounded spatial domains in Theorems 2.2.3 and 2.2.4.

The third result on well-posedness for the viscous model is the result for the existence of smooth solutions locally in time from Theorem 2.3.1. This theorem states the regularity and compatibility conditions on the initial data that will ensure the existence of smooth solutions for some finite time. This results relies upon the showing the existence of solutions to the linear NSS system and then taking the limit.

5.1.2 Viscous Case–Singular Limits

The next set of results in the dissertation involves approximating solutions to the NSS model for compressible fluids. In this case, the low Mach number regime is considered, that is, when the speed of the fluid is much smaller than the speed of sound in the fluid. In the low stratification case, it is shown that the solutions can be approximated by solutions to a corresponding model for incompressible fluids (see Theorem 3.1.1). Secondly, the strong stratification case is explored. Here, instead of an incompressibility condition for the approximating model, an anelastic approximation is argued as valid (see Theorem 3.2.1).

5.1.3 Inviscid Case

After the work on the viscous model, attention is turned toward the inviscid model. This model is the same as the viscous model save for the fact that the viscosity coefficients μ and λ are taken to be zero. In this dissertation, Theorem 4.1.3 shows the existence of smooth solutions to the ES model for compressible fluids for finite time. This is done by transforming the system to a symmetric hyperbolic system coupled with the Smoluchowski equation.

5.2 Future Work

The topics of this dissertation lend themselves to continuing work. The first line of work continues the result on the existence of smooth solutions to the NSS system. First, the work here does not consider the case of unbounded domains. However, the result of Theorem 2.3.1 should be able to be extended to unbounded spatial domains Ω much like it is argued for heat-conducting flows in [19].

Next, and perhaps most glaring, is continuing the results of Theorems 2.3.1 and 4.1.3 to unbounded time domains. For the NSS system, examining the global-in-time existence will likely involve the development of blow-up conditions. In other words, it remains to find the quantities that either blow up in finite time, or if they do not, ensure the existence of global-in-time solutions. This would follow the methods of [18], [28], and [29] for compressible Navier-Stokes and heat conducting flows.

For the Euler system of fluid flow without a forcing term, a key feature is the blow-up of solutions in finite time for general initial data (see [16], [17], and [48]). Thus, as in [48] global-in-time existence of smooth solutions for the inviscid case will rely upon conditions on the external forcing term. One such proposal is the weak dissipation condition defined below.

Definition 5.2.1 (Weak Dissipation Condition)**.** The ES system (4.1)-(4.3) is said to obey a **weak dissipation condition** if and only if

$$-\int_{\mathbb{R}^3} (\beta\varrho + \eta)\nabla_x \Phi \cdot \mathbf{u} \leq 0$$

on $\mathbb{R}^3$ for all times t in the time domain.

Remark 5.2.1. Using Lemma 4.1.1, this definition is equivalent to requiring

$$-\int_{\mathbb{R}^3} (\beta + f(w)\eta)\nabla_x \Phi \cdot \mathbf{u} \, \mathrm{d}x \leq 0$$

for all t in the time domain.

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