AN EXPLORATION OF MATHEMATICAL KNOWLEDGE FOR TEACHING GEOMETRIC PROOFS PhD (Mathematics Education) Thesis Submitted to the Department of Curriculum and Teaching Studies, School of...

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AN EXPLORATION OF MATHEMATICAL KNOWLEDGE FOR TEACHING GEOMETRIC PROOFS

PhD (Mathematics Education) Thesis

By

LISNET ELIZABETH NAMPINGA MWADZAANGATI

MSc. (Applied Educational Research) -The University of Strathclyde

Submitted to the Department of Curriculum and Teaching Studies, School of Education, in fulfilment of the requirements for the degree of Doctor of Philosophy in Curriculum and Teaching Studies (Mathematics Education).

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DECLARATION

I, the undersigned, hereby declare that this thesis/dissertation is my own original work which has not been submitted to any other institution for similar purposes. Where other people's work has been used, acknowledgements have been made.

LISNET ELIZABETH NAMPINGA MWADZAANGATI

Full Legal Name

Signature

27th July, 2017

Date

Certificate of Approval/Certification

The undersigned certify that this thesis represents the student's own work and effort and has been submitted with our approval.

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Dedication

To my late father, Mr Shelbunny Hamilton Nampinga who sacrificed the little that he earned to ensure that I got better education. And to my late mother, Miss Elizabeth Chirombo who suffered a lot of humiliation for sending me, her only child to school instead of encouraging me to get married and give her grandchildren.

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ABSTRACT

Previous studies showed that secondary school students fail to understand geometric proof development because they are not offered effective learning experiences. However, the studies do not describe the knowledge required by teachers to conduct effective geometric proving lessons. This study explored mathematical knowledge for teaching geometric proofs (MKT-GP) with an aim of providing insights into its content knowledge (CK) and pedagogical content knowledge (PCK). The categories of cognitive activation (COACTIV) model by Baumert and Kunter (2013) were used as overarching theory to inform the study in formulation of research questions, data analysis and presentation of findings. Qualitative case study design was utilised to generate and analyse data. Data were generated from four secondary school teachers through pencil and paper tests, individual interviews, and lesson observations. Both deductive and inductive thematic analyses were conducted on the data to provide CK and PCK for teaching geometric proving. The study has proposed the following categories of CK for geometric proof development: Geometry content knowledge, geometric reasoning, geometric deductive reasoning, problem solving skills and algebraic reasoning. The study has also proposed several sub-categories of PCK relevant for good assessment of students' thinking, implementation of cognitively activating tasks, for explaining and representing geometric proofs. These include to knowledge of identifying causes of mistakes, knowledge of providing good guidance to students, knowledge of exploratory activities, knowledge of problem solving skills, and knowledge of good teaching and learning materials.

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LIST OF ACRONYMS AND ABBREVIATIONS

CCK:	Common Content Knowledge.
CK:	Content Knowledge.
COACTIV:	Cognitive Activation.
JCE:	Junior Certificate Examinations.
KAT:	Knowledge for Algebra Teaching.
KCT:	Knowledge of Content and Teaching.
MANEB:	Malawi National Examinations Board.
MCK:	Mathematical Content knowledge.
MKT:	Mathematical Knowledge for Teaching.
MKT-G:	Mathematical Knowledge for Teaching High School Geometry.
MKT-GP:	Mathematical Knowledge for Teaching Geometric Proofs.
MoEST:	Ministry of Education, Science and Technology.
MPCK:	Mathematical Pedagogical Content Knowledge.
MSCE:	Malawi School Certificate Examination.

NCTM: National Council of Teachers of Mathematics.

NESP: National Education Sector Plan.

PCK: Pedagogical Content Knowledge.

PK: Pedagogical Knowledge.

PME: Psychology of Mathematics Education.

QUANTUM: Quality mathematical education for teachers in South Africa.

SCK: Specialised Content Knowledge.

SMK: Subject Matter Knowledge.

TEDS-M: Teacher Education Development Study in Mathematics.

UK: United Kingdom.

USA: United States of America.

CHAPTER 1

INTRODUCTION TO THE STUDY

1.1. Chapter Overview

In this study, I explore the mathematical knowledge for teaching secondary school geometric proofs (MKT-GP). This chapter introduces the study by presenting the background of the study, the problem statement, the purpose of the study, the research questions, significance of the study, definition of the key words, and the thesis outline.

1.2. Background of the study

The Ministry of Education, Science and Technology (MoEST) justifies secondary Mathematics as a vehicle for developing and improving a person’s intellectual competence in logical reasoning, spatial visualisation, analysis and abstract thought (MoEST, 2013). As such, one of the objectives of teaching secondary school Mathematics is that by the end of secondary education, Malawian students should develop computational, reasoning, critical thinking and problem-solving skills through the learning of Mathematics. In Malawian Secondary School Mathematics Syllabus, Euclidean Geometry is the main area of Mathematics that offers students opportunities to achieve the rationale and objectives stated by MoEST (2013).

However, the reports from Malawi National Examinations Board (MANEB) Mathematics chief examiners indicate that Malawian secondary school students experience challenges to develop geometric proofs during national examinations (MANEB, 2013). After marking Junior Certificate Examinations (JCE) which are written by form two (grade 10) students and Malawi School Certificate Examination (MSCE) written by form four (grade 12) students, chief examiners write reports. The reports are based on analysis of the level of difficulty of each task on the examination paper, and how the students performed each task. The Chief Examiners reports for Mathematics for five consecutive years (from 2009 to 2013) indicate that students have been consistently performing poorly on tasks that required them to develop either narrative or computation geometric proofs. For example, the 2013 Mathematics examiner's report indicates that despite the tasks being fair, candidates faced challenges in answering Geometry questions that required proof development. This is expressed in the following extract from MANEB (2013) MSCE Mathematics Chief Examiners' report:

> Many candidates performed very well in questions on sets, matrices, and simple vectors in paper 1 and surds and speed time graphs in question paper 2. It was noted that students still lacked sound knowledge in Geometry application of concepts to real life situation as were the case with questions 6 and 8 for paper 1 and 3b, 8b for paper 2. Few candidates attempted such questions (MANEB, 2013; p. 6).

The tasks mentioned in the extract required candidates to apply properties of lines, angles, triangles and quadrilaterals to develop either narrative or computation proofs.

The following mistakes have been consistently reported on Geometry tasks by the Mathematics Chief Examiners in the reports from 2009 to 2013:

(i) Failure to name angles; for example naming angles using 2 letters, or using one capital letter, or using a wrong symbol like that of triangle (Δ abc) instead of angle (<abc or $a\hat{b}c$).

(ii) Justifying geometric statements with wrong reasons; for example writing that two angles are equal because they are alternate angles instead of corresponding angles.

(iii) Using alternate, allied and corresponding angle properties on lines which are not parallel.

(iv) Writing geometric statements without justifications.

(v) Using a wrong theorem when developing a geometric proof.

(vi) Failing to identify equal sides and angles when they are marked or labelled in a figure.

(vii) Developing a geometric proof using notations that are not shown in the diagram.

The mistakes indicated by the Chief Examiners imply that students fail to produce correct proving sentences because of lack of good interaction with geometric diagrams.

The Chief Examiners' reports generally assume that lack of teacher knowledge is the major factor that contributes to students' mistakes during geometric proof development. They explain that secondary school teachers do not teach geometric proofs effectively because they lack skills for explaining the proofs clearly to students (MANEB, 2013). This assumption concurs with the Malawi National Education

Sector Plan (NESP), which stresses that the education sector is facing challenges of limited human capacity in terms of capability and quantity in Mathematics and Science (Government of Malawi, 2008).

The problem of students' challenges in geometric proof developmentis reported by several scholars (Battista, 2007; Chinnappan, Ekanayake & Brown, 2012). Several reasons have been advanced for this problem. Usiskin (1982) argued that students do not succeed in secondary geometric proof because their coming in knowledge in Geometry course is poor. He claimed that the students come to learn secondary Geometry before they have reached the level of formal deduction that was proposed by van Hiele (1999). Jones (2002) noted three reasons for students' difficulties in learning to develop geometric proofs. Firstly, the learning of geometric proving is complex because it requires co-ordination of a range of competencies. Secondly, the teaching approaches used during geometric proving lessons tend to concentrate on verification and devalue, or omit exploration and explanation. Thirdly, learning to prove involves students making the difficult transition from computational geometric reasoning to abstract geometric reasoning. He argued that these reasons imply that teachers find it difficult to provide students with meaningful experiences to enable them understand geometric proof development.

Battista (2007) explained two reasons why geometric proof development continues to be challenging to students. Firstly, geometric proving involves geometric reasoning which mainly requires spatial reasoning. As such, students face challenges in using spatial reasoning that involves the ability to see, inspect, and reflect on spatial objects, images and relationships (Battista, 2007). This argument supports the initial claim of this study that Malawian students have trouble in developing geometric proofs because they fail to interact with geometric diagrams successfully. Secondly, researchers have paid little attention to the concept of proof and proof development in the study of Geometry. This implies that little is known concerning the effective teaching and learning of geometric proofs. Battista (2007) suggested that research on teaching of geometric proofs should focus on investigating of the knowledge components that enable students to understand and develop of proofs. Even (1990) contends that students fail to develop proofs because they are asked to memorise proofs without being convinced about why they are true. When students are taught the rules of developing geometric proofs without understanding, they are able to reproduce similar proofs but they cannot apply the principles to develop a different proof (Ding & Jones, 2009; Herbst, 2004).

This brief review has shown that researchers appreciate that geometric proof development is a complex domain, hence its teaching and learning continues to be challenging. The review also shows that the remedy for this challenge lies in teachers' ability to provide students with meaningful experiences for understanding geometric proof development. I argue that the experiences that teachers are able to provide to their students are determined by their MKT. The quality of teachers' mathematical practices is directly related to their MKT (Hill, Rowan & Ball, 2005; Stylianides & Ball, 2008). This implies that teachers need to know both geometric proof CK and PCK to be able to involve students in meaningful experiences for understanding geometric proof development. However, while the studies have provided reasons for students' challenges in learning of geometric proof development, they have not elaborated on MKT required for addressing such challenges. This study therefore, seeks to fill the gap by exploring MKT-GP.

1.3. Problem statement

Competence to reason and prove in Mathematics involves ability to reason coherently and systematically, to identify valid mathematical arguments and to establish a correct mathematical proof (Heinze & Reiss, 2007). Since secondary school Geometry is rich in proofs (Jones, 2002), the students' inability to develop geometric proofs imply that they exit secondary education without achieving the objective of developing competence in mathematical reasoning. The problem of students' difficulties in acquiring competency in geometric proving is attributed to the nature of the domain and to teaching practices. As a domain, geometric proof development is regarded as an area that requires a lot of persistence because students have to recall and apply several previously learnt geometric concepts, as well as to show deductive reasoning (Chinnappan et al., 2012). In relation to teaching practice, the studies show that teaching strategies used in classrooms during geometric proof development do not provide students with powerful opportunities to understand geometric proofs (Herbst et al., 2009; Jones et al., 2009). Although many frameworks have been developed for understanding MKT, the problem of students' challenges to develop geometric proofs persist.

In literature review on studies that have been conducted on MKT, Hoover, Mosvold, Ball and Lai (2016) found that although there is a rising interest on the mathematical knowledge that is specific to teaching, there is lack of theoretically grounded, well-defined and shared conception of MKT. They, therefore argue that, "mathematical knowledge for teaching needs to be elaborated for specific mathematical topics and tasks of teaching, across educational levels," (p. 17).

Kaarstein (2014) analysed the differences and similarities among three frameworks that build on Shulman's (1986) work and putsemphasis on CK and PCK. The frameworks that were compared are MKT framework by Ball, Thames and Phelps (2008), COACTIV framework by Krauss, Neubrand, Blum and Baumert (2008), and Knowledge for teaching Mathematics framework by Tatoo et al. (2008). Kaarstein (2014) found that all these three frameworks divided PCK into three sub-categories in which all the elements of Shulman's PCK are present. But the names of the sub-categories differed because they originated from three projects with different contexts, different expert members and different aims and goals. The differences in the names of the sub-categories implied that the operationalisation of the sub-categories was going to be different (Kaarstein, 2014).

The study by Kaarstein reveals that the basic level categories of PCK are understood and operationalised differently. Similar findings were also reported by Depiepe, Vershaffel and Kelchtermans (2013) in their review of how PCK was conceptualised and (empirically) studied in Mathematics Education. They reviewed 60 articles from different databases and found that researchers agreed on the four general characteristics of PCK. Firstly, they agreed that PCK connects at least two forms of teacher knowledge which are CK and PK. Secondly they agreed that PCK is a form of teachers' knowledge which is concerned with making content comprehensible to students. Thirdly, they agreed that PCK is specific to a particular subject. Fourthly, they agree that CK is the prerequisite of PCK.

Despite the agreements on general characteristics of PCK, there was disagreement on the components covered by PCK (Depaepe et al., 2013). The authors found that there were about eight different conceptualisations of PCK that in turn had some degree of influence on the methods used in the study of PCK. Regarding the mathematical domain in which PCK is studied, the authors found that most of the studies were mainly conducted among elementary school and secondary school pre-service teachers. Mathematics topics which were popularly studied included fractions at elementary level, and algebra and functions at secondary school level (Depaepe et al., 2013). This review shows that although geometric proving is regarded as a difficult mathematical domain it has not received much attention by scholars who studied MKT.

In another review on frameworks developed for understanding MKT, Scheiner (2015) found that the current generic frameworks for Mathematics teachers' knowledge portray differences of opinion and lack of clarity about the nature of teachers' knowledge. Scheiner (2015), therefore, suggested that research on knowledge for teaching should focus on specific mathematical concepts. These suggestions agree with Battista's (2007) view that although there are general processes applicable to most mathematical proofs, success on most geometric proofs depends on understanding specific concepts and situations. These studies suggest that improving teaching of geometric proof development lies in understanding MKT specific for the teaching of geometric proving. As such, the problem that this study addresses is; what mathematical knowledge do teachers need in order to teach geometric proof development in a comprehensive manner.

The problem addressed in this study is concerned with proposing a framework for understanding MKT geometric proof development. To be able to do this, I examined the knowledge demands of geometric proof development and its teaching. In that case, I had to examine knowledge that the teachers used when developing and teaching geometric proofs.

1.4. Purpose of the study

The aim of the present study was to explore MKT-GP by analysing different types of data that were generated from secondary school in-service teachers through pencil and paper tests, interviews and lesson observations. The study was guided by the Cognitively Activating (COACTIV) model which was developed under Professional Competence of Teachers, Cognitively Activating Instruction, and Development of Students' Mathematical Literacy project (Baumert & Kunter, 2010). The COACTIV model guided development of the research questions, analysis of some of the data, and presenting the findings of the study. I am in agreement with the model's assumption that teacher's professional knowledge creates a better environment for students' learning of Mathematics. Considering that Mathematics is multi-domain and that the work involved in the teaching of each domain is different, I used the COACTIV model to inform my study in exploring the professional knowledge specific for teaching geometric proofs (Battista, 2007; Scheiner, 2015).

I used the philosophical underpinnings of social constructivism theory of Mathematics Education as a guide for the research methodology of the study (Ernest, 1994). As a researcher, I view teaching and learning of Mathematics as socially constructed. This also implies that teachers construct their own realities about their work of teaching Mathematics. Therefore, it makes sense that the study about the teaching and learning of Mathematics should be guided by social theoretical perspectives. To be able to understand the teachers' conceptualisations and actions in relation to geometric proof development, I used qualitative case study design as my research methodology for the study.

1.5 Research questions

This qualitative case study aimed at answering four research questions. The COACTIV model guided the development of the research questions. Each research question addressed a specific category of the COACTIV model. The research questions guided data collection and data analysis for the study. The questions are as follows:

1) How do secondary school teachers conceptualise geometric proof development and its teaching?
2) How do secondary school teachers select and implement tasks during the teaching and learning of geometric proof development?
3) How do secondary school teachers assess students' thinking in geometric proving?
4) How are geometric proofs explained and represented to secondary school students?

1.6. Significance of the study

Despite being difficult to teach and learn, Proof is one of the most important aspects of school Mathematics (Kim & Ju, 2012). Herbst et al. (2009) claimed that the reason for teaching students to develop proofs is to provide them with a valuable tool for mathematical problem solving. This is done by developing students' capacity and disposition to infer necessary conclusions from the given possibilities. This study sheds some light on what Malawian teachers need to know in order to teach geometric proofs effectively. The study also seeks to fill the gap in research on MKT-GP. Specifically, the study suggests some categories of geometric proof content knowledge (CK) and pedagogical content knowledge (PCK).

Usiskin, Peressini, Marchisotto and Stanley (2003) explained how the power of deduction is illustrated in Geometry. They noted that first the power of deduction is illustrated whenever we deduce a property held by all members of an infinite set. For example, the proof that some of interior angles of a triangle is 360^0 applies to any type of triangle and everywhere on earth, moon, sun as well as triangles in chemical bonds (Usiskin et al., 2003). Thus the number of triangles that one can draw is infinite but the sum of interior angles of each triangle is the same. The second aspect of the power of deduction is that theorems may be proved whose truth is hard to believe (Usiskin et al., 2003). For example, the authors explained that it is obvious to believe that base angles of isosceles triangle are congruent but it would not be easy to believe the Pythagoras Theorem unless you see its proof. Thirdly, deduction does not only show that the statement is true but it also justifies why the statement is true (Usiskin et al., 2003). These authors maintain that the logical linguistic structure of the proof provides a basis for one to believe why the statement is true. Fourthly, they hold that deduction provides a universally accepted criterion for the establishment of mathematical truth. This is because when a new theorem is discovered, it needs to be checked against universally agreed upon mathematical principles.

The power of deductions as explained by Usiskin et al. (2003) implies that geometric proving provide students opportunities to develop logical thinking and discovery skills that are not only necessary in Mathematics but also in real life situations. This agrees with Battista's (2007) observation that deductive reasoning promotes logical thinking because its ability to justify an argument enhances critical thinking. These arguments agree with the rationale for teaching Mathematics in Malawi which is to improve students' competence in logical reasoning and critical thinking. The study on MKT-GP can help to enhance students' logical reasoning and critical thinking by providing insights on how teachers can provide effective experiences for students to understand proof development. As pointed out in the previous sections, the role of teachers has emerged as a key influence in students' achievement in many studies on geometric proof development (Ding, Fujita & Jones, 2005).

The fact that MKT is a key element of their effectiveness has been agreed upon by many prominent scholars, mathematicians and policy makers (Hill, et al., 2005; Shulman, 1986). As Jones (2000) pointed out, Mathematics teachers need to have a deep understanding of the Geometry that is appropriate for school Mathematics if they are going to teach it well. As a teacher educator, I will use the findings from this study to inform myself and other teacher educators how to educate teachers so that they can be equipped with good knowledge and skills for teaching geometric proof development. The study will, therefore, help to improve the quality of teaching geometric proofs in Malawian schools. This means that as a teacher educator and a researcher, I am the right person to intervene than any other in the Ministry of Education in Malawi. Skemp (1986) claimed that teaching can better be studied by an educator and not anyone else. He emphasised that a person who intervenes in educational issues without an adequate mental image of what is going on inside is likely to do harm than good.

Furthermore, most of the research on MKT has focused on elementary Mathematics, and more recently, middle school Mathematics pre-service teachers (Ball, et al., 2008; McCrory, 2012). Very little has been done in researching in-service secondary teachers' knowledge (McCrory, 2012). As such, this study will help to fill the gaps of secondary Mathematics teacher knowledge.

1.7. Definition of key words

1.7.1. Geometric proof development

Cheng and Lin (2009) distinguished two types of geometric tasks, namely narrative and computation tasks. The narrative task asks the students to produce a formal proof using given conditions. The written form of a narrative task mainly includes geometric statements and their reasons. The computation task asks students to use given conditions to calculate sizes of angles or lengths of lines using geometric arguments. In this study, the proof that is produced from a narrative task will be called a narrative proof. Since the mode of argumentation for both narrative and computation tasks is the same, then the solution that is produced from the computation task will be called a computation proof. As such, geometric proof development means use of given conditions to develop logical sequenced geometric statements for justifying a conclusion.

1.7.2. Mathematical knowledge for teaching

Ball et al. (2008) consider mathematical knowledge for teaching (MKT) as, "the mathematical knowledge needed to perform the recurrent tasks of teaching Mathematics to students," (p. 399). In this study, MKT-GP means mathematical knowledge needed by teachers to perform meaningful classroom activities for effective teaching of geometric proof development.

1.8. Thesis outline

The thesis is presented in six chapters. Chapter 1 provides the background to the study. Chapter 2 presents a review of some relevant literature related to MKT. The chapter also presents a review of some theoretical frameworks related to teacher knowledge and MKT, geometric reasoning and mathematical problem solving. The theoretical frameworks used for various purposes in this study have also been presented in chapter two. Chapter 3 presents the research methodology that was used in this study. This includes the epistemological perspectives of the study, the research design, issues of trustworthiness of the work, the details of the participants, data collection methods, data analysis procedures, and ethical issues that were considered during the study.

Chapter 4 presents the findings of the study in relation to each research question. The findings are presented in relation to the theoretical framework that was used for analysing each set of data. Chapter 5 presents a discussion of the findings in relation to various studies on teacher knowledge and geometric proof development. The findings have been classified into different categories of teacher knowledge for teaching geometric proofs. Chapter 6 presents a summary of the major findings and the implications of the study. The chapter also presents the limitations of the study and concluding remarks.

CHAPTER 2

LITERATURE REVIEW AND THEORETICAL FRAMEWORKS

2.1. Chapter overview

In this chapter, I present a review of literature relevant to MKT. Since different researchers from different places have studied MKT, it is necessary to start the chapter by presenting a review of studies on MKT from different parts of the world. This is followed by a review of studies on MKT that have been done in Southern Africa for purposes of contextualising the study. Then I present a review on MKT studies related to teaching of mathematical proofs in general, and those specifically related to the teaching of geometric proofs. Lastly, I review different theoretical frameworks; some of these were used for different purposes like development of research questions and data analysis.

2.2. Studies on MKT from different parts of the world

The idea of teacher knowledge was introduced by Shulman (1986) in his seminal presentation on findings of the research programme which aimed at exploring knowledge issues related to teacher development and teacher education. The presentation was based on findings from review of tests which were used for examining teachers in America for over a hundred years (1875-1980s). He found that all tests that were used between 1875 and 1975 were dominated by subject matter with very few questions on theory and practice of teaching. But there was a reverse in focus on the tests that were used between the late 1970s to early 1985 in the sense that they were dominated by teaching pedagogy and had few questions which focused on subject matter. Shulman (1986) explained that the absence of focus on subject matter in the tests meant that little attention was being paid to the importance of content in teaching. He argued that content knowledge is equally important as pedagogical knowledge. Hence, he proposed that teacher effectiveness should be viewed as a combination of content knowledge and pedagogical knowledge.

Several researchers have conducted studies on Mathematics Education in responding to Shulman's (1986) notion of teacher knowledge. Ball and her colleagues studied MKT for about two decades in two projects which run concurrently. The first project focused on developing a MKT theoretical framework for understanding the work that teachers do in teaching Mathematics, and the second project focused on constructing instruments for measuring the different categories of teachers' MKT (Ball et al., 2008). They defined MKT as the mathematical knowledge needed to carry out the work of teaching Mathematics (Ball, et al., 2008). Their argument was that teaching has to be seen as a form of mathematical work with several aspects, and that every aspect of Mathematics teaching involves mathematical problem solving (Ball, Bass & Hill, 2004). The MKT framework developed by Ball et al. (2008) comprises of two of the categories of knowledge suggested by Shulman and his colleagues: PCK and CK. They represented the MKT framework with an egg shape in which the CK and PCK categories of teacher knowledge are further divided into three sub-categories. Each of the subcategories of MKT framework is discussed in Section 2.4.2.

Other studies on MKT aimed at establishing evidence of whether teachers' mathematical knowledge influences students' achievement. Hill, Rowan and Ball (2005) conducted a large scale survey in which they followed teachers' and learners' achievements over a period of time from first to third grade. Their studies found that teachers' mathematical knowledge is directly related to students' mathematical achievements. This means that teachers with high MKT score provide better instructional experiences for their students. They therefore suggested that improvements on teachers' mathematical knowledge can result into improvements in students' mathematical achievements.

Hill et al. (2008) observed that early studies on MKT took two approaches; the deficit approach and the affordance approach. The studies which used the deficit approach aimed at establishing links between a teacher's lack of mathematical understanding and patterns in his/her mathematics instruction. The studies which used the affordance approach aimed at highlighting MKT that was useful during instruction. Hill et al. (2008) explained that although these studies have generated more knowledge, they have several shortcomings. The studies are fine grained in the sense that they are qualitative studies which mainly focused on one topic in the context of only one lesson. The studies also analysed the relationships within one teacher, and they have not linked teacher knowledge to students' achievement. As such the aim of their studies was to continue establishing links between MKT and instruction.

Hill et al. (2008) used mixed methods of data collection in USA to examine the link between teacher's MKT and quality of instructional practice. Their studies found that there is a strong, significant and positive association between MKT and teachers' mathematical quality of instruction. They explained that strong MKT helps teachers to play different roles in their teaching of the subject appropriately. These include providing explanations for thinking and general conceptual discussion of procedures, appropriate selection and sequencing of mathematical tasks, use of multiple representations, making connections between informal and formal mathematical ways, good use of mathematical language, and several others.

Another MKT study concerning teaching of secondary school Mathematics was conducted using COACTIV model (Krauss et al., 2008) The components of COACTIV were Shulman's (1986) teacher CK and PCK teacher knowledge categories. The details of CK and PCK categories of the COACTIV model that were used to develop the tests are discussed in Section 2.7.2. The CK test was developed using the secondary school Mathematics curriculum, and the PCK test was developed from its subcategories (Krauss et al., 2008). After analysing the tests that were administered to secondary school teachers, they found that there was a high correlation between CK and PCK. Krauss et al. (2008) concluded that CK supports the development of PCK.

This review shows that the studies on MKT took different approaches. The common aspect of these studies was that they studied MKT with respect to whole Mathematics curriculum. This implied that the researchers assumed that it is possible to generalise MKT for all primary and secondary school mathematical fields.

2.3. Studies on MKT in Southern Africa

There is little research on Mathematics teacher knowledge that has been conducted in Africa. Lin and Rowland (2016) were commissioned to review Psychology of Mathematics Education (PME) studies related to pre-service and in-service teachers' knowledge and teaching development. The aim of the review was to find out how much research had been conducted in the field from 2006 to 2016. The review focused on several Mathematics teacher education issues including teacher knowledge, teacher beliefs, teacher education, educator education, professional development, and professional growth. Lin and Rowland (2016) found that there were two hundred and twenty papers submitted to PME with a focus on Mathematics teacher knowledge. They also found that most of the research on Mathematics teacher knowledge was conducted in USA and Europe and there was only one paper that originated from Africa. The authors argued these finding simply that Shulman's (1986) powerful categories of teacher knowledge have not yet been conceptualised in African countries.

The largest studies on mathematical work for teaching in Africa were conducted in South Africa. The studies were conducted by Adler and her colleagues under the project called quality mathematical education for teachers (QUANTUM). One of the goals of the QUANTUM project was to elaborate on MKT. As part of the project, Adler (2005) studied how knowledge is being assessed in Mathematics teacher education programmes in South Africa. Her justification for the study on Mathematics Education was that there were fewer people taking up advanced study of Mathematics (Adler, 2005). She viewed this as a threat to the development of the discipline of Mathematics itself as well as to the provision of scientists, engineers and mathematically well qualified teachers for the South African schools.

The study focused on the kind and quality of Mathematics being taught in South African secondary schools. Adler wanted to find out if the notion of unpacking is being valued as part of mathematical competence needed by teachers. She identified all teacher education courses and analysed their formal assessment tasks. The results indicated that the Mathematics in the teacher education course was highly compressed. This implied that teachers did not have an opportunity to learn to unpack Mathematics.

Analysis of assessment tasks in Geometry course showed that most tasks demanded unpacking of Mathematics and they did not reflect the concept of teaching. She argued that the absence of unpacking in the courses is a clear indication that this is a challenging mathematical work. She suggested that there is need for further detailed study of the actual teaching practice to improve understanding of unpacking. Although the study depicts unpacking of Mathematics as general mathematical work, the later studies in relation to the QUANTUM project were conducted on specific topics.

As part of the QUANTUM project, Adler and Davis (2006) studied mathematical practices revealed in formal assessments across a range of Mathematics teacher education courses in South Africa. The authors assumed that there is specificity to the Mathematics that teachers need to know and know how to use. They view unpacking of mathematical ideas as the most important element of knowledge for the work of teaching Mathematics. Their analysis was based on assessment tasks. One of their aims was to find out the mathematical knowledge as well as the teaching practices embedded in the assessment tasks. Thus the questions that guided the analysis of assessment tasks focused on the primary and secondary objects of the content (whether Mathematics and/or teaching) and the mathematical knowledge revealed by the tasks. For the tasks that revealed both Mathematics and teaching objects, Adler and Davis (2006) were also interested in determining the object which was prioritised in a particular assessment task. The study findings showed that despite being specifically designed for teachers, Mathematics Education courses were dominated by mathematical knowledge which involved the ability to demonstrate mastery of mathematical concepts and their procedures. Adler & Davis (2006) characterised the courses as compressed and unelaborated Mathematics which does not demand any display of understanding. Furthermore, the study findings showed that most of the tasks that were presented in a compressed form required some sort of unpacking. They argued that the prevalence of compressed mathematical tasks in the assessment is a sign of lack of enough knowledge of Mathematics for teaching. They suggested that further studies be conducted to elaborate the idea of Mathematics for teaching.

In further exploration of the idea of Mathematics for teaching under QUANTUM project, Kazima, Pillay and Adler (2008) investigated mathematical work demanded by teachers during teaching of probability and functions. They used the eight aspects of problem solving suggested by Ball et al. (2004) as a theoretical framework for their study. Kazima et al. (2008) condensed the eight aspects into the following six categories of problem solving; definitions, explanations, representations, working with learners' ideas, restructuring tasks and questioning. Their study mainly focused on the mathematical work of teaching that is performed by teachers when introducing and explaining probability and function concepts, and the resources used.

They observed and analysed lessons by two teachers, one was teaching probability and the other was teaching functions in South African secondary schools. The study findings showed that the teachers used different mathematical approaches in introducing concepts, ideas or procedures to the learners. This implied that the teachers dealt with different mathematical work. The results also indicated that the teachers differed in their areas of focus and emphasis during the lessons. The teacher who was teaching probability captured all categories of mathematical work in different degrees while the one who was teaching functions captured only some of the categories. Kazima et al. (2008) argued that the differences in approaches used by the teachers implied that different topics demand different mathematical work. In conclusion, they argued that the tasks of problem solving suggested by Ball et al. (2004) have specific meanings across mathematical topics and teaching approaches. They suggested, therefore, that research on Mathematics for teaching should be investigated further and be specific to mathematical topics.

Another study which was part of the QUANTUM project was conducted by Adler (2010). She used Shulman's (1986) theoretical framework of teacher knowledge to further explore the notion of Mathematics for teaching and its importance in teaching and learning of Mathematics, and in teacher education. She argued that, "strengthening our understanding of the mathematical work of teaching, what some refer to as Mathematics for teaching, is a critical dimension of enhancing its teaching and learning," (Adler, 2010, p. 1). She justified this argument using two examples of Geometry lessons from school Mathematics classrooms under the QUANTUM project. In the first example, a teacher gave students a task with an aim of developing angle properties of a triangle. In the second example, a teacher asked students to work in groups to find number of diagonals in a 700-sided polygon. In both examples, Adler was interested in illustrating and illuminating four components of mathematical work; designing the task, mediating learner progress, valuing and evaluating learner responses, and managing the integration of mathematical content and mathematical processes as foci in the lesson.

The results of the study indicated that the tasks constructed by the two teachers were not similar to the usual tasks found in South African Mathematics textbooks. This meant that there was a mathematical work that was involved during development of the tasks. Adler (2010) noticed that mathematical reasoning was in focus in both examples and that the learners developed multiple solution paths for each task. This implied that the teachers were required to analyse the learners' responses and do mathematical judgments. In conclusion, the author argued that Mathematics for teaching matters, especially that of designing and mediating a task. She, therefore, suggested that educators need to embrace deeper understanding of the complexities of teaching, and know their work in teacher education (Adler, 2010).

Furthermore as part of the QUANTUM project, Bowie (2013) analysed a grade 10 South African textbook chapter on quadrilateral to find out how it managed tensions inherent in transition between informal and formal Geometry. She found that in some instances, the book used tightly prescribed investigations, generalisations and definitions to manage the transition. The author argued that these approaches offered very little help in developing geometric reasoning (Bowie, 2013). This implied that the textbooks struggled with the transition from informal to formal Geometry.

The studies by Adler and her colleagues under the QUANTUM project agree with Ball et al. (2004) that teaching of Mathematics involves carrying out some mathematical tasks. The studies have shed more light on what is involved in several tasks of teaching Mathematics in relation to different topics. Regarding teaching of geometric proofs, the study by Adler (2010) has clarified two tasks; designing tasks and evaluating learner responses, but they have not clarified on mathematical knowledge required for carrying out the tasks. Ball et al. (2004) explained that tasks of teaching Mathematics depend on mathematical knowledge. This means that teachers are only able to carry out the tasks of teaching geometric proof development if they have the relevant mathematical knowledge. As such, my study was necessary to understand knowledge involved in carrying out various tasks for teaching geometric proofs.

Still in South Africa, MacKay (2011) investigated students’ performance on computational type and proof type problems. The study was done by asking students to answer two sets of test which comprised matched items on geometric computation and narrative proof. The test items were matched because students were supposed to draw from same type of geometric propositions when answering both sets of questions. The findings indicated that although both types of problems drew on same propositions, pupils exhibited a greater facility with techniques for computation proofs than making deductive arguments to develop a narrative proof.

Furthermore, MacKay (2013) examined the difficulties that students faced when solving one of the geometric tasks that were administered in an earlier study. The study sought to find out what students did more carefully when solving matched computation-type and narrative-type task. The students were supposed to relate several properties of geometric theorems to answer both tasks. The findings showed that the students were able to use their visualisations to come up with correct theorems to be used in both questions. In the computation-type task, students applied the proof correctly to find the value of an angle. But in the narrative-type task, students were only able to use visualisations to come up with several geometric statements but were unable to connect the sentences in a coherent manner to develop a deductive geometric proof. The findings agree with the worldwide claim that students face challenges in developing geometric proofs.

Jakobsen and Mosvold (2015) conducted a review of empirical research on MKT in Africa. The review was done with an aim of shedding light on what has already been investigated on MKT, and how MKT has been approached in African countries. There are several issues highlighted in the review in relation to amount of studies done, the methodology used and the target groups. Firstly, they pointed out that there are very few studies (only 7) conducted on MKT in Africa. Five studies were conducted in South Africa, one study in South Africa and Botswana, and one in Mozambique. Secondly, most of the studies were qualitative, hence did not use any measures of MKT. Thirdly, the studies were mainly conducted with in-service secondary school teachers. The authors therefore suggest that more studies should be done in different African countries using existing measures of MKT, especially in pre-service and on primary teachers.

The review by Jakobsen and Mosvold (2015) concur with Lin and Rowland's (2016) observation that there is limited research on MKT that has been conducted in Africa. This suggests that the work that is involved in the teaching of Mathematics is not clearly understood in most of the African countries. Although Jacobsen and Mosvold (2015) have emphasised that research on MKT should focus on measuring MKT using existing measures, it might also be necessary to focus on understanding MKT for African countries. This is because the contexts in which the MKT frameworks and its measures were developed are different from the African context. As such, there might be some issues and interest concerning African countries that do not concern the countries in which the frameworks were developed and vice versa. So, to successfully adapt the existing MKT measures to suit a particular African country, researchers might need to begin from a framework that is sensitive to some contextual issues.

2.4. Studies on MKT of mathematical proofs

Harel and Sowder (2007) defined a mathematical proof as an argument that aims at convincing people that something is true. They explained that proving takes two processes; ascertaining and persuading. Ascertaining is a process of removing one's or the community's doubt about the truth of an assertion, while persuading is a process that one or the community takes in order to remove other's doubt about the truth of an assertion. Harel and Sowder (2007) pointed that the persuading process emerges as a response to cognitive-social need, rather than only exclusive to either cognitive or social need. Their definition limits the function of proof to removal of doubt by verifying the truth of statements. However, de Villiers (1999) claimed that mathematical proof has other functions apart from verification. He provided six functions of proof as follows:

(i) Verification (concerned with the truth of a statement)

(ii) Explanation (providing insight into why it is true)

(iii) Systematisation (the organisation of various results into a deductive system of axioms, major concepts and theorems)

(iv) Discovery (the discovery or invention of new results)

(v) Communication (the transmission of mathematical knowledge)

(vi) Intellectual challenge (the self-realisation/fulfilment derived from constructing a proof) (de Villiers, 1999, p. 3).

de Villiers (1999) explained that although the functions are distinguishable from one another, they are interwoven in specific cases. In some cases, certain functions dominate others, and in other cases, certain functions may not feature at all. de Villiers (1999) suggested that the first five functions of proof need to be introduced earlier to students, but the last function which is of self-realisation might be attained in a later stage perhaps at tertiary level after some interesting experiences with proving. Stylianides (2009) clarified four purposes of a mathematical proof which partially overlap de Villiers' six purposes of proofs:

(i) Explanation, when the proof provides insight into why a claim is true or false.

(ii) Verification, when it establishes the truth of a given claim.

(iii) Falsification, when it establishes the falseness of a given claim.

(iv) Generation of new knowledge, when it contributes to the development of new results used to describe products that solvers in a particular community add to their knowledge base as a result of constructing a proof (Stylianides, 2009, p. 269).

Stylianides (2009) argued that apart from confirming truth or falsifying a statement, mathematical proofs also have to contribute to the construction of new knowledge. This implies that the importance of the connection between development of new mathematical knowledge and proofs should be emphasised during teaching of mathematical proofs.

Stylianides and Ball (2008) defined a mathematical proof as a mathematical argument that fulfills three criteria:

(i) It uses statements accepted by the classroom community (set of accepted statements) that are true and available without further justification;

(ii) It employs forms of reasoning (modes of argumentation) that are valid and known to, or within the conceptual reach of, the classroom community; and

(iii) It is communicated with forms of expression (modes of argument representation) that are appropriate and known to, or within the conceptual reach of, the classroom community (Stylianides and Ball, 2008, pp. 309).

Stylianides and Ball's (2008) definition of proof breaks each mathematical argument into three major components: the set of accepted statements, the modes of argumentation, and the modes of argument representation. They explained that the terms 'true,' 'valid,' and 'appropriate' are concerned with Mathematics as a field, while the terms 'what is accepted,' 'known,' or 'conceptually accessible to a classroom community at a given time', are concerned with students as mathematical learners (p. 309). This implies that the authors' definition of a mathematical proof also takes a social view of leaning of proofs.

Stylianides and stylianides (2006) developed a framework of content knowledge for teaching mathematical proofs to students of elementary school Mathematics. The framework is structured around the connection of four ideas; patterns, conjectures, arguments, and proof. They argued that patterns give rise to conjectures that motivate the development of arguments that may or may not qualify as proofs (Stylianides & Stylianides, 2006). In their further studies on knowledge for teaching mathematical proofs, Stylianides and Ball (2008) investigated mathematical knowledge for engaging students in the activity of proving. They claimed that in addition to knowledge of logical-linguistic aspects of proof, teachers also need knowledge of situations for proving. Their study identified two sub-components of knowledge of situations for proving: knowledge of different kinds of proving tasks and knowledge of the relationship between proving tasks and proving activity. They suggested that teachers must develop different proving tasks to provide students with opportunities for understanding of different proving strategies and reasoning skills, and to discuss the difference between empirical arguments and proofs.

Stylianides (2011) asserted that it is unclear how discussion of the difference between an empirical argument and proof can be organised to help students to overcome their deeply rooted misconception that empirical arguments are proofs. He, therefore, proposed a comprehensive knowledge package that is necessary for teaching mathematical proof effectively. The knowledge package emphasises the importance of teachers' subject-matter knowledge about proofs, pedagogical content knowledge about students' understanding of mathematical proofs, and pedagogical knowledge for teaching proofs in classrooms. He suggested that teachers need to have good understanding of the distinction between an empirical argument and a formal proof to overcome the students' misconceptions about proofs.

Students' difficulties in development of a mathematical proof are reported in many countries. Heinze (2004) conducted a large-scale quantitative national survey on proof and argumentation with 669 grade 8 students in Germany. He also interviewed ten of the students. The results of the study showed that there are three main students' difficulties in proof and argumentation: (a) insufficient knowledge of facts; (b) deficits in methodological knowledge about mathematical proofs; and (c) lack of knowledge with respect to developing and implementing a proof strategy. He concluded that the students' difficulties are influenced by classroom factors. To find out these factors, Heinze and Reiss (2004) analysed classroom instruction using 20-videotaped lessons. They examined steps that were either emphasised or underemphasised during teaching of mathematical proofs. The results of the study showed that the teachers neglected essential phases in proof process. This is because the teachers provided hints for all critical phases of proof process like generation of arguments, and elaborated the proof process step by step. They argued that the teachers' practices prevented students from exploring the problem situation and providing additional information.

Jones (2000) studied university Mathematics Education students' conceptions of proof. He found that most of the students were unable to understand mathematical proofs even upon completion of undergraduate Mathematics and initial teacher education course. This implied that the Mathematics Education students entered and exited teacher education without good understanding of mathematical proofs. He suggested that the lack of understanding of mathematical proofs by Mathematics Education students has an effect on the teaching of proof in high school. He argued that the findings of the study imply that students continue to face challenges in mathematical proofs because the problem lies in the Mathematics Education system. The students who are not taught well mathematical proofs are the ones who become teachers of mathematical proofs. As such the problem of students' challenges in learning of mathematical proofs continues to repeat itself. Jones (2000) therefore suggested that to break the vicious circle, attention should be paid to subject matter knowledge and pedagogical content knowledge for teaching mathematical proofs.

The studies on mathematical proof have revealed several advantages of proof development and the way of viewing teaching and learning of proof development. The studies have also attempted to develop general frameworks for understanding knowledge for teaching mathematical proofs based on elementary Mathematics. Since the frameworks are general and based on elementary Mathematics, they might not have captured the demands of teaching secondary school mathematical proofs. Furthermore, as the frameworks are general, they might not have considered the differences in demands of teaching proofs for different domains of Mathematics. As such, it was necessary to develop a framework specific for understanding content knowledge for teaching geometric proofs.

2.5. Studies on MKT of Geometry and geometric proofs

Several investigations have been done in mathematical knowledge for teaching Geometry and geometric proofs. Herbst and Kosko (2012) reported on efforts to develop an instrument to measure MKT of high school Geometry (MKT-G). Their work was developed using Ball’s MKT theoretical framework. They developed items dealing with Common Content Knowledge (CCK) by using the curriculum. Herbst and Kosko (2012) decided to develop items for knowledge of content and teaching (KCT), knowledge of students and teaching (KST) and specialised content knowledge (SCK) by listing tasks entailed in each category. They managed to come up with list of tasks for KCT and KST without problems but failed to come up with an exhaustive list of tasks of for SCK. They concluded that the work that is required in different areas of Geometry such as Euclid and non-Euclid Geometry is different. The authors argued that Ball’s framework is generic and suggested that tasks for teaching Geometry be developed by differentiation of domains of CCK and SCK with respect to nature of concepts. Since tasks of teaching depend on MKT, the findings by Herbst and Kosko (2012) also imply that the knowledge for teaching different fields of Geometry is different.

Several reasons have been advanced as to why students experience difficulties in developing and understanding geometric proofs. Most of these reasons are related to the complexity of the field of geometric proof development and to teaching practices. Various researchers have conducted studies to address the challenges associated with the problem of teaching and learning of geometric proof development. Chinnappan et al. (2012) investigated the nature of geometric knowledge used by students in geometric proof development. The authors analysed geometric proofs that were developed by students to find out knowledge embedded in them. They found that Geometry content knowledge was an essential component of knowledge for the development of proofs. In addition to content knowledge, the authors found that the geometric proof development also involved problem-solving skills and Geometry reasoning skills. Chinnappan et al. (2012) observed that all these three components of knowledge were necessary and they influenced students' ability to develop geometric proofs. Their study has provided an illumination on knowledge that students use when developing geometric proofs, but it has not provided insight into teacher knowledge involved in helping students to develop abilities in geometric proofs development.

Knuth (2002) studied how teachers understood the concept of geometric proof development. After analysing interviews conducted with several secondary school teachers, he found that all the teachers understood the concept of proof development but differed in their opinions about the values of geometric proofs. This implied that the teachers would not focus on providing students meaningful experiences for understanding the value of geometric proofs (Knuth, 2002). He, therefore, suggested that there is need to enhance teachers' conceptions about geometric proofs so that they can be successful in enhancing the role of geometric proof development during instruction.

Regarding classroom instructional practices, Herbstet al. (2009) focused on investigating how the work of teaching geometric proof is shared between teachers and students in classroom. Their studies were based on the argument that the teaching of geometric proof development involves division of labour between the teacher and the students. As such, there are specific norms that guide the responsibilities for teachers and students during development of geometric proofs in the classrooms. They called the teaching of geometric proof development, “the situations for doing proofs,” (p. 254).

During an exploration of the norms, Herbst (2002) analysed a classroom episode to investigate what teachers do to create a task in which students can produce a proof, and what teachers do to get students to prove a proposition. He found that the teacher’s decisions in the classroom were mainly influenced by the conception of proof as a two-column proving. He argued that the conception of a geometric proof as two-column places contradictory demands on the teacher concerning how the proof should be developed in classroom. This is because, on one hand, the teacher has to help the students to develop the proof on their own, while, on the other hand, the teacher has to ensure that students have developed the formal proof by the end of the lesson. He, therefore, argued further that due to this contradiction, the situation of proving places high demands on the teacher. As a result, teachers mainly concentrate on helping students only to understand the proof development process, but not to appreciate the value of learning geometric proving. Consequently, Herbst (2002) suggested that teachers should not only view geometric proving as a process of developing a formal proof, but also as a process of discovering Mathematics. This implies that effective teaching of geometric proof development requires teachers to understand the value of proving in Mathematics.

In continuing to explore norms of situations for doing proofs, Herbst (2004) introduced four distinctive ways in which students interact with diagrams in Geometry during geometric proof development. These are empirical, representational, descriptive, and generative modes of interaction. He explained that these modes of interaction with diagrams support the work of doing proofs in different ways. In empirical interaction with diagrams, a student is allowed to make a variety of operations on the diagram (measuring, looking at, and drawing in) according to the actual features of the physical drawing and the instruments available. Hands on activities dominate this mode of interaction and the deductions are drawn based on empirical basis. In representational interaction, the student follows prescribed rules to draw the diagram and does not make any alterations on it. Students are involved in this mode of interaction when proving through geometric constructions. In descriptive interaction, the diagram contains features like marks and labels which represent all objects mentioned in the problem statement as well as those that are implied by the problem statements. The students' responsibility is to use the diagram to complete a proof by lifting corresponding features and giving reasons.

Herbst (2004) noted that during his observation of classroom instruction, he found that teachers mainly engaged students in descriptive interaction with diagrams. He argued that this type of interaction with diagrams is not meaningful for students because the presence of features in the diagram limits students' responsibility for producing the proof. The features also portray learning of geometric proving as a process of acquiring good logic rather than discovering of new Mathematics. He, therefore, suggested that teachers should engage students in generative interaction with diagrams to give them an opportunity to anticipate operations and add features into the diagram depending on the given proof task. Herbst (2004) further argued that such mode of interaction with diagram provides students meaningful experiences to involve in exploration of proof and to understand the value of proving in Mathematics. Although the author acknowledges that creating and involving students in activities that support generative interaction with diagrams is challenging, he did not clarify on what a teacher need to know in order to support students during this kind of instruction.

Herbst and Brach (2006) **s**tudied students' experiences with geometric proving and to what extent those experiences enable them to use mathematical reasoning when working on geometric tasks. They presented to students different tasks in which a narrative proof was supposed to be produced. The students were supposed to explain whether the tasks were normal for their classrooms or not. For the tasks that were not normal, the students were supposed to modify them into suitable tasks for their classes. The results of the study showed that students expected the teacher to give them a statement in which the given information and the statement to be proved were clearly stated. The students also expected the teacher to give them a diagram that had all points labelled, and congruent segments and angles marked. The students expected that their responsibility was to use the diagram to lift congruent lines and angles for developing the proof. This implied that the students expected to be involved in descriptive interaction with diagrams but not in generative interaction. Herbst and Brach (2006) observed that the students expected this mode of interaction with diagrams because it is what they are accustomed to do. They held that the students' views might have been different if they were being exposed to a different type task during instruction.

The studies by Herbst and his colleagues agree with Ball et al. (2008) that teaching of Mathematics involves some mathematical work that consist of a collection of actions. According to Herbst et al. (2009), the actions in situations for doing proofs are regulated by the norms that describe how the work of teaching geometric proofs should be shared between the teacher and the students. This means that norms of situations for doing proofs depend on mathematical knowledge just as tasks of teaching Mathematics do (Ball et al., 2004). However, Herbst and his colleagues were mainly concerned with understanding how the work of teaching Mathematics is shared between the teacher and the students but not on understanding mathematical knowledge for carrying out the work.

Still on classroom instruction, Jones et al. (2009) conducted several studies which focused on two aims. Firstly, to identify good models of pedagogy for helping students to understand the development of geometric proofs through abstract reasoning. Secondly, to develop new pedagogic principles that aimed at helping secondary school students not only to develop deductive proof, but also to understand its discovery function in Mathematics. The studies were conducted by comparing teaching strategies in lower secondary schools in China, Japan and the UK.

Ding, Fujita and Jones (2005) studied how expert teachers structured geometric proof lessons in China and Japan. Their aim was to come up with PCK for teaching geometric reasoning. They found that the way teachers structured their lessons was influenced by the pattern expected by their country. The Chinese teachers structured their lessons around questioning. This is because Chinese education system conceptualise questioning as a key part in Mathematics learning and expect teachers to use good questions in motivating students to explore new tasks. Thus they introduced the new content and used considerable number of short tasks and questions to achieve each lesson objective.

The Japanese teachers structured their lessons according to specification of the Mathematics curriculum, the design of textbooks, lesson studies and research into the learning and teaching of Mathematics. The Japanese lessons had a three-part structure where a problem was introduced in the first part and developed in the second part, and main teacher explanation was given in the third part. Ding et al. (2005) observed that Japanese teachers had collaboratively developed their own conception of good Mathematics lessons through lesson studies. This is a professional development for Japanese teachers that involves teachers working in small teams collaboratively to plan, teach and review lessons. They concluded that the pedagogies were different in each country because of different educational expectations by each country and suggested that pedagogy for Geometry has to take into account the context in which the teaching is taking place.

In another study, the focus was on the teaching of Geometry in lower secondary schools in China by expert teachers (Ding & Jones, 2006). In this study, several lessons were observed in different schools to explore how the teachers engaged their students successfully in geometric proof development. It was found that the teachers used a common instructional model to teach new geometric theorems. The strategy involved experimenting, guessing a possible fact of a geometric figure, drawing a figure to represent a fact, using accurate mathematical language to represent the theorem, highlighting key words in the theorem, and reciting the theorem for writing proof. Ding and Jones (2006) concluded that the Chinese teachers provided meaningful experiences by mutually reinforcing visual and deductive approaches in order to develop students' geometrical insights in the learning of geometric proofs.

In another study, Jones et al. (2009) analysed different lessons by Japanese expert teachers with an aim of examining principles that guided the structuring of geometric proof lessons. These authors aimed at understanding how the principles used by the teachers might help students to appreciate the value of formal proofs. They found that in most of the lessons, the teachers involved students in problem solving where students first investigated theorems/properties of geometrical figures through construction activities. They noticed that the construction activities helped the students to understand why the construction worked. After the construction activity, the teachers guided the students in proving. They concluded that problem solving helped the students to experience some important processes that acted as a bridge between conjecturing and proving.

To explore how problem-solving strategy could enhance students' understanding and appreciation of geometric proof development, Ding and Jones (2009) analysed how teachers conducted the first two phases of problem solving suggested by Polya (1945). These phases include understanding of a problem and devising a plan. They found that the phase of understanding of problem was facilitated by variation of mathematical problems, while phase of devising of a plan was facilitated by variation of teaching questions. Basing on these findings, Ding and Jones (2009) proposed a shift from strategies linked to van Hiele levels to the ones that involve problem solving. They suggested that during problem solving, teachers should present a proof problem as an experimental problem, modify teaching tasks and vary teaching questions to help students cope with cooperation between experiment and proof.

In another study, Jones, Fujita and Kunimune (2012) reviewed studies in the teaching and learning of Geometry in relation to three issues: mathematical definitions, mathematical representations, and the form of teacher's instructions. On definitions, the studies showed that teachers were only using Euclidean properties to define a concept. For example, a square could only be defined using its sides and angles but not using other properties like diagonals and lines of symmetry (Jones et al., 2012).

They claimed that these findings implied that students' reasoning and proving processes depended on the type of definitions the teacher provided during classroom instruction. The authors called for research that focuses on mathematical definitions that can be used when formulating geometrical problems and how the teacher might introduce such different definitions during problem solving.

On representations, Jones et al., (2012) found that some representations seemed more typical to students than other representations. This is because the students were more oriented in some types of representations than others. As a result, a minor twist on the representation made students to be unable to apply knowledge that they could use if the figure was not twisted. This meant that the students' types of reasoning depended on the representation that was used during instruction. These authors suggested that there is a need for further research to investigate the influence of mathematical representations on students' decision-making, conjecture production, and proof development processes in the classroom. They also called for more studies on how teachers can utilise representations to develop students' productive reasoning process.

On teachers' instructions during geometric proof lessons, Jones et al. (2012) found that some teachers redirected students' focus by pointing to them what they should focus on. This finding agrees with Heinze and Reiss (2004) who found that teachers provided most critical hints during teaching of mathematical proofs. Jones et al. (2012) argued that the practice of providing much guidance might restrict students from thinking of other methods for proving a task. They suggested that there is need for further research on how different interactional strategies and interventions by teachers influence and shape students' proving products during teaching of geometric proof development.

The studies by Jones and his colleagues have revealed essential issues in teaching and learning of geometric proofs. Firstly, the studies have identified context as a contributing factor to how teachers structure and present their lessons. This implies that contextual issues contribute to the work and demands of teaching Mathematics. Secondly, the studies have introduced problem solving as an important pedagogical strategy for helping students to succeed in understanding the development of formal proofs and appreciating their discovery value. However, basing on the observation by Herbst (2002) that teaching of geometric proofs is demanding for a teacher, and that it requires different competences, I argue that in addition to problem solving, there might be several categories of mathematical knowledge that are necessary for effective teaching of geometric proof development. Therefore, my study aimed at building on the previous studies related to teaching and learning of geometric proof development by exploring other categories of MKT-GP.

2.6. Theoretical frameworks for teacher knowledge and geometric thinking

Teachers ‘content knowledge has been an important area of study in response to Shulman’s notion of teacher knowledge. There are various conceptualisations of Mathematics teacher knowledge which build on Shulman’s (1986) framework of teacher knowledge. As such several frameworks of Mathematics teacher knowledge have been developed by different scholars based on their conceptualisation of teacher knowledge. The following sections present Shulman's theoretical framework of teacher knowledge and some of the frameworks that build on it in relation to Mathematics Education. Some of the frameworks that are related to geometric thinking in general and the teaching of geometric proofs are also presented in the following sections.

2.6.1. Shulman's Framework for teacher knowledge

Shulman (1986) proposed a framework for understanding teacher knowledge. He argued that there is a specialised form of knowledge that is unique to teachers and teaching. This special form of knowledge distinguishes teachers from subject matter specialists. The categories of teacher knowledge suggested by Shulman (1986) include subject matter content knowledge (SMK), pedagogical knowledge (PK), pedagogical content knowledge (PCK) and curricular knowledge (CK). SMK, "refers to the amount and organisation of knowledge per se in the mind of the teacher," (Shulman, 1986, p. 9). Shulman emphasised that apart from knowing how an algorithm works, a teacher also needs to know why the algorithm works, and on what grounds its warrant can be either accepted or denied. This agrees with suggestions by Herbst et al. (2009) and Jones et al. (2009) that the teachers' responsibility is not only to show students how to prove a geometric statement but also to help them to understand why the proof works.

Shulman (1986) described PCK as a blend between content knowledge and pedagogical knowledge. He pointed out that this type of knowledge is necessary for making content understandable to students. PCK includes knowledge of powerful representations, analogies, illustrations, examples, explanations, and demonstrations, as well as knowledge of students' misconceptions in relation to a particular topic. He proposed that more studies should be conducted on PCK to better understand it in relation to a particular subject. Curricular knowledge is the knowledge of the full range of programmes designed for the teaching of particular subjects and topics at a given level. It also includes knowledge of different kinds of instructional materials available in relation to those programmes and their use.

In continuing with studying teacher knowledge, Shulman (1987) further categorised teacher knowledge as follows.

(i) Content knowledge.

(ii) General pedagogical knowledge, with special reference to those broad principles and strategies of classroom management and organisation that appear to transcend subject matter.

(iii) Curriculum knowledge, with particular grasp of the materials and programs that serve as "tools of the trade" for teachers.

(iv) Pedagogical content knowledge, that special amalgam of content and pedagogy that is uniquely the province of teachers, their own special form of professional understanding.

(v) Knowledge of learners and their characteristics.

(vi) Knowledge of educational contexts, ranging from workings of the group or classroom, the governance and financing of school districts, to the character of communities and cultures.

(vii) Knowledge of educational ends, purposes, and values, and their philosophical and historical grounds (Shulman, 1987, p. 8).

Shulman (1987) argued that PCK is of special interest among the seven categories of teacher knowledge because it represents a blending of content and pedagogy into an understanding of how particular topics or issues are organised, represented, and adapted to the diverse needs and abilities of learners.

Shulman's work offered useful characteristics of content knowledge for effective teaching in general. The work also shifted researchers' attention from pedagogy only to both CK and PCK. Ball et al. (2001) justifies PCK as an essential idea in teacher knowledge for three reasons. Firstly, it fills the gap left when the focus is only on teacher's credentials. Secondly, it improves understanding of the knowledge required for teaching to a larger extent. It is through Shulman's framework that educators and researchers realised that teachers' mathematical knowledge is different from that of a mathematician. Thirdly, it has produced new methods of teacher study that have resulted in generation of rich knowledge (Ball et al., 2001). Ball et al. (2008) acknowledged two contributions of Shulman's framework. Firstly, it shifted researchers' attention from general aspects of teaching to the role of content in teaching. Secondly, it represents content understanding as a special component to the profession of teaching.

However, Jones, Mooney and Hurries (2002) argued that Shulman's model may be too simplistic because it does not distinguish between the nature of different school subjects. They noted that the teaching of Mathematics is complex and different from the teaching of other subjects, hence need for specialised framework. Ball et al. (2001) also observed that the framework might be very difficult to use because it does not provide specific guides for teaching Mathematics well. Furthermore, Ball et al. (2005) argued that Shulman's framework might be difficult to operationalise because it does not clarify the difference between CK and PCK.

My view is that Shulman (1986) does not present his framework as a final tool for understanding knowledge for teaching any subject but as a lens for guiding further research on knowledge for teaching a particular subject. The framework acts as a revelation for researchers to realise that there are different dimensions of teacher knowledge for every subject. By suggesting that more research be conducted to ensure deep understanding of PCK for specific subjects, Shulman was acknowledging that the demands for teaching of different subjects are not the same. This could be the reason why despite the criticisms, several studies have used Shulman's framework as a lens for understanding and developing categories of teacher knowledge for different subjects including Mathematics.

2.6.2. Mathematical knowledge for teaching (MKT) framework

Ball et al. (2008) developed MKT framework by building on Shulman's (1986) SMK and PCK categories of teacher knowledge. They emphasised that the issues of teacher knowledge identified by Shulman are important to research on teaching and teacher education. They redefined Mathematics PCK as a special form of knowledge that bundles mathematical knowledge with knowledge of learners, learning and pedagogy. They explained that the bundles are a very important resource in teaching Mathematics as they might help the teachers to anticipate students' misconceptions and their remedies in a particular topic in advance. Figure 1 presents an egg representation of Ball et al. (2008) MKT framework.

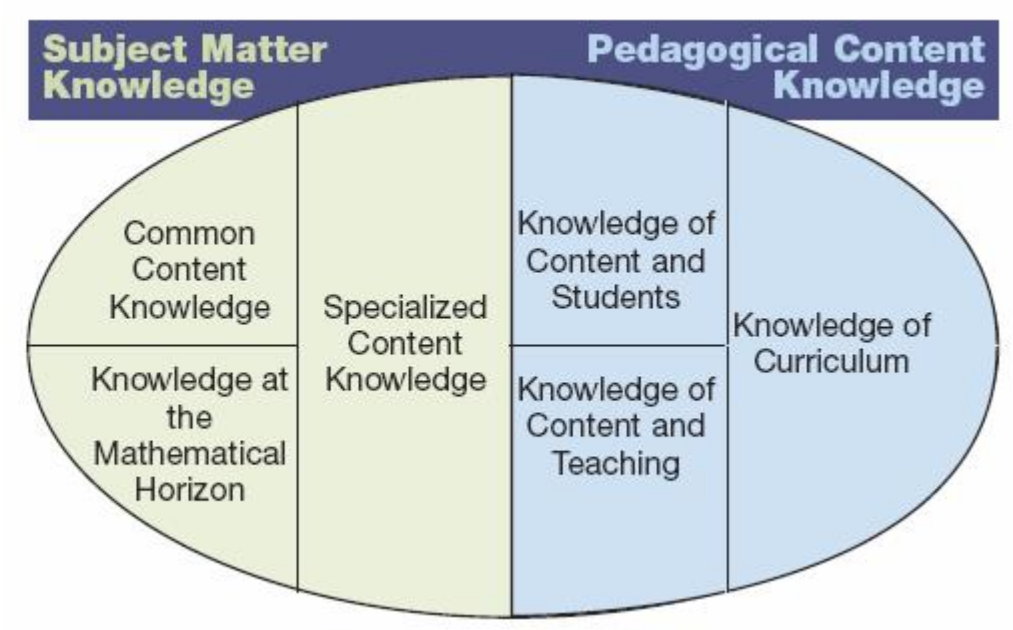


Figure 1: MKT Framework adopted from Ball, Thames and Phelps (2008)

MKT framework divides SMK and PCK further into three different domains for each. SMK is divided into common content knowledge (CCK), specialised content knowledge (SCK), and horizon knowledge of Mathematics. PCK is divided into knowledge of content and students (KCS), knowledge of content and teaching (KCT), and knowledge of content and curriculum (KCC).

CCK is the mathematical knowledge and skill used in settings other than teaching. Examples of CCK include knowledge of how to perform an operation and recognising a wrong answer or wrong information from the textbooks (Ball et al., 2008). SCK is the mathematical knowledge and skill unique to teaching (Ball et al., 2008). SCK is regarded as the new contribution in the framework and it is sorely needed for teaching purposes. It includes knowledge for analysing and making patterns of students' errors and determining whether a nonstandard approach would work in general. It is the kind of knowledge that teachers use to accomplish their daily problem solving tasks described by Ball et al. (2004).

KCS combines knowing of content and knowing of students (Ball et al., 2008). It includes knowledge about how students will likely think or reason during the teaching of a particular concept, and about students' conceptions or misconceptions related to a particular topic. KCT combines knowing about Mathematics, and knowing about teaching (Ball et al., 2008). It includes knowledge of good sequencing of examples in an instruction, order of presenting instruction, when to pose a question, and appropriate methods for a particular instruction. MKT framework has provided insights into the teaching of Mathematics by emphasising that teaching of Mathematics involves some mathematical work. The framework also acknowledges the usefulness of classroom interactions between the teacher and students. This means that the framework acknowledges that Mathematics is socially constructed. However, McCrory et al. (2012) argued that Ball's MKT framework does not take into account the complexities of secondary school Mathematics teaching because it was developed based on elementary and middle school Mathematics. I argue that despite being developed based on elementary and middle school Mathematics, the MKT framework enhances understanding of teacher knowledge in relation to the subject. The further division of SMK and PCK into sub-domains sheds light onto the complexities and demands of the teaching of Mathematics.

I concur with Herbst and Kosko (2012) that the framework is too general because it assumes that knowledge for teaching of different mathematical fields is the same. However, Mathematics has several fields that require different content, pedagogic and reasoning skills. Hence, researchers might mainly use the framework as a guiding tool for exploring and developing MKT frameworks for specific fields of Mathematics.

2.6.3. Knowledge for Algebra Teaching (KAT) Framework

Building on Ball's framework, McCrory et al. (2012) developed a framework of knowledge for Algebra teaching (KAT). The framework was designed for middle and secondary school Algebra and it was developed after conducting extensive studies on two main topics in Algebra; algebraic equations and expressions, and linear relationships. McCrory et al. (2012) sought to understand teacher knowledge demands for these topics because they are foundational to school Algebra and they pose challenges to students. Their argument was that despite knowing that advanced mathematical knowledge and mathematical knowledge closer to the practices of teaching is necessary for secondary teacher effectiveness, there is little evidence to inform decisions about what to emphasise during preparation of Algebra teachers. This is because there are no measures of teacher knowledge that would capture advanced Mathematics and Mathematics closer to teaching. As such, their work attempted to develop a theory that would test teachers’ MKT of Algebra.

The KAT framework comprises three categories of knowledge: knowledge of school Algebra, knowledge of advanced Mathematics, and Mathematics for teaching knowledge. These categories reflect three perspectives of knowledge teachers needed to be taught in pre-service for effective teaching of Algebra. Knowledge of school Algebra includes proficiency in what the teachers are going to teach (McCrory et al., 2012). Ball et al., (2008) called this CCK while Shulman (1986) called it SMK. Knowledge of Advanced Mathematics includes other mathematical knowledge like College Mathematics. This agrees with Shulman who suggested that the level of content level of secondary school Mathematics teachers should be the same as that of Mathematics majors. McCrory et al. (2012) argued that this category of knowledge gives a teacher broader and deeper understanding of school Algebra. Mathematics for teaching knowledge is Mathematics that is useful in teaching, but not included in the teachers’ formal mathematical education (McCrory et al., 2012). Examples of this knowledge include knowledge of different definitions of a particular mathematical concept, and how those definitions influence understanding of the Mathematics that follows. They explained that this type of knowledge is related to SCK suggested by Ball et al. (2008).

Apart from the three categories of teacher knowledge, the KAT framework also contains categories of work that teachers do when teaching Algebra. These include decompressing, trimming and bridging. McCrory et al. (2012) argued that a teacher could only accomplish the three activities of the work of teaching Mathematics (decompressing, trimming and bridging) if they have enough knowledge of school Mathematics, knowledge of Advanced Mathematics and MKT.

Decompressing means working backwards to unpack content from mature and complex understanding to simple and understandable form. They suggested that decompressing could be relevant in analysing students' mathematical work and thinking, as well as designing, modifying and selecting mathematical tasks. Ball et al. (2004) also suggested that decompressing or unpacking should be one of the tasks of a Mathematics teacher. Trimming involves reducing the complexity of Mathematics content into ways that make the content accessible to students while maintaining its integrity (McCrory et al. 2012). It includes interpreting and transforming Mathematics into a form that is understandable to students while preserving rigor and originality of the concept. This task was also suggested by Shulman's (1986) framework of teacher knowledge under PCK category. Bridging involves making various kinds of links and connections among mathematical concepts between students' thinking and the lesson objectives. Bridging activity is also one of the tasks of teaching suggested by Ball at al. (2004).

There are several strengths of the KAT framework. Firstly, it was developed based on analysis of secondary school teachers' subject matter knowledge of Mathematics. This means that the framework was developed based on in-depth analysis of the issues that concern teaching of secondary school Mathematics. The other strength is that the KAT framework agrees with both Shulman (1986) and Ball et al. (2008) that teaching of Mathematics involves doing some work which requires special kind of knowledge. However, I argue that the KAT framework does not strengthen insights into understanding of the work of teaching Mathematics. This is because several scholars like Shulman (1986), Ball et al. (2004) and Ball et al. (2008) already suggested the categories of teacher knowledge and the activities of teaching suggested by the framework. Furthermore, McCrory et al. (2012) assumed that despite being designed specifically for teaching of Algebra, the KAT framework could still be used in any other field of Mathematics because it is widespread and general. As already argued, MKT is specific to mathematical fields, hence an algebraic MKT framework might not be applicable to Euclidean Geometry.

2.6.4. Teacher Education Development Study in Mathematics (TEDS-M) Framework

Tatto et al. (2008) developed the TEDS-M framework which was used to develop instruments for investigating and evaluating both primary and lower secondary teacher education preparation around the world. The authors aimed at developing test items for enabling different countries to analyse their education systems and policies, and compare their expectations about future Mathematics teachers. The TEDS-M theoretical framework of teachers professional competence regards MKT as comprising of Shulman's (1986) two teacher knowledge categories; mathematical content knowledge (MCK) and Mathematics pedagogical content knowledge (MPCK) (Tatto et al., 2008).The domains of MCK include the mathematical content that the teachers are required to teach, and Mathematics beyond the level that the teachers are required to teach. The domains of MPCK included Mathematics curricular knowledge, knowledge of planning, and knowledge of enacting Mathematics (Tatto et al., 2008). The assumptions of the TEDS-M framework are as follows:

(i). Teacher education is understood and implemented differently across national settings and even between institutions within the same country.

(ii). Teacher education and teacher learning are complex, contested processes.

(iii). Knowledge of the content to be taught is a crucial factor influencing the quality of teaching.

(iv). Teacher education requires understanding of and addressing how teachers should think about Mathematics, teaching, and learning.

(v). Teacher education embodies a developmental logic of how teachers acquire professional knowledge for the teaching of Mathematics and other subjects.

(vi). Knowledge for teaching involves consideration of the situational contexts whereteachers will teach.

(vii). Teacher education is assumed to be linked to student achievement, but this Relationship is poorly understood.

The assumptions of the TEDS-M project agree with those of different scholars. For instance, the assumption that teachers' content knowledge influences quality of teaching agrees with several authors who argue that the successful teaching of mathematical proof depends crucially on the subject knowledge of Mathematics teachers (Charalambous, 2010; Hill et al., 2008; Jones, 1997). The assumption that teacher education and teacher learning are complex agree with Jones (2000) who argued that the complexity in the issues of teacher education in Geometry UK curriculum imply lack of consensus regarding what should be offered. Jones' (2000) observation is consonant with the TEDS-M assumption that teacher education is understood and implemented differently between or among different institutions in a country. Assumption that teacher education is linked to student achievement agrees with Jones (2000) who argued that, teachers face challenges to teach mathematical proofs because its content is not well captured in teacher education curriculum. The assumption also support Hill et al. (2005) who argued that teacher practices influence students' achievements in Mathematics. The assumption that knowledge for teaching involves consideration of the situational contexts where teachers will teach agrees with Ding et al. (2005) who found that the contexts within which mathematical proof and proving are taught around the world vary in terms of curriculum specification, student age-level, teacher knowledge, and so on.

The TEDS-M framework was developed based on assumptions that agree with many scholars. This implies that the framework was developed by considering issues that concern teacher knowledge. This study agrees with the assumptions of the TEDS-M project because they hinge on the importance and complexity of teacher education. Lin and Rowland (2016) argue that the TEDS-M framework ignores the power of Mathematics classrooms in advancing teachers' professional development by assuming that teacher knowledge can be evaluated only through pencil and paper tests taken outside Mathematics classroom. This means that the framework ignores the usefulness of teacher-students interaction in Mathematics Education. Besides, I argue that the framework also ignores the importance of assessing students' thinking in Mathematics Education. Like some mathematical teacher knowledge frameworks, the PCK categories of TEDS-M framework describe general knowledge for both primary and secondary school pre-service teachers. This implies that the framework does not recognise that PCK is determined by specific subject matter knowledge.

2.7. Frameworks used to guide this study

I used several frameworks to guide me at different stages of the study including research question development, data generation, data analysis, and presentation of findings. These are; COACTIV model, the framework for analysing cognitive level of tasks, the problem-solving framework, and the situations for doing proofs model. In the following sections, I describe the theoretical roots of these models, and how each of them guided my study.

2.7.1. Theoretical roots of the frameworks which guided the study

Although the frameworks that I used to guide the study were developed by different scholars, they stem from common theoretical roots of social constructivism. The COACTIV model and the situations for doing proofs models were developed from qualitative studies. Data that was analysed for developing the frameworks was generated by observing Mathematics lessons and interviewing secondary school teachers (Baumert et al., 2010). This means that the frameworks took into consideration context, teacher-students interactions, as well as teachers' perspectives. The framework for analysing cognitive level of tasks and the problem-solving framework recommend that teachers involve students in constructing their own understanding of mathematical concepts through exploration and explanations (Polya, 1945; Smith & Stein, 1998). This also implies that these frameworks are informed by social constructivism theory.

2.7.2. The COACTIV model

Building on Shulman's notion of teacher knowledge, Baumert et al. (2010) developed the COACTIV model for conceptualising teachers' professional knowledge for teaching secondary school Mathematics (Baumert et al., 2010). According to Baumert and Kunter (2013), the main objective of COACTIV model was to identify the professional qualities that teachers need in order to meet the demands of their profession. They mainly focused on mathematical knowledge for effective classroom instruction.

The COACTIV model adopted three categories of teacher knowledge suggested by Shulman (1986), namely, CK, PCK and PK. Baumert et al., (2010) explained that although PK is included in the model, it has not been tested empirically. As such, for the purposes of this study, I only describe the two categories of knowledge (CK and PCK) because they comprised the main conceptual framework for my study. I did not regard PK category because it is mainly concerned with general classroom management issues that were not a concern for this study.

2.7.2.1. Content Knowledge

COACTIV model conceptualise CK needed for teaching as a deep or profound understanding of the Mathematics taught in the secondary school (Baumert & Kunter, 2013). This type of knowledge is found in the school curriculum and is continually developed basing on instructional practice (Krauss et al., 2008). The definition of CK for the COACTIV model at secondary Mathematics level is similar to Ma's (1999) definition of CK at primary Mathematics level (Baumert et al., 2010). Ma (1999) defined CK as profound understanding of fundamental Mathematics (PUFM). She argued that although profound is often considered to mean intellectual depth, its three connotations; deep, broad and thorough are interconnected. She, therefore, defined understanding a topic with depth as, "connecting it with more conceptually powerful ideas of the subject," (Ma, 1999, p. 121). She explained that a topic which is closely connected to the subject is more powerful and has greater potential of supporting other topics. Ma defined understanding a topic with breadth as ability to connect topics with equivalent or different conceptual powers. Thoroughness is the capability to pass through all parts of the field (Ma, 1999).

Although CK includes content of the secondary school Mathematics, it is not enough to equip teachers with skills that are necessary for coping with the mathematical challenges facing them during the preparation and implementation of instruction (Krauss et al., 2008). As such, teachers need to have PCK as well in order to be able to conduct their work of teaching Mathematics. However, the authors acknowledged that PCK cannot be conceptualised without CK (Krauss et al., 2008). Therefore, they regard CK as a foundation of PCK and suggest that teachers need strong CK to develop PCK. The authors further argued that teachers' knowledge of the mathematical content covered in the school curriculum should be deeper than that of their students so that they are able to assist them to learn Mathematics successfully (Krauss et al., 2008). This agrees with Shulman (1986) who argues that, "We expect that the subject matter content understanding of the teacher be at least equal to that of his or her lay colleague, the mere subject matter major," (p. 9). Furthermore, Ball et al. (2004) also argued that if teachers have acquired enough mathematical content knowledge, they would be able to unpack mathematical concepts into forms that are understandable to their students.

2.7.2.2. Pedagogical content knowledge

The COACTIV model distinguishes three categories of PCK; (i) knowledge of cognitive activating tasks, (ii) knowledge of student's cognitions and ways of assessing students' knowledge and comprehension processes, and (iii) knowledge of explanations and multiple representations. I explain the PCK categories in detail in the following sections.

2.7.2.2.1. Knowledge of cognitive activating tasks

Krauss et al. (2008) pointed out that tasks play a central role in teaching Mathematics to the extent that much of the time allocated to Mathematics lessons is devoted to tasks and their solutions. Knowledge of cognitive activating tasks involves appropriate selection and implementation of tasks that lay a foundation of students' mathematical knowledge construction and present students with powerful learning opportunities (Baumert & Kunter, 2013). Krauss et al. (2008) identified three sub-categories of knowledge of cognitively activating tasks: knowledge of multiple solution paths to tasks, knowledge of sequencing of the tasks to meet the desired effect in the classroom, and knowledge of prior knowledge required for the selected tasks.

2.7.2.2.2. Knowledge of student's cognitions and ways of assessing students' knowledge and comprehension processes

It includes knowledge of students' misconceptions, typical errors and difficulties in relation to a particular subject or topic, and ways of overcoming them (Baumert & Kunter, 2013). Teachers need to work with students' existing conceptions and prior knowledge either to build on them if they are mathematically correct or to correct them if they are wrong (Krauss et al., 2008). In the COACTIV model, mistakes are viewed as valuable because they provide useful insights into teaching of Mathematics. As such, it is important for teachers to be aware of typical student misconceptions and difficulties (Baumert & Kunter, 2013).

2.7.2.2.3. Knowledge of explanations and multiple representations

This involves knowledge of supporting and guiding students' learning processes. Krauss et al. (2008) argued that students' construction of knowledge is often only successful with appropriate instructional support and guidance in the form of explanations and representations.

The COACTIV model has several strengths that are relevant to this study. Firstly, the model was developed based on secondary school classroom empirical data. This implies that the model took into consideration the context of teaching secondary school Mathematics. Secondly, the model comprises of few categories, hence easy to use for further understanding of teacher knowledge. The other strength of the COACTIV model is that it values teacher pupil interaction during instruction. This implies that the model adheres to the values of social constructivism theory of learning which guide the epistemological perspectives of this study. The other strength of the model is the emphasis on the value of cognitively activating tasks in the Mathematics classroom. I agree that teachers spend most of their time in solving tasks in a Mathematics classroom. This implies that the success of a Mathematics learning depends on the teachers' ability to involve students in exploring tasks that will challenge them to use high levels of thinking and to be innovative in coming up with solutions. This implies that the model emphasises on developing of students' critical thinking through the teaching of Mathematics. This emphasis supports the objective of secondary school Mathematics Education in Malawi, which is to develop students' competence in critical thinking. Although the model was developed mainly for testing secondary school teachers' MKT, I found it appropriate for conceptualising MKT for geometric proof development because of the strengths highlighted above.

Like some of the models that I discussed earlier, the weakness of the COACTIV model is its generality in assuming that knowledge demands for the teaching of all fields of secondary school Mathematics are the same. This might be the reason why the model does not categorise CK. Despite these weaknesses, I still used the model as a lens for developing research questions and analysing some of the data for the study. The issue of generality is observed in most of the models of teacher knowledge, and it might be the cause of lack of understanding of knowledge for teaching geometric proofs. The absence of categorisation of CK in the COACTIV model was regarded as a gap which my study aimed to fill in relation to teaching of geometric proof development.

In this study, I used the COACTIV model to guide development of research questions. Each of the research questions was developed from a particular category of the COACTIV model. I also used the categories of the COACTIV model as the pre-ordinate themes to guide me in interview data analysis and in presenting the findings.

2.7.3. The Framework for analysing cognitive demands of a task

Smith and Stein (1998) developed a model for classifying mathematical tasks. They described four categories of cognitive demands of a task: (i) memorisation, (ii) procedures without connections to concepts or meaning, (iii) procedures with connections to concepts and meaning, and (iv) doing Mathematics. The authors described the categories of tasks depending on the level of students' thinking involved in each category. Memorisation tasks involve exact reproduction of previously learnt facts without showing of their algorithms. Procedures without connections to concepts or meaning are tasks that require use of an algorithm without showing understanding of how the algorithm works (Smith & Stein, 1998). Memorisation and procedures without connections are classified under lower-level tasks because they place little demands on students' thinking and explanations.

Procedures with connections to concepts require some degree of cognitive effort as they involve thinking of how to apply a procedure to a task. Sometimes, procedures with connections to concepts may require more than one representation to show how and why an algorithm works (Smith & Stein, 1998). Doing Mathematics tasks also demand considerable cognitive effort because the procedure is not known, so they require the student to explore and understand the nature of mathematical concepts, processes, or relationships to be used in solving the task.

By focusing on cognitive demands of a task, the framework agrees with the COACTIV model that students should be involved in cognitive activating tasks. The other strength of the model is that it focuses on both selection and implementation of the tasks. According to this model, a high cognitive level selected task maintains its level when it is implemented using explanation and exploratory methods. This also agrees with the COACTIV model that teachers are supposed to know how to select and implement tasks in a manner that enhances productive learning. Because of these strengths, I used the framework by Smith and Stein (1998) for analysing the cognitive level of both the narrative and computation tasks that were used by the teachers during teaching of geometric proving.

2.7.4. The problem-solving framework

Another analytical framework that I used for data analysis is the problem-solving framework suggested by Polya (1945) in his seminal book. This framework was developed on the basis that problem solving is the main theme of Mathematics Education. As such, the purpose of the framework was to act as a guide on how teachers are supposed to help students unobtrusively to acquire much experience and become independent thinkers during problem solving in Mathematics. According to

Polya (1945), one of the important tasks of the teacher is to help the students. He argued that helping students to learn Mathematics is not an easy task because it requires time, practice, devotion, and sound principles. This is because a teacher is supposed to know how to give appropriate support that is neither too little nor too much for the students. He suggested that in order to be able help the students appropriately, the teacher is supposed to help the student naturally. This can be done if the teacher puts himself in the place of the student by trying to understand what is going on in the mind of the student and ask questions that might occur to the student (Polya, 1945).

Polya (1945) distinguished four phases of problem solving that teachers must carry out with their students during teaching of Mathematics. These are: (i) understanding the problem, (ii) devising a plan, (iii) carrying out the plan, and (iv) looking back. The four phases of problem solving are represented in the following cyclic figure 2.

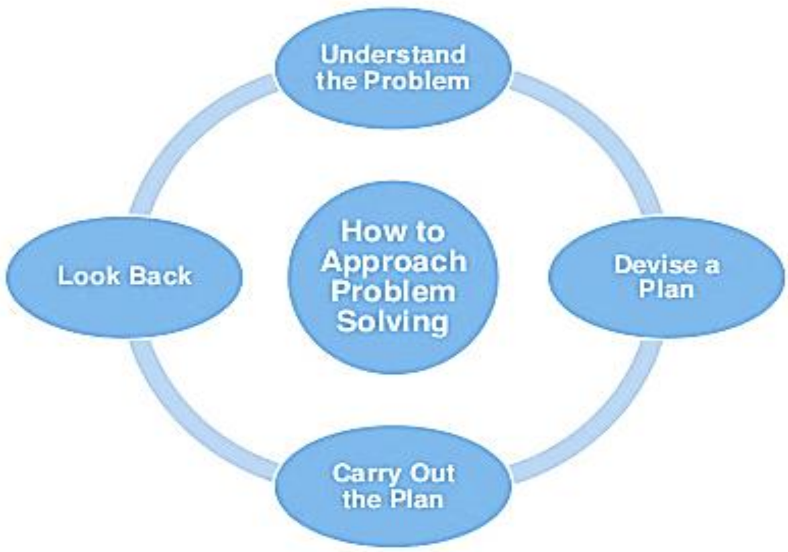


Figure 2: Polya's cyclic problem-solving framework adopted from

2.7.4.1. Understanding the problem

Polya (1945) emphasised that when beginning to answer a problem, “start when this problem is so clear,” (p.33). This phase of problem solving involves helping students to understand the verbal statement of the problem and its principle parts. He explained that if the student is failing to understand the problem, the fault does not lie with the student but the selected problem. As such, a teacher is supposed to select a problem that is not too difficult and not too easy, natural and interesting, and allow sometime for the student to present their understanding of the problem. Polya’s argument about the selection of a problem agrees with van Hiele (1999) who argued that if students are not able to develop and understand a geometric proof, the problem does not lie with the student but the mathematical content that the teacher has selected.

Polya (1945) explained that when students are presenting their understanding of the problem, they are supposed to point out the principle parts of the problem, which include the unknown, the data and the condition. He explained that for a proving problem, the principle parts are the hypothesis and the conclusion. Hypothesis is the part of the statement that contains the given information while conclusion is the part of the statement that contains what needs to be proved. He also suggested that when a proving problem is connected to a figure, the teacher must help the students to draw the figure, introduce suitable notations, and label on the figure the hypothesis and conclusion. This means that a teacher is supposed to help the students to understand the problem in relation to both the language of the problem statement and the figure drawn.

2.7.4.2. Devising plan

There are several aspects involved at this phase of problem solving. They include analysing the given information to see if it is sufficient to determine the conclusion. For a problem to prove, this phase involve helping the students to analyse the given information on the diagram and to decide what to do with the diagram in order to come up with proving statements that are necessary for the proof. The phase also involves finding the connection between hypothesis and conclusion, deciding on the theorem to use and making decisions whether to introduce auxiliary elements to enable solving of the problem (Polya, 1945). Some of the auxiliary elements for a problem to prove involve introducing lines and letters into the diagram to enable it to provide more information to be used for proof production. He suggested that when devising a plan, the teacher must help students to seek contact with their prior knowledge. This implies that the phase of devising the plan involves recalling principles and theorems that were learnt and identifying the ones that are useful for doing the task.

Polya (1945) noted that problem solving might be time consuming because students explore different ways when devising a plan for finding solution of the problem. This means that a teacher needs to be patient in order provide good guidance to students when exploring the plan. In support of Polya’s suggestion, Ding and Jones (2009) proposed that varying of questions can help the students to understand the problem to prove. They suggested that teachers must ask different types of questions to either probe or redirect students’ thinking. This means that supporting of students’ learning processes also requires knowledge of Questioning. When the teacher asks thought provoking questions, he creates an opportunity for students to develop independent thinking in Mathematics. As Polya (1945) explained:

> Thus a teacher of Mathematics has a great opportunity. If he fills his allotted time with drilling his students in routine operations, he kills their interest, hampers their intellectual development, and misuses his opportunity. But if he challenges the curiosity of his students by setting them problems proportionate to their knowledge, and helps them to solve their problems with stimulating questions, he may give them a taste for, and some means of, independent thinking (Polya, 1945, p. v.).

The extract implies students' interest and intellectual development in Mathematics increases when the phases of problem solving are carried out in a good manner.

2.7.4.3. Carrying out the plan

This phase includes putting the plan into practice by doing the task step by step and checking that each step is correct. In geometric proof development, it means writing of the proving statements in a logical sequence, each statement accompanied by a valid and abstract reason. Polya (1945) explained that when writing the geometric statements, the aim should be to convince yourself why the statement is correct. For the students, this means that they need not only to make logical arguments but also to understand why what they are writing is true.

2.7.4.4. Looking back

This phase of problem solving involve analysing the plan and solution of the problem to check if the arguments are correct, and if the solution can be derived using a different way. Polya (1945) suggested that teachers must give students an opportunity to analyse their solutions and discuss other ways of solving the problem. He argued that looking back to the solution path and thinking of other ways of finding the solution can enhance the students' ability to understand the solution, to consolidate knowledge and to develop problem solving skills.

During geometric proving, looking back involves checking if the geometric statements are mathematically true, logically connected, and appropriately justified. In a problem to prove, looking back phase also includes either proving the problem using a different approach or using the theorem that has been proved to solve other problems. According to Polya (1973), students can update their knowledge and develop their ability to solve problems by looking back at the completed solution, reconsidering and re-examining the solution path. He also explained that looking back can help students to be able to make mathematical connections, come up with a new and better solution, and to discover new facts. He suggested that teachers should encourage students to imagine or come up with new problems in which the same procedure could be utilised.

Although Polya's framework was developed some several decades ago, it has been reproduced several times over the years. I found Polya's (1945) framework useful for this study for several reasons. One reason is that it resonates with the epistemological considerations of social constructivism by encouraging interactive learning through problem solving. As Jaworski (2016) argued in the opening remarks of the second handbook of research on the PME, Polya's (1945) seminal book, "How to solve it" has been very influential in promoting of the doing of Mathematics through problem solving and the use of heuristics of problem solving. Secondly, the framework views the learning of geometric proofs as an exploration approach. Ding and Jones (2009) recommended that using problem-solving pedagogical strategy suggested by Polya (1945) could help to enhance students' understanding and appreciation of geometric proof development. They argued that since problem solving is exploratory, it affords students an opportunity to develop inquiry skills and to appreciate the value of geometric proof development in Mathematics (Ding & Jones, 2009).

A third reason is that although the phases of problem solving are general, Polya (1945) managed to illustrate how to apply them to various mathematical topics including geometric proofs. This implies that the phases of problem solving are flexible, hence capable of being applied to any field of Mathematics. Furthermore, the phases of problem solving are clearly explained and exemplified using classroom experiences. This implies that the framework is practical and easy to operationalise when lesson observations are used to study teaching of Mathematics. A fourth reason is that Polya's phases of problem solving are dynamic and cyclic in nature. Dynamic in the sense that if the student is unable to complete or carry out a plan for a problem, they can go back to the problem and try to understand it better or pose a new problem and then develop a new plan (Polya, 1973). The cyclic property of the problem-solving framework implies that during looking back, when a student comes up with a new problem, they can start again the process by trying to understand the problem, devise a plan, carry out the plan and look back again to develop another problem.

The drawback of framework is that it mainly focuses on how Mathematics should be taught. This might be because the framework was developed before the idea of teacher knowledge was discovered. Still more, the framework does contain some aspects of teacher knowledge including questioning which is recommended by Ball et al. (2004), and multiple solution paths for tasks recommended by the COACTIV model. Based on these reasons, I used the framework to analyse how geometric proofs are explained and represented to students.

2.7.5. The situations for doing proofs model

Herbst et al. (2009) developed a model for understanding the situations of doing proofs in Geometry classrooms. They used the model to inspect teachers' views of geometric proofs through interviews and classroom proof development practices. Herbst et al. (2009) regarded teaching of geometric proof development as a kind of mathematical work consisting of steps and actions. They argued that teaching of proofs is an instructional situation that consists of a collection of several actions. These actions are regulated by norms that describe the work to be done and shared between the teacher and the students. The first part of the model describes the process of developing geometric proofs in a classroom. The second part describes how work should be divided between the teacher and the students during teaching and learning of geometric proof development. Only the first part of the model has been discussed in this section because it was suitable for analysing the proofs developed by the teachers. Herbst et al. (2009) explained that the work of developing proofs consist of the following norms:

> (1)Writing a sequence of steps (each consisting of a "statement" and a "reason"), where (2) the first statement is the assertion of one or more "given" properties of a geometric figure, (3) each other statement asserts a fact about a given specific figure in a diagrammatic or generic register, and (4) the last step is the assertion property identified earlier as the "prove"; during which (5) each of these asserted statements are tracked on a diagram by way of standard marks, (6) the reasons listed for each of those statements are previously studied definitions, theorems, or postulates, as well as "given", and (7) each of those reasons are stated in a conceptual (abstract) register. (8) Students' successful production of a proof exchanges for the claim that they know how to do proofs

(and can reason logically, justifying their steps). (Herbst et al., 2009; pp. 254-255).

The first norm about writing a sequence of steps accompanied with reasons means following a certain order when writing a geometric proof. The norm also means that a proof is a collection of statements that are accompanied with reasons. For example, to prove that two triangles are congruent, one has to write three connected statements that are justified with reasons to show that some sets of angles and sides are equal, and then make a conclusion that the triangles are congruent.

In those set of statements, the first statement must contain elements that are given in the geometric diagram (norm 2). This means that the reason for justifying the first statement should be written as ‘given’. Norm 3 means that the second statement and the other remaining statements should be written based on interpretation of the geometrical properties of the diagram. This norm suggests use of prior knowledge of geometrical concepts during geometric proof development. Norm 4 means that the concluding statement in geometric proof development should be the assertion being proved. Before writing the set of proving statements for the proof, all elements of the statements must be clearly marked in the geometric diagram (norm 5). Prior knowledge should be combined with the given information when developing the geometric proof statements (norm 6). Norm 7 about stating each reason in an abstract manner means that the reason for each statement in the proof should be based on theoretical geometrical properties and not empirical properties. Norm 8 is about analysing the whole proof to make judgement of whether it is correct or not. When the student’s proof is correct, it means that the student is able to reason logically and to justify each step.

I did not consider norm 8 during analysing of the proofs developed by the teachers because the aim was not to examine teachers' abilities to develop a correct geometric proof, but to understand knowledge used when developing the proofs. Furthermore, justifying of each step with a reason is also mentioned in norm 1, so including 8 in the analysis would have been redundant. The weakness of the model of situations for doing proofs is that it does not clarify what knowledge teachers need in order to conduct the norms successfully when teaching geometric proofs. This is a gap that I intended to fill in this study. I regarded the norms as suitable pre-ordinate themes for thematic analysis because they were developed based on empirical studies on teaching and learning of secondary school geometric proof development (Herbst et al., 2009). Furthermore, the norms were developed based on the assumption that teaching of geometric proof development involves some work. This implies that use of the norms could afford an opportunity to understand teacher knowledge required for conducting that particular work.

The literature review has revealed that in trying to respond to Shulman's (1986) notion of teacher knowledge, researchers took different directions to study knowledge for teaching Mathematics. Some researchers focused on developing both tasks of teaching and framework for understanding teaching of Mathematics in general, (Ball, et al., 2008; Baumert & Kunter, 2013). Others focused on developing tasks for teaching and frameworks for understanding knowledge for teaching specific branches of Mathematics (Herbst & Kosko, 2012; McCrory et al., 2012). The studies focused on different aspects because the researchers had different study aims and assumptions about the teaching of Mathematics.

The studies that focused on tasks of teaching Mathematics were based on the assumption that the work of teaching Mathematics involves performing of several defined tasks (Ball et al., 2004; Herbst & Kosko, 2012). The other assumption of focusing on tasks was that teacher knowledge is demonstrated through performance of some tasks that can be categorised (Drageset, 2013). The focus on description of general domains of knowledge was based on the assumption that there is general knowledge of Mathematics that is used by teachers for different situations and tasks (Drageset, 2013). The focus on specific branches of Mathematics is due to the assumption that domains of knowledge needed for teaching different mathematical disciplines are different (Even, 1990; McCrory et al., 2012).

This review supports the claims that MKT is viewed and understood from different perspectives by different researchers (Hoover et al., 2016; Scheiner, 2015). My position on the perspectives of MKT is that it is domain specific, meaning that teaching of different domains of Mathematics involve performing of specific tasks rely on specific knowledge. This is the reason why my study aimed at exploring knowledge required for teaching geometric proofs.

2.8. Chapter Summary

This literature review has addressed several important issues in this study of MKT-GP. The first aspect of the review was concerned with how MKT has been studied in different parts of the world. The review showed that the first studies on MKT took several perspectives (Hill et al., 2005). Some studies were concerned with identifying tasks of teaching Mathematics (Ball et al., 2004), others on developing frameworks for understanding teacher knowledge (Ball et al., 2008) and other studies were concerned with establishing relationships between teachers' MKT and quality of instruction (Hill et al., 2008). The second aspect of the review was concerned with reviewing of MKT studies in Africa. The review has shown that most of the studies were done by Adler and her colleagues in South Africa under the QUANTUM project. The scholars were interested in exploring Mathematics for teaching which is relevant to the Southern African context. The studies found that Mathematics for teaching involved problem solving (Adler, 2005), decompressing (Adler, 2005; Adler & Davis, 2006), and is topic specific (Kazima et al., 2008).

The third aspect of the review involved studies on conceptualisation of MKT of mathematical proofs in general, and geometric proofs in particular. The studies showed that students experience difficulties in developing mathematical proofs because they are mainly taught the process of proof development without understanding why the proof works (Herbst, 2004; Herbst & Brach, 2006, Jones et al., 2009, Ding & Jones, 2009). The studies suggested that teaching of geometric proofs should involve problem solving to help students to understand why a proof works, and to appreciate the value of developing formal proofs in Mathematics.

The last aspect of the review involved a discussion on different theoretical frameworks for teacher knowledge, MKT, and teaching and learning of geometric proofs. The review revealed that most of the theoretical frameworks on MKT are general. However, the fact that demands for teaching different mathematical fields is not the same implies that there is a need of topic specific frameworks of teacher knowledge. Some of the frameworks that I have discussed in this chapter were used to analyse data for this study. In Chapter 3, I present the methodology of the study to show how the study was conducted.

CHAPTER 3

RESEARCH DESIGN AND METHODOLOGY

3.1. Chapter overview

The purpose of this research was to explore MKT-GP. In this chapter, I present the methods that I used to carry out the study. The presentation includes a description of the research design, the selection of participants, data collection techniques, and the methods of data analysis. I have also explained my role as a researcher, and the ethical issues that I followed during data generation and analysis.

3.2. Research paradigm

I conducted the study from an interpretivist epistemological position that what we know is always negotiated in cultural and social context. This implies that people construct meanings of reality according to their social settings. According to Cohen, Manion and Morrison (2007), the interpretivist paradigm attempts to understand the subjective world of human experience and to preserve the integrity of the phenomena under investigation. The interpretivist approach seeks to interpret and understand the world in terms of its actors (Cohen et al., 2007). As such, interpretivist researchers make efforts to get inside the context and to understand from within. This approach regard meanings constructed by the researcher as the original meanings from the peoples' action. As such, researchers guided by interpretivism are required to employ naturalistic methodologies like observation and interviewing. These methods ensure an adequate dialogue between the researchers and those with whom they interact in order to collaboratively construct a meaningful reality and to avoid misinterpreting the original meanings (Schwandt, 2003). The epistemological perspectives of interpretivism lead me to conduct the research using social constructivist theories of research methodology. The assumption of social constructivism is that human beings construct knowledge through their interactions with each other as well as through their individual processes (Ernest, 2010). Social constructivism emphasises that research on teaching and learning of Mathematics must focus on two areas. These are the importance of all aspects of the social context and interpersonal relations like teacher-learner and learner-learner interactions in learning situations, and the role of language, texts and symbols in the teaching and learning of Mathematics (Ernest, 2010). These suggestions imply that study of teaching and learning of Mathematics is better understood through observation of classroom practices. Ernest (1998) argued that social constructivism is a better approach for studying Mathematics Education because mathematical knowledge is manifested in different linguistic behaviours and symbolic conversations situated in a variety of different social contexts including the classroom. This means that that one can learn many things by observing classroom interaction between teachers and students.

3.3. Qualitative research approach

The study was guided by a qualitative research approach that has its roots in social constructivism (Baxter & Jack, 2008). Bryman (2008) describes four characteristics of qualitative research design. Firstly, qualitative research puts emphasis on words rather than numbers in data collection and analysis, thus the aim is to penetrate into an issue and gain adequate information and deep understanding of the issue being studied.

Secondly, qualitative research uses inductive approaches to relationship between theory and research, thus the purpose of qualitative research is to generate theories or hypotheses. Thirdly, qualitative research emphasises individuals' interpretation of the social world. Fourthly, qualitative research mostly involves collection of data from very small samples by using semi-structured or unstructured interviews or observations, and documentary data. A qualitative research design was suited to my study because I needed to understand in depth the interpretations that teachers make in relation to geometric proof development and its teaching. Thus, the qualitative research approach helped me to regard human beings as social actors whose behaviour is deeply rooted in context.

Bryman (2008) explained that there are three major criticisms against qualitative research. The first criticism is that qualitative research is too subjective. He explained that qualitative research findings are usually criticised of relying too much on the researcher's views about what is significant, and upon the close personal relationship that the researcher establishes with the respondent. The second criticism is that qualitative research is difficult to replicate because there are no standard procedures to be followed and data is generated from unstructured interviews and observations (Bryman, 2008). Bryman (2008) observed that this criticism arises from the claim that the investigator himself is the main instrument of data collection, as such they choose what to concentrate on and what to observe. Thirdly, he noted that findings from qualitative research are criticised of being generated from a very small sample, as such, it is impossible to generalise the findings to other settings because they are not a representative of the population being studied. However, Bryman (2008) reacted to these criticisms by arguing that the aim of qualitative research is not to generalise to population but to generate theory through understanding of behaviour in the context in which the study is conducted. This implies that qualitative research is mainly concerned with the quality of the theoretical inferences that are made out of the data and not the representativeness of the sample to its population.

3.3.1. Issues of trustworthiness

There were several basic elements to the study design that I followed to enhance the trustworthiness of the study (Baxter & Jack, 2008). The first element was to ensure establishment of credibility of the findings (Bryman, 2008). This meant ensuring that research was carried out using good practices in data collection and analysis. Credibility of the study was achieved through triangulation of data collection methods. Baxter and Jack (2008) explain that triangulation of data sources or data types is a primary strategy that can be used and would support the principle of credibility by ensuring that the phenomena is viewed and explored from multiple perspectives. Triangulation helps to enhance credibility of the study by enabling comparison of findings from different data sources for confirmation.

The second element for ensuring trustworthiness of the findings was through transferability (Bryman, 2008). This involved production of thick descriptions of every stage that I carried out during data generation and data analysis. I also enhanced transferability of the study by explaining clearly the purpose of the study and the research questions. The aim was to provide in-depth information for replication of the study.

The third element for trustworthiness was by ensuring dependability of the findings. This was done by keeping records of all phases of the research process in accessible form to enhance dependability of the theoretical inferences made in the study. The records included problem formulation, selection of research participants, field notes, interview transcripts, video observation transcripts, and data analysis decisions.

3.4. Qualitative case study research design

The fourth characteristic of qualitative research is about use of small sample in a contextualised setting, This led me to use case study research design. Cohen et al. (2007) pointed out that case studies are suited to the paradigm of interpretivism. Yin (2009) defined a case study research as, “an empirical inquiry that investigates a contemporary phenomenon in depth and within a real-life context especially when the boundaries between phenomenon and context are not clearly evident,” (p.18). Yin’s definition of case study limits application of case study designs only to situations where the researcher has little or no control on boundaries between phenomena and real-life situations. However, Woodside (2010) defined a case study as, “an inquiry that focuses on describing, understanding, predicting, and or controlling the individual (i.e., process, animal, person, household, organisation, group, industry, culture, or nationality),” (p.1). This definition means that a case study research design can be used to study any situation as long as the research issues focus on a specified unit.

Cohen et al. (2007) explained that the principle objective of case study research is to achieve deep understanding of the actors, interactions, sentiments, or behaviours occurring for a specific process through time. Deep understanding of phenomena in case studies is achieved through triangulation of methods not only in one-step, but also across the period of study. According to Woodside (2010), triangulation often includes (1) direct observation by the researcher within the environment of the case, (2) probing by asking participants for explanations and interpretations of operational data, and (3) analyses of documents written by the participant. The methods that are used in case studies afford a researcher to penetrate situations in ways that are not susceptible to numerical analysis (Cohen et al., 2007).

Yin (2009) explained three major criticisms that have been made on case studies. These include lack of rigor, lack of basis for generalisation, and taking long time to be completed. Lack of rigor could be due to unsystematic procedures employed by some researchers, like altering of cases by teachers in schools, and influencing of direction of findings and conclusions due to researcher's biases. Regarding the criticism on case alteration by teachers in schools, Yin (2009) argues that such criticisms exist perhaps because people have confused case study teaching and case study research. Teachers can alter case study materials to effectively demonstrate a particular point, but such alterations are forbidden in case study research. He suggested that all work and evidence be reported fairly in case study research to ensure rigor.

Yin (2009) reacted to the criticism about researcher bias, by arguing that bias can enter into any type of research design at any stage, the problems are not different whether in experiments or historic research. The difference may be that researcher bias may have been more frequently encountered than overcome in case study research. On the criticism of providing little basis for generalisation, Yin (2009) argued that issues of external validity are of little concern in case studies as the goal is not on statistical generalisations but on theory development. On the same issue of generalisation, Woodside (2010) argued that the goal of case studies is to expand understanding of social issues in their context and generate or generalise theories. The arguments by Yin (2009) and Woodside (2010) support those made by Bryman (2008) concerning aim of qualitative research. Thus, case study is applicable to my study because its aim is to emphasise optimising understanding of MKT and not to generalise to some population.

3.5. Research methods

Data generation for the study was conducted in three phases. The first phase involved observing lessons, which were video-recorded and followed up by short interviews with teachers about their lesson plan and presentation. The second phase involved administering of pencil and paper tests to teachers, and the third phase involved in-depth interviews with the teachers. Table 1 presents a summary of the types of data that I generated and their frequencies.

Table 1: Summary of data generation methods

Method of data generation	**John**	**Kim**	**Paul**	**Pike**
Lesson observation	14 lessons	16 lessons	7 lessons	18 lessons
Pencil and paper tests	2	2	2	2
Interviews	1 long interview 14short interviews	1 long interview 16 short interviews	1 long interview 7 short interviews	1 long interview 18 short interviews

I explain the details of data generation and data analysis in the following sections.

3.5.1. Lesson observation

Case studies are supposed to take place in a natural setting, hence observation is an important source of case study information (Yin, 2009). I conducted lesson observations with an aim of taking a naturalistic stance by going into the school to study knowledge for teaching geometric proofs in classroom settings. Cohen et al. (2007) argued that observational data is sensitive to contexts, and demonstrates strong ecological validity. They also point out that observation data enable researchers to see things that might otherwise be unconsciously missed, and it provides opportunity to discover things that participants might not freely talk about in interviews.

Lesson observations provided me with an opportunity to collect first hand data regarding exact things that teachers do when they are teaching geometric proof. I also found observation data to be suitable for the study because it provided additional information on PCK for geometric proofs. As Yin (2009) argued, “observational evidence is often useful in providing additional information about the topic being studied,” (p. 110). Considering the assumption that PCK is practical knowledge, it implies that study of knowledge for teaching geometric proof development needed to be conducted through observing lessons (Shulman, 1986). Furthermore, since the aim of this study was to explore the mathematical knowledge that the teachers were using in their work, it was necessary to consider the importance of context by observing classroom instruction. As Ball et al. (2001) argued, research approaches that focus on teachers only rather than on teaching could exclude opportunities to examine closely the mathematical territory and demands for the work of teaching. Through observational data, I was able to examine the demands involved in teaching geometric proofs.

I conducted lesson observation with the aim of exploring research Questions 2, 3 and 4, which were concerned with the nature of the tasks selected and used to enhance students' geometric thinking, how teachers assessed students thinking in geometric proof, and how teachers represented and explained geometric proof concepts to students.

During lesson observation, I did video-recording of the lessons. Sherin (2004) explains the advantage of a video as follows: "watching video affords the opportunity to develop a different kind of knowledge for teaching knowledge not of what to do next, but rather, knowledge of how to interpret and reflect on classroom practices," (p. 13–14). This implies that video lessons offer an opportunity for a researcher to recall the classroom scenarios and interpret them properly. Through watching of a video-recording one can identify different kinds of knowledge that were being used during the lesson.

I observed all lessons by each participant. Three teachers (Kim, Paul and Pike) were teaching circle geometry topic in form 3 in three different schools. There were two streams for each form (Form 3A and 3B). The students in these streams were learning same topic of circle geometry. One teacher (Pike) was teaching in form 2which had two streams as well (Form 2A and Form 2B). In both streams, Pike was teaching same topic of polygons. I observed all lessons under each topic in each stream.

Mertens (2010) describes some negative effects of observation on the data that is generated. The effects are divided into two categories; reactive and investigator effects. Reactive effects can occur when participants know that they are being observed; they may decide to behave in a different way either to conform to the desired behaviour, or to portray worse behaviour (Mertens, 2010). This effect was avoided by observing the lessons for a long time to get the teacher and students used to observation and act normally. Investigator effects can occur due to personal biases and researcher's selective perceptions (Mertens, 2010). This happens when the investigator only observes either the positive, or the negative side of behaviour. The investigator effect was minimised by recording all lesson scenarios and transcribing them without any pre-determined themes. Furthermore, Mertens (2010) explained that observation data can also be difficult to analyse and interpret as seen behaviours can be complex to categorise. This effect was minimised by using a priori themes from different frameworks for categorising and interpreting the data after transcription.

3.5.2. Pencil and paper tests

It is generally agreed that content knowledge is a fundamental requirement for teaching of Mathematics. The National Council of Teachers of Mathematics (NCTM) (2000) argued that, "effective teaching requires knowing and understanding Mathematics, students as learners, and pedagogical strategies," (p. 17). Baumert and Kunter (2013) claim that PCK cannot be conceived in the absence of CK. Ball et al. (2001) encouraged researchers involved in exploring the work of teaching Mathematics to use methods that probe into both teachers' content knowledge as well as teaching. These scholars suggest that researchers should use questions that respond to students' confusion, explain a mathematical procedure whose meaning is buried inside rules of thumb, or consider connections among ideas. They argued that, "such questions create conditions where teachers have to make explicit their understanding of the mathematical ideas and procedures behind the questions," (Ball et al., 2001, p. 444). Therefore, in trying to explore knowledge for teaching geometric proofs, I asked teachers to answer two questionnaires with an aim of probing knowledge used for geometric proof development and its teaching. As Manouchehri (2008) has pointed out that asking teachers to explain, justify, and defend their choice of representations, assessment, and intervention, can provide them opportunities to engage in both mathematical and pedagogical reasoning.

In relation to Ball et al. (2001) suggestion, Questionnaire 1 (see Appendix 3) required participants to develop both narrative and computation proofs. The questionnaire aimed at exploring content knowledge required for teaching geometric proofs. The questionnaire contained two narrative proof tasks and three computation proof tasks. The tasks were taken from the senior secondary school students' Mathematics textbook and from the Malawi National Examination Board (MANEB) Question papers.

Questionnaire 2 (see Appendix 4) contained students' responses to the three computational proof questions in Questionnaire 1. The teachers were asked to analyse the questions to identify the site of error and its cause, and to explain what they would do to eradicate the errors that they identified. The aim for administering Questionnaire 2 was to generate data for examining knowledge for analysing students thinking. This is because being able to analyse students' thinking is a vital element of PCK (Shulman, 1986, Ball et al., 2008, Baurmert et al., 2013). As Shulman (1987) argued, one way of using PCK is task evaluation. Therefore, the aim of administering the second questionnaire was to explore knowledge that is needed for assessing students' thinking in geometric proof development.

The questionnaires were administered to each participant separately for two hours. At the beginning of each test, I explained the aims of the test to the participant and asked for informed consent. Asking the teachers to develop geometric proofs was a sensitive mode of data generation because it meant revealing their competences in geometric proof development. As such, I emphasised to the participants that the aim of the two questionnaires was not to evaluate their competences but to learn from their responses.

3.5.3. Qualitative interviews

Qualitative interviews are regarded as a powerful method of producing knowledge of human actions and situations. Yin (2009) explained that interviews are one of the most important sources of case study information. Shulman (1987) argued that teaching necessarily begins with a teacher's understanding of what is to be learnt and how it is to be taught. The aim of conducting interviews with the teachers was to gain first-hand information regarding their understanding of geometric proof development and its teaching. As Cohen et al. (2007) explained, interviews are strong instruments in the sense that they provide in-depth information about phenomena. Mertens (2010) suggested that, when you want to fully understand someone's impressions or experiences, or learn more about their answers to the questionnaires, you could use interviews. Interviews also allow for probing and asking of follow up questions for clarity and more information. By probing into teachers' conceptualisations about knowledge for geometric proof development and geometric proof teaching, I was able to generate insights into knowledge required for teaching geometric proof development.

There were two types of interviews conducted with the teachers: short and long interviews. I conducted the short interviews with each teacher after observing the lesson. I did not use any interview guide during short interviews because the questions that I asked the teachers depended on the lesson. The aim of these interviews was to follow up on some important issues identified during the lesson. The length of each short interview was between five to ten minutes.

Long interview is an intensive questioning of informants selected because of their special knowledge, experience and insights (or ignorance) of the topic under study (Woodside, 2010). Woodside (2010) observed that objectives of long interviews include learning the thinking, feeling and doing processes of the informants, and understanding of the informants' worldview of the topic under study in their own language. The selected teachers were regarded as key informants because they have long teaching experience in Mathematics. This implies that the teachers have special knowledge for teaching geometric proofs, which they have accumulated over the years.

Woodside (2010) explained that long interviews have several characteristics: (1) they have a duration of 2 to 6 hour conversation between interviewer and respondent, (2) the respondents are interviewed in their own life space (environment related to the topic under study), (3) they use open-ended or semi-structured questions with deeper exploration of unexpected topics related to the study, (4) the responses are tape-recorded during the interview, (5) the responses are verified using triangulation of research methods. In this study, the minimum duration of interviews with teachers was 2 hours. I conducted the long interviews with an interview guide (see Appendix 5) which contained open-ended questions. I also used follow-up and probing questions to prompt deeper into the teachers' views (Yin, 2009). Each interview conversation was tape-recorded with the informed consent from the respondent.

The major weakness of interviews is that they can have reactive and investigator effects (Woodside, 2010). As the interviewer is more of a guest in the field of study like the school, as such, the interviewees may only try to give the socially desirable behaviour or they may become emotionally sensitive during the interviews (Woodside, 2010). Woodside (2010) noted that investigator effects like personal biases may occur due to poor interviewing skills whereby interviewer can cause interviewee bias. I argue that for my study, reactive effects were greatly reduced because I spent long a time with the participants before conducting the interviews. My long-time presence in the participants' classrooms during lesson observation helped to establish trust with the interviewee and to minimise reactive effects.

To reduce the issue of investigator effects I tried to establish investigator integrity and use an interview guide. I was aware that knowledge produced during interviews depended very much on social relationship between the interviewer and the interviewee. As such, I took the responsibility of seeking informed consent, assuring confidentiality, and creating an environment where the interviewee would feel free and safe to talk of private life. This is because interviews have an ethical dimension as they concern interpersonal interaction and produce information about human condition (Cohen et al., 2007). Therefore, informed consent, confidentiality and emotional harm were regarded as important ethical issues to be addressed during the interviews. Towards the end of the interviews, I shared with the participant the main points captured from their views with an aim of checking on the accuracy of my interpretations and to make sure that I captured the participant's perspectives accurately. Lastly, I asked for the respondent's feedback on the main points captured during the interview with an aim of allowing respondents to make corrections on what they did not want to say and add on what they forgot to say.

3.6. The pilot case study

Yin (2009) suggested that it is important to conduct a pilot case study before doing the main case study. He explained that a pilot case study helps a researcher to refine their data collection plans with respect to both the content of the data collection instruments and the procedure. I used Yin's (2009) criteria of convenience, access and geographic proximity to select a sample for the pilot case study. A sample of four first year Master of Science Education students who were majoring in Mathematics Education participated in the pilot case study. As these students were already on campus, it was easy to have access to them and establish rapport. Furthermore, the selection criteria for these students into the programme of Master of Education is that they must have a minimum of three years of teaching secondary school Mathematics, and they are supposed to have obtained at least a credit in their Bachelor of Education degree. Therefore, I decided to conduct the pilot study with these student because of their long teaching experience, and because they are assumed to be above average teachers due to their quality of first degrees. The assumption was that due to their experience and knowledge, these teachers could provide constructive and objective feedback in terms of the quality of the instruments.

Firstly, the pilot sample took the two pencil and paper tests individually. The duration for each test was 2 hours. Secondly, they were involved in focus group discussions using the interview guide. The interviews were audio-recorded and transcribed for further analysis. The main reason for piloting the instruments was to ensure validity of the data collection instruments in terms of generating the data that I needed (Bryman, 2008). The other reason for piloting the instruments was to find out if they were realistic in terms of level of difficulty. As such, I analysed the pilot data with an aim of finding out if it was relevant for achieving the aims of the study. I maintained the items that seemed to make sense to the pilot sample and those that provided the intended data, and revised those that seemed to be ambiguous and providing irrelevant information.

3.7. Selection and description of participants for the study

I conducted the study with four male secondary school teachers who were purposively selected from three schools in one of the cities in Malawi. Like in Kazima et al. (2008), the identification of appropriate participants began with finding a reputably successful, qualified and experienced secondary school Mathematics teacher who was willing to take part in the study. Therefore, the first requirement was that the participants should be sought in either national or conventional secondary schools. These are secondary schools which normally have good quality teachers in terms of qualification and teaching practice. The second requirement was that the teachers to participate in the study must have at least four years of teaching secondary school Mathematics.

There were two assumptions for selecting teachers with long teaching experience. Firstly, I assumed that by the end of four years' experience, the teachers would be familiar with the whole secondary school Mathematics curriculum. Secondly, I assumed that teachers with long experience have rich information about knowledge for teaching geometric proofs which they have accumulated over the years. This argument is based on a study conducted by Herbst and Kosko (2012) who found that teachers with different experiences held different levels of knowledge for teaching Geometry and suggest that MKT-Geometry may be learnt from experience of teaching Geometry. Table 2 presents a summary of the information about qualification and experience of the teachers.

Table 2: Information about the teachers

Pseudonym	Qualification	Major/ minor teaching subjects	Teaching experience	Level they were teaching	Type of secondary school
John	Bachelor of Education Degree	Agriculture/ Mathematics	Thirteen years	Form 3 (grade 11)	Conventional secondary school
Kim	Primary Teacher Certificate and Diploma in Education.	Mathematics	Eighteen years	Form 3 (grade 11)	National secondary school
Paul	Bachelor of Social Science degree	Economics/sociology	Six years	Form 3 (grade 11)	National secondary school
Pike	Bachelor of Education Degree	Mathematics	Six years	Form 2 (grade 10)	Conventional secondary school

Paul is not qualified to teach Mathematics at secondary school. However his economics courses required him to take some Mathematics courses up to second year of university education. It can, therefore, be argued that Paul had enough SMK for secondary school Mathematics. Kim first taught at primary school for 14 years. Malawian primary school teachers do not specialise in any teaching subject at teacher education college, as such, they teach all primary school subjects including

Mathematics. Later on Kim upgraded to diploma in education with major in Mathematics in 2012, and has been teaching secondary Mathematics for 4 years.

Pike and John teach at the same conventional secondary school. These are secondary schools which mainly consist of students with average performance at standard eight primary school leaving certificate examinations (PSLCE). These are national examinations that students write for selection to various secondary schools in Malawi. Paul and Kim teach at different national secondary schools in the city where I conducted the study. These are secondary schools whose students score best grades at PSLCE. The teachers at national secondary school are regarded as best teachers because their students score best grades at both JCE and MSCE.

3.8. Data analysis procedures

Ritchie et al. (2003) provided a detailed explanation of activities to be carried out during qualitative data analysis. They proposed that qualitative data analysis should be conducted in two key stages; managing the data and making sense of the data through either descriptive or explanatory accounts. Qualitative data has to go through the process of data management because it is produced in large volumes (Ritchie et al., 2003). One of the methods of managing and describing qualitative data is by applying a thematic framework. This is a framework that is used to classify, organise and describe data according to key themes, concepts and emergent categories (Ritchie et al., 2003). There are two ways of deriving themes for data analysis; inductive and deductive theme development (Fereday & Muir-Cochrane, 2006). Inductive theme development involves developing of initial themes from the data. It involves reading and re-reading of the data several times to identify patterns from the data and allow themes to emerge direct from the data. The emerging themes become the categories for data analysis (Fereday & Muir-Cochrane, 2006). Deductive approach involves the use of a priori template of codes to be applied as a means of organising data for subsequent interpretation (Fereday & Muir-Cochrane, 2006). An analyst is supposed to first define the codes of the template before commencing in-depth analysis of the data. The template might be drawn from the data or as a priori themes from either research questions or analytical framework.

Cohen et al. (2007) presented five methods of managing and describing qualitative data during thematic analysis. These are by group, individual, an issue, research question and instrument. I used research question method to organise and present data analysis. Cohen et al. (2007) state that organising data analysis by research question is a very useful method for several reasons. Firstly, because it draws together relevant data from all sources and collate it to the research question. Secondly, it links research questions to the data there by closing the 'loop' on the research questions that were typically raised in the early part of an inquiry. Thirdly, the method contains a degree of systematisation in the sense that it is possible to draw patterns, relationships and comparisons across data types. This implies that analysis by question allows convenient and clear exploration as data from different sources can be organised consecutively.

Research method of organising and analysing data allowed me to connect research questions and results clearly, hence ensuring that the data analysis is geared towards answering the research questions and achieving study aims. The method also allowed me to compare results from different sources to search for consistencies and inconsistencies. This created an opportunity to confirm data from different sources and look for meanings of either consistencies or inconsistencies in the results. I conducted both deductive and inductive thematic analysis on each type of data because some data could not fit into the a priori themes during deductive thematic analysis.

Cohen et al. (2007) suggest that qualitative data analysis must begin as soon as data generation begins. As such, I started conducting data analysis soon after collecting the first data. This helped me to identify and understand recurrent themes exhibited by the data. Analysis of data during data collection also helped me to notice the moment I reached saturation point. This towards the end of the topics where no new themes emerged from the data.

During deductive thematic analysis, I used a priori themes from different theoretical frameworks to analyse the data because it was difficult to identify initial themes in some of the data. This meant that use of inductive reasoning only, for example, in analysing geometric proofs would have led into leaving out some important themes that were difficult to identify. As Cohen et al. (2007) explained, one of the advantages of using a priori themes is that it offers opportunity of finding themes or examples that do not appear in an orderly way in the data. But Cohen et al. (2007) argued that while deductive thematic analysis is an economical approach to handling, summarising and presenting data, it raises three main concerns:

(i). The integrity and wholeness of each individual can be lost, such that comparisons across the whole picture from each individual are almost impossible.

(ii). The data can become decontextualised. This may occur in two ways: first, in terms of their place in the emerging sequence and content of the interview or the questionnaire and second, in terms of the overall picture of the relatedness of the issues, as this approach can fragment the data into relatively discrete chunks, thereby losing their interconnectedness.

(iii). Having had its framework and areas of interest already decided pre-ordinately, the analysis may be unresponsive to additional relevant factors that could emerge responsively in the data (Cohen et al., 2007, pp. 347-348).

The first concern of losing integrity and coherence of an individual is applicable if a researcher is interested in foregrounding descriptions of respondents rather than issues. The second concern about creating of disjoints among different types of data might be experienced when issues across respondents are presented across all sources of data, which might be cumbersome as pointed out earlier on by Cohen et al. (2007). In the case of this study, the same pre-ordinates were applied separately to each type of data that aimed at answering the same question. The results from the different types of data were presented together after being analysed further to search for consistencies and inconsistencies. The third concern about leaving out important issues that might not have been captured in the a priori themes was addressed in two ways. Firstly, by taking note of other issues that were emerging in the course of data analysis and adding them to the pre-ordinate themes. Secondly, by conducting inductive thematic analysis on data that could not fit into a priori themes.

3.8.1. General activities carried out during thematic analysis

The first activity in data management that involves transcription was mainly carried out on lesson observation data and interview data. This is because these types of data were very huge as compared to pencil and paper test data that was already in a manageable form. The second activity in data management was a familiarisation process and was carried on all types of data. This involved reading and re-reading the transcribed data several times to be acquainted with the data and to decide on the method of organising and presenting it (Ritchie et al., 2003). The third activity was to code and sort the data using a priori themes. The data that could not fit into a priori themes were read further to identify themes for its coding and sorting. Coding of the data involved identifying parts of the data that applied to each them. Sorting involved putting together pieces of data with similar themes for descriptive analysis. Descriptive analysis involved looking within a theme across all cases to synthesise consistencies and inconsistencies (Cohen et al., 2007).

3.8.2. Analysis of geometric proofs developed by the teachers (Questionnaire 1 data)

Analysis of Questionnaire 1 data aimed at answering the first part of research Question1 that was concerned with how secondary school teachers conceptualise geometric proof development. Firstly, I conducted deductive thematic analysis on the narrative geometric proofs that were developed by the teachers. I used the framework of norms for situations of doing proofs by Hebst et al. (2009) as a priori themes for analysing the proofs.

I conducted the process of identifying and capturing segments related to a particular theme by reading each proof several times to capture areas that involved a particular norm. I analysed each proof that was against all norms with an aim of capturing information which represented a particular norm. To develop descriptions of how each norm was captured in the proofs, I conducted a cross-sectional analysis of each norm. This was done by analysing the grouped results under each norm to find consistencies and inconsistencies. Analysis of each norm across all proofs helped me to find out what was happening within each norm and to identify connections and differences on how each norm was conceptualised. Secondly, I conducted inductive thematic analysis on the proofs with an aim of analysing the proofs further to identify additional themes that were not reflected by Herbst et al.'s (2009) framework.

3.8.3. Analysis of interview data

Analysis of interview data also involved both deductive and inductive thematic analysis. I began by conducting deductive thematic analysis using the COACTIV model. Firstly, I coded and sorted the data under the two categories of COACTIV model (CK and PCK). Then I did further coding and sorting of the PCK data using the three categories of PCK in the COACTIV model as a priori themes (see appendix 6). I analysed the PCK data that could not be coded and sorted using the PCK categories of the COACTIV model inductive thematic analysis. This involved reading the data further to identify themes for data coding and sorting.

As earlier explained, one of the weaknesses of the COACTIV model is that it does not categorise CK. As such, further analysis of the data under CK category involved conducting inductive analysis. I read the data several times and identified themes for coding, sorting and classifying the data for descriptive analysis.

3.8.4. Analysis of lesson observation data

Lesson observation provided data for answering research Question 1 about task selection and implementation, and research Question 2 about teachers' ways of explaining and representing geometric proofs. To be able to answer these questions, I transcribed all lessons and analysed them in full as expressed by (Kazima et al., 2008). The first phase of lesson transcript analysis was done with an aim of choosing transcripts that could be used for each research question. The second phase of lesson transcript analysis involved identification of units of analysis from the lesson transcripts. I needed to divide each lesson transcript into units of analysis in a way that would enable me to analyse the data and answer each question appropriately (Kazima et al., 2008).

Herbst and Chazan (2009) describe identification of unit of analysis as an important conceptual element for establishing data set of classroom interactions. They defined units of analysis as the units on which observations are aggregated and claims are made. Another use of units of analysis or sub-units is to segment the data. According to Herbst and Chazan (2009), the purpose of research acts as the guide in defining a unit of analysis. It can be a whole lesson, lesson section (like introduction, body and conclusion), or according to instructional objective. In the following sections, I explain how I identified and analysed units of analysis for both sets of data.

3.8.4.1. Analysis of task selection and implementation

For the lessons that were chosen for analysing the teachers' selection and implementation of tasks in classroom, the identification of unit of analysis was according to the purpose of the task (Herbst & Chazan, 2009). The teachers used the tasks for three purposes during geometric proving, for narrative proving, exemplification and assessment. As such, I divided the lesson transcripts into three units depending on the purpose that the task was intended to serve in the classroom. I analysed each unit separately to determine the cognitive level of the task, and the mode of implementation of each task.

After identifying the units of analysis, I used the four categories of cognitive demands of a task suggested by Smith and Stein (1998) to analyse the cognitive level of the tasks. During analysis of task implementation, I concentrated on analysing the teachers' and students contributions to the implementation of the task. The aim was to find out if the teacher provided appropriate support for students' involvement in exploring the task. As such, I analysed the questions asked by the teachers, and the suggestions that were made either by the teacher or by the students.

3.8.5. Analysis of teachers' ways of assessing students' thinking

I analysed Questionnaire 2 by examining how the teachers identified mistakes from a flawed student's solution and what they proposed to do to address the mistake. The analysis was guided by PCK category of COACTIV model about knowledge of assessing students' thinking. This category of PCK contains three aspects of assessing students thinking. They include identifying of students' errors, figuring out the causes of the errors, and devising appropriate strategies for addressing the errors (Krauss et al., 2008). I used the three aspects of PCK as a priori themes to guide me in examining how a teacher analysed students' thinking.

3.8.6. Analysis of teachers' explanations and representations of geometric proofs

I divided the lessons that were selected for analysing teachers' explanations and representations into units of analysis according to the purpose of activities that were being conducted at each stage of the lesson (Herbst & Chazan, 2009). The change in purpose of a lesson activity implied a demarcation between units of analysis. I used the phases of problem solving suggested by Polya (1945) as a priori themes for identifying the purpose of each lesson activity. I regarded each unit of the lesson as a phase of problem solving and analysed it using Polya's description of what is involved at each phase.

3.9. Ethical issues and my role as a researcher

I observed several school based research ethical principles proposed by Cohen et al. (2007) and Berg (2001) during data generation and data analysis. I followed official protocol to gain access and acceptance into the school (Cohen, et al., 2007). Before visiting the schools, I got an ethical clearance from the faculty of education, granting me an approval to do a study with the secondary school teachers and observe their geometric proof lessons. To get entry into the schools, my college wrote an official introductory letter to the Education Division Manager to seek permission to allow me to conduct research in some secondary schools in the city (See appendix 1). In response to the letter, the Education Division Manager wrote a letter introducing me to head teachers of secondary schools in the city (see Appendix 2). The head teachers who were also the gatekeepers of the schools helped me to gain access to Mathematics teachers and their classes.

Upon meeting each teacher, I addressed the issue of informed consent. Berg (2001) described informed consent as the participant's agreement to participate in the research knowingly and by their choice. I addressed several issues to gain informed consent from each teacher. Firstly, I introduced myself as a researcher and explained to the teacher the study aims, the benefits of the study, the procedures of data generation, data analysis and reporting of the findings. As part of gaining informed consent, I also discussed with the teacher the issue of confidentiality. Berg (2001) described confidentiality as an active attempt to remove from the research records any elements that might indicate the subjects' identities including the name and the school in this case. I explained to each teacher that I was going to use pseudonyms instead of their real names during data analysis, and that I would not request them to write their names or name of their school on any questionnaire. The teachers had to express verbally their willingness to participate in the study.

Another ethical consideration lay on my personal influences on the study as a researcher. In qualitative research, researchers are regarded as data collection instruments in the sense that they decide the questions to ask, the order of questions, what to observe and what to record or write down. Mertens (2010) suggested that researchers must reflect on their own values, interests, assumptions, experiences, beliefs and biases, and monitor them to minimise their influence on the study. As a researcher, I realised that my role, views and values could have an impact on how I generate and interpret data. In this study, I addressed the issue of researcher effects and tried to control the biases that I was aware of by making and keeping detailed descriptions of daily actions.

Concerning my role as a researcher, I collected data as an observer as participant. The participants knew me as a researcher. During lesson observation, I had less contact with the teacher and the students whom I was observing (Cohen, et al., 2007). This is because I did not aim at manipulating any of the events, but to observe the events independently in their natural occurrence (Yin, 2009). Complete detachment from participation in the activities that I was observing helped enabled me to observe objectively, and afforded me an opportunity to record the lessons and to write some notes.

3.10. Chapter summary

In this chapter, I have presented the methodology that was used to conduct the study. By taking an interpretivist view that people construct knowledge through their experiences, I used social constructivism approaches to conduct the study. I utilised qualitative case study design with an aim of capturing secondary school teachers' experiences regarding the teaching and learning of geometric proof development. I generated data for the study through lesson observations, interviews and pencil and paper tests. I analysed the data by using both deductive and inductive thematic analysis. In Chapter 4, I present the results from analysis of the three types of data.

CHAPTER 4

RESULTS

4.1. Chapter overview

In this chapter, I present the findings from analysis of the three types of data that were generated for the study. The findings are presented against each research question. I used different analytical frameworks to analyse the data. Questionnaire 1 data was analysed to examine how teachers conceptualise geometric proof development using framework of norms of situations for doing proofs by Herbst et al. (2009) as a priori themes. The interview data was analysed using categories of the COACTIV model as a priori themes to examine how teachers conceptualise geometric proof development and the teaching of geometric proving. Smith and Stein's (1998) framework for analysing tasks was used to analyse lesson observation data to find out how teachers select and implement tasks during teaching of geometric proving. The sub-categories of knowledge of students' thinking from COACTIV model were used as guiding themes for analysing Questionnaire 2 data to learn how teachers assessed students' thinking in geometric proving. The phases of problem solving by Polya (1945) were used as guiding themes for analysing lesson observation data to find out how teachers explain and represent geometric proofs to students.

4.2. Teachers' conceptualisations of geometric proof development and its teaching

In this section, I intend to approach research Question1, which aimed at exploring teachers' conceptualisations about geometric proof development and the teaching of geometric proving. I started data analysis by examining the first part of the question that is concerned with how teachers conceptualise geometric proof development and then analysed the data that was about how teachers conceptualise the teaching of geometric proving. The teachers' conceptualisations of geometric proving were explored through analysis of Questionnaire 1 and interview data, while those of teaching of geometric proving were analysed from interview data only. In Sections 4.2.1., I present findings on how teachers conceptualise geometric proving and in Sections 4.2.2., I present findings on how the teachers conceptualise the teaching of geometric proving.

4.2.1. Teachers' conceptualisations of geometric proof development

Two narrative proof geometric tasks were proved by the teachers. The tasks were as follows;

Task (1). AB is the diameter of a circle with centre O and AC is a chord. OD is perpendicular to AC. Prove that BC is two times OD.

Task (2). MNOP is a parallelogram. H is a point on MN. HO = NO. Prove that MHOP is a cyclic quadrilateral.

I asked the teachers to prove the tasks using several approaches. My assumption was that the different approaches for developing the proofs would require different reasoning skills, hence providing in-depth data for exploring how the teachers conceptualise geometric proof development. Furthermore, as Herbst et al. (2009) noted that not all norms might be enforced in every instant of doing proof. Some norms are implicit while others are explicit in a particular proof. Asking teachers to develop proofs using different types of approaches would ensure that the norms that were implicit in one approach could be explicit in another approach. The proofs for both tasks showed that before starting to prove the tasks, all teachers represented the word task into a diagram. The findings also show that all teachers used several symbols to write the proving statements. The proving statements that were written by the teachers were logically sequenced, and justified with reasons. During interviews, all teachers said that geometric proof development involved following a stepwise process of identifying given information and the statement to be proved, adding a construction to the diagram if necessary, deciding on the theorem to be used, and writing of proving statements in sequential order to connect the given information and the conclusion. Table 3 contains a summary of results from the four teachers in relation to how they developed the geometric proofs for both tasks, and their conceptions about geometric proof development.

Table 3: Summary of teachers' conceptualisations of geometric proof development

Category	John	Kim	Paul	Pike
Abstract reasoning.	- Justified all proving statements with abstract reasons. -Some of the abstract reasons were valid while others were not.	- Justifiedall proving statements with abstract reasons.	-Justified all proving statements with abstract reasons.	-Justified some proving statements with abstract reasons - Justified one proof with measurements.
Use of labels.	-Introduced some labels in	-Introduced labels in the	-Did not introduce any	-Did not

	the diagrams but did not use them in the proof.	diagrams and used them for developing the proofs.	labels in the diagrams.	introduce any labels in the diagrams.
Multiple solution paths.	-Came up with two approaches for developing proofs for each task. -The two proofs for task 1 were not correct but those of task 1 were correct.	- Came up with two approaches for developing proofs for each task. - One proof involved algebraic reasoning.	- Came up with two approaches for developing proofs for each task. - Used algebraic reasoning in one of the proofs.	-Came up with two proofs for task 1, and only one proof for task 2. -The second proof for task 1 was based on measurements.
Views about geometric proof development.	-Geometric proving is a process of developing of convincing logical statements. -Proving statements must be written using given information and prior knowledge of geometric properties.	-Geometric proving involves writing of logically sequenced statements. -The aim of proving is to show that a geometric statement is true. -Geometric proving involves generating and	-Geometric proving needs understanding of the given information and the statement to be proved. -Geometric proving involves developing of geometric statements using information	- Geometric proving must start from the given information and end with the statement required to prove. - Writing of the proving statements has to be done

		putting together statements related to the given information and the statement to be proved. -Geometric proving is an opportunity for convincing the students that Geometry theorems are true.	from the problem statement, and generating new statements from the diagram.	while focusing on the statement to be proved. - Development of geometric proof is only possible when there is a geometric diagram. -A geometric proof is supposed to use abstract reasons and not measurements.

In the following sections, I have used the norms for situations of doing proofs by Herbst et al. (2009) to discuss in detail the results that have been summarised in Table 3.

4.2.1.1. Writing a sequence of steps (each consisting of a "statement" and a "reason") – norm 1

Both the interview and questionnaire data analyses results have shown that the teachers conceptualise geometric proving as a process of writing a set of proving statements that are accompanied by reasons. During interviews, the teachers explained that they regard geometric proving as a way of arguing which aims at convincing the reader that something is true. Because of this view, the teachers explained that proof must be developed in a convincing manner, by ensuring that arguments are connected and justified with valid reasons. John said that development of convincing arguments require knowledge of making logical arguments and geometrical reasoning during proving. John emphasised this point in the following ways:

> *For example, to prove that those angles are equal is not easy because when you are proving you need to use reasoning. You need to know why the angles are equal or what makes such angles to be equal. So you need to come up with connected statements that can help you to show that the angles or sides are equal. The statements are supposed to be accompanied by reasons. So you do not just say this is equal to that but you need to give a reason for your argument...When you are proving you think of the sequence, so that whoever reads your proof must see that the angles or lines are equal because of abc, and you make sure that your arguments are connected from the start to the end, that is logical sequencing of arguments, so that the reader must be able to connect your arguments.*

The extract suggests that when developing a geometric proof, one must think of how to connect the proving statements so that they are logically sequenced. This helps the reader to follow the proof and understand it. There is also an emphasis on use of reasoning during proving to justify the geometric statements. The views expressed by John in the extract were also reflected in most of the proofs that the teachers developed. The proofs consisted of statements that were linked together in a logical manner. There was progression from one statement to the other. Most of the statements in each proof were supported with reasons. The statements that were not supported with reasons were simplified versions of the statements that were supported with reasons, so the same reason applied to these statements. Figure 3 is a proof developed by John in response to task 2.

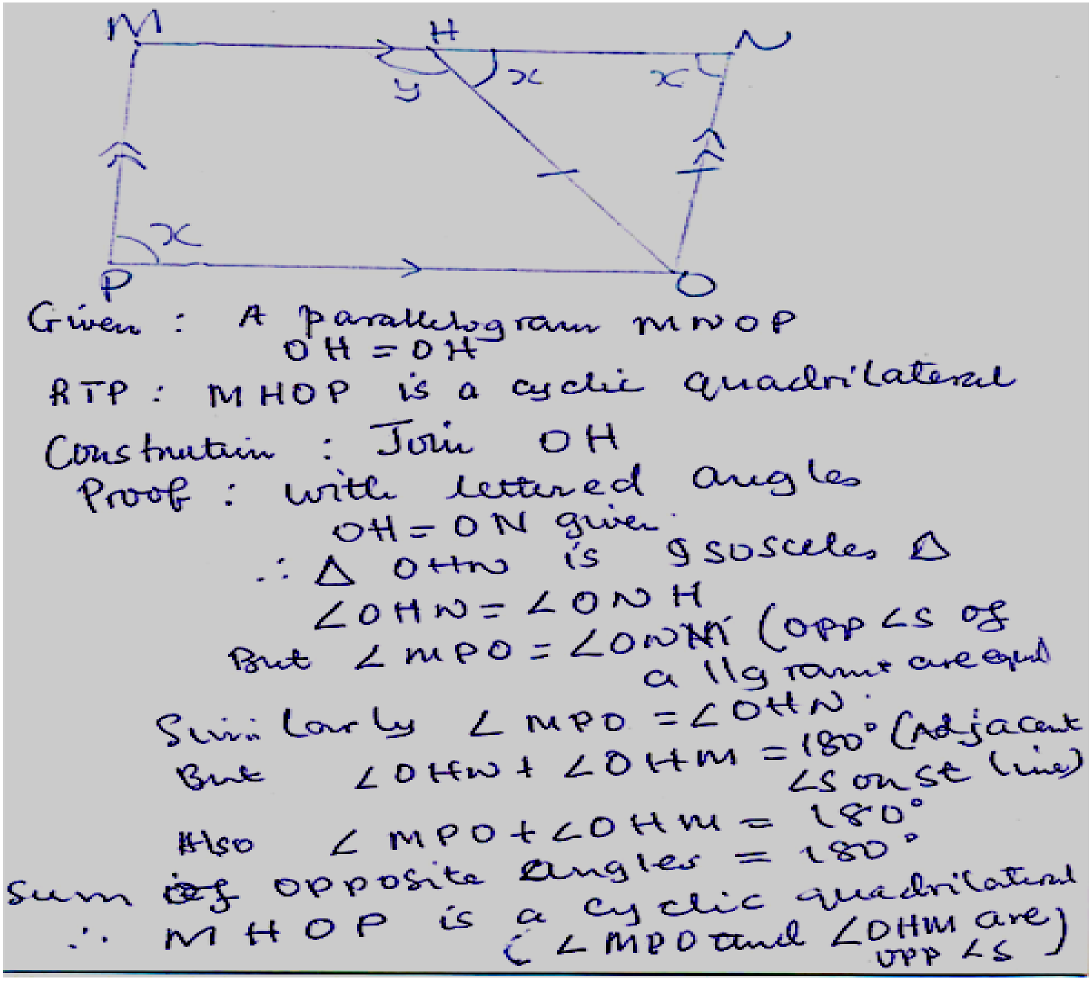


Figure 3: John's proof of tasks focusing on sequence of statements and reasons

The proof in Figure 3 comprises of statements that are connected to each other. Each statement is supported with a reason. The statements are also written in a logical order. Statement 1 and 2 of the proof justify why OHN is an isosceles triangle. The properties of an isosceles triangle are used to show that angle OHN is equal to angle ONH. The property of opposite angles of a parallelogram is used for `showing that angles OHN and OPM are equal. This argument is used to show connection between opposite angles of a cyclic quadrilateral that is a condition that the teacher decided to use in order to show that MHOP is a cyclic quadrilateral.

However, there were also some proofs which contained statements whose reasons required further justification. Figure 4 which is a proof developed by John is an example of proofs containing such statements.

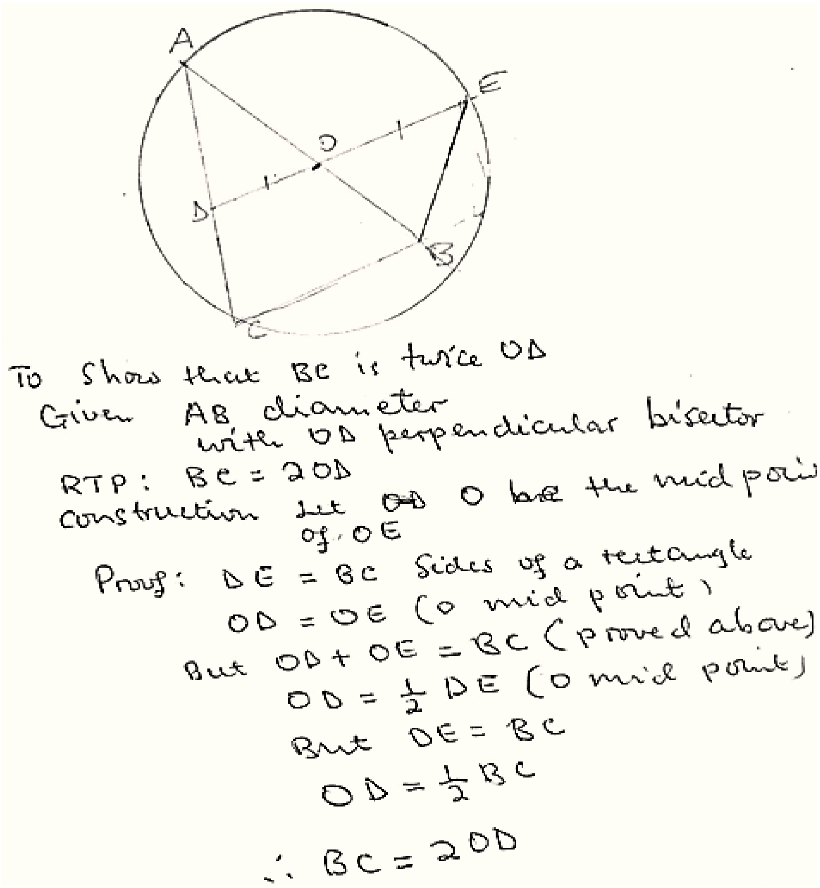


Figure 4: John's proof for task 1 focusing on statements requiring further justification

The proof in Figure 4 comprises a set of statements that are supported with reasons. However, the reasons for all statements are not valid. For example, the statement that DE = BC, because these are sides of a rectangle is not valid. The proof contains no justification that quadrilateral DEBC is a rectangle. Likewise, statement that OD = OE because O is mid-point of DE is not correct. OE is radius because it is distance from the centre to the circumference, while OD is less than radius because point D is hanging inside the circle.

4.2.1.2. The first statement is the assertion of one or more "given" properties of a geometric figure – norm 2

During interviews, the teachers did not explain the details of information that should be contained in the first statement of the proof. They only emphasised that the proving statements should be logically connected. Findings from the proof analysis also show that there were very few statements that contained the reason "given" in the first statement. The statement that was justified by a "given" reason could be found anywhere; either in second or third statement but not as conclusion of the proof. In some proofs, the given information was combined with other geometric properties to develop other proving statements that were used for developing the proofs. What seemed to be emphasised in the teachers' explanations was the connectedness of the proving statements. An example of a proof that contains 'given' as the reason for the first statement is the proof by John in Figure 3 on page 103.

4.2.3. Each other statement asserts a fact about a given specific figure in a diagrammatic or generic register – norm 3

The findings from both interviews and proof analysis show that the teachers view geometric proving as an activity that involves developing of proving statements from a diagram. During interviews, the teachers emphasised that a diagram must be considered as a source of proving statements. John and Paul said that proving involves developing geometric statements from a diagram. John said that it is very important for teachers to know how to use the diagram for developing geometric statements during proving. Paul explained that all information that is supposed to be used for developing the proof is always provided in the given statement and on the diagram. He explains as follows:

> *Proving is like answering a comprehension passage whereby the answers are right there on the statement and the diagram that you are given. So when given a circle, the answers are right there in that circle, the lines that are in there, the angles that are in there, and there is information that is given, so what is it that you are given, and what is it that you are asked to do? So once you understand that you are required to find this, then you need to relate it to the rest of the given information and the information that you already know to come up with the statements for the proof that you are asked to make.*

The extract shows that developing of geometric statements has to be done through interaction with the diagram. What is required during proving is to understand the given information and the problem to be proved. Once these are understood, the next step is to relate the given information to what is required to prove using the diagram. This is done by finding relationship between the given information on the diagram and other geometric properties and theorems. The teachers' views were also reflected in the statements that were used for developing the proofs. The proofs show that all proving statements were developed from diagrams that were drawn by the teachers. The statements were developed by combining the given information and the prior knowledge of geometric properties of the diagram. This illustrates that during proving, knowledge retrieval was involved. Figure 5 is a proof by Paul illustrating how proving statements were drawn from the geometric diagram.

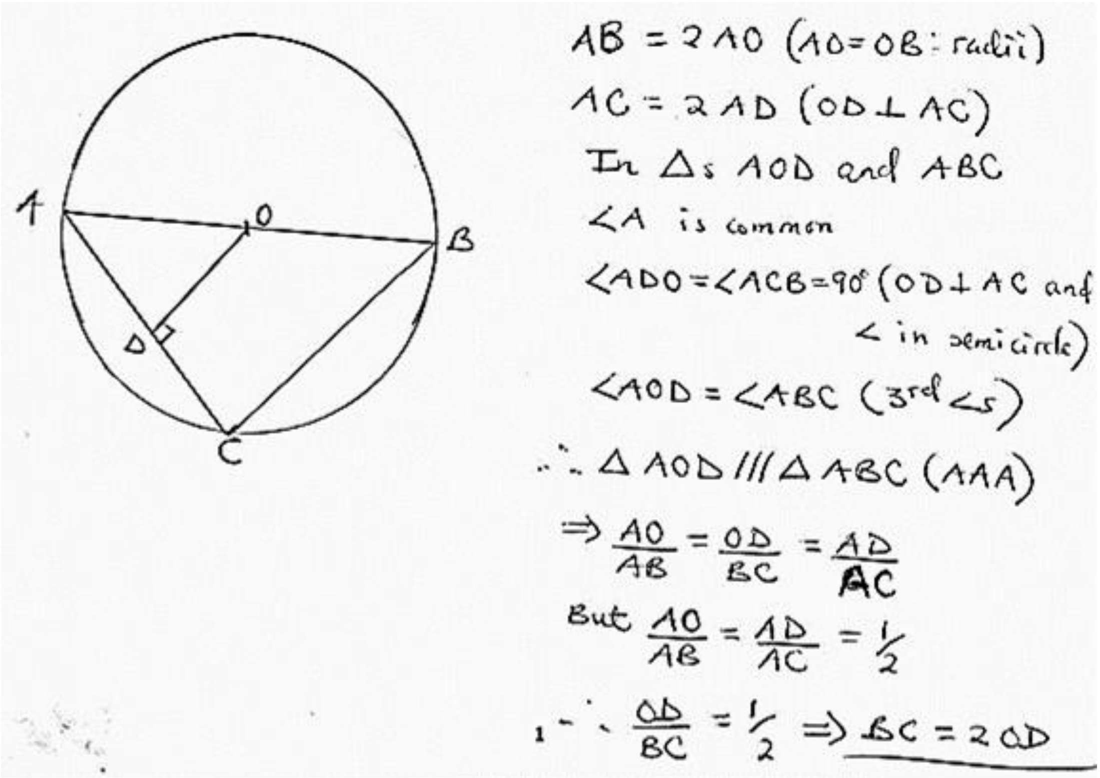


Figure 5: Paul’s proof for task 1 focusing on developing statements from diagram

The proof in Figure 5 shows that developing of the proving statements involved use of both the given information and facts generated from the geometric properties of the diagram. For example, the second statement; AC = 2AD (OD is perpendicular to AC) was generated based on the given information that OD is perpendicular to AC and using geometric property of perpendicular bisector. In the statement that angle ADO = angle ACB = 90^0 (OD is perpendicular to AC and angle in a semicircle), the second reason (angle in a semicircle) applies to angle ACB. This fact was also generated from the given information that AB is a diameter and AC is a chord.

4.2.1.4. The last step is the assertion property identified earlier as the "prove"-norm 4

Both interviews and proof analysis showed that the teachers conceptualise the statement to prove as the conclusion of the proof. During interviews, the teachers explained that they always tell their students to ensure that they prove with an aim of arriving at the destination that is the statement to be proved. As such, they emphasise to the students that just as the destination is the end of a journey, the statement to be proved must only appear at the end of the proof not in the middle of the proof. Pike explained that he tells his students to consider proving as a circular journey that starts and end at the same point. That point is the question required to be proved. The teacher emphasised that since proof aims at arriving at a required conclusion, then the focus point should always be the question or theorem that is being proved. This means that to have a clear point of focus, the statement to be proved has to be properly understood during proof development.

The teachers explained three aspects of focus point that are supposed to be considered during geometric proof development. The first point is to understand the problem to be proved. The second point is to regard the problem to be proved as a focus point and develop the proof with an aim of arriving at the focus point. This means that the problem to be proved should be regarded as the goal for the proof development. The third point is to regard the problem to be proved as a destination. This means that the problem to be proved should not appear as one of the proving statements, but it must appear as a concluding statement for the proof. The teachers' views were consistent with proofs that they developed. All proofs ended with the statement that was required to be proved. Analysis of the proofs also showed that the proving statements were sequenced with an aim of arriving at the statement to be proved. Figure 6 is a proof by Pike illustrating this point.

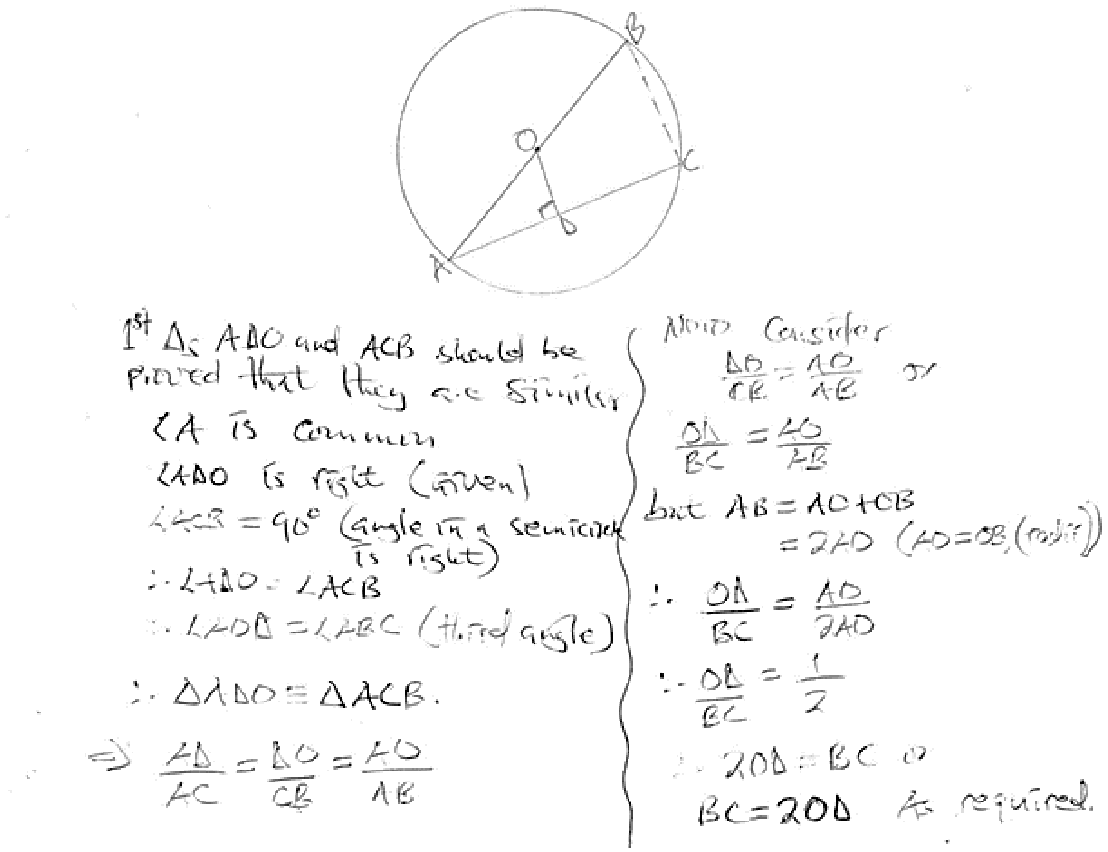


Figure 6: Pike’s proof for task 1 focusing on last statement as the problem to prove

The proof in Figure 6 shows that the last statement in the proof is BC = 2OD. This statement was required to be proved. The progression of the proving statements shows that the aim was to produce the statement required to be proved. After proving that triangle ADO and triangle ACB are similar, Pike came up with sets of corresponding sides of the two triangles. The sets of the sides that are not required in the conclusion are used to come up with the scale factor. The scale factor is applied to the set of required sides to deduce the conclusion. As suggested by the teachers during interviews, the proof shows that the statement to be proved was not only regarded as the conclusion but also as a focus point.

4.2.1.5. Each of these asserted statements are tracked on a diagram by way of standard marks - norm 5

Interview analysis shows that this norm was not mentioned by any of the teachers. The teachers only explained about the importance of understanding the geometrical properties embedded in a diagram. They said that ability to develop geometric statements from diagrams could only be possible if one understands the diagram. This means that the teachers do conceptualise geometric diagrams as sources of information for proving. During proof analysis, it was observed that in all the proofs, the teachers marked on the diagram the information that was given in the problem statement. For example, in the proof in which a diagram represented a parallelogram, the teachers used properties that opposite angles of a parallelogram are equal to mark equal angles on the diagram. This was done by labelling the opposite angles of a parallelogram with the same small letter. The small letters were used instead of capital letters to develop proving statements. The teachers introduced marks on the diagram based on the interpretation of the geometric concepts represented by the diagram. Figure 7 is a proof by Kim containing standard marks introduced in the diagram and used for developing proving statements.

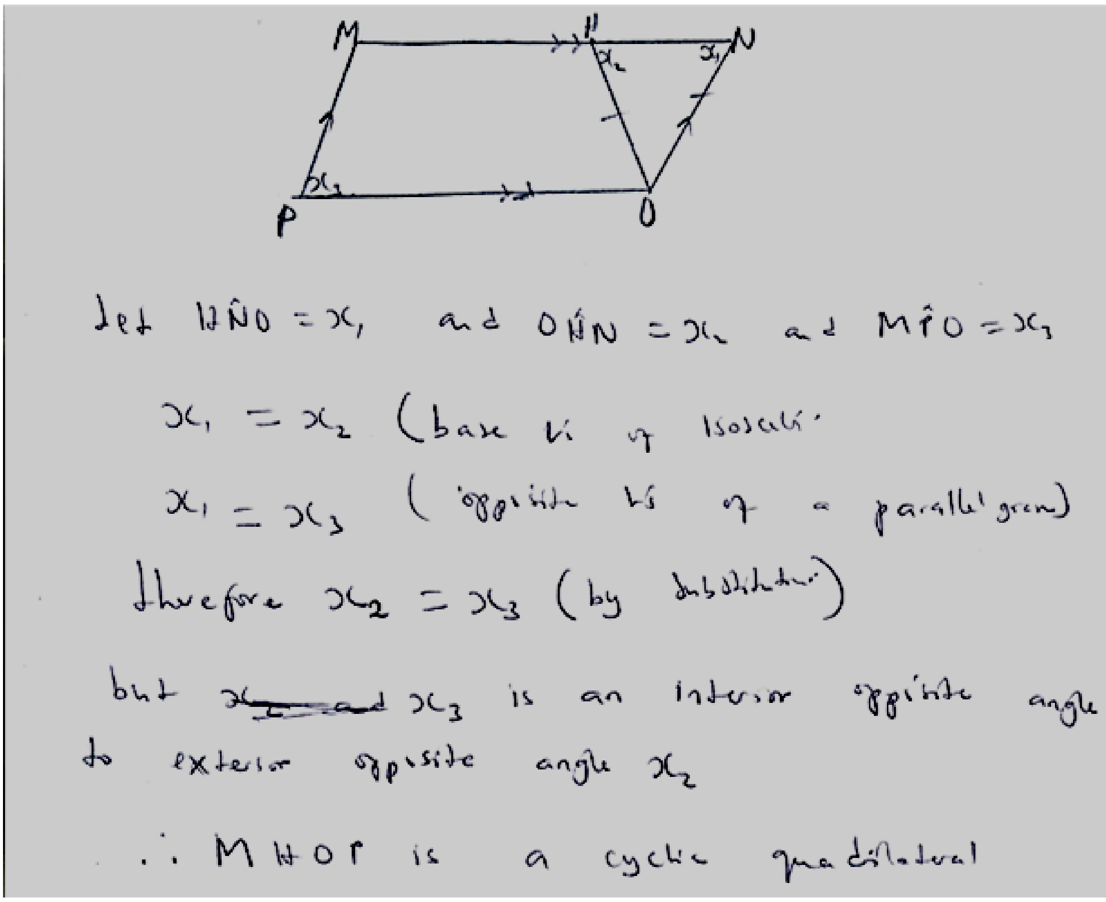


Figure 7: Kim’s proof for task 2 focusing on use of standard marks

The diagram in Figure 7 contains marks on opposite sides to show that they are parallel sides. Other marks are also introduced on lines OH and ON to show that they are equal. Some of the angles in the diagrams are also marked with same small letters to show that they are equal. The letters were introduced into the diagram by combining given information that OH = ON to the properties of isosceles triangles. This resulted into concluding that angles OHN and ONH are equal, hence the introduction of label x on both angles. As it can be noted from the proof that was developed by the teacher, the small letters that were introduced in the diagram are used to simplify the naming of the angles in the proving statements.

However, in some proofs, although John used small letters to represent equal angles in the diagram, he developed geometric proofs using three capital letters that represented vertices. An example of the diagram that contains letters that are not used in the proof is in Figure 4 (page 104). The small letters that John used for labelling angles in the diagram were not used for naming angles in the proving statements. On one hand, the reason might be that it could be problematic for John to use the small letters because he did not distinguish them with subscripts. As a result, it would be difficult to differentiate the angles represented by the letters. On the other hand, John might have used the small letters with an aim of only indicating and identifying equal angles but not for naming them in the proving statements.

4.2.1.6. The reasons listed for each of those statements are previously studied definitions, theorems, or postulates, as well as "given" – norm 6

The results of both interview and proof analysis show that the teachers conceptualise geometric proving as involving use of both given information and prior knowledge. During interviews, the teachers said that development of a geometric proof involves use of either available information or already known information to come up with proving statements. The available information is the given information while already known information includes prior knowledge. Some of the already known information mentioned by the teachers includes knowledge of properties of lines and angles, circle and polygons. The teachers said that it is not possible for students to develop geometric proofs if they do not have the required prior knowledge. The teachers explained that if learners have poor knowledge of these geometric properties, they are not able to understand geometric proof development. This shows that the teachers regard prior knowledge of geometric facts as a basic requirement for proof production. John, Paul and Kim also explained that when they are teaching geometric proofs at grade 12, they assume that students have acquired foundation knowledge of geometric concepts in grade grades 9, 10 and 11. They said that this assumption must also apply to the teacher who is teaching geometric proofs at that level. Kim explained as follows:

> *For example, in Question2, we are told that MNOP is a parallelogram. So it is important to recall some of the properties of a parallelogram, like opposite angles are equal, because that is what is important for this question. And because we are also told that ON is equal to OH, then we must think of the properties of the triangle OHN that is formed when we join OH. So in this case it is an isosceles triangle. So what are the properties of isosceles triangle? So to prove the question, it is important to know some of the properties such as: the base angles are equal. And it is important to remember that adjacent angles on a straight line add up to*180^0*. Then because the triangle formed is also an isosceles triangle, then you can make some substitutions to concentrate on the part that is needed. Now the question is that we should prove that MHOP is a cyclic quadrilateral, which means it is also important to remember the properties of a cyclic quadrilateral for us to answer the question.*

In the extract, Kim has explained what is required in geometric proving by referring to task 2. The extract shows that in order for the teacher to develop the proof, he had to make some properties of some geometrical concepts to make connections. The geometric properties (isosceles triangles and adjacent angles) which Kim applied in proving the senior secondary mathematical task are learnt in lower secondary Mathematics. This implies that Kim was not only supposed to know the content he was teaching at that particular level, but also for the lower secondary school

Geometry. The other teachers also expressed the points that Kim has explained in the extract. This shows that the teachers regard geometric proving as a process of making connection between different geometric concepts.

Likewise, the proofs that were written by the teachers show that several theorems were used for proof development like theorems about similarity, opposite angles of parallelogram and angle in a semi-circle. The proofs also showed that the prior knowledge that was involved for developing proving statements was based on the information that was given. For example, in the Question 1 which asked the teacher to prove that BC = 2OD, prior knowledge of chord properties of a circle was used based on given information that line OD is perpendicular to line AC. This information triggered the use of the theorem that states that if a line drawn from the centre of the circle is perpendicular to the chord, it bisects the chord. The product of combination of the given information and the prior knowledge is a proving statement that states that AC is equal to 2 times AD. Likewise, the given information that AB is the diameter of the circle triggered the use of geometric property that diameter is equal to two times radius. A proving statement which emerged from this combination of given information and prior knowledge is that AB is equal to 2 times AO. This shows that the teachers had to combine given information and their prior knowledge to develop proving statements for the proof.

4.2.1.7. Each of those reasons is stated in a conceptual (abstract) register-norm 7

All teachers explained about use of valid reasons for justifying geometric statements. They said that each geometric statement has to be justified by a property of the argument raised. For example, John said that if one writes that a figure is a parallelogram, he has to justify the statement by using the property of parallelogram that is represented in the figure. The teacher's view suggests use of abstract reasoning during geometric proof development. The view made by John was not consistent with some of the statements and reasons that he wrote down during the proving task. For example, he wrote that sides DE and BC are equal because they are opposite sides of a rectangle without justifying why he assumes that DECB is a rectangle. The reason that John gave is abstract but it is not developed based on any given or retrieved geometric property.

Kim emphasised that the reason supporting a claim should be a true property of the claim. This suggests that the reason should be formulated in a conceptual manner, using geometrical properties of the claim. The results of proof analysis showed that all proofs were developed based on abstract reasoning. For example, the reasons used by Paul in proving that BC = 2OD in Figure 4 (page 104) are based on given information that AB is diameter of the circle and OD is perpendicular to AC. The given information was combined with prior knowledge of chord properties to come up with reasons for justifying why AD is equal to DC. This illustrates that the reasons were developed from geometric reasoning.

Examples of reasons developed from perception are given by John in Figure 3 (Page 103). He made a judgement that figure DEBC is a rectangle based on his perception of the figure and not based on its geometric properties. Despite being developed from perceptions, John stated the reasons in an abstract manner. There was also one proof developed by Pike for task 2, which contained empirical evidence. Pike came up with his own values for the radius of the circle and constructed the circle according to the given information. He then measured the lengths of OD and BC and found that BC was 2 times OD. However, during interviews, when Pike was asked to comment on the proof, he explained that what he had written should not be considered as a proof because he used measurements instead of geometric reasons. Pike explained that he realised that what he had written is not a proof when he went to check in the textbooks and found that there was no proof that was developed using measurements.

The following are the additional themes that were developed through inductive analysis of both the interview data and the proofs.

4.2.1.8. Representing the word problem into a diagram – additional theme 1

The interview analysis showed that the teachers regarded drawing of geometric diagrams as the first step in geometric proof development when given a word problem. The teachers said that most of the theorems and geometric proof problems are in words, so it is not possible to start developing geometric proof direct from a word problem. Although there were differences regarding who should draw a diagram during teaching of geometric proofs, there was a consensus among the teachers that proving statements should be developed from a diagram and not from a word problem. John explained that knowing how to make a sketch from a word problem is a critical point for the required proof to be possible. John explained as follows:

> *Geometry also contains diagrams, so there is need to know how to draw a diagram that is a true interpretation of the word problem and how to develop concepts from the diagram. Understanding what the diagram means is necessary for proving and solving geometrical problems… In Geometry, it is not possible to prove a word problem without interpreting it into a diagram.*

The extract shows that when given a word problem, the first thing to be done is to represent it into a diagram. The extract also shows that to be able to develop a required proof, one must make sure that the diagram that is drawn should be a true representation of the given problem. John also emphasised that it is necessary to understand the word problem before representing it into a diagram. He said that understanding of the word problem can contribute highly to drawing of a diagram that is a true representation of the word problem. John further said that if a geometric diagram that is drawn is different from the word problem, one might end up proving a scenario that is different from what is required. The findings of the interview data analysis correspond to those of proof analysis. Proofs analyses show that before writing the proving statements, the teachers represented the word problem into a diagram. The diagrams were drawn based on the given information and contained the given information as well. Some of the given information was represented in the diagram in form of marks and symbols. In addition to representing the given information, some diagrams also represented the information that was generated from interpretation of given information and prior knowledge.

4.2.1.9. Identifying of proving steps - additional theme 2

Both the interviews and proof data analysis showed that the teachers conceptualise geometric proving as a stepwise procedure. In the following extract, Kim explains in an interview the steps to be carried out during proof development:

> *There are some steps that we follow when we are constructing proofs in Geometry. The first step is to make sure that we are clear on what we have been given. Then, secondly, we also need to be clear on what we are asked to prove. When we are sure of what we are asked to prove, then, thirdly, we are supposed to decide if there is need to add a construction to our diagram. From there, we now decide on how to construct the proof. This step involves deciding whether*

you need to use a bridge theorem or just use several geometric properties. Then the last step is to start the proving itself, if they have said that we should prove that this is equal to that, we need to prove logically up to the last point.

In the extract, Kim explains steps that are supposed to be followed during geometric proving. The first step is to know what is given, the second step is to understand what is required to prove, the third step is to decide if construction is required, and the fourth step is to write the proving statement. Although representation of word problem into a diagram is not listed as one of the steps for geometric proof development, it was observed that the teachers began solving the proof task by drawing a diagram that represented the word problem. Figure 8 is a proof by John illustrating the steps that were carried out in developing most of the proofs.

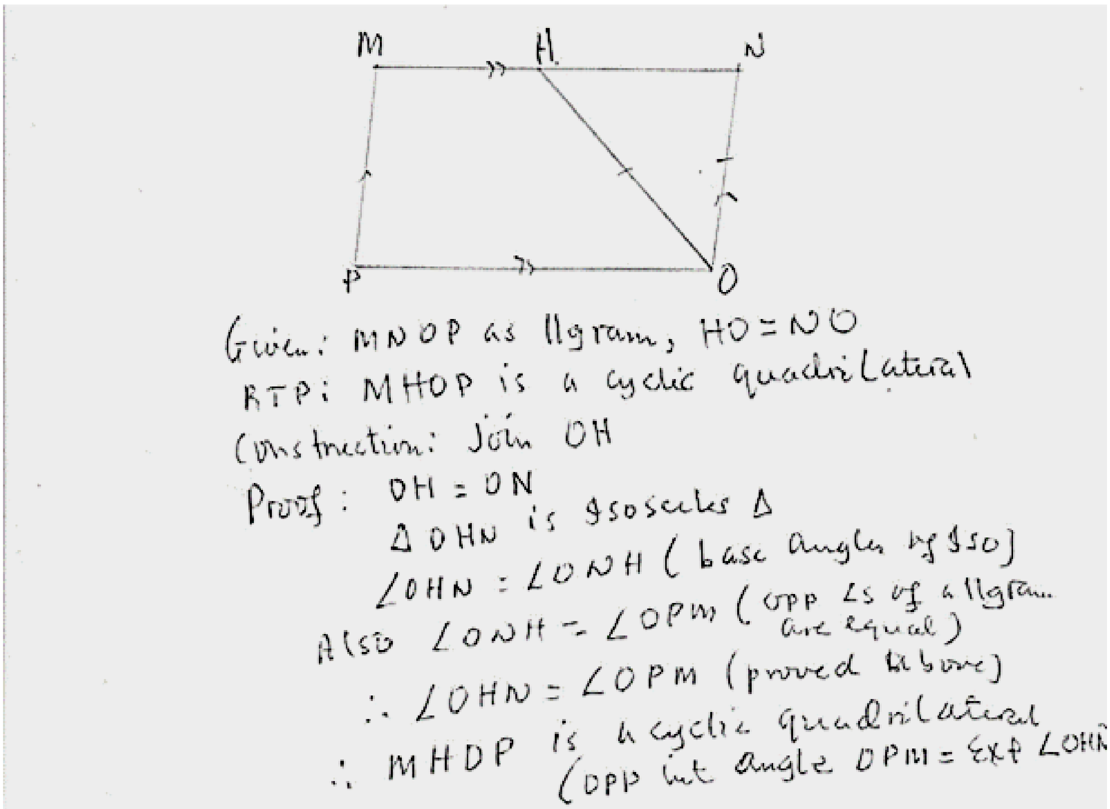


Figure 8: John’s proof for task 2 focusing on steps of geometric proof development

There are several steps that were carried out during development of the proof in Figure 8. The first step was to represent the word problem into a diagram. The second step was to identify the given information which is; MNOP is a parallelogram, HO = NO. The third step was to understand the statement to be proved. Although the second step is not explicit in the proof, I argue that the fact the required proof was developed illustrate that the statement to be proved was understood. The fourth step was to identify some construction to be added to the diagram that is to join OH. The fifth step was to decide on the property of cyclic quadrilateral to be used as a condition for proof development. The sixth step was to write the proving statements. The steps reflected in the proof were observed in most of the proofs that the teachers developed. However, in most of the proofs, the steps were not clearly stated as indicated in the proof above.

4.2.1.10 Proving an in-between theorem - additional theme 3

The interview data analysis shows that the teachers think that when developing geometric proofs, sometimes there is need to prove another theorem that can help to reach the conclusion. They said that the theorem to be proved is supposed to be suitable for the proof being developed. In the following extract, Kim explains what he thinks is necessary to do when developing a geometric proof:

> *I think that the first thing is to analyse the problem and think whether a bridge is necessary or not. When I was at secondary school, I would analyse the question and once I saw that yes, a bridge was necessary then I proved the bridge. When proving the bridge, I also needed to decide if a construction was necessary, so I would ask myself; what do I want to do with the bridge? And what do I want to do if I draw a line here? In other words what will the line bring into the*

drawing? There are instances where students make constructions without saying why they have made the construction.

Kim calls the in-between theorem “a bridge”. He suggests that before starting proving, it is necessary to analyse the problem and decide if there is a need to construct a bridge. When thinking about the bridge, it is also necessary to think of the purpose the in-between theorem will serve during geometric proving. After making the decision about in-between theorem, then it is necessary to decide if there is a need to add a construction to the diagram. Kim emphasises that when thinking of construction, it is also necessary to think of why the construction is necessary. Kim’s explanations showed that they regard the process of deciding a theorem to be used for proving a given conjecture as the most difficult and critical step in geometric proof development. This is because the theorem has to be developed from what is given and what is required to prove.

What the teachers said during interviews was consistent with the results of proof analysis. There was development of an in-between proof in most of the proofs that were developed by the teachers. In task 1 where the teachers were proving that BC = 2OD, the teachers first proved that the two triangles are similar and then developed arguments using property of similar triangles. In task 2 where the teachers were proving that MHOP is a cyclic quadrilateral, the teachers proved two theorems about cyclic quadrilaterals with an aim of showing that the quadrilateral met conditions of property of cyclic quadrilaterals.

4.2.1.11. Simplifying algebraic expressions - additional theme 4

The teachers also mentioned knowledge of simplifying algebraic expressions as an important element of content knowledge for geometric proof development. Some of the proofs that the teachers developed also reflected the use of knowledge of simplifying algebraic expressions. For instance, during interviews Pike explained that sometimes the development of geometric proofs require algebraic thinking like substitution and simplifying of algebraic expressions. However, analysis of the proofs developed by Pike in response to the two tasks showed that he did not use an approach that reflected the use of algebraic thinking. This implies that apart from explaining content knowledge required specifically for developing the proofs, Pike was also explaining his conceptions about geometric proof development in general.

Paul explained that sometimes students find it difficult to develop geometric proofs because their algebraic reasoning is not up to standard. He also said that teaching of geometric proofs is sometimes a challenge because it does not only require knowledge of geometric properties but algebra as well. The results of proof analysis show that some of the proofs involved simplifying of algebraic expressions. For example, one way of finding solution to task 1 was by use of Pythagoras Theorem. Figure 9 shows the proof by Kim that used algebraic reasoning.

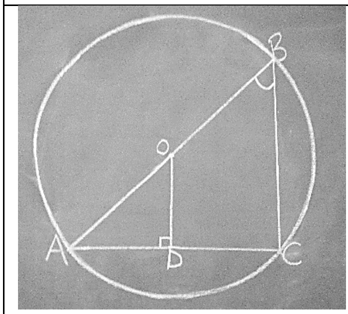
Diagram	**Statements**
	AO = ½ AB (2 times radii = diameter) AD = ½ AC (perpendicular from centre bisects the cho $A\hat{D}O = 90^\circ$ (given) $A\hat{C}B = 90^\circ$ (angle in a semi-circle) let AO = y and AD = x hence AB = $2y$ and AC = $2x$ In Δ ADO $AD^2 + OD^2 = AO^2$ (pythagoras theorem) $x^2 + OD^2 = y^2$ $OD^2 = y^2 - x^2$ $OD = \sqrt{y^2 - x^2}$ In Δ ACB. $(AC)^2 + (BC)^2 = (AB)^2$ (pythagoras theorem) $(2x)^2 + BC^2 = (2y)^2$ $BC^2 = 4y^2 - 4x^2$ $BC = \sqrt{4(y^2 - x^2}$ $BC = 2\sqrt{y^2 - x^2}$ $BC = 2 \times OD$

Figure 9: Kim's proof for task 1, focusing on simplifying algebraic expressions

The proof in Figure 9 shows that in triangle ADO, Pythagoras theorem and algebraic reasoning were used to come up with statement representing OD which states that OD = $\sqrt{AO^2 - AD^2}$. In triangle ABC, there was also use of Pythagoras theorem and algebraic reasoning to come up with statement representing BC which states that BC = $2\sqrt{AO^2 - AD^2}$. Then using algebraic reasoning the term $\sqrt{AO^2 - AD^2}$ was substituted with OD to arrive at the conclusion.

4.2.1.12. Use of geometric symbols and abbreviations - additional theme 5

During interviews, the teachers did not mention anything concerning symbols and abbreviation. However, the proofs that the teachers developed contained different types of symbols and abbreviations. There are standard symbols with conventional meanings in Mathematics. There were some symbols that were common in all proofs and other symbols that were noticed in proofs that were developed for same question.

Some of the common symbols which were noticed in all proofs include use of = to represent equality, and < to represent angle. The examples of symbols that were specific to a proof included // to represent parallel lines in proof that MHOP is a cyclic quadrilateral by John. In proof that BC = 2OD, there were three symbols ⊥ to represent perpendicular lines, + to represent plus, /// to represent similarity. In proof for task 1, Pike used the triple horizontal bar symbol (≡) to represent similarity of triangles but the other three teachers represented similarity using the triple vertical bar symbol ///. However, the conventional meaning of the horizontal triple bar that Pike used is usually used for congruency because it represents equivalence. This might mean that Pike confused the conventional meaning of the symbol.

4.2.2. Teachers' conceptualisations of the teaching of geometric proving

The PCK categories of COACTIV model were used as pre-ordinate themes for analysing teachers' conceptualisation of the teaching of geometric proving. These are: knowledge of cognitive activating tasks, knowledge of student's cognitions and ways of assessing students' knowledge and comprehension processes and Knowledge of explanations and multiple representations. As in sub-section 4.1.1., the data that did not fit into these categories was analysed further using inductive thematic analysis and additional themes were developed. All teachers explained that teaching of geometric proving involves guiding students to understand how to develop a proof. The teachers also said that this might require involving students in different proving activities, and assessing their thinking. Table 4 presents a summary of findings from all teachers in relation to their conceptions about what is involved during the teaching of geometric proof development.

Table 4: Summary of teachers' conceptions of the teaching of geometric proving

Category	John	Kim	Paul	Pike
Proving activities.	-Doing proving activities with students. -Making connections between proving activity and formal proof.	-Involving students in activities to deduce a theorem. -Guiding students in doing activities that can help them understand proof development.	-Involving students in hands-on activities during proving.	-Involving students in an activity that can help them predict the theorem.
Explanation and representation of proofs.	-Discussing with students how to show that a given statement is true. -Showing students how to prove a given task by following the proving steps. -Helping	-Guiding students in making connections between given information and prior knowledge. -Helping the students to understand geometric	-Demonstrating to students how to prove a theorem or any geometric statement. -Showing students how to apply a geometric theorem when solving	-Showing students how to prove a geometric statement. -Showing students connections among different geometric concepts.

	students to make connections among different geometric concepts.	theorem and its application.	computation problems.	
Tasks.	-Doing examples to show students how to apply a theorem to different tasks.	-Letting students practice different types of examples to be conversant with theorem application.	-Giving students evaluation and practice tasks.	- Giving students different types of tasks for evaluation and practice.
Assessing students' thinking	- Giving students exercise or homework to evaluate their understanding. -Giving the students feedback about their work. -Making correction on the students' work.	-Asking students to explain and analyse their work. - Knowing prior knowledge of the students. -Giving students an opportunity to ask Questions to reveal their errors. -Responding to	-Judging students' understanding of geometric theorems and proofs. -Knowing the background of the students. -Revising students' homework by explaining the	-Knowing the characteristic s of the students. -Asking questions to check students' progress. -Identifying mistakes made by students and

	-Responding to students' questions.	students' questions.	steps which were supposed to be followed.	correcting them
Value of proof	- Explaining to students the real life benefits of learning geometric proof development.	- Helping students to understand and appreciate the importance of proof development.		- Helping students appreciate the value of geometric proving.
Use of materials.	-Know the materials to use when proving a particular proof.	- Preparing teaching materials that are suitable for proving a theorem.		-planning and preparing materials for involving students in geometric proof development.

The summary findings about teachers' views about teaching of geometric proving have been discussed in detail in the following sections.

4.2.2.1 Knowledge of cognitive activating tasks – category 1

All teachers mentioned that the teaching of geometric proving involved solving of geometric problems. The problems described by the teachers are considered as tasks in the COACTIV model. Sometimes the teachers also referred to the tasks as questions or tasks as well. The teachers' explanations showed that tasks serve various purposes during the teaching and learning of geometric proofs. The first purpose is for explaining or showing how to perform the procedure of geometric proving. Paul explained that he starts teaching geometric proofs by drawing a sketch of the theorem and then shows his students how to develop the proof. One of the aims of making the sketch is to turn the theorem into a diagrammatic question so that proof development can be possible. Paul further explained that during teaching of geometric proofs, he always reminds his students to regard a proof question as a comprehension question that contains all information that is necessary for proof development. The second purpose of geometric tasks is to help students understand how to apply the theorem to solving different situations. All teachers explained that after teaching students how to develop a proof of a theorem, they do some examples with the students. Paul said that after proving the theorem, he usually solve two to three questions with an aim of showing students how to apply the theorem to different problems. He explained as follows:

> *I first show the students how to prove the theorem. I usually do not spend a lot of time on showing students how to prove the theorem like I said because not all students are interested in knowing the proof, some are only interested in knowing how to apply the theorem. After proving, I show students how to use the theorem to solve a variety of questions.*

The extract shows that Paul mainly has two objectives when teaching geometric proofs. The first objective is to show students how to prove the theorem, and the second objective is to show students how to apply the theorem to solving different tasks. Both aims are met using tasks. The extract also shows that while there might be only one task for achieving the first aim, the second aim is achieved by solving different types of tasks.

The third purpose of geometric tasks is to evaluate students' understanding of the theorem. The teachers also said that they give students tasks to be solved in order to evaluate their understanding of the theorem. John suggested that questions that are asked to evaluate students understanding of proof must cover all areas that were emphasised during proving and must enable students apply proof to different situations. The fourth purpose of geometric proof tasks is for practice. All teachers said that they give students tasks to practice solving either in groups or individually to enhance their mathematical skills. They called them "practice questions". The teachers described practice questions as routine tasks given to students after assessing the evaluation tasks and doing correction. It was, however, noticed that the teachers had different aims for giving students practice tasks. Pike and Kim explained that they give students different practice tasks with an aim of helping the students to become conversant with different ways of applying the proof. According to these teachers, students must first understand the proof before they are given practice questions. Pike said the following:

> *One way of helping students become conversant with proving is to provide them with opportunities for proving by giving the more questions for practice... After proving a theorem, teachers must give students different types of questions which they can solve by applying that particular proof...Giving students different types of questions after proving a theorem will help the students to be able to apply the theorem to different situations.*

In the extract, Pike explains that after teaching geometric proving, the teacher must give students some questions which will require them to develop proofs and also to apply the theorem to other situations. Pike emphasises that if students are given different types of practice questions, they will be able to use the theorem to solve different problems. Pike's emphasis agrees with Kim who explained that the aim of giving students practice questions is to help them to become independent thinkers who are able to apply theorems to different situations.

Pike's extract also shows that the teacher is supposed to analyse the practice questions that he gives students to make sure that they demand application of the theorem to different situations. John and Paul think that practice questions aim at clearing misconceptions. These two teachers stated that when they have assessed evaluation questions and done correction, they give students practice questions to solve in groups with an aim of clearing students' misconceptions. Paul explained that he ensures that students are forced to solve the practice tasks by giving them a deadline for handing in their work.

4.2.2.2 Knowledge of student's cognitions and ways of assessing students' knowledge and comprehension processes – category 2

All teachers explained that a teacher is supposed to understand what their students know and think. This involves knowing the nature of the students they are teaching including their mathematical background and abilities. The teachers said that before teaching geometric proving, a teacher must first find out their students' prior knowledge. This is because geometric proving involves use of basic geometric properties like lines, angles and polygons. These concepts are taught in Form one (grade 9) Geometry in preparation for geometric proofs that are taught in forms 2, 3 and 4 (grade 10, 11 and 12). The teachers said that they assume that if students understand these concepts, they will not have problems in understanding geometric proofs. The teachers further regard students who do not know the basic geometric concepts as having poor background for learning geometric proofs. John explained that it is impossible for such students to understand geometric proofs. He, therefore, suggested that the teacher must address the gaps of prior knowledge before teaching geometric proofs.

The teachers also pointed out that sometimes students make mistakes when they are solving geometric proof problems, so knowledge of students also includes knowing the process of analysing students' solutions. This involves being able to identify the mistakes that students are making, figuring out the root cause of the mistakes, and deciding how to clear the mistakes. Pike, John and Paul explained that during marking of students' assignments or tests, they do not concentrate on finding out mistakes made by a particular student because they have many students in their classes. As such, they only note the mistakes that are common and address them during correction. Pike and John said that they correct the mistakes by explaining what was supposed to be done when answering the question. Paul noted that usually, he does not make corrections by answering the questions he asked but he explains the general procedure implied in the question. In the following extract, Paul explains what he does when doing correction in class:

> *When I want to make corrections, I do not tackle the same question but I revise in general while tackling the area that students were unable to do. This is because the aim is to make students understand the concept in general and not only the question they failed to tackle. The other reason is that I realised that normally when we come back to class, most of the students have already copied*

the correct solution from their friends and they do not pay attention to what I say.

There are several points that Paul has explained in the extract. Firstly, that he does not revise the same tasks that he gave the students as an assignment but he chooses other related tasks. He also explains that he does the revision by explaining the general procedure that was supposed to be followed. This is because his aim is to help the students to understand the concept in general, but not only to understand the question that he asked. The other reason for explaining the general procedure is that he noticed that his students do not pay attention when he is making corrections. Paul believes that students do not pay attention because they have already copied the correct solutions from their friends. This means that Paul assesses the students when he is explaining to see if they are interested in his explanation. However, the reason given by Paul that the students do not pay attention because they copy solutions from their friends might not be accurate. This is because copying a correct solution does not mean understanding the solution process. Therefore, even if students copy solutions, they are still supposed to be interested in understanding the solution and to know why their solution was not correct. This might mean that the approach that Paul uses does not motivate the students to pay attention because he does not help the students to realise the mistakes in their solutions.

In the following extract, Kim explains what he does to correct students' mistakes:

I like group work, I give students an assignment, students discuss in groups and present their work to the whole class, so if the group or if a student has made an error, his friends come in to point it out or correct it. So if an error has been made by a student and another students recognises and corrects it, then it means *that other students who made similar errors will learn from that and will know that they were not the only ones who made errors. Even if when fellow students do not recognise the error when the student is presenting, I try to follow and see how a student understood what I was teaching, in that case, I compile the errors as well. After the presentation, I ask the whole class; do you find any problem or mistake with his answer? This makes the students to analyse their friend's work and find if there is something to correct.*

Kim explains that he normally give students tasks to discuss in groups and report their solutions to the whole class. This provides an opportunity for the other students to analyse the reported solutions, and correct mistakes where necessary. The analysis of tasks in class helps students who made similar mistakes to realise why their solution is not correct and to make corrections. Kim pointed out that if the students do not notice mistakes made by their friends, he tries to ask questions to probe their thinking and to encourage them to analyse the work that has been presented. Kim observed that when he has given individual assignment, he also compiles mistakes made by the students and identify students' solutions to be presented in class for analysis. After students have pointed out the mistakes and suggested ways of correcting them, he explains and focuses on the mistakes that the students made and how to avoid them.

Apart from knowing students' background and analysing students' solutions, John explained that the teaching of geometric proving involves responding to students' questions. He said that it is very important that teachers plan their lessons so that they are ready to answer students' questions appropriately in class. John believes that when the teacher responds well to students' questions, he builds confidence in the students.

The way in which John thinks about responding to students' questions was different from what Kim thought. The following extract presents ideas from Kim:

> *I always encourage students to ask questions in my class. That is one way of knowing what they are thinking. So, after discussing a proof with students, I give them an opportunity to ask questions or comments. So, when one student asks a question, I throw it back to the class that this is what your friend has asked, how can you assist him? So when the other students are responding to the question, I also pay attention to analyse their ideas. When the students are agreeing or disagreeing on a point, I ask questions that can help them to realise what is right.*

The extract shows that Kim thinks that students' questions expose what the students are thinking. Therefore, he let students ask questions with an aim of knowing how the students understood what was being taught. The extract also shows that Kim thinks that he can know what the other students are thinking concerning the question by giving them opportunities to answer the question. This indicates that Kim uses students' questions as opportunities for understanding their thinking. The extract also shows that Kim consolidates students' responses with an aim of clarifying either concepts or procedures and clearing misconceptions.

4.2.2.3. Knowledge of explanations and multiple representations – category 3

All teachers said that it is important for teachers to ensure that students understand theorems and the process of developing geometric proofs. The teachers differed in their ways of helping students to understand theorems and proofs. John and Paul said that they carry out the proving steps with the students, while Pike and Kim said that they would first involve students in proving activities to help them understand the theorem and its proof and then later on concentrate on the procedure of proof development.

Paul explained that the teaching of geometric proofs is challenging and time consuming. He explained that unlike other topics where he just shows the procedure once and let the students practice it, in geometric theorems, he has to show students how to prove every theorem. Furthermore, Paul said that the other challenge is that theorems seem to be simple from his point of view, so he teaches with an assumption that the students would easily understand, but to his surprise, he finds that some students have difficulties to understand what is happening.

This is the procedure that Paul follows when explaining to his students how to prove a particular theorem.

> *When I am teaching I usually move with the students in steps so that they know how to construct the proof for the theorem that we aim to prove at that particular day. Because most of the theorems are in words or statements then first step is that I make a sketch or a diagram. Then I ask the students to analyse the diagram to identify given information and the information that we are required to prove. Then I do some constructions if it is necessary. Lastly, I ask the students questions that would help us to come up with arguments for our proof. So, for example, I would say which angle is equal to angle y? When the students identify the angle, I ask them to give the reason.*

Paul explains that he begins by making a sketch and then uses question and answer method to show the students how to develop the proof. The extract shows that there is sharing of responsibilities between the teacher and the students during proving. The teacher provides the statement and the diagram and asks probing questions to help the students come up with proving statements and the reasons. The students use the diagram to answer questions asked by the teacher. Paul's views about the approach for teaching of geometric proofs seem to be different from those of Pike and Kim. Pike and Kim suggest that students should be given an opportunity to come up with their own proofs during teaching of geometric proofs. Pike explained that it is good to firstly involve students in activities to help them develop the proof on their own before guiding them into procedure for developing the formal proof.

Pike explains as follows:

> *When proving a theorem, teachers must provide students with opportunities to engage in a proving activity. After the activity, the teacher must connect the activity to the theorem... Sometimes activities that students do are easy like when students measure angles to make a guess like for sum of interior angles of a regular polygon, it is very easy for them to come up with the proof and formulae using figures. But students get confused during formal proving because sometimes the activities that we give students are simple and through the activity, students are able to make correct conclusions, but when we try to connect the activity to the formal way of proving, you find that the formal way is quite complicated.*

Pike explains that students should first be given an activity in which they should come up with predictions. Then the teacher needs to connect the students' predictions to the theorem to be proved. Pike recognises that students might not develop the proof using standardised approaches, so he suggests that he would consolidate the students' proofs with a formal way of proving. Pike also acknowledges that the formal way of proving is sometimes difficult than the empirical way. The extract shows that when the students have done the empirical activity and made their conclusions, Pike comes in to show the students how to develop the formal proof. This approach is also slightly different from the one suggested by Kim in the following extract:

> *What actually happen is that students get confused with simple things. So, if you just start proving without engaging them in an activity like measuring or discussions on how to prove, they just memorise the theorem and the proof. So, to avoid memorisation, it is necessary that the students be involved in activities. It is those activities like measuring and discussion that can instil the knowledge or make the knowledge be established in their brain. This helps the students to know that these theorems did not just come or were not just created from nowhere but that they are there and they are true. So asking students to do measurements before proof construction helps them to have tangible evidence that the theorem is true. So apart from measuring I also ask them to prove on their own with my assistance of course, this helps to develop independent thinking, because if I do it for them and they memorise, then the moment the same question comes in a different situation they get confused.*

In the extract, Kim believes that it is not good to tell the students how to develop geometric proofs because students might only memorise the proofs without understanding them. Kim suggests that it is necessary that students be engaged in either a measuring, or a discussion activity to help them to understand the proofs. Kim also thinks that the measuring and discussion activity helps the students to understand why the proofs are true. This means that Kim is suggesting that teachers must firstly involve students in empirical reasoning before deductive reasoning. The extract shows that Kim thinks that when students develop a proof on their own they become independent thinkers, hence able to apply the theorem to solving different tasks.

The following are additional themes that were developed from the data that could not fit into the categories of COACTIV model using inductive thematic analysis.

4.2.2.4 Knowledge of relevance and contexts – additional theme 1

The teachers explained that it is also necessary to help students appreciate the rationale for learning geometric proof development. They said that students become motivated to learn something when they understand its usefulness. John said that sometimes students do not pay attention when learning how to prove Geometry theorems because they do not see a link between the proofs and their daily lives. This is emphasised in the following extract:

> *Of course, for students to be motivated to learn something, they must be convinced on how that particular thing is useful in their real world. It is necessary for students to have a chance of relating what they have learnt to the real world. The other reason why students do not perform well on Geometry proof questions is lack of motivation. If the students were taught in such a way that they realise the importance of geometric proofs in their daily life then they would begin to like it for its usefulness and not for exam purpose only.*

John thinks that in order for students to be motivated to learn geometric proofs, they must be convinced about why proofs are useful to their life. He suggests that the teacher must use real life examples during proving, and ask real life questions for evaluation and practice to enhance students' involvement in geometric proof development. Nevertheless, he observed that being able to ask real life questions is a challenge to most of the teachers because the textbooks that they use do not contain such examples. He further explained that as a result, although he knows that they are supposed to use real life examples to motivate the students to understand the importance of geometric proving, he fails to do this due to lack of textbooks which contains such examples.

However, Paul did not explain that it is necessary to explain to students the usefulness of learning to prove geometric theorems. During teaching of geometric theorems, after writing the theorem on the chalkboard, Paul used to ask students if they wanted to learn its proof. When the students replied that they were not interested to learn the proof, Paul could not encourage the students to be motivated to know why the proof was necessary. This implies that Paul thought that the students were supposed to be motivated to learn how to prove a particular geometric theorem on their own.

4.2.2.1.5. Knowledge of teaching materials – additional theme 2

The teachers also explained that to teach geometric proofs effectively, a teacher must know the type of materials to be used for teaching a particular theorem, and how to use the materials. The materials that they mentioned include textbooks, models, mathematical tools like ruler and pair of compasses, paper and any other teaching aids. They said that the teacher must choose a textbook that explains the proofs in detail and that can be easily understood. The teacher must also know how to use every tool in the mathematical instrument box in order to guide the students to use them properly.

Pike and Paul said that when they were marking National examinations they noticed that many students were drawing circles using free hand instead of using a pair of compasses. He thinks that this could be a result of lack of good guidance on how to use mathematical instruments. Apart from drawing and measuring materials, the teachers also maintained that a teacher needs to know the type of materials to use when proving a particular theorem. John explained this point as follows:

> *The other thing is that in other topics we do use teaching and learning aids to remove misconceptions of students using hands on activities but you find out that in Geometry we neglect the use of teaching and learning aids because we are not sure of the type of teaching and learning aids that we can use to construct a geometric proof. So, we teachers are sometimes not resourceful to say what type of material should I find and use in class so that students should understand this proof.*

There are several points emphasised by John. The first point is that he thinks that students' misconceptions in geometric proofs can be removed by using teaching and learning materials. However, John notes that most of the times, teachers do not use teaching materials when teaching geometric proofs because they do not know the type of materials that they can use. John also mentions that sometimes teachers do not use teaching materials because of lack of resourcefulness. However, in might be difficult for the teachers to be resourceful if they are not sure of the materials that they need to use. This implies that there is need for the teachers to be guided very well in terms of the teaching and learning materials that they can use.

4.2.2.6. Knowledge of proving activities– additional theme 3

The teachers explained that they think that when proving a theorem, teachers are responsible for providing students with opportunities to engage in meaningful activities. Pike explained that for the activity to be meaningful to the students, it must be connected to the formal proof. Pike explained challenges that he faces when engaging students in a proving activity in the following extract:

The other thing is that sometimes I do have a very good activity that I ask my students to do in class. At the end of that activity students are supposed to make conclusions, and you as a teacher has to bridge the activity and the theorem, that is to connect the two is not easy sometimes...In Geometry, students must prove theorems formally even after proving through an activity. Sometimes the formal way of proving is a bit tougher than the activity; in that case students get confused.

The extract above shows that Pike does not only face challenges in coming up with proving activities but also in bridging the activity to the theorem. The other challenge is that students get confused when formal proof is tougher than activity. In this situation, the teacher is responsible for coming up with a remedy. One way would be to represent the proof in a simpler manner without losing its meaning. Pike said that if he has no solution for one or both of challenges, he decides not to involve students in a proving activity.

4.2.2.7 Knowledge of teaching geometrical concepts as connected entities – additional theme 4

The teachers also explained that teaching of geometric proofs involves making connections between different geometric concepts. Pike stated that there is need to recall and relate concepts during proving, as such, students need to understand properties of several geometric concepts before learning geometric proving. John said that teaching of geometric concepts as connected entities could help students be able to apply theorems correctly during proving or solving of problems. He said that teachers must teach geometric proofs while bearing in mind that Geometry is connected so the learning of geometric theorems is dependent on knowledge of different geometrical concepts. In the following extract, Kim explains his views about making of connections when teaching geometric proofs:

> *A teacher must teach Geometry while bearing in mind that what students learn this year will be used next year and also that what the students are learning this year will be affected by what they learnt last year. So, for example, when you are teaching about congruency in Form 2, you need to know what the students learnt in form 1 that can affect their understanding of the theorem, and you also need to know what the congruency theorem will be used for in Form 3 and Form 4. In that case, a teacher should know what the students will need in order to connect this year's Geometry to next year's Geometry. If we were teaching geometric concepts and proofs while bearing in mind what the students will learn in future, then we would make sure that we clear all students' misconceptions on what they learn today so that they do not suffer tomorrow.*

The extract by Kim suggests that teaching of geometric proofs requires knowledge of whole secondary school Euclidean Geometry. Kim explains that a teacher must teach Geometry while bearing in mind about the topic's prior knowledge and the requisite knowledge. He thinks that teaching geometric concepts and proofs while focusing on how they will be used in future would ensure clearing of students' misconceptions. Kim thinks that students would find it easier if previous knowledge is well understood.

4.2.2.8. Knowledge of areas to emphasise during teaching of geometric proofs – additional theme 5

The teachers mentioned different areas that need to be emphasised during teaching of geometric proofs. Although the teachers differed on some areas of emphasis, there were three main areas that the teachers' explanations agreed. Firstly, the teachers said that it is very important to emphasise on understanding the theorem. Paul explains this point as follows:

> *First and foremost, we emphasise that these are theorems, which means that they are always true, that means that when we are teaching the theorems, the students should understand the theorem statements so that they are able to use them to answer different questions that may arise from the same theorem. So that is the first point, they must understand the theorem.*

Like Paul, all the other teachers described application of theorem to solving geometric questions as the main aim of emphasising on understanding of the statement. Secondly, the teachers mentioned that it is important to encourage students to develop the proof while focusing on what they have been asked to prove since this would ensure arriving at the required conclusion. In addition, Pike and John also said that it is important to emphasise to students that the statement that they are proving should only appear as a closing statement of the proof and not as part of the proving statements. Thirdly, the teachers explained that during teaching of geometric proofs, the teacher must make sure that students come up with connected arguments that will enable making of clear link between the given information and the statement to be proved.

Apart from understanding the theorem, proving while focusing on the statement to be proved, and making connected arguments, Pike, Paul and Kim also explained that they also emphasise that students must justify their proving statements with reasons. Pike and Kim also explained that they also emphasise that students should understand the given information. This is because the students use this information to develop proving statements. In addition to the areas that have been mentioned, Kim also observed that he emphasises that students must analyse the given information on the diagram and the statement to be proved and decide if there is need to develop an in-between theorem to connect the two.

4.3. How teachers select and implement geometric proof tasks

In this section, I intend to approach research Question2 concerning selection and implementation of tasks during geometric proving. Most of the tasks that were used by the teachers were taken from secondary school Mathematics textbooks and were presented in form of either a statement or a diagram, or both statement and diagram. There were two main textbooks that were used for both junior and senior secondary Mathematics. The tasks for proving different theorems were similar in both textbooks for both levels. As a result, the tasks selected by John, Kim and Paul who were teaching Form 3 geometric proofs were similar. However, the tasks used by Pike who was teaching Form 2 geometric proofs were different from those of John, Paul and Pike because they were using different textbooks.

Analysis of the cognitive level of the tasks that were used by the teachers for teaching geometric proofs showed that all teachers selected a combination of both low and high level tasks. The difference was in the level of implementation of the tasks. In Table 5, I present a summary of how the tasks were implemented during the teaching of geometric proof development.

Table 5: Summary of task implementation

Category	John	Kim	Paul	Pike
Level of implementation of tasks.	-Implemented both low and high cognitive level tasks at low level of cognitive demand. - Suggested most of the steps to be carried during proving. -He also suggested most of the proving statements for each proof. - The students were mainly involved in providing reasons for the statements.	-Implemented most of the tasks at high level of cognitive demand. -Involved students in exploration of theorems to be used for developing the proofs. -The students discussed in groups how to prove the tasks. -Involved the students in presenting and discussing their solutions.	-Implemented both the low and high cognitive level tasks at low cognitive level of cognitive demand. - Mainly showed the students how to develop the proofs. -For each task, Paul suggested the construction to be made, the theorem to be used and the proving statements. -Sometimes he	-Implemented most of the low and high cognitive level tasks at low level of cognitive demand. -Provided much guidance regarding the development of the proofs. -Some of the tasks were implemented at high cognitive level as the teacher involved the students in exploration and

		-Students were given an opportunity to analyse and comment on solutions presented by their friends.	involved the students in giving reasons to justify the proving statements.	explanation.

In the following sections, I provide a detailed discussion on some of the tasks that were used for proving different theorems. I start by analysing the cognitive demands of the narrative tasks and how they were implemented and then continue with analysis of cognitive demands of computational tasks and how they were implemented. To ensure in-depth analysis of the tasks, I developed sub-questions for the research question using description of knowledge of cognitively activating tasks by the COACTIV model (Baurmert & Kunter, 2013). These are:

(i) What is the cognitive level of tasks that are chosen and used by teachers during the teaching of geometric proving?

(ii) How are the tasks implemented during geometric proving lessons?

(iii) What knowledge is involved in implementation of the task during geometric proving lessons?

4.3.1. Analysis of narrative task by Pike

Figure 10 is an example of a task that Pike selected for teaching geometric proof development.

Given: parallelogram PQRS in which PR and QS are diagonals that meet at point T. Prove that the diagonals are bisecting each other at a Point T.

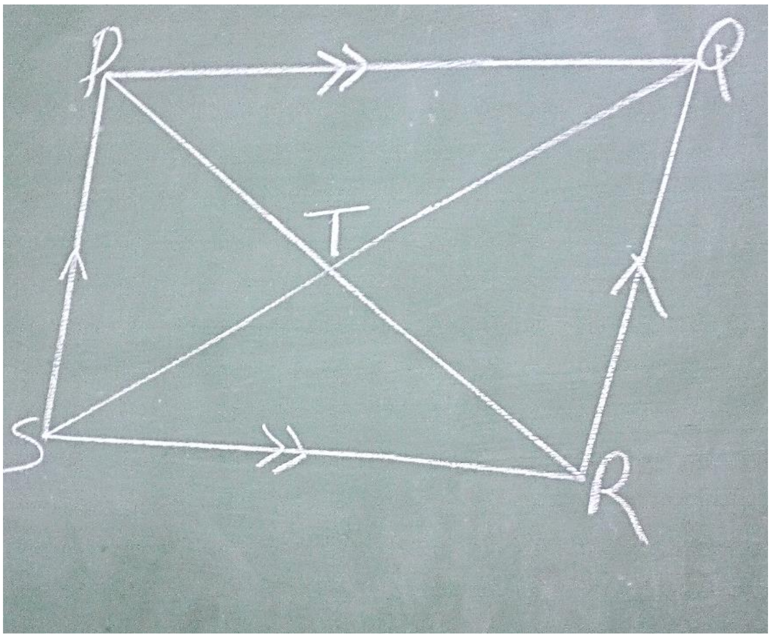


Figure 10: Pike's Task for narrative proof

4.3.1.1. Cognitive level of the task

The task in Figure 10 demands production of a proof to show that line PT is equal to line RT and line ST is equal to line QT. The students are required to show understanding of several concepts in order to develop the proof. Firstly, the students have to understand the meaning of diagonals in order to identify the lines that need to be proved that they are equal. Secondly, students have to consider properties of parallelogram so that they identify equal lines and angles. Thirdly, students have to consider properties of parallel lines and transversals to come up with alternate interior angles. Then the students have to consider set of triangles that can be used for developing an in-between proof that is congruency in this case. Lastly, the students have to consider properties of congruent triangles to come up with the sets of lines that are equal. This, therefore, shows that the task involves a lot of geometric thinking, hence it can be considered as a cognitively demanding task.

4.3.1.2. Task implementation by Pike

15. Pike:	We are asked to prove that PT = RT, and QT = ST. Now let me find out from you, how can we prove this? (Silence 9 seconds) Okay let's see what we have. How many triangles do we have in the diagram?
16. Student:	4.
17. Pike:	4? No, we have more than that, let us see, we have these triangles (pointing at triangle PTQ, PTS, STR, RTQ, QRS, RST, SPQ and PQR) so how many triangles do we have?
18. Student2:	8.
19. Pike:	So, we will prove using triangle PTS and triangle QTR. We will use the same reasoning and procedure as yesterday, that is we must prove that triangle PTS and QTR are congruent. Since we have been given that this whole figure (pointing at PQRS) is a parallelogram, then we consider the relationships of the opposite sides and the opposite angles. So let's start with angle SPT, what is it equal to?
20. Student:	Angle QRT.
21. Pike:	Yes, angle SPT is equal to angle QRT. What's the reason?
22. Student1:	Alternate angles.
23. Pike:	Why are angles SPT and angle QRT alternate angles?
24. Student 2:	Because PS is parallel to QR.

25. Pike: Yes, PS is parallel to QR. What else can we say as another point for our

proof? (silence for 8 seconds) we can also use what we learnt yesterday, that is to say that PS = QR and the reason is that opposite sides of a parallelogram are equal. So, we have considered an angle and sides. What else can we consider?

26. Student 1: PQ is equal to SR.

27. Pike: Okay, but we are considering triangles PTS and QTR so in these triangle we do not have PQ and SR.

28. Student 2: ST is equal to QT.

29. Pike: No that is what we want to handle. We want to prove that ST equals QT so we cannot include it as our argument. So what other statement can we come up with from the diagram?

30. Student 3: Angle PTS is equal to angle QTR.

31. Pike: What's the reason?

32. Student 3: Vertically opposite angle.

33. Pike: Yes, we can take that one. So we can write (while writing on chalkboard) that angle PTS is equal to angle RTR and the season is that they are vertically opposite angles. Okay therefore, we can conclude that triangle PST is congruent to which triangle?

The extract shows that Pike attempted to involve students in deciding how to develop the proof. When the students showed that they had no idea through their silence (15), Pike makes suggestions on how to develop the proof (19). He suggests on the triangles to be used for developing the proof (PTS and QTR), the theorem to be used (congruency) and the proving statements to be developed (opposite sides and opposite angles of a parallelogram) and the angle to be used for the first proving statement (angle SPT) (19). The extract of the dialogue shows that most of the critical considerations for the proof production were made by Pike. He came up with all the proving statements except one statement (30). This shows that Pike provided more suggestions to how the proving task should be done than the students. The connections between the congruent triangles and the conclusion were also made by Pike. This shows that students were not involved to a greater extent in deciding, explaining and justifying how the proof should be developed. The way the task was implemented contradicts Smith and Stein's (1998) description of high cognitive demand tasks. This is because the students were not involved in exploring the proof and in explaining their answers. Therefore, although the task was of high cognitive level, it was implemented as a low cognitive level task.

Analysis of the interview that I conducted with Pike after the lesson shows that he provided much guidance to the student because he thought that they could not manage to come up with the proof on their own. This is what he said in response to why he did not include the students in exploring and explaining how to prove the task:

> *As you might have noticed earlier in the lesson, the students were unable to come up with a correct number of triangles in that figure, I had to show them the triangles first. So asking them to figure out the theorem to use and the statements on their own would have been very difficult for them. I don't think they could have managed to do that. But when you as a teacher give them an angle and tell them to identify its corresponding angle, then the students do not struggle (Pike).*

In the extract, Pike explains that he decided to tell the students the theorem to be used for proving the task based on his view that if they failed to tackle the simple question of identifying the number of triangles in the figure, then they would not manage to come up with a plan of proving the task on their own. Similarly, he asked the students to complete the proving statements to provide them with a reference point so that they do not have challenges in identifying corresponding angles. This implies that Pike thought that he was providing good guidance to the students by telling them the theorem to use and initiating the proving statements.

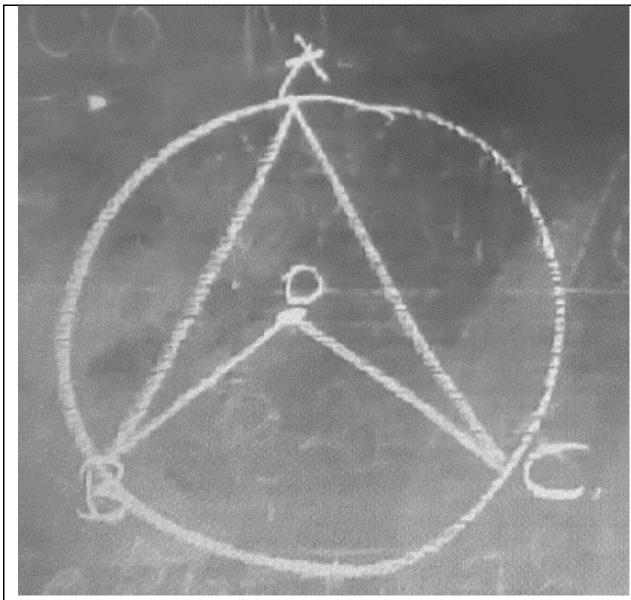
4.3.1.3. Teacher knowledge involved during task implementation

We observe that Pike attempted to prompt students to think of how the theorem can be used (15-19). Despite his inability to prompt the students' thinking further by asking probing questions, Pike attempted to provide students an opportunity to think of how they could prove the theorem. Pike is encouraging the students to think of another proving statement that can be developed from the diagram (29-32). One student mentions a statement and its reason that Pike accepts to be correct (30-32). The student developed the statement by using properties of the diagram and prior knowledge of geometric concepts of line and angles. This might suggest that Pike prompted the students to develop proving statements because he was aware of their prior knowledge.

I identified two aspects of knowledge from the extract. These are knowledge of prompting students to develop proving statements from the diagram and knowledge of prior knowledge required for the task. Pike is responding to students' suggestions that are not correct (26-29) and he explains to the students why their suggestions are not correct. In doing so, Pike focuses the students' attention to the task being proved and the parts of the diagram where the proving statements are being developed. The extract of the dialogue illustrate that the task implementation involved knowledge of analysing and responding to students' thinking. The lesson episode shows that the proving task was done by identifying two triangles, developing proving statements to show that the triangles are congruent, and using the property of congruency to come up with corresponding sides. This also illustrates the use of knowledge of geometric theorems and geometric properties.

4.3.2. Analysis of narrative tasks by John, Kim and Paul

Most of the tasks that the three teachers who were teaching Form 3 circle Geometry used when proving theorems were similar. This might be because there are mainly two types of secondary Mathematics books that the Ministry of Education recommend to the teachers and students. The similarity of the tasks might suggest that the teachers made either few or no alterations on the textbook tasks. Figures 11, 12 and 13 present examples of narrative tasks that were used for proving a theorem that states that an angle subtended by an arc at the centre is equal to two times an angle subtended by the same arc at the circumference.



Theorem: prove that angle at the centre is equal to two times angle at the circumference.

Figure 11: John's task for narrative proof

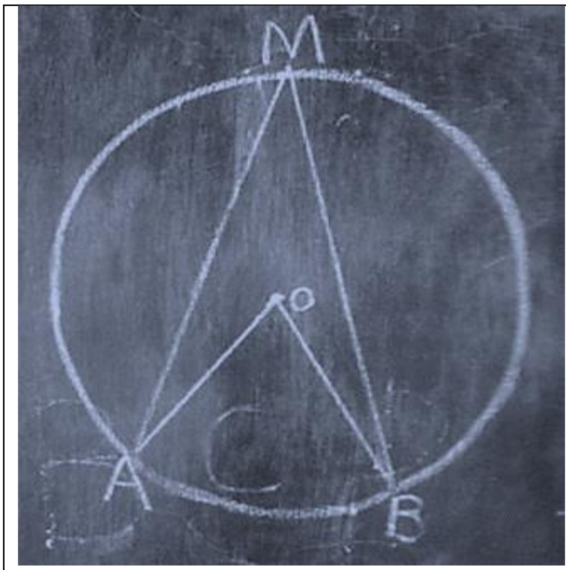
	Given, a circle with centre O, with arc AB subtending angle AOB at the centre and angle AMB at the circumference. Prove that the angle subtended by an arc at the centre is two times the angle subtended by the same arc at the circumference.

Figure 12: Kim's task for narrative proof

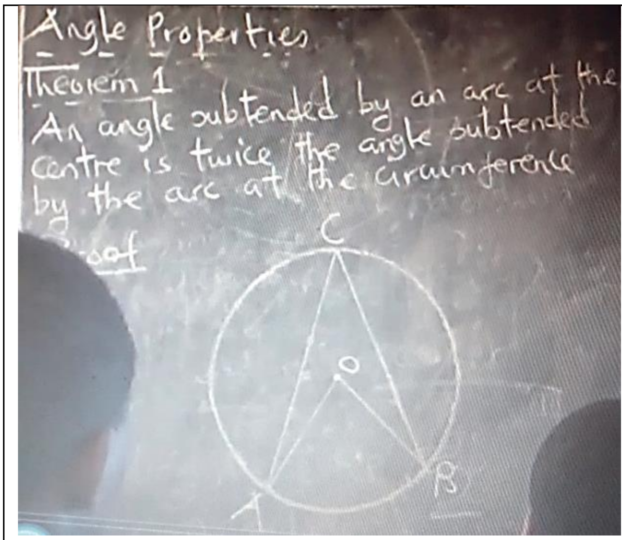
	Theorem: An angle subtended by an arc at the center is twice an angle subtended by the arc at the circumference.

Figure 13: Paul's task for narrative proof

Figures 11, 12 and 13 show that despite differences in the letters that were used, the diagrams were similar. The statement by Paul is similar to the one in the Form 3 Mathematics textbook. Paul has paraphrased the statement from the Mathematics textbook good because he has indicated that the same arc should subtend the angles at the centre and at the circumference. However, John's theorem statement is not good because it has left out an important aspect concerning the arc.

4.3.2.1. Cognitive level of the tasks

Using the diagrams and the statement in Figures 11, 12 and 13, students were supposed to begin by identifying angles subtended by the same arc at the centre and at the circumference. The angle at the circumference could be identified easily because the figure only contains one angle subtended by each arc at the circumference. However, it would be slightly difficult for students to identify the angle at the centre because the diagram contains two angles at the centre (the reflex angle and an obtuse angle). This means that students were supposed to use exploration to figure out the angle subtended by arc AB at the centre and to develop the proof as well.

There are at least two approaches that could be used to develop a proof using the given information. The first approach would be to join MO (using Figure 12) and use both the reflex angle and obtuse angle at the centre to come up with proving statements. This approach would mainly be used if students were unable to identify a correct angle at the centre. The second approach would be to construct a line from M to pass through the centre to make exterior angles at the centre and come up with proving statements. Students would use this approach if they were able to notice that the angle at the centre is the obtuse angle. Since both approaches required students' ability to make connections among different geometric concepts like radii, isosceles triangles, sum of interior angles of a triangle and sum of angles at a point, this shows that the task is of high cognitive level.

The diagram that was used by all the teachers involved a minor arc that was facing one direction. This means that the students might have understood the theorem as specific to the diagram. To help the students to generalise the theorem to different types of geometric diagrams, the teachers could have also used diagrams involving a major arc. They could also have rotated the arcs to show that students the angles can face different directions.

4.3.2.2. Task implementation by Paul

Paul explained everything that was supposed to be done to developthe proof. He labelled the diagram as ACB with angle ACB at the circumference and angle AOB at the centre. Then he joined CO and extended it to create exterior angles to triangles ACO and BCO. Lastly, he introduced letters into the diagram and explained how the proof was supposed to be developed. This means that Paul did not implement the task as a high cognitive task because he did not involve the students in any form of explanation and exploration.

4.3.2.3. Task implementation by John

John started by telling the students that the lesson was about proof development and they were going to prove that angle at the centre is two times angle at the circumference. He told the students that the angle at the circumference is any angle whose vertex touches the circumference while the angle at the centre is any angle that is outside the diagram in which the angle at the circumference is contained. Then he explained that they were going to prove using angle BAC at the circumference and obtuse angle BOC at the centre. The following extract shows how the teacher implemented the task.

20. John: So, we start with the first step, what have we been given?
21. Student: Circle with centre O.
22. John: Yes, this is what we have been given (and he writes on the

board). So, what is it that we want to prove? What is RTP?

23. Student: Angle at the centre is twice angle at the circumference.

24. John: Yes, this is what we want to prove but what are these angles in this diagram (he points at the diagram). Can someone tell us?

25. Student: Angle BOC should be twice angle BAC.

26. John: Yes, that is true, we need to prove that angle BOC is 2 times Angle BAC (then he writes RTP: BOC = 2BAC). For us to prove this do we need extra information? Do we need to construct anything?

27. Student: We need to label the angles BOC and BAC on the diagram, let angle BOC be equal to 2*x* and angle BAC be equal to *x*.

28. John: Thank you very much for trying but this is what we want to prove. By saying that angle BOC is 2*x* and angle BAC is *x*, we are saying that we have already verified that angle at the centre is twice angle at the circumference. But what we want to do here is to try to verify if this is true, so we are not yet there, we are still on the way of trying to find out whether this is true or not. (Then he rubs the labelling suggested by the student). So, do we need to add a construction? Yes, for this to be possible, we need to produce line AO to K. To produce means to go beyond O. (Then he draws the line AOK). What do we know about AO, OB and OC?

29. Student: Radii.

30. John: Okay, what do we know about radii?

31. Student: They are equal.

32. John: Yes, they are equal this means that AO, OB and OC are equal (he indicates equal signs on the diagram). So when you have equal sides what can you say about the base angles of triangle AOB and triangle AOC?

33. Student: They are equal.

34. John: Yes, they are equal so it means if we label base angles of triangle AOB as y_1 and y_2 then $y_1 = y_2$. Similarly, if we label base angles of triangle AOC as x_1 and x_2 then $x_1 = x_2$. Fine, aaa! now we want to go to the last step. We want to present the whole information that we have been discussing in a written form. So proof, what can you say about adding x_1 and x_2? It will be equal to what?

35. Student: Angle KOC.

36. John: (While pointing on the diagram), if we add x_1 and x_2 we will have angle KOC and similarly if you add y_1 and y_2 we are going to get KOB. What is the reason?

37. Student: Sum of opposite interiors angles of a triangle is equal to exterior angle.

The extract of the dialogue shows that John implemented the task using question and answer method. The questions that John asked were guided by his view of geometric proof development as a stepwise process. Firstly, he asked the students to identify the hypothesis (20-21). Secondly, John asked the students to identify the conclusion (23-26). Thirdly, he attempted to involve students in figuring out the construction to be made (26-27). However, when one student failed to come up with a correct suggestion, John suggested the construction to be made (28). Utterances (28-35) show that John involved students in generating proving sentences. However, the questions that John used in these utterances were of low cognitive level. The students were mainly involved in giving reasons and completing a proving statement that was given by the teacher (28-33).

The critical part of geometric proof development is to find the intermediary conciliator that is the theorem to be used for developing the proof (Chen & Lin (2009). The lesson extract shows that students did not struggle to come up with the intermediary conciliator because John asked leading questions (34-36). This implies that the students were not involved in high cognitive level thinking as John made the critical suggestions for the development of the proof. The cognitive level of the task implementation was also low because the students were not involved in making their own explorations regarding the proof development.

The lesson episode also shows that John paraphrased the theorem as: angle at the centre is twice angle at the circumference. The implication of this paraphrasing in John's lesson was that there was no emphasis on the condition that the angles should be subtended by the same arc at both the circumference and the centre. This could lead to students' misunderstanding of the theorem and its proof.

4.3.2.3. Task implementation by Kim

Kim asked the students to work in small groups to discuss how to develop the proof. As the students were developing the proof, Kim went around the groups to check their work and to provide some guidance. In the following extract, I present a discussion between Kim and students in group 6.

25. Kim:	Do you understand what you should do?
26. Students:	Yes.

27. Kim: Tell me.

29. Student 1: We should show that this angle (pointing at reflex angle at AOB) is2 times this angle (pointing at angle AMB).

30. Kim: Okay, so what are you going to do, have you discussed?

31. Student 1: Yes, we will join MO and prove that these two triangles (pointing at triangle AMO and BMO) are congruent. Then relate the corresponding angles.

32. Kim: Can you show me how you will relate the angles?

33. Student1: First, AO = BO (radii), OM is common, and AM = BM (third side) AOM is congruent to BOM. Then angle a = angle b, the two angle here are also equal (pointing at the reflex angle at AOB) and the two angles here are equal (pointing at M). (The student is silent).

34. Kim: Go ahead.

35. Student 2: Then we add angles here (pointing at the reflex angle at AOB) and angles here (pointing at M) uhhh… (silence 4 seconds).

36. Kim: Yes, go ahead what about the other group members, how do you proceed from here to the theorem? (silence 6 seconds), how do you arrive at the question that you have been asked using that theorem? Do you know the angle at the centre referred in the theorem?

37. Student 3: Yes, this one (pointing at the reflex angle at AOB).

38. Student 4: No, this one (pointing at the obtuse angle at AOB).

39. Kim: Can you try to measure the angles and see if it is the upper or lower angle which is twice the angle at M? After that think of another way, this one might not work.

After utterance (39) Kim went to check students in other groups, he listened to their suggestions and asked questions. When he noticed that most of the groups were not focusing on a correct angle at the centre, he interrupted the activity and asked the students to measure the three angles in their diagrams to find out the correct angle. The following segment is a continuation of dialogue between Kim and group 6 students.

40. Kim: What did you find after measuring the angles? Which is the correct angle?

41. Student 3: This one (pointing at the obtuse angle at AOB).

42. Kim: Did you all agree that it is this angle?

43. Students: Yes.

44. Kim: Why?

45. Student 3: This angle (angle at M) was 52^0 while this one (the obtuse angle) was104^0. So this (angle at M) is twice this (obtuse angle).

46. Kim: Okay, so how are you going to prove the theorem?

47. Student 5: We tried similarity but we found that it was going to be difficult as well because it was not saying anything about this angle (pointing at obtuse angle at O) it was only saying about this one (pointing at the reflex angle at O). So, since this angle is outside these two triangles, we agreed to use the property of exterior

angle of triangle. So we extended MO to N to create exterior angles here (pointing at obtuse angle at the centre).

The extract shows that Kim involved the students in both exploration of the proof and explaining of their procedure. When the students were exploring the proof using the diagram, they were focusing on a wrong angle at the centre (27-39). To prolong students' exploration, Kim suggests that they measure the angle at the centre and the two angles at the circumference and compare their values (39). Through this activity, the students managed to conclude on their own that the angle that was subtended by the arc at the centre was the obtuse angle (41). Utterance (47) shows that after the measuring activity, the students were able to explain their decisions regarding the theorem they used for proof development. The student's explanation in utterance (47) also shows that the students were involved in exploring theorems to be used for developing the proof. This indicates that Kim implemented the task in a manner that involved students' high levels of cognitive activation.

4.3.2.4. Teacher knowledge involved during task implementation

There were several aspects of knowledge for task implementation identified from the lesson extracts. The teachers gave students different levels of guidance and support during task implementation. John and Paul provided much guidance to the students as they always suggested what was supposed to be done on the diagram, and they developed the proving statements for the proof. Kim provided appropriate guidance as he only suggested to students the proving activity and involved the students in deciding what to do on the diagram and developing proving statements from the diagram. The type of guidance that was given to the students had an effect on the cognitive level at which the task was pitched. The teachers who provided much guidance made the cognitive level of the task to diminish while those who provided good guidance maintained the high cognitive level of the task. This implies that the teachers need knowledge of good guidance when supporting students to perform a task. This might help either to increase the cognitive level of a low level task, or to maintain the cognitive level of a high level task.

Secondly, there was the use of knowledge of asking probing questions. This was also observed from the two extracts in different degrees. John displayed this knowledge when he asked the students to explain what they knew about different types of geometric concepts like radii and exterior angles. However, he provided much guidance in the questions that he asked. For example, John asks the following question, "*When you have equal sides what can you say about the base angles of triangle AOB and triangle AOC*?" (34). By mentioning base angles of triangle, he had provided the clue to the answer, as such, the students might have used memorisation to answer the question. Similarly in utterance (37) John asks the following question, *"What can you say about adding x_1 and x_2? It will be equal to what?"*John provides a clue for an answer in the question upon mentioning addition of the angles, so in this case, the students might have also used memorisation to come up with a correct answer. This shows that the questions needed to be phrased in a way that could provoke students to think deeply to come up with an answer.

The questions that were asked by Kim can be regarded as cognitively demanding because they influenced students to evaluate their thinking. Examples of such questions are in utterance (36) where he asks the students to explain how they would proceed from congruency proof to the theorem that they were asked to prove. The question guided the students to think of how to make connections between the given information and the conclusion. This is evidenced in utterance (47) where a student explains why their group decided to use exterior angle property instead of similarity property. This indicates that the questions that Kim asked prompted the students to think of why a certain theorem was either suitable or not suitable to the proof.

During post-lesson interviews, Kim explained that he asked the students to measure the angles instead of telling them the correct angle at the centre because he wanted the students to understand the theorem and its proof, and to believe that it is true. He also said that he asked the students to discuss how to prove the theorem because he wanted the students to internalise the proof and be able to apply it to different situations. This means that during task implementation, when the students were unable to prove the theorem as expected, Kim had to think of how to guide the students in a manner that they could identify the solution to their challenge. This led to observation that he used knowledge of guiding and fostering students' exploration opportunities during task implementation.

However, although the task had multiple solution paths, all teachers only guided the students to developthe proof using one approach. This is the proof that is available in the teachers' and students' textbooks. Nevertheless, it would be possible for the teachers to guide the students to develop the proof by only joining MO, finding the sum of the two interior angles of the two triangles (AMO and BMO) and compare it to the sum of angles at a point. This method would have also led to the proof of the theorem. Focusing on one solution path for this task suggest lack of knowledge of multiple solution paths. For example, if Kim had knowledge of multiple solution paths to the task, he would probably not have postponed the first group discussion in which students had agreed to join MO, he would have asked them to explore different theorems apart from congruency. Alternatively, Kim would have asked the students questions that would guide them into thinking of the sum of interior angles of a triangle and angles at a point. The measuring activity could have been utilised for discussing an alternative way of developing proof for the theorem.

4.3.2.5. Provision of opportunities for multiple solution paths

The findings on the proof solutions that were developed by both teachers and students in the class show that Pike, John and Paul mainly involved the students in proving only one proof for each theorem despite that some theorems could have multiple proofs. However, some of the theorems proved by Kim and his students had multiple proofs. The following lesson extract presents an example of the theorem where students developed multiple proofs. In this lesson, Kim was helping students to prove that an angle in a semicircle is a right angle. He drew a circle with diameter AB subtending angle ACB at the circumference. He asked the students to go into their usual groups to discuss how to prove the theorem. During reporting, most of the students explained that they developed the proof using the theorem that an angle subtended by an arc at the centre is equal to two times an angle subtended by the same arc at the circumference. Since an angle at the centre is a straight angle, then its value is 180^0, so the angle at the circumference is half of the angle at the centre that is 90^0 in this case. Then the students concluded that an angle in a semicircle is a right angle.

This proof is related to the proof in the students' textbook. This is also the proof that John and Paul developed for the theorem in their classes. The theorem that the students used for the first proof was proved in the previous lesson. The proof shows that there were few geometric connections that the students had to make when developing the proof. In addition to this approach, a student from group 2 explained another approach that they used to prove the theorem using the diagram in Figure 14 below.

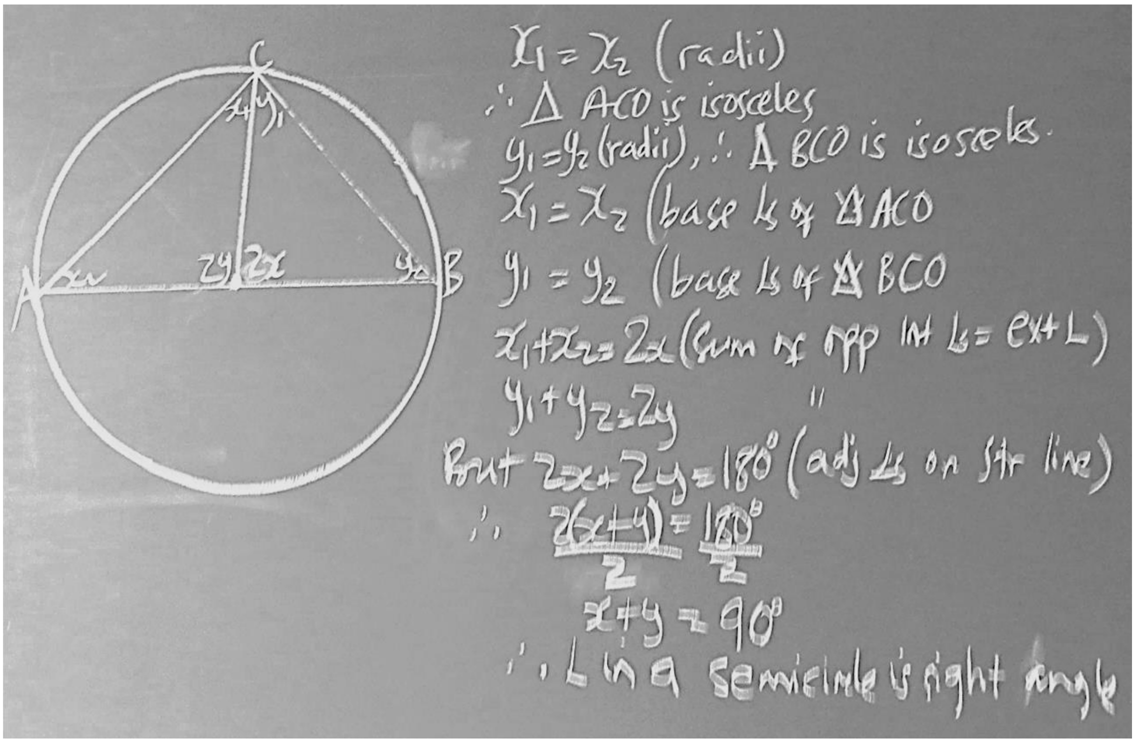


Figure 14: Diagram and proof by group 2 students

50. Student: (Explains while referring to the diagram). We used theorem of exterior angle of a triangle equals sum of opposite interior angles. So, we joined AO. So $x_1 + x_2$ equals $2x$ because these three lines are radii AO, CO and BO so these two triangles are isosceles triangles (pointing at the two triangles formed after joining C to the centre).

51. Kim: Can you please indicate O on the diagram, we cannot see where O is.

52. Student: Okay sorry, O is here (he indicates O at the centre of the circle).

Same here in this triangle (pointing at triangle CBO) $y_1 + y_2$ equals $2y$ same reason, base angles of isosceles triangle. So, if we add angles here on the straight line, the sum is 180^0 , so, $2x$ plus $2y$ equals 180^0. Factor out 2 equals 2open brackets x plus y close bracket equals 180^0. Divide 2 both sides, x plus y equals 90^0. So, angle x is here and angle y is here (pointing at angle at the circumference). So, we add these two we get 90^0, it means this angle is a right angle.

53. Kim: Okay that is how your friends constructed the proof. Do you have comments?

54. Students: No (chorus answer).

55. Kim: Do you agree with this approach?

56. Students: Yes (chorus answer).

57. Kim: Okay all of us have come up with correct proofs. This means that you understood what you were asked to do. But remember to indicate the given information, the statement you are proving and the construction you have made on the diagram before starting writing the proof statements. I think we should clap hands for this group for coming up with a method which is different from the rest of us. It means they involved a lot of thinking during the discussion not so?

The proof presented by a student in group 2 students in Figure 14 is not available in the textbooks that are used by the teachers and the students. The dialogue in the lesson extract shows that it was the students who devised plans for developing the proof for the theorem. The students drew the diagram, then they added features and labels into the diagram and decided on the theorems to be used for developing the proof (52). The students had to make several geometric connections to develop the proof. This included use of properties of isosceles triangles, exterior angle of triangle and adjacent angles (52). Kim does not seem to provide much guidance, the students are working independently. The extract also shows that Kim did not shift the students' focus to the approach of proof that is presented in the textbook. This indicates that Kim had knowledge of multiple solution paths to the development of the proof for the theorem.

The students might be able to develop this proof because they were involved in exploration. This implies that the students were involved in high cognitive level thinking when developing the proof. This enabled the students to relate several geometric concepts to come up with the proving statements. This implies that the students had opportunities to try several ways of developing the proof using different geometric theorems and concepts. Kim commended the group 2 students for thinking deeply to come up with the proof (57). In this regard, the other students in the class realised that apart from the approach they used in their groups, there was also another approach for proving the theorem.

4.3.3. Analysis of computation tasks by Pike

The lesson observation data shows that computation tasks were used for two purposes. The first purpose was to help the students to understand the theorem that was proved and how to apply it to solve different problems. The second purpose was to evaluate students' understanding of the theorem and its application. Tasks which aimed at supporting the learning of theorems were used in a form of examples and were done after implementation of narrative tasks. It was found that the number of examples used in each lesson depended on the teaching methods that were used during the lesson.

Lessons which involved group discussions had few examples than those that mainly used question and answer methods. For example, after proving that an angle subtended by an arc at the centre is equal to two times an angle subtended by the same arc at the circumference, there was only one example that was done in Kim’s lesson, while four examples were done in Paul’s lesson.

Figure 15 presents an example of computation task that was written on the chalkboard by Pike after proving a theorem which states that the sum of opposite interior angles is equal to the exterior angle of a triangle.

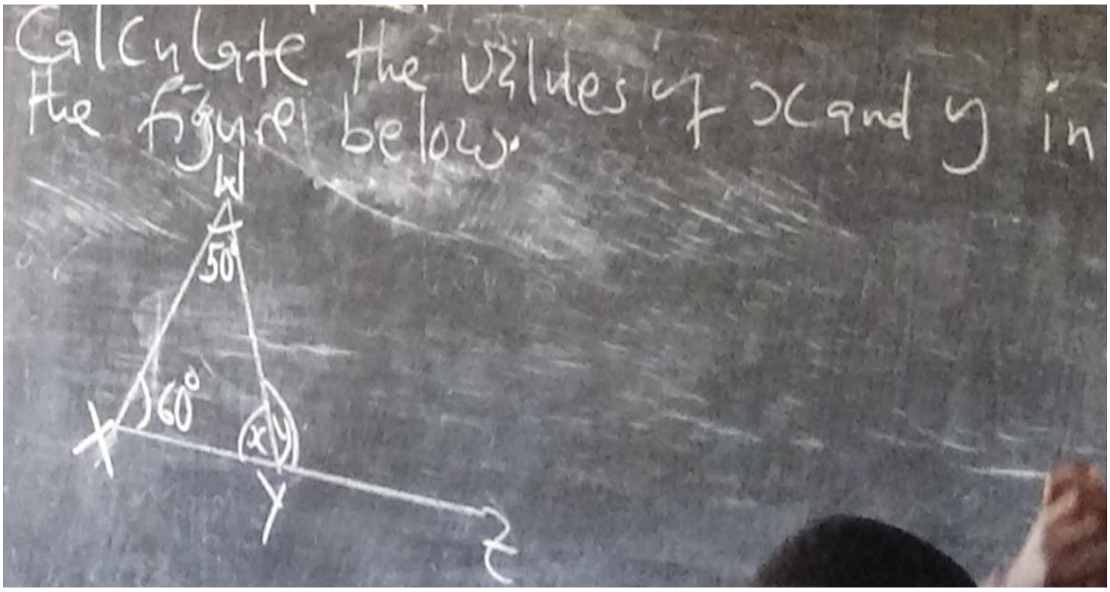


Figure 15: Computation task used by Pike to show students how to apply a theorem

4.3.3.1. Cognitive level of the task

Pike selected the task in Figure 15 with an aim of helping the students to understand how to use the theorem to solve problems. The task can be regarded as involving high cognitive demands because it required students to make connections between the new knowledge and prior knowledge of either interior sum of angles in a triangle or adjacent angles on a straight line to find angle x. The task could also be regarded as cognitively demanding because it had multiple solution paths, so the teacher could use the task to challenge students to explore different ways of finding the solution to the task.

4.3.3.2. Task implementation

Pike asked the students to find the solution for the task individually. After 10 minutes of individual work, Pike involved the students in a class discussion in which the students explained how they found the solutions for the task. In the following extract, I present the dialogue between Pike and students during reporting of the solutions.

58. Pike: So you are asked to calculate the value of *x* and *y* in that figure. So what did you do to find the value of *x*?
59. Student: $60^0 + 50^0 + x$ equals 180^0.
60. Pike: What is the reason?
61. Student: Sum of angles in a triangle.
62. Pike: So, if we simplify that, it gives us 110^0 plus x equals 180^0 so what is the value of *x*?
63. Student: 180^0- 110^0 which is 70^0.
64. Pike: Yes ,*x* is 70^0 what about *y*? How did you find *y*?
65. Student 1: I found *y* by subtracting 70^0 from180^0 because *x* plus y is equal to180^0adjacent angles on a straight line.
66. Pike: No, we need to find y using the theorem we have proved today and not just any other property. So how can we find *y*?
67. Student 2: 60^0 plus 50^0 equals *y*
68. Pike: Yes, 60^0 plus 50^0 equals *y*, let’s use what we have found today,

let's put it into practice, of course we know that x plus y equals180^0 because those are adjacent angle but let us apply the theorem we have learnt today. So 60^0 plus 50^0 equals y, what's the reason?

69. Student: Exterior angle equals sum of opposite interior angles.

70. Pike: Yes, so what is the value of y?

71. Student: 110^0(chorus answer).

The extract shows that the students were involved in exploring the solution for the task and explaining their answers. Pike accepted the use of sum of interior angles of a triangle property to find angle$x$ (58-64). But he only expects the students to find y by using the theorem that they had just proved (sum of opposite interior angles is equal to the exterior angle). As such, in utterance (66) he rejects the student's suggestion to use property of adjacent angles on a straight line (65). Pike insisted that the students use the theorem they proved during the lesson despite that what the student suggested was correct.

To avoid discouraging the student, Pike could have accepted the student's suggestion as one way of solving for y then later on ask if there were other students who used a different approach to find the value of angle y. This could have helped the students to compare the values of y that they found using the different methods, and appreciate that there are multiple ways of finding solutions to geometric tasks. The way the task was implemented suggests that during planning Pike did not analyse the task and think of several ways of solving it. If he had analysed the task, he would have modified it to suit his objective by asking students to find y only. In that case, it would have been easier for the students to use the theorem that they had just learnt.

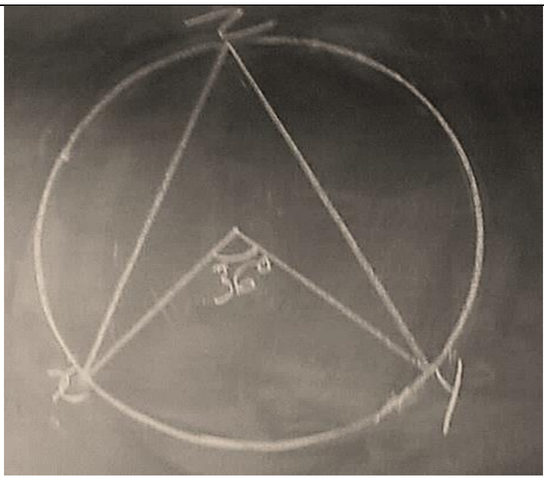
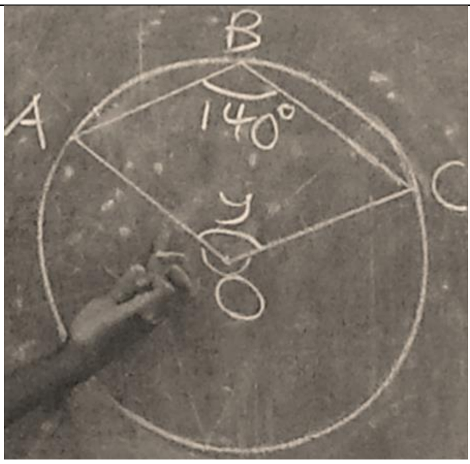
4.3.3.3. Teacher knowledge involved during task implementation

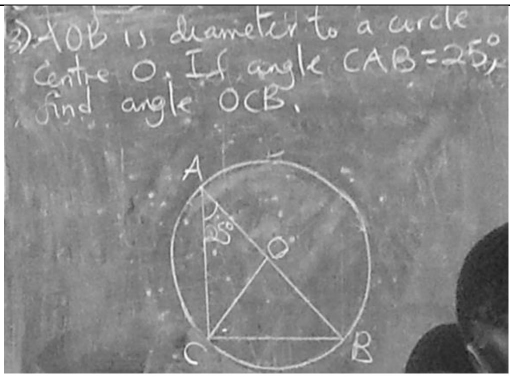
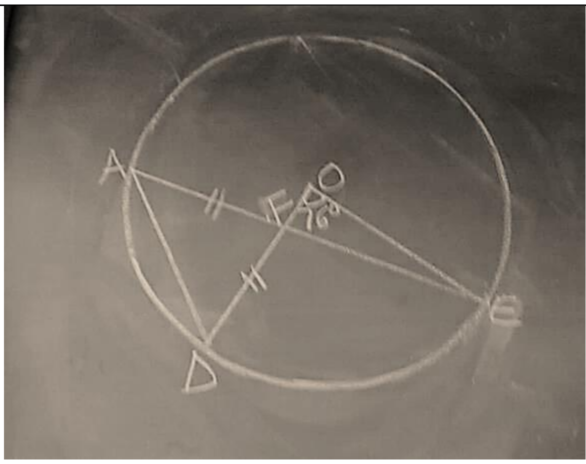
Pike asked the students to explain what they did to find the solution for the task (58). When a student explained what they did (59), Pike provided feedback by challenging the students to do some thinking (60). This shows that Pike utilised knowledge of analysing students thinking during the conversation. The extract shows that Pike only guided the students to use one solution path despite that there were different ways of solving the computation proof task. I identified two types of knowledge that could have been appropriate for the implementation of the task. Knowledge of multiple solution paths could have helped the teacher to accept the suggestion made by the student in paragraph 66. The student's suggestion was correct despite the fact that it was not what the teacher desired. Pike displayed lack of knowledge of multiple solution paths to geometric computation proof tasks by denying a student's suggestion which was correct. Another type of knowledge that was identified as appropriate but missing in the extract is knowledge of selecting tasks according to the intended purpose. This knowledge could help Pike to select tasks that could enable students only to use the new knowledge to meet the teacher's aim of the task. In this case, Pike would have presented a task with value of the exterior angle and one opposite interior angle and asked the students to find the value of the other opposite interior angle. However, considering that the COACTIV model regard knowledge of cognitively activating tasks as ability to formulate multiple solutions for the task, the better suggestion was to allow students come up with different ways of solving the task.

4.3.4. Analysis of computation proof tasks by Paul

Table 6 presents the four examples that were done by Paul after proving that an angle subtended by an arc at the centre is equal to two times an angle subtended by the same arc at the circumference.

Table 6: Examples of computation tasks used by Paul.

Task number	Diagram	Statement
1		The following figure is a circle centre O. lf angle XOY is 36^0, find angle XZY.
2		Calculate angle y in the following figure if the circle ABC has centre O.

3		AOB is diameter to circle centre O. if angle CAB = 25^0, find angle OCB.
4		The following figure is a circle with centre O. If angle DOE = 76^0 and AF = DF, find angle ADO

4.3.4.1. Cognitive level of the tasks

In Table 5, task 1 aimed at showing students how to apply the theorem when given angle at the centre and required to find an angle at the circumference. The minor arc is the one that is subtending the angles at the centre and at the circumference. The diagram of the task is similar to the one that was used in the narrative task. As such, the task can be regarded as involving low level cognitive demands because it does not require students to show their understanding of the theorem to find the value of the angle.

In the diagram for task 1, the angle at the centre is clearly more than an acute angle, hence it was supposed to be more than 60^0. Paul might not be concerned with drawing accurate angles because the lesson was not about geometric constructions. However, to avoid confusing the students, it would have been appropriate if the angle that he drew was approximate to the given size. This is because at this stage, the students have done some geometric constructions and they can identify different types of angles through visualisation. As such, the angle that Paul drew could contradict with their prior knowledge of acute and obtuse angles.

Task 2 aimed at showing students how to use the theorem to find an angle at the centre when the angle at the circumference is given, and also to use the theorem as a bridge for finding other angles. The diagram seems different from the one that was used for the narrative task because it is the major arc that is subtending the angles at the centre and at the circumference. The task can be regarded as involving high levels of cognitive demands because it would require students to demonstrate their understanding that the theorem also applies to a situation where the angles are subtended by the major arc. The task is also of high cognitive level because it involves making of several geometric connections. The student has to first of all find the value of the reflex angle at the centre before finding angle *y*.

Task 3 is the same as task 2, it aims at showing students how to use the theorem to find the angle at the centre when the angle at the circumference is given. Despite involving a minor arc, the task is different from task 2 because one of the radii that is forming the angle at the centre is combined with the chord that is forming the angle at the circumference. The task requires students to first find obtuse angle COB and then use properties of isosceles triangles to find angle OCB. This means that the task involves making of geometric connections. Furthermore, the diagram might be confusing for students to identify angle at the centre, which means that only the students who understood the theorem and the proof are able to find the value of obtuse angle COB. Therefore, the task can be regarded as a high cognitive level task.

Task 4 is similar to task 1. It would be regarded as a low cognitive level task if it required the students to find the angle at the circumference (angle EAD). Since the task required students to use the theorem that they learnt as a bridge to find the value of an angle which could be used for finding the required angle, then the task can be regarded as a high cognitive level task. However, it is observed that three out of the four tasks involved a minor arc. This might make students understand the theorem as if it mainly involves the minor arc, hence having problems in applying it to solving problems involving the major arc.

4.3.4.2. Task implementation

Paul implemented the four computation tasks in the same way. He explained to the students what to do when they are given a particular task. The following extract shows how Paul implemented task 2.

3.	Paul:	We have another situation here; we have a circle with centre O and angle at the circumference here (pointing at angle ABC) is equal to 140^0. And we are required to find angle *y*. this angle at the circumference (pointing at angle ABC) is facing which arc? (silence for (6 seconds) it is facing this arc (pointing from A to C), it is facing which arc?
4.	Students:	AC. (Chorus answer).
5.	Paul:	Yes it is facing arc AC because as you can see here (pointing at lines AB and BC) these lines are these far ends (pointing at A and C on the circumference). That means that arc AC is subtending this angle (pointing at angle ABC). And this angle (pointing at angle ABC) is facing the same arc that the angle at the centre is

facing here. So, the angle at the centre that is facing arc AC is this angle (pointing at reflex angle AOC. This other angle (pointing at angle *y*) is facing a different arc, it is facing the smaller arc up here (pointing at minor arc AC) which is also arc AC. That means this angle (pointing at angle *y*) is not related to this angle (pointing at angle ABC but it is related to the angle which is out here (pointing at reflex angle AOC) because the angle out here is facing the major arc AC and also angle ABC here is facing the major arc AC. So, it means that we can be able to find the angle which is out here (pointing at reflex angle AOC) and the value of this angle will help us to find angle *y*. That means when we are finding this angle (pointing at reflex angle AOC). We must name it appropriately to show that we do not mean angle *y*. So, we will name this angle as reflex angle AOC. So, (explains while writing) reflex angle AOC is equal to 2 x ABC. So reflex angle AOC is 2 x 140^0 for the same reason angle at the centre is twice angle at the circumference. Multiplication what do we have?

6. Student: 280^0. (Chorus answer).

7. Paul: 280^0. Okay, now we know the angle which is out here (pointing at reflex angle AOC) and we know that these two angles (pointing at angle *x* and *y*) are angles at a point and we should recall their sum. So how can we use sum of angles at a point to find angle *y*?

8. Student: *y* = 360^0minus280^0.

9. Paul: She is saying angle *y* equals 360^0- 280^0. What is the reason?

10. Student: Angles at a point. (Chorus answer).

11. Paul: So what is the answer?

12. Student: 80^0. (Chorus answer).

13. Paul: Yes, angle $y = 80^0$. Anyquestion? (Silence for 4 seconds). Okay since you have no question, then we will move on to the third example.

Extract for example 2 shows that Paul provides both the procedure and explanation for finding the two angles. He starts by explaining to the students how to identify angles that are subtended by the same arc (3). Then he shows the students that the angle at the centre is related to the angle at the circumference (5). He also explains why it is the reflex angle at the centre that is related to the angle at the circumference (5). The students are involved in answering low cognitive level questions like multiplying 140^0 by 2 and subtracting 280^0 from 360^0 (6-8). Although the task was regarded as a high cognitive level task, the mode of implementation did not provide students an opportunity to explore the solution and explain the procedure. The same way of task implementation was used for all the remaining three example tasks by Paul. This shows that all tasks were implemented at low cognitive level.

Analysis of the interview that I conducted with Paul after the lesson shows that he implemented the task in this manner because of the way he conceptualises geometric proof development. He said that teaching of geometric proof development for each theorem involves repeating of the same procedure; as such, most of the students are not interested in knowing how to prove the theorem, but they are interested in understanding how to apply the theorem to solve different geometry problems. It is due to this reason that he usually decided not to spend more time on involving the students in exploring ways of proving the theorem.

4.3.4.3 Teacher knowledge involved during task selection and implementation

The two extracts show that Paul aimed at drilling the students on how they could solve certain types of tasks using the theorem that was proved. There are three aspects of knowledge that could have been appropriate for good task selection and implementation. These are knowledge selecting appropriate tasks, knowledge of good implementation of the tasks, and knowledge of making clear explanations. Knowledge of selecting appropriate tasks could have helped Paul to realise that since he used the minor arc only when proving the theorem, then it would have been appropriate if he used several tasks involving the major arc during exemplification. Knowledge of good implementation of the tasks could have helped Paul to involve the students more in finding the solutions to the tasks and to enhance their understanding of the theorem and the proof. Knowledge of making clear explanations is observed when Paul points out areas of focus when doing the calculations like how to identify angles subtended by the same arc. However, Paul could have explained the solution after students had attempted the tasks on their own.

4.3.5. Analysis of computation tasks by Kim

This task is also an example of an evaluation task. After discussing how to prove a theorem that states that an angle subtended by an arc at the centre is equal to two times an angle subtended by the same arc at the circumference, Kim gave students the tasks in Figure 16 for homework.

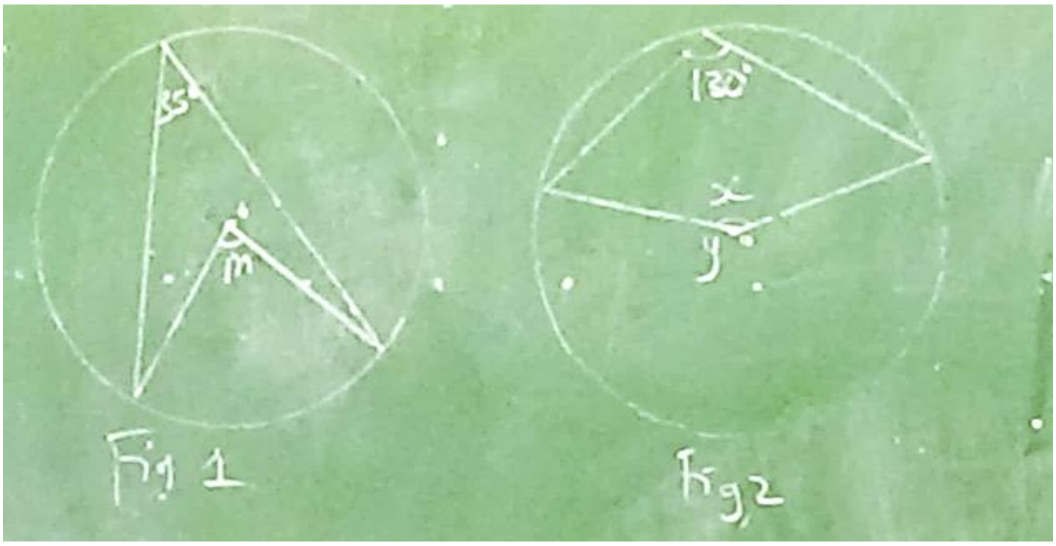


Figure 16: Tasks by Kim focusing on evaluating students' understanding of a theorem

4.3.5.1. Cognitive level of the tasks

In Figure 16, Task 1involves finding of angle *m*. The task is similar to the one that was used for proof development and to the one that was used for an exemplification after proof development. Students were only required to apply the theorem by multiplying the angle at the circumference by 2 to find angle *m* at the centre. This shows that the task would mainly require students to use the procedure that they had learnt or memorised. So, the task is regarded as of low cognitive level the one used by Paul as an example. Task 2 which involves finding of angle *x* and *y* looks slightly different from Task 1 because the angles at the centre and circumference are subtended by a major arc. This means that the students were supposed to show their understanding of the theorem in order to find the solution of the task. The main trick in the task lies in the way the radii are drawn, it would require much thinking for the students to realise that the angle at the centre which is related to the one at the circumference is angle *y* and not angle x.

4.3.5.2. Task implementation

The following day, as we were going to class, Kim explained that he was going to revise the theorem about angle at the centre and angle at the circumference because most of the students were unable to solve task 2. Kim said that he suspected that the students were mainly unable to solve task 2 because it was different from the ones he used for narrative and computational proving in the previous lesson. So, when we went into the class Kim drew the homework tasks on the chalkboard. He used question and answer to find angle *m* and the students were able to mention the value of the angle and the reason without difficulties. Then for values of *x* and *y*, Kim asked for volunteers to write the solutions on the board. Table 7 show the solutions that were written by two students who volunteered to write their solutions on the board.

Table 7: Proofs developed by students

Student 1	**Student 2**
$130° = X$ (opposite angles of a kite) $X = 130°$ $X + Y = 360°$ (angles at a point) $Y = 360° – 130°$ $Y = 230°$	$Y = 2\ (130°)$ (angle at the centre is twice the angle at the circumference) $Y = 260°$ $X + Y = 360°$ (angles at a point) $X = 360° – 260°$ $X = 100°$

The following extract of a lesson episode presents a conversation between Kim and the students in connection to the answers that were presented on the board.

3. Kim: Let us start with the first one: is this correct?
4. Students: Yes (some of them), no (others).
5. Kim: Is there anywhere given that this (pointing at the quadrilateral in

figure 1) is a kite?

6. Students: No (chorus answer).
7. Kim: So, why are you indicating the property of a kite?
8. Student 1: It's because of the way it is drawn.
9. Kim: Are you telling me that you just look at a diagram and judge that it is a kite?
10. Students: No (chorus answer).
11. Kim: You need to either use the information that you are given, or prove first and convince us that it is a kite. But here we are not told that this is a kite and you haven't proved to convince us that this is a kite. Why did you not use yesterday's theorem to find *x* and *y*?
12. Student 1: There is no angle at the centre which is connected to the angle at the circumference.
13. Kim: Class is it true that there is no angle at the centre in diagram 2?
14. Student 3: There is angle *x* and *y* at the centre.

Then Kim asked the students in the class to analyse solution 2 and state if it is correct. Only a few students said that it was correct; most students remained silent. The segment below presents Kim's explanation in trying to help the students to understand the theorem and its application to the question.

15. Kim: These two angles here (pointing at angle x and y) are at the centre. You needed not to be confused because this diagram (pointing at the second diagram) is the same as this diagram

(pointing at first diagram). The only difference is that the radii in this diagram (pointing at second diagram) are facing upwards and that makes the angle at the centre to be bigger or to be a reflex angle while the radii in this diagram (pointing at first diagram) are facing downwards making the angle at the centre to be smaller or to be an obtuse angle. So you find the value of x the same way you found value of m and then proceed to finding x using property of angles at a point. Any questions?

16. Student 5: Yes, what if somebody takes y as x? What I am trying to say is that if we compare the radii, the M angle in Question 1 looks like the x angle in question. So, since we take the one which is smaller, I am thinking of x as equal to 2 times 130°.

17. Kim: No, we take the one which is facing the direction the angle at the circumference is facing. It can either be the obtuse angle or the reflex angle. Had it been that there was an angle down here (pointing at the bottom of the circle) facing x then we would say that x is 2 times that angle.

In the lesson extract, it is observed that Kim maintained the cognitive levels of both tasks. For Task 1 which was of low cognitive level, Kim only asked the students to give the answer and its reason. Kim might have decided to spend very little time (about 5minutes out of 40 minutes) on task 1 because he found that it was not very challenging for the students. He spent the rest of the time (about 35 minutes) on task 2 which was very challenging to the students. Kim started reviewing the task by asking students to write their solutions on the chalkboard and then engaged all students in a class discussion using the solutions on the chalkboard. He began the discussion by asking the students to analyse the solutions and check if they were correct. Then he mentioned the misconception that he observed in solution 2 and asked the student to explain their thinking about it. The student's explanation exposed the cause of misconception making judgement based on the appearance of the diagram. Kim addressed the misconception by explaining to the students how they are supposed to judge a diagram. He continued with the discussion to get to the misconception that he identified as the main cause of the students' inability to solve the task. This was done by asking the students to explain why they did not use the theorem that they learnt in the previous lesson.

The response that was given by student 1 helped the teacher to understand their thinking about the task (12). The teacher addressed the misconception by explaining the properties of the diagram in relation to the theorem (15). Despite Kim's explanation about the direction of radii (15), student 3 still held a misconception that the smaller angle should be the angle at the centre (16). This led, Kim to describe angles subtended by the same arc using direction that the angles face (17). The misconception exposed by student 3 led Kim to realise that he needed to explain the meaning of the theorem in general not only in relation to the task that the students failed to solve. The lesson extract indicates that Kim involved students in analysing the solutions, explaining the procedures and justifying their arguments. This shows that the Kim maintained the cognitive level of the task during task implementation.

4.3.5.3. Teacher knowledge involved during task implementation

In the lesson extract, it is observed that the teacher was asking questions aiming at exposing students' misconceptions in relation to the theorem learnt in the previous lesson. The questions that the teacher asked did elicit the misconceptions that the teacher had anticipated during evaluation of the students' solutions. It is observed from the extract that the teacher used knowledge of questioning to prompt students' thinking. When a student exposed his thinking, the teacher involved the class in analysing and discussing the student's thinking. This illustrates that the teacher used knowledge of working with students' ideas. After students exposed and discussed their misconceptions, the teacher provided some specific and general guidance on how the students are supposed to approach proof tasks. This illustrates that the teacher used knowledge of explaining important points of the geometric concepts and procedures.

4.4. Assessment of students' thinking in geometric proving

In this section, I intend to approach research Question 3 which involves how the teachers analysed students' thinking. Results for this question were generated through analysis of Questionnaire 2 in which teachers were engaged in practical experience of analysing and responding to students' real life scenarios. The results indicate that all teachers identified statement that they thought were not correct in the proofs. Pike, John and Paul, mentioned the statements which they thought were not correct while Kim mainly wrote the causes of the mistakes that he seemed to have identified in the solutions. However, some of the statements that were identified by Pike, John and Paul were not entirely wrong. The results also showed that all teachers were not able to identify one of the statements that was wrong in a solution by student 1. Table 8 present a summary of the results of data analysis.

Table 8: Summary of how teachers assessed students' thinking

Category	John	Kim	Paul	Pike
Reasons why a statement is wrong.	-Did not explain why he thought that some statements were wrong.	-Explained the reasons why he thought that the solutions were wrong.	- Did not explain why he thought that some statements were wrong	-Explained why he thought that a particular statement was wrong.
Causes of the mistakes	-Mentioned poor background of geometrical concepts as the main cause of all the identified mistakes in the proofs.	- Attributed the causes of the mistakes to both lack of prior knowledge of geometric concepts and lack of ability to make connections among geometric concepts.	-Wrote that the identified mistakes were a result of lack of knowledge of procedure for writing geometric proofs.	-Attributed the students' mistakes mainly to lack of knowledge of the procedure for writing a geometric proof
Ways of addressing the mistakes.	-Show the students how they were supposed to	- Let the students explain their solutions in order to reveal	- Re-teach the geometric concepts. - Give the	- Help the students to understand how to use

	find solutions for the tasks. - Explain again the geometrical concepts with the aid of teaching materials.	more mistakes. - Ask the students to analyse and correct each other's mistakes. - Give the students more tasks for practice.	students more tasks for practice.	given information when solving a geometric task. - Help the students to understand the geometric concepts required in each proof.

In the following sections, I have discussed in detail how each teacher assessed students' thinking during teaching of geometric proving.

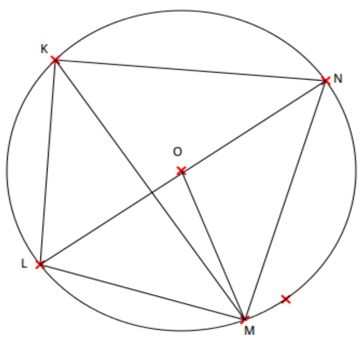
4.4.1. How the teachers identified mistakes in the students' scenarios

Analysis of the teachers' responses to the students' solutions show that the teachers identified students' mistakes by analysing each statement of the solution to find the statement where the students made mistakes. Some teachers continued to analyse the solution after spotting the first mistake while others did not. As such, some teachers only reported one mistake in a solution while others reported several mistakes. The findings also show that some teachers concentrated on reporting the specific statements that were wrong in the solutions while others concentrated on reporting general issues that were not correct in the solutions. Table 9 presents task 1 and the students' solutions.

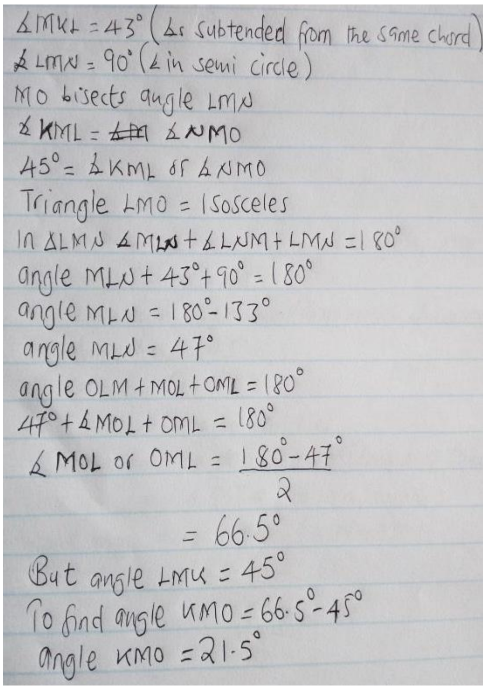
Table 9: Students' flawed solutions to task 1

Task 1

Figure below shows a circle centre O. Line LON is a diameter, KL = LM and angle LNM = 43^0. Calculate angle KMO.

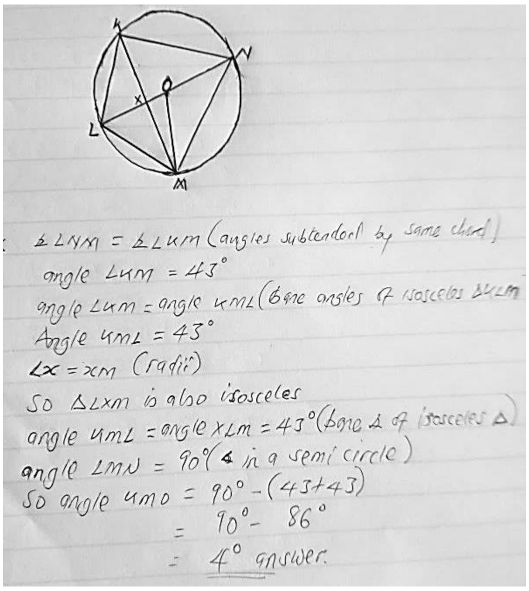


Solution 1



∡MKL = 43° (∡s subtended from the same chord)
∡LMN = 90° (∡ in semi circle)
MO bisects angle LMN
∡KML = ~~∡M~~ ∡NMO
45° = ∡KML or ∡NMO
Triangle LMO = Isosceles
In ∆LMN ∡MLN + ∡LNM + LMN = 180°
angle MLN + 43° + 90° = 180°
angle MLN = 180° − 133°
angle MLN = 47°
angle OLM + MOL + OML = 180°
47° + ∡MOL + OML = 180°
∡MOL or OML = $\frac{180° - 47°}{2}$
= 66.5°
But angle LMK = 45°
To find angle KMO = 66.5° − 45°
angle KMO = 21.5°

Solution 2



∡LNM = ∡LKM (angles subtended by same chord)
angle LKM = 43°
angle LKM = angle KML (base angles of isosceles ∆KLM)
Angle KML = 43°
∠X = XM (radii)
So ∆LXM is also isosceles
angle KML = angle XLM = 43° (base ∡ of isosceles ∆)
angle LMN = 90° (∡ in a semi circle)
So angle KMO = 90° − (43+43)
= 90° − 86°
= 4° answer.

Pike, John and Paul claimed that the statement that angle KML = angle NMO is wrong in solution 1. Only Pike gave a reason why he thought that the statement is wrong by arguing that OM does not bisect angle LMN. This means that Pike considers the statement wrong because he thinks that the reason that is supporting the statement is not true. About causes of the mistake, Pike stated that the student forgot to use the given information that KL = LM to come up with isosceles triangle KLM, and conclude that angles LKM and LMK are equal. This implies that Pike assumes that the student did not make the mistake because of lack of understanding of the concept of isosceles triangles but because he/she did not follow a good procedure.

John and Paul did not explain why they thought that the statement angle KML = angle NMO is wrong but they explained the causes of the mistake. John observed that the student had poor background of isosceles triangles and that the student's learning of geometrical concepts of angles in the same segment was not good. This might imply that he assumes that the students made the mistake because of lack of understanding of the concept of isosceles triangles and angles in the same segment. Paul noted that the student did not have any problem with content but he/she committed the mistake because of two reasons, first because he/she was writing in a hurry hence made mistakes, secondly because he/she thought that angle KM is the diameter of the circle. Paul's response assumes that the student understands the concepts involved in answering the question but he/she made the mistake of using the wrong procedure. Paul's response suggest that the student considered angle KLM as an angle in the semicircle, hence equal to 90^0. Then using the given information that LM = LK, it means triangle KLM is isosceles hence angle KML = 45^0. The reason given by Paul can be assumed as one of the causes of the student's thinking that angle KML = 45^0.

However, the three teachers (Pike, John and Paul) are not correct that KML is not equal to angle NMO. The statement is true because triangles LKM and OMN are isosceles; hence, their base angles are equal. So, since angle LKM = angle LNM as these are angles in the same segment, it follows that the four base angles in the two isosceles triangle are equal (LKM = KML = LNM = NMO). In that case it would not be appropriate to assume that the student has poor background of isosceles triangles as suggested by John, but to assume that the student was writing the solution in a hurry, so he/she made a mistake of writing that angle KML = 45^0 instead 43^0.

Kim did not mention statements that are wrong in the solution but he wrote several observations to justify why the solution is wrong. For example, regarding solution 1, Kim wrote that the solution is wrong due to the following observations: the student was not able to recognise number of angles sharing the angle in the semicircle, and the student did not link the relationship of isosceles triangles and angles in the same segment to show that angle LNM = LKM = LMK = 43^0. Kim might have made the first observation about number of angles sharing the same angle in the semicircle basing on the statement that 45^0 = angle KML or angle NMO. Kim assumed that the student came up with the value of45^0 because he/she disregarded angle KMO among the angles at LMK. The second observation concerning the relationship of angles LNM, LKM, LMK, NMO to 43^0 means that Kim thought that the student was unable to make good connections among equal angles in the diagram.

The other mistake that not all teachers were able to identify in solution 1 is the student's use of isosceles triangle OML. The student considered that angle MOL = OML. This means that the student regarded lines LO and LM as equal instead of lines LO and MO. This might be regarded as a procedural mistake but not a conceptual mistake because in the other triangle (MNO), the student was able to apply the property of isosceles triangle. After finding that angle MLN = 47^0, it was not necessary for the student to calculate angle OML as these are base angles of isosceles triangle OML. So the student could have concluded that angle OML = 47^0 as well. Then he/she would have subtracted angle KML (which is 43^0) from angle OML to get angle KMO. This suggests that one of the causes of the student's mistakes in this solution is inability to make geometric connections as suggested by Kim.

On solution 2, Pike and John pointed out that statement LX = LM is wrong. The teachers wrote the statement is wrong because lines LX and LM are not radii as X is not the centre of the circle. Paul wrote that there is nothing wrong with the solution. This might be because the answer was correct; so Paul did not notice any mistakes in the solution. Kim was not clear on identifying what is wrong with the solution. He wrote that the students did not mention the arc subtending the angles and has not given valid reasons why triangles KLM and LXM are isosceles.

Regarding the causes of the mistakes, Pike noted that the student did not use information that KL = ML. The mistake identified by Pike and the reason are not directly linked. John pointed out that the student did not grasp well the concept of radii. This explanation is linked to the identified mistake of equal lines. However, Kim observed that the student did not know what an isosceles triangle is. This assumption might not be true because the student had indicated that the triangles (KML, LXM and OMN) are isosceles because their base angles are equal, the student has also deduced that triangle LXM is isosceles because LX = LM. Although the student used wrong statement (LX = LM), there is evidence that the student understood the angle and line properties of isosceles triangles. The statement that LX = LM led to use of isosceles property on triangle LXM. This led to statement that angle KML = angle XLM = 43^0 because these are base angles of isosceles triangle. So in this statement, the student came up with the value of angle KML which was used further to find angle KMO. But the value of angle KMO was already found earlier on in the solution using properties of isosceles triangle KLM. This means that the three statements (statements angle KML = 43^0, LX = XM (radii), So, ∆LXM is also isosceles, angle KML = angle XLM) were not necessary in the solution and they did not have any impact on the value of angle KMO. This might be the reason why Paul claimed that there was nothing wrong with the solution. However, the point that the solution contains irrelevant statements was supposed to be explained by the teachers.

4.4.2. What the teachers propose to do to help students cope with the mistakes

The teachers mentioned several ways in which they would address students' mistakes. Some strategies were in general while others were specific to the mistake made by the student. For the solutions provided for task 1, Pike suggested that he would help the students to understand how to use the given information. He proposed this suggestion because he assumed that both students did not use the information that KL = LM in their solutions. Pike also wrote that he would give the students more tasks to practice.

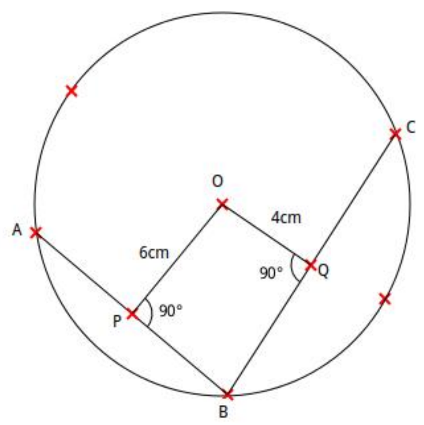
John observed that he would explain again the concept of isosceles triangles using teaching and learning aids to assist the students grasp the information, show the students how they were supposed to answer the question, and he would also involve the students in group work to practice solving tasks related to the concepts embedded in the solutions. Paul mentioned that he would help the students to identify the diameter, equal lengths, isosceles triangles involved and recall necessary theorems. The strategies suggested by Pike, John and Paul are examples of general approaches that the teachers proposed to use in addressing the mistakes. The strategies are general because they do not point to a specific mistake made by the students but they aim at explaining the concepts in general and showing the students how the task was supposed to be done.

Kim noted that he would make corrections by allowing the students to talk more about their solutions to discover more mistakes, and prepare more problems similar to those for students to practice. The suggestions mentioned by the teachers are related to what the teachers think as the causes of the mistakes. Table 10 presents task 2 and the students' solutions.

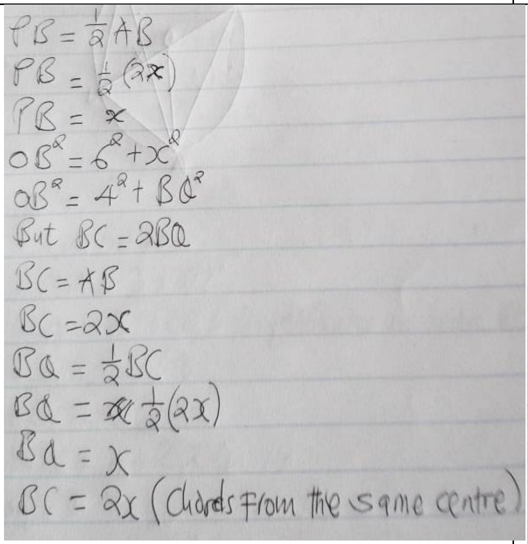
Table 10: Students' flawed solutions to Task 2

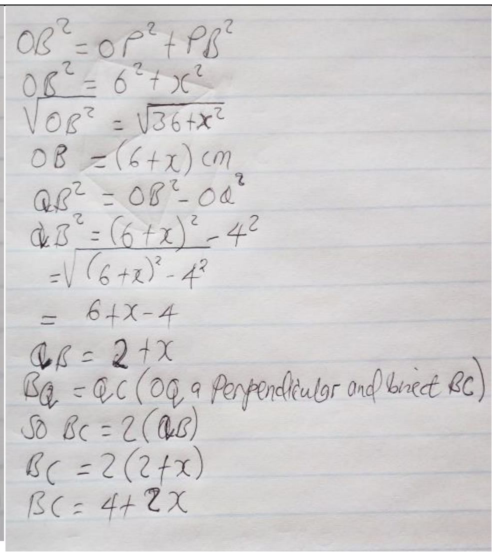
Question 2

Figure below shows a circle centre O. lines AB and BC are chords, AB = $2x$ and OQ = 4cm and OP = 6cm. Find BC in terms of$x$.



Solution 1	Solution 2





On solution 1, Pike and Paul wrote that the statements BC = AB is wrong because the chords are not equal since the lines that join the chords from the centre are not equal. John and Kim did not identify the statement that is wrong in the solution but they only claimed that the student is not giving reasons for the statements. On reasons why the solution is not correct, John stated that the student had poor background of geometrical facts. This statement is also not specific because the teacher did not identify the geometric facts in which the student has poor background. Kim explained that the student lacks understanding of chord properties and their application. The causes of mistakes identified by Kim suggest that the student's mistake resulted from lack of both understanding of concepts and procedure for the solution. This shows that Kim identified the statement that is the source of the student's mistakes although he did not specifically mention it.

On how they would address the students' mistakes, Pike stated that he would help the students see the relationships of the two equations related to OB, emphasise on use of chord properties and give the students more tasks to practice. This shows that Pike thinks that the solution was wrong mainly because the student was unable to relate the two equations through the common line OB. Pike's suggestion also shows that he thinks that there is only one way of solving the problem that is by joining OB, then forming 2 equations using Pythagoras theorem and making substitutions. John suggested that he would re-teach the geometrical facts to correct the mistakes and give the student a similar exercise to solve. Paul did not write what he was going to do to address the mistakes in both solutions.

Kim said that he was going to address the mistake by making corrections using varied methods for teaching Mathematics and by teaching the students from known concepts to unknown concepts. Kim's suggestion to make correction using different teaching methods means that he is proposing to use specific approaches to address the mistakes. Kim has also suggested same approaches for addressing mistakes in solution 2. On solution 2, all teachers were able to identify statements that are wrong as simplifying of$\sqrt{36 + x^2}$ as 6 + x, and $\sqrt{4^{2+BQ^2}}$as 4 + BQ. The teachers stated that the cause of the error is lack of understanding of simplifying algebraic expressions.

On how they would address the mistakes, for solution 2, Pike said that he would do correction to help the student understand that we do not find the square root of the sum of terms, and also that he would give the students several tasks for practice so that they are familiar with content. Although Pike has not specified on how he would implement the suggestions, the way he has expressed the suggestion shows that he would show the students what they were supposed to do and what they were not supposed to do. John claimed that he would teach the students how to use square roots. This suggestion also shows that John would show the students how they are supposed to simplify expressions involving square roots. These suggestions imply that the type of remedy to be provided by the teachers is general and not specific to the mistake.

4.5. Explanation and representation of geometric proofs by the teachers

In this section, I intend to approach research Question4 that concerns how the teachers explained and represented geometric proofs to their students. The research question was approached by analysing transcribed lesson observation data using problem solving phases suggested by Polya (1945). The results of the analysis show that all teachers emphasised on the stepwise proving process during the lessons. However, the way the process was carried out differed among the teachers. Table 11 presents a summary of the findings on how the teachers explained and represented geometric proofs in their classrooms.

Table 11: Summary of how teachers explained and represented geometric proofs

Categories	John	Kim	Paul	Pike
Engaging students in empirical activities.	-Did not engage students in empirical proving activities.	-Started every geometric proving lesson with an empirical activity.	-Did not engage students in empirical proving activities.	-Engaged students in empirical proving activities in three out of the twelve lessons.
Construction of formal	-Started the lesson by	-Involved students in	-Wrote the theorem on	- The formal proving was

proof.	writing a theorem on the chalkboard. -Then he would ask the students to explain the meaning of the theorem. -Consolidated students' suggestions using a diagram. -This was followed by proving of the theorem using question and answer method, and by emphasising on proving process.	group activity where they discussed how to develop a formal proof for the theorem. -During the group discussion, the students devised a plan for developing the proof. - Gave the students an opportunity to analyse and comment on proofs written by their colleagues on the chalkboard. - Lastly he made	the chalkboard, then read it out and explained its meaning to the students. -From there he would draw a diagram on the chalkboard and then show the students all stages of developing a particular proof. - Spent little time (about 10 out of 40 minutes) on explaining	done by the teacher together with students through Question and answer. -The plan for developing the proof was mainly devised by the teacher. -Guided the students in carrying out each step of the proving process through question and answer. -Only one proof was developed for each theorem although there were some theorems which could be proved

		comments and some corrections on the proofs by emphasising on proving steps.	about proof development.	using several approaches.
Example and evaluation tasks	-After the formal proof, mainly gave students evaluation tasks which involved application of the theorem that was proved.	-Involved students in discussing examples of how to apply the theorem to some computation tasks. - Then students would write either a group or individual exercise or homework.	-Explained several examples to show students how to apply the theorem to different situations. -Spent more time (about 30 out of forty minutes lesson) on explaining different ways of applying the theorem to	-After developing a formal proof, Pike wrote a computation proofs task and used question and answer method to discuss how to apply the theorem in solving it. - Lastly, Pike gave students some tasks to evaluate their understanding of the theorem and

			solve computation tasks. -Lastly, he gave students some evaluation tasks for homework.	its proof.

In the following sections, I provide detailed discussion on the results summarised in Table 8 concerning how teachers explained and represented geometric proofs during teaching of geometric proving.

4.5.1. Understanding the problem

The lesson observation data show that there were three main ways through which teachers helped the students to understand the problem. The first way, which was through explanation, was mainly used by Paul. In all lessons, Paul began by introducing the theorem to be proved during the lesson. Then he drew a diagram to be used for developing the proof. In trying to help the students to understand the problem, he used the diagram to explain the given information (hypothesis) and the conclusion of the problem to prove. This indicates that Paul gave much help by providing and explaining all information required for understanding the problem. The second way, which was through questions and answers, was mainly applied by John and Pike. The third way, which was through activity, was used by both Kim and sometimes by Pike. The following extract is an example of how John used question and answer method to help the students to understand the problem.

30. John:	Can someone read the theorem for us?
31. Student:	The sum of opposite angles of a cyclic quadrilateral are supplementary.
32. John:	What do we mean when we say the sum?
33. Student:	Addition.
34. John:	What about supplementary?
35. Student:	180^0.
36. John:	Okay, let's see the following figure.

Figure 17 is a diagram drawn by John on the chalkboard

Figure 17: Diagram by John focusing on meaning of theorem.

37. John:	So, when we say the sum of opposite angles, which are the opposite angles?

38. Student: Angle x and angle *y*.

39. John: Yes, angle x and angle *y*. Which set as well?

40. Student: Angle *a* and angle *b*.

41. John: Yes, angle a and angle b. So, how can we express the theorem; the opposite angles of a cyclic quadrilateral in relation to the diagram on the board?

42. Student: We add angle a and angle b and the sum is 180^0

43. John: Yes, angle a plus angle b equal 180^0 (he wrote on the board; $a + b = 180^0$) and angle *x* plus angle *y* equals 180^0(he wrote on the board; $x + y = 180^0$). Okay, so now, how did they come up with the theorem? We need to show why they are saying that opposite angles of a cyclic quadrilateral are supplementary. So, let us see how we can prove the theorem.

The extract by John shows that he asked students several questions to help them to understand the theorem which was to be proved. Before utterance (30), John asked the students to define some geometric concepts like circle, quadrilateral and cyclic quadrilateral. The students defined a cyclic quadrilateral as a quadrilateral drawn inside a circle. John challenged the students to improve their definition by using two diagrams. First diagram contained a quadrilateral inscribed in a circle with vertices not touching the circumference. Second diagram contained a quadrilateral inscribed in a circle with vertices touching the circumference. The diagrams helped the students to understand that not all quadrilaterals that are inside a circle are cyclic quadrilaterals. Therefore, the students improved their definition of cyclic quadrilateral by taking into account the condition displayed in the diagrams. After consolidating the definition of cyclic quadrilaterals, John wrote down the theorem to be proved on the chalkboard and asked a student to read it.

To enhance students' understanding of the theorem, John drew the diagram in Figure 17 he introduced letters into the diagram and asked the students to mention pairs of opposite angles (30-34). The extract shows that work was shared between the teacher and the students. John's responsibilities involved drawing diagrams, asking questions and assessing students' thinking. The students were involved in analysing the diagrams and responding to questions asked by John. The questions that were asked by John helped the students to understand both the meaning of a cyclic quadrilateral and the theorem. This was observed when the students improved on their definition of cyclic quadrilateral and when they gave a correct symbolic representation of the theorem (35).

The third approach used by the teachers to help the students to understand the problem was through activity. This approach was used in most of Kim's lessons and in some of Pike's lessons. Kim started each proving lesson by asking students to do either a measuring or a construction activity from which they would deduce a theorem. For example, during the lesson, which aimed at proving that angles in the same segment are equal, Kim drew a diagram containing three angles at the circumference. Figure 18 presents the diagram drawn by Kim.

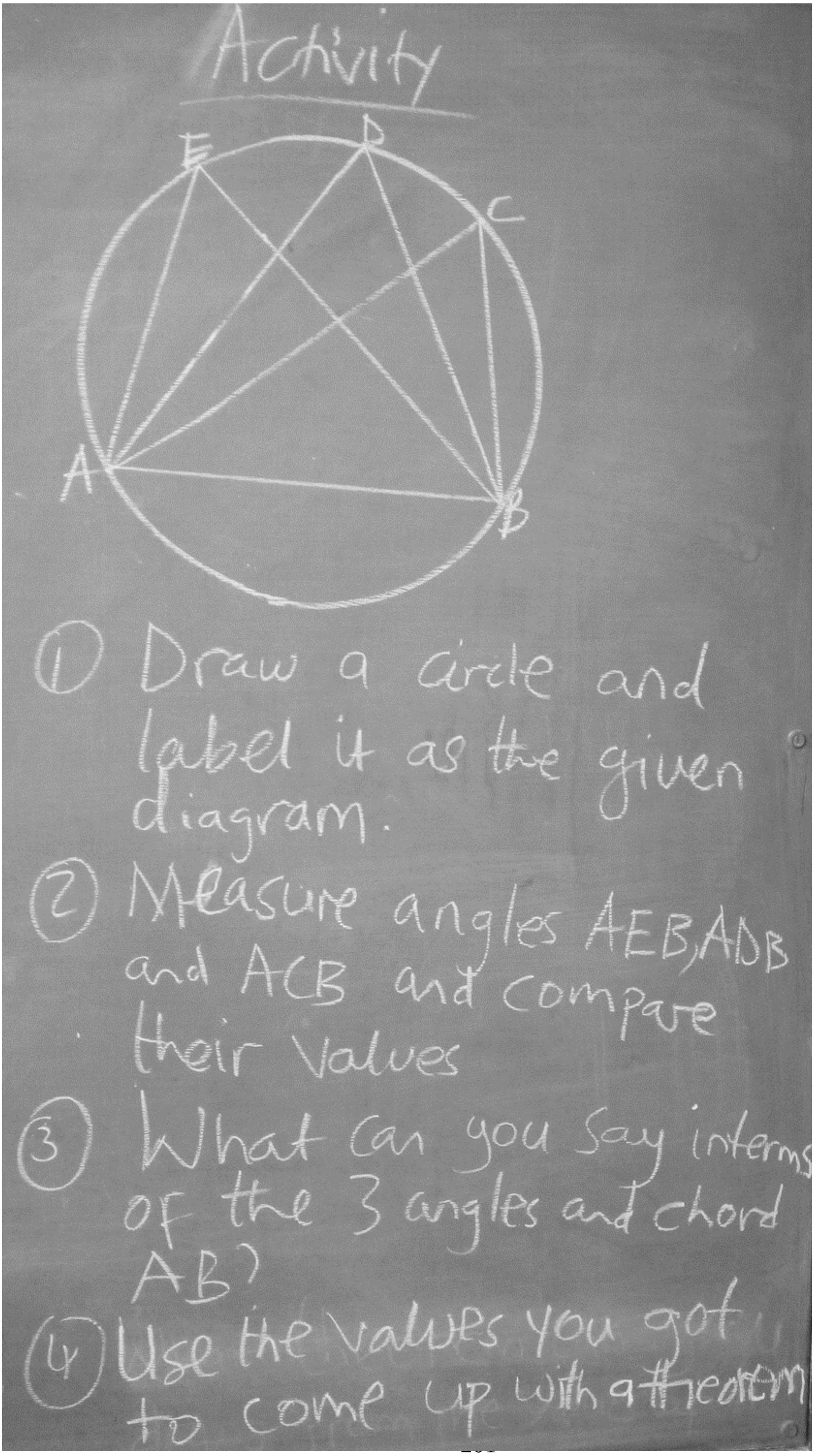

Activity
E
D
C
A
B
① Draw a circle and label it as the given diagram.
② Measure angles AEB, ADB and ACB and compare their Values
③ What can you say interms of the 3 angles and chord AB?
④ Use the values you got to come up with a theorem

Figure 18: Diagram by Kim focusing on developing of theorem

The diagram in Figure 18is followed by some instructions where Kim asks the students to draw a similar diagram in their respective groups, then to measure the angles, and discuss the results to come up with a theorem. Table 12 presents values found by students after measuring the angles. This is followed by a lesson extract presenting dialogue between Kim and the students in relation to the values they were presented.

Table 12: Values of angles reported by the students

Group 1	**Group 2**	**Group 3**	**Group 4**	**Group 5**	**Group 6**
E = 52^0	E = 30^0	E = 35^0	E = 30^0	E = 40^0	E = 29^0
D = 52^0	D = 30^0	D = 35^0	D = 30^0	D = 40^0	D = 29^0
C = 52^0	C = 30^0	C = 35^0	C = 30^0	C = 40^0	C = 29^0

3. Kim: Okay, so that's what you came up with. So can you study the sizes of angles for each group individually? What do you notice from the result?
4. Student 1: I noticed that the angles at the circumference are equal.
5. Kim: Yes, but we need to be specific here, which angles at the circumference? Any angle as long as it is at the circumference?
6. Student 2: The angles subtended by the same arc are equal.
7. Kim: You are close to the theorem, yes angles subtended by the same arc, but do you also notice that if we join AO and OB, we will have an angle subtended by the same arc AB? So how would angle AOB be different to angles E, D and C?
8. Student 3: Angle AOB lie at the centre while angles E, D and C are at the

circumference.

9. Kim: So, how can we rephrase the theorem to ensure that it only refer to angles at the circumference?

10. Student 4: Angles subtended by the same arc at the circumference are equal.

11. Kim: Very good, angles subtended by the same arc at the circumference are equal. We can also say angles in the same segment are equal. This is the theorem that we are going to prove today. So, we will use the same diagram on the board. We are given a circle with centre O, chord AB subtending angles AEB, ADB and angle ACD at the circumference. We want to prove that angles in the same segment are equal. So in our case we want to prove that which angles are equal?

12. Student: Angle ACB, angle ADC and angle AEB.

13. Kim: Yes, that is our task: we want to prove that these angles (pointing at the three angles at the circumference) are equal.

The dialogue in the extract shows that students are guided from empirical understanding of the theorem to abstract understanding. The extract also shows that students are not only helped to understand the theorem and the problem to prove but also to discover a theorem from their empirical work. After measuring the angles and presenting their values, Kim asks the students to analyse the values and to deduce a theorem. This indicates that Kim's aim was to help the students to derive a theorem on their own through an empirical activity. The extract also shows that Kim helps the students to improve on the phrasing of the theorem by asking them thought provoking questions (5). After agreeing on the phrasing of the theorem, Kim helps the students to understand the problem to prove. He does this by identifying the hypothesis and asking the students to identify the conclusion (11-12).

This approach of helping students to understand the problem was also observed in two lessons taught by Pike. When Pike was teaching students the theorem of exterior angle of a triangle and opposite interior angles, he asked the students to measure angles and to deduce the relationship between exterior angle and opposite interior angles of a triangle. When Pike was teaching theorem of sum of interior angles of a polygon he gave the students paper which contained diagrams of different types of polygons and asked them to find the sum of interior angles of different polygons. The students were first of all supposed to draw lines from one vertex of the polygon to other vertices, then count number of triangles in the polygon, and find sum of interior angles using the number of triangles. Then Pike asked the students to identify a pattern from their findings and deduce the formula for sum of interior angles of a polygon.

4.5.2. Devising the plan

Devising of plan for the task to prove was done in different ways by the different teachers. Paul mainly used explanation approach where he explained to the students how a theorem was supposed to be proved. Sometimes he would involve students in either suggesting proving statements of providing reasons for proving statements. Pike and John used question and answer approach while Kim used exploratory approach. The following extract is an example of how Pike helped the students to devise a plan for a problem to prove that opposite angles of a parallelogram are equal using question and answer approach.

12. Pike: Now our aim is to prove that opposite angles of a parallelogram

are equal. So we need to prove that angle ABC is equal to angle ADC, and angle BAC is equal to angle BCD. How can we prove these two statements? (silence for 6 seconds). Okay we need to come up with two triangles and prove that they are congruent. So what can we do to have two triangles in that parallelogram?

13. Student: We can join two vertices for example A and C or B and D.
14. Pike: Yes, correct, so who can come and join one of the pairs of vertices?
15. Student: (He joined AC).
16. Pike: Okay, after drawing diagonal AC, we have come up with triangle ABC and triangle ADC. So let us see, who can come and label angles that are equal using the small letters as usual?
17. Student: (Label angle BCA as x_1 and angle CAD x_2)
18. Pike: What is the reason?
19. Student: Alternate angles.
20. Pike: Yes, alternate angles, how do you know thatx_1 and x_2 are alternate angles?
21. Student 3: Because the angles lie on parallel lines?
22. Pike: Which are the parallel lines?
23. Student: Side AD and BC.
24. Pike: Yes, that’s what I am looking for: BC is parallel to AD. So when you are saying this angle is alternate or corresponding or allied to this angle, you have to mention the lines that are parallel. So; who can show another set of equal angles on the diagram?
25. Student: (Labels angle BAC asy_1 and angle ACD and y_2.

26. Pike: What is the reason?

27. Student: Alternate angles, AB is parallel to DC.

28. Pike: Yes, AB is parallel to DC.

Figure 19 shows a diagram that was being used in this extract.

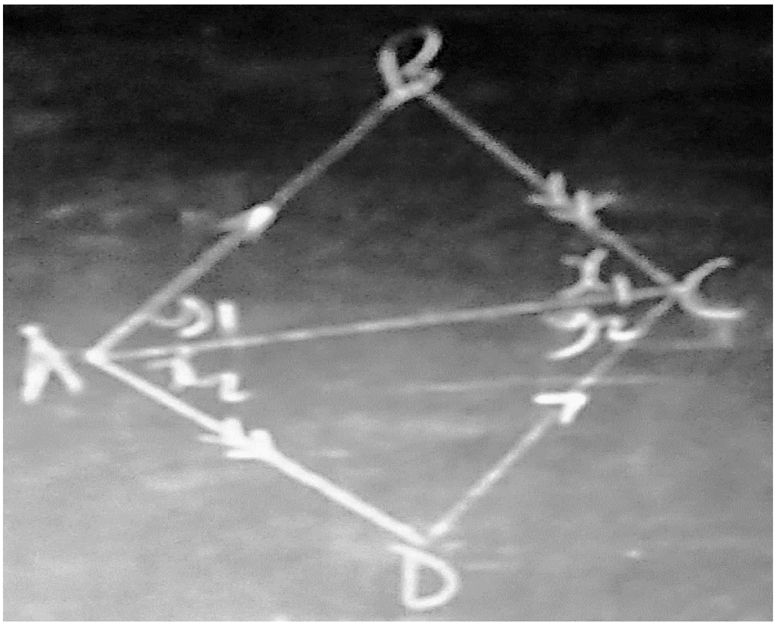


Figure 19: Diagram by Pike focusing on use of labels

29. Teacher: But remember that our aim is to prove that triangle ABC is equal to triangle CDA. We have managed to come up with two arguments; $x_1 = x_2$ and $y_1 = y_2$. What is the other argument that we can use? (Silence for 5 seconds). Try to remember theorem we proved yesterday.

30. Student: Line AD = line BC and the reason is opposite angles of a parallelogram.

The extract shows that Pike is devising the plan with the students. The students are involved in adding features into the diagram like drawing auxiliary lines and adding labels for equal angles (13-17). The extract also shows that students are adding information into the diagram by labelling equal angles and identifying equal lines. The equal angles are labelled using prior knowledge that alternate angles are equal (18-21). Equal lines have been identified using knowledge that opposite sides of a parallelogram are equal (23-28). This shows that the students engaged their prior knowledge when devising the plan for developing the proof.

However, despite involving students in answering questions and adding features into the diagram when devising the plan, Pike provided too much guidance on important aspects of the proof. This is evidenced in utterance (12) where Pike explains that they will develop the proof by forming two triangles and by using congruency theorem. Pike did a lot of thinking for the students by explaining the triangles and the theorem to be used for developing the proof. As pointed out earlier on, deciding on the theorem to be used as a bridge for developing a geometric proof is the most challenging part in geometric proof development (Chen & Lin, 2009). Furthermore, by asking students to label angles that are equal on the diagram, Pike had also done some thinking on how to come up with the proving statements for the proof.

Kim used exploratory method in which he asked the students to go into their groups to discuss how to prove the theorem. During the discussion activity, Kim went into the groups to ask the students to explain how they were going to develop the proof. This implies that Kim gave students an opportunity to decide their own ways of developing the proof. When devising the plan for the problem to prove that angles in the same segment are equal, all groups came up with same plan of using the theorem which states that an angle subtended by an arc at the circumference is equal to two times an angle subtended by the same arc at the centre. This might have happened for two reasons. Firstly, because during the phase of understanding the problem Kim might have provided the clue for devising a plan when he mentioned joining AO and BO to form an angle at the centre. Secondly, because the students learnt this theorem and its application in the previous two lessons. This means that the students were connecting the problem to prove with their prior knowledge.

4.5.3. Carrying out the plan

The lesson observation data shows that like the other phases of problem solving, this phase was also carried out differently by the teachers. The lesson observation data shows that Kim involved the students in writing the proving statements in their groups and then reporting to the whole class. Each group chose a representative who explained to the whole class how they developed the proof. The students were given an opportunity to analyse and comment on the proofs presented by each group.

Pike, John and Paul did much of explanation and writing on the chalkboard. The following lesson extract illustrates how Paul carried out a plan for proving a theorem which states that a perpendicular of a chord drawn from the centre will bisect the chord. The plan that the teacher had devised for developing the proof was to draw a circle with centre O, then draw perpendicular line OD, and join the radii to form two triangles (triangle ODA and ODB) and prove that they are congruent. Figure 20 shows the diagram that was drawn by Paul. The figure is followed by a dialogue between Paul and the students.

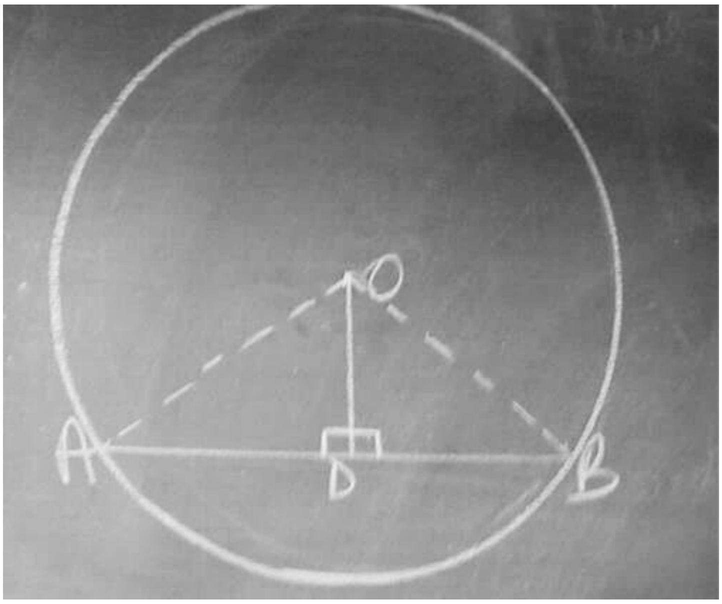


Figure 20: Diagram by Paul for developing the proof

5. Paul: So, let's show the congruency in triangles ODA and ODB. So we start with what we are given. Look at the diagram and mention what we are given.
6. Student: A circle with centre O.
7. Paul: Yes, we are given a circle, what else are we given?
8. Student: Line OD and chord AB.
9. Paul: Okay, I said that OD is a perpendicular bisector of AB. So we are given circle with centre O and OD perpendicular bisector of chord AB (he writes on the chalkboard). So what are we asked to prove?
10. Student 1: Angle ADO and angle BDO are 90^0 each.
11. Paul: (Silence for about 7seconds) It means you are not paying attention, why are you not paying attention? (Silence for 6 seconds). What do we know about these triangles? What same

things can we notice in both triangles? (Silence for 5 seconds) who has a better answer?

12. Student 2: ODA is congruent to triangle ODB.

13. Paul: No, I did not say that, I said that we want to prove that line AD is equal to line BD (He writes on the board). You must pay attention when I am speaking. Okay, what construction did I make on the diagram? We need to write the construction.

14. Student: Join AO and BO.

15. Paul: Yes, join AO and BO (He writes on the board). So proof, what are the equal angles that you can notice on the diagram?

16. Student: Angle ODA is equal to angle ODB.

17. Paul: Yes, angle ODA equals angle ODB. (He writes angle ODA = Angle ODB on the chalk board). What is the reason?

18. Student: Given.

19. Paul: Yes, we are given (He writes the reason). What else can we say?

20. Student: Side OA is equal to side OB.

21. Paul: Yes, OA is equal to OB (he writes OA = OB on the chalkboard) What is the reason? (Silence for 4 second), what are OA and OB?

22. Student: Radius.

23. Paul: Yes, OA and OB are radii, so what do we know about radii?

24. Student: They are equal.

25. Paul: Yes, they are equal, they must have same length and the reason is radii (He writes the reason on the chalk board). What else?

26. Student: OD is common.

27. Paul: Yes, OD is common, so from here we can confidently say that the

Two triangles are congruent so how do we write the congruency statement?

28. Student: Therefore, ODA is congruent to ODB.

29. Paul: Yes, triangle ODA is congruent to triangle ODB. (He writes both statements on the chalkboard). So from here we can
conclude to say that therefore DA is equal to DB as these are corresponding sides of the congruent triangles. Now, we have proved the theorem which says that a perpendicular of a chord drawn through the centre will bisect the chord.

The extract shows that when carrying out the plan, Paul was responsible for writing the statements on the chalkboard while students were responsible for providing the proving statements and their reasons. Paul started by asking the students to mention the given information, what was required to be proved and construction made on the diagram (9-14). The students had difficulties in providing the statement that was required prove (15-17). This might mean the students did not understand the problem to prove when Paul was explaining to them. After repeating the statement to be proved, Paul asked students questions that required them to come up with the proving statements. The extract shows that the students did not seem to experience difficulties in suggesting the proving statements (20-33). It can be argued that this is because Paul had already made some thinking for the students when he was devising a plan of how the proof was going to be developed. The diagram that is provided by Paul contains all the information for proving that the two triangles are congruent. As a result the students were mainly involved in lifting the statements from the diagram. But the fact that the students were supposed to give reasons for the suggested statements could mean that the students were involved in connecting the diagram to their prior knowledge. For example, the statement that OA = OB as these are radii (26) means that the students used their knowledge of lines of the circle.

4.5.4. Looking back

Analysis of the lesson observation data shows that for most of the teachers, the looking back phase of problem solving mainly involved doing of examples which aimed at helping students to understand how to apply a theorem to computation proof problems. The teachers did not involve the students in reviewing of the solution to explore other ways of developing the proof. As a result, most theorems were mainly proved using one approach despite that some theorems had multiple approaches for developing the proofs. The teachers mainly used the approach that is given in the secondary school textbooks. For example, Pike only guided the students in using one approach to prove that opposite angles of a parallelogram are equal. But there was also a different approach for proving the same theorem which did not require making of constructions. Using the same parallelogram ABCD in Figure 20, where AB and CD are parallel lines, if side BC was regarded as a transversal then the sum of interior angles could be 180^0 and again if CD was regarded as a transversal, the sum of interior angles could also be 180^0. Since the angle at vertex C is common, then angle at vertex B is equal to the angle at vertex C, hence opposite angles of a parallelogram are equal.

In the lessons by John, Paul and Kim there were also some theorems which could have been proven using different approaches. For example, the theorem that an angle subtended by an arc at the centre is equal to 2 times an angle subtended by the same arc at the circumference was only proven by using exterior angle of triangle property. However, the theorem could also be proven using the sum of interior angles of triangle and the sum of angles at a point. It was noted that students in Kim's class would have used this property if they received good help during group discussions. Similarly, the theorem that opposite angles of a cyclic quadrilateral are supplementary was proven only by using theorem that an angle subtended by an arc at the centre is equal to two times an angle subtended by the same arc at the circumference. But the theorem could also be proven using theorems of angles in the same segment and the sum of interior angles of a triangle.

However, there was a difference in the way Kim reviewed the proofs. Before discussing different examples with students, Kim and the students analysed proofs that they wrote on the chalkboard. Then the students made their comments on the proofs. Although the teacher did not ask students to think of other ways of developing the proof, the presentations helped the students to understand the proof in different ways as some of the statements for the proofs were not similar. In one instance, it was observed that when analysing proofs on the chalkboard, the students learnt that there was another approach of proving that the angle in a semicircle is a right angle.

4.6. Chapter summary

This chapter has presented the findings of the study. Analysis of both Questionnaire 1 and interview data shows that teachers conceptualise geometric proof development as a stepwise process. The success of geometric proof development depends on ability to make connected arguments using prior knowledge of properties of several geometrical concepts. Geometric proof development steps include understanding the hypothesis and conclusion, making construction if necessary, deciding on the theorem to be proved, and writing connected arguments while focusing on the conclusion. The teachers' conceptions of teaching of geometric proofs varied. Some teachers thought that teaching of geometric proofs involve showing students how to follow the stepwise process of geometric proof development. Other teachers thought that it includes involving students in geometric proving activities that can convince them about the validity of a theorem and value of formal proof development.

As regards the tasks, the findings show that teachers selected both low and high level tasks for teaching geometric proof development. Most of the teachers implemented all tasks at low cognitive level because they gave too much guidance on how the tasks were supposed to be proved. Only one teacher implemented most of the tasks at high cognitive level as he involved the students in exploring and explaining solutions to the tasks. In relation to assessing the students' thinking, the findings show that the teachers concentrated on different issues. Most teachers assumed that the students' mistakes were due to lack of procedural knowledge while one teacher thought that the mistakes were due to lack of conceptual understanding of geometric concepts. The suggested ways of helping the students to overcome their mistakes depended on the teachers' views about the cause of the mistakes. The teachers who pointed out lack of procedural knowledge, suggested showing the students how to write the computation proof appropriately. The teacher who thought that the mistakes were due to lack of conceptual understanding, suggested involving the students in class discussions where they could have opportunities to explain their answers and to analyse the solutions.

There were differences in the way the teachers explained and represented geometric proofs to their students. Some teachers only demonstrated to the students how to develop the formal proof. Other teachers started by involving students in empirical activities so that they could use the findings to deduce the theorem. This was followed by a group discussion activity where students explored how to develop a proof for a theorem. In both cases, after development of the formal proof, examples were done with an aim of helping the students to understand how to apply a theorem in developing different computation proofs. In the next chapter, I discuss the findings further and in relation to literature.

CHAPTER 5

DISCUSSION OF FINDINGS

5.1. Chapter overview

In this chapter, I present a discussion of the findings according to each of the four research questions in relation to literature on MKT and teaching of geometric proof development. I begin by discussing findings on research Question 1 about how teachers conceptualise geometric proving and its teaching. Secondly, I discuss findings on research Question 2 about how teachers select and implement tasks during teaching of geometric proving. Thirdly, I discuss findings on research Question 3 about how teachers assess students' thinking in geometric proving. Then lastly, I discuss findings on research Question 4 about how teachers explain and represent geometric proofs to students. I conclude the chapter by proving a brief summary of the discussion of findings and an overview of what is contained in the next chapter.

5.2. Discussion of findings on teachers' conceptualisation of geometric proving and its teaching

In this section, I provide a discussion on findings from data analysis that was conducted with an aim of answering research Question1 that is about teachers' conceptualisations of geometric proving and its teaching. The results show that teachers view geometric proving and its teaching as a stepwise process that requires several aspects of CK and PCK. In the following sections, I discuss what is involved in the categories of content knowledge and pedagogical content knowledge for teaching geometric proof development.

5.2.1. Content knowledge for teaching geometric proofs

Several studies agree that CK enhances teachers' capacity to explain and represent mathematical concepts. Ball et al. (2008) describe CK as subject matter knowledge used in the work of teaching Mathematics. Charalambous (2010) argues that the teacher's way of explaining and representing mathematical tasks largely depends on the breadth and depth of their conceptual understanding of Mathematics. Ma (1999) found that a profound understanding of fundamental Mathematics was reflected in the different pedagogical strategies that were used by Chinese teachers over a range of mathematical topics. She, therefore, argued that the breadth, depth, and flexibility of Chinese teachers' understanding of the Mathematics they teach provided them a broader and more varied range of strategies for representing and explaining mathematical content. Chinnappan et al. (2012) argue that Geometry proof problems are domain-specific hence their problem-solving process may be content knowledge-driven. This agrees with Putnam, Heaton, Prawat, and Remillard (1992) who found that teachers with limited conceptual understanding failed to provide students with powerful mathematical experiences in Geometry. This is in agreement with the COACTIV model that assumes that PCK is inconceivable without CK (Baumert et al., 2010). These arguments imply that if teachers do not have CK for geometric theorems and proofs, they cannot provide good guidance to students on how to develop geometric proofs. Thus, a thorough understanding of the components of CK of geometric proofs can enhance the teaching of geometric proof development.

This study has revealed several components (sub-categories) of CK for teaching geometric proofs. These aspects of content knowledge include (i) knowledge of Geometry content, (ii) knowledge of geometric deductive reasoning, (iii) knowledge of geometric proof problem solving, and (iv) knowledge of algebraic reasoning. In the following sections, I discuss these components of CK for geometric proof development in detail.

5.2.1.1. Knowledge of Geometry content

This includes knowledge of geometric properties, geometric reasoning and geometric language. According to Chinnappan et al. (2012), content knowledge related to geometric proving consist of knowledge of geometric concepts, geometric relationships and the visual representation of such relationships in appropriate diagrams. This is similar to what Ball et al. (2008) referred to as CCK. The teachers in this study refer to this as, "knowledge of the basic geometric concepts and their applications". During interviews, the teachers emphasised that this type of knowledge is crucial for geometric proving because one cannot develop a proof successfully without good understanding of the basic geometric concepts. It was also noticed that the teachers also considered the theorems and proofs that are learnt at one level as prior knowledge for the next level. This was reflected in John's explanation that when he is teaching geometric proofs in Form 4 he expects the students to know and to apply geometric concepts and theorems learnt in Forms 1, 2 and 3.

Knowledge of geometric reasoning involves establishing of good link between the hypothesis and the conclusion. Geometric reasoning also involves supporting each geometric proving statement with a reason that can be traced from the diagram. This implies that geometric reasoning also involves making good interaction with geometric diagrams. The findings from proof analysis show that most of the proving statements were developed from the diagram and from the word problem statement. This supports Battista's (2007) explanation that geometric reasoning consists of the invention and use of formal conceptual systems to investigate shape and space. This implies that geometric reasoning involves spatial reasoning which is the ability to interact with diagrams mentally to come up with the proving statements and their reasons (Chinnappan et al., 2012). According to Chen and Lin (2009), geometric reasons are acceptable when they are derived from acceptable theorems. This means that spatial reasoning also involves deciding on the theorems that are appropriate for a particular proof. Chen and Lin (2009) referred to this as hypothetical bridging and emphasised that this is the most critical stage in geometric proof development. The hinted that the success of choosing a correct hypothetical bridge depends on understanding of the hypothesis and the conclusion. Heinze et al. (2008) also argue that hypothetical bridging is one of the aspects that make geometric proof development difficult. This is because there is no single way of bridging the hypothesis and the conclusion. As such, students are supposed to decide a hypothetical bridge depending on the problem that they are proving.

The proofs that were developed by the teachers contained different types of hypothetical bridging. For example, two types of hypothetical bridging were used to show different ways of proving task 1. Paul developed a proof of similarity as a hypothetical bridge for proving that BC = 2OD in Figure 4 (page104). Kim developed a hypothetical bridge using Pythagoras theorem in Figure 9 (page 122) to prove that BC = 2OD. In task 2where teachers were proving that figure MHOP is a cyclic quadrilateral, they used two theorems of cyclic quadrilaterals as hypothetical bridge. In Figure 3 (page 103), John developed a hypothetical bridge by proving that the opposite angles of the quadrilateral are supplementary. In Figure 7 (page 111), Kim developed a hypothetical bridge by proving that the exterior angle is equal to the opposite interior angle of the quadrilateral.

In both tasks, the teachers' choices of the hypothetical bridge depended on the problem that they were proving and the diagram that they were using for developing the proof. This suggests that geometric reasoning is problem specific, as such, it has to be developed based on the problem to prove. This suggestion agrees with Chinnappan et al. (2012) who explain that geometric reasoning skills, involve the development and testing of an argument that is based on understanding the structure of the given Geometry problem context. They argue that geometric proof development is not algorithmic in nature but it is problem specific. The suggestion also consonant with Battista (2007) who argues that although there are general processes applicable to most geometric proofs, success on most proofs depends on understanding specific concepts and situations.

Knowledge of the language of Geometry includes knowledge of definitions of concepts, symbols and abbreviations that are commonly used for developing geometric proofs. Definitions are a basis for mathematical proofs and arguments because they help students to develop a shared understanding of mathematical concepts (Stylianides & Stylianides, 2006). This implies that that teachers need to develop an understanding of what makes a good definition and the ability to tailor these definitions to their students' existing knowledge. The teachers used different types of concepts, symbols and abbreviations in the proofs that they developed. This means that the teachers required good understanding of some geometric language to develop the proofs.

It was, however, noticed that in proving that BC = 2OD, Pike wrote the symbol of congruency instead of symbol of similarity. This shows that geometric symbols are sometimes confusing. The findings support Somayajulu (2012) who observes that the use of definitions and symbols in Geometry is ambiguous because Geometry uses its own set of symbols and notations and its own language. He explains that knowledge of geometric language is essential to teachers because some definitions, symbols and notations can be confusing to the learners. This is because some of the concepts have different meanings in Geometry and in everyday life, for example, the concept of similarity. Ball et al. (2004) defined mathematical definitions as precise statements of the nature of objects, procedures and properties. They explain that mathematical definitions are crucial in supporting mathematical reasoning as they help to clarify and to communicate mathematical concepts effectively. They argue that knowledge of use and choice of good definitions demands a flexible and serious understanding of mathematical language and precision.

Defining of mathematical concepts can be part of the first step in geometric proving which involves understanding the statement to be proved. For instance, John explained that before starting to develop a geometric proof, the teacher must make sure that what they are asked to prove is clear to the students. This implies that when teachers are proving a theorem with their students, they have to first of all emphasise on helping their students to understand the language and meaning of the theorem. As Polya (1945) suggested, a teacher must not move to the phase of devising the plan unless the problem is clearly understood by the students. He emphasised that if students devise a plan without understanding the problem, they either focus on a wrong goal, or get stuck and fail to complete the plan. As a result, the students might get frustrated upon realising that their efforts have become meaningless.

The findings of this study show that defining was not only done on geometric concepts but on theorems as well. Before starting to prove any theorem, the teachers explained to the students the meaning of the theorem using different approaches. Paul mostly explained to the students the meaning of the theorem either verbally, or with the aid of a diagram. Pike and John involved the students in answering questions with reference to a diagram. Kim involved the students in an activity that could help them to understand the meaning of the theorem. Although the approaches for explaining the theorem were different, all teachers had a similar aim of helping the students to understand the meaning of the theorem.

5.2.1.2. Knowledge of geometric deductive reasoning

This is knowledge that is required for the development of a logical geometric proof. It involves combining of general deductive reasoning and specific geometric reasoning skills. Deductive reasoning is required in development of any mathematical proof (Hanna et al., 2009). Deductive reasoning is used to establish truth or falsity of the hypothesis that is developed during inductive reasoning (Stylianides, 2005). Hanna et al. (2009) explain that the general principle of any mathematical proof, "is to specify clearly the assumptions made and to provide an appropriate argument supported by valid reasoning so as to draw necessary conclusions," (p. xix). This implies that, every mathematical proof is supposed to involve deductive reasoning.

Chen and Lin (2009) define geometric proving as a process of constructing a sequence of argumentation from X to Y with supportive reasons. X is the given information while Y is the statement to prove. The definition shows that deductive reasoning is regarded as main feature of a geometric proof as well. Although knowledge of deductive reasoning is regarded as general knowledge required for any mathematical proof, but geometric proving requires special type of deductive reasoning. This was noted during both analysis of proofs written by the teachers as well as the geometric proving lesson episodes. The proofs were developed using proving statements that were constructed by expressing geometrical relationships among different geometric concepts of shape and space. Deductive reasoning was used to connect the proving statements in a logical sequence. This means that the sequence of the proofs was structured by combining deductive reasoning and geometric reasoning. This implies that development of geometric proofs requires competence in combining both geometric reasoning and deductive reasoning.

In this study, I refer to this type of reasoning as geometric deductive reasoning to mean that geometric proof development requires competence in combining deductive and geometric reasoning. According to van Hiele's (1999), students can only develop formal proofs if they reach a level of formal deduction. This is a level where the students are able to see relationships among properties of geometric diagrams. It is this type of geometric reasoning that helps in developing of logical geometric statements.

5.2.1.3. Knowledge of geometric proof problem solving

This involves knowledge of the general phases of problem solving. Chinnappan et al. (2012) point out that the non-algorithmic nature of the geometric proof development process requires the use of general problem-solving strategies. There are different approaches of problem solving suggested for geometric proving. Some approaches can be categorised as PCK while others as CK. Problem solving suggested by Polya (1945) can be categorised as PCK because it is a teaching pedagogy while that of Chinnappan et al. (2012) can be categorised under CK because it reflects on what students do when they are developing a geometric proof. The steps that are involved in this type of problem solving include: analysis, representation, planning and use of knowledge retrieval. These problem solving steps are similar to the steps of geometric proof development explained by the teachers and to those that were reflected in the proofs developed by the teachers.

In some of the proofs, the steps were explicitly shown while in other proofs, they were implicit. For example, in the proof by John to show that figure MHOP is a cyclic quadrilateral, the phases were explicit (Figure 3, page 103). On the analysis step, the proof showed the given information and the statement that was required to prove. On representation step, the teacher interpreted the word problem into a geometric diagram. All teachers regarded the representation step as the first step in geometric proof development, but Chinnappan et al. (2012) regard it as the second step. This might be because the teachers assumed that the given information and the statement to be proved might only be understood and identified when a word problem is interpreted into a diagram. In the case of proving a geometric theorem, the representation step has to be done first because the theorems are usually in general statement form that is supposed to be interpreted into a diagram to make analysis step possible.

Barmby, Boldon and Thompson (2014) also regard the formulation of a diagram from word problem as a form of mathematical representation. They explain that to represent can be defined as to reformulate an original concept or scenario in a different way. They describe two types of representations, internal and external representations. They explain that internal representations are the mathematical images stimulated in the mind, hence difficult to identify. External representations are easier to identify because they are in real sense to a significant amount (Barmby et al., 2014). Examples of external representations are symbols, pictures, diagrams and written or spoken words. Barmby et al. (2014) explain that external representations such as diagrams and drawings can aid problem solving by helping students to understand mathematical concepts and word problems.

The teachers' view that development of a geometric proof cannot be done directly from a word problem agrees with Chinnappan et al. (2012) who emphasise that representation of a word problem into a diagram is required for every geometric proof development. This view may be the reason the teachers emphasised that ability to develop a correct geometric proof depends on constructing of diagrams that are true representations of the word problems. This point is also emphasised by Panaoura (2014) who explains that in Geometry students have to interpret geometrical figures and in many cases they have to construct the appropriate figure in order to translate the verbal information and solve a geometric task appropriately. Similarly, Usiskin et al. (2003) explain that although diagrams are often useful in proofs, when not drawn carefully and correctly, they can lead to invalid assumptions and false conclusions.

Last planning step involves what the teachers called, "deciding of the construction to be made and the theorems to be used for developing the proof". The teachers believe that this step can only be accomplished if students are able to retrieve their prior knowledge and connect it to the given information. This supports Chinnappan et al. (2012) who suggest that knowledge retrieval involves combining of the given information on the diagram and geometrical properties related to the diagram to develop proving statements.

5.2.1.4. Knowledge of algebraic reasoning

This includes knowledge of identifying patterns, simplifying algebraic expressions and using algebraic representations. Algebra offers Geometry a powerful form of symbolic representation (Dindyal, 2007). In the proofs that were developed by the teachers, algebraic reasoning was noticed in proofs that involved the use of Pythagoras theorem. The lengths of the triangles were represented algebraically using letters. This led to easy formulation of algebraic expressions that were simplified by changing the subject of formula and using substitution to arrive at the conclusion. There were also some proofs in which angles were represented by algebraic variables like a and x. This was done with an aim of making writing of the proofs simpler. This suggests that algebraic representations assist in simplifying the process of geometric proof development. The findings concur with Dindyal (2007) who reports that students' thinking in Geometry also requires facility with algebra. Dindyal (2007) argues that algebraic thinking has strong connections to thinking Geometry, as such, students studying Geometry need to be well prepared in Algebra. Although the claims made by Dindyal (2007) are based on computation algebraic and geometric tasks, they seem to apply to narrative proof tasks as well.

5.2.2. Pedagogical content knowledge for teaching geometric proofs

As already explained, I used the PCK categories of the COACTIV model as a priori themes for analysing PCK for geometric proof development. The study revealed several additional sub-categories in addition to those suggested by the COACTIV model. In the following sections, I discuss the sub-categories of PCK for teaching geometric proofs in relation to the COACTIV model as well as those that the study revealed. They include (i) knowledge of cognitively activating tasks, (ii) knowledge of students' cognitions and ways of assessing students' knowledge and comprehension processes, (iii) knowledge of explanations and multiple representations, and (iv) knowledge of pedagogic approaches for teaching geometric proofs.

5.2.2.1. Knowledge of cognitively activating tasks

According to the COACTIV model, knowledge of cognitively activating tasks involves knowledge of the level of cognitive demands of tasks, prior knowledge required by the tasks, their effective implementation in the classroom, and appropriate sequence in the curriculum and during instruction (Baumert & Kunter 2013). These aspects of knowledge were also identified during analysis of teachers' conceptualisations about knowledge for teaching geometric proofs. In the COACTIV model, sequencing of the tasks mainly involves level of cognitive demand of the tasks, but findings from this study show that the teachers also sequence their tasks according to the purpose of the task during instruction. The teachers identified four purposes of geometric proof tasks: proving, applying the proof, evaluating students understanding and practice. When teaching geometric proofs, the teachers start with tasks that aim at showing students how to develop the proof. These tasks are called narrative tasks (Chen & Lin, 2009). After developing the proof, the teachers show students how to apply the proof to different situation using different types of computation tasks (Chen & Lin, 2009). The computation tasks are also used for evaluating and enhancing students understanding of the proof, and for providing students with opportunities to practice application of the theorem to different situations. The findings illustrate that the teachers select tasks according to their instructional purposes. This implies that teachers need to know the purpose that a specific task will serve during instruction.

5.2.2.2. Knowledge of students' cognitions and ways of assessing students' knowledge and comprehension processes

According to the COACTIV model, this category of knowledge involves knowledge of working with students' existing conceptions, misconceptions, prior knowledge and difficulties, and ways of overcoming those (Krauss et al., 2008). These aspects of knowledge were also identified from analysis of findings from interview data in this study. The findings of this study also agree with Ball et al. (2004) who explain that teaching involves sizing up of students' typical wrong answers, and analysing the source and cause of the mistake. There were differences in the way teachers explained about how they address students' mistakes and misconceptions. John said that when he is marking students' work, he tries to find the root cause of the mistakes that are being made by the students and explain their related concepts again in class. Paul explained that he does not do correction of the tasks that he gave the students but address the mistakes by explaining again the main concepts implied in the tasks. This shows that John and Paul make assumptions on why the students made the mistakes and act according to their assumptions. Kim explained that he addresses students' misconceptions through classroom discussions. He let the students write their answers on the board, then he asks other students to analyse the solutions and identify mistakes. This is followed by a discussion on the cause of the mistakes and ways of preventing the mistake. The way of addressing mistakes explained by Kim is supported by Legutko (2008) who recommends that teachers should create situations in which students reveal their mistakes so that teachers are able to methodologically correct them.

On knowledge of responding to students' questions, the teachers explained that when they are teaching geometric proofs, they are also expected to answer questions asked by the students. John explained that the teacher's ability to respond to students questions appropriately boost students' confidence. This might mean that John thought that a teacher is the only one who is supposed to respond to students' questions. However, Kim explained that he gives students an opportunity to ask questions during the lesson in order to uncover their misconceptions and address them. Kim also said that he gives an opportunity for the other students to attempt a question asked by their fellow student with an aim of finding out misconceptions held by them. Kim's conceptualisations of students' questions agree with Shulman (1986) who explains that when students ask questions, they reveal some of their misconceptions. Shulman (1986) suggests that after teaching a topic for some time, a teacher should have a collection of some of the misconceptions that students might bring to the learning of a particular topic. Ball et al. (2004) point out that one of the tasks of teaching Mathematics is interpreting and making mathematical and pedagogical judgments about students' questions. They also suggest that the teacher's task is not only to respond productively to expected questions only, but also to the questions which are unpredictable. In this case, a teacher is supposed to devise strategy for responding productively to both predictable and unpredictable questions. Shulman (1986) also suggested this kind of teacher knowledge.

5.2.2.3. Knowledge of explanations and multiple representations

This includes knowledge of guiding students in developing geometric proofs using good pedagogical strategies, knowledge of areas to be emphasised during teaching of geometric proving, and knowledge of explaining geometric proof concepts as connected entities. The category also involves knowledge of using different forms of representing geometric proofs and knowledge of teaching materials.

The proving steps that were mentioned by the teachers imply that geometric proof development involves problem-solving. This suggests that, pedagogical strategies to be used for teaching geometric proving should also be guided by problem-solving process. The study findings showed that the teachers differed in their explanations of how they conduct the proving steps with their students in classroom. John said that he uses question and answer to do the steps with the students. Paul said that he demonstrates to the students how to do the steps. Pike and Kim said that they give students an opportunity to discuss in groups how to do the proving steps. The teachers' explanations imply that during problem solving, John and Paul involve the students less, while Pike and Kim involve students more. There could be several reasons for less involvement of students in problem solving process. Paul mentioned two of these reasons:(ii) that geometric proofs are easy to understand as such, he is always surprised when students are unable to understand them, (ii) that explaining how to develop the proofs for each geometric theorem is like repeating to students the same stages of geometric proof development over and over. He claimed that he mainly rushes through proving the theorem and concentrate on showing students how to apply the theorem in solving different tasks.

Paul's explanations make several assumptions. Firstly, he assumes that the teaching of geometric proving is not cognitively demanding. This could be the reason why he rushes through geometric proving and concentrate on examples. This assumption contradicts with Chinnapppan (2012) and Batista (2007) who argue that teaching of geometric proofs is very challenging. The second assumption is that the teaching of geometric proving is mainly time consuming as it involves repeating same procedure for each theorem. This might imply that Paul does not know the value of teaching students geometric proof development. The way Paul explained the geometric proofs to the student may be the reason why some students said no when Paul asked them if they wanted to know how to prove some of the theorems. Patkin (2012) argues that the sense of needing a formal proof can be developed if students are given an opportunity to observe patterns, present assumptions and enhance the need for generalisation leading to a formal proof. Furthermore, Ding and Jones (2009) suggest that teachers should know strategies that can help students to understand both the deductive proof development process and appreciate the values of learning how to prove. But my view is that the teachers can only help students appreciate the value of geometric proving if they have knowledge of the importance of geometric proof development. Thirdly, Paul assumes that the stages of geometric proof development that he mentioned do not require problem-solving pedagogic strategies. This suggests that knowledge of making clear explanations and representations during teaching of geometric proof development require teachers' understanding of geometric proving as a problem-solving process.

The findings from interviews analysis also show that there are areas that are supposed to be emphasised during teaching of geometric proofs. These include understanding the theorem, proving with an aim of arriving at the conclusion, making connected arguments that are supported with reasons, and ensuring that the statement to be proved only appears in the proof as a conclusion but not as part of the proving statements. The points of emphasis are like guidelines for developing a geometric proof. Some of the points of emphasis mentioned by the teachers resemble the norms of doing proofs suggested by Herbst et al. (2009). For example, making connected arguments that are supported with reasons is similar to the norm of writing a sequence of steps (each consisting of a "statement" and a "reason"). The emphasis point of proving with an aim of arriving at the conclusion and ensuring that statement to be proved is conclusion of the proof are similar to the norm that the last step is the assertion property identified earlier as the “prove”. This shows that teaching of geometric proof development involves laying emphasis on important techniques for writing the proofs.

The teachers also explained that teaching of geometric proofs is one of the areas in Mathematics that require teachers to help students to make connections. This is because when students are developing geometric proofs, they are required to show a network of relationships between, or among lines and angles by using their prior knowledge of basic properties of geometric concepts. Kim said that the teacher who is teaching Geometry in lower secondary school levels must do so by bearing in mind that the students will be required to apply the concepts to formulate geometric proofs in the later years. As such, the teacher is supposed to make sure that students understand the concepts and clear all misconceptions that might arise when the students use the concepts in future. Kim also said that teachers are also supposed to know the pre-requisite knowledge of the proofs that they are teaching at a certain level. John emphasised that a teacher is not supposed to start teaching geometric proofs unless the students have the relevant pre-requisite knowledge. This suggests that the teacher is supposed to analyse learners' understandings of particular geometrical concepts before starting to teach geometric proof development. The findings imply that during teaching of geometric proofs, a teacher should aim at guiding the students to show the mathematical relatedness of several geometric concepts.

These findings agree with Businskas (2008) who argues that making of mathematical connections is a cognitive process that involves recognising links between mathematical ideas. The findings also support Battista's (2007) argument that learning of geometric proofs is complex because Geometry is interwoven, hence it requires ability to make connections between, or among several geometric concepts. As such, students are expected to make a web of relationships of shape and space. According to the NCTM (2000), mathematical connections provide students an opportunity to develop deep and more lasting understanding of mathematical concepts. This implies that providing students with experiences that can enhance their opportunities to make connections can enhance understanding of geometric proof development.

The NCTM (2000) suggests the following general pedagogies as some of the ways of helping learners make connections in Mathematics: building on students' previous experiences; paying attention to students' responses to assess the connections students bring to their situation; selecting problems that connect mathematical ideas; capitalise on unexpected learning opportunities; and asking thought-provoking questions. Making of connections among the given information, the statement to be proved, the diagram and prior knowledge enable making of good decision on the theorems to be used and the proving statements to be developed for the proof. The suggestions made by the NCTM were also mentioned by the teachers when they were explaining their views about what is involved in teaching of geometric proof development.

Representations help to portray, clarify, or extend a mathematical idea by focusing on its essential features (NCTM, 2000). However, representations cannot describe fully a mathematical content on their own, as such a teacher is supposed to interpret them so that they can be understood by the students (NCTM, 2000). Panaoura (2014) argues that in Geometry, the understanding of Mathematics requires that there should not be any confusion between mathematical objects and their respective representation. The teachers in this study identified two types of representations to be done during teaching of geometric proving. The first type of representation involves sketching of a diagram from the word problem. They claimed that a diagram is supposed to be used for clarifying a problem by enhancing identification of the hypothesis and the conclusion, and for developing proving statements. The teachers' explanations about aims of sketching a diagram are supported by Wong et al. (2011) who explain that teaching of geometric proofs involves moving with students in three forms of representations. These are word problem representation (understanding the language of word problem), visual representation (making inferences between given information and conclusion using the geometric diagram), and proof representation (formatting sequences of deductive statements).

The second type of representation identified during interview data analysis involves the different approaches for expressing the proof of a theorem. This involved inductive and deductive ways of representing the proofs. Deductive representation was identified as the formal way of representing geometric proofs by all the teachers. Teaching of deductive geometric proof development was explained and emphasised by all the teachers as a stepwise process. Pike and Kim also mentioned inductive proof process of geometric proof development. Kim's explanations showed that he values both deductive and inductive proof process despite that inductive representation is an informal way of representing geometric proofs. This agrees with Wong et al. (2011) who argue that proving of geometric theorem involves skills that are difficult to learn, therefore, instead of working with abstract and complicated representations, students might start with concrete, graphical representations. Stylianides and Stylianides (2005) argue that inductive proof is a good way of helping students to understand deductive proof. They explain that inductive proof enhances students' understanding of why proof works. According to Ding and Jones (2009), helping students to understand why a proof works is one way of encouraging students to understand and appreciate the discovery function of mathematical proofs. This implies that students can be encouraged to value learning of formal proof development through inductive proof activities.

This study has also found that the use of teaching materials aid in developing different forms of representations for geometric proofs. As such, knowledge of teaching materials is necessary for enabling teachers to develop different forms of representations for geometric proofs. The findings from interviews data analysis show that the teachers do not use different types of representations because of lack of teaching materials and also because of lack of knowledge of the materials to be used for teaching geometric proof development. Examples of teaching materials mentioned by the teachers include books, models and mathematical instruments like pair of compasses and ruler. The teachers' explanations showed that they do not face challenges with use of mathematical instruments but on creation and use of models. This is because the models are not available and also because the teachers do not know the type of models to be made and used for proving a particular theorem.

Heinze et al. (2008) recommended use of materials for improving students' abilities to develop geometric proofs. They report of high abilities in development of geometric proofs among students when materials were used to show the process of proving on a diagram. Wong et al. (2011) argue that static figures can serve the role of constraining an interpretation. The authors explain that when students are only involved in abstract interaction with the diagram, they might end up over generalising the proof to any other figure similar to the one being used for developing the proof. They, therefore, suggest that teachers involve students in interacting with dynamic geometric figures. This involves manipulating of a geometric figure on a computer through object dragging and observing the changes. Wong et al. (2011) argue that this type of interaction with diagrams can help students to avoid over-generalisation of theorems from typical images. To be able to involve students in the use of computer-assisted resources require knowledge of the use of the computer and knowledge of application of the software being used.

The teachers that I studied might have not mentioned this resource due to lack of knowledge of the resource as they explained during interviews. Although the use of computer assisted materials is expensive and might not be possible in a Malawian context, it is necessary that teachers have knowledge of using Geometry software materials as well as physical materials. This knowledge can help the teachers to know the type of improvisations that they can make using locally available resources where possible. Knowledge of using Geometry software materials can also help the teachers to be able to make clear explanation about how a particular proof works.

5.2.2.4. Knowledge of pedagogic approaches for teaching geometric proofs

This includes knowledge of relevance of teaching geometric proofs, knowledge of proving activities and knowledge of problem solving pedagogies. Knowledge of relevance of geometric proving includes understanding the importance of proof in both Mathematics as a field and in everyday life. Jones (2002) explains that teaching Geometry well involves knowing how to recognise interesting geometrical problems and theorems and understanding different uses of Geometry. He also explains that teaching of Geometry involves understanding the many and varied uses to which Geometry is put. Herbst (2002) argues that if proving is to play in the classroom the instrumental role for knowing Mathematics that it plays in the discipline, alternative ways of engaging students in proving must be found. This implies that students have to be offered opportunities for appreciating what Geometry can offer them. Even (1990) describes this as the strength of the concept. She argues that concepts are regarded as powerful when they have some special and unique powerful characteristics that open new opportunities. This implies that if teachers have knowledge of the strength of the concept, they might help the students to understand the learning geometric proof development.

Jones et al. (2009) argue that apart from helping students to develop geometric proofs in a good way, teachers also need to know how to help the students to understand the functions of deductive arguments. In my study, the teachers said that sometimes students are not motivated to learn geometric proofs because they do not see a link between the proofs and their daily life activities. This implies that the way geometric proof development is taught does not afford students opportunities to acknowledge the value of proving in their lives. Kim and Ju (2012) argue that proof is core to development of Mathematics, therefore it should be designed to help students understand its significance and cultivate their competence. This implies that teachers must view proof as a tool for learning and discovering Mathematics. Hanna (1990) suggests that to capitalise on proof as a tool for learning, teachers need to better understand how proof can serve as a vehicle for explaining why something is true and providing insight into underlying mathematical concepts. Jones et al. (2009) claim that being able to help students to be able to proceed with deductive proof in Geometry is not enough, it is also important to help students understand why such deductive arguments are necessary. This means that students might be motivated to learn geometric proof development if they understand the value of deductive formal proof.

As such, it is necessary for teachers to understand the relevance of geometric proof development not only to students' daily lives but to the field of Mathematics as well. Ding and Jones (2009) explain that some of the ways of helping the students to understand the discovery function of proof are by presenting a proof problem as an experimental problem, and by varying mathematical problems and teaching questions. In addition to this suggestion, this study has found that when planning or coming up with different teaching problems, a teacher has to consider using real life examples. For example, if a proving task is abstract, a teacher should to know how to paraphrase the task in a manner that will capture real life situations but without losing its rigor.

Knowledge of proving activities includes developing and involving students in activities that can help them understand geometric proof development and appreciate its value. As explained in the introduction chapter, one of the causes of students' challenges in effective learning of geometric proof development is that teachers do not provide the students experiences that can help them to understand the process of developing proofs (Jones, 2002). This claim agrees with the teachers' view that teacher's lack of knowledge for developing proving activities is another factor that contributes to students' lack of motivation to learn geometric proving. The findings in my study reveal several challenges that teachers face in relation to proving activities. The first challenge is lack of knowledge of developing proving activities appropriate for a particular theorem. This forces the teachers to simply explain the steps of geometric proof development to students without engaging them in any activity. The second challenge is lack of knowledge for bridging the proving activity and the theorem to be proved. This makes students to fail to notice the link between activity and the theorem being proved. The third challenge is that sometimes it is very easy for students to understand the proving activity but very difficult to understand the formal proof. As a result, the students are only motivated to participate in the proving activity but not in the development of formal proof. This implies that lack of teacher knowledge for proving activities contributes to the challenges in students' ability to understand geometric proof development.

The teachers' view that they are responsible for providing students with proving activities also concurs with Jones et al.'s (2000) proposal that teaching of geometric proof should also include involving students in exploration activities where proving is perceived as an activity associated with the search for a proof. Stylianides and Ball (2008) suggest that besides the formulation of arguments and proofs, proving activity can include empirical explorations to generate conjectures, and reasoning by analogy to develop possible ideas for the formulation of arguments. They explain that there are two forms of knowledge for involving students in situations for proving; knowledge of different kinds of proving tasks and knowledge of the relationship between proving tasks and proving activity. They argue that it is very important for teachers to have knowledge of different kinds of proving tasks so that they are able to afford students different opportunities to experience proving. Stylianides and Ball (2008) further argue that it is important for teachers to have knowledge of relationship between proving tasks and proving activity, "that is, understanding of critical mathematical aspects of the proving activity that can (potentially) be provoked by certain kinds of tasks when they are implemented in classroom settings," (p. 314). My findings, that teachers view the teaching of geometric proofs as requiring knowledge of bridging proving activity and the theorem, support this argument. Meaning that, a teacher should be able to identify aspects of the proving activity that can be related to the proof that is being developed.

Knowledge of problem-solving pedagogies includes ability to engage students in exploratory strategies when performing the problem-solving process. Ding and Jones (2009) propose that teaching of geometric proving should involve problem solving strategies to help the students to contribute actively to their learning of proofs. They argue that presenting a proof problem as an experimental problem represents an important didactical opportunity for enhancing students' capability of understanding the collaboration between experiment and proof (Ding & Jones, 2009). As the students manipulate geometric objects mentally, they learn to derive the results from previously learnt geometric properties and not from measurements or experiments. Polya (1945) suggests that teachers should know what is involved and what type of questions to ask at each phase of problem-solving so that they are able to guide students unobtrusively when learning Mathematics. This study found that teachers viewed Questioning as one of the activities that is involved in the teaching of geometric proof development. This view is consistent with the findings from lesson observation regarding how the teachers explained and represented geometric proofs to their students. The teachers asked different types of questions when they were teaching geometric proof development for different purposes. The teachers asked questions for different purposes like helping the students to understand the problem, devising the plan and carrying out the plan. This agrees with Polya's (1945) suggestion about asking questions depending on the phase of problem solving.

However, the findings show that the type of questions that the teachers asked had several implications on the students' participation in geometric proof development. For example, some of the questions that John and Paul asked were recall and leading type of questions which could not help the students to explain and explore the answers. This might imply that the questions were obtrusive to students' opportunities to engage in high cognitive level thinking during instruction. But the questions that Kim asked helped the students to explain and explore answers during the learning of geometric proof development. This implies that the questions provided the students with opportunities to understand the process of geometric proof development in unobtrusive manner. As a result the students might have developed appreciation for development of geometric proofs.

5.3. Discussion of findings on teachers' selection and implementation of tasks

In this section, I provide a discussion on findings from data analysis that was conducted with an aim of answering research question2 that is about how teachers select and implement tasks during teaching of geometric proofs. Analysis of the lesson extracts show that teachers presented tasks of different cognitive levels during teaching of geometric proofs. The findings also show that the teachers' practices determined the level at which students thinking was involved during task implementation. When teachers provided much guidance, students were involved in low level thinking in the sense that they mainly gave short answer questions and reasons without explanations. Good guidance enabled students to involve in high level thinking by exploring how to develop a proof, explaining their ideas and justifying their arguments. The teachers who provided much guidance might have done so due to lack of CK and those who provided good guidance might have high CK as suggested by Charalambous (2010). The argument about the teachers' levels of CK will not be explained further as the aim of the study was not to compare teachers' CK, but to explore knowledge involved in teaching of geometric proofs. As such, the discussions of findings have mainly focused on different aspects of PCK knowledge that were identified during task implementation in classroom. The aspects of teacher knowledge that I have discussed in the following sections include, (i) knowledge of effective implementation of tasks, (ii) knowledge of students' thinking, and (iii) knowledge of multiple solution paths for tasks.

5.3.1. Knowledge of effective implementation of tasks

Findings show that there were differences in terms of how teachers implemented tasks for geometric proving. In some cases, the teacher provided too much guidance to students while in other cases teachers provided sufficient guidance on how to perform a task. According to Polya (1945), teachers should be able to help their students to acquire much experience of independent work. He suggests that a teacher should know the level of help that students would need. He says that giving insufficient help will result into lack of progress in the work while giving student too much help will result in having no reasonable share of responsibility of the work. Sears and Chávez (2014) argue that teachers' practices directly influence students' opportunity to engage with proof tasks. They suggest that during task implementation, teachers need to strike a balance between giving too little, or too much support to students. They argue that excessive guidance offer limited opportunities for students to reflect on possible solutions for the tasks while very little guidance can provide limited opportunity for students to reflect on possible solutions for the task.

In their study of the nature of proof tasks in two Geometry textbooks and its influence on enacted lessons, Sears and Chávez (2014) found that most of the tasks in the books were of high cognitive demand. However, the cognitive level of the tasks declined during enactment of the tasks due to the teachers' practices. In a similar study, Charalambous (2010) found that depending on knowledge, teachers could either make a task elevate, or decline in its cognitive demand during task presentation and enactment. A low-level task selected and presented by a teacher could appreciate in terms of cognitive demands when the teacher involved students in exploration and explanation of their procedures during task enactment. In contrast, a high-level task could decline in terms of cognitive demand if enacted by students involving exploration and explanation of procedures.

In this study, the findings on selection and implementation of tasks support studies by both Sears and Chávez (2014) and Charalambous (2010). The two narrative tasks that were presented by Pike were of high cognitive demand because they were capable of offering students opportunities to practice proving and to carry out procedures with connections to geometric concepts and reasoning. The cognitive level of the tasks could have been maintained if students were given an opportunity of exploring and discussing their procedure. But during task implementation, Pike made all suggestions on how the proof was to be developed and the arguments to be made. The challenge of the task was not sustained during task implementation because the students were not involved in much thinking and explanation. The teachers' practices resulted in decline of the cognitive value of the task (Charalambous, 2010; Sears & Chávez, 2014). The lesson extracts also show that John and Paul did not maintain the high cognitive level of the narrative tasks that were presented for proving that angle subtended by an arc at the centre is two times an angle subtended by the same arc at the circumference. John and Paul provided much guidance to the students in terms of the construction to be made, the theorem to be used for developing the proof and also suggested the proving statements. The teachers' actions imply that they were thinking for the students during task implementation (Charalambous, 2010). This implies that the students were not involved in high order thinking, as a result, the cognitive level of the tasks declined as well (Charalambous, 2010). The teachers' practices are consistent with their explanation concerning knowledge for proving activities. The teachers said that due to lack of knowledge of proving activities, they just show the students how to develop a geometric proof without involving them in any hands-on activities. This implies that the teachers require knowledge of proving activities in order to involve their students in high cognitive level experiences like exploration and explanation.

Kim used the same task that was used by John and Paul to involve students in explorations and discussions on how to develop the proof. Students were involved in exploring the proof and explaining their procedure. The teacher's practices maintained the cognitive level of the task. It was, however, observed that Kim dismissed a student's suggestion that could have been appropriate for a different proving path. Kim suggested to the students a measuring task to redirect their thinking because he thought that the students were proving using a wrong theorem and focusing at a wrong angle at the centre. On one hand, it can be argued that the guidance given by the teacher was appropriate for two reasons. Firstly, because the teacher did not provide a solution to the students, then he did not think for the students (Charalambous, 2010). Secondly, because the guidance helped the students to get involved in further explorations and explanation of their proof, then it maintained the cognitive level of the tasks. While, on the other hand, it can also be argued that although the guidance was appropriate, it might have led the students to think that focusing on the reflex angle was completely a wrong idea. As already discussed, Kim might have suggested to the students to measure the angles because he had only one solution path in mind. If Kim had several solution paths for the theorem, he would have guided the students to complete their proofs using the initial approach before asking them to do the empirical activity.

This discussion lead to observation of two factors which affected the teachers' ways of providing guidance to the students during task implementation; knowledge of proving activities and knowledge of multiple proving paths to geometric theorems. In the COACTIV model, the category of knowledge of cognitively activating tasks include: knowledge of the cognitive demands of the task, prior knowledge required by the tasks, knowledge of effective orchestration of the tasks in the classroom, and knowledge of the long-term sequencing of learning content in the curriculum. This study has revealed that in order for teachers to implement tasks effectively in classroom they require knowledge of providing good guidance during task implementation. The study has also shown that ability to provide good guidance during task implementation requires knowledge of multiple proving paths and knowledge of proving activities. This implies that some of the sub-categories of PCK in the COACTIV model are interactive.

5.3.2. Knowledge of students' thinking

This includes knowledge of the students' prior knowledge for doing the task and knowledge of students' misconceptions. The teachers mainly examined students' prior knowledge for developing geometric proofs using the question and answer method. Before starting to prove a task, the teachers asked the students to explain the meaning of several geometric concepts and theorems that were going to be applied in development of a particular proof. If the students were not able to either define, or explain the meaning of a geometric concept or a theorem, the teachers did some revision before starting to prove the theorem. This practice was consistent with the teachers' view that before teaching geometric proof development, a teacher should check their students' prior knowledge. During interviews, John emphasised that a teacher should not start proving a theorem when students do not have the required prior knowledge. This implies that a teacher is supposed to check if the students are ready to learn a particular proof by examining their prior knowledge. This suggestion supports the argument that students cannot understand secondary school geometric proofs if they do not have the required prior knowledge (Crowley, 1987; Usiskin, 1982).

The lesson extracts that were analysed under tasks showed that teachers identified students' misconceptions and dealt with them in different ways. One of the ways in which teachers identified students' misconceptions involved asking questions. There is a common misconception observed in the students' explanations that is lack of understanding of the task that was being proved (comprehension challenge). Pike and John addressed this misconception by reminding the students what the task is about. For example, in a narrative proof taught by Pike, a student suggested that one of the proving statements should be the statement that they were proving. Pike responded by saying that the statement suggested by the student was supposed to be the conclusion of the proof because it is what they were asked to prove. A similar incident was also noticed in one of the lessons by John where a student suggested that proving statements should consist of a statement which was being proved. John also responded by reminding the student that the statement which is being proved is not supposed to be part of the proving statement but it is supposed to come at the end of the proof. The students might have given this suggestion due to either lack of understanding of the problem which was being proved, or lack of knowledge of developing the proving statements. But the teachers' responses to the students' suggestions mainly focused on emphasising rules for developing a geometric proof. The response given by both Pike and John agrees with the norms for situations for doing proofs suggested by Herbst et al. (2009). But before reminding the students about the rule of developing geometric proofs, the teachers would have asked several questions to examine why the students responded in that manner. This could have helped them to find out whether the student's responses were due to lack of understanding of the problem which was being proved, or lack of understanding of the procedure for developing the proof.

The approach by Pike and John was different from the one used by Kim during implementation of the computation proof task. When Kim noticed that the students did not understand the theorem due to the proving tasks that he used, he decided to let the students expose their mistakes and misconceptions in the classroom so that other students would analyse them. Kim also asked questions which prompted the students to explain the reasons for their misconceptions. The computation task implementation extract shows that the teacher's technique of allowing the students to expose their solutions to the whole class exposed several misconceptions. The first misconception was that of judging, or naming a figure according to its appearance instead of properties, the second misconception was that angles at the centre and circumference are only formed by a minor arc, and the third misconception was that the angle which an arc subtends at the centre is always the smaller angle and not the larger angle.

The second and third misconception rose because the teacher only used a minor arc and small angle at the centre during development of the proof for the theorem. As such, the students might have concluded that the theorem only applies to the smaller angle at the centre and small arc. The second and third misconception is consonant with Moru and Qhobela (2013) who argue that the knowledge used in making a misconception may be identifiable as prior knowledge that has been successful in other context. Kim addressed each misconception separately by clarifying on the areas that he identified as cause of the misconception. The approach of analysing students' thinking which was used by Kim helped him to further understand the causes of the students' comprehension and application challenges of the theorem. This indicates that good implementation of proving tasks requires knowledge of prompting students to expose their misconceptions in order to address them effectively.

5.3.3. Knowledge of multiple solution paths for tasks

It was observed that all teachers mainly guided students to solve the tasks using only one solution path which was available in the textbooks. But Hanna (2000) suggests that proof development involves learners having to explore different paths to the solution outcome by using a combination of deductive and inductive reasoning processes. Analysis of lesson extracts from Pike, John and Paul showed that they only developed one proof for each theorem. But in some of Kim's lessons, students developed several proofs for a theorem. This was done because Kim involved the students in exploratory activities. The students explored ways of developing proof for a theorem on their own. However, in some instances, the students were not able to develop several proofs because Kim had only one solution path in mind and he redirected students' focus to the proof that he expected them to develop. This implies that when the teacher only knows one solution path to a task, he might not be able to guide students to complete a task using different correct paths. The same scenario was also noticed in John's lesson where students developed a computation proof using an approach that was correct but not expected by the teacher. Although John acknowledged that the approach was correct, he opted that the students use a different approach by applying the theorem that was learnt during the lesson. By ignoring a correct approach suggested by his students, John illustrated to the students that it was not necessary for the students to know different ways of developing the proof. This suggests that John limited the students' thinking and ability to make several geometric connections during development of the proofs.

5.4. Discussion of findings on teachers' ways of assessing students' thinking in geometric proving

In the following sections, I present discussion on findings on Question 3 that focuses on how secondary school teachers assess students' thinking in geometric proofs. The discussion is centred on how the teachers identify students' mistakes, how they examine the causes of the mistakes in a written geometric proof, and how they plan to assist the students to overcome the mistakes.

5.4.1. Knowledge for identifying students' mistakes and their causes

The computation proof tasks in students' flawed scenarios (see Appendix 3) required demonstration of conceptual understanding by showing relationship and properties of several geometric concepts. In geometric proofs, conceptual understanding is demonstrated by carrying out the norms of situations for doing proofs properly (Herbst et al., 2009). In task 1 (see Appendix 3), the students were supposed to come up with connected statements supported with reasons using several geometric properties like angles in the same segments, base angles of isosceles triangles, angle in a semi-circle and radii. In task 2 (see Appendix 3), the students were supposed to answer the task using Pythagoras Theorem and simplifying of algebraic expressions. This means that the teachers were supposed to analyse the mathematical correctness of each solution by checking whether the students applied the geometric properties appropriately using valid reasons. This implies that the teachers were supposed to have adequate geometric content knowledge to be able to make good judgement about the correctness of the students' solutions.

The findings show that the teachers reported on different issues during analysis of the students' solutions. The teachers were able to identify wrong statements and give good reasons to some of the mistakes in the solutions. Some of the statements which were mentioned by Pike, John and Paul were wrong while others were not wrong. The findings show that these teachers were mainly able to identify mistakes in statements that were easy to understand. These were statements that did not require making of several geometric connections to be understood on the diagram. For example, in solution 1 for Question 2 (see Appendix 4), Pike, John and Paul identified BC = AB as a wrong statement. The teachers explained that the statement is not correct because the lines that are joining both BC and AB to the centre are not equal. The teachers identified the mistake by using chord property of the circle that states that equal chords are equidistant from the centre. In solution 2 for Question 1 (see Appendix 4), the teachers identified a sentence which is not true as LX = XM using their knowledge of radii. In both cases, only one property was applied in identifying the statements that were wrong in the solution. The findings provide an insight that the teachers justified the correctness or incorrectness of the sentences by analysing the validity of the reasons for each statement. This agrees with Stylianides and Ball (2008) who suggest that a true statement is supposed to be justified by a valid reason. Herbst et al. (2009) also points out supporting of statements with reasons as one of the norms in situations for doing proofs.

In another solution, Pike, John and Paul identified a statement as wrong although the statement was correct. Only Pike gave a reason why he thought that the statement was wrong. The reason given by Pike was not related to any of the angles in the statement. There might be several reasons why the teachers claimed that the statement that angle KML = angle NMO is not correct. The first reason is that the statement is not justified by a reason, as such the teachers might have judged the statement based on what they emphasised to their students that every geometric statement must be justified with a reason. During interviews, the teachers explained that they emphasise to their students that each geometric statement must be supported with a reason. But this assumption is not reflected in the causes of mistakes that were given by the teachers. The causes reported by the teachers are related to lack of understanding of geometric concepts like isosceles triangle and lack of using given information. Secondly, the teachers might have thought that the statement is not correct because they were unable to use their knowledge of connections among geometric concepts reflected in the diagram to notice that the statement is true. The diagram that was used for developing the geometric statement shows that ability to notice that the angles are equal required making several geometric connections. This might mean that the teachers arrived at the decision that the angles are not equal without making a thorough analysis on the diagram. This might also be the reason why the teachers made wrong assumptions that the student does not have knowledge of some geometric concepts like properties of isosceles triangles.

The findings also imply that the teachers did not make relationships between the given information and the angles on the diagram. This was noted when the teachers only pointed out one mistake even in solutions which contained several mistakes. Son (2011) refers to this approach of analysing students' work as procedural- based approach. He explains that in procedural-based approach of analysing students' work, the teachers explain individual concept which is either correct, or incorrect without relating it to the whole solution. Since the teachers were analysing each sentence separately, and relating each sentence to the diagram separately, it was difficult for them to notice that the sentence was correct. As already pointed out, ability to recognise that statement (angle KML = angle NMO) is correct requires both understanding of several concepts in the diagram and making conceptual connections. This type of analysis is called conceptual-based analysis (Son, 2011). The findings suggest that good assessment of students' thinking requires making connections between, or among statements within a solution, and also between each statements and the information on the diagram.

Kim concentrated on explaining why the solutions were wrong instead of pointing out the statements that were wrong. For example, in solution 1 for task 1, Kim pointed out two reasons related to why the solution as whole was not correct. In solution 2 of task 1 Kim also explained some of general reasons regarding what was wrong in the solution. The findings might mean that Kim analysed the solutions based on all three criteria for defining a mathematical proof as suggested by Stylianides, and Ball (2008). For example, the reason given by Kim which states that the student did not link the relationship of being isosceles on angle LNM = LKM = LMK = 43^0 has several implications. The statement implies that the students were supposed to show that the three angles are equal. This may be done using connection of several geometric properties like isosceles triangles and angles in the same segment. This implies use of valid argumentation and valid modes of argument representation suggested by Stylianides and Ball (2008). Secondly, it implies that the values of the mentioned angles should be 43^0. This means that the statement in the solution which states that 43^0 = angle KML, or angle NMO is not true as angle LKM (or MKL) is supposed to be 43^0. The other observation concerning noticing that there are three angles forming an angle in a semicircle LMN implies that the student was supposed to find the value of angle KMO by subtracting the sum of the two angles from 90^0. This illustrates that Kim was analysing the solutions while considering the conceptual understanding that the students were supposed show in finding the solution. The findings illustrate that Kim used a conceptual-based approach to analyse the solution (Son, 2011).

The results also show that there was one statement which was not clearly identified to be wrong by all the teachers. The statement is that 45^0 = angle KML, or angle NMO. Ability to recognise that this statement is not correct required making several conceptual connections like using properties of isosceles triangles KLM and MNO, and angles in the same segment. Kim might have recognised the mistake but did not point it out because he mainly concentrated on explaining the correct conceptual understanding that the students were supposed to demonstrate. This is evidenced when Kim claimed that one of the problems with the solution is that the student is not able to recognise number of angles sharing the angle in the semicircle. This means that the students might have ignored one of the angles, especially angle KMO and divided the value of the angle in a semicircle (90^0) by 2 to come up with the values of angle KML and angle NMO. In this case, it may be argued that Kim used conceptual-based approach to analyse the solution. The fact that the statement is wrong was not reflected in responses given by Pike, John and Paul. This might mean that the teachers did not recognise that the statement is wrong. These teachers might not have recognised a mistake in the sentence due to their approach to analysis of the solutions. As the teachers were only identifying a single wrong statement in each solution, it can be assumed that they stopped analysing the solution upon identifying the first mistake in the solution. The teachers might have assumed that the remaining part of the solution was wrong due to only one mistake. This finding provides an insight that analysis of the students' solutions should be conducted while focusing on conceptual understanding of the whole students' solution and how it was supposed to be developed. By doing this, the teachers can capture all mistakes as well as find out their causes and better ways of helping the students to understand and overcome them.

On causes of the mistakes, the results show that the teachers explained either what the student was thinking, or the knowledge that the student was lacking. The teachers explained the causes of the mistakes according to the way they identified the mistakes. For example, Pike, John and Paul who identified mistakes using procedural-based approach mainly attributed the causes of the students' mistakes to lack of understanding of either a procedure, or a concept but not to both. For example, these teachers explained that in solution 2 for task 2 the cause of the mistake is lack of knowledge of simplifying algebraic expressions. This means that the students that made the mistakes do not understand what is involved in simplifying of algebraic expressions. Regarding solution 1 for task 1, Pike mentioned that the student made the mistake because of failure to use given information that KL = LM. Pike regarded the mistake made by the student as a result of not following a correct procedure. In this case, the correct procedure was to use the given information and conclude that triangle KLM is isosceles. This implies that during analysis Pike did not focus on whether the student shows understanding of the concept of isosceles triangles when developing the solution. Paul claimed that the student does not have problems with content but he/she was writing in a hurry as a result the student regarded AB as diameter of the circle. This suggests that Paul assumed that the student understood the concepts but used a wrong procedure. The findings show that during analysis of the solutions, Pike and Paul focused on either understanding of the procedure, or the concept but not on both. The findings agree with Son (2011) who explains that teachers who use procedural-based approaches in evaluating proofs tend to think that a student has understanding of the concepts but does not know the procedure.

There were some mistakes in the solutions that were a result of wrong procedure. For example, the mistake in solution 2 for task 1 is a procedural-based mistake. The solution shows that the student understands the concepts of base angles of isosceles triangle and applied them correctly. But the mistake rose because the student regarded point x as the centre of the circle instead of point O. Since the student was able to apply the concept of radii and base angles of isosceles triangle appropriately, this might mean that he/she understood the concepts but used a wrong procedure for identifying the equal sides. Although the student was able to apply the properties of radii and isosceles triangles appropriately to the solutions, all teachers regarded the student mistake as a result of lack of understanding of the concept of radii. This might imply that in some of the solutions, the teachers were not able to recognise whether the mistake was a result of lack of understanding of either the concept or the procedure.

For Kim who used conceptual-based approach in identifying the mistakes, the results show that he described the causes of the mistakes in terms of the geometric concepts, which the students were not competent, in the connection of the concepts and in terms of the procedure that was supposed to be used. For solution 1 of task 1, Kim suggested that the solution was wrong because the student did not make good connections among different angles in the diagram. This implies that Kim thinks that the mistake was caused by lack of conceptual understanding. According to Kilpatrick et al. (2001), conceptual understanding includes ability to make connections between, or among different mathematical concepts. In solution 1 of task 2, Kim claimed that the student made a mistake because of lack of understanding of chord properties of a circle and their application. Ability to understand and apply a concept involves making of connections between, or among several mathematical concepts. This shows that apart from not being competent in some of geometric concepts, Kim also thinks that the student did not make correct connections in the solution. This implies that Kim is also referring to lack of conceptual understanding as a cause of the student's mistake in the solution. The results suggest that analysis of causes of mistakes should be done by regarding not only the mistake that the student has committed but also what the student was supposed to do. By doing so the teachers would identify the missing gaps in the students' solutions and think of how they can address them. The results also suggest that good assessment of causes of students' mistakes requires knowledge of classifying the mistakes.

5.4.2. Knowledge for addressing students' mistakes

The results show that Pike, John and Paul came up with general strategies for addressing the mistakes while Kim came up with both specific and general strategies for addressing the mistakes. Some of the general ways proposed by Pike, John and Paul included showing the students how the task was supposed to be answered and by re-defining some geometric concepts to the students like diameter, radius and isosceles triangles. For example, on addressing mistakes in solution 2 of task 2, Pike and John wrote that they would show the students how they are supposed to simplify algebraic expressions. The teachers also proposed that after re-explaining the concepts and the solution of the task they would give the students similar tasks for either individual, or group practice.

The ways of addressing the students' mistakes suggested by Pike, John and Paul are regarded as general strategies for addressing mistakes because they do not focus on addressing a specific mistake made by the students. The idea of showing the students how to do the task and giving them similar types of tasks for practice might lead to making the students be competent in applying the geometric concepts to only a single type of situation but unable to apply it to different situations. Furthermore, John's suggestion to explain the procedure for doing the task in addressing the students' mistakes confirms that he used procedure-based approach in analysing causes of the mistakes. This is because his suggestions are not based on helping the students to understand the concepts but on showing how to perform the task. Son (2011) classifies this type of intervention as procedural-based intervention. This is the type of intervention in which the teachers focus on delivering information to the students rather than listening to the students' views. The proposed suggestions show that the teachers will not give students an opportunity to analyse their solutions. This might mean that the suggestions might not target at understanding students' thinking from their point of view. As a result, the intervention might mainly be based on the teachers' assumptions on the causes of the students' mistakes. Son (2011) explains that this type of disconnect might create a disjoint between the students' mistakes, teacher's interpretation and teacher's intervention. This illustrates that some of the students' misconceptions that lead to the mistakes might not be addressed during the process of re-explaining.

Kim mainly suggested having discussions with the students. He said that he would ask some students to write their solutions on the chalkboard and give them an opportunity to explain their solutions so that they expose more mistakes for whole class discussion. Kim suggested the same strategy for addressing mistakes for all solutions. This means that class discussion is a general strategy that Kim would use in helping the students to assess their solutions. As the students' solutions would be the subject of discussion, it implies that the strategy proposed by Kim can address both specific mistakes in the solution and general mistakes regarding students' understanding of different geometric concepts. This strategy is called conceptual-based strategy by Son (2011) because it focuses on both understanding of the geometric concepts and the procedure. According to Son (2011), conceptual-based intervention involves problem solving in a way that the teacher asks students questions that would help them to identify their mistakes and redirect their thinking. Kim's strategy of letting students expose their mistakes is supported by Legutko (2008) as one of the good strategies of addressing students' mistakes. He argues that a conversation with students often clarifies more than a long analysis of their written creations and a long search for the error origins.

The findings have revealed that the teachers suggested their ways of addressing the students' mistakes depending on their views concerning the causes of the mistakes. The teachers who regarded the mistakes as procedural-based suggested a showing and telling approach while those who viewed the mistake as a conceptual-based suggested a class discussion on the mistakes. The findings provide an insight that a correct assessment of students' mistakes can lead to good decisions on ways of addressing the mistakes, hence it is important knowledge for teaching.

5.5. Discussion of how teachers explain and represent geometric proofs to students

In the following sections, I will discuss findings from data analysis that aimed at answering research Question 4 that focuses at how teachers explained and represented geometric proofs to their students. In the COACTIV model, knowledge of explanations and multiple representations involves knowledge of providing appropriate guidance and support to students during instruction (Krauss et al., 2008). Polya (1945) describes appropriate guidance and support as making sure that the students have a fair share of the work being carried out. He explains that to do this, the teacher should ask questions that will help the students get acquainted with problem-solving. In a problem to prove, such questions include what is the hypothesis? What is the conclusion? Can you find a similar, or same problem? These questions help students to understand the problem and to devise a plan. Findings from my study show that teachers used different approaches for explaining the steps of geometric proving to their students. These are discussed in the following sections.

5.5.1. Explaining geometric proof development through demonstration.

Findings show that in all steps of developing a proof, Paul mainly showed to the students what was supposed to be done. In helping the students to understand the problem, he explained the definitions of geometric concepts and meanings of theorems to the students. After explaining, he asked the students to repeat the definition. If the students failed to provide a correct answer, he repeated the definition. During developing of the proof, Paul suggested the construction to be made and the theorem to be used for developing the proof. He also usually gave the proving sentences but students only gave reasons. For example, he wrote on the chalkboard a statement showing that either two angles, or lines are equal and ask the students to give a reason. When he realised that the students did not understand the proof, he explained again the stages of developing the proof. The guidance that he gave hindered the students from taking active part in thinking about the problem and its plan. Paul's actions imply that he assumed that the students could not develop the proofs on their own.

The consequence of Paul's approach is that the students might only learnt logical reasoning but not discovery skills. Herbst (2004) claims that this approach can make the student memorise the proof but not understand its meaning and own the proof. As a result, the students are only able to develop a similar proof but not able to develop any other related geometric proof. Even (1990) explains that forcing the students to memorise the proof but not to understand it is one of the causes of students' challenges in learning to develop mathematical proofs. These findings support Jones et al. (2009) who argue that their studies found that teachers provided students an experience to memorise and reproduce geometric proofs, but the students could not appreciate the value of developing a formal proof. Ding and Jones (2009) argue, providing students an opportunity to engage in activities where they explore ways of developing geometric proofs help the students to understand the proof and appreciate its discovery function in Mathematics.

5.5.2. Explaining geometric proof development through questioning

Pike and John mainly used question and answer method to develop the proof with the students. They would start by asking students to define geometric concepts that are related to the theorem to be proved. For example, when teaching students how to prove that opposite angles of a cyclic quadrilateral are supplementary, John started by asking the students to define a circle, a quadrilateral and then a cyclic quadrilateral. After clarifying on the definitions given by the students, he wrote the theorem on the chalkboard and asked one student to read it. According to Polya (1945), asking students to read a problem is one way of helping the students to understand the problem. Both Pike and John asked questions that required different levels of cognitive demands from the students. Some questions were low level while other questions were high cognitive level questions. The difference was that Pike did not probe further when students had given a wrong answer, but John did. For example, when teaching cyclic quadrilateral theorems, John drew two diagrams on the chalkboard and asked the students to choose a diagram that represented a cyclic quadrilateral. When the students identified the correct diagram, he asked them to use the diagram to improve their definition of a cyclic quadrilateral. This led the students to rephrase their definitions of cyclic quadrilateral in a good way. John's diagrams and questions asked helped the students to understand that a cyclic quadrilateral has all vertices lying on the circumference.

Similarly, when proving a theorem that states that angle subtended by an arc at the centre is equal to two times an angle subtended by the same arc at the circumference, John drew a circle containing the two angles and asked the students varied questions which helped them to understand the meaning of the theorem. This was evidentwhen John started to develop the proof with his students. He drew a diagram with angle BOC at the center and angle BAC at the circumference. When he asked the students to mention the statement which was to be proved they responded correctly by giving the sentence; BOC = 2angle BAC. The evidence that the students had understood the statement which was being proven was also shown when another student suggested that as part of adding a construction to the diagram the teacher should label angle BOC as $2x$ and angle BAC as x. Although the student's suggestion was wrong, it does show that the student understood the meaning of the theorem. The student's mistake does not arise from lack of understanding of the problem but the procedure for developing the proof. As such, John had to remind the student that they were trying to prove that angle BOC = angle 2BAC, so labelling the mentioned angles in that way would mean that the statement is already proved. It can, therefore, be argued that John's questions helped the students to understand the meaning of the theorem.

The approach of asking questions was also used by Pike and John when they were carrying out the other steps of geometric proof development. The teachers were the ones who wrote the proving statements on the chalkboard. They started by asking the students to refer to the diagram and mention the given information, the statement to be proved and the construction to be made. Pike and John also allowed the students to decide the theorem to be used and the features to be added to the diagram. According to Herbst (2004), giving students an opportunity to decide on features to be added to the diagram enhances both understanding of the proof and its meaning. However, in other lessons, it was observed that Pike was quick to suggest the plan for developing the proof before rephrasing the question, or asking probing questions. Polya (1945) explains that the process of devising a plan can be time consuming and gradual, hence a teacher need to help the students hunt for a better idea by encouraging them to relate the problem to their prior knowledge. This suggests that Pike could have continued to probe students' thinking by asking questions that would help them discover the plan.

Kim also used questioning when explaining of geometric proof development. For example, when the students failed to provide a correct plan, Kim asked the students different types of questions that would help them to think of how such plan could be linked to the problem that they were asked to prove. This means that in his lessons, variation of questions helped the students to hunt for a better idea (Jones et al., 2009). The students were able to realise that a better plan is the one that would help them make connections between the given information and the conclusion. This reveals that in geometric proving knowledge of explanations and representations also involves knowledge of varying teaching questions as suggested by Jones et al. (2009). The findings are supported by Ding and Jones (2009) who suggest that varying of questions can help students to understand the problem to prove.

5.5.3. Explaining geometric proof development through activities

Kim and John also involved students in several proving activities. Firstly, they involved the students in activities that could help them to deduce a theorem. Jones et al. (2009) explain that activities that encourage students to formulate a guess promote students understanding of the discovery functions of a proof. After coming up with the theorem, Kim guided the students to discuss and develop the proof in their usual groups. When he noticed that the students were not doing the right thing, he asked them questions, or suggested an activity that would help them come up with a correct plan for developing the proof.

After the discussions, a representative from each group wrote their proofs on the chalkboard and explained them to the whole class. Then Kim gave the students from other groups a chance to analyse and comment on the proof. Lastly, Kim made comments on the proofs presented by the students and highlighted the proving steps that were either correct, or wrong in each proof. This shows that Kim concentrated both on helping the students to understand how to develop the proof and to value proof as a tool for discovering Mathematics. Kim's teaching practices were consistent with his conceptualisations of geometric proving as an opportunity for students to discover and understand theorems and proofs. This is the reason why, he does not introduce the theorem to the students, but he asks them to do an activity that can help them to come up with a guess. The teacher's practices support Jones et al. (2009) who suggest that apart from enhancing logical reasoning skills, teaching of geometric proving should also aim at helping students to understand and appreciate the discovery function of proof. In Kim's classes, the students' discovery skills were enhanced when Kim involved them in measuring angles and deducing theorems from the measurements. Logical reasoning skills were developed when the students discussed and developed the proof in groups.

Most of the activities that Pike and Kim used were not available in the Mathematics textbooks. For example, the activity that Pike used for students to come up with a guess concerning the exterior angle and sum of opposite interior angles of a triangle is not in the junior secondary textbooks. Likewise, that Kim used to involve students in coming up the guess concerning cord properties of a circle is not in senior secondary Mathematics textbooks. This means that the teacher had to formulate the activities. This illustrates that knowledge of explanation and multiple representations also involve knowledge of activities for proving. Stylianides and Ball (2008) define proving activities as actions that students conduct when solving proving tasks. They explain that different proving tasks require different proving activities. This implies that a teacher is supposed to know a proving activity that is appropriate for a particular proving task.

Ding and Jones (2009) suggest that teachers vary proving activities in a lesson to enable the students to devise a good plan for their proofs. Some proving activities involve exploration with different materials in search of either the hypothesis, or the empirical proof. This means that when planning the proving activity, a teacher also needs to know the type of teaching materials that will be required for the activity. Ding and Jones (2009) suggest that proving activities can help students to understand geometric proofs and to appreciate discovery value in Mathematics. Herbst (2004) argue that involvement of students in proving activities can help them to understand and own the proof in such a way that they can apply it to different situations.

Knowledge on areas to emphasise during teaching of geometric proving involves knowledge of areas that are crucial for development of geometric proofs. Some of the aspects of this knowledge are similar to the norms for situations of doing proofs suggested by Herbst et al. (2009). This includes ensuring that the proving statements are logically sequenced, supporting every proving statement with valid reasons, and the statement to be proved always comes as last statement, or conclusion of the proof. In addition to these areas, the teacher should also emphasise on ensuring that the students understand the problem to prove and develop a correct hypothetical bridge.

5.5.4. Inductive and deductive representation of geometric proofs

The results from analysis of the lesson observation data show that there were two types of representations that were made during the teaching and learning of geometric proving. The first type of representation was made on the problem to prove. All the theorems that were being proved were in word statements. The teachers represented the theorems into a diagram form before beginning to prove the theorem. This type of representation has been discussed in the section on knowledge of problem solving stages and is considered to be under CK. The second type of representations was observed according to forms in which proofs were represented in class. Two forms of proof representation were observed from the results of lesson observation data analysis. These are inductive and deductive forms of proof representation.

Inductive forms of representation were done empirically while deductive forms of representation were conducted in an abstract manner. During inductive representation, the students were asked to measure either angles, or sides of a diagram and deduce a theorem. For example, the students measured several angles at the circumference in Kim’s lesson and deduced the theorem that angles in the same segment are equal. This means that the students started working from empirical form of the proof to abstract form of the proof. It was in Kim's lessons and in some of Pike's lessons that students were asked to deduce theorems from the values of either angles, or sides. Stylianides and Stylianides (2006) support inductive form of representing proofs. They argue that to be able to engage students successfully in inductive proving activities, a teacher is supposed to involve students in proving activities that enhance their development of patterns, conjectures and arguments. Furthermore, they observe that the teacher is supposed to know the connection among patterns, conjectures, arguments, and proofs in order to teach proofs effectively.

Inductive forms of proof representations provide students with opportunities to involve in formation of patterns and conjectures, while deductive reasoning provides students opportunities to develop arguments. Stylianides and Stylianides (2006) explain that the development of geometric proofs in high school Geometry is often treated as a formal process and isolated from other mathematical activities. They argue that this treatment of proof is problematic because it does not offer students opportunities to make sense of and establish mathematical truth. The students are not valued and treated as mathematicians during this type of teaching of geometric proving. Hanna (2000) suggests that teaching of proof development should be conducted in a manner that provides learners with opportunities to explore different paths to the solution outcome by using a combination of both deductive and inductive reasoning processes.

The results of lesson observation data from John and Paul show that geometric proofs were mainly represented to students in a deductive form. The teachers guided the students in formulating logically sequenced geometric statements that aimed at verifying the truth of the statement that was being proved. These were the teachers who, during interviews, explained that the teaching of geometric proofs is challenging. This implies that the teachers might have been presenting the proofs to the students in this way due to lack of knowledge of relevant approaches and activities for teaching geometric proof development. Furthermore, the teachers might also have presented formal proofs only to the students due to lack of knowledge of why students are supposed to learn geometric proofs. For example, Paul said that he does not spend much time on showing students how to develop proofs of geometric theorems but he mainly emphasises on showing them how to use the theorem to solve different types of tasks. The teacher explained that he does this because students are not motivated to learn how to develop the proof. As a result he mainly concentrates on helping the students to know how to apply the theorem to different situations so that they can pass national examinations. This implies that Paul assumes that the only reason for teaching geometric proofs to students is to help them pass examinations. As a result of this, he might not think that his way of teaching geometric proofs might have an impact on how the students experience and value geometric proving. This was evidenced in some of the lessons where the students would say no to Paul when asked if they were interested in knowing the proof of a particular theorem. In that case Paul went ahead and show the students how to apply the theorem to different tasks without developing its proof.

Paul's way of teaching geometric proofs might be influenced by his lack of pre-service teacher education. As explained earlier on, Paul is not a qualified teacher because he did not study education courses at university. He was employed to teach secondary school Mathematics because he studied university Mathematics, and is regarded to have acquired enough content knowledge for the subject. However, the way Paul explained the geometric proof to the students implies that he struggles to help students understand proofs. This is consistent with his explanation that he views geometric proofs as easy to learn but wonders why students do not understand and are not interested to learn them.

Paul might not know that his way of explaining affects students' ability to understand the proofs. The findings support one of the assumptions of the COACTIV model that CK is not enough for teaching Mathematics, hence teachers need to have PCK as well (Baurmert & Kunter, 2013). According to Shulman (1986), the sources of PCK are research, pre-service teacher education and experience. However, this study has revealed that experience might only help the teacher to build on the PCK that they acquire during pre-service. Despite teaching secondary Mathematics for six years, Paul seems to struggle to help his students to understand geometric proof development. This might imply that Paul has not acquired the relevant PCK for teaching geometric proofs through his long teaching experience. The findings suggest that the primary source of PCK is the pre-service teacher education.

Furthermore, the findings show that although John and Pike went through teacher education, their teaching practices were not very different from those of Paul. In all the lessons, John did not involve his students in exploration and explanations. Pike only involved the students in exploration and explanation in only two out of about twelve lessons that I observed. The findings have shown despite attending teacher education, both John's and Pike's lessons were dominated by teacher guidance. The teachers decided the essential decisions in the lessons like the construction to be added to the diagram, and the theorem to be used as a bridge for the proof. This might imply that the type of content that John and Pike learnt at teacher education is different from that of Kim. This agrees with one of the assumptions of TEDS-M framework that teacher education is understood and implemented differently between among institutions in a country (Tatto et al., 2008). The findings imply that the type of teacher education that teachers attend influence their teaching practices. The findings also support Tatto et al. (2008) that teacher education is a complex issue, as such, there is need for a comprehensive framework that can help teacher educators to analyse their education systems.

5.6. Chapter summary

This chapter presented a discussion of the findings for each research question in relation to literature. Several categories of CK and PCK for teaching geometric proofs have been revealed and discussed in the chapter. The findings of the study support Battista (2007) and Chinnappan et al. (2012) who argue that geometric proof development involves special type of deductive reasoning. The proving statements for a geometric proof are developed from relationships of several geometric concepts using geometric reasoning. This implies that geometric proof development requires special type of deductive reasoning which is related to specific geometric concepts.

Regarding PCK for teaching of geometric proofs, the findings showed that effective implementation of geometric proof tasks by teachers require knowledge of the purpose of the task, knowledge good guidance to students, knowledge of teaching and learning materials, knowledge of proving activities, knowledge of problem-solving pedagogies, knowledge of questioning and knowledge of relevance of geometric proof development. In addition to what the COACTIV model suggests regarding knowledge of assessing students' thinking, the findings also suggest that this category of PCK also involves knowledge of responding to students 'questions appropriately. This finding supports Shulman (1986) and Ball et al. (2004). The findings also suggest that knowledge of explanations and representations of geometric proofs includes knowledge of proving activities, knowledge of problem-solving pedagogies and knowledge of questioning. It can be argued that the sub-categories of knowledge of explanations and representations overlap with those of knowledge of cognitively activating tasks because teaching of geometric proofs involves explanation and representation of tasks as explained by Krauss et al. (2008). The categories of CK and PCK that have been discussed in this chapter are grouped and summarised in the next chapter that presents the conclusion of the study.

CHAPTER 6

CONCLUSION

6.1. Chapter overview

In this chapter, I present a summary of the major findings of the present study proposing CK and PCK categories of understanding MKT-GP. I also present the implications of the findings, the study limitations and the conclusion for this study.

6.2. Major findings of the study

The findings of this study support those of other researchers that the teaching and learning of geometric proofs is challenging and complex for several reasons. Firstly, geometric proof development is done by making relationships among several geometric concepts, some of which might have been forgotten because they are learnt in early years of secondary school Mathematics. Secondly, strategies of helping students to understand geometric proof development and appreciate the value of developing a formal proof are not fully conceptualised by teachers. Thirdly, research conducted on the teaching and learning of geometric proofs has not been interested in understanding MKT this particular field of Mathematics.

The present study has made a contribution on understanding of knowledge for teaching geometric proofs. The study has proposed categories of both CK and PCK for teaching geometric proof development. The categories of teacher knowledge proposed by this study are based on findings from analysis of data generated from secondary school teachers through lesson observations, interviews, and pencil and paper tests. The data was analysed using both deductive and inductive thematic analysis. During deductive thematic analysis, a priori themes from different frameworks were used to categories and interpret different types of data. The data that could not fit into a priori themes was analysed using inductive thematic analysis by identifying initial themes and categories from the data, grouping the data under the identified categories, and then interpreting the data. The different themes were further analysed to come up with common themes that could be grouped under the CK and PCK categories of COACTIV model.

I have summarised the major findings of this study by using categories of the COACTIV model (Baumert & Kunter, 2013). Firstly, I present Table 13, containing the major findings under each category of the COACTV model and then provide a brief summary of the points listed under each category.

Table 13: Summary of major findings

Teacher knowledge category	**Sub-category**	**Components**
Content Knowledge	Knowledge of Geometry content knowledge.	• Geometric concepts. • Geometric language. • Geometric reasoning (geometric connections and spatial reasoning).

	Knowledge of geometric deductive reasoning	• Deductive reasoning. • Combination of deductive and geometric reasoning.
	Knowledge of geometric proof problem-solving skills.	• Problem-solving process.
	Knowledge of algebraic reasoning.	• Identifying patterns. • Simplifying algebraic expressions. • Using algebraic representations.
Pedagogical Content Knowledge	Knowledge of cognitive activating tasks	• Level of cognitive demands of a task. • Relevant prior knowledge for a task. • Effective implementation of tasks (good guidance, proving activities, questioning, multiple proving paths). • Appropriate sequence of tasks (in terms of cognitive level and

		purpose)
	Knowledge of student's cognitions and ways of assessing students' knowledge and comprehension processes.	• Knowing students' misconceptions and their causes. • Identifying the mistake and its causes. • Classifying of the mistake. • Analysing the impact of mistake to the whole solution. • Ways of handling students' mistakes. • Responding to students' questions.
	Knowledge of explanations and multiple representations	• Problem solving pedagogies. • Proving activities. • Teaching and learning materials. • Questioning. • Areas of emphasis. • Relevance of teaching geometric proofs.

6.2.1. Content knowledge

Findings of the study suggest that content knowledge for teaching geometric proving includes several categories. As already pointed out, the COACTIV model does not clearly specify the categories of content knowledge for teaching Mathematics. But Ball et al. (2008) argue that categorising of CK is important to better understand its components. As such, one of the purposes of this study was to develop categories of content knowledge for teaching geometric proofs. The categories of content knowledge revealed from the findings of this study include: Geometry content knowledge, geometric reasoning, geometric deductive reasoning, problem solving skills and algebraic reasoning. Geometry content knowledge is knowledge of Euclidean Geometry which includes shape and space, geometric reasoning and geometric language. Knowledge of geometric deductive reasoning is a combination of deductive reasoning and geometric reasoning to come up with a logically sequenced geometric proof. Knowledge of geometric proof problem solving skills includes knowledge of the steps of developing a geometric proof. Knowledge of algebraic reasoning involves identifying patterns, simplifying algebraic expressions and using algebraic representations.

There are several advantages of geometric content knowledge that have been observed in this study. They included good task implementation, good analysis of students' thinking and ability to use different representations.

6.2.2. Pedagogical content knowledge

Using the PCK categories of the COACTIV model, this study has tried to provide descriptions of what is involved in each category of PCK for geometric proofs. These are presented in the following sections.

6.2.2.1. Knowledge of cognitive activating tasks

According to the COACTIV model, this category includes knowledge of multiple solution paths to tasks, knowledge of sequencing of the tasks to meet the desired effect in the classroom, knowledge of prior knowledge required for the selected tasks, and knowledge of good implementation of the tasks (Krauss et al., 2008). This study has found that these aspects of knowledge are also necessary for teaching geometric proofs. On sequencing of the tasks, the COACTIV model suggests that teachers are supposed to select tasks that present students with powerful learning opportunities in terms of cognitive activation. According to the model, selection of the task must mainly depend on the cognitive level of students' conceptual involvement. This study has found that selection of the task must also depend on the purpose to be accomplished. The teacher is supposed to select a task according to the purpose it would serve during teaching of geometric proving. Some of the purposes that tasks might serve in geometric proofs are: developing proofs; exemplifying proof application; assessing understanding of either proof development, or application; and enhancing understanding of the theorem and proof through practice.

The study has found that there are several aspects of knowledge which can contribute to good implementation of tasks. These include knowledge of guiding students appropriately, knowledge of asking thought provoking questions, knowledge of providing students with activities that can afford them opportunities to explore the task, and knowledge of teaching and learning materials that can enhance students' participation during task implementation.

6.2.2.2. Knowledge of student's cognitions and ways of assessing students' knowledge and comprehension processes

In the COACTIV model, this category includes knowledge of student misconceptions, typical errors and difficulties in relation to a particular subject or topic, and ways of overcoming them (Krauss et al., 2008). This study has found that in addition to these, this category of PCK also involves knowledge of classifying mistakes, knowledge of analysing impact of the mistake to the whole solution, and knowledge of responding to students 'questions. Ability to classify students' mistakes involves knowing whether a mistake happened due to lack of procedural, or conceptual understanding. The study has found that the teachers proposed methods of addressing the students' mistakes according to how they categorised the mistake. This agrees with Son (2011) who explains that if teachers classify the mistake in a good way, they come up with good ways of addressing the mistake. The findings suggest that ability to analyse the impact of the mistake to the whole solution can help to see if there are some connections among the mistakes identified in the solution and to come up with good ways of addressing the mistakes.

6.2.2.3. Knowledge of explanations and multiple representations

In the COACTIV model, this category involves knowledge of supporting and guiding students' learning processes. The COACTIV model has not elaborated on knowledge that is involved in supporting and guiding students' learning processes during learning of Mathematics. This study has found that supporting and guiding of students' learning processes during teaching of geometric proving, requires several aspects of knowledge like problem solving pedagogies, proving activities and materials, questioning, areas of emphasis and relevance of teaching geometric proofs.

Knowledge of problem solving strategies for geometric proving involves application of general problem solving skills in conducting the steps of geometric proving.

6.3. Implications of the study

The findings of this study have several implications to different areas including professional development and future research. I explain these implications in the following sections.

6.3.1. Implications for professional development

The purpose of this study was to explore categories of MKT-GP. Literature review showed that there is a gap in terms of studies that have been done to improve the teaching and learning of geometric proofs. The area of MKT-GP has not been given much attention by researchers. The study was conducted with an aim of filling the gap in order to improve the teaching and learning of geometric proofs. The findings have revealed knowledge that was being used by the teachers in developing geometric proofs and proving tasks, assessing students' thinking, and explaining and representing geometric proofs to students. Using these findings, the study has developed categories of teacher knowledge that might be used for creating effective learning experiences for helping students to be successful in developing geometric proofs.

One of the major implications of this study is that it will inform my teaching practice as a teacher educator. Apart from achieving its aim of developing categories of teacher knowledge for teaching geometric proofs, the study has also revealed that the teachers think that the type of pre-service education they receive during their teacher education does not equip them with good knowledge for teaching geometric proofs. Therefore, the findings from this study, especially the CK and PCK categories for teaching geometric proofs will be used as a basis for improving secondary school teacher education. Pre-service is the primary source of teacher knowledge (Baumert & Kunter, 2013; Shulman, 1986). This means that improvement in teaching and learning of geometric proofs needs to start with improving pre-service curriculum. Jones (1997) argues that although mathematical proof is an essential component of Mathematics, providing a Mathematics curriculum that makes proof accessible to school students appears to be difficult. This implies that there is need to improve mathematical proof content of the education curriculum. Therefore, since geometric proofs occupy most the secondary school mathematical proofs, this implies that teaching of geometric proofs should be a component of secondary school teacher education. As I have already argued in the previous sections, improving teacher education curriculum can help to improve the quality of teaching of geometric proofs, which will result in improvement in students' abilities in understanding of geometric proof development.

Furthermore, the categories of CK and PCK that I have proposed in this study can be used for developing curriculum for in-service teachers' professional development courses. Teachers' lack of knowledge has been pointed out in the earlier chapters as one of the causes of students' failure to understand geometric proofs. This implies that the teachers have not acquired relevant MKT-GP through their teaching experience. This means that improving teaching and learning of geometric proof development requires paying attention to in-service teachers as well. As such, there is need for development of professional development courses that can help to improve in-service teachers' knowledge for teaching geometric proofs. As Jones (2000) argues, improvement of teaching of Geometry requires paying attention to both initial and continuing education of teachers of Mathematics in terms of their background and understanding of Geometry.

6.3.2. Implications for future research

The reflections from the study findings and discussions show that there is still need for some further research in teaching of geometric proofs. Some of the suggestions for future research are as follows.

- Although the study was undertaken with experienced teachers, there seems to be challenges that the teachers are facing in terms of both CK and PCK. The blame is put on the pre-service education for not concentrating on these aspects of knowledge during teacher preparation. But experience is also regarded as a rich source of the teachers' knowledge base (Shulman, 1986). This shows that there is need to explore why the teachers are unable to acquire enough CK and PCK in geometric proofs from their teaching experience.
- The study can be repeated with pre-service teachers to find out if there are some aspects of knowledge that were not revealed by the in-service teachers. Undertaking the study with pre-service teachers might also reveal categories of teacher knowledge which are acquired from pre-service education and those which are acquired through experience.
- The study can also be done with students to capture their views about geometric proof development and their interactions during geometric proof development lessons. This could complement the findings by revealing what is involved in geometric proof development and its learning.
- As observing of lessons was mainly done on one topic for each teacher, it would also be interesting to extend the data generation period and observe more geometric proof topics by each teacher. This would help to reveal other categories of teacher knowledge that were not manifested in the observed topics

as well as to find out if knowledge for teaching geometric proofs is specific to a particular geometric topic.

- Knowledge of teaching and learning materials has been highlighted as one of the sub-categories of knowledge of explaining and representing geometric proofs PCK category. Since the textbook is one of such resource that the teachers usually rely upon, it is also necessary to analyse the mathematics textbook content in relation to the teachers practices.
- The findings have shown that the teaching of Paul, who did not undergo teacher education was not very different from some of the teachers who participated in the study (John and Paul). This implies that the problem might be the quality of teacher education program which these teachers underwent. This might mean that the teachers were not offered relevant content to prepare them for teaching mathematics. As such, there is need to examine the content being offered by various teacher education colleges in Malawi.

6.4. Limitations of the study

This qualitative study has several limitations in terms of both data generation and analysis. The first limitation is that data for the study was only generated from four participants. This means that the observations made in this study are based on a very small sample and they might not have captured all categories of teacher knowledge required for teaching geometric proof development. Therefore, the categories of teacher knowledge proposed by this study might not be assumed to be comprehensive, but they can be regarded as a basis for understanding knowledge for teaching geometric proofs.

The second limitation is that this study was conducted over a short period of time compared to the studies that aimed at building theory for mathematical teacher knowledge like those conducted by Adler and Davis (2006), Ball et al. (2008) and Herbst et al. (2009). These studies observed Mathematics lessons taught by several teachers over a long period of time and on different mathematical topics. This means that these studies generated comprehensive data for development of the theories. But the theory that I have developed in this study was based on data from two topics as I only observed one geometric proof lesson from each teacher. I am aware that observing of same teachers, teaching different geometric proof lessons in different classes could have provided me with comprehensive data for developing the theory. However, this was not possible due to time limit and lack of resources as this is a PhD study that was conducted in a limited period and with few resources. However, although data generation covered a narrow range in terms of Geometry content, the data that I generated might be sufficient for theory proposition because of several reasons. Firstly, the data is thick and rich for making descriptions and propositions for I generated it from real classroom settings (Yin, 2009). Secondly, because I observed all lessons under each topic and each teacher was teaching the same topic in several streams at the same level. This made me to collect large volumes of data over a short period of time. Furthermore, the lessons that I observed towards the end of each lesson did not yield new results when I was analysing them, this meant that a saturation point had been reached and data collection could be stopped (Cohen et al., 2007). Furthermore, the triangulation of the lesson observation data with interview and pencil and paper data afforded an opportunity to complement and confirm the findings.

Another limitation of the study is that it did not take into consideration students' views about geometric proof development. This means that the categories of knowledge revealed by the study might not have captured enough information regarding what students go through when developing geometric proofs. Through analysis of the students' views, the classroom interactions between and among students, and proofs developed by the students, I would have captured more insights regarding the complexities of geometric proof development and MKT-GP.

6.5. Concluding remarks

In this study, I have discussed how I generated and analysed data for exploring what is involved in the teaching of geometric proof development. The study has revealed that teaching of geometric proof development is complex and challenging on the part of the teacher. As such, there is need for a comprehensive framework for understanding MKT-GP.

As a contribution to knowledge for teaching, I have proposed a framework for understanding teacher knowledge for geometric proof development. As an improvement to the COACTIV model in relation to geometric proof development, I have proposed several sub-categories of CK for teaching geometric proofs and some additional sub-categories of PCK for teaching geometric proof development. These are interesting findings that can be followed up and elaborated in future research.

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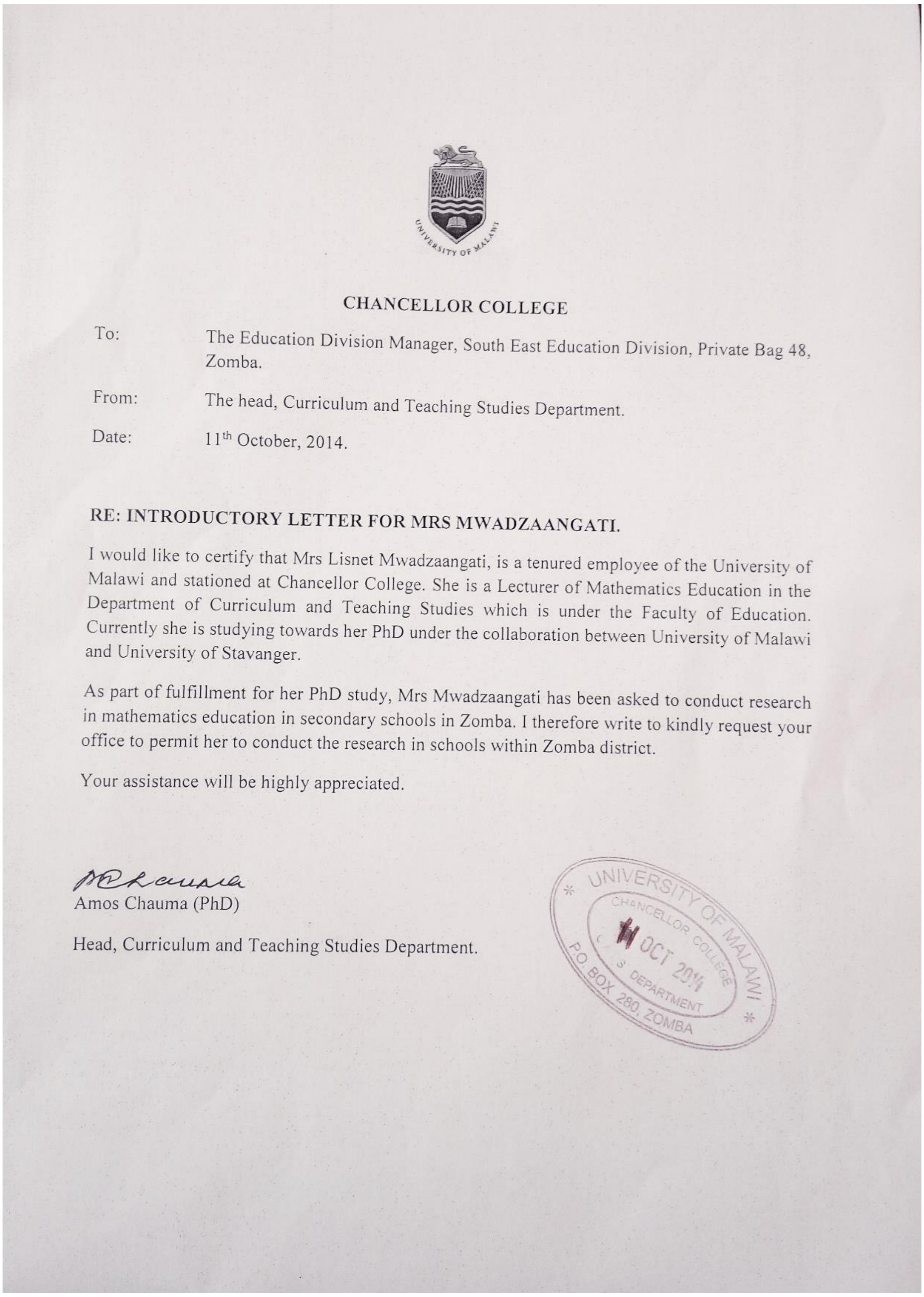
APPENDICES

Appendix 1: Permission letter from Chancellor College to the Education Division Office.



CHANCELLOR COLLEGE

To:	The Education Division Manager, South East Education Division, Private Bag 48, Zomba.
From:	The head, Curriculum and Teaching Studies Department.
Date:	11th October, 2014.

RE: INTRODUCTORY LETTER FOR MRS MWADZAANGATI.

I would like to certify that Mrs Lisnet Mwadzaangati, is a tenured employee of the University of Malawi and stationed at Chancellor College. She is a Lecturer of Mathematics Education in the Department of Curriculum and Teaching Studies which is under the Faculty of Education. Currently she is studying towards her PhD under the collaboration between University of Malawi and University of Stavanger.

As part of fulfillment for her PhD study, Mrs Mwadzaangati has been asked to conduct research in mathematics education in secondary schools in Zomba. I therefore write to kindly request your office to permit her to conduct the research in schools within Zomba district.

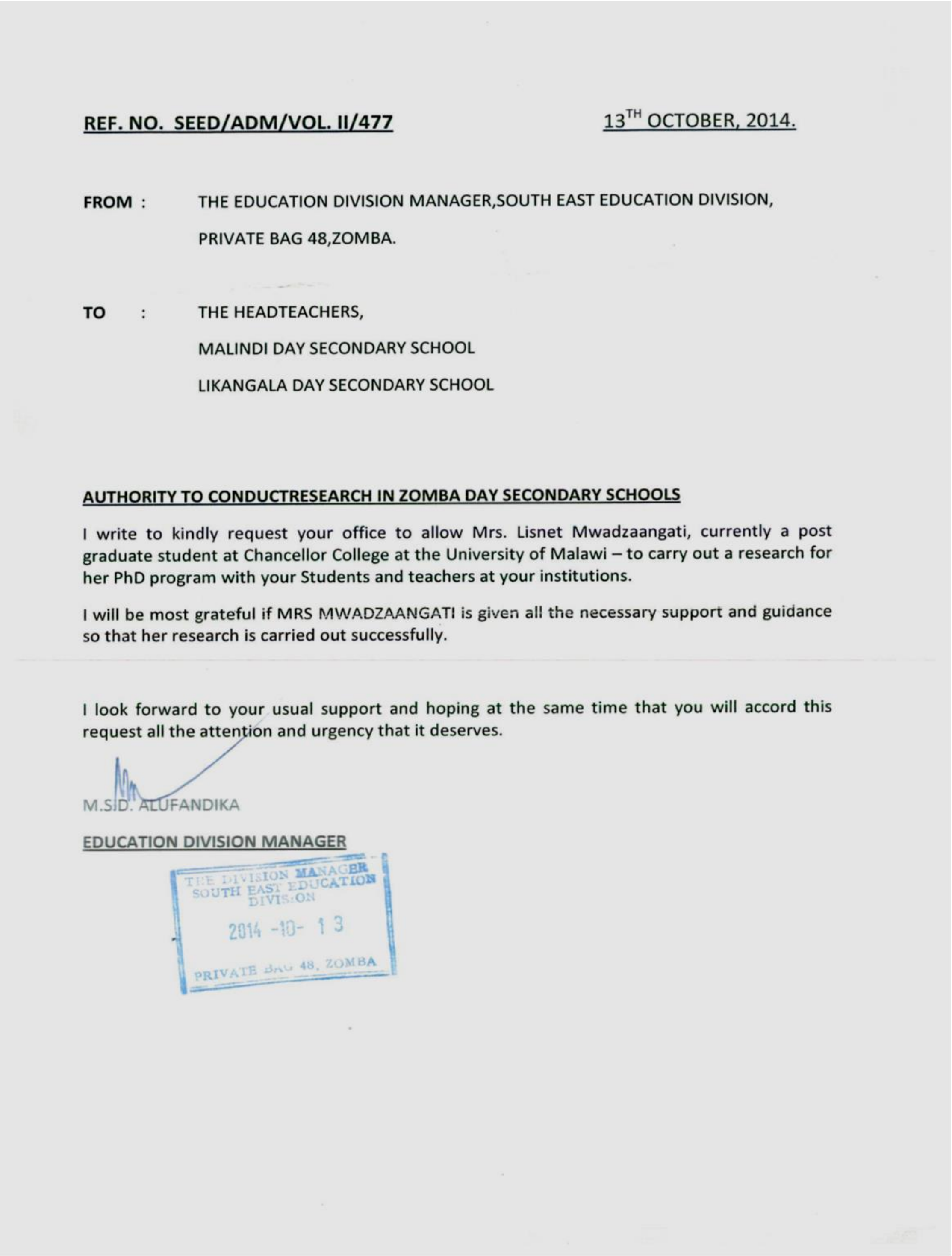
Your assistance will be highly appreciated.

Amos Chauma (PhD)

Head, Curriculum and Teaching Studies Department.

UNIVERSITY OF MALAWI
CHANCELLOR COLLEGE
14 OCT 2014
DEPARTMENT
P.O. BOX 280, ZOMBA

Appendix 2: Permission letter from the Education Division Office to the head teachers.



REF. NO. SEED/ADM/VOL. II/477 13TH OCTOBER, 2014.

FROM : THE EDUCATION DIVISION MANAGER,SOUTH EAST EDUCATION DIVISION,
PRIVATE BAG 48,ZOMBA.

TO : THE HEADTEACHERS,
MALINDI DAY SECONDARY SCHOOL
LIKANGALA DAY SECONDARY SCHOOL

AUTHORITY TO CONDUCTRESEARCH IN ZOMBA DAY SECONDARY SCHOOLS

I write to kindly request your office to allow Mrs. Lisnet Mwadzaangati, currently a post graduate student at Chancellor College at the University of Malawi – to carry out a research for her PhD program with your Students and teachers at your institutions.

I will be most grateful if MRS MWADZAANGATI is given all the necessary support and guidance so that her research is carried out successfully.

I look forward to your usual support and hoping at the same time that you will accord this request all the attention and urgency that it deserves.

M.S.D. ALUFANDIKA

EDUCATION DIVISION MANAGER

THE DIVISION MANAGER
SOUTH EAST EDUCATION
DIVISION
2014 -10- 13
PRIVATE BAG 48, ZOMBA

Appendix 3: Questionnaire 1

Questionnaire 1

Geometry proof content knowledge questionnaire

Dear participant,

You are being asked to respond to some mathematics content questions. The questions were taken from the senior secondary school Mathematics textbook for students and from Malawi National examination Board (MANEB) past Question papers. Please be free to answer the questions, the aim is not to evaluate your knowledge but to collect information that will help in getting in-depth understanding of knowledge for teaching Geometry proof. Should you find some questions difficult, feel free to attempt them even if you are not sure of being correct, that will help me to explore further into the content demands for teaching Geometry proof.

Question 1

a).AB is the diameter of a circle with centre O and AC is a chord. OD is perpendicular to AC. Prove that BC is two times OD.

b). Apart from the method of argumentation that you have used, what other ways could you use to prove the statement above?

c).Reflect on your own solution processes above. What knowledge of geometric concepts did you need in order to solve the question?

Question 2

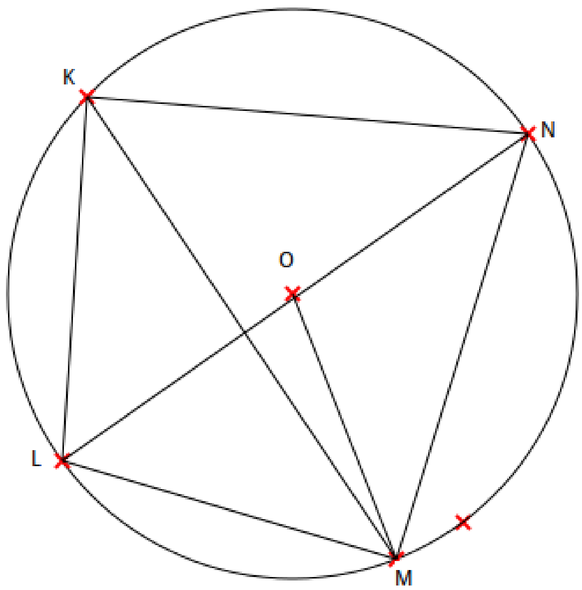
a). MNOP is a parallelogram. H is a point on MN. HO = NO. Prove that MHOP is a cyclic quadrilateral.

b). Apart from the method of argumentation that you have used, what other ways could you use to prove the statement above?

c). Reflect on your own solution processes above. What knowledge of geometric concepts did you need in order to solve the question?

Question 3

a). Figure below shows a circle centre O. Line LON is a diameter, KL = LM and angle LNM = 43^0. Calculate angle KMO.

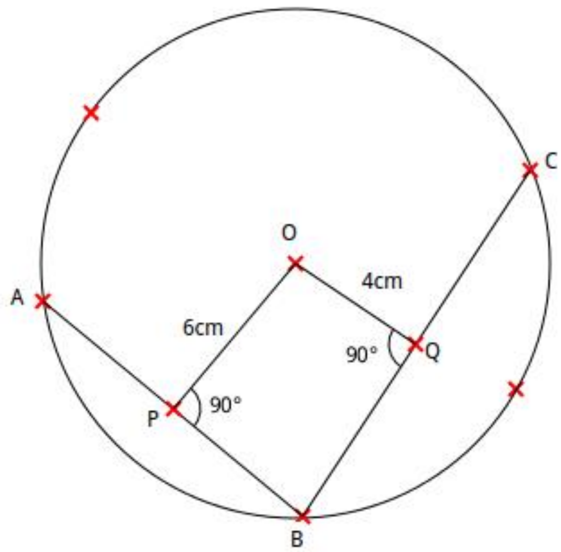


b). How else could you solve the problem above?

c). Reflect on your own solution processes above. What knowledge of geometric concepts did you need in order to solve the question?

Question 4

a). Figure below shows a circle centre O. lines AB and BC are chords, AB = $2x$ and OQ = 4cm and OP = 6cm. Find BC in terms of x.

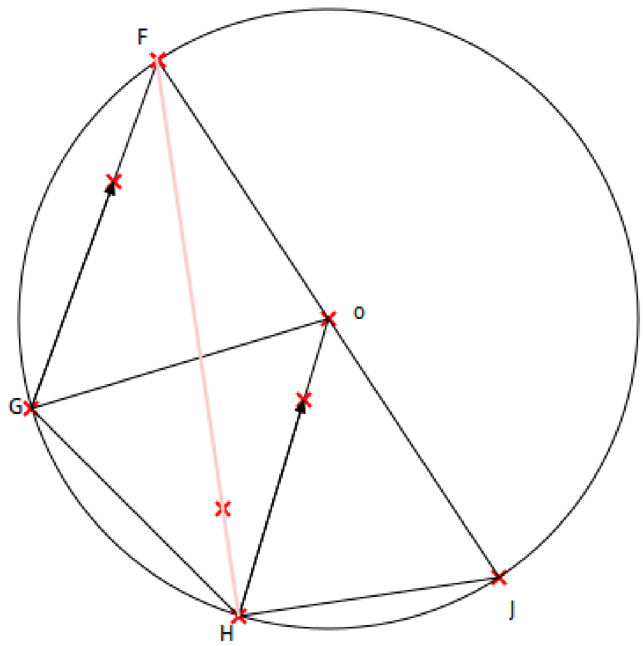


b). How else could you solve the question above?

c). Reflect on your own solution processes above. What knowledge of geometric concepts did you need in order to solve the question?

Question 5

a). Figure below shows a circle FGHJ with centre O. FG is parallel to OH and FOJ is a diameter. If angle HFJ is 37^0 calculate angle GHF.



b). How else could you solve the question above?

c). Reflect on your own solution processes above. What knowledge of geometric concepts did you need in order to solve the question?

Appendix 4: Questionnaire 2

Questionnaire 2

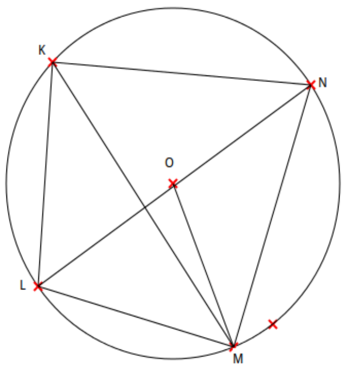
Geometry proof content knowledge questionnaire

Dear participant

You have been provided with vignettes of solutions to some of the Questions that you answered in the first questionnaire. You are kindly asked to examine the solutions and answer questions that follow each of them. Your responses to the questions will help me to develop understanding of content knowledge for teaching Geometry proof. Please be free to answer the questions, the aim is not to evaluate your knowledge but to collect information that will help in getting in-depth understanding of knowledge for teaching Geometry proof. Should you find some questions difficult, feel free to attempt them even if you are not sure of being correct, that will help me to explore further into the content demands for teaching Geometry proof.

Question 1

Figure below shows a circle centre O. Line LON is a diameter, KL = LM and angle LNM = 43^0. Calculate angle KMO.



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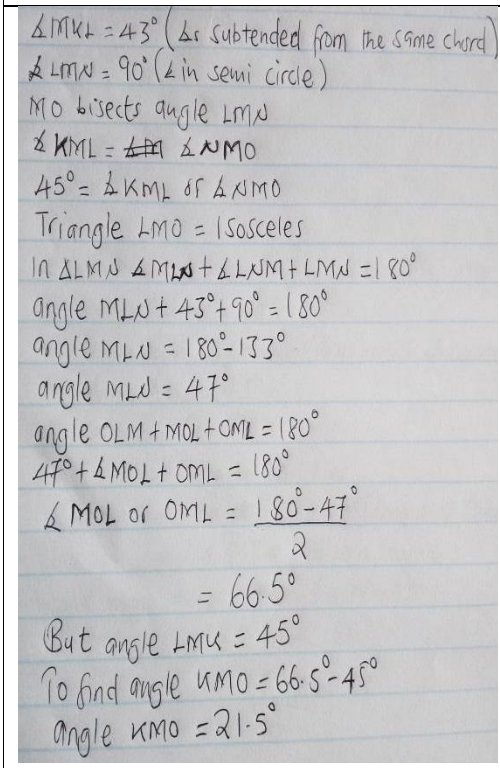
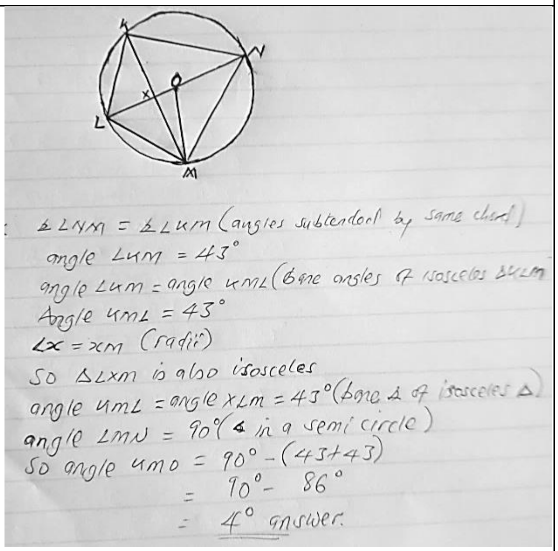
<tr>
<td>
∡MKL = 43° (∡s subtended from the same chord)
∡LMN = 90° (∡ in semi circle)
MO bisects angle LMN
∡KML = ~~∡M~~ ∡NMO
45° = ∡KML or ∡NMO
Triangle LMO = isosceles
In ΔLMN ∡MLN + ∡LNM + LMN = 180°
angle MLN + 43° + 90° = 180°
angle MLN = 180° − 133°
angle MLN = 47°
angle OLM + MOL + OML = 180°
47° + ∡MOL + OML = 180°
∡MOL or OML = (180° − 47°) / 2
= 66.5°
But angle LMK = 45°
To find angle KMO = 66.5° − 45°
Angle KMO = 21.5°</td>
<td>
∡LNM = ∡LKM (angles subtended by same chord)
angle LKM = 43°
angle LKM = angle KML (base angles of isosceles ΔKLM)
Angle KML = 43°
∠x = xm (radii)
So ΔLxm is also isosceles
angle KML = angle xLm = 43° (base ∡ of isosceles Δ)
angle LMN = 90° (∡ in a semi circle)
So angle KMO = 90° − (43+43)
= 90° − 86°
= 4° answer.</td>
</tr>
<tr>
<td>a). Do you see anything wrong with the student’s solution? Explain.
b). Suggest reasons for the student’s solution?
c). What would you do to correct the</td>
<td>a).Do you see anything wrong with the student’s solution? Explain.
b). Suggest reasons for the student’s solution?
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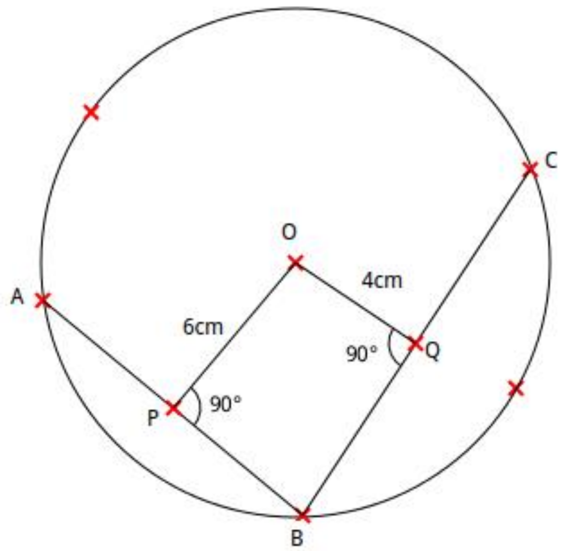
error(s) if any, and help the student understand the concepts? d). What skills would you require in addressing the students errors (if any)?	error(s) if any, and help the student understand the concepts? d). What skills would you require in addressing the students errors (if any)?

Question 2

Figure below shows a circle centre O. lines AB and BC are chords, AB = $2x$ and OQ = 4cm and OP = 6cm. Find BC in terms of x.



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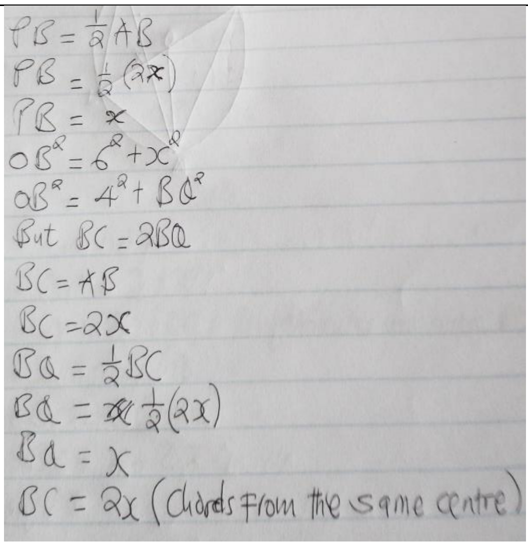
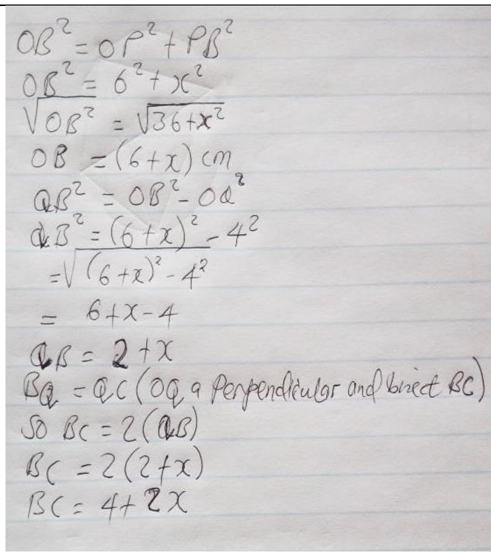


$PB = \frac{1}{2}AB$
$PB = \frac{1}{2}(2x)$
$PB = x$
$OB^2 = 6^2 + x^2$
$OB^2 = 4^2 + BQ^2$
But $BC = 2BQ$
$BC = AB$
$BC = 2x$
$BQ = \frac{1}{2}BC$
$BQ = \cancel{x}\frac{1}{2}(2x)$
$BQ = x$
$BC = 2x$ (Chords from the same centre)
</td><td>


$OB^2 = OP^2 + PB^2$
$OB^2 = 6^2 + x^2$
$\sqrt{OB^2} = \sqrt{36+x^2}$
$OB = (6+x)$ cm
$QB^2 = OB^2 - OQ^2$
$QB^2 = (6+x)^2 - 4^2$
$= \sqrt{(6+x)^2 - 4^2}$
$= 6+x-4$
$QB = 2+x$
$BQ = QC$ (OQ a Perpendicular and bisect BC)
So $BC = 2(QB)$
$BC = 2(2+x)$
$BC = 4+2x$
</td></tr>
<tr><td>
a). Do you see anything wrong with the student's solution? Explain.
b). Suggest reasons for the student's solution?
c). What would you do to correct the error(s) if any, and help the student understand the concepts?
</td><td>
a). Do you see anything wrong with the student's solution? Explain.
b). Suggest reasons for the student's solution?
c). What would you do to correct the error(s) if any, and help the student understand the concepts?
</td></tr>
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d). What skills would you require in addressing the students errors (if any)?	d). What skills would you require in addressing the students errors (if any)?

Appendix 5: Interview guide

Interview Guide

Having responded to Questionnaire 1 and 2, I would like to learn from you further through an oral conversation. I am optimistic that from our discussion, I will continue to learn the knowledge requirements for teaching geometric proof.

1. How did you experience the questions from Questionnaire 1? Is there anything you wish to share?
2. Was there anything that you needed to know in order for you to develop solutions for the Questions in Questionnaire 1?
3. How about Questionnaire 2, how did you experience the questions from it?
4. Now from your own experience as a Mathematics teacher, what do you think about the teaching of geometric proof? How is it similar or different from the teaching of other types of Geometry, or algebra or arithmetic?
5. What do you think a teacher need to know in order for him/her to be able to teach geometric proof in a way that students will understand?
6. Let's now talk about your teaching, what do you do to ensure that students understand Geometry theorems?
7. If you realise that students still hold some misconceptions even after doing what you have just discussed, what else do you do?
8. In your opinion, what should teachers emphasise when they are teaching Geometry theorems?
9. What teacher knowledge challenges arise when you are either planning or teaching geometric proof?

10. How do you address the knowledge challenges that you encounter during planning and teaching geometric proof?
11. Where and how did you learn about teaching Geometry (and geometric proof)?

Appendix 6: Coded interview transcript

Coded interview with Pike

Text	**Codes**
Interviewer: Firstly, I would like to thank you very much for answering the questions from the two questionnaires and for accepting to be engaged in an oral conversation. I think we need to start by commenting on the questionnaires. How did you experience the questions? May be let us start with the first questionnaire. **Participant:**aaa of course some questions were a little bit difficult, for example in each question there is part a, part b and part c. for part b, it is saying apart from the method of argumentation above what other ways could you use to prove the statement above. **In this case, it was sometimes difficult to remember what else can be done and in this case I was just doing trial and error to see what could be done, to see if it was going to work or not.** In this case here I used construction, I tried to sketch the diagram according to scale but it was difficult to express that this method is called this or that as we said here we are using argumentation. Although I constructed the second proof using construction and measurement, I do not consider it as a correct proof because in geometry, a proof is supposed to use geometric properties not measurements. On part c, it	**Multiple solution paths** Abstract geometric reasoning

was on reflect on your solution process above, here what I was doing, I was just considering the first part not the second part. That the solution for question 1a, 2a up to the last question. **For geometric concepts I was thinking of the pre-requisite knowledge, what should the students know for them to solve the question, so I was just trying to think about some of the concepts that can be used. Because in geometry most of the concepts are related. First of all you have to understand this before you understand the other.**	**-Prior knowledge of geometric concepts** **-making connections**
Interviewer: Ok thank you very much, I think we can start with the first part of the questions that is question a, did you experience any difficulties with these?	
Participant: for question 1a this was ok aaaya this was not difficult. For 2a, what was needed here was to think about the properties of isosceles triangles, and how it can be used. Of course here we are supposed to recall many concepts so for a student that does not easily connect things this could be a difficult question. But me I did not find any difficulties but to some especially learners, this could be a difficult question.	-geometric reasoning. -geometric concepts -geometric relationships.
Interviewer: Why do you think it would be a difficult question?	
Participant: because it involves a lot of pre-requisite knowledge, they have to **think about adjacent angles,**	

how can we connect these, similar triangles were also needed. And once you remember the properties of similar triangles, obviously you would be in a position to know that the sides are proportional, so you just concentrate on the sides which are required. So its important to do that, and also apart from that, in addition you can look at the relationship between diameter and radius. Because you can make some substitutions, if AB is equal to AO, and OB but these are the same so you can just concentrate on one side and be able to find the right answer. For 2c aa can I go to 2c? **Interviewer:** Yes please **Participant:** the knowledge that is needed, because this is a parallelogram, it is important to recall some of the properties of a parallelogram. And because once we joined OH a triangle formed we must also remember the properties of that triangle because we are told that two sides are equal therefore it is an isosceles triangle. So it is important to know some of the properties such as: the base angles are equal. And also it is important to remember adjacent angles that they add up to 180^0. Then because the triangle formed is also an isosceles then you can make some substitutions to concentrate on the part that is needed. Now I was also saying it is important to remember	**-geometric reasoning.** **-geometric concepts** **-geometric relationships.** -geometric reasoning. -geometric concepts -geometric relationships.

the properties of a cyclic quadrilateral for you to answer the question, because if you don't recall the properties of the cyclic quadrilateral then it is easy to lose the direction. Number 4c, again here I was saying that it is important to remember the properties of chord theory. Once you remember that you must also recall the properties of right angled triangles given the sides how can you find the other side. And it is also important to consider, aaa once we have formed triangles using the common side, **in this case we have OB, we can make some substitutions. That is once we find the side OB considering the triangle OPB, and we also find OB using triangle OQB, then it is important to remember that OB is common and therefore the other sides are equal.** **Interviewer:** which means there is some algebra coming in? **Participant:** yes, so you can make some substitutions and simplifications. **Interviewer:** all these questions are on circle geometry as you have seen. And considering the different types of knowledge of theorems needed to solve these, then what does this tell us as teachers about geometric proofs? For our students to be able to understand and apply the proofs what knowledge demands are put on the teacher? **Participant:** I am thinking about the basics, it's very	**-algebraic reasoning**

important to help the students understand the basics. Because as we can see, the concepts are related and you just repeat some of the concepts. So if the students do not understand especially the part that is mostly used in other concept developments, then the student will not understand the concept being developed. So it's important to help them understand the basics. So how can we do that? I think we can try to find the right materials, to be used in the lessons and also we can develop them by giving students a lot of work to practice so that they can understand the concepts.	Geometric concepts
Interviewer: ok thank you very much, let us now talk about the teaching of geometric proofs, is it the same as that of algebra or arithmetic? I remember you were teaching about polygons, was that the same way you teach algebra and arithmetic?	
Participant: no it is not the same, it is a little bit different. In geometry, most of the **times we must use materials, teaching aids must be available.** In addition you mainly deal with mathematical operations like addition of numbers, no letters are used, in algebra we sometimes combine letters and numbers, in geometry there is some sought of algebra, we have letters or variables and number just like in algebra but in addition to these, **there are diagrams which are not many in algebra. So it is**	**Teaching materials** Algebraic reasoning **Interpreting the**

important to help the students to look at the diagrams, how we are using the diagrams, how we are developing concepts from the diagrams, unlike just having the word problem then formulate equation from them in algebra. But here in geometry we cannot do that unless we have the diagrams and they must be proper diagrams not just any diagram. It must be a diagram that represent what has been asked in the word problem. So sometimes students are given a word problem like in question 1, they must know how to translate that into a diagram? Participant: ya exactly, in this case this student cannot answer this question without sketching. But in algebra it is possible, in arithmetic it is possible, you can do that without drawing. **Interviewer:** so which one is easier to teach, geometry, arithmetic or algebra or it's the same? **Participant:** no it is not the same, it depends because geometry to some extent is not easy to teach. **Especially when you have to develop the basic concepts, you reach a point where by aaaa the diagrams, of course you can sketch, but to visualise, to find the correct angle, sides it will involve a lot,** you must put much effort on that for you to help the students. But geometry is enjoyable to teach because as you draw the diagrams, as you try to bring materials and models to class, so geometry is	**diagram.** Representing word problem into a diagram. **Interpreting the diagram**

enjoyable though it pauses a lot of challenges. But if you have the knowledge of this, if you understand then geometry is a branch of mathematics that you can enjoy when teaching. **Interviewer:** I am asking this question because when I was trying to compare the students' performance in geometry, arithmetic and algebra, I found that they are able to perform well or to answer algebra and arithmetic questions better than geometry questions. So I was wondering why this is the case, may be you may have answers? **Participant**: yes of course aaa some of the reasons could be teachers. We as teachers you know as I said that some of the geometry topics require the teacher to draw diagrams, you must find the resources, so mostly most of the students do not come to class with the required materials, you can tell them that tomorrow or next week we are starting geometry so you must come with mathematical instrument, and all the materials must be available in mathematical instrument don't just bring the box. But you find that some will not come with the materials, so they depend on their friends. On the part of the teacher, may be the school does not provide you with the required resources, and you are not creative enough, because if you are creative, then you would be able to	Teaching materials

come up with correct models that would help you to teach the concepts. So the problem could be both the teachers and the students. As teachers, sometimes **we do not give students much work to practice and if we dogive the work to the students sometimes we do not give feedback may be due to other engagements. But as a teacher, I believe that one way of helping students become conversant with proving is to provide them with opportunities for proving by giving the more questions for practice. When the students are practicing to construct the proofs for example, they learn different way of geometric proof construction. After proving a theorem, teachers must give students different types of questions which they can solve by applying that particular proof. The teacher is supposed to make sure that there is variety in the questions that he asks. Some questions can require students to solve and provide a value of an angle using the theorem learnt while others can involve production of proofs using the theorem that they proved. Giving students different types of questions after proving a theorem will help the students to be able to apply the theorem to different situations.** In addition to that, if we give the work some students don't write. Only those who are interested in mathematics strive to write but the rest don't	**Tasks (Practice questions)**

write, as a result come examinations, they fail. In addition there is a problem in our school programme may be I don't know if the school curriculum has too much stuff so our teaching target examinations. The aim of teaching is to make students pass examinations, due to this reason, there are some concepts which are emphasised and others which are not. There are also some questions which are option that is they come in section B. So you look at the trend of the students, what they like to answer during examinations in that section so when you find that they do not like to answer geometry questions then you do not put much emphasis in it. For example 3 dimensions come in section B and is optional so most teacher don't teach it. So teachers also look at what has already been asked in the previous years and concentrate on topics which were not asked because they know that those have a high probability of being asked that year. So you just consider that part forgetting that geometry is connected.	Geometric relationships
Interviewer: So I will still go back to the teaching of geometric theorems. As a teacher what do you emphasise when you are teaching geometric theorems, or you can imagine that you would like to teach students who by the end of the topic have to answer the questions that you have just answered in questionnaire 1, what would you emphasise?	Understand the problem

Participant: aaaaa I think I would emphasise on the **understanding of the theorem itself. Because most of the theorems are in words or statements but for the students to understand the statements it is really a challenge. So I think I would start by trying to help the students understand the concepts by trying to bring all the required resources for understanding the concepts.** I would also make sure not to skip any part by thinking that the students already know it. I would also make sure that I explain each and every point clearly and in detail. I don't know if I have answered you correctly.	**Teaching resources** Explaining
Interviewer: yes you have answered correctly because what I am interested in is the way you are teaching geometric proofs. For example let us talk about the one of the theorems embedded in one of the questions that I gave you. This is the theorem which says that if the line from the centre is perpendicular to the chord, then it will bisect the chord or the other way round. So how would you teach your students how to prove this theorem?	
Participant: I will begin by emphasising that when proving a theorem, teachers must provide students with opportunities to engage in a proving activity. After the activity, the teacher must connect the activity to the theorem. So for example for the question that you have asked me, I would let the students do some	**Proving activity**

constructions and measurements. The major problem when doing this is that sometimes activities that students do are easy like when students measure angles to make a guess like for sum of interior angles of a regular polygon, it is very easy for them to come up with the proof and formulae using figures. But students get confused during formal proving because sometimes the activities that we give students are simple and through the activity, students are able to make correct conclusions, but when we try to connect the activity to the formal way of proving, you find that the formal way is quite complicated. But let me finish answering your question, of course I kind of diverted, so in this case I would for the first time aaa because I know that before this the students have already done construction, then I would aaa first of all **I can draw a line, or tell students to draw a line without constructing the circle. Then I would tell the students to construct an angle by taking the line as or divide the line into two equal parts, may be before dividing the line I would ask the students to construct a 90 degree angle by considering either end. So we have this end and the other end so that we have a right angle. Then they would measure form one end to where we have the angle and from the other end to where we have the angle. Then I would ask them to relate the lengths.**	Bridging empirical and abstract proof. **Proving activity**

Then once we have done this activity, I can take that to the circle so that we can try to draw the circle and take that line as a chord now and also do the same and see if that would help. Can I add another thing? **Interviewer:** Yes go ahead please **Participant:** Okay, the other thing is that sometimes I do have a very good activity that I ask my students to do in class. At the end of that activity students are supposed to make conclusions, and you as a teacher has to bridge the activity and the theorem, that is to connect the two is not easy sometimes. When we fail to make connections between the activity and the theorem, students don't appreciate the value of learning either the activity of the proof. In Geometry, students must prove theorems formally even after proving through an activity. Sometimes the formal way of proving is a bit tougher than the activity; in that case students get confused **Interviewer:** Ok I will now tell you what I was doing when I was teaching geometry, I would firstly write the statement, then draw the diagram, then I would ask the students to mention what we have been given, what we are required to find, then start proving. What I observed was that students could also use what we have been asked to prove in the course of proving. For example if they are required to prove that AB is equal to CD, they would	Bridging empirical and abstract proof.

mention this in they argumentation as well. I saw the same thing in your lessons, so why do you think did students do that or did you notice it yourself? How do we improve on that? **Participant:** Yes it happens aaa as I said that most of the times we teachers rush to cover the syllabus, and we assume that students remember some of the concepts they did in other classes. So that assumption, I think that's the one that brings problems. I think its important to let **them first do what you know is important under that part.** So once you try to find out what the students already know, I think that will be the stepping stone to the development of the other concepts because once they have remembered what they have done, and have also been told what is going to be done, I think they will not be able to say something like what you have said. And also it's important to tell the students that when you are proving, the thing that you have been asked to prove should come at the end. You can give them an example to say like when you have been given a term to define, you do not repeat the same word in the definition; we try to find means to simplify it. So again if we try to let them know that if for definition we do this, in proving we also do not state the statement that we have been asked to prove in the process but it has to come at the end.	**Prior knowledge for a proving task** Last statement is the statement to prove.

Interviewer: Ok so it can come to the students as a rule that what you are required to find should not come in the process of argumentation unless you are concluding the proof? **Participant:** yes but most of the times that's what students really do, they use the required to find in their argumentation process. **Interviewer:** Ok so do you think teachers face any knowledge challenges in geometric proof in terms of content or pedagogy? Of course you have explained why students fail geometry but now in terms of the teacher are there any challenges teachers face? **Participant:** knowledge challenges, yes, because you look at the way we have been trained. The problems come in because if we look at the way we have been trained, from secondary school to college, the way we have been I don't remember if I have been trained much in this**. But in other concepts I think I was trained well but here it is just the way of the teacher to go through the text books, look at the examples,** read them and study them, know the way they have been tackled, just like that but it is not like you have somewhere been told how to do them. If there was that training then it was very minimal. So that the problem, the training the teachers have gone through in terms of content. Then the other problem that we can talk	**Teaching materials**

about is the teacher's interest; some teachers are not very interested in teaching geometry, because geometry teaching involves use of materials. Sometimes you find that students mock you when you walk around with different teaching materials, they try to give you lots of names and this and that so some teachers would not prefer to bring the required materials to the class because of that. And also because the materials are not available then it will be difficult for you to may be deepen your understanding of the concepts, because if you have the materials and you use then it will help to build your understanding and have the required knowledge to teach geometric proof. May be aaaa, no I think this will be like a solution so I should not go into that. **Interviewer:** No nono, go ahead please there is no problem with that. **Participant:** I was thinking about trying to help the teachers especially in geometry by training them on how to **teach geometric proof and other things since you know that the way we have been trained we haven't covered this so it is important to help one another through in-service training. If we could be helped to understand how we are supposed to present geometric proofs in different ways to the students and how to explain them clearly, it would be better for us.**	Teaching materials **-Explaining** **-Representing**

Interviewer: Ok, so you are saying that teachers aer not well prepared to teach geometric proof during pre-service. But still more, could there be anything that you still remember about how you were trained during mathematics education courses that you can share? **Participant:** in mathematics education courses, of course I can remember although we were not paying much attention but there was this other lecture, she was trying to aaaa she was bringing a lot of material to class. I remember when we were trying to prove the Pythagoras theorem, she brought lots of things to class but I could not remember exactly what was been done but because of the materials, we were like trying to come up with the Pythagoras theorem. This is what I can remember, but not of the other theorems. That's the only thing I can remember because it's a long time ago. **Interviewer:** I think we can now talk about questionnaire 2. This one is about students' responses to some of the questions, I also took these from MANEB 2013. So what I gave you was how the students responded to some of the questions that you answered in questionnaire 1 and I asked you to comment on them to see if they answered well or not and what you could do to remedy the errors if you find and. So in general how was it? **Participant:** I will start with the solutions, I found that	Teaching materials

almost all of them were wrong except one which is question one and student 2, that I found to be correct. The answer that the student found was correct though there were some steps that were off target. But s/he found the answer, it seems that the student did not use the wrong information further. **So the answer was correct but some part of the procedure was wrong. The rest of the solutions were wrong.**	**Procedure analysis**
Interviewer: Ok so what do you do when you meet such problems? I know for sure that you do meet errors, what do you do?	
Participant: you know problems in our schools are many, you have problem of lack of teaching and learning resources, large classes, so mostly to be honestly do not do well. Of course some do well but some do not. So with the problems that I have highlighted we try to explain again but its not easy, so you just make corrections and emphases on areas where students errored most.	Teaching materials
Interviewer: Do you do any error analysis, that is do you try to find out the site where the student started to error? How do you mark exercises and tests and what do you do thereafter?	
Participant: as I am marking, I try to find the root to the challenge that students are facing. For example some would write clearly but some would not. For those who	

haven't written, I ask them why they did not write. For **those who write then I try to find the common errors they have made and when doing correction I emphasise on how to avoid such errors by clearly explaining how they were supposed to tackle the questions.** Since as I said already that if it is a form three work, we expect the students to apply knowledge of form one and form two geometry. So when we are marking we don't concentrate on the form one or form three work, we concentrate on what they were supposed to learn at that particular time. So we try to find out if they have grasped what we were teaching them. We do not really emphasise on who got this and who got that, or who did this error and who did that, we just make the corrections. Sometimes I ask the one who got the question right to come and explain to his friends on how they got it. and normally we just concentrate on the solution and not the process. But when it is a geometric proof then we try to follow every step to see how the student is arguing up to the conclusion. We do not interact with the student one by one because they are many. Since we do revision after we have distributed the exercise books, sometimes we find that students are not concentrating during revision. This is because after you have given back the exercise books, students just copy from the ones who got the right answer, so they do not take	**Analysing students' thinking (-Source of error and -Addressing the errors)** Analysing students' thinking (-probing)

the revision time seriously. That is why you could not see me doing the revision most of the times because the students takes it as a worst of time. And also with the problem of time, you find that we do not interact much with the students. For those students who like mathematics, they try to come to staffroom or to book you an appointment and ask you where they did not understand. But for those who do not like it then we do not interact much with them because we have very limited time. **Interviewer:** I will go back to question 2b where we have a student who just conclude that the chords are equal without giving a reason what do you do in that case. Is it necessary to say the reason, or how do you teach them about proving. Do they need to justify every claim or it's not necessary? **Participant:** last year I participated in marking of national examinations. We went through training of how to mark the papers. Mostly they emphasised on the process not the answer at the end, the steps, such that a student can pass mathematics even if all the solutions are wrong but if the steps are correct then what they lose is just a single mark like they get 4 out of 5. So its also important as we are teaching for us teachers to emphasise on the steps. Because you need to know if the student reach to the correct answer	Analysing students' thinking (whole procedure)

by doing right things, **you try to find out if there is logical argument in their solution as sometimes they** may just copy the answer from their friends by peeping. So it is important to train the students to write or explain fully, in that case to give reasons for every claim they make instead of making shortcuts. Of course sometimes we do consider such solutions but it is good to emphases to students to always write reasons for their claims at every step.	**Analysing students' thinking (whole procedure)** -Explaining -Geometric reasoning
Interviewer: You have said that when you are teaching form three you assume that the students know the basics in geometry, that is form one and two geometry which mainly consist of lines and angles, triangles and polygons. What happens when you find out that they do not have that knowledge?	
Participant: in that case then we face challenges because the student cannot understand how to prove circle theorems for example without that knowledge. They fail to **connect ideas because of that gap. So as a teacher it is very difficult to even if you try to explain this or that way, they still don't understand. This means that you have to go back and teach the form one and two work before you start the current work.**	**-Prior knowledge** **-Geometric relationships**
Interviewer: how was your experience like when you were marking the national examinations?	

Participant: of course the in terms of the students presentation, that is the way they were asking questions, there are still more problems. The way they were answering the questions showed that they were not exposed to problem solving as a result the students don't know what to use on a particular question. For example whether to use chord theory, parallel gram, or triangle property, so they just experiment with the theorems and get stuck on the way. The question that requires may be for the students to construct, that is construction may be you find them drawing using may be coins for a circle which has specific length of radius, sometimes they draw using free hand. **Interviewer:** so what did you discuss and wrote in the recent report. I am asking you this because in the past reports, the blame is pushed to the students, saying that they ignore their form 1 and 2 work when they are in form three. So my question was can this really be true, can students decide to ignore information when they know that it is required in that question? **Participant:** not necessarily ignoring the form 1 and 2 work but the issue is that they do **not know how to connect what they learned in those classes because they did not understand it. These students want to pass examinations and they cannot choose to ignore**	Problem solving-skills **Geometric relationships**

information that can help them pass examinations. But the fact is that the students fail to connect what they did at JCE and MSCE and because they are unable to make those connections then we assume that they ignore it. We may say that there is a disjoint between the JCE and MSCE work, we do not teach the concepts as connected things.	
Interviewer	
So what does that mean to you as a mathematics teacher or in this case a geometric proof teacher?	
Participant: I think aaa for us teachers to help our teachers to understand geometric proofs, we must try to teach them the concepts clearly without rushing. So instead of saying this is this is how we do it, we must give them a chance to do activities, try to prove on their own, then we consolidate with the formal way of proving.	Explaining Exploring
After proving a theorem, we also need to give them a chance to practice it, we can give them several questions that demand them to different things with the theorem they have proved.	**Analysing students' thinking**
In that way then students will understand the concepts. We must help them understand the theorem first and then let them apply it in solving problem of different kinds. We also need to help our students to be resourceful. Sometimes you can just give them the theorem and ask them to bring resources which will help them prove the	Tasks

theorem. **For us teachers it is also good that we use resources, if you know that your students will understand better if I use this resource, then it is good that you use the resource.** **Interviewer:** Now you have made to remember this: when you were teaching, you used to your students some questions to answer. The students used to complain that you have given them a different work which is not related to what they had learnt. I hope you remember those situations. Then in that case, what did that mean? **Participant:** yes I remember very well, students mostly do that, they do not like to be given questions that are different from the example that you have given them. So as I said already, we teachers assume that as we are proving then we assume that the students are following the steps, so after proving we giving them questions so that they can apply the proof. The aim is that they should apply the new situation to the questions. We feel that giving them questions related that resemble the example will not make the students think about other situations the theorem might be applied, as you know we cannot do many examples in class since we have limited time. That is why students complain that we have asked them to do something they haven't leant. **Interviewer:** Ok thank you very much, now my last	**Teaching materials** Tasks

question is on in-service training. You said that one of the solutions to improving the teaching of geometric proof is to conduct in-service training on how to teach geometric proof. So imagine that you are asked to explain what you would like to learn in that in-service training, what are you going to say? **Participant:** I think I would like to learn other means of proving theorems that is other means of representing theorems apart from what I do. Because sometimes we just follow how it has been presented in the text books and we do not have other options to choose from. The staff should be the one that has tried other means of proving the theorem, **so this will help me to know different ways of proving theorems so that when students don't understand one approach then I should be able to change the way of proving instead of repeating the same way.** Out of this I will be able to improve and become innovative and come up with different materials for teaching. Sometimes we have questions which need to be translated into diagrams, on this one **I think I also need knowledge of how I can help my students draw diagrams that are a true representation of such statements.** To me I think the main issue is how to present the content in way that students can understand and make connections.	**-Explaining** **-Representing** Teaching materials **Representing word problem into a diagram**