Bands with High Symmetry and Uniform Bands

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BANDS WITH HIGH SYMMETRY AND UNIFORM BANDS

by

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ABSTRACT

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In this dissertation we will be focused on determining classes of bands which are embeddable into some band with high symmetry. It is known that rectangular bands have high symmetry and every semilattice is embeddable into a semilattice with high symmetry. We will try to expand on these classes as much as possible.

We first discuss properties of classes of semigroups in which every semigroup either has high symmetry or is embeddable into a semigroup with high symmetry. We show that normal bands are embeddable into normal bands with high symmetry and also that the bands, free in the class of all bands that can be embedded in some band with high symmetry, are precisely the free bands.

In accordance with techniques in [38], we show an embedding of a normal band into a normal band with high symmetry that preserves much of the original structure. This allows us to look at an embedding of orthodox semigroups for which the band of idempotents is embeddable into a band with high symmetry.

We finish the dissertation by showing the result that every band is embeddable into a uniform band. From this, it will then follow that every orthodox semigroup is embeddable into a bisimple orthodox semigroup.

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Contents

Chapter 1

Introduction

In this dissertation we will be looking at classes of semigroups that are embeddable into semigroups with high symmetry. In the introduction we will start by looking at what it means for a semigroup to have high symmetry. We are interested not only in the definition, but also in the consequences for the structure of the semigroup. In particular, we immediately see that a semigroup with high symmetry is either idempotent free or is a band. We acknowledge the fact that there are idempotent free semigroups with high symmetry, but we will focus on bands throughout the rest of the dissertation.

It is shown in [38] that all semilattices are embeddable into semilattices with high symmetry. In Chapter 2 we will expand on this to show that all normal bands are embeddable into normal bands with high symmetry and all free bands are embeddable into bands with high symmetry. Here we will be finding particular examples of bands which have high symmetry and then using them to show that the aforementioned bands are embeddable into bands with high symmetry.

In [38] it was not just shown that semilattices were embeddable into semilattices with high symmetry, but that it was also possible to maintain much of the structure of the original semilattice through this embedding. In Chapter 3 we revisit normal bands in order to do a similar embedding. While we will have already shown that normal bands are embeddable into bands with high symmetry, here we will be interested in more than just the fact that the new band has high symmetry. That is, we will want that through our embedding we will also get an embedding of the automorphism group of the original band into the automorphism group of the new band and an embedding of the hull of the original band into the hull of the new band. Being able to embed the hulls of the bands allows us to then use our results on orthodox semigroups. In particular, since the new band will have high symmetry, it will be uniform and according to Hall [16],[17],[18],[19] the hull of the new band will be bisimple. We then are able to show that any fundamental generalized inverse semigroup is embeddable into a bisimple fundamental generalized inverse semigroup.

In Chapter 4 we focus on uniformity. In Chapter 3 we will have shown that every normal band is embeddable into a uniform band, but in Chapter 4 we will expand this result to all bands. We use a different approach in Chapter 4 since the approach in Chapter 3 is not applicable to bands other than normal bands. We will again be able to use the results by Hall to establish an embedding of an orthodox semigroup into a bisimple orthodox semigroup.

For all notions in the area of universal algebra we refer to the references [13] and [30]. The standard references for semigroup theory are [3], [4],[21], [22],[28] and [49]. In what follows we recall some of the basic notions involved in this dissertation.

1.1 Automorphisms and high symmetry

Throughout this dissertation we shall be interested in automorphisms of semigroups. It is of interest to look at what we can derive from the fact that $a\alpha = b$ for some automorphism α of S and $a, b \in S$.

We first note that if $a\alpha = b$, then $a^n\alpha = b^n$, so α will induce an isomorphism between the semigroups $\langle a\rangle$ and $\langle b\rangle$, generated by a and b respectively. Such one-generated semigroups are called *cyclic semigroups*. In order to see the relevance of this we recall information about cyclic semigroups.

There exists, up to isomorphism, a unique infinite cyclic semigroup. This infinite cyclic semigroup is isomorphic to the additive semigroup of positive integers, where the number 1 is its (unique) generator. This infinite cyclic semigroup has no idempotents, in other words is *idempotent free*. Otherwise, if the cyclic semigroup $\langle a\rangle$ with generator a is finite, then this cyclic semigroup contains a unique maximal subgroup, which happens to be a cyclic group. The size m of this cyclic group is called the *period* of a and the *index* r of a is the smallest positive integer such that a^r belongs to the maximal subgroup of $\langle a\rangle$. Thus, if a has period m and index r, then a^{mr} is an idempotent, that is, the identity element of the maximal subgroup of $\langle a\rangle$. In conclusion, the cyclic semigroup $\langle a\rangle$ generated by a is infinite if and only if $\langle a\rangle$ is idempotent free. Otherwise $\langle a\rangle$ has a unique idempotent, namely the identity element of its unique maximal cyclic subgroup.

If S is any semigroup, $\alpha \in \mathrm{Aut}S$, where $\mathrm{Aut}S$ is the automorphism group of S, and $a\alpha = b$ for some $a, b \in S$, then the cyclic semigroups $\langle a\rangle$ and $\langle b\rangle$ are isomorphic, so either both $\langle a\rangle$ and $\langle b\rangle$ are infinite and idempotent free or are finite and a and b have the same index and the same period. In the latter case, if r and m are the index and period of a and b, then α maps the unique idempotent a^{mr} of $\langle a\rangle$ to the unique idempotent b^{mr} of $\langle b\rangle$.

We say that S has a *transitive automorphism group* if for any $a, b \in S$ there exists $\alpha \in \mathrm{Aut}S$ such that $a\alpha = b$ (and thus $b\alpha^{-1} = a$). If this is the case, we say that S has *high symmetry*. Let S be a semigroup which has high symmetry and $a \in S$. By what we have seen, for any $b \in S$ we need to have that the cyclic subsemigroups $\langle a\rangle$ and $\langle b\rangle$ of S are isomorphic. Thus if S has an idempotent, say a, then every element b of S is an idempotent, and S consists of idempotents only. A semigroup which consists of idempotents only is called a *band*. Otherwise S is idempotent free.

We summarize our findings in the following result.

Result 1.1.1. *A semigroup with high symmetry is either a band or idempotent free.* □

Since we shall be interested in semigroups which have high symmetry, we hasten to give an example.

Example 1.1.2. Let $(\mathbb{Q}^+, +)$ be the set of positive rational numbers equipped with the usual addition. Clearly $(\mathbb{Q}^+, +)$ is an idempotent free commutative semigroup. For any

$c \in \mathbb{Q}^+$, the mapping

$$\alpha_c : \mathbb{Q}^+ \to \mathbb{Q}^+, \qquad a \to ac,$$

ac being the usual product of rational numbers, is an automorphism of $(\mathbb{Q}^+, +)$ with $\alpha_{\frac{1}{c}}$ being the inverse of α_c in $\mathrm{Aut}\mathbb{Q}^+$. Indeed, if $a, b \in \mathbb{Q}^+$, then $\alpha_{\frac{b}{a}}$ maps a to b. Therefore, the automorphism group $\mathrm{Aut}\mathbb{Q}^+$ acts in a transitive way.

We have given an example of an idempotent free semigroup with high symmetry.

1.2 Prevarieties, quasivarieties, varieties

Our goal will not only be to find semigroups with high symmetry, but also to find classes of semigroups which are embeddable into semigroups with high symmetry. To this end, we give definitions and results pertaining to classes of semigroups.

A (nonempty) class of semigroups is called an *isomorphism class* if this class is closed for the taking of isomorphic copies. For any isomorphism class $\mathbf{K}$ of semigroups, let $\mathbf{H}(\mathbf{K})$ [$\mathbf{S}(\mathbf{K})$] denote the isomorphism class consisting of all the semigroups isomorphic to a homomorphic image [subsemigroup] of a member of $\mathbf{K}$, $\mathbf{P}(\mathbf{K})$ the isomorphism class of all semigroups isomorphic to a semigroup which is the direct product of members of $\mathbf{K}$.

A *prevariety* [*variety*] of semigroups is an isomorphism class $\mathbf{K}$ *closed for the operations* $\mathbf{S}$ and $\mathbf{P}$ [and $\mathbf{H}$], that is, $\mathbf{S}(\mathbf{K}) \subseteq \mathbf{K}$ and $\mathbf{P}(\mathbf{K}) \subseteq \mathbf{K}$ [and $\mathbf{H}(\mathbf{K}) \subseteq \mathbf{K}$]. Using the operators (in sequence) we see that for any isomorphism class $\mathbf{K}$, $\mathbf{SP}(\mathbf{K})$ [$\mathbf{HSP}(\mathbf{K})$] is the smallest prevariety [variety] containing $\mathbf{K}$. If $\mathbf{K}$ is the isomorphism class consisting of all semigroups isomorphic to a given semigroup S, we write $\mathbf{SP}(S)$ and $\mathbf{HSP}(S)$ instead of $\mathbf{SP}(\mathbf{K})$ and $\mathbf{HSP}(\mathbf{K})$. There are several ways to define quasivarieties. We refer to the survey by W. Taylor, which appears as Appendix 4 in [13], for more details. One useful way to define quasivarieties is as follows: a *quasivariety* is a prevariety which is closed for the taking of direct limits.

The notions of prevariety, variety, and quasivariety are useful for classifying classes of semigroups. We now want to show that the class of semigroups that are embeddable into a semigroup with high symmetry is in fact a prevariety. Before we get to this we have the following lemmas.

Lemma 1.2.1. *The class of all semigroups which have high symmetry is closed under* **P**.

Proof. Let $(A_i, i \in I)$ be a family of semigroups with high symmetry and $(a_i : i \in I)$ and $(b_i : i \in I)$ elements of the direct product $\prod_{i \in I} A_i$. Then for every $i \in I$ there exists $\alpha_i \in \mathrm{Aut} A_i$ such that $a_i \alpha_i = b_i$. Then

$$\beta : \prod_{i \in I} A_i \to \prod_{i \in I} A_i, \qquad (c_i : i \in I) \to (c_i \alpha_i : i \in I)$$

is an isomorphism of $\prod_{i \in I} A_i$ which maps $(a_i : i \in I)$ to $(b_i : i \in I)$. $\square$

We will henceforth denote the class of all semigroups embeddable into a semigroup with high symmetry as $\mathcal{E}$.

Lemma 1.2.2. *$\mathcal{E}$ is closed under* **S**.

Proof. Suppose that $A \in \mathcal{E}$. Then there exists an embedding $\varphi : A \to A'$ where A' is a semigroup with high symmetry. Now suppose B is a subsemigroup of A, then the restriction of φ to B, $\varphi|_B : B \to A'$, is an embedding of B into A'. $\square$

Lemma 1.2.3. *$\mathcal{E}$ is closed under* **P**.

Proof. Let $A_i \in \mathcal{E}$ for all $i \in I$. Then for every $i \in I$, A_i can be embedded into A'_i which has high symmetry. Then $\prod_{i \in I} A_i$ is embeddable into $\prod_{i \in I} A'_i$ which has high symmetry by Lemma 1.2.1. $\square$

Result 1.2.4. *$\mathcal{E}$ is a prevariety.*

Proof. Follows directly from Lemmas 1.2.2 and 1.2.3. $\square$

Before discussing if this class of semigroups is a variety or quasivariety we recall the following information.

If $\mathbf{V}$ and $\mathbf{W}$ are semigroup varieties, we then put $\mathbf{V} \leq \mathbf{W}$ if $\mathbf{V}$ is included in $\mathbf{W}$. The semigroup varieties constitute a lattice for this partial order, $\leq$, where for any given semigroup $\mathbf{V}$ and $\mathbf{W}$, $\mathbf{V} \cap \mathbf{W}$ [$\mathbf{V} \vee \mathbf{W} = \mathbf{HSP}(\mathbf{V} \cup \mathbf{W})$] is the *meet* [*join*] of $\mathbf{V}$ and $\mathbf{W}$ in this lattice. For any semigroup variety $\mathbf{V}$, $\mathcal{L}(\mathbf{V})$ denotes the principal ideal generated by $\mathbf{V}$ in this lattice, in other words the *lattice of all subvarieties of* $\mathbf{V}$.

If A is any nonempty set, we let A^+ be the semigroup of all nonempty words over A, the operation being the concatenation of words. The mapping $\iota : A \to A^+$ associates to every $a \in A$ the one-letter word a of A^+. We call (A^+, ι) a *free semigroup on* A since, given any semigroup S and $\varphi : A \to S$ a mapping, there exists a (unique) homomorphism $\overline{\varphi} : A^+ \to S$ such that $\iota\overline{\varphi} = \varphi$. Every prevariety $\mathbf{V}$ of semigroups has free objects: if $\rho_{\mathbf{V}}$ is the smallest congruence on A^+ such that $A^+/\rho_{\mathbf{V}} \in \mathbf{V}$, and $\iota_{\mathbf{V}}$ associates with $a \in A$ the $\rho_{\mathbf{V}}$-class of a, $a\rho_{\mathbf{V}}$, then $(A^+/\rho_{\mathbf{V}}, \iota_{\mathbf{V}})$ is free in $\mathbf{V}$ in the obvious sense.

A relation ρ on a semigroup S is said to be *fully invariant* if for every $a, b \in S$ and any endomorphism φ of S, $a\rho b$ implies $a\varphi\rho b\varphi$. A *fully invariant congruence* would then be a congruence which is also a fully invariant relation. We will now let X denote a countably infinite set whose elements will be called *variables.* It turns out that there exists a one-to-one correspondence between the lattice of semigroup varieties and the lattice of fully invariant congruences on X^+: we associate with any semigroup variety $\mathbf{V}$ the corresponding fully invariant congruence $\rho_{\mathbf{V}}$ on X^+. Then $\mathbf{V} \to \rho_{\mathbf{V}}$ yields an inclusion reversing one-to-one mapping of the lattice of semigroup varieties onto the lattice of fully invariant congruences of X^+. Accordingly, for $u, v \in X^+$ we have that $(u, v) \in \rho_{\mathbf{V}}$ if and only if for every semigroup S and every $\varphi : X \to S$, the above mentioned homomorphism $\overline{\varphi} : X^+ \to S$ for which $\iota\overline{\varphi} = \varphi$ yields a true equality $u\overline{\varphi} = v\overline{\varphi}$ in S. We express this by saying that $u \approx v$ is an *identity satisfied* in $\mathbf{V}$. If $(u_i \approx v_i, i \in I)$ is a family where $u_i, v_i \in X^+$ for every i in the index set I, and $\mathbf{V}$ the variety such that $\rho_{\mathbf{V}}$ is the smallest fully invariant congruence containing the $(u_i, v_i), i \in I$, then we say that $\mathbf{V}$ is *determined by the identities* $u_i \approx v_i, i \in I$. Thus for instance, if $x \in X$, then the variety $\mathbf{B}$ of bands is determined by the single identity $x^2 \approx x$.

A class of semigroups is a semigroup variety if and only if it is determined by a set of identities. The class of all semigroups satisfying a given family of identities is called a *equational class.* We just emphasized that every variety of semigroups is an equational class, and vice versa. This was first shown by Birkhoff. We shall henceforth use the concepts of variety and equational class of semigroups interchangeably.

While a semigroup variety is an equational class, a class of semigroups is a quasivariety if and only if it is determined by a set of implications. An implication is of the form

$$u_1 \approx v_1, ..., u_n \approx v_n \Rightarrow u \approx v$$

where n is finite, $u_1, ..., u_n, v_1, ..., v_n, u, v$ are in X^+. A semigroup S satisfies this implication if for any $\varphi : X \to S$ and homomorphism $\overline{\varphi} : X^+ \to S$ such that $\iota\overline{\varphi} = \varphi$, we have that $u\overline{\varphi} = v\overline{\varphi}$ in S whenever $u_1\overline{\varphi} = v_1\overline{\varphi}, ..., u_n\overline{\varphi} = v_n\overline{\varphi}$ in S. Thus for instance, the quasivariety consisting of all the cancellative semigroups is determined by the implications $xy \approx xz \Rightarrow y \approx z, yx \approx zx \Rightarrow y \approx z$. For our purpose, the implication $x^2 \approx x \Rightarrow y^2 \approx y$ is important: a semigroup S satisfies this implication if $b = b^2$ for all $b \in S$ if there exists $a \in S$ such that $a = a^2$, that is, S is either idempotent free or otherwise a band. Thus the implication $x^2 \approx x \Rightarrow y^2 \approx y$ determines the class consisting of all semigroups which are either idempotent free or bands.

From the results obtained so far we can thus state the next result.

Result 1.2.5. *$\mathcal{E}$ is contained in the quasivariety determined by $x^2 \approx x \Rightarrow y^2 \approx y$.*

It remains to be seen whether the prevariety and the quasivariety mentioned in Result 1.2.5 coincide, and this dissertation does not provide any counterexample to prove that they do not.

However, at this point we can show that $\mathcal{E}$ is not a variety.

Result 1.2.6. *$\mathcal{E}$ is not a variety.*

Proof. Note that $(\mathbb{Q}^+, +) \in \mathcal{E}$ as shown in Example 1.1.2. However, every cyclic group is a homomorphic image of $(\mathbb{Z}^+, +)$, the positive integers under the usual addition and $(\mathbb{Z}^+, +)$ is a subsemigroup $(\mathbb{Q}^+, +)$. Hence if $\mathcal{E}$ were a variety every cyclic group would be embeddable into a semigroup with high symmetry. However, this is not the case since any nontrivial cyclic group contains at least one idempotent and one nonidempotent. Therefore any nontrivial cyclic group is not contained in the quasivariety determined by $x^2 \approx x \Rightarrow y^2 \approx y$. □

We also have the following.

Result 1.2.7. *Suppose that* $\mathbf{V}$ *is a variety which is contained in* $\mathcal{E}$*. Then* $\mathbf{V}$ *is a subvariety of the variety of bands.*

Proof. By way of contradiction suppose that A is a semigroup in $\mathbf{V}$ such that A contains an element, a, which is not idempotent. Then if $\langle a \rangle$ is finite we have already seen that $\langle a \rangle$ contains an idempotent, hence $\langle a \rangle$ and therefore $\mathbf{V}$ is not contained in $\mathcal{E}$. Otherwise $\langle a \rangle$ is isomorphic to the positive integers under the usual addition, so, by the proof of Result 1.2.6, we see that $\mathbf{V}$ is not contained in $\mathcal{E}$. □

Result 1.2.7 therefore motivates us to focus on bands.

1.3 Bands

In this section we recall useful information pertaining to bands.

1.3.1 Properties and descriptions of bands

For S a semigroup, let $E(S)$ be the set of idempotents of the semigroup S. The relation $\leq$ on $E(S)$ defined by: for $a, b \in E(S)$,

$$a \leq b \Leftrightarrow ab = ba = a,$$

yields a partially ordered set $(E(S), \leq)$. If S is a band, then clearly $E(S) = S$, so $(S, \leq)$ yields a partially ordered set. We call $\leq$ the *natural partial order* on S.

An element $e \in E(S)$ is called *primitive* if e is minimal with respect to the natural partial order on $E(S)$. A band for which every element is primitive is called a *rectangular* *band*. We will denote the class of all rectangular bands by **RB**. Note that for a band $aba \leq a$, so in a rectangular band $aba = a$ for all a and b. In fact we get that S is a rectangular band if and only if $aba = a$ for all $a, b \in S$.

A commutative band is called a *semilattice*. Therefore a semigroup S is a semilattice if $a^2 = a$ and $ab = ba$ for all $a, b \in S$. The class of all semilattices will be denoted **SL**. One easily verifies that if S is a semilattice, then $(S, \leq)$ is a partially ordered set for which greatest common lower bounds always exist, namely the greatest common lower bound of a and b is ab. Similarly if we have a partially ordered set $(S, \leq)$ for which greatest common lower bounds always exist we have a corresponding semilattice $(S, \cdot)$ where $a \cdot b =$ greatest common lower bound of a and b.

We call b a *right zero* of the semigroup S if $ab = b$ for all $a \in S$. Note that if a is a right zero of S, then $a^2 = a$, so a is idempotent. If S consists entirely of right zero elements, we call S a *right zero band*. We define *left zero* and *left zero band* in a left right dual way. We will denote the class of all right [left] zero bands by **RZ** [**LZ**].

We shall define the relations $\leq_l, \leq_r$ on a band S by: for $a, b \in S$

$$a \leq_l b \Leftrightarrow ab = a,$$

$$a \leq_r b \Leftrightarrow ba = a.$$

Note that $\leq = \leq_r \cap \leq_l$. We define the relations $\mathcal{R} = \leq_r \cap (\leq_r)^{-1}$ and $\mathcal{L} = \leq_l \cap (\leq_l)^{-1}$. We then define the join of $\mathcal{L}$ and $\mathcal{R}$ in the lattice of equivalence relations to be $\mathcal{D}$. The relations $\mathcal{R}, \mathcal{L}$ and $\mathcal{D}$ are traditionally called the *Green relations* on the band S. Notice that S is a right [left] zero band if and only if $\mathcal{R}[\mathcal{L}] = S \times S$. From this it can be seen that $\mathcal{R}-[\mathcal{L}-]$classes of any band S are precisely the maximum right [left] zero subbands of S. A very nice feature concerning bands is as follows.

Result 1.3.1. *For a band S,*

(i) $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$ where $\circ$ denotes the composition of equivalence relations,

(ii) $\mathcal{D}$ is a congruence relation on S such that $S/\mathcal{D}$ is a semilattice,

(iii) every $\mathcal{D}$-class of S constitutes a rectangular band,

(iv) if ρ is a congruence relation on S such that S/ρ is a semilattice, then $\mathcal{D} \subseteq \rho$.

Proof. We note that it is well known that the Green's relations $\mathcal{L}$ and $\mathcal{R}$ commute. $(ii) - (iv)$ are a specialization of Clifford's structure theorem for completely regular semigroups [5]. We provide a proof for completeness.

(i) Suppose that $x(\mathcal{L} \circ \mathcal{R})y$ in S, then there exists $z \in S$ such that $x\mathcal{L}z\mathcal{R}y$. Note that $x = xz = x(yz)$. Then $(xy)x = xy(xyz) = (xy)z = x$ and $x(xy) = xy$ therefore $x\mathcal{R}xy$. Furthermore $y = zy = (zx)y$. Now $(xy)y = xy$ and $y(xy) = (zxy)xy = z(xy) = y$, so $xy\mathcal{L}y$ and $x\mathcal{R}xy\mathcal{L}y$. By duality we see that $\mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$. Since $\mathcal{L}$ and $\mathcal{R}$ commute and $\mathcal{D}$ is the join of $\mathcal{L}$ and $\mathcal{R}$, (i) follows.

(ii) Note that if $a, b \in S$, then $ab\mathcal{R}aba\mathcal{L}ba$, so $ab\mathcal{D}ba$. Now suppose that $a\mathcal{D}c$ and $b\mathcal{D}d$ with $a, b, c, d \in S$. Then $ab = acabdb\mathcal{D}cacdbd = cd$, so $\mathcal{D}$ is a congruence. Then, since $a^2 = a\mathcal{D}a$, we have (ii).

(iii) Let $a\mathcal{D}b$ such that $b \leq a$. Recall that $a = aba$. However, $aba = b$ since $b \leq a$, hence $b = a$ and every element of the $\mathcal{D}$-class is primitive.

(iv) Here we must show that if ρ is a congruence such that $xy\rho yx$ and if $a\mathcal{D}b$, then $a\rho b$. Note that since $a\mathcal{D}b$, we get that $a = aba\rho bab = b$. □

By virtue of Result 1.3.1, we see that the Green relation $\mathcal{D}$ on the band S is the *least semilattice congruence* on S and the $\mathcal{D}-$classes are precisely the maximal rectangular subbands. Putting $Y = S/\mathcal{D}$, we shall write $S = S(Y, \mathcal{D}_\alpha)$ where Y is the greatest semilattice homomorphic image of S and the $\mathcal{D}_\alpha$, $\alpha \in Y$, the $\mathcal{D}-$classes of S. We call Y the *structure semilattice* of S, and we shall say that S is a semilattice of rectangular bands $\mathcal{D}_\alpha$, $\alpha \in Y$. In general it is not possible to reconstruct the original band S with only knowledge of the structure semilattice and the $\mathcal{D}-$classes, we shall show an example of this in Figure 1.1. However, knowing the structure semilattice and $\mathcal{D}-$classes of S does tell us much about the band S, as we will see throughout this section.

A *characteristic relation* on S is a relation on S which is invariant for every automorphism of S. That is, if $\alpha \in \mathrm{Aut}S$ and θ a characteristic relation, then $a\theta b \Rightarrow a\alpha\theta b\alpha$.

Result 1.3.2. *$\leq, \mathcal{R}, \mathcal{L}$ and $\mathcal{D}$ are all characteristic relations.*

Proof. This follows from routine verification. We will therefore supply the proof for $\mathcal{R}$ and leave the others to the reader. Let $a\mathcal{R}b$, and let $\alpha \in \mathrm{Aut}S$. Then $a\alpha b\alpha = ab\alpha = b\alpha$ and $b\alpha a\alpha = ba\alpha = a\alpha$, so $a\alpha\mathcal{R}b\alpha$. □

In particular, since $\mathcal{D}$ is characteristic, we can see that an automorphism α of S will induce an automorphism on the structure semilattice $S/\mathcal{D}$. Thus we have

Result 1.3.3. *If a band has high symmetry, then its structure semilattice has high symmetry.*

We will say that a band is a *regular band* if both $\mathcal{L}$ and $\mathcal{R}$ are congruence relations and will denote the class of all regular bands by **ReB**. From the definitions of the Green relations $\mathcal{L}$ and $\mathcal{R}$ on a band S it follows that $\mathcal{L} \cap \mathcal{R} = \iota_S$ is the equality on S. If S is a regular band, then S is a subdirect product of $S/\mathcal{L}$ and $S/\mathcal{R}$. More can be said about this subdirect product however. For any band S denote the $\mathcal{L} - [\mathcal{R}-, \mathcal{D}-]$ classes of $a \in S$ by $L_a[R_a, D_a]$. Thus, if S is a regular band

$$\begin{tikzcd} & S/\mathcal{L} \arrow[dr] & \\ S \arrow[ur] \arrow[dr] & & S/\mathcal{D}, \\ & S/\mathcal{R} \arrow[ur] & \end{tikzcd} \qquad \begin{tikzcd} & L_a \arrow[dr] & \\ a \arrow[ur] \arrow[dr] & & D_a \\ & R_a \arrow[ur] & \end{tikzcd}$$

is a commutative diagram with

$$S \to S/\mathcal{L} \times S/\mathcal{R}, \qquad a \to (L_a, R_a)$$

an embedding of S into $(S/\mathcal{L}) \times (S/\mathcal{R})$. The image of the embedding is easily identified: (L_a, R_b) belongs to the image under the above mapping if and only if $\mathcal{D}_a = \mathcal{D}_b$. We express this fact by saying that the regular band S is the *spined product* of $(S/\mathcal{L}) \times (S/\mathcal{R})$ *over the structure semilattice* $S/\mathcal{D}$ of S and write $S \cong (S/\mathcal{L}) \rtimes (S/\mathcal{R})$. In general, if S is a semigroup, ρ_1, ρ_2 congruences on S such that $\rho_1 \cap \rho_2$ is the equality relation on

S, and $\rho = \rho_1 \vee \rho_2$ in the lattice of congruences Con$S$, then $S \to S/\rho_1 \times S/\rho_2$ is an embedding. We say that S is a *spined product of* S/ρ_1 and S/ρ_2 *over* S/ρ if the image of this embedding consists precisely of the $(a\rho_1, b\rho_2)$ such that $a\rho = b\rho$.

If S is a regular band, then $S/\mathcal{L}$ and $S/\mathcal{R}$ are bands which have trivial $\mathcal{L}-$ and $\mathcal{R}-$relations respectively, that is, bands for which respectively $\mathcal{R} = \mathcal{D}$ and $\mathcal{L} = \mathcal{D}$. Such bands are called *right regular bands* and *left regular bands* and the class of all such bands will be denoted **RRB** and **LRB**, respectively.

A relation ρ on a semigroup S is called *left [right] compatible* if for every a, b, c, with $a\rho b$, we have that $ca\rho cb[ac\rho bc]$. A relation is *compatible* if it is both left and right compatible. We will call S a *normal band* if $\leq$ is compatible. The class of all normal bands will be denoted **NB**. We shall note in Section 1.3.2 that all normal bands are regular bands. We then define *right [left] normal bands* to be normal bands that are also right [left] regular. We denote this class of bands by **RNB** [**LNB**]. From what we had seen for regular bands, we get that every normal band is the spined product of the left normal band $S/\mathcal{R}$ and the right normal band $S/\mathcal{L}$ over the structure semilattice of S.

Let $S = S(Y; D_\alpha)$ be a normal band which is a semilattice Y of the rectangular bands D_α, $\alpha \in Y$. For $a \in S$, aSa is a semilattice, and it is easy to see that this semilattice intersects each $\mathcal{D}$-class in at most one element. In fact, if $a \in D_\alpha$ and $\beta \leq \alpha$ in Y, then D_β contains a unique $b \in aSa$. This allows us to define a mapping $\varphi_{\alpha,\beta} : D_\alpha \to D_\beta$, $a \to b$. Such a mapping $\varphi_{\alpha,\beta}$, $\beta \leq \alpha$ in Y, is a homomorphism, called a *structure homomorphism* of S. This system $(\varphi_{\alpha,\beta}, \beta \leq \alpha \text{ in } Y)$ is a transitive system, that is,

(i) $\varphi_{\alpha,\alpha}$ is the identity transformation on D_α for every $\alpha \in Y$,

(ii) if $\gamma \leq \beta \leq \alpha$ in Y, then $\varphi_{\alpha,\gamma} = \varphi_{\alpha,\beta}\varphi_{\beta,\gamma}$.

Conversely, given a semilattice Y, pairwise disjoint rectangular bands D_α, $\alpha \in Y$ and a transitive system $(\varphi_{\alpha,\beta}, \beta \leq \alpha \text{ in } Y)$ of homomorphisms $\varphi_{\alpha,\beta} : D_\alpha \to D_\beta$, we then can define a multiplication on the disjoint union $S = \cup_{\alpha \in Y} D_\alpha$. We define this multiplication by putting $ab = a\varphi_{\alpha,\alpha\beta} b\varphi_{\beta,\alpha\beta}$, for $a \in D_\alpha$ and $b \in D_\beta$, where the product in the right hand side is to be performed within the rectangular band $D_{\alpha\beta}$. As a result $S = S(Y, D_\alpha)$

is a normal band which is a semilattice Y of its maximal rectangular subbands $D_\alpha, \alpha \in Y$, and the structure homomorphisms for S are precisely the given $\varphi_{\alpha,\beta}$. The normal band which results from the construction described here is called a *strong composition* and is denoted $S = S[Y; D_\alpha, \varphi_{\alpha,\beta}]$. Thus, a band is a normal band if and only if it is (isomorphic to) a strong composition of rectangular bands. We also refer to [51], [52] for the notion of a strong composition in a broader context: strong compositions are therefore also called *Płonka sums*.

At this point we will introduce notation that will become useful in Chapter 3. As we saw above a normal band will have a partial order such that if $a \in D_\alpha$ and $\beta \leq \alpha$ in Y, then D_β contains a unique $b \leq a$. In this way we can define a normal band by giving its structure semilattice, $\mathcal{D}$-classes and an appropriate natural order. We give this more formally in the following result.

Result 1.3.4. *Let S be a band with structure semilattice Y and $\mathcal{D}$-classes $D_\alpha, \alpha \in Y$. If there is a natural partial order defined on S such that for every $a \in D_\alpha$ and $\beta \leq \alpha$ (in Y) there exists a unique $b \in D_\beta$ such that $b \leq a$, then S is a normal band.*

Proof. We already have the structure semilattice and the $\mathcal{D}-$classes defined, so we need only define the structure homomorphism. For $\beta \leq \alpha$ (in Y) and $a \in D_\alpha$ let $a\varphi_{\alpha,\beta} = b$ such that $b \leq a$ and $b \in D_\beta$. That these structure homomorphisms are well-defined follows directly from the assumptions of the result. A routine verification will show that the $\varphi_{\alpha,\beta}$ form a transitive system. $\square$

Due to Result 1.3.4 we can state the natural partial order to define a normal band. If we do this we will denote this normal band as $S = S[Y; D_\alpha, \leq]$. If we further assume that the $\mathcal{D}-$classes are in fact $\mathcal{R}-$classes, that is, the $\mathcal{L}-$relation is trivial, then S will be a right normal band. Similarly if the $\mathcal{D}-$classes are in fact $\mathcal{L}-$classes, then S will be a left normal band. A useful consequence of this notation is that we can draw an equivalent of a *Hasse diagram* for normal bands. In a Hasse diagram the elements are denoted by dots (or are labeled). We then note that one element, a, is less than another, b, if it is connected by slanted lines in such a way that we can move from b to a over

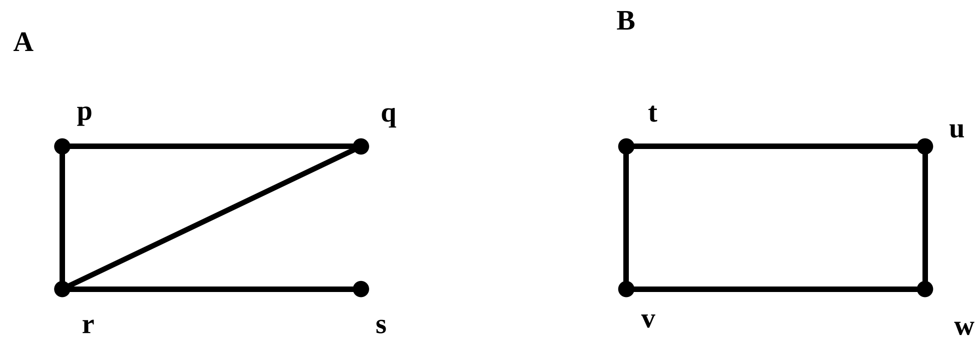

Figure 1.1

these slanted lines always moving down. For right normal bands we use horizontal lines to show $\mathcal{R}-$related elements. That is, $a\mathcal{R}b$ if there is a sequence of all horizontal lines connecting a and b. Suppose that $a \in R_\alpha$ and $b \in R_\beta$, then to find ab, we first find $\alpha\beta$ by finding the greatest common lower bound of α and β as we usually would from the Hasse diagram and then we find $c \in R_{\alpha\beta}$ such that $c \leq b$ in R. Figure 1.1 shows the Hasse diagrams of two right normal bands, A and B. In A and in B we have the same $\mathcal{R}-$classes and structure semilattice. However, in A we have that $sq = r$ whereas in B we have $wu = w$.

Note that if we instead let horizontal lines denote the $\mathcal{L}-$classes the same diagrams would denote left normal bands. In Figure 1.1 we would then have $qs = r$ in A and $uw = w$ in B. In order to draw a normal band we would have to introduce a different technique of drawing lines for both $\mathcal{R}$ and $\mathcal{L}$ classes. Because of this, the diagram will become more cumbersome, and we will therefore refrain from doing so here.

1.3.2 Varieties of bands

We will begin by defining some bands which will prove useful throughout this dissertation. We define T to be the trivial semigroup and Y_2 to be the semigroup consisting of two distinct elements, an identity element and a zero. Also, R_2 $[L_2]$ is the semigroup consisting of two right [left] zero elements. For any semigroup S let S^1 be the semigroup with identity element 1 where $S^1 = S$ if S has an identity element, otherwise S^1 is S with the identity element 1 adjoined. We define S^0 in a similar manner for the zero element 0.

We denote by $\mathbf{T}$ the variety of all trivial semigroups. Then many of the classes of bands mentioned in the previous subsection are varieties. In fact we get that

$$\begin{aligned}
&\mathbf{T} = \mathbf{HSP}(T) = [x \approx y],\\
&\mathbf{RZ} = \mathbf{HSP}(R_2) = [x \approx yx],\\
&\mathbf{LZ} = \mathbf{HSP}(L_2) = [x \approx xy],\\
&\mathbf{SL} = \mathbf{HSP}(Y_2) = [x^2 \approx x, xy \approx yx],\\
&\mathbf{RB} = \mathbf{HSP}(R_2 \times L_2) = [x \approx xyx],\\
&\mathbf{RNB} = \mathbf{HSP}(R_2^0) = [x \approx x^2, xyz \approx yxz],\\
&\mathbf{LNB} = \mathbf{HSP}(L_2^0) = [x \approx x^2, xyz \approx xzy],\\
&\mathbf{NB} = \mathbf{HSP}(R_2^0 \times L_2^0) = [x^2 \approx x, xyzx \approx xzyx],\\
&\mathbf{RRB} = \mathbf{HSP}(R_2^1) = [x \approx x^2, xy \approx yxy],\\
&\mathbf{LRB} = \mathbf{HSP}(L_2^1) = [x \approx x^2, xy \approx xyx],\\
&\mathbf{RQNB} = \mathbf{HSP}(R_2^1 \times L_2) = [x \approx x^2, xyz \approx xzyz],\\
&\mathbf{LQNB} = \mathbf{HSP}(L_2^1 \times R_2) = [x \approx x^2, xyz \approx xyxz],\\
&\mathbf{ReB} = \mathbf{HSP}(R_2^1 \times L_2^1) = [x \approx x^2, xyzx \approx xyxzx].
\end{aligned}$$

The Hasse diagram in Figure 1.2 exhibits inclusion for the varieties of regular bands.

Proof. We will not show all of these, so we refer the reader to [50] for details. We will, however, show that $\mathbf{RRB} = [x \approx x^2, xy \approx yxy]$ and $\mathbf{NB} = [x^2 \approx x, xyzx \approx xzyx]$.

First recall that we defined a right regular band to be a band in which $\mathcal{L}$ and $\mathcal{R}$ are congruences and $\mathcal{L}$ is trivial. If S is a right regular band, then $x^2 = x$ for all $x \in S$, since S a band. Now let $x, y \in S$. Then note that $(xy)(yxy) = xyxy = xy$

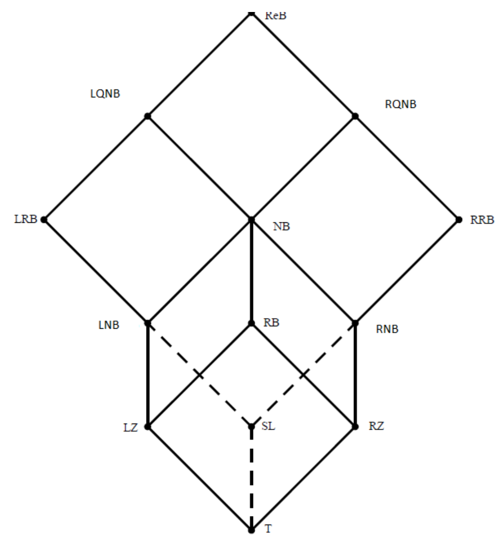

Figure 1.2

and $(yxy)(xy) = y(xy)(xy) = yxy$, hence $xy\mathcal{L}yxy$. However, the $\mathcal{L}-$relation is trivial, so $xy = yxy$. Hence $\mathbf{RRB} \subseteq [x^2 \approx x, xy \approx yxy]$. Now let S be in the equational class $[x^2 \approx x, xy \approx yxy]$. First note S is a band. If $x\mathcal{L}y$, then $x\mathcal{D}y$, so $y = yxy = xy = x$, hence $\mathcal{L}$ is trivial. Now since $\mathcal{D} = \mathcal{L} \circ \mathcal{R}$ it follows that $\mathcal{R} = \mathcal{D}$, and, since $\mathcal{D}$ is a congruence on all bands, it follows that $\mathcal{R}$ is a congruence, so S is a right regular band.

Now recall that we defined a normal band as a band in which $\leq$ is a compatible relation. Let S be a normal band. Then $x^2 = x$ for all $x \in S$ since S a band. Furthermore, for any $x, y, z \in S$, $xyzx \leq x$ and $xzyx \leq x$ and $xyzx\mathcal{D}xzyx$. But then since $\leq$ is compatible, $xyzx = xyzx(xzyx)xyzx \leq (x)xzyx(x) = xzyx$. Since the $\mathcal{D}-$classes are rectangular bands, it follows that $xyzx = xzyx$. Now let S be in the equational class $[x^2 \approx x, xyzx \approx xzyx]$. Then S is a band. Now let $x, y, z \in S$ and $y \leq z$. We need to show that $xy \leq xz$ and $yx \leq zx$. Note that $(xy)(xz) = xxyz = xy = xxzy = (xz)(xy)$, so $xy \leq xz$. Similarly $yx \leq zx$, so $\leq$ is a compatible relation and S is a normal band. $\square$

We will refer to normal bands extensively in Chapters 2 and 3 and will talk more about regular bands in Chapter 2. While the complete lattice of varieties of bands was described independently in [1], [7],[8], [9], we will instead refer to [46] for a description of the lattice of varieties of bands using Mal'cev products.

The notion of a *Mal'cev Product* of varieties allows for the construction of quasivarieties from varieties. If $\mathbf{U}$ and $\mathbf{V}$ are varieties of bands, then the *Mal'cev product* $\mathbf{U} \circ \mathbf{V}$ *of* $\mathbf{U}$ *and* $\mathbf{V}$ *within* $\mathbf{B}$ is the isomorphism class consisting of all bands S on which there exists a congruence relation ρ such that $S/\rho \in \mathbf{V}$ and such that ρ-classes belong to $\mathbf{U}$. Thus, $\mathbf{B} = \mathbf{RB} \circ \mathbf{SL}$ since every band is a semilattice of rectangular bands. It turns out that for any varieties $\mathbf{V}$ and $\mathbf{W}$ of bands, $\mathbf{V} \circ \mathbf{W}$ is a quasivariety of bands. An alternative to our definition of the Mal'cev product is to find the smallest variety containing the Mal'cev product as we defined it. In [25] it is shown that $\mathbf{V} \circ \mathbf{W}$ is a variety if $\mathbf{V} \subseteq \mathbf{RB}$. Since we will be interested in Mal'cev products of the type, the two definitions will coincide.

Starting with the previously defined varieties $\mathbf{T}, \mathbf{LZ}, \mathbf{RZ}, \mathbf{SL}$ and using the join $\vee$ and the Mal'cev product, $\circ$, one finds all band varieties properly contained in $\mathbf{B}$: the following are the *join irreducible* elements containing $\mathbf{SL}$ in the lattice $\mathcal{L}(\mathbf{B})$ of all band varieties, that is, the band varieties that cannot be written as a finite join of band varieties properly contained in them (see [46]):

$$\begin{aligned}&\mathbf{SL}, \mathbf{LZ} \circ \mathbf{SL}, \mathbf{RZ} \circ \mathbf{SL}, \mathbf{LZ} \circ (\mathbf{RZ} \circ \mathbf{SL}), \mathbf{RZ} \circ (\mathbf{LZ} \circ \mathbf{SL}),\\ &\qquad \mathbf{LZ} \circ (\mathbf{RZ} \circ (\mathbf{LZ} \circ \mathbf{SL})), \mathbf{RZ} \circ (\mathbf{LZ} \circ (\mathbf{RZ} \circ \mathbf{SL})), \ldots\end{aligned} \tag{1.1}$$

$$\begin{aligned}&\mathbf{LZ} \circ (\mathbf{RZ} \vee \mathbf{SL}), \mathbf{RZ} \circ (\mathbf{LZ} \vee \mathbf{SL}), \mathbf{LZ} \circ (\mathbf{RZ} \circ (\mathbf{LZ} \vee \mathbf{SL})),\\ &\qquad \mathbf{RZ} \circ (\mathbf{LZ} \circ (\mathbf{RZ} \vee \mathbf{SL})), \ldots\end{aligned} \tag{1.2}$$

Here $\mathbf{LZ}\vee\mathbf{SL}$ [$\mathbf{RZ}\vee\mathbf{SL}$] is the variety of left [right] normal bands, $\mathbf{LZ}\vee\mathbf{SL}\vee\mathbf{RZ}$ the variety of normal bands, $\mathbf{LZ}\circ\mathbf{SL}$ [$\mathbf{RZ}\circ\mathbf{SL}$] the variety of left [right] regular bands, $(\mathbf{LZ}\circ\mathbf{SL})\vee(\mathbf{RZ}\circ\mathbf{SL})$ the variety of regular bands.

We note that the Mal'cev product is neither commutative nor associative even if all products are varieties. An example that Mal'cev products are not commutative is that while we know $\mathbf{B} = \mathbf{RB}\circ\mathbf{SL}$, we instead get that $\mathbf{NB} = \mathbf{SL}\circ\mathbf{RB}$. To show that $\mathbf{NB} \subsetneq \mathbf{B}$

note that R_2^1 is a band but is not normal since, if we denote by a and b the two elements of the right zero band, we notice that $1ab1 = ab \neq ba = 1ba1$. For associativity, we look at $\mathbf{LZ} \circ (\mathbf{RZ} \circ \mathbf{SL}) = \mathbf{LZ} \circ \mathbf{RRB} = [zxy \approx zxyzyxy]$ the variety of left semiregular bands. However, $(\mathbf{LZ} \circ \mathbf{RZ}) \circ \mathbf{SL} = \mathbf{RB} \circ \mathbf{SL} = \mathbf{B}$. We refer to Figure 1.3 and [46] for verification that these are not equal.

In Figure 1.3 we show the lattice of varieties of bands using a Hasse diagram. Note that although we do not mark all varieties of bands, those not labeled can be found by taking the join of at most two of the varieties that are labeled. This figure can also be found in [46].

Amalgamation of bands provides for a construction of new bands in terms of given ones and will play an important role into our investigations in Chapter 3. An isomorphism class $\mathbf{K}$ of semigroups is said to have the *strong amalgamation property* if, for any family of semigroups $(A_i, i \in I)$ of $\mathbf{K}$ and U a subsemigroup of each A_i, there exists a semigroup B in $\mathbf{K}$ and one-to-one homomorphisms $\varphi_i : A_i \to B$, $i \in I$, such that

(i) the restrictions $\varphi_i|_U = \varphi_j|_U$ coincide for all $i, j \in I$,

(ii) $A_i\varphi_i \cap A_j\varphi_j = U\varphi_i$ for all $i, j \in I$, with $i \neq j$, in I.

Following [24] we know that the variety $\mathbf{NB}$ of normal bands is the largest variety of bands which has the strong amalgamation property and every subvariety of $\mathbf{NB}$ has the strong amalgamation property. This property has been put to use in [38] for the purpose of constructing semilattices which have a transitive automorphism group, and will be used again in Chapter 3 of the present dissertation when we deal with normal bands. We need not rely on [24] for what we shall do when constructing normal bands in Chapter 3. Our proofs will be self-sufficient, and we shall use [24] only when we declare that our methods do not extend to varieties beyond the variety of normal bands.

1.3.3 Uniformity

In this section we look at what it means for a band to be uniform. In order to do this, we will need the following definitions.

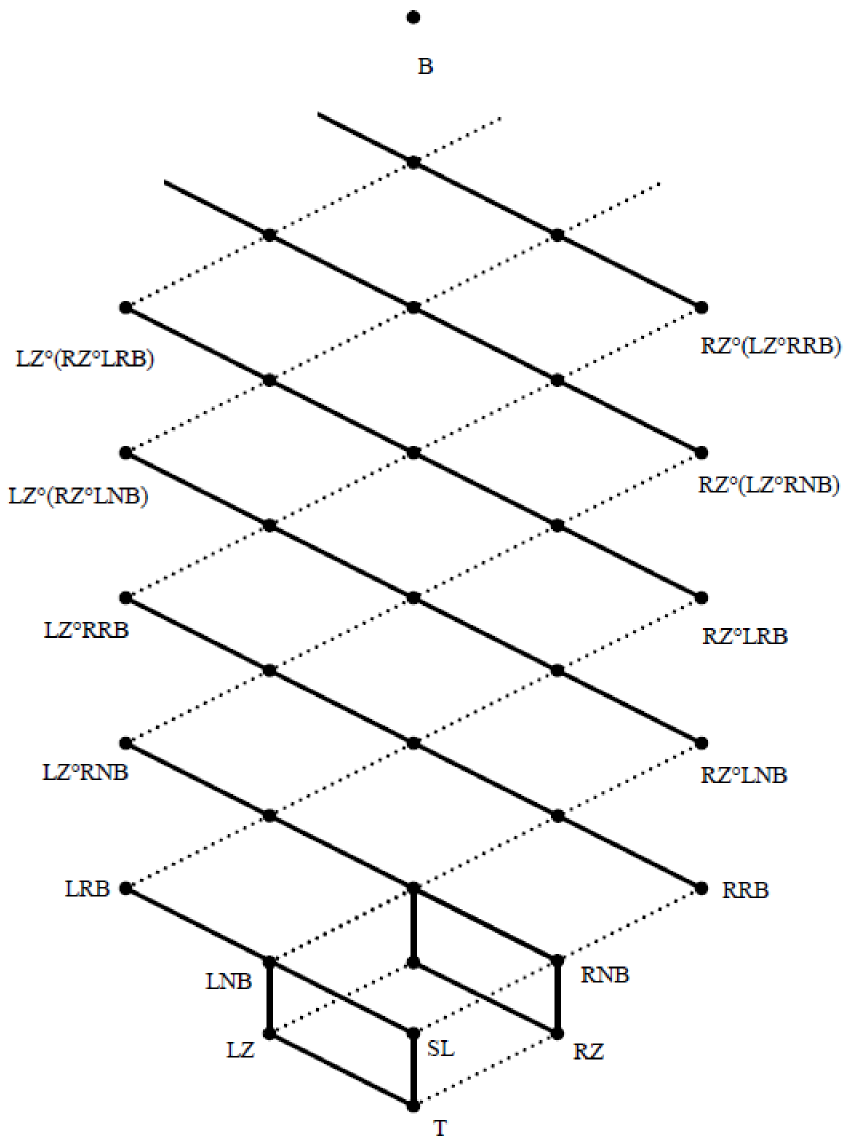
B
LZ°(RZ°LRB)
RZ°(LZ°RRB)
LZ°(RZ°LNB)
RZ°(LZ°RNB)
LZ°RRB
RZ°LRB
LZ°RNB
RZ°LNB
LRB
RRB
LNB
RNB
LZ
SL
RZ
T

Figure 1.3

Let S be a band and $\leq$ the natural order on S. A nonempty subset I of S is called a *filter* if whenever $a, b \in S$ with $a \in I$ and $a \leq b$, then $b \in I$. A nonempty subset I of S is called a *convex subset* of S if whenever $a, b, c \in S$ with $a, b \in I$ and $a \leq c \leq b$, then $c \in I$. The concept of an *order ideal* is the dual of the concept of a filter, but we shall in addition require that order ideals are subsemigroups. More precisely, a subsemigroup I of S is called an order ideal of S if whenever $a, b \in S$ with $a \in I$ and $b \leq a$, then $b \in I$. If $a \in S$, then the smallest order ideal containing a is

$$\begin{aligned} aSa &= \{aba | b \in S\} \\ &= \{c \in S | c \leq a\}. \end{aligned}$$

We also use the notation $(a] = aSa$ and call $(a]$ the *principal order ideal generated by a.* Since $(a] = aSa$ is a subsemigroup of S which has identity element a, $(a]$ is sometimes also called a *local submonoid* of S.

An ideal of the band S is also an order ideal of S, but the converse is not true. Indeed, if we let a and b denote the two elements of R_2, then $\{a\}$ is the smallest order ideal of R_2 containing a but is not an ideal of R_2: the smallest ideal of R_2 containing a is R_2 itself. Accordingly, the notions of *principal ideal* and *principal order ideal* do not coincide in general. The following emphasizes that for semilattices, however, the two concepts coincide.

Result 1.3.5. *For semilattices the notions of principal ideal and principal order ideal coincide.*

Proof. Let S be a semilattice and $a \in S$. The principal ideal generated by a is SaS and the principal order ideal generated by a is aSa. That $SaS = aSa$ follows from the commutativity of S and from the fact that a is an idempotent. $\square$

Result 1.3.6. *For a semilattice the notions of ideal and order ideal coincide.*

Proof. Let S be a semilattice. We already noted that every ideal of S is an order ideal of S. Let I be an order ideal of S, $a \in I$ and $b \in S$. Then $ab = ba = aba \leq a$, thus $ab = ba \in I$. It follows that I is an ideal of S. $\square$

If I, J are order ideals of S and $\varphi : I \to J$ is an isomorphism, we call φ an *order ideal isomorphism* (*oi-isomorphism*) of S. If $a, b \in S$ and $(a] \cong (b]$, then an isomorphism $\varphi : (a] \to (b]$ is called a *partial isomorphism.* We now define the *uniformity relation* on S to be $\mathcal{U}_S = \{(a, b) \in S : (a] \cong (b]\}$, and we say a band S is *uniform* if $\mathcal{U}_S = S \times S$. At this point we make the following observation.

Result 1.3.7. *If S has high symmetry, then S is uniform.*

Proof. Suppose that $a, b \in S$. Since S has high symmetry there exists an automorphism $\varphi : S \to S$ which maps a to b. Note that $\leq$ is characteristic, so $\varphi|_{(a]}$ embeds $(a]$ into $(b]$. Furthermore $\varphi^{-1}|_{(b]}$ embeds $(b]$ into $(a]$ and $\varphi|_{(a]}$ and $\varphi^{-1}|_{(b]}$ are inverse embeddings, so $(a] \cong (b]$. Since a and b arbitrary in S, it follows that $\mathcal{U}_S = S \times S$. $\square$

The converse of Result 1.3.7 is not necessarily true. In order to show this we have the following example.

Example 1.3.8. $(\mathbb{Z}^-, \min)$ is a uniform semilattice, but does not have high symmetry.

Proof. Note that $(j] \cong (\mathbb{Z}^-, \min)$ for every $j \in (\mathbb{Z}^-, \min)$, so $(\mathbb{Z}^-, \min)$ is uniform. However; the only automorphism of $(\mathbb{Z}^-, \min)$ is the identity mapping, so if $j \neq k$ there does not exist an automorphism of $(\mathbb{Z}^-, \min)$ that maps j to k. Hence $(\mathbb{Z}^-, \min)$ does not have high symmetry. $\square$

In Chapter 3 we shall need the notion of a retract ideal of a semilattice L. If S is a semigroup and φ an endomorphism of S such that φ fixes every element of $S\varphi$, then we say that $S\varphi$ is a *retract* of S. A *retract ideal* of a semilattice L is an (order) ideal of L that is also a retract of L.

Result 1.3.9. *An ideal I of a semilattice L is a retract ideal of L if and only if for every $i \in L$ there exists $i' \in L$ such that $iL \cap I = i'L$.*

Proof. First suppose that I is a retract ideal of L. Then there exists $\varphi : L \to I$ such that $j\varphi = j$ for every $j \in I$. Now let $i \in L$. Then $j \in iL \cap I$ if and only if $j \leq i$ and $j\varphi = j$. But if this is the case, $j = j\varphi = ij\varphi = i\varphi j\varphi = i\varphi j$, so $j \in iL \cap I$ if and only if $j \leq i\varphi$. Hence $iL \cap I = (i\varphi)L$. Letting $i' = i\varphi$ we have our result.

Now suppose that I is an ideal of L such that for every $i \in L$ there exists $i' \in L$ such that $iL \cap I = i'L$. Then $\varphi : S \to I$ that maps $i \to i'$ is routinely shown to be an endomorphism which fixes every element of I, so I is a retract ideal. $\square$

Every principal ideal gL of L is a retract ideal of L since for every $i \in L$, $iL \cap gL = igL$. Also L itself is a retract ideal of L since for every $i \in L$, $iL \cap L = iL$. For any band S we denote by $\underline{R}_S$ the set of oi-isomorphisms α of S such that the oi-isomorphism $\widetilde{\alpha}$ induced by α on the structure semilattice L of S has the property that both dom$\widetilde{\alpha}$ and im$\widetilde{\alpha}$ are retract ideals of L. If α is a partial isomorphism of S, then $\widetilde{\alpha}$ is a partial isomorphism of L, so dom$\widetilde{\alpha}$ and im$\widetilde{\alpha}$ are principal ideals of L, therefore $\alpha \in \underline{R}_S$. Further, if $\alpha \in \mathrm{Aut}S$, then $\widetilde{\alpha} \in \mathrm{Aut}L$ and $L = \mathrm{dom}\widetilde{\alpha} = \mathrm{im}\widetilde{\alpha}$, so that again $\alpha \in \underline{R}_S$. In other words, $\mathrm{Aut}S \subseteq \underline{R}_S$.

If I and J are retract ideals of the semilattice L, then $I \cap J$ is again a retract ideal of L. To see this, first notice that $I \cap J \neq \emptyset$ since $IJ = I \cap J$ with $I \neq \emptyset$ and $J \neq \emptyset$. Also, given $i \in L$, there exists j and $j' \in L$ such that $iL \cap I = jL$ and $iL \cap J = j'L$, so that $iL \cap (I \cap J) = (iL \cap I) \cap (iL \cap J) = jL \cap j'L = jj'L$ is indeed a principal ideal of L.

We shall need the following elementary result. As always, juxtaposition denotes composition of partial transformations.

Lemma 1.3.10. *If B is a band, $\alpha \in \underline{R}_B$ and $\gamma \in \mathrm{Aut}B$, then $\alpha\gamma \in \underline{R}_B$.*

Proof. Since the composition of oi-isomorphisms of B is an oi-isomorphism of B, it follows that $\alpha\gamma$ is an oi-isomorphism of B. If suffices to prove that dom$\widetilde{\alpha\gamma}$ and im$\widetilde{\alpha\gamma}$ are retract ideals of the structure semilattice L of B. Certainly domα =dom$\alpha\gamma$, thus dom$\widetilde{\alpha}$ =dom$\widetilde{\alpha\gamma}$ is a retract of ideal of L since $\alpha \in \underline{R}_B$. We proceed to investigate im$\widetilde{\alpha\gamma}$.

We have

$$\begin{aligned} j \in \mathrm{im}\widetilde{\alpha\gamma} &\Leftrightarrow D_y = j \qquad \text{for some } y \in \mathrm{im}\alpha\gamma \\ &\Leftrightarrow D_{x\gamma} = j \qquad \text{for some } x \in \mathrm{im}\alpha \\ &\Leftrightarrow j = i\widetilde{\gamma} \qquad \text{for } i = D_x \text{ and some } x \in \mathrm{im}\alpha \\ &\Leftrightarrow j = i\widetilde{\gamma} \qquad \text{for some } i \in \mathrm{im}\widetilde{\alpha} \\ &\Leftrightarrow j \in (\mathrm{im}\widetilde{\alpha})\widetilde{\gamma}. \end{aligned}$$

Therefore $\mathrm{im}\widetilde{\alpha\gamma} = (\mathrm{im}\widetilde{\alpha})\widetilde{\gamma} = \mathrm{im}\widetilde{\alpha}\widetilde{\gamma}$.

For $j, l \in L$, put $i = j\widetilde{\gamma}^{-1}$ and $k = l\widetilde{\gamma}^{-1}$. Since $\mathrm{im}\widetilde{\alpha}$ is a retract ideal of L, there exists $m \in L$ such that $kL \cap \mathrm{im}\widetilde{\alpha} = mL$. Then

$$\begin{aligned} j \in lL \cap \mathrm{im}\widetilde{\alpha\gamma} &\Leftrightarrow j \in lL \cap \mathrm{im}\widetilde{\alpha}\widetilde{\gamma} \\ &\Leftrightarrow i \in kL \cap \mathrm{im}\widetilde{\alpha} \\ &\Leftrightarrow i \in mL \\ &\Leftrightarrow j \in (m\widetilde{\gamma})L, \end{aligned}$$

so that $lL \cap \mathrm{im}\widetilde{\alpha\gamma} = m\widetilde{\gamma}L$. We conclude that $\mathrm{im}\widetilde{\alpha\gamma}$ is a retract ideal of L, as required. $\square$

1.4 Orthodox semigroups

Recall that a semigroup S is called a *regular semigroup* if for all $a \in S$, there exists at least one $a' \in S$ such that $aa'a = a$; if this is the case, then for every $a \in S$, there exists $a' \in S$ such that $aa'a = a$ and $a'aa' = a$, in which case we call a and a' *mutually inverse elements of* S. Regular semigroups contain idempotents: if S is a regular semigroup, $a \in S$, and a, a' mutually inverse in S, then aa' and $a'a$ are idempotents. Thus certainly the set $E(S)$ of idempotents of S is nonempty. An *orthodox semigroup* is a regular semigroup S for which $E(S)$ constitutes a subsemigroup (a subband). An orthodox semigroup S is an *inverse semigroup* if $E(S)$ is a semilattice and is a *[left, right] generalized inverse* *semigroup* if $E(S)$ forms a [left, right] normal band. Furthermore an orthodox semigroup in which the set of idempotents forms a rectangular band is a *rectangular group*.

Given any orthodox semigroup S and $a \in S$, $Sa = \{sa|s \in S\}$ is the smallest left ideal of S containing a. We put $a\mathcal{L}b$ for $a, b \in S$ if $Sa = Sb$. Then $\mathcal{L}$ is an equivalence relation on S. We hasten to assert that this terminology and notation conforms with what we did in Section 1.3.1: if S is a band, then S is in particular an orthodox semigroup, that is, an orthodox semigroup for which $S = E(S)$, and in this band (orthodox semigroup) S, we have for any $a, b \in S$:

$$\begin{aligned} a\mathcal{L}b \text{ in } S &\Leftrightarrow Sa = Sb, \\ &\Leftrightarrow ab = a, ba = b, \\ &\Leftrightarrow a \leq_l b \text{ and } b \leq_l a. \end{aligned}$$

For the orthodox semigroup S, we define the left-right dual of $\mathcal{L}$ by $\mathcal{R}$. Just as $\mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$ for bands, it is also well known that $\mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$ for all orthodox semigroups. The relations $\mathcal{L}$ and $\mathcal{R}$ on an orthodox semigroup S are called the *Green relations* $\mathcal{L}$ and $\mathcal{R}$ on S and $\mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L} = \mathcal{D}$ is called the Green relation $\mathcal{D}$ on the orthodox semigroup S. One verifies that if S is an orthodox semigroup, then the $\mathcal{L}-[\mathcal{R}-]$relation on S induces the $\mathcal{L}-[\mathcal{R}-]$relation on the band $E(S)$, and therefore the $\mathcal{D}$-relation on $E(S)$ is contained in the $\mathcal{D}$-relation on S.

An orthodox semigroup S is called *bisimple* if $\mathcal{D} = S \times S$. If B is the band of idempotents of an orthodox semigroup S and $e\mathcal{D}f$ in S, then there exist mutually inverse elements $a, a' \in S$ such that $e = aa'$ and $f = a'a$ and $eBe \to fBf$, $x \to a'xa$ is an isomorphism of bands. Therefore, if $B = E(S)$ is the band of idempotents of a bisimple orthodox semigroup S, B is uniform. In order to prove that, conversely, every uniform band is the band of idempotents of some bisimple orthodox semigroup, we shall make use of the hull introduced by Hall, as reviewed below. Then in particular, every band which has high symmetry will be the band of idempotents of some bisimple orthodox semigroup.

In what follows, B will be a band. We shall give an outline of Hall's representation [16],[17],[18],[19] which he generalized for regular semigroups in [20]. We follow the notation of [42]. Hall's work generalizes that of Munn [31],[32] and for regular semigroups is equivalent to work done by Grillet [14],[15] and Nambooripad [34].

We denote the set of partial isomorphisms of B by $\underline{T}_B$. If B is a band and α, β oi-isomorphisms of B such that $\alpha\beta$ is nonempty, then $\alpha\beta$ is an oi-isomorphism of B.

For $e\mathcal{D}f$ in B we denote by $\pi(e,f)$ the partial isomorphism

$$\pi(e,f) : eBe \to fBf, \qquad x \to fxf.$$

Thus if $e\mathcal{R}f$ in B, then

$$\pi(e,f) : eBe \to fBf, \qquad x \to xf,$$

and if $e\mathcal{L}f$ in B, then

$$\pi(e,f) : eBe \to fBf, \qquad x \to fx.$$

Thus, if $e\mathcal{D}f$ in B, then $e\mathcal{R}ef\mathcal{L}f\mathcal{R}fe\mathcal{L}e$ in B and

$$\pi(e,f) = \pi(e,ef)\pi(ef,f) = \pi(e,fe)\pi(fe,e)$$

belongs to $\underline{T}_B$. In general, for $e,f \in B$, $\pi_{e,f} \in \underline{T}_B$ where

$$\begin{aligned}\pi_{e,f} &= \pi(efe,fef)\\ &= \pi(efe,ef)\pi(ef,fef)\\ &= \pi(efe,fe)\pi(fe,fef),\end{aligned}$$

and

$$\mathrm{dom}\pi_{e,f} = efBfe, \qquad \mathrm{im}\pi_{e,f} = feBef.$$

If B is a band, then define a multiplication $\cdot$ on $\underline{T}_B$ by the following. If $\alpha, \beta \in \underline{T}_B$ with $\alpha : eBe \to fBf$ and $\beta : gBg \to hBh$, then

$$\alpha \cdot \beta = \alpha\pi_{f,g}\beta,$$

where again the juxtaposition of partial isomorphisms denotes the composition of partial transformations. If B is a band, then $\underline{T}_B$ is an orthodox semigroup whose idempotents form the band consisting of the $\pi_{e,f}$, $e, f \in B$ (see [17], [19], [42]). In fact, the set of idempotents of $\underline{T}_B$ is given by

$$\{\pi_{e,f} | e, f \in B\} = \{\pi(e,f) | e\mathcal{D}f \text{ in } B\}.$$

Let B be a band, $B/\mathcal{D} = L$ the structure semilattice of B, and D_e, $e \in B$, the rectangular components of B, that is, the elements of L. For any oi-isomorphism α of B,

$$\widetilde{\alpha} : L \to L, \qquad D_e \to D_{e\alpha}, \qquad e \in \mathrm{dom}\alpha$$

is an oi-isomorphism of L, and in particular, if $\alpha \in \underline{T}_B$, then $\widetilde{\alpha} \in \underline{T}_L$, and if $\alpha \in \mathrm{Aut}B$, then $\widetilde{\alpha} \in \mathrm{Aut}L$. The mappings

$$\underline{T}_B \to \underline{T}_L \qquad \alpha \to \widetilde{\alpha},$$

$$\mathrm{Aut}B \to \mathrm{Aut}L, \qquad \alpha \to \widetilde{\alpha},$$

are homomorphisms.

Let B be a band, and introduce an equivalence relation κ_B on $\underline{T}_B$ as follows. For

$\alpha, \beta \in \underline{T}_B$ with $\alpha : eBe \to fBf$ and $\beta : gBg \to hBh$ put

$$\alpha \kappa_B \beta \Leftrightarrow e\mathcal{R}g, f\mathcal{L}h \text{ and } \widetilde{\alpha} = \widetilde{\beta}.$$

Then κ_B is a congruence relation on the orthodox semigroup $\underline{T}_B$ and we use the notation $T_B = \underline{T}_B/\kappa_B$. The κ_B-class of $\alpha \in \underline{T}_B$ will be denoted $\overline{\alpha}$ (see [17], [19], [42]). The set of idempotents of the orthodox semigroup T_B is $\left\{\overline{\pi(e,e)} | e \in B\right\}$, and $e \to \overline{\pi(e,e)}$ yields an isomorphism of B onto the band of idempotents of T_B. If in particular $B = L$ is a semilattice, then κ_L is the equality on L and $T_L \cong \underline{T}_L$; we prefer to write T_L instead of $\underline{T}_L$, the inverse semigroup also known as the *Munn semigroup* of L (see [31]). If in general B is a band, then we call $T_B = \underline{T}_B/\kappa_B$ the *hull of the band* B which is due to Hall.

A *full regular subsemigroup* of a regular semigroup S is a regular subsemigroup of S which contains all the idempotents of S. A regular semigroup S is called *fundamental* if the equality on S is the greatest idempotent separating congruence on S. Let S be an orthodox semigroup which has B as its band of idempotents. For any $a \in S$ and a' an inverse of a in S, the mapping

$$\theta_{a',a} : aa'Baa' \to a'aBa'a, \qquad e \to a'ea$$

belongs to $\underline{T}_B$. Hence $\overline{\theta_{a',a}} \in T_B$, and

$$\theta : S \to T_B, \qquad a \to \overline{\theta_{a',a}}$$

is a well defined homomorphism which induces the greatest idempotent separating congruence on S and $S\theta$ is a full regular subsemigroup of T_B (see [17], [19], [42]). The mapping θ is called the *fundamental representation* of S, and the result quoted here is called the *Hall representation theorem* for orthodox semigroups. This representation is faithful if S is fundamental. In fact, an orthodox semigroup which has the band B as its band of idempotents is fundamental if and only if it can be embedded as a full regular subsemigroup into T_B. Further, $\underline{T}_B$ and T_B are fundamental themselves (see [19], [42]). Accordingly, T_B is called the *fundamental hull* of the band B. We note that with the notation given above, for every $e \in B$, $\overline{\pi_{e,e}} = \overline{\pi(e,e)} = \overline{\theta_{e,e}}$, and $B \to T_B$, $e \to \overline{\pi(e,e)}$ is an isomorphism of B onto the band of idempotents of T_B.

For any $e, f \in B$ we have

$$\begin{aligned} e\mathcal{U}_B f &\Leftrightarrow eBe \cong fBf \\ &\Leftrightarrow eBe = \mathrm{dom}\alpha, \qquad fBf = \mathrm{im}\alpha \qquad \text{for some } \alpha \in \underline{T}_B \\ &\Leftrightarrow \overline{\pi(e,e)}\mathcal{D}\overline{\pi(f,f)} \qquad \text{in } T_B. \end{aligned}$$

In particular, B is uniform if and only if T_B is bisimple [17]. Furthermore, if B has high symmetry, then B is uniform and T_B is bisimple.

Hall's fundamental representation is the crucial ingredient for structure theorems of orthodox semigroups, the investigation of which was opened up in Yamada's papers [62], [63].

Chapter 2

Embedding Techniques

In this chapter we give several techniques for finding bands which have a transitive automorphism group. We then show that every band free in a variety generated by a band with identity element can be embedded into a band which has a transitive automorphism group and which generates the same band variety. We also show every normal band can be embedded into a normal band which has a transitive automorphism group.

In Section 2.1 we prove that every [left, right] normal band can be embedded into a [left, right] normal band with high symmetry. This generalizes what can be done for semilattices. Then in Section 2.2 we give examples of regular bands with high symmetry and show that every free [left, right] regular band can be embedded into a [left,right] regular band with high symmetry. We then define $\mathcal{D}$-*transitivity* and prove that $\mathcal{D}$-transitivity can only hold in the context of regular bands. In Section 2.3 we show every band variety generated by a band with identity element is generated by a band with high symmetry. Furthermore we show that every free band is embeddable in a band with high symmetry and we identify an infinity of band varieties, $\mathbf{V}$, where every free object in $\mathbf{V}$ can be embedded into a band of $\mathbf{V}$ which has high symmetry.

2.1 Normal bands

Recall that in Section 1.3.2 we saw that R_2^0 [L_2^0] generates the variety of right [left] normal bands. We now set out to prove that R_2^0 [L_2^0] can be embedded into a right [left] normal band with high symmetry.

The following construction is interesting in its own right and is more powerful than need be for our purpose. $\mathbb{Z}$ will stand for the chain of integers with the natural ordering.

Construction 1. We let B be a nontrivial band with zero 0 and q be a fixed nonzero element of B. We let $\overline{B}$ consist of mappings of the following kind. For every $\overline{a} \in \overline{B}$ there exists $i \in \mathbb{Z}$, called the *mark* of $\overline{a}$, and $\overline{a} : \{i - n | n \in \mathbb{Z}, n \geq 0\} \to B \setminus \{0\}$, such that only finitely many of the values $\overline{a}(i-n)$ are distinct from q. If we denote the $\overline{a}(i-n)$ by a_{i-n}, then we can briefly denote $\overline{a}$ by the *marked sequence* $\overline{a} = (a_{i-n})_i = (a_i, a_{i-1}, ...)_i$: the index i reminds us of the mark of $\overline{a}$. Note that if $\overline{a} = (a_i, a_{i-1}, ...)$ the mark can easily be seen to be i, so using the mark here is redundant. However, we use this notation to avoid ambiguity about whether (a_{i-n}) is being used to denote a term in $\overline{a}$ or the element $\overline{a}$.

We now define a product in $\overline{B}$. Let $\overline{a} = (a_{i-n})_i$ and $\overline{b} = (b_{j-n})_j$ be elements of $\overline{B}$. There exists a largest $k \leq min(i, j)$ such that $a_{k-n} = b_{k-n}$ for all $n \geq 1$ since only a finite number of entries in $\overline{a}$ and $\overline{b}$ are different from q. We consider two cases, depending on whether or not $a_k b_k$ equals 0 in B.

$$\text{If } a_k b_k \neq 0 \text{ in } B, \text{ then}$$
$$\overline{a}\overline{b} = (d_{k-n})_k$$
$$\text{where } d_k = a_k b_k$$
$$\text{and } d_{k-n} = a_{k-n} = b_{k-n}, \qquad n \geq 1. \tag{2.1}$$

$$\text{If } a_k b_k = 0 \text{ in } B, \text{ then}$$
$$\overline{a}\overline{b} = (d_{k-1-n})_{k-1}$$
$$\text{where } d_{k-1-n} = a_{k-1-n} = b_{k-1-n}, \qquad n \geq 0. \tag{2.2}$$

Thus if $a_k b_k \neq 0$ in B, then the mark of $\overline{a}\overline{b}$ is k, whereas if $a_k b_k = 0$ in B, then the mark of $\overline{a}\overline{b}$ is $k-1$.

Lemma 2.1.1. *Let B be a nontrivial band with zero, 0, and let $\overline{B}$ be as in Construction 1. Let the multiplication on $\overline{B}$ be given by* (2.1) *and* (2.2)*. Then*

(i) $\overline{B}$ is a band,

(ii) B can be embedded into $\overline{B}$,

(iii) B and $\overline{B}$ generate the same band variety.

Proof. In $\overline{B}$ we consider the elements $\overline{a} = (a_{i-n})_i$, $\overline{b} = (b_{j-n})_j$ and $\overline{c} = (c_{l-n})_l$ with marks i, j, and l, respectively. We let $m \leq \min(i,j,l)$ be the largest integer such that $a_{m-n} = b_{m-n} = c_{m-n}$ for all $n \geq 1$. To prove associativity of the multiplication in $\overline{B}$, it suffices to show that

$$\begin{aligned} (\overline{a}\overline{b})\overline{c} = \overline{a}(\overline{b}\overline{c}) = (e_{m-n})_m \\ \text{where } e_m = a_m b_m c_m \qquad \text{in } B \\ e_{m-n} = a_{m-n} = b_{m-n} = c_{m-n}, \qquad n \geq 1 \\ \text{if } a_m b_m c_m \neq 0 \qquad \text{in } B, \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} (\overline{a}\overline{b})\overline{c} = \overline{a}(\overline{b}\overline{c}) = (e_{m-1-n})_{m-1} \\ \text{where } e_{m-1-n} = a_{m-1-n} = b_{m-1-n} = c_{m-1-n}, \qquad n \geq 0 \\ \text{if } a_m b_m c_m = 0 \qquad \text{in } B. \end{aligned} \tag{2.4}$$

We first consider the situation where $a_m b_m c_m = 0$ in B. If $a_m b_m = 0$ in B, then $a_m \neq b_m$, whereas $a_{m-n} = b_{m-n} = c_{m-n}$ for all $n \geq 1$. In this case we have $\overline{a}\overline{b} = (\overline{a}\overline{b})\overline{c} = \overline{e}$, where $\overline{e} = (e_{m-1-n})_{m-1}$ is as described in (2.4). Otherwise, $a_m b_m \neq 0$ in B and then

$a_m b_m \neq c_m$ because $a_m b_m c_m = 0$. Then again $(\overline{a}\overline{b})\overline{c} = \overline{e}$, where $\overline{e} = (e_{m-1-n})_{m-1}$ is as described in (2.4). By symmetry, we proved that (2.4) holds true.

We next consider the situation where $a_m b_m c_m \neq 0$ in B. Let k be as in Construction 1. If $k > m$, then $a_m = b_m \neq c_m$: this follows from the way m is defined. In this case $(\overline{a}\overline{b})\overline{c} = \overline{e}$ where $\overline{e} = (e_{m-n})_m$ as described in (2.3). Otherwise $k = m$ and $\overline{a}\overline{b} = \overline{d} = (d_{m-n})_m$ as described in (2.1). Clearly then $(\overline{a}\overline{b})\overline{c} = \overline{e}$ where $\overline{e} = (e_{m-n})_m$ as in (2.3). By symmetry, we proved that (2.3) holds true.

We proved associativity. Applying (2.1) we see that $\overline{B}$ is a band. We define a mapping $\varphi : B \to \overline{B}$ by the following. For $a \neq 0$ in B, let $a\varphi = \overline{a} = (a_{1-n})_1$ where $a_1 = a$ and $a_{1-n} = q$ for all $n \geq 1$. Also put $0\varphi = (q, q, q, ...)_0$, that is, the element of $\overline{B}$ which has mark 0 and where all entries equal q. One readily verifies that φ embeds B isomorphically into $\overline{B}$. We proved (i) and (ii).

In order to prove (iii) it suffices to prove that B and $\overline{B}$ satisfy the same identities. Since we already know that B is a subband of $\overline{B}$, it suffices to show that every semigroup identity $u \approx v$ which is satisfied in B is also satisfied in $\overline{B}$. In the following we take such an identity $u \approx v$ which is satisfied in B, and we note that, since B is a nontrivial band with zero 0, $u \approx v$ is a regular identity. We need the following generalization of (2.3) and (2.4).

Let $\overline{a}_1, ..., \overline{a}_p \in \overline{B}$ be such that $\overline{a}_l = (a^{(l)}_{i_l}, a^{(l)}_{i_l-1}, \ldots, a^{(l)}_{i_l-n}, \ldots)_{i_l} = (a^{(l)}_{i_l-n})_{i_l}$ for $1 \leq l \leq p$. Let $m \leq \min(i_1, ..., i_p)$ be the largest integer such that $a^{(l)}_{m-n} = a^{(l')}_{m-n}$ for all $1 \leq l, l' \leq p$ and all $n \geq 1$. Then

$$\begin{aligned} \overline{a}_1 ... \overline{a}_p = & (e_{m-n})_m \\ \text{where } e_m = & a^{(1)}_m ... a^{(p)}_m \qquad \text{in } B \\ e_{m-n} = & a^{(l)}_{m-n} \qquad \text{for all } n \geq 1, 1 \leq l \leq p \\ \text{if } & a^{(1)}_m ... a^{(p)}_m \neq 0 \qquad \text{in } B, \end{aligned} \tag{2.5}$$

$$\begin{aligned} \overline{a}_1 ... \overline{a}_p = & (e_{m-1-n})_{m-1} \\ & \text{where } e_{m-1-n} = a^{(l)}_{m-1-n} \qquad \text{for all } n \geq 0, 1 \leq l \leq p \\ & \quad \text{if } a^{(1)}_m ... a^{(p)}_m = 0 \qquad \text{in } B. \end{aligned} \tag{2.6}$$

To prove (2.5) and (2.6) we use induction on p and the fact that associativity has been established already. The basis of our proof by induction (the case p=2) follows from the definition of the product in $\overline{B}$. Now that associativity has been established, the induction step can be proved following the case by case proof for $p = 3$ from the case $p = 2$ as given above.

Let $u \approx v$ be any semigroup identity which is satisfied in B, and let $x_1, ..., x_p$ be the variables which occur in u. Recall that $x_1, ..., x_p$ are then also the variables which occur in v. Let $\overline{a}_1, ..., \overline{a}_p$ be *any* elements of $\overline{B}$. In order to verify that $u \approx v$, or explicitly $u(x_1, ..., x_p) \approx v(x_1, ..., x_p)$, is satisfied in $\overline{B}$, it suffices to show that in $\overline{B}$ we have $\overline{u} = \overline{v}$ where $\overline{u} = u(\overline{a}_1, ..., \overline{a}_p)$ and $\overline{v} = v(\overline{a}_1, ..., \overline{a}_p)$ are obtained from u and v by substituting the x_l by the corresponding $\overline{a}_l$. Let us adopt the notation for the $\overline{a}_1, ..., \overline{a}_p$ which was used for (2.5) and (2.6). Then in particular,

$$\begin{aligned} & u(a^{(1)}_m, \ldots, a^{(p)}_m) \neq 0 \qquad \text{in } B \\ & \quad \text{iff } a^{(1)}_m \ldots a^{(p)}_m \neq 0 \qquad \text{in } B \\ & \quad \text{iff } v(a^{(1)}_m, \ldots, a^{(p)}_m) \neq 0 \qquad \text{in } B \end{aligned}$$

since these three elements of B are $\mathcal{D}$-related in B. If this is the case, then by (2.5),

$$\overline{u} = \overline{u}(\overline{a}_1, ..., \overline{a}_p) = \overline{e} = \overline{v}(\overline{a}_1, ..., \overline{a}_p) = \overline{v}$$

where

$$\overline{e} = (e_{m-n})_m$$
$$\text{where } e_m = u(a_m^{(1)}, \ldots, a_m^{(p)}) = v(a_m^{(1)}, \ldots, a_m^{(p)})$$
$$(\text{since } u \approx v \text{ is satisfied in } B)$$
$$\text{and } e_{m-n} = a_{m-n}^{(l)}, \qquad \text{for all } n \geq 1, 1 \leq l \leq p.$$

Otherwise, we have in B that

$$u(a_m^{(1)}, \ldots, a_m^{(p)}) = 0 = v(a_m^{(1)}, \ldots, a_m^{(p)}),$$

and then according to (2.6),

$$\overline{u} = u(\overline{a}_1, \ldots, \overline{a}_p) = \overline{e} = v(\overline{a}_1, \ldots, \overline{a}_p) = \overline{v}$$
$$\text{where } \overline{e} = (e_{m-1-n})_{m-1}$$
$$\text{with } e_{m-1-n} = a_{m-1-n}^{(l)} \qquad \text{for all } 1 \leq l \leq p \text{ and } n \geq 0.$$

Therefore $\overline{B}$ satisfies $u \approx v$, and we conclude that B and $\overline{B}$ generate the same band variety. We proved (iii). □

Lemma 2.1.2. *Let B and $\overline{B}$ be as in Construction 1. Then the following holds.*

(i) For every $l \in \mathbb{Z}$, let α_l be an automorphism of B, such that for only finitely many $l \in \mathbb{Z}$, $q\alpha_l \neq q$. Then

$$\alpha : \overline{B} \to \overline{B}, \qquad \overline{a} = (a_{i-n})_i \to \overline{a}\alpha = (a_{i-n}\alpha_{i-n})_i \tag{2.7}$$

is an automorphism of $\overline{B}$.

(ii) The mapping

$$\beta : \overline{B} \to \overline{B}, \qquad \overline{a} = (a_i, ..., a_{i-n}, ...)_i = (a_{i-n})_i$$
$$\to \overline{a}\beta = (e_{i+1}, ..., e_{i+1-m}, ..)_{i+1}$$
$$\textit{where } e_{i+1-n} = a_{i-n} \qquad \textit{for all } n \geq 0 \tag{2.8}$$

is an automorphism of $\overline{B}$.

Proof. Let $\overline{a} = (a_{i-n})_i$, $\overline{b} = (b_{j-n})_j \in \overline{B}$, and let $k \leq \min(i, j)$ be the largest integer such that $a_{k-n} = b_{k-n}$ for all $n \geq 1$. We consider the mapping α given by (2.7). Certainly α is well defined since $q\alpha_n \neq q$ for only finitely many $n \in \mathbb{Z}$ and each α_n fixes the zero 0 of B. Clearly then α is a permutation of $\overline{B}$. We have $\overline{a}\alpha = (a_{i-n}\alpha_{i-n})_i$, $\overline{b}\alpha = (b_{j-n}\alpha_{j-n})_j$ and $k \leq min(i, j)$ is the largest integer such that $a_{k-n}\alpha_{k-n} = b_{k-n}\alpha_{k-n}$ for all $n \geq 1$. Therefore (2.1) applies if and only if $(a_k\alpha_k)(b_k\alpha_k) \neq 0$ in B, and then

$$(\overline{a}\alpha)(\overline{b}\alpha) = (d_{k-n}\alpha_{k-n})_k = (\overline{a}\overline{b})\alpha$$

where the d_k and d_{k-n}, $n \geq 1$, are as described in (2.1). Alternatively, case (2.2) applies if and only if $(a_k\alpha_k)(b_k\alpha_k) = 0$ in B, and then

$$(\overline{a}\alpha)(\overline{b}\alpha) = (d_{k-1-n}\alpha_{k-1-n})_{k-1} = (\overline{a}\overline{b})\alpha$$

where the d_{k-1-n}, $n \geq 0$, are as described in (2.2). Therefore, α is an automorphism of $\overline{B}$.

The proof that the mapping β given by (2.8) is an automorphism of $\overline{B}$ is routine. □

Theorem 2.1.3. *Let B be a nontrivial band with zero 0 such that the automorphism group of B acts transitively on the set of nonzero elements of B. Then B can be embedded into a band $\overline{B}$ which has a transitive automorphism group such that B and $\overline{B}$ generate the same band variety.*

Proof. We let $\overline{B}$ be as in Construction 1. By Lemma 2.1.1, $\overline{B}$ is a band which generates the same band variety as B. Let $\overline{a} = (a_{i-n})_i$ and $\overline{b} = (b_{j-n})_j$ be any two elements of $\overline{B}$ and put $m = j - i$. Then, no matter whether m is positive, negative or zero, we have with the notation of (2.8),

$$\beta^m : \overline{B} \to \overline{B}, \qquad \overline{a} = (a_{i-n})_i \to \overline{e} = (e_{j-n})_j$$

where $a_{i-n} = e_{j-n}$ for every $n \geq 0$. For any $l \in \mathbb{Z}$ choose an automorphism α_l of B subject to the condition that for all $l = j - n$, $n \geq 0$, we have that $e_l \alpha_l = b_l$, whereas $q\alpha_l = q$ for all $l > j$. If we then define α as in (2.8) we have that $\overline{e}\alpha = \overline{b}$, thus $\overline{a}\beta^m\alpha = \overline{b}$. According to Lemma 2.1.2, $\beta^m\alpha$ is an automorphism of $\overline{B}$. Thus the automorphism group of $\overline{B}$ acts transitively. $\square$

Remark Let B be a nontrivial band with zero such that the automorphism group of B acts transitively on the set of nonzeros of B. Let $\overline{B}$ be the band obtained as in Construction 1 after the choice of the nonzero element q in B. If we choose a nonzero q' in B instead and then construct the band B' following the procedure outlined in Construction 1, do we obtain a band $\overline{B}'$ which is essentially different from $\overline{B}$? And if not, are the embeddings of B into $\overline{B}$ and into $\overline{B}'$, respectively, equivalent?

Let ζ be a fixed automorphism of B such that $q\zeta = q'$ and define

$$\varphi : \overline{B} \to \overline{B}', \qquad (a_{i-n})_i \to (a'_{i-n})_i$$

where $a'_1 = a_1$ if this case arises, otherwise $a'_l = a_l\zeta$. One verifies that φ is a well-defined isomorphism. Moreover, if $\iota : B \to \overline{B}$ is the embedding considered in the proof of Lemma 2.1.1 (ii) and $\iota' : B \to \overline{B}'$ the corresponding embedding of B into $\overline{B}'$, then these embeddings are equivalent, meaning that the diagram

$$\begin{tikzcd} \overline{B} \arrow[r, "\zeta"] & \overline{B}' \\ B \arrow[u, "\iota"] \arrow[ur, "\iota'"'] & \end{tikzcd}$$

is commutative. For this reason we can say that if the automorphism group of B acts transitively on the nonempty set of nonzeros of B, the procedure of Construction 1 is a *standard* procedure which indeed does not depend on the choice of $q \in B \setminus \{0\}$.

Corollary 2.1.4. *Let B be a band which has a transitive automorphism group and B^0 be the band obtained from B by adjoining an extra zero. Then B^0 can be embedded into a band $\overline{B^0}$ which has a transitive automorphism group such that B^0 and $\overline{B^0}$ generate the same band variety.*

It should now be obvious that starting from a band B satisfying the conditions as stipulated in the statement of Theorem 2.1.3, one can construct a class of bands which have a transitive automorphism group and which each generate the same band variety as B. Indeed, given any infinite cardinality κ, then using transfinite induction and invoking direct limits one obtains a band of cardinality κ which has high symmetry, contains B as a subband, and which generates the same variety as B.

For Construction 1 and the associated Theorem 2.1.3 we have only a modest application in mind. From Corollary 2.1.4 we obtain the following.

Corollary 2.1.5. *R_2^0 can be embedded into a right normal band $\overline{R_2^0}$ which has a transitive automorphism group.*

Proof. Recall that R_2^0 generates the variety of right normal bands. If $\overline{R_2^0}$ is constructed from R_2^0 along the lines of Construction 1, then from Corollary 2.1.4 we have that R_2^0 can be embedded into the band $\overline{R_2^0}$ which has high symmetry and also generates the variety of the right normal bands. $\square$

Let $\overline{R_2^0}$ be the band obtained from R_2^0; according to Construction 1, $\overline{R_2^0}$ is a right normal band which has a transitive automorphism group. Figure 2.1 depicts this band $\overline{R_2^0}$ according to the techniques discussed in Section 1.3.1. Two elements are on the same "level" if and only if they have the same mark. Each $\mathcal{R}$-class has two elements. We leave the details to the reader.

We conclude this section with the following theorem.

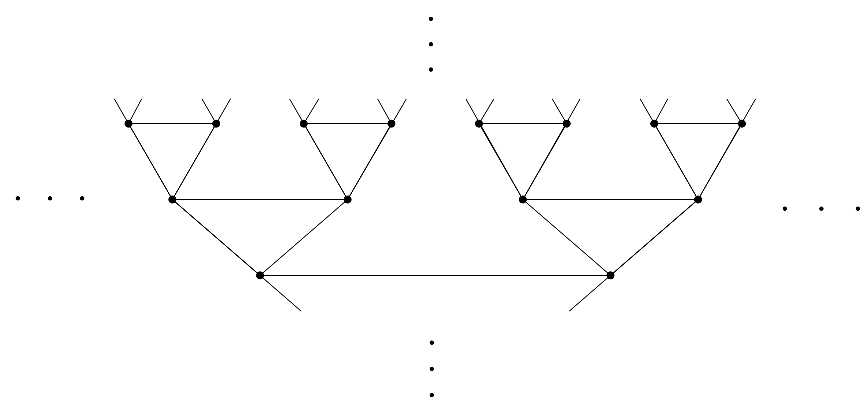

Figure 2.1

Theorem 2.1.6. *Every normal band B can be embedded into a normal band which has a transitive automorphism group and which generates the same normal band variety as B.*

Proof. The situation is clear from the remarks made in the Introduction for the cases when B is a left or right zero band, a rectangular band or a semilattice. If B generates the variety of right normal bands then B can be written as the subdirect product of the subdirectly irreducible right normal bands, Y_2, R_2, and R_2^0. Each of these can be embedded into R_2^0 which can then be embedded into the right normal band depicted in Figure 2.1. It then follows that every right normal band can be embedded into a power of this band $\overline{R}_2^0$. This band generates the variety of right normal bands and has a transitive automorphism group. If B generates the variety of left normal bands, the result follows by duality. If B generates the variety of all normal bands, then B can be embedded into a direct product with terms of the form R_2^0 or L_2^0. This direct product can then be embedded into a direct product with terms $\overline{R}_2^0$ and $\overline{L}_2^0$. The resulting direct product yields a band which generates the variety of all normal bands and has a transitive automorphism group. $\square$

2.2 Regular bands

For normal bands we were in the fortunate situation that there were only finitely many subdirectly irreducible normal bands. This is not the case for any variety not contained within the variety of normal bands. Therefore, we cannot expect to extend the strategy leading up to Theorem 2.1.6 for regular bands. When dealing with right normal bands we only had to deal with the problem of embedding the subdirectly irreducible R_2^0 into a right normal band with high symmetry. However, these techniques are adaptable to embedding [relatively] free bands. This is due to the following.

Result 2.2.1. *Let F be free for* $\mathbf{V}$*, and* $\mathbf{V}$ *generated by the band B, that is,* $\mathbf{V} = \mathbf{HSP}(B)$*, then* $F \in \mathbf{SP}(B)$

Proof. See section 4.11 of [30]. □

The variety of right regular bands is generated by R_2^1. Therefore every free right regular band can be embedded into a power of R_2^1. Thus, in order to show that every free right regular band can be embedded into a right regular band with high symmetry it suffices to show that the same can be done for R_2^1. This is not difficult as we shall see.

Construction 2. Let B be a band and $\overline{B}$ the set $\overline{B} = B \times \mathbb{Z}$. On $\overline{B}$ define a product in $\overline{B}$ by: for $(a,i),(b,j) \in \overline{B}$ put

$$(a,i)(b,j) = \begin{cases} (a,i) \text{ if } i < j \\ (b,j) \text{ if } j < i \\ (ab,i) \text{ if } i = j. \end{cases} \tag{2.9}$$

Lemma 2.2.2. *Let B be a band and $\overline{B}$ be as in Construction 2. Then*

(i) $\overline{B}$ is a band,

(ii) B^1 can be embedded into $\overline{B}$,

(iii) B^1 and $\overline{B}$ generate the same band variety.

Proof. To show that $\overline{B}$ is a band we first explicitly define the multiplication. If $a_1, \ldots, a_p \in B$, let $(a_i, l_i) \in \overline{B}$, $1 \le i \le p$. Let $j = \min(l_1, \ldots, l_p)$ and $(a'_1, ..., a'_s)$ the s-tuple obtained from the p-tuple $a_1, ..., a_p$ by deleting the entries a_i where $j < l_i$. Let $a'_1...a'_s$ be the product of the $a'_1, ..., a'_s$ in B. Then note that $(a_1, l_1)...(a_p, l_p) = (a'_1...a'_s, j)$ in $\overline{B}$. It should now be clear that if $(a_1, l_1), (a_2, l_2), (a_3, l_3) \in \overline{B}$, then $[(a_1, l_1)(a_2, l_2)](a_3, l_3) = (a_1, l_1)(a_2, l_2)(a_3, l_3) = (a_1, l_1)[(a_2, l_2)(a_3, l_3)]$ and that $(a_1, l_1)(a_1, l_1) = (a_1 a_1, l_1) = (a_1, l_1)$. Hence the multiplication is associative and idempotent, so $\overline{B}$ is a band.

If B is trivial then $\overline{B} \cong \mathbb{Z}$, so $B^1 \cong Y_2$ is embeddable in $\overline{B}$. Otherwise let $c \in B \backslash 1$, then the mapping $B^1 \to \overline{B}$ given by $a \to (a, 0)$ if $a \neq 1$ and $1 \to (c, 1)$ is an embedding of the band B^1 into the band $\overline{B}$.

We need to verify that an identity $u \approx v$ which is satisfied in B^1 is also satisfied in $\overline{B}$. We note that an identity $u \approx v$ in $\mathbf{HSP}(B^1)$ is a regular identity since $\mathbf{SL} \subseteq \mathbf{HSP}(B^1)$, therefore a variable occurs in u if and only if it occurs in v. Let $x_1, ...x_p$ be the variables which occur in both u and v, and, using the notation used above, consider a substitution of the x_i by the (a_i, l_i). Let $u' \in B$ $v' \in B$ be obtained from u and v by substituting x_i by a_i if $l_i = j$ and otherwise by 1. Then $u' = v'$ in B because $u \approx v$ is satisfied in B^1. Further $u((a_1, l_1), ..., (a_p, l_p)) = (u', j) = (v', j) = v((a_1, l_1), ..., (a_p, l_p))$, so $u \approx v$ is satisfied in $\overline{B}$. $\square$

Lemma 2.2.3. *Let B and $\overline{B}$ be as in Construction 2. Then the following holds.*

(i) For every $l \in \mathbb{Z}$, let α_l be an automorphism of B. Then

$$\alpha : \overline{B} \to \overline{B}, (a, i) \to (a\alpha_i, i) \tag{2.10}$$

is an automorphism of $\overline{B}$.

(ii) The mapping

$$\beta : \overline{B} \to \overline{B}, (a, i) \to (a, i + 1) \tag{2.11}$$

is an automorphism of $\overline{B}$.

Proof. The proof is routine and will be omitted. □

Theorem 2.2.4. *Let B be a band which has a transitive automorphism group. Then B^1 can be embedded into a band $\overline{B}$ which has a transitive automorphism group such that B^1 and $\overline{B}$ generate the same band variety.*

Proof. We let $\overline{B}$ be as in Construction 2. Let (a, i) and (b, j) be any elements of $\overline{B}$. Put $m = j - i$ and choose α as in (2.10) with $a\alpha_j = b$. Then, with β as in (2.11) we have that $\beta^m\alpha$ is an automorphism of $\overline{B}$ which maps (a, i) to (b, j). □

Corollary 2.2.5. *R_2^1 can be embedded into a right regular band which has a transitive automorphism group.*

Proof. Recall that R_2^1 generates the variety of all right regular bands. We let $\overline{R_2}$ be constructed from R_2 as in Construction 2 and apply Theorem 2.2.4. □

Using the same conventions as before, we depict the right regular band $\overline{R_2}$ described in the proof of Corollary 2.2.5 in Figure 2.2. The multiplication is obvious from the information given in Figure 2.2.

Theorem 2.2.6. *Every band F, free in some regular band variety, can be embedded into a band which has a transitive automorphism group and which generates the same band variety as F.*

Proof. In view of Theorem 2.1.6 we need to prove the theorem only in the case where F does not generate a normal band variety, that is, if F contains a copy of R_2^1 or its dual as a subband. Also since F is a subdirect product of at most two bands, each free in some join irreducible regular band variety, we are reduced, by duality, to the case where F generates the variety of all right regular bands. Since the variety of all right regular bands is generated by R_2^1, F can be embedded into a power of R_2^1. The result now follows from Corollary 2.2.5. □

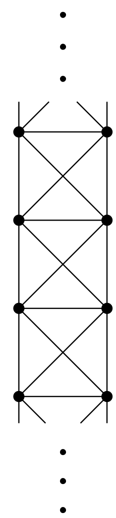

Figure 2.2: $\overline{R_2}$

2.2.1 D-transitivity

Call an automorphism α of a band B *$\mathcal{D}$-preserving* if $a\mathcal{D}a\alpha$ for every $a \in B$. We say that the automorphism group of B *acts $\mathcal{D}$-transitively* if the $\mathcal{D}$-classes of B are the orbits for the group of $\mathcal{D}$-preserving automorphisms of B. Recall in this context that $\mathcal{D}$ is a characteristic congruence on B, so the $\mathcal{D}$-preserving automorphisms of B form a normal subgroup of the automorphism group. We shall soon prove that the condition the automorphism group of B acts $\mathcal{D}$-transitively implies that B is a regular band. Here we note that the band in Figure 2.2 has a transitive automorphism group that acts $\mathcal{D}$-transitively, whereas, for the band of Figure 2.1, the identity transformation is the only $\mathcal{D}$-preserving automorphism. For the Green relations $\mathcal{L}$ and $\mathcal{R}$ we can define in an analogous way the notions of $\mathcal{L}$-[$\mathcal{R}$-]*preserving automorphism* and $\mathcal{L}$-[$\mathcal{R}$-]*transitivity.*

Looking at $\mathcal{D}$-transitivity was motivated by the following example which was suggested by Pastijn.

Example 2.2.7. We use the terminology of [23]. We let $\mathcal{A}$ be an affine plane, P its set of points, L its set of lines and $B = P \cup L$. We define a multiplication on B by: for any $p, q \in P$ and $l, m \in L$,

$$pq = q, \qquad lm = m, \qquad pl = l,$$

and lp is the line through p and parallel to l.

Then B is a right regular band and P, L its two $\mathcal{R}$-classes. For any line $l \in L$ and any point $p \in P$ we have that $l \leq p$ for the natural order in the band B if and only if p is on l. From this it follows that the automorphism group of B is precisely the automorphism group of the affine plane $\mathcal{A}$. Under the right circumstances this group acts $\mathcal{D}$-transitively on B. Thus for instance, if $\mathcal{A}$ is desarguesian (that is, over a skewfield), then this automorphism group is doubly transitive on P and thus transitive on L (see Theorem 2.12 of [23]). The converse is true in the finite case by the Ostrom-Wagner Theorem (Theorem 14.13 of [23]). If $\mathcal{A}$ is a finite affine plane and the automorphism group acts doubly transitive on P then the plane is finite and desarguesian, and thus also pappian (that is, over a finite field).

We now proceed to show that if a band has $\mathcal{D}$-transitivity, then it is a regular band. We will therefore need to show that both the $\mathcal{L}$ and $\mathcal{R}$ relations are in fact congruences.

Lemma 2.2.8. *Let $S = S(Y, D_\alpha)$ be a band such that for every $\beta \leq \alpha$ in Y and $h \in D_\beta$, there exists $k \in D_\alpha$ such that $kh = h$. Then $\mathcal{R}$ is a congruence on S.*

Proof. Since $\mathcal{R}$ is a left congruence relation, it suffices to show that $\mathcal{R}$ is a right congruence on S. Therefore, let $e, f, g \in S$ with $e\mathcal{R}g$. We need to show that $ef\mathcal{R}gf$. We can assume that $e, g \in D_\alpha$ whereas $ef, gf \in D_\beta$ for some $\beta \leq \alpha$ in Y. Put $h = fe \in D_\beta$ and choose $k' \in D_\alpha$ such that $k'h = h$. Choose $k \in D_\alpha$ such that $k'\mathcal{R}k\mathcal{L}g$. Then $gk = g$, $kh = h$, so $eh = geh = gkekh = gh$. Thus, since $h = fe$,

$$ef\mathcal{R}efe{=}gfe\mathcal{R}gf$$

as required. □

Proposition 2.2.9. *Let $S = S(Y, D_\alpha)$ be a band. If the automorphism group of S acts $\mathcal{D}$-transitively, we then have that for every $\beta \leq \alpha$ in Y and $h \in D_\beta$, there exists $k \in D_\alpha$ such that $khk = h$. Furthermore S is a regular band.*

Proof. Suppose that the automorphism group of S acts $\mathcal{D}$-transitively. Now let $\beta \leq \alpha$ in Y and $h \in D_\beta$. Since $D_\alpha \neq \emptyset$ there exists an $e \in D_\alpha$. Since $\beta \leq \alpha$ it follows that $ehe\mathcal{D}h$, so there exists a $\mathcal{D}$-preserving automorphism θ such that $(ehe)\theta = h$. Now since $\leq$ a characteristic relation it follows that $h = (ehe)\theta \leq e\theta$ since $ehe \leq e$. Therefore $(e\theta)h(e\theta) = h$ and, since θ is $\mathcal{D}$-preserving, it follows that $e\theta \in D_\alpha$, so we have the first part of our result.

Now note that $(e\theta)h = h = h(e\theta)$, so, by Lemma 2.2.8 and its dual, both $\mathcal{R}$ and $\mathcal{L}$ are congruences, so S is regular. □

Theorem 2.2.10. *Let S be a band. The following are equivalent:*

(i) the automorphism group of S acts $\mathcal{D}$-transitively,

(ii) the automorphism group of S acts $\mathcal{L}$-transitively and $\mathcal{R}$-transitively.

If this is the case, then S is a regular band.

Proof. If (ii) holds, then (i) holds since $\mathcal{D} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}$.

If (i) holds, then by Lemma 2.2.9 both $\mathcal{L}$ and $\mathcal{R}$ are congruences on S. Since $\mathcal{L}$ and $\mathcal{R}$ are then necessarily characteristic congruences on S, every $\mathcal{D}$-preserving automorphism γ on S induces an $\mathcal{L}$-preserving automorphism

$$\gamma_l : S/\mathcal{R} \to S/\mathcal{R}, \qquad R_e \to R_{e\gamma}$$

on the left regular band $S/\mathcal{R}$ and an $\mathcal{R}$-preserving automorphism

$$\gamma_r : S/\mathcal{L} \to S/\mathcal{L}, \qquad L_e \to L_{e\gamma}$$

on the right regular band $S/\mathcal{L}$. Since the automorphism group of S acts $\mathcal{D}$-transitively, it follows that the automorphism group of $S/\mathcal{R}$ [$S/\mathcal{L}$] acts $\mathcal{L}$-[$\mathcal{R}$-]transitively.

Recall that

$$\varphi : S \to (S/\mathcal{R}) \times (S/\mathcal{L}), \qquad e \to (R_e, L_e)$$

is an embedding. Thus, if σ is an $\mathcal{L}$-preserving automorphism on $S/\mathcal{R}$, then

$$S\varphi \to S\varphi, \qquad (R_e, L_g) \to (R_e\sigma, L_g)$$

is an $\mathcal{L}$-preserving automorphism of $S\varphi$, and it follows that the automorphism group of $S\varphi$, and thus also of S itself, acts $\mathcal{L}$-transitively. One shows that dually, the automorphism group of S acts $\mathcal{R}$-transitively. □

Remark For any band B, the group $A_{\mathcal{L}}(B)$ $[A_{\mathcal{R}}(B)]$ of $\mathcal{L}$-[$\mathcal{R}$-]preserving automorphisms of B is a normal subgroup of the automorphism group $A(B)$ of B, since $\mathcal{L}$ and $\mathcal{R}$ are characteristic equivalence relations of B. Moreover $A_{\mathcal{L}}(B) \cap A_{\mathcal{R}}(B)$ is trivial since $\mathcal{L} \cap \mathcal{R}$ is the equality relation on B. If $A_{\mathcal{D}}(B)$ is the group of $\mathcal{D}$-preserving automorphisms, then following the same reasoning as in the proof of Theorem 2.2.10 one sees that $A_{\mathcal{D}}(B) = A_{\mathcal{L}}(B)A_{\mathcal{R}}(B)$ is a normal subgroup of $A(B)$ which is (isomorphic to) the direct product of $A_{\mathcal{L}}(B)$ and $A_{\mathcal{R}}(B)$. Every $\mathcal{D}$-preserving automorphism can be written uniquely as the composition of an $\mathcal{L}$-preserving and an $\mathcal{R}$-preserving automorphism.

Corollary 2.2.11. *If $S = S(\mathbb{Z}, D_i)$ has a transitive automorphism group, then it is a regular band.*

Proof. This follows from Theorem etthm3.9 and the fact that the only automorphism of $\mathbb{Z}$ that fixes any point of $\mathbb{Z}$ is the identity mapping. □

2.3 Beyond regular bands

From Corollary 2.2.11 it follows that if we want to produce an example of a band which is not regular and which has high symmetry, then it cannot be a $\mathbb{Z}$-chain of rectangular bands. In this section we find examples of such bands which each generate a variety which is arbitrarily "high up" in the lattice of band varieties. The structure semilattice for each of these examples will be a scattered ordered chain of order type $\zeta\gamma$, ζ being the order type of $\mathbb{Z}$ and γ some ordinal (see Theorem 8.5 of [58], and also [44]). From this, it will easily follow that every free band is embeddable into a band with high symmetry.

In the following construction our notation is similar to that of Construction 1, but we will be adding integers instead of subtracting them. That is, we now have $a^l = (a_{i+n})_i$ instead of $\overline{a} = (a_{i-n})_i$. We mention this with the hope of avoiding confusion.

Construction 3. Let S be a band and let S^l consist of mappings of the following kind. For every $a^l \in S^l$, there exists $i \in \mathbb{Z}$, called the *mark* of a^l, and $a^l : \{i+n | n \in \mathbb{Z}, n \geq 0\} \to S$. If we denote $a^l(i+n)$ by a_{i+n} for all $n \geq 0$, then we can briefly denote a^l by the marked sequence $a^l = (a_{i+n})_i = (a_i, a_{i+1}, ...)_i$: all the entries of the sequence belong to S and the index i reminds us of the mark of a^l. If a^l has mark i, then $a^l(i) = a_i$ will be called the *leading entry* of a^l.

We define a multiplication in S^l by the following. Let $a^l = (a_{i+n})_i$ and $b^l = (b_{j+n})_j$ respectively. The mark of $a^l b^l$ will be $min(i,j)$. We put

$$a^l b^l = \begin{cases} a^l \text{ if } i < j \\ d^l = (d_{j+n})_j \text{ if } j \leq i \end{cases} \tag{2.12}$$

where

$$d_{j+n} = \begin{cases} b_{j+n} \text{ for } j+n < i \\ a_i b_i \text{ (as in S) for } j+n = i \\ a_{j+n} \text{ for } j+n > i. \end{cases} \tag{2.13}$$

Lemma 2.3.1. *Let S be a band and S^l be as in Construction 3. Then S^l is a band.*

Proof. Obviously the multiplication given by (2.12) is idempotent: $(a^l)^2 = a^l$ for every $a^l \in S^l$. We need only verify associativity. Therefore, let $a^l = (a_{i+n})_i$, $b^l = (b_{j+n})_j$ and $c^l = (c_{k+n})_k$ be any elements of S^l with marks i, j, and k respectively. We need to verify that $(a^l b^l)c^l = a^l(b^l c^l)$ with the multiplication as in (2.12) and (2.13).

One verifies that if $k > \min(i,j)$ then $(a^l b^l)c^l = a^l(b^l c^l) = a^l b^l$. Otherwise $k \leq \min(i,j)$ and the mark of both $(a^l b^l)c^l$ and $a^l(b^l c^l)$ is k. We put $(a^l b^l)c^l = d^l = (d_{k+n})_k$ and $a^l(b^l c^l) = d'^l = (d'_{k+n})_k$. If $k+n < \min(i,j)$ then $d_{k+n} = d'_{k+n} = c_{k+n}$, and, if

$k+n > \min(i,j)$, then $d_{k+n} = d'_{k+n}$ is the $(k+n)$th entry of $a^l b^l$. If suffices to investigate whether $d_{k+n} = d'_{k+n}$ when $k \leq \min(i,j)$ and $k+n = \min(i,j)$. One verifies that in this case

$$d_{k+n} = d'_{k+n} = \begin{cases} a_i c_i \text{ (as in S) if } i < j \\ a_i b_i c_i \text{ (as in S) if } i = j \\ b_j c_j \text{ (as in S) if } j < i. \end{cases}$$

□

Lemma 2.3.2. *Let S be a band, S^l be as in Construction 3, and $\overline{S} = S \times \mathbb{Z}$ as in Construction 2. Then*

(i) for $a^l, b^l \in S^l$, $a^l \mathcal{L} b^l$ in S^l if and only if a^l and b^l have the same mark and the leading entries of a^l and b^l are $\mathcal{L}$-related in S,

(ii) the mapping

$$\varphi : S^l \to \overline{S}, \qquad a^l = (a_{i+n})_i \to (a^l(i), i) = (a_i, i)$$

is a homomorphism of S^l onto $\overline{S}$,

(iii) the homomorphism φ induces a congruence relation which is contained in the $\mathcal{L}$-relation on S^l, so $S^l \in \mathbf{LZ} \circ \mathbf{HSP}(S)$.

Proof. (i) follows immediately from (2.12) and (2.13), (ii) from (2.9) and (2.12), and (iii) follows from (i) and (ii). □

Lemma 2.3.3. *Let S be a band and S^l be as in Construction 3. If S has a transitive automorphism group, then so does S^l.*

Proof. Let $c^l = (c_{k+n})_k$ and $d^l = (d_{m+n})_m$ be any elements of S^l with marks k and m, respectively. We need to show that there exists an automorphism γ of S^l such that $c^l\gamma = d^l$. Our proof is similar to the proof for Lemma 2.1.2.

For every $i \in \mathbb{Z}$, let α_i be an automorphism of S. Then from (2.12) and (2.13) it follows immediately that

$$\alpha : S^l \to S^l, \qquad a^l = (a_{i+n})_i \to (a_{i+n}\alpha_{i+n})_i$$

is an automorphism of S^l. Also the "shift"

$$\beta : S^l \to S^l, \qquad a^l = (a_{i+n})_i \to (d_{i+1+n})_{i+1}$$

where $d_{i+1+n} = a_{i+n}$ in S for every $n \geq 0$ is an automorphism of S^l. If in the above one chooses α_{k+n} such that $c_{k+n}\alpha_{k+n} = d_{m+n}$ for every $n \geq 0$, then with $\gamma = \alpha\beta^{m-k}$ we have $c^l\gamma = d^l$, as required. □

Construction 4. Let B be a band and B^L the band generated by the elements of $B \cup \{e\}$ where $e \notin B$, subject to the defining relations

$$\begin{aligned} ab =& c \qquad \text{in } B^L \text{ if } a, b, c \in B \text{ and } ab = c \text{ in } B, \\ ea =& e \qquad \text{for every } a \in B, \\ e^2 =& e. \end{aligned}$$

Then B^L is the disjoint union of B and $Be \cup \{e\}$, where $Be \cup \{e\}$ is an $\mathcal{L}$-class of B^L. If $a \neq b$ in B, then $a \neq b$ in B^L, $ae \neq be$ in B^L, and $ae \neq e$ in B^L.

The band B^R is constructed from B in a dual way.

Lemma 2.3.4. *Let B be a band and B^L be the band as in Construction 4. If the band variety $\boldsymbol{HSP}(B)$ generated by B contains $\boldsymbol{SL}$, then $\boldsymbol{HSP}(B^L)$=$\boldsymbol{LZ}\circ\boldsymbol{HSP}(B)$.*

Proof. With the notation of Construction 4 and letting B^0 be B with an *extra* 0 adjoined,

$$\begin{aligned} B^L \to B^0, \qquad & a \to a \text{ if } a \in B \\ & e \to 0 \\ & ae \to 0 \text{ if } a \in B \end{aligned}$$

is a homomorphism of B^L to the band B^0. Since $\mathbf{SL}$ is contained in $\mathbf{HSP}(B)$, we have that $\mathbf{HSP}(B)$=$\mathbf{HSP}(B^0)$. Since $\{e, ae|a \in B\}$ is the only nontrivial congruence class of the congruence relation induced by this homomorphism and since this congruence class forms a left zero semigroup, it follows that $\mathbf{HSP}(B^L)\subseteq\mathbf{LZ}\circ\mathbf{HSP}(B)$. Since B is a subsemigroup of B^L, we thus have

$$\mathbf{HSP}(B) \subseteq \mathbf{HSP}(B^L) \subseteq \mathbf{LZ} \circ \mathbf{HSP}(B).$$

In order to show that the second inclusion is actually an equality, it suffices to show that if an identity $u \approx v$ is satisfied in $\mathbf{HSP}(B^L)$, then it is satisfied in $\mathbf{LZ}\circ\mathbf{HSP}(B)$.

We let $u \approx v$ be any identity satisfied in $\mathbf{HSP}(B^L)$. Since $\mathbf{SL}\subseteq\mathbf{HSP}(B^L)$, the *content* of u, $c(u)$, (that is, the set of variables which occur in u) is the same as the content of v, $c(v)$. We put $c(u) = c(v) = X$ and we consider u and v as belonging to the set X^+ of nonempty words over X. We show that $u \approx v$ is satisfied in $\mathbf{LZ}\circ\mathbf{HSP}(B)$ by induction on $|c(u)| = |c(v)|$. The claim is evident if $|c(u)| = |c(v)| = 1$ and we henceforth assume that $|c(u)| = |c(v)| \geq 2$.

From the definition of B^L we have that $u \approx v$ is satisfied in B^L if and only if, using the terminology of I.6.6 of [49],

(i) $u \approx v$ is satisfied in B,

(ii) the *head* of u, $h(u)$, (the first variable from the left to occur in u) equals the head of v, $h(v)$,

(iii) if $x \in X$ and $x \neq h(u) = h(v)$, and $l_x(u) = u_1x$ $[l_x(v) = v_1x]$ the shortest left cut of u $[v]$ such that $x \notin c(u_1)$ $[x \notin c(v_1)]$, then $u_1 \approx v_1$ is satisfied in B.

In particular then, since $\mathbf{SL}\subseteq\mathbf{HSP}(B)$, $x \in X$ is such that $c(l_x(u)) = X$ if and only if $c(l_x(v)) = X$, then $u_1x = l_x(u)$ and $l_x(v) = v_1x$ with $u_1 \approx v_1$ satisfied in B. From the above description of the identities satisfied in B^L, it follows that $u_1 \approx v_1$ is satisfied in B^L, so by the induction hypothesis $u_1 \approx v_1$ and thus also $u_1x \approx v_1x$ are satisfied in $\mathbf{LZ}\circ\mathbf{HSP}(B)$.

Let u_2 $[v_2]$ be the shortest right cut of u $[v]$ such that $c(u_2) = X = c(v_2)$. Then $u \approx u_1xu_2$ and $v \approx v_1xv_2$ are identities valid in the variety of all bands. We only need to show that u_1xu_2 and v_1xv_2 represent the same element in the relatively free band F on X in $\mathbf{LZ}\circ\mathbf{HSP}(B)$. Certainly $u_1xu_2\mathcal{R}v_1xv_2$ in F since u_1x and v_1x represent the same element of F as we have seen above, and since $c(u_1x) = c(v_1x) = c(u_2) = c(v_2) = X$, u_1x, v_1x, u_2 and v_2 are $\mathcal{D}$-related in F when considered as representing elements of F. Since $u_2v \approx u_2v_1xv_2 \approx u_2v_2$, $u_2u \approx u_2u_1xu_2 \approx u_2$ hold in the variety of all bands and $u \approx v$ in $\mathbf{HSP}(B^L)$, $u_2v_2 \approx u_2$ is satisfied in $\mathbf{HSP}(B^L)$. By symmetry, $v_2u_2 \approx v_2$ is satisfied in $\mathbf{HSP}(B^L)$, so u_2 and v_2 represent $\mathcal{L}$-related elements in the band relatively free in $\mathbf{HSP}(B^L)$ on X. Since the canonical homomorphism from F onto the band relatively free in $\mathbf{HSP}(B^L)$ on X induces a congruence contained in the $\mathcal{L}$-relation on F, it follows that u_2 and v_2 represent $\mathcal{L}$-related elements in F. We can now conclude that $u_1xu_2 = v_1xv_2$ in F, thus $u_1xu_2 \approx v_1xv_2$ and consequently $u \approx v$ hold in $\mathbf{LZ}\circ\mathbf{HSP}(B)$, as required. $\square$

Lemma 2.3.5. *Let B be a band such that the band variety $\boldsymbol{HSP}(B)$ contains $\boldsymbol{SL}$. Let $S = \overline{B}$ be the band obtained from B as in Construction 2, S^l the band obtained from S as in Construction 3, and B^L the band obtained from B as in Construction 4. Then*

(i) $(B^1)^L$ can be embedded into S^l,

(ii) $\boldsymbol{HSP}(S^l)$=$\boldsymbol{LZ}\circ\boldsymbol{HSP}(B^1)$,

(iii) if B has a transitive automorphism group, then so does S^l.

Proof. (i) By Lemma 2.2.2 we can embed B^1 into S. We shall henceforth identify B^1 with its image under this embedding. Following the details of this embedding, there exists $f \in S$ such that $f \notin B^1$ and such that $fb = b = bf$ for every $b \in B^1$. Then for $a \in B^1$ and using the notation of Construction 3,

$$a \to (a, f, f, f, ...)_1,$$

$$e \to (f, f, f, f, ...)_0,$$

$$ae \to (f, a, f, f, ...)_0 \tag{2.14}$$

is an embedding of $(B^1)^L$ into S^l.

(ii) By Lemma 2.2.2, $B^1, \overline{B} = S$ and $\overline{S}$ generate the same band variety, and by Lemma 2.3.2(iii), $\mathbf{HSP}(S^l) \subseteq \mathbf{LZ} \circ \mathbf{HSP}(\overline{S})$. Therefore $\mathbf{HSP}(S^l) \subseteq \mathbf{LZ} \circ \mathbf{HSP}(B^1)$. By (i) and Lemma 2.3.4, we have that $\mathbf{HSP}(S^l) = \mathbf{LZ} \circ \mathbf{HSP}(B^1)$.

(iii) This statement follows from Theorem 2.2.4 (and its proof) and Lemma 2.3.3. □

We shall make a distinction between band monoids and bands which have an identity element. A band monoid is an algebra of type $\langle 2, 0 \rangle$ where the nullary operation corresponds to the selection of the identity element, whereas a band with identity element is considered here as an algebra of type $\langle 2 \rangle$. There exists a natural embedding of the lattice of varieties of band monoids into the lattice of band varieties, which associates with every variety $\mathbf{W}$ of band monoids the band variety $\mathbf{V}$ which is generated by the members of $\mathbf{W}$ when considered as bands. The image of this embedding is a copy L of the lattice of all band monoid varieties which was described in [61]. As it turns out, this copy L consists precisely of the trivial band variety together with the band varieties in the complete lattice generated by the varieties in the list (1.1). Further, a band variety belongs to this copy of L if and only if it is generated by a band with identity element. Equivalently, a band variety $\mathbf{V}$ belongs to L if whenever $B \in \mathbf{V}$, then also $B^1 \in \mathbf{V}$.

Theorem 2.3.6. *Each band variety in the complete lattice of band varieties generated by the varieties in the list* (1.1) *is generated by a band which has a transitive automorphism group.*

Proof. We first prove by induction that each variety listed in (1.1) is generated by a band which has a transitive automorphism group. The induction will be on the number of Mal'cev products involved. By the remarks preceding the statement of this theorem, if $\mathbf{V}$ is any variety listed in (1.1) and $\mathbf{V}$=$\mathbf{HSP}(B)$ for some band B, then $\mathbf{V}$=$\mathbf{HSP}(B^1)$.

As for the basis for our proof by induction the variety $\mathbf{SL}$ of all semilattices is generated by the $\mathbb{Z}$-chain, considered as a semilattice. Assume that $k \geq 0$, that $\mathbf{V}$ is a variety in the list (1.1) where k Mal'cev products are involved, and that $\mathbf{V}$ is generated by a band which has a transitive automorphism group. Then $\mathbf{LZ}\circ\mathbf{V}$ is generated by a band which has a transitive automorphism group by Lemma 2.3.5, and the dual of Lemma 2.3.5 entails that $\mathbf{RZ}\circ\mathbf{V}$ is generated by a band which has a transitive automorphism group.

Every band variety in the complete lattice generated by the varieties listed in (1.1) is either the join of at most two distinct varieties in (1.1), or is the variety $\mathbf{B}$ of all bands. In the latter case, the variety $\mathbf{B}$ is the join of all the varieties in (1.1). If $\mathbf{V} = \mathbf{V}_1 \vee \mathbf{V}_2$ with $\mathbf{V}_1$, $\mathbf{V}_2$ in (1.1), then $\mathbf{V}_1 = \mathbf{HSP}(B_1)$ and $\mathbf{V}_2 = \mathbf{HSP}(B_2)$ for some bands B_1 and B_2 which have a transitive automorphism group. Then $\mathbf{V} = \mathbf{HSP}(B_1 \times B_2)$ where $B_1 \times B_2$ has a transitive automorphism group. Otherwise, if $\mathbf{V}_i$, $i < \omega$, is another way to the list (1.1), then for every $i < \omega$, $\mathbf{V}_i = \mathbf{HSP}(B_i)$ for a band B_i which has a transitive automorphism group, then $\mathbf{B} = \mathbf{HSP}(\prod_{i<\omega} B_i)$ where $\prod_{i<\omega} B_i$ has a transitive automorphism group. □

Corollary 2.3.7. *Any band variety which is generated by a band which has an identity element is generated by a band which has a transitive automorphism group.*

Theorem 2.3.8. *Every relatively free band, free in a band variety generated by a band which has an identity element, can be embedded into a band which has a transitive automorphism group and which generates the same band variety.*

Proof. We note that we have already shown this result when the variety is a subvariety of regular bands in Theorem 2.2.6, so we have an overlap for the varieties $\mathbf{SL}, \mathbf{RRB}, \mathbf{LLB}$ and $\mathbf{ReB}$.

Note that if $\mathbf{V}$ is a generated by a band which has an identity element then $\mathbf{V}$ is generated by a band with high symmetry by Corollary 2.3.7. Call this band B. Now if F is free for $\mathbf{V}$, then $F \in \mathbf{SP}(B)$ by Result 2.2.1, so the result follows. $\square$

The following theorem now follows as a special case.

Theorem 2.3.9. *Every free band can be embedded into a band which has a transitive automorphism group.*

Chapter 3

Normal Bands

In Chapter 2 we showed that every normal band is embeddable in a normal band with high symmetry. In this chapter we revisit embedding normal bands into normal bands with high symmetry; however, we will do so in such a way that we are able to maintain much of the structure of the original band. That is, through the embedding we will be able to extend partial isomorphisms of the original band to partial isomorphisms of the new band in such a way that we have an embedding of the hulls. Also we will be able to extend automorphisms of the original band to automorphisms of the new band in such a way that we get an embedding of automorphism groups. Further our original band will be embedded as a convex subset into our new band. In [38] a similar procedure was given for semilattices. It will follow that every fundamental generalized inverse semigroup can be embedded into a bisimple fundamental generalized inverse semigroup.

Section 3.1 is concerned with the embedding of any normal band into a normal band which is uniform in such a way that we preserve the properties mentioned above. After the intermediate Section 3.2, we shall prove the result stated above.

Another avenue for proving our main result is suggested by the work of Szendrei in [59]. There the author proves that every orthodox semigroup has a E-unitary cover which is embeddable into a semidirect product of a band by a group. Applying this to the hull of an arbitrary band, B, it may be possible to show that B can be embedded into a band B' such that every partial isomorphism of B extends to an automorphism of B' which would give an alternative to the method used in Section 3.2. We shall not pursue this line of investigation here.

3.1 Uniform normal bands

The following summarizes Lemma 1, Lemma 2 and Lemma 4 of [39]. We use the notation as defined in Section 1.4.

Lemma 3.1.1. *Any semilattice L can be embedded as a filter into a semilattice K in such a way that*

(i) $L \times L \subseteq \mathcal{U}_K$,

(ii) every partial isomorphism $\alpha : eL \to fL$ of L can be extended to a partial isomorphism $\alpha_K : eK \to fK$ of K such that the mapping

$$T_L \to T_K, \qquad \alpha \to \alpha_K$$

embeds T_L isomorphically into T_K,

(iii) every automorphism γ of L can be extended to an automorphism γ_K of K such that the mapping

$$\mathrm{Aut}L \to \mathrm{Aut}K, \qquad \gamma \to \gamma_K$$

embeds $\mathrm{Aut}L$ *isomorphically into* $\mathrm{Aut}K$.

Our first task will be to generalize this result for right normal bands.

Lemma 3.1.2. *Any right normal band B can be embedded as a filter (for the natural partial order) into a right normal band M in such a way that*

(i) $B \times B \subseteq \mathcal{U}_M$,

(ii) every partial isomorphism $\alpha : Be \to Bf$ of B can be extended to a partial isomorphism $\alpha_M : Me \to Mf$ of M such that the mappings

$$\underline{T}_B \to \underline{T}_M, \qquad \alpha \to \alpha_M$$
$$T_B \to T_M, \qquad \overline{\alpha} \to \overline{\alpha}_M$$

are embeddings, and the diagram

$$\begin{tikzcd} \underline{T}_B \arrow[r, "\kappa_B^\natural"] \arrow[d] & T_B \arrow[d] \\ \underline{T}_M \arrow[r, "\kappa_M^\natural"'] & T_M \end{tikzcd} \quad , \quad \begin{tikzcd} \alpha \arrow[r, "\kappa_B^\natural"] \arrow[d] & \overline{\alpha} \arrow[d] \\ \alpha_M \arrow[r, "\kappa_M^\natural"'] & \overline{\alpha}_M \end{tikzcd}$$

commutes,

(iii) every automorphism γ of B can be extended to an automorphism of γ_M of M such that

$$\mathrm{Aut}B \to \mathrm{Aut}M, \qquad \gamma \to \gamma_M$$

embeds Aut$B$ *isomorphically into* Aut$M$*,*

(iv) the structure semilattice of M is the semilattice obtained from the structure semilattice of B as in Lemma 3.1.1.

Proof. Let $B = B[L; B_i; \varphi_{i,j}]$ and K be a semilattice such that the conditions in the statement of Lemma 3.1.1 are satisfied with regards to the structure semilattice L . We let $M = B \cup (K \backslash L)$, and we can always assume that B and $K \backslash L$ are disjoint. We note that $K \backslash L$, if nonempty, is an ideal of K since L is a filter of K. We define a multiplication on M which extends the given ones on B and on $K \backslash L$, such that, if $e \in B_i$ and $k \in K \backslash L$, then $ek = ke = ik$ where the product ik is as in K. It is routine to verify that $M = M[K : M_i, \psi_{i,j}]$ is a strong semilattice of right zero bands where

(i) $K = M/\mathcal{D}$ is the structure semilattice of M,

(ii) $M_i = B_i$ if $i \in L$ and $M_i = \{i\}$ if $i \in K \backslash L$ are right zero bands which are the $\mathcal{D}$-classes of M,

(iii) the structure homomorphisms $\psi_{i,j}$ of M are given by

$$\psi_{i,j} = \varphi_{i,j} \qquad \text{if } j \leq i \text{ in } L,$$
$$\psi_{i,j} : B_i \to \{j\} \qquad \text{if } i \in L,\ j \in K \backslash L,\ j \leq i \text{ in } K,$$
$$\psi_{i,j} : i \to j \qquad \text{if } j \leq i \text{ in } K \backslash L.$$

It readily follows that B is embedded as a subsemigroup of the right normal band M, that B is, for the natural partial order, a filter of M, and that $M \backslash B = K \backslash L$, if nonempty, is a semilattice which is an ideal of M.

Let $e, f \in B$, with $e \in B_i$, $f \in B_j$, where $i, j \in L$. From the definition of the multiplication in M it follows that $Me \cong iK$ and $Mf \cong jK$ are pairs of isomorphic semilattices. Since $i, j \in L$ and $L \times L \in \mathcal{U}_K$, it follows that $Me \cong iK \cong jK \cong Mf$. We conclude that (i) in the statement of the lemma is satisfied.

We set out to prove statement (ii). Let $e, f \in B$ with $e \in B_i$, $f \in B_j$, and $i, j \in L$. Let $\alpha : Be \to Bf$ be a partial isomorphism of B and $\widetilde{\alpha} : iL \to jL$ induced on L by α. By Lemma 3.1.1, $\widetilde{\alpha}$ can be extended to a partial isomorphism $\widetilde{\alpha}_K : iK \to jK$ such that the mapping

$$T_L \to T_K, \qquad \widetilde{\alpha} \to \widetilde{\alpha}_K$$

embeds the inverse semigroup T_L isomorphically into the inverse semigroup T_K. We define $\alpha_M : Me \to Mf$ by :

$$\begin{aligned} x\alpha_M &= x\alpha \text{ if } x \in Be, \\ &= x\widetilde{\alpha}_K \text{ if } x \in Me \backslash Be. \end{aligned}$$

It follows from the definition of the multiplication on M that α_M is a partial isomorphism of M which extends α.

Let $e, f, g, h \in B$ with $e \in B_i$, $f \in B_j$, $g \in B_k$, and $h \in B_l$ where $i, j, k, l \in L$. Let

$\alpha : Be \to Bf$ and $\beta : Bg \to Bh$ be partial isomorphisms of B and $\widetilde{\alpha}$, $\widetilde{\beta}$ the partial isomorphisms on L which are induced on L by α and β respectively. Let $\widetilde{\alpha}_K$ and $\widetilde{\beta}_K$ be the extensions to partial isomorphisms of K of $\widetilde{\alpha}$ and $\widetilde{\beta}$ as indicated above. Then let α_M and β_M be the corresponding partial isomorphisms of M. In order to prove that $\underline{T}_B \to \underline{T}_M$, $\alpha \to \alpha_M$ is an embedding, it suffices to prove that $(\alpha \cdot \beta)_M = \alpha_M \cdot \beta_M$. We already know that $(\widetilde{\alpha} \cdot \widetilde{\beta})_K = \widetilde{\alpha}_K \widetilde{\beta}_K$ since $T_L \to T_K$, $\widetilde{\alpha} \to \widetilde{\alpha}_K$ is an embedding of inverse semigroups.

Recall from Section 1.4 that in $\underline{T}_B$,

$$
\begin{aligned}
\alpha \cdot \beta &= \alpha \pi_{f,g} \beta \\
&= \alpha \pi(fgf, gfg) \beta \\
&= \alpha \pi(gf, fg) \beta
\end{aligned}
$$

is a partial isomorphism of B mapping $B(gf)\alpha^{-1}$ onto $B(fg)\beta$. Further, $\widetilde{\alpha \cdot \beta} = \widetilde{\alpha}\widetilde{\beta}$ since $\underline{T}_B \to T_L$, $\alpha \to \widetilde{\alpha}$ is a homomorphism. Thus $(\widetilde{\alpha \cdot \beta})_K = (\widetilde{\alpha}\widetilde{\beta})_K = \widetilde{\alpha}_K \widetilde{\beta}_K$. Thus, according to our construction, the domain of $(\alpha \cdot \beta)_M$ is $M(gf)\alpha^{-1}$ which is the disjoint union of $B(gf)\alpha^{-1}$ and $(K \backslash L) \cap M(gf)\alpha^{-1}$ where

$$
(K \backslash L) \cap M(gf)\alpha^{-1} = ((K \backslash L) \cap Mgf)\widetilde{\alpha}_K^{-1}.
$$

If $x \in B(gf)\alpha^{-1}$, then

$$
x(\alpha \cdot \beta)_M = x(\alpha \cdot \beta) = ((x\alpha)g)\beta,
$$

and if $x \in (K \backslash L) \cap M(gf)\alpha^{-1}$, then

$$
x(\alpha \cdot \beta)_M = ((x\widetilde{\alpha}_K)g)\widetilde{\beta}_K = x\widetilde{\alpha}_K \widetilde{\beta}_K
$$

where the second equality follows from

$$x(\alpha \cdot \beta)_M = x\widetilde{(\alpha \cdot \beta)}_K$$
$$= x\widetilde{\alpha}_K\widetilde{\beta}_K.$$

The domain of $\alpha_M \cdot \beta_M$ is $M(gf)\alpha_M^{-1} = M(gf)\alpha^{-1}$. Thus, $(\alpha \cdot \beta)_M$ and $\alpha_M \cdot \beta_M$ have the same domain. If $x \in B(gf)\alpha^{-1}$, then clearly

$$x(\alpha_M\beta_M) = ((x\alpha_M)g)\beta_M$$
$$= ((x\alpha)g)\beta$$
$$= x(\alpha \cdot \beta)_M.$$

If $x \in (K \backslash L) \cap M(gf)\alpha^{-1}$, then

$$x(\alpha_M \cdot \beta_M) = ((x\widetilde{\alpha}_K)g)\widetilde{\beta}_K = x\widetilde{\alpha}_K\widetilde{\beta}_K$$

where the second equality follows from

$$x\widetilde{\alpha}_K \in (K \backslash L) \cap Mgf,$$
$$gf, fg \in B_{jk},$$
$$x\widetilde{\alpha}_K \leq jk \text{ in } K,$$
$$x\widetilde{\alpha}_K \leq fg \leq g \text{ in } M.$$

We conclude that indeed $(\alpha \cdot \beta)_M = \alpha_M \cdot \beta_M$ for every $\alpha, \beta \in \underline{T}_B$, thus $\underline{T}_B \to \underline{T}_M$, $\alpha \to \alpha_M$ is an embedding.

In order to show the remaining part of the statement (ii) it suffices to show that κ_B is the restriction to $\underline{T}_B$ of κ_M, when identifying $\underline{T}_B$ with its image under the above considered embedding. Therefore, given any $\alpha, \beta \in \underline{T}_B$, we need to show that $\alpha\kappa_B\beta$ if and only if $\alpha_M\kappa_M\beta_M$. Again we let $\alpha : Be \to Bf$ and $\beta : Bg \to Bh$ with $e \in B_i$, $f \in B_j$,

$g \in B_k$, and $h \in B_l$ where $i, j, k, l \in L$. We have

$$\begin{aligned}
\alpha\kappa_B\beta &\Leftrightarrow e\mathcal{R}g \text{ and } f\mathcal{L}h \text{ in } B, \widetilde{\alpha} = \widetilde{\beta} \\
&\Leftrightarrow f = h \text{ and } \widetilde{\alpha} = \widetilde{\beta} \\
&\Leftrightarrow f = h \text{ and } \widetilde{\alpha}_K = \widetilde{\beta}_K \\
&\Leftrightarrow e\mathcal{R}g \text{ and } f\mathcal{L}h \text{ in } M \text{ and } \widetilde{\alpha}_M = \widetilde{\beta}_M \\
&\Leftrightarrow \alpha_M\kappa_M\beta_M
\end{aligned}$$

since $\widetilde{\alpha}_K = \widetilde{\alpha_M}$ and $\widetilde{\beta}_K = \widetilde{\beta_M}$. The proof of statement (ii) is complete.

We proceed to prove statement (iii) of the lemma. Therefore, let γ be an automorphism of the right normal band B and $\widetilde{\gamma}$ the automorphism of L induced by γ. By Lemma 3.1.1 there exists an extension $\widetilde{\gamma}_K$ of $\widetilde{\gamma}$ and an embedding $\mathrm{Aut}L \to \mathrm{Aut}K$ which maps $\widetilde{\gamma}$ to $\widetilde{\gamma}_K$. We define $\gamma_M : M \to M$ by:

$$\begin{aligned}
x\gamma_M &= x\gamma \text{ if } x \in B \\
&= x\widetilde{\gamma}_K \text{ if } x \in M \backslash B.
\end{aligned}$$

It is now routine to verify that γ_M is an automorphism of M and that $\mathrm{Aut}B \to \mathrm{Aut}M$, $\gamma \to \gamma_M$ is an embedding of groups. $\square$

Construction 5. We let B be a right normal band. Let $B = B_0, B_1, \ldots, B_i, \ldots$ be right normal bands such that for every $i < \omega$, B_{i+1} is constructed from B_i in the same way as M was constructed from B in the proof of Lemma 3.1.2. Let

$$\begin{aligned}
\theta_i :& B_i \to B_{i+1}, \\
\underline{\tau}_i :& \underline{T}_{B_i} \to \underline{T}_{B_{i+1}}, \\
\tau_i :& T_{B_i} \to T_{B_{i+1}}, \\
\sigma_i :& \mathrm{Aut}B_i \to \mathrm{Aut}B_{i+1}
\end{aligned}$$

be the embeddings which correspond to the embeddings described in the statement of Lemma 3.1.2 and its proof. Let

$$M = \bigcup_{i<\omega} B_i, \qquad \theta = \bigcup_{i<\omega} \theta_i,$$
$$\underline{\tau} = \bigcup_{i<\omega} \underline{\tau}_i, \qquad \tau = \bigcup_{i<\omega} \tau_i, \qquad \sigma = \bigcup_{i<\omega} \sigma_i.$$

Theorem 3.1.3. *Let B be a right normal band. Then there exists an embedding $\theta : B \to M$ of B into a uniform right normal band M such that $B\theta$ is a filter (for the natural partial order) of M, and such that*

(i) every partial isomorphism $\alpha : Be \to Bf$ of B can be extended to a partial isomorphism $\alpha_M : Me \to Mf$ of M such that the mappings

$$\underline{\tau} : \underline{T}_B \to \underline{T}_M, \qquad \alpha \to \alpha_M,$$
$$\tau : T_B \to T_M, \qquad \overline{\alpha} \to \overline{\alpha}_M$$

are embeddings and the diagram

$$\begin{tikzcd} \underline{T}_B \arrow[r, "\kappa^\natural_B"] \arrow[d, "\underline{\tau}"'] & T_B \arrow[d, "\tau"] \\ \underline{T}_M \arrow[r, "\kappa^\natural_M"'] & T_M \end{tikzcd} \quad , \quad \begin{tikzcd} \alpha \arrow[r, "\kappa^\natural_B"] \arrow[d, "\underline{\tau}"'] & \overline{\alpha} \arrow[d, "\tau"] \\ \alpha_M \arrow[r, "\kappa^\natural_M"'] & \overline{\alpha}_M \end{tikzcd}$$

commutes,

(ii) every automorphism γ of B can be extended to an automorphism γ_M of M such that

$$\sigma : \mathrm{Aut}B \to \mathrm{Aut}M, \qquad \gamma \to \gamma_M$$

embeds $\mathrm{Aut}B$ *isomorphically into* $\mathrm{Aut}M$.

Proof. We let $M, \theta, \underline{\tau}, \tau$, and σ be as in Construction 5. By Lemma 3.1.2 and Construction

5, M is the direct limit of the directed system (B_i, θ_i) of right normal bands B_i, $i < \omega$, therefore M is again a right normal band and θ is an embedding. From Lemma 3.1.2 we have that $B_i\theta_i$ is a filter of B_{i+1} for every $i < \omega$, hence $B\theta$ is a filter of M.

For any $i < \omega$ and $\alpha \in \underline{T}_B$, define $\alpha_{B_{i+1}}$ inductively by : $\alpha_{B_0} = \alpha$, whereas $\alpha_{B_{i+1}}$ is obtained from α_{B_i} in the same way as α_M was obtained from α in the proof of Lemma 3.1.2. Also put $\alpha_M = \bigcup \alpha_{B_i}$. An inductive argument using Lemma 3.1.2 and its proof shows that $\underline{T}_B \to \underline{T}_{B_i}$, $\alpha \to \alpha_{B_i}$ is an embedding for every $i < \omega$, and $\underline{\tau} : \underline{T}_B \to \underline{T}_M$, $\alpha \to \alpha_M$ is an embedding. Let $L = L_0, L_1, \ldots, L_i, \ldots$ be the structure semilattices of $B = B_0, B_1, \ldots, B_i, \ldots$ respectively: for each $i < \omega$, L_{i+1} is obtained from L_i in the same way as K was obtained from L in Lemma 3.1.1. Putting $K = \bigcup_{i<\omega} L_i$, we see that K is the structure semilattice of M. Given $\alpha \in \underline{T}_B$, let $\widetilde{\alpha} \in T_L$ be the partial isomorphism of L induced by α and, for every $i < \omega$, let $\widetilde{\alpha}_{L_i} = \widetilde{\alpha_{B_i}}$ be the partial isomorphism of L_i induced by α_{B_i}. Using induction we see that $T_L \to T_{L_i}$, $\widetilde{\alpha} \to \widetilde{\alpha}_{L_i}$ is an embedding and that

$$
\begin{tikzcd}
\underline{T}_B \arrow[r] \arrow[d] & T_L \arrow[d] \\
\underline{T}_{B_i} \arrow[r] & T_{L_i}
\end{tikzcd}
\quad , \quad
\begin{tikzcd}
\alpha \arrow[r] \arrow[d] & \widetilde{\alpha} \arrow[d] \\
\alpha_{B_i} \arrow[r] & \widetilde{\alpha}_{L_i} = \widetilde{\alpha_{B_i}}
\end{tikzcd}
$$

yields a commutative diagram. Putting $\widetilde{\alpha_M} = \widetilde{\alpha}_K = \bigcup_{i<\omega} \widetilde{\alpha}_{L_i}$, we then have that

$$
\begin{tikzcd}
\underline{T}_B \arrow[r] \arrow[d, "\underline{\tau}"'] & T_L \arrow[d] \\
\underline{T}_M \arrow[r] & T_K
\end{tikzcd}
\quad , \quad
\begin{tikzcd}
\alpha \arrow[r] \arrow[d, "\underline{\tau}"'] & \widetilde{\alpha} \arrow[d] \\
\alpha_M \arrow[r] & \widetilde{\alpha}_K = \widetilde{\alpha_M}
\end{tikzcd}
$$

is a commutative diagram.

Let $\alpha, \beta \in \underline{T}_B$ with $\alpha : Be \to Bf$ and $\beta : Bg \to Bh$. Then, using induction and the details of the proof of Lemma 3.1.2, we have for $i < \omega$ that

$$\begin{aligned}\alpha\kappa_B\beta &\Leftrightarrow e\mathcal{R}g \text{ and } f\mathcal{L}h \text{ in } B, \widetilde{\alpha} = \widetilde{\beta}\\ &\Leftrightarrow f = h \text{ and } \widetilde{\alpha} = \widetilde{\beta}\\ &\Leftrightarrow f = h \text{ and } \widetilde{\alpha}_{L_i} = \widetilde{\beta}_{L_i}\\ &\Leftrightarrow e\mathcal{R}g \text{ and } f\mathcal{L}h \text{ in } B_i \text{ and } \widetilde{\alpha_{B_i}} = \widetilde{\beta_{B_i}}\\ &\Leftrightarrow \alpha_{Bi}\kappa_{B_i}\beta_{B_i},\end{aligned}$$

hence

$$\alpha\kappa_B\beta \Leftrightarrow \alpha_M\kappa_M\beta_M,$$

otherwise stated,

$$\kappa_B = (\underline{T}_B \times \underline{T}_B) \cap \kappa_M.$$

It follows that the diagram mentioned in the statement (i) of the theorem is commutative and $\tau : T_B \to T_M$, $\overline{\alpha} \to \overline{\alpha}_M$ is an embedding.

Let $e, f \in M$. By Construction 5 there exists $i < \omega$ such that $e, f \in B_i$, then $(e, f) \in \mathcal{U}_{B_{i+1}}$ by Lemma 3.1.2. Using Lemma 3.1.2 and induction we find that $(e, f) \in \mathcal{U}_{B_j}$ for every $i < j < \omega$ and also $(e, f) \in \mathcal{U}_M$. Thus M is a uniform band.

Let γ be an automorphism of B and let $\widetilde{\gamma}$ be the automorphism of L induced by γ. Using Lemma 3.1.2 (iii) and induction we show that γ can be extended to an automorphism γ_{B_i} on B_i for every $i < \omega$ such that

$$\mathrm{Aut}B \to \mathrm{Aut}B_i, \qquad \gamma \to \gamma_{B_i}$$

is an embedding. Putting $\gamma_M = \bigcup_{i<\omega} \gamma_{B_i}$ it is now routine to verify that (ii) in the statement of the theorem holds true. $\square$

Theorem 3.1.4. *Let B be a right normal band. Every fundamental orthodox semigroup which has B as its band of idempotents can be embedded into a fundamental right generalized inverse semigroup.*

Proof. Let S be a fundamental orthodox semigroup whose band of idempotents is B. Then S can be embedded as a full regular subsemigroup into the orthodox semigroup T_B. Using Theorem 3.1.3, we have that T_B can be embedded into the fundamental orthodox semigroup T_M whose band of idempotents is (isomorphic to) the uniform right normal band M. Since M is uniform, it follows that T_M is bisimple. □

We now give a two-sided version of Theorems 3.1.3 and 3.1.4.

Theorem 3.1.5. *The statements of Theorem 3.1.3 are true when "right normal band" is replaced by "normal band" and "right generalized inverse semigroup" is replaced by "generalized inverse semigroup."*

Proof. Let B be a normal band and L its structure semilattice. There exists a right normal band B_1 and a left normal band B_2, each with the same structure semilattice L, such that B is (isomorphic to) the spined product $B_1 \rtimes B_2$ of B_1 and B_2 over the semilattice L. We henceforth identify B with $B_1 \rtimes B_2$. Let M_1 be the right normal band obtained from B_1 in the same way as M was obtained from B in the proof of Theorem 3.1.3 and M_2 be obtained from B_2 in a dual way. Let K be obtained from L as in the proof of Theorem 3.1.3. Then K is the structure semilattice of the right normal band M_1 and of the left normal band M_2. We let $M = M_1 \rtimes M_2$ be the spined product of M_1 and M_2 over K. Then M is a normal band which has K as its structure semilattice, and, from Theorem 3.1.3 and its dual, it follows that M is uniform and contains $B = B_1 \rtimes B_2$ as a subband and as a filter for the natural partial order.

Recall that for $i = 1, 2$ and $\alpha_i \in \underline{T}_{B_i}$, $\widetilde{\alpha}_i \in T$ denotes the partial isomorphism of L induced by α_i where $\underline{T}_{B_i} \to T_L$, $\alpha_i \to \widetilde{\alpha}_i$ induces the least inverse semigroup congruence on the orthodox semigroups $\underline{T}_{B_i}$, $i = 1, 2$. The identification of B with $B_1 \rtimes B_2$ leads us to identifying $\underline{T}_B$ with $\underline{T}_{B_1} \rtimes \underline{T}_{B_2}$, the spined product of $\underline{T}_{B_1}$ and $\underline{T}_{B_2}$

over T_L. Thus, $\underline{T}_B$ consists of the partial isomorphisms $\alpha = (\alpha_1, \alpha_2)$ of $B = B_1 \rtimes B_2$ where $\widetilde{\alpha}_1 = \widetilde{\alpha_2}$ in T_L. If $(\alpha_1)_{M_1}$ is constructed from α_1 in the same way as α_M was constructed from α in the proof of Theorem 3.1.3 and $(\alpha_2)_{M_2}$ is constructed from α_2 in a dual way, then $\alpha_M = ((\alpha_1)_{M_1}, (\alpha_2)_{M_2})$ is a partial isomorphism of $M = M_1 \rtimes M_2$, where $\widetilde{(\alpha_1)_{M_1}} = (\widetilde{\alpha_1})_K = (\widetilde{\alpha_2})_K = \widetilde{(\alpha_2)_{M_2}}$: here $\widetilde{(\alpha_1)_{M_1}}$ is the partial isomorphism induced by $(\alpha_1)_{M_1}$ on K, $(\widetilde{\alpha}_1)_K$ is the partial isomorphism of K obtained from α_1 according to the notation adopted in Theorem 3.1.3 and its proof, and the meanings of $\widetilde{(\alpha_2)_{M_2}}$ and $(\widetilde{\alpha}_2)_K$ follow by duality. Again, since $\underline{T}_M = \underline{T}_{M_1} \rtimes \underline{T}_{M_2}$ is the spined product of $\underline{T}_{M_1}$ and $\underline{T}_{M_2}$ over T_K, it follows from the above that

$$\underline{\tau} : \underline{T}_B \to \underline{T}_M, \qquad \alpha = (\alpha_1, \alpha_2) \to ((\alpha_1)_{M_1}, (\alpha_2)_{M_2})$$

is an embedding.

We let $\alpha = (\alpha_1, \alpha_2)$, $\beta = (\beta_1, \beta_2) \in \underline{T}_B = \underline{T}_{B_1} \rtimes \underline{T}_{B_2}$ and accordingly $\alpha_M = ((\alpha_1)_{M_1}, (\alpha_2)_{M_2})$, $\beta_M = ((\beta_1)_{M_1}, (\beta_2)_{M_2}) \in \underline{T}_M = \underline{T}_{M_1} \rtimes \underline{T}_{M_2}$. We assume that

$$\alpha : eBe \to fBf, \qquad \beta : gBg \to hBh$$

where $e = (e_1, e_2)$, $f = (f_1, f_2)$, $g = (g_1, g_2)$, and $h = (h_1, h_2)$. We then have from Theorem 3.1.3 and its dual,

$$\begin{aligned}
\alpha_M \kappa_M \beta_M &\Leftrightarrow e\mathcal{R}g, f\mathcal{L}h \text{ in } M \text{ and } \widetilde{\alpha_M} = \widetilde{\beta_M} \\
&\Leftrightarrow e_2 = g_2, f_1 = h_1, \widetilde{(\alpha_1)_{M_1}} = \widetilde{(\alpha_2)_{M_2}} = \widetilde{(\beta_1)_{M_1}} = \widetilde{(\beta_2)_{M_2}} \\
&\Leftrightarrow e_2 = g_2, f_1 = h_1, (\widetilde{\alpha}_1)_K = (\widetilde{\alpha}_2)_K = (\widetilde{\beta}_1)_K = (\widetilde{\beta}_2)_K \\
&\Leftrightarrow e_2 = g_2, f_1 = h_1, \widetilde{\alpha}_1 = \widetilde{\alpha}_2 = \widetilde{\beta}_1 = \widetilde{\beta}_2 \\
&\Leftrightarrow e\mathcal{R}g, f\mathcal{L}h, \widetilde{\alpha} = \widetilde{\beta} \\
&\Leftrightarrow \alpha \kappa_B \beta.
\end{aligned}$$

It follows that κ_B is the restriction to $\underline{T}_B$ of κ_M, so the diagram mentioned in Theorem 3.1.3 (i) is commutative.

Every automorphism γ of $B = B_1 \rtimes B_2$ can be written as $\gamma = (\gamma_1, \gamma_2)$ where for $i = 1, 2$, $\gamma_i : B_i \to B_i$ is an automorphism: for $e = (e_1, e_2) \in B$, $e\gamma = (e_1\gamma_1, e_2\gamma_2)$. For $i = 1, 2$, let $(\gamma_i)_{M_i}$ be the automorphism of M_i obtained from γ_i as in the proof of Theorem 3.1.3 (ii) and its dual. Then $\gamma_M = ((\gamma_1)_{M_1}, (\gamma_2)_{M_2})$ is an automorphism of $M = M_1 \rtimes M_2$ and $\mathrm{Aut}B \to \mathrm{Aut}M$, $\gamma \to \gamma_M$ is an embedding by Theorem 3.1.3 (ii). $\square$

Theorem 3.1.6. *The statement of Theorem 3.1.4 is true when "right normal band" is replaced by "normal band" and "right generalized inverse semigroup" is replaced by "generalized inverse semigroup."*

3.2 Extending partial isomorphisms

In this section we prove that every normal band B can be embedded into a normal band B' such that again T_B embeds into $T_{B'}$ and such that every partial isomorphism of B can be extended to an automorphism of B'. We moreover prove that every automorphism of B extends to an automorphism of B'. The results obtained in this section set the stage for finding an interesting embedding of any normal band into a normal band with high symmetry. We shall follow the same strategy as for Lemma 2 and Theorem 3 of [38], though with considerable modifications.

Construction 6. Let B be a right normal band and L its structure semilattice.

Let $\alpha \in \underline{R}_B$ (as defined in Section 1.3.3), B'_α an isomorphic copy of B, and $B \to B'_\alpha$, $e \to e'$ an isomorphism. Here we may as well assume that B and B'_α are disjoint. Let $(\mathrm{dom}\alpha)'$ be the image of $\mathrm{dom}\alpha$ under this isomorphism. We remove $(\mathrm{dom}\alpha)'$ from B'_α and replace it by the isomorphic copy $\mathrm{im}\alpha$: for every $e \in \mathrm{dom}\alpha$, we replace e' by $e\alpha$. We put $B_\alpha = (B'_\alpha \backslash (\mathrm{dom}\alpha)') \cup (\mathrm{im}\alpha)$ and define a multiplication on B_α in the same way as in B'_α under the provision that every element e', with $e \in \mathrm{dom}\alpha$, has been renamed $e\alpha$.

Then

$$\widehat{\alpha} : B \to B_\alpha, \qquad e \to e' \qquad \text{if } e \in B\backslash \text{dom}\alpha,$$
$$e \to e\alpha \qquad \text{if } e \in \text{dom}\alpha$$

is an isomorphism of B onto B_α such that $B \cap B_\alpha = \text{im}\alpha$, and such that $\widehat{\alpha}$ extends α. In particular, if $\alpha \in \text{Aut}B$, then $B = B_\alpha$ and $\widehat{\alpha} = \alpha$. We shall let ι stand for the identity transformation of B and we thus have $B = B_\iota$ and $\widehat{\iota} = \iota$.

We shall do the above described construction for every $\alpha \in \underline{R}_B$, and, when doing so, we may as well assume that if $\alpha \neq \beta$ in $\underline{R}_B$, then $(B_\alpha\backslash B)\cap(B_\beta\backslash B) = (B_\alpha\backslash\text{im}\alpha)\cap(B_\beta\backslash\text{im}\beta) = \emptyset$. Thus for any $\alpha \neq \beta$ in $\underline{R}_B$, $B_\alpha \cap B_\beta = B_\alpha \cap B_\beta \cap B = (B_\alpha \cap B) \cap (B_\beta \cap B) = \text{im}\alpha \cap \text{im}\beta$ which may well be empty. We shall put $B^{(1)} = \bigcup_{\alpha \in \underline{R}_B} B_\alpha$, and we proceed to define a multiplication on $B^{(1)}$ which will turn $B^{(1)}$ into a right normal band which contains B as a subsemigroup. We will do this by specifying the $\mathcal{R}-$relation, structure semilattice, and partial order of $B^{(1)}$.

We set out to construct the structure semilattice $L^{(1)}$ of our future right normal band $B^{(1)}$. This construction of $L^{(1)}$ parallels what we have done above.

Let $(L, \preceq)$ be the structure semilattice of B. Recall that for every $\alpha \in \underline{R}_B$, $\widetilde{\alpha}$ denotes the oi-isomorphism induced by α on L and that both dom$\widetilde{\alpha}$ and im$\widetilde{\alpha}$ are retract ideals of L. For any $\alpha \in \underline{R}_B$, let $\mathcal{R}'_\alpha$ $[\mathcal{R}_\alpha]$ be the $\mathcal{R}$-relation on B'_α $[B_\alpha]$ and $L'_\alpha = B'_\alpha/\mathcal{R}'_\alpha$ $[L_\alpha = B_\alpha/\mathcal{R}_\alpha]$ the structure semilattice of the right normal band B'_α $[B_\alpha]$. In particular $\mathcal{R}_\iota$ is the $\mathcal{R}$-relation on B. We will denote by $e'\mathcal{R}'{}_\alpha$ $[e\mathcal{R}_\alpha]$ the $\mathcal{R}$-class of e' $[e]$ in B'_α $[B_\alpha]$. If $B \to B'_\alpha$, $e \to e'$ is the isomorphism considered before, then $L \to L'_\alpha$, $e\mathcal{R}_\iota \to e'\mathcal{R}'_\alpha$ is the induced semilattice isomorphism. As before we may as well assume that L and L'_α, $\alpha \in \underline{R}_B$ are mutually disjoint.

For $\alpha \in \underline{R}_B$, let $L \to L'_\alpha$, $e\mathcal{R}_\iota \to e'\mathcal{R}'_\alpha$ be the above considered semilattice homomorphism and $(\text{dom}\widetilde{\alpha})'$ be the image of dom$\widetilde{\alpha}$ under this isomorphism. Remove $(\text{dom}\widetilde{\alpha})'$ from L'_α and replace it by the isomorphic copy im$\widetilde{\alpha}$: for every $e \in \text{dom}\alpha$ we replace $e'\mathcal{R}'_\alpha$

by $(e\alpha)\mathcal{R}_\iota$. Then, as before,

$$\widehat{\widetilde{\alpha}} : L \to L_\alpha, \qquad \begin{array}{ll} e\mathcal{R}_\iota \to e'\mathcal{R}'_\alpha & \text{if } e\mathcal{R}_\iota \in L\backslash \mathrm{dom}\widetilde{\alpha}, \\ e\mathcal{R}_\iota \to (e\alpha)\mathcal{R}_\iota & \text{if } e\mathcal{R}_\iota \in \mathrm{dom}\widetilde{\alpha} \end{array}$$

is an isomorphism of L onto L_α such that $L \cap L_\alpha = \mathrm{im}\widetilde{\alpha}$ and $\widehat{\widetilde{\alpha}}$ extends $\widetilde{\alpha}$.

We do this construction for every $\alpha \in \underline{R}_B$ and we may as well assume that $L_\alpha \cap L = \mathrm{im}\widetilde{\alpha}$, and, for $\alpha \neq \beta$ in $\underline{R}_B$, we have that

$$(L_\alpha \backslash L) \cap (L_\beta \backslash L) = (L_\alpha \backslash \mathrm{im}\widetilde{\alpha}) \cap (L_\beta \backslash \widetilde{\beta}) = \emptyset.$$

Thus for $\alpha \neq \beta$ in $\underline{R}_B$, $L_\alpha \cap L_\beta = L_\alpha \cap L_\beta \cap L = \mathrm{im}\widetilde{\alpha} \cap \mathrm{im}\widetilde{\beta}$ is a retract ideal of L.

We let $L^{(1)} = \cup_{\alpha \in \underline{R}_B} L_\alpha$ and $\preceq_\alpha$ be the semilattice order on L_α which corresponds to the semilattice order $\preceq$ on L under the isomorphism $\widehat{\widetilde{\alpha}}$. We then put $\preceq^{(1)} = \cup_{\alpha \in \underline{R}_B} \preceq_\alpha$, a relation on $L^{(1)}$.

Claim 3.2.1. $\preceq^{(1)}$ is a semilattice order on $L^{(1)}$ which induces the semilattice order $\preceq_\alpha$ on each L_α, $\alpha \in \underline{R}_B$.

We do not want to interrupt our construction of the right normal band $B^{(1)}$ and shall therefore postpone the proofs of the above and subsequent claims. However, it seems appropriate to remark here that the introduction of the notion of retract ideals is essential to proving Claim 3.2.1.

In the following we consider the mapping

$$\nu : B^{(1)} \to L^{(1)}, \qquad f \to f\mathcal{R}_\alpha, \qquad f \in B_\alpha.$$

We hasten to show that this is indeed a well defined mapping. Suppose that for some $\alpha \neq \beta$, $f \in B_\alpha \cap B_\beta$, then $f \in \mathrm{im}\alpha \cap \mathrm{im}\beta \subseteq B$ and $f\mathcal{R}_\alpha = f\mathcal{R}_\iota = f\mathcal{R}_\beta$ in $L^{(1)}$.

We let $\mathcal{R}^{(1)}$ be the equivalence relation induced on $B^{(1)}$ by the above considered mapping ν. The relation $\mathcal{R}^{(1)}$ will be the $\mathcal{R}$-relation on our future right normal band $B^{(1)}$. The following claim gives a more explicit description of the relation $\mathcal{R}^{(1)}$.

Claim 3.2.2. The equivalence relation $\mathcal{R}^{(1)}$ consists of the pairs $(x_\alpha, y_\beta) \in B_\alpha \times B_\beta$ for $\alpha, \beta \in \underline{R}_B$ where either $\alpha = \beta$ and $x_\alpha \mathcal{R}_\alpha y_\alpha$ or $\alpha \neq \beta$ and for some $z_\alpha \in B \cap B_\alpha = \mathrm{im}\alpha$ and $z_\beta \in B \cap B_\beta = \mathrm{im}\beta$, $x_\alpha \mathcal{R}_\alpha z_\alpha \mathcal{R}_\iota z_\beta \mathcal{R}_\beta y_\beta$ holds true.

Therefore,

Claim 3.2.3. $\mathcal{R}^{(1)}$ is the least equivalence relation on $B^{(1)}$ which induces R_α on each B_α, $\alpha \in \underline{R}_B$.

On $B^{(1)}$ we introduced the equivalence relation $\mathcal{R}^{(1)}$, and on $L^{(1)} \cong B^{(1)}/\mathcal{R}^{(1)}$ we considered the semilattice order $\preceq^{(1)}$. For every $\alpha \in \underline{R}_B$ we let $\leq_\alpha$ be the natural order on B_α, that is, the partial order which corresponds to the natural partial order $\leq$ under the isomorphism $\widehat{\alpha} : B \to B_\alpha$. We put $\leq^{(1)} = \cup_{\alpha \in \underline{R}_B} \leq_\alpha$. Then

Claim 3.2.4. $\leq^{(1)}$ is a partial order on $B^{(1)}$ which induces $\leq_\alpha$ on each B_α,

and

Claim 3.2.5. For any $r_1, r_2 \in L^{(1)}$, $r_1 \preceq^{(1)} r_2$ in $L^{(1)}$ if and only if there exists $\alpha \in \underline{R}_B$, $x_\alpha \leq_\alpha y_\alpha$ in B_α such that $r_1 = x_\alpha \nu$, $r_2 = y_\alpha \nu$.

We shall identify $B^{(1)}/R^{(1)}$ with $L^{(1)}$ and when doing so also identify the $\mathcal{R}^{(1)}$-class $x\mathcal{R}^{(1)}$ of $x \in B^{(1)}$ with $x\nu$. We take these $\mathcal{R}^{(1)}$-classes to be right zero semigroups, and, by Claim 3.2.3, they intersect each B_α according to an $\mathcal{R}_\alpha$-class, that is, a maximal right zero subband of B_α.

Claim 3.2.6. For $r_1 \preceq^{(1)} r_2$ in $L^{(1)}$ and $y \in B^{(1)}$ such that $y\nu = r_2$ there exists a unique $x \in B^{(1)}$ such that $x \leq^{(1)} y$ and $x\nu = r_1$.

Proposition 3.2.7. $B^{(1)} = B^{(1)}[L^{(1)}; R_r^{(1)}, \leq^{(1)}]$ *is a right normal band which contains each* B_α *as an order ideal.*

Proof. Now for $B^{(1)}$ we have a structure semilattice $L^{(1)}$, right zero bands $R_r^{(1)}$ for each $r \in L^{(1)}$ and a partial order $\leq^{(1)}$. From Claim 3.2.6 we see that $\leq^{(1)}$ is an appropriate partial order in terms of Result 1.3.4 and therefore $B^{(1)} = B^{(1)}[L^{(1)}; R_r^{(1)}, \leq^{(1)}]$ is a right normal band. That each B_α, and in particular $B = B_\iota$, is an order ideal of $B^{(1)}$ for each $\alpha \in \underline{R}_B$ then follows from Claims 3.2.1, 3.2.3, 3.2.4. $\square$

We proceed to give a proof of the previously stated claims.

Proof of Claim 3.2.1. Let $r_1 \preceq^{(1)} r_2$ for some $r_1, r_2 \in L^{(1)}$ with $r_2 \in L_\beta$. There exists $\alpha \in \underline{R}_B$ such that $r_1 \preceq_\alpha r_2$. If $\alpha \neq \beta$, then $r_2 \in L_\alpha \cap L_\beta = \operatorname{im}\widetilde{\alpha} \cap \operatorname{im}\widetilde{\beta}$ which is a retract ideal of L, L_α, and L_β. The restrictions of $\preceq, \preceq_\alpha$, and $\preceq_\beta$ to this retract ideal are the same. Therefore $r_1 \in L_\beta$ and $r_1 \preceq_\beta r_2$. We conclude that in any case,

$$r_1 \preceq^{(1)} r_2, r_2 \in L_\beta \Rightarrow r_1 \preceq_\beta r_2.$$

In particular, $\preceq^{(1)}$ induces the semilattice order $\preceq_\beta$ on L_β.

If, for some $r_1, r_2, r_3 \in L^{(1)}$, we have that $r_1 \preceq^{(1)} r_2 \preceq^{(1)} r_3$ where $r_3 \in L_\beta$ for some $\beta \in \underline{R}_B$, it then follows from the foregoing that $r_1 \preceq_\beta r_2 \preceq_\beta r_3$, thus $r_1 \preceq r_3$, so $r_1 \preceq^{(1)} r_3$. Therefore $\preceq^{(1)}$ is a transitive relation. It routinely follows that $\preceq^{(1)}$ is reflexive and antisymmetric and is therefore a partial order.

It remains to be shown that $\preceq^{(1)}$ is a semilattice order on $L^{(1)}$. That is, if $r_1, r_2 \in L^{(1)}$, we need to prove that r_1 and r_2 have a greatest common lower bound for $\preceq^{(1)}$ in $L^{(1)}$. First suppose that $r_1, r_2 \in L_\alpha$ for some $\alpha \in \underline{R}_B$. Then any lower bound of r_1 or r_2 is in L_α. Therefore any common lower bound of r_1, r_2 with respect to $\preceq^{(1)}$ is a common lower bound with respect to $\preceq_\alpha$, so the greatest common lower bound of r_1, r_2 in L_α is the greatest common lower bound of r_1, r_2 in $L^{(1)}$. Next suppose that $r_1 \in L_\alpha$, $r_2 \in L_\beta$ with $\alpha \neq \beta$. Then it follows that any common lower bound of r_1 and r_2 must be in $L_\alpha \cap L_\beta = \operatorname{im}\widetilde{\alpha} \cap \operatorname{im}\widetilde{\beta}$ which is a retract ideal of L, L_α and L_β. Let $r_3[r_4]$ be the greatest element of $\operatorname{im}\widetilde{\alpha} \cap \operatorname{im}\widetilde{\beta}$ which is less than or equal to r_1 $[r_2]$: this element, r_3 $[r_4]$, exists because $\operatorname{im}\widetilde{\alpha} \cap \operatorname{im}\widetilde{\beta}$ is a retract ideal of L, L_α, and L_β. Now let r be the greatest common lower bound of r_3 and r_4 in L. Since $\mathrm{im}\widetilde{\alpha} \cap \mathrm{im}\widetilde{\beta} \subseteq L$, it now follows that the greatest common lower bound of r_1 and r_2 in $L^{(1)}$ is r. $\square$

Proof of Claim 3.2.2. Let $f, g \in B^{(1)}$ such that $f\nu = g\nu$. There exist $\alpha, \beta \in \underline{R}_B$ such that $f \in B_\alpha$ and $g \in B_\beta$, then $f\mathcal{R}_\alpha$ and $g\mathcal{R}_\beta$ represent the same element of $L^{(1)}$. Clearly if $\alpha = \beta$, then $f\mathcal{R}_\alpha g$ in B_α. We now assume that $\alpha \neq \beta$. By our construction $f\mathcal{R}_\alpha = g\mathcal{R}_\beta$ is in $\mathrm{im}\widetilde{\alpha} \cap \mathrm{im}\widetilde{\beta} \subseteq L$, thus $f\mathcal{R}_\alpha = z\mathcal{R}_\iota = g\mathcal{R}_\beta$ for some $z \in L$. Therefore there exists $z_\alpha \in \mathrm{im}\alpha$ and $z_\beta \in \mathrm{im}\beta$ such that $f\mathcal{R}_\alpha z_\alpha \mathcal{R}_\iota z \mathcal{R}_\iota z_\beta \mathcal{R}_\beta g$. Conversely, if $f\mathcal{R}_\alpha z_\alpha \mathcal{R}_\iota z_\beta \mathcal{R}_\beta g$ for some $z_\alpha \in \mathrm{im}\alpha$ and $z_\beta \in \mathrm{im}\beta$, then $f\mathcal{R}_\alpha$, $z_\alpha\mathcal{R}_\iota$, $z_\beta\mathcal{R}_\iota$ and $g\mathcal{R}_\beta$ represent the same element of $L^{(1)}$, so $f\nu = g\nu$. $\square$

Claim 3.2.3 is an immediate consequence of Claim 3.2.2.

Proof of Claim 3.2.4. The proof here parallels the proof for Claim 3.2.1.

Let $x \leq^{(1)} y$ for some $x, y \in B^{(1)}$ and $y \in B_\beta$. There exists $\alpha \in \underline{R}_B$ such that $x \leq_\alpha y$. If $\alpha \neq \beta$, then $y \in B_\alpha \cap B_\beta = \mathrm{im}\alpha \cap \mathrm{im}\beta$ which is an order ideal of B, B_α and B_β. The restrictions of $\leq$, $\leq_\alpha$ and $\leq_\beta$ to this order ideal are the same, then, in particular, $x \in B_\beta$ and $x \leq_\beta y$. In any case,

$$x \leq^{(1)} y, y \in B_\beta \Rightarrow x \leq_\beta y,$$

so $\leq^{(1)}$ induces $\leq_\beta$ on B_β.

If for some $x, y, z \in B^{(1)}$ we have $x \leq^{(1)} y \leq^{(1)} z$ and $z \in B_\beta$, then $x \leq_\beta y \leq_\beta z$, by the above, so $x \leq_\beta z$ and $x \leq^{(1)} z$. Therefore $\leq^{(1)}$ is transitive. It is also routinely verified that $\leq^{(1)}$ is antisymmetric and reflexive and is therefore a partial order on $B^{(1)}$. $\square$

Proof of Claim 3.2.5. If $x_\alpha \leq_\alpha y_\alpha$ in B_α, then $x_\alpha\nu = x_\alpha\mathcal{R}_\alpha \preceq y_\alpha\mathcal{R}_\alpha = y_\alpha\nu$ in L_α, thus $x_\alpha\nu \preceq y_\alpha\nu$ in the semilattice $L^{(1)}$.

Conversely, suppose that $r_1, r_2 \in L^{(1)}$ such that $r_1 \preceq^{(1)} r_2$ in $L^{(1)}$. There exist $\alpha \in \underline{R}_B$ and $y_\alpha \in B_\alpha$ such that $r_2 = y_\alpha\nu = y_\alpha\mathcal{R}_\alpha \in L_\alpha$. Thus $r_1 = x\nu = x\mathcal{R}_\alpha$ for some $x \in B_\alpha$ and $x\mathcal{R}_\alpha \preceq_\alpha y_\alpha\mathcal{R}_\alpha$ in the semilattice $L_\alpha = B_\alpha/\mathcal{R}_\alpha$. In the right normal band B_α there exists a (unique) x_α such that $x\mathcal{R}_\alpha x_\alpha$ and $x_\alpha \leq_\alpha y_\alpha$. Then $x_\alpha \leq_\alpha y_\alpha$ in B_α such that $x_\alpha \nu = r_1$ and $y_\alpha \nu = r_2$. $\square$

Proof of Claim 3.2.6. In view of Claim 3.2.5 we only need to prove uniqueness.

Let $r_1 \preceq^{(1)} r_2$ in $L^{(1)}$ and $y \in B^1$ such that $y\nu = r_2$. There exists $\alpha \in \underline{R}_B$ such that $y \in B_\alpha$ and then $r_2 = y\mathcal{R}_\alpha \in L_\alpha$. As in the proof of Claim 3.2.5 there exists a unique $x \in B_\alpha$ such that $x\nu = r_1$ and $x \leq_\alpha y$ in B_α, where the uniqueness follows from the fact that B_α is a right normal band. Suppose that $z \in B^{(1)}$, $z\nu = x\nu = r_1$ and $z \leq^{(1)} y$. From the proof of Claim 3.2.4 we have that

$$z \leq^{(1)} y, y \in B_\alpha \Rightarrow z \leq_\alpha y,$$

so $z \in B_\alpha$. Then, by Claim 3.2.3, $x\nu = z\nu$ implies $x\mathcal{R}_\alpha z$, so, in the right normal band B_α,

$$x\mathcal{R}_\alpha z, x \leq_\alpha y, z \leq_\alpha y \Rightarrow x = z.$$

$\square$

In the following we shall adopt the notation established in Construction 6.

Lemma 3.2.8. *Let B be a right normal band and let $B^{(1)}$ be the right normal band obtained from B as in Construction 6.*
Then

(i) $B^{(1)}$ is a right normal band which contains B as an order ideal.

(ii) If $\alpha \in \underline{R}_B$, then α can be extended to oi-isomorphisms $\widehat{\alpha}$, $(\widehat{(\alpha^{-1})})^{-1} \in \underline{R}_{B^{(1)}}$ where $B \subseteq dom\widehat{\alpha}$ and $B \subseteq im(\widehat{(\alpha^{-1})})^{-1}$.

(iii) Every automorphism γ of B can be extended to an automorphism $\gamma^{(1)}$ of $B^{(1)}$ such that the mapping $\mathrm{Aut}B \to \mathrm{Aut}B^{(1)}$, $\gamma \to \gamma^{(1)}$ is an embedding of groups.

Proof. (i) follows directly from Proposition 3.2.7 since $B = B_\iota$.

We proceed to prove (ii). For $\alpha \in \underline{R}_B$ we construct the isomorphism $\widehat{\alpha} : B \to B_\alpha$ which extends α as in Construction 6. Clearly $\widehat{\alpha}$ is an oi-isomorphism of $B^{(1)}$ which induces on $L^{(1)} = B^{(1)}/\mathcal{R}^{(1)}$ an oi-isomorphism $\widetilde{\widehat{\alpha}}$ where $\mathrm{dom}\widetilde{\widehat{\alpha}} = L$ and $\mathrm{im}\widetilde{\widehat{\alpha}} = L_\alpha$. In order to show that $\widehat{\alpha} \in \underline{R}_{B^{(1)}}$, it suffices to show that $\mathrm{dom}\widetilde{\widehat{\alpha}} = L$ and $\mathrm{im}\widetilde{\widehat{\alpha}} = L_\alpha$ are retract ideals of $L^{(1)} = B^{(1)}/\mathcal{R}^{(1)}$. Since $L = L_\iota$ for the identity transformation ι on B, it suffices to show that L_α is a retract ideal of the semilattice $L^{(1)}$ for any $\alpha \in \underline{R}_B$. Therefore, let $i \in L^{(1)}$ and $\alpha \in \underline{R}_B$. If $i \in L_\alpha$, then $iL^{(1)} \cap L_\alpha = iL^{(1)} = iL_\alpha$ is a principal ideal of $L^{(1)}$ since L_α is an order ideal of $L^{(1)}$. If $i \in L_\beta$ with $\alpha \neq \beta$, then $iL^{(1)} \cap L_\alpha = iL_\beta \cap L_\alpha = iL_\beta \cap \mathrm{im}\widetilde{\alpha} \cap \mathrm{im}\widetilde{\beta}$ is a principal ideal of L_β and of $L^{(1)}$ since L_β is an order ideal of $L^{(1)}$, $\mathrm{im}\widetilde{\beta}$ a retract ideal of L_β and $\mathrm{im}\widetilde{\alpha} \cap \mathrm{im}\widetilde{\beta}$ a retract ideal of $\mathrm{im}\widetilde{\beta}$. We conclude that indeed L_α is a retract ideal of $L^{(1)}$ and $\widehat{\alpha} \in \underline{R}_{B^{(1)}}$. Similarly, $\widehat{(\alpha^{-1})}$ and $(\widehat{(\alpha^{-1})})^{-1}$ belong to $\underline{R}_{B^{(1)}}$.

Clearly $\widehat{\alpha}$ extends α by Construction 6. Also, $\mathrm{dom}\widehat{(\alpha^{-1})} = B$ and $\mathrm{im}\widehat{(\alpha^{-1})} = B_{\alpha^{-1}}$, thus $\mathrm{dom}(\widehat{(\alpha^{-1})})^{-1} = \mathrm{im}\widehat{(\alpha^{-1})} = B_{\alpha^{-1}}$ contains $\mathrm{im}\alpha^{-1} = \mathrm{dom}\alpha$. Let $x \in \mathrm{dom}\alpha$ and $x\alpha = y$. Then $x = y\alpha^{-1}$, so $x = y(\widehat{(\alpha^{-1})})$ since $\widehat{(\alpha^{-1})}$ extends α^{-1}. Therefore $y = x(\widehat{(\alpha^{-1})})^{-1}$, so $(\widehat{(\alpha^{-1})})^{-1}$ extends α, as required. Also $B = \mathrm{dom}\widehat{\alpha} = \mathrm{im}(\widehat{(\alpha^{-1})})^{-1}$ by Construction 6.

To prove (iii), let $\gamma \in \mathrm{Aut}B$. We define

$$\gamma^{(1)} : B^{(1)} \to B^{(1)}, \qquad x \to x(\widehat{\alpha})^{-1}(\widehat{(\alpha\gamma)}) \qquad \text{if } x \in B_\alpha, \alpha \in \underline{R}_B.$$

We need to show that $\gamma^{(1)}$ is a well-defined transformation of $B^{(1)}$, that is, if $\alpha \neq \beta$ in $\underline{R}_B$ and $x \in B_\alpha \cap B_\beta$, then $x(\widehat{\alpha})^{-1}(\widehat{(\alpha\gamma)}) = x(\widehat{\beta})^{-1}(\widehat{(\beta\gamma)})$. Indeed, if this is the case, then $x \in \mathrm{im}\alpha \cap \mathrm{im}\beta \subseteq B$, thus

$$\begin{aligned} x(\widehat{\alpha})^{-1}(\widehat{(\alpha\gamma)}) &= x\alpha^{-1}(\widehat{(\alpha\gamma)}) && \text{since } \widehat{\alpha} \text{ extends } \alpha \\ &= x\alpha^{-1}\alpha\gamma && \text{since } \alpha\gamma \in \underline{R}_B \text{ and } \mathrm{dom}\alpha\gamma = \mathrm{dom}\alpha \\ &= x\gamma, \end{aligned}$$

and, similarly, $x(\widehat{\beta})^{-1}(\widehat{(\beta\gamma)}) = x\gamma$, so $x(\widehat{\alpha})^{-1}(\widehat{(\alpha\gamma)}) = x\gamma = x(\widehat{\beta})^{-1}(\widehat{(\beta\gamma)})$, as required. In particular, for every $x \in B = B_\iota$, $x\gamma^{(1)} = x(\widehat{\iota})^{-1}(\widehat{(\iota\gamma)}) = x\gamma$, thus $\gamma^{(1)}$ extends γ. It is now clear that if $\gamma \neq \delta$ in Aut$B$, then $\gamma^{(1)} \neq \delta^{(1)}$.

We show that for $\gamma \in \mathrm{Aut}B$, $\gamma^{(1)}$ is a permutation of $B^{(1)}$. Certainly the restriction of $\gamma^{(1)}$ to B is the permutation γ of B as we have seen. Let $\alpha \in \underline{R}_B$. Since $\widehat{\alpha}$ is an isomorphism of B onto B_α which extends α, $(\widehat{\alpha})^{-1}$ is an isomorphism of B_α to B which restricts to the isomorphism α^{-1} of imα onto domα and to a bijection of $B_\alpha \backslash B = B_\alpha \backslash \mathrm{im}\alpha$ onto $B \backslash \mathrm{dom}\alpha = B \backslash \mathrm{dom}\alpha\gamma$. Similarly, $\widehat{(\alpha\gamma)}$ is an isomorphism of B onto $B_{\alpha\gamma}$ which restricts to the isomorphism $\alpha\gamma$ of domα =dom$\alpha\gamma$ onto im$\alpha\gamma$ and to a bijection of $B \backslash \mathrm{dom}\alpha = B \backslash \mathrm{dom}\alpha\gamma$ onto $B \backslash \mathrm{im}\alpha\gamma$. We conclude that $\gamma^{(1)}$ restricts to an isomorphism of B_α onto $B_{\alpha\gamma}$ which maps $\mathrm{im}\alpha = B \cap B_\alpha$ isomorphically to $\mathrm{im}\alpha\gamma = B \cap B_{\alpha\gamma}$, and which restricts to a bijective mapping of $B_\alpha \backslash B = B_\alpha \backslash \mathrm{im}\alpha$ onto $B_{\alpha\gamma} \backslash B = B_{\alpha\gamma} \backslash \mathrm{im}\alpha\gamma$. Similarly, $\gamma^{-1} \in \mathrm{Aut}B$ and for every $\beta \in \underline{R}_B$, $(\gamma^{-1})^{(1)}$ restricts to an isomorphism of B_β onto $B_{\beta\gamma^{-1}}$ which maps imβ isomorphically to $\mathrm{im}\beta\gamma^{-1} = B \cap B_{\beta\gamma^{-1}}$ and which restricts to a bijective mapping of $B_\beta \backslash B = B_\beta \backslash \mathrm{im}\beta$ onto $B_{\beta\gamma^{-1}} \backslash B = B_{\beta\gamma^{-1}} \backslash \mathrm{im}\beta\gamma^{-1}$. In particular, for $\beta = \alpha\gamma$, $(\gamma^{-1})^{(1)}$ maps $B_{\alpha\gamma}$ isomorphically to $B_{\alpha\gamma\gamma^{-1}} = B_\alpha$, im$\alpha\gamma$ isomorphically to imα =im$\alpha\gamma\gamma^{-1}$, and restricts to a bijective mapping of $B_{\alpha\gamma} \backslash B = B_{\alpha\gamma} \backslash \mathrm{im}\alpha\gamma$ onto $B_{\alpha\gamma\gamma^{-1}} \backslash B = B_\alpha \backslash \mathrm{im}\alpha$. Therefore, for $x \in B_\alpha$, we have $x\gamma^{(1)}(\gamma^{-1})^{(1)} = x\gamma\gamma^{-1} = x$ if $x \in B_\alpha \cap B = \mathrm{im}\alpha$, otherwise, $x \in B_\alpha \backslash B = B \backslash \mathrm{im}\alpha$ and

$$\begin{aligned} x\gamma^{(1)}(\gamma^{-1})^{(1)} &= x(\widehat{\alpha})^{-1}(\widehat{(\alpha\gamma)})((\gamma^{-1})^{(1)}) \\ &= x(\widehat{\alpha})^{-1}(\widehat{(\alpha\gamma)})(\widehat{(\alpha\gamma)})^{-1}(\widehat{(\alpha\gamma\gamma^{-1})}) \\ &= x(\widehat{\alpha})^{-1}\widehat{\alpha} = x. \end{aligned}$$

Similarly, for every $\beta \in \underline{R}_B$ and $x \in B_\beta$, $x(\gamma^{-1})^{(1)}\gamma^{(1)} = x$. We conclude that $\gamma^{(1)}$ and $(\gamma^{-1})^{(1)} = (\gamma^{(1)})^{-1}$ are pairwise inverse permutations of B.

We set out to prove that $\gamma^{(1)}$ is an automorphism of $B^{(1)}$. For any $\alpha \in \underline{R}_B$, the restriction of $\gamma^{(1)}$ to B_α is $(\widehat{\alpha})^{-1}(\widehat{(\alpha\gamma)})$ which is the composition of the isomorphisms $(\widehat{\alpha})^{-1} : B_\alpha \to B$ and $\widehat{(\alpha\gamma)} : B \to B_{\alpha\gamma}$. Therefore, the restriction of $\gamma^{(1)}$ to B_α is an isomorphism of B_α onto $B_{\alpha\gamma}$. In particular, if $x\mathcal{R}_\alpha y$ in B_α then $x\gamma^{(1)}\mathcal{R}_{\alpha\gamma}y\gamma^{(1)}$, and if $x \leq_\alpha y$ in B_α, then $x\gamma^{(1)} \leq_{\alpha\gamma} y\gamma^{(1)}$. Since the $\mathcal{R}$-relation $\mathcal{R}^{(1)}$ on $B^{(1)}$ is the transitive closure of $\bigcup_{\alpha\in \underline{R}_B} \mathcal{R}_\alpha$, it follows that $\gamma^{(1)}$ maps $\mathcal{R}^{(1)}$-related elements to $\mathcal{R}^{(1)}$-related elements. Since the natural partial order $\leq^{(1)}$ on $B^{(1)}$ is given by $\bigcup_{\alpha\in \underline{R}_B} \leq_\alpha$, it follows that $\gamma^{(1)}$ maps $\leq^{(1)}$-related elements to $\leq^{(1)}$-related elements. The same observations apply for $(\gamma^{-1})^{(1)} = (\gamma^{(1)})^{-1}$. Therefore $x\mathcal{R}^{(1)}y$ in $B^{(1)}$ if and only if $x\gamma^{(1)}\mathcal{R}^{(1)}y\gamma^{(1)}$ in $B^{(1)}$, and $x \leq^{(1)} y$ in $B^{(1)}$ if and only if $x\gamma^{(1)} \leq^{(1)} y\gamma^{(1)}$ in $B^{(1)}$. Since the multiplication for the right normal band $B^{(1)}$ is determined uniquely by $\leq^{(1)}$ and $\mathcal{R}^{(1)}$ it follows that $\gamma^{(1)}$ is an automorphism of $B^{(1)}$.

We next show that $\mathrm{Aut}B \to \mathrm{Aut}B^{(1)}$, $\gamma \to \gamma^{(1)}$ is an embedding of groups. As remarked before, this mapping is certainly injective, since γ is the restriction to B of $\gamma^{(1)}$ for every $\gamma \in \mathrm{Aut}B$. Let $\gamma, \delta \in \mathrm{Aut}B$. We need to show that $(\gamma\delta)^{(1)} = \gamma^{(1)}\delta^{(1)}$. Therefore, let $\alpha \in \underline{R}_B$ and $x \in B_\alpha$. Then

$$\begin{aligned}
x\gamma^{(1)}\delta^{(1)} &= x(\widehat{\alpha})^{-1}(\widehat{(\alpha\gamma)})\delta^{(1)} \\
&= x(\widehat{\alpha})^{-1}(\widehat{(\alpha\gamma)})(\widehat{(\alpha\gamma)})^{-1}(\widehat{(\alpha\gamma\delta)}) && \text{(since } x\gamma^{(1)} \in B_{\alpha\gamma}) \\
&= x(\widehat{\alpha})^{-1}(\widehat{(\alpha\gamma\delta)}) \\
&= x(\gamma\delta)^{(1)}.
\end{aligned}$$

□

The proof of the following theorem follows the same argument as the proof of Theorem 3 of [38]. We shall need to return to the details of the proof when we prove a two-sided version.

Theorem 3.2.9. *Every right normal band B can be embedded as an order ideal and as a subsemigroup into a right normal band B' in such a way that*

(i) every partial isomorphism $\alpha : B'e \to B'f$ of B' can be extended to an automorphism α' of B',

(ii) every automorphism γ of B can be extended to an automorphism γ' of B' such that $\mathrm{Aut}B \to \mathrm{Aut}B'$, $\gamma \to \gamma'$ *is an embedding of groups.*

Proof. We consider the sequence of right normal bands, $B = B^{(0)}, B^{(1)}, \ldots, B^{(i)}, \ldots$, where $B^{(i+1)}$ is constructed from $B^{(i)}$ in the same way as $B^{(1)}$ is constructed from B in Construction 6. We let $B' = \bigcup_{i<\omega} B^{(i)}$ be the direct limit of this sequence, and it follows from Lemma 3.2.8 (i) that B' is a right normal band which contains every $B^{(i)}$ as an order ideal and as a subsemigroup.

Let $\alpha : B'e \to B'f$ be a partial isomorphism of B'. There exists $i < \omega$ such that $e, f \in B^{(i)}$, and since $B^{(i)}$ is an order ideal of B', we have that $B^{(i)}e = B'e$ and $B^{(i)}f = B'f$. Therefore α is a partial isomorphism of $B^{(i)}$, and $\alpha \in \underline{R}_{B^{(i)}}$. We use Lemma 3.2.8 (ii) to construct $\alpha_j \in \underline{R}_{B^{(i+j)}}$ inductively as follows:

$$\begin{aligned} \alpha_0 &= \alpha, & \\ \alpha_{j+1} &= \widehat{\alpha_j} & \text{if } j \text{ is even,} \\ \alpha_{j+1} &= (\widehat{(\alpha_j^{-1})})^{-1} & \text{if } j \text{ is odd.} \end{aligned}$$

We shall put $\alpha' = \bigcup_{j<\omega} \alpha_j$. Since by Lemma 3.2.8 (ii) we have

$$\alpha = \alpha_0 \subseteq \alpha_1 \subseteq \ldots \subseteq \alpha_j \subseteq \ldots \subseteq \alpha'$$

and $\alpha_j \in \underline{R}_{B^{(i+j)}}$, it follows that α' is a partial transformation of B' which maps $\mathrm{dom}\alpha'$ isomorphically onto $\mathrm{im}\alpha'$. If $x \in B'$, then for some $i \leq i+j$, $x \in B^{(i+j)}$. From our construction of the α_j and Lemma 3.2.8 (ii) we have that $B^{(i+j)} \subseteq \mathrm{dom}\alpha_{j+1}$ and $B^{(i+j)} \subseteq \mathrm{im}\alpha_{j+2}$, thus $x \in \mathrm{dom}\alpha'$ and $x \in \mathrm{im}\alpha'$. Therefore $\mathrm{dom}\alpha' = B' = \mathrm{im}\alpha'$, and α' is an automorphism of B' which extends α. We proved (i).

Let $\gamma \in \mathrm{Aut}B$. We use Lemma 3.2.8 (iii) to construct $\gamma_j \in \mathrm{Aut}B^{(j)}$ inductively as follows: $\gamma_0 = \gamma$, and γ_{j+1} is constructed from γ_j in the same way as $\gamma^{(1)}$ is constructed from γ in the course of the proof of Lemma 3.2.8. We put $\gamma' = \bigcup_{j<\omega} \gamma_j$ and find that γ'

is an automorphism of B' which extends γ. Using an inductive argument it follows from Lemma 3.2.8 (iii) that $\mathrm{Aut}B \to \mathrm{Aut}B'$, $\gamma \to \gamma'$ is an embedding of groups. $\square$

Before we give a two-sided version of Theorem 3.2.9, we need the following.

Construction 7. Let B be a normal band with structure semilattice L. We shall assume that $B = B_1 \rtimes B_2$ is the spined product of the right normal band B_1 and the left normal band B_2 over the semilattice L. We let A_1 be a right normal band and A_2 a left normal band such that $A_1 \to B_2$, $e \to e^*$ and $B_1 \to A_2$, $f \to f^*$ are anti-isomorphisms. We shall assume here that $A_1 \cap B_1 = \emptyset = A_2 \cap B_2$, and we put $C_1 = A_1 \cup B_1$ and $C_2 = A_2 \cup B_2$. Thus the mapping $^* : C_1 \to C_2$, $e \to e^*$ is a bijection and the restrictions of * to A_1 and B_1 are anti-isomorphisms. We define multiplication on C_1 in such a way that C_1 becomes a right normal band which contains A_1 and B_1 as subsemigroups. In order to do so we need to define an appropriate Green $\mathcal{R}$-relation, $\mathcal{R}_{C_1}$, on C_1 and an appropriate natural partial order $\leq_{C_1}$ on C_1. We simply put $\leq_{C_1} = \leq_{A_1} \cup \leq_{B_1}$, the (disjoint) union of the natural partial order $\leq_{A_1}$ on A_1 and the natural partial order $\leq_{B_1}$ on B_1. We let $\mathcal{R}_{C_1}$ be the equivalence relation on C_1 such that each $\mathcal{R}_{C_1}$-class is the (disjoint) union of an $\mathcal{R}$-class of A_1 and an $\mathcal{R}$-class of B_1; more specifically, for $e \in A_1$ and $f \in B_1$ we shall put $e\mathcal{R}_{C_1}f$ if and only if $(f, e^*) \in B_1 \rtimes B_2 = B$. One then introduces a multiplication on C_1 such that $\leq_{C_1}$ becomes the natural partial order on C_1 and $\mathcal{R}_{C_1}$ the Green $\mathcal{R}$-relation on C_1: this multiplication is uniquely defined, A_1 and B_1 are subsemigroups and order ideals of C_1, and L is the structure semilattice of A_1, B_1, and C_1. In a left-right dual way we introduce a multiplication on C_2 which turns C_2 into a left normal band for which L is the structure semilattice. We let $C = C_1 \rtimes C_2$ be the spined product of C_1 and C_2 over L. Then C is a normal band which contains $B = B_1 \rtimes B_2$ as an order ideal and as a subsemigroup. We see that $^* : C_1 \to C_2$, $e \to e^*$ is an anti-isomorphism of C_1 onto C_2. This allows us to define an anti-automorphism $^* : C \to C$ where $(e, f^*)^* = (f, e^*)$ for every $e, f \in C_1$, with $e\mathcal{R}_{C_1}f$. This anti-automorphism * is $\mathcal{D}$-class preserving:$(e, f^*)^*\mathcal{D}(e, f^*)$ in C.

Lemma 3.2.10. *Let B be a normal band. Then B can be embedded into a normal band*

$B^{(1)}$ such that the assertions of Lemma 3.2.8 (i), (ii), and (iii) are true when replacing "right normal band" by "normal band."

Proof. We let $B = B_1 \rtimes B_2$ and $C = C_1 \rtimes C_2$ as in Construction 7. We construct $C_1^{(1)}$ from C_1 in the same way as $B^{(1)}$ was constructed from B in Construction 6, and we construct $C_2^{(1)}$ from C_2 in a dual way. If β is an oi-isomorphism of C_1, then β^*, given by

$$\begin{aligned} &\mathrm{dom}\beta^* = (\mathrm{dom}\beta)^* = \{e^* | e \in \mathrm{dom}\beta\}, \\ &e^*\beta^* = (e\beta)^* \qquad\qquad \text{for every } e \in \mathrm{dom}\beta, \end{aligned}$$

is an oi-isomorphism of C_2, and

$$\mathrm{im}\beta^* = (\mathrm{im}\beta)^* = \{e^* | e \in \mathrm{im}\beta\}.$$

Moreover, since L is the structure semilattice of both C_1 and C_2 and, for every $e \in \mathrm{dom}\beta$, $e\beta$ and $e^*\beta = (e\beta)^*$ correspond to the same element of L, we have that β and β^* induce the same oi-isomorphism $\widetilde{\beta} = \widetilde{\beta^*}$ on L. In particular, $\mathrm{dom}\widetilde{\beta}$ and $\mathrm{im}\widetilde{\beta}$ are retract ideals of L if and only if $\mathrm{dom}\widetilde{\beta^*}$ and $\mathrm{im}\widetilde{\beta^*}$ are retract ideals of L. It follows that $\underline{R}_{C_1} \to \underline{R}_{C_2}$, $\beta \to \beta^*$ is a bijection. Following the notation of Construction 6 and its left-right dual counterpart, for every $\beta \in \underline{R}_{C_1}$ and $\widehat{\beta} : C_1 \to (C_1)_{\beta_1}$, the isomorphism which extends β, we can extend the given * to $^* : (C_1)_\beta \to (C_2)_{\beta^*}$ by putting $(e(\widehat{\beta}))^* = e^*\widehat{(\beta^*)}$ for every $e \in C_1$. Then according to Construction 6 and its left-right dual, $^* : C_1^{(1)} \to C_2^{(1)}$ is an anti-isomorphism. The structure semilattice $L^{(1)}$ of $C_1^{(1)}$ and $C_2^{(1)}$ is the same in view of the above considered bijection $\underline{R}_{C_1} \to \underline{R}_{C_2}$. We put $B^{(1)} = C_1^{(1)} \rtimes C_2^{(1)}$, the spined product of $C_1^{(1)}$ and $C_2^{(1)}$ over $L^{(1)}$. By Lemma 3.2.8 (i) and its dual, $C = C_1 \rtimes C_2$ is a subsemigroup and an order ideal of $B^{(1)}$ and since $B = B_1 \rtimes B_2$ is a subsemigroup and an order ideal of C, it follows that B is a subsemigroup and an order ideal of $B^{(1)}$. We proved (i).

We set out to prove (ii). Therefore, let $\alpha \in \underline{R}_B$. If $(e, f^*), (e, f_1^*) \in \mathrm{dom}\alpha$ with $e \in B_1$, $f, f_1 \in A_1$, and $(e, f^*)\alpha = (e', f'^*)$,$(e, f_1^*)\alpha = (e'', f_1''^*)$, then, since $(e, f^*)\mathcal{L}(e, f_1^*)$ in B,

$(e', f'^*)\mathcal{L}(e'', f_1''^*)$ in B, hence $e' = e''$. From this and its dual, it follows that there exist partial transformations α_1 and α_2 of B_1 and A_1 such that $(e, f^*)\alpha = (e\alpha_1, (f\alpha_2)^*)$ for every $(e, f^*) \in$ domα. Since $\alpha \in \underline{R}_B$, it follows that $\alpha_1 \in \underline{R}_{B_1}$, $\alpha_2 \in \underline{R}_{A_1}$, and $\widetilde{\alpha_1} = \widetilde{\alpha_2} = \widetilde{\alpha} = \widetilde{\alpha_1^*} = \widetilde{\alpha_2^*}$ the same oi-isomorphism induced on L. Further, dom$\alpha =$dom$\alpha_1 \rtimes$domα_2^*, the spined product of domα_1 and domα_2^* over the retract ideal dom$\widetilde{\alpha_1}$ =dom$\widetilde{\alpha_2^*}$ of L, and imα =im$\alpha_1 \rtimes$imα_2^*, the spined product of imα_1 and imα_2^* over the retract ideal im$\widetilde{\alpha_1}$ =im$\widetilde{\alpha_2^*}$ of L. It will be convenient to write $\alpha = (\alpha_1, \alpha_2^*)$, since for any $(e, f^*) \in$ domα, we have $(e, f^*)\alpha = (e\alpha_1, f^*\alpha_2^*)$.

We have that $\alpha_1 \cup \alpha_2 \in \underline{R}_{C_1}$ and therefore also $(\alpha_1 \cup \alpha_2)^* = \alpha_1^* \cup \alpha_2^* \in \underline{R}_{C_2}$. With the notation of Construction 6 and its dual, $((\widehat{\alpha_1 \cup \alpha_2})) : C_1 \to (C_1)_{\alpha_1 \cup \alpha_2}$ is an oi-isomorphism of $C_1^{(1)}$ which, according to Lemma 3.2.8 (ii), belongs to $\underline{R}_{C_1^{(1)}}$, and $((\widehat{\alpha_1 \cup \alpha_2})^*) : C_2 \to (C_2)_{(\alpha_1 \cup \alpha_2)^*}$ is an oi-isomorphism of $C_2^{(1)}$ which belongs to $\underline{R}_{C_2^{(1)}}$. We denote by $\widehat{\alpha}$ the oi-isomorphism of $B^{(1)} = C_1^{(1)} \rtimes C_2^{(2)}$ such that dom$\widehat{\alpha} = C$ and such that for every $(e, f^*) \in C = C_1 \rtimes C_2$ with $e, f \in C_1$, we have

$$(e, f^*)(\widehat{\alpha}) = (e((\widehat{\alpha_1 \cup \alpha_2})), f^*(((\widehat{\alpha_1 \cup \alpha_2})^*)).$$

Then $\widehat{\alpha}$ induces an oi-isomorphism $\widetilde{\widehat{\alpha}}$ on the structure semilattice $L^{(1)}$ of $B^{(1)}$ such that dom$\widetilde{\widehat{\alpha}} = L$ is a retract ideal of $L^{(1)}$ and im$\widetilde{\widehat{\alpha}}$ is the structure semilattice of $(C_1)_{\alpha_1 \cup \alpha_2}$, $(C_2)_{(\alpha_1 \cup \alpha_2)^*}$ and im$\widehat{\alpha} = (C_1)_{\alpha_1 \cup \alpha_2} \rtimes (C_2)_{(\alpha_1 \cup \alpha_2)^*}$, a retract ideal of $L^{(1)}$. Therefore, $\widehat{\alpha} \in \underline{R}_{B^{(1)}}$ and $B \subseteq$dom$\widehat{\alpha}$. For every $(e, f^*) \in B$, with $e \in B_1$ and $f \in A_1$, we have

$$\begin{aligned}
(e, f^*)(\widehat{\alpha}) &= (e((\widehat{\alpha_1 \cup \alpha_2})), f^*(((\widehat{\alpha_1 \cup \alpha_2})^*))) \\
&= (e(\alpha_1 \cup \alpha_2), f^*(\alpha_1^* \cup \alpha_2^*)) \\
&\qquad (\text{ since } ((\widehat{\alpha_1 \cup \alpha_2})) \text{ extends } \alpha_1 \cup \alpha_2 \\
&\qquad \text{and } (((\widehat{\alpha_1 \cup \alpha_2})^*) \text{ extends } (\alpha_1 \cup \alpha_2)^* = \alpha_1^* \cup \alpha_2^*) \\
&= (e\alpha_1, f^*\alpha_2^*) \\
&= (e, f^*)\alpha,
\end{aligned}$$

therefore $\widehat{\alpha}$ extends α. Similarly, $(\widehat{(\alpha^{-1})})^{-1} \in \underline{R}_{B^{(1)}}$, $B \subseteq \mathrm{im}(\widehat{(\alpha^{-1})})^{-1}$, and $(\widehat{(\alpha^{-1})})^{-1}$ extends α. We proved (ii).

To prove (iii), let γ be an automorphism of B. As before, there exist automorphisms γ_1 of B_1 and γ_2 of A_1 such that for every $(e, f^*) \in B$, with $e \in B_1$ and $f \in A_1$, we have $(e, f^*)\gamma = (e\gamma_1, f^*\gamma_2^*) = (e\gamma_1, (f\gamma_2)^*)$. Then $\gamma_1 \cup \gamma_2 \in \mathrm{Aut}C_1$ and $(\gamma_1 \cup \gamma_2)^* \in \mathrm{Aut}C_2$. By Lemma 3.2.8 (iii) and its dual, both these automorphisms can be extended to automorphisms $(\gamma_1 \cup \gamma_2)^{(1)}$ and $(\gamma_1 \cup \gamma_2)^{*(1)}$ of $C_1^{(1)}$ and $C_2^{(1)}$, respectively. Define

$$\gamma^{(1)} : B^{(1)} \to C_1^{(1)} \times C_2^{(1)},$$
$$(x, y^*) \to (x(\gamma_1 \cup \gamma_2)^{(1)}, y^*(\gamma_1 \cup \gamma_2)^{*(1)}), \qquad x, y \in C_1^{(1)}.$$

Following the details of the constructions of $(\gamma_1 \cup \gamma_2)^{(1)}$ and $(\gamma_1 \cup \gamma_2)^{*(1)}$ as outlined in the proof of Lemma 3.2.8 (iii), a lengthy but routine verification shows that $\gamma^{(1)}$ maps $B^{(1)}$ onto itself and is an automorphism of $B^{(1)}$ which extends $\gamma = (\gamma_1, \gamma_2^*)$. □

Using Lemma 3.2.10 we are now in the position to prove the following theorem. The proof of Theorem 3.2.11 is otherwise verbatim the proof of Theorem 3.2.9.

Theorem 3.2.11. *The statements of Theorem 3.2.9 are true when "right normal band" is replaced by "normal band."*

We conclude this section with some remarks about the constructions. The first remark concerns the use of retract ideals in Construction 6, and the second remark explains why Construction 7 was necessary at all.

The construction of $B^{(1)}$ from the right normal band B in Construction 6 is an analogue of the corresponding construction of the semilattice $L^{(1)}$ from the semilattice L in [38]. The latter one needs partial isomorphisms only. In both situations we want to extend a partial isomorphism of the given band B or semilattice L to an appropriate oi-isomorphism of the larger $B^{(1)}$ or $L^{(1)}$ which contains B or L in its domain. For a semilattice L one can restrict oneself to partial isomorphisms only because every oi-isomorphism of the $L^{(1)}$ in [38] which has domain L can be extended to a partial isomorphism of $L^{(1)}$ which has domain L^1. This, however, cannot be done when we instead start with a right normal band B which is not a semilattice, because in this case B^1 is not a normal band anymore. We nevertheless want to construct a right normal band $B^{(1)}$ which contains B as an order ideal and as a subsemigroup. Thus the structure semilattice of B should be an ideal of the structure semilattice of $B^{(1)}$. It is therefore natural to make use of the concept of a retract ideal, since this concept is an essential tool in the construction of ideal extensions of semilattices (see [48]).

Construction 7 appears to lead to a roundabout way to obtain the two-sided version in Theorem 3.2.11. However, given B as the spined product $B = B_1 \rtimes B_2$ of the right normal band B_1 and the left normal band B_2 over the semilattice L and $B_1^{(1)}$ and $B_2^{(1)}$ constructed from B_1 and B_2 following the procedure of Construction 6 and its dual, it is in general not possible to form the spined product of $B_1^{(1)}$ and $B_2^{(1)}$ since these two need not have the same structure semilattice.

3.3 Normal bands with a transitive automorphism group

In this section we combine the results obtained in Section 3.1 and 3.2 to prove our main result.

Theorem 3.3.1. *Every normal band B can be embedded as a subsemigroup and as a convex subset (for the natural partial order) into a normal band N which has a transitive automorphism group and such that*

(i) N is a right normal band [left normal band, semilattice] if the same holds true for B,

(ii) every partial isomorphism $\alpha_B : eBe \to fBf$ of B can be extended to a partial isomorphism $\alpha_N : eNe \to fNf$ of N in such a way that the mapping $T_B \to T_N$, $\overline{\alpha_B} \to \overline{\alpha_N}$ is an embedding of T_B into T_N,

(iii) every partial isomorphism of N can be extended to an automorphism of N,

(iv) every automorphism γ_B of B can be extended to an automorphism γ_N of N such that $\mathrm{Aut}B \to \mathrm{Aut}N$, $\gamma_B \to \gamma_N$ *is an embedding of groups.*

Proof. If B is a semilattice, then the statement is precisely the statement of Theorem 4 of [38]. The proof for the remaining cases is similar to the proof given for Theorem 4 in [38] and is now based on the main results obtained earlier in Sections 3.1 and 3.2. If B is a right normal band, then we use our Theorems 3.1.3 and 3.2.9. If B is a left normal band, then the proof follows by duality. For the two-sided version we shall apply our Theorems 3.1.5 and 3.2.11. We shall give a proof for the two-sided version only.

We let B be a normal band and we consider the sequence of normal bands

$$B = B_0, M_0, B_1, M_1, \ldots, B_j, M_j, \ldots$$

where for every j, M_j is constructed from B_j in the same way as M was constructed from B in Theorem 3.1.5 and B_{j+1} is constructed from M_j in the same way as B' was constructed from B in Theorem 3.2.11. We let N be the direct limit of this sequence. When identifying each member of this sequence with the corresponding subsemigroup of its successor, we may as well assume that $N = \bigcup_{j<\omega} B_j = \bigcup_{j<\omega} M_j$. It immediately follows from Theorems 3.1.5 and 3.2.11 that N is a normal band which contains B as a subsemigroup and as a convex subset.

Let γ_j be an automorphism of B_j. By Theorems 3.1.5 (ii) and 3.2.11 (ii), γ_j can be extended to an automorphism of M_j which in turn can be extended to an automorphism γ_{j+1} of B_j such that $\mathrm{Aut}B_j \to \mathrm{Aut}B_{j+1}$, $\gamma_j \to \gamma_{j+1}$ is an embedding of groups. Continuing in this way we construct a sequence

$$\gamma_j, \gamma_{j+1}, \ldots, \gamma_{j+k}, \ldots$$

where $\gamma_{j+k} \in \mathrm{Aut}B_{j+k}$. Then $\bigcup_{k<\omega} \gamma_{j+k} \in \mathrm{Aut}N$ and $\mathrm{Aut}B_j \to \mathrm{Aut}N$, $\gamma_j \to \bigcup_{k<\omega} \gamma_{j+k}$ is an embedding of groups. In particular, every automorphism $\gamma_B = \gamma_0$ of $B = B_0$ can be extended to an automorphism $\gamma_N = \bigcup_{k<\omega} \gamma_k$ of N such that $\mathrm{Aut}B \to \mathrm{Aut}N$, $\gamma_B \to \gamma_N$ is an embedding of groups. We proved that (iv) is satisfied.

Let α_j be a partial isomorphism of B_j. Then α_j can be extended to a partial isomorphism of M_j by Theorem 3.1.5 (i) and this partial isomorphism of M_j can be extended to an automorphism γ_{j+1} of B_{j+1} by Theorem 3.2.11 (i). Thus by the result obtained in the previous paragraph, α_j can be extended to an automorphism of N. In particular we proved (iii).

Let $e, f \in N$. There exist $j < \omega$ such that $e, f \in B_j$. By Theorem 3.1.5, M_j is uniform, so there exists a partial isomorphism of M_j which maps e to f. This same partial isomorphism of M_j is also a partial isomorphism α_{j+1} of B_{j+1} since, by Theorem 3.2.11, M_j is a subsemigroup and an order ideal of B_{j+1}. By what we have seen above, α_{j+1} extends to an automorphism γ of N. Hence since $e\alpha_{j+1} = f$ we have that $e\gamma = f$ for some $\gamma \in \mathrm{Aut}N$. We proved that N has a transitive automorphism group.

Let α_j be a partial isomorphism of B_j. By Theorem 3.1.5 (i), α_j can be extended to a partial isomorphism of M_j, and this very same partial isomorphism of M_j is a partial isomorphism α_{j+1} of B_{j+1} because M_j is a subsemigroup and an order ideal of B_{j+1} by Theorem 3.2.11. Continuing in this way we construct a sequence

$$\alpha_j, \alpha_{j+1}, \ldots, \alpha_{j+k}, \alpha_{j+k+1}, \ldots$$

where for some $e, f \in B_j$, $\alpha_j : eB_je \to fB_jf$ and $\alpha_{j+k+1} : eB_{j+k+1}e \to fB_{j+k+1}f$ is a partial isomorphism of B_{j+k+1} which extends $\alpha_{j+k} : eB_{j+k}e \to fB_{j+k}f$. Putting $\alpha = \bigcup_{k<\omega} \alpha_{j+k}$ we have that $\alpha : eNe \to fNf$ is a partial isomorphism of N which extends α_j. From Theorems 3.1.5 (i) and 3.2.11, an inductive argument shows that

$$\underline{T}_{B_j} \to \underline{T}_{B_{j+k}}, \qquad \alpha_j \to \alpha_{j+k}$$

and

$$T_{B_j} \to T_{B_{j+k}}, \qquad \overline{\alpha}_j \to \overline{\alpha}_{j+k}$$

are embeddings, and that consequently

$$\underline{T}_{B_j} \to \underline{T}_N, \qquad \alpha_j \to \alpha$$

and

$$T_{B_j} \to T_N, \qquad \overline{\alpha}_j \to \overline{\alpha}$$

are embeddings. In particular, for $j = 0$ the statement (ii) follows. $\square$

In view of Theorem 3.3.1 there now is an obvious analogue to Theorems 3.1.4 and 3.1.6 that every fundamental [left, right] generalized inverse semigroup can be embedded into a bisimple [left,right] generalized inverse semigroup which has a transitive automorphism group. The particular case where S is an inverse semigroup has given rise in [40] to some interesting division theorems for inverse and locally inverse semigroups. It is likely that the embedding theorems for normal bands that we have considered here may lead to interesting division theorems for generalized inverse semigroups or, in general, for regular semigroups whose idempotents generate a normal band of groups.

Chapter 4

Uniform Bands

In this chapter we will show every band B can be embedded into a uniform band B' such that B and B' generate the same band variety. We also show that every orthodox semigroup S can be embedded into a bisimple orthodox semigroup S' such that the bands $E(S)$ and $E(S')$ generate the same band variety. Note that in Chapter 3 we showed this result for normal bands. That is, the results of this chapter can be specialized to show the results in Section 3.1. We have retained Section 3.1 because we use specifics of the embedding written there in Section 3.3.

In the following we recall the relevant facts. Every semigroup can be embedded into a bisimple semigroup [54] (see also §8.6 of [4]) and every inverse semigroup can be embedded into a bisimple inverse semigroup [57]. In fact, every inverse semigroup can be embedded into a bisimple inverse semigroup which has no nontrivial congruences [27]. In particular, every semilattice can be embedded into a uniform semilattice and every fundamental inverse semigroup can be embedded into a bisimple fundamental inverse semigroup. We shall generalize the latter results for orthodox semigroups.

4.1 An embedding of bands

Let B be a band. We denote by B^0 the band B with an *extra* zero 0 adjoined: $0 \notin B$, and $a0 = 0a = 0$ for every $a \in B^0$. $\mathbb{N} = \{0, 1, \dots\}$ is the set of natural numbers and $\mathbb{Z}^+$ the set of positive integers.

The *power* $(B^0)^{\mathbb{N}\times B}$ consists of all the mappings $\alpha : \mathbb{N} \times B \to B^0$ endowed with a pointwise multiplication. We denote this multiplication by " $\cdot$ " and define it as: for any

$\alpha_1, \alpha_2 \in (B^0)^{\mathbb{N}\times B}$, $\alpha_1 \cdot \alpha_2 \in (B^0)^{\mathbb{N}\times B}$ such that, for any $(i, e) \in \mathbb{N} \times B$,

$$(i, e)(\alpha_1 \cdot \alpha_2) = ((i, e)\alpha_1)((i, e)\alpha_2)$$

is the product of $(i, e)\alpha_1$ and $(i, e)\alpha_2$ in B^0. We let B_1 be the set of all $\alpha \in (B^0)^{\mathbb{N}\times B}$ satisfying the following conditions:

$$\begin{aligned} &(i) && (0, e)\alpha = (0, g)\alpha && \text{for all } e, g \in B, \\ &(ii) && (i, e)\alpha \le e \text{ in } B^0 && \text{for all } e \in B, i \in \mathbb{Z}^+, \\ &(iii) && (i, e)\alpha \ne e && \text{for only finitely many } (i, e) \in \mathbb{Z}^+ \times B. \end{aligned} \tag{4.1}$$

It is easy to see that $(B^0)^{\mathbb{N}\times B}$ is a band, and B_1 a subband of $(B^0)^{\mathbb{N}\times B}$.

For every $e \in B^0$ we let $\epsilon_e \in B_1$ be defined by

$$\begin{aligned} (0, g)\epsilon_e &= e && \text{for every } g \in B \\ (i, g)\epsilon_e &= g && \text{for every } (i, g) \in \mathbb{Z}^+ \times B \end{aligned} \tag{4.2}$$

Lemma 4.1.1. *(i) The mapping*

$$\iota_1 : B \to B_1, \qquad e \to \epsilon_e \tag{4.3}$$

is an embedding of bands.

(ii) For every $e \in B^0$, $\epsilon_e B_1 \epsilon_e$ consists of the $\alpha \in (B^0)^{\mathbb{N}\times B}$ such that

(a) $(0, e)\alpha = (0, g)\alpha \le e$ in B^0 for every $g \in B$,

(b) $(i, g)\alpha \le g$ for every $(i, g) \in \mathbb{Z}^+ \times B$,

(c) $(i, g)\alpha \ne g$ for only finitely many $(i, g) \in \mathbb{Z}^+ \times B$,

(iii) $B\iota_1$ is a filter of B_1.

Proof. The proof follows a routine verification. We provide some details concerning (iii).

Therefore, let $e \in B$, $\alpha \in B_1$ and suppose that $\epsilon_e \leq \alpha$ in B_1. Let $(0,e)\alpha = (0,g)\alpha = f$ for all $g \in B$. Then $e = (0,e)\epsilon_e \leq (0,e)\alpha = f$ in B^0, hence $f \in B$. Further, for every $(i,g) \in \mathbb{Z}^+ \times B$, $g = (i,g)\epsilon_e \leq (i,g)\alpha$, whereas $(i,g)\alpha \leq g$ in B^0. It follows that $(i,g)\alpha = g$ for every $(i,g) \in \mathbb{Z}^+ \times B$. Thus, $\alpha = \epsilon_f \in B\iota_1$. $\square$

Lemma 4.1.2. *(i) For every $e \in B$, let the mapping $\varphi_e : \epsilon_e B_1 \epsilon_e \to \epsilon_0 B_1 \epsilon_0$ be given by: for $\alpha \in \epsilon_e B_1 \epsilon_e$*

$$(0,g)(\alpha\varphi_e) = 0 \qquad \textit{for every } g \in B,$$

$$(i,e)(\alpha\varphi_e) = (i-1,e)\alpha \qquad \textit{for every } i \in \mathbb{Z}^+,$$

$$(i,g)(\alpha\varphi_e) = (i,g)\alpha \qquad \textit{for every } i \in \mathbb{Z}^+ \textit{ and } g \neq e \textit{ in } B.$$

Then φ_e is a partial isomorphism which maps $\epsilon_e B_1 \epsilon_e$ isomorphically onto $\epsilon_0 B_1 \epsilon_0$.

(ii) Let $\theta : eBe \to gBg$ be a partial isomorphism of B. Then the partial isomorphism $\iota_1^{-1}\theta\iota_1 : \epsilon_e(B\iota_1)\epsilon_e \to \epsilon_g(B\iota_1)\epsilon_g$ of $B\iota_1$ can be extended to a partial isomorphism $\theta_1 : \epsilon_e B_1 \epsilon_e \to \epsilon_g B_1 \epsilon_g$.

(iii) $(B\iota_1) \times (B\iota_1) \subseteq \mathcal{U}_{B_1}$.

Proof. (i) Using Lemma 4.1.1 (ii), one routinely verifies that for every $\alpha \in \epsilon_e B_1 \epsilon_e$ we have that $\alpha\varphi_e \in \epsilon_0 B_1 \epsilon_0$. We prove that φ_e is one-to-one. If $\alpha_1, \alpha_2 \in \epsilon_e B_1 \epsilon_e$ and $(i,e)\alpha_1 \neq (i,e)\alpha_2$ for some $i \in \mathbb{N}$, then $(i+1,e)(\alpha_1, \varphi_e) \neq (i+1,e)(\alpha_2\varphi_e)$. Furthermore, if $(i,g)\alpha_1 \neq (i,g)\alpha_2$ for some $i \in \mathbb{Z}^+$ and $g \neq e$ in B, then $(i,g)(\alpha_1\varphi_e) \neq (i,g)(\alpha_2\varphi_e)$. We next prove that φ_e is onto. Therefore let $\beta \in \epsilon_0 B_1 \epsilon_0$. Define $\alpha \in (B^0)^{\mathbb{N}\times B}$ by :

$$(0,g)\alpha = (1,e)\beta \qquad \text{for every } g \in B,$$

$$(i,e)\alpha = (i+1,e)\beta \qquad \text{for every } i \in \mathbb{N},$$

$$(i,g)\alpha = (i,g)\beta \qquad \text{for every } i \in \mathbb{Z}^+ \text{ and } g \neq e \text{ in } B.$$

One verifies that $\alpha \in \epsilon_e B_1 \epsilon_e$ and $\alpha\varphi_e = \beta$. We conclude that φ_e is a bijection of $\epsilon_e B_1 \epsilon_e$ onto $\epsilon_0 B_1 \epsilon_0$.

In order to prove (i) it suffices to prove that φ_e is a band homomorphism. Therefore, let $\alpha_1, \alpha_2 \in \epsilon_e B_1 \epsilon_e$ and let us calculate $(\alpha_1 \cdot \alpha_2)\varphi_e$ and $(\alpha_1\varphi_e) \cdot (\alpha_2\varphi_e)$: for any $g \in B$,

$$\begin{aligned}(0,g)((\alpha_1\alpha_2)\varphi_e) &= 0 = 00 \\ &= (0,g)((\alpha_1\varphi_e) \cdot (\alpha_2\varphi_e)),\end{aligned}$$

for any $i \in \mathbb{Z}^+$,

$$\begin{aligned}(i,e)((\alpha_1 \cdot \alpha_2)\varphi_e) &= (i-1,e)(\alpha_1 \cdot \alpha_2) \\ &= (i,e)((\alpha_1\varphi_e) \cdot (\alpha_2\varphi_e)),\end{aligned}$$

and for every $i \in \mathbb{Z}^+$ and $g \neq e$ in B,

$$\begin{aligned}(i,g)((\alpha_1 \cdot \alpha_2)\varphi_e) &= (i,g)(\alpha_1\alpha_2) \\ &= (i,g)((\alpha_1\varphi_e) \cdot (\alpha_2\varphi_e)).\end{aligned}$$

Therefore $(\alpha_1 \cdot \alpha_2)\varphi_e = (\alpha_1\varphi_e) \cdot (\alpha_2\varphi_e)$ and we conclude that φ_e is a partial isomorphism of B_1.

(ii) For the partial isomorphism $\theta : eBe \to gBg$ of B, define $\theta_1 : \epsilon_e B_1 \epsilon_e \to \epsilon_g B_1 \epsilon_g$ by: for $\alpha \in \epsilon_e B_1 \epsilon_e$, $\alpha\theta_1$ is given by

$$\begin{aligned}(0,g)(\alpha\theta_1) &= ((0,g)\alpha)\theta \qquad \text{for every } g \in B, \\ (i,g)(\alpha\theta_1) &= (i,g)\alpha \qquad \text{for every } i \in \mathbb{Z}^+, g \in B. \end{aligned} \tag{4.4}$$

Using Lemma 4.1.1 (ii), one routinely verifies that θ_1 is a partial isomorphism of B_1. Further, if $\epsilon_f \in \epsilon_e B_1 \epsilon_e$, that is, $f \in eBe$, then

$$\begin{aligned}(0,g)(\epsilon_f\theta_1) &= ((0,g)\epsilon_f)\theta = f\theta \qquad \text{for every } g \in B, \\ (i,g)(\epsilon_f\theta_1) &= (i,g)\epsilon_f = g \qquad \text{for every } i \in \mathbb{Z}^+, g \in B,\end{aligned}$$

thus $\epsilon_f\theta_1 = \epsilon_{f\theta}$. Therefore θ_1 extends $\iota_1^{-1}\theta\iota_1$.

(iii) From (i) it follows that $\epsilon_e \mathcal{U}_{B_1} \epsilon_0$ for every $e \in B$. Therefore $(B\iota_1) \times (B\iota_1) \subseteq \mathcal{U}_{B_1}$. □

We consider the sequence of bands

$$B = B_0, B_1, \ldots, B_j, B_{j+1}, \ldots \tag{4.5}$$

and embeddings $\iota_{j+1} : B_j \to B_{j+1}$ where B_{j+1} is obtained from B as in the foregoing discussion, and the embedding ι_{j+1} is defined along the same lines as $\iota_1 : B \to B_1$ was given by (4.3). We thus obtain a direct family of bands B_j, $j < \omega$, and we let B' be the direct limit of this direct family (in the sense of §21 of [13]). For notational convenience we shall identify B_j with $B_j\iota_{j+1}$. When doing so, we have that $B' = \bigcup_{j<\omega} B_j$ is a band and the B_j form a chain of subbands of B'. In the following we shall also consider the sequence of sets

$$\underline{T}_B = \underline{T}_{B_0}, \underline{T}_{B_1}, \ldots, \underline{T}_{B_j}, \underline{T}_{B_{j+1}}, \ldots \tag{4.6}$$

of partial isomorphisms of the respective bands in (4.5). For $\theta_j \in \underline{T}_{B_j}$, we denote by $\theta_{j+1} \in \underline{T}_{B_{j+1}}$ the partial isomorphism obtained from θ_j in the same way as θ_1 was obtained from θ in (4.4). In view of the identification of B_j with $B_j\iota_{j+1}$ mentioned in the preceding paragraph, we have $\theta_j \subseteq \theta_{j+1}$ by Lemma 4.1.2 (ii).

If $\mathbf{K}$ is an algebraic class of bands which is closed under adding an extra zero, subdirect powers, and direct limits (see §20,21 of [13]) and $B \in \mathbf{K}$, then, if we follow the construction of B', we see that B' also belongs to $\mathbf{K}$. This is in particular the case if $\mathbf{K}$ is a variety of bands which contains the variety of semilattices.

Theorem 4.1.3. *Every band B can be embedded into a uniform band B' such that B and B' generate the same band variety.*

Proof. If B is a rectangular band, we take $B = B'$ and the result follows. We henceforth assume that B is not a rectangular band. Then the variety $\mathbf{K}$ generated by B contains the variety of all semilattices. Let B' be constructed from B as described in this section. Since B is a subband of B', it follows from the remark made in the preceding paragraph that B and B' generate the same band variety $\mathbf{K}$.

Let $e, g \in B'$. There exists $j < \omega$ such that $e, g \in B_{j-1}$. By Lemma 4.1.2 (iii) there exists $\theta_j \in \underline{T}_{B_j}$ which maps eB_je isomorphically onto gB_jg. Consider the sequence of partial isomorphisms

$$\theta_j \subseteq \theta_{j+1} \subseteq \ldots \subseteq \theta_{j+k} \subseteq \theta_{j+k+1} \subseteq \ldots \tag{4.7}$$

where, for $\theta_{j+k} \in \underline{T}_{B_{j+k}}$, θ_{j+k+1} is obtained from θ_{j+k} as θ_1 was obtained from θ in (4.4). Put $\theta'_j = \bigcup_{k<\omega} \theta_{j+k}$. Then $\theta'_j : eB'e \to gB'g$ is a partial isomorphism of B', where $e\mathcal{U}_{B'}g$. We conclude that B' is uniform. $\square$

We conclude this section with some additional properties which are satisfied by the embedding of the band B into the band B' in Theorem 4.1.3.

Theorem 4.1.4. *Let B and B' be bands, as in Theorem 4.1.3. Then*

(i) if B is not a rectangular band, then B' is countably infinite if B is finite, otherwise B and B' have the same cardinality,

(ii) B is a filter of B',

(iii) every endomorphism γ of B can be extended to an endomorphism γ' of B' such that $\mathrm{End}B \to \mathrm{End}B'$, $\gamma \to \gamma'$ is an embedding of endomorphism monoids which induces an embedding $\mathrm{Aut}B \to \mathrm{Aut}B'$ of automorphism groups,

(iv) every congruence ρ on B is the restriction to B of a congruence ρ' on B' such that $\mathrm{Con}B \to \mathrm{Con}B'$, $\rho \to \rho'$ embeds the congruence lattice of B as a complete sublattice of the congruence lattice of B'.

Proof. (i) This property is guaranteed by the condition (4.1) (iii).

(ii) This property follows from Lemma 4.1.1 (iii).

(iii) In the following we adopt the notation of Lemma 4.1.1. For $\gamma \in \mathrm{End}B$, let $\iota_1^{-1}\gamma\iota_1 : \epsilon_e \to \epsilon_{e\gamma}$ be the corresponding endomorphism in $B\iota_1$. This endomorphism of $B\iota_1$ can be extended to the endomorphism γ_1 of B_1 where, for every $\alpha \in B_1$, $\alpha\gamma_1$ is given by

$$
\begin{aligned}
&(0,g)(\alpha\gamma_1) = ((0,g)\alpha)\gamma \qquad \text{for every } g \in B \text{ such that } (0,g)\alpha \neq 0\\
&(i,g)(\alpha\gamma_1) = (i,g)\alpha \qquad \text{otherwise.}
\end{aligned}
$$

It should be clear that $\mathrm{End}B \to \mathrm{End}B_1$, $\gamma \to \gamma_1$ is an embedding of endomorphism monoids. If we adopt the convention that B is identified with its isomorphic image $B\iota_1$, then $\gamma \subseteq \gamma_1$ for every $\gamma \in \mathrm{End}B$. We note that if $\gamma \subseteq \mathrm{Aut}B$, then $\gamma_1 \in \mathrm{Aut}B_1$, thus $\mathrm{Aut}B \to \mathrm{Aut}B_1$, $\gamma \to \gamma_1$ is an embedding of automorphism groups.

We now consider the sequence (4.5) of bands B_j, $j < \omega$, whose direct limit is B' and the corresponding sequence

$$\mathrm{End}B = \mathrm{End}B_0, \mathrm{End}B_1, \ldots, \mathrm{End}B_j, \mathrm{End}B_{j+1}, \ldots$$

of endomorphism monoids. For $\gamma \in \mathrm{End}B$, we construct the $\gamma_j \in \mathrm{End}B_j$ inductively by

$$
\begin{aligned}
&\gamma_0 = \gamma,\\
&\gamma_{j+1} \text{ is constructed from } \gamma_j \text{ as } \gamma_1 \text{ is constructed from } \gamma.
\end{aligned}
$$

We thus obtain a sequence of endomorphisms

$$\gamma = \gamma_0 \subseteq \gamma_1 \subseteq \ldots \subseteq \gamma_j \subseteq \gamma_{j+1} \subseteq \ldots$$

and we put $\gamma' = \bigcup_{j<\omega} \gamma_j$. One verifies that $\gamma' \in \mathrm{End}B'$ and $\mathrm{End}B \to \mathrm{End}B'$, $\gamma \to \gamma'$ is an embedding of endomorphism monoids.

(iv) The proof of (iv) follows the same lines as the proof of (iii). We only indicate here how to construct $\rho_1 \in \mathrm{Con}B_1$ from a given $\rho \in \mathrm{Con}B$. For $\alpha_1, \alpha_2 \in B_1$ we put

$(\alpha_1, \alpha_2) \in \rho_1$ if and only if

$$((0,g)\alpha_1, (0,g)\alpha_2) \in \rho \qquad \text{for every } g \in B \text{ with } (0,g)\alpha_1 \neq 0 \neq (0,g)\alpha_2,$$
$$(i,g)\alpha_1 = (i,g)\alpha_2 \qquad \text{otherwise.}$$

□

Following our procedure for constructing the uniform band B' from the band B, one can set up a faithful functor from the category of bands to the category of uniform bands in a straightforward way. We refrain from exploring this line of investigation here.

4.2 An embedding of orthodox semigroups

For any band B, we adopt the notation of Section 4.1: B_1 is the band constructed from B as in (4.1), and again we shall adopt the convention that in the sequence of bands (4.5) we have $B = B_0$, $B_j \subseteq B_{j+1}$, and $B' = \bigcup_{j<\omega} B_j$ corresponding to the sequence (4.5) is the sequence (4.6) of sets of partial isomorphisms of the respective bands of (4.5). For $\theta_j \in \underline{T}_{B_j}$, let $\theta_{j+k} \in \underline{T}_{B_{j+k}}$ be as in the sequence (4.7). Then, as in the proof of Theorem 4.1.3, we put $\theta'_j = \bigcup_{k<\omega} \theta_{j+k} \in \underline{T}_{B'}$ a partial isomorphism of $B' = \bigcup_{k<\omega} B_k$. In particular, any $\theta = \theta_0 \in \underline{T}_B = \underline{T}_{B_0}$ extends to a partial isomorphism $\theta' = \cup\theta_j \in \underline{T}_{B'}$.

Lemma 4.2.1. *Let $e, g \in B_j$. Define $\pi_{j,k}(e,g)$ inductively by: $\pi_{j,k+1}(e,g)$ is obtained from $\pi_{j,k}(e,g)$ as $\theta_1 \in \underline{T}_{B_1}$ is obtained from $\theta \in \underline{T}_B$ in* (4.4)*. Then $\pi'(e,g) = \bigcup_{k<\omega} \pi_{j,k}(e,g) \in \underline{T}_{B'}$ where*

$$\pi'(e,g) : egeB'ege \to gegB'geg, \qquad d \to gdg.$$

Proof. The proof easily follows from an inductive argument and the details of (4.4). □

We now have

Lemma 4.2.2. *For any j, k, and $\theta_j \in \underline{T}_{B_j}$, let $\theta_{j,k} \in \underline{T}_{B_{j+k}}$ be inductively defined by:*

$$\theta_{j,0} = \theta_j,$$

$$\theta_{j,k+1} \in \underline{T}_{B_{j+k+1}} \text{ is obtained from } \theta_{j,k} \in \underline{T}_{B_{j+k}}$$
$$\text{as } \theta_1 \text{ is obtained from } \theta \text{ as in } (4.4).$$

Then

$$\underline{\tau}_{j,k} : \underline{T}_{B_j} \to \underline{T}_{B_{j+k}}, \qquad \theta_j \to \theta_{j,k} \tag{4.8}$$

is an embedding of $(\underline{T}_{B_j}, \cdot)$ into $(\underline{T}_{B_{j+k}}, \cdot)$.

Proof. The proof follows from Lemma 4.2.1 and the details of (4.4). □

Lemma 4.2.3. *With the notation of Lemma 4.2.2, $\theta_j \in \underline{T}_{B_j}$, and $\theta'_j = \bigcup_{k<\omega} \theta_{j,k}$, we then have*

$$\underline{\tau}'_j : \underline{T}_{B_j} \to \underline{T}_{B'}, \qquad \theta_j \to \theta'_j \tag{4.9}$$

is an embedding of $(\underline{T}_{B_j}, \cdot)$ into $(\underline{T}_{B'}, \cdot)$.

Proof. The proof follows from Lemma 4.2.1 and a direct verification. □

From Lemma 4.2.2 and 4.2.3 then follows

Corollary 4.2.4. *The direct limit of the directed system of orthodox semigroups $(\underline{T}_{B_{j+k}}, \cdot)$ given by (4.8) is an orthodox subsemigroup of $(\underline{T}_{B'}, \cdot)$, and the mapping (4.9) embeds each orthodox semigroup $(\underline{T}_{B_j}, \cdot)$ isomorphically into the orthodox semigroup $(\underline{T}_{B'}, \cdot)$.*

Proof. That each $(\underline{T}_{B_j}, \cdot)$ is a subsemigroup of $(\underline{T}_{B'}, \cdot)$ follows from Lemma 4.2.3. It then follows that direct limit of the $(\underline{T}_{B_{j+k}}, \cdot)$ is a subsemigroup of $(\underline{T}_{B'}, \cdot)$. □

Using Lemmas 4.2.1, 4.2.2, we obtain the following in sequence.

Lemma 4.2.5. *For any $j, k < \omega$,*

(i) for every $\sigma_j, \theta_j \in \underline{T}_{B_j}$,

$$\sigma_{j,k}\kappa_{j+k}\theta_{j,k} \Leftrightarrow \sigma_{j,k+1}\kappa_{j+k+1}\theta_{j,k+1},$$

(ii) for every $\sigma_j, \theta_j \in \underline{T}_{B_j}$,

$$\sigma_j\kappa_j\theta_j \Leftrightarrow \sigma'_j\kappa'\theta'_j,$$

(iii)

$$\begin{tikzcd} \underline{T}_{B_j} \arrow[r, "\underline{\tau}_{j,k}"] \arrow[d, "\kappa_j^\natural"'] & \underline{T}_{B_{j+k}} \arrow[d, "\kappa_{j+k}^\natural"] \\ T_{B_j} \arrow[r, "\tau_{j+k}"'] & T_{B_{j+k}} \end{tikzcd} \quad , \quad \begin{tikzcd} \theta_j \arrow[r, "\underline{\tau}_{j,k}"] \arrow[d, "\kappa_j^\natural"'] & \theta_{j,k} \arrow[d, "\kappa_{j+k}^\natural"] \\ \overline{\theta}_j \arrow[r, "\tau_{j,k}"'] & \overline{\theta}_{j,k} \end{tikzcd} \tag{4.10}$$

and

$$\begin{tikzcd} \underline{T}_{B_j} \arrow[r, "\underline{\tau}'_j"] \arrow[d, "\kappa_j^\natural"'] & \underline{T}_{B'} \arrow[d, "\kappa'^\natural"] \\ T_{B_j} \arrow[r, "\tau'_j"'] & T_{B'} \end{tikzcd} \quad , \quad \begin{tikzcd} \theta_j \arrow[r, "\underline{\tau}'_j"] \arrow[d, "\kappa_j^\natural"'] & \theta'_j \arrow[d, "\kappa'^\natural"] \\ \overline{\theta}_j \arrow[r, "\tau'_j"'] & \overline{\theta}'_j \end{tikzcd} \tag{4.11}$$

are commuting diagrams.

□

Therefore,

Corollary 4.2.6. *The directed system of orthodox semigroups $T_{B_{j+k}}$ given by* (4.10) *is* *an orthodox subsemigroup of $T_{B'}$ and the mapping τ_j' given by* (4.11) *embeds T_{B_j} isomorphically into $T_{B'}$.* □

We mention the following intermediate result for clarity.

Proposition 4.2.7. *Let B be a band which is not a rectangular band and S any fundamental orthodox semigroup such that $E(S) = B$ is the band of idempotents of S. Let B' be the band constructed from B as in Theorem 4.1.3. Then S can be embedded into the orthodox semigroup $T_{B'}$ which is bisimple and fundamental where B and $B' \cong E(T_{B'})$ generate the same band variety.*

Proof. We put $B = B_0$ as in (4.5). Following Corollary 4.2.6, with $j = 0$, T_B can be embedded into $T_{B'}$. Since $B = B_0 \cong E(T_B)$ and $B' \cong E(T_{B'})$, we have that $E(T_B)$ and $E(T_{B'})$ generate the same band variety by Theorem 4.1.3. By Theorem 1.5 of [42], there exists an idempotent separating homomorphism of S into T_B which induces an isomorphism of bands. This homomorphism is one-to-one since S is assumed to be fundamental. Thus, S embeds isomorphically into $T_{B'}$. The orthodox semigroup $T_{B'}$ is bisimple and fundamental by Lemmas 1.8 and 6.4 of [42]. □

In order to prove our final theorem we will need the following well known lemma. We provide a proof for completeness.

Lemma 4.2.8. *If S_1 and S_2 are fundamental orthodox semigroups, then $S_1 \times S_2$ is a fundamental orthodox semigroup.*

Proof. Suppose that (a_1, a_2), (b_1, b_2) are μ-related elements of $S_1 \times S_2$. Then a routine verification using the description of the greatest idempotent separating congruence μ on an orthodox semigroup S (see Section 4 of [20]) shows that $a_1 \mu b_1$. However, S_1 is fundamental, so $a_1 = b_1$. Similarly we have that $a_2 = b_2$. Hence μ is the equality relation on $S_1 \times S_2$, so $S_1 \times S_2$ is fundamental. □

The proof of the following theorem refers to the primary references, but it may be useful to consult instead the survey paper [41] or [43].

Theorem 4.2.9. *Let S be an orthodox semigroup. Then S can be embedded into an orthodox semigroup S' which is bisimple and such that the bands $E(S)$ and $E(S')$ of idempotents of S and S' generate the same band variety. Moreover, if S is not a rectangular group, then S' can be chosen to be fundamental.*

Proof. If S is a rectangular group, then take $S' = S$. We henceforth assume that S is not a rectangular group, that is, the variety of bands generated by $E(S)$ contains the variety of all semilattices. By Proposition 4.2.7 it suffices to embed the given orthodox semigroup S into a fundamental orthodox semigroup S_0 whose band $B = E(S_0)$ generates the same band variety as $E(S)$.

We let $\mathcal{Y}$ be the least inverse congruence on the orthodox semigroup S as described in Section 6.2 of [22]. We next embed $S/\mathcal{Y}$ into a fundamental inverse semigroup: this can for instance be done using the Vagner-Preston representation which embeds $S/\mathcal{Y}$ isomorphically into an appropriate symmetric inverse semigroup (see Theorem 5.1.7 and Exercise 22 in Chapter 5 of [22]). We will denote the fundamental inverse semigroup from this embedding as I.

Let μ be the greatest idempotent separating congruence on S. From [6] and Section 6.2 of [22] it follows that $\mu \cap \mathcal{Y}$ is the equality on S. Therefore S can be embedded into the direct product, S_0, of the fundamental orthodox semigroup S/μ and the fundamental inverse semigroup I. The band of idempotents of S_0 is the direct product of the band $E(S)$ and the semilattice $E(I)$, therefore $E(S)$ and $E(S_0)$ generate the same band variety. That S_0 is fundamental follows from Lemma 4.2.8. $\square$

Chapter 5

Final Remarks

We now know more about $\mathcal{E}$. In particular we know that $\mathcal{E}$ is a prevariety, but is not a variety. Furthermore $\mathcal{E}$ is contained within the quasivariety determined by the implication $x^2 \approx x \Rightarrow y^2 \approx y$, but it is still not determined whether this is a proper inclusion or if the two coincide. This dissertation focuses on bands, so to answer the preceding question it may be useful to look at idempotent free semigroups that are embeddable into semigroups with high symmetry.

We then came up with examples of useful bands which had high symmetry. From these examples we were able to make constructions which then gave us more bands with high symmetry. Then we were able to show that all normal bands and all free bands are embeddable into bands with high symmetry. Furthermore normal bands and free bands within the complete sublattice of the lattice of variety of bands generated by the varieties in the list (1.1) can be embedded into a band within the same variety which has high symmetry. It is still left to be seen whether or not this holds true for free bands within the varieties in the list (1.2).

Then we revisited normal bands. We already knew at this point that every normal band was embeddable in a band with high symmetry, but here we wanted more than just to show there was an embedding. We found a useful embedding that maintained much of the structure of the original band. In particular, because we were able to embed the hull of the original band into the hull of the new band, we were able to find an embedding of any fundamental generalized inverse semigroup into a bisimple fundamental generalized inverse semigroup. The embedding we constructed is similar to that found in [39]. The techniques there led to division theorems for inverse semigroups. It is hoped that future research will be able to use the embedding from Chapter 3 to find division theorems for generalized inverse semigroups as well.

In Chapter 3 our first major step toward finding a useful embedding of a normal band into a normal band with high symmetry was to find an embedding of a normal band into a uniform normal band. In Chapter 4 we were able to do this not just for normal bands, but also for all bands. This embedding again maintained much of the structure of the original band in that we had an embedding of endomorphism monoids and automorphism groups. We also had an embedding of the hull of the original band into the hull of the new band, and both bands were within the same band variety. This then allowed us to expand upon our findings about generalized inverse semigroups. That is, we were able to show that any orthodox semigroup was embeddable into a bisimple orthodox semigroup in which the set of idempotents of both orthodox semigroups were in the same band variety. Left open here is whether or not we can then extend partial isomorphisms into automorphisms so that we can then have an embedding of any band into a band with high symmetry.

Chapter 6

Index

6.1 Index of terminology

6.2 Index of symbols

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