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comparing graphical method and a modified method to fit weibull distribution

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COMPARING GRAPHICAL METHOD AND A MODIFIED METHOD TO FIT WEIBULL DISTRIBUTION

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ABSTRACT

This thesis concerns the Weibull distribution for lifetime data analysis, studies the statistical properties of the distribution, and emphasizes parameters estimation methods. It has been known for more than four decades that the mixed Weibull distribution is a proper distribution to use in modeling the lifetimes. Parameter estimation is critical for a statistical model to be used and is a challenging problem, especially for a Weibull distribution with more than two parameters.

In the thesis, both graphical estimation methods and analytical methods are studied in detail. Using Weibull probability paper, a typical graphical estimation method, has been accepted and used for a long time. An analytical method proposed by Dimitri B, Kececioglu is also implemented and tested. Three case studies are conducted and compared with the two methods by using the Kolmogorov-Smirnov goodness of fit test. The result shows that the two methods in general give good estimations when they are applied for fitting a Weibull distribution to the failure times in the cases. The Weibull probability paper method is a quick approach but will produce a crude estimate. Kececioglu's estimation method is able to provide high accuracy and is easy to use by following the computation given in the thesis. An extension of Kececioglu's estimation method for 3-subpopluation Weibull distributions is made. An example is also conducted in order to verify its feasibility. The result shows that the Kececioglu estimation method can also provide a high accuracy for 3-subpopluation Weibull distributions parameter estimation.

Chapter 1 Introduction

The Weibull distribution has attracted the attention of statisticians for half a century. It is named for Waloddi Weibull (1887-1979). Thousands of papers have been written on this distribution and it is still drawing broad attention. It is of importance to statisticians because of its ability to fit to data from various areas, ranging from life data to observations made in economics, biology or materials reliability studied in the thesis.

In the early 1920s, there were three groups of scientists working on the derivation of the distribution independently with different purposes. Waloddi Weibull was one of them. The distribution bears his name because he promoted this distribution both internationally and interdisciplinarily. His discoveries lead the distribution to be productive in engineering practice, statistical modeling and probability theory.[1,2]

The aim of this chapter is to review the properties of the distribution. Then the interpretation of the parameters and their physical meaning will be introduced. The parameter estimation will be primarily explained in the following chapters.

1.1 Two parameter Weibull distribution

1.1.1 Two parameter Weibull distribution function

The Two parameter Weibull distribution has a density function (PDF)[3,4]

$$f(x|\alpha,\beta)=\frac{\alpha}{\beta}(\frac{x}{\beta})^{\alpha-1}\exp\left\{-(\frac{x}{\beta})^{\alpha}\right\};\ \alpha,\beta\in(0,\infty) \quad (1.1)$$

Cumulative distribution function (CDF)

$$F(x|\alpha,\beta)=\int_0^x f(x|\alpha,\beta)du=1-\exp\left\{-(\frac{x}{\beta})^{\alpha}\right\} \quad (1.2)$$

And hazard rate (HR)

$$h(x|\alpha,\beta)=\frac{f(x|\alpha,\beta)}{1-F(x|\alpha,\beta)}=\frac{\alpha}{\beta}(\frac{x}{\beta})^{\alpha-1} \quad (1.3)$$

α is the shape parameter, also known as the Weibull slope.

β is the scale parameter.

1.1.2 Weibull distribution shape parameter, α

The value of α is equal to the slope of the line in a probability plot on Weibull probability paper. The value of shape parameter has remarkable effect on the behavior of Weibull distribution[5,6]. The following plot shows the effect of different values of the shape parameter, α.

Table 1 Weibull distribution shape parameter α properties

Shape Parameter	PDF
$0<\alpha<1$	Exponentially decay from infinity
$\alpha=1$	Exponentially decay from 1/mean
$1<\alpha<2$	Rises to peak and then decreases
$\alpha=2$	Rayleigh distribution
$3\le\alpha\le4$	Has normal bell shape appearance
$\alpha>10$	Has shape very similar to type 1 extreme value distribution

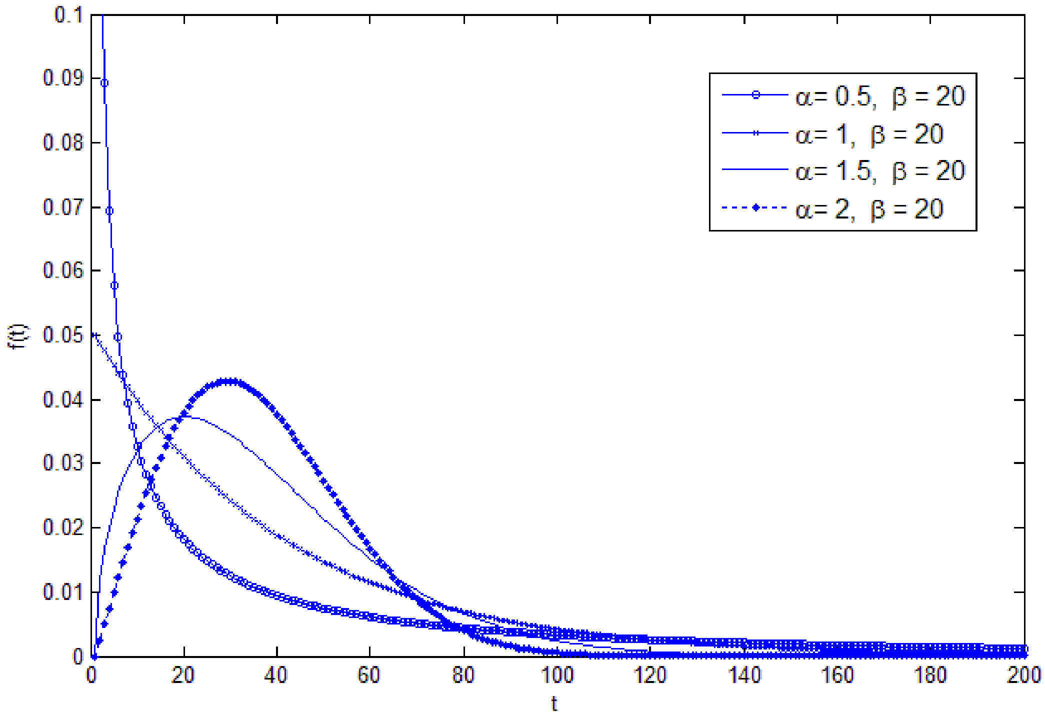

Figure 1 Weibull distribution profile at various α

1.1.3 Weibull distribution scale parameter, β

The value of scale parameter β has a different effect on the Weibull distribution. It is related to the location of the central portion along the abscissa scale.

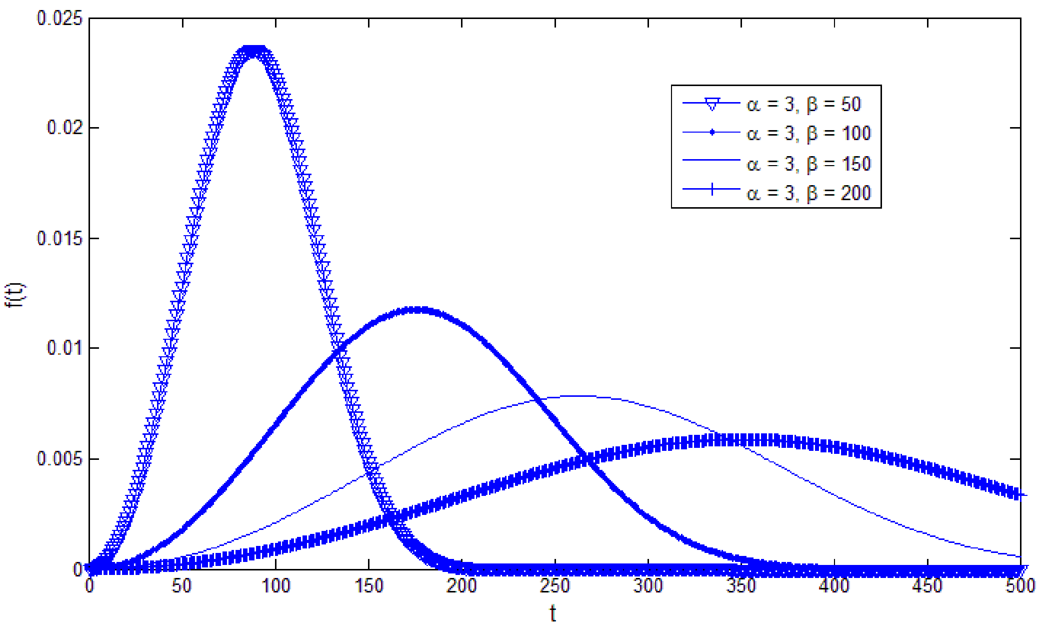

Figure 2 Weibull distribution profile at various β

From Figure 2, the conclusions are

- If β is increased, while α is constant, the Weibull distribution gets stretched out to the right and its height lowers.
- If β is decreased, while α is constant, the Weibull distribution gets pushed in towards the left, and its height increases.

This is because the area under the density must be unity.

1.2 Mixed Weibull distribution

The Mixed Weibull probability density function is defined as[7,8]

$$f(x) = \sum_{i=1}^{k} P_i f_i(x) \,; \quad \sum_{i=1}^{k} P_i = 1 \tag{1.5}$$

where

$f_i(x)$ is for the i th subpopulation

P_i is the proportion of subpopulation i known as the mixture parameter

The bimodal (five-parameter) Weibull distribution is

$$f(x) = P_1 f_1(x) + P_2 f_2(x) \,; \quad P_1 + P_2 = 1 \tag{1.6}$$

$$f(x) = P f_1(x) + (1-P) f_2(x) \tag{1.7}$$

$$f(x) = P \frac{\alpha_1}{\beta_1} \left(\frac{x}{\beta_1}\right)^{\alpha_1 - 1} \exp\left[-\left(\frac{x}{\beta_1}\right)^{\alpha_1}\right] + (1-P) \frac{\alpha_2}{\beta_2} \left(\frac{x}{\beta_2}\right)^{\alpha_2 - 1} \exp\left[-\left(\frac{x}{\beta_2}\right)^{\alpha_2}\right] \tag{1.8}$$

Where $\alpha_1, \beta_1, \alpha_2, \beta_2, p > 0$.

This Mixed Weibull distribution is known as a bimodal mixtures model. Its CDF is defined as

$$F(x) = P F_1(x) + (1-P) F_2(x)$$

The Figure 3 shows how subpopulation distributions effect the Mixed Weibull distribution.

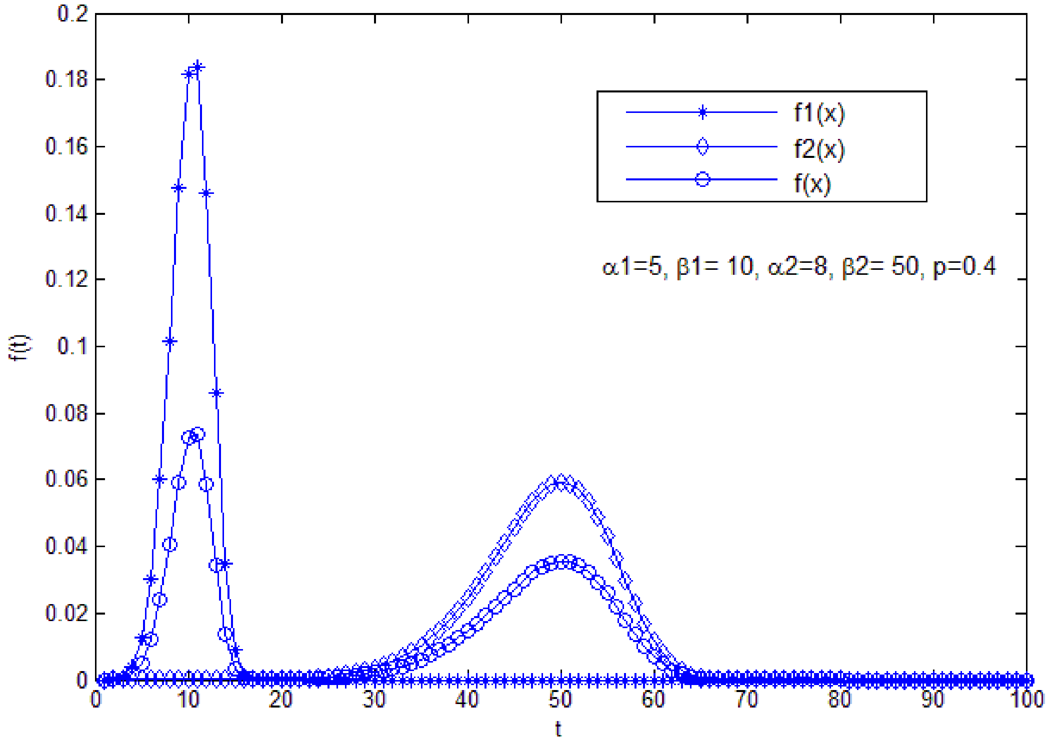

Figure 3 Mixed Weibull distribution plot with subpopulations

1.3 Current parameter estimation methods

Now, the parameter estimation methods considered here can be classified into two categories: 1) the graphical estimation method and 2) An analytical estimation method.[9]

1.3.1 Graphical Methods

The graphical methods have been used for some time because of their simplicity; however, they generate a bias because of the need of plotting points.

1.3.1.1 Hazard Plotting Technique

The hazard plotting technique is an estimation approach for the Weibull parameters by plotting the cumulative hazard function $H(x)$ against failure times on hazard paper.

The hazard rate is expressed by[10,11]:

$$h(x) = \frac{\alpha}{\beta}(\frac{x}{\beta})^{\alpha-1} \tag{1.9}$$

The cumulative hazard function is

$$H(x) = \int h(x) = \left(\frac{x}{\beta}\right)^{\alpha} \tag{1.10}$$

Taking the logarithm yields

$$\ln H(x) = \alpha\{\ln x - \ln \beta\} \tag{1.11}$$

$$\ln x = \frac{1}{\alpha}\ln H(x) + \ln \beta \tag{1.12}$$

From the equations above, we can plot the cumulative hazard function by following procedure.

1) Rank the failure times
2) Calculate $\Delta H_i = \frac{1}{(n+1)-1}$ for each failure
3) Calculate $H = \Delta H_1 + \Delta H_2 + ...\Delta H_i$
4) Plot lnH vs. lnx
5) Obtain curve by fitting points

The estimated parameters will be as follows

$$\alpha = \frac{1}{slope} \tag{1.13}$$

$$\beta = x, \text{ at } H = 1 \tag{1.14}$$

1.3.1.2 Weibull Probability Plotting

Weibull Probability Plotting will be thoroughly explained and used in the following chapter.

1.3.2 Analytical Methods

Due to bias in using graphical method, analytical methods have been used more generally. In the following, I will introduce some of the analytical methods used in estimating Weibull distribution parameters.

1.3.2.1 Maximum Likelihood Estimator (MLE)

Generally speaking, the likelihood of a set of data is the probability of obtaining that particular set of data, given the chosen probability distribution model. This expression comprises the unknown model parameters. The values of these parameters can be estimated by maximizing the sample likelihood; this method is known as the Maximum Likelihood Estimator (MLE). [12,13]

The cdf is

$$F(t)=1-e^{-(\frac{t}{\beta})^{\alpha}} \tag{1.15}$$

The pdf is

$$f(t)=\frac{\alpha}{\beta}(\frac{t}{\beta})^{\alpha-1}e^{-(\frac{t}{\beta})^{\alpha}} \tag{1.16}$$

And the likelihood function is

$$\Lambda=\sum_{i=1}^{N}\ln f(t_i;\alpha,\beta) \tag{1.17}$$

For a complete sample of size N.

Using Maximum Likelihood Estimation (MLE) to calculate the Parameters (α,β) of the Weibull Distribution[14,15]

$$\frac{\partial\Lambda}{\partial\alpha}=\frac{N}{\alpha}+\sum_{i=1}^{N}\ln(\frac{t_i}{\eta})-\sum_{i=1}^{N}(\frac{t_i}{\eta})^{\alpha}\ln(\frac{t_i}{\eta})=0 \tag{1.18}$$

$$\frac{\partial\Lambda}{\partial\beta}=\frac{-\alpha}{\beta}\bullet N+\frac{\alpha}{\beta}\sum_{i=1}^{N}\ln(\frac{t_i}{\beta})^{\alpha}=0 \tag{1.19}$$

1.3.2.2 Least Squares Method

The least square method is wildly used in estimating the parameters of a Weibull distribution. We assume that two variables (α,β) have a linear relation[16,17,18]. From the Weibull distribution, it can be seen that

$$\ln\ln[\frac{1}{1-F(x)}]=\alpha(\ln x-\ln\beta) \tag{1.20}$$

Because the equation above is linear in $\ln\ln[\frac{1}{1-F(x)}]$ versus $\ln x$, it can be rewritten as

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} \ln\left\{\ln\left[\frac{1}{(1-\frac{i}{n+1})}\right]\right\} \tag{1.21}$$

$$\bar{y} = \frac{1}{n}\sum_{i=1}^{n} \ln x_i \tag{1.22}$$

$$\hat{\alpha} = \frac{\left\{n\cdot\sum_{i=1}^{n}(\ln x_i)\cdot\left(\ln(\ln[\frac{1}{1-\frac{i}{n+1}}])\right)\right\} - \left\{\sum_{i=1}^{n}\ln(\ln[\frac{1}{1-\frac{i}{n+1}}])\cdot\sum_{i=1}^{n}\ln x_i\right\}}{\left\{n\cdot\sum_{i=1}^{n}(\ln x_i)^2\right\} - \left\{\sum_{i=1}^{n}(\ln x_i)\right\}^2} \tag{1.23}$$

$$\hat{\beta} = e^{(\bar{y}-\bar{x}/\hat{\alpha})} \tag{1.24}$$

From equations above, we can obtain parameters α, β. Where the nonparametric estimate for F(x) is the plotting point $P_i = \frac{i}{n+1}$; i is the rank of the data and n is the sample size.

1.3.2.3 Method of Moments

The method of moments is another technique broadly used in estimating parameters. Suppose that *the* numbers $x_1, x_2, \ldots x_n$, represent a set of data[19,20], the unbiased estimator for the k^{th} moment about the origin is

$$s_k = \frac{1}{n}\sum_{i=1}^{n} x_i^k \tag{1.25}$$

In the Weibull distribution, the k^{th} moment can be expressed as

$$m_k = \beta^k \Gamma(1+\frac{k}{\alpha}) \tag{1.26}$$

where $\Gamma = \int_0^\infty x^{j-1} e^{-x} dx$

Specifically the first and second moments are [21]

$$s_1 = m_1 = \beta\Gamma(1+\frac{1}{\alpha}) \tag{1.27}$$

The variance is

$$s_2 = u_k^2 + \sigma_k^2 = \beta^2 \left\{ \Gamma(1+\frac{2}{\alpha}) - \left(\Gamma(1+\frac{1}{\alpha}) \right)^2 \right\} \tag{1.28}$$

Introduce the coefficient of variation

$$CV = \frac{\sqrt{\Gamma(1+\frac{2}{\alpha}) - \Gamma^2(1+\frac{1}{\alpha})}}{\Gamma(1+\frac{1}{\alpha})} \tag{1.29}$$

By maximizing the coefficient of variation, we can determine α, then β can be expressed as:

$$\beta = (\frac{\bar{x}}{\Gamma(1+\frac{1}{\alpha})})^{\alpha} \tag{1.30}$$

In general, we have two major methods to estimate Weibull distribution parameters. In the thesis, I will mainly use the graphical method using Weibull probability paper and Kececioglu's Method proposed in "Parameter Estimation for mixed-Weibull Distribution". Furthermore I will compare results of their application in estimating the parameters of a 2-subpopulation Mixed-Weibull Distribution. Finally, I will extend the Kececioglu's method into 3-subpopulation Mixed-Weibull Distributions.

Chapter 2 Graphical estimation method

The mixed Weibull distribution parameters can be estimated by fitting the curve on Weibull probability paper. The following steps provide a method for separating the Mixed Weibull distribution and estimating the parameters for each subpopulation.[22,23]

Step1: Calculate the median rank (MR): the MR is given by

$$MR = \frac{N_F - 0.3}{N + 0.4} \tag{2.1}$$

where N_F total number of components failed at the time t_i.

N total number of components in the test

This is a nonparametric estimate for the distribution function.

Step2: plot the ordered data and median ranks on Weibull probability paper.

Step3: determine points that fall into distinct subpopulations by visual judgment and obtain the value of p.

Step4: draw the best fit straight line representing each subpopulation and note the number of points belonging to each subpopulation, N_i.

Step5: plot each subpopulation in another Weibull distribution by the following equation, estimate parameters for each subpopulation.

$$MR = \frac{N_F(T) - 0.3}{N_i + 0.4} \tag{2.2}$$

where $N_F(T)$ total number of components failed at the time t_i in each subpopulation

N_i = total number of items belonging to each subpopulation.

By following the above process, we can get the Weibull distribution parameters for each subpopulation.

Chapter 3 Kececioglu's estimation method

In the paper "Parameter Estimation For Mixed-Weibull Distribution", Dimitri B. Kececioglu proposed his method which combines Bayes' theorem and the Least-Square Method. In this chapter, I will explain it and indicate how to program it in Matlab.

A Mixed Weibull distribution has two failure modes, the time-to-failure sample {t_i , i=1,2,3,…,N} is present. Suppose that the data are ordered $t_1 < t_2 < ....t_N$. At time t_i , the failure can be split by two failure modes called the j th subpopulation (j =1,2). [24,25] The equation is

$$P_j(t_i) = P\{T \in f_j(t) \middle| t_i - \frac{1}{2}\Delta t < T < t_i + \frac{1}{2}\Delta t\} \tag{3.1}$$

where j=1,2; i=1,2,…N

Applying Bayes' Theorem, Eq. (1.1) can be written:

$$P_j = \frac{P\{t_i - \frac{1}{2}\Delta t < T < t_i + \frac{1}{2}\Delta t \middle| T \in f_j(t)\} \bullet P\{T \in f_j(t)\}}{\sum_j P\{t_i - \frac{1}{2}\Delta t < T < t_i + \frac{1}{2}\Delta t \middle| T \in f_j(t)\} \bullet P\{T \in f_j(t)\}} \tag{3.2}$$

The probabilities of failure occurred at the time t_i that belong to subpopulation 1 and subpopulation 2 are

$$P_1(t_i) = \frac{pf_1(t_i)}{pf_1(t_i) + (1-p)f_2(t_i)} = \frac{pf_1(t_i)}{f(t_i)} \tag{3.3}$$

$$P_2(t_i) = \frac{(1-p)f_2(t_i)}{pf_1(t_i) + (1-p)f_2(t_i)} = \frac{(1-p)f_2(t_i)}{f(t_i)} \tag{3.4}$$

where i=1,2,3…,N. p is the proportion of subpopulation 1.

For each point at time t_i , the sum of probabilities belonging to two subpopulations must be equal to 1,

$$P_1(t_i) + P_2(t_i) = 1 \tag{3.5}$$

So the failure occurring at time t_i can be divided into two portions: $P_1(t_i)$ of failure can fall in subpopulation 1 and $P_2(t_i)$ of failure belong to subpopulation 2. The size of subpopulation 1 is $N \cdot p$ and the size of subpopulation 2 is $N \cdot (1-p)$. So the Mixed Weibull distribution yields the following two subsamples:

Subsample1: $\{(t_1, P_1(t_1)), (t_2, P_1(t_2)), ..., (t_N, P_1(t_N))\}$;

Subsample2: $\{(t_1, P_2(t_1)), (t_2, P_2(t_2)), ..., (t_N, P_2(t_N))\}$;

For each subpopulation, its corresponding subsample can be solved by the conventional estimation method- the Rank Regression method. So the Mean Order Number (MON) of the i th failure in the j th subpopulation will be

$$MON_1(t_i) = \sum_{k=1}^{i} P_1(t_k), i = 1, 2, ..., N \tag{3.6}$$

$$MON_2(t_i) = \sum_{k=1}^{i} P_2(t_k), i = 1, 2, ..., N \tag{3.7}$$

The corresponding Median Ranks $MR_j(t_i)$ is:

Subpopulation 1
$$MR_1(t_i) = \frac{MON_1(t_i)}{MON_1(t_N) + 0.4} \tag{3.8}$$

Subpopulation 2
$$MR_2(t_i) = \frac{MON_2(t_i)}{MON_2(t_N) + 0.4} \tag{3.9}$$

So the subsamples could be written as

Subpopulation 1 $\{(t_1, MR_1(t_1)), (t_2, MR_1(t_2)), ..., (t_N, MR_1(t_N))\}$

Subpopulation 2 $\{(t_1, MR_2(t_1)), (t_2, MR_2(t_2)), ..., (t_N, MR_2(t_N))\}$

The CDF of Weibull distribution can be given in the form of

$$\log_e\{\log_e \frac{1}{1 - MR_j(t_i)}\} = \alpha_j(\log_e t_i - \log_e \beta_j) \tag{3.10}$$

The linearized form of

$$Y_j(i) = \alpha_j X(i) + b_j \tag{3.11}$$

where $Y_j(i) = \log_e\{-\log_e[1 - MR_j(t_i)]\}$,

$X(i) = \log_e t_i$,

$b_j = -\alpha_j \log_e \beta_j$.

Applying the least-square method, the distribution parameter are determined by

$$\alpha_j = \frac{\sum_{i=1}^{N} X(i)Y_j(i) - \frac{1}{N}\left[\sum_{i=1}^{N} X(i) \cdot \sum_{i=1}^{N} Y_j(i)\right]}{\sum_{i=1}^{N} X^2(i) - \frac{1}{N}[\sum_{i=1}^{N} X(i)]^2}, \ j = 1,2 \qquad (3.12)$$

$$b_j = \frac{1}{N}\sum_{i=1}^{N} Y_j(i) - \alpha_j \frac{1}{N}\sum_{i=1}^{N} X(i), \ j = 1,2 \qquad (3.13)$$

$$\beta_j = \exp(-\frac{b_j}{\alpha_j}), \ j = 1,2 \qquad (3.14)$$

The mixing portion is found to be

$$p = \frac{MON_1(t_N)}{N} = \frac{1}{N}\sum_{i=1}^{N} P_1(t_i) = 1 - \frac{MON_2(t_N)}{N} = 1 - \frac{1}{N}\sum_{i=1}^{N} P_2(t_i) \qquad (3.15)$$

The least-square method aims at finding the 'best' fit. The best fit is to minimize the residual variation around the line that is defined by correlation coefficient ρ. The larger the absolute value of ρ is, the better the fitted line is. Therefore, the five parameters can be obtained by applying the least square method to iterate on the $\alpha_1, \alpha_2, \beta_1, \beta_2, p$ values to minimize the deviations from the points to the line or maximize the correlation coefficient. The correlation coefficient can be given by[26,27]

$$\rho_j = \frac{\sum_{i=1}^{N} X(i)Y_j(i) - \frac{1}{N}\left[\sum_{i=1}^{N} X(i) \cdot \sum_{i=1}^{N} Y_j(i)\right]}{[\sum_{i=1}^{N} X^2(i) - \frac{1}{N}[\sum_{i=1}^{N} X(i)]^2] \cdot \sum_{i=1}^{N} Y^2(i) - \frac{1}{N}[\sum_{i=1}^{N} Y(i)]^2}, \ j = 1,2 \qquad (3.16)$$

The two subpopulation correlation coefficients can be attained from the above equation. Every parameter has an effect on both correlation coefficients. The sum of two subpopulation correlation coefficients can be best measure for the degree of fitting. The coefficient is positive, $0 < \rho_j < 1, \ j = 1,2$. The target correlation coefficient is:

$$\rho = \rho_1 + \rho_2 \tag{3.17}$$

So, applying the iterative procedure, the estimation of $\alpha_1, \alpha_2, \beta_1, \beta_2, p$ can be attained by maximizing the value of ρ. The iteration starts with a proper initial point($\alpha_1^0, \alpha_2^0, \beta_1^0, \beta_2^0, p^0$). The program flow chart in Matlab is given in Figure 4.

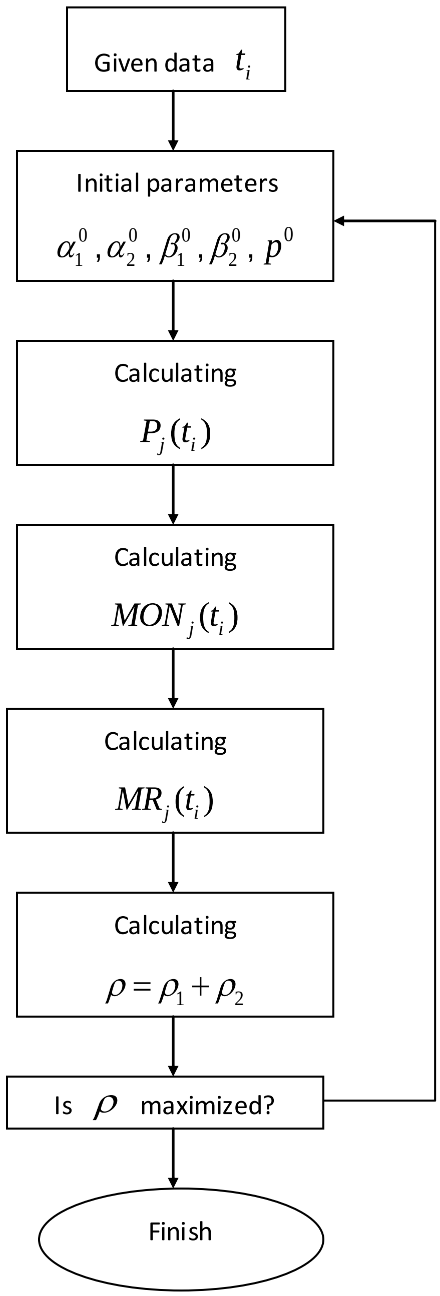

Figure 4 Kececioglu's Method's computing flow chart

Chapter 4 Case studies

In this chapter, I will use the both of graphical estimation method and Kececioglu's method to analyze three sets of data and correspondingly get the five Mixed Weibull distribution parameters. Three sets of data are shown below.

Table 2 Testing data

Case No.	Case 1 Burst Stress (MPa)		Case 2 BMG pressure 400 MPa (cycles)	Case 3 BMG pressure 600 MPa (cycles)
	3197.10	4230.18	9800	620
	3751.83	4230.18	11800	710
	3904.95	4272.69	12000	1040
	3904.95	4272.69	12100	1250
	4023.87	4272.69	13600	1430
	4105.16	4315.64	13400	2220
	4105.16	4315.64	14500	3030
	4105.16	4315.64	20100	3510
	4146.42	4359.01	29700	3810
	4146.42	4359.01		
	4188.09	4402.82		
	4188.09	4402.82		
	4188.09	4402.82		
	4188.09	4402.82		
	4188.09	4447.07		
	4230.18	4447.07		
	4230.18	4491.76		

4.1 Graphical estimation method

I follow steps mentioned in the chapter 2, compute the median rank for entire data at first, then plot all data on Weibull probability paper, eventually separate data into two subpopulations and then determine value of parameters α , β for each subpopulation.

Case 1

Table 3 Grouped failure data in Case 1 and the associated median ranks

Group Number	Time To Failure	Failures in each Group N_F	Cumulative failures by end of group $\sum N_F$	Median Rank, MR,%
1	3197.1	1	1	2.0
2	3751.83	1	2	4.9
3	3904.95	2	4	10.8
4	4023.87	1	5	13.7
5	4105.16	3	8	22.4
6	4146.42	2	10	28.2
7	4188.09	5	15	42.7
8	4230.18	4	19	54.4
9	4272.69	3	22	63.1
10	4315.64	3	25	71.8
11	4359.01	2	27	77.6
12	4402.82	4	31	89.2
13	4447.07	2	33	95.1
14	4491.76	1	34	98.0

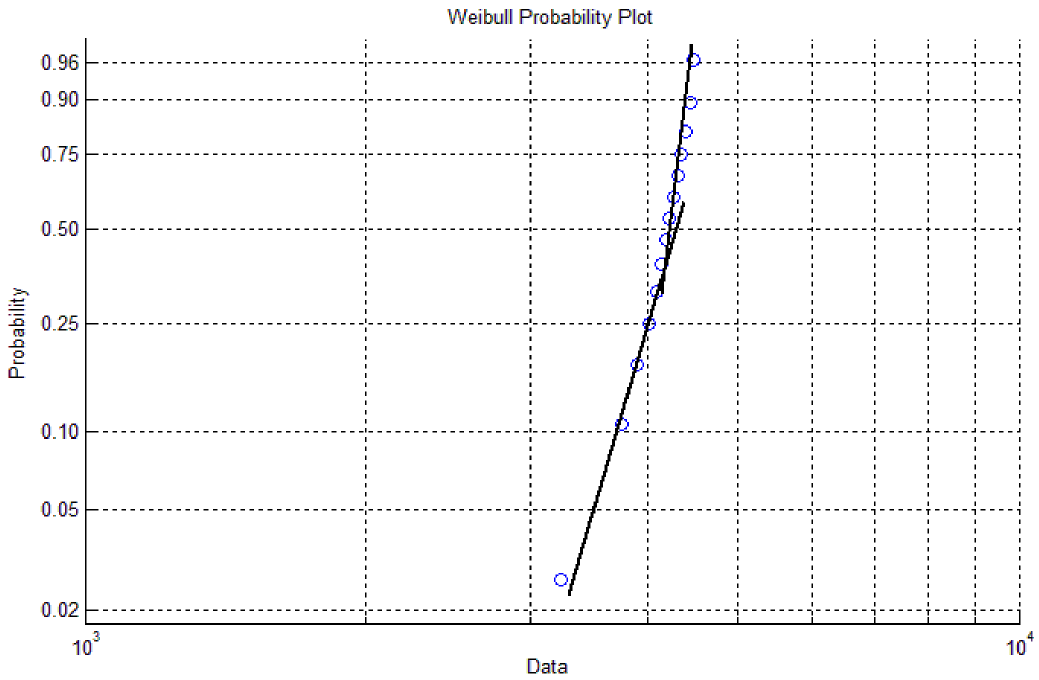

Figure 5 Plot of data in Table 3 to identify the two subpopulations in the data

After trying to fit those data into two straight lines, I put two subpopulation data on Weibull Probability paper to estimate parameters for each subpopulation.

Table 4 Failure data in case 1 grouped into two subpopulations to determine their parameters

Subpopulation	Subpopulation size	Point Number	Time To Failure	Cumulative failures by end of group $\sum N_F$	Median Rank, MR,%
1	N_1=10	1	3197.19	1	6.731
		2	3751.83	2	16.346
		3	3904.95	4	35.576
		4	4023.87	5	45.192
		5	4105.16	8	74.038
		6	4146.42	10	93.269
2	N_2=24	7	4188.09	5	19.262
		8	4230.18	9	35.656

		9	4272.69	12	47.951
		10	4315.64	15	60.246
		11	4359.01	17	68.443
		12	4402.82	21	84.836
		13	4447.07	23	93.033
		14	4491.76	24	97.131

These Median Ranks are plotted versus the time to failure on Weibull probability paper, as shown in Figure 6, separately for two subpopulations, yielding the following parameters

$p = 0.43$, $\alpha_1 = 28.5$, $\beta_1 = 4095.3$, $\alpha_2 = 29.5$, $\beta_2 = 4321.3$

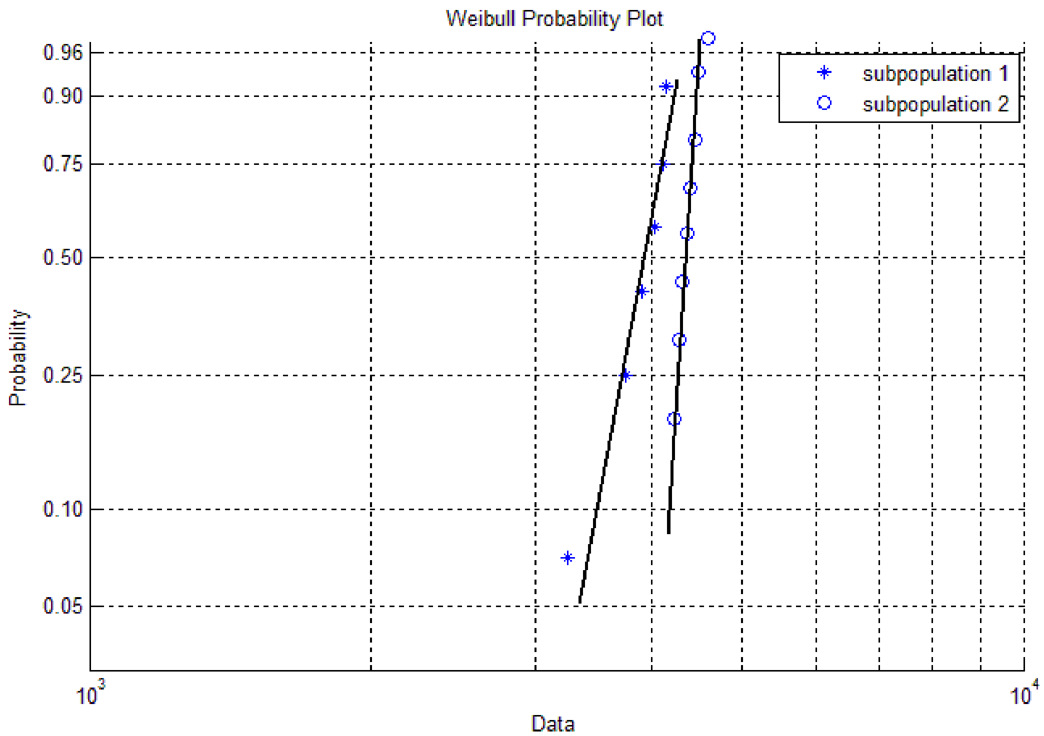

Figure 6 Two subpopulations drawn separately to determine Weibull distribution parameters in Case 1

Case 2

Table 5 Grouped failure data in Case 2 and the associated median ranks

Group Number	Time To Failure	Failures in each Group N_F	Cumulative failures by end of group $\sum N_F$	Median Rank, MR,%
1	9800	1	1	7.4
2	11800	1	2	18.1
3	12000	1	3	28.7
4	12100	1	4	39.4
5	13400	1	5	50.0
6	13600	1	6	60.6
7	14500	1	7	71.3
8	20100	1	8	81.9
9	29700	1	9	92.6

Table 6 Failure data in case 2 grouped into two subpopulations to determine their parameters

Subpopulation	Subpopulation size	Point Number	Time To Failure	Cumulative failures by end of group $\sum N_F$	Median Rank, MR,%
1	N_1=6	1	9800	1	10.9
		2	11800	2	26.6
		3	12000	3	42.2
		4	12100	4	57.8
		5	13400	5	73.4
		6	13600	6	89.1
2	N_2=3	7	14500	1	20.6
		8	20100	2	50.0
		9	29700	3	79.4

The mixed Weibull parameters in Case 2 are:

$p = 0.67, \quad \alpha_1 = 9.1, \quad \beta_1 = 12012.5, \quad \alpha_2 = 2.9, \quad \beta_2 = 27476$

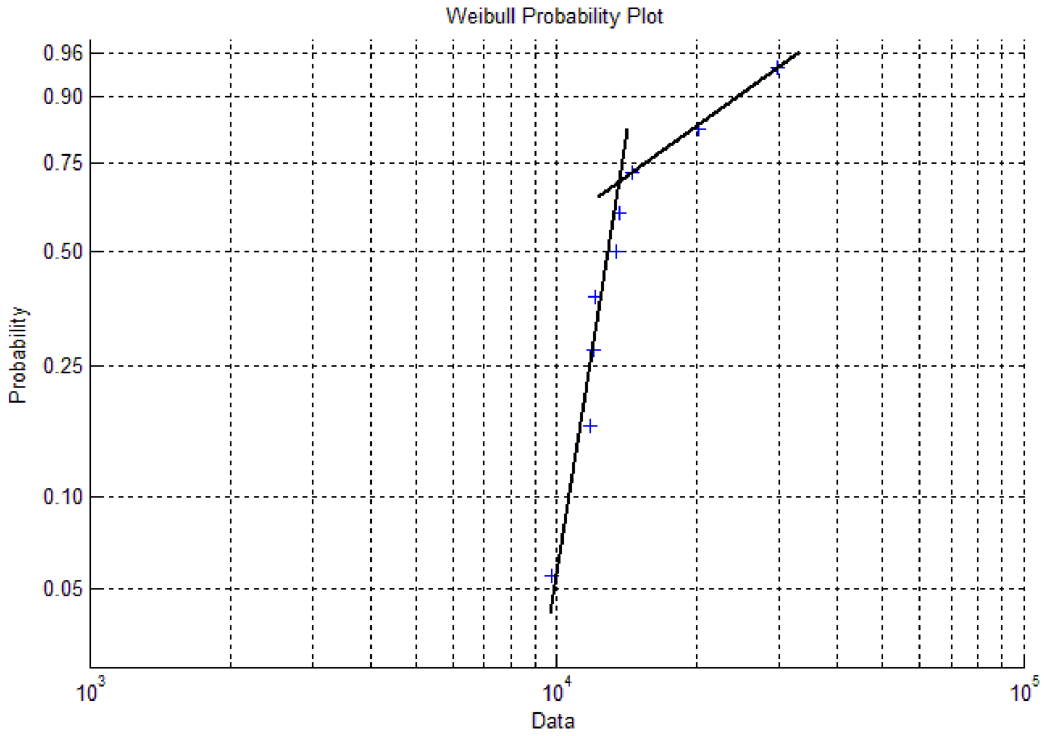

Figure 7 Plot of data in Table 4.4 to identify the two subpopulations in the data

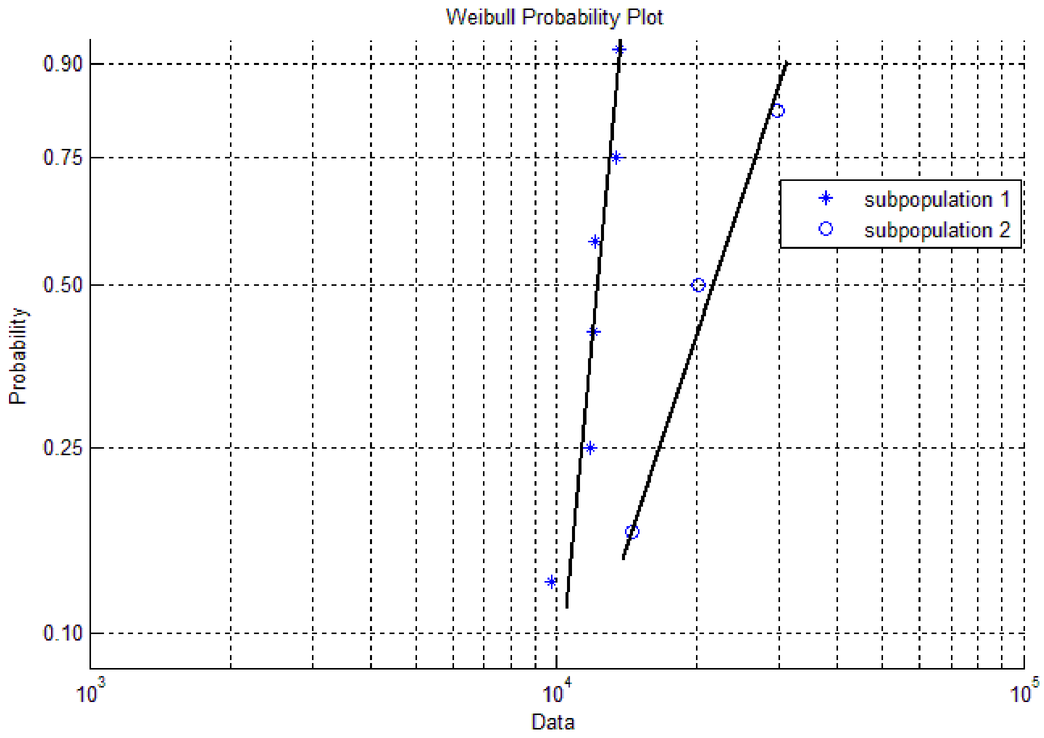

Figure 8 Two subpopulations drawn separately to determine Weibull distribution parameters in Case 2

Case 3

Table 7 Grouped failure data in Case 3 and the associated median ranks

Group Number	Time To Failure	Failures in each Group N_F	Cumulative failures by end of group $\sum N_F$	Median Rank, MR,%
1	620	1	1	7.4
2	710	1	2	18.1
3	1040	1	3	28.7
4	1250	1	4	39.4
5	1430	1	5	50.0
6	2220	1	6	60.6
7	3030	1	7	71.3
8	3510	1	8	81.9
9	3810	1	9	92.6

Table 8 Failure data in Case 3 grouped into two subpopulations to determine their parameters

Subpopulation	Subpopulation size	Point Number	Time To Failure	Cumulative failures by end of group $\sum N_F$	Median Rank, MR,%
1	N_1=5	1	620	1	12.963
		2	710	2	31.481
		3	1040	3	50.0
		4	1250	4	68.516
		5	1430	5	87.037
2	N_2=4	6	2220	1	15.909
		7	3030	2	38.636
		8	3510	3	61.363
		9	3810	4	84.091

The mixed Weibull parameters in Case 3 are:

$p = 0.56, \quad \alpha_1 = 3.56, \quad \beta_1 = 1205.4, \quad \alpha_2 = 6.2, \quad \beta_2 = 3589.7$

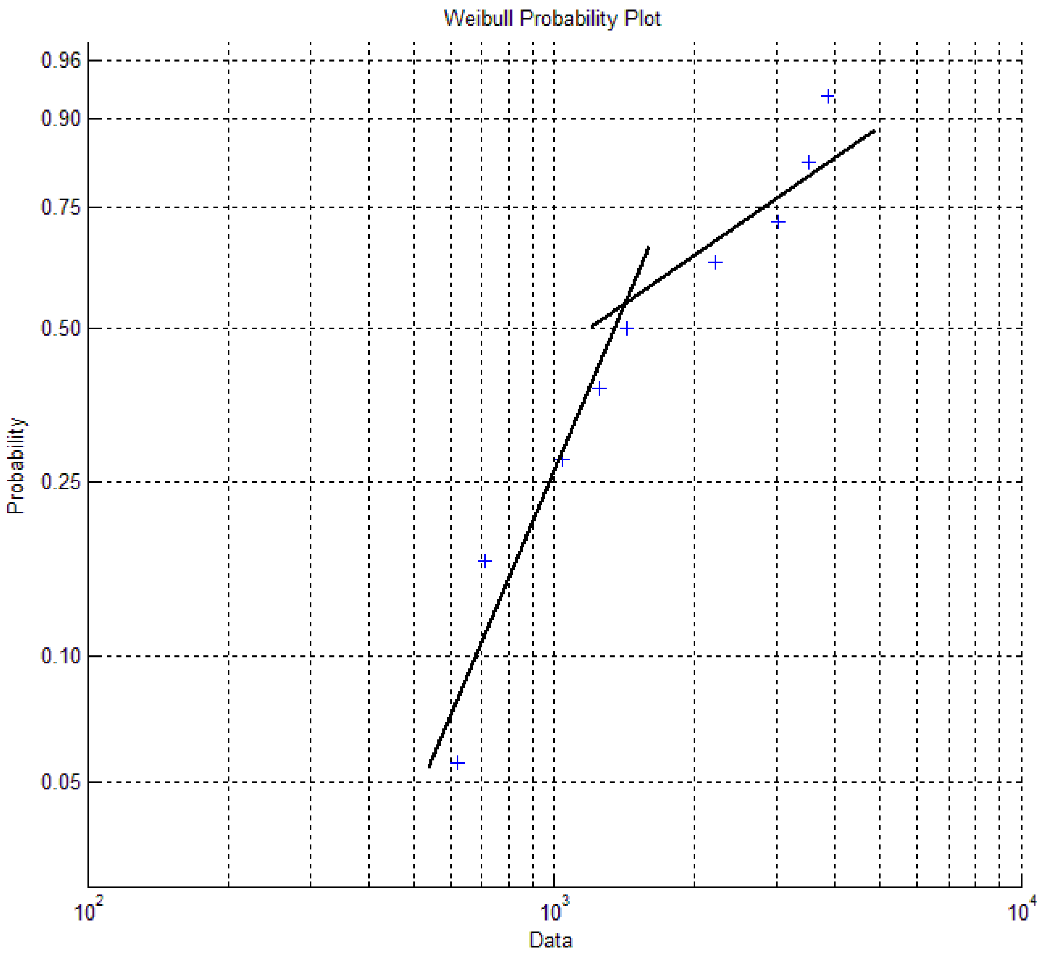

Figure 9 Plot of data in Table 8 to identify the two subpopulations in the data

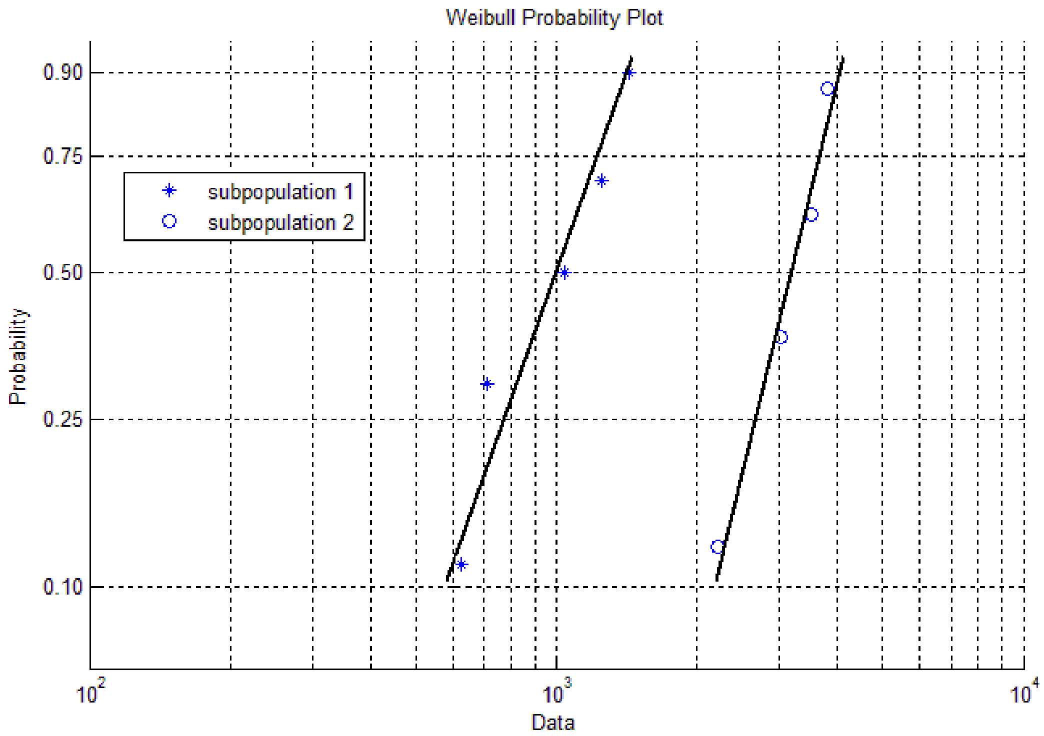

Figure 10 Two subpopulations drawn separately to determine Weibull distribution parameters in Case 3

4.2 Kececioglu's method

Kececioglu's method is used below. In this project, I will use Matlab to aid the computation. The Mixed Weibull distribution will be plotted on the specific coordinates, which is Weibull probability paper with transformed coordinates. Specifically the coordinate for the X axis represents $\ln t_{[i]}$ and the Y axis stands for $\ln[-\ln(1-P_i)]$, where $t_{[i]}$ is the ordered time to failure and P_i is cumulative probability estimated by MR. The linear least squares is applied for fitting the curve, and the slope b and incept m can be expressed by

$$b = -\alpha \ln \beta \tag{4.1}$$

$$m = \alpha \tag{4.2}$$

$$\beta = e^{(-\frac{b}{m})} \tag{4.3}$$

Case 1

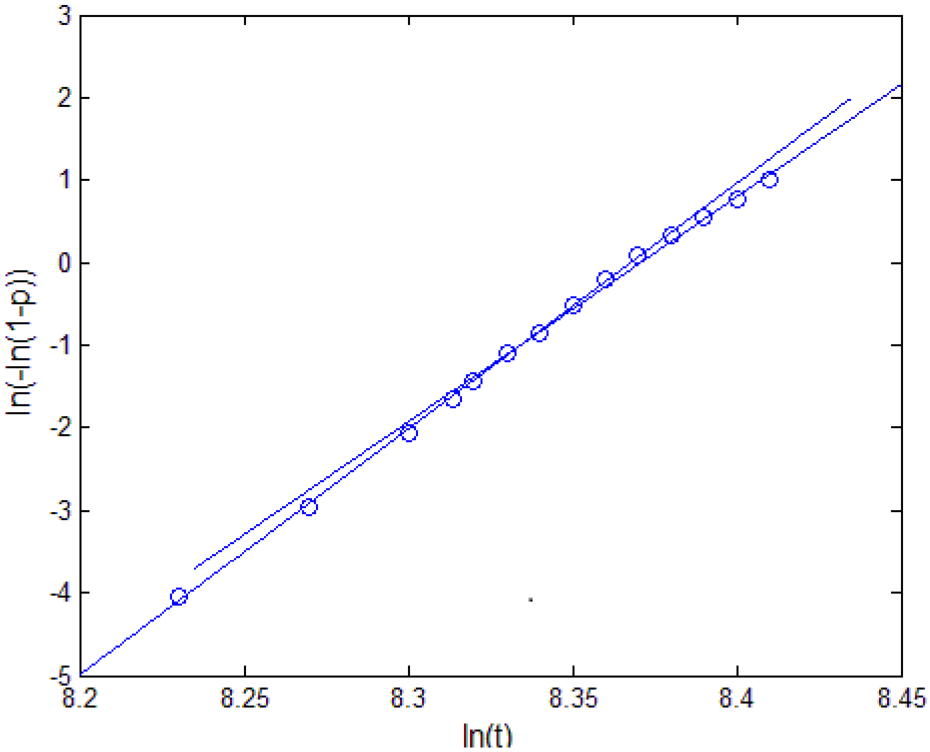

Figure 11 Case 1 Weibull distribution Plot

$$F(T)=\begin{cases}1-e^{-(\frac{T}{4235.4})^{39.9}}, \ln T \le 8.353 \\ 1-e^{-(\frac{T}{4327.5})^{26.2}}, \ln T \ge 8.353\end{cases} \tag{4.4}$$

The mixed Weibull distribution parameters are

p = 0.43, α_1 = 39.9, β_1 = 4235.4, α_2 = 26.2, β_2 = 4327.5

Case 2

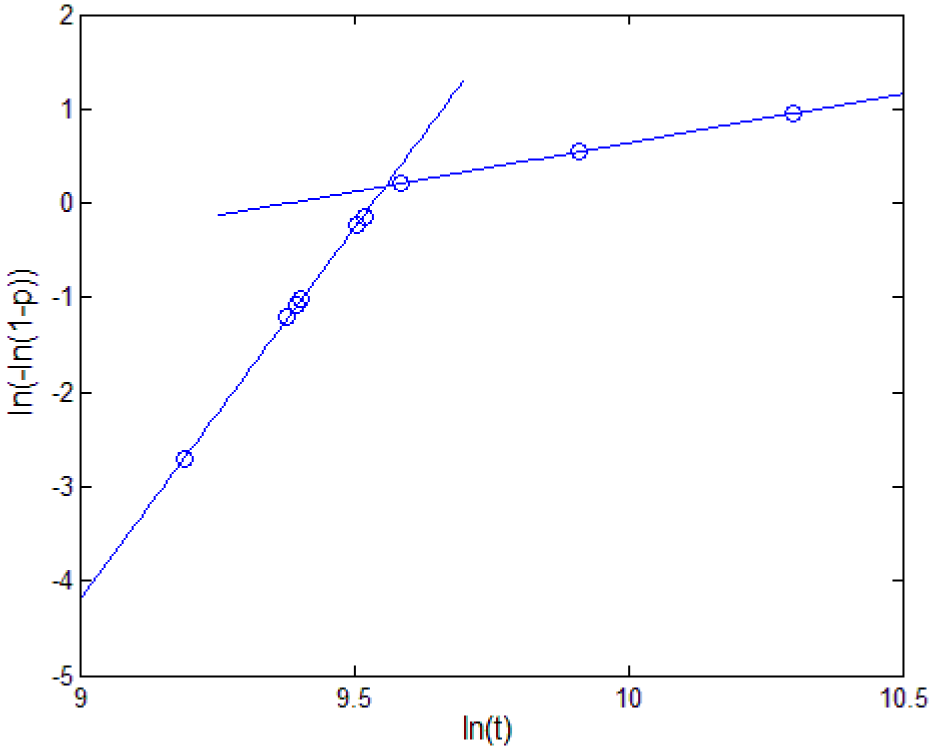

Figure 12 Case2 Weibull distribution Plot

$$F(T)=\begin{cases}1-e^{-(\frac{T}{13127.4})^{8.9}}, \ln T\leq 9.55\\ 1-e^{-(\frac{T}{28516.2})^{3.1}}, \ln T\geq 9.55\end{cases} \tag{4.5}$$

The mixed Weibull distribution parameters are

$p = 0.67$, $\alpha_1 = 8.9$, $\beta_1 = 13127.4$, $\alpha_2 = 3.1$, $\beta_2 = 28516.2$

Case 3

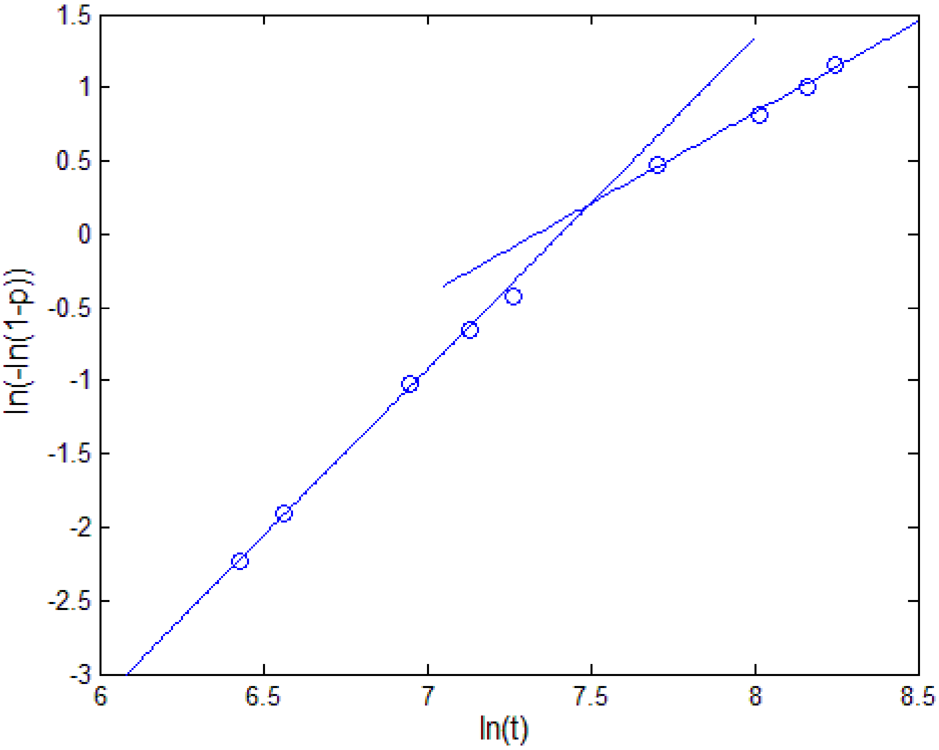

Figure 13 Case 3 Weibull distribution Plot

$$F(T)=\begin{cases}1-e^{-(\frac{T}{1034.4})^{2.6}}, \ln T \le 7.51 \\ 1-e^{-(\frac{T}{3719.7})^{7.5}}, \ln T \ge 7.51\end{cases} \qquad (4.6)$$

The Mixed Weibull distribution parameters are

p = 0.56, α_1 = 2.6, β_1 = 1034.4, α_2 = 7.5, β_2 = 3719.7

4.3 Comparison and conclusion

Two methods indicated above assume Mixed Weibull distribution by derivation of their results. Now, the Kolmogorov-Smirnov(K-S) Goodness of Fit (GoF) Test is applied in this section to assess those two methods' feasibility and accuracy. This test is based on the empirical distribution function (ECDF). Given N ordered data points H_1, H_2,..., H_N the empirical distribution function is defined by

$$E_N = n(i)/N \qquad (4.7)$$

where $n(i)$ is the number of data smaller than H_i, and the H_i are ordered from the smallest to largest. The test is a step function that increases by 1/N.

The feature of the test is that the distribution of the test itself does not rely on the underlying cumulative distribution being tested. Another advantage is that it is an accurate test compared with chi-square goodness of fit test which requires a sufficient size in order to make valid approximations. The GoF tests are mainly based on either of two distributions: the probability density function (PDF) and cumulative distribution function (CDF) which is used in this section. To implement the K-S test, we usually analyze at the data, the absolute difference between the ECDF and the estimated distribution we are trying to assess, so the K-S GoF test can also be considered as distance test. The distance D_n can be defined as

$$D_n = \max_{1 \le i \le n} \left| \frac{i}{n} - \hat{F}(x_{[i]}) \right| \tag{4.8}$$

$$\hat{F}(x_{[i]}) = P(X \le X_i) = CDF(X_i) \tag{4.9}$$

where n is the amount of data.

$\hat{F}(x_{[i]})$ is the cumulative distribution function being tested.

Three Comparisons are made by conducting the K-S GoF test. The results are shown below.

Table 9 K-S goodness –of-fit test on the parameter estimates for case 1

Times to Failure, t_i	Graphical estimation method D^g	Kececioglu's method D^p
3197.1	0.0302	0.0128
3751.83	0.0741	0.0475
3904.95	0.0687	0.0375
4023.87	0.0249	0.0141
4105.16	0.0382	0.0389
4146.42	0.0438	0.0206
4188.09	0.0521	0.0120

4230.18	0.0595	0.0137
4272.79	0.0637	0.0342
4315.64	0.0629	0.0227
4359.01	0.0566	0.0392
4402.82	0.0422	0.0125
4447.07	0.0162	0.0287
4491.76	0.0249	0.0256

Table 10 K-S GoF test on the parameter estimates for case 2

Times to Failure, t_i	Graphical estimation method D^g	Kececioglu's method D^p
9800	0.0141	0.0374
11800	0.0566	0.0203
12000	0.0698	0.0350
12100	0.0147	0.0141
13400	0.0497	0.0276
13600	0.0017	0.0206
14500	0.0375	0.0407
20100	0.0655	0.0368
29700	0.0565	0.0371

Table 11 K-S GoF test on the parameter estimates for case 3

Times to Failure, t_i	Graphical estimation method D^g	Kececioglu's method D^p
620	0.0293	0.0382
710	0.0724	0.0363
1040	0.0399	0.0349
1250	0.0304	0.0188
1430	0.0401	0.0221

2220	0.0408	0.0183
3030	0.0422	0.0102
3510	0.0351	0.0287
3810	0.0497	0.0136

$$D^g = \left| D_O(t_i) - D_E(t_i) \right|$$

where $D_O(t_i)$ is observed probability of failure or unreliability

$D_E(t_i)$ is expected probability of failure or unreliability

From the Table 12 below, it can be seen that Kececioglu's method yields a value of $D_{\max}$ smaller than value obtained from the graphical method.

Table 12 Comparison of $D_{\max}$ in two methods

Case No	$D^g(\max)$	$D^p(\max)$
Case 1	0.0741	0.0475
Case2	0.0698	0.0407
Case3	0.0724	0.0382

The Kececioglu's method, which combines the Bayesian method with the least-square method, can yield smaller distance difference than graphical estimation method. Therefore, this method is more accurate than graphical estimation method and also is easy to program.

Chapter 5 Extension of Kececioglu's method in 3-subpopluation mixed Weibull distribution

Though the application of Kececioglu's method for parameter estimation of Mixed Weibull distribution has been shown in the above chapters, its feasibility for a three-subpopulation Weibull distribution is not validated yet. In this chapter, I will extend its application for three subpopulation Weibull distribution parameter estimation.[28,29]

Assume that the three subpopulation Weibull distribution's time-to-failure sample is {t_i, i=1,2,3,…,N}. Suppose that the data are ordered $t_1 < t_2 < ....t_N$. At time t_i, the failure at j th subpopulation (j =1,2,3) is

$$P_j(t_i) = P\{T \in f_j(t) \Big| t_i - \frac{1}{2}\Delta t < T < t_i + \frac{1}{2}\Delta t\} \tag{5.1}$$

where j=1,2,3; i=1,2,…N

The probabilities of failure occurred at the time t_i belongs to subpopulation 1, subpopulation 2 and subpopulation 3, respectively, are

$$P_1(t_i) = \frac{pf_1(t_i)}{pf_1(t_i) + qf_2(t_i) + (1-p-q)\, f_3(t_i)} = \frac{pf_1(t_i)}{f(t_i)} \tag{5.2}$$

$$P_2(t_i) = \frac{qf_2(t_i)}{pf_1(t_i) + qf_2(t_i) + (1-p-q)\, f_3(t_i)} = \frac{qf_2(t_i)}{f(t_i)} \tag{5.3}$$

$$P_3(t_i) = \frac{(1-p-q)\, f_3(t_i)}{pf_1(t_i) + qf_2(t_i) + (1-p-q)\, f_3(t_i)} = \frac{(1-p-q)\, f_3(t_i)}{f(t_i)} \tag{5.4}$$

For each failure point, three equations should conform to equation below

$$P_1(t_i) + P_2(t_i) + P_3(t_i) = 1 \tag{5.5}$$

So the failure point occurring at time t_i can be divided into three possibilities: $P_1(t_i)$

of failure can fall in subpopulation 1, $P_2(t_i)$ of failure belong to subpopulation 2 and $P_3(t_i)$ of failure belong to subpopulation 3. The size of subpopulation 1 is $N \cdot p$, the size of subpopulation 2 is $N.q$ and the size of subpopulation 3 is $N(1-p-q)$ .So the Weibull distribution yields the following three subsamples:

Subsample1: $\{(t_1, P_1(t_1)),(t_2, P_1(t_2)),...,(t_N, P_1(t_N))\}$;

Subsample2: $\{(t_1, P_2(t_1)),(t_2, P_2(t_2)),...,(t_N, P_2(t_N))\}$;

Subsample2: $\{(t_1, P_3(t_1)),(t_2, P_3(t_2)),...,(t_N, P_3(t_N))\}$

For each subpopulation, its corresponding subsample can be solved by the Rank Regression method. So the Mean Order Number (MON) of the i th failure in the j th subpopulation will be

$$MON_1(t_i) = \sum_{k=1}^{i} P_1(t_k), i = 1, 2, ..., N \tag{5.6}$$

$$MON_2(t_i) = \sum_{k=1}^{i} P_2(t_k), i = 1, 2, ..., N \tag{5.7}$$

$$MON_3(t_i) = \sum_{k=1}^{i} P_3(t_k), i = 1, 2, ..., N \tag{5.8}$$

The corresponding Median Ranks $MR_j(t_i)$ is:

Subpopulation 1
$$MR_1(t_i) = \frac{MON_1(t_i)}{MON_1(t_N) + 0.4} \tag{5.9}$$

Subpopulation 2
$$MR_2(t_i) = \frac{MON_2(t_i)}{MON_2(t_N) + 0.4} \tag{5.10}$$

Subpopulation 3
$$MR_3(t_i) = \frac{MON_3(t_i)}{MON_3(t_N) + 0.4} \tag{5.11}$$

So the subsamples could be written as

Subpopulation 1 $\{(t_1, MR_1(t_1)),(t_2, MR_1(t_2)),...,(t_N, MR_1(t_N))\}$

Subpopulation 2 $\{(t_1, MR_2(t_1)),(t_2, MR_2(t_2)),...,(t_N, MR_2(t_N))\}$

Subpopulation 3 $\{(t_1, MR_3(t_1)), (t_2, MR_3(t_2)), ..., (t_N, MR_3(t_N))\}$

The CDF of Weibull distribution can be given in the form of

$$\log_e\{\log_e \frac{1}{1-MR_j(t_i)}\} = \alpha_j(\log_e t_i - \log_e \beta_j) \tag{5.12}$$

The linearized form of

$$Y_j(i) = \alpha_j X(i) + b_j \tag{5.13}$$

where $Y_j(i) = \log_e\{-\log_e[1-MR_j(t_i)]\}$,

$X(i) = \log_e t_i$,

$b_j = -\alpha_j \log_e \beta$.

The Weibull distribution parameters can be expressed by

$$p = \frac{MON_1(t_N)}{N} = \frac{1}{N}\sum_{i=1}^{N} P_1(t_i) \tag{5.14}$$

$$q = \frac{MON_2(t_N)}{N} = \frac{1}{N}\sum_{i=1}^{N} P_2(t_i) \tag{5.15}$$

$$\alpha_j = \frac{\sum_{i=1}^{N} X(i)Y_j(i) - \frac{1}{N}\left[\sum_{i=1}^{N} X(i) \cdot \sum_{i=1}^{N} Y_j(i)\right]}{\sum_{i=1}^{N} X^2(i) - \frac{1}{N}[\sum_{i=1}^{N} X(i)]^2}, \ j = 1, 2, 3 \tag{5.16}$$

$$b_j = \frac{1}{N}\sum_{i=1}^{N} Y_j(i) - \alpha_j \frac{1}{N}\sum_{i=1}^{N} X(i), \ j = 1, 2, 3 \tag{5.17}$$

$$\beta_j = \exp(-\frac{b_j}{\alpha_j}), \ j = 1, 2, 3 \tag{5.18}$$

Use these parameters estimation to maximize the correlation coefficient ρ . The correlation coefficient cab be given by

$$\rho_j = \frac{\sum_{i=1}^{N} X(i)Y_j(i) - \frac{1}{N}\left[\sum_{i=1}^{N} X(i) \cdot \sum_{i=1}^{N} Y_j(i)\right]}{[\sum_{i=1}^{N} X^2(i) - \frac{1}{N}[\sum_{i=1}^{N} X(i)]^2] \cdot \sum_{i=1}^{N} Y^2(i) - \frac{1}{N}[\sum_{i=1}^{N} Y(i)]^2} \quad j = 1, 2, 3 \tag{5.19}$$

Every parameter has an influence on correlation coefficients. The sum of three subpopulation correlation coefficients can be obtained for the degree of fitting. The coefficient is positive, $0 < \rho_j < 1$, $j = 1, 2, 3$. The target correlation coefficient is:

$$\rho = \rho_1 + \rho_2 + \rho_3 \qquad (5.20)$$

The APPENDIX contains life test data. After applying the method above, the parameters are found to be

p = 0.25, q = 0.48, α_1 = 2.93, β_1 = 2.89, α_2 = 1.86, β_2 = 6.61, α_3 = 1.42, β_3 =18.2

Figure 14 is each subpopulation plot. The function is

$$F(T) = \begin{cases} 1 - e^{-(\frac{T}{2.89})^{2.93}}, \ln T \le 1.15 \\ 1 - e^{-(\frac{T}{6.61})^{1.86}}, 1.15 \le \ln T \le 2.18 \\ 1 - e^{-(\frac{T}{18.2})^{1.42}}, 2.18 \le \ln T \le 5.14 \end{cases} \qquad (5.21)$$

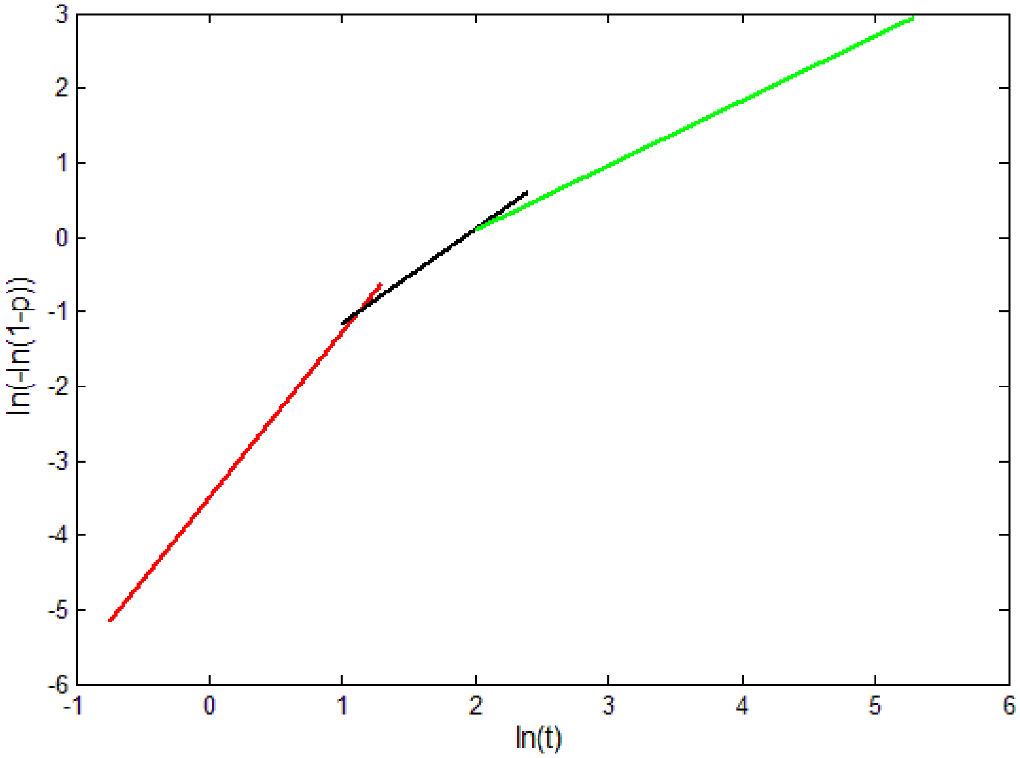

Figure 14 3-subpopulation Weibull distributions plot

Conclusion

The Figure 15 is the 3 subpopulation Weibull distribution plot with the data (circle). We can find the plot closely fit the data. Due to high volume of data, ten random data were selected and conducted by the K-S GoF test, shown in the Table 13. We can see the error is acceptable, so the Kececioglu's method can also be extended to 3-subpopulation Weibull distributions.

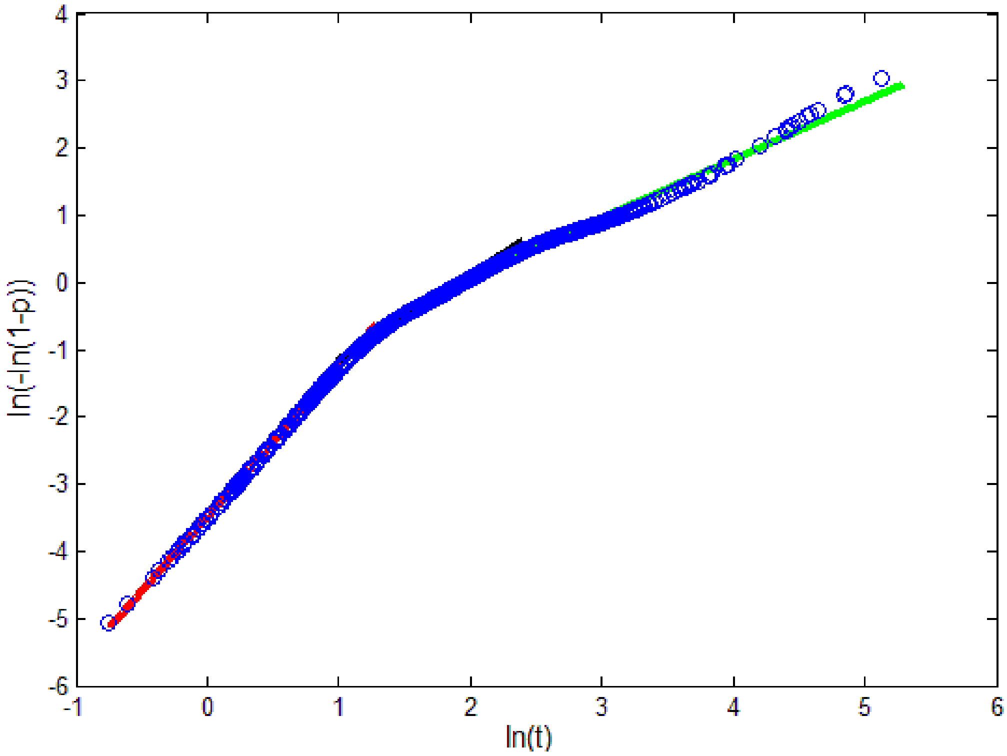

Figure 15 3-subpopulation Weibull distribution plot with data

Table 13 K-S goodness –of-fit test on the parameter estimates

Times to Failure, t_i	Proposed method D^p
0.9283	0.0349
1.1816	0.0385
1.246	0.0157
1.8822	0.0412
2.084	0.0245
2.3983	0.0329
2.634	0.0187

2.7821	0.0305
2.9787	0.0281

Chapter 6 Conclusion

The main focus of the work presented in the thesis is to study Weibull distribution parameter estimation methods which have been wildly used in lifetime analysis. This chapter summarizes the results of research work in the thesis and their implications.

In the thesis, detailed descriptions of graphical estimation method using Weibull probability paper and Kececioglu's method are presented. Graphical estimation methods are straightforward and convenient; however, Kececioglu's estimation method which combines Bayes' Theorem and the Least-Squares Method can produce less error. The mixed Weibull distribution consists of several subpopulations, each characterized by a Weibull distribution. At first, Kececioglu's method splits the data into distinct subpopulations by taking the posterior probability of each observation belonging to each subpopulation. Then Kececioglu's method uses Fracture Failure and Mean Order Number to estimate the parameters of each subpopulation.

In Chapter 4, three case studies have been carried out by comparing the accuracy of the two estimation methods. By using the Kolmogorov-Smirnov Goodness of Fit Test, it was found that generally Kececioglu's method provides a more accurate parameter estimation for the Mixed Weibull distribution in both small sample size data and median sample size data. It is therefore concluded with a recommendation of using Kececioglu's method for Mixed Weibull distributions.

Furthermore, an extension of Kececioglu's method into a 3 subpopulation Weibull distribution also has been attempted and verified. An example was conducted and the result shows that the error is in the acceptable range.

Concerning these conclusions, it is of importance to point out the following considerations in the future work:

In the thesis, Kececioglu's method has been proved by small size sample data, so this method is also need to be tested by large size sample data.

An extension of Kececioglu’s method into 3 subpopulation Weibull distribution is made, so an extension into n-subpopulation Weibull distribution can be tested in the future research.

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Appendix

200% Fatigue Life Expended 7075-T6 Aluminum LS surface; high stress double circular hole test

0.4734	1.7225	2.4619	3.0901	3.8427	4.8785	6.3909	8.2286	10.9835	17.077
0.5413	1.7225	2.4744	3.1043	3.849	4.967	6.3924	8.272	10.9835	17.1157
0.6625	1.7792	2.486	3.1066	3.8901	5.0073	6.4045	8.3139	11.0262	17.1723
0.6947	1.7888	2.492	3.1106	3.8901	5.0189	6.415	8.3218	11.0513	17.4127
0.7482	1.8061	2.4979	3.1221	3.8901	5.0189	6.4285	8.3463	11.0522	17.5926
0.7666	1.8061	2.4979	3.1459	3.8926	5.0213	6.445	8.3648	11.1536	18.02
0.7951	1.8098	2.5	3.1495	3.8926	5.0218	6.445	8.3701	11.227	18.4194
0.8115	1.8238	2.5008	3.1618	3.8926	5.0269	6.445	8.4339	11.2904	18.4324
0.8303	1.8274	2.5008	3.1647	3.9124	5.028	6.4596	8.4749	11.2954	18.5775
0.837	1.8419	2.5008	3.1754	3.9507	5.0496	6.4596	8.482	11.3227	18.5938
0.837	1.8453	2.5008	3.1767	3.9516	5.0572	6.4672	8.4931	11.3855	18.6706
0.838	1.8536	2.5162	3.191	3.9541	5.0789	6.4685	8.4976	11.4213	18.9903
0.8473	1.8597	2.5162	3.1954	3.975	5.0929	6.5195	8.5305	11.4229	19.1066
0.8873	1.8597	2.5292	3.1968	3.9784	5.1026	6.5431	8.5305	11.4676	19.2635
0.8888	1.8797	2.5314	3.1985	3.9898	5.132	6.5667	8.5345	11.5389	19.2855
0.8944	1.8822	2.5353	3.2044	3.9929	5.1573	6.5864	8.5426	11.6669	19.3544
0.9283	1.9115	2.5503	3.2075	4.0002	5.1675	6.5872	8.5587	11.6777	19.3898
0.9632	1.9301	2.559	3.2225	4.0099	5.1692	6.6061	8.586	11.7353	19.8133
0.9725	1.945	2.5618	3.2225	4.0164	5.1704	6.7048	8.6438	11.7574	20.1452
0.9725	1.945	2.5618	3.2523	4.0281	5.1909	6.7226	8.654	11.8125	20.2263
0.9918	1.945	2.5618	3.2587	4.029	5.2211	6.7478	8.686	11.8405	20.2309
1.0115	1.95	2.5793	3.2619	4.0314	5.2267	6.7548	8.6958	11.8526	20.552
1.0349	1.9641	2.5955	3.2701	4.0314	5.2525	6.7548	8.7001	11.8691	20.8225

1.0826	1.9836	2.6096	3.2847	4.0386	5.2794	6.7575	8.7703	11.9514	20.8351
1.1115	1.9867	2.6099	3.2926	4.0458	5.2794	6.7646	8.7802	11.9514	20.9558
1.1115	1.9892	2.6209	3.2926	4.0572	5.2813	6.7825	8.7889	11.974	20.9723
1.1201	1.9892	2.6257	3.2926	4.0885	5.2867	6.8054	8.8105	11.9817	21.0247
1.1201	2.0037	2.634	3.3024	4.1485	5.3353	6.8091	8.8186	12.003	21.1291
1.1428	2.0229	2.634	3.3057	4.168	5.368	6.8798	8.8402	12.0943	21.2662
1.1816	2.0229	2.6361	3.3057	4.1703	5.3765	6.9292	8.8927	12.1054	21.5363
1.1816	2.0443	2.6397	3.308	4.1703	5.3806	6.9415	8.909	12.267	21.7734
1.1987	2.0483	2.6397	3.3152	4.1772	5.384	6.9425	8.9188	12.3157	21.834
1.2104	2.0483	2.6397	3.3152	4.2144	5.3904	6.948	8.9317	12.4148	21.9956
1.2216	2.0822	2.6397	3.3348	4.2183	5.4201	6.9483	8.944	12.5039	22.1116
1.2384	2.084	2.6434	3.3402	4.2255	5.4255	6.9591	8.9791	12.6428	22.1686
1.2384	2.0856	2.6434	3.3459	4.2505	5.4294	6.9779	9.0402	12.6436	22.4598
1.2384	2.0886	2.6434	3.3479	4.2677	5.4344	6.989	9.0402	12.6459	22.5108
1.2427	2.0886	2.6467	3.3574	4.2743	5.4821	6.9943	9.0813	12.6596	22.5281
1.246	2.1162	2.6533	3.3603	4.2744	5.5038	7.0161	9.177	12.6954	22.7633
1.2525	2.1207	2.6583	3.3611	4.3008	5.5056	7.0174	9.2231	12.758	22.9023
1.2584	2.1241	2.6688	3.3804	4.3008	5.5275	7.0344	9.3095	12.7783	22.9133
1.2626	2.1241	2.6724	3.3804	4.3159	5.5313	7.0855	9.3178	12.7938	23.0778
1.2704	2.1253	2.6724	3.3804	4.327	5.5389	7.0974	9.3343	12.7958	23.1988
1.2809	2.1253	2.6831	3.4034	4.3905	5.5467	7.155	9.3565	12.8214	23.2615
1.2809	2.1568	2.6953	3.406	4.4053	5.5573	7.166	9.3921	12.9458	23.283
1.293	2.1673	2.6957	3.4063	4.4458	5.5642	7.2084	9.4637	12.9604	23.3244
1.305	2.1732	2.6976	3.4066	4.4489	5.5642	7.2205	9.497	12.981	23.3335
1.3194	2.1756	2.6976	3.4134	4.4545	5.5792	7.2272	9.5731	13.0745	23.348
1.325	2.189	2.7065	3.4497	4.4545	5.5902	7.2272	9.5873	13.2111	23.3932
1.3707	2.1967	2.7217	3.4646	4.4545	5.6005	7.2578	9.5954	13.344	23.9196
1.3893	2.1967	2.7554	3.465	4.4653	5.6391	7.2939	9.6024	13.3453	24.2792

1.3893	2.1995	2.7667	3.4844	4.4698	5.6431	7.2939	9.6357	13.3497	24.3702
1.3893	2.2058	2.7682	3.5079	4.4737	5.6691	7.3169	9.6474	13.3621	24.6079
1.3893	2.2132	2.7819	3.5175	4.5537	5.6745	7.3647	9.6854	13.4016	24.6474
1.4029	2.2229	2.7821	3.5349	4.5869	5.6979	7.3843	9.7342	13.5412	25.497
1.4099	2.2273	2.7821	3.5421	4.5869	5.6979	7.394	9.75	13.6414	25.9896
1.4168	2.2273	2.7825	3.5449	4.5932	5.703	7.4274	9.7693	13.7462	26.1492
1.4168	2.2273	2.7836	3.5503	4.6025	5.7307	7.432	9.7833	13.755	26.3125
1.4168	2.2318	2.7908	3.552	4.6118	5.7569	7.4353	9.8246	13.7718	26.3244
1.4168	2.2357	2.8132	3.5534	4.6183	5.7807	7.462	9.8263	13.9106	26.5724
1.4168	2.2357	2.8337	3.5534	4.6371	5.7807	7.5023	9.8446	14.0409	26.8229
1.4168	2.2447	2.834	3.5719	4.6375	5.7822	7.5036	9.8652	14.0517	27.7963
1.4168	2.2472	2.8369	3.5775	4.6599	5.8052	7.5036	9.8681	14.0658	28.4966
1.4441	2.2525	2.8372	3.5881	4.6599	5.809	7.5106	9.9108	14.0724	28.9534
1.45	2.2617	2.8608	3.6069	4.6599	5.8245	7.5139	9.912	14.0761	29.0592
1.4587	2.2863	2.8642	3.6149	4.6636	5.8245	7.5229	9.9431	14.2549	29.1269
1.4635	2.2913	2.9136	3.6229	4.6636	5.8931	7.5344	9.984	14.3127	29.8862
1.4647	2.2913	2.915	3.6362	4.6777	5.8944	7.5744	10.0185	14.3128	30.3076
1.4678	2.3064	2.9439	3.6362	4.6777	5.8977	7.5744	10.0925	14.3161	31.3986
1.5283	2.3065	2.9472	3.6431	4.6803	5.9156	7.5943	10.2012	14.4581	33.5746
1.5283	2.3289	2.9495	3.6477	4.7105	5.9166	7.5943	10.2169	14.4857	34.3232
1.531	2.3289	2.9495	3.656	4.7105	5.9352	7.6092	10.2231	14.8716	35.3696
1.5332	2.3289	2.9495	3.6587	4.7176	5.9805	7.6526	10.2939	14.9072	36.4302
1.5384	2.3391	2.9498	3.6784	4.7237	6.0111	7.6728	10.3206	15.1205	37.6547
1.5503	2.3391	2.9498	3.7082	4.7257	6.032	7.7814	10.338	15.2099	37.7602
1.5533	2.3455	2.9639	3.7124	4.7286	6.032	7.7814	10.3744	15.2825	39.0461
1.5685	2.3619	2.9787	3.7312	4.7289	6.0384	7.7926	10.3744	15.4199	39.7174
1.5754	2.3631	2.9893	3.7409	4.7291	6.0576	7.8111	10.3938	15.4615	40.3767
1.5841	2.3659	3.0051	3.743	4.7319	6.0702	7.843	10.4212	15.6013	41.8206

1.5841	2.3659	3.0056	3.7537	4.7319	6.094	7.9033	10.4582	15.635	45.0913
1.5841	2.3668	3.0056	3.7557	4.7421	6.101	7.9203	10.5221	15.6993	45.373
1.5971	2.3891	3.0056	3.7614	4.7522	6.1146	7.924	10.5429	15.7248	46.236
1.5996	2.3983	3.0271	3.7614	4.7563	6.1899	7.95	10.5588	15.9433	46.4933
1.6672	2.3983	3.0504	3.7614	4.7563	6.1927	8.0078	10.5598	16.0409	51.1547
1.673	2.4086	3.0649	3.7614	4.7967	6.2023	8.1132	10.7059	16.2486	52.0917
1.673	2.4086	3.0733	3.7743	4.8237	6.2742	8.1144	10.7365	16.2574	52.3051
1.673	2.4128	3.0754	3.7905	4.8467	6.2766	8.126	10.7876	16.2992	55.8502
1.6902	2.4128	3.0754	3.8117	4.8626	6.3144	8.1473	10.8099	16.4416	67.1228
1.7067	2.4252	3.0754	3.8255	4.8626	6.3363	8.2076	10.975	16.9073	75.1628
1.7185	2.441	3.0865	3.8303	4.8706	6.3758	8.2264	10.9792	17.0609	81.2206

Vita

Zhen Xu was born in Jinhua, Zhejiang, China, on 6 June 1985, the son of Jianhua Xu and Xiuzhen Cao. He completed his Bachelor degree in Hunan University majoring in Mechanical Engineering in 2007. For the next three years he went to Zhejiang University to pursue Master degree. He published two patents in machine design. In 2010, he entered Mechanical Engineering and Mechanics in Lehigh University. He worked with Prof. Harlow studying Reliability Engineering.