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The Steady Aerodynamics of Airfoils with Uniform Porosity

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The Steady Aerodynamics of Airfoils with Uniform Porosity

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Abstract

The steady aerodynamic loads on a flat plate with uniform porosity in a uniform background flow are determined in closed form by an extension of classical thin airfoil theory. The porous boundary condition on the airfoil surface assumes a linear Darcy law relationship, which furnishes a Fredholm integral equation for the bound vorticity distribution over the airfoil. The solution to this singular integral equation yields a single dimesionless group that determines when porosity effects are important. The pressure distribution, integrated lift, and pitching moment for the uniformily-porous airfoil are shown to be the product of the corresponding impermeable airfoil results and a simple function of the new dimensionless group.

Chapter 1

Introduction

The aerodynamic performance of fluid-loaded blades is of paramount importance in engineered structures such as aircraft wings, rotorcraft blades, and wind turbines. However, these applications also produce sound as a consequence of the blade passing through the air. Many researchers [1] have noted that changing the acoustical impedance of the wing can lead to significant reductions in turbulance-generated noise. One of the ways to affect acoustical impedance is the introduction of porosity at the surface or through the thickness of a wing or blade. Recently, Jaworski and Peake [2] show analytically that trailing-edge porosity and elasticity can be tuned to effectively eliminate the predominant scattering mechanism of trailing edge noise. However, for such an approach to be viable in application, one must quantify the tradeoff between the acoustical advantages of porosity versus its potentially negative impact on aerodynamic performance. The present work seeks to begin to address this tradeoff on the aerodynamic side by extending classical thin airfoil theory in steady flow to include a uniform porosity distribution across its chord.

The present work first reviews the derivation of aerodynamic forces acting on an impermeable airfoil in a steady flow. To determine uniquely the vorticity distribution across the airfoil before solving those forces, we have recourse to Kutta's hypothesis of smooth flow off of the sharp trailing edge. Continuous motion of the fluid in that region exists only if no pressure discontinuity is met by the flow when passing rearward off of the airfoil. With Kutta's condition, we can solve the aerodynamic problem by appropriate Fourier series substitutions for the known and unknown quantities [3].

The analysis for the limiting case of an uncambered, thin airfoil in steady flow is reviewed in §2 to motivate its extension to porous airfoils. The solution to the steady aerodynamic problem hinges on the ability to perform an inversion of an integral equation, which in the impermeable case is handled by the method of Söhngen [4]. In §3, the introduction of porosity changes the character of the essential equation involving Cauchy-type singular integrals to a Fredholm integral equation of the second kind. We use the inversion procedure of Ioakimidis [5] to solve this equation exactly, which furnishes a general result for thin airfoils with uniform porosity. As in the case of an impermeable airfoil, we obtain its chordwise pressure distribution, the lift, and pitching moment. A comparison between the impermeable and porous coefficients of lift and moment, as well as the pressure distribution across the airfoil, are made in §4. This comparison demonstrates the impact of uniform porosity on an airfoil in steady flow and identifies a new dimensionless parameter and important parameter regimes. Furthermore, the differences between impermeable and porous airfoil and how constant porosity affects pressure distribution, lift and moment are highlighted. Conclusions from this work and avenues for future research are presented and in §4

1.1 Research questions

This thesis seeks to answer the following research questions. Full details of the analysis to address these questions are given in §3.

1. What are the governing equations of permeable airfoils in a uniform flow?

2. Can the vorticity and pressure distributions for a uniformily-porous airfoils be found in closed form?

3. Can new result for the porous airfoil be linked to the integrated aerodynamic loads of impermeable airfoils?

Chapter 2

Impermeable thin airfoil theory in steady flow

Aircraft structures rely on many different wing configurations, such as delta, swept, straight, and low-aspect-ratio wings, where the wings can be idealized as a collection of two-dimensional airfoils. In this chapter, the analysis of impermeable airfoils is reviewed, following closely the presentation by Bisplinghoff, Ashley, and Halfman [6] (cf. pp. 208-218), to establish known results and motivate the extension to porous airfoils made in §3.

2.1 Governing equations and boundary conditions

The standard airfoil nomenclature is sketched in figure 2.1 [7], which introduces the principal geometric elements of the airfoil, such as mean camberline, chord line, leading edge, trailing edge, camber, and chord. A typical airfoil cross-section and its coordinate system are illustrated in figure 2.2 [7].

Based on the assumption of small disturbances, the so-called thin airfoil theory is established. Suppose a thin airfoil at small incidence to a uniform flow associated

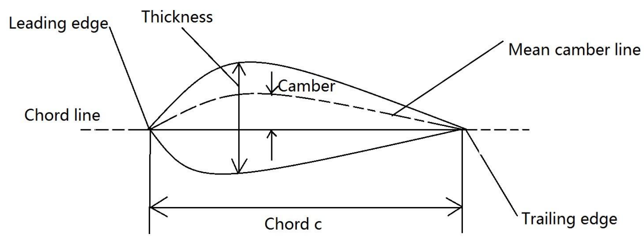

Figure 2.1: Airfoil nomenclature

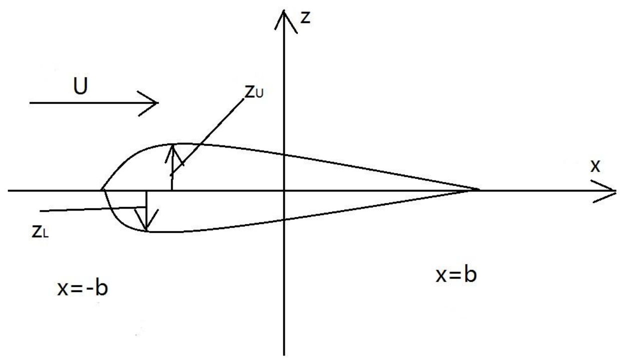

Figure 2.2: Coordinate system definition of the thin airfoil

with a steady, incompressible flow with a linear disturbance velocity potential ϕ. The governing equation for this two-dimensional potenial is Laplace's equation [6]:

$$\nabla^2\phi = 0. \tag{2.1}$$

For steady flow, the flow tangency conditions at the upper and lower airfoil surfaces $z = z_U(x)$ and $z = z_L(x)$ are, respectively,

$$\frac{w}{U} = \frac{dz_U}{dx}, \quad -b \leq x \leq b, \tag{2.2}$$

and

$$\frac{w}{U} = \frac{dz_L}{dx}, \quad -b \leq x \leq b. \tag{2.3}$$

for where $c = 2b$ is the chord.

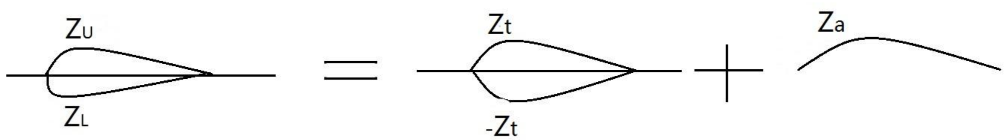

Figure 2.3: Thin airfoil illustrates as the superposition of a symmetrical body at zero incidence and a cambered and inclined mean line

z_U and z_L are functions of x only since the airfoil is two-dimensional. The boundary conditions for a thin airfoil in the x-direction in a uniform flow of speed U can be simplified to

$$w = U\frac{dz_U}{dx}, \tag{2.4}$$

for

$$z = 0^+, \quad -b \leq x \leq b, \tag{2.5}$$

and

$$w = U\frac{dz_L}{dx}, \tag{2.6}$$

for

$$z = 0^-, \quad -b \leq x \leq b. \tag{2.7}$$

We can split z_U and z_L into an even part z_a and an odd part z_t,

$$z_U = z_a + z_t, z_L = z_a - z_t \tag{2.8}$$

where z_a contains the angle of attack and camber, where z_t gives the chordwise distribution of thickness and describes a shape symmetrical about the x-axis,as pictured figure 2.3 [7]. We now have

$$w = U\frac{dz_U}{dx}, \quad z_U = z_a + z_t, \tag{2.9}$$

$$w = U\frac{dz_L}{dx}, \quad z_L = z_a - z_t. \tag{2.10}$$

Conditions (2.9) and (2.10) can also be applied to a finite wing in steady motion. The functions z_U, z_L, z_a and z_t are related to both x and y, and the aforementioned derivatives become partials with respect to x. The linear small disturbance assumption enables us to specify the boundary conditions on $z = 0^{\pm}$, provided that the slopes of the airfoil surface in the chordwise direction are sufficiently small. The flow field due to the airfoil thickness can be found by solving Laplace's equation (2.1) subject to boundary conditions.

$$w = \frac{\partial \phi}{\partial z} = U\frac{\partial z_t}{\partial x}; \quad \text{for } z = 0^+,\ (x, y) \in R_a \tag{2.11}$$

$$w = \frac{\partial \phi}{\partial z} = -U\frac{\partial z_t}{\partial x}; \quad \text{for } z = 0^-,\ (x, y) \in R_a, \tag{2.12}$$

where R_a is a region consisting of the projection of the planform for the xy-plane. Since we are not presently interested in the effects of variations in airfoil thickness, we shall not pursue this further here and will now focus on the lifting solution.

2.2 Vorticity and pressure distribution

The flow field is continuous in velocity and pressure across the xy-plane, except for the jump of $2U\frac{\partial z_t}{\partial x}$ in w specified over R_a [6]. Following the procedure outlined in [6], the bound vorticity on the airfoil can be found using a continuous distribution of point-source solutions to Laplace's equation. A single, concentrated source centered at $x = \xi$, $y = \eta$, $z = \zeta$ has the velocity potential

$$\phi_s = \frac{-H}{4\pi}\frac{1}{\sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}}, \tag{2.13}$$

where H is the strength of the source. The source produces a volume of liquid per unit time H into the surrounding space with spherical symmetry. A sheet having strength $H(\xi, \eta)$ per unit area in the neighborhood of the point (ξ,η,0) over the surface R_a has the disturbance potential

$$\phi(x, y, z) = \frac{-1}{4\pi} \iint_{R_a} \frac{H(\xi, \eta) d\xi d\eta}{\sqrt{(x-\xi)^2 + (y-\eta)^2 + z^2}}. \tag{2.14}$$

In order to satsify the condition

$$w = \frac{\partial \phi}{\partial z} = U \frac{\partial z_t}{\partial x}; \text{ for } z = 0^+, \ (x, y) \in R_a, \tag{2.15}$$

we must calculate

$$w(x, y, 0^+) = \frac{\partial \phi}{\partial z}(x, y, 0^+) \tag{2.16}$$

$$= \frac{-1}{4\pi} \lim_{z \to 0^+} \frac{\partial}{\partial z} \iint_{R_a} \frac{H(\xi, \eta) d\xi d\eta}{\sqrt{(x-\xi)^2 + (y-\eta)^2 + z^2}} \tag{2.17}$$

$$= \frac{1}{4\pi} \lim_{z \to 0^+} z \iint_{R_a} \frac{H(\xi, \eta) d\xi d\eta}{((x-\xi)^2 + (y-\eta)^2 + z^2)^{3/2}}. \tag{2.18}$$

The integral vanishes when z goes smaller positive values, except in the vicinity of point $\xi = x$, $\eta = y$, where the integrand tends to infinity. Isolating this region with a small square of side 2ϵ, (2.18) becomes

$$w(x, y, 0^+) = \frac{1}{4\pi} \lim_{z \to 0^+} z \int_{y-\epsilon}^{y+\epsilon} \int_{x-\epsilon}^{x+\epsilon} \frac{H(\xi, \eta) d\xi d\eta}{((x-\xi)^2 + (y-\eta)^2 + z^2)^{3/2}} \tag{2.19}$$

$H(\xi, \eta)$ over the entire square and its center value $H(x, y)$ differ by an amount of order ϵ, since H is a continuous function. Neglecting these $O(\epsilon)$ variations and applying the temporary integration variables $\xi' = (x-\xi)$, $\eta' = (y-\eta)$, we have

$$w(x,y,0^+) = \frac{H(x,y)}{4\pi} \lim_{z\to 0^+} z \int_{-\epsilon}^{\epsilon}\int_{-\epsilon}^{\epsilon} \frac{d\xi' d\eta'}{[\xi'^2+\eta'^2+z^2]^{3/2}} \tag{2.20}$$

$$= \frac{H(x,y)}{4\pi} \lim_{z\to 0^+} \left[2\arctan\frac{\epsilon^2}{z\sqrt{2\epsilon^2+z^2}} - 2\arctan\frac{-\epsilon^2}{z\sqrt{2\epsilon^2+z^2}}\right]. \tag{2.21}$$

The inverse tangents approach $-\pi/2$ and $\pi/2$, leading to

$$w(x,y,0^+) = \frac{H(x,y)}{2} \tag{2.22}$$

$$= w(x,y,0^-). \tag{2.23}$$

Now it is easy to see that the symmetrical discontinuity specified by

$$w = \frac{\partial\phi}{\partial z} = U\frac{\partial z_t}{\partial x};\ \text{for } z=0^+,\ (x,y)\in R_a \tag{2.24}$$

and

$$w = \frac{\partial\phi}{\partial z} = -U\frac{\partial z_t}{\partial x};\ \text{for } z=0^-,\ (x,y)\in R_a \tag{2.25}$$

can both be satisfied by setting

$$H(x,y) = 2U\frac{\partial z_t(x,y)}{\partial x}. \tag{2.26}$$

The solution for the velocity potential due to a distribution of sources R_a is completed by substituting (2.26) into (2.15) to get

$$\phi(x,y,z) = \frac{U}{2\pi}\iint_{R_a} \frac{\partial z_t(\xi,\eta)}{\partial x} \frac{d\xi d\eta}{\sqrt{(x-\xi)^2+(y-\eta)^2+z^2}}. \tag{2.27}$$

From this equation we can deduce all properties of the flow. For two-dimensional flow, a similar procedure yields the following solution to the disturbance velocity potential.

$$\phi(x,z) = \frac{U}{2\pi}\int_{-b}^{b}\frac{dz_t(\xi)}{dx}\ln[(x-\xi)^2+z^2]d\xi. \tag{2.28}$$

In general the boundary conditions can depend on time, in which case (2.26) must be changed to

$$H(x,y,t) = 2\frac{Dz_t(x,y,t)}{Dt}. \tag{2.29}$$

where is the D/Dt is the convective derivative.

The velocity must drop off inversely with distance by large circles surrounding a body with circulation, and the remote flow pattern approximates that due to a line vortex. whose velocity potential in cartesian coordinates is

$$\phi_v = -\frac{\gamma}{2\pi}\arctan\left[\frac{z-\zeta}{x-\xi}\right]. \tag{2.30}$$

A distributed potential along $\zeta = 0$ with circulation $\gamma_a(\xi)$ per unit length is

$$\phi(x,z) = \frac{1}{2\pi}\int_{-b}^{b}\gamma_a(\xi)\arctan\frac{z}{x-\xi}d\xi. \tag{2.31}$$

We make use of the relationship between γ_a and the local disturbance velocity u to decide what should be the extent of the vortex sheet [6]. This can be done by differentiating the last equation with respect to x and letting z approach zero. If the streamline distribution velocity volume is $u = u_U$ on the upper surface $z = 0^+$, then by antisymmetry it must be $u = u_L = -u_U$ on $z = 0^-$. Calculating the circulation using a rectangular contour constructed about a differential length of airfoil dx yields

$$(U+u_U)dx - wdz - (U-u_U)dx + wdz = 2u_U dx. \tag{2.32}$$

The circulation must also equal $\gamma_a dx$ with the definition of γ_a, so that $u_U = \gamma_a/2$. Thus, the pressure discontinuity across the vortex sheet and the bound vorticity are locally related by

$$\frac{p_U - p_L}{1/2\rho U^2} = \frac{-2\gamma_a}{U}. \tag{2.33}$$

2.3 Integrated lift and pitching moment for an impermeable airfoil

An integral equation for γ_a is derived by calculating w on the airfoil surface and applying the boundary condition

$$w = \frac{\partial \phi}{\partial z} = U\frac{dz_a}{dx};\ \text{for } z = 0,\ -b \leq x \leq b \tag{2.34}$$

to get

$$w(x,z) = \frac{\partial \phi}{\partial z} = \frac{-1}{2\pi}\int_{-b}^{b} \frac{[x-\xi]\gamma_a(\xi)d\xi}{(x-\xi)^2 + z^2}. \tag{2.35}$$

As $z \to 0$, the singularity of the integrand at $\xi = x$ leads this integral to be mathematically undefined. However, as before, we require on physical grounds that w must remain a continuous function of z throughout the limiting process. Thus,fr the uniquely correct w is found by taking the Cauchy principal value of the integral:

$$w(x,0) = U\frac{dz_a}{dx} = \frac{-1}{2\pi}\oint_{-b}^{b} \frac{\gamma_a(\xi)}{x-\xi}d\xi. \tag{2.36}$$

Cauchy's principal value is calculated by isolating the singular point $\xi = x$ with a small interval symmetrical about the point (i.e. $x - \epsilon \leq \xi \leq x + \epsilon$), evaluating each piece of the integral, and then letting ϵ approach zero.

We now introduce the following dimensionless variables for convenience,

$$x^* = \frac{x}{b}, \quad \xi^* = \frac{\xi}{b}, \tag{2.37}$$

which lead to

$$w(x^*, 0) = \frac{-1}{2\pi} \oint_{-1}^{1} \frac{\gamma_a(\xi^*)}{x^* - \xi^*} d\xi^*. \tag{2.38}$$

From [6], it is recalled that arbitrary amount of circulation can be placed around the airfoil without violating the boundary conditions, and by defining the vorticity distribution as

$$\gamma_{ac}(\xi*) = \frac{\Gamma_c}{b\pi} \sqrt{1 - \xi^{*2}}, \tag{2.39}$$

it can be shown that

$$\oint_{-1}^{1} \frac{\gamma_{ac}(\xi^*)}{x^* - \xi^*} d\xi^* = 0. \tag{2.40}$$

Thus, the solution for (2.39) matches the boundary conditions. We note that this solution puts a circulation Γ_c on the airfoil, which is found by summing the contributions of all bound vortices along the chord around the airfoil, which is

$$\Gamma = \int_{-b}^{b} \gamma_a d\xi = b \int_{-1}^{1} \gamma_{ac} d\xi^* = \frac{\Gamma_c}{\pi} \int_{-1}^{1} \frac{d\xi^*}{\sqrt{1 - \xi^{*2}}} = \frac{\Gamma_c}{\pi} \int_{0}^{\pi} d\theta = \Gamma_c. \tag{2.41}$$

We employ Kutta's hypothesis of smooth flow off the sharp trailing edge determine the exact magnitude of Γ_c. If no pressure discontinuity is met when passing rearward of the airfoil, continuous motion of the liquid can occur in that region. Hence,

$$\frac{2(P_U - P_L)}{\rho U^2} = \frac{-2u_U}{U} + \frac{2(-u'_U)}{U} = \frac{-2\gamma_a}{U} \tag{2.42}$$

turns out to be

$$\gamma_a = 0; \text{ for } x = b \ (x^* = 1). \tag{2.43}$$

With condition (2.43) at the trailing edge, (2.38) can be solved by appropriate Fourier-series substitutions for the known and unknown quantities [6]. However, Söhngen [4] proved that for any two functions f and g of engineering interest, the unique solution to the integral equation

$$g(x^*) = \frac{1}{2\pi} \oint_{-1}^{1} \frac{f(\xi^*)}{x^* - \xi^*} d\xi^*, \tag{2.44}$$

for which $f(1)$ is finite or zero, is

$$f(x^*) = \frac{-2}{\pi} \sqrt{\frac{1-x^*}{1+x^*}} \oint_{-1}^{1} \sqrt{\frac{1+x^*}{1-x^*}} \frac{g(\xi^*)}{x^* - \xi^*} d\xi^*. \tag{2.45}$$

Since f can be identified with γ_a, and g with $-w$, the inverted form of (2.38) becomes [6]

$$\gamma_a(x^*) = \frac{-2}{\pi} \sqrt{\frac{1-x^*}{1+x^*}} \oint_{-1}^{1} \sqrt{\frac{1+x^*}{1-x^*}} \frac{w(\xi^*, 0)}{x^* - \xi^*} d\xi^* \tag{2.46}$$

$$= \frac{2U}{\pi} \sqrt{\frac{1-x^*}{1+x^*}} \oint_{-1}^{1} \sqrt{\frac{1+x^*}{1-x^*}} \frac{dz_a(\xi^*)/dx}{x^* - \xi^*} d\xi. \tag{2.47}$$

We are able to integrate this equation and insert the answer into

$$g(x^*) = \frac{1}{2\pi} \oint_{-1}^{1} \frac{f(\xi^*)}{x^* - \xi^*} d\xi^* \tag{2.48}$$

to find the chordwise pressure distribution once we know the camber line and angle of attack of the airfoil. The lift and pitching moment (about an axis at $x = ba$) per unit span are then computed directly using

$$L = -\int_{-b}^{b} [p_U - p_L] dx = \rho U \int_{-b}^{b} \gamma_a dx = \rho U \Gamma, \tag{2.49}$$

$$M_y = \int_{-b}^{b} [p_U - p_L][x - ba] dx. \tag{2.50}$$

Γ therefore denotes the actual bound circulation on the airfoil. Substituting (2.46) and (2.49) gives the general lift result

$$L = -2\rho U^2 b \int_{-1}^{1} \sqrt{\frac{1+\xi^*}{1-\xi^*}} \frac{dz_a(\xi^*)}{dx} d\xi^* \tag{2.51}$$

in turns of prescribed airfoil geometry. For the simple case of a flat plate at angle of attack α,

$$z_a = -\alpha x = -\alpha b x^*. \tag{2.52}$$

Inserting this equation into (2.46) yields the flat-plate chordwise loading

$$\gamma_a = 2U\alpha\sqrt{\frac{1-x^*}{1+x^*}}. \tag{2.53}$$

This equation exhibits the well-known singularity at the leading edge, $x^* = -1$. The leading edge singularity is integrable, by elementry integration (2.49) and (2.50) give the following lift and pitching moment coefficients per unit span and pressure distribution:

$$(c_L)_{NP} = \frac{L}{\frac{1}{2}\rho U^2 c} = 2\pi\alpha, \tag{2.54}$$

$$\left(c_{M_y}\right)_{NP} = \frac{M_y}{\frac{1}{2}\rho^2 U^2 c^2} = -\pi\alpha\left[a + \frac{1}{2}\right] \tag{2.55}$$

$$\left(\frac{\Delta p}{\frac{1}{2}\rho U^2}\right)_{NP} = 4\alpha\sqrt{\frac{1-x^*}{1+x^*}}. \tag{2.56}$$

The subscript NP denotes solutions for the non-porous flat plate, which will be later compared with the porous airfoil results.

Chapter 3

Analytical analysis of a porous airfoil in steady flow

We now move on to the analysis of an airfoil of uniform porosity in a steady, uniform flow. As shown in the previous chapter, the mathematical problem for the flow around an impermeable, thin airfoil problem is reduced to a linear, one-dimensional, singular Cauchy integral equation of the first kind. In that case, the central technical challenge was the solution of the integral equation by inversion, which for the impermeable case was accomplished by Söhngen [4]. However, the problem of the flow around a permeable thin airfoil can lead generally to the solution of a singular, variable-coefficient integral equation of the second kind on a segment, or to a system of such equations [8]. In this chapter, the essential Fredholm integral equation is derived and inverted to solve the permeable airfoil problem, which appears by assuming a linear Darcy-type porosity boundary condition. We define a porosity constant C, which when $C = 0$, the impermeable case discussed in §2 is recovered. For increasing values of C, the airfoil is effectively more porous. For the case of the uniformily-porous airfoil, we build upon the theory for an impermeable airfoil, where careful attention is paid to the edge conditions for the porous case. The inversion solution of the Fredholm integral equation then furnishes the pressure distribution as well as the integrated lift and momment, which are proportional to the impermeable results by a function of a new dimensonless porosity parameter.

3.1 Governing equations and boundary conditions

Suppose a thin, and uniform porous airfoil immersed in a uniform, ideal, incompressible flow with a velocity U at infinity. As in §2 the perturbed velocity potential ϕ due to the pressure of the airfoil obeys Laplace's equation (2.1) at any point. The boundary condition for a permeable airfoil,

$$(\nabla\phi + \mathbf{U}) \cdot \hat{n} = W_s \tag{3.1}$$

is a generalization of the problem of a flow about its impermeable counterpart, where W_s is a local flow rate directed along the unit normal to the airfoil surface $\hat{n}$. If $W_s = 0$, we recover the impermeable limit examined in §2.

To close the problem considered, it is assumed that the flow rate W_s depends only on the pressure difference on the surface:

$$W_s = f_0(\Delta p). \tag{3.2}$$

We solve this problem with a method of singularities, by placing on the airfoil section surface an attached vorticity layer whose velocity potential automatically obeys Laplace's equation and satisfies the extinction condition at infinity.

From the Zhukovsky theorem [9], after linearization, it follows that:

$$\Delta p = \rho U_0 \gamma_a. \tag{3.3}$$

Here ρ is the fluid density, and γ_a is the intensity of the bound vorticity layer on airfoil. If we write the problem in two dimension, it will be

$$(\nabla\phi + \mathbf{U}) \cdot \hat{n} = (\frac{\partial\phi}{\partial x} + U, \frac{\partial\phi}{\partial z} + W)\cdot(\frac{dz}{dx}, 1) \tag{3.4}$$

$$= U\frac{dz_a}{dx} + W + H.O.T_o \tag{3.5}$$

and then

$$U\frac{dz_a}{dx} + W = W_s. \tag{3.6}$$

We will assume here only a streamwise background flow IE:$W = 0$. From the last chapter, we are already aware of that (2.54) and the order of $\frac{-\partial\phi}{\partial x}\frac{dz_a}{dx}$ is negligible to linear approximation. We now assume W is a linear function of the pressure gradient:

$$W = f_0(\Delta p) \tag{3.7}$$

$$= -C(p_U - p_L) \tag{3.8}$$

$$= C(\rho U \gamma_a). \tag{3.9}$$

Here C denotes the degree of porosity, which we will compare to Darcy's law to determine its value. Darcy's law is

$$q = \frac{-k}{\mu}\nabla p \tag{3.10}$$

$$v = \frac{q}{n} \tag{3.11}$$

where q is the flow rate (discharge per unit area), n is the open area fraction, v is the velocity, μ is the viscosity, k is the intrinsic permeablility of the medium, and ∇p is the pressure gradient in the z-direction across a uniform airfoil thickness l.

Combining (3.10) and (3.11), yields

$$v = \frac{-k}{\mu n} \nabla p, \tag{3.12}$$

which when compared to (3.7), identifies the porosity parameter C in terms of physical quantities, which is

$$C = \frac{kl}{\mu n}. \tag{3.13}$$

Recall that the airfoil is impermeable when $C = 0$. In general, C is an arbitrary positive constant. Then we can revise the boundary condition (3.6) to furnish the essential integral equation for a uniformily porous airfoil.

$$\frac{-1}{2\pi} \int_{-b}^{b} \frac{\gamma_a(\xi) d\xi}{(x-\xi)^2} = C(\rho U \gamma_a) + U \frac{dz_a}{dx}. \tag{3.14}$$

3.2 Fredholm integral equation

To solve the integral equation (3.14), we have recourse to the theory of Cauchy-type singular integral equations [10]. The Cauchy principal value representation of a function is [6]

$$F(x) = \int_a^b \frac{f(t)}{t-x} dt = \lim_{\epsilon \to 0} \left(\int_a^{x-\epsilon} \frac{f(t)}{t-x} dt + \int_{x+\epsilon}^{b} \frac{f(t)}{t-x} dt \right), \quad a < x < b. \tag{3.15}$$

The function $f(t)$ is assumed to be Hölder-continuous in the neighborhood of point x [11]. A more general definition of the same integral has been formulated by Ioakimidis [12] as

$$F(x) = \lim_{\epsilon', \epsilon'' \to 0} \left(\int_a^{x-\epsilon'} \frac{f(t)}{t-x} dt + \int_{x+\epsilon''}^{b} \frac{f(t)}{t-x} dt \right), \quad a < x < b. \tag{3.16}$$

This definition does not give a unique value to $F(x)$, as its value depends on the ratio ϵ'/ϵ'', or, more precisely, on the limit

$$k = \lim_{\epsilon', \epsilon'' \to 0} \frac{\epsilon'}{\epsilon''}, \quad k > 0. \tag{3.17}$$

If $k = 1$, we obtain the principal value of the integral. If not, we obtain some secondary value of the same integral, depending on the value of k, a positive number [12].

We identify (3.14) as a Fredholm integral equation, of which there are two kinds. The first kind is written as:

$$\phi(x) = \int_a^b K(x,t)\phi(t)dt, \tag{3.18}$$

from the form of we obtained last section. Our focus is on Fredholm equation of second kind, with the form [13]

$$\phi(x) = f(x) + \lambda \int_a^b K(x,t)\phi(t)dt. \tag{3.19}$$

Given the kernel $K(x,t)$, and the function $f(x)$, the problem is typically to find the function $\phi(x)$.

Now consider a general singular integral equation,

$$Aw(x)g(x) + \frac{B}{\pi}\int_{-1}^{1} \frac{w(t)g(t)}{t-x}dt = h(x), \tag{3.20}$$

where

$$w(x)g(x) = f(x), \tag{3.21}$$

and A and B are constants. The solution of this equation has the form frequently called the inversion formula [14, 15], which is

$$(A^2 + B^2)g(t) = A\frac{h(t)}{w(t)} - \frac{B}{\pi}\int_{-1}^{1} \frac{h(x)}{w(x)(x-t)}dx. \tag{3.22}$$

Using the relation

$$^{(s)}\int_{-1}^{1} \frac{f(t)}{t-x}dt = \int_{-1}^{1} \frac{f(t)}{t-x} + sf(x), \tag{3.23}$$

where $s = \pi A/B$, that results from using the generalized Cauchy-type singular integral [12]

$$F(x) = \int_{-1}^{1} \frac{f(t) - f(x)}{t-x}dt + f(x)\left(\ln\frac{1-x}{x+1} + s\right), \quad -1 < x < 1 \tag{3.24}$$

we are able to rewrite (3.22) as

$$(A^2 + B^2)g(t) = -\frac{B}{\pi}{}^{(-s)}\int_{-1}^{1} \frac{h(x)}{w(x)(x-t)}dx. \tag{3.25}$$

We can now rewrite (3.14) as

$$2C\pi\rho U\gamma_a - \int_{-b}^{b} \frac{\gamma_a(\xi)}{\xi - x}d\xi = -2\pi U\frac{dz_a}{dx}, \tag{3.26}$$

into the form for which an inversion solution exists,

$$Aw(x)g(x) + \frac{B}{\pi}\int_{-1}^{1} \frac{w(t)g(t)}{t-x}dt = h(x). \tag{3.27}$$

However, in order to use this formula, we still need further details about the edge conditions, which will be addressed in the next section.

3.3 Edge conditions for inversion

From §3.2, if $h(x)$ is a Hölder-continuous function along $x \in$[-1,1], then the solution of $f(x)$ of the function behaves near the endpoints of the integration interval [-1,1] like We can then assume $f(t) = w(t)g(t)$, where $g(t)$ remains bounded for $t \to \pm 1$ and $w(t)$ is given by [16]

$$w(t) = (1-t)^{\alpha}(1+t)^{\beta}. \tag{3.28}$$

Let $\kappa_i = -(\alpha_i + \beta_i)$(cf.[[15]]). In order to have integrable singularities, which are required on physical grounds, κ must be restricted to values $\kappa \in [-1, 0, 1]$. Each value of κ implies the possible values for α and β.

$$\kappa = -1 \quad \alpha < 0, \quad \beta < 0, \tag{3.29}$$

$$= 0 \quad \alpha = \beta, \tag{3.30}$$

$$= 1 \quad \alpha > 0, \quad \beta > 0. \tag{3.31}$$

Now compare equation (3.26) with (3.27) to get $A = 2C\pi\rho U, B = -\pi, h(x) = -2\pi U \frac{dz_a}{dx}$, and $f(x) = w(x)g(x) = \gamma_a(x)$. However, α and β need to be determined. A set of possible values of α and β can be determined by considering the special case of an impermeable airfoil. In that case, $C = 0$ implies $A = 0$, and the integral equation (3.27) simplies to

$$\frac{B}{\pi}\int_{-1}^{1}\frac{w(t)g(t)}{t-x}dt = h(x), \tag{3.32}$$

and the solution changes to

$$g(t) = -\frac{1}{\pi B}\int_{-1}^{1}\frac{h(x)}{w(x)(x-t)}dx. \tag{3.33}$$

Therefore, the possible exponent values for (3.28) are $\alpha = \pm 1/2$ and $\beta = \pm 1/2$ where the specific values values of α and β follow from κ. If $\kappa = 1$, then $\alpha = \beta = -1/2$, if $\kappa = 0$; then $\alpha = -\beta = \pm 1/2$; if $\kappa = -1$, then $\alpha = \beta = 1/2$ [16, 17]. We must choose the correct value of κ to recover the vorticity distribution in the impermeable case,

$$\gamma_a(x^*) = \frac{2U}{\pi}\sqrt{\frac{1-x^*}{1+x^*}} \oint_{-1}^{1} \sqrt{\frac{1+\xi^*}{1-\xi^*}} \frac{dz_a(\xi^*)/dx}{x^* - \xi^*} d\xi^*, \tag{3.34}$$

The choice of $\kappa \in [-1, 1]$ leads to an expression that is consistent with (3.34). For $\kappa = 0$, α and β can be either 1/2 or -1/2. From the equation (2.43), the Kutta condition implies zero pressure jump (zero vorticity) at the trailing edge, and $\gamma_a(x^*)$ is not singular. These conditions are met only when $\alpha = -1/2$ and $\beta = 1/2$, leading to $w(x^*) = \sqrt{\frac{1+x^*}{1-x^*}}$, and $g(\xi^*) = \gamma_a(\xi^*)\sqrt{\frac{1-\xi^*}{1+\xi^*}}$, and noting the leading-edge singularity at $x^* = -1$. To check that this choice is correct, substitute these values for α and β to get

$$\gamma_a(x^*)\sqrt{\frac{1-x^*}{1+x^*}} = \frac{2}{\pi} \int_{-1}^{1} \sqrt{\frac{1+\xi^*}{1-\xi^*}} \frac{dz_a/dx}{x^* - \xi^*} d\xi^*. \tag{3.35}$$

Thus, we recover the same vorticity distribution equation as before (2.47) for the impermeable case, with the proper behavior of the leading and trailing edges. We can now use the inversion equation (3.33) to solve the porous case, for when $C \neq 0$, using the following substitutions.

$$A = 2C\pi\rho U; \tag{3.36}$$

$$B = -\pi; \tag{3.37}$$

$$h(x) = -2\pi U \frac{dz_a}{dx}; \tag{3.38}$$

$$w(x^*) = \sqrt{\frac{1-x^*}{1+x^*}}; \tag{3.39}$$

$$g(\xi^*) = \gamma_a(\xi^*)\sqrt{\frac{1+\xi^*}{1-\xi^*}}. \tag{3.40}$$

Direct substitution of these relations into (3.25) yields

$$((2C\pi\rho U)^2 + \pi^2)\sqrt{\frac{1+x^*}{1-x^*}}\gamma_a(x^*) = 2\pi U \int_{-1}^{1}\sqrt{\frac{1+\xi^*}{1-\xi^*}}\frac{dz_a/dx}{x^*-\xi^*}d\xi^* - \frac{4C\rho U^2}{4C^2\rho^2U^2+1}\frac{dz_a}{dx}. \tag{3.41}$$

Defining the dimensionless group

$$\psi = 2C\rho U = 2\frac{\rho U k l}{un}, \tag{3.42}$$

(3.41) can be rearranged to get the vorticity distribution $\gamma_a(x^*)$ for the uniformily porous airfoil

$$\gamma_a(x^*) = \frac{2U}{(1+\psi^2)\pi}\sqrt{\frac{1-x^*}{1+x^*}}\int_{-1}^{1}\sqrt{\frac{1+\xi^*}{1-\xi^*}}\frac{dz_a/dx}{x^*-\xi^*}d\xi^* - \frac{2\psi U}{\psi^2+1}\frac{dz_a(x^*)}{dx}. \tag{3.43}$$

3.4 Integrated lift and pitching moment for a permeable airfoil

We observe that the solution (3.43) for the vorticity distribution differs from the impermeable case (2.47) by the constant factor $(1+\psi^2)^{-1}$ and by an addition terms of the right hand side involing the dimensionless porosity parameter ψ and the airfoil shape. In the limit of impermeable case, $\psi \to 0$ and this term is of no consequence. However, for airfoil with finite porosity this term leads to a violation of the Kutta condition and must be removed by this requirement. Further work [18] is underway to examine any potential issues with special porous edge conditions, which at present are not expected to affect the results of this thesis.

With this caveat, the pressure distribution, lift, and moment are all directly proportional to the vorticity distribution, we anticipate that these metrics for the impermeable case will also be modified by the factor $(1+\psi^2)^{-1}$ given the same airfoil shape. It is now sufficient to state the pressure distribution and aerodynamic loads for the porous airfoil in terms of the impermeable results (2.54) and (2.55).

$$(c_L)_P = \frac{1}{\psi^2+1}(c_L)_{NP} = \frac{2\pi\alpha}{\psi^2+1}, \tag{3.44}$$

$$\left(c_{M_y}\right)_P = \frac{1}{\psi^2+1}\left(c_{M_y}\right)_{NP} = \frac{-\pi\alpha\left[a+\frac{1}{2}\right]}{\psi^2+1}, \tag{3.45}$$

$$\left(\frac{\Delta p}{\frac{1}{2}\rho U^2}\right)_P = \frac{1}{\psi^2+1}\left(\frac{\Delta p}{\frac{1}{2}\rho U^2}\right)_{NP} = \frac{4\alpha}{\psi^2+1}\sqrt{\frac{1-x^*}{1+x^*}}. \tag{3.46}$$

The crucial porosity function $f(\psi) = (1+\psi^2)^{-1}$, relating the results for the impermeable airfoil to those of the porous airfoil, is shown in figure 3.1. Simple asymptotic analysis on this function indicates four parameter regimes of interest.

$$f(\psi) = 1 \quad \text{as} \quad \psi \to 0, \tag{3.47}$$

$$f(\psi) = 1-\psi^2 \quad \text{for} \quad \psi^2 \ll 1, \tag{3.48}$$

$$f(\psi) = \psi^{-2} \quad \text{for} \quad \psi^2 \gg 1, \tag{3.49}$$

$$f(\psi) = 0 \quad \text{as} \quad \psi \to \infty. \tag{3.50}$$

Clearly, the impermeable limit is recovered in (3.47), and no pressure jump across the airfoil can be sustained in the limit of zero solid fraction in (3.50). However, the results (3.48) and (3.49) indicate intermediate ranges of interest to the porous airfoil problem. For $\psi^2 \ll 1$, the effect of porosity is simply a second-order correction to the impermeable case. In the opposite limit, $\psi^2 \gg 1$ (but not infinite), it is clear to see the dominant effect of porosity, as illustrated in figure 3.1.

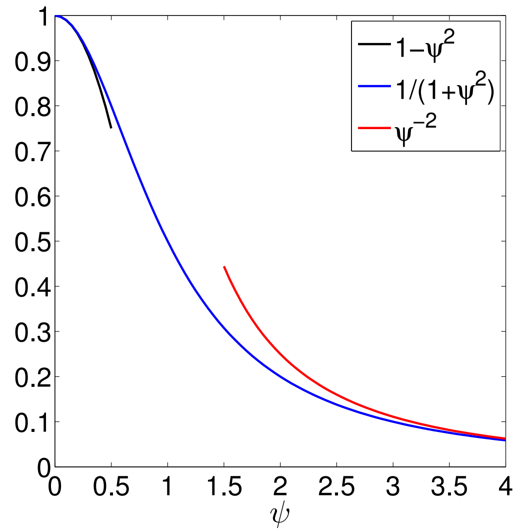

Figure 3.1: Porosity function relating the impermeable airfoil results to those of a uniformily porous airfoil, with asymptotic approximations for low and high effective porosity ψ.

Chapter 4

Conclusion and future work

The aerodynamic loads on a flat plate with a uniform porosity distribution are predicted by extension of classical thin airfoil theory. The established analysis of the impermeable case is reviewed in §2 to motivate the extension to the porous case in §3, where the existence of porosity furnishes a Fredholm integral equation for the bound vorticity distribution. This equation is solved by appeal to an inversion formula, and it is found that the pressure distribution, as well as the integrated lift and moment coefficient, of the porous airfoil are directly proportional to the results for the impermeable case. The derived proportionality constant is a function of a dimensionless porosity parameter, which can be related to measurable physical quantities.

Future study is needed to analyze an extra term that arises during the inversion process for a porous airfoil, which if retained leads to a contradiction with the Kutta condition. Future work is also warranted to investigate functional gradients of porosity on the airfoil aerodynamics and how such a distribution can be optimized for noise reduction by complementary aeroacoustic analysis.

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Vita

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