Analysis of three Classes of Cross Diffusion Systems

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Dedicated to

My family

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Abstract

A mathematical and numerical analysis has been undertaken for three cross diffusion systems which arise in the modelling of biological systems. The first system appears in modelling the movement of multiple interacting cell populations whose kinetics are of competition type. The second model is the mechanical tumor-growth model of Jackson and Byrne that consists of nonlinear parabolic cross-diffusion equations in one space dimension for the volume fractions of tumor cells and an extracellular matrix (ECM), and describes tumor encapsulation influenced by a cell-induced pressure coefficient. The third system is the Keller-Segel model in multiple-space dimensions with an additional cross-diffusion term in the elliptic equation for the chemical signal.

A fully practical piecewise linear finite element approximation for each system is proposed and studied. With the aid of a fixed point theorem, existence of fully discrete solution is shown. By using entropy type inequalities and compactness arguments, the convergence of each approximation is proved and hence existence of a global weak solution is obtained. In the case of the Keller-Segel model, we were able to obtain additional regularity to provide an improved weak formulation. Further, for the Keller-Segel model we established uniqueness results and error estimates. Finally, a practical algorithm for computing the numerical solutions of each system is described and some numerical experiments are performed to illustrate and verify the theoretical results.

Declaration

The work in this thesis is based on research carried out at the Numerical Analysis Group, Department of Mathematical Sciences, University of Durham, UK. No part of this thesis has been submitted elsewhere for any other degree or qualification and it all my own work unless referenced to the contrary in the text.

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Contents

List of Figures

List of Tables

Chapter 1

Introduction

1.1 Introduction

This thesis concerns the analysis of cross-diffusion systems. In order that we can describe what we mean by cross-diffusion, we first begin by describing diffusion and then self-diffusion. The term diffusion (diffusion, direct diffusion, ordinary diffusion) implies material moving from a high concentration to a low-concentration region. In the case of self diffusion, the rate depends on the local concentration. The term cross-diffusion means that a flow of one species occurs in the gradient of other substances. Cross-diffusion coefficients may be positive, negative, or zero. A positive coefficient suggests motion towards a region with low concentration of other substances; a negative coefficient indicates that motion occurs towards a region with a high concentration of other substances. The simplest example at the population level is a parasite (first object) moving by diffusion of a host (second object). Systems with cross-diffusion are rather widespread in nature and play an important role, especially in biophysical and biomedical situations. They have been the subject of active research for many years due to their wide applicability in biology, see for example [42, 74, 76, 78, 93] and the references therein. Earlier studies on modelling cross diffusion systems have been made in [78,88] and more recent work on modelling cross diffusion systems can be found in [45, 57, 68, 79]. In addition, we refer to [8,35,50,75,97] for some mathematical studies of a number of cross diffusion models of Lotka-Volterra type. Other mathematical studies of cross diffusion systems can be found in the literature, cf. [24, 34, 58, 69].

In this thesis, we use the finite element method as a technique to study three classes of strongly coupled cross diffusion systems arising in certain biological and physical applications. The first is a population model of competition type arising in biological study of the movement of multi-interacting cell populations. The second is the tumor-growth model which can provide biologists with complementary insight into the chemical and biological mechanisms which influence the development of solid tumors. The third is the Keller-Segel model arising in biological fields, such as embryogenesis, immunology, cancer growth and wound healing.

1.2 Introduction to the population model

We study the mathematical aspects of the multi-dimensional version of a cross-diffusion model with homogeneous Neumann boundary conditions and appropriate initial data. Up to now, the research has chiefly been concerned with Lotka-Volterra ODEs and their qualitative analysis such as persistence, permanence and attractability [1, 37, 64]. We consider the m-species cross-diffusion model: (P) Find $\{u_i(\mathbf{x},t)\}_{i=1}^m \in \mathbb{R}^{\geq 0} \times ... \times \mathbb{R}^{\geq 0}$ such that

$$\partial_t u_i - \nabla \cdot [D_i \nabla u_i + u_i \sum_{j=1}^m \nabla u_j] = g_i(\mathbf{u}), \qquad \text{in} \quad Q_T, \tag{1.2.1}$$

$$[(D_i \nabla u_i + u_i \sum_{j=1}^m \nabla u_j)] \cdot \nu = 0, \qquad \text{on} \quad S_T, \tag{1.2.2}$$

$$u_i(\cdot, 0) = u_i^0, \qquad \text{in} \quad \Omega, \tag{1.2.3}$$

where Ω is an open bounded domain in $\mathbb{R}^n (n \geq 1)$, with smooth boundary $\partial\Omega$. Here T is a positive number, $Q_T = \Omega \times (0,T)$, $S_T = \partial\Omega \times (0,T)$, $\mathbb{R}^{\geq 0} = \{x \in \mathbb{R} : x \geq 0\}$, ν denotes the exterior unit normal to $\partial\Omega$. $D_i \geq 0, i = 1, .., m$ are the constant diffusion rates. Furthermore, the source form is given by a Lotka-Volterra form where

$$g_i(\mathbf{u}) = \gamma_i u_i - u_i \sum_{j=1}^m u_j, \qquad i = 1, ..., m,$$

where the competition coefficients γ_i, $i = 1, ..., m$ represent a growth advantage of populations.

In Chapter 2, we introduce an extended study of the problem (P). The existence of a global weak solution of the system (1.2.1)-(1.2.3) is studied. To this end, we introduce and analyze a fully discrete finite element approximation of (P). The main features of the system are explicitly reflected in the analysis of the fully discrete problem. For this purpose, we have to derive an entropy inequality of the problem as this is the key in our analysis of the discrete problem. By testing the equations (1.2.1) with $\ln u_i, i = 1, ..., m$, integration over Ω and using integration by parts we can derive the entropy inequality of the problem (P):

$$\frac{d}{dt}\int_\Omega [u_i \ln u_i - u_i]dx + \int_\Omega (\frac{D_i}{u_i}|\nabla u_i|^2 + \sum_{j=1}^{m} \nabla u_i \nabla u_j)dx \le \int_\Omega g_i(\mathbf{u}) \ \ln u_i \, dx,$$

and summing $i = 1, ..., m$, yields

$$\frac{d}{dt}\sum_{i=1}^{m}\int_\Omega [u_i \ln u_i - u_i]dx + \int_\Omega (\sum_{i=1}^{m} \frac{D_i}{u_i}|\nabla u_i|^2 + |\sum_{i=1}^{m} \nabla u_i|^2)dx \le \sum_{i=1}^{m}\int_\Omega g_i(\mathbf{u}) \ \ln u_i \, dx.$$

However, owing to the singular nature of the derived inequality we have to go through a regularization procedure in order that we treat this problem. Hence, we establish a well defined entropy inequality of a regularized version of (P) and derive bounds on the regularized functions which are independent of the regularization parameter. The entropy inequality and the uniform bounds of the regularized problem provide the foundation of a discrete analogue of the entropy inequality and uniform estimates of the corresponding approximation problem. Such estimates are needed to prove the convergence of the regularized fully discrete problem as the regularization parameter and the discretization parameters simultaneously tend to zero, and therefore we obtain existence of a weak solution to the system (1.2.1)-(1.2.3).

For the study of different types of partial differential equations, the idea of defining and exploiting an entropy inequality has been used. For instance, in [9, 11], the entropy inequality is considered to study a thin film equation. In [8, 35, 36, 50, 51] the entropy inequality is used to study the cross diffusion systems. The approach adopted in this thesis uses the standard piecewise linear finite element method. For references that use this approach, or employ similar arguments and tools to our own, see for example [6,8–11,52,92]. For the theoretical tools, techniques and results used in this thesis see e.g. [2, 39, 49, 70, 83, 84].

In conclusion, the finite element approach used to show the existence of a non-negative global weak solution of (P) mainly contains five steps. The first step is to regularize the problem (P) and then establish its entropy inequality. Secondly, we introduce a fully discrete finite element approximation of the regularized problem and prove the existence of the approximate solutions at each time step using appropriate initial data. Thirdly, a discrete analogue of the entropy inequality is derived and then we establish some bounds of the approximate solutions. In the fourth step, the convergence of the fully discrete problem is studied as $h \to 0$. Finally, we study the convergence of the discrete problem which results from the fourth step as $\Delta t \to 0$.

1.3 Introduction to the cross-diffusion Tumor-growth model

The modelling and simulation of tumor growth may provide biologists with complementary insight into the chemical and biological mechanisms which influence the development of solid tumors. In [63], Jackson and Byrne have developed a continuous mechanical model which gives some insight into tumor encapsulation and transcapsular spread. The model consists of strongly nonlinear cross-diffusion equations for the volume fractions of the tumor cells and the extracellular matrix (ECM). A particular feature of the model is tumor encapsulation which is triggered by the increase of the pressure of the ECM due to tumor growth. This increase is modelled by the cell-induced pressure coefficient $\theta \geq 0$. When $\theta > 0$, the ECM becomes more compressed as the tumor cell fraction increases. For this problem, we are interested in a mathematical analysis of this model.

Tumor growth can be very roughly classified into three stages. The first stage is the avascular growth which is mostly governed by the proliferation of tumor cells. When the tumor grows, less and less nutrition is available for the cells in the tumor center, and the tumor starts developing its own blood supply (vascular stage). Later, the tumor cells are able to escape from the tumor via the circulatory system and lead to secondary tumors in the body (metastatic stage). The model considered in this problem describes the avascular stage only.

Most models for avascular tumor growth fall into two categories: discrete cell population models that track the individual cell behavior and continuum models that formulate the average behavior of tumor cells and their interactions with the tissue structure [23]. In the following, we concentrate on continuum models and in particular only on those which contain cross diffusion.

A possible continuum model ansatz is the use of reaction-diffusion equations. The system is then composed of mass balance equations for the cellular components, coupled to a system of reaction-diffusion equations for the concentrations of the extracellular substances [23]. The mass balance equations need to be closed by defining (or deriving) equations for the corresponding velocities. Roughly speaking, there are two classes of models: phenomenological and mechanical models (see Section 4 in [23]).

In phenomenological models, it is assumed that the cells or the ECM do not move or that they move due to diffusion [95], chemotaxis [32] or other mechanisms. Mechanical models differ from phenomenological ones by the fact that the latter ones do not take into account mechanical causes of cell movement due to pressure produced by proliferating tumor cells to the surrounding tissue [23]. An example of such a model is given by Casciari et al. [30]. When the cells are considered as an elastic fluid within a rigid ECM, the velocity may be closed according to the Darcy law, i.e., the velocity is proportional to the negative gradient of the pressure (see Formula (7) in [33] or Formula (4.4) in [23]). Alternatively, the cell-matrix system may be supposed to behave as a viscous fluid, in which the stress depends on the viscosity [28], as a viscoelastic fluid [61], or as a cell mixture in a porous medium made of the ECM filled with extracellular liquid [53]. More details can be found in the review of Roose et al. [86].

The mechanical model of Jackson and Byrne [63] describes the growth and encapsulation of solid tumors. The mass balance equations for the volume fractions of the tumor cell, the ECM, and the water phases are supplemented by equations for the velocities, depending on the gradient of the corresponding pressure. It is assumed in [63] that the pressure of the tumor cells and the ECM increases with the respective volume fraction and that the presence of tumor cells induces an increase in the ECM pressure, which leads to a nonlinear term in the ECM pressure. The model is given by the following scaled equations in one space dimension for the volume fractions of the tumor cells, c, and the ECM, m :

(W) Find $\{c(x,t), m(x,t)\} \in \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0}$ such that

$$\frac{\partial}{\partial t}\begin{pmatrix} c \\ m \end{pmatrix} - \nabla\left[D(c,m)\begin{pmatrix} \nabla c \\ \nabla m \end{pmatrix}\right] = R(c,m) \quad \text{in } \Omega, \quad t > 0, \tag{1.3.4}$$

where $\Omega = (0,1)$, subject to the Neumann boundary and initial conditions

$$\nabla c = \nabla m = 0 \quad \text{on } \partial\Omega, \quad t > 0, \quad c(\cdot,0) = c_0, \quad m(\cdot,0) = m_0 \quad \text{in } \Omega. \tag{1.3.5}$$

The mixture is supposed to be saturated, i.e., the volume fractions of the tumor cells c, the ECM m and water w sum up to one. Therefore, the volume fraction of water can be computed from $w = 1 - c - m$. Assuming that cell proliferation is proportional to the cell and water fractions (with rate γ), the tumor cells die with rate δ, and that the ECM production is proportional to all three fractions (with rate α), the net production rate is given by

$$R(c,m) = \begin{pmatrix} R_c(c,m) \\ R_m(c,m) \end{pmatrix} = \begin{pmatrix} \gamma c(1-c-m) - \delta c \\ \alpha c m(1-c-m) \end{pmatrix}. \tag{1.3.6}$$

The diffusion matrix

$$D(c,m) = \begin{pmatrix} 2c(1-c) - \beta\theta c m^2 & -2\beta c m(1+\theta c) \\ -2cm + \beta\theta(1-m)m^2 & 2\beta m(1-m)(1+\theta c) \end{pmatrix}, \tag{1.3.7}$$

with the pressure coefficients $\beta > 0$ and $\theta \geq 0$ is generally neither symmetric nor positive definite, which makes the analysis of the above system challenging.

A key observation is that system (1.3.4)-(1.3.7) possesses an entropy functional if $\theta < \theta^* := 4/\sqrt{\beta}$. To explain this, we introduce the logarithmic entropy

$$H(c,m) = \int_\Omega \Big(c(\ln c - 1) + m(\ln m - 1) + (1-c-m)(\ln(1-c-m) - 1) \Big) dx.$$

By testing the equations $(1.3.4)_1$ with $\ln c - \ln(1-c-m)$ and $(1.3.4)_2$ with $\ln m - \ln(1-c-m)$, integrating over Ω and using integration by parts we can drive the entropy inequality of the problem (W):

$$\frac{dH}{dt} + \int_\Omega (2(\nabla c)^2 + \beta\theta m \nabla c \nabla m + 2\beta(1+\theta c)(\nabla c)^2) dx$$

$$\leq \int_\Omega \big(R_c(c,m) \ln \frac{c}{1-c-m} + R_m(c,m) \ln \frac{m}{1-c-m}\big) dx.$$

For $c, m > 0$ and $c + m < 1$, it is easy to show that the right-hand side is bounded. It turns out that the integrand of the second term on the left-hand side is a positive definite quadratic form in c_x and m_x if $\theta < \theta^*$, which provides gradient estimates for c and m. This result can be strengthened: If $0 < \theta < 4/\sqrt{\beta}$, then we have

$$\int_\Omega (2(\nabla c)^2 + \beta\theta m \nabla c \nabla m + 2\beta(1+\theta c)(\nabla m)^2) dx \geq K_\theta \int_\Omega ((\nabla c)^2 + (\nabla m)^2) dx.$$

Here, we have used the properties $c, m > 0$, and $c + m < 1$.

1.4 Introduction to the Keller-Segel model

Chemotaxis, the directed movement of cells in response to chemical gradients, plays an important role in many biological fields, such as embryogenesis, immunology, cancer growth, and wound healing [60, 81]. The mathematical modeling of chemotaxis dates to the pioneering works of Patlak [80] and Keller and Segel [67]. The original model equations have been reduced to describe the evolution of the cell density $e(\mathbf{x}, t)$ and the concentration of the chemical signal $s(\mathbf{x}, t)$, and it is given, in its general form by:

(Q) Find $\{e, s\} \in \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 0}$ such that

$$\partial_t e - \nabla \cdot [\nabla e - e \nabla s] = 0, \qquad \text{in} \quad Q_T, \tag{1.4.8}$$

$$\alpha \partial_t s - \Delta s - \delta \Delta e - \mu e + s = 0, \qquad \text{in} \quad Q_T, \tag{1.4.9}$$

$$\nabla e \cdot \nu = 0, \qquad \nabla s \cdot \nu = 0, \qquad \text{on} \quad S_T, \tag{1.4.10}$$

$$e(\cdot, 0) = e^0, \qquad s(\cdot, 0) = s^0, \qquad \text{in} \quad \Omega, \tag{1.4.11}$$

where Ω is an open bounded domain in $\mathbb{R}^n (n \geq 1)$, with smooth boundary $\partial\Omega$. Here T is a positive number, $Q_T = \Omega \times (0, T)$, $S_T = \partial\Omega \times (0, T)$, ν denotes the exterior unit normal to $\partial\Omega$. The parameter $\alpha \geq 0$ is a measure of the ratio of the time scales of the cell movement and the distribution of the chemical, $\mu > 0$ is the secretion or production rate at which the chemical substance is emitted by the cells and δ is a positive constant. When $\alpha = 1$, the above system is of parabolic-parabolic type, whereas in the case $\alpha = 0$, it is parabolic-elliptic. The rigorous derivation of the classical Keller-Segel model from an interacting stochastic many-particle system has been performed by Stevens [89].

For the Keller-Segel model, we developed a finite element analysis. As both systems (1.2.1)-(1.2.3) and (1.4.8)-(1.4.11) belong to a similar class of equations, the analysis of problem (P) will significantly contribute to our study of the problem (Q). In particular, similar arguments used for (P) will be employed to prove the existence of a global weak solution of the system (1.4.8)-(1.4.11). Our analysis involves a discussion of the uniqueness of the weak solution of (Q) and a derivation of some fully discrete error estimates. By testing the equations (1.4.8) with $\ln e$ and (1.4.9) with s, integrating over Ω and using integration by parts we can derive the entropy inequality of the problem (Q):

$$\frac{d}{dt}\int_\Omega \left(e(\ln e - 1) + \frac{\alpha}{2\delta}s^2\right)dx + \int_\Omega \left(4|\nabla\sqrt{e}|^2 + \frac{1}{\delta}|\nabla s|^2 + \frac{1}{\delta}s^2\right)dx$$

$$\leq C(\mu,\delta)\|e\|_{L^1(\Omega)}^{5/2} + \int_\Omega \left(2|\nabla\sqrt{e}|^2 + \frac{1}{2\delta}|\nabla s|^2 + \frac{1}{2\delta}s^2\right)dx,$$

where $C(\mu,\delta)$ is a constant depend on μ and δ.

1.5 Research objectives and outline

We now give a brief description of each chapter for this thesis. Each of these descriptions is followed by the methodology that has been used.

In Chapter 2, the population model (P) is considered. A truncated alternative "equivalent" solvable problem to (P) is introduced. A regularized problem of the truncated system is studied and some a priori estimates of the regularized functions are obtained. A practical fully discrete approximation of the regularized problem is presented using a finite element method, with piecewise linear basis functions, to discretise in space and using backward Euler method to discretise in time. Then, some technical lemmata necessary for the analysis of the approximate problem are discussed. Finally, existence of the approximate solution at each time level is proven using the Schauder fixed point theorem.

In Chapter 3, the analysis of the population model (1.2.1)-(1.2.3) is continued. Some stability bounds on the fully discrete approximations, defined in Chapter 2, are derived. Using classical compactness arguments, the convergence of the approximate problem to (P) is studied. Existence of a global weak solution of the system (1.2.1)-(1.2.3) is shown.

In Chapter 4, we pass to the limit $M \to \infty$ in the discrete problem to deduce the existence of solutions to (P). To do this, we derive bounds on the approximate solution of (P), independent of M. The approximate model includes "microscopic cut-off" parameter M, where $M > 1$ is a (fixed, but otherwise arbitrary) cut-off parameter. Our ultimate objective is to pass to the limits $M \to \infty$ and $\Delta t \to 0$ in the discrete model, with M and Δt linked by the condition $\Delta t = o(M^{-1})$, as $M \to \infty$. To that end, we need to develop various bounds on sequences of weak solutions of the discrete problem that are uniform in the cut-off parameter M and thus permit the extraction of weakly convergent subsequences, as $M \to \infty$, through the use of a weak compactness argument.

In Chapter 5, some practical algorithms for computing the numerical solutions of problem (P) are described. Some numerical simulations in one and two spaces dimensions are performed and discussed.

The mechanical tumor-growth model of Jackson and Byrne is approximated using a finite element scheme in Chapter 6. The model consists of nonlinear parabolic cross-diffusion equations in one space dimension for the volume fractions of the tumor cells and the extracellular matrix (ECM). It describes tumor encapsulation influenced by a cell-induced pressure coefficient. The global-in-time existence of bounded weak solutions to the initial-boundary-value problem is proved when the cell-induced pressure coefficient is smaller than a certain explicit critical value.

In Chapter 7, a practical algorithm for solving the finite element problem of (W) at each time step is introduced. Some numerical results are presented to illustrate the tumor-growth behaviour.

Chapter 8 will be devoted to the analysis of the problem (Q). As both systems (1.2.1)-(1.2.3) and (1.4.8)-(1.4.11) belong to a similar class of equations, the analysis of problem (Q) is similar to the extent that we are able to prove the existence of a global weak solution of the system (1.4.8)-(1.4.11).

Ideally, one would like to pass to the limit $M \to \infty$ in the discrete problem to deduce the existence of solutions to (Q). Of course, our aim is to show existence of weak solutions to the Problem (Q), and that demands passing to the limits $\Delta t \to 0^+$ and $M \to \infty$, this then brings us to the next step in our argument. In Chapter 9, we shall link the time step Δt to the cut-off parameter $M > 1$ by demanding that $\Delta t = o(M^{-1})$, as $M \to \infty$, so that the only parameter in the approximate problem is the cut-off parameter. We shall show that the approximate problem can be bounded, independent of the cut-off parameter M. The collection of $M-$independent bounds enables us to extract some convergent subsequences of solutions to problem as $M \to \infty$. Due to the structure of (Q), the second part of this thesis will also involve a discussion of the uniqueness of the weak solution of (Q) as well as a derivation of some fully discrete error estimates. Some uniqueness results for weak solution have been discussed. An error bound between the fully discrete and weak solutions of (Q) has been proved.

A practical algorithm for computing the numerical solutions of the Keller-Segel model is given at the beginning of Chapter 10. We then perform numerical experiments in one space dimension demonstrating the fully-discrete error bound and the growth behaviour of the numerical approximation. Furthermore, simulations in two space dimensions are performed.

Finally, in Chapter 11, some concluding remarks are given and some possible future work is suggested.

Chapter 2

The population model: A fully discrete approximation of a regularized truncated problem

In Section 2.1 we mention the basic notation adopted in the thesis, regarding the Sobolev spaces, and recall and show some auxiliary results. In Section 2.2 we make a significant step towards showing the existence of a global in-time weak solution of the problem (P). Our approach in proving existence is based on the idea of defining an entropy inequality that leads us to obtain energy estimates. Thus in Section 2.2, we introduce a truncated alternative problem to (P). In Section 2.3 we introduce a regularized problem of the problem (P). Next, we derive a well defined entropy inequality of the regularized problem. In Section 2.4 we present some finite element notation which will be used in the current and the following chapters. A practical fully discrete finite element approximation of the regularized problem is proposed then we present some necessary lemmata. Finally, the existence of the approximate solutions are discussed by using a fixed point theorem.

2.1 Notation and auxiliary results

Throughout this study Ω denotes a bounded domain in $\mathbb{R}^d$, $d \leq 3$, with a Lipschitz boundary $\partial\Omega$. We use the usual Sobolev spaces $W^{m,p}(\Omega), m \in N, p \in [1, \infty]$ with the associated norms and semi-norms, denoted by $\|\cdot\|_{m,p}$ and $|\cdot|_{m,p}$ respectively. In particular, for $p = 2, W^{m,2}(\Omega)$ will be denoted by $H^m(\Omega)$ with norm $\|\cdot\|_m$ and semi-norm $|\cdot|_m$ and if $m = 0, W^{0,2}(\Omega) = L^2(\Omega)$. The $L^2(\Omega)$ inner product over Ω with norm $\|\cdot\|_0 = |\cdot|_0$ is denoted by $(\cdot,\cdot)$. In addition, $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $(H^1(\Omega))'$ and $H^1(\Omega)$ where $(H^1(\Omega))'$ is the dual space of $H^1(\Omega)$. A norm on $(H^1(\Omega))'$ is given by

$$\|f\|_{(H^1(\Omega))'} := \sup_{v\neq 0}\frac{|\langle f,v\rangle|}{\|v\|_1} \equiv \sup_{\|v\|_1=1}|\langle f,v\rangle|. \tag{2.1.1}$$

We also introduce the function spaces depending on time and space $L^p(0,T;X)$ $(1 \le p \le \infty)$ where X is a Banach space, consisting of all functions u such that for $a.e.\ t \in (0,T)$ $u \in X$ and the following norm is finite

$$\|u(t)\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p dt\right)^{\frac{1}{p}},$$

$$\|u(t)\|_{L^\infty(0,T;X)} = \text{ess}\sup_{t\in(0,T)}\|u(t)\|_X.$$

We also define $L^p(\Omega_T) = L^p(0,T;L^p(\Omega)),\ p \in [1,\infty]$. Furthermore, we define $C([0,T];X)$, the space of continuous functions from $[0,T]$ into X, which consists of those $u(t) : [0,T] \to X$ such that $u(t) \to u(t_0)$ in X as $t \to t_0$. We recall that $C([0,T];X)$ is a Banach space with the associated norm (see [91] page 43):

We also recall the following well-known Sobolev results

$$H^1(\Omega) \overset{c}{\hookrightarrow} L^r(\Omega) \hookrightarrow (H^1(\Omega))' \text{ holds for } r \in \begin{cases} [1,\infty] & \text{if } d = 1,\\ [1,\infty) & \text{if } d = 2,\\ [1,6] & \text{if } d = 3,\end{cases} \tag{2.1.2}$$

$$\langle f,v\rangle = (f,v) \quad \forall f \in L^2(\Omega) \text{ and } v \in H^1(\Omega), \tag{2.1.3}$$

where $\hookrightarrow$ denotes the continuous embedding. Further, we have from the Rellich-Kondrachov theorem, e.g. see [39] page 114 and [31] page 8, that the embedding in (2.1.2) is compact with $r \in [1,6]$ replaced by $r \in [1,6)$ in the case $d = 3$. The compact embedding will be denoted by the symbol $\overset{c}{\hookrightarrow}$.

For later use we recall the Gagliardo-Nirenberg inequality, see e.g. Adams [2]: Let $p \in [1, \infty]$, $k \geq 1$ and $v \in W^{k,p}(\Omega)$. Then there are constants C and $\varpi = \frac{d}{k}\left(\frac{1}{p} - \frac{1}{r}\right)$ such that the inequality

$$\|v\|_{0,r} \leq C\|v\|_{0,p}^{1-\varpi}\|v\|_{k,p}^{\varpi}, \qquad \text{holds for } r \in \begin{cases} [p, \infty] & \text{if } k - \frac{d}{p} > 0, \\ [p, \infty) & \text{if } k - \frac{d}{p} = 0, \\ [p, -\frac{d}{k-d/p}] & \text{if } k - \frac{d}{p} < 0. \end{cases} \tag{2.1.4}$$

We also need the following version of the Sobolev interpolation result: Let $v \in H^1(\Omega)$ then there are constants C and $\theta = \frac{2d(r-1)}{r(d+2)}$ such that the following inequality holds

$$\|v\|_{0,r} \leq C\|v\|_{0,1}^{1-\theta}\|v\|_1^{\theta}, \qquad \text{holds for } r \in \begin{cases} [1, \infty] & \text{if } d = 1, \\ [1, \infty) & \text{if } d = 2, \\ [1, 6] & \text{if } d = 3. \end{cases} \tag{2.1.5}$$

For later use, we recall the following embedding compactness result (see [72], page 58): Let X, Y and Z be three Banach spaces with X and Z being reflexive and $X \overset{c}{\hookrightarrow} Y \hookrightarrow Z$. Also let

$$W = \{v : v \in L^r(0,T;X), \frac{\partial v}{\partial t} \in L^s(0,T;Z)\},$$

where $T < \infty$ and $1 < r, s < \infty$. Then

$$W \overset{c}{\hookrightarrow} L^r(0,T;Y). \tag{2.1.6}$$

For later purpose we mention the Hölder's inequality: For $1 \leq p, q \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$, if $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$ then $f\,g \in L^1(\Omega)$ and

$$|f\,g|_{0,1} = \int_\Omega |f\,g|dx \leq \left(\int_\Omega |f|^p dx\right)^{\frac{1}{p}}\left(\int_\Omega |g|^q dx\right)^{\frac{1}{q}} = |f|_{0,p}|g|_{0,q}. \tag{2.1.7}$$

One can generalise this inequality by applying it for example twice to yield

$$|f\,g\,h|_{0,1} = \int_\Omega |f\,g\,h|dx$$
$$\leq \left(\int_\Omega |f|^p dx\right)^{\frac{1}{p}}\left(\int_\Omega |g|^q dx\right)^{\frac{1}{q}}\left(\int_\Omega |h|^r dx\right)^{\frac{1}{r}} = |f|_{0,p}|g|_{0,q}|h|_{0,r}, \tag{2.1.8}$$

for $1 \leq p, q, r \leq \infty$ such that $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$.

Another well-known inequality we need is the Poincaré inequality (e.g. see Wloka [96], page 117)

$$\|u\|_0^2 \leq C_p(|u|_1^2 + |(u,1)|^2), \qquad \forall u \in H^1(\Omega), \tag{2.1.9}$$

where C_p is a positive constant that depends on the domain Ω.
We shall frequently need the following simple version of Young's inequality

$$ab \leq \varepsilon^{p_1}\frac{a^{p_1}}{p_1} + \varepsilon^{-p_2}\frac{b^{p_2}}{p_2}, \quad \frac{1}{p_1} + \frac{1}{p_2} = 1,$$

valid for any $a, b \geq 0, \varepsilon > 0$ and $p_1, p_2 > 1$.

We shall also need the following simple inequality

$$(a-b)^2 \geq \frac{a^2}{2} - b^2, \qquad \forall\, a, b \in \mathbb{R}, \tag{2.1.10}$$

which follows from a direct application of the Young's inequality.

Another useful consequence of the Young's inequality is the following

$$ab \geq -\varepsilon\frac{a^2}{2} - \varepsilon^{-1}\frac{b^2}{2}, \qquad \forall\, a, b \in \mathbb{R}, \forall \varepsilon > 0. \tag{2.1.11}$$

We note the following elementary inequalities, valid for any $a \in \mathbb{R}$:

$$(1-a) = [1-a]_+ + [1-a]_- \leq [1-a]_+ \leq 1 - [a]_-, \tag{2.1.12}$$

$$(1-a) = [1-a]_+ + [1-a]_- \geq [1-a]_- \leq [a]_- - 1, \tag{2.1.13}$$

where $[a]_+ = \max\{a, 0\}$ and $[a]_- = \min\{a, 0\}$. Finally, for later reference we define the mean integral as

$$⨍ \eta := \frac{1}{|\Omega|}(\eta, 1) \quad \forall\, \eta \,\in\, L^1(\Omega). \tag{2.1.14}$$

Throughout C represents a generic positive constant, independent of any regularization and discretization parameter, which may change from one expression to another. In addition, $C(c_1, ..., c_n)$ denotes a constant depending on $\{c_i\}_{i=1}^n$.

2.2 A truncated alternative problem

One of the main difficulties of (P) is how to deal with the diffusion terms to derive H^1-norm bounds of the solutions $\{u_i\}_{i=1}^m$. To deal with this difficulty, from a biological point of view, we note that one does not expect all solutions $\{u_i\}_{i=1}^m$, to be unbounded. For $\gamma_j > \gamma_i$ we have an advantage of the u_j cells over the u_i cells. Thus, for the mathematical analysis of (P), we replace the term $u_i \sum_{j=1}^m \nabla u_j$ in (1.2.1) by $\phi(u_i) \sum_{j=1}^m \nabla u_j$ for $i = 1, ..., m$ and to replace the reaction terms $g_i(\mathbf{u}), i = 1, ..., m$ by $g_{i,M}(\mathbf{u}), i = 1, ..., m$, where

$$\phi(u_i) = [u_i - M]_- + M, \tag{2.2.15}$$

$$g_{i,M}(\mathbf{u}) = \gamma_i u_i - \phi(u_i) \sum_{j=1}^m u_j, \qquad i = 1, ..., m. \tag{2.2.16}$$

Here M is fixed positive number, and for later computational purposes we choose $M \geq e$. Without loss of generality, such a replacement can be considered even if for $\gamma_j > \gamma_i$, u_j does not have advantage over u_i. Thus the modified problem is:

(P$_M$) Find $\{u_{i,M}(x,t)\}_{i=1}^m \in \mathbb{R}^{\geq 0} \times ... \times \mathbb{R}^{\geq 0}$ such that

$$\partial_t u_i - \nabla \cdot [D_i \nabla u_i + \phi(u_i) \sum_{j=1}^m \nabla u_j] = g_{i,M}(\mathbf{u}), \qquad \text{in} \quad Q_T, \tag{2.2.17}$$

$$[D_i \nabla u_i + \phi(u_i) \sum_{j=1}^m \nabla u_j] \cdot \nu = 0, \qquad \text{on} \quad S_T, \tag{2.2.18}$$

$$u_i(\cdot, 0) = u_i^0, \qquad \text{in} \qquad \Omega, \quad i = 1, ..., m. \tag{2.2.19}$$

Before we go through the analysis of the problem (P$_M$), we first demonstrate the point of considering such a problem as an alternative to the model (P). In particular, we clarify the relation between a solution of (P$_M$) and a solution of (P). On noting the system (1.2.1)-(1.2.3) and the system (2.2.17)-(2.2.19), it can be seen clearly that the problem (P$_M$) is equivalent to (P), if the number M is chosen large enough such that $u_i < M$. This equivalence has meaning since the values of u_i, in (P), represent densities of multi types of cell populations, which are expected in the biological literature to be bounded (see Painter and Sherratt [79]). We finally mention that our analysis of the problem (P) will be also restricted to the assumption $D_i > 0$ as in the analysis of the problem (P).

2.3 A regularized problem

A key step of the multi-dimensional existence proof is to establish and exploit an entropy inequality. This will play a central role in our finite element approximation of (P). In order to make the entropy inequality of problem (P_M) well defined, we adopt the approach which has been used in [8–11]. Firstly, we introduce a function $\mathcal{F}^M \in C^2(\mathbb{R}^{>0})$ such that $\phi(s)(\mathcal{F}^M)''(s) = 1$ and $\mathcal{F}^M(1) = 0$ that is $\mathcal{F}^M : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ given by

$$\mathcal{F}^M(s) := \begin{cases} (\ln s - 1)s + 1, & 0 \leq s \leq M, \\ \frac{s^2 - M^2}{2M} + (\ln M - 1)s + 1, & M \leq s, \end{cases} \tag{2.3.20}$$

with the first two derivative of $\mathcal{F}^M$ given by

$$(\mathcal{F}^M)'(s) := \begin{cases} \ln s, & 0 < s \leq M, \\ \frac{s}{M} + \ln M - 1, & M \leq s, \end{cases} \tag{2.3.21}$$

and

$$(\mathcal{F}^M)''(s) := \begin{cases} \frac{1}{s}, & 0 < s \leq M, \\ \frac{1}{M}, & M \leq s. \end{cases} \tag{2.3.22}$$

Assuming positive values of the population densities, $\{u_{i,M}\}_{i=1}^m$, one can define the non-negative entropy functional

$$E(t) = \sum_{i=1}^m \int_\Omega \mathcal{F}^M(u_{i,M})dx. \tag{2.3.23}$$

Now, multiplying (2.2.17) by $(\mathcal{F}^M)'(u_{i,M})$, integrating by parts over Ω and summing the resulting equations, after recalling (2.3.20) and (2.2.18), we have the following entropy inequality

$$E(t) + \int_0^t (\sum_{i=1}^m \frac{D_i}{M} \|\nabla u_{i,M}\|_0^2 + \|\sum_{i=1}^m \nabla u_{i,M}\|_0^2) dt$$

$$\leq E(0) + \int_{Q_t} \sum_{i=1}^m g_{i,M}(\mathbf{u}_M)\ (\mathcal{F}^M)'(u_{i,M})\, dx\, dt. \tag{2.3.24}$$

Obviously, the bound (2.3.24) is only formal since e.g. a priori we do not know that $u_i(\mathbf{x}, t) \in \mathbb{R}^{>0}$ for $\mathcal{F}^M$ to be well defined. To make this bound rigorous, and in constructing our numerical approximation of (P), one has to go through a regularization procedure. Following the approach of Barrett and Blowey [8], we introduce an alternative regularization procedure, which we believe to be more transparent, to that employed in [35]. We replace $\mathcal{F}^M \in C^2(\mathbb{R}^{>0})$ for any $\varepsilon \in (0, e^{-1})$ by the regularized function $F_\varepsilon : \mathbb{R} \to \mathbb{R}^{\geq 0}$ such that

$$F_\varepsilon(s) := \begin{cases} \frac{s^2-\varepsilon^2}{2\varepsilon} + (\ln\varepsilon - 1)s + 1, & s \leq \varepsilon, \\ (\ln s - 1)s + 1, & \varepsilon \leq s \leq M, \\ \frac{s^2-M^2}{2M} + (\ln M - 1)s + 1, & M \leq s. \end{cases} \tag{2.3.25}$$

Hence $F_\varepsilon \in C^{2,1}(\mathbb{R})$ with the first two derivatives of F_ε given by

$$F'_\varepsilon(s) := \begin{cases} \varepsilon^{-1}s + \ln\varepsilon - 1, & s \leq \varepsilon, \\ \ln s, & \varepsilon \leq s \leq M, \\ \frac{s}{M} + \ln M - 1, & M \leq s, \end{cases} \tag{2.3.26}$$

and

$$F''_\varepsilon(s) := \begin{cases} \frac{1}{\varepsilon}, & s \leq \varepsilon, \\ \frac{1}{s}, & \varepsilon \leq s \leq M, \\ \frac{1}{M}, & M \leq s, \end{cases} \tag{2.3.27}$$

respectively. We introduce also the regularized function $\phi_\varepsilon : \mathbb{R} \to [\varepsilon, M]$ defined by

$$\phi_\varepsilon(s) := [F''_\varepsilon(s)]^{-1} := \begin{cases} \varepsilon, & s \leq \varepsilon, \\ s, & \varepsilon \leq s \leq M, \\ M, & M \leq s. \end{cases} \tag{2.3.28}$$

It is easily established from (2.3.25), (2.3.26) and (2.3.27) that for $\varepsilon \in (0, e^{-1})$ (see [11] for more details)

$$F_\varepsilon(s) \geq \frac{s^2}{2\varepsilon} \quad \forall s \leq 0, \tag{2.3.29}$$

$$F_\varepsilon(s) \geq \frac{s^2}{4M} - \frac{3M}{2} \quad \forall s \geq 0, \tag{2.3.30}$$

$$sF'_\varepsilon(s) \leq 2F_\varepsilon(s) + 1 \quad \forall s \in \mathbb{R}, \tag{2.3.31}$$

and

$$sF'_\varepsilon(s) \geq \phi_\varepsilon(s)\ F'_\varepsilon(s) \geq s - 1 \quad \forall s \in \mathbb{R}. \tag{2.3.32}$$

From Taylor's theorem for any $F \in C^2(\mathbb{R})$ we have

$$(s-r)F'(s) = F(s) - F(r) + \frac{(s-r)^2}{2}F''(\xi), \quad \text{for some } \xi \text{ between } s \text{ and } r. \tag{2.3.33}$$

We now introduce for $\varepsilon \in (0, e^{-1})$ the corresponding regularized version of the problem (P_M):

$(\mathrm{P}_{M,\varepsilon})$ For fixed $M \geq e$ Find $\{u_{i,\varepsilon}(x,t)\}_{i=1}^m \in \mathbb{R} \times ... \times \mathbb{R}$ such that

$$\partial_t u_{i,\varepsilon} - \nabla \cdot [D_i \nabla u_{i,\varepsilon} + \phi_\varepsilon(u_{i,\varepsilon}) \sum_{j=1}^m \nabla u_{j,\varepsilon}] = g_{i,\varepsilon}(\mathbf{u}_\varepsilon), \qquad \text{in} \quad Q_T, \quad i = 1, ..., m, \tag{2.3.34}$$

$$[D_i \nabla u_{i,\varepsilon} + \phi_\varepsilon(u_{i,\varepsilon}) \sum_{j=1}^m \nabla u_{j,\varepsilon}] \cdot \nu = 0, \qquad \text{on} \quad S_T, \tag{2.3.35}$$

$$u_{i,\varepsilon}(x,0) = u_i^0, \qquad \forall x \in \Omega, \tag{2.3.36}$$

where

$$g_{i,\varepsilon}(\mathbf{u}_\varepsilon) = \gamma_i u_{i,\varepsilon} - \phi_\varepsilon(u_{i,\varepsilon}) \sum_{j=1}^m \phi_\varepsilon(u_{j,\varepsilon}), \qquad i = 1, ..., m. \tag{2.3.37}$$

In the next lemma we prove an entropy inequality for the system (2.3.34)-(2.3.37) which is very important in the numerical analysis that follows.

Lemma 2.3.1 Let $\{u_i^0(x)\}_{i=1}^m$ be non-negative bounded functions. There exists a positive $C(u_1^0, ..., u_m^0, M, \gamma_1, ..., \gamma_m)$ independent of ε such that any solution of $(\mathrm{P}_{M,\varepsilon})$ satisfies

$$\sup_{0 \leq t \leq T} \int_\Omega \sum_{i=1}^m F_\varepsilon(u_{i,\varepsilon}) dx + \int_0^t (\sum_{i=1}^m \frac{D_i}{M} \|\nabla u_{i,\varepsilon}\|_0^2 + \| \sum_{i=1}^m \nabla u_{i,\epsilon}\|_0^2) dt \leq C. \tag{2.3.38}$$

In addition,

$$\sup_{0 \leq t \leq T} \int_\Omega \sum_{i=1}^m |[u_{i,\varepsilon}]_-|^2 dx \leq C\varepsilon. \tag{2.3.39}$$

Proof: Testing (2.3.34) with $F'_\varepsilon(u_{i,\varepsilon}), i = 1, ..., m$ and summing the resulting equations yields, after using (2.3.28) and the boundary conditions (2.3.35) that

$$\frac{d}{dt} \int_\Omega \sum_{i=1}^m F_\varepsilon(u_{i,\varepsilon}) dx + \int_\Omega \sum_{i=1}^m \frac{D_i}{\phi_\varepsilon(u_{i,\varepsilon})} |\nabla u_{i,\varepsilon}|^2 dx + \| \sum_{i=1}^m \nabla u_{i,\epsilon}\|_0^2$$
$$\leq \int_\Omega \sum_{i=1}^m g_{i,M,\varepsilon}(\mathbf{u}_\varepsilon) \; F'_\varepsilon(u_{i,\varepsilon}) dx, \tag{2.3.40}$$

where we have noticed that

$$\phi_\varepsilon(u_{i,\varepsilon}) \nabla [F'_\varepsilon(u_{i,\varepsilon})] = \nabla u_{i,\varepsilon}. \tag{2.3.41}$$

We now obtain from (2.3.28), (2.3.31), (2.3.32), (2.1.12), Young's inequality and (2.3.29) that for i=1,...,m

$$
\begin{aligned}
g_{i,M,\varepsilon}(\mathbf{u}_\varepsilon)\ F'_\varepsilon(u_{i,\varepsilon}) &= [\gamma_i u_{i,\varepsilon} - \ \phi_\varepsilon(u_{i,\varepsilon})\sum_{j=1}^m \phi_\varepsilon(u_{j,\varepsilon})]F'_\varepsilon(u_{i,\varepsilon}) \\
&= \gamma_i u_{i,\varepsilon} F'_\varepsilon(u_{i,\varepsilon}) - \ \phi_\varepsilon(u_{i,\varepsilon})F'_\varepsilon(u_{i,\varepsilon})\sum_{j=1}^m \phi_\varepsilon(u_{j,\varepsilon}) \\
&\leq \gamma_i(2F_\varepsilon(u_{i,\varepsilon})+1) + \ (1-u_{i,\varepsilon})\sum_{j=1}^m \phi_\varepsilon(u_{j,\varepsilon}) \\
&\leq \gamma_i(2F_\varepsilon(u_{i,\varepsilon})+1) + \ (1-[u_{i,\varepsilon}]_-)\sum_{j=1}^m \phi_\varepsilon(u_{j,\varepsilon}) \\
&= \gamma_i(2F_\varepsilon(u_{i,\varepsilon})+1) + \sum_{j=1}^m \phi_\varepsilon(u_{j,\varepsilon}) - [u_{i,\varepsilon}]_- \sum_{j=1}^m \phi_\varepsilon(u_{j,\varepsilon}) \\
&\leq 2\gamma_i F_\varepsilon(u_{i,\varepsilon}) + \frac{m}{2\varepsilon}[u_{i,\varepsilon}]_-^2 + \frac{\varepsilon}{2}\sum_{j=1}^m \phi_\varepsilon^2(u_{j,\varepsilon}) + C(M,\gamma_i,m) \\
&\leq (2\gamma_i + m)F_\varepsilon(u_{i,\varepsilon}) + C(M,\gamma_i,m).
\end{aligned} \tag{2.3.42}
$$

Combining (2.3.40) and (2.3.42) and noting (2.3.28), leads to

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega \sum_{i=1}^m F_\varepsilon(u_{i,\varepsilon})dx + \sum_{i=1}^m \frac{D_i}{M}\|\nabla u_{i,\varepsilon}\|_0^2 + \|\sum_{i=1}^m \nabla u_{i,\epsilon}\|_0^2 \\
&\qquad \leq C(M,\gamma_i)\Big(1 + \int_\Omega \sum_{i=1}^m F_\varepsilon(u_{i,\varepsilon})dx\Big).
\end{aligned} \tag{2.3.43}
$$

Hence, on noting the assumptions on the initial conditions (2.3.36) and the assumption on u_i^0, the the desired result follows from (2.3.43) after a simple application of the Grönwall lemma (A.1.3).

Then, from (2.3.38) we have

$$
\sup_{0\leq t\leq T}\int_\Omega \sum_{i=1}^m F_\varepsilon(u_{i,\varepsilon})dx \leq C.
$$

Finally, it follows from (2.3.29), that

$$
\sup_{0\leq t\leq T}\int_\Omega \sum_{i=1}^m [u_{i,\varepsilon}]_-^2 dx \leq C\varepsilon.
$$

□

The regularized entropy inequality (2.3.38) and the estimate (2.3.39) can be used to pass to the limit $\varepsilon \longrightarrow 0$ in $(\mathrm{P}_{M,\varepsilon})$ in order to obtain existence of a non-negative solution to problem (P_M). In the following section we formulate and analyse a fully discrete finite element approximation of the regularized system (2.3.34)-(2.3.37).

2.4 A fully discrete finite element approximation

In this section we formulate a fully discrete approximation to the solution of the continuous problem $(\mathrm{P}_{M,\varepsilon})$ where we discretise in the spatial variable using a finite element method.

In Section 2.4.1 we briefly cover the assumptions and results needed for the subsequent analysis and present a fully-discrete in time, finite element approximation. We also define some necessary operators and mention briefly their associated properties. In addition, we recall definitions of different types of partitioning in space. We state the required assumptions on the partitioning of Ω and $(0, T)$. We also define the s-tandard piecewise linear finite element space and discuss some associated results. In Section 2.4.2 we formulate a practical fully discrete finite element approximation of the system $(\mathrm{P}_{M,\varepsilon})$ and prove some technical lemmata. Then, in Subsection 2.4.3, we prove existence of the finite element approximations under appropriate assumptions on the discretization parameters.

2.4.1 Notation and associated results

Let $\Omega \in \mathbb{R}^d, d = 1, 2, 3,$ be a convex polygonal domain in $d = 2$ and a convex polyhedral domain in $d = 3$. Let $\mathcal{T}^h$ be a quasi-uniform partitioning of Ω, into disjoint open simplices τ with $h_\tau := \mathrm{diam}\tau$ and $h := \max h_\tau$ so that $\bar{\Omega} = \cup_{\tau \in \mathcal{T}^h} \bar{\tau}$. The parameter h indicates the maximal diameter of the simplices of the partitioning. We recall that a partitioning $\mathcal{T}^h$ is said to be "quasi-uniform" if there exists a positive constant β such that

$$\frac{\varrho_\tau}{h_\tau} \geq \beta, \qquad \forall \tau \in \mathcal{T}^h,$$

where ϱ_τ denotes the diameter of the sphere inscribed in τ. For instance, in the case $d = 2$, the quasi-uniform condition means that the angles of the triangles $\tau \in \mathcal{T}^h$

are not allowed to be arbitrarily small; see Johnson [65] page 85. Additionally, we assume $\mathcal{T}^h$ is weakly acute.

We also recall that a partitioning $\mathcal{T}^h$ is said to be "acute" for $d = 2$ if all the angles of the triangles are less than or equal to $\pi/2$, and for $d = 3$ if the angles made by any two faces of the same tetrahedron are less than or equal to $\pi/2$. Another type of partitioning is the "right-angled" that is, in the case $d = 2$, if all triangles are right-angled; and in the case $d = 3$, if all tetrahedra have a vertex at which all the edges meet at right angles. From the definitions, we note that the right-angled partitioning is acute.

In the work that follows we consider a finite element approximation of $(\mathrm{P}_{M,\varepsilon})$ under the following assumptions on the spacial and temporal meshes:
(A) Let $\Omega \in \mathbb{R}^d, d = 1, 2, 3$, be a polygonal domain in $d = 2$ and a polyhedral domain in $d = 3$. Let $\mathcal{T}^h$ be a quasi-uniform and right-angled partitioning of Ω into disjoint open simplices $\{\tau\}$ with $h_\tau := \mathrm{diam}\tau$ and $h := \max_{\tau\in\mathcal{T}^h} h_\tau$, so that $\overline{\Omega} = \bigcup_{\tau\in\mathcal{T}^h} \overline{\tau}$. Let $0 = t_0 < t_1 < ... < t_{N-1} < t_N = T$ be a partitioning of $(0, T)$ into time steps $\Delta t_n = t_n - t_{n-1}$, $n = 1, ..., N$ with $\Delta t = \max_{n=1,...,N} \Delta t_n$. Let $S^h \subset H^1(\Omega)$, we define the standard finite element space consisting of the continuous piecewise linear functions

$$S^h := \{\chi \in C(\overline{\Omega}) : \chi|_\tau \text{ is linear } \forall\tau \in \mathcal{T}^h\}.$$

Let $\{\varphi_j\}_{j=0}^J$ be the standard basis functions for S^h, satisfying $\varphi_j(p_i) = \delta_{ij}$ for $i, j = 0, ..., J$ where $\mathcal{N}^h := \{p_j\}_{j=0}^J$ the set of nodes of the partitioning $\mathcal{T}^h$. We also introduce

$$S^h_{\geq 0} := \{\chi \in S^h : \chi(p_j) \geq 0, j = 0, ..., J\}$$
$$\subset H^1_{\geq 0} := \{\eta \in H^1(\Omega) : \eta \geq 0 \ a.e. \in \Omega\}.$$

Let $\pi^h : C(\overline{\Omega}) \to S^h$ be the Lagrange interpolation operator (alternatively, piecewise linear interpolant) such that

$$\pi^h\eta(p_j) := \eta(p_j), \quad \text{for } j = 0, ..., J.$$

In addition, we define a discrete L^2 inner (semi-inner) product on $S^h(C(\overline{\Omega}))$ as

$$(u,v)^h := \int_\Omega \pi^h(u(x)v(x))dx = \sum_{j=0}^{J} \widehat{M}_{jj}\ u(p_j)\, v(p_j), \tag{2.4.44}$$

where $\widehat{M}_{jj} = (\varphi_j, \varphi_j)^h = (1, \varphi_j) > 0$. On noting (2.4.44) it is easy to verify that

$$(\eta_1, \eta_2)^h = (\pi^h \eta_1, \eta_2)^h = (\pi^h \eta_1, \pi^h \eta_2)^h \quad \forall \eta_1, \eta_2 \in C(\overline{\Omega}). \tag{2.4.45}$$

Below we mention some well-known results concerning the finite element space S^h. The induced discrete semi-norm on $C(\overline{\Omega})$, and norm on S^h, is $|\cdot|_h := [(\cdot,\cdot)^h]^{1/2}$. It is well-known that $|\cdot|_h$ is equivalent to the norm $\|\cdot\|_0 := [(\cdot,\cdot)]^{1/2}$ (e.g. Raviart [82]) via,

$$\|\chi\|_0^2 \le |\chi|_h^2 \le (d+2)\|\chi\|_0^2 \quad \forall \chi \in S^h. \tag{2.4.46}$$

The discrete inner product (2.4.44) approximating the continuous L^2 inner product is exact for all piecewise polynomials $u\,v$ of degree less than or equal to one. For future reference we also define

$$M_{ij} = (\varphi_i, \varphi_j), \quad K_{ij} = (\nabla\varphi_i, \nabla\varphi_j), \quad \widehat{M}_{ij} = (\varphi_i, \varphi_j)^h,$$

corresponding to the mass matrix M, stiffness matrix K and lumped mass matrix $\widehat{M}$ respectively. Note that $\widehat{M}$ is a diagonal matrix. Notice that

$$\widehat{M}_{ii} = \sum_{j=0}^{J} M_{ij}, \quad i = 0, ..., J,$$

i.e., the elements of the lumped mass matrix $\widehat{M}$ are obtained by adding the off diagonal elements of M in any row to the diagonal element of that row. This is easily proved via

$$\sum_{j=0}^{J} M_{ij} = \sum_{j=0}^{J} \int_\Omega \varphi_i \varphi_j dx = \int_\Omega \varphi_i \sum_{j=0}^{J} \varphi_j dx = (\varphi_i, 1) = \widehat{M}_{ii},$$

using that $\sum_{j=0}^{J} \varphi_j = 1$. The use of the discrete inner product to approximate the mass matrix is often called "lumped mass integration" (e.g., Strang and Fix [90] , page 118). One advantage of mass lumping is that the (diagonal) mass matrix is trivially inverted.

As the partitioning $\mathcal{T}^h$ is acute, we have that (see [77] page 49)

$$K_{jj} > 0, \forall j \quad \text{and} \quad K_{ij} \leq 0, \forall i \neq j. \tag{2.4.47}$$

Using the fact $\sum_{j=0}^{J} \varphi_j = 1$, we also have

$$\sum_{j=0}^{J} K_{ij} = (\nabla \varphi_i, \nabla \sum_{j=0}^{J} \varphi_j) = 0. \tag{2.4.48}$$

Providing that the partitioning $\mathcal{T}^h$ is acute, we state the following lemma about the regularized functions $\phi_\varepsilon(s)$ which will be important in deriving later stability estimates and is a consequence of the weak acuteness property.

Lemma 2.4.1 Assume the partitioning $\mathcal{T}^h$ is weakly acute and $U(\chi) \in S^h$ is a monotonic function for all $\chi \in S^h$. Then

$$(\nabla \chi, \nabla \pi^h[U(\chi)]) = \frac{1}{2} \sum_{i=0}^{J} \sum_{j=0 j\neq i}^{J} (-K_{ij})(\chi_i - \chi_j)(U(\chi_i) - U(\chi_j)) \geq 0. \tag{2.4.49}$$

Proof: Recall the weak acuteness properties (2.4.47) and the fact that $K_{ii} > 0$. Set $\chi = \sum_{j=0}^{J} \chi_j \varphi_j$ where $\chi_j = \chi(x_j)$ and note that $\pi^h U(\chi) = \sum_{j=0}^{J} U(\chi_j)\varphi_j$, thus

$$\begin{aligned}
(\nabla \chi, \nabla \pi^h[U(\chi)]) &= \sum_{i=0}^{J} \sum_{j=0}^{J} K_{ij}\, \chi_j\, U(\chi_i) \\
&= \sum_{i=0}^{J} \Big(\sum_{j=0 j\neq i}^{J} K_{ij}\, \chi_j\, U(\chi_i) + K_{ii}\, \chi_i\, U(\chi_i) \Big) \\
&= \sum_{i=0}^{J} \Big(\sum_{j=0 j\neq i}^{J} K_{ij}\, \chi_j\, U(\chi_i) - \sum_{j=0 j\neq i}^{J} K_{ij}\, \chi_i\, U(\chi_i) \Big) \\
&= \sum_{i=0}^{J} \sum_{j=0 j\neq i}^{J} K_{ij}\, (\chi_j - \chi_i)\, U(\chi_i).
\end{aligned} \tag{2.4.50}$$

Additionally,

$$\begin{aligned}
\sum_{i=0}^{J} \sum_{j=0 j\neq i}^{J} K_{ij}(\chi_j - \chi_i)U(\chi_i) &= \sum_{j=0}^{J} \sum_{i=0 i\neq j}^{J} K_{ij}(\chi_j - \chi_i)U(\chi_i) \\
&= \sum_{i=0}^{J} \sum_{j=0 j\neq i}^{J} K_{ij}(\chi_i - \chi_j)U(\chi_j),
\end{aligned} \tag{2.4.51}$$

as $\sum_{i=0}^{J} \sum_{j=0 j\neq i}^{J}(\cdot) = \sum_{j=0}^{J} \sum_{i=0 i\neq j}^{J}(\cdot)$, $K_{ij} = K_{ji}$ and swapping the indices i and j. Thus from (2.4.50) and (2.4.51) yields the desired result (2.4.49). □

Lemma 2.4.2 Let the assumptions (A) hold. Then for all $\chi \in S^h$

$$\|\nabla \pi^h[\phi_\varepsilon(\chi)]\|_0^2 \leq (\nabla \chi, \nabla \pi^h[\phi_\varepsilon(\chi)]). \tag{2.4.52}$$

Proof: The proof of this Lemma follows from (2.4.49) on noting that the functions ϕ_ε are Lipschitz continuous and non-decreasing functions. □

We now recall some well-known results about the space S^h under our assumption that $\mathcal{T}^h$ is a quasi-uniform partitioning:
For any $\tau \in \mathcal{T}^h, \chi \in S^h, 1 \leq p, q \leq \infty$ and $m, l \in \{0, 1\}$ with $l \leq m$, we have

$$\|\chi\|_{m,p,"\tau"} \leq C h_{"\tau"}^{l-m+d\min(0,\frac{1}{p}-\frac{1}{q})} \|\chi\|_{l,q,"\tau"}, \tag{2.4.53}$$

where the abbreviation "τ" means "with" or "without" τ. The above inequality is known as "the inverse inequality", see [49] page 75-77, and it also holds with $\|\cdot\|$ replaced by $|\cdot|$, see [39] page 140-142.
For later purpose we introduce the following inverse inequalities which follow from the quasi-uniform condition (see Theorem 3.2.6, in Ciarlet [39])

$$|\chi|_{1,p,"\tau"} \leq C h_{"\tau"}^{-1} |\chi|_{0,p,"\tau"}, \quad 1 \leq p \leq \infty, \tag{2.4.54}$$

$$|\chi|_{m,p,"\tau"} \leq C h_{"\tau"}^{-d(\frac{1}{q}-\frac{1}{p})} |\chi|_{m,q,"\tau"}, \quad 1 \leq q \leq p \leq \infty, m \in \{0, 1\}. \tag{2.4.55}$$

We also require the following interpolation results for all $\eta \in W^{1,s}(\Omega), s \in [2, \infty]$ if $d = 1$ and $s \in (d, \infty]$ if $d = 2$ or 3:

$$|(I - \pi^h)\eta|_{m,s} \leq C h^{1-m} |\eta|_{1,s}, \quad m \in \{0, 1\}, \tag{2.4.56}$$

$$\lim_{h \to 0} |(I - \pi^h)\eta|_{1,s} = 0, \tag{2.4.57}$$

(see Theorem 1.103 and Corollary 1.110 in [49] respectively). In addition, the following interpolation error estimates (Theorem 5, in Ciarlet and Raviart [40]) holds

$$\|(I - \pi^h)\eta\|_{0,1} \leq C h^2 |\eta|_{2,1}, \quad \forall\, \eta \in W^{2,1}(\Omega). \tag{2.4.58}$$

We also recall the following useful result (e.g. Ciavaldini [41]), for all $\chi_1, \chi_2 \in S^h$, that

$$|(\chi_1, \chi_2) - (\chi_1, \chi_2)^h| \leq C h^{1+m} |\chi_1|_{m,n_1} |\chi_2|_{1,n_2}, \tag{2.4.59}$$

for $m \in \{0,1\}$ and $1 \le n_1, n_2 \le \infty$ with $\frac{1}{n_1} + \frac{1}{n_2} = 1$.

For later purposes, we introduce the following generalized version of the estimate (2.4.59). For all $\chi_1, \chi_2, \chi_3 \in S^h$

$$|(\chi_1, \chi_2, \chi_3) - (\chi_1, \chi_2, \chi_3)^h| \le Ch^2 |\chi_1|_{1,n_1} |\chi_2|_{1,n_2} |\chi_3|_{1,n_3}, \tag{2.4.60}$$

where $1 \le n_1, n_2, n_3 \le \infty$ with $\frac{1}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} = 1$.

We are now in a position to formulate a practical fully discrete finite element approximation of the system ($\mathrm{P}_{M,\varepsilon}$).

2.4.2 A practical fully discrete approximation

Similarly to the approach in [98] and [54], we introduce, for any $\varepsilon \in (0, e^{-1})$, $\Lambda_\varepsilon : S^h \to [L^\infty(\Omega)]^{d\times d}$ such that for all $\chi \in S^h$ and $a.e.$ in Ω

$$\Lambda_\varepsilon \text{ is symmetric and positive definite} \tag{2.4.61}$$

$$\Lambda_\varepsilon(\chi)\, \nabla \pi^h [F'_\varepsilon(\chi)] = \nabla \chi, \tag{2.4.62}$$

that is, the discrete analogue to (2.3.41). Firstly, we give the construction of Λ_ε in the simple case when $d = 1$. Given $\chi \in S^h$ and $\tau \in \mathcal{T}^h$ having vertices p_j and p_k, we set

$$\Lambda_\varepsilon(\chi)|_\tau := \begin{cases} \frac{\chi(p_k)-\chi(p_j)}{F'_\varepsilon(\chi(p_k))-F'_\varepsilon(\chi(p_j))} = \frac{1}{F''_\varepsilon(\chi(\zeta))} \quad \text{for some } \zeta \in \tau & \text{if} \quad \chi(p_k) \ne \chi(p_j), \\ \frac{1}{F''_\varepsilon(\chi(p_k))} & \text{if} \quad \chi(p_k) = \chi(p_j). \end{cases} \tag{2.4.63}$$

Since $F''_\varepsilon(s) > 0$ and $\sum_{j=0}^{J} \nabla \varphi_j = 0$, it can be easily seen that the piecewise constant function Λ_ε satisfies the conditions (2.4.61) and (2.4.62). Following [54] we extend the above construction to $d = 2$ or 3. Let $\{e_i\}_{i=1}^d$ be the orthonormal vectors in $\mathbb{R}^d$, such that the j-th component of e_i is $\delta_{ij}, i, j = 1 \to d$. Given non-zero constants $\alpha_i, i = 1 \to d$, let $\widehat{\tau}(\{\alpha_i\}_{i=1}^d)$ be a reference simplex in $\mathbb{R}^d$ with vertices $\{\widehat{p}_i\}_{i=1}^d$, where $\widehat{p}_0$ is the origin and $\widehat{p}_i = \widehat{p}_{i-1} + \alpha_i e_i, i = 1 \to d$. Given a $\tau \in \mathcal{T}^h$ with vertices $\{p_{j_i}\}_{i=0}^d$, such that p_{j_0} is not a right-angled vertex, then there exists a rotation/reflection matrix R_τ and non-zero constants $\{\widehat{p}_i\}_{i=1}^d$ such that the mapping

$R_\tau : \widehat{x} \in \mathbb{R}^d \to p_{j_0} + R_\tau \widehat{x} \in \mathbb{R}^d$ maps the vertex $\widehat{p}_i$ to p_{j_i}, and hence $\widehat{\tau} \equiv \widehat{\tau}(\{\alpha_i\}_{i=1}^d)$ to τ. For all $\tau \in \mathcal{T}^h$ and $\chi \in S^h$, we set

$$\Lambda_\varepsilon(\chi)|_\tau := R_\tau \widehat{\Lambda}_\varepsilon(\widehat{\chi})|_{\widehat{\tau}} R_\tau^T, \tag{2.4.64}$$

where $\widehat{\chi}(\widehat{x}) \equiv \chi(R_\tau \widehat{x})$ for all $\widehat{x} \in \widehat{\tau}$ and $\widehat{\Lambda}_\varepsilon(\widehat{\chi})|_{\widehat{\tau}}$ is the $d \times d$ diagonal matrix with diagonal entries, $k = 1, ..., d$,

$$[\widehat{\Lambda}_\varepsilon(\widehat{\chi})|_{\widehat{\tau}}]_{kk} := \begin{cases} \frac{\widehat{\chi}(\widehat{p}_k) - \widehat{\chi}(\widehat{p}_j)}{F'_\varepsilon(\widehat{\chi}(\widehat{p}_k)) - F'_\varepsilon(\widehat{\chi}(\widehat{p}_j))} = \frac{\chi(p_{j_k}) - \chi(p_{j_0})}{F'_\varepsilon(\chi(p_{j_k})) - F'_\varepsilon(\chi(p_{j_0}))} & \\ \qquad = \frac{1}{F''_\varepsilon(\chi(\zeta))} & \text{for some } \zeta \text{ between } p_{j_k} \text{ and } p_{j_0} \\ & \text{if } \chi(p_{j_k}) \neq \chi(p_{j_0}), \\ \frac{1}{F''_\varepsilon(\widehat{\chi}(\widehat{p}_0))} = \frac{1}{F''_\varepsilon(\chi(p_{j_0}))} & \text{if} \chi(p_{j_k}) = \chi(p_{j_0}). \end{cases} \tag{2.4.65}$$

As $R_\tau^T = R_\tau^{-1}$, we have that

$$\nabla \chi|_\tau := R_\tau \widehat{\nabla} \widehat{\chi}|_{\widehat{\tau}}, \tag{2.4.66}$$

where $\widehat{\nabla}$ is the gradient on $\widehat{\tau}$. On noting (2.4.64), (2.4.65), (2.4.66), the positivity of $F''_\varepsilon(s)$ and the fact $\sum_{j=0}^J \nabla \varphi_j = 0$, one can easily show that Λ_ε satisfies the conditions (2.4.61) and (2.4.62).

Under the assumptions (A), for any given $\varepsilon \in (0, e^{-1})$ we consider the following fully discrete finite element approximation of $(\mathrm{P}_{M,\varepsilon})$:

$(\mathrm{P}_{M,\varepsilon}^{h,\Delta t})$ For $n \geq 1$ find $\{U_{1,\varepsilon}^n, ..., U_{m,\varepsilon}^n\} \in [S^h]^m$ such that for all $\chi \in S^h$

$$\left(\frac{U_{i,\varepsilon}^n - U_{i,\varepsilon}^{n-1}}{\Delta t_n}, \chi \right)^h + \left(D_i \nabla U_{i,\varepsilon}^n + \Lambda_\varepsilon(U_{i,\varepsilon}^n) \sum_{j=1}^m \nabla U_{j,\varepsilon}^n, \nabla \chi \right)$$

$$= \left(\gamma_i U_{i,\varepsilon}^n - \phi_\varepsilon(U_{i,\varepsilon}^n) \sum_{j=1}^m \phi_\varepsilon(U_{j,\varepsilon}^{n-1}), \chi \right)^h, \qquad i = 1, ..., m, \tag{2.4.67}$$

where $\{U_{i,\varepsilon}^0\}_{i=1}^m \in S^h$ are given approximations of $\{u_i^0\}_{i=1}^m$ respectively.

Lemma 2.4.3 Let the assumptions (A) hold. Then for any given $\varepsilon \in (0, e^{-1})$ the function $\Lambda_\varepsilon : S^h \to [L^\infty(\Omega)]^{d \times d}$ satisfies, for *a.e.* in Ω

$$\varepsilon \xi^T \xi \leq \xi^T \Lambda_\varepsilon(\chi) \xi \leq M \xi^T \xi \quad \forall \xi \in \mathbb{R}^d, \forall \chi \in S^h. \tag{2.4.68}$$

Lemma 2.4.4 Let the assumptions (A) hold. Then for any given $\varepsilon \in (0, e^{-1})$ the function $\Lambda_\varepsilon : S^h \to [L^\infty(\Omega)]^{d\times d}$ is continuous in the following sense. For all $\chi_1, \chi_2 \in S^h$ and $\tau \in \mathcal{T}^h$

$$\|(\Lambda_\varepsilon(\chi_1) - \Lambda_\varepsilon(\chi_2))|_\tau\| \leq \frac{2M}{\varepsilon}\|\chi_1 - \chi_2\|_{0,\infty}, \tag{2.4.69}$$

Lemma 2.4.5 Let the assumptions (A) hold. Then for any given $\varepsilon \in (0, e^{-1})$ the function $\Lambda_\varepsilon : S^h \to [L^\infty(\Omega)]^{d\times d}$ satisfies

$$\max_{x\in\tau} \|(\Lambda_\varepsilon(\chi(x)) - \phi_\varepsilon(\chi(x))\mathcal{I}\| \leq h_\tau |\nabla\chi|_\tau, \tag{2.4.70}$$

where $\mathcal{I}$ is the $d \times d$ identity matrix.

2.4.3 Existence of the approximations

In order to prove the existence of a solution $\{U^n_{i,\varepsilon}\}^m_{i=1}, n \geq 1$, of the system (2.4.67) for given $\{U^{n-1}_{i,\varepsilon}\}^m_{i=1}$, it is convenient to define the functions $A_i : [S^h]^m \to S^h, i = 1, ..., m$ such that for all $\chi \in S^h$

$$(A_i(\mathbf{U}), \chi)^h = (U_i - U^{n-1}_{i,\varepsilon}, \chi)^h + \Delta t_n(D_i\nabla U_i + \Lambda_\varepsilon(U_i)\sum_{j=1}^m \nabla U_j, \nabla\chi)$$

$$-\Delta t_n(\gamma_i U_i - \phi_\varepsilon(U_i)\sum_{j=1}^m \phi_\varepsilon(U^{n-1}_{j,\varepsilon}), \chi)^h, \qquad i = 1, ..., m. \tag{2.4.71}$$

We first note that the continuous piecewise linear functions $A_i(\mathbf{U})$ can be defined uniquely in terms of their values at the nodal points $\mathcal{N}^h$. This can be seen by setting $\chi \equiv \varphi_j$, for $j = 0, ..., J$, in (2.4.71) and then obtaining the following solvable square matrix systems

$$\widehat{M} A_i(\mathbf{U}) = S_i, \quad i = 1, ..., m,$$

where $\widehat{M}$ is the lumped mass matrix introduced in Subsection 2.4.1 , and S_i are given vectors in terms of the nodal values of $\{U_i\}^m_{i=1}$ and $\{U^{n-1}_{i,\varepsilon}\}^m_{i=1}$. Thus, the functions A_i are well defined.

It is clear that solving the system (2.4.71) is equivalent to finding $\{U^n_{i,\varepsilon}\}^m_{i=1} \in [S^h]^m, n \geq 1$, such that

$$A_i(\mathbf{U}^n_\varepsilon) = 0, \quad i = 1, ..., m, \tag{2.4.72}$$

for given $\{U^0_{i,\varepsilon}\}^m_{i=1} \in [S^h]^m$.

Lemma 2.4.6 For any given $R > 0$, the functions $A_i : [S^h]^m_R \to S^h$ are continuous, where

$$[S^h]^m_R = \{\{\chi_1, ..., \chi_m\} \in [S^h]^m : \sum_{i=1}^{m} |\chi_i|_h^2 \le R^2\}.$$

Proof: Let $\{U_i^1\}_{i=1}^m, \{U_i^2\}_{i=1}^m \in [S^h]^m_R$. It follows from (2.4.71) that for all $\chi \in S^h$

$$(A_i(\mathbf{U}^1) - A_i(\mathbf{U}^2), \chi)^h = (U_i^1 - U_i^2, \chi)^h + \Delta t_n (D_i \nabla (U_i^1 - U_i^2)$$
$$+\Lambda_\varepsilon(U_i^1) \sum_{j=1}^{m} \nabla U_j^1 - \Lambda_\varepsilon(U_i^2) \sum_{j=1}^{m} \nabla U_j^2, \nabla \chi) - \Delta t_n (\gamma_i (U_i^1 - U_i^2)$$
$$- (\phi_\varepsilon(U_i^1) - \phi_\varepsilon(U_i^2)) \sum_{j=1}^{m} \phi_\varepsilon(U_{j,\varepsilon}^{n-1}), \chi)^h, \qquad i = 1, ..., m. \tag{2.4.73}$$

Choosing $\chi = A_i(\mathbf{U}^1) - A_i(\mathbf{U}^2)$ in (2.4.73) yields on noting the Cauchy- Schwarz inequality, (2.4.54), (2.4.46) and the Lipschitz continuity of ϕ_ε that

$$|A_i(\mathbf{U}^1) - A_i(\mathbf{U}^2)|_h \le C(M, h^{-1}, \Delta t_n, D_i, \gamma_i) |U_i^1 - U_i^2|_h$$
$$+C(h^{-1}, \Delta t_n) \|\Lambda_\varepsilon(U_i^1) \sum_{j=1}^{m} \nabla U_j^1 - \Lambda_\varepsilon(U_i^2) \sum_{j=1}^{m} \nabla U_j^2\|_0 \qquad i = 1, ..., m. \tag{2.4.74}$$

We also have from (2.4.54), (2.4.46), (2.4.69), (2.4.68) and (2.4.55) that

$$\|\Lambda_\varepsilon(U_i^1) \sum_{j=1}^{m} \nabla U_j^1 - \Lambda_\varepsilon(U_i^2) \sum_{j=1}^{m} \nabla U_j^2\|_0$$

$$= \|\Lambda_\varepsilon(U_i^1) \sum_{j=1}^{m} \nabla U_j^1 - \Lambda_\varepsilon(U_i^2) \sum_{j=1}^{m} \nabla U_j^1 + \Lambda_\varepsilon(U_i^2) \sum_{j=1}^{m} \nabla U_j^1 - \Lambda_\varepsilon(U_i^2) \sum_{j=1}^{m} \nabla U_j^2\|_0$$
$$\le \|(\Lambda_\varepsilon(U_i^1) - \Lambda_\varepsilon(U_i^2)) \sum_{j=1}^{m} \nabla U_j^1\|_0 + \|\Lambda_\varepsilon(U_i^2)(\sum_{j=1}^{m} \nabla U_j^1 - \sum_{j=1}^{m} \nabla U_j^2)\|_0$$
$$\le \|(\Lambda_\varepsilon(U_i^1) - \Lambda_\varepsilon(U_i^2))\|_{0,\infty} \sum_{j=1}^{m} |U_j^1|_1 + \|\Lambda_\varepsilon(U_i^2)\|_{0,\infty} \sum_{j=1}^{m} |U_j^1 - U_j^2|_1$$
$$\le Ch^{-1} \|(\Lambda_\varepsilon(U_i^1) - \Lambda_\varepsilon(U_i^2))\|_{0,\infty} \sum_{j=1}^{m} |U_j^1|_h + Ch^{-1} \|\Lambda_\varepsilon(U_i^2)\|_{0,\infty} \sum_{j=1}^{m} |U_j^1 - U_j^2|_h$$
$$\le C(h^{-1}, M, \varepsilon^{-1}) \|U_i^1 - U_i^2\|_{0,\infty} \sum_{j=1}^{m} |U_j^1|_h + C(h^{-1}, M) \sum_{j=1}^{m} |U_j^1 - U_j^2|_h$$
$$\le C(h^{-1}, M, \varepsilon^{-1}, R) \|U_i^1 - U_i^2\|_0 + C(h^{-1}, M) \sum_{j=1}^{m} |U_j^1 - U_j^2|_h$$

$$\leq C(h^{-1}, M, \varepsilon^{-1}, R) \sum_{j=1}^{m} |U_j^1 - U_j^2|_h. \tag{2.4.75}$$

Combining (2.4.74) and (2.4.75) yields that for $i = 1, ..., m$, A_i is Lipchitz continuous. $\square$

We now show the main result of this chapter where we establish the existence of a solution $\{U_{i,\varepsilon}^n\}_{i=1}^m$ to $(\mathrm{P}_{M,\varepsilon}^{h,\Delta t})$.

Theorem 2.4.7 Let the assumptions (A) hold and let $\{U_{i,\varepsilon}^{n-1}\}_{i=1}^m \in [S^h]^m$ be a given solution to the $(n-1)$-th step of $(\mathrm{P}_{M,\varepsilon}^{h,\Delta t})$ for some $n = 1, ..., N$. Then for all $\varepsilon \in (0, e^{-1})$, for all $h > 0$ and for all Δt_n such that $\Delta t_n \leq \frac{1}{2\gamma_i + m}, \forall i = 1, ..., m$, there exists a solution $\{U_{i,\varepsilon}^n\}_{i=1}^m \in [S^h]^m$ to the n-th step of $(P_{M,\varepsilon}^{h,\Delta t})$.

Proof:

By contradiction, let $R > 0$ and assume that there does not exist $\{U_{i,\varepsilon}^n\}_{i=1}^m \in [S^h]_R^m$ with $A_i(\mathbf{U}) = 0$. Hence, on noting the continuity of the functions $A_i(\mathbf{U})$ on $[S^h]_R^m$, we define the continuous function $B : [S^h]_R^m \to [S^h]_R^m$ given by

$$B(\mathbf{U}) = (B_1(\mathbf{U}), ..., B_m(\mathbf{U})),$$

where $B_i(\mathbf{U}), i = 1, ..., m$ are given by

$$B_i(\mathbf{U}) := \frac{-R\, A_i(\mathbf{U})}{|(A_1(\mathbf{U}), ..., A_m(\mathbf{U}))|_{S^h \times ... \times S^h}}, \tag{2.4.76}$$

where $|(\cdot, ..., \cdot)|_{S^h \times ... \times S^h}$ is the standard norm on $[S^h]_R^m$ defined by

$$|(\chi_1, ..., \chi_m)|_{S^h \times ... \times S^h} = (\sum_{i=1}^{m} |\chi_i|_h^2)^{\frac{1}{2}}.$$

We note from the continuity of $\{A_i\}_{i=1}^m$, see Lemma 2.4.6, that the function B is continuous. Hence, on recalling that $[S^h]_R^m$ is a convex and compact subset of $S^h \times ... \times S^h$, it follows from Schauder's theorem (see Appendix A.1.1) that there exists $\{U_i\}_{i=1}^m \in [S^h]_R^m$ which is fixed point of B, that is

$$B(\mathbf{U}) = (B_1(\mathbf{U}), ..., B_m(\mathbf{U})) = (U_1, ..., U_m).$$

We deduce from Schauder's theorem, see Appendix A.1.1, that there exists $\{U_i\}_{i=1}^m \in [S^h]_R^m$ that is a fixed point of B such that

$$\sum_{i=1}^{m} |U_i|_h^2 = \sum_{i=1}^{m} |B_i(\mathbf{U})|_h^2 = R^2. \tag{2.4.77}$$

To prove a contradiction for R sufficiently large, we choose $\chi \equiv \pi^h[F'_\varepsilon(U_i)], i = 1, ..., m$, in (2.4.71) yielding on noting (2.4.45), (2.4.62) and (2.4.68) that

$$(A_i(\mathbf{U}), F'_\varepsilon(U_i))^h = (U_i - U_{i,\varepsilon}^{n-1}, F'_\varepsilon(U_i))^h + \Delta t_n (D_i [\Lambda_\varepsilon(U_i)]^{-1} \nabla U_i + \sum_{j=1}^m \nabla U_j, \nabla U_i)$$

$$-\Delta t_n (\gamma_i U_i - \phi_\varepsilon(U_i) (\sum_{j=1}^m \phi_\varepsilon(U_{j,\varepsilon}^{n-1}), F'_\varepsilon(U_i))^h$$

$$\geq (U_i - U_{i,\varepsilon}^{n-1}, F'_\varepsilon(U_i))^h + \Delta t_n \frac{D_i}{M} |U_i|_1^2 + \Delta t_n \sum_{j=1}^m (\nabla U_j, \nabla U_i)$$

$$-\Delta t_n (\gamma_i U_i - \phi_\varepsilon(U_i) \sum_{j=1}^m \phi_\varepsilon(U_{j,\varepsilon}^{n-1}), F'_\varepsilon(U_i))^h, \qquad i = 1, ..., m. \tag{2.4.78}$$

We obtain from (2.3.28), (2.3.33) and (2.1.10) that

$$(U_i - U_{i,\varepsilon}^{n-1}, F'_\varepsilon(U_i))^h \geq (F_\varepsilon(U_i) - F_\varepsilon(U_{i,\varepsilon}^{n-1}), 1)^h + \frac{1}{2} ((U_i - U_{i,\varepsilon}^{n-1})^2, F''_\varepsilon(\xi))^h$$

$$\geq (F_\varepsilon(U_i) - F_\varepsilon(U_{i,\varepsilon}^{n-1}), 1)^h + \frac{1}{2M} |U_i - U_{i,\varepsilon}^{n-1}|_h^2$$

$$\geq (F_\varepsilon(U_i) - F_\varepsilon(U_{i,\varepsilon}^{n-1}), 1)^h + \frac{1}{4M} |U_i|_h^2 - \frac{1}{2M} |U_{i,\varepsilon}^{n-1}|_h^2. \tag{2.4.79}$$

It follows from (2.3.31), (2.3.32), (2.1.13), (2.1.11) and (2.3.29) that

$$-\Delta t_n (\gamma_i U_i - \phi_\varepsilon(U_i) \sum_{j=1}^m \phi_\varepsilon(U_{j,\varepsilon}^{n-1}), F'_\varepsilon(U_i))^h$$

$$\geq -\gamma_i \Delta t_n (2 F_\varepsilon(U_i) + 1, 1)^h + \Delta t_n \left(U_i - 1, \sum_{j=1}^m \phi_\varepsilon(U_{j,\varepsilon}^{n-1})\right)^h$$

$$\geq -2 \Delta t_n \gamma_i (F_\varepsilon(U_i), 1)^h + \Delta t_n \left([U_i]_-, \sum_{j=1}^m \phi_\varepsilon(U_{j,\varepsilon}^{n-1})\right)^h - C(\mathbf{U}_\varepsilon^{n-1})$$

$$\geq -2 \Delta t_n \gamma_i (F_\varepsilon(U_i), 1)^h - \frac{m \Delta t_n}{2\varepsilon} [U_i]_-^2 - \frac{\varepsilon \Delta t_n}{2} \sum_{j=1}^m |\phi_\varepsilon(U_{j,\varepsilon}^{n-1})|_h^2 - C(\mathbf{U}_\varepsilon^{n-1})$$

$$\geq -\Delta t_n (2\gamma_i + m)(F_\varepsilon(U_i), 1)^h - C(\mathbf{U}_\varepsilon^{n-1}), \qquad i = 1, ..., m. \tag{2.4.80}$$

Combining (2.4.78) for $i = 1, ..., m$ and noting (2.4.79), (2.4.80), and the stated assumption on Δt_n yields for R sufficiently large that

$$\sum_{i=1}^m (A_i(\mathbf{U}), F'_\varepsilon(U_i))^h \geq \sum_{i=1}^m (F_\varepsilon(U_i), 1)^h + \frac{1}{4M} \sum_{i=1}^m |U_i|_h^2$$

$$-\sum_{i=1}^{m} \Delta t_n (2\gamma_i + m)(F_\varepsilon(U_i), 1)^h + \Delta t_n \sum_{i=1}^{m}\sum_{j=1}^{m} (\nabla U_j, \nabla U_i) - C(\mathbf{U}_\varepsilon^{n-1})$$

$$\geq \frac{1}{4M}\sum_{i=1}^{m} |U_i|_h^2 + \sum_{i=1}^{m}[1-\Delta t_n(2\gamma_i+m)](F_\varepsilon(U_i),1)^h + \Delta t_n \sum_{i=1}^{m}\sum_{j=1}^{m}(\nabla U_j, \nabla U_i) - C(\mathbf{U}_\varepsilon^{n-1})$$

$$\geq \frac{R^2}{4M} + \Delta t_n |\sum_{j=1}^{m} U_j|_1^2 - C(\mathbf{U}_\varepsilon^{n-1}) > 0. \tag{2.4.81}$$

Further, for R sufficiently large, we have from (2.4.76) and (2.4.81), since $\{U_i\}_{i=1}^m$ is fixed point of B, that

$$\sum_{i=1}^{m} (U_i, F_\varepsilon'(U_i))^h = \sum_{i=1}^{m} (B_i(\mathbf{U}), F_\varepsilon'(U_i)))^h = \frac{-R \sum_{i=1}^{m} (A_i(\mathbf{U}), F_\varepsilon'(U_i))^h}{|(A_1(\mathbf{U}), ..., A_m(\mathbf{U}))|_{S^h \times ... \times S^h}} < 0. \tag{2.4.82}$$

Once again, it follows from (2.3.33) and (2.3.28) that

$$(U_i, F_\varepsilon'(U_i))^h \geq (F_\varepsilon(U_i) - F_\varepsilon(0), 1)^h + \frac{1}{2M}|U_i|_h^2, \quad i = 1, ..., m, \tag{2.4.83}$$

and from (2.4.83) and the non-negativity of $F_\varepsilon(s)$, we have that

$$\sum_{i=1}^{m} (U_i, F_\varepsilon'(U_i))^h \geq \frac{R^2}{2M} - m(1 - \frac{\varepsilon}{2})|\Omega| > 0, \tag{2.4.84}$$

which contradicts (2.4.82). As a result, we conclude that there exists $\{U_{i,\varepsilon}^n\}_{i=1}^m \in S^h \times ... \times S^h$ that satisfies $A_i(\mathbf{U}_\varepsilon^n) = 0$. Thus, we have existence of a solution to the n-th step of $(\mathrm{P}_{M,\varepsilon}^{h,\Delta t})$. □

Chapter 3

The population model: Convergence and existence of a weak solution

In this chapter we prove the existence of a global weak solution to the system $(\mathrm{P}_M^{\Delta t})$ by analysing the convergence of the fully discrete approximate problem $(\mathrm{P}_{M,\varepsilon}^{h,\Delta t})$. In Section 3.1, additional notation to that presented in Chapter 2 previously is also included. A discrete analogue of the entropy inequality is derived and some stability bounds on the approximate solution are shown in Section 3.2. In Section 3.3, the convergence of our approximation is established and hence existence of a global weak solution to the system $(\mathrm{P}_M^{\Delta t})$ is shown. The argument in Section 3.3 will consist of three main steps. We first utilize the stability estimates derived in Section 3.2. Then we prove the existence of non-negative functions $\{U_i\}_{i=1}^m$ bounded in various time-dependent spaces using classical sequential compactness arguments (see the results collected in A.1.11 $\rightarrow$ A.1.16). Finally, we prove that the functions $\{U_i\}_{i=1}^m$ represent a global weak solution of the system $(\mathrm{P}_M^{\Delta t})$ via passage to the limit $\varepsilon, h \rightarrow 0$ of the approximate system. In Chapter 4, we will let $\Delta t \rightarrow 0$ in $(\mathrm{P}_M^{\Delta t})$.

3.1 Notation

For dealing with the initial data of the fully-discrete approximation, given η we introduce the discrete L^2-projection $P^h : L^2(\Omega) \to S^h$ defined by

$$(P^h\eta, \chi)^h = (\eta, \chi), \quad \forall \chi \in S^h. \tag{3.1.1}$$

The above projection satisfies the following important results (see, e.g., [11]):

$$\|P^h\eta\|_{0,\infty} \leq \|\eta\|_{0,\infty}, \quad \forall \eta \in L^\infty(\Omega), \tag{3.1.2}$$

$$\|(I-P^h)\eta\|_{m,s} \leq Ch^{1-m}\|\eta\|_{1,s}, \quad \forall \eta \in W^{1,s}(\Omega) \quad \text{for any } s \in [2,\infty] \text{ and } m \in \{0,1\}. \tag{3.1.3}$$

For $q \in (1,2]$, let $(W^{1,q'}(\Omega))'$ denote the dual of $W^{1,q'}(\Omega)$. It is convenient to introduce the inverse Laplacian operator $\mathcal{G}_q : (W^{1,q'}(\Omega))' \to W^{1,q}(\Omega),\ q' = \frac{q}{q-1}$ such that

$$(\nabla\mathcal{G}_q v, \nabla\eta) + (\mathcal{G}_q v, \eta) = \langle v, \eta\rangle_{q'} \quad \forall \eta \in W^{1,q'}(\Omega), \tag{3.1.4}$$

and $\langle\cdot,\cdot\rangle_{q'}$ denotes the duality pairing between $(W^{1,q'}(\Omega))'$ and $W^{1,q'}(\Omega)$ that satisfies (see Appendix A.1.17):

$$\langle v, \eta\rangle_{q'} = (v, \eta) \quad \forall v \in L^2(\Omega),\ \eta \in W^{1,q'}(\Omega). \tag{3.1.5}$$

The well-posedness of the operator $\mathcal{G}_q$ follows from the generalized Lax-Milgram theorem, see Appendix A.1.4, which additionally asserts the existence of a positive constant C such that

$$\|\mathcal{G}_q v\|_{1,q} \leq C\|v\|_{(W^{1,q'}(\Omega))'} \quad \forall v \in (W^{1,q'}(\Omega))'. \tag{3.1.6}$$

For consistency of notation, when $q = 2$ the indices q and q' will be dropped in the above operator and duality pairing, that is $\mathcal{G} : (H^1(\Omega))' \to H^1(\Omega)$ defined by

$$(\nabla\mathcal{G}v, \nabla\eta) + (\mathcal{G}v, \eta) = \langle v, \eta\rangle \quad \forall \eta \in H^1(\Omega), \tag{3.1.7}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $(H^1(\Omega))'$ and $H^1(\Omega)$ such that

$$\langle v, \eta\rangle = (v, \eta) \quad \forall v \in L^2(\Omega),\ \eta \in H^1(\Omega). \tag{3.1.8}$$

Also, it is important to introduce the norm

$$\|f\|_{-1} := |\mathcal{G}f|_1 \equiv \langle f, \mathcal{G}f\rangle^{\frac{1}{2}}, \forall f \in (H^1(\Omega))'. \tag{3.1.9}$$

Lemma 3.1.1 For given f, the norms $\|f\|_{(H^1(\Omega))'}$ and $\|f\|_{-1}$ are equivalent on $(H^1(\Omega))'$

$$\|f\|_{(H^1(\Omega))'} \leq \|f\|_{-1} \leq C\|f\|_{(H^1(\Omega))'}. \tag{3.1.10}$$

Proof: Let $0 \neq f \in (H^1(\Omega))'$. From (2.1.1) and (3.1.7) we have that

$$\|f\|_{(H^1(\Omega))'} = \sup_{\|v\|_1=1} |\langle f, v\rangle| = \sup_{\|v\|_1=1} |(\nabla \mathcal{G}f, \nabla v) + (\mathcal{G}f, v)|$$

$$\leq \sup_{\|v\|_1=1} (|\mathcal{G}f|_1|v|_1 + \|\mathcal{G}f\|_0\|v\|_0) = \sup_{\|v\|_1=1} \|\mathcal{G}f\|_1\|v\|_1 = \|\mathcal{G}f\|_1 \leq C|\mathcal{G}f|_1 = C\|f\|_{-1}.$$

Now by taking $v = \frac{\mathcal{G}f}{\|\mathcal{G}f\|_1} \in H^1(\Omega)$ we deduce using (3.1.9) that

$$\|f\|_{(H^1(\Omega))'} \geq |\langle f, v\rangle| = \frac{|\langle f, \mathcal{G}f\rangle|}{\|\mathcal{G}f\|_1} = \frac{|\mathcal{G}f|_1^2}{\|\mathcal{G}f\|_1} \geq C\frac{|\mathcal{G}f|_1^2}{|\mathcal{G}f|_1} = C|\mathcal{G}f|_1 = C\|f\|_{-1},$$

where we have applied Poincaré inequality (2.1.9) to give $\|\mathcal{G}f\|_1^2 = |\mathcal{G}f|_0^2 + |\mathcal{G}f|_1^2 \leq (C_p^2 + 1)|\mathcal{G}f|_1^2$. □

We finally recall the following lemma, about the operator $\mathcal{G}_q$ for $q \in (1, 2]$, which is a consequence of the quasi-uniform partitioning of $\mathcal{T}^h$:

Lemma 3.1.2 For any $q \in (1, 2]$, it holds that

$$\|\eta\|_{0,q} \leq Ch^{-1}\|\mathcal{G}_q\eta\|_{1,q} \quad \forall \eta \in S^h. \tag{3.1.11}$$

Proof: On noting (3.1.5), (3.1.4), Hölder's inequality, Young's inequality and (2.4.53), we have for any $\eta \in S^h$ and for any $\alpha > 0$ that

$$\|\eta\|_0^2 \leq \langle \eta, \eta\rangle_{q'} \leq (\nabla \mathcal{G}_q\eta, \nabla\eta) + (\mathcal{G}_q\eta, \eta)$$

$$\leq 2\|\mathcal{G}_q\eta\|_{1,q}\|\eta\|_{1,q'}$$

$$\leq \alpha\|\mathcal{G}_q\eta\|_{1,q}^2 + \frac{C}{\alpha}h^{-2(1+d(\frac{1}{2}-\frac{1}{q'}))}\|\eta\|_0^2. \tag{3.1.12}$$

It follows from choosing $\alpha = 2\,C\,h^{-2(1+d(\frac{1}{2}-\frac{1}{q'}))}$ in (3.1.12) and (2.4.53), that

$$\|\eta\|_{0,q} \leq Ch^{d(\frac{1}{q}-\frac{1}{2})}\|\eta\|_0 \leq Ch^{d(\frac{1}{q}-\frac{1}{2})-(1+d(\frac{1}{2}-\frac{1}{q'}))}\|\mathcal{G}_q\eta\|_{1,q} \leq Ch^{-1}\|\mathcal{G}_q\eta\|_{1,q}.$$

□

3.2 Stability estimates

In this section we establish some uniform bounds on the solution $\{U^n_{i,\varepsilon}\}^m_{i=1}$, independent of the parameters ε and h, which will be used to prove the convergence of the approximate problem.

Lemma 3.2.1 Let the assumptions of Theorem 2.4.7 hold and let $\Delta t_n \leq \frac{1}{2\gamma+1}$, $D_i > 0, \forall i$ and $\{U^{n-1}_{i,\varepsilon}\}^m_{i=1} \in S^h \times ... \times S^h, n \geq 1$. Then a solution $\{U^n_{i,\varepsilon}\}^m_{i=1} \in S^h \times ... \times S^h, n \geq 1$ to the $n-$th step of $(\mathrm{P}^{h,\Delta t}_{M,\varepsilon})$ satisfies

$$[1 - \Delta t_n(2\gamma + m)](\sum_{i=1}^m F_\varepsilon(U^n_{i,\varepsilon}), 1)^h + \frac{D}{M}\Delta t_n \sum_{i=1}^m |U^n_{i,\varepsilon}|^2_1 + \Delta t_n |\sum_{i=1}^m U^n_{i,\varepsilon}|^2_1$$

$$\leq (\sum_{i=1}^m F_\varepsilon(U^{n-1}_{i,\varepsilon}), 1)^h + C\Delta t_n, \qquad (3.2.13)$$

where $D = \min_i D_i, \;\; \gamma = \max_i \gamma_i$.

Proof: Choosing $\chi \equiv \Delta t_n \pi^h[F'_\varepsilon(U^n_{i,\varepsilon})]$ as a test function in (2.4.67) yields, on noting (2.4.61), (2.4.62) and (2.4.45), that

$$(U^n_{i,\varepsilon} - U^{n-1}_{i,\varepsilon}, F'_\varepsilon(U^n_{i,\varepsilon}))^h + \Delta t_n(D_i[\Lambda_\varepsilon(U^n_{i,\varepsilon})]^{-1}\nabla U^n_{i,\varepsilon} + \sum_{j=1}^m \nabla U^n_{j,\varepsilon}, \nabla U^n_{i,\varepsilon})$$

$$= \Delta t_n(\gamma_i U^n_{i,\varepsilon} - \; \phi_\varepsilon(U^n_{i,\varepsilon}) \sum_{j=1}^m \phi_\varepsilon(U^{n-1}_{j,\varepsilon}), F'_\varepsilon(U^n_{i,\varepsilon}))^h, \qquad i = 1, ..., m. \qquad (3.2.14)$$

Using (2.3.28), (2.3.32), (2.1.12), Young's inequality, (2.3.29) and the fact that $F_\varepsilon(\cdot) \geq 0$ yields

$$\Delta t_n(\gamma_i U^n_{i,\varepsilon} - \; \phi_\varepsilon(U^n_{i,\varepsilon}) \sum_{j=1}^m \phi_\varepsilon(U^{n-1}_{j,\varepsilon}), F'_\varepsilon(U^n_{i,\varepsilon}))^h$$

$$= \Delta t_n(\gamma_i U^n_i, F'_\varepsilon(U^n_{i,\varepsilon}))^h - \Delta t_n \left(\phi_\varepsilon(U^n_{i,\varepsilon}) \sum_{j=1}^m \phi_\varepsilon(U^{n-1}_{j,\varepsilon}), F'_\varepsilon(U^n_{i,\varepsilon})\right)^h$$

$$\leq \Delta t_n \gamma_i (2F_\varepsilon(U^n_{i,\varepsilon}) + 1, 1)^h + \Delta t_n \left(1 - U^n_{i,\varepsilon}, \sum_{j=1}^m \phi_\varepsilon(U^{n-1}_{j,\varepsilon})\right)^h$$

$$\leq \Delta t_n \gamma_i (2F_\varepsilon(U^n_{i,\varepsilon}) + 1, 1)^h + \Delta t_n \left(1 - [U^n_{i,\varepsilon}]_-, \sum_{j=1}^m \phi_\varepsilon(U^{n-1}_{j,\varepsilon})\right)^h$$

$$\leq 2\Delta t_n \gamma_i (F_\varepsilon(U^n_{i,\varepsilon}), 1)^h - \Delta t_n \left([U^n_{i,\varepsilon}]_-, \sum_{j=1}^m \phi_\varepsilon(U^{n-1}_{j,\varepsilon})\right)^h + C(M, |\Omega|, \gamma_i)\Delta t_n$$

$$\leq 2\Delta t_n \gamma_i (F_\varepsilon(U^n_{i,\varepsilon}), 1)^h + \frac{m\Delta t_n}{2\varepsilon}[U^n_{i,\varepsilon}]^2_- + \frac{\varepsilon \Delta t_n}{2}(\sum_{j=1}^m |\phi_\varepsilon(U^{n-1}_{j,\varepsilon})|^2_h) + C(M, |\Omega|, \gamma_i)\Delta t_n$$

$$\leq \Delta t_n(2\gamma + m)(F_\varepsilon(U^n_{i,\varepsilon}), 1)^h + C(M, |\Omega|, \gamma_i)\Delta t_n, \quad i = 1, ..., m. \tag{3.2.15}$$

It follows from (3.2.14), (3.2.15) and the first inequality in (2.4.79) that

$$[1 - \Delta t_n(2\gamma_i + m)](F_\varepsilon(U^n_{i,\varepsilon}), 1)^h + \Delta t_n(D_i[\Lambda_\varepsilon(U^n_{i,\varepsilon})]^{-1}\nabla U^n_i + \sum_{j=1}^m \nabla U^n_{j,\varepsilon}, \nabla U^n_{i,\varepsilon})$$

$$\leq (F_\varepsilon(U^{n-1}_{i,\varepsilon}), 1)^h + C(M, |\Omega|, \gamma_i)\Delta t_n, \qquad i = 1, ..., m. \tag{3.2.16}$$

Summing (3.2.16) for $i = 1, ..., m$ yield

$$\Big(\sum_{i=1}^m [1 - \Delta t_n(2\gamma_i + m)](F_\varepsilon(U^n_{i,\varepsilon}), 1)^h, 1\Big)^h + \Delta t_n \sum_{i=1}^m \big(D_i[\Lambda_\varepsilon(U^n_{i,\varepsilon})]^{-1}\nabla U^n_{i,\varepsilon}, \nabla U^n_{i,\varepsilon}\big)$$

$$+\Delta t_n |\sum_{i=1}^m U^n_{i,\varepsilon}|^2_1 \leq \Big(\sum_{i=1}^m F_\varepsilon(U^{n-1}_{i,\varepsilon}), 1\Big)^h + C\Delta t_n,$$

and then using (2.4.68), $F_\varepsilon(s) \geq 0$ and that $\gamma_i > 1$ yields the desired result. □

Lemma 3.2.2 Let the assumptions of Lemma 3.2.1 hold and let $\{u^0_i\}^m_{i=1} \in L^\infty(\Omega)$ with $u^0_i(x) \geq 0, i = 1, ..., m$ for *a.e.* $x \in \Omega$. Let either $U^0_{i,\varepsilon} \equiv P^h u^0_i$; or $U^0_{i,\varepsilon} \equiv \pi^h u^0_i$ if $\{u^0_i\}^m_{i=1} \in C(\overline{\Omega})$. Then for all $\varepsilon \in (0, e^{-1})$, for all $h > 0$ and for all $\Delta t_n \leq \frac{1-\delta}{2\gamma+m}$, for some $\delta \in (0, 1)$, the problem $(\mathrm{P}^{h,\Delta t}_{M,\varepsilon})$ possesses a solution $\{U^n_{i,\varepsilon}\}^m_{i=1}, n = 1, ..., N$ satisfying

$$\max_{n=1,...,N}[(\sum_{i=1}^m F_\varepsilon(U^n_{i,\varepsilon}), 1)^h + \varepsilon^{-1}\sum_{i=1}^m \|\pi^h[U^n_{i,\varepsilon}]_-\|^2_0 + \sum_{i=1}^m \|U^n_{i,\varepsilon}\|^2_0]$$

$$+ \sum_{n=1}^N \Delta t_n \|\sum_{i=1}^m U^n_{i,\varepsilon}\|^2_1 + \sum_{n=1}^N \Delta t_n \sum_{i=1}^m \|U^n_{i,\varepsilon}\|^2_1 \leq C. \tag{3.2.17}$$

Proof: Firstly, we note that, $\Delta t_n \leq \frac{1-\delta}{2\gamma+m}$ and thus we have

$$\delta \leq 1 - \Delta t_n(2\gamma + m) \leq 1 - \Delta t(2\gamma + m). \tag{3.2.18}$$

Also,

$$\frac{1}{1 - \Delta t_n(2\gamma + m)} = 1 + \frac{(2\gamma + m)\Delta t_n}{1 - \Delta t_n(2\gamma + m)} \leq 1 + \frac{(2\gamma + m)\Delta t_n}{\delta}. \tag{3.2.19}$$

From (3.2.13), (3.2.18), (3.2.19), we deduce for $n = 1, ..., N$, that

$$
\begin{aligned}
&(\sum_{i=1}^{m} F_\varepsilon(U_{i,\varepsilon}^n), 1)^h \\
&\leq \frac{1}{1 - \Delta t_n(2\gamma + m)}(\sum_{i=1}^{m} F_\varepsilon(U_{i,\varepsilon}^{n-1}), 1)^h + \frac{C\Delta t_n}{1 - \Delta t_n(2\gamma_m + m)} \\
&\leq (1 + \frac{(2\gamma + m)\Delta t_n}{\delta})(\sum_{i=1}^{m} F_\varepsilon(U_{i,\varepsilon}^{n-1}), 1)^h + \frac{C\Delta t_n}{\delta} \\
&\leq e^{\frac{(2\gamma+m)\Delta t_n}{\delta}}(\sum_{i=1}^{m} F_\varepsilon(U_{i,\varepsilon}^{n-1}), 1)^h + \frac{C\Delta t_n}{\delta} \\
&\leq Ce^{\frac{(2\gamma+m)\Delta t_n}{\delta}}\big[(\sum_{i=1}^{m} F_\varepsilon(U_{i,\varepsilon}^{n-1}), 1)^h + e^{-\frac{(2\gamma+m)\Delta t_n}{\delta}}\Delta t_n\big].
\end{aligned}
\tag{3.2.20}
$$

Next, with the use of the assumptions on the initial data $\{U_i^0\}_{i=1}^m$, (2.3.25), the definition of π^h and (3.1.2), it follows that

$$
(\sum_{i=1}^{m} F_\varepsilon(U_{i,\varepsilon}^0), 1)^h \leq C. \tag{3.2.21}
$$

Therefore, (3.2.20) and (3.2.21) imply that

$$
\begin{aligned}
&\max_{n=1,\dots,N}[(\sum_{i=1}^{m} F_\varepsilon(U_{i,\varepsilon}^n), 1)^h] \\
&\leq Ce^{\frac{(2\gamma+m)\Delta t_n}{\delta}}[(\sum_{i=1}^{m} F_\varepsilon(U_{i,\varepsilon}^{n-1}), 1)^h + e^{-\frac{(2\gamma+m)\Delta t_n}{\delta}}\Delta t_n] \\
&\leq Ce^{\frac{(2\gamma+m)(\Delta t_n+\Delta t_{n-1})}{\delta}}[(\sum_{i=1}^{m} F_\varepsilon(U_{i,\varepsilon}^{n-2}), 1)^h \\
&+e^{-\frac{(2\gamma+m)\Delta t_n}{\delta}}\Delta t_n + e^{-\frac{(2\gamma+m)\Delta t_{n-1}}{\delta}}\Delta t_{n-1}] \\
&\leq Ce^{\frac{(2\gamma+m)(\Delta t_n+\ldots+\Delta t_1)}{\delta}}[(\sum_{i=1}^{m} F_\varepsilon(U_{i,\varepsilon}^0), 1)^h \\
&+e^{-\frac{(2\gamma+m)\Delta t_n}{\delta}}\Delta t_n + \ldots + e^{-\frac{(2\gamma+m)\Delta t_1}{\delta}}\Delta t_1] \leq C.
\end{aligned}
\tag{3.2.22}
$$

From this result, with the aid of (2.4.46), (2.3.29) and (2.3.30) we obtain, for $n = 1, ..., N$ with $i = 1, ..., m$, that

$$
\|U_{i,\varepsilon}^n\|_0^2 \leq |U_{i,\varepsilon}^n|_h^2 = ((U_{i,\varepsilon}^n)^2, 1)^h \leq C((F_\varepsilon(U_{i,\varepsilon}^n), 1)^h + 1) \leq C. \tag{3.2.23}
$$

Now using (2.4.46), (2.4.45) and (3.2.22) and noting the facts $s = [s]_+ + [s]_-$ and $F(s) \geq 0$, yields for $n = 1, ..., N$

$$\|\pi^h[U^n_{i,\varepsilon}]_-\|_0^2 \leq \|\pi^h[U^n_{i,\varepsilon}]_-\|_h^2 = ([U^n_{i,\varepsilon}]_-^2, 1)^h \leq 2\varepsilon(F_\varepsilon(U^n_{i,\varepsilon}), 1)^h \leq C\varepsilon. \tag{3.2.24}$$

We now note that the bounds $1 \to 3$ in (3.2.17) follow by combining (3.2.22), (3.2.23) and (3.2.24). Now, to prove the fourth and the fifth bounds in (3.2.17), firstly, we sum (3.2.13) over n, next we use (3.2.21), (3.2.22), to get

$$\sum_{n=1}^{N} \Delta t_n \sum_{i=1}^{m} |U^n_{i,\varepsilon}|_1^2 + \sum_{n=1}^{N} \Delta t_n |\sum_{i=1}^{m} U^n_{i,\varepsilon}|_1^2 \leq C. \tag{3.2.25}$$

From the third bound in (3.2.17), we have

$$\sum_{n=1}^{N} \Delta t_n \sum_{i=1}^{m} \|U^n_{i,\varepsilon}\|_0^2, \leq C, \tag{3.2.26}$$

then the fifth bound follow from (3.2.25) and (3.2.26). Now, On noting Poincaré inequality and the second and third bounds in (3.2.17), we have

$$\|\sum_{i=1}^{m} U^n_{i,\varepsilon}\|_0^2 \leq C(|\sum_{i=1}^{m} U^n_{i,\varepsilon}|_1^2 + |(\sum_{i=1}^{m} U^n_{i,\varepsilon}, 1)|^2) \leq C, \tag{3.2.27}$$

then the fourth bound follow from (3.2.25) and (3.2.27).

□

Theorem 3.2.3 Let the assumptions of Lemma 3.2.2 hold. Let $\alpha = \frac{2(d+2)}{d}$ and $\{\Delta t_n\}_{n=1}^N$ be such that

$$\Delta t_n \leq \Delta t_{n-1}, \quad \forall n = 2, ..., N.$$

Then a solution $\{U^n_{i,\varepsilon}\}_{i=1}^m, n = 1, ..., N$ to $(\mathrm{P}^{h,\Delta t}_{M,\varepsilon})$ satisfies

$$\sum_{n=1}^{N} \Delta t_n \Big[\sum_{i=1}^{m} \|U^n_{i,\varepsilon}\|_{0,\alpha}^\alpha + \sum_{i=1}^{m} \|\frac{U^n_{i,\varepsilon} - U^{n-1}_{i,\varepsilon}}{\Delta t_n}\|_{(H^1(\Omega))'}^2 + \sum_{i=1}^{m} \|\mathcal{G}[\frac{U^n_{i,\varepsilon} - U^{n-1}_{i,\varepsilon}}{\Delta t_n}]\|_1^2 \Big] \leq C. \tag{3.2.28}$$

Proof: Using the Sobolev interpolation theorem (2.1.4) and the third and fifth bounds in (3.2.17) gives for $n = 1, ..., N$,

$$\|U^n_{i,\varepsilon}\|_{0,\alpha}^\alpha \leq C\|U^n_{i,\varepsilon}\|_0^{\alpha-2}\|U^n_{i,\varepsilon}\|_1^2 \leq C, \quad i = 1, ..., m, \tag{3.2.29}$$

where $\alpha d(\frac{1}{2}-\frac{1}{\alpha})=2$, that is $\alpha=\frac{2(d+2)}{d}$.

It is crucial to note from the definition of π^h, (3.2.13) and the assumptions on $\{u_i^0\}_{i=1}^m$ that

$$\sum_{i=1}^m \|U_{i,\varepsilon}^0\|_0 = \sum_{i=1}^m (\int_\Omega (U_{i,\varepsilon}^0)^2 dx)^{\frac{1}{2}} = \sum_{i=1}^m (\int_\Omega (\pi^h u_i^0)^2 dx)^{\frac{1}{2}}$$

$$\leq C \sum_{i=1}^m \|\pi^h u_i^0\|_{0,\infty}^2 \leq C \sum_{i=1}^m \|u_i^0\|_{0,\infty}^2 \leq C. \qquad (3.2.30)$$

Next, it follows from (3.1.8), (3.1.1), (2.4.67), (2.4.46), (2.3.28), (3.1.3), (2.4.64) and (2.4.65) for any $\eta \in H^1(\Omega)$ and for $n=1,...,N$ that

$$\langle \frac{U_{i,\varepsilon}^n - U_{i,\varepsilon}^{n-1}}{\Delta t_n}, \eta \rangle = (\frac{U_{i,\varepsilon}^n - U_{i,\varepsilon}^{n-1}}{\Delta t_n}, \eta) = (\frac{U_{i,\varepsilon}^n - U_{i,\varepsilon}^{n-1}}{\Delta t_n}, P^h \eta)^h$$

$$= (\gamma_i U_{i,\varepsilon}^n - \phi_\varepsilon(U_{i,\varepsilon}^n)(\sum_{j=1}^m \phi_\varepsilon(U_{j,\varepsilon}^{n-1})), P^h\eta)^h - (D_i \nabla U_{i,\varepsilon}^n + \Lambda_\varepsilon(U_{i,\varepsilon}^n) \sum_{j=1}^m \nabla U_{j,\varepsilon}^n, \nabla P^h \eta)$$

$$\leq C[1 + |U_{i,\varepsilon}^n|_h + C[\ |U_{i,\varepsilon}^n|_1 + |\sum_{j=1}^m U_{j,\varepsilon}^n|_1\,]]\ |P^h\eta|_1$$

$$\leq C[\,1 + \|U_{i,\varepsilon}^n\|_0 + C[\ |U_{i,\varepsilon}^n|_1 + |\sum_{j=1}^m U_{j,\varepsilon}^n|_1\,]]\ |P^h\eta|_1$$

$$\leq C[\,1 + \|U_{i,\varepsilon}^n\|_0]\ \|\eta\|_1 + C[\ |U_{i,\varepsilon}^n|_1 + |\sum_{j=1}^m U_{j,\varepsilon}^n|_1\,]]\ \|\eta\|_1$$

$$\leq C[\,1 + \|U_{i,\varepsilon}^n\|_1 + \|\sum_{j=1}^m U_{j,\varepsilon}^n\|_1\,]\ \|\eta\|_1, \qquad i=1,...,m, \qquad (3.2.31)$$

to arrive at the following bound,

$$\|\frac{U_{i,\varepsilon}^n - U_{i,\varepsilon}^{n-1}}{\Delta t_n}\|_{(H^1(\Omega))'}^2 \leq C[\,1 + \|U_{i,\varepsilon}^n\|_1^2 + \|\sum_{j=1}^m U_{j,\varepsilon}^n\|_1^2\,], \qquad i=1,...,m. \qquad (3.2.32)$$

If we use this result with (3.2.17), our assumption on the time steps and (3.2.30), we find

$$\sum_{n=1}^N \Delta t_n \|\frac{U_{i,\varepsilon}^n - U_{i,\varepsilon}^{n-1}}{\Delta t_n}\|_{(H^1(\Omega))'}^2$$

$$\leq C \sum_{n=1}^N \Delta t_n [\,1 + \|U_{i,\varepsilon}^n\|_1^2 + \|\sum_{j=1}^m U_{j,\varepsilon}^n\|_1^2\,] \leq C, \qquad i=1,...,m. \qquad (3.2.33)$$

To complete the proof of the theorem, we note that the last bounds in (3.2.28) follow from the bounds in (3.2.33) and on recalling (3.1.9). □

Lemma 3.2.4 Let the assumptions of (A) hold and let $\{u_i^0\}_{i=1}^m \in H^1_{\geq 0}(\Omega)$. Let either $U^0_{i,\varepsilon} \equiv P^h u_i^0$; or $U^0_{i,\varepsilon} \equiv \pi^h u_i^0$ if either $d = 1$ or $u_i^0 \in W^{1,r}(\Omega)$ with $r > d$, it follows that $U^0_{i,\varepsilon} \in S^h_{\geq 0}$, for $i = 1, ..., m$ and

$$\sum_{i=1}^m \|U^0_{i,\varepsilon}\|_1^2 \leq C. \tag{3.2.34}$$

Proof: We first mention that $\pi^h u_i^0, i = 1, ..., m$, are well defined as the Sobolev embedding result (see Ciarlet [39], page 114):

$$W^{m,r}(\Omega) \overset{c}{\hookrightarrow} C(\overline{\Omega}) \ \text{ holds for } r \in [1, \infty] \ \text{ if } \ m > \frac{d}{r}.$$

It can be seen clearly from the definitions of the projection operator P^h and the interpolation operator π^h that $\{U^0_{i,\varepsilon}\}_{i=1}^m \in S^h_{\geq 0}$. Now, to drive the bound (3.2.34), we use (2.4.56), (3.1.3) and the assumptions on $\{u_i^0\}_{i=1}^m$ as follows:

$$\begin{aligned}
\sum_{i=1}^m \|U^0_{i,\varepsilon}\|_1^2 &= \sum_{i=1}^m \|\pi^h u^0_{i,\varepsilon}\|_1^2 = \sum_{i=1}^m [\ \|\pi^h u^0_{i,\varepsilon}\|_0^2 + |\pi^h u^0_{i,\varepsilon}|_1^2\ ] \\
&= \sum_{i=1}^m [\|u^0_{i,\varepsilon} - u^0_{i,\varepsilon} + \pi^h u^0_{i,\varepsilon}\|_0^2 + |u^0_{i,\varepsilon} - u^0_{i,\varepsilon} + \pi^h u^0_{i,\varepsilon}|_1^2] \\
&\leq C \sum_{i=1}^m [\|u^0_{i,\varepsilon}\|_0^2 + \|(I - \pi^h) u^0_{i,\varepsilon}\|_0^2 + |u^0_{i,\varepsilon}|_1^2 + |(I - \pi^h) u^0_{i,\varepsilon}|_1^2] \\
&\leq C \sum_{i=1}^m \|u^0_{i,\varepsilon}\|_1^2 \leq C.
\end{aligned}$$

□

3.3 Existence of a weak solution

In this section, we establish convergence of our approximation (2.4.67) in one, two and three space dimensions; and hence existence of a solution to the problem $(\mathrm{P}^{\Delta t}_M)$. This is achieved by taking the limit of the regularization and discretization parameters of the problem $(\mathrm{P}^{h,\Delta t}_{M,\varepsilon})$. The condition $U_i^0 \in H^1(\Omega), i = 1, ..., m$ will be essential in the analysis of this section.

We shall first consider the following definitions:

$$U_{i,\varepsilon}(t) = (\frac{t - t_n}{\Delta t_n}) U^n_{i,\varepsilon} + (\frac{t_n - t}{\Delta t_n}) U^{n-1}_{i,\varepsilon}, \qquad t \in [t_{n-1}, t_n],\ \ n \geq 1, \qquad i = 1, ..., m, \tag{3.3.35}$$

and

$$U^+_{i,\varepsilon}(t) = U^n_{i,\varepsilon}, \qquad U^-_{i,\varepsilon}(t) = U^{n-1}_{i,\varepsilon}, \qquad t \in [t_{n-1}, t_n], \;\; n \geq 1, \qquad i = 1, ..., m. \quad (3.3.36)$$

We also have that for $t \in (t_{n-1}, t_n)$

$$\frac{\partial U_{i,\varepsilon}}{\partial t} = \frac{U^+_{i,\varepsilon} - U^-_{i,\varepsilon}}{\Delta t_n} = \frac{U^+_{i,\varepsilon} - U_{i,\varepsilon}}{t_n - t} = \frac{U_{i,\varepsilon} - U^-_{i,\varepsilon}}{t - t_{n-1}}, \qquad t \in [t_{n-1}, t_n], \;\; n \geq 1, \qquad i = 1, ..., m. \quad (3.3.37)$$

Using the above we can restate the problem $(\mathrm{P}^{h,\Delta t}_{M,\varepsilon})$ as follows:
Find $U_{i,\varepsilon} \in C([0,T]; S^h) \times C([0,T]; S^h), i = 1, ..., m$ such that for all $\chi \in L^2(0,T; S^h)$

$$\int_0^T [(\frac{\partial U_{i,\varepsilon}}{\partial t}, \chi)^h + D_i(\nabla U^+_{i,\varepsilon}, \nabla\chi) + (\Lambda_\varepsilon(U^+_{i,\varepsilon}) \sum_{j=1}^m \nabla U^+_{j,\varepsilon}, \nabla\chi)] d\, t$$

$$= \int_0^T [(\gamma_i U^+_{i,\varepsilon} - \; \phi_\varepsilon(U^+_{i,\varepsilon})(\sum_{j=1}^m \phi_\varepsilon(U^-_{j,\varepsilon}), \chi)^h] d\, t, \qquad i = 1, ..., m. \quad (3.3.38)$$

We now show the main theorem in this chapter which deals with the existence of a global weak solution to the system $(\mathrm{P}^{\Delta t}_M)$.

Theorem 3.3.1 Let the assumptions (A) hold, $D_i > 0, \;\; \gamma_i > 1, \;\; \forall i$, and $\{U^0_i\}^m_{i=1} \in H^1_{\geq 0}(\Omega) \cap L^\infty(\Omega)$. In addition, let $\varepsilon, h, \{\Delta t_n\}^N_{n=1}, \{U^0_{i,\varepsilon}\}^m_{i=1}$ be such that

(i) either $U^0_{i,\varepsilon} \equiv P^h U^0_i$; or $U^0_{i,\varepsilon} \equiv \pi^h U^0_i$ if either $d = 1$ or $U^0_i \in W^{1,r}(\Omega)$ with $r > d$.

(ii) $\Delta t_n \leq \frac{1-\delta}{2\gamma+m}$, for some $\delta \in (0,1)$.

(iii) $\Delta t_n \leq C\Delta t_{n-1}, \quad \forall n = 2, ..., N$.

(iv) $\varepsilon \to 0$ as $h \to 0$.

Then there exists a subsequence of $\{U_{i,\varepsilon}\}^m_{i=1}$, solving (2.4.67), and functions

$$U_i \in L^2(0,T; H^1(\Omega)) \cap L^\alpha(\Omega_T) \cap L^\infty(0,T; L^2(\Omega)) \cap H^1(0,T; (H^1(\Omega))'), \quad i = 1, ..., m, \quad (3.3.39)$$

where $\alpha = \frac{2(d+2)}{d}$, with $U^\pm_i(x,t) \geq 0, i = 1, ..., m$ almost everywhere and

$$U_i(\cdot, 0) = u^0_i(\cdot), \quad \text{in} \quad L^2(\Omega). \quad (3.3.40)$$

Moreover, it holds as $h \to 0$ that for $i = 1, ..., m$

$$U_{i,\varepsilon}, U^\pm_{i,\varepsilon} \rightharpoonup U_i, U^\pm_i \qquad \text{in} \quad L^2(0,T; H^1(\Omega)) \cap L^\alpha(\Omega_T), \quad (3.3.41)$$

$$U_{i,\varepsilon}, U_{i,\varepsilon}^{\pm} \rightharpoonup^{*} U_i, U_{i,}^{\pm} \qquad \text{in} \qquad L^{\infty}(0,T;L^2(\Omega)), \tag{3.3.42}$$

$$\frac{\partial U_{i,\varepsilon}}{\partial t} \rightharpoonup \frac{\partial U_i}{\partial t} \qquad \text{in} \qquad L^2(0,T;(H^1(\Omega))'), \tag{3.3.43}$$

$$U_{i,\varepsilon}, U_{i,\varepsilon}^{\pm} \to U_i, U_i^{\pm} \qquad \text{in} \qquad L^2(0,T;L^s(\Omega)), \tag{3.3.44}$$

$$\phi_\varepsilon(U_{i,\varepsilon}^{\pm}) \to \phi(U_i^{\pm}) \qquad \text{in} \qquad L^2(0,T;L^s(\Omega)), \tag{3.3.45}$$

$$\pi^h\phi_\varepsilon(U_{i,\varepsilon}^{\pm}) \to \phi(U_i^{\pm}) \qquad \text{in} \qquad L^2(0,T;L^s(\Omega)), \tag{3.3.46}$$

$$\Lambda_\varepsilon(U_{i,\varepsilon}^{\pm}) \to \phi(U_i^{\pm})I \qquad \text{in} \qquad L^2(0,T;L^s(\Omega)), \tag{3.3.47}$$

for any

$$s \in \begin{cases} [2,\infty] & \text{if } d = 1, \\ [2,\infty) & \text{if } d = 2, \\ [2,6] & \text{if } d = 3, \end{cases}$$

where the symbols $\to$, $\rightharpoonup$, and $\rightharpoonup^{*}$ represent strong, weak and weak-star convergence respectively (see A.1.11 $\to$ A.1.13).

Proof: From the assumptions (i)$\to$(iii), (3.2.17), (3.2.28), (2.3.28), (2.4.64), (2.4.65), (3.3.35), (3.3.36), (3.3.37) and (3.2.34) one may establishes the following uniform bounds independently of the parameters ε, h and Δt

$$\|U_{i,\varepsilon}^{\pm}\|_{L^2(0,T;H^1(\Omega))} + \|U_{i,\varepsilon}^{\pm}\|_{L^{\alpha}(\Omega_T)} + \|U_{i,\varepsilon}^{\pm}\|_{L^{\infty}(0,T;L^2(\Omega))} + \varepsilon^{-\frac{1}{2}}\|\pi^h[U_{i,\varepsilon}^{\pm}]_-\|_{L^{\infty}(0,T;L^2(\Omega))}$$
$$+\|\frac{\partial U_{i,\varepsilon}}{\partial t}\|_{L^2(0,T;(H^1(\Omega))')} + \|\mathcal{G}\frac{\partial U_{i,\varepsilon}}{\partial t}\|_{L^2(0,T;H^1(\Omega))} + \|\phi_\varepsilon(U_{i,\varepsilon}^{\pm})\|_{L^{\infty}(\Omega_T)} + \|\pi^h\phi_\varepsilon(U_{i,\varepsilon}^{\pm})\|_{L^{\infty}(\Omega_T)}$$
$$+\|\Lambda_\varepsilon(U_{i,\varepsilon}^{\pm})\|_{L^{\infty}(\Omega_T)} \leq C, \qquad i = 1, ..., m. \tag{3.3.48}$$

In the above, and throughout, the notation $U_{i,\varepsilon}^{\pm}$ means with and without the superscript $\pm$. Although $U_{i,\varepsilon}$ can go negative, the amount it can is controlled by the regularization parameter ε through the fourth term in (3.3.48).

Also,we have

$$\|U_{i,\varepsilon}\|_{L^2(0,T;H^1(\Omega))}^2 = \int_0^T \|U_{i,\varepsilon}\|_{H^1(\Omega)}^2 dt$$
$$\leq \sum_{n=1}^{N} \frac{2}{(\Delta t_n)^2} \int_{t_{n-1}}^{t_n} [|t - t^+|^2 \|U_{i,\varepsilon}^+\|_{H^1(\Omega)}^2 + |t^+ - t|^2 \|U_{i,\varepsilon}^-\|_{H^1(\Omega)}^2] dt$$
$$\leq \sum_{n=1}^{N} \frac{2(\Delta t)^2}{(\Delta t_n)^2} \int_{t_{n-1}}^{t_n} [|U_{i,\varepsilon}^+\|_{H^1(\Omega)}^2 + |U_{i,\varepsilon}^-\|_{H^1(\Omega)}^2] dt$$

$$\leq C\ [\|U^+_{i,\varepsilon}\|^2_{L^2(0,T;H^1(\Omega))} + \|U^-_{i,\varepsilon}\|^2_{L^2(0,T;H^1(\Omega))}] \leq C, \qquad i = 1, ..., m, \tag{3.3.49}$$

and

$$\|U_{i,\varepsilon}\|^\alpha_{L^\alpha(\Omega_T)} = \int_0^T \|U_{i,\varepsilon}\|^\alpha_{L^\alpha(\Omega)} dt$$

$$\leq \sum_{n=1}^N \frac{C}{(\Delta t_n)^\alpha} \int_{t_{n-1}}^{t_n} [|t - t^+|^\alpha \|U^+_{i,\varepsilon}\|^\alpha_{L^\alpha(\Omega)} + |t^+ - t|^\alpha \|U^-_{i,\varepsilon}\|^\alpha_{L^\alpha(\Omega)}] dt$$

$$\leq \sum_{n=1}^N \frac{C(\Delta t)^\alpha}{(\Delta t_n)^\alpha} \int_{t_{n-1}}^{t_n} [\|U^+_{i,\varepsilon}\|^\alpha_{L^\alpha(\Omega)} + \|U^-_{i,\varepsilon}\|^\alpha_{L^\alpha(\Omega)}] dt$$

$$\leq C\ [\|U^+_{i,\varepsilon}\|_{L^\alpha(\Omega_T)} + \|U^-_{i,\varepsilon}\|_{L^\alpha(\Omega_T)}] \leq C, \qquad i = 1, ..., m. \tag{3.3.50}$$

Moreover, we can get

$$\|U_{i,\varepsilon}\|^2_{L^\infty(0,T;L^2(\Omega))} = \operatorname{ess\,sup} \|U_{i,\varepsilon}\|^2_{L^2(\Omega)}$$

$$= \max_{n=1,...,N} \big[\frac{2}{(\Delta t_n)^2} \big[|t - t^+|^2 \|U^+_{i,\varepsilon}\|^2_{L^2(\Omega)} + |t^+ - t|^2 \|U^-_{i,\varepsilon}\|^2_{L^2(\Omega)}\big]\big]$$

$$= \max_{n=1,...,N} \big[\frac{2(\Delta t)^2}{(\Delta t_n)^2} \big[\|U^+_{i,\varepsilon}\|^2_{L^2(\Omega)} + \|U^-_{i,\varepsilon}\|^2_{L^2(\Omega)}\big]\big] dt$$

$$\leq C[\|U^+_{i,\varepsilon}\|_{L^\infty(0,T;L^2(\Omega))} + \|U^-_{i,\varepsilon}\|_{L^\infty(0,T;L^2(\Omega))}] \leq C. \tag{3.3.51}$$

Before moving onto the passage to the limit step of the proof we recall that $L^\infty(0, T, L^2(\Omega))$ is the dual space of $L^1(0, T, L^2(\Omega))$, which is a separable Banach space but not reflexive, while the Banach spaces $L^2(0, T, H^1(\Omega)), L^\alpha(\Omega_T)$ are reflexive. Thus, by compactness arguments (see A.1.6 and A.1.8) and the bounds (3.3.48) we can extract subsequences, still denoted $\{U^\pm_{i,\varepsilon}\}_h, \{U_{i,\varepsilon}\}_h$, such that as $h \to 0$ we have

$$U^\pm_{i,\varepsilon}, U_{i,\varepsilon} \rightharpoonup U^\pm_i, U_i \qquad \text{in } L^2(0, T, H^1(\Omega)) \cap L^\alpha(\Omega_T),$$

$$U^\pm_{i,\varepsilon}, U_{i,\varepsilon} \rightharpoonup^* U^\pm_i, U_i \qquad \text{in } L^\infty(0, T, L^2(\Omega)),$$

and thus the convergence results (3.3.41) and (3.3.42) were satisfied. Then, since $\{\frac{\partial U_{i,\varepsilon}}{\partial t}\}_h \in L^2(0, T, (H^1(\Omega))')$ and $L^2(0, T, (H^1(\Omega))')$ are reflexive Banach spaces then according to the weak compactness theorem, there exist a subsequences $\{\frac{\partial U_{i,\varepsilon}}{\partial t}\}_h \in L^2(0, T, (H^1(\Omega))')$ and a functions $\widetilde{\eta} \in L^2(0, T, (H^1(\Omega))')$ such that

$$\frac{\partial U_{i,\varepsilon}}{\partial t} \rightharpoonup \widetilde{\eta} \qquad \text{in } L^2(0, T, (H^1(\Omega))').$$

A well known argument can be easily adapted to show that $\widetilde{\eta} = \frac{\partial U_i}{\partial t}$, (see Robinson [84], page 204). Thus, the result (3.3.43) holds.

Note that from (3.3.41) and (3.3.42) we have $U_i \in L^2(0,T;H^1(\Omega)) \cap L^\alpha(\Omega_T) \cap L^\infty(0,T;L^2(\Omega))$, thus to prove (3.3.39) we need to prove that $U_i \in H^1(0,T;(H^1(\Omega))')$. From the embedding $L^2(0,T;H^1(\Omega)) \hookrightarrow L^2(0,T;(H^1(\Omega))')$, we conclude that $U_i \in L^2(0,T;(H^1(\Omega))')$, and from (3.3.43) we have that $\frac{\partial U_i}{\partial t} \in L^2(0,T;(H^1(\Omega))')$, thus

$$\|U_i\|_{H^1(0,T;(H^1(\Omega))')} = \|U_i\|_{L^2(0,T;(H^1(\Omega))')} + \|\frac{\partial U_i}{\partial t}\|_{L^2(0,T;(H^1(\Omega))')} \leq C.$$

Thus, (3.3.39) has been proved.

From an application of the Lions-Aubin theorem, see (2.1.6), on noting the following embedding results

$$H^1(\Omega) \overset{c}{\hookrightarrow} L^s(\Omega) \hookrightarrow (H^1(\Omega))',$$

which hold from the Rellich-Kondrachov theorem under the stated choice of s, we find that

$$W_u = \{\eta : \eta \in L^2(0,T;H^1(\Omega)), \frac{\partial \eta}{\partial t} \in L^2(0,T;(H^1(\Omega))'\} \overset{c}{\hookrightarrow} L^2(0,T;L^s(\Omega)).$$

As $U_{i,\varepsilon} \in L^2(0,T;H^1(\Omega))$ and $\frac{\partial U_{i,\varepsilon}}{\partial t} \in L^2(0,T;(H^1(\Omega))')$, thus, $U_{i,\varepsilon} \in W_u$, then we can extract a subsequence, still denoted U_i, such that the convergence result (3.3.44) holds.

Using the strong convergence of $U_{i,\varepsilon}$ to U_i in $L^2(0,T;L^s(\Omega))$ and the fourth bound in (3.3.48), we can extract a subsequence, still denoted $U_{i,\varepsilon}$, such that as $h \to 0$ (see Appendix A.1.17)

$$U_{i,\varepsilon} \to U_i \qquad \text{and} \qquad \pi^h[U_{i,\varepsilon}]_- \to 0 \qquad a.e. \text{ in } \Omega \times (0,T). \tag{3.3.52}$$

But we have from the definition of π^h that

$$U_{i,\varepsilon} = \pi^h[U_{i,\varepsilon}]_+ + \pi^h[U_{i,\varepsilon}]_-. \tag{3.3.53}$$

Therefore, we deduce from (3.3.52) and (3.3.53) that $U_i \geq 0$ almost everywhere.

Noting (2.2.15), (2.3.28), the non-negativity of the function U_i and the assumption (iv) yields that

$$\|\phi_\varepsilon(U_i^\pm) - \phi(U_i^\pm)\|_{L^2(0,T;L^s(\Omega))} \leq C\varepsilon \to 0 \quad \text{as} \quad h \to 0. \tag{3.3.54}$$

With the aid of the Lipschitz continuity of the function ϕ_ε and (3.3.44) we have

$$\|\phi_\varepsilon(U_{i,\varepsilon}^\pm) - \phi_\varepsilon(U_i^\pm)\|_{L^2(0,T;L^s(\Omega))} \leq \|U_{i,\varepsilon}^\pm - U_i^\pm\|_{L^2(0,T;L^s(\Omega))} \to 0 \quad \text{as} \quad h \to 0. \tag{3.3.55}$$

Therefore, in order to establish (3.3.45) we find that

$$\begin{aligned}
&\|\phi_\varepsilon(U_{i,\varepsilon}^\pm) - \phi(U_i^\pm)\|_{L^2(0,T;L^s(\Omega))} \\
&\leq \|\phi_\varepsilon(U_{i,\varepsilon}^\pm) - \phi_\varepsilon U_i^\pm)\|_{L^2(0,T;L^s(\Omega))} + \|\phi_\varepsilon(U_i^\pm) - \phi(U_i^\pm)\|_{L^2(0,T;L^s(\Omega))} \\
&\to 0 \quad \text{as} \quad h \to 0.
\end{aligned} \tag{3.3.56}$$

Next, employ (2.4.56), (2.4.52), (2.4.55) and the first bound in (3.3.48) to see that

$$\begin{aligned}
\|(I - \pi^h)\phi_\varepsilon(U_{i,\varepsilon}^\pm)\|_{L^2(0,T;L^s(\Omega))} &\leq Ch\|\nabla\phi_\varepsilon(U_{i,\varepsilon}^\pm)\|_{L^2(0,T;L^s(\Omega))} \\
&\leq Ch\|\nabla U_{i,\varepsilon}^\pm\|_{L^2(0,T;L^s(\Omega))} \\
&\leq Ch^{1-d(\frac{1}{2}-\frac{1}{s})}\|U_{i,\varepsilon}^\pm\|_{L^2(0,T;H^1(\Omega))} \\
&\leq Ch^{1-d(\frac{1}{2}-\frac{1}{s})} \quad \to 0 \quad \text{as} \quad h \to 0.
\end{aligned} \tag{3.3.57}$$

Next, use (2.4.70), (2.4.55), the first bound in (3.3.48) and (3.3.45) to derive

$$\begin{aligned}
&\|\Lambda_\varepsilon(U_{i,\varepsilon}^\pm) - \phi(U_i^\pm)\mathcal{I}\|_{L^2(0,T;L^s(\Omega))} \\
&= \|\Lambda_\varepsilon(U_{i,\varepsilon}^\pm) - \phi_\varepsilon(U_{i,\varepsilon}^\pm)\mathcal{I} + \phi_\varepsilon(U_{i,\varepsilon}^\pm)\mathcal{I} - \phi(U_i^\pm)\mathcal{I}\|_{L^2(0,T;L^s(\Omega))} \\
&\leq \|\Lambda_\varepsilon(U_{i,\varepsilon}^\pm) - \phi_\varepsilon(U_{i,\varepsilon}^\pm)\mathcal{I}\|_{L^2(0,T;L^s(\Omega))} + \|\phi_\varepsilon(U_{i,\varepsilon}^\pm) - \phi(U_i^\pm)\|_{L^2(0,T;L^s(\Omega))} \\
&\leq h\|\nabla U_{i,\varepsilon}^\pm\|_{L^2(0,T;L^s(\Omega))} + \|\phi_\varepsilon(U_{i,\varepsilon}^\pm) - \phi(U_i^\pm)\|_{L^2(0,T;L^s(\Omega))} \\
&\leq Ch^{1-d(\frac{1}{2}-\frac{1}{s})}\|U_{i,\varepsilon}^\pm\|_{L^2(0,T;H^1(\Omega))} + \|\phi_\varepsilon(U_{i,\varepsilon}^\pm) - \phi(U_i^\pm)\|_{L^2(0,T;L^s(\Omega))} \\
&\leq Ch^{1-d(\frac{1}{2}-\frac{1}{s})} + \|\phi_\varepsilon(U_{i,\varepsilon}^\pm) - \phi(U_i^\pm)\|_{L^2(0,T;L^s(\Omega))} \to 0 \quad \text{as} \quad h \to 0.
\end{aligned} \tag{3.3.58}$$

Hence the result (3.3.47) holds from (3.3.58).

To complete the proof of the theorem, we still have to deal with the initial approximations and show that the solution $\{u_i\}_{i=1}^m$ satisfies (3.3.40). We first note from the error estimates (3.1.3) and (2.4.56) and the stated assumptions on the initial data, $\{u_i^0\}_{i=1}^m$, that for $i = 1, ..., m$

$$\|u_i^0 - P^h u_i^0\|_0 \leq Ch|u_i^0|_1 \leq Ch,$$

and

$$\|u_i^0 - \pi^h u_i^0\|_0 \leq \begin{cases} Ch|u_i^0|_1 \leq Ch & \text{for } d = 1, \\ Ch|u_i^0|_{1,r} \leq Ch & \text{for } d = 2 \text{ or } 3, \end{cases}$$

which provide the following strong convergence results as $h \to 0$

$$U_i^0 \to u_i^0 \quad \text{in } L^2(\Omega), \quad i = 1, ..., m. \tag{3.3.59}$$

It follows from (3.3.43) and (3.3.44) that for *a.e.* (see Theorem A.1.11)

$$U_i(t) \to u_i(t) \quad \text{in } L^2(\Omega) \quad \text{as } \Delta t \to 0, \quad i = 1, ..., m. \tag{3.3.60}$$

We comment that (3.3.59) and (3.3.60) are not sufficient to prove the equalities in (3.3.40) since if $t = 0$ belongs to the null-set of the almost everywhere statement for (3.3.60) then possibly $u_i(0) \neq u_i^0, i = 1, ..., m$ (see Robinson [58], Section 7.4.4, for further discussion). In addition to (3.3.59) and (3.3.60), we actually exploit other properties of the solutions $\{U_i\}_{i=1}^m$ and the functions $\{u_i\}_{i=1}^m$ in order to conclude that (3.3.40) holds.

We note that since

$$U_i, u_i \in L^2(0, T; H^1(\Omega)) \quad \text{and} \quad \frac{\partial U_i}{\partial t}, \frac{\partial u_i}{\partial t} \in L^2(0, T; (H^1(\Omega))'), \quad i = 1, ..., m,$$

it follows that

$$U_i, u_i \in C([0, T]; L^2(\Omega)), \quad i = 1, ..., m, \tag{3.3.61}$$

see Theorem 7.2 and Proposition 7.1 in Robinson [84], respectively. Therefore, the desired result (3.3.40) follows easily by combining (3.3.59), (3.3.60) and (3.3.61).

□

Lemma 3.3.2 Let the assumptions of Theorem 3.3.1 hold. Then the following convergence results are valid as $h \to 0$:

$$\pi^h \phi_\varepsilon(U_{i,\varepsilon}^+)\, \pi^h \phi_\varepsilon(U_{i,\varepsilon}^-) \to \phi(U_i^+)\phi(U_i^-) \qquad \text{in } L^2(\Omega_T), \quad i = 1, ..., m. \tag{3.3.62}$$

Proof: From (2.2.15), (3.3.48), the Hölder's inequality and the embedding result $L^2(0,T;L^s(\Omega)) \hookrightarrow L^2(\Omega_T)$ one shows

$$\begin{aligned}
&\|\pi^h\phi_\varepsilon(U^+_{i,\varepsilon})\ \pi^h\phi_\varepsilon(U^-_{i,\varepsilon}) - \ \phi(U^+_i)\phi(U^-_i)\|_{L^2(\Omega_T)} \\
&\le \|\pi^h\phi_\varepsilon(U^+_{i,\varepsilon})\ \pi^h\phi_\varepsilon(U^-_{i,\varepsilon}) - \pi^h\phi_\varepsilon(U^+_{i,\varepsilon})\ \phi(U^-_i)\|_{L^2(\Omega_T)} \\
&\quad + \|\pi^h\phi_\varepsilon(U^+_{i,\varepsilon})\ \phi(U^-_i) - \phi(U^+_i)\phi(U^-_i)\|_{L^2(\Omega_T)} \\
&\le \|\pi^h\phi_\varepsilon(U^-_{i,\varepsilon}) - \ \phi(U^-_i)\|_{L^2(\Omega_T)}\|\pi^h\phi_\varepsilon(U^+_{i,\varepsilon})\|_{L^\infty(\Omega_T)} \\
&\quad + \|\pi^h\phi_\varepsilon(U^+_{i,\varepsilon}) \ - \phi(U^+_i)\ \|_{L^2(\Omega_T)}\|\phi(U^-_i)\|_{L^\infty(\Omega_T)} \\
&\le C(\|\pi^h\phi_\varepsilon(U^-_{i,\varepsilon}) - \ \phi(U^-_i)\|_{L^2(\Omega_T)} + \|\pi^h\phi_\varepsilon(U^+_{i,\varepsilon}) \ - \phi(U^+_i)\ \|_{L^2(\Omega_T)}) \\
&\le C(\|\pi^h\phi_\varepsilon(U^-_{i,\varepsilon}) - \ \phi(U^-_i)\|_{L^2(0,T,L^s(\Omega)} + \|\pi^h\phi_\varepsilon(U^+_{i,\varepsilon}) \ - \phi(U^+_i)\ \|_{L^2(0,T,L^s(\Omega))}) \\
&\to 0 \quad \text{as} \quad h \to 0, \qquad\qquad i = 1, ..., m. \qquad (3.3.63)
\end{aligned}$$

□

Theorem 3.3.3 Let the assumptions of Theorem 3.3.1 hold. Then there exists a subsequence of $\{U_{i,\varepsilon}\}_{h>0}, i = 1, ..., m$, where $\{U_{i,\varepsilon}\}, i = 1, ..., m$ solves (3.3.38), and nonnegative functions $\{U_i\}, i = 1, ..., m$ satisfying (3.3.39). In addition, as $h \to 0$ the convergence results (3.3.41)-(3.3.47) and (3.3.62) hold. Furthermore, the functions $\{U_i\}, i = 1, ..., m$ represent a global weak solution of the problem ($\mathrm{P}^{\Delta t}_M$) in the sense that

$$\begin{aligned}
&\int_0^T [\langle \frac{\partial U_i}{\partial t}, \eta\rangle + D_i(\nabla U^+_i, \nabla\eta) + (\phi(U^+_i)\sum_{j=1}^m \nabla U^+_j, \nabla\eta)]dt \\
&= \int_0^T [(\gamma_i U^+_i - \phi(U^+_i)(\sum_{j=1}^m \phi(U^-_j), \eta)]dt, \quad \forall\eta \in L^2(0,T;H^1(\Omega)), \quad i = 1, ..., m.
\end{aligned} \qquad (3.3.64)$$

Proof: The first and second parts of the theorem follow from Theorem 3.3.1. To show that $\{U_i\}^m_{i=1}$ is a weak solution of ($\mathrm{P}^{\Delta t}_M$) in sense that (3.3.64) are satisfied, we set $\chi \equiv \pi^h\eta$ as a test function in (3.3.38) and then pass to the limit $\varepsilon, h \to 0$.

For any $\eta \in L^2(0,T;H^1(\Omega))$, we set $\chi \equiv \pi^h\eta$ as a test function in (3.3.38) yielding

$$\int_0^T [(\frac{\partial U_{i,\varepsilon}}{\partial t}, \pi^h\eta)^h + D_i(\nabla U^+_{i,\varepsilon}, \nabla\pi^h\eta) + (\Lambda_\varepsilon(U^+_{i,\varepsilon})\sum_{j=1}^m \nabla U^+_{j,\varepsilon}, \nabla\pi^h\eta)]dt$$

$$= \int_0^T [(\gamma_i U_{i,\varepsilon}^+ - \phi_\varepsilon(U_{i,\varepsilon}^+)(\sum_{j=1}^m \phi_\varepsilon(U_{j,\varepsilon}^-), \pi^h \eta)^h] d\,t, \qquad i = 1, ..., m. \tag{3.3.65}$$

We shall now study the convergence of each term in (3.3.65) separately. For all $\eta \in L^2(0,T;H^1(\Omega))$ and for all $\widetilde{\eta} \in H^1(0,T;H^1(\Omega))$ we have that

$$\begin{aligned}
\int_0^T (\frac{\partial U_{i,\varepsilon}}{\partial t}, \pi^h \eta)^h = & \int_0^T \big[(\frac{\partial U_{i,\varepsilon}}{\partial t}, \pi^h[\eta - \widetilde{\eta}])^h - (\frac{\partial U_{i,\varepsilon}}{\partial t}, \pi^h[\eta - \widetilde{\eta}])\big] dt \\
& + \int_0^T \big[(\frac{\partial U_{i,\varepsilon}}{\partial t}, \pi^h \widetilde{\eta})^h - (\frac{\partial U_{i,\varepsilon}}{\partial t}, \pi^h \widetilde{\eta})\big] dt \\
& + \int_0^T \big(\frac{\partial U_{i,\varepsilon}}{\partial t}, (\pi^h - I)\eta\big) dt \\
& + \int_0^T \big(\frac{\partial U_{i,\varepsilon}}{\partial t}, \eta\big) dt \\
:= & \, I_{1,1} + I_{1,2} + I_{1,3} + I_{1,4}.
\end{aligned} \tag{3.3.66}$$

Then from (2.4.59), (3.1.11), (2.4.56), Hölder's inequality, the denseness of $H^1(0,T;H^1(\Omega))$ in $L^2(0,T;H^1(\Omega))$ and (3.3.48) we may derive

$$\begin{aligned}
|I_{1,1}| \equiv & \Big| \int_0^T \big[(\frac{\partial U_{i,\varepsilon}}{\partial t}, \pi^h[\eta - \widetilde{\eta}])^h - (\frac{\partial U_{i,\varepsilon}}{\partial t}, \pi^h[\eta - \widetilde{\eta}])\big] dt \Big| \\
\leq & \, Ch \int_0^T \|\frac{\partial U_{i,\varepsilon}}{\partial t}\|_0 \; |\pi^h[\eta - \widetilde{\eta}]|_1 dt \\
\leq & \, C \int_0^T \|\mathcal{G}\frac{\partial U_{i,\varepsilon}}{\partial t}\|_1 \; |\eta - \widetilde{\eta} + (\pi^h - I)\eta - (\pi^h - I)\widetilde{\eta}|_1 dt \\
\leq & \, C \int_0^T \|\mathcal{G}\frac{\partial U_{i,\varepsilon}}{\partial t}\|_1 \; \big[\|\eta - \widetilde{\eta}\|_1 + \|(\pi^h - I)\eta\|_1 + \|(\pi^h - I)\widetilde{\eta}\|_1\big] dt \\
\leq & \, C\|\mathcal{G}\frac{\partial U_{i,\varepsilon}}{\partial t}\|_{L^2(0,T;H^1(\Omega))} \; \big[\|\eta - \widetilde{\eta}\|_{L^2(0,T;H^1(\Omega))} + \|(\pi^h - I)\eta\|_{L^2(0,T;H^1(\Omega))} + \|(\pi^h - I)\widetilde{\eta}\|_{L^2(0,T;H^1(\Omega))}\big] \\
\leq & \, C\big[\|\eta - \widetilde{\eta}\|_{L^2(0,T;H^1(\Omega))} + \|(\pi^h - I)\eta\|_{L^2(0,T;H^1(\Omega))} + \|(\pi^h - I)\widetilde{\eta}\|_{L^2(0,T;H^1(\Omega))}\big] \\
\rightarrow & \quad C\|\eta - \widetilde{\eta}\|_{L^2(0,T;H^1(\Omega))} \quad \text{as} \quad h \rightarrow 0.
\end{aligned} \tag{3.3.67}$$

We now see from (2.4.59), (2.4.56), Hölder's inequality and (3.3.48) that

$$\begin{aligned}
|I_{1,2}| \equiv & \Big| \int_0^T \big[(\frac{\partial U_{i,\varepsilon}}{\partial t}, \pi^h \widetilde{\eta})^h - (\frac{\partial U_{i,\varepsilon}}{\partial t}, \pi^h \widetilde{\eta})\big] dt \Big| \\
\leq & \Big| \int_0^T \big[(U_{i,\varepsilon}, \frac{\partial \pi^h \widetilde{\eta}}{\partial t})^h - (U_{i,\varepsilon}, \frac{\partial \pi^h \widetilde{\eta}}{\partial t})\big] dt \Big| \\
& + \big| (U_{i,\varepsilon}(\cdot, T), \pi^h \widetilde{\eta}(\cdot, T))^h - (U_{i,\varepsilon}(\cdot, T), \pi^h \widetilde{\eta}(\cdot, T)) \big|
\end{aligned}$$

$$+\big|(U_{i,\varepsilon}(\cdot,0),\pi^h\widetilde{\eta}(\cdot,0))^h-(U_{i,\varepsilon}(\cdot,0),\pi^h\widetilde{\eta}(\cdot,0))\big|$$
$$\le Ch\int_0^T \|U_{i,\varepsilon}\|_0|\frac{\partial\pi^h\widetilde{\eta}}{\partial t}|_1dt+Ch\|U_{i,\varepsilon}(\cdot,T)\|_0|\pi^h\widetilde{\eta}(\cdot,T)|_1+Ch\|U_{i,\varepsilon}(\cdot,0)\|_0|\pi^h\widetilde{\eta}(\cdot,0)|_1$$
$$\le Ch\|U_{i,\varepsilon}\|_{L^\infty(0,T,L^2(\Omega))}\|\pi^h\widetilde{\eta}\|_{H^1(0,T,H^1(\Omega))}+Ch\|U_{i,\varepsilon}(\cdot,T)\|_0|\pi^h\widetilde{\eta}(\cdot,T)|_1$$
$$+Ch\|U_{i,\varepsilon}(\cdot,0)\|_0|\pi^h\widetilde{\eta}(\cdot,0)|_1$$
$$\le Ch[\|\widetilde{\eta}\|_{H^1(0,T,H^1(\Omega))}+\|(\pi^h-I)\widetilde{\eta}\|_{H^1(0,T,H^1(\Omega))}]$$
$$+Ch\|U_{i,\varepsilon}(\cdot,T)\|_0|\pi^h\widetilde{\eta}(\cdot,T)|_1+Ch\|U_{i,\varepsilon}(\cdot,0)\|_0|\pi^h\widetilde{\eta}(\cdot,0)|_1 \quad\to\quad 0 \quad\text{as}\quad h\to 0, \tag{3.3.68}$$

where the fourth inequality was obtained from (2.4.56) and exploiting the continuous embedding (see Robinson [84] page 190):

$$W^{1,p}(0,T;X)\hookrightarrow C([0,T];X) \qquad 1\le p\le\infty,$$

namely,

$$\sup_{t\in[0,T]}\|\zeta(t)\|_X\le\|\zeta\|_{W^{1,p}(0,T;X)} \qquad \text{for}\quad \zeta\in W^{1,p}(0,T;X). \tag{3.3.69}$$

To treat the term $I_{1,3}$, we observe using (3.1.8), Hölder's inequality and the fifth bound in (3.3.48) that

$$|I_{1,3}|=\big|\int_0^T\big(\frac{\partial U_{i,\varepsilon}}{\partial t},(\pi^h-I)\eta\big)dt\big|=\big|\int_0^T\big\langle\frac{\partial U_{i,\varepsilon}}{\partial t},(\pi^h-I)\eta\big\rangle dt\big|$$
$$\le\int_0^T\big|\big\langle\frac{\partial U_{i,\varepsilon}}{\partial t},(\pi^h-I)\eta\big\rangle\big|dt$$
$$\le\int_0^T|\frac{\partial U_{i,\varepsilon}}{\partial t}|_{(H^1(\Omega))'}|(\pi^h-I)\eta|_1dt$$
$$\le\|\frac{\partial U_{i,\varepsilon}}{\partial t}\|_{L^2(0,T;(H^1(\Omega))')}\|(\pi^h-I)\eta\|_{L^2(0,T;H^1(\Omega))}$$
$$\le C\|(\pi^h-I)\eta\|_{L^2(0,T;H^1(\Omega))} \quad\to\quad 0 \quad\text{as}\quad h\to 0. \tag{3.3.70}$$

Next we use (3.1.8) and the weak convergence result (3.3.43) to arrive for all $\eta\in L^2(0,T;H^1(\Omega))$,

$$I_{1,4}\equiv\int_0^T\big(\frac{\partial U_{i,\varepsilon}}{\partial t},\eta\big)dt=\int_0^T\big\langle\frac{\partial U_{i,\varepsilon}}{\partial t},\eta\big\rangle dt\to\int_0^T\big\langle\frac{\partial U_i}{\partial t},\eta\big\rangle dt \quad\text{as}\quad h\to 0. \tag{3.3.71}$$

Combining (3.3.66)-(3.3.68), (3.3.70), (3.3.71), (2.4.57) and the denseness of $H^1(0,T;H^1(\Omega))$ in $L^2(0,T;H^1(\Omega))$ one then obtains for all $\eta \in L^2(0,T;H^1(\Omega))$

$$\int_0^T \big(\frac{\partial U_{i,\varepsilon}}{\partial t}, \pi^h \eta\big)^h \to \int_0^T \langle \frac{\partial U_i}{\partial t}, \eta\rangle dt \quad \text{as} \quad h \to 0. \tag{3.3.72}$$

We employ Hölder's inequality, (3.3.48) and (2.4.57) to now see for all $\eta \in L^2(0,T;H^1(\Omega))$ that

$$\begin{aligned} \Big|\int_0^T (\nabla U^+_{i,\varepsilon}, \nabla(\pi^h - I)\eta) dt\Big| &\leq \int_0^T \big|(\nabla U^+_{i,\varepsilon}, \nabla(\pi^h - I)\eta)\big| dt \\ &\leq \int_0^T |U^+_{i,\varepsilon}|_1 \quad |(\pi^h - I)\eta|_1 dt \\ &\leq \|U^+_{i,\varepsilon}\|_{L^2(0,T,H^1(\Omega))} \quad \|(\pi^h - I)\eta\|_{L^2(0,T,H^1(\Omega))} \\ &\leq C\, \|(\pi^h - I)\eta\|_{L^2(0,T,H^1(\Omega))} \\ &\to 0 \quad \text{as} \quad h \to 0. \end{aligned} \tag{3.3.73}$$

It follows from (3.3.73) and (3.3.41) for all $\eta \in L^2(0,T;H^1(\Omega))$ that

$$\begin{aligned} \int_0^T (\nabla U^+_{i,\varepsilon}, \nabla \pi^h \eta) dt &= \int_0^T (\nabla U^+_{i,\varepsilon}, \nabla(\pi^h - I)\eta) dt + \int_0^T (\nabla U^+_{i,\varepsilon}, \nabla \eta) dt \\ &\to \int_0^T (\nabla U^+_i, \nabla \eta) dt \quad \text{as} \quad h \to 0. \end{aligned} \tag{3.3.74}$$

We obtain for all $\eta \in L^2(0,T;H^1(\Omega))$ and for all $\widetilde{\eta} \in H^1(0,T;H^1(\Omega))$ that

$$\begin{aligned} &\int_0^T (\Lambda_\varepsilon(U^+_{i,\varepsilon}) \sum_{j=1}^m \nabla U^+_{j,\varepsilon}, \nabla \pi^h \eta) d\,t \\ &= \int_0^T (\Lambda_\varepsilon(U^+_{i,\varepsilon}) \sum_{j=1}^m \nabla U^+_{j,\varepsilon}, \nabla(\pi^h - I)\eta) d\,t \\ &+ \int_0^T ([\Lambda_\varepsilon(U^+_{i,\varepsilon}) - \phi(U^+_i)\mathcal{I}] \sum_{j=1}^m \nabla U^+_{j,\varepsilon}, \nabla(\eta - \widetilde{\eta})) d\,t \\ &+ \int_0^T ([\Lambda_\varepsilon(U^+_{i,\varepsilon}) - \phi(U^+_i)\mathcal{I}] \sum_{j=1}^m \nabla U^+_{j,\varepsilon}, \nabla \widetilde{\eta}) d\,t \\ &+ \int_0^T (\phi(U^+_i) \sum_{j=1}^m \nabla U^+_{j,\varepsilon}, \nabla \eta) d\,t \\ &:= I_{2,1} + I_{2,2} + I_{2,3} + I_{2,4}. \end{aligned} \tag{3.3.75}$$

Now, the generalized Hölder's inequality and (3.3.48) are used to find

$$
\begin{aligned}
|I_{2,1}| &\equiv \Big| \int_0^T (\Lambda_\varepsilon(U_{i,\varepsilon}^+) \sum_{j=1}^m \nabla U_{j,\varepsilon}^+, \nabla(\pi^h - I)\eta) d\,t \Big| \\
&\leq \int_0^T \|\Lambda_\varepsilon(U_{i,\varepsilon}^+)\|_\infty \ \sum_{j=1}^m |U_{j,\varepsilon}^+|_1 \ |(\pi^h - I)\eta|_1 d\,t \\
&\leq \|\Lambda_\varepsilon(U_{i,\varepsilon}^+)\|_{L^\infty(\Omega_T)} \ \sum_{j=1}^m \|U_{j,\varepsilon}^+\|_{L^2(0,T,H^1(\Omega))} \ \|(\pi^h - I)\eta\|_{L^2(0,T,H^1(\Omega))} \\
&\leq C \, \|(\pi^h - I)\eta\|_{L^2(0,T,H^1(\Omega))} \to 0 \quad \text{as} \quad h \to 0. \qquad (3.3.76)
\end{aligned}
$$

Similarly to the treatment of the term $I_{2,1}$, the generalized Hölder's inequality, (3.3.48), the denseness of the space $H^1(0,T;H^1(\Omega))$ in $L^2(0,T;H^1(\Omega))$ and (2.2.15) are employed to see that

$$
\begin{aligned}
|I_{2,2}| &\equiv \Big| \int_0^T ([\Lambda_\varepsilon(U_{i,\varepsilon}^+) - \phi(U_i^+)\mathcal{I}] \sum_{j=1}^m \nabla U_{j,\varepsilon}^+, \nabla(\eta - \widetilde{\eta})) d\,t \Big| \\
&\leq \|\Lambda_\varepsilon(U_{i,\varepsilon}^+) - \phi(U_i^+)\mathcal{I}\|_{L^\infty(\Omega_T)} \ \sum_{j=1}^m \|U_{j,\varepsilon}^+\|_{L^2(0,T,H^1(\Omega))} \ \|\eta - \widetilde{\eta}\|_{L^2(0,T,H^1(\Omega))} \\
&\leq C \ \|\eta - \widetilde{\eta}\|_{L^2(0,T,H^1(\Omega))} \to 0 \quad \text{as} \quad h \to 0. \qquad (3.3.77)
\end{aligned}
$$

In addition, we have that

$$
\begin{aligned}
|I_{2,3}| &\equiv \Big| \int_0^T ([\Lambda_\varepsilon(U_{i,\varepsilon}^+) - \phi(U_i^+)\mathcal{I}] \sum_{j=1}^m \nabla U_{j,\varepsilon}^+, \nabla\widetilde{\eta}) d\,t \Big| \\
&\leq \|\Lambda_\varepsilon(U_{i,\varepsilon}^+) - \phi(U_i^+)\mathcal{I}\|_{L^2(\Omega_T))} \ \sum_{j=1}^m \|U_{j,\varepsilon}^+\|_{L^2(0,T,H^1(\Omega))} \ \|\nabla\widetilde{\eta}\|_{L^\infty(\Omega_T))} \\
&\leq C \|\Lambda_\varepsilon(U_{i,\varepsilon}^+) - \phi(U_i^+)\mathcal{I}\|_{L^2(\Omega_T)} \ \|\widetilde{\eta}\|_{L^\infty(0,T;W^{1,\infty}(\Omega)} \\
&\leq C \|\Lambda_\varepsilon(U_{i,\varepsilon}^+) - \phi(U_i^+)\mathcal{I}\|_{L^2(0,T,L^s(\Omega)} \ \to 0 \quad \text{as} \quad h \to 0. \qquad (3.3.78)
\end{aligned}
$$

It follows from (3.3.41) for all $\eta \in L^2(0,T;H^1(\Omega))$ that

$$
I_{2,4} \equiv \int_0^T (\phi(U_i^+) \sum_{j=1}^m \nabla U_{j,\varepsilon}^+, \nabla\eta) d\,t \to \int_0^T (\phi(U_i^+) \sum_{j=1}^m \nabla U_i^+, \nabla\eta) d\,t \quad \text{as} \quad h \to 0, \qquad (3.3.79)
$$

where we used the fact that the function $\phi(s)$ is bounded. Now, combining (3.3.75)-(3.3.79), (2.4.57) and (3.3.47) leads for all $\eta \in L^2(0,T;H^1(\Omega))$ that

$$\int_0^T (\Lambda_\varepsilon(U_{i,\varepsilon}^+)\sum_{j=1}^m \nabla U_{j,\varepsilon}^+, \nabla\pi^h\eta)d\,t \to \int_0^T (\phi(U_i^+)\sum_{j=1}^m \nabla U_i^+, \nabla\eta)d\,t \quad \text{as} \quad h \to 0, \tag{3.3.80}$$

It remains to show the convergence of the reaction term in (3.3.65). On noting (2.4.59), Hölder's inequality, (2.4.56) and (3.3.48) yields for all $\eta \in L^2(0,T;H^1(\Omega))$ that

$$\begin{aligned}
&\Big|\int_0^T [(U_{i,\varepsilon}^+, \pi^h\eta)^h - (U_{i,\varepsilon}^+, \pi^h\eta)]d\,t + \int_0^T (U_{i,\varepsilon}^+, (\pi^h - I)\eta)d\,t\Big| \\
&\leq \Big|\int_0^T [(U_{i,\varepsilon}^+, \pi^h\eta)^h - (U_{i,\varepsilon}^+, \pi^h\eta)]d\,t\Big| + \Big|\int_0^T (U_{i,\varepsilon}^+, (\pi^h - I)\eta)d\,t\Big| \\
&\leq Ch\int_0^T \|U_{i,\varepsilon}^+\|_0|\pi^h\eta|_1 d\,t + \int_0^T \|U_{i,\varepsilon}^+\|_0\|(\pi^h - I)\eta\|_0 d\,t \\
&\leq Ch\int_0^T \|U_{i,\varepsilon}^+\|_0|(\pi^h - I)\eta + \eta|_1 d\,t + Ch\int_0^T \|U_{i,\varepsilon}^+\|_0|\eta|_1 d\,t \\
&\leq Ch\int_0^T \|U_{i,\varepsilon}^+\|_0|\eta|_1 d\,t \\
&\leq Ch\|U_{i,\varepsilon}^+\|_{L^2(\Omega_T))}\|\eta\|_{L^2(0,T,H^1(\Omega))} \\
&\leq Ch\|U_{i,\varepsilon}^+\|_{L^\alpha(\Omega_T))}\|\eta\|_{L^2(0,T,H^1(\Omega))} \\
&\leq Ch\|\eta\|_{L^2(0,T,H^1(\Omega))} \to 0 \quad \text{as} \quad h \to 0.
\end{aligned} \tag{3.3.81}$$

Combining (3.3.81) and (3.3.41) leads for all $\eta \in L^2(0,T;H^1(\Omega))$ that

$$\begin{aligned}
\int_0^T (U_{i,\varepsilon}^+, \pi^h\eta)^h d\,t &= \int_0^T [(U_{i,\varepsilon}^+, \pi^h\eta)^h - (U_{i,\varepsilon}^+, \pi^h\eta)]d\,t + \int_0^T (U_{i,\varepsilon}^+, (\pi^h - I)\eta)d\,t \\
&+ \int_0^T (U_{i,\varepsilon}^+, \eta)d\,t \to \int_0^T (U_i^+, \eta)d\,t \quad \text{as} \quad h \to 0.
\end{aligned} \tag{3.3.82}$$

Now, we deal with the convergence of the non-linear reaction terms in (3.3.65). Firstly, it follows from (2.4.45) for all $\eta \in L^2(0,T;H^1(\Omega))$ that

$$\begin{aligned}
\int_0^T (\phi_\varepsilon(U_{i,\varepsilon}^+)\phi_\varepsilon(U_{j,\varepsilon}^-), \pi^h\eta)^h d\,t &= \int_0^T (\pi^h\phi_\varepsilon(U_{i,\varepsilon}^+)\pi^h\phi_\varepsilon(U_{j,\varepsilon}^-), \pi^h\eta)^h d\,t \\
= \int_0^T [(\pi^h\phi_\varepsilon(U_{i,\varepsilon}^+)\pi^h\phi_\varepsilon(U_{j,\varepsilon}^-), \pi^h\eta)^h &- (\pi^h\phi_\varepsilon(U_{i,\varepsilon}^+)\pi^h\phi_\varepsilon(U_{j,\varepsilon}^-), \pi^h\eta)]d\,t \\
+ \int_0^T (\pi^h\phi_\varepsilon(U_{i,\varepsilon}^+)\pi^h\phi_\varepsilon(U_{j,\varepsilon}^-) &- \phi(U_i^+)\,\phi(U_j^-), \pi^h[\eta - \widetilde{\eta}])d\,t
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^T (\pi^h\phi_\varepsilon(U_{i,\varepsilon}^+)\pi^h\phi_\varepsilon(U_{j,\varepsilon}^-) - \phi(U_i^+)\,\phi(U_j^-), \pi^h\widetilde{\eta})d\,t \\
&\quad+\int_0^T (\phi(U_i^+)\,\phi(U_j^-), (\pi^h - I)\eta)d\,t \\
&\quad\quad+\int_0^T (\phi(U_i^+)\,\phi(U_j^-), \eta)d\,t \\
&:= I_{3,1} + I_{3,2} + I_{3,3} + I_{3,4} + I_{3,5}. \qquad (3.3.83)
\end{aligned}
$$

Using (2.4.60), (2.4.54), (2.4.52), Hölder's inequality, (2.4.56), (3.3.48) gives that

$$
\begin{aligned}
|I_{3,1}| &\equiv \Big|\int_0^T \big[(\pi^h\phi_\varepsilon(U_{i,\varepsilon}^+)\pi^h\phi_\varepsilon(U_{j,\varepsilon}^-), \pi^h\eta)^h - (\pi^h\phi_\varepsilon(U_{i,\varepsilon}^+)\pi^h\phi_\varepsilon(U_{j,\varepsilon}^-), \pi^h\eta)\big]d\,t\Big| \\
&\le Ch^2\int_0^T \|\pi^h\phi_\varepsilon(U_{i,\varepsilon}^+)\|_{1,\infty}\|\pi^h\phi_\varepsilon(U_{j,\varepsilon}^-)\|_1\|\pi^h\eta\|_1 d\,t \\
&\le Ch\int_0^T \|\pi^h\phi_\varepsilon(U_{i,\varepsilon}^+)\|_{0,\infty}\big[\|\pi^h\phi_\varepsilon(U_{j,\varepsilon}^-)\|_0 + |\pi^h\phi_\varepsilon(U_{j,\varepsilon}^-)|_1\big]\,\|\pi^h\eta\|_1 d\,t \\
&\le Ch\|\pi^h\phi_\varepsilon(U_{i,\varepsilon}^+)\|_{L^\infty(\Omega_T)}\big[\|\pi^h\phi_\varepsilon(U_{j,\varepsilon}^-)\|_{L^2(\Omega_T)} + \|\pi^h\phi_\varepsilon(U_{j,\varepsilon}^-)\|_{L^2(0,T;H^1(\Omega))}\big]\,\|\pi^h\eta\|_{L^2(0,T;H^1(\Omega))} \\
&\le Ch\big[\|\pi^h\phi_\varepsilon(U_{j,\varepsilon}^-)\|_{L^2(\Omega_T)} + \|U_{j,\varepsilon}^-\|_{L^2(0,T;H^1(\Omega)}\big]\,\|(\pi^h - I)\eta + \eta\|_{L^2(0,T;H^1(\Omega))} \\
&\le Ch\|\eta\|_{L^2(0,T;H^1(\Omega))} \\
&\to 0 \quad \text{as} \quad h \to 0. \qquad (3.3.84)
\end{aligned}
$$

Using Hölder's inequality, (2.4.56), (2.2.15), (3.3.48) and the denseness of $H^1(0,T;H^1(\Omega))$ in $L^2(0,T;H^1(\Omega))$ gives that

$$
\begin{aligned}
|I_{3,2}| &\equiv \Big|\int_0^T (\pi^h\phi_\varepsilon(U_{i,\varepsilon}^+)\pi^h\phi_\varepsilon(U_{j,\varepsilon}^-) - \phi(U_i^+)\,\phi(U_j^-), \pi^h[\eta - \widetilde{\eta}])d\,t\Big| \\
&\le \|\pi^h\phi_\varepsilon(U_{i,\varepsilon}^+)\pi^h\phi_\varepsilon(U_{j,\varepsilon}^-) - \phi(U_i^+)\,\phi(U_j^-)\|_{L^2(\Omega_T)}\ \|\pi^h[\eta - \widetilde{\eta}]\|_{L^2(\Omega_T)} \\
&\le \|\pi^h\phi_\varepsilon(U_{i,\varepsilon}^+)\pi^h\phi_\varepsilon(U_{j,\varepsilon}^-) - \phi(U_i^+)\,\phi(U_j^-)\|_{L^2(\Omega_T)}\ \|(\pi^h - I)(\eta - \widetilde{\eta}) + \eta - \widetilde{\eta}\|_{L^2(\Omega_T)} \\
&\le \|\pi^h\phi_\varepsilon(U_{i,\varepsilon}^+)\pi^h\phi_\varepsilon(U_{j,\varepsilon}^-) - \phi(U_i^+)\,\phi(U_j^-)\|_{L^2(\Omega_T)} \\
&\quad\times\big[\|(\pi^h - I)(\eta - \widetilde{\eta})\|_{L^2(\Omega_T)} + \|\eta - \widetilde{\eta}\|_{L^2(\Omega_T)}\ \big] \\
&\le \|\pi^h\phi_\varepsilon(U_{i,\varepsilon}^+)\pi^h\phi_\varepsilon(U_{j,\varepsilon}^-) - \phi(U_i^+)\,\phi(U_j^-)\|_{L^2(\Omega_T)} \\
&\quad\times\big[\ \|(\eta - \widetilde{\eta})\|_{L^2(0,T;H^1(\Omega))} + \|\eta - \widetilde{\eta}\|_{L^2(\Omega_T)}\ \big] \\
&\le \|\pi^h\phi_\varepsilon(U_{i,\varepsilon}^+)\pi^h\phi_\varepsilon(U_{j,\varepsilon}^-) - \phi(U_i^+)\,\phi(U_j^-)\|_{L^2(\Omega_T)}\|(\eta - \widetilde{\eta})\|_{L^2(0,T;H^1(\Omega))} \\
&\to 0 \quad \text{as} \quad h \to 0. \qquad (3.3.85)
\end{aligned}
$$

With the aid of Hölder's inequality and (2.4.56) we have

$$
\begin{aligned}
|I_{3,3}| &\equiv \Big| \int_0^T (\pi^h \phi_\varepsilon(U^+_{i,\varepsilon}) \pi^h \phi_\varepsilon(U^-_{j,\varepsilon}) - \phi(U^+_i)\, \phi(U^-_j), \pi^h \widetilde{\eta}) d\,t \Big| \\
&\leq \|\pi^h \phi_\varepsilon(U^+_{i,\varepsilon}) \pi^h \phi_\varepsilon(U^-_{j,\varepsilon}) - \phi(U^+_i)\, \phi(U^-_j)\|_{L^2(\Omega_T)} \|\pi^h \widetilde{\eta}\|_{L^2(\Omega_T)} \\
&\leq \|\pi^h \phi_\varepsilon(U^+_{i,\varepsilon}) \pi^h \phi_\varepsilon(U^-_{j,\varepsilon}) - \phi(U^+_i)\, \phi(U^-_j)\|_{L^2(\Omega_T)} \|(\pi^h - I)\widetilde{\eta} + \widetilde{\eta}\|_{L^2(\Omega_T)} \\
&\leq \|\pi^h \phi_\varepsilon(U^+_{i,\varepsilon}) \pi^h \phi_\varepsilon(U^-_{j,\varepsilon}) - \phi(U^+_i)\, \phi(U^-_j)\|_{L^2(\Omega_T)} \big[\, \|(\pi^h - I)\widetilde{\eta}\|_{L^2(\Omega_T)} + \|\widetilde{\eta}\|_{L^2(\Omega_T)} \big] \\
&\leq \|\pi^h \phi_\varepsilon(U^+_{i,\varepsilon}) \pi^h \phi_\varepsilon(U^-_{j,\varepsilon}) - \phi(U^+_i)\, \phi(U^-_j)\|_{L^2(\Omega_T)} \ \|\widetilde{\eta}\|_{L^2(0,T;H^1(\Omega))} \\
&\to 0 \quad \text{as} \quad h \to 0.
\end{aligned} \tag{3.3.86}
$$

From equations (2.2.15) and (2.4.56), we have

$$
\begin{aligned}
|I_{3,4}| &\equiv \Big| \int_0^T (\phi(U^+_i)\, \phi(U^-_j), (\pi^h - I)\eta) d\,t \Big| \\
&\leq C \int_0^T \|\phi(U^-_j)\|_0 \ \|(\pi^h - I)\eta\|_0 \\
&\leq Ch \int_0^T \|\phi(U^-_j)\|_0 \ \|\eta\|_1 d\,t \\
&\leq Ch \|\phi(U^-_j)\|_{L^2(\Omega_T)} \ \|\eta\|_{L^2(0,T;H^1(\Omega))} \\
&\leq Ch \|\phi(U^-_j)\|_{L^2(0,T;L^s(\Omega))} \ \|\eta\|_{L^2(0,T;H^1(\Omega))} \\
&\to 0 \quad \text{as} \quad h \to 0.
\end{aligned} \tag{3.3.87}
$$

Upon use of (3.3.83)-(3.3.87) we see for all $\eta \in L^2(0,T;H^1(\Omega))$ that

$$
\int_0^T \big(\phi_\varepsilon(U^+_{i,\varepsilon}) \phi_\varepsilon(U^-_{j,\varepsilon}), \pi^h \eta\big)^h d\,t \to \int_0^T \big(\phi(U^+_i)\, \phi(U^-_j), \eta\big) d\,t \quad \text{as} \quad h \to 0. \tag{3.3.88}
$$

If we combine the results on (3.3.65), (3.3.72), (3.3.74), (3.3.80), (3.3.82) and (3.3.88) we find the desired result (3.3.64).

This completes the proof of the main theorem in this chapter. □

Chapter 4

Time convergence

4.1 Introduction

Our starting point for the analysis here is the final result of the previous chapter, which concerns the existence of a solution to the discrete-in-time problem $(\mathrm{P}_M^{\Delta t})$. The model $(\mathrm{P}_M^{\Delta t})$ includes "microscopic cut-off" in some terms in problem $(\mathrm{P}_M^{\Delta t})$, where $M > 1$ is a (fixed, but otherwise arbitrary,) cut-off parameter. Our ultimate objective is to pass to the limits $M \to \infty$ and $\Delta t \to 0$ in the model $(\mathrm{P}_M^{\Delta t})$, with M and Δt linked by the condition $\Delta t = o(M^{-1})$, as $M \to \infty$. To that end, we need to develop bounds on sequences of weak solutions of $(\mathrm{P}_M^{\Delta t})$ that are uniform in the cut-off parameter M and thus permit the extraction of weakly convergent subsequences, as $M \to \infty$, through the use of a weak-compactness argument. This approach has been adopted in [13–21]

Now, we consider the following cut-off version $\mathcal{F}^M$ of the entropy function $\mathcal{F}$: $s \in \mathbb{R}^{\geq 0} \to \mathcal{F}(s) = (\ln s - 1)s + 1 \in \mathbb{R}^{\geq 0}$ which is given by

$$\mathcal{F}^M(s) = \begin{cases} (\ln s - 1)s + 1, & 0 \leq s \leq M, \\ \frac{s^2 - M^2}{2M} + (\ln M - 1)s + 1, & M \leq s. \end{cases} \tag{4.1.1}$$

Note that

$$(\mathcal{F}^M)'(s) = \begin{cases} \ln s, & 0 < s \leq M, \\ \frac{s}{M} + \ln M - 1, & M \leq s, \end{cases} \tag{4.1.2}$$

and

$$(\mathcal{F}^M)''(s) := \begin{cases} \frac{1}{s}, & 0 < s \le M, \\ \frac{1}{M}, & M \le s. \end{cases} \tag{4.1.3}$$

Hence, we define the function ϕ as follows

$$\phi(s) = [(\mathcal{F}^M)''(s)]^{-1} = \begin{cases} s, & 0 < s \le M, \\ M, & M \le s, \end{cases} \tag{4.1.4}$$

with the convention $1/\infty := 0$ when $s = 0$, and

$$(\mathcal{F}^M)''(s) \ge (\mathcal{F})''(s) = s^{-1}, \quad s \in \mathbb{R}^{>0}. \tag{4.1.5}$$

We shall also require the following inequality, relating $\mathcal{F}^M$ to $\mathcal{F}$:

$$\mathcal{F}^M(s) \ge \mathcal{F}(s), \quad s \in \mathbb{R}^{\ge 0}. \tag{4.1.6}$$

For $s > 1$, (4.1.6) follows from (4.1.5), with s replaced by a dummy variable σ, after integrating twice over $\sigma \in [1, s]$, and noting that $(\mathcal{F}^M)'(1) = (\mathcal{F})'(1)$ and $\mathcal{F}^M(1) = \mathcal{F}(1)$. For $s \in [0, 1]$, we have $\mathcal{F}^M(s) = \mathcal{F}(s)$ by definition.

4.2 M-independent bounds on the derivatives

We are now ready to embark on the derivation of the required bounds, uniform in the cut-off parameter M, on norms of $U_i^+, i = 1, ..., m$. The appropriate choice of test function in (3.3.64) for this purpose is $\eta = \chi_{[0,t]}(\mathcal{F}^M)'(U_i^+), i = 1, ..., m$ with $t = t_n$, $n \in \{1, ..., N\}$, and $\chi_{[0,t]}$ denoting the characteristic function of the interval $[0, t]$. While Theorem 3.3.1 guarantees that $U_i^+(., t), i = 1, ..., m$ is nonnegative a.e. on $\Omega \times [0, T]$, there is unfortunately no reason why $U_i^+, i = 1, ..., m$ should be strictly positive on $\Omega \times [0, T]$, and therefore the expression $(\mathcal{F}^M)'(U_i^+), i = 1, ..., m$ may in general be undefined; the same is true of $(\mathcal{F}^M)''(U_i^+), i = 1, ..., m$ which also appears in the algebraic manipulations. We shall circumvent this problem by working with $(\mathcal{F}^M)'(U_i^+ + \epsilon), i = 1, ..., m$ instead of $(\mathcal{F}^M)'(U_i^+), i = 1, ..., m$, where $\epsilon > 0$; since $U_i^+, i = 1, ..., m$ are known to be nonnegative from Theorem 3.3.1, $(\mathcal{F}^M)'(U_i^+ + \epsilon), i = 1, ..., m$ and $(\mathcal{F}^M)''(U_i^+ + \epsilon), i = 1, ..., m$ are well-defined. After deriving the relevant bounds, which will involve $\mathcal{F}^M(U_i^+ + \epsilon), i = 1, ..., m$ only, we shall pass to the limit $\epsilon \to 0^+$, noting that, unlike $(\mathcal{F}^M)'(U_i^+), i = 1, ..., m$ and $(\mathcal{F}^M)''(U_i^+), i = 1, ..., m$, the function $\mathcal{F}^M(U_i^+ + \epsilon), i = 1, ..., m$ is well-defined for any nonnegative $U_i^+, i = 1, ..., m$.

Before we prove the bounds on the approximate solutions, in the next Lemma, we provide a result which will be important in the analysis of the approximation problem ($\mathrm{P}_M^{\Delta t}$).

Lemma 4.2.1

$$\int_\Omega \mathcal{F}^M(\phi(U_i^0) + \epsilon)dx \leq \frac{3}{2}\,\epsilon|\Omega| + \int_\Omega \mathcal{F}(U_i^0 + \epsilon)dx.$$

Proof:

We label $\mathcal{Q}(\epsilon)$ and express as follows:

$$\mathcal{Q}(\epsilon) = \int_\Omega \mathcal{F}^M(\phi(U_i^0) + \epsilon)dx$$

$$= \int_{\mathbb{Y}_{M,\epsilon}} \mathcal{F}^M(\phi(U_i^0) + \epsilon)dx + \int_{\mathcal{Y}_{M,\epsilon}} \mathcal{F}^M(\phi(U_i^0) + \epsilon)dx,$$

where

$$\mathbb{Y}_{M,\epsilon} = \{\mathbf{x} \in \Omega : 0 \leq \phi(U_i^0(\mathbf{x})) \leq M - \epsilon\},$$

$$\mathcal{Y}_{M,\epsilon} = \{\mathbf{x} \in \Omega : M - \epsilon < \phi(U_i^0(\mathbf{x})) \leq M\}.$$

We begin by noting that

$$\int_{\mathbb{Y}_{M,\epsilon}} \mathcal{F}^M(\phi(U_i^0) + \epsilon)dx = \int_{\mathbb{Y}_{M,\epsilon}} \mathcal{F}(\phi(U_i^0) + \epsilon)dx.$$

For the integral over $\mathcal{Y}_{M,\epsilon}$ we have

$$\int_{\mathcal{Y}_{M,\epsilon}} \mathcal{F}^M(\phi(U_i^0) + \epsilon)dx$$

$$= \int_{\mathcal{Y}_{M,\epsilon}} [\frac{(\phi(U_i^0) + \epsilon)^2 - M^2}{2M} + (\phi(U_i^0) + \epsilon)(\log M - 1) + 1]dx$$

$$\leq \int_{\mathcal{Y}_{M,\epsilon}} [\frac{(M + \epsilon)^2 - M^2}{2M} + (\phi(U_i^0) + \epsilon)(\log(\phi(U_i^0) + \epsilon) - 1) + 1]dx$$

$$= \int_{\mathcal{Y}_{M,\epsilon}} \frac{(2\epsilon M + \epsilon^2)}{2M}dx + \int_{\mathcal{Y}_{M,\epsilon}} \mathcal{F}(\phi(U_i^0) + \epsilon)dx$$

$$\leq \frac{3}{2}\epsilon|\Omega| + \int_{\mathcal{Y}_{M,\epsilon}} \mathcal{F}(\phi(U_i^0) + \epsilon)dx.$$

Thus we have shown that

$$\mathcal{Q}(\epsilon) \leq \frac{3}{2}\epsilon|\Omega| + \int_{\Omega} \mathcal{F}(\phi(U_i^0) + \epsilon)dx.$$

Now, there are two possibilities:

1. If $\phi(U_i^0) + \epsilon \leq 1$, then $0 \leq \phi(U_i^0) \leq 1 - \epsilon$. Since $M > 1$ it follows that $0 \leq \phi(s) \leq 1$ if, and only if, $\phi(s) = s$. Thus we deduce that in this case $\phi(U_i^0) = U_i^0$, and therefore $0 \leq \mathcal{F}(\phi(U_i^0) + \epsilon) = \mathcal{F}(U_i^0 + \epsilon)$.

2. Alternatively, if $\phi(U_i^0) + \epsilon > 1$, then, on noting that $\phi(s) \leq s$ for all $s \in [0, \infty)$, it follows that $1 < \phi(U_i^0) + \epsilon \leq U_i^0 + \epsilon$. However, the function $\mathcal{F}$ is strictly monotonic increasing on the interval $[1, \infty)$, which then implies that $0 = F(1) \leq \mathcal{F}(\phi(U_i^0) + \epsilon) \leq \mathcal{F}(U_i^0 + \epsilon)$.

The conclusion we draw is that, either way,

$$0 \leq \mathcal{F}(\phi(U_i^0) + \epsilon) \leq \mathcal{F}(U_i^0 + \epsilon).$$

Hence,

$$\mathcal{Q}(\epsilon) \leq \frac{3}{2}\epsilon|\Omega| + \int_{\Omega} \mathcal{F}(U_i^0 + \epsilon)dx.$$

□

Theorem 4.2.2 Suppose that we impose the condition of relating Δt to M, is such that $\Delta t M = o(1)$ as $\Delta t \to 0$ (or, equivalently, $\Delta t = o(M^{-1})$ as $M \to \infty$). Then, the solutions $\{U_i^{\pm}, i = 1, ..., m\}$ satisfy the following bounds

$$\sum_{i=1}^{m} \int_{\Omega} \mathcal{F}^M(U_i^+)dx + (\frac{1}{2M\Delta t} - 2^{m-1}) \int_0^t \int_{\Omega} (U_i^+ - U_i^-)^2 dxdt$$

$$+2D^* \int_0^t \int_{\Omega} \sum_{i=1}^{m} |\nabla \sqrt{U_i^+}|^2 \, dxdt + \frac{1}{2} \int_0^t \int_{\Omega} (\sum_{i=1}^{m} \nabla U_i^+)^2 \, dxdt \leq B_1(U_i^0), \qquad (4.2.7)$$

where D^* is a constant and $B_1(U_i^0) = [1 + 2\gamma(1 + 2\gamma\Delta t)^k] \sum_{i=1}^{m} \int_{\Omega} \mathcal{F}(U_i^0)dx + C$.

Proof: For any $\epsilon \in (0, 1)$, whereby $0 < \epsilon < 1 < M$, we choose $\eta = \chi_{[0,t]}(\mathcal{F}^M)'(U_i^+ + \epsilon)$, $i = 1, ..., m$ with $t = t_n$, $n \in \{1, ..., N\}$, as test function in (3.3.64):

$$\int_0^T [\langle \frac{\partial U_i}{\partial t}, \chi_{[0,t]}(\mathcal{F}^M)'(U_i^+ + \epsilon)\rangle + D_i(\nabla U_i^+, \nabla \chi_{[0,t]}(\mathcal{F}^M)'(U_i^+ + \epsilon))$$

$$+(\phi(U_i^+)\sum_{j=1}^{m}\nabla U_j^+,\nabla\chi_{[0,t]}(\mathcal{F}^M)'(U_i^++\epsilon))]dt$$

$$=\int_0^T[(\gamma_i U_i^+ - \phi(U_i^+)\sum_{j=1}^{m}\phi(U_j^-),\chi_{[0,t]}(\mathcal{F}^M)'(U_i^++\epsilon))]dt,\qquad i=1,...,m. \quad (4.2.8)$$

Then, we start by considering the first term in (4.2.8). Clearly $\mathcal{F}^M(U_i^++\epsilon)$ is twice continuously differentiable on the interval $(-\epsilon,\infty)$ for any $\epsilon>0$. Thus, by Taylor series expansion of $s\in[0,\infty)\to\mathcal{F}^M(s+\epsilon)\in[0,\infty)$ with remainder, and $c\in[0,\infty)$,

$$(s-c)(\mathcal{F}^M)'(s+\epsilon)=\mathcal{F}^M(s+\epsilon)-\mathcal{F}^M(c+\epsilon)+\frac{1}{2}(s-c)^2(\mathcal{F}^M)''(\theta s+(1-\theta)c+\epsilon),$$

with $\theta\in(0,1)$. Hence, on noting that $t\in[0,T]\to U_i^+(.,t)$ is piecewise linear relative to the partition $\{0=t_0,t_1,...,t_N=T\}$ of the interval $[0,T]$,

$$\begin{aligned}\widetilde{T}_1&=\int_0^T\int_\Omega\frac{\partial U_i}{\partial t}\chi_{[0,t]}(\mathcal{F}^M)'(U_i^++\epsilon)dxdt=\int_0^t\int_\Omega\frac{\partial U_i}{\partial t}(\mathcal{F}^M)'(U_i^++\epsilon)dxdt\\&=\frac{1}{\Delta t}\int_0^t\int_\Omega(U_i^+-U_i^-)(\mathcal{F}^M)'(U_i^++\epsilon)dxdt\\&=\frac{1}{\Delta t}\int_0^t\int_\Omega(\mathcal{F}^M)(U_i^++\epsilon)dxdt-\frac{1}{\Delta t}\int_0^t\int_\Omega(\mathcal{F}^M)(U_i^-+\epsilon)dxdt\\&+\frac{1}{2\Delta t}\int_0^t\int_\Omega(U_i^+-U_i^-)^2(\mathcal{F}^M)''(\theta U_i^++(1-\theta)U_i^-+\epsilon)dxdt.\end{aligned}\quad (4.2.9)$$

Noting from (4.1.3) that $(\mathcal{F}^M)''(s+\epsilon)\geq 1/M$ for all $s\in[0,\infty)$ and all $\epsilon>0$, this then implies, with $t=t_n$, $n\in\{1,...,N\}$, that

$$\begin{aligned}\widetilde{T}_1\geq&\frac{1}{\Delta t}\int_0^t\int_\Omega(\mathcal{F}^M)(U_i^++\epsilon)dxdt-\frac{1}{\Delta t}\int_0^t\int_\Omega(\mathcal{F}^M)(U_i^-+\epsilon)dxdt\\&+\frac{1}{2M\Delta t}\int_0^t\int_\Omega(U_i^+-U_i^-)^2dxdt.\end{aligned}\quad (4.2.10)$$

The denominator in the prefactor of the last integral motivates us to link Δt to M so that $\Delta tM=o(1)$ as $\Delta t\to 0$ (or, equivalently, $\Delta t=o(M^{-1})$ as $M\to\infty$), in order to drive the integral multiplied by the prefactor to 0 in the limit of $M\to\infty$, once the product of the two has been bounded above by a constant, independent of M.

Next we consider the second term in (4.2.8). From (4.1.5), we have for all $i=1,...,m$ that

$$\widetilde{T}_2=D_i\int_0^T\int_\Omega\nabla U_i^+\nabla\chi_{[0,t]}(\mathcal{F}^M)'(U_i^++\epsilon)dxdt$$

$$= D_i \int_0^t \int_\Omega |\nabla U_i^+|^2 (\mathcal{F}^M)''(U_i^+ + \epsilon) dx dt = D_i \int_0^t \int_\Omega \frac{|\nabla U_i^+|^2}{\phi(U_i^+ + \epsilon)} dx dt. \qquad (4.2.11)$$

With the aid of (4.1.5), the third term in (4.2.8) can be simplified as follows

$$\widetilde{T}_3 = \int_0^T \int_\Omega \phi(U_i^+) \sum_{j=1}^m \nabla U_j^+ \ \nabla \chi_{[0,t]} (\mathcal{F}^M)'(U_i^+ + \epsilon) dx dt$$

$$= \int_0^t \int_\Omega \frac{\phi(U_i^+)}{\phi(U_i^+ + \epsilon)} \nabla U_i^+ \ \sum_{j=1}^m \nabla U_j^+ \ dx dt. \qquad (4.2.12)$$

Now, we deal with the fourth term in (4.2.8). It follows from (2.3.31), (2.3.32) and $\phi(s) \leq s, \forall s$ that

$$\widetilde{T}_4 = \gamma_i \int_0^T \int_\Omega U_i^+ \ \chi_{[0,t]} (\mathcal{F}^M)'(U_i^+ + \epsilon) dx dt$$

$$= \gamma_i \int_0^t \int_\Omega (U_i^+ + \epsilon) \ (\mathcal{F}^M)'(U_i^+ + \epsilon) dx dt - \epsilon \ \gamma_i \int_0^t \int_\Omega (\mathcal{F}^M)'(U_i^+ + \epsilon) dx dt$$

$$\leq \gamma_i \int_0^t \int_\Omega (2\mathcal{F}^M(U_i^+ + \epsilon) + 1) dx dt - \epsilon \ \gamma_i \int_0^t \int_\Omega \frac{1}{\phi(U_i^+ + \epsilon)} \phi(U_i^+ + \epsilon)(\mathcal{F}^M)'(U_i^+ + \epsilon) dx dt$$

$$\leq 2\gamma_i \int_0^t \int_\Omega \mathcal{F}^M(U_i^+ + \epsilon) dx dt + \gamma_i T|\Omega| - \epsilon \ \gamma_i \int_0^t \int_\Omega \frac{(U_i^+ + \epsilon - 1)}{\phi(U_i^+ + \epsilon)} dx dt$$

$$\leq 2\gamma_i \int_0^t \int_\Omega \mathcal{F}^M(U_i^+ + \epsilon) dx dt + \gamma_i T|\Omega| - \epsilon \ \gamma_i \int_0^t \int_\Omega \frac{(U_i^+ + \epsilon)}{\phi(U_i^+ + \epsilon)} dx dt$$

$$+ \gamma_i \int_0^t \int_\Omega \frac{\epsilon}{\phi(U_i^+ + \epsilon)} dx dt \leq 2\gamma \int_0^t \int_\Omega \mathcal{F}^M(U_i^+ + \epsilon) dx dt + 2\gamma T|\Omega|, \qquad (4.2.13)$$

where $\gamma = \max_{i=1}^m \gamma_i$.

Next, we consider the last term in (4.2.8). On noting (2.3.32) and $\phi(s) \leq \phi(\hat{s})$ for $s \leq \hat{s}$ yields that

$$\widetilde{T}_5 = \int_0^T \int_\Omega \phi(U_i^+) \sum_{j=1}^m \phi(U_j^-) \ \chi_{[0,t]} (\mathcal{F}^M)'(U_i^+ + \epsilon) dx dt$$

$$= \int_0^t \int_\Omega \phi(U_i^+) \sum_{j=1}^m \phi(U_j^-) \ (\mathcal{F}^M)'(U_i^+ + \epsilon) dx dt$$

$$= \int_0^t \int_\Omega \frac{\phi(U_i^+)}{\phi(U_i^+ + \epsilon)} \sum_{j=1}^m \phi(U_j^-) \ \phi(U_i^+ + \epsilon)(\mathcal{F}^M)'(U_i^+ + \epsilon) dx dt$$

$$\geq \int_0^t \int_\Omega \frac{\phi(U_i^+)}{\phi(U_i^+ + \epsilon)} \sum_{j=1}^m \phi(U_j^-) \ (U_i^+ + \epsilon - 1) dx dt$$

$$\geq \int_0^t \int_\Omega \frac{(U_i^+ + \epsilon)\phi(U_i^+)}{\phi(U_i^+ + \epsilon)} \sum_{j=1}^m \phi(U_j^-) dxdt - \int_0^t \int_\Omega \frac{\phi(U_i^+)}{\phi(U_i^+ + \epsilon)} \sum_{j=1}^m \phi(U_j^-) dxdt$$

$$\geq \int_0^t \int_\Omega \phi(U_i^+) \sum_{j=1}^m \phi(U_j^-) dxdt - \int_0^t \int_\Omega \sum_{j=1}^m \phi(U_j^-) dxdt. \quad (4.2.14)$$

Combining (4.2.10)-(4.2.14) and (4.2.8), then summing the final results for $i = 1, ..., m$, leads that

$$\sum_{i=1}^m \frac{1}{\Delta t} \int_0^t \int_\Omega \mathcal{F}^M(U_i^+ + \epsilon) dxdt - \sum_{i=1}^m \frac{1}{\Delta t} \int_0^t \int_\Omega \mathcal{F}^M(U_i^- + \epsilon) dxdt$$
$$+ \frac{1}{2M\Delta t} \sum_{i=1}^m \int_0^t \int_\Omega (U_i^+ - U_i^-)^2 dxdt + \sum_{i=1}^m D_i \int_0^t \int_\Omega \frac{|\nabla U_i^+|^2}{\phi(U_i^+ + \epsilon)} dxdt$$
$$+ \int_0^t \int_\Omega \sum_{i=1}^m \frac{\phi(U_i^+)}{\phi(U_i^+ + \epsilon)} \nabla U_i^+ \sum_{j=1}^m \nabla U_j^+ \, dxdt \leq 2\gamma \sum_{i=1}^m \int_0^t \int_\Omega \mathcal{F}^M(U_i^+ + \epsilon) dxdt$$
$$+ m \int_0^t \int_\Omega \sum_{i=1}^m \phi(U_i^-) dxdt - \int_0^t \int_\Omega \sum_{i=1}^m \phi(U_i^+) \sum_{i=1}^m \phi(U_i^-) dxdt + 2\hat{\gamma} T |\Omega|, \quad (4.2.15)$$

where $\hat{\gamma} = \sum_{i=1}^m \gamma_i$. By using Young's inequality, we deal with the second term in the right side of (4.2.15) as follows,

$$m \int_0^t \int_\Omega \sum_{i=1}^{m-1} \phi(U_i^-) dxdt \leq \frac{m^2}{2} T |\Omega| + \frac{1}{2} \int_0^t \int_\Omega \left[\sum_{i=1}^m \phi(U_i^-) \right]^2 dxdt. \quad (4.2.16)$$

With the aid of the Lipschitz continuity of the function ϕ and Young's inequality, we have

$$- \int_0^t \int_\Omega \sum_{i=1}^m \phi(U_i^+) \sum_{i=1}^m \phi(U_i^-) dxdt$$
$$= - \int_0^t \int_\Omega \left[\sum_{i=1}^m \phi(U_i^+) - \sum_{i=1}^m \phi(U_i^-) + \sum_{i=1}^m \phi(U_i^-) \right] \sum_{i=1}^m \phi(U_i^-) dxdt$$
$$= - \int_0^t \int_\Omega \left[\sum_{i=1}^m \phi(U_i^+) - \sum_{i=1}^m \phi(U_i^-) \right] \sum_{i=1}^m \phi(U_i^-) dxdt - \int_0^t \int_\Omega \left[\sum_{i=1}^m \phi(U_i^-) \right]^2 dxdt$$
$$\leq 2^{m-1} \sum_{i=1}^m \int_0^t \int_\Omega \left(\phi(U_i^+) - \phi(U_i^-) \right)^2 dxdt - \frac{1}{2} \int_0^t \int_\Omega \left[\sum_{i=1}^m \phi(U_i^-) \right]^2 dxdt$$
$$\leq 2^{m-1} \sum_{i=1}^m \int_0^t \int_\Omega \left(U_i^+ - U_i^- \right)^2 dxdt - \frac{1}{2} \int_0^t \int_\Omega \left[\sum_{i=1}^m \phi(U_i^-) \right]^2 dxdt. \quad (4.2.17)$$

From the Lipschitz continuity of the function ϕ, Young's inequality and $\phi(s+\epsilon) \geq \epsilon$, it follows that

$$\int_0^t \int_\Omega \sum_{i=1}^m \frac{\phi(U_i^+)}{\phi(U_i^+ + \epsilon)} \nabla U_i^+ \sum_{j=1}^m \nabla U_j^+ \, dxdt$$

$$= \int_0^t \int_\Omega (\sum_{i=1}^m \nabla U_i^+)^2 \, dxdt + \int_0^t \int_\Omega \sum_{i=1}^m \frac{\phi(U_i^+) - \phi(U_i^+ + \epsilon)}{\phi(U_i^+ + \epsilon)} \nabla U_i^+ \sum_{j=1}^m \nabla U_j^+ \, dxdt$$

$$\geq \int_0^t \int_\Omega (\sum_{i=1}^m \nabla U_i^+)^2 \, dxdt - 2^{m-2} \int_0^t \int_\Omega \sum_{i=1}^m \frac{(\phi(U_i^+) - \phi(U_i^+ + \epsilon))^2}{\phi(U_i^+ + \epsilon)} \frac{|\nabla U_i^+|^2}{\phi(U_i^+ + \epsilon)} \, dxdt$$

$$-\frac{1}{2} \int_0^t \int_\Omega (\sum_{i=1}^m \nabla U_i^+)^2 \, dxdt$$

$$\geq \frac{1}{2} \int_0^t \int_\Omega (\sum_{i=1}^m \nabla U_i^+)^2 \, dxdt - 2^{m-2}\epsilon \int_0^t \int_\Omega \sum_{i=1}^m \frac{|\nabla U_i^+|^2}{\phi(U_i^+ + \epsilon)} \, dxdt. \quad (4.2.18)$$

Substituting (4.2.16), (4.2.17) and (4.2.18) in (4.2.15) we have

$$(\frac{1}{\Delta t} - 2\gamma) \sum_{i=1}^m \int_0^t \int_\Omega \mathcal{F}^M(U_i^+ + \epsilon) dxdt - \sum_{i=1}^m \frac{1}{\Delta t} \int_0^t \int_\Omega \mathcal{F}^M(U_i^- + \epsilon) dxdt$$

$$+(\frac{1}{2M\Delta t} - 2^{m-1}) \int_0^t \int_\Omega (U_i^+ - U_i^-)^2 dxdt + \frac{D^*}{2} \int_0^t \int_\Omega \sum_{i=1}^m \frac{|\nabla U_i^+|^2}{\phi(U_i^+ + \epsilon)} \, dxdt$$

$$+\frac{1}{2} \int_0^t \int_\Omega (\sum_{i=1}^m \nabla U_i^+)^2 \, dxdt \leq \frac{m^2}{4} T|\Omega| + 2\hat{\gamma} T|\Omega| \quad (4.2.19)$$

In the above inequality we use $D^* - 2^{m-2}\epsilon \geq D^*/2$ and this holds for $\epsilon \leq D^*/2^{m-1}$ where $D^* = \min_{i=1,..m} D_i$. From (4.2.19) we conclude that

$$(\frac{1}{\Delta t} - 2\gamma) \sum_{i=1}^m \int_0^t \int_\Omega \mathcal{F}^M(U_i^+ + \epsilon) dxdt - \sum_{i=1}^m \frac{1}{\Delta t} \int_0^t \int_\Omega \mathcal{F}^M(U_i^- + \epsilon) dxdt \leq C, \quad (4.2.20)$$

where $C = (\frac{m^2}{4} + 2\hat{\gamma})T|\Omega|$. Now, let

$$v_k = \sum_{i=1}^m \int_0^t \int_\Omega \mathcal{F}^M(U_i^+ + \epsilon) dxdt = \sum_{i=1}^m \sum_{j=0}^{k-1} \Delta t \int_\Omega \mathcal{F}^M(U_i^{j+1} + \epsilon) dx,$$

then we can write (4.2.20) as follows

$$v_k \leq \frac{v_{k-1}}{(1 - 2\gamma\Delta t)} + \frac{\Delta t C}{(1 - 2\gamma\Delta t)} \leq (1 + 2\gamma\Delta t) v_{k-1} + \Delta t(1 + 2\gamma\Delta t) C. \quad (4.2.21)$$

Finally, using induction we arrive at the following inequality

$$v_k \leq (1 + 2\gamma\Delta t)^k v_0 + \Delta t C \sum_{l=1}^k (1 + 2\gamma\Delta t)^l$$

$$\leq (1+2\gamma\Delta t)^k v_0 + \frac{C}{2\gamma}(1+2\gamma\Delta t)^{k+1}$$
$$\leq (1+2\gamma\Delta t)^k v_0 + \frac{C}{2\gamma}e^{2\gamma\Delta t(k+1)}.$$

That is

$$\sum_{i=1}^{m}\int_0^t\int_\Omega \mathcal{F}^M(U_i^+ + \epsilon)dxdt \leq (1+2\gamma\Delta t)^k \sum_{i=1}^{m}\int_\Omega \mathcal{F}^M(U_i^0+\epsilon)dxdt + \frac{C}{2\gamma}e^{2\gamma\Delta t(k+1)}. \tag{4.2.22}$$

Moreover, noting $\phi(s) \leq s$ yields that

$$\int_0^t\int_\Omega \frac{|\nabla U_i^+|^2}{\phi(U_i^+ + \epsilon)}dxdt \geq \int_0^t\int_\Omega \frac{|\nabla U_i^+|^2}{U_i^+ + \epsilon}dxdt = 4\int_0^t\int_\Omega |\nabla\sqrt{U_i^+ + \epsilon}|^2 dxdt. \tag{4.2.23}$$

Now, by substituting (4.2.22) and (4.2.23) in (4.2.19), and using induction we arrive at

$$\sum_{i=1}^{m}\int_\Omega \mathcal{F}^M(U_i^+ + \epsilon)dx + (\frac{1}{2M\Delta t} - 2^{m-1})\int_0^t\int_\Omega (U_i^+ - U_i^-)^2 dxdt$$
$$+2D^*\int_0^t\int_\Omega \sum_{i=1}^{m}|\nabla\sqrt{U_i^+ + \epsilon}|^2\ dxdt + \frac{1}{2}\int_0^t\int_\Omega \left[\sum_{i=1}^{m}\nabla U_i^+\right]^2 dxdt$$
$$\leq [1+2\gamma(1+2\gamma\Delta t)^k]\sum_{i=1}^{m}\int_\Omega \mathcal{F}^M(U_i^0+\epsilon)dx + C$$
$$\leq [1+2\gamma(1+2\gamma\Delta t)^k]\sum_{i=1}^{m}\int_\Omega \mathcal{F}^M(\phi(U_i^0)+\epsilon)dx + C$$
$$\leq [1+2\gamma(1+2\gamma\Delta t)^k]\sum_{i=1}^{m}\int_\Omega \mathcal{F}(U_i^0+\epsilon)dx + C. \tag{4.2.24}$$

We use in the second inequality a simple fact that, clearly, if there exists $M > 0$ such that $0 \leq U_i^0 \leq M$, then $\phi(U_i^0) = U_i^0$. Henceforth $M > 1$ is assumed. Then in the last inequality, we use the results of Lemma 4.2.1.

We shall tidy up the bound (4.2.24) by passing to the limit $\epsilon \to 0^+$. Concerning the ϵ-dependent term on the right-hand side, Lebesgue's dominated convergence theorem implies that

$$\lim_{\epsilon\to 0^+}\int_\Omega \mathcal{F}(U_i^0+\epsilon)dx = \int_\Omega \mathcal{F}(U_i^0)dx$$

We can easily pass to the limit on the left-hand side of (4.2.24). By applying Fatou's lemma to the first and third terms on the left-hand side of (4.2.24) we get, for $t = t_n, n \in \{1, ..., N\}$, that

$$\liminf_{\epsilon \to 0^+} \int_\Omega \mathcal{F}^M(U_i^+ + \epsilon) dxdt \geq \int_\Omega \mathcal{F}^M(U_i^+) dxdt,$$

$$\liminf_{\epsilon \to 0^+} \sum_{i=1}^m \int_0^t \int_\Omega |\nabla \sqrt{U_i^+ + \epsilon}|^2 dxdt \geq \sum_{i=1}^m \int_0^t \int_\Omega |\nabla \sqrt{U_i^+}|^2 dxdt.$$

Thus, after passage to the limit $\epsilon \to 0^+$, we have, for all $t = t_n, n \in \{1, ..., N\}$, that

$$\sum_{i=1}^m \int_\Omega \mathcal{F}^M(U_i^+) dxdt + (\frac{1}{2M\Delta t} - 2^{m-1}) \int_0^t \int_\Omega (U_i^+ - U_i^-)^2 dxdt$$

$$+2D^* \int_0^t \int_\Omega \sum_{i=1}^m |\nabla \sqrt{U_i^+}|^2 \; dxdt + \frac{1}{2} \int_0^t \int_\Omega \left[\sum_{i=1}^m \nabla U_i^+ \right]^2 \; dxdt$$

$$\leq [1 + 2\gamma(1 + 2\gamma\Delta t)^k] \sum_{i=1}^m \int_\Omega \mathcal{F}^M(U_i^0) dx + C. \tag{4.2.25}$$

□

Additional regularity, more than we have been able to prove, is required to complete the analysis of problem ($\mathrm{P}_M^{\Delta t}$). Unfortunately, we have been unable to prove the regularity requirement which is essential to establish the convergence results. However, in order to proceed with the convergence analysis we adopt an alternative technique to prove that $U_i^{\pm}(\mathbf{x}, t) \in L^\infty(\Omega_T)$.

Lemma 4.2.3 Let us divide the region Ω into two regions such that $\Omega = \Omega_M(t) \bigcup \Omega_0(t)$ and these regions be defined as follows:

$$\Omega_M(t) = \{x \in \Omega : U_i^+(\mathbf{x}, t) \geq M\},$$

$$\Omega_0(t) = \{x \in \Omega : U_i^+(\mathbf{x}, t) < M\}.$$

Then we have $|\Omega_M(t)| \to 0$ as $M \to \infty$, $a.e.$ in $\Omega \times [0, T]$.

Proof: We note from (4.1.1) that (when $s \geq M$)

$$\mathcal{F}^M(s) = \frac{s^2 - M^2}{2M} + (\ln M - 1)s + 1 \geq (\ln M - 1)s.$$

Then, using the first bound in (4.2.7), we have

$$\int_\Omega \mathcal{F}^M(U_i^+) dx = \int_{\Omega_M(t)} \mathcal{F}^M(U_i^+) dx + \int_{\Omega_0(t)} \mathcal{F}^M(U_i^+) dx \leq C. \tag{4.2.26}$$

Then, (4.2.26) lead to the following inequality:

$$C \geq \int_{\Omega_M(t)} \mathcal{F}^M(U_i^+)dx \geq (\ln M - 1)\int_{\Omega_M(t)} U_i^+ dx \geq M(\ln M - 1)|\Omega_M(t)|. \quad (4.2.27)$$

So for each i

$$|\Omega_M(t)| \leq \frac{C}{M(\ln M - 1)} \quad \to \quad 0 \quad \text{as} \quad M \to \quad \infty. \quad (4.2.28)$$

□

Assumption 4.2.1 From Lemma 4.2.3 we will assume that: $U_i^+ \leq \Upsilon,\ \ a.e.$ in $\Omega \times [0,T]$ for M sufficiently large, i.e

$$\|U_i^+\|_{L^\infty(\Omega_T)} \leq \Upsilon \ \text{ where } \ \Upsilon \in \mathbb{R} < \infty. \quad (4.2.29)$$

Theorem 4.2.4 Suppose that the condition of relating Δt to M, is such that $\Delta t M = o(1)$ as $\Delta t \to 0$ (or, equivalently, $\Delta t = o(M^{-1})$ as $M \to \infty$). Moreover, if $\Delta t < 1/4\gamma$, then, the solutions $\{U_i^\pm, i = 1, ..., m\}$ satisfy the following bounds

$$\sum_{i=1}^m \int_\Omega (U_i^+)^2 dx + \frac{1}{\Delta t}\sum_{i=1}^m \int_0^t \int_\Omega (U_i^+ - U_i^-)^2 dxdt + 2\breve{D}\sum_{i=1}^m \int_0^t \int_\Omega |\nabla U_i^+|^2 dxdt$$

$$+\sum_{i=1}^m \int_0^t \int_\Omega U_i^+ \phi(U_i^+) \sum_{j=1}^m \phi(U_j^-) dxdt \leq B_2(U_i^0), \quad (4.2.30)$$

where $\breve{D} = D - 2^{m-2}\Upsilon^2 > 0$ and $B_2(U_i^0) = 2[1 + 2\gamma(1 + 2\gamma\Delta t)^k]\sum_{i=1}^m \int_\Omega (U_i^0)^2 dx + CB_1(U_i^0)$.

Proof: For any $\epsilon \in (0,1)$, whereby $0 < \epsilon < 1 < M$, we choose $\eta = \chi_{[0,t]}U_i^+$, $i = 1, ..., m$ with $t = t_n$, $n \in \{1, ..., N\}$, as the test function in (3.3.64), to obtain

$$(\frac{1}{2\Delta t} - 2\gamma)\sum_{i=1}^m \int_0^t \int_\Omega (U_i^+)^2 dxdt - \frac{1}{2\Delta t}\sum_{i=1}^m \int_0^t \int_\Omega (U_i^-)^2 dxdt + \frac{1}{2\Delta t}\sum_{i=1}^m \int_0^t \int_\Omega (U_i^+ - U_i^-)^2 dxdt$$

$$+D\sum_{i=1}^m \int_0^t \int_\Omega |\nabla U_i^+|^2 dxdt + \sum_{i=1}^m \int_0^t \int_\Omega U_i^+ \phi(U_i^+) \sum_{j=1}^m \phi(U_j^-) dxdt$$

$$\leq -\int_0^t \int_\Omega \sum_{i=1}^m \phi(U_i^+)\nabla U_i^+ \sum_{j=1}^m \nabla U_j^+ dxdt, \quad (4.2.31)$$

where $D = \min_{i=1,..m} D_i$ and $\gamma = \max_{i=1}^{m} \gamma_i$. We also have from Young's inequality, the fourth bound in (4.2.7) and the bound (4.2.29), that

$$
\begin{aligned}
&-\int_0^t \int_\Omega \sum_{i=1}^m \phi(U_i^+)\nabla U_i^+ \sum_{j=1}^m \nabla U_j^+ dxdt \\
&\le \frac{\rho}{2}\int_0^t \int_\Omega \Big[\sum_{i=1}^m \phi(U_i^+)\nabla U_i^+\Big]^2 dxdt + \frac{1}{2\rho}\int_0^t \int_\Omega \Big[\sum_{i=1}^m \nabla U_i^+\Big]^2 dxdt \\
&\le 2^{m-2}\rho \int_0^t \int_\Omega \sum_{i=1}^m \phi^2(U_i^+)|\nabla U_i^+|^2 dxdt + \frac{B_1(U_i^0)}{2\rho} \\
&\le 2^{m-2}\Upsilon^2\rho \int_0^t \int_\Omega \sum_{i=1}^m |\nabla U_i^+|^2 dxdt + \frac{B_1(U_i^0)}{2\rho}. \qquad (4.2.32)
\end{aligned}
$$

Combining (4.2.31), (4.2.32) yields that

$$
\begin{aligned}
&(\frac{1}{2\Delta t} - 2\gamma)\sum_{i=1}^m \int_0^t \int_\Omega (U_i^+)^2 dxdt - \frac{1}{2\Delta t}\sum_{i=1}^m \int_0^t \int_\Omega (U_i^-)^2 dxdt + \frac{1}{2\Delta t}\sum_{i=1}^m \int_0^t \int_\Omega (U_i^+ - U_i^-)^2 dxdt \\
&+\breve{D}\sum_{i=1}^m \int_0^t \int_\Omega |\nabla U_i^+|^2 dxdt + \sum_{i=1}^m \int_0^t \int_\Omega U_i^+\phi(U_i^+)\sum_{j=1}^m \phi(U_j^-) dxdt \le \frac{B_1(U_i^0)}{2\rho}. \qquad (4.2.33)
\end{aligned}
$$

Similarly to (4.2.22), we have from (4.2.32) that

$$
\sum_{i=1}^m \int_0^t \int_\Omega (U_i^+)^2 dxdt \le (1+2\gamma\Delta t)^k \sum_{i=1}^m \int_\Omega (U_i^0)^2 dxdt + \frac{B_1(U_i^0)}{4\rho\gamma} e^{2\gamma\Delta t(k+1)}. \qquad (4.2.34)
$$

Now, by substituting (4.2.34) in (4.2.33), and using induction we arrive to the required result. □

4.3 M-independent bounds on the time-derivatives

We begin by bounding the time-derivative of $U_i, i = 1, ..., m$ using (3.3.64). It follows from (3.3.64) that

$$
\begin{aligned}
&|\int_0^T \int_\Omega \frac{\partial U_i}{\partial t}\eta\, d\, xd\, t| \le |D_i \int_0^T \int_\Omega \nabla U_i^+ \cdot \nabla \eta\, d\, xd\, t| + |\int_0^T \int_\Omega \phi(U_i^+)\sum_{j=1}^m \nabla U_j^+ \cdot \nabla\eta\, d\, xd\, t| \\
&+|\gamma_i \int_0^T \int_\Omega U_i^+\, \eta\, d\, xd\, t| + |\int_0^T \int_\Omega \phi(U_i^+)\sum_{j=1}^m \phi(U_j^-)\,\eta\, d\, xd\, t| \\
&:= \hat{S}_1 + \hat{S}_2 + \hat{S}_3 + \hat{S}_4, \qquad i = 1, ..., m. \qquad (4.3.35)
\end{aligned}
$$

We proceed to bound each of the terms $\hat{S}_1, ..., \hat{S}_4$. We shall use throughout the rest of this section test functions η such that

$$\eta \in L^2(0,T;H^1(\Omega)).$$

We begin by considering $\hat{S}_1$. We the Hölder's inequality and (4.2.7) to arrive for all $\eta \in L^2(0,T;H^1(\Omega))$,

$$\hat{S}_1 = |D_i \int_0^T \int_\Omega \nabla U_i^+ \cdot \nabla \eta \, d\,x d\,t| \le D_i \int_0^T |U_i^+|_1 |\eta|_1 \, d\,t$$

$$\le D_i \|U_i^+\|_{L^2(0,T;H^1(\Omega))} \|\eta\|_{L^2(0,T;H^1(\Omega))} \le C\|\eta\|_{L^2(0,T;H^1(\Omega))}. \tag{4.3.36}$$

Next, we consider term $\hat{S}_2$. We observe using the Hölder's inequality, (4.2.30) and (4.2.29) that

$$\hat{S}_2 = |\int_0^T \int_\Omega \phi(U_i^+) \sum_{j=1}^m \nabla U_j^+ \cdot \nabla \eta \, d\,x d\,t| \le \int_0^T \|\phi(U_i^+)\|_\infty \sum_{j=1}^m \|\nabla U_j^+\| \|\nabla \eta\| d\,t$$

$$\le \int_0^T \|U_i^+\|_\infty \sum_{j=1}^m \|\nabla U_j^+\| \|\nabla \eta\| d\,t$$

$$\le \|U_i^+\|_{L^\infty(\Omega_T)} \sum_{j=1}^m \|U_j^+\|_{L^2(0,T;H^1(\Omega))} \|\eta\|_{L^2(0,T;H^1(\Omega))} \le C\|\eta\|_{L^2(0,T;H^1(\Omega))}. \tag{4.3.37}$$

We are ready to consider $\hat{S}_3$. Employing the Hölder's inequality, (4.2.7), (4.2.29) and the embedding result $L^\infty(\Omega_T) \hookrightarrow L^2(\Omega_T)$ yields

$$\hat{S}_3 = |\gamma_i \int_0^T \int_\Omega U_i^+ \, \eta \, d\,x d\,t| \le \gamma_i \|U_j^+\|_{L^2(\Omega_T)} \|\eta\|_{L^2(\Omega_T)} \le C\|\eta\|_{L^2(\Omega_T)}. \tag{4.3.38}$$

Now, we consider term $\hat{S}_4$. We employ, the Hölder's inequality, (4.2.29) to see for all $\eta \in L^2(0,T;W^{1,\infty}(\Omega))$ that

$$\hat{S}_4 = |\int_0^T \int_\Omega \phi(U_i^+) \sum_{j=1}^m \phi(U_j^-) \, \eta \, d\,x d\,t| \le \int_0^T \|\phi(U_i^+)\|_\infty \sum_{j=1}^m \|\phi(U_j^-)\| \, \|\eta\| \, d\,t$$

$$\le \int_0^T \|U_i^+\|_\infty \sum_{j=1}^m \|U_j^-\| \, \|\eta\| \, d\,t$$

$$\le \|U_i^+\|_{L^\infty(\Omega_T)} \sum_{j=1}^m \|U_j^-\|_{L^2(\Omega_T)} \, \|\eta\|_{L^2(\Omega_T)} \le C\|\eta\|_{L^2(\Omega_T)}. \tag{4.3.39}$$

Upon substituting the bounds on the terms $\hat{S}_1$ to $\hat{S}_4$ into (4.3.35), with $\eta \in L^2(0,T;W^{1,\infty}(\Omega))$, and noting the embedding results $L^2(0,T;H^1(\Omega)) \hookrightarrow L^2(\Omega_T)$, we deduce from (4.3.35) that

$$|\int_0^T \int_\Omega \frac{\partial U_i}{\partial t}\,\eta\,dx\,dt| \leq C\|\eta\|_{L^2(0,T;H^1(\Omega))}, \qquad i = 1,...,m. \tag{4.3.40}$$

Thus, we deduce that

$$\left\|\frac{\partial U_i}{\partial t}\right\|^2_{L^2(0,T;(H^1(\Omega))')} \leq C. \qquad i = 1,...,m. \tag{4.3.41}$$

4.4 Passage to the limit $M \to \infty$

We note that we have had to assume that U_i is in $L^\infty(\Omega_T)$, but as this is an artificial assumption, in the following theorem, we exclude any convergence properties associated with this assumption.

Theorem 4.4.1 Suppose that $\Delta t = o(M^{-1})$, then, there exists a subsequence of $\{U_i^\pm, i = 1,...,m\}_{M>1}$ (denoted by the same sequence), and functions $\{u_i, i = 1,...,m\}$ such that

$$u_i \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;L^s(\Omega)), \tag{4.4.42}$$

with $u_i(\mathbf{x},t) \geq 0, i = 1,...,m$ almost everywhere. Moreover, it holds as $M \to \infty$ (and thereby $\Delta t \to 0^+$), that for $i = 1,...,m$

$$U_i, U_i^\pm \rightharpoonup u_i \qquad \text{in} \;\; L^2(0,T;H^1(\Omega)), \tag{4.4.43}$$

$$U_i, U_i^\pm \rightharpoonup^* u_i, \qquad \text{in} \qquad L^\infty(0,T;L^2(\Omega)), \tag{4.4.44}$$

$$\frac{\partial U_i}{\partial t} \rightharpoonup \frac{\partial u_i}{\partial t} \qquad \text{in} \;\; L^2(0,T;(H^1(\Omega))'), \tag{4.4.45}$$

$$U_i, U_i^\pm \to u_i, \qquad \text{in} \;\; L^2(0,T;L^s(\Omega)), \tag{4.4.46}$$

$$\phi(U_i^\pm) \to u_i \qquad \text{in} \;\; L^2(0,T;L^s(\Omega)), \tag{4.4.47}$$

for any

$$s \in \begin{cases} [2, \infty] & \text{if } d = 1, \\ [2, \infty) & \text{if } d = 2, \\ [2, 6] & \text{if } d = 3. \end{cases}$$

The function $\{u_i, i = 1, ..., m\}$ is a global weak solution to problem (P) in the sense that

$$\int_0^T [\langle \frac{\partial u_i}{\partial t}, \eta \rangle_{H^1(\Omega)} + D_i(\nabla u_i, \nabla \eta) + (u_i \sum_{j=1}^m \nabla u_j, \nabla \eta)] dt$$

$$= \int_0^T [(\gamma_i u_i - u_i \sum_{j=1}^m u_j, \eta)] dt, \quad \forall \eta \in L^2(0, T; H^1(\Omega)), \quad i = 1, ..., m. \quad (4.4.48)$$

Proof: On recalling the weak* compactness of bounded balls in the Banach space $L^\infty(0, T; L^2(\Omega))$ and noting the first bound on (4.2.30), upon three successive extractions of subsequences, we deduce the existence of an unbounded index set $\mathcal{M} \subset (1, \infty)$ such that each of the three sequences $\{U_i, U_i^\pm\}$ converges to its respective weak* limit in $L^\infty(0, T; L^2(\Omega))$ as $M \to \infty$ with $M \in \mathcal{M}$. Thanks to (3.3.35), (3.3.36) and (3.3.37)

$$\int_0^T \|U_i - U_i^+\|^2 ds = \frac{1}{3} \int_0^T \|U_i^+ - U_i^-\|^2 ds \leq \frac{1}{3} C \Delta t, \quad (4.4.49)$$

where the last inequality is a consequence of the second bound in (4.2.30). On passing to the limit $\Delta t \to 0$ and using (4.2.30) we thus deduce that the weak* limits of the sequences $\{U_i, U_i^\pm\}$ coincide. We label this common limit by u_i; by construction then, $u_i \in L^\infty(0, T; L^2(\Omega))$. Thus we have shown (4.4.44).

Upon further successive extraction of subsequences from $\{U_i, U_i^\pm\}$, and noting the third bound on (4.2.30), the limits (4.4.43) follow directly from the weak compactness of bounded balls in the Hilbert spaces $L^2(0, T; H^1(\Omega))$ and the uniqueness of limits of sequences in the weak topology of $L^2(0, T; H^1(\Omega))$. Thus, the result (4.4.43) holds.

Next, since $\{\frac{\partial U_i}{\partial t}\}_h \in L^2(0, T, (H^1(\Omega))')$ and $L^2(0, T, (H^1(\Omega))')$ are reflexive Banach spaces then according to the weak compactness theorem, there exist a subsequences $\{\frac{\partial U_i}{\partial t}\}_M \in L^2(0, T, (H^1(\Omega))')$ and a functions $\widetilde{\eta} \in L^2(0, T, (H^1(\Omega))')$ such that

$$\frac{\partial U_i}{\partial t} \rightharpoonup \widetilde{\eta} \qquad \text{in } L^2(0,T,(H^1(\Omega))').$$

A well known argument can be easily adapted to show that $\widetilde{\eta} = \frac{\partial u_i}{\partial t}$, (see Robinson [84], page 204). Thus, the result (4.4.45) holds.

From an application of the Lions-Aubin theorem, see (2.1.6), on noting the following embedding results

$$H^1(\Omega) \overset{c}{\hookrightarrow} L^s(\Omega) \hookrightarrow (H^1(\Omega))',$$

which hold from the Rellich-Kondrachov theorem under the stated choice of s, we find that

$$W_u = \{\eta : \eta \in L^2(0,T;H^1(\Omega)), \frac{\partial \eta}{\partial t} \in L^2(0,T;(H^1(\Omega))'\} \overset{c}{\hookrightarrow} L^2(0,T;L^s(\Omega)).$$

As $U_i \in L^2(0,T;H^1(\Omega))$ and $\frac{\partial U_i}{\partial t} \in L^2(0,T;(H^1(\Omega))'$, thus, $U_i \in W_u$, then we can extract a subsequence, still denoted u_i, such that the convergence result (4.4.46) holds.

Next from the Lipschitz continuity of ϕ, we obtain for any $s < \infty$ that

$$\begin{aligned} \|\phi(U_i^{\pm}) - u_i\|_{L^2(0,T;L^s(\Omega))} &\leq \|u_i - \phi(u_i)\|_{L^2(0,T;L^s(\Omega))} + \|\phi(u_i) - \phi(U_i^{\pm})\|_{L^2(0,T;L^s(\Omega))} \\ &\leq \|u_i - \phi(u_i)\|_{L^2(0,T;L^s(\Omega))} + C\|u_i - U_i^{\pm}\|_{L^2(0,T;L^s(\Omega))}. \end{aligned} \tag{4.4.50}$$

The first term on the right-hand side of (4.4.50) converges to zero as $M \to \infty$ on noting that $\phi(u_i)$ converges to u_i almost everywhere on $\Omega \times [0,T]$ and applying Lebesgues dominated convergence theorem, see Appendix A.1.20. The second term converges to 0 on noting (4.4.46). That yields the desired result (4.4.47).

For any $\eta \in L^\infty(0,T;W^{1,\infty}(\Omega))$, we set $\chi \equiv \eta$ as a test function in (3.3.64) yielding

$$\begin{aligned} &\int_0^T [(\frac{\partial U_i}{\partial t}, \eta) + D_i(\nabla U_i^+, \nabla \eta) + (\phi(U_i^+) \sum_{j=1}^m \nabla U_j^+, \nabla \eta)] dt \\ &= \int_0^T [(\gamma_i U_i^+ - \phi(U_i^+) \sum_{j=1}^m \phi(U_j^-), \eta)] dt, \qquad i = 1, ..., m. \end{aligned} \tag{4.4.51}$$

We shall now study the convergence of each term in (4.4.51) separately. Passing to the limit on the first term in (4.4.51) is easy. Using (4.4.45) we immediately have that

$$\int_0^T \int_\Omega \frac{\partial U_i}{\partial t}\, \eta \, dx\, dt = \int_0^T \langle \frac{\partial U_i}{\partial t}\, \eta \rangle_{H^1(\Omega)} \, dt \to \int_0^T \langle \frac{\partial u_i}{\partial t}\, \eta \rangle_{H^1(\Omega)} \, dt, \qquad i = 1, ..., m, \tag{4.4.52}$$

as $M \to \infty$ (and $\Delta t \to 0_+$), for $\eta \in L^\infty(0, T; W^{1,\infty}(\Omega))$, as required.

The second term in (4.4.51) is dealt with as follows:

$$D_i \int_0^T \int_\Omega \nabla U_i^+ \, \nabla \eta d x\, dt \to D_i \int_0^T \int_\Omega \nabla u_i \, \nabla \eta d x\, dt. \tag{4.4.53}$$

The third term in (4.4.51) will be dealt with by decomposing it into two further terms, the first of which tends to 0, while the second converges to the expected limiting value. We proceed as follows:

$$\begin{aligned} &\int_0^T \int_\Omega \phi(U_i^+) \sum_{j=1}^m \nabla U_j^+ \cdot \nabla \eta d x\, dt \\ &= \int_0^T \int_\Omega (\phi(U_i^+) - u_i) \sum_{j=1}^m \nabla U_j^+ \cdot \nabla \eta d x\, dt + \int_0^T \int_\Omega u_i \sum_{j=1}^m \nabla U_j^+ \cdot \nabla \eta d x\, dt \\ &=: V_1 + V_2. \end{aligned} \tag{4.4.54}$$

We shall show that V_1 converges to 0 and that V_2 converges to the expected limit.

$$\begin{aligned} |V_1| &\le \int_0^T \int_\Omega |\phi(U_i^+) - u_i| \sum_{j=1}^m |\nabla U_j^+| \, |\nabla \eta| d x\, dt \\ &\le \|\phi(U_i^+) - u_i\|_{L^2(\Omega_T)} \sum_{j=1}^m \|U_j^+\|_{L^2(0,T;H^1(\Omega))} \, \|\eta\|_{L^\infty(0,T;W^{1,\infty}(\Omega))} d x\, dt. \end{aligned}$$

The second term is bounded by (4.4.43). The norm of the difference of the bound on V_1 is known to converge to 0 as $M \to \infty$ (and $\Delta t \to 0_+$), on noting (4.4.47) and the embedding result $L^2(0, T; L^s(\Omega)) \hookrightarrow L^2(\Omega_T)$. This then implies that the term V_1 converges to 0 as $M \to \infty$ (and $\Delta t \to 0_+$).

Concerning the term V_2, we have that

$$V_2 = \int_0^T \int_\Omega u_i \sum_{j=1}^m \nabla U_j^+ \cdot \nabla \eta d x\, dt \to \int_0^T \int_\Omega u_i \sum_{j=1}^m \nabla u_j \cdot \nabla \eta d x\, dt, \qquad i = 1, ..., m, \tag{4.4.55}$$

as $M \to \infty$ (and $\Delta t \to 0_+$).

The fourth term in (4.4.51) is easily shown to converge to following limit:

$$\gamma_i \int_0^T \int_\Omega U_i^+ \eta d\,x\,d\,t \to \gamma_i \int_0^T \int_\Omega u_i \eta d\,x\,d\,t, \qquad i = 1, ..., m. \tag{4.4.56}$$

Next, the last term in (4.4.51) can be divided into two part as follows

$$\begin{aligned} &- \int_0^T \int_\Omega \phi(U_i^+) \sum_{j=1}^m \phi(U_j^-) \eta d\,x\,d\,t \\ &= - \int_0^T \int_\Omega (\phi(U_i^+) - u_i) \sum_{j=1}^m \phi(U_j^-) \eta d\,x\,d\,t \\ &- \int_0^T \int_\Omega u_i \sum_{j=1}^m \phi(U_j^-) \eta d\,x\,d\,t =: V_3 + V_4. \end{aligned} \tag{4.4.57}$$

With the aid of the Hölder's inequality, we have

$$\begin{aligned} |V_3| &= | \int_0^T \int_\Omega (\phi(U_i^+) - u_i) \sum_{j=1}^m \phi(U_j^-) \eta d\,x\,d\,t| \\ &\leq \int_0^T \int_\Omega |\phi(U_i^+) - u_i| \sum_{j=1}^m |\phi(U_j^-)|\, |\eta| d\,x\,d\,t \\ &\leq \|\phi(U_i^+) - u_i\|_{L^2(\Omega_T)} \sum_{j=1}^m \|U_j^-\|_{L^2(\Omega_T)} \|\eta\|_{L^\infty(\Omega_T)}. \end{aligned}$$

Thus, we deduce that term V_3 converges to 0 as $M \to \infty$ (and $\Delta t \to 0_+$) on noting the embedding $L^2(0, T; L^s(\Omega)) \hookrightarrow L^2(\Omega_T)$ and $L^\infty(0, T; W^{1,\infty}(\Omega)) \hookrightarrow L^\infty(\Omega_T)$. It is clear that the second part converge to the expected limit. This ends the proof of the theorem.

Chapter 5

The population model: Numerical experiments

In this chapter we shall perform numerical experiments in one and two space dimensions which verify the theoretical results derived above and to show the growth behaviour of the solutions. All simulations were run by programs written in the Matlab programming language. In Section 5.1 we present a practical algorithm for computing the numerical solution. We then introduce the numerical experiments in one and two space dimensions in Sections 5.1.1 and 5.1.2, respectively. In Section 5.2.1 we discuss computational results of the fully-discrete scheme in one space dimension. Finally, the results of two dimensional simulations are presented in Section 5.2.2.

5.1 The population model: Numerical experiments

We first introduce the following practical algorithm to solve the nonlinear algebraic system arising from the approximate problem $(\mathrm{P}^{h\,\Delta t}_{M,\varepsilon})$ at each time level:

$(\mathrm{P}^{h\,\Delta t,k}_{M,\varepsilon})$: Given $\{U^{n,0}_{i,\varepsilon}, i = 1,...,m\} \in S^h \times ... \times S^h$ for $k \geq 1$ find $\{U^{n,k}_{i,\varepsilon}, i = 1,...,m\} \in S^h \times ... \times S^h$ such that for all $\chi \in S^h$

$$(\frac{U^{n,k}_{i,\varepsilon} - U^{n-1}_{i,\varepsilon}}{\Delta t_n}, \chi)^h + (D_i \nabla U^{n,k}_{i,\varepsilon} + \Lambda_\varepsilon(U^{n,k-1}_{i,\varepsilon}) \sum_{j=1}^{m} \nabla U^{n,k}_{j,\varepsilon}, \nabla \chi)$$

$$= (\gamma_i U_{i,\varepsilon}^{n,k} - \phi_\varepsilon(U_{i,\varepsilon}^{n,k-1}) \sum_{j=1}^{m} \phi_\varepsilon(U_{j,\varepsilon}^{n-1}), \chi)^h, \qquad i = 1, ..., m. \tag{5.1.1}$$

We start with $U_{i,\varepsilon}^0 \equiv \pi^h u_i^0$ and we set, for $n \geq 1$, $U_{i,\varepsilon}^{n,0} \equiv U_{i,\varepsilon}^{n-1}$. We can write (5.1.1) as a system of $m \times (J+1)^d, d = 1, 2, 3$ linear equations, simply by testing (5.1.1) with $\varphi_j, j = 0, ..., J$. For our numerical results, we set $TOL = 10^{-6}$ and adopt the stopping criteria

$$|U_{i,\varepsilon}^{n,k} - U_{i,\varepsilon}^{n,k-1}|_{0,\infty} < TOL, \tag{5.1.2}$$

i.e. for k satisfying (5.1.2) we set $U_{i,\varepsilon}^n \equiv U_{i,\varepsilon}^{n,k}, i = 1, ..., m$.

Programs were written in Matlab. The resulting linear systems were solved directly with sparse matrix facilities in Matlab. Although, we have been unable to prove convergence of $U_{i,\varepsilon}^{n,k}, i = 1, ..., m$ to $U_{i,\varepsilon}^n, i = 1, ..., m$ for n fixed, good convergence properties have been observed in practise. We found that the iterative method always converged well (only a few steps were required to fulfill the stopping criteria at each time level).

5.1.1 One-dimensional simulations

Numerical simulations in one space dimension were performed with $\Omega = [0, L]$, for $0 \leq t \leq T$ with mesh points $x_j = jh,\ j = 0, ..., J$ where $h = L/J$. In all simulations we take $J = 200$. Thus, the equation is posed on the interval $\Omega = [0, L] = [0, 2]$ with $\Delta t = 0.001$ and $h = 0.01$. We consider the initial boundary conditions:

$$u_1(x, 0) = 0.2 - 0.1\cos(2\pi x), \quad u_2(x, 0) = 0.3 - 0.3\cos(2\pi x), \quad u_3(x, 0) = 0.5 - 0.5\cos(2\pi x). \tag{5.1.3}$$

5.1.2 Two-dimensional simulations

We take $\Omega = [0, L]^2$ and a square uniform mesh with vertices $(x_i, y_j) = (ih, jh)$, where $i, j = 0, ..., J$ (see Figure 5.1). Note $h = L/J$, i.e., we used the same space step in both the x and y directions. We employ a 'right-angled' triangulation where each square is bisected by a diagonal running from the top-right corner to the bottom-left corner. Nodes are ordered in the 'natural way', that is, we number the nodes consecutively left to right starting with the bottom row. We implemented the fully-discrete finite element approximation, except now we have $m\times(J+1)^2$ unknowns and the resulting linear system has a block matrix structure. As in the one dimensional case, the linear system is strictly diagonally dominant for Δt sufficiently small and so no partial pivoting is required. We consider the initial boundary conditions:

$$u_1(x,y,0) = 0.5 + 0.25\cos(2\eta\pi x) + 0.25\cos(2\eta\pi y),$$
$$u_2(x,y,0) = 0.5 - 0.25\cos(2\eta\pi x) - 0.25\cos(2\eta\pi y). \quad (5.1.4)$$

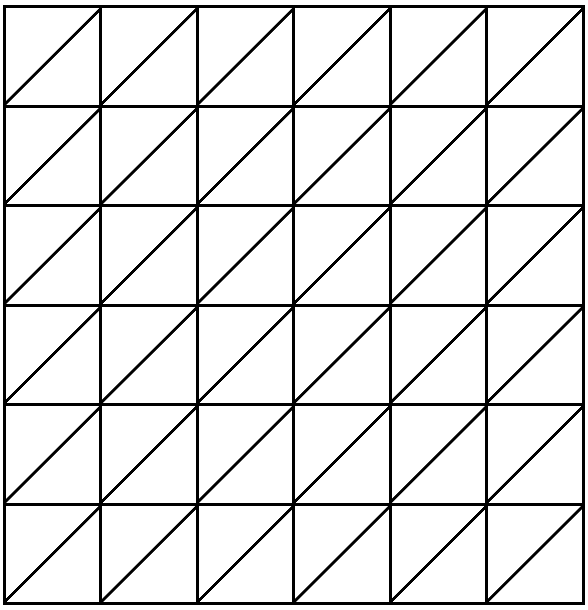


Figure 5.1: Right-angled uniform mesh for two dimensional simulations.

5.2 Numerical results

5.2.1 One-dimensional experiments

Firstly, we considered the dynamics of three interacting cell populations in one dimensional space. We choose the parameters such that $D = D_1 = D_2 = D_3 = 1$ and $M = 10$. To discuss how the parameters could reflect a competitive advantage of certain cell populations over the others, firstly we performed the experiment for

$\gamma_1 = \gamma_2 = \gamma_3 = 1$ then secondly we select $\gamma_1 = 1, \gamma_2 = 2$ and $\gamma_3 = 4$. At several times, the results of numerical solution of $(\mathrm{P}_{M,\varepsilon}^{h,\Delta t})$ are plotted in Figure 5.2 and 5.3. We selected these times carefully to show the evolution of the interacting cells as t increases. We see that the solution arrives to a steady state for sufficiently large time. For $\gamma_1 = \gamma_2 = \gamma_3 = 1$, the cells evolve to form a homogeneous distribution, see Figure 5.2. The same behaviour is observed when $\gamma_1 = 1, \gamma_2 = 2$ and $\gamma_3 = 4$, but with a distinct advantage of the u_3 cells, see Figure 5.3.

Next, we repeated the same experiment but for $D = D_1 = D_2 = D_3 = 100$. In general, the behaviour was very similar, however, we arrive to the stationary solutions earlier than the case $D = 1$ for both $\gamma_1 = 1, \gamma_2 = 2$ and $\gamma_3 = 4$ and $\gamma_1 = \gamma_2 = \gamma_3 = 1$ see Figure 5.4.

In the previous experiments each population moves down its own gradient as claimed in [79]. Also, we observed that due to the large diffusivity in the case $D = D_1 = D_2 = D_3 = 100$, the movement to the direction of lower concentrations is faster than the case when $D = D_1 = D_2 = D_3 = 1$.

In all our previous experiments, the computed solution did not exceed the value M. Also, we repeated the above experiments for different values of $M > 10$ and obtained the same results. The question is: How we can choose a suitable value of M which leads to an accurate numerical solution to (P) a priori? Firstly, we can initially start with a value M which satisfies $\max_i \|U_{i,\varepsilon}^0\| \le M$ then we use the following criterion in the solver: For fixed n and k, if $\max_i \|U_{i,\varepsilon}^{n,k}\| > M$ then set $M = \max_i \|U_{i,\varepsilon}^{n,k}\|$ and recompute $\{U_{i,\varepsilon}^{n,k}\}_{i=1}^m$. This approach was used successfully throughout.

We note that the steady-state solution of (P) in space and time, denoted by $\{u_{1,c}, u_{2,c}, u_{3,c}\}$, is determined by the following equations

$$u_{1,c}(\gamma_1 - u_{1,c} - u_{2,c} - u_{3,c}) = 0,$$

$$u_{2,c}(\gamma_2 - u_{1,c} - u_{2,c} - u_{3,c}) = 0,$$

$$u_{3,c}(\gamma_3 - u_{1,c} - u_{2,c} - u_{3,c}) = 0.$$

For $\gamma_1 < \gamma_2 < \gamma_3$, the u_1 and u_2 cells will vanish in (P) due to the advantage of the u_3 cells, Therefore, the expected steady state solutions will be $u_{1,c} = 0$, $u_{2,c} = 0$ and

$u_{3,c} = \gamma_3$. In the case of $\gamma_1 = \gamma_2 = \gamma_3$, we clearly have either $u_{1,c} = u_{2,c} = u_{3,c} = 0$ or $u_{1,c} + u_{2,c} + u_{3,c} = \gamma_1 = \gamma_2 = \gamma_3$, and this has been satisfied by all numerical steady-state solutions in our experiments.

The rapid change of the solutions in Figure 5.5 is a point of interest. As an attempt to investigate whether such behaviour is due to the existence of a singularity when $D = 0$, we have repeated the experiment in Figure 5.5 for $D =$ 0.5, 0.2, 0.1, 0.01, 0.001 and 0 with a finer mesh (we took $h = 0.005$). The solutions at $t = 2$ for $U_{1,\varepsilon}$, $U_{2,\varepsilon}$ and $U_{3,\varepsilon}$ are plotted in Figure 5.5 (a), (b) and (c), respectively. As D decreases to zero, the solutions change rapidly at $x = 0$, 0.5, 1, 1.5 and 2. The solutions appear to be continuous but we expect there will be limited regularity when $D = 0$, i.e. $u_i \notin C^{0,1}$. We also note that the solutions behave smoothly outside the small neighborhoods of $x = 0$, 0.5, 1, 1.5 and 2. It may be possible in future work to investigate the behaviour of the solution around points of rapid change by performing small-parameter expansions(see the techniques used in [25]).

5.2.2 Two-dimensional experiment

In the second experiment, we considered the dynamics of two interacting cell populations in two-dimensional space. Due to the lack of an exact solution for the cross-diffusion equations, we compute errors in different norms using a numerical solution on a fine mesh as reference. To measure errors between such a reference solution z_{ref} and an approximate solution z_h, at time t_n, we will use normalized L^p-errors:

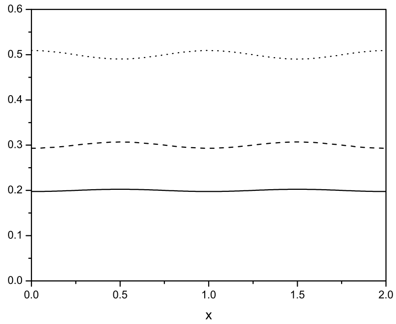
$$e_p^n = \frac{\|z_{ref}^n - z_h^n\|_p}{\|z_{ref}^n\|_p}, \quad p = 1, 2, \infty,$$

where

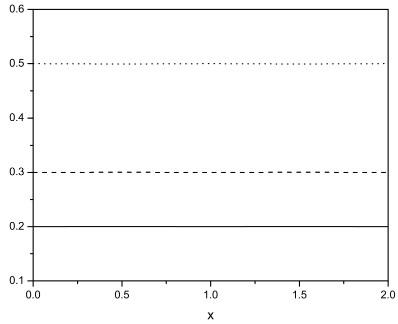
$$\|z_{ref}^n - z_h^n\|_\infty = \max_{i,j=0,\ldots,J} |z_{ref,i,j}^n - z_{h,i,j}^n|,$$

$$\|z_{ref}^n - z_h^n\|_p = \left(\frac{L}{(J+1)^2} \sum_{i=0}^{J} \sum_{j=0}^{J} |z_{ref,i,j}^n - z_{h,i,j}^n|^p \right)^{\frac{1}{p}}, \ \ p = 1, 2.$$

Here $z_{ref,i,j}^n$ stands for the projection of the reference solution onto the point i, j. Note that we don't use the exact norms in these computation as they can be difficult to calculate so instead use these approximate measures.



(a)



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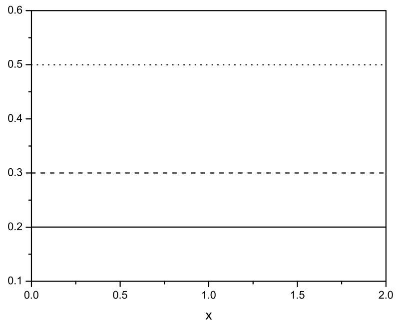

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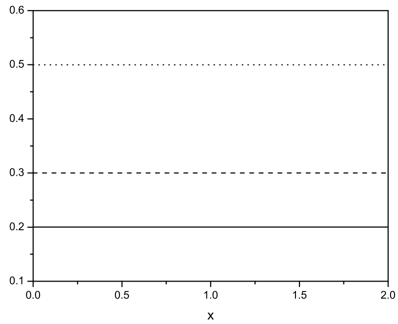

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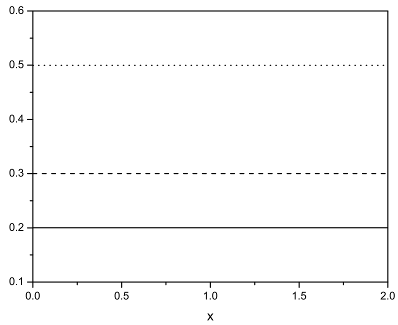


(e)

Figure 5.2: Numerical solutions of $(P_{M,\varepsilon}^{h\Delta t})$ in one dimension plotted at several times. The initial data are $u_1(x,0) = 0.2 - 0.1\cos(2\pi x)$, $u_2(x,0) = 0.3 - 0.3\cos(2\pi x)$ and $u_3(x,0) = 0.5 - 0.5\cos(2\pi x)$. The parameter values are: $D = 1$, $M = 10$, $\gamma_1 = \gamma_2 = \gamma_3 = 1$. The solid, dash, dot lines represent u_1, u_2, u_3, respectively. (a) $t = 0.1$, (b) $t = 0.2$, (c) $t = 0.5$, (d) $t = 1$, (e) $t = 4$.

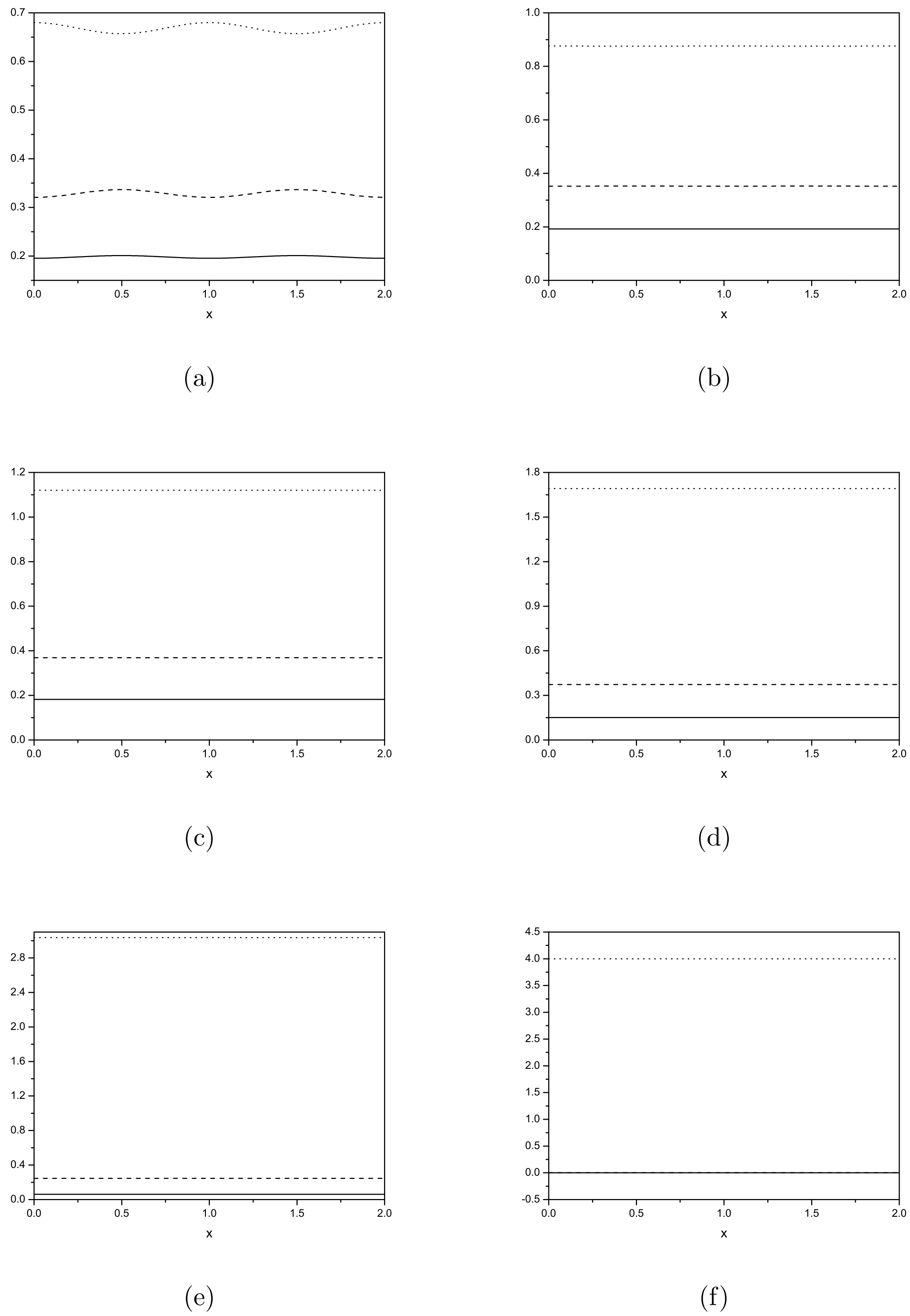


Figure 5.3: Numerical solutions of $(P_{M,\varepsilon}^{h\,\Delta t})$ in one dimension plotted at several times. The initial data are $u_1(x,0) = 0.2 - 0.1\cos(2\pi x)$, $u_2(x,0) = 0.3 - 0.3\cos(2\pi x)$ and $u_3(x,0) = 0.5 - 0.5\cos(2\pi x)$. The parameter values are: $D = 1$, $M = 10$, $\gamma_1 = 1$, $\gamma_2 = 2$ and $\gamma_3 = 4$. The solid, dash, dot lines represent u_1, u_2, u_3, respectively. (a) $t = 0.1$, (b) $t = 0.2$, (c) $t = 0.3$, (d) $t = 0.5$, (e) $t = 1$, (f) $t = 4$.

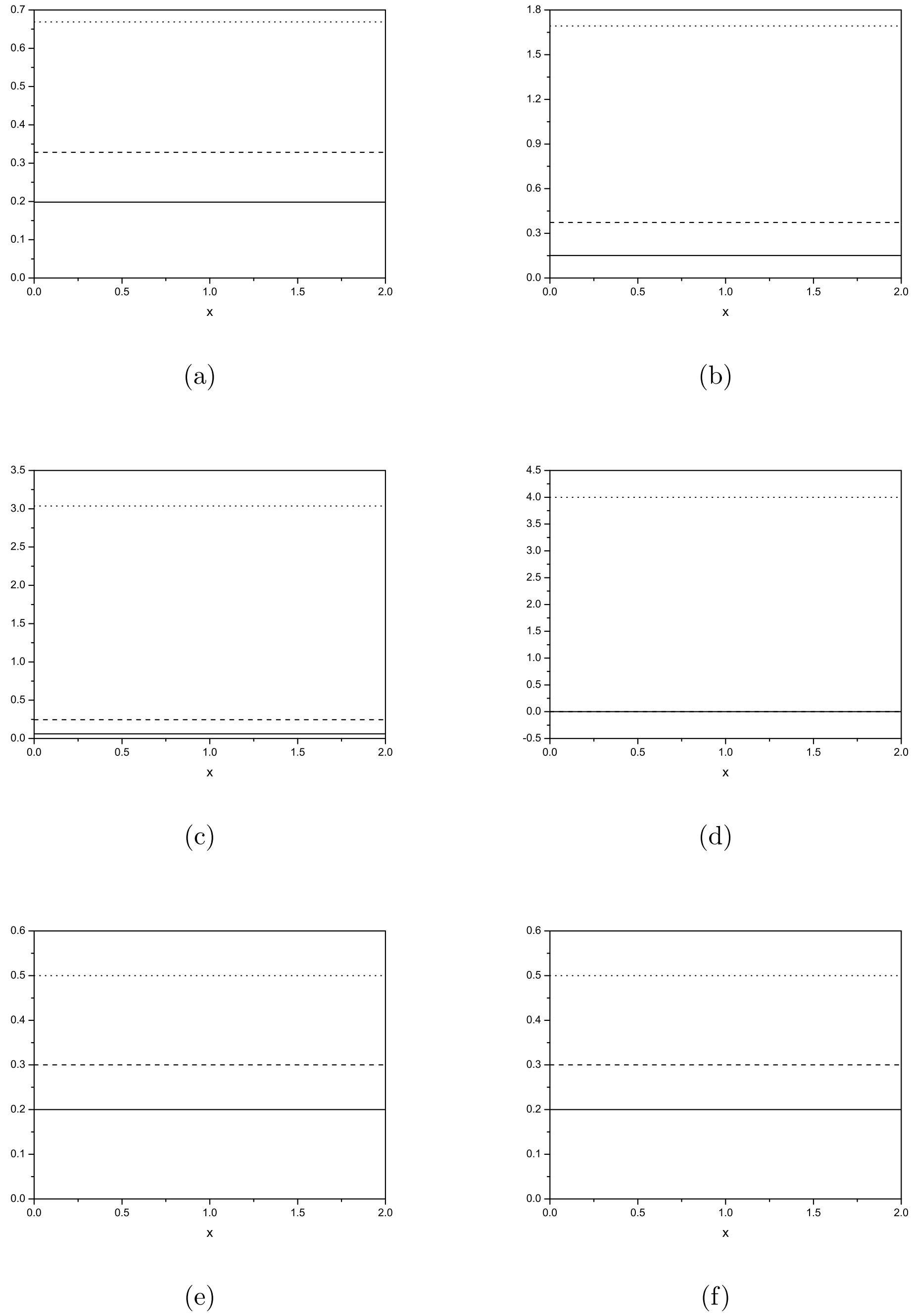


Figure 5.4: Numerical solutions of $(P^{h\,\Delta t}_{M,\varepsilon})$ in one dimension plotted at several times. The initial data are $u_1(x,0) = 0.2 - 0.1\cos(2\pi x)$, $u_2(x,0) = 0.3 - 0.3\cos(2\pi x)$ and $u_3(x,0) = 0.5 - 0.5\cos(2\pi x)$. The parameter values are: $D = 100$, $M = 10$, $\gamma_1 = 1$, $\gamma_2 = 2$ and $\gamma_3 = 4$ in (a), (b), (c) and (d) while $\gamma_1 = \gamma_2 = \gamma_3 = 1$ in (e) and (f). The solid, dash, dot lines represent u_1, u_2, u_3, respectively. (a) $t = 0.1$, (b) $t = 0.5$, (c) $t = 1$, (d) $t = 4$, (e) $t = 0.1$, (f) $t = 4$.

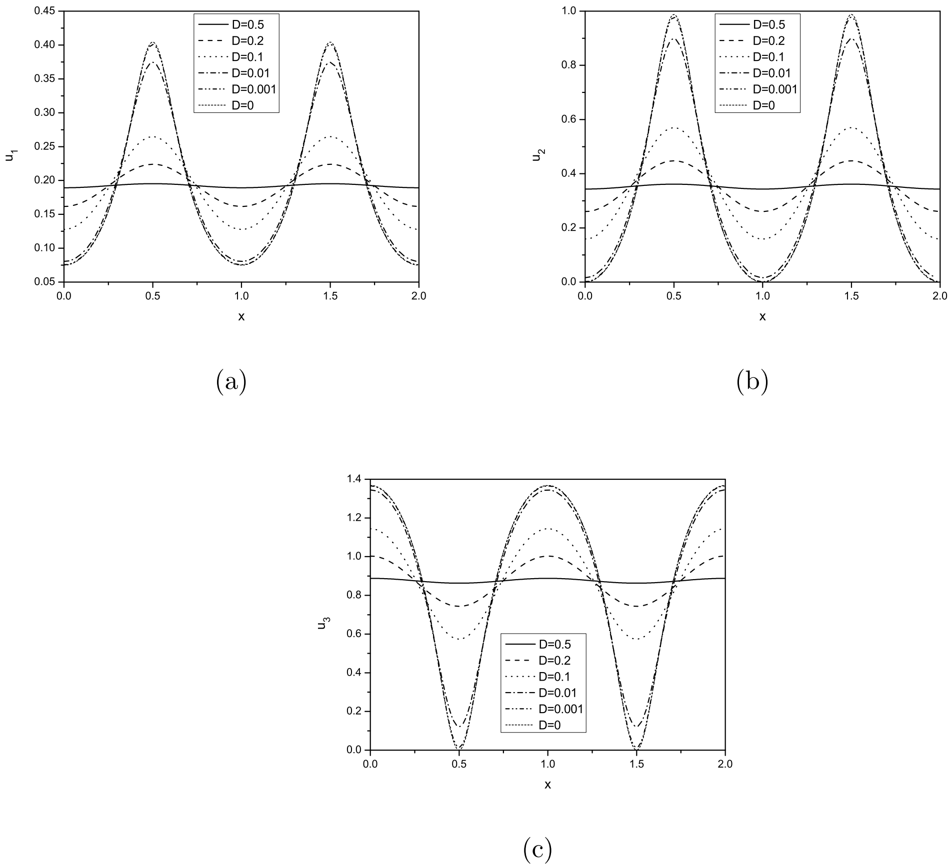
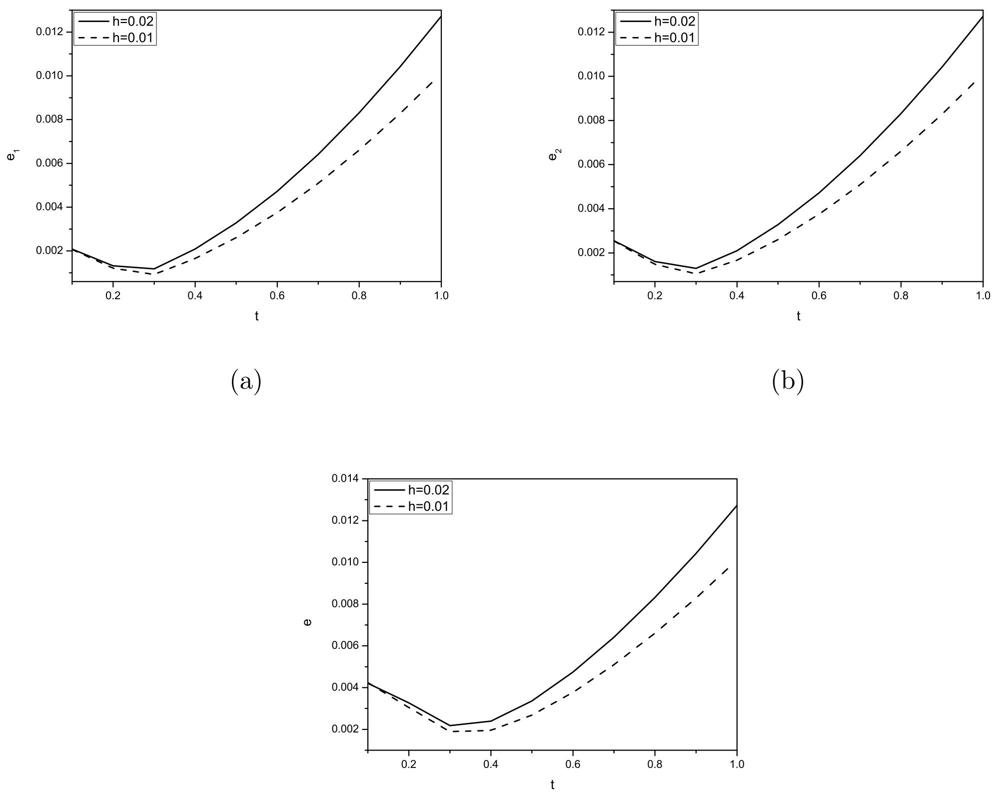


Figure 5.5: Numerical solutions of $(P^{h\,\Delta t}_{M,\varepsilon})$ in one dimension plotted at time $t = 2$. The initial data are $u_1(x, 0) = 0.2 - 0.1\cos(2\pi x)$, $u_2(x, 0) = 0.3 - 0.3\cos(2\pi x)$ and $u_3(x, 0) = 0.5 - 0.5\cos(2\pi x)$. The solutions are plotted for different parameter values of D with $M = 10$, $\gamma_1 = 1$, $\gamma_2 = 2$ and $\gamma_3 = 4$. (a) u_1, (b) u_2, (c) u_3.

The numerical test corresponds to the two dimensional cross-diffusion model endowed with zero-flux boundary conditions. The spatial domain is $\Omega = [0, L]^2$ and to perform the numerical simulations we adopt a set of parameters as follows: $\Delta t = 0.000002$, $L = 0.5$, $\eta = 2$, $D = D_1 = D_2 = 0.1$, $\gamma_1 = \gamma_2 = 1$ and $h = 0.02, 0.01$ and 0.005. We computed the reference solution z_{ref} with $h = 0.005$, then we compare this solution with the approximated solutions at $h = 0.02$ and $h = 0.01$. The corresponding error results for this example is given in Figure 5.6. This figure shows that the errors in the numerical solutions decrease roughly as the space-steps are decreased. Also, we notice from Figure 5.6 that the error in the approximate solutions increases with increasing the time.

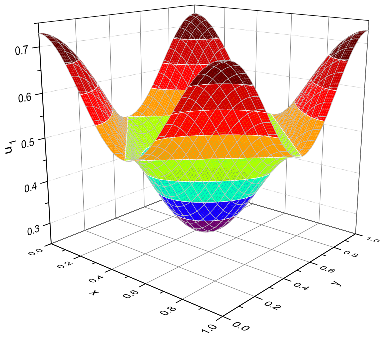
Next, we solve our problem in two dimensions with the following selections: $\Delta t = 0.00001$, $L = 1$, $\eta = 1$, $D_1 = 1 = D_2 = 1$, $\gamma_1 = \gamma_2 = 1$ and $h = 0.01$. In order to display the numerical results clearly, the solutions are plotted in Figures 5.7 and 5.8. A comparison of the species' behaviour can be analyzed from Figure 5.9, where we display profiles of the numerical solutions at time $t = 0.2, 0.4, 0.6, 1$ and 2 in a one dimensional slice of the domain, namely the level $y = 0.5$.



(a) (b)

(c)

Figure 5.6: Errors for u_1 in different norms versus the simulated time for different the mesh size h (a) L^1-norm, (b) L^2-norm, (c) L^∞-norm.



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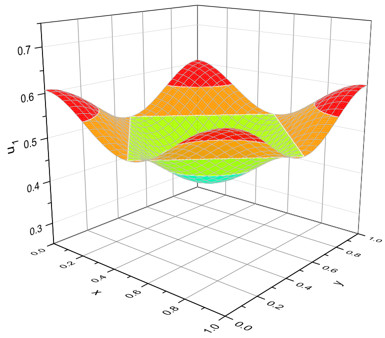

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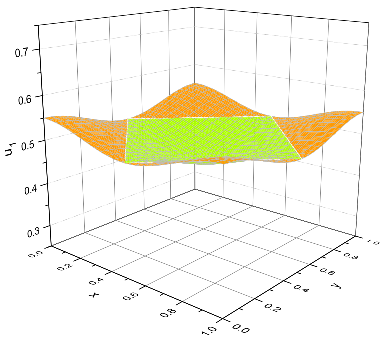

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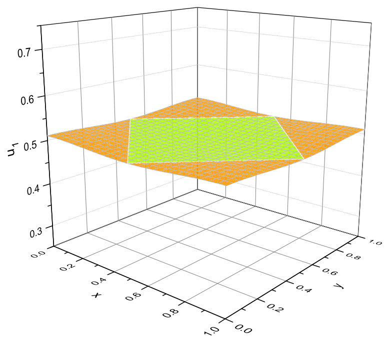

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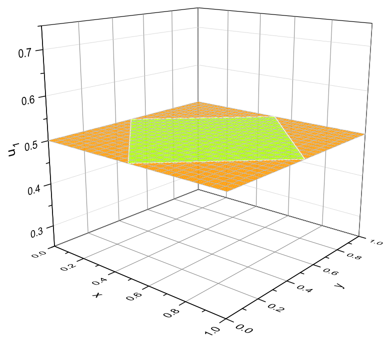
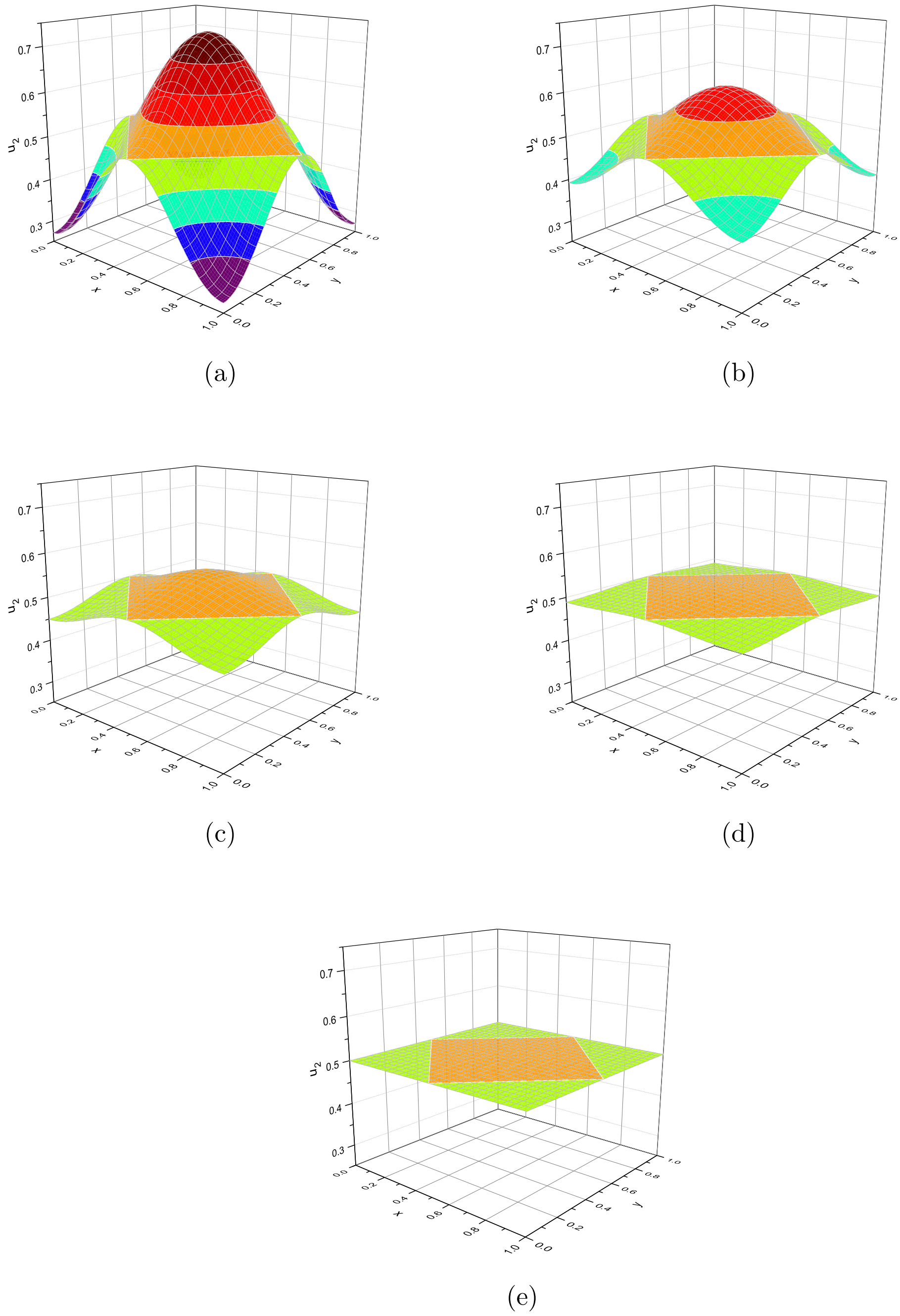


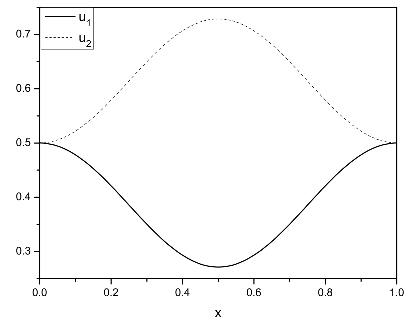
(e)

Figure 5.7: Model with cross-diffusion: Spread of a population for species u_1 at times (a) $t = 0.2$, (b) $t = 0.4$, (c) $t = 0.6$, (d) $t = 1$, (e) $t = 2$.

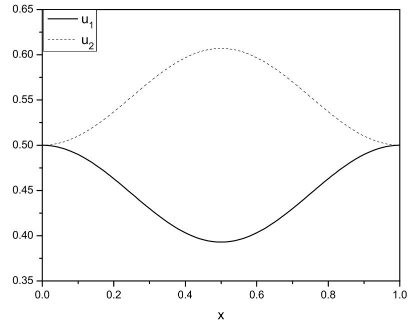


(a) (b) (c) (d) (e)

Figure 5.8: Model with cross-diffusion: Spread of a population for species u_2 at times (a) $t = 0.2$, (b) $t = 0.4$, (c) $t = 0.6$, (d) $t = 1$, (e) $t = 2$.



(a)



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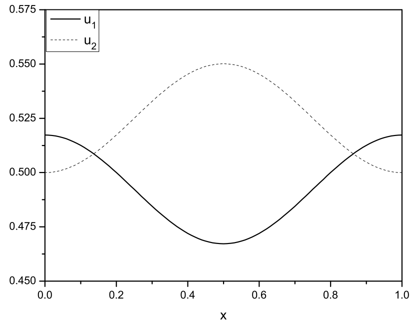

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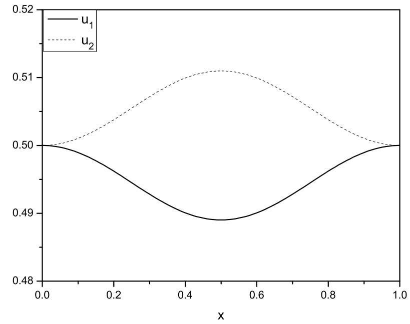

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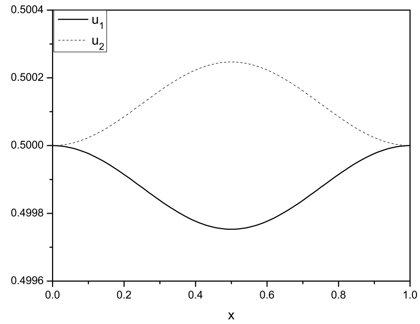


(e)

Figure 5.9: Model with cross-diffusion: Profile view at $y = 0.5$ of the spread of a population for species u_1 and u_2 at times (a) $t = 0.2$, (b) $t = 0.4$, (c) $t = 0.6$, (d) $t = 1$, (e) $t = 2$.

Chapter 6

Fully discrete approximation for a cross-diffusion tumor-growth model

In this chapter we discretise the cross-diffusion Tumor-growth model in space using a finite element method and discretise in time using finite differences. In Section 6.1, we present a fully discrete finite element approximation of problem (W). In Section 6.2 we prove the existence and uniqueness of the fully discrete approximations, while in Section 6.3 some stability estimates are proved.

6.1 A fully discrete approximation

The corresponding fully discrete regularized version of the problem (W) is:

$(W^{h,\Delta t}_{M,\varepsilon})$ For $n \geq 1$ find $\{\mathcal{C}^n_\varepsilon, \mathcal{M}^n_\varepsilon\} \in S^h \times S^h$ such that for all $\chi \in S^h$

$$(\frac{\mathcal{C}^n_\varepsilon - \mathcal{C}^{n-1}_\varepsilon}{\Delta t}, \chi)^h + (D^n_{11}\nabla\mathcal{C}^n_\varepsilon + D^n_{12}\nabla\mathcal{M}^n_\varepsilon, \nabla\chi)$$
$$= (-\gamma\phi_\varepsilon(\mathcal{C}^n_\varepsilon)\,\phi_\varepsilon(\mathcal{C}^{n-1}_\varepsilon + \mathcal{M}^{n-1}_\varepsilon - 1) - \delta\phi_\varepsilon(\mathcal{C}^n_\varepsilon), \chi)^h, \tag{6.1.1}$$

$$(\frac{\mathcal{M}^n_\varepsilon - \mathcal{M}^{n-1}_\varepsilon}{\Delta t}, \chi)^h + (D^n_{21}\nabla\mathcal{C}^n_\varepsilon + D^n_{22}\nabla\mathcal{M}^n_\varepsilon, \nabla\chi)$$
$$= -(\alpha\phi_\varepsilon(\mathcal{C}^{n-1}_\varepsilon)\,\phi_\varepsilon(\mathcal{M}^n_\varepsilon)\,\phi_\varepsilon(\mathcal{C}^{n-1}_\varepsilon + \mathcal{M}^{n-1}_\varepsilon - 1), \chi)^h, \tag{6.1.2}$$

where

$$D^n_{11}[\nabla\pi^h F'_\varepsilon(\mathcal{C}^n_\varepsilon)-\nabla\pi^h F'_\varepsilon(1-\mathcal{C}^n_\varepsilon-\mathcal{M}^n_\varepsilon)] = \frac{1}{\tau}\int_\tau \pi^h(2(1-\phi_\varepsilon(\mathcal{C}^n_\varepsilon))-\beta\theta\phi^2_\varepsilon(\mathcal{M}^n_\varepsilon))\ dx\ \nabla\mathcal{C}^n_\varepsilon$$

$$+\frac{1}{\tau}\int_\tau \pi^h(\frac{2\phi_\varepsilon(\mathcal{C}^n_\varepsilon)(1-\phi_\varepsilon(\mathcal{C}^n_\varepsilon))-\beta\theta(1-\phi_\varepsilon(\mathcal{M}^n_\varepsilon)-\phi_\varepsilon(1-\mathcal{C}^{n-1}_\varepsilon-\mathcal{M}^{n-1}_\varepsilon))\phi^2_\varepsilon(\mathcal{M}^n_\varepsilon)}{\phi_\varepsilon(1-\mathcal{C}^{n-1}_\varepsilon-\mathcal{M}^{n-1}_\varepsilon)}dx$$
$$\times(\nabla\mathcal{C}^n_\varepsilon+\nabla\mathcal{M}^n_\varepsilon), \tag{6.1.3}$$

$$D^n_{12}[\nabla\pi^h F'_\varepsilon(\mathcal{C}^n_\varepsilon)-\nabla\pi^h F'_\varepsilon(1-\mathcal{C}^n_\varepsilon-\mathcal{M}^n_\varepsilon)] = -\frac{2\beta}{\tau}\int_\tau \pi^h(\phi_\varepsilon(\mathcal{M}^n_\varepsilon)(1+\theta\phi_\varepsilon(\mathcal{C}^n_\varepsilon)))\ dx\ \nabla\mathcal{C}^n_\varepsilon$$

$$-\frac{2\beta}{\tau}\int_\tau \pi^h(\frac{(1-\phi_\varepsilon(\mathcal{M}^n_\varepsilon)-\phi_\varepsilon(1-\mathcal{C}^{n-1}_\varepsilon-\mathcal{M}^{n-1}_\varepsilon))\phi_\varepsilon(\mathcal{M}_\varepsilon)(1+\theta\phi_\varepsilon(\mathcal{C}^n_\varepsilon))}{\phi_\varepsilon(1-\mathcal{C}^{n-1}_\varepsilon-\mathcal{M}^{n-1}_\varepsilon)})dx$$
$$\times(\nabla\mathcal{C}^n_\varepsilon+\nabla\mathcal{M}^n_\varepsilon), \tag{6.1.4}$$

$$D^n_{21}[\nabla\pi^h F'_\varepsilon(\mathcal{M}^n_\varepsilon)-\nabla\pi^h F'_\varepsilon(1-\mathcal{C}^n_\varepsilon-\mathcal{M}^n_\varepsilon)]$$
$$=\frac{1}{\tau}\int_\tau \pi^h(-2\phi_\varepsilon(\mathcal{C}^n_\varepsilon)+\beta\theta(1-\phi_\varepsilon(\mathcal{M}^n_\varepsilon))\phi_\varepsilon(\mathcal{M}^n_\varepsilon))\ dx\ \nabla\mathcal{M}^n_\varepsilon$$

$$+\frac{1}{\tau}\int_\tau \pi^h(\frac{-2\phi_\varepsilon(\mathcal{C}^n_\varepsilon)(1-\phi_\varepsilon(\mathcal{C}^n_\varepsilon)-\phi_\varepsilon(1-\mathcal{C}^{n-1}_\varepsilon-\mathcal{M}^{n-1}_\varepsilon))+\beta\theta(1-\phi_\varepsilon(\mathcal{M}^n_\varepsilon))\phi^2_\varepsilon(\mathcal{M}^n_\varepsilon)}{\phi_\varepsilon(1-\mathcal{C}^{n-1}_\varepsilon-\mathcal{M}^{n-1}_\varepsilon)})dx$$
$$\times(\nabla\mathcal{C}^n_\varepsilon+\nabla\mathcal{M}^n_\varepsilon), \tag{6.1.5}$$

and

$$D^n_{22}[\nabla\pi^h F'_\varepsilon(\mathcal{M}^n_\varepsilon)-\nabla\pi^h F'_\varepsilon(1-\mathcal{C}^n_\varepsilon-\mathcal{M}^n_\varepsilon)] = \frac{2\beta}{\tau}\int_\tau(1-\phi_\varepsilon(\mathcal{M}^n_\varepsilon))(1+\theta\phi_\varepsilon(\mathcal{C}^n_\varepsilon))\ dx\ \nabla\mathcal{M}^n_\varepsilon$$

$$+\frac{2\beta}{\tau}\int_\tau \pi^h(\frac{\phi_\varepsilon(\mathcal{M}^n_\varepsilon)(1-\phi_\varepsilon(\mathcal{M}^n_\varepsilon))(1+\theta\phi_\varepsilon(\mathcal{C}^n_\varepsilon))}{\phi_\varepsilon(1-\mathcal{C}^{n-1}_\varepsilon-\mathcal{M}^{n-1}_\varepsilon)})\ dx\ (\nabla\mathcal{C}^n_\varepsilon+\nabla\mathcal{M}^n_\varepsilon), \tag{6.1.6}$$

subject to the initial conditions

$$C^0_\varepsilon=\pi^h c^0 \text{ or } P^h c^0, \quad C^0_\varepsilon=\pi^h m^0 \text{ or } P^h m^0 \quad \text{in } \Omega. \tag{6.1.7}$$

In the above equations, the regularized functions ϕ_ε and F_ε, and the parameter ε have been defined in Chapter 2. Here, the functions on the right hand side of (6.1.1)-(6.1.2) are considered to be appropriate to control the nonlinearity and obtain the intended entropy results. In the following lemma we derive the entropy inequality for the regularized problem $(W^{h,\Delta t}_{M,\varepsilon})$ which will provide us with some uniform bounds on the regularized solutions $\{\mathcal{C}^n_\varepsilon,\mathcal{M}^n_\varepsilon\}\in S^h\times S^h$.

Lemma 6.1.1 Let $\{\mathcal{C}_\varepsilon^{n-1}, \mathcal{M}_\varepsilon^{n-1}\} \in S^h \times S^h$ be given for some $n = 1, ..., N$. Then for all $\varepsilon \in (0, e^{-1})$, for all $h > 0$ such that

$$\Delta t \leq \frac{1}{2},$$

there exists a solution $\{\mathcal{C}_\varepsilon^n, \mathcal{M}_\varepsilon^n\} \in S^h \times S^h$ to the n-th step of $(W_{M,\varepsilon}^{h,\Delta t})$ such that

$$[1-2\Delta t]E(\mathcal{C}_\varepsilon^n, \mathcal{M}_\varepsilon^n)+K_\theta \Delta t|\mathcal{C}_\varepsilon^n|_1^2+K_\theta \Delta t|\mathcal{M}_\varepsilon^n|_1^2+\frac{1}{4M}|\mathcal{C}_\varepsilon^n|_h^2+\frac{1}{4M}|\mathcal{M}_\varepsilon^n|_h^2+\frac{1}{4M}|1-\mathcal{C}_\varepsilon^n-\mathcal{M}_\varepsilon^n|_h^2$$

$$\leq E(\mathcal{C}_\varepsilon^{n-1}, \mathcal{M}_\varepsilon^{n-1}) + \Delta t C(\delta, \gamma, \alpha, \varepsilon, M, |\Omega|) + \frac{1}{2M}\left[|\mathcal{C}_\varepsilon^{n-1}|_h^2 + |\mathcal{M}_\varepsilon^{n-1}|_h^2\right], \quad (6.1.8)$$

where $E(\mathcal{C}_\varepsilon^n, \mathcal{M}_\varepsilon^n) = (F_\varepsilon(\mathcal{C}_\varepsilon^n) + F_\varepsilon(\mathcal{M}_\varepsilon^n) + F_\varepsilon(1 - \mathcal{C}_\varepsilon^n - \mathcal{M}_\varepsilon^n), 1)^h$, and K_θ is a constant depending on θ, β and M.

Proof: Choosing $\chi \equiv \Delta t \pi^h F_\varepsilon'(\mathcal{C}_\varepsilon^n) - \Delta t \pi^h F_\varepsilon'(1 - \mathcal{C}_\varepsilon^n - \mathcal{M}_\varepsilon^n)$ as a test function in (6.1.1) and $\chi \equiv \Delta t \pi^h F_\varepsilon'(\mathcal{M}_\varepsilon^n) - \Delta t \pi^h F_\varepsilon'(1 - \mathcal{C}_\varepsilon^n - \mathcal{M}_\varepsilon^n)$ as a test function in (6.1.2) yields,

$$(\mathcal{C}_\varepsilon^n - \mathcal{C}_\varepsilon^{n-1}, \pi^h F_\varepsilon'(\mathcal{C}_\varepsilon^n) - \pi^h F_\varepsilon'(1 - \mathcal{C}_\varepsilon^n - \mathcal{M}_\varepsilon^n))^h$$

$$+(D_{11}^n \nabla \mathcal{C}_\varepsilon^n + D_{12}^n \nabla \mathcal{M}_\varepsilon^n, \Delta t \nabla \pi^h F_\varepsilon'(\mathcal{C}_\varepsilon^n) - \Delta t \nabla \pi^h F_\varepsilon'(1 - \mathcal{C}_\varepsilon^n - \mathcal{M}_\varepsilon^n))$$

$$= -(\gamma \phi_\varepsilon(\mathcal{C}_\varepsilon^n)\ \phi_\varepsilon(\mathcal{C}_\varepsilon^{n-1} + \mathcal{M}_\varepsilon^{n-1} - 1), \Delta t \pi^h F_\varepsilon'(\mathcal{C}_\varepsilon^n) - \Delta t \pi^h F_\varepsilon'(1 - \mathcal{C}_\varepsilon^n - \mathcal{M}_\varepsilon^n))^h$$

$$-(\delta \phi_\varepsilon(\mathcal{C}_\varepsilon^n), \Delta t \pi^h F_\varepsilon'(\mathcal{C}_\varepsilon^n))^h + (\delta \phi_\varepsilon(\mathcal{C}_\varepsilon^n), \Delta t \pi^h F_\varepsilon'(1 - \mathcal{C}_\varepsilon^n - \mathcal{M}_\varepsilon^n))^h, \quad (6.1.9)$$

and

$$(\mathcal{M}_\varepsilon^n - \mathcal{M}_\varepsilon^{n-1}, \pi^h F_\varepsilon'(\mathcal{M}_\varepsilon^n) - \pi^h F_\varepsilon'(1 - \mathcal{C}_\varepsilon^n - \mathcal{M}_\varepsilon^n))^h$$

$$+(D_{21}^n \nabla \mathcal{C}_\varepsilon^n + D_{22}^n \nabla \mathcal{M}_\varepsilon^n, \Delta t \nabla \pi^h F_\varepsilon'(\mathcal{M}_\varepsilon^n) - \Delta t \nabla \pi^h F_\varepsilon'(1 - \mathcal{C}_\varepsilon^n - \mathcal{M}_\varepsilon^n))$$

$$= -(\alpha \phi_\varepsilon(\mathcal{C}_\varepsilon^{n-1})\ \phi_\varepsilon(\mathcal{M}_\varepsilon^n)\ \phi_\varepsilon(\mathcal{C}_\varepsilon^{n-1}+\mathcal{M}_\varepsilon^{n-1}-1), \Delta t \pi^h F_\varepsilon'(\mathcal{M}_\varepsilon^n)-\Delta t \pi^h F_\varepsilon'(1-\mathcal{C}_\varepsilon^n-\mathcal{M}_\varepsilon^n))^h. \quad (6.1.10)$$

Firstly, it follows from (2.3.32) and (2.1.10) that

$$(\mathcal{C}_\varepsilon^n - \mathcal{C}_\varepsilon^{n-1}, F_\varepsilon'(\mathcal{C}_\varepsilon^n))^h \geq (F_\varepsilon(\mathcal{C}_\varepsilon^n) - F_\varepsilon(\mathcal{C}_\varepsilon^{n-1}), 1)^h + (\frac{1}{2}(\mathcal{C}_\varepsilon^n - \mathcal{C}_\varepsilon^{n-1})^2, F_\varepsilon''(\xi))^h$$

$$\geq (F_\varepsilon(\mathcal{C}_\varepsilon^n) - F_\varepsilon(\mathcal{C}_\varepsilon^{n-1}), 1)^h + \frac{1}{2M}|\mathcal{C}_\varepsilon^n - \mathcal{C}_\varepsilon^{n-1}|_h^2$$

$$\geq (F_\varepsilon(\mathcal{C}_\varepsilon^n) - F_\varepsilon(\mathcal{C}_\varepsilon^{n-1}), 1)^h + \frac{1}{4M}|\mathcal{C}_\varepsilon^n|_h^2 - \frac{1}{2M}|\mathcal{C}_\varepsilon^{n-1}|_h^2. \quad (6.1.11)$$

Similarly,

$$(\mathcal{M}_\varepsilon^n - \mathcal{M}_\varepsilon^{n-1}, F_\varepsilon'(\mathcal{M}_\varepsilon^n))^h \geq (F_\varepsilon(\mathcal{M}_\varepsilon^n) - F_\varepsilon(\mathcal{M}_\varepsilon^{n-1}), 1)^h + \frac{1}{4M}|\mathcal{M}_\varepsilon^n|_h^2 - \frac{1}{2M}|\mathcal{M}_\varepsilon^{n-1}|_h^2, \tag{6.1.12}$$

and

$$(-\mathcal{C}_\varepsilon^n - \mathcal{M}_\varepsilon^n + \mathcal{C}_\varepsilon^{n-1} + \mathcal{M}_\varepsilon^{n-1}, F_\varepsilon'(1 - \mathcal{C}_\varepsilon^n - \mathcal{M}_\varepsilon^n))^h$$
$$\geq (F_\varepsilon(1-\mathcal{C}_\varepsilon^n-\mathcal{M}_\varepsilon^n) - F_\varepsilon(1-\mathcal{C}_\varepsilon^{n-1}-\mathcal{M}_\varepsilon^{n-1}), 1)^h + \frac{1}{4M}|1-\mathcal{C}_\varepsilon^n-\mathcal{M}_\varepsilon^n|_h^2 - \frac{1}{2M}|1-\mathcal{C}_\varepsilon^{n-1}-\mathcal{M}_\varepsilon^{n-1}|_h^2. \tag{6.1.13}$$

Next, we are going to find a bound on the first term of right-hand side of (6.1.9). It follows from (2.3.32), (2.1.12), Young's inequality and (2.3.29) that

$$-\gamma(\phi_\varepsilon(\mathcal{C}_\varepsilon^n)\phi_\varepsilon(\mathcal{C}_\varepsilon^{n-1} + \mathcal{M}_\varepsilon^{n-1} - 1), \pi^h F_\varepsilon'(\mathcal{C}_\varepsilon^n) - \pi^h F_\varepsilon'(1 - \mathcal{C}_\varepsilon^n - \mathcal{M}_\varepsilon^n))^h$$
$$= -\gamma(\phi_\varepsilon(\mathcal{C}_\varepsilon^{n-1}+\mathcal{M}_\varepsilon^{n-1}-1), \phi_\varepsilon(\mathcal{C}_\varepsilon^n)\, F_\varepsilon'(\mathcal{C}_\varepsilon^n))^h + \gamma(\phi_\varepsilon(\mathcal{C}_\varepsilon^n)\phi_\varepsilon(\mathcal{C}_\varepsilon^{n-1}+\mathcal{M}_\varepsilon^{n-1}-1), F_\varepsilon'(1-\mathcal{C}_\varepsilon^n-\mathcal{M}_\varepsilon^n))^h$$
$$\leq \gamma(\phi_\varepsilon(\mathcal{C}_\varepsilon^{n-1}+\mathcal{M}_\varepsilon^{n-1}-1), 1-\mathcal{C}_\varepsilon^n)^h + \gamma(\phi_\varepsilon(\mathcal{C}_\varepsilon^n)\phi_\varepsilon(\mathcal{C}_\varepsilon^{n-1}+\mathcal{M}_\varepsilon^{n-1}-1), F_\varepsilon'(1-\mathcal{C}_\varepsilon^n-\mathcal{M}_\varepsilon^n))^h$$
$$\leq \frac{1}{2\varepsilon}([\mathcal{C}_\varepsilon^n]_-^2, 1)^h + \gamma(\phi_\varepsilon(\mathcal{C}_\varepsilon^n)\phi_\varepsilon(\mathcal{C}_\varepsilon^{n-1} + \mathcal{M}_\varepsilon^{n-1} - 1), F_\varepsilon'(1 - \mathcal{C}_\varepsilon^n - \mathcal{M}_\varepsilon^n))^h$$
$$+C(M, \varepsilon, \gamma, |\Omega|)$$
$$\leq (F_\varepsilon(\mathcal{C}_\varepsilon^n), 1)^h + \gamma(\phi_\varepsilon(\mathcal{C}_\varepsilon^n)\phi_\varepsilon(\mathcal{C}_\varepsilon^{n-1} + \mathcal{M}_\varepsilon^{n-1} - 1), F_\varepsilon'(1 - \mathcal{C}_\varepsilon^n - \mathcal{M}_\varepsilon^n))^h + C(M, \varepsilon, \gamma, |\Omega|). \tag{6.1.14}$$

To deal with the second term on the right-hand side of inequality (6.1.14), we partition the interval Ω as follows

$$\Omega = \mathfrak{I}_+ \bigcup \mathfrak{I}_-,$$

where

$$\mathfrak{I}_+ = \{i : \quad F_\varepsilon'((1 - \mathcal{C}_\varepsilon^n - \mathcal{M}_\varepsilon^n)(x_i)) \geq 0\},$$

and

$$\mathfrak{I}_- = \{i : \quad F_\varepsilon'((1 - \mathcal{C}_\varepsilon^n - \mathcal{M}_\varepsilon^n)(x_i)) < 0\},$$

then, we arrive

$$\gamma(\phi_\varepsilon(\mathcal{C}_\varepsilon^n)\phi_\varepsilon(\mathcal{C}_\varepsilon^{n-1} + \mathcal{M}_\varepsilon^{n-1} - 1), F_\varepsilon'(1 - \mathcal{C}_\varepsilon^n - \mathcal{M}_\varepsilon^n))^h$$
$$= \gamma \sum_{i\in\mathfrak{I}_+} \widehat{M}_{ii}\phi_\varepsilon(\mathcal{C}_\varepsilon^n(x_i))\phi_\varepsilon((\mathcal{C}_\varepsilon^{n-1} + \mathcal{M}_\varepsilon^{n-1} - 1)(x_i))\, F_\varepsilon'((1 - \mathcal{C}_\varepsilon^n - \mathcal{M}_\varepsilon^n)(x_i))$$

$$+\gamma\sum_{i\in\mathfrak{I}_-}\widehat{M}_{ii}\phi_\varepsilon(\mathcal{C}_\varepsilon^n(x_i))\phi_\varepsilon((\mathcal{C}_\varepsilon^{n-1}+\mathcal{M}_\varepsilon^{n-1}-1)(x_i))\,F'_\varepsilon((1-\mathcal{C}_\varepsilon^n-\mathcal{M}_\varepsilon^n)(x_i))$$

$$\le\gamma\sum_{i\in\mathfrak{I}_+}\widehat{M}_{ii}\phi_\varepsilon(\mathcal{C}_\varepsilon^n(x_i))\phi_\varepsilon((\mathcal{C}_\varepsilon^{n-1}+\mathcal{M}_\varepsilon^{n-1}-1)(x_i))\,F'_\varepsilon((1-\mathcal{C}_\varepsilon^n-\mathcal{M}_\varepsilon^n)(x_i))$$

$$=\gamma\sum_{i\in\mathfrak{I}_+}\widehat{M}_{ii}\frac{\phi_\varepsilon(\mathcal{C}_\varepsilon^n(x_i))\phi_\varepsilon((\mathcal{C}_\varepsilon^{n-1}+\mathcal{M}_\varepsilon^{n-1}-1)(x_i))}{\phi_\varepsilon((1-\mathcal{C}_\varepsilon^n-\mathcal{M}_\varepsilon^n)(x_i))}$$

$$\times\phi_\varepsilon((1-\mathcal{C}_\varepsilon^n-\mathcal{M}_\varepsilon^n)(x_i))F'_\varepsilon((1-\mathcal{C}_\varepsilon^n-\mathcal{M}_\varepsilon^n)(x_i)). \tag{6.1.15}$$

Now, since $(1-\mathcal{C}_\varepsilon^n-\mathcal{M}_\varepsilon^n)(x_i)\ge 1$ then $\phi_\varepsilon((1-\mathcal{C}_\varepsilon^n-\mathcal{M}_\varepsilon^n)(x_i))\ge\phi_\varepsilon(1)=1$, then, on noting (2.3.28), we have

$$\frac{\phi_\varepsilon((\mathcal{C}_\varepsilon^n)(x_i))\phi_\varepsilon((\mathcal{C}_\varepsilon^{n-1}+\mathcal{M}_\varepsilon^{n-1}-1)(x_i))}{\phi_\varepsilon((1-\mathcal{C}_\varepsilon^n-\mathcal{M}_\varepsilon^n)(x_i))}\le M^2.$$

Now, it follows from (2.3.32), Young's inequality and (2.3.29), that

$$\gamma(\phi_\varepsilon(\mathcal{C}_\varepsilon^n)\phi_\varepsilon(1-\mathcal{C}_\varepsilon^{n-1}-\mathcal{M}_\varepsilon^{n-1}),F'_\varepsilon(1-\mathcal{C}_\varepsilon^n-\mathcal{M}_\varepsilon^n))^h$$

$$\le\gamma M^2\sum_{i\in\mathfrak{I}_+}\widehat{M}_{ii}\phi_\varepsilon((1-\mathcal{C}_\varepsilon^n-\mathcal{M}_\varepsilon^n)(x_i))F'_\varepsilon((1-\mathcal{C}_\varepsilon^n-\mathcal{M}_\varepsilon^n)(x_i))$$

$$\le\gamma M^2\sum_{i\in\mathfrak{I}_+}\widehat{M}_{ii}(1-[(1-\mathcal{C}_\varepsilon^n-\mathcal{M}_\varepsilon^n)(x_i)]_-)$$

$$\le(F_\varepsilon(1-\mathcal{C}_\varepsilon^n-\mathcal{M}_\varepsilon^n),1)^h+C(\gamma,M,\varepsilon,|\Omega|). \tag{6.1.16}$$

Finally, combining (6.1.14) and (6.1.16), we have

$$-\gamma(\phi_\varepsilon(\mathcal{C}_\varepsilon^n)\phi_\varepsilon(\mathcal{C}_\varepsilon^{n-1}+\mathcal{M}_\varepsilon^{n-1}-1),\pi^hF'_\varepsilon(\mathcal{C}_\varepsilon^n)-\pi^hF'_\varepsilon(1-\mathcal{C}_\varepsilon^n-\mathcal{M}_\varepsilon^n))^h$$

$$\le(F_\varepsilon(\mathcal{C}_\varepsilon^n),1)^h+(F_\varepsilon(1-\mathcal{C}_\varepsilon^n-\mathcal{M}_\varepsilon^n),1)^h+C(\gamma,M,\varepsilon,|\Omega|). \tag{6.1.17}$$

Now to bound the terms in right-hand side of (6.1.10), we can use a similar technique which was used in bounding the second term in the right-hand side of (6.1.14), and noting (2.3.28), (2.3.31), (2.3.32), (2.1.12), Young's inequality and (2.3.29), to obtain

$$-\alpha(\phi_\varepsilon(\mathcal{C}_\varepsilon^{n-1})\phi_\varepsilon(\mathcal{M}_\varepsilon^n)\phi_\varepsilon(\mathcal{C}_\varepsilon^{n-1}+\mathcal{M}_\varepsilon^{n-1}-1),\,\pi^hF'_\varepsilon(\mathcal{M}_\varepsilon^n)-\pi^hF'_\varepsilon(1-\mathcal{C}_\varepsilon^n-\mathcal{M}_\varepsilon^n))^h$$

$$=-\alpha(\phi_\varepsilon(\mathcal{C}_\varepsilon^{n-1})\phi_\varepsilon(\mathcal{C}_\varepsilon^{n-1}+\mathcal{M}_\varepsilon^{n-1}-1),\phi_\varepsilon(\mathcal{M}_\varepsilon^n)\,\pi^hF'_\varepsilon(\mathcal{M}_\varepsilon^n))^h$$

$$+\alpha(\phi_\varepsilon(\mathcal{M}_\varepsilon^n)\phi_\varepsilon(\mathcal{C}_\varepsilon^{n-1})\phi_\varepsilon(\mathcal{C}_\varepsilon^{n-1}+\mathcal{M}_\varepsilon^{n-1}-1),\,\pi^hF'_\varepsilon(1-\mathcal{C}_\varepsilon^n-\mathcal{M}_\varepsilon^n))^h$$

$$\le(F_\varepsilon(\mathcal{M}_\varepsilon^n),1)^h+(F_\varepsilon(1-\mathcal{C}_\varepsilon^n-\mathcal{M}_\varepsilon^n),1)^h+C(\alpha,M,\varepsilon,|\Omega|). \tag{6.1.18}$$

The second term in the right-hand side of (6.1.9) can be easily bound as follows:

$$-\delta(\phi_\varepsilon(\mathcal{C}_\varepsilon^n), \pi^h F_\varepsilon'(\mathcal{C}_\varepsilon^n))^h \leq \delta(1-\mathcal{C}_\varepsilon^n, 1)^h \leq \frac{1}{2\varepsilon}([\mathcal{C}_\varepsilon^n]_-^2, 1)^h + C(\delta, |\Omega|, \varepsilon)$$
$$\leq (F_\varepsilon(\mathcal{C}_\varepsilon^n), 1)^h + C(\delta, |\Omega|, \varepsilon). \tag{6.1.19}$$

To deal with the third term in the right-hand side of (6.1.9), we use similar technique which was used in bounding the second term in the right-hand side of (6.1.14). Firstly, we divide the interval Ω as before, then we obtain

$$\delta(\phi_\varepsilon(\mathcal{C}_\varepsilon^n), \pi^h F_\varepsilon'(1-\mathcal{C}_\varepsilon^n-\mathcal{M}_\varepsilon^n))^h$$
$$= \delta \sum_{i\in\mathfrak{J}_+} \widehat{M}_{ii}\phi_\varepsilon(\mathcal{C}_\varepsilon^n(x_i))F_\varepsilon'((1-\mathcal{C}_\varepsilon^n-\mathcal{M}_\varepsilon^n)(x_i))$$
$$+\delta \sum_{i\in\mathfrak{J}_-} \widehat{M}_{ii}\phi_\varepsilon((\mathcal{C}_\varepsilon^n(x_i))F_\varepsilon'((1-\mathcal{C}_\varepsilon^n-\mathcal{M}_\varepsilon^n)(x_i))$$
$$\leq \delta \sum_{i\in\mathfrak{J}_+} \widehat{M}_{ii}\phi_\varepsilon(\mathcal{C}_\varepsilon^n(x_i))F_\varepsilon'((1-\mathcal{C}_\varepsilon^n-\mathcal{M}_\varepsilon^n)(x_i))$$
$$= \delta \sum_{i\in\mathfrak{J}_+} \widehat{M}_{ii}\frac{\phi_\varepsilon(\mathcal{C}_\varepsilon^n(x_i))}{\phi_\varepsilon((1-\mathcal{C}_\varepsilon^n-\mathcal{M}_\varepsilon^n)(x_i))}\phi_\varepsilon((1-\mathcal{C}_\varepsilon^n-\mathcal{M}_\varepsilon^n)(x_i))F_\varepsilon'((1-\mathcal{C}_\varepsilon^n-\mathcal{M}_\varepsilon^n)(x_i)). \tag{6.1.20}$$

Now, since $\phi_\varepsilon((1-\mathcal{C}_\varepsilon^n-\mathcal{M}_\varepsilon^n)(x_i)) \geq 1$ on $\mathfrak{J}_+$ and $\phi_\varepsilon(s) \leq M,\ \forall s$, we have

$$\frac{\phi_\varepsilon((\mathcal{C}_\varepsilon^n)(x_i))}{\phi_\varepsilon((1-\mathcal{C}_\varepsilon^n-\mathcal{M}_\varepsilon^n)(x_i))} \leq M.$$

Finally, using (2.3.32), we have

$$\delta(\phi_\varepsilon(\mathcal{C}_\varepsilon^n), \pi^h F_\varepsilon'(1-\mathcal{C}_\varepsilon^n-\mathcal{M}_\varepsilon^n))^h$$
$$\leq M\delta \sum_{i\in\mathfrak{J}_+} \widehat{M}_{ii}\phi_\varepsilon((1-\mathcal{C}_\varepsilon^n-\mathcal{M}_\varepsilon^n)(x_i))F_\varepsilon'((1-\mathcal{C}_\varepsilon^n-\mathcal{M}_\varepsilon^n)(x_i))$$
$$\leq \sum_{i\in\mathfrak{J}_+} \widehat{M}_{ii}F_\varepsilon((1-\mathcal{C}_\varepsilon^n-\mathcal{M}_\varepsilon^n)(x_i)) + C(\delta, M)$$
$$\leq (F_\varepsilon(1-\mathcal{C}_\varepsilon^n-\mathcal{M}_\varepsilon^n), 1)^h + C(\delta, M). \tag{6.1.21}$$

Combining (6.1.9), (6.1.11), (6.1.17), (6.1.19) and (6.1.21), gives

$$(1-2\Delta t)(F_\varepsilon(\mathcal{C}_\varepsilon^n), 1)^h - 2\Delta t(F_\varepsilon(1-\mathcal{C}_\varepsilon^n-\mathcal{M}_\varepsilon^n), 1)^h + \frac{1}{4M}|\mathcal{C}_\varepsilon^n|_h^2$$
$$-(\mathcal{C}_\varepsilon^n-\mathcal{C}_\varepsilon^{n-1}, \pi^h F_\varepsilon'(1-\mathcal{C}_\varepsilon^n-\mathcal{M}_\varepsilon^n))^h + (D_{11}^n\nabla\mathcal{C}_\varepsilon^n + D_{12}^n\nabla\mathcal{M}_\varepsilon^n, \Delta t\nabla\pi^h F_\varepsilon'(\mathcal{C}_\varepsilon^n)$$

$$-\Delta t\nabla\pi^h F'_\varepsilon(1-\mathcal{C}^n_\varepsilon-\mathcal{M}^n_\varepsilon)) \leq (\, F_\varepsilon(\mathcal{C}^{n-1}_\varepsilon),1)^h + \frac{1}{2M}|\mathcal{C}^{n-1}_\varepsilon|^2_h + \Delta t C(M,\gamma,\delta,\varepsilon,|\Omega|). \tag{6.1.22}$$

Combining (6.1.10), (6.1.12) and (6.1.18) gives

$$(1-\Delta t)(F_\varepsilon(\mathcal{M}^n_\varepsilon),1)^h - \Delta t(F_\varepsilon(1-\mathcal{C}^n_\varepsilon-\mathcal{M}^n_\varepsilon),1)^h + \frac{1}{4M}|\mathcal{M}^n_\varepsilon|^2_h$$
$$-(\mathcal{M}^n_\varepsilon-\mathcal{M}^{n-1}_\varepsilon,\pi^h F'_\varepsilon(1-\mathcal{C}^n_\varepsilon-\mathcal{M}^n_\varepsilon))^h + (D^n_{21}\nabla\mathcal{C}^n_\varepsilon + D^n_{22}\nabla\mathcal{M}^n_\varepsilon,\Delta t\nabla\pi^h F'_\varepsilon(\mathcal{M}^n_\varepsilon)$$
$$-\Delta t\nabla\pi^h F'_\varepsilon(1-\mathcal{C}^n_\varepsilon-\mathcal{M}^n_\varepsilon)) \leq F_\varepsilon(\mathcal{M}^{n-1}_\varepsilon),1)^h + \frac{1}{2M}|\mathcal{M}^{n-1}_\varepsilon|^2_h + \Delta t C(M,\alpha,\varepsilon,|\Omega|). \tag{6.1.23}$$

Now, by adding (6.1.22) and (6.1.23) and noting (6.1.13), we have

$$[1-2\Delta t]E(\mathcal{C}^n_\varepsilon,\mathcal{M}^n_\varepsilon) + (D^n_{11}\nabla\mathcal{C}^n_\varepsilon + D^n_{12}\nabla\mathcal{M}^n_\varepsilon,\Delta t\nabla\pi^h F'_\varepsilon(\mathcal{C}^n_\varepsilon)$$
$$-\Delta t\nabla\pi^h F'_\varepsilon(1-\mathcal{C}^n_\varepsilon-\mathcal{M}^n_\varepsilon)) + (D^n_{21}\nabla\mathcal{C}^n_\varepsilon + D^n_{22}\nabla\mathcal{M}^n_\varepsilon,\Delta t\nabla\pi^h F'_\varepsilon(\mathcal{M}^n_\varepsilon)$$
$$-\Delta t\nabla\pi^h F'_\varepsilon(1-\mathcal{C}^n_\varepsilon-\mathcal{M}^n_\varepsilon)) + \frac{1}{4M}|\mathcal{C}^n_\varepsilon|^2_h + \frac{1}{4M}|\mathcal{M}^n_\varepsilon|^2_h + \frac{1}{4M}|1-\mathcal{C}^n_\varepsilon-\mathcal{M}^n_\varepsilon|^2_h$$
$$\leq E(\mathcal{C}^{n-1}_\varepsilon,\mathcal{M}^{n-1}_\varepsilon) + \frac{1}{2M}\left[|\mathcal{C}^{n-1}_\varepsilon|^2_h + |\mathcal{M}^{n-1}_\varepsilon|^2_h\right] + \Delta t C(\delta,\gamma,\alpha,M,\varepsilon,|\Omega|). \tag{6.1.24}$$

Next, we can simplify the second and the third terms in (6.1.24) as follows

$$(D^n_{11}\nabla\mathcal{C}^n_\varepsilon + D^n_{12}\nabla\mathcal{M}^n_\varepsilon,\nabla\pi^h F'_\varepsilon(\mathcal{C}^n_\varepsilon) - \nabla\pi^h F'_\varepsilon(1-\mathcal{C}^n_\varepsilon-\mathcal{M}^n_\varepsilon))$$
$$+(D^n_{21}\nabla\mathcal{C}^n_\varepsilon + D^n_{22}\nabla\mathcal{M}^n_\varepsilon,\nabla\pi^h F'_\varepsilon(\mathcal{M}^n_\varepsilon) - \nabla\pi^h F'_\varepsilon(1-\mathcal{C}^n_\varepsilon-\mathcal{M}^n_\varepsilon))$$
$$=\sum_\tau \frac{1}{\tau}\int_\tau\int_\tau \pi^h(2(1-\phi_\varepsilon(\mathcal{C}^n_\varepsilon)) - \beta\theta\phi^2_\varepsilon(\mathcal{M}^n_\varepsilon))\ dx\ |\nabla\mathcal{C}^n_\varepsilon|^2\ dx'$$
$$+\sum_\tau \frac{1}{\tau}\int_\tau\int_\tau \pi^h(\frac{2\phi_\varepsilon(\mathcal{C}^n_\varepsilon)(1-\phi_\varepsilon(\mathcal{C}^n_\varepsilon)) - \beta\theta(1-\phi_\varepsilon(\mathcal{M}^n_\varepsilon)-\phi_\varepsilon(1-\mathcal{C}^{n-1}_\varepsilon-\mathcal{M}^{n-1}_\varepsilon))\phi^2_\varepsilon(\mathcal{M}^n_\varepsilon)}{\phi_\varepsilon(1-\mathcal{C}^{n-1}_\varepsilon-\mathcal{M}^{n-1}_\varepsilon)})$$
$$\times dx\ (|\nabla\mathcal{C}^n_\varepsilon|^2 + \nabla\mathcal{C}^n_\varepsilon\cdot\nabla\mathcal{M}^n_\varepsilon)\ dx'$$
$$-\sum_\tau \frac{2\beta}{\tau}\int_\tau\int_\tau \pi^h(\phi_\varepsilon(\mathcal{M}^n_\varepsilon)(1+\theta\phi_\varepsilon(\mathcal{C}^n_\varepsilon)))\ dx\ \nabla\mathcal{C}^n_\varepsilon\nabla\mathcal{M}^n_\varepsilon\ dx'$$
$$-\sum_\tau \frac{2\beta}{\tau}\int_\tau\int_\tau \pi^h(\frac{(1-\phi_\varepsilon(\mathcal{M}^n_\varepsilon)-\phi_\varepsilon(1-\mathcal{C}^{n-1}_\varepsilon-\mathcal{M}^{n-1}_\varepsilon))\phi_\varepsilon(\mathcal{M}^n_\varepsilon)(1+\theta\phi_\varepsilon(\mathcal{C}^n_\varepsilon)))}{\phi_\varepsilon(1-\mathcal{C}^{n-1}_\varepsilon-\mathcal{M}^{n-1}_\varepsilon)})$$
$$\times\ dx\ (|\nabla\mathcal{M}^n_\varepsilon|^2 + \nabla\mathcal{C}^n_\varepsilon\cdot\nabla\mathcal{M}^n_\varepsilon)\ dx'$$
$$+\sum_\tau\int_\tau\frac{1}{\tau}\int_\tau \pi^h(-2\phi_\varepsilon(\mathcal{C}^n_\varepsilon) + \beta\theta(1-\phi_\varepsilon(\mathcal{M}^n_\varepsilon))\phi_\varepsilon(\mathcal{M}^n_\varepsilon))\ dx\ \nabla\mathcal{C}^n_\varepsilon\nabla\mathcal{M}^n_\varepsilon$$

$$+\sum_{\tau}\frac{1}{\tau}\int_{\tau}\int_{\tau}\pi^{h}\Big(\frac{-2\phi_\varepsilon(\mathcal{C}_\varepsilon^n)(1-\phi_\varepsilon(\mathcal{C}_\varepsilon^n)-\phi_\varepsilon(1-\mathcal{C}_\varepsilon^{n-1}-\mathcal{M}_\varepsilon^{n-1}))+\beta\theta(1-\phi_\varepsilon(\mathcal{M}_\varepsilon^n))\phi_\varepsilon^2(\mathcal{M}_\varepsilon^n)}{\phi_\varepsilon(1-\mathcal{C}_\varepsilon^{n-1}-\mathcal{M}_\varepsilon^{n-1})}\Big)$$

$$\times\ dx\ (|\nabla\mathcal{C}_\varepsilon^n|^2+\nabla\mathcal{C}_\varepsilon^n\cdot\nabla\mathcal{M}_\varepsilon^n)\ dx'$$

$$+\sum_{\tau}\frac{2\beta}{\tau}\int_{\tau}\int_{\tau}(1-\phi_\varepsilon(\mathcal{M}_\varepsilon^n))(1+\theta\phi_\varepsilon(\mathcal{C}_\varepsilon^n))\ dx\ |\nabla\mathcal{M}_\varepsilon^n|^2\ dx'$$

$$+\sum_{\tau}\frac{2\beta}{\tau}\int_{\tau}\int_{\tau}\pi^{h}\Big(\frac{\phi_\varepsilon(\mathcal{M}_\varepsilon^n)(1-\phi_\varepsilon(\mathcal{M}_\varepsilon^n))(1+\theta\phi_\varepsilon(\mathcal{C}_\varepsilon^n))}{\phi_\varepsilon(1-\mathcal{C}_\varepsilon^{n-1}-\mathcal{M}_\varepsilon^{n-1})}\Big)\ dx\ (|\nabla\mathcal{M}_\varepsilon^n|^2+\nabla\mathcal{C}_\varepsilon^n\cdot\nabla\mathcal{M}_\varepsilon^n)\ dx'$$

$$=\int_\Omega(2|\nabla\mathcal{C}_\varepsilon^n|^2+\beta\theta\phi_\varepsilon(\mathcal{M}_\varepsilon^n)\ \nabla\mathcal{C}_\varepsilon^n\cdot\nabla\mathcal{M}_\varepsilon^n+2\beta(1+\theta\phi_\varepsilon(\mathcal{C}_\varepsilon^n))|\nabla\mathcal{M}_\varepsilon^n|^2)dx$$

$$\geq\int_\Omega((2-\frac{\beta\theta^2}{8}\phi_\varepsilon^2(\mathcal{M}_\varepsilon^n))|\nabla\mathcal{C}_\varepsilon^n|^2+2\beta\theta\phi_\varepsilon(\mathcal{C}_\varepsilon^n)|\nabla\mathcal{M}_\varepsilon^n|^2)dx$$

$$\geq\int_\Omega((2-\frac{\beta\theta^2}{8}M^2)|\nabla\mathcal{C}_\varepsilon^n|^2+2\beta\theta\phi_\varepsilon(\mathcal{C}_\varepsilon^n)|\nabla\mathcal{M}_\varepsilon^n|^2)dx. \tag{6.1.25}$$

It is clear that the last integral is nonnegative if $\theta\leq 4/M\sqrt{\beta}$. This result can be strengthened: If $0<\theta<4/M\sqrt{\beta}$, then we have

$$\int_\Omega((2-\frac{\beta\theta^2}{8}M^2)|\nabla\mathcal{C}_\varepsilon^n|^2+2\beta\theta\phi_\varepsilon(\mathcal{C}_\varepsilon^n)|\nabla\mathcal{M}_\varepsilon^n|^2)dx$$

$$\geq K_\theta\int_\Omega(|\nabla\mathcal{C}_\varepsilon^n|^2+|\nabla\mathcal{M}_\varepsilon^n|^2)dx. \tag{6.1.26}$$

where $K_\theta=\min\{2-\frac{\beta\theta^2}{8}M^2,2\beta\theta\}$. Combining (6.1.24), (6.1.25) and (6.1.26) then we arrive at the required result. □

6.2 Existence of the approximations

In this section we establish existence of a solution to the problem $(W_{M,\varepsilon}^{h\,\Delta t})$ by adapting a similar approach applied in [8] to prove existence of a finite element approximation of a cross diffusion equation. The approach relies on constructing a contradiction to the Schauder fixed point theorem (see Appendix A.1.1).

In order to prove the existence of a solution $\{\mathcal{C}_\varepsilon^n,\mathcal{M}_\varepsilon^n\}, n\geq 1$, of the system (6.1.1) and (6.1.2) for given $\{\mathcal{C}_\varepsilon^{n-1},\mathcal{M}_\varepsilon^{n-1}\}$, it is convenient to define the functions $A_c: S^h\times S^h\to S^h$ and $A_m: S^h\times S^h\to S^h$ such that for all $\chi\in S^h$

$$(A_c(\mathcal{C},\mathcal{M}),\chi)^h=(\mathcal{C}-\mathcal{C}_\varepsilon^{n-1},\chi)^h+\Delta t(D_{11}\nabla\mathcal{C}+D_{12}\nabla\mathcal{M},\nabla\chi)$$

$$+\Delta t(\gamma\phi_\varepsilon(\mathcal{C})\ \phi_\varepsilon(\mathcal{C}_\varepsilon^{n-1}+\mathcal{M}_\varepsilon^{n-1}-1)+\delta\phi_\varepsilon(\mathcal{C}),\chi)^h, \tag{6.2.27}$$

$$(A_m(\mathcal{C},\mathcal{M}),\chi)^h = (\mathcal{M}-\mathcal{M}_\varepsilon^{n-1},\chi)^h + \Delta t(D_{21}\nabla\mathcal{C}+D_{22}\nabla\mathcal{M},\nabla\chi)$$
$$+\Delta t(\alpha\phi_\varepsilon(\mathcal{C}_\varepsilon^{n-1})\ \phi_\varepsilon(\mathcal{M})\ \phi_\varepsilon(\mathcal{C}_\varepsilon^{n-1}+\mathcal{M}_\varepsilon^{n-1}-1),\chi)^h, \tag{6.2.28}$$

respectively. Therefore, from (6.2.27) and (6.2.28) we have that (6.1.1) and (6.1.2) at the $n-th$ step is equivalent to the problem:
For $n\geq 1$ find $\{\mathcal{C}_\varepsilon^n,\mathcal{M}_\varepsilon^n\}\in S^h\times S^h$ such that for all $\chi\in S^h$

$$(A_c(\mathcal{C},\mathcal{M}),\chi)^h = 0, \qquad (A_m(\mathcal{C},\mathcal{M}),\chi)^h = 0. \tag{6.2.29}$$

Before we prove existence of the approximate solutions, in the following subsection, we provide some lemmata which will be important in the analysis of the approximation problem $(W_{M,\varepsilon}^{h\,\Delta t})$. Firstly, we shall prove that $A_c(\mathcal{C},\mathcal{M})$ and $A_m(\mathcal{C},\mathcal{M})$ are well defined, then we note that the continuous piecewise linear functions $A_c(\mathcal{C},\mathcal{M})$ and $A_m(\mathcal{C},\mathcal{M})$ can be defined uniquely in terms of their values at the nodal points $\mathcal{N}^h$.

Lemma 6.2.1 The definitions of $D_{11},...,D_{22}$ are well defined. Moreover,

$$\nabla\pi^h F_\varepsilon'(\mathcal{C}) - \nabla\pi^h F_\varepsilon'(1-\mathcal{C}-\mathcal{M}) = F_\varepsilon''(\xi^c)\nabla\mathcal{C} + F_\varepsilon''(\xi^{c,m})(\nabla\mathcal{C}+\nabla\mathcal{M}),$$
$$\nabla\pi^h F_\varepsilon'(\mathcal{M}) - \nabla\pi^h F_\varepsilon'(1-\mathcal{C}-\mathcal{M}) = F_\varepsilon''(\xi^m)\nabla\mathcal{M} + F_\varepsilon''(\xi^{c,m})(\nabla\mathcal{C}+\nabla\mathcal{M}).$$

Proof: Firstly, by using the mean value theorem, we can derive

$$\nabla\pi^h F_\varepsilon'(\mathcal{C}) - \nabla\pi^h F_\varepsilon'(1-\mathcal{C}-\mathcal{M})$$
$$= \frac{1}{h}[F_\varepsilon'(\mathcal{C}_{i+1}) - F_\varepsilon'(\mathcal{C}_i)] - \frac{1}{h}[F_\varepsilon'(1-\mathcal{C}_{i+1}-\mathcal{M}_{i+1}) - F_\varepsilon'(1-\mathcal{C}_i-\mathcal{M}_i)]$$
$$= \frac{1}{h}\int_{\mathcal{C}_i}^{\mathcal{C}_{i+1}} F_\varepsilon''(s)ds - \frac{1}{h}\int_{1-\mathcal{C}_i-\mathcal{M}_i}^{1-\mathcal{C}_{i+1}-\mathcal{M}_{i+1}} F_\varepsilon''(s)ds$$
$$= F_\varepsilon''(\xi^c)\nabla\mathcal{C} + F_\varepsilon''(\xi^{c,m})(\nabla\mathcal{C}+\nabla\mathcal{M}), \tag{6.2.30}$$

where $\xi^c\in[\mathcal{C}_i,\mathcal{C}_{i+1}]$ and $\xi^{c,m}\in[1-\mathcal{C}_i-\mathcal{M}_i, 1-\mathcal{C}_{i+1}-\mathcal{M}_{i+1}]$. Similarly, one can show that

$$\nabla\pi^h F_\varepsilon'(\mathcal{M}) - \nabla\pi^h F_\varepsilon'(1-\mathcal{C}-\mathcal{M}) = F_\varepsilon''(\xi^m)\nabla\mathcal{M} + F_\varepsilon''(\xi^{c,m})(\nabla\mathcal{C}+\nabla\mathcal{M}),$$

We work with (6.1.1) and (6.1.2), we are left with the question of whether $D_{11},...,D_{22}$ are well defined. We will now discuss how we find D_{11} and the same ideas are used for finding D_{12}, D_{21} and D_{22}. For n fixed on each triangle, in the case that the constant $\nabla\pi^h F'_\varepsilon(\mathcal{C}) - \nabla\pi^h F'_\varepsilon(1-\mathcal{C}-\mathcal{M})$ is non-zero, D^n_{11} is easily found by division. In the case that it is zero, we can conclude that

- If $\nabla\mathcal{C} = 0$, this means that $\mathcal{C}$ is constant, and hence $\nabla\pi^h F'_\varepsilon(\mathcal{C}) = 0$ then as $\nabla\pi^h F'_\varepsilon(\mathcal{C}) - \nabla\pi^h F'_\varepsilon(1-\mathcal{C}-\mathcal{M}) = 0$, this implies that $\nabla\pi^h F'_\varepsilon(1-\mathcal{C}-\mathcal{M}) = 0$, and $1-\mathcal{C}-\mathcal{M}$ is constant, but as $\mathcal{C}$ is constant, then $\mathcal{M}$ is constant so $\nabla(\mathcal{C}+\mathcal{M}) = 0$ thus both sides of (6.1.3) are equal to zero and we can choose D^n_{11} as desired.

- If $\nabla(\mathcal{C}+\mathcal{M}) = 0$, this mean that $1-\mathcal{C}-\mathcal{M}$ is constant, and hence $\nabla\pi^h F'_\varepsilon(1-\mathcal{C}-\mathcal{M}) = 0$ then as $\nabla\pi^h F'_\varepsilon(\mathcal{C}) - \nabla\pi^h F'_\varepsilon(1-\mathcal{C}-\mathcal{M}) = 0$, and this implies that $\nabla\pi^h F'_\varepsilon(\mathcal{C}) = 0$, and thus $\mathcal{C}$ is constant, and hence $\nabla\mathcal{C} = 0$. Thus, both sides of (6.1.3) are equal to zero and we can choose D^n_{11} as desired.

- If $\nabla\mathcal{C} \neq 0$ and $\nabla(\mathcal{C}+\mathcal{M}) \neq 0$. In this case, it is clear that $\nabla\pi^h F'_\varepsilon(\mathcal{C}) \neq 0$ and $\nabla\pi^h F'_\varepsilon(1-\mathcal{C}-\mathcal{M}) \neq 0$. Here, we have four possibilities:

  1. If $\nabla\mathcal{C} > 0$ and $\nabla(\mathcal{C}+\mathcal{M}) > 0$, then, it is clear that $\nabla\pi^h F'_\varepsilon(\mathcal{C}) - \nabla\pi^h F'_\varepsilon(1-\mathcal{C}-\mathcal{M}) > 0$, so this cannot occur.
  2. If $\nabla\mathcal{C} < 0$ and $\nabla(\mathcal{C}+\mathcal{M}) < 0$, then, similarly to the first case, we have $\nabla\pi^h F'_\varepsilon(\mathcal{C}) - \nabla\pi^h F'_\varepsilon(1-\mathcal{C}-\mathcal{M}) < 0$, so this cannot occur.
  3. If $\nabla\mathcal{C} < 0$ and $\nabla(\mathcal{C}+\mathcal{M}) > 0$, in this case, we are unable to prove the well-posedness of any D^n_{11}. In Chapter 7, we discuss the algorithm and we overcome this ill-posedness by adding $\omega D^n_{11}\nabla\mathcal{C}^n$ to the left-hand side of (6.1.3) at points where $\nabla\pi^h F'_\varepsilon(\mathcal{C}) - \nabla\pi^h F'_\varepsilon(1-\mathcal{C}-\mathcal{M}) = 0$, where ω is small, but retaining the same right hand side for that single iteration.
  4. If $\nabla\mathcal{C} > 0$ and $\nabla(\mathcal{C}+\mathcal{M}) < 0$, then we use a similar treatment which has been used in 3.

Large values for ω will make our experiments less accurate. Therefore, to make the effect of ω on the numerical results small, we select $\omega = \varepsilon$ at those points where 3 and 4 above occurs.

□

In the next Lemma we prove some preliminary results that will be useful for the existence proof. We investigate the continuous dependence of D_{11}^n on $\mathcal{C}^n$ and $\mathcal{M}^n$ and we temporarily drop the index n for ease of exposition. Also, we denote $D_{11}^1 = D_{11}(\mathcal{C}_1, \mathcal{M}_1)$ and $D_{11}^2 = D_{11}(\mathcal{C}_2, \mathcal{M}_2)$.

Lemma 6.2.2 Assume that $|\nabla\pi^h F_\varepsilon'(\mathcal{C}) - \nabla\pi^h F_\varepsilon'(1-\mathcal{C}-\mathcal{M})|_\infty \geq \nu > 0$ and $\|D_{11}^l\|_0$, $\|D_{12}^l\|_0$, $\|D_{21}^l\|_0$, $\|D_{22}^l\|_0 \leq \varsigma$, l=1, 2, where ν and ς are constant. Let $[S^h]_R^2 = \{\{\chi_1, \chi_2\} \in S^h \times S^h : |\chi_1|_h^2 + |\chi_2|_h^2 \leq R^2\}$ and $\{\mathcal{C}_1, \mathcal{M}_1\}, \{\mathcal{C}_2, \mathcal{M}_2\} \in [S^h]_R^2$ be two solutions of (6.1.1) and (6.1.2), then we have the following bounds

$$\|D_{11}^1\nabla\mathcal{C}_1 - D_{11}^2\nabla\mathcal{C}_2\|_0 \leq C(R, M, \beta, \theta, h^{-1}, \varepsilon^{-1}, \nu^{-1}, \varsigma)[|\mathcal{C}_1 - \mathcal{C}_2|_h + |\mathcal{M}_1 - \mathcal{M}_2|_h], \tag{6.2.31}$$

$$\|D_{12}^1\nabla\mathcal{M}_1 - D_{12}^2\nabla\mathcal{M}_2\|_0 \leq C(R, M, \beta, \theta, h^{-1}, \varepsilon^{-1}, \nu^{-1}, \varsigma)[|\mathcal{C}_1 - \mathcal{C}_2|_h + |\mathcal{M}_1 - \mathcal{M}_2|_h], \tag{6.2.32}$$

$$\|D_{21}^1\nabla\mathcal{C}_1 - D_{21}^2\nabla\mathcal{C}_2\|_0 \leq C(R, M, \beta, \theta, h^{-1}, \varepsilon^{-1}, \nu^{-1}, \varsigma)[|\mathcal{C}_1 - \mathcal{C}_2|_h + |\mathcal{M}_1 - \mathcal{M}_2|_h], \tag{6.2.33}$$

$$\|D_{22}^1\nabla\mathcal{M}_1 - D_{22}^2\nabla\mathcal{M}_2\|_0 \leq C(R, M, \beta, \theta, h^{-1}, \varepsilon^{-1}, \nu^{-1}, \varsigma)[|\mathcal{C}_1 - \mathcal{C}_2|_h + |\mathcal{M}_1 - \mathcal{M}_2|_h]. \tag{6.2.34}$$

Proof:

We have from (2.4.54) and (2.4.46) that

$$\begin{aligned}
\|D_{11}^1\nabla\mathcal{C}_1 - D_{11}^2\nabla\mathcal{C}_2\|_0 &= \|(D_{11}^1 - D_{11}^2)\nabla\mathcal{C}_1 + D_{11}^2(\nabla\mathcal{C}_1 - \nabla\mathcal{C}_2)\|_0 \\
&\leq \|(D_{11}^1 - D_{11}^2)\nabla\mathcal{C}_1\|_0 + \|D_{11}^2(\nabla\mathcal{C}_1 - \nabla\mathcal{C}_2)\|_0 \\
&\leq \|D_{11}^1 - D_{11}^2\|_{0,\infty}\ |\mathcal{C}_1|_1 + \|D_{11}^2\|_0|\mathcal{C}_1 - \mathcal{C}_2|_{1,\infty} \\
&\leq C(h^{-1})\|D_{11}^1 - D_{11}^2\|_0\ |\mathcal{C}_1|_1 + C(h^{-1}, \varsigma)\ |\mathcal{C}_1 - \mathcal{C}_2|_h.
\end{aligned} \tag{6.2.35}$$

To deal with the term $D_{11}^1 - D_{11}^2$ and to make our proof more simple, we use Lemma 6.2.1 and suppose that

$$\Upsilon_i(\mathbf{x}, t) = \nabla\pi^h F_\varepsilon'(\mathcal{C}_i) - \nabla\pi^h F_\varepsilon'(1 - \mathcal{C}_i - \mathcal{M}_i), \quad i = 1, 2,$$

$$\psi_i(\mathbf{x}, t) = \frac{1}{\tau}\int_\tau \pi^h(2(1 - \phi_\varepsilon(\mathcal{C}_i)) - \beta\theta\phi_\varepsilon^2(\mathcal{M}_i))\ dx\ \nabla\mathcal{C}_i, \quad i = 1, 2,$$

and

$$\Psi_i(\mathbf{x},t) = \frac{1}{\tau}\int_\tau \pi^h(\frac{2\phi_\varepsilon(\mathcal{C}_i)(1-\phi_\varepsilon(\mathcal{C}_i)) - \beta\theta(1-\phi_\varepsilon(\mathcal{M}_i)-\phi_\varepsilon(1-\mathcal{C}_\varepsilon^{n-1}-\mathcal{M}_\varepsilon^{n-1}))\phi_\varepsilon^2(\mathcal{M}_i)}{\phi_\varepsilon(1-\mathcal{C}_\varepsilon^{n-1}-\mathcal{M}_\varepsilon^{n-1})})dx$$

$$\times(\nabla\mathcal{C}_i + \nabla\mathcal{M}_i), \quad i = 1,2,$$

then, we have

$$D_{11}^1 - D_{11}^2 = \frac{\psi_1+\Psi_1}{\Upsilon_1} - \frac{\psi_2+\Psi_2}{\Upsilon_2} = \frac{\Upsilon_2(\psi_1+\Psi_1) - \Upsilon_1(\psi_2+\Psi_2)}{\Upsilon_1\Upsilon_2}$$

$$= \frac{\Upsilon_2(\psi_1-\psi_2)+\psi_2(\Upsilon_2-\Upsilon_1)}{\Upsilon_1\Upsilon_2} + \frac{\Upsilon_2(\Psi_1-\Psi_2)+\Psi_2(\Upsilon_2-\Upsilon_1)}{\Upsilon_1\Upsilon_2}. \tag{6.2.36}$$

Now let $g_i = \frac{1}{\tau}\int_\tau \pi^h(2(1-\phi_\varepsilon(\mathcal{C}_i)) - \beta\theta\phi_\varepsilon^2(\mathcal{M}_i))dx$ and $f_i = \nabla\mathcal{C}_i$, $i = 1,2$, then it follows from the definition of the function ψ and on noting the Cauchy-Schwarz inequality, (2.3.28), (2.4.54) and (2.4.46), that

$$\|\psi_1-\psi_2\|_0 = \|f_1g_1 - f_2g_2\|_0 \le \|f_1-f_2\|_0\|g_1\|_{0,\infty} + \|g_1-g_2\|_{0,\infty}\|f_2\|_0. \tag{6.2.37}$$

Using the Cauchy-Schwarz inequality, the Lipschitz continuity of ϕ_ε, (2.3.28), (2.4.54) and (2.4.46), we can deal with term $\|g_1-g_2\|_{0,\infty}$ as follows:

$$\|g_1-g_2\|_{0,\infty} \le \|\phi_\varepsilon(\mathcal{C}_1)-\phi_\varepsilon(\mathcal{C}_2)\|_{0,\infty} + C(\beta,\theta)\|\phi_\varepsilon^2(\mathcal{M}_1)-\phi_\varepsilon^2(\mathcal{M}_2)\|_{0,\infty}$$

$$\le \|\mathcal{C}_1-\mathcal{C}_2\|_{0,\infty} + C(M,\beta,\theta)\|\mathcal{M}_1-\mathcal{M}_2\|_{0,\infty}$$

$$\le C(h^{-1},M,\beta,\theta)\big[|\mathcal{C}_1-\mathcal{C}_2|_h + |\mathcal{M}_1-\mathcal{M}_2|_h\big]. \tag{6.2.38}$$

Substituting (6.2.38) into (6.2.37) and using the definition of R, (2.4.54) and (2.4.46) we have that

$$\|\psi_1-\psi_2\|_0 \le C(M)|\mathcal{C}_1-\mathcal{C}_2|_1 + C(h^{-1},M,\beta,\theta)\big[|\mathcal{C}_1-\mathcal{C}_2|_h + |\mathcal{M}_1-\mathcal{M}_2|_h\big]|\mathcal{C}_2|_1$$

$$\le C(h^{-1},M)|\mathcal{C}_1-\mathcal{C}_2|_h + C(h^{-1},M,\beta,\theta,R)\big[|\mathcal{C}_1-\mathcal{C}_2|_h + |\mathcal{M}_1-\mathcal{M}_2|_h\big]$$

$$\le C(h^{-1},M,\beta,\theta,R)\big[|\mathcal{C}_1-\mathcal{C}_2|_h + |\mathcal{M}_1-\mathcal{M}_2|_h\big]. \tag{6.2.39}$$

Similarly, we can deal with term $\Psi_1-\Psi_2$, using (2.3.28), (2.4.54) and (2.4.46), to arrive at

$$\|\Psi_1-\Psi_2\|_0 \le C(R,M,\beta,\theta,\varepsilon^{-1})[|\mathcal{C}_1-\mathcal{C}_2|_1 + |\mathcal{M}_1-\mathcal{M}_2|_1]$$

$$\leq C(R, M, \beta, \theta, h^{-1}, \varepsilon^{-1})[|\mathcal{C}_1 - \mathcal{C}_2|_h + |\mathcal{M}_1 - \mathcal{M}_2|_h]. \tag{6.2.40}$$

Finally, to deal with the term $\Upsilon_2 - \Upsilon_1$, we apply the Cauchy-Schwarz inequality and (2.4.46), to obtain

$$\|\Upsilon_2 - \Upsilon_1\|_0 = \|\nabla\pi^h F'_\varepsilon(\mathcal{C}_2) - \nabla\pi^h F'_\varepsilon(1-\mathcal{C}_2-\mathcal{M}_2) - \nabla\pi^h F'_\varepsilon(\mathcal{C}_1) + \nabla\pi^h F'_\varepsilon(1-\mathcal{C}_1-\mathcal{M}_1)\|_0$$

$$\leq \|\nabla\pi^h F'_\varepsilon(\mathcal{C}_2) - \nabla\pi^h F'_\varepsilon(\mathcal{C}_1)\|_0 + \|\nabla\pi^h F'_\varepsilon(1-\mathcal{C}_2-\mathcal{M}_2) - \nabla\pi^h F'_\varepsilon(1-\mathcal{C}_1-\mathcal{M}_1)\|_0$$

$$\leq |\nabla\pi^h F'_\varepsilon(\mathcal{C}_2) - \nabla\pi^h F'_\varepsilon(\mathcal{C}_1)|_h + |\nabla\pi^h F'_\varepsilon(1-\mathcal{C}_2-\mathcal{M}_2) - \nabla\pi^h F'_\varepsilon(1-\mathcal{C}_1-\mathcal{M}_1)|_h. \tag{6.2.41}$$

To deal with the first term on the right-hand side of (6.2.41), firstly, we note from the definition of π^h and mean value theorem that

$$\begin{aligned}
&\nabla\pi^h F'_\varepsilon(\mathcal{C}_2) - \nabla\pi^h F'_\varepsilon(\mathcal{C}_1)\\
&= \frac{1}{h}[F'_\varepsilon(\mathcal{C}_2(x_{i+1})) - F'_\varepsilon(\mathcal{C}_2(x_i))] - \frac{1}{h}[F'_\varepsilon(\mathcal{C}_1(x_{i+1})) - F'_\varepsilon(\mathcal{C}_1(x_i))]\\
&= \frac{1}{h}\int_{\mathcal{C}_2(x_i)}^{\mathcal{C}_2(x_{i+1})} F''_\varepsilon(s)ds - \frac{1}{h}\int_{\mathcal{C}_1(x_i)}^{\mathcal{C}_1(x_{i+1})} F''_\varepsilon(s)ds\\
&= \frac{1}{h}\int_{\mathcal{C}_2(x_i)}^{\mathcal{C}_2(x_{i+1})} F''_\varepsilon(s)ds - \frac{1}{h}\left[\int_{\mathcal{C}_1(x_i)}^{\mathcal{C}_2(x_i)} F''_\varepsilon(s)ds + \int_{\mathcal{C}_2(x_i)}^{\mathcal{C}_1(x_{i+1})} F''_\varepsilon(s)ds\right]\\
&= \frac{1}{h}\left[\int_{\mathcal{C}_1(x_{i+1})}^{\mathcal{C}_2(x_i)} F''_\varepsilon(s)ds + \int_{\mathcal{C}_2(x_i)}^{\mathcal{C}_2(x_{i+1})} F''_\varepsilon(s)ds\right] - \frac{1}{h}\int_{\mathcal{C}_1(x_i)}^{\mathcal{C}_2(x_i)} F''_\varepsilon(s)ds\\
&= \frac{1}{h}\int_{\mathcal{C}_1(x_{i+1})}^{\mathcal{C}_2(x_{i+1})} F''_\varepsilon(s)ds - \frac{1}{h}\int_{\mathcal{C}_1(x_i)}^{\mathcal{C}_2(x_i)} F''_\varepsilon(s)ds\\
&= \frac{1}{h}[\mathcal{C}_2(x_{i+1})) - \mathcal{C}_1(x_{i+1})]F''_\varepsilon(\xi^1) - \frac{1}{h}[\mathcal{C}_2(x_i)) - \mathcal{C}_1(x_i)]F''_\varepsilon(\xi^2),
\end{aligned} \tag{6.2.42}$$

where ξ^1 is between $\mathcal{C}_1(x_{i+1})$ and $\mathcal{C}_2(x_{i+1})$ and ξ^2 is between $\mathcal{C}_1(x_i)$ and $\mathcal{C}_2(x_i)$. As a consequence of (6.2.42), on noting (2.3.28) and (2.4.46) we have

$$|\nabla\pi^h F'_\varepsilon(\mathcal{C}_2) - \nabla\pi^h F'_\varepsilon(\mathcal{C}_1)|_h \leq C(h^{-1}, \varepsilon^{-1})|\mathcal{C}_2 - \mathcal{C}_1|_h. \tag{6.2.43}$$

Next, we can use a similar technique to that employed in (6.2.43) to obtain

$$|\nabla\pi^h F'_\varepsilon(1-\mathcal{C}_2-\mathcal{M}_2) - \nabla\pi^h F'_\varepsilon(1-\mathcal{C}_1-\mathcal{M}_1)|_h \leq C(h^{-1}, \varepsilon^{-1})\big[|\mathcal{C}_2-\mathcal{C}_1|_h + |\mathcal{M}_2-\mathcal{M}_1|_h\big]. \tag{6.2.44}$$

Combining (6.2.41), (6.2.43) and (6.2.44), gives

$$\|\Upsilon_2 - \Upsilon_1\|_0 \leq C(h^{-1}, \varepsilon^{-1})\big[|\mathcal{C}_2 - \mathcal{C}_1|_h + |\mathcal{M}_2 - \mathcal{M}_1|_h\big]. \tag{6.2.45}$$

To arrive to the required result (6.2.31), we combine (6.2.35)-(6.2.45). Similarly, by applying the same techniques we arrive at (6.2.32)-(6.2.34) on noting (2.3.28) and (2.4.46). □

Lemma 6.2.3 Let the assumptions of Lemma 6.2.2 hold. Then, for any given $R > 0$, the functions $A_c : [S^h]^2_R \to S^h$ and $A_m : [S^h]^2_R \to S^h$ are continuous.

Proof: Let $\{\mathcal{C}_1, \mathcal{M}_1\}, \{\mathcal{C}_2, \mathcal{M}_2\} \in [S^h]^2_R$ be two solutions of (6.1.1) and (6.1.2), it follows from (6.2.27) that for all $\chi \in S^h$

$$(A_c(\mathcal{C}_1, \mathcal{M}_1) - A_c(\mathcal{C}_2, \mathcal{M}_2), \chi)^h = (\mathcal{C}_1 - \mathcal{C}_2, \chi)^h$$

$$+\Delta t(D^1_{11}\nabla\mathcal{C}_1 - D^2_{11}\nabla\mathcal{C}_2 + D^1_{12}\nabla\mathcal{M}_1 - D^2_{12}\nabla\mathcal{M}_2, \nabla\chi)$$

$$+\Delta t(\gamma\ \phi_\varepsilon(\mathcal{C}^{n-1}_\varepsilon + \mathcal{M}^{n-1}_\varepsilon - 1)(\phi_\varepsilon(\mathcal{C}_1) - \phi_\varepsilon(\mathcal{C}_2)) + \delta(\phi_\varepsilon(\mathcal{C}_1) - \phi_\varepsilon(\mathcal{C}_2)), \chi)^h. \quad (6.2.46)$$

Choosing $\chi = A_c(\mathcal{C}_1, \mathcal{M}_1) - A_c(\mathcal{C}_2, \mathcal{M}_2)$ in (6.2.46) yields on noting the Cauchy-Schwarz inequality, (2.4.54), (2.4.46), (6.2.31), (6.2.32) and the Lipschitz continuity of ϕ_ε that

$$|A_c(\mathcal{C}_1, \mathcal{M}_1) - A_c(\mathcal{C}_2, \mathcal{M}_2)|_h \leq |\mathcal{C}_1 - \mathcal{C}_2|_h$$

$$+C(\Delta t,\ h^{-1})\|D^1_{11}\nabla\mathcal{C}_1 - D^2_{11}\nabla\mathcal{C}_2 + D^1_{12}\nabla\mathcal{M}_1 - D^2_{12}\nabla\mathcal{M}_2\|_0$$

$$+\Delta t|(\gamma\ \phi_\varepsilon(\mathcal{C}^{n-1}_\varepsilon + \mathcal{M}^{n-1}_\varepsilon - 1) + \delta)(\phi_\varepsilon(\mathcal{C}_1) - \phi_\varepsilon(\mathcal{C}_2))|_h$$

$$\leq C(M, \Delta t, \gamma, \delta)|\mathcal{C}_1 - \mathcal{C}_2|_h$$

$$+C(\Delta t,\ h^{-1})\|D^1_{11}\nabla\mathcal{C}_1 - D^2_{11}\nabla\mathcal{C}_2 + D^1_{12}\nabla\mathcal{M}_1 - D^2_{12}\nabla\mathcal{M}_2\|_0$$

$$\leq C(M, \beta, \theta, h^{-1}, \varepsilon^{-1}, \gamma, \delta, \Delta t, R, \nu^{-1}, \varsigma)[|\mathcal{C}_1 - \mathcal{C}_2|_h + |\mathcal{M}_1 - \mathcal{M}_2|_h]. \quad (6.2.47)$$

This proves the continuity of A_c and the continuity of A_m follows similarly to A_c.

Theorem 6.2.4 Let the assumptions (A) hold, and let $\{\mathcal{C}^{n-1}_\varepsilon, \mathcal{M}^{n-1}_\varepsilon\} \in S^h \times S^h$ be a given solution to the $(n-1)$-th step of $(W^{h,\Delta t}_{M,\varepsilon})$ for some $n = 1, ..., N$. Then for all $\varepsilon \in (0, e^{-1})$, for all $h > 0$ and for all Δt such that $\Delta t \leq \frac{1}{3}$, there exists a solution $\{\mathcal{C}^n_\varepsilon, \mathcal{M}^n_\varepsilon\} \in S^h \times S^h$ to the n-th step of $(W^{h,\Delta t}_{M,\varepsilon})$.

Proof: By using a proof by contradiction argument, for a given $R \in \mathbb{R}^{>0}$ sufficiently large we prove existence of at least one solution to (6.2.29). For this purpose, we assume that for all $R \in \mathbb{R}^{>0}$ there does not exist $\{\mathcal{C}, \mathcal{M}\} \in S^h \times S^h$ with $A_c(\mathcal{C}, \mathcal{M}) = A_m(\mathcal{C}, \mathcal{M}) = 0$. It has been proved in Lemma 6.2.3 that $A_c(\mathcal{C}, \mathcal{M})$ and $A_m(\mathcal{C}, \mathcal{M})$ are continuous on $[S^h]^2_R$ and hence one can define a continuous function $B : [S^h]^2_R \to [S^h]^2_R$ such that

$$B(\mathcal{C}, \mathcal{M}) = (B_c(\mathcal{C}, \mathcal{M}), B_m(\mathcal{C}, \mathcal{M})),$$

where $B_c(\mathcal{C}, \mathcal{M})$ and $B_m(\mathcal{C}, \mathcal{M})$ are given by

$$B_c(\mathcal{C}, \mathcal{M}) := \frac{-R\, A_c(\mathcal{C}, \mathcal{M})}{|(A_c(\mathcal{C}, \mathcal{M}), A_m(\mathcal{C}, \mathcal{M}))|_{S^h \times S^h}}, \tag{6.2.48}$$

$$B_m(\mathcal{C}, \mathcal{M}) := \frac{-R\, A_m(\mathcal{C}, \mathcal{M})}{|(A_c(\mathcal{C}, \mathcal{M}), A_m(\mathcal{C}, \mathcal{M}))|_{S^h \times S^h}}, \tag{6.2.49}$$

where $|(\cdot,\cdot)|_{S^h \times S^h}$ is the standard norm on $S^h \times S^h$ defined by

$$|(\chi_1, \chi_2)|_{S^h \times S^h} = \sqrt{|\chi_1|_h^2 + |\chi_2|_h^2}.$$

Since $[S^h]^2_R$ is a convex and compact subset of the finite dimensional space $S^h \times S^h$, the Schauder fixed point theorem shows that there exists a pair $\{\mathcal{C}, \mathcal{M}\} \in [S^h]^2_R$ such that

$$B(\mathcal{C}, \mathcal{M}) = (B_c(\mathcal{C}, \mathcal{M}), B_m(\mathcal{C}, \mathcal{M})) = (\mathcal{C}, \mathcal{M}).$$

Hence, it follows from (6.2.48) and (6.2.49) that

$$|\mathcal{C}|_h^2 + |\mathcal{M}|_h^2 = |B_c(\mathcal{C}, \mathcal{M})|_h^2 + |B_m(\mathcal{C}, \mathcal{M})|_h^2 = R^2. \tag{6.2.50}$$

Choosing $\chi \equiv \pi^h F'_\varepsilon(\mathcal{C}) - \pi^h F'_\varepsilon(1 - \mathcal{C} - \mathcal{M})$ as a test function in (6.2.27) and $\chi \equiv \pi^h F'_\varepsilon(\mathcal{M}) - \pi^h F'_\varepsilon(1 - \mathcal{C} - \mathcal{M})$ as a test function in (6.2.28) yields, on noting (2.4.45), that

$$\begin{aligned}(A_c(\mathcal{C}, \mathcal{M}), F'_\varepsilon(\mathcal{C}) - F'_\varepsilon(1 - \mathcal{C} - \mathcal{M}))^h &= (\mathcal{C} - \mathcal{C}_\varepsilon^{n-1}, F'_\varepsilon(\mathcal{C}) - F'_\varepsilon(1 - \mathcal{C} - \mathcal{M}))^h \\ &+ (D_{11}^n \nabla \mathcal{C} + D_{12}^n \nabla \mathcal{M}, \Delta t \nabla F'_\varepsilon(\mathcal{C}) - \Delta t \nabla F'_\varepsilon(1 - \mathcal{C} - \mathcal{M})) \\ &+ (-\gamma \phi_\varepsilon(\mathcal{C})\, \phi_\varepsilon(\mathcal{C}_\varepsilon^{n-1} + \mathcal{M}_\varepsilon^{n-1} - 1) - \delta \phi_\varepsilon(\mathcal{C}), \Delta t F'_\varepsilon(\mathcal{C}) - \Delta t F'_\varepsilon(1 - \mathcal{C} - \mathcal{M}))^h,\end{aligned} \tag{6.2.51}$$

and

$$(\mathcal{M} - \mathcal{M}_\varepsilon^{n-1}, F_\varepsilon'(\mathcal{M}) - F_\varepsilon'(1 - \mathcal{C} - \mathcal{M}))^h$$

$$= (D_{21}^n \nabla c + D_{22}^n \nabla \mathcal{M}, \Delta t \nabla F_\varepsilon'(\mathcal{M}) - \Delta t \nabla F_\varepsilon'(1 - \mathcal{C} - \mathcal{M}))$$

$$+(-\alpha\phi_\varepsilon(\mathcal{C}_\varepsilon^{n-1})\ \phi_\varepsilon(\mathcal{M})\ \phi_\varepsilon(\mathcal{C}_\varepsilon^{n-1}+\mathcal{M}_\varepsilon^{n-1}-1), \Delta t F_\varepsilon'(\mathcal{M})-\Delta t F_\varepsilon'(1-\mathcal{C}-\mathcal{M}))^h. \quad (6.2.52)$$

Here, we can use the similar technique which is used to prove the entropy inequality (see (6.1.25)) to arrive to the following inequality:

$$(A_c(\mathcal{C}, \mathcal{M}), F_\varepsilon'(\mathcal{C}) - F_\varepsilon'(1 - \mathcal{C} - \mathcal{M}))^h + (A_m(\mathcal{C}, \mathcal{M}), F_\varepsilon'(\mathcal{M}) - F_\varepsilon'(1 - \mathcal{C} - \mathcal{M}))^h$$

$$\geq (1-2\Delta t)(F_\varepsilon(\mathcal{C})+F_\varepsilon(\mathcal{M})+F_\varepsilon(1-\mathcal{C}-\mathcal{M}))+K_\theta\Delta t|\mathcal{C}|_1^2+\Delta t K_\theta|\mathcal{M}|_1^2+\frac{1}{4M}|\mathcal{C}|_h^2+\frac{1}{4M}|\mathcal{M}|_h^2$$

$$+\frac{1}{4M}|1 - \mathcal{C} - \mathcal{M}|_h^2 - C(\mathcal{C}_\varepsilon^{n-1}, \mathcal{M}_\varepsilon^{n-1}) \geq \frac{R^2}{4M} - C(\mathcal{C}_\varepsilon^{n-1}, \mathcal{M}_\varepsilon^{n-1}) > 0. \quad (6.2.53)$$

On noting that $\{\mathcal{C}, \mathcal{M}\}$ is fixed point of the function B, (6.2.48), (6.2.49) and (6.2.53) we obtain for R sufficiently large that

$$(\mathcal{C}, F_\varepsilon'(\mathcal{C}) - F_\varepsilon'(1 - \mathcal{C} - \mathcal{M}))^h + (\mathcal{M}, F_\varepsilon'(\mathcal{M}) - F_\varepsilon'(1 - \mathcal{C} - \mathcal{M}))^h$$

$$= (B_c(\mathcal{C}, \mathcal{M}), F_\varepsilon'(\mathcal{C}) - F_\varepsilon'(1 - \mathcal{C} - \mathcal{M}))^h + (B_m(\mathcal{C}, \mathcal{M}), F_\varepsilon'(\mathcal{M}) - F_\varepsilon'(1 - \mathcal{C} - \mathcal{M}))^h$$
$$= \frac{-R\left[(A_c(\mathcal{C}, \mathcal{M}), F_\varepsilon'(\mathcal{C}) - F_\varepsilon'(1 - \mathcal{C} - \mathcal{M})) + (A_m(\mathcal{C}, \mathcal{M}), F_\varepsilon'(\mathcal{M}) - F_\varepsilon'(1 - \mathcal{C} - \mathcal{M}))\right]}{|(A_c(\mathcal{C}, \mathcal{M}), A_m(\mathcal{C}, \mathcal{M}))|_{S^h \times S^h}} < 0. \quad (6.2.54)$$

Once again, it follows from (2.3.33) and (2.3.28) that

$$(\mathcal{C}, F_\varepsilon'(\mathcal{C}) - F_\varepsilon'(1 - \mathcal{C} - \mathcal{M}))^h \geq (F_\varepsilon(\mathcal{C}) - F_\varepsilon(0))^h + \frac{1}{2M}|\mathcal{C}|_h^2 - (\mathcal{C}, F_\varepsilon'(1 - \mathcal{C} - \mathcal{M}))^h, \quad (6.2.55)$$

$$(\mathcal{M}, F_\varepsilon'(\mathcal{M})-F_\varepsilon'(1-\mathcal{C}-\mathcal{M}))^h \geq (F_\varepsilon(\mathcal{M})-F_\varepsilon(0))^h+\frac{1}{2M}|\mathcal{M}|_h^2-(\mathcal{M}, F_\varepsilon'(1-\mathcal{C}-\mathcal{M}))^h. \quad (6.2.56)$$

On the contrary, combining (6.2.55) and (6.2.56) yields on noting the non-negativity of $F_\varepsilon(s)$ that

$$(\mathcal{C}, F_\varepsilon'(\mathcal{C}) - F_\varepsilon'(1 - \mathcal{C} - \mathcal{M}))^h + (\mathcal{M}, F_\varepsilon'(\mathcal{M}) - F_\varepsilon'(1 - \mathcal{C} - \mathcal{M}))^h$$

$$\geq \frac{1}{2M}R^2 - (2 - \varepsilon)|\Omega| - (\mathcal{C} + \mathcal{M}, F_\varepsilon'(1 - \mathcal{C} - \mathcal{M}))^h$$

$$\geq \frac{1}{2M}R^2 - (2 - \varepsilon)|\Omega| + (1 - \mathcal{C} - \mathcal{M} - 1, F_\varepsilon'(1 - \mathcal{C} - \mathcal{M}))^h$$

$$\geq \frac{1}{2M}R^2 - (2-\varepsilon)|\Omega| + (F_\varepsilon(1-\mathcal{C}-\mathcal{M}) - F_\varepsilon(1), 1)^h + \frac{1}{2M}|\mathcal{C}+\mathcal{M}|_h^2,$$

$$\geq \frac{1}{2M}R^2 - (2-\varepsilon)|\Omega| > 0, \tag{6.2.57}$$

which will be positive for R sufficiently large. Therefore, this is a contradiction and so guarantees the existence of $\{\mathcal{C}_\varepsilon^n, \mathcal{M}_\varepsilon^n\} \in S^h \times S^h$ satisfying $A_c(\mathcal{C}_\varepsilon^n, \mathcal{M}_\varepsilon^n) = A_m(\mathcal{C}_\varepsilon^n, \mathcal{M}_\varepsilon^n) = 0$. Equivalently, we have existence of a solution, which is $\{\mathcal{C}_\varepsilon^n, \mathcal{M}_\varepsilon^n\}$, to the n-th step of $(W_{M,\varepsilon}^{h,\Delta t})$. □

6.3 Stability bounds

In this section we derive stability estimates for the regularized approximations $\{\mathcal{C}_\varepsilon^n, \mathcal{M}_\varepsilon^n\}$ under the assumptions of Theorem 6.2.4.

Lemma 6.3.1 let $\{c^0, m^0\} \in L^\infty(\Omega)$ with $c^0, m^0 \geq 0$ for *a.e.* $x \in \Omega$. Let either $\mathcal{C}_\varepsilon^0 \equiv P^h c^0$ and $\mathcal{M}_\varepsilon^0 \equiv P^h m^0$; or $\mathcal{C}_\varepsilon^0 \equiv \pi^h c^0$ and $\mathcal{M}_\varepsilon^0 \equiv \pi^h m^0$ if $c^0, m^0 \in C(\overline{\Omega})$. Then for all $\varepsilon \in (0, e^{-1})$, for all $h > 0$ and for all Δt such that $\Delta t \leq \frac{1-\varrho}{2}$, for some $\varrho \in (0,1)$ and K_θ, the problem $(W_{M,\varepsilon}^{h,\Delta t})$ possesses a solution $\{\mathcal{C}_\varepsilon^n, \mathcal{M}_\varepsilon^n\}, n = 1, ..., N$ satisfying

$$\max_{n=1,\dots,N}\left[(F_\varepsilon(\mathcal{C}_\varepsilon^n)+F_\varepsilon(\mathcal{M}_\varepsilon^n),1)^h+\varepsilon^{-1}\|\pi^h[\mathcal{C}_\varepsilon^n]_-\|_0^2+\varepsilon^{-1}\|\pi^h[\mathcal{M}_\varepsilon^n]_-\|_0^2+\|\mathcal{C}_\varepsilon^n\|_0^2+\|\mathcal{M}_\varepsilon^n\|_0^2\right]$$

$$+\sum_{n=1}^{N}\left[\Delta t\|\mathcal{C}_\varepsilon^n\|_1^2+\Delta t\|\mathcal{M}_\varepsilon^n\|_1^2+\|\mathcal{C}_\varepsilon^n\|_{0,\alpha}^\alpha+\|\mathcal{M}_\varepsilon^n\|_{0,\alpha}^\alpha\right]+\|\mathcal{C}_\varepsilon^0\|_0+\|\mathcal{M}_\varepsilon^0\|_0\leq C, \tag{6.3.58}$$

where $\alpha = \frac{2(d+2)}{d}$.

Proof: It follows immediately from (6.1.8), for $n = 1, ..., N$, that

$$\begin{aligned}(F_\varepsilon(\mathcal{C}_\varepsilon^n) + F_\varepsilon(\mathcal{M}_\varepsilon^n), 1)^h &\leq \frac{1}{1-2\Delta t}(F_\varepsilon(\mathcal{C}_\varepsilon^{n-1}) + F_\varepsilon(\mathcal{M}_\varepsilon^{n-1}), 1)^h + \frac{C\Delta t}{\varrho}\\ &\leq (1+\frac{2\Delta t}{\varrho})(F_\varepsilon(\mathcal{C}_\varepsilon^{n-1}) + F_\varepsilon(\mathcal{M}_\varepsilon^{n-1}), 1)^h + \frac{C\Delta t}{\varrho}\\ &\leq e^{\frac{2\Delta t}{\varrho}}(F_\varepsilon(\mathcal{C}_\varepsilon^{n-1}) + F_\varepsilon(\mathcal{M}_\varepsilon^{n-1}), 1)^h + \frac{C\Delta t}{\varrho}.\end{aligned} \tag{6.3.59}$$

It follows from the assumptions on the initial data $\{c^0, m^0\}$, (2.3.25), the definition of π^h and (3.1.2), that

$$(F_\varepsilon(\mathcal{C}_\varepsilon^0) + F_\varepsilon(\mathcal{M}_\varepsilon^0), 1)^h \leq C. \tag{6.3.60}$$

Combining (6.3.59), (6.3.60) yields that

$$\max_{n=1,...,N}[(F_\varepsilon(\mathcal{C}_\varepsilon^n)+F_\varepsilon(\mathcal{M}_\varepsilon^n),1)^h] \leq Ce^{\frac{2\Delta t}{\varrho}}[(F_\varepsilon(\mathcal{C}_\varepsilon^{n-1})+F_\varepsilon(\mathcal{M}_\varepsilon^{n-1}),1)^h+\Delta t]$$

$$\leq Ce^{\frac{2\Delta t}{\varrho}}[(F_\varepsilon(\mathcal{C}_\varepsilon^{n-2})+F_\varepsilon(\mathcal{M}_\varepsilon^{n-2}),1)^h+2\Delta t]$$

$$\leq Ce^{\frac{2T}{\varrho}}[(F_\varepsilon(\mathcal{C}_\varepsilon^{0})+F_\varepsilon(\mathcal{M}_\varepsilon^{0}),1)^h+T] \leq C. \tag{6.3.61}$$

On noting (2.4.46), (2.3.29), (2.3.30) and (6.3.61) we have for $n = 1, ..., N$ that

$$\|\mathcal{C}_\varepsilon^n\|_0^2 \leq |\mathcal{C}_\varepsilon^n|_h^2 \leq ((\mathcal{C}_\varepsilon^n)^2,1)^h \leq C((F_\varepsilon(\mathcal{C}_\varepsilon^n),1)^h+1) \leq C, \tag{6.3.62}$$

and similarly,

$$\|\mathcal{M}_\varepsilon^n\|_0^2 \leq |\mathcal{M}_\varepsilon^n|_h^2 \leq ((\mathcal{M}_\varepsilon^n)^2,1)^h \leq C((F_\varepsilon(\mathcal{M}_\varepsilon^n),1)^h+1) \leq C. \tag{6.3.63}$$

From (2.4.46), (2.4.45), (2.3.30) and (6.3.61) we obtain, after recalling that $s = [s]_+ + [s]_-$ and $F_\varepsilon(s) \geq 0$, that for $n = 1, ..., N$

$$\|\pi^h[\mathcal{C}_\varepsilon^n]_-\|_0^2 \leq |\pi^h[\mathcal{C}_\varepsilon^n]_-|_h^2 = ([\mathcal{C}_\varepsilon^n]_-^2,1)^h \leq 2\varepsilon(F_\varepsilon(\mathcal{C}_\varepsilon^n),1)^h \leq C\varepsilon, \tag{6.3.64}$$

$$\|\pi^h[\mathcal{M}_\varepsilon^n]_-\|_0^2 \leq |\pi^h[\mathcal{M}_\varepsilon^n]_-|_h^2 = ([\mathcal{M}_\varepsilon^n]_-^2,1)^h \leq 2\varepsilon(F_\varepsilon(\mathcal{M}_\varepsilon^n),1)^h \leq C\varepsilon. \tag{6.3.65}$$

Now, to prove the sixth and the seventh bounds in (6.3.58), firstly, we sum (6.1.8) over n, next we use (6.3.60), (6.3.61), to get

$$\Delta t\sum_{n=1}^{N}|\mathcal{C}_\varepsilon^n|_1^2 + \Delta t\sum_{n=1}^{N}|\mathcal{M}_\varepsilon^n|_1^2 \leq C. \tag{6.3.66}$$

Due to the fourth and fifth bounds in (6.3.58), the following bound holds

$$\Delta t\sum_{n=1}^{N}\|\mathcal{C}_\varepsilon^n\|_1^2 + \Delta t\sum_{n=1}^{N}\|\mathcal{M}_\varepsilon^n\|_1^2 \leq C. \tag{6.3.67}$$

Then use of the Sobolev interpolation theorem (2.1.4) and the bounds (6.3.62), (6.3.63) and (6.3.67) yields for $n = 1, ..., N$

$$\|\mathcal{C}_\varepsilon^n\|_{0,\alpha}^\alpha \leq C\|\mathcal{C}_\varepsilon^n\|_0^{\alpha-2}\|\mathcal{C}_\varepsilon^n\|_1^2 \leq C, \tag{6.3.68}$$

$$\|\mathcal{M}_\varepsilon^n\|_{0,\alpha}^\alpha \leq C\|\mathcal{M}_\varepsilon^n\|_0^{\alpha-2}\|\mathcal{M}_\varepsilon^n\|_1^2 \leq C, \tag{6.3.69}$$

where $\alpha d(\frac{1}{2}-\frac{1}{\alpha}) = 2$; that is $\alpha = \frac{2(d+2)}{d}$.

□

Chapter 7

Numerical results of a cross-diffusion Tumor-growth model

This chapter is devoted to the discussion of some numerical experiments for the model (W) in one space dimension. We introduce an iterative approach to solve our fully discrete finite element approximation to problem $(W^{h,\Delta t}_{M,\varepsilon})$. We then discuss some numerical solutions for different choices of parameters. We also introduce a modified iterative scheme for the problem $(\hat{W}^{h,\Delta t}_{M,\varepsilon})$. Further, we make an experimental comparison between the solutions of $(W^{h,\Delta t}_{M,\varepsilon})$ and $(\hat{W}^{h,\Delta t}_{M,\varepsilon})$. All programs were written in Matlab to generate the numerical results and Originlab 8.5 to plot the graphs.

7.1 Numerical results

We remark that Jackson and Byrne [63] and Jüngel and Stelzer [66] have employed a different scaling to our model to arrive to the following scaled system

$$\frac{\partial}{\partial t}\begin{pmatrix} c \\ m \end{pmatrix} - \left[D^{JB}(c,m)\begin{pmatrix} c_x \\ m_x \end{pmatrix}\right]_x = \begin{pmatrix} \gamma c(1-c-m) - \delta c \\ \alpha c m(1-c-m) \end{pmatrix}, \qquad (7.1.1)$$

where the diffusion matrix

$$D^{JB}(c,m) = \begin{pmatrix} 2\beta_c c(1-c) - \beta_m \theta c m^2 & -2\beta_m c m(1+\theta c) \\ -2\beta_c c m + \beta_m \theta(1-m)m^2 & 2\beta_m m(1-m)(1+\theta c) \end{pmatrix}. \quad (7.1.2)$$

The results proved in the previous chapter are with $\beta = \beta_m/\beta_c$, a rescaling of time and redefining γ, α and δ, and this formulation is used in the numerical experiments. We introduce the following practical algorithm to solve the nonlinear algebraic system arising from the approximate problem $(\hat{W}^{h,\Delta t}_{M,\varepsilon})$ at each time level:

Given $\{\mathcal{C}^{n,0}_\varepsilon, \mathcal{M}^{n,0}_\varepsilon\} \in S^h \times S^h$ for $k \geq 1$ find $\{\mathcal{C}^{n,k}_\varepsilon, \mathcal{M}^{n,k}_\varepsilon\} \in S^h \times S^h$ such that for all $\chi \in S^h$

$$\begin{aligned} &(\frac{\mathcal{C}^{n,k}_\varepsilon - \mathcal{C}^{n-1}_\varepsilon}{\Delta t}, \chi)^h + (D^{JB\ n,k-1}_{11}\ \nabla \mathcal{C}^{n,k}_\varepsilon + D^{JB\ n,k-1}_{12} \nabla \mathcal{M}^{n,k}_\varepsilon, \nabla \chi) \\ &\quad = (\gamma \phi_\varepsilon(\mathcal{C}^{n,k-1}_\varepsilon)\ \phi_\varepsilon(1 - \mathcal{C}^{n-1}_\varepsilon - \mathcal{M}^{n-1}_\varepsilon) - \delta \phi_\varepsilon(\mathcal{C}^{n,k-1}_\varepsilon), \chi)^h, \quad (7.1.3) \\ &(\frac{\mathcal{M}^{n,k}_\varepsilon - \mathcal{M}^{n-1}_\varepsilon}{\Delta t}, \chi)^h + (D^{JB\ n,k-1}_{21} \nabla \mathcal{C}^{n,k}_\varepsilon + D^{JB\ n,k-1}_{22} \nabla \mathcal{M}^{n,k}_\varepsilon, \nabla \chi) \\ &\quad = (\alpha \phi_\varepsilon(\mathcal{C}^{n-1}_\varepsilon)\ \phi_\varepsilon(\mathcal{M}^{n,k-1}_\varepsilon)\ \phi_\varepsilon(1 - \mathcal{C}^{n-1}_\varepsilon - \mathcal{M}^{n-1}_\varepsilon), \chi)^h, \quad (7.1.4) \end{aligned}$$

where we start with $\mathcal{C}^0_\varepsilon \equiv \pi^h c^0$ and $\mathcal{M}^0_\varepsilon \equiv \pi^h m^0$ and we set, for $n \geq 1$, $\mathcal{C}^{n,0}_\varepsilon \equiv \mathcal{C}^{n-1}_\varepsilon$ and $\mathcal{M}^{n,0}_\varepsilon \equiv \mathcal{M}^{n-1}_\varepsilon$. As the system (7.1.3) and (7.1.4) is linear, existence of $\{\mathcal{C}^{n,k}_\varepsilon, \mathcal{M}^{n,k}_\varepsilon\}$ follows from uniqueness. The standard method to solve the system (7.1.3) and (7.1.4) at each iteration is by testing the equations (7.1.3) and (7.1.4) with $\varphi_j, j = 0, ..., J$, which is the standard hat function, to obtain a $(2J+2)\times(2J+2)$ linear system, in terms of the nodal values of $\mathcal{C}^{n,k}_\varepsilon$ and $\mathcal{M}^{n,k}_\varepsilon$, which can be solved using linear programming. For our numerical results, we set $TOL = 10^{-6}$ and adopt the stopping criteria

$$|\mathcal{C}^{n,k}_\varepsilon - \mathcal{C}^{n,k-1}_\varepsilon|_{0,\infty} < \text{TOL} \quad \text{and} \quad |\mathcal{M}^{n,k}_\varepsilon - \mathcal{M}^{n,k-1}_\varepsilon|_{0,\infty} < \text{TOL}, \quad (7.1.5)$$

i.e. for k satisfying (7.1.5) we set $\mathcal{C}^n_\varepsilon \equiv \mathcal{C}^{n,k}_\varepsilon$ and $\mathcal{M}^n_\varepsilon \equiv \mathcal{M}^{n,k}_\varepsilon$. We have been unable to prove convergence of $\{\mathcal{C}^{n,k}, \mathcal{M}^{n,k}\}^\infty_{k=1}$ to $\{\mathcal{C}^n, \mathcal{M}^n\}$ for n fixed. However, in practice we found that the iterative method always converged well (only a few steps were required to fulfill the stopping criteria at each time level).

We now present some numerical results in one space dimension. Unless otherwise specified, in all experiments we consider a uniform partitioning of $\Omega = (0,2)$ into

400 subintervals, i.e. $J = 400$ and $h = 0.005$), and choose $\Delta t = 0.001$, $n \geq 1$, and $\varepsilon = 10^{-31}$. The initial data are defined as in [63] and [66]

$$c^0(x) = \frac{C_1}{2}(1 + \tanh(\frac{x_0 - x}{\eta})) + \mu,$$

$$m^0(x) = \frac{M_1}{2}(1 - \tanh(\frac{x_0 - x}{\eta})),$$

where $C_1 = M_1 = 0.25, x_0 = 0.1, \mu = 0.0002$ and $\eta = 0.05$. The diffusion coefficients are taken as in [63] and [66]:

$$\beta_c = 0.2, \qquad \beta_m = 0.0015.$$

Firstly, we plot the entropy E which is defined

$$E = (F_\varepsilon(\mathcal{C}_\varepsilon^n) + F_\varepsilon(\mathcal{M}_\varepsilon^n) + F_\varepsilon(1 - \mathcal{C}_\varepsilon^n - \mathcal{M}_\varepsilon^n), 1)^h,$$

versus time in Figure 7.1 for $R_c = R_m = 0$. We see in Figure 7.1 that the entropy E decreases as t increases when the pressure coefficient θ is smaller than the theoretical critical value $\theta^* = \frac{4}{M}\sqrt{\beta_c/\beta_m} = 9.23$, with $\beta = \beta_m/\beta_c$ and $M = 5$, see the proof of Lemma 6.1.1 in Chapter 6. Also, this behaviour is illustrated in Figure 7.2 for $R_c \neq R_m \neq 0$. However, we performed additional experiments for θ beyond this threshold and found that the entropy is decreasing for larger values of θ, too. Figure 7.2 shows that the entropy E is uniformly bounded in time for $\theta = 0$ and the curves for $\theta = 100$ and $\theta = 800$ are graphically indistinguishable. The entropy is decreasing rapidly up to $t = 15$ then the entropy value decreases very slowly.

Now, we consider the case of vanishing production rates, $R_c = R_m = 0$. Figures 7.3-7.6 show the volume fractions of the tumor cells and the extracellular matrix at various times, where we have used the cell-induced pressure coefficient $\theta = 100, 200, .., 900$. The cross-diffusion term $D_{21}^{JB} c_x$ causes a drift of the extracellular matrix to the right boundary, induced by variations of the tumor volume. The diffusion D_{22}^{JB} of the extracellular matrix outside of the tumor is very small, such that the extracellular matrix cannot diffuse and forms a peak. However, the peak indicates a loss of regularity of m, and we conjecture that global classical solutions to the tumor-growth model do not exist. With increasing times, the tumor cell front

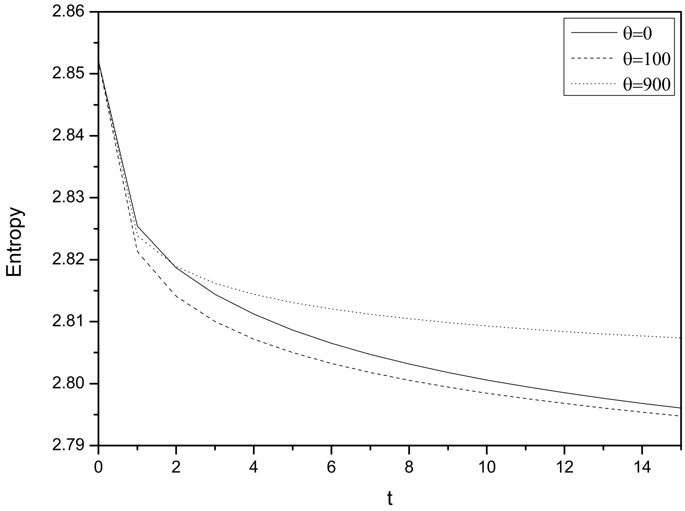


Figure 7.1: Entropy versus time at $\Delta t = 0.001$. The production rates vanish, $R_c = R_m = 0$.

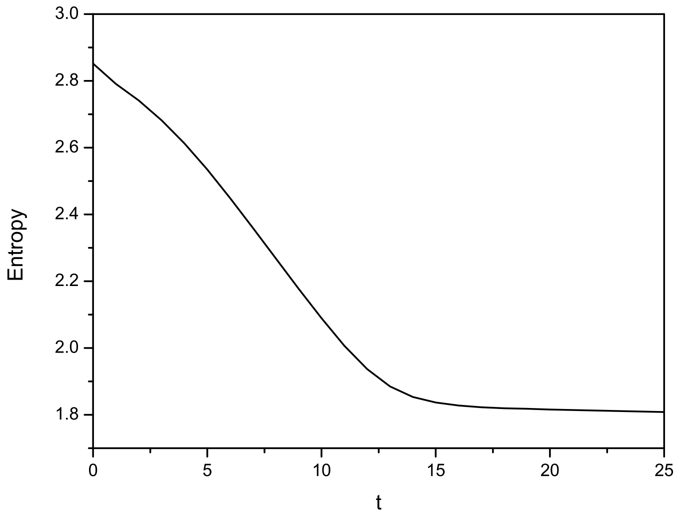


Figure 7.2: Entropy versus time using $\theta = 0$, $\Delta t = 0.001$. The production rates are $\alpha = 0.1, \gamma = 1$, $\delta = 0.35$.

moves to the right boundary, i.e., the tumor penetrates the surrounding extracellular matrix. The tumor cell fraction at the left boundary $x = 0$ is decreasing in time since the total volume fraction $\int_0^2 c dx$ is constant in time and we have drift to the right. In Figures 7.5 and 7.6, we see that the m is close to 0.25 at all times. We see the shape of the curve spreading. The height of the wave is larger for greater values of θ. We conjecture that the maximum height of the travelling wave above 0.25 is directly proportional to θ.

Next, we include the production terms in the equations. In Figures 7.7, we plot the volume fractions of the tumor cells $\theta = 0$ and 800. We find that the qualitative features of the solution for other values of θ are very similar. The right hand boundary plays a role from early times as the solution lifts away from the constant value of $c = 0$, taken by the initial data, to $c_b(t)$, where $c_b(t)$ denotes the constant value of c close to the boundary at time t. Moreover, the shape of c decreases and dips down below $c_b(t)$ before recovering to $c_b(t)$ which becomes more prominent the larger θ becomes. In this experiment we observe that even at small times, e.g. $T = 4$, the solution lifts away from 0.

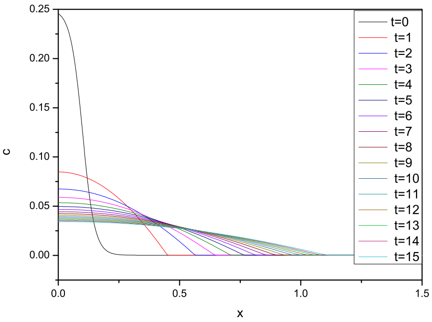
In Figures 7.8 and 7.9, we plot the the extracellular matrix at various times, where the values of pressure coefficient were $\theta = 100, .., 800$. When we compare Figures 7.7-7.9 to Figures 7.3-7.6 we can see that the cell front and the extracellular matrix peaks are moving much faster. Also, the tumor cell volume is increasing (because of the production rates). The height of the peak becomes smaller for smaller values of θ. This behavior has also been observed by Jackson and Byrne [63] and Jüngel and Stelzer [66]. Moreover, we see in Figures 7.8 and 7.9 that wave appears to move with constant velocity but the shape of the wave spreads and elongates with time.

In Figure 7.8, we saw that after a short amount of time, a wave has formed on the line $m = 0.25$ which moves to the right as the time increases. We define $x_F(t)$ such that

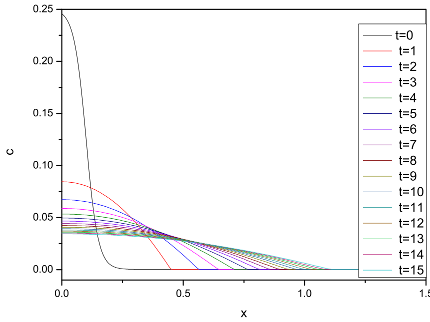
$$x_F(t) = \{x \in [0, 2] : \mathcal{M}_\varepsilon^n(x, t) \geq \mathcal{M}_\varepsilon^n(y, t); \forall y \in [0, 2]\}.$$

Assuming a linear velocity so that

$$x_F(t) = \beta_1 + \beta_2 t, \tag{7.1.6}$$



(a)



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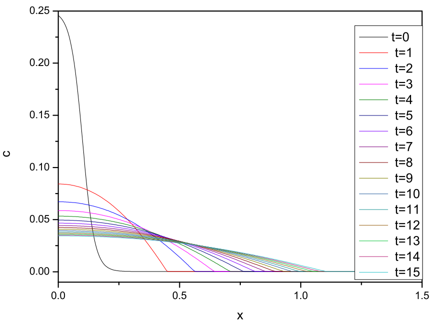

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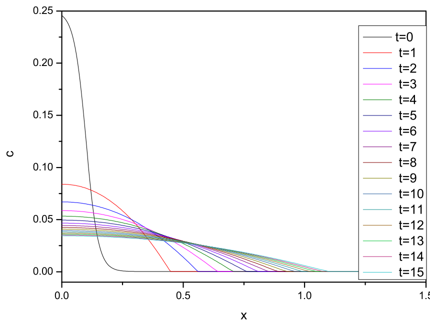

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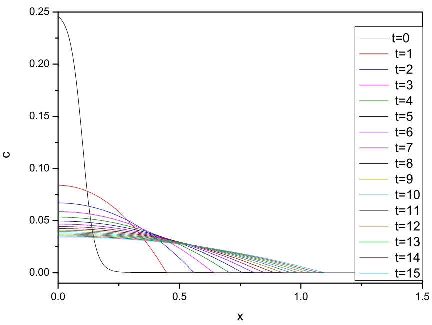

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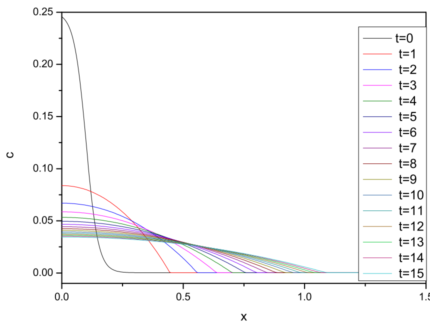
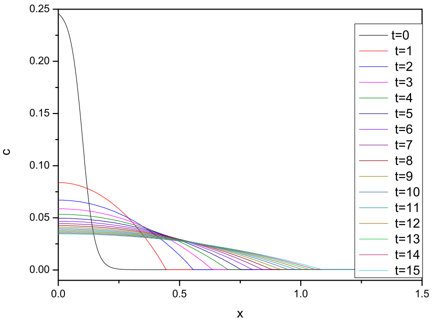
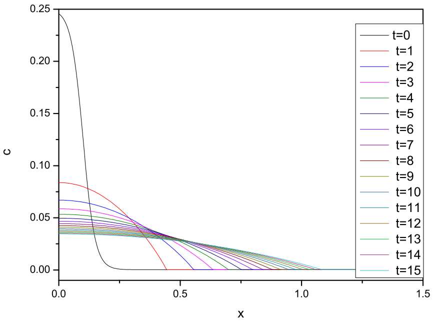


(f)

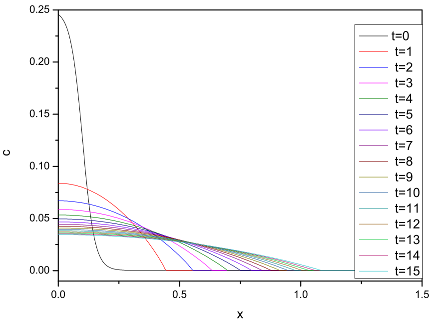
Figure 7.3: Volume fractions of the tumor cells c versus position at times $t = 0, .., 15$ and $\Delta t = 0.001$. The production rates vanish, $R_c = R_m = 0$. (a) $\theta = 0$, (b) $\theta = 50$, (c) $\theta = 100$, (d) $\theta = 200$, (e) $\theta = 300$, (f) $\theta = 400$.



(a)



(b)



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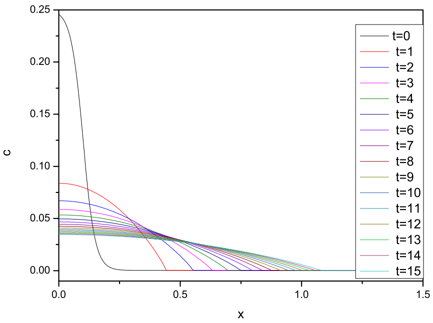

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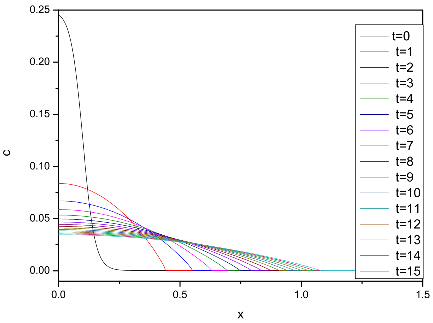


(e)

Figure 7.4: Volume fractions of the tumor cells c versus position at times $t = 0, .., 15$ and $\Delta t = 0.001$. The production rates vanish, $R_c = R_m = 0$. (a) $\theta = 500$, (b) $\theta = 600$, (c) $\theta = 700$, (d) $\theta = 800$, (e) $\theta = 900$.

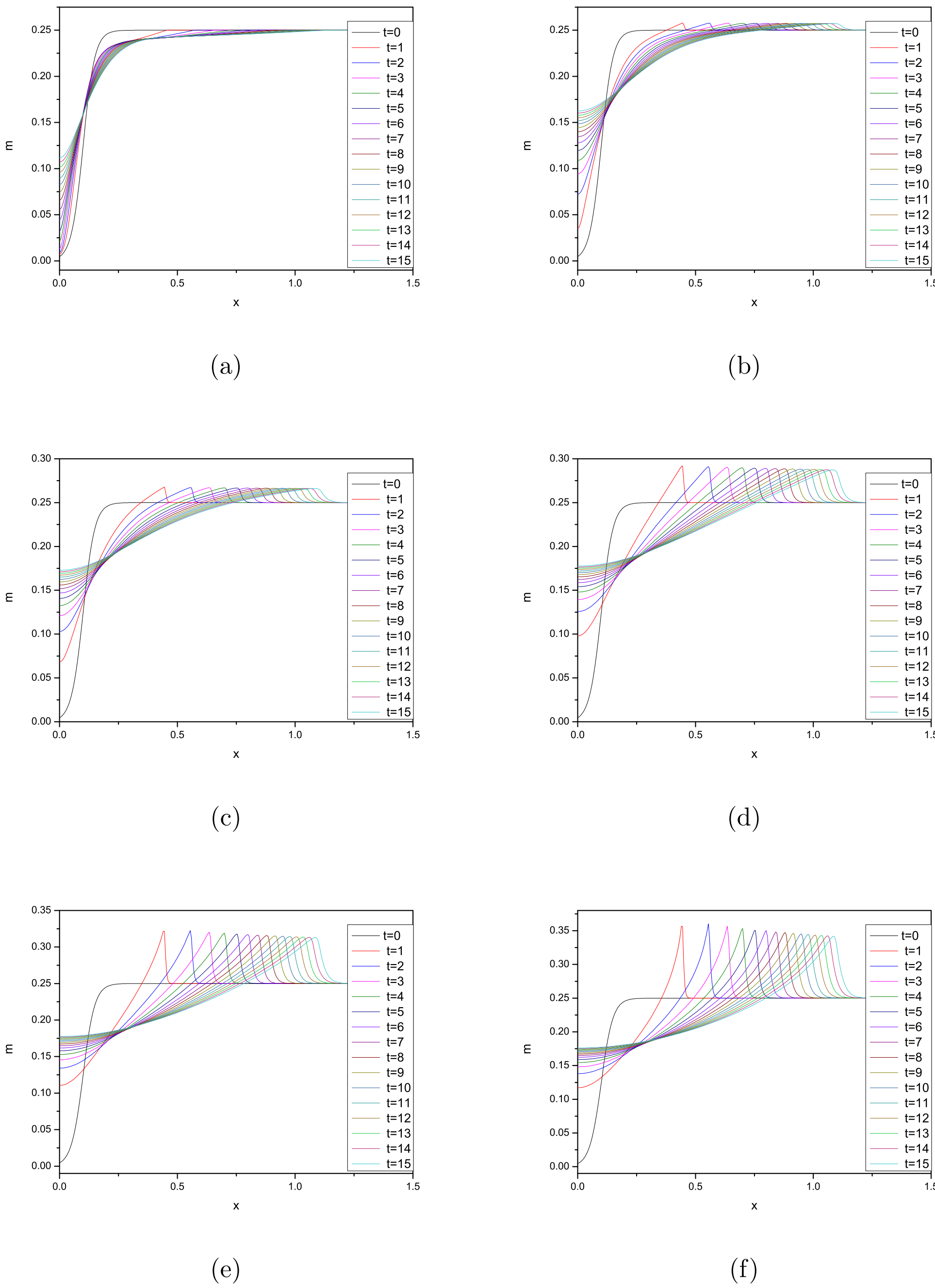
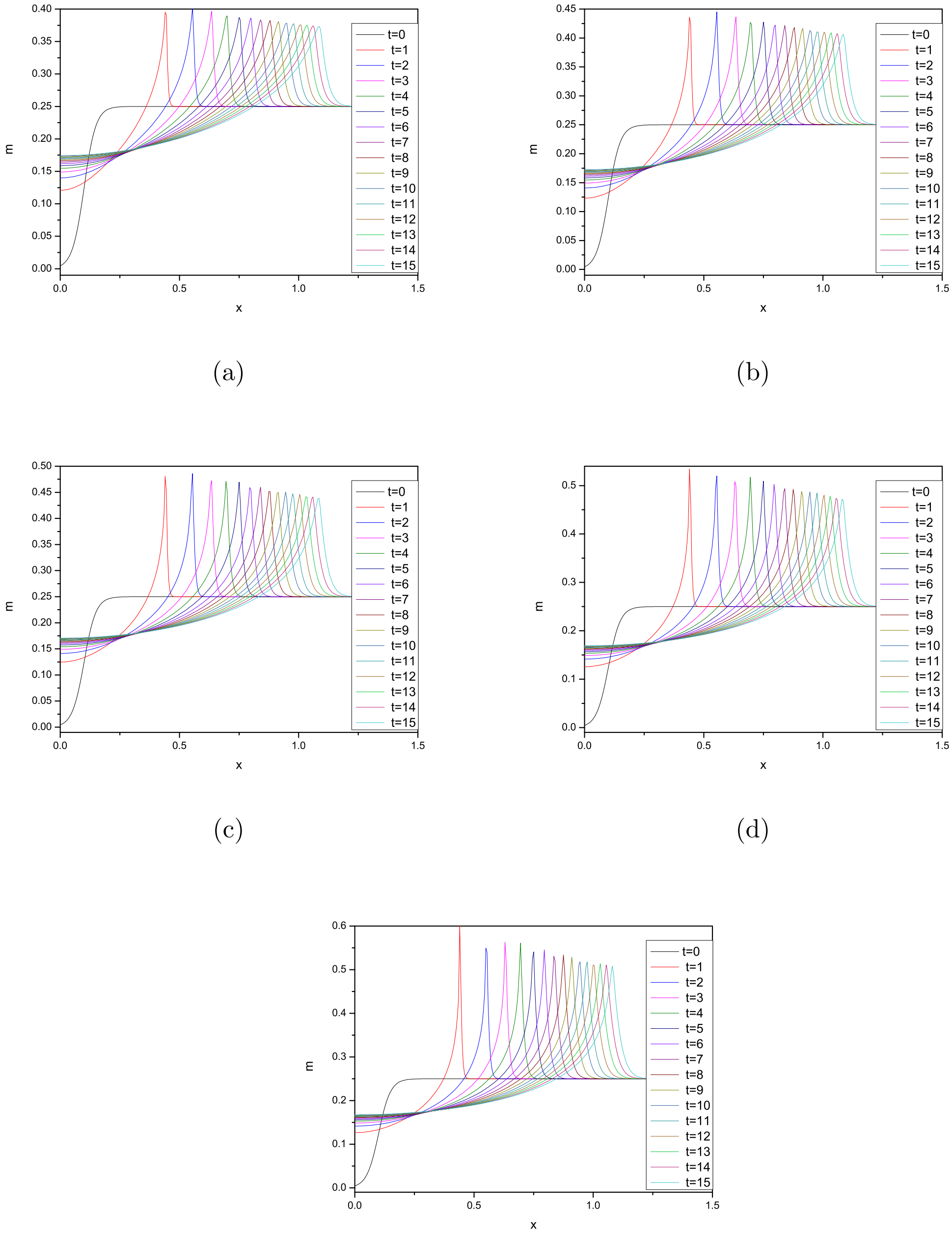


Figure 7.5: Volume fractions of the Extracellular matrix m versus position at times $t = 0, .., 15$ and $\Delta t = 0.001$. The production rates vanish, $R_c = R_m = 0$. (a) $\theta = 0$, (b) $\theta = 50$, (c) $\theta = 100$, (d) $\theta = 200$, (e) $\theta = 300$, (f) $\theta = 400$.

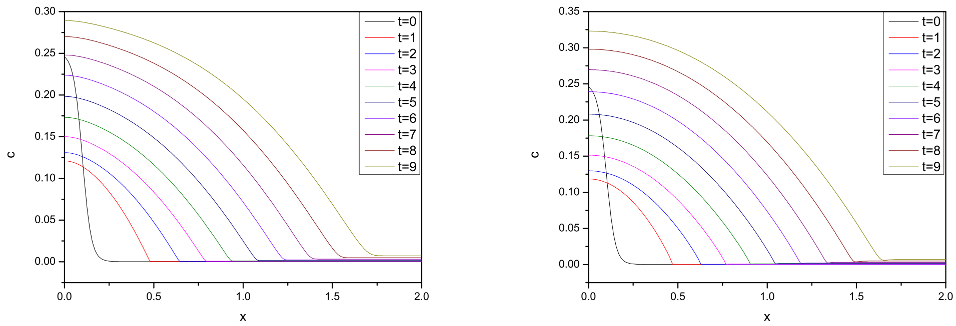


(a) (b)

(c) (d)

(e)

Figure 7.6: Volume fractions of the Extracellular matrix m versus position at times $t = 0, .., 15$ and $\Delta t = 0.001$. The production rates vanish, $R_c = R_m = 0$. (a) $\theta = 500$, (b) $\theta = 600$, (c) $\theta = 700$, (d) $\theta = 800$, (e) $\theta = 900$.



(a) (b)

Figure 7.7: Volume fractions of the tumor cells c versus position at times $t = 0, ..., 9$. The production rates are $\alpha = 0.1, \gamma = 1$, $\delta = 0.35$ and $\Delta t = 0.001$ (a) $\theta = 0$, (b) $\theta = 800$.

for the position of the peak, we performed a least-square fit to find β_2 given in Figure 7.10 and Tables 7.1 and 7.2.

In all of these experiments, increasing h by a factor of two resulted in little change for the calculated velocity to the extent that we feel confident that the calculated values are correct to 3 decimal places. There is a period where the maximum value moves its position a constant speed when $|\Omega| = 2$ and $|\Omega| = 4$. For large values of θ (see Figure 7.11), the velocity varies very little, but, the shape of the wave and its amplitude changes significantly. In Tables 7.1 and 7.2, we list the position which corresponding to the maximum of the Extracellular matrix m for each time level, when $R_c = R_m = 0$ and $\alpha = 0.1, \gamma = 1$, and $\delta = 0.35$. Also, in Figure 7.10, we plot a graph when $R_c = R_m = 0$ and $\alpha = 0.1, \gamma = 1$, and $\delta = 0.35$. Note that in the second experiment x_F hits the right hand boundary shortly after $t = 11$.

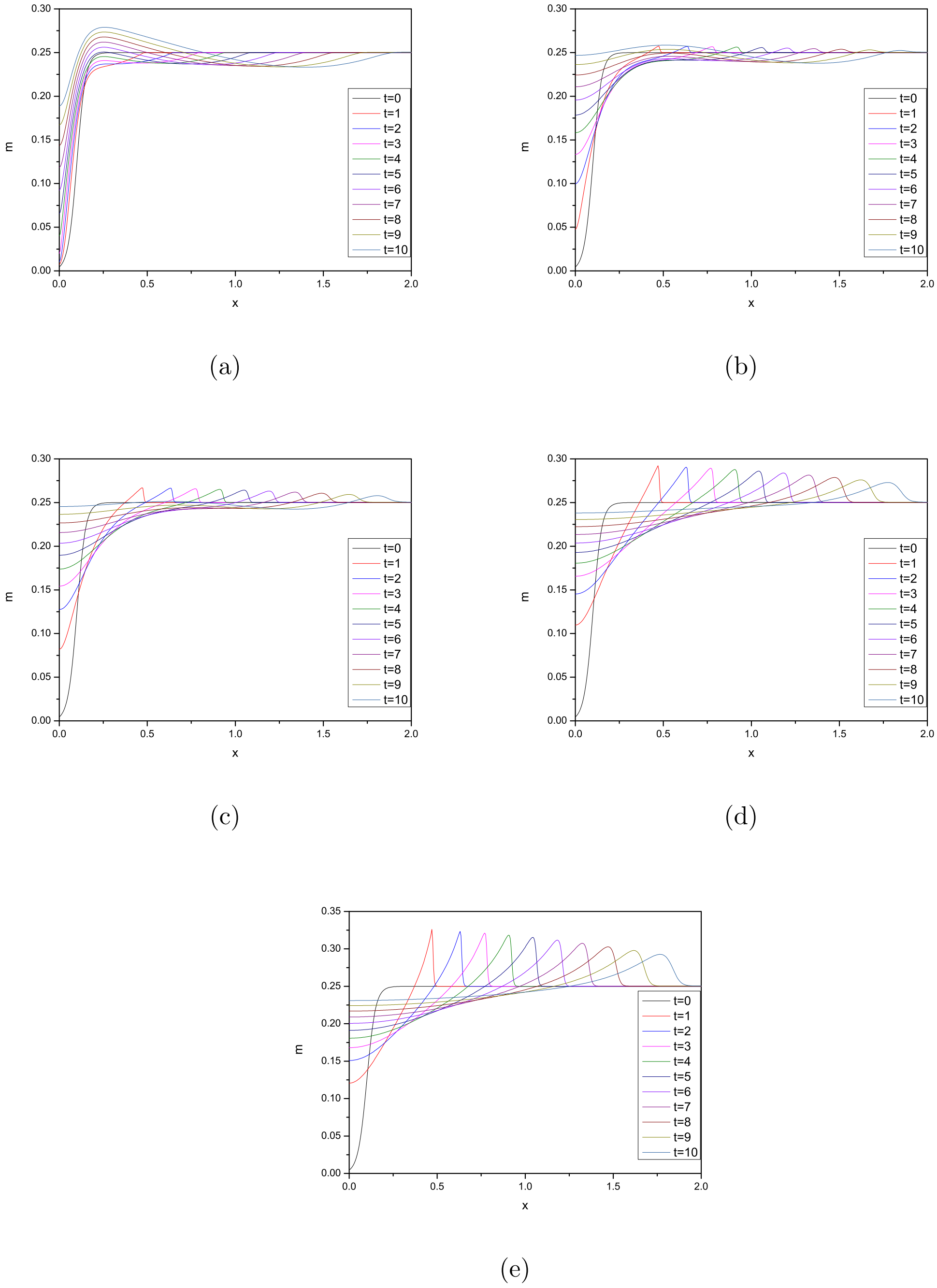
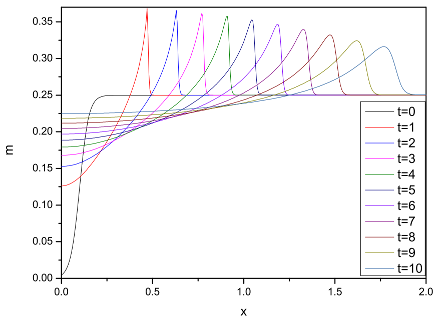
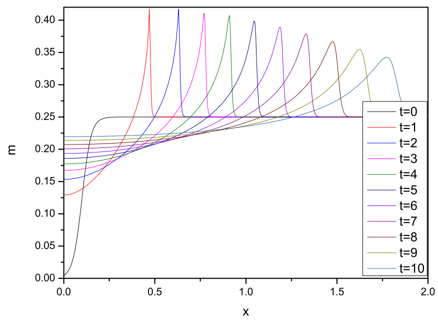


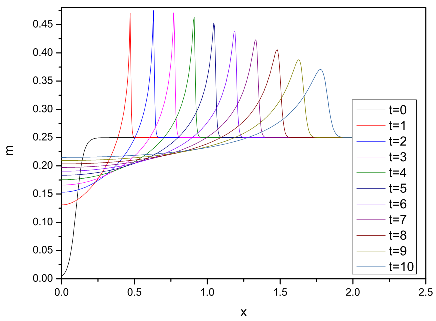
Figure 7.8: Volume fractions of the Extracellular matrix m versus position at times $t = 0, .., 10$. The production rates are $\alpha = 0.1, \gamma = 1$, $\delta = 0.35$ and $\Delta t = 0.001$ (a) $\theta = 0$, (b) $\theta = 50$, (c) $\theta = 100$, (d) $\theta = 200$, (e) $\theta = 300$.



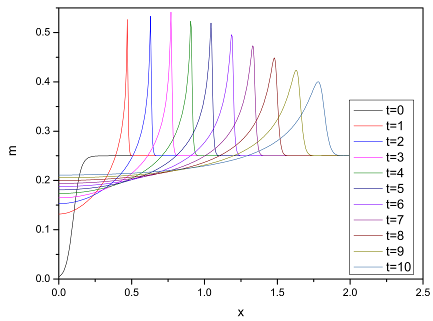
(a)



(b)



(c)



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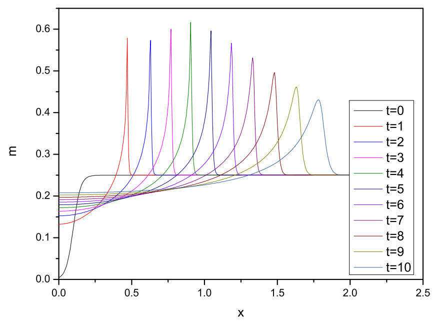
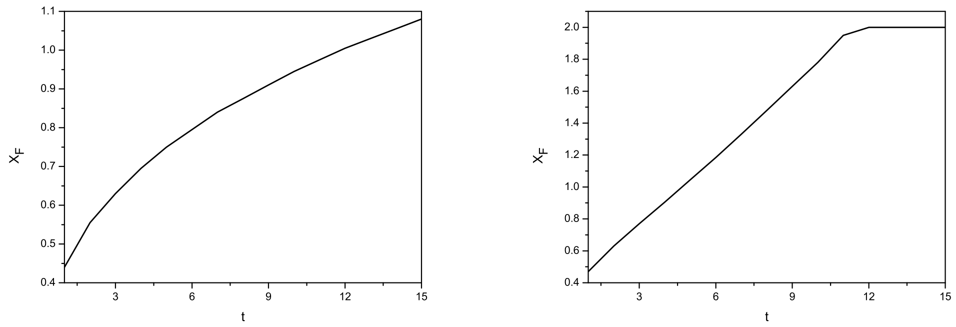


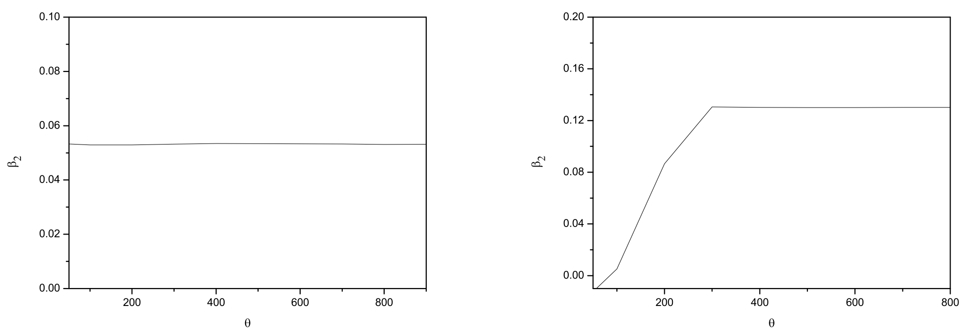
(e)

Figure 7.9: Volume fractions of the Extracellular matrix m versus position at times $t = 0, .., 10$. The production rates are $\alpha = 0.1, \gamma = 1$, $\delta = 0.35$ and $\Delta t = 0.001$ (a) $\theta = 400$, (b) $\theta = 500$, (c) $\theta = 600$, (d) $\theta = 700$. (e) $\theta = 800$.



(a) (b)

Figure 7.10: The position which corresponding to the maximum of the Extracellular matrix m for each time level, i.e. $x_F(t) = \{x \in [0,2] : \mathcal{M}^n_\varepsilon(x,t) \geq \mathcal{M}^n_\varepsilon(y,t); \forall y \in [0,2]\}$ for $\theta = 800$ (a) The production rates vanish, $R_c = R_m = 0$. (b) The production rates are $\alpha = 0.1, \gamma = 1$, and $\delta = 0.35$.



(a) (b)

Figure 7.11: β_2 in equation (7.1.6) versus θ. (a) The production rates vanish, $R_c = R_m = 0$. (b) The production rates are $\alpha = 0.1, \gamma = 1$, and $\delta = 0.35$.

	θ										
t	0	50	100	200	300	400	500	600	700	800	900
1	1.015	0.445	0.445	0.445	0.445	0.440	0.440	0.440	0.440	0.440	0.440
2	1.005	0.560	0.555	0.555	0.555	0.555	0.555	0.555	0.555	0.555	0.550
3	1.000	0.640	0.635	0.635	0.635	0.635	0.635	0.635	0.635	0.630	0.630
4	1.765	0.700	0.700	0.695	0.700	0.700	0.700	0.695	0.695	0.695	0.695
5	1.170	0.755	0.750	0.750	0.750	0.750	0.750	0.750	0.750	0.750	0.750
6	1.765	0.800	0.800	0.795	0.800	0.800	0.800	0.800	0.795	0.795	0.795
7	1.765	0.845	0.840	0.840	0.840	0.840	0.840	0.840	0.840	0.840	0.835
8	1.420	0.885	0.880	0.875	0.880	0.880	0.880	0.880	0.875	0.875	0.875
9	1.565	0.920	0.915	0.910	0.915	0.915	0.915	0.915	0.915	0.910	0.910
10	1.565	0.950	0.945	0.945	0.945	0.950	0.950	0.945	0.945	0.945	0.945
11	1.565	0.980	0.975	0.975	0.980	0.980	0.980	0.975	0.975	0.975	0.975
12	1.565	1.010	1.005	1.005	1.005	1.010	1.005	1.005	1.005	1.005	1.000
13	1.565	1.035	1.030	1.030	1.035	1.035	1.035	1.035	1.030	1.030	1.030
14	1.565	1.065	1.055	1.055	1.060	1.060	1.060	1.060	1.060	1.055	1.055
15	1.565	1.085	1.080	1.080	1.085	1.085	1.085	1.085	1.085	1.080	1.080

Table 7.1: The position corresponding to the maximum of the Extracellular matrix m for each time level, i.e. $x_F(t) = \{x \in [0,2] : \mathcal{M}_\varepsilon^n(x,t) \geq \mathcal{M}_\varepsilon^n(y,t); \forall y \in [0,2]\}$. The production rates vanish, $R_c = R_m = 0$ and $\Delta t = 0.001$.

	θ									
t	0	50	100	200	300	400	500	600	700	800
1	1.760	0.475	0.470	0.470	0.470	0.470	0.470	0.470	0.470	0.470
2	1.705	0.635	0.630	0.630	0.630	0.630	0.630	0.630	0.630	0.630
3	1.695	0.780	0.775	0.770	0.770	0.770	0.770	0.770	0.770	0.770
4	1.700	0.915	0.910	0.905	0.905	0.910	0.910	0.910	0.905	0.905
5	0.255	1.060	1.050	1.040	1.045	1.045	1.045	1.045	1.045	1.045
6	0.255	1.205	1.190	1.185	1.185	1.185	1.185	1.190	1.185	1.185
7	0.255	1.355	1.340	1.325	1.325	1.330	1.330	1.330	1.330	1.330
8	0.255	1.510	1.490	1.475	1.470	1.475	1.475	1.480	1.480	1.480
9	0.255	0.530	1.645	1.625	1.615	1.620	1.625	1.625	1.630	1.630
10	0.260	0.515	1.805	1.775	1.765	1.770	1.775	1.775	1.780	1.780
11	0.260	0.500	2	2	2	1.965	1.950	1.950	1.950	1.950
12	0.265	0.480	0.480	2	2	2	2	2	2	2
13	0.270	0.450	0.370	2	2	2	2	2	2	2
14	0.275	0.400	0.170	2	2	2	2	2	2	2
15	0.280	0.315	0	0	2	2	2	2	2	2

Table 7.2: The position corresponding to the maximum of the Extracellular matrix m for each time level, i.e. $x_F(t) = \{x \in [0,2] : \mathcal{M}_\varepsilon^n(x,t) \geq \mathcal{M}_\varepsilon^n(y,t); \forall y \in [0,2]\}$. The production rates are $\alpha = 0.1, \gamma = 1$, $\delta = 0.35$ and $\Delta t = 0.001$.

θ	β_2
50	0.05327206
100	0.05289706
200	0.05294118
300	0.05323529
400	0.05343382
500	0.05336765
600	0.05333088
700	0.05326471
800	0.05310294
900	0.05312500

Table 7.3: The values of velocity β_2. The production rates vanish, $R_c = R_m = 0$, $\Delta t = 0.001$.

θ	β_2
50	-0.01247794
100	0.00515441
200	0.08652941
300	0.13048529
400	0.13013235
500	0.13003676
600	0.13002206
700	0.13015441
800	0.13015441

Table 7.4: The values of velocity β_2. The production rates are $\alpha = 0.1, \gamma = 1$, and $\delta = 0.35$, $\Delta t = 0.001$.

Chapter 8

Approximation of the Keller-Segel Model

In this chapter a finite element scheme for the Keller-Segel model with an additional cross-diffusion term in the elliptic equation for the chemical signal is analyzed. In Section 8.1 we introduce a regularized problem of the truncated system. Then we obtain some a priori estimates of the regularized functions, independent of the regularization parameter, via deriving a well defined entropy inequality of the regularized problem. In Section 8.2.1, we propose a practical fully discrete finite element approximation of the regularized problem. Next, in Section 8.2.2, we use a fixed point theorem to show the existence of the approximate solutions. In Section 8.2.3 we derive a discrete entropy inequality and some stability bounds on the solutions of regularized problem. In Section 8.2.4, the uniqueness of the fully discrete approximations is discussed. Finally, in Section 8.3, we discuss the convergence to the semi-discrete problem.

8.1 A regularized problem

The key step of our analysis in proving existence of a global weak solution of the system (1.4.8)-(1.4.11) is to derive a priori estimates. To achieve this, we use a mathematical approach that deals with an entropy inequality of the problem (Q). Such an approach has been employed in studying different kinds of partial differential equations, e.g. see [8], [9], [10] and [11]. However, although the methodology we will use has been utilized before, we include all details here for completeness. By using an appropriate entropy functional, we first obtain some a priori estimates on a solution of the model (Q).

For illustrative purposes, we now introduce for $\varepsilon \in (0, e^{-1})$ the corresponding regularized version of the problem ($\mathrm{Q}_{M,\varepsilon}$):

Find $\{e_\varepsilon, s_\varepsilon\} \in \mathbb{R} \times \mathbb{R}$ such that

$$\partial_t e_\varepsilon - \nabla \cdot [\nabla e_\varepsilon - \phi_\varepsilon(e_\varepsilon)\nabla s_\varepsilon] = 0, \qquad \text{in} \quad Q_T, \tag{8.1.1}$$

$$\alpha \partial_t s_\varepsilon - \Delta s_\varepsilon - \delta \Delta e_\varepsilon - \mu e_\varepsilon + s_\varepsilon = 0, \qquad \text{in} \quad Q_T, \tag{8.1.2}$$

$$\nabla e_\varepsilon \cdot \nu = 0, \qquad \nabla s_\varepsilon \cdot \nu = 0, \qquad \text{on} \quad S_T, \tag{8.1.3}$$

$$e_\varepsilon(\cdot, 0) = e^0, \qquad s_\varepsilon(\cdot, 0) = s^0, \qquad \text{in} \qquad \Omega, \tag{8.1.4}$$

where the regularized function ϕ_ε and the parameter ε have been defined in Chapter 2. In the following lemma we derive the entropy inequality for the regularized problem ($\mathrm{Q}_{M,\varepsilon}$) which will provide us with some uniform bounds on the regularized solutions e_ε and s_ε.

Lemma 8.1.1 Let $\{e^0(\mathbf{x}), s^0(\mathbf{x})\} \in L^2(\Omega) \times L^2(\Omega)$. Then, there exists a positive $C(e^0, s^0, \delta, \mu, C)$ independent of ε such that any solution of $\mathrm{Q}_{M,\varepsilon}$ satisfies

$$\sup_{0 \le t \le T} \int_\Omega (F_\varepsilon(e_\varepsilon) + \frac{\alpha}{2\delta} s_\varepsilon^2) dx + \int_0^T \left(\frac{|\nabla e_\varepsilon|^2}{2\phi_\varepsilon(e_\varepsilon)} + \frac{1}{2\delta}\|s_\varepsilon\|_0^2 + \frac{1}{2\delta}|s_\varepsilon|_1^2 \right) \le C(M). \tag{8.1.5}$$

where F_ε is defined in (2.3.25). In addition,

$$\sup_{0 \le t \le T} \int_\Omega |[e_\varepsilon]_-|^2 dx \le C(M)\varepsilon. \tag{8.1.6}$$

Proof: Multiplying (8.1.1) and (8.1.2) by $F'_\varepsilon(e_\varepsilon)$ and $\frac{1}{\delta}s_\varepsilon$ respectively, integrating by parts over the domain Ω, summing the resulting equations yields, after recalling the boundary conditions (8.1.3), that

$$\frac{d}{dt}\int_\Omega (F_\varepsilon(e_\varepsilon) + \frac{\alpha}{2\delta}s_\varepsilon^2) dx + \int_\Omega \left(\frac{|\nabla e_\varepsilon|^2}{\phi_\varepsilon(e_\varepsilon)} + \frac{1}{\delta}|\nabla s_\varepsilon|^2 + \frac{1}{\delta}s_\varepsilon^2 \right) dx \le \frac{\mu}{\delta}\int_\Omega e_\varepsilon s_\varepsilon dx, \tag{8.1.7}$$

where we used the relation

$$\phi_\varepsilon(e_\varepsilon)\nabla[F'_\varepsilon(e_\varepsilon)] = \nabla e_\varepsilon. \tag{8.1.8}$$

We note that testing (8.1.1) with $\chi \equiv 1$ gives for $a.e.\ t \in (0,T)$ that

$$(e_\varepsilon(.,t),1) = (e_\varepsilon(.,0),1) = (e^0,1) \leq C. \tag{8.1.9}$$

It follows immediately from (8.1.2) for $a.e.\ t \in (0,T)$ that

$$\alpha \frac{d}{dt} \fint s_\varepsilon - \mu \fint e_\varepsilon + \fint s_\varepsilon = 0. \tag{8.1.10}$$

Therefore, on noting the assumptions on the initial data and the bound (8.1.9), integrating (8.1.10) over $(0,T)$ leads to

$$\fint s_\varepsilon \leq C. \tag{8.1.11}$$

We estimate the right-hand side of (8.1.7) using the Hölder inequality:

$$\frac{\mu}{\delta} \int_\Omega e_\varepsilon s_\varepsilon dx \leq \frac{\mu}{\delta} \big\| e_\varepsilon - \fint e_\varepsilon \big\| \, \big\| s_\varepsilon \big\|_0 + \mathfrak{C}, \tag{8.1.12}$$

where $\mathfrak{C} = |\Omega| \fint e_\varepsilon \fint s_\varepsilon$. Then, the term $\| e_\varepsilon - \fint e_\varepsilon \|$ can be bounded by use of the Poincaŕe inequality:

$$\big\| e_\varepsilon - \fint e_\varepsilon \big\|^2 \leq C_p \big| e_\varepsilon \big|_1^2 \leq C_p \int_\Omega \frac{|\nabla e_\varepsilon|^2}{\phi_\varepsilon(e_\varepsilon)} \phi_\varepsilon(e_\varepsilon) dx \leq C_p M \int_\Omega \frac{|\nabla e_\varepsilon|^2}{\phi_\varepsilon(e_\varepsilon)} dx. \tag{8.1.13}$$

Combining (8.1.12) and (8.1.13) and using Young's inequality leads to

$$\frac{\mu}{\delta} \int_\Omega e_\varepsilon s_\varepsilon dx \leq \frac{1}{2} \int_\Omega \frac{|\nabla e_\varepsilon|^2}{\phi_\varepsilon(e_\varepsilon)} dx + C_p M \frac{\mu^2}{2\delta^2} \big\| s_\varepsilon \big\|_0^2 + \mathfrak{C}. \tag{8.1.14}$$

Hence, the result (8.1.5) follows from (8.1.7) on noting (8.1.14), (8.1.9) and (8.1.11). Finally, the result (8.1.6) follows immediately from the first bound in (8.1.5) and (2.3.29). □

The existence of a solution of problem (Q) could be shown by passing to the limit $\varepsilon \longrightarrow 0$. However, this can only be performed in the case that we have existence of a solution to the regularized problem $(\mathrm{Q}_{M,\varepsilon})$. To deal with this issue, in our study of problem (Q), we use the power of the finite element method.

We now formulate a fully discrete finite element approximation of $(\mathrm{Q}_{M,\varepsilon})$ and prove existence of fully discrete approximate solutions using discretization parameters h and Δt. In actual fact, to prove existence for (Q) we let $h \to 0$ to yield a semi-discrete problem $(\mathrm{Q}_M^{\Delta t})$ and then let $\frac{1}{M}, \Delta t \to 0$.

8.2 A fully discrete approximation of the Keller-Segel Model

8.2.1 An approximation problem

In order to introduce a fully discrete approximation that is consistent with the regularized problem $(\mathrm{Q}_{M,\varepsilon})$, we adapt a technique developed in [54] for studying a degenerate nonlinear fourth order parabolic equation modelling the height of thin films of viscous fluids driven by surface tension. This technique has been also adapted and employed in a number of numerical studies, see for example [8], [9], [10], [11] and [12].

Now, we propose the following fully discrete finite element approximation of $(\mathrm{Q}_{M,\varepsilon})$ for any $\varepsilon \in (0, e^{-1})$:

$(\mathrm{Q}^{h,\Delta t}_{M,\varepsilon})$ For $n \geq 1$ find $\{E^n_\varepsilon, S^n_\varepsilon\} \in [S^h]^2$ such that for all $\chi \in S^h$

$$\Big(\frac{E^n_\varepsilon - E^{n-1}_\varepsilon}{\Delta t}, \chi\Big)^h + (\nabla E^n_\varepsilon - \Lambda_\varepsilon(E^n_\varepsilon)\nabla S^n_\varepsilon, \nabla\chi) = 0, \tag{8.2.15}$$

$$\alpha\Big(\frac{S^n_\varepsilon - S^{n-1}_\varepsilon}{\Delta t}, \chi\Big)^h + (S^n_\varepsilon, \chi)^h + (\nabla S^n_\varepsilon, \nabla\chi) + \delta(\nabla E^n_\varepsilon, \nabla\chi) = \mu(E^n_\varepsilon, \chi)^h, \tag{8.2.16}$$

where E^0_ε and $S^0_\varepsilon \in S^h$ are given approximations of e^0_ε and s^0_ε, respectively, and Λ_ε is given by (2.4.65).

8.2.2 Existence of the approximations

In order to prove the existence of solution E^n_ε and $S^n_\varepsilon, n \geq 1$, of the system (8.2.15)-(8.2.16) for given E^{n-1}_ε and S^{n-1}_ε, it is convenient to define the functions $A_e : S^h \times S^h \to S^h$ and $A_s : S^h \times S^h \to S^h$ such that for all $\chi \in S^h$

$$(A_e(E,S), \chi)^h = (E - E^{n-1}_\varepsilon, \chi)^h + \Delta t(\nabla E - \Lambda_\varepsilon(E)\nabla S, \nabla\chi), \tag{8.2.17}$$

$$\begin{aligned}(A_s(E,S), \chi)^h = \alpha(S - S^{n-1}_\varepsilon, \chi)^h + \Delta t(S, \chi)^h + \Delta t(\nabla S, \nabla\chi) + \delta\Delta t(\nabla E, \nabla\chi)\\ -\Delta t\mu(E, \chi)^h,\end{aligned} \tag{8.2.18}$$

respectively. We first note that the continuous piecewise linear functions A_e and A_s can be defined uniquely in terms of their values at the nodal points $\mathcal{N}^h$. This can be seen by setting $\chi \equiv \varphi_j$, for $j = 0, ..., J$, in (8.2.17) and (8.2.18) and then obtaining the following solvable square matrix systems

$$\widehat{M} A_e(E, S) = S_1,$$

$$\widehat{M} A_s(E, S) = S_2,$$

where $\widehat{M}$ is the lumped mass matrix, and S_1 and S_2 are given vectors in terms of the nodal values of $E, S, E_\varepsilon^{n-1}$ and S_ε^{n-1}. Thus, the functions A_e and A_s are well defined.

From (8.2.17) and (8.2.18) we note that the problem $(\mathrm{Q}_{M,\varepsilon}^{h,\Delta t})$ can be restated as:
For given $\{E_\varepsilon^0, S_\varepsilon^0\} \in S^h \times S^h$, find $\{E_\varepsilon^n, S_\varepsilon^n\} \in S^h \times S^h, n \geq 1$, such that

$$A_e(E, S) = 0, \qquad A_s(E, S) = 0. \tag{8.2.19}$$

Lemma 8.2.1 For any given $R > 0$, the functions $A_e : [S^h]_R^2 \to S^h$ and $A_s : [S^h]_R^2 \to S^h$ are continuous, where

$$[S^h]_R^2 = \Big\{ \{\chi_1, \chi_2\} \in S^h \times S^h : |\chi_1|_h^2 + |\chi_2|_h^2 \leq R^2 \Big\}.$$

Proof: Let $\{E_1, S_1\}, \{E_2, S_2\} \in [S^h]_R^2$. It follows from (8.2.17) that for all $\chi \in S^h$

$$(A_e(E_1, S_1) - A_e(E_2, S_2), \chi)^h = (E_1 - E_2, \chi)^h + \Delta t(\nabla(E_1 - E_2) - \Lambda_\varepsilon(E_1)\nabla S_1$$
$$+\Lambda_\varepsilon(E_2)\nabla S_2, \nabla\chi). \tag{8.2.20}$$

Choosing $\chi = A_e(E_1, S_1) - A_e(E_2, S_2)$ in (8.2.20) yields on noting the Cauchy-Schwarz inequality, (2.4.54) and (2.4.46), that

$$|A_e(E_1, S_1) - A_e(E_2, S_2)|_h \leq C(h^{-1}, \Delta t)|E_1 - E_2|_h + C(h^{-1}, \Delta t)\|\Lambda_\varepsilon(E_1)\nabla S_1 - \Lambda_\varepsilon(E_2)\nabla S_2\|_0. \tag{8.2.21}$$

It follows from (2.4.54), (2.4.46), (2.4.69), (2.4.68) and (2.4.55) that

$$\|\Lambda_\varepsilon(E_1)\nabla S_1 - \Lambda_\varepsilon(E_2)\nabla S_2\|_0 = \|\Lambda_\varepsilon(E_1)\nabla S_1 - \Lambda_\varepsilon(E_2)\nabla S_1 + \Lambda_\varepsilon(E_2)\nabla S_1 - \Lambda_\varepsilon(E_2)\nabla S_2\|_0$$

$$\leq \|(\Lambda_\varepsilon(E_1) - \Lambda_\varepsilon(E_2))\nabla S_1\|_0 + \|\Lambda_\varepsilon(E_2)(\nabla S_1 - \nabla S_2)\|_0$$

$$\leq \|(\Lambda_\varepsilon(E_1) - \Lambda_\varepsilon(E_2))\|_{0,\infty}|S_1|_1 + \|\Lambda_\varepsilon(E_2)\|_{0,\infty}|S_1 - S_2|_1$$

$$\leq Ch^{-1}\|(\Lambda_\varepsilon(E_1) - \Lambda_\varepsilon(E_2))\|_{0,\infty}|S_1|_h + Ch^{-1}\|\Lambda_\varepsilon(E_2)\|_{0,\infty}|S_1 - S_2|_h$$

$$\leq C(h^{-1}, M, \varepsilon^{-1})\|E_1 - E_2\|_{0,\infty}|S_1|_h + C(h^{-1}, M)|S_1 - S_2|_h$$
$$\leq C(h^{-1}, M, \varepsilon^{-1}, R)\|E_1 - E_2\|_0 + C(h^{-1}, M)|S_1 - S_2|_h$$
$$\leq C(h^{-1}, M, \varepsilon^{-1}, R)\big(\|E_1 - E_2\|_0 + |S_1 - S_2|_h\big). \qquad (8.2.22)$$

Combining (8.2.20), (8.2.21) and (8.2.22) yields that for A_e is Lipchitz continuous. The proof of the continuity of A_s follows similarly to the proof of the continuity of A_e.

□

We now show the main result of this chapter where we establish the existence of a solution $\{E^n_\varepsilon, S^n_\varepsilon\}$ to $(\mathrm{Q}^{h,\Delta t}_{M,\varepsilon})$.

Theorem 8.2.2 Let $\{E^{n-1}_\varepsilon, S^{n-1}_\varepsilon\} \in S^h \times S^h$ be a given solution to the $(n-1)$-th step of $(\mathrm{Q}^{h,\Delta t}_{M,\varepsilon})$ for some $n = 1, ..., N$. Then for all $\varepsilon \in (0, e^{-1})$, for all $h > 0$ and for all $\Delta t \leq \frac{\delta}{2M\mu^2}$, there exists a solution $\{E^n_\varepsilon, S^n_\varepsilon\} \in S^h \times S^h$ to the n-th step of $(\mathrm{Q}^{h,\Delta t}_{M,\varepsilon})$.

Proof: Now, we recall that the proof is equivalent to the proof of existence of $\{E^n_\varepsilon, S^n_\varepsilon\} \in [S^h]^2_R$ satisfies (8.2.19). One approach is to use a proof by contradiction. Let R be a fixed positive number and assume that there does not exist $\{E, S\} \in [S^h]^2_R$ with $A_e(E, S) = A_s(E, S) = 0$. This assumption enables us to define a function $B : [S^h]^2_R \to [S^h]^2_R$ such that

$$B(E, S) = (B_e(E, S), B_s(E, S)),$$

where $B_e(E, S)$ and $B_s(E, S)$ are given by

$$B_e(E, S) := \frac{-R\, A_e(E, S)}{|(A_e(E, S), A_s(E, S))|_{S^h \times S^h}},$$
$$B_s(E, S) := \frac{-R\, A_s(E, S)}{|(A_e(E, S), A_s(E, S))|_{S^h \times S^h}}, \qquad (8.2.23)$$

where $|(\cdot, \cdot)|_{[S^h]^2_R}$ is the standard norm on $[S^h]^2_R$ defined by

$$|(\chi_1, \chi_2)|_{S^h \times S^h} = \left(\sum_{i=1}^{2} |\chi_i|^2_h\right)^{\frac{1}{2}}.$$

We note from the continuity of A_e and A_s, see Lemma 8.2.1, that the function B is continuous. Hence, on recalling that $[S^h]^2_R$ is a convex and compact subset of

$S^h \times S^h$, it follows from the Schauder's theorem (see Appendix A.1.1) that there exists $E, S \in [S^h]^2_R$ which is a fixed point of B; that is

$$B(E,S) = (B_e(E,S), B_s(E,S)) = (E,S).$$

We also note from (8.2.23) that the fixed point $\{E, S\}$ satisfies

$$|E|_h^2 + |S|_h^2 = |B_e(E,S)|_h^2 + |B_s(E,S)|_h^2 = R^2. \tag{8.2.24}$$

We now prove a contradiction for R sufficiently large. Choosing $\chi \equiv \pi^h[F'_\varepsilon(E)]$, in (8.2.17) yields on noting (2.4.45), (2.4.62) and (2.4.68) that

$$\begin{aligned}(A_e(E,S), F'_\varepsilon(E))^h &= (E - E_\varepsilon^{n-1}, F'_\varepsilon(E))^h + \Delta t(\Lambda_\varepsilon^{-1}(E)\nabla E - \nabla S, \nabla E)\\ &\geq (E - E_\varepsilon^{n-1}, F'_\varepsilon(E))^h + \frac{\Delta t}{M}|E|_1^2 - \Delta t(\nabla S, \nabla E),\end{aligned} \tag{8.2.25}$$

and $\chi \equiv \frac{S}{\delta}$ in (8.2.18)

$$\begin{aligned}(A_s(E,S), \frac{S}{\delta})^h = \frac{\alpha}{\delta}(S - S_\varepsilon^{n-1}, S)^h + \frac{\Delta t}{\delta}|S|_h^2 + \frac{\Delta t}{\delta}|S|_1^2 + \Delta t(\nabla E, \nabla S)\\ - \frac{\Delta t \mu}{\delta}(E,S)^h.\end{aligned} \tag{8.2.26}$$

We obtain from (2.3.28), (2.3.33) and (2.1.10) that

$$\begin{aligned}(E - E_\varepsilon^{n-1}, F'_\varepsilon(E))^h &\geq (F_\varepsilon(E) - F_\varepsilon(E_\varepsilon^{n-1}), 1)^h + \frac{1}{2}((E - E_\varepsilon^{n-1})^2, F''_\varepsilon(\xi))^h\\ &\geq (F_\varepsilon(E) - F_\varepsilon(E_\varepsilon^{n-1}), 1)^h + \frac{1}{2M}|E - E_\varepsilon^{n-1}|_h^2\\ &\geq (F_\varepsilon(E) - F_\varepsilon(E_\varepsilon^{n-1}), 1)^h + \frac{1}{4M}|E|_h^2 - \frac{1}{2M}|E_\varepsilon^{n-1}|_h^2.\end{aligned} \tag{8.2.27}$$

Using the simple identity

$$2\varphi(\varphi - \kappa) = \varphi^2 - \kappa^2 + (\varphi - \kappa)^2, \qquad \forall\ \varphi,\ \kappa\ \in \mathbb{R},$$

we obtain that

$$\frac{\alpha}{\delta}(S - S_\varepsilon^{n-1}, S)^h \geq \frac{\alpha}{2\delta}|S|_h^2 - \frac{\alpha}{2\delta}|S_\varepsilon^{n-1}|_h^2. \tag{8.2.28}$$

The last term of (8.2.26) can be bound using Young's inequality, as follows:

$$\frac{\Delta t \mu}{\delta}\big(E,S\big)^h \leq \frac{\Delta t \mu^2}{2\delta}|E|_h^2 + \frac{\Delta t}{2\delta}|S|_h^2. \tag{8.2.29}$$

Adding (8.2.25) and (8.2.26) and noting (8.2.27)-(8.2.29), (8.2.24), the non-negativity of $F_\varepsilon(s)$, and the stated assumption on Δt yields for sufficiently large R that

$$
\begin{aligned}
&(A_e(E,S),F'_\varepsilon(E))^h+\frac{1}{\delta}(A_s(E,S),S)^h\\
&\geq (F_\varepsilon(E),1)^h+\big(\frac{1}{4M}-\frac{\mu^2}{2\delta}\Delta t\big)|E|_h^2+\big(\frac{\alpha}{2\delta}+\frac{\Delta t}{2\delta}\big)|S|_h^2+\frac{\Delta t}{M}|E|_1^2+\frac{\Delta t}{\delta}|S|_1^2-C(E_\varepsilon^{n-1},S_\varepsilon^{n-1})\\
&\geq (F_\varepsilon(E),1)^h+\min\left\{\big(\frac{1}{4M}-\frac{\mu^2}{2\delta}\Delta t\big),\big(\frac{\alpha}{2\delta}+\frac{\Delta t}{2\delta}\big)\right\}R^2+\frac{\Delta t}{M}|E|_1^2+\frac{\Delta t}{\delta}|S|_1^2-C(E_\varepsilon^{n-1},S_\varepsilon^{n-1}).
\end{aligned}
\tag{8.2.30}
$$

Noting that $\{E,S\}$ is a fixed point of the function B, (8.2.23) and (8.2.30) yields for R sufficiently large that

$$
\begin{aligned}
(E,F'_\varepsilon(E)^h+\frac{1}{\delta}(S,S)^h&=(B_e(E,S),F'_\varepsilon(E))^h+\frac{1}{\delta}(B_s(E,S),S)^h\\
&=\frac{-R\left[(A_e(E,S),F'_\varepsilon(E,S)^h+\frac{1}{\delta}(A_s(E,S),S)^h\right]}{|(A_e(E,S),A_s(E,S))|_{S^h\times S^h}}<0.
\end{aligned}
\tag{8.2.31}
$$

Once again, it follows from (2.3.33) and (2.3.28) that

$$
(E,F'_\varepsilon(E))^h\geq (F_\varepsilon(E)-F_\varepsilon(0),1)^h+\frac{1}{2M}|E|_h^2. \tag{8.2.32}
$$

Thus, using (8.2.32) yields on noting the non-negativity of $F_\varepsilon(s)$ for R sufficiently large that

$$
(E,F'_\varepsilon(E))^h+\frac{1}{\delta}(S,S)^h\geq R^2\min\{\frac{1}{2M},\frac{1}{\delta}\}-(1-\frac{\varepsilon}{2})|\Omega|>0, \tag{8.2.33}
$$

which contradicts (8.2.31). This contradiction ensures that there exists $\{E_\varepsilon^n,S_\varepsilon^n\}\in S^h\times S^h$ satisfying $A_e(E_\varepsilon^n,S_\varepsilon^n)=A_s(E_\varepsilon^n,S_\varepsilon^n)=0$. Equivalently, we have existence of a solution, which is $\{E_\varepsilon^n,S_\varepsilon^n\}$, to the n-th step of $(\mathrm{Q}_{M,\varepsilon}^{h,\Delta t})$.

□

8.2.3 Discrete entropy inequality and stability bounds

In this section we obtain a discrete analogue of the a priori estimates in Lemma 8.1.1. We also prove some uniform bounds on the solution $\{E_\varepsilon^n,S_\varepsilon^n\}$, independent of the parameters ε,h and Δt, which are necessary to prove the convergence of the approximate problem. The following estimate is discrete analogue of (8.1.5), and plays a key role in obtaining important stability bounds of various norms of the approximate solutions.

Lemma 8.2.3 Let the assumptions of Theorem 8.2.2 hold. Let $\{E_\varepsilon^{n-1}, S_\varepsilon^{n-1}\} \in S^h \times S^h$ be given for some $n = 1, ..., N$. Then for all $\varepsilon \in (0, e^{-1})$ and for all $h > 0$, there exists a solution $\{E_\varepsilon^n, S_\varepsilon^n\} \in S^h \times S^h$ to the n-th step of $(\mathrm{Q}_{M,\varepsilon}^{h,\Delta t})$ such that

$$\begin{aligned}\big(F_\varepsilon(E_\varepsilon^n), 1\big)^h + \frac{\Delta t}{2M}|E_\varepsilon^n|_1^2 + \left(\frac{\alpha}{2\delta} + \frac{\Delta t}{\delta} - \frac{\mu^2}{2\delta^2}\varsigma\Delta t\right)|S_\varepsilon^n|_h^2 + \frac{\Delta t}{\delta}|S_\varepsilon^n|_1^2 \\ \leq \big(F_\varepsilon(E_\varepsilon^{n-1}), 1\big)^h + \frac{\alpha}{2\delta}|S_\varepsilon^{n-1}|_h^2 + C\Delta t|(E_\varepsilon^0, 1)|^2, \qquad (8.2.34)\end{aligned}$$

where $\varsigma = M(d+2)C_p$.

Proof: The existence was demonstrated in Theorem 8.2.2. We now show that the solution $\{E_\varepsilon^n, S_\varepsilon^n\} \in S^h \times S^h$ satisfies (8.2.34). Choosing $\chi \equiv \Delta t \pi^h[F'_\varepsilon(E_\varepsilon^n)]$ as a test function in (8.2.15) and $\chi \equiv \frac{\Delta t}{\delta} S_\varepsilon^n$ as a test function in (8.2.16) yields, on noting (8.2.27), (8.2.25) and (8.2.28), the discrete analogue of (8.1.7)

$$\big(F_\varepsilon(E_\varepsilon^n), 1\big)^h + \frac{\Delta t}{M}|E_\varepsilon^n|_1^2 - \Delta t(\nabla S_\varepsilon^n, \nabla E_\varepsilon^n) \leq \big(F_\varepsilon(E_\varepsilon^{n-1}), 1\big)^h, \qquad (8.2.35)$$

$$\frac{\alpha}{2\delta}|S_\varepsilon^n|_h^2 + \frac{\Delta t}{\delta}|S_\varepsilon^n|_1^2 + \frac{\Delta t}{\delta}|S_\varepsilon^n|_h^2 + \Delta t(\nabla S_\varepsilon^n, \nabla E_\varepsilon^n) \leq \frac{\alpha}{2\delta}|S_\varepsilon^{n-1}|_h^2 + \frac{\mu\Delta t}{\delta}(E_\varepsilon^n, S_\varepsilon^n)^h. \qquad (8.2.36)$$

It follows immediately from (8.2.15) with $n = 1, ..., N$, that

$$(E_\varepsilon^n, 1) = (E_\varepsilon^0, 1). \qquad (8.2.37)$$

It follows from the Young's inequality, the Poincaré inequality, (2.4.46), and (8.2.37) that

$$\begin{aligned}\frac{\mu\Delta t}{\delta}(E_\varepsilon^n, S_\varepsilon^n)^h &\leq \frac{\Delta t}{2C_pM(d+2)}|E_\varepsilon^n|_h^2 + \frac{\mu^2}{2\delta^2}C_pM(d+2)\Delta t|S_\varepsilon^n|_h^2 \\ &\leq \frac{\Delta t}{2C_pM}\|E_\varepsilon^n\|_0^2 + \frac{\mu^2}{2\delta^2}C_pM(d+2)\Delta t|S_\varepsilon^n|_h^2 \\ &\leq \frac{\Delta t}{2M}|E_\varepsilon^n|_1^2 + \frac{\mu^2}{2\delta^2}C_pM(d+2)\Delta t|S_\varepsilon^n|_h^2 + C\Delta t|(E_\varepsilon^0, 1)|^2. \qquad (8.2.38)\end{aligned}$$

Combining (8.2.35), (8.2.36) and noting (8.2.38), leads to the desired result (8.2.34). □

In the following theorem we derive a discrete entropy inequality of the system (8.2.15)-(8.2.16) that is consistent with the entropy inequality obtained in Lemma 8.1.1.

Theorem 8.2.4 Let $e^0, s^0 \in L^2(\Omega)$ with $|e^0(\cdot)| \leq 1$ *a.e.* in Ω. Let $E^0_\varepsilon \in L^1(\Omega)$. Further, let either $E^0_\varepsilon \equiv P^h e^0$, $S^0_\varepsilon \equiv P^h s^0$; or $E^0_\varepsilon \equiv \pi^h e^0$, $S^0_\varepsilon \equiv \pi^h s^0$ if $e^0, s^0 \in C(\overline{\Omega})$. Then for all $\varepsilon \in (0, e^{-1})$, for all $h > 0$ and for all $\Delta t > 0$ such that

$$\Delta t \leq \begin{cases} \delta/2M\mu^2 & \text{if } \varsigma \leq \frac{2\delta}{\mu^2}, \\ \delta(\frac{\alpha}{2\delta} - \rho)/(\frac{\mu^2}{2\delta}\varsigma - 1) & \text{if } \varsigma > \frac{2\delta}{\mu^2}. \end{cases}$$

Then, the problem $(\mathrm{Q}^{h,\Delta t}_{M,\varepsilon})$ possesses a solution $\{E^n_\varepsilon, S^n_\varepsilon\}, n = 1, ..., N$ satisfying

$$\max_{n=1,\dots,N} \left[(F_\varepsilon(E^n_\varepsilon) + \|S^n_\varepsilon\|_0^2\right] + \sum_{n=1}^{N} \Delta t [\frac{1}{M}\|E^n_\varepsilon\|_1^2 + \|S^n_\varepsilon\|_1^2 + \varepsilon^{-1}\|\pi^h[E^n_\varepsilon]_-\|_0^2 + \|\pi^h[S^n_\varepsilon]_-\|_0^2] \leq C. \tag{8.2.39}$$

Furthermore,

$$\sum_{n=1}^{N} \Delta t \left[\left\|\frac{E^n_\varepsilon - E^{n-1}_\varepsilon}{\Delta t}\right\|^2_{(H^1(\Omega))'} + \left\|\frac{S^n_\varepsilon - S^{n-1}_\varepsilon}{\Delta t}\right\|^2_{(H^1(\Omega))'}\right] +$$

$$\sum_{n=1}^{N} \Delta t \left[\left\|\mathcal{G}[\frac{E^n_\varepsilon - E^{n-1}_\varepsilon}{\Delta t}]\right\|_1^2 + \left\|\mathcal{G}[\frac{S^n_\varepsilon - S^{n-1}_\varepsilon}{\Delta t}]\right\|_1^2\right] \leq C. \tag{8.2.40}$$

Proof: We consider the case when $\varsigma > \frac{2\delta}{\mu^2}$ and we comment later on the simple case $\varsigma \leq \frac{2\delta}{\mu^2}$. Using (3.1.1), the definition of the interpolation operator and (3.1.2) and our assumptions on the initial data, we obtain that

$$\|E^0_\varepsilon\|_0 + \|S^0_\varepsilon\|_0 \leq C, \tag{8.2.41}$$

It follows from our assumptions on the initial and (2.3.25) that

$$(F_\varepsilon(E^0_\varepsilon), 1)^h \leq C. \tag{8.2.42}$$

Moreover, it holds from (8.2.37) and (8.2.41) with $n = 1, ..., N$, that

$$(E^n_\varepsilon, 1) = (E^0_\varepsilon, 1) \leq C. \tag{8.2.43}$$

Since $F_\varepsilon(E^n_\varepsilon) \geq 0$, we have from (8.2.34) and (8.2.43) for $n = 1, ..., N$ that

$$\left(\frac{\alpha}{2\delta} - (\frac{\mu^2}{2\delta}\varsigma - 1)\frac{\Delta t}{\delta}\right)\left[\frac{2\delta}{\alpha}\big(F_\varepsilon(E^n_\varepsilon), 1\big)^h + |S^n_\varepsilon|_h^2\right] + \frac{\Delta t}{2M}|E^n_\varepsilon|_1^2$$

$$\leq \left(\frac{\alpha}{2\delta} + (\frac{\mu^2}{2\delta}\varsigma - 1)\frac{\Delta t}{\delta}\right)\left[\frac{2\delta}{\alpha}\big(F_\varepsilon(E^{n-1}_\varepsilon), 1\big)^h + |S^{n-1}_\varepsilon|_h^2\right] + C\Delta t|(E^0_\varepsilon, 1)|^2. \tag{8.2.44}$$

On noting (8.2.44), we have that

$$\frac{2\delta}{\alpha}\big(F_\varepsilon(E_\varepsilon^n),1\big)^h + |S_\varepsilon^n|_h^2 + \frac{\Delta t}{2M\bigg(\frac{\alpha}{2\delta}-(\frac{\mu^2}{2\delta}\varsigma-1)\frac{\Delta t}{\delta}\bigg)}|E_\varepsilon^n|_1^2$$

$$\leq \bigg(1+\frac{2}{\rho}(\frac{\mu^2}{\delta}\varsigma-1)\frac{\Delta t}{\delta}\bigg)\bigg[\frac{2\delta}{\alpha}\big(F_\varepsilon(E_\varepsilon^{n-1}),1\big)^h + |S_\varepsilon^{n-1}|_h^2\bigg] + C\Delta t$$

$$\leq e^{\frac{2}{\rho}(\frac{\mu^2}{\delta}\varsigma-1)\frac{\Delta t}{\delta}}\bigg[\frac{2\delta}{\alpha}\big(F_\varepsilon(E_\varepsilon^{n-1}),1\big)^h + |S_\varepsilon^{n-1}|_h^2\bigg] + C\Delta t, \quad (8.2.45)$$

where $\rho = \frac{\alpha}{2\delta} - (\frac{\mu^2}{2\delta}\varsigma - 1)\frac{\Delta t}{\delta}$. Therefore, the first two bounds in (8.2.39) flows form (8.2.45) and noting (8.2.41) and (2.4.46). When $\varsigma \leq \frac{2\delta}{\mu^2}$, we can rewrite (8.2.44) as follow

$$\frac{\alpha}{2\delta}\bigg[\frac{2\delta}{\alpha}\big(F_\varepsilon(E_\varepsilon^n),1\big)^h + |S_\varepsilon^n|_h^2\bigg] + \frac{\Delta t}{2M}|E_\varepsilon^n|_1^2$$

$$\leq \frac{\alpha}{2\delta}\bigg[\frac{2\delta}{\alpha}\big(F_\varepsilon(E_\varepsilon^{n-1}),1\big)^h + |S_\varepsilon^{n-1}|_h^2\bigg] + C\Delta t|(E_\varepsilon^0,1)|^2. \quad (8.2.46)$$

Thus, the proof will follow the same steps of the case when $\varsigma > \frac{2\delta}{\mu^2}$. The only difference is that the constant $e^{\frac{2}{\rho}(\frac{\mu^2}{2\delta}\varsigma-1)\frac{\Delta t}{\delta}}$ will be changed to 1 and hence the proof will be much easier.

The third and fourth bounds in (8.2.39) can be obtained easily by summing (8.2.34) over n on noting (8.2.41), (8.2.43) and the second bound in (8.2.39). From (2.4.46), (2.4.45), (2.3.29) and the first two bounds in (8.2.39) we obtain, after recalling that $s = [s]_+ + [s]_-$ and $F_\varepsilon(s) \geq 0$, that for $n = 1, ..., N$

$$\|\pi^h[E_\varepsilon^n]_-\|_0^2 \leq \|\pi^h[E_\varepsilon^n]_-\|_h^2 = ([E_\varepsilon^n]_-^2,1)^h \leq 2\varepsilon(F_\varepsilon(E_\varepsilon^n),1)^h \leq C\varepsilon, \quad (8.2.47)$$

$$\|\pi^h[S_\varepsilon^n]_-\|_0^2 \leq \|\pi^h[S_\varepsilon^n]_-\|_h^2 = ([S_\varepsilon^n]_-^2,1)^h \leq ((S_\varepsilon^n)^2,1)^h \leq C. \quad (8.2.48)$$

Now, from (3.1.1), (8.2.15), (2.4.68), (3.1.3) and (2.4.46) we obtain for any $\eta \in H^1(\Omega)$ and for $n = 1, ..., N$ that

$$\langle\frac{E_\varepsilon^n - E_\varepsilon^{n-1}}{\Delta t},\eta\rangle = (\frac{E_\varepsilon^n - E_\varepsilon^{n-1}}{\Delta t},\eta) = (\frac{E_\varepsilon^n - E_\varepsilon^{n-1}}{\Delta t},P^h\eta)^h$$

$$= (\Lambda_\varepsilon(E_\varepsilon^n)\nabla S_\varepsilon^n - \nabla E_\varepsilon^n, \nabla P^h\eta)$$

$$\leq C\big(|E_\varepsilon^n|_1 + |S_\varepsilon^n|_1\big)|P^h\eta|_1$$

$$\leq C\big(\|E_\varepsilon^n\|_1 + \|S_\varepsilon^n\|_1\big)\|\eta\|_1, \quad (8.2.49)$$

and therefore,

$$\left\|\frac{E_\varepsilon^n - E_\varepsilon^{n-1}}{\Delta t}\right\|^2_{(H^1(\Omega))'} \le C\big(\|E_\varepsilon^n\|_1^2 + \|S_\varepsilon^n\|_1^2\big). \tag{8.2.50}$$

Hence, we have from (8.2.39) that

$$\sum_{n=1}^N \Delta t \left\|\frac{E_\varepsilon^n - E_\varepsilon^{n-1}}{\Delta t}\right\|^2_{(H^1(\Omega))'} \le C \sum_{n=1}^N \Delta t\big(\|E_\varepsilon^n\|_1^2 + \|S_\varepsilon^n\|_1^2\big) \le C. \tag{8.2.51}$$

Similarly to (8.2.49), it follows from (3.1.1), (8.2.16), (3.1.3) and (2.4.46) we obtain for any $\eta \in H^1(\Omega)$ and for $n = 1, ..., N$ that

$$\begin{aligned}
\langle \frac{S_\varepsilon^n - S_\varepsilon^{n-1}}{\Delta t}, \eta \rangle = (\frac{S_\varepsilon^n - S_\varepsilon^{n-1}}{\Delta t}, \eta) = (\frac{S_\varepsilon^n - S_\varepsilon^{n-1}}{\Delta t}, P^h\eta)^h \\
= \mu(E_\varepsilon^n, P^h\eta)^h - (S_\varepsilon^n, P^h\eta)^h - (\nabla S_\varepsilon^n, \nabla P^h\eta) - \delta(\nabla E_\varepsilon^n, \nabla P^h\eta) \\
\le C\big(\|E_\varepsilon^n\|_1 + \|S_\varepsilon^n\|_1\big)\|P^h\eta\|_1 \\
\le C\big(\|E_\varepsilon^n\|_1 + \|S_\varepsilon^n\|_1\big)\|\eta\|_1.
\end{aligned} \tag{8.2.52}$$

Thus, (8.2.52) implies

$$\left\|\frac{S_\varepsilon^n - S_\varepsilon^{n-1}}{\Delta t}\right\|^2_{(H^1(\Omega))'} \le C\big(\|E_\varepsilon^n\|_1^2 + \|S_\varepsilon^n\|_1^2\big). \tag{8.2.53}$$

Hence we have from (8.2.39), that

$$\sum_{n=1}^N \Delta t \left\|\frac{S_\varepsilon^n - S_\varepsilon^{n-1}}{\Delta t}\right\|^2_{(H^1(\Omega))'} \le C \sum_{n=1}^N \Delta t\big(\|E_\varepsilon^n\|_1^2 + \|S_\varepsilon^n\|_1^2\big) \le C. \tag{8.2.54}$$

To complete the proof of the theorem, we note that the last two bounds in (8.2.40) follow from the the first two bounds in (8.2.40), respectively, on recalling (3.1.10).

Remark 8.2.1 *As M is a non-physical parameter, we could have taken $M > 2\delta/(\mu^2(d+2)C_p)$.*

8.2.4 Uniqueness of the approximation

Theorem 8.2.5 Let the assumptions of Theorem 8.2.4 hold. Let $\{E_\varepsilon^n, S_\varepsilon^n\}, n = 1, ..., N$ be a solution of the problem $(\mathrm{Q}_{M,\varepsilon}^{h,\Delta t})$. If $C_b = \max_{n=1,...,N} \|S_\varepsilon^n\|_0^2$ and $\Delta t \in (0, \tau_1)$, where the values of τ_1 is stated in the proof (8.2.63), then, the solution $\{E_\varepsilon^n, S_\varepsilon^n\}, n = 1, ..., N$ is unique.

Proof: We perform the proof by induction. Assume there are two discrete solutions $\{E^n_{\varepsilon,1}, S^n_{\varepsilon,1}\}$ and $\{E^n_{\varepsilon,2}, S^n_{\varepsilon,2}\}, n = 1, ..., N$ to the problem $(\mathrm{Q}^{h,\Delta t}_{M,\varepsilon})$ such that

$$\max_{n=1,\ldots,N} \left\{ \|S^n_{\varepsilon,1}\|^2_0, \|S^n_{\varepsilon,2}\|^2_0 \right\} \leq C_b. \tag{8.2.55}$$

Firstly, we note that the approximation solutions are unique at time $t = 0$, then we assume that the approximations are unique at the $(n-1)$-time step of $(\mathrm{Q}^{h,\Delta t}_{M,\varepsilon})$. Secondly, we set $\mathcal{E}^n_\varepsilon = E^n_{\varepsilon,1} - E^n_{\varepsilon,2}$ and $\mathcal{S}^n_\varepsilon = S^n_{\varepsilon,1} - S^n_{\varepsilon,2}$. On subtracting the fully discrete approximations gives for all $\chi \in S^h$ that

$$\frac{1}{\Delta t}\big(\mathcal{E}^n_\varepsilon, \chi\big)^h + (\nabla \mathcal{E}^n_\varepsilon, \nabla \chi) = (\Lambda_\varepsilon(E^n_{\varepsilon,1})\nabla S^n_{\varepsilon,1} - \Lambda_\varepsilon(E^n_{\varepsilon,2})\nabla S^n_{\varepsilon,2}, \nabla \chi), \tag{8.2.56}$$

$$\frac{\alpha}{\Delta t}\big(\mathcal{S}^n_\varepsilon, \chi\big)^h + (\mathcal{S}^n_\varepsilon, \chi)^h + (\nabla \mathcal{S}^n_\varepsilon, \nabla \chi) + \delta(\nabla \mathcal{E}^n_\varepsilon, \nabla \chi) = \mu(\mathcal{E}^n_\varepsilon, \chi)^h. \tag{8.2.57}$$

Next, we set $\chi \equiv \mathcal{E}^n_\varepsilon$ in (8.2.56) and $\chi \equiv \frac{1}{\delta}\mathcal{S}^n_\varepsilon$ in (8.2.57) as a test function and adding the resulting equations yields, on using the Hölder's inequality, (2.4.68), (2.4.69), (2.4.54) and (8.2.55), that

$$\begin{aligned}
&\frac{1}{\Delta t}|\mathcal{E}^n_\varepsilon|^2_h + |\mathcal{E}^n_\varepsilon|^2_1 + \frac{\alpha}{\delta \Delta t}|\mathcal{S}^n_\varepsilon|^2_h + \frac{1}{\delta}|\mathcal{S}^n_\varepsilon|^2_h + \frac{1}{\delta}|\mathcal{S}^n_\varepsilon|^2_1 \\
&= (\Lambda_\varepsilon(E^n_{\varepsilon,1})\nabla S^n_{\varepsilon,1} - \Lambda_\varepsilon(E^n_{\varepsilon,2})\nabla S^n_{\varepsilon,2}, \nabla \mathcal{E}^n_\varepsilon) + \frac{\mu}{\delta}(\mathcal{E}^n_\varepsilon, \mathcal{S}^n_\varepsilon)^h - (\nabla \mathcal{E}^n_\varepsilon, \nabla \mathcal{S}^n_\varepsilon) \\
&= \big([\Lambda_\varepsilon(E^n_{\varepsilon,1}) - 1]\nabla \mathcal{E}^n_\varepsilon, \nabla \mathcal{S}^n_\varepsilon\big) + \big([\Lambda_\varepsilon(E^n_{\varepsilon,1}) - \Lambda_\varepsilon(E^n_{\varepsilon,2})]\nabla S^n_{\varepsilon,2}, \nabla \mathcal{E}^n_\varepsilon\big) + \frac{\mu}{\delta}(\mathcal{E}^n_\varepsilon, \mathcal{S}^n_\varepsilon)^h \\
&\leq C_1|\mathcal{E}^n_\varepsilon|_1 \; |\mathcal{S}^n_\varepsilon|_1 + \|\Lambda_\varepsilon(E^n_{\varepsilon,1}) - \Lambda_\varepsilon(E^n_{\varepsilon,2})\|_0 |S^n_{\varepsilon,2}|_{1,\infty} \; |\mathcal{E}^n_\varepsilon|_1 + \frac{\mu}{\delta}|\mathcal{E}^n_\varepsilon|_h \; |\mathcal{S}^n_\varepsilon|_h \\
&\leq C_1|\mathcal{E}^n_\varepsilon|_1 \; |\mathcal{S}^n_\varepsilon|_1 + \frac{C_2}{h}\|\Lambda_\varepsilon(E^n_{\varepsilon,1}) - \Lambda_\varepsilon(E^n_{\varepsilon,2})\|_0 \|S^n_{\varepsilon,2}\|_{0,\infty} \; |\mathcal{E}^n_\varepsilon|_1 + \frac{\mu}{\delta}|\mathcal{E}^n_\varepsilon|_h \; |\mathcal{S}^n_\varepsilon|_h \\
&\leq C_1|\mathcal{E}^n_\varepsilon|_1 \; |\mathcal{S}^n_\varepsilon|_1 + \frac{C_2 C_b}{h}\|\Lambda_\varepsilon(E^n_{\varepsilon,1}) - \Lambda_\varepsilon(E^n_{\varepsilon,2})\|_0 |\mathcal{E}^n_\varepsilon|_1 + \frac{\mu}{\delta}|\mathcal{E}^n_\varepsilon|_h \; |\mathcal{S}^n_\varepsilon|_h \\
&\leq C_1|\mathcal{E}^n_\varepsilon|_1 \; |\mathcal{S}^n_\varepsilon|_1 + \frac{2MC_2C_b}{h\varepsilon}\|\mathcal{E}^n_\varepsilon\|_{0,\infty}|\mathcal{E}^n_\varepsilon|_1 + \frac{\mu}{\delta}|\mathcal{E}^n_\varepsilon|_h \; |\mathcal{S}^n_\varepsilon|_h := I_1 + I_2 + I_3,
\end{aligned} \tag{8.2.58}$$

where C_2 is the positive constant, independent of the parameters h and ε, that is generated from applying (2.4.54) and $C_1 = M + 1$.

Next, we obtain from the Young's inequality, (2.4.54) and (2.4.55) that

$$I_1 \le \frac{\delta C_1^2}{4}|\mathcal{E}_\varepsilon^n|_1^2 + \frac{1}{\delta}|\mathcal{S}_\varepsilon^n|_1^2 \le \frac{\delta C_1^2 C_2^2}{4h^2}\|\mathcal{E}_\varepsilon^n\|_0^2 + \frac{1}{\delta}|\mathcal{S}_\varepsilon^n|_1^2 = a_4|\mathcal{E}_\varepsilon^n|_h^2 + \frac{1}{\delta}|\mathcal{S}_\varepsilon^n|_1^2, \tag{8.2.59}$$

$$I_2 \le \frac{2MC_2C_3C_b}{h^{3/2}\varepsilon}\|\mathcal{E}_\varepsilon^n\|_0|\mathcal{E}_\varepsilon^n|_1 \le \frac{(MC_2C_3C_b)^2}{h^3\varepsilon^2}\|\mathcal{E}_\varepsilon^n\|_0^2 + |\mathcal{E}_\varepsilon^n|_1^2 \le a_5\|\mathcal{E}_\varepsilon^n\|_h^2 + |\mathcal{E}_\varepsilon^n|_1^2, \tag{8.2.60}$$

$$I_3 \le a_6|\mathcal{E}_\varepsilon^n|_h^2 + \big(\frac{\alpha}{\delta\Delta t} + \frac{1}{\delta} - \beta\big)|\mathcal{S}_\varepsilon^n|_h^2, \tag{8.2.61}$$

where

$$a_4 = \frac{\delta C_1^2 C_2^2}{4h^2}, \quad a_5 = \frac{(MC_2C_3C_b)^2}{h^3\varepsilon^2} \quad \text{and} \quad a_6 = \frac{\mu^2}{4\delta^2(\frac{\alpha}{\delta\Delta t} + \frac{1}{\delta} - \beta)}.$$

and $0 < \beta < \frac{\alpha}{\delta\Delta t} + \frac{1}{\delta}$. Combining (8.2.58) and (8.2.59)-(8.2.61) yields on noting the equivalence (2.4.46) that

$$\big(\frac{1}{\Delta t} - (a_4 + a_5 + a_6)\big)|\mathcal{E}_\varepsilon^n|_h^2 + \beta|\mathcal{S}_\varepsilon^n|_h^2 \le 0. \tag{8.2.62}$$

Now, we set

$$\tau_1 < 1/(a_4 + a_5 + a_6). \tag{8.2.63}$$

It follows from (8.2.62), for any $\Delta t \in (0, \tau_1)$ that

$$|\mathcal{E}_\varepsilon^n|_h^2 + |\mathcal{S}_\varepsilon^n|_h^2 \le 0,$$

leading to $E_{\varepsilon,1}^n = E_{\varepsilon,2}^n$ and $S_{\varepsilon,1}^n = S_{\varepsilon,2}^n$, $n = 1, ..., N$ as required. □

8.3 A semi-discrete approximation of the Keller-Segel Model

By extending the notation (3.3.35)-(3.3.37) to E_ε and S_ε and noting (8.2.15)-(8.2.16), we can rewrite the problem $(\mathrm{Q}_{M,\varepsilon}^{h,\Delta t})$ as:

Find $\{E_\varepsilon, S_\varepsilon\} \in C([0,T]; S^h) \times C([0,T]; S^h)$ such that for all $\chi \in L^2(0,T;S^h)$

$$\int_0^T \Big[\big(\frac{\partial E_\varepsilon}{\partial t}, \chi\big)^h + (\nabla E_\varepsilon^+, \nabla\chi)\Big]dt = \int_0^T (\Lambda_\varepsilon(E_\varepsilon^+)\nabla S_\varepsilon^+, \nabla\chi)dt, \tag{8.3.64}$$

$$\int_0^T \left[\alpha(\frac{\partial S_\varepsilon}{\partial t},\chi)^h + (S_\varepsilon^+,\chi)^h + (\nabla S_\varepsilon^+,\nabla\chi) + \delta(\nabla E_\varepsilon^+,\nabla\chi)\right] dt = \mu\int_0^T (E_\varepsilon^+,\chi)^h dt. \tag{8.3.65}$$

Theorem 8.3.1 Let $e^0, s^0 \in H^1(\Omega)$ and ε, h, e^0, s^0 be such that

(i) $E_\varepsilon^0 \equiv P^h e^0$, $S_\varepsilon^0 \equiv P^h s^0$; or $E_\varepsilon^0 \equiv \pi^h e^0$, $S_\varepsilon^0 \equiv \pi^h s^0$ if $e^0, s^0 \in C(\overline{\Omega})$

(ii) $\varepsilon \to 0$ as $h \to 0$.

Then there exists a subsequence of $\{E_\varepsilon, S_\varepsilon\}$, solving (8.3.64) and (8.3.65), and functions

$$E, S \in L^2(0,T;H^1(\Omega)) \cap L^\infty(0,T;L^2(\Omega)) \cap H^1(0,T;(H^1(\Omega))'), \tag{8.3.66}$$

and

$$E(\cdot,0) = e^0(\cdot) \quad \text{and} \quad S(\cdot,0) = s^0(\cdot) \quad \text{in} \quad L^2(\Omega), \tag{8.3.67}$$

$$E \geq 0, \qquad a.e. \text{ on } \Omega. \tag{8.3.68}$$

Moreover, it holds as $h \to 0$ that

$$E_\varepsilon, E_\varepsilon^\pm \rightharpoonup E, E^\pm \quad \text{and} \quad S_\varepsilon, S_\varepsilon^\pm \rightharpoonup S, S^\pm \quad \text{in} \quad L^2(0,T;H^1(\Omega)), \tag{8.3.69}$$

$$E_\varepsilon, E_\varepsilon^\pm \rightharpoonup^* E, E^\pm \quad \text{and} \quad S_\varepsilon, S_\varepsilon^\pm \rightharpoonup^* S, S^\pm \quad \text{in} \quad L^\infty(0,T;L^2(\Omega)), \tag{8.3.70}$$

$$\frac{\partial E_\varepsilon}{\partial t} \rightharpoonup \frac{\partial E}{\partial t} \quad \text{and} \quad \frac{\partial S_\varepsilon}{\partial t} \rightharpoonup \frac{\partial S}{\partial t} \quad \text{in} \quad L^2(0,T;(H^1(\Omega))'), \tag{8.3.71}$$

$$E_\varepsilon, E_\varepsilon^\pm \to E, E^\pm \quad \text{and} \quad S_\varepsilon, S_\varepsilon^\pm \to S, S^\pm \quad \text{in} \quad L^2(0,T;L^s(\Omega)), \tag{8.3.72}$$

$$\phi_\varepsilon(E_\varepsilon^\pm) \to \phi(E^\pm) \quad \text{in} \quad L^2(0,T;L^s(\Omega)), \tag{8.3.73}$$

$$\pi^h\phi_\varepsilon(E_\varepsilon^\pm) \to \phi(E^\pm) \quad \text{in} \quad L^2(0,T;L^s(\Omega)), \tag{8.3.74}$$

$$\Lambda_\varepsilon(E_\varepsilon^\pm) \to \phi(E^\pm)I \quad \text{in} \quad L^2(0,T;L^s(\Omega)), \tag{8.3.75}$$

for any

$$s \in \begin{cases} [2,\infty] & \text{if } d = 1, \\ [2,\infty) & \text{if } d = 2, \\ [2,6] & \text{if } d = 3. \end{cases}$$

Proof: First of all, we note from (3.1.3), (2.4.56) and the stated assumptions on the initial data that

$$\|E_\varepsilon^0\|_1 + \|S_\varepsilon^0\|_1 \leq C, \tag{8.3.76}$$

and

$$E_\varepsilon^0 \to e^0 \qquad \text{and} \qquad S_\varepsilon^0 \to s^0 \quad \text{in} \quad L^2(\Omega). \tag{8.3.77}$$

By using (2.3.28), (2.4.64), (2.4.65), (3.3.35), (3.3.36), (3.3.37), (8.3.76), (2.4.68), (8.2.39) and (8.2.40) we obtain the following uniform bounds independently of the parameters ε and h

$$\|E_\varepsilon^\pm\|_{L^2(0,T;H^1(\Omega))} + \|E_\varepsilon^\pm\|_{L^\infty(0,T;L^2(\Omega))} + \|\frac{\partial E_\varepsilon}{\partial t}\|_{L^2(0,T;(H^1(\Omega))')} + \varepsilon^{-\frac{1}{2}}\|\pi^h[E_\varepsilon^\pm]_-\|_{L^\infty(0,T;L^2(\Omega))}$$

$$+\|\mathcal{G}\frac{\partial E_\varepsilon}{\partial t}\|_{L^2(0,T;H^1(\Omega))} + \|\phi_\varepsilon(E_\varepsilon^\pm)\|_{L^\infty(\Omega_T)} + \|\Lambda_\varepsilon(E_\varepsilon^\pm)\|_{L^\infty(\Omega_T)} \leq C, \tag{8.3.78}$$

and

$$\|S_\varepsilon^\pm\|_{L^2(0,T;H^1(\Omega))} + \|S_\varepsilon^\pm\|_{L^\infty(0,T;L^2(\Omega))} + \|\frac{\partial S_\varepsilon}{\partial t}\|_{L^2(0,T;(H^1(\Omega))')} + \|\mathcal{G}\frac{\partial S_\varepsilon}{\partial t}\|_{L^2(0,T;H^1(\Omega))} \leq C. \tag{8.3.79}$$

Furthermore, we have from the third bounds in (8.3.78) and (8.3.79), respectively, that

$$\|E_\varepsilon^\pm - E_\varepsilon\|^2_{L^2(0,T;(H^1(\Omega))')} + \|S_\varepsilon^\pm - S_\varepsilon\|^2_{L^2(0,T;(H^1(\Omega))')}$$

$$\leq (\Delta t)^2\|\frac{\partial E_\varepsilon}{\partial t}\|^2_{L^2(0,T;(H^1(\Omega))')} + (\Delta t)^2\|\frac{\partial S_\varepsilon}{\partial t}\|^2_{L^2(0,T;(H^1(\Omega))')} \leq C(\Delta t)^2. \tag{8.3.80}$$

From (8.3.78), (8.3.79), (8.3.80), (2.1.6) and the compact embedding $H^1(\Omega) \overset{c}{\hookrightarrow} L^2(\Omega) \hookrightarrow (H^1(\Omega))'$, one can obtain using sequential compactness arguments the existence of a subsequence of $\{E_\varepsilon, S_\varepsilon\}_h$, still denoted $\{E_\varepsilon, S_\varepsilon\}_h$, and functions $\{E, S\}$ such that the results (8.3.66) and (8.3.69)-(8.3.72) hold. We note that since

$$E_\varepsilon, S_\varepsilon, E, S \in \{\eta : \eta \in L^2(0,T;H^1(\Omega)), \frac{\partial \eta}{\partial t} \in L^2(0,T;(H^1(\Omega))'\},$$

it follows that

$$E_\varepsilon, S_\varepsilon, E, S \in C([0,T];L^2(\Omega)), \tag{8.3.81}$$

see Theorem 7.2 in Robinson [84]. Thus, (8.3.67) follows from (8.3.72), (8.3.77) and (8.3.81).

Using the strong convergence of E_ε to E in $L^2(0,T;L^s(\Omega))$ and the fourth bound in (8.3.78), we can extract a subsequence, still denoted E_ε, such that as $h \to 0$ (see Appendix A.1.17)

$$E_\varepsilon \to E \qquad \text{and} \qquad \pi^h[E_\varepsilon]_- \to 0 \qquad a.e. \text{ in } \Omega_T. \tag{8.3.82}$$

But we have from the definition of π^h that

$$E_\varepsilon = \pi^h[E_\varepsilon]_+ + \pi^h[E_\varepsilon]_-. \tag{8.3.83}$$

Therefore, we deduce from (8.3.82) and (8.3.83) that $E \geq 0$ almost everywhere.

We obtain from (2.3.28), the non-negativity of the function E and the assumption (ii), on using the dominated convergence theorem, that

$$\|\phi_\varepsilon(E^\pm) - \phi(E^\pm)\|_{L^2(0,T;L^s(\Omega))} \leq C\varepsilon \to 0 \quad \text{as} \quad h \to 0. \tag{8.3.84}$$

From the Lipschitz continuity of the function ϕ_ε and (8.3.69), it follows that

$$\|\phi_\varepsilon(E^\pm_\varepsilon) - \phi_\varepsilon(E^\pm)\|_{L^2(0,T;L^s(\Omega))} \leq \|E^\pm_\varepsilon - E^\pm\|_{L^2(0,T;L^s(\Omega))} \to 0 \quad \text{as} \quad h \to 0. \tag{8.3.85}$$

Thus, in order to prove (8.3.73) we note that

$$\begin{aligned} &\|\phi_\varepsilon(E^\pm_\varepsilon) - \phi(E^\pm)\|_{L^2(0,T;L^s(\Omega))} \\ &\leq \|\phi_\varepsilon(E^\pm_\varepsilon) - \phi_\varepsilon(E^\pm)\|_{L^2(0,T;L^s(\Omega))} + \|\phi_\varepsilon(E^\pm) - \phi(E^\pm)\|_{L^2(0,T;L^s(\Omega))} \\ &\to 0 \quad \text{as} \quad h \to 0. \end{aligned} \tag{8.3.86}$$

We also have from (2.4.56), (2.4.52), (2.4.55) and the first bound in (8.3.78) that

$$\begin{aligned} &\|(I - \pi^h)\phi_\varepsilon(E^\pm_\varepsilon)\|_{L^2(0,T;L^s(\Omega))} \leq Ch\|\nabla\phi_\varepsilon(E^\pm_\varepsilon)\|_{L^2(0,T;L^s(\Omega))} \\ &\leq Ch\|\nabla E^\pm_\varepsilon\|_{L^2(0,T;L^s(\Omega))} \leq Ch^{1-d(\frac{1}{2}-\frac{1}{s})}\|E^\pm_\varepsilon\|_{L^2(0,T;H^1(\Omega))} \\ &\leq Ch^{1-d(\frac{1}{2}-\frac{1}{s})} \qquad \to 0 \quad \text{as} \quad h \to 0. \end{aligned} \tag{8.3.87}$$

We obtain from (2.4.70), (2.4.55), the first bound in (8.3.78) and (8.3.73) that

$$\begin{aligned} &\|\Lambda_\varepsilon(E^\pm_\varepsilon) - \phi(E^\pm)\mathcal{I}\|_{L^2(0,T;L^s(\Omega))} \\ &= \|\Lambda_\varepsilon(E^\pm_\varepsilon) - \phi_\varepsilon(E^\pm_\varepsilon)\mathcal{I} + \phi_\varepsilon(E^\pm_\varepsilon)\mathcal{I} - \phi(E^\pm)\mathcal{I}\|_{L^2(0,T;L^s(\Omega))} \end{aligned}$$

$$\leq \|\Lambda_\varepsilon(E_\varepsilon^\pm) - \phi_\varepsilon(E_\varepsilon^\pm)\mathcal{I}\|_{L^2(0,T;L^s(\Omega))} + \|\phi_\varepsilon(E_\varepsilon^\pm) - \phi(E^\pm)\|_{L^2(0,T;L^s(\Omega))}$$

$$\leq h\|\nabla E_\varepsilon^\pm\|_{L^2(0,T;L^s(\Omega))} + \|\phi_\varepsilon(E_\varepsilon^\pm) - \phi(E^\pm)\|_{L^2(0,T;L^s(\Omega))}$$

$$\leq Ch^{1-d(\frac{1}{2}-\frac{1}{s})}\|E_\varepsilon^\pm\|_{L^2(0,T;H^1(\Omega))} + \|\phi_\varepsilon(E_\varepsilon^\pm) - \phi(E^\pm)\|_{L^2(0,T;L^s(\Omega))}$$

$$\leq Ch^{1-d(\frac{1}{2}-\frac{1}{s})} + \|\phi_\varepsilon(E_\varepsilon^\pm) - \phi(E^\pm)\|_{L^2(0,T;L^s(\Omega))} \to 0 \quad \text{as} \quad h \to 0. \qquad (8.3.88)$$

Hence the result (8.3.75) holds from (8.3.88).

Theorem 8.3.2 Let the assumptions of Theorem 8.3.1 hold. Then, the functions $\{E, S\}$ represent a global weak solution in sense that for all $\eta \in L^2(0,T;H^1(\Omega))$

$$\int_0^T \big[\langle\frac{\partial E}{\partial t}, \eta\rangle + (\nabla E^+, \nabla\eta)\big]\,dt = \int_0^T (\phi(E^+)\nabla S^+, \nabla\eta)dt, \qquad (8.3.89)$$

$$\int_0^T \big[\alpha\langle\frac{\partial S}{\partial t}, \eta\rangle + (S^+, \eta) + (\nabla S^+, \nabla\eta) + \delta(\nabla E^+, \nabla\eta)\big]\,dt = \mu\int_0^T (E^+, \eta)dt. \quad (8.3.90)$$

Proof: For any $\eta \in L^2(0,T;H^1(\Omega))$, we set $\chi \equiv \pi^h\eta$ in (8.3.64) and (8.3.65) and then we analyse the convergence of the resulting terms as $h \to 0$. On setting $Y_\varepsilon = E_\varepsilon$ and S_ε, respectively, we have for all $\eta \in L^\infty(0,T;W^{1,\infty}(\Omega))$ and for all $\widetilde{\eta} \in W^{1,\infty}(0,T;W^{1,\infty}(\Omega))$ that

$$\begin{aligned}\int_0^T (\frac{\partial Y_\varepsilon}{\partial t}, \pi^h\eta)^h &= \int_0^T [(\frac{\partial Y_\varepsilon}{\partial t}, \pi^h[\eta - \widetilde{\eta}])^h - (\frac{\partial Y_\varepsilon}{\partial t}, \pi^h[\eta - \widetilde{\eta}])]dt \\ &+ \int_0^T [(\frac{\partial Y_\varepsilon}{\partial t}, \pi^h\widetilde{\eta})^h - (\frac{\partial Y_\varepsilon}{\partial t}, \pi^h\widetilde{\eta})]dt \\ &+ \int_0^T (\frac{\partial Y_\varepsilon}{\partial t}, (\pi^h - I)\eta)dt \\ &+ \int_0^T (\frac{\partial Y_\varepsilon}{\partial t}, \eta)dt \\ &:= I_{1,1} + I_{1,2} + I_{1,3} + I_{1,4}.\end{aligned} \qquad (8.3.91)$$

Using (2.4.59), (3.1.11), (2.4.56), Hölder's inequality, the denseness of $W^{1,\infty}(0,T;W^{1,\infty}(\Omega))$ in $L^\infty(0,T;W^{1,\infty}(\Omega))$, (8.3.78) and (8.3.79) gives that

$$|I_{1,1}| \equiv |\int_0^T [(\frac{\partial Y_\varepsilon}{\partial t}, \pi^h[\eta - \widetilde{\eta}])^h - (\frac{\partial Y_\varepsilon}{\partial t}, \pi^h[\eta - \widetilde{\eta}])]dt|$$

$$\leq \int_0^T |(\frac{\partial Y_\varepsilon}{\partial t}, \pi^h[\eta - \widetilde{\eta})^h - (\frac{\partial Y_\varepsilon}{\partial t}, \pi^h[\eta - \widetilde{\eta}])|dt$$

$$
\begin{aligned}
&\leq Ch \int_0^T \|\frac{\partial Y_\varepsilon}{\partial t}\|_0 \quad |\pi^h[\eta - \widetilde{\eta}]|_1 dt \\
&\leq C \int_0^T \|\mathcal{G}\frac{\partial Y_\varepsilon}{\partial t}\|_1 \quad \|\eta - \widetilde{\eta}\|_1 dt \\
&\leq C\|\mathcal{G}\frac{\partial Y_\varepsilon}{\partial t}\|_{L^2(0,T;H^1(\Omega))} \quad \|\eta - \widetilde{\eta}\|_{L^2(0,T;H^1(\Omega))} \\
&\leq C\|\eta - \widetilde{\eta}\|_{L^2(0,T;H^1(\Omega))}. \qquad (8.3.92)
\end{aligned}
$$

It also follows from (2.4.59), (2.4.56), Hölder's inequality, (8.3.78) and (8.3.79) that

$$
\begin{aligned}
|I_{1,2}| &\equiv |\int_0^T [(\frac{\partial Y_\varepsilon}{\partial t}, \pi^h\widetilde{\eta})^h - (\frac{\partial Y_\varepsilon}{\partial t}, \pi^h\widetilde{\eta})]dt| \\
&\leq |\int_0^T [(Y_\varepsilon, \frac{\partial \pi^h\widetilde{\eta}}{\partial t})^h - (Y_\varepsilon, \frac{\partial \pi^h\widetilde{\eta}}{\partial t})]dt| \\
&+|(Y_\varepsilon(\cdot,T), \pi^h\widetilde{\eta}(\cdot,T))^h - (Y_\varepsilon(\cdot,T), \pi^h\widetilde{\eta}(\cdot,T))| \\
&+|(Y_\varepsilon(\cdot,0), \pi^h\widetilde{\eta}(\cdot,0))^h - (Y_\varepsilon(\cdot,0), \pi^h\widetilde{\eta}(\cdot,0))| \\
&\leq Ch \int_0^T \|Y_\varepsilon\|_0 |\frac{\partial \pi^h\widetilde{\eta}}{\partial t}|_1 dt + Ch\|Y_\varepsilon(\cdot,T)\|_0 |\pi^h\widetilde{\eta}(\cdot,T)|_1 + Ch\|Y_\varepsilon(\cdot,0)\|_0 |\pi^h\widetilde{\eta}(\cdot,0)|_1 \\
&\leq Ch\|Y_\varepsilon\|_{L^\infty(0,T,L^2(\Omega))}\|\pi^h\widetilde{\eta}\|_{H^1(0,T,H^1(\Omega))} + Ch\|Y_\varepsilon(\cdot,T)\|_0|\pi^h\widetilde{\eta}(\cdot,T)|_1 \\
&+Ch\|Y_\varepsilon(\cdot,0)\|_0|\pi^h\widetilde{\eta}(\cdot,0)|_1 \\
&\leq Ch\|\widetilde{\eta}\|_{H^1(0,T,H^1(\Omega))} \quad \to \quad 0 \quad \text{as} \quad h \to 0. \qquad (8.3.93)
\end{aligned}
$$

To treat the term $I_{1,3}$, we observe using (3.1.8), Hölder's inequality and the fifth bound in (8.3.78) and (8.3.79) that

$$
\begin{aligned}
|I_{1,3}| &= |\int_0^T (\frac{\partial Y_\varepsilon}{\partial t}, (\pi^h - I)\eta)dt| = |\int_0^T \langle\frac{\partial Y_\varepsilon}{\partial t}, (\pi^h - I)\eta\rangle dt| \\
&\leq \int_0^T |\langle\frac{\partial Y_\varepsilon}{\partial t}, (\pi^h - I)\eta\rangle| dt \\
&\leq \int_0^T |\frac{\partial Y_\varepsilon}{\partial t}|_{(H^1(\Omega))'} |(\pi^h - I)\eta|_1 dt \\
&\leq \|\frac{\partial Y_\varepsilon}{\partial t}\|_{L^2(0,T;(H^1(\Omega))')} \|(\pi^h - I)\eta\|_{L^2(0,T;H^1(\Omega))} \\
&\leq C\|(\pi^h - I)\eta\|_{L^2(0,T;H^1(\Omega))}. \qquad (8.3.94)
\end{aligned}
$$

From (3.1.8) and the weak convergence result (8.3.71) we have, for all $\eta \in L^\infty(0,T;W^{1,\infty}(\Omega))$, that

$$
I_{1,4} \equiv \int_0^T (\frac{\partial Y_\varepsilon}{\partial t}, \eta)dt = \int_0^T \langle\frac{\partial Y_\varepsilon}{\partial t}, \eta\rangle dt \to \int_0^T \langle\frac{\partial Y}{\partial t}, \eta\rangle dt \quad \text{as} \quad h \to 0. \qquad (8.3.95)
$$

Combining (8.3.91)-(8.3.95), (2.4.57) and the denseness of $W^{1,\infty}(0,T;W^{1,\infty}(\Omega))$ in $L^\infty(0,T;W^{1,\infty}(\Omega))$ yields for all $\eta \in L^2(0,T;H^1(\Omega))$ that

$$\int_0^T (\frac{\partial Y_\varepsilon}{\partial t}, \pi^h \eta)^h \to \int_0^T \langle \frac{\partial Y}{\partial t}, \eta \rangle dt \quad \text{as} \quad h \to 0. \tag{8.3.96}$$

With the aid of Hölder's inequality, (8.3.78), (8.3.79) and (2.4.57) we obtain for all $\eta \in L^\infty(0,T;W^{1,\infty}(\Omega))$ that

$$\begin{aligned} |\int_0^T (\nabla Y_\varepsilon^+, \nabla(\pi^h - I)\eta) dt| &\leq \int_0^T |(\nabla Y_\varepsilon^+, \nabla(\pi^h - I)\eta)| dt \\ &\leq \int_0^T |Y_\varepsilon^+|_1 \quad |(\pi^h - I)\eta|_1 dt \\ &\leq \|Y_\varepsilon^+\|_{L^2(0,T,H^1(\Omega))} \quad \|(\pi^h - I)\eta\|_{L^2(0,T,H^1(\Omega))} \\ &\leq C \,\|(\pi^h - I)\eta\|_{L^2(0,T,H^1(\Omega))} \\ &\to 0 \quad \text{as} \quad h \to 0. \end{aligned} \tag{8.3.97}$$

Noting (8.3.97) and (8.3.69) yields for all $\eta \in L^\infty(0,T;W^{1,\infty}(\Omega))$ that

$$\begin{aligned} \int_0^T (\nabla Y_\varepsilon^+, \nabla \pi^h \eta) dt &= \int_0^T (\nabla Y_\varepsilon^+, \nabla(\pi^h - I)\eta) dt + \int_0^T (\nabla Y_\varepsilon^+, \nabla \eta) dt \\ &\to \int_0^T (\nabla Y^+, \nabla \eta) dt \quad \text{as} \quad h \to 0. \end{aligned} \tag{8.3.98}$$

and similarly

$$\int_0^T (Y_\varepsilon^+, \pi^h \eta)^h dt \to \int_0^T (Y^+, \eta) dt \quad \text{as} \quad h \to 0. \tag{8.3.99}$$

We have for all $\eta \in L^\infty(0,T;W^{1,\infty}(\Omega))$ and for all $\widetilde{\eta} \in W^{1,\infty}(0,T;W^{1,\infty}(\Omega))$ that

$$\begin{aligned} &\int_0^T (\Lambda_\varepsilon(E_\varepsilon^+) \nabla S_\varepsilon^+, \nabla \pi^h \eta) d\,t \\ &= \int_0^T (\Lambda_\varepsilon(E_\varepsilon^+) \nabla S_\varepsilon^+, \nabla(\pi^h - I)\eta) d\,t \\ &+ \int_0^T ([\Lambda_\varepsilon(E_\varepsilon^+) - \phi(E^+)\mathcal{I}] \nabla S_\varepsilon^+, \nabla(\eta - \widetilde{\eta})) d\,t \\ &+ \int_0^T ([\Lambda_\varepsilon(E_\varepsilon^+) - \phi(E^+)\mathcal{I}] \nabla S_\varepsilon^+, \nabla \widetilde{\eta}) d\,t \\ &+ \int_0^T (\phi(E^+) \nabla S_\varepsilon^+, \nabla \eta) d\,t \\ &:= I_{2,1} + I_{2,2} + I_{2,3} + I_{2,4}. \end{aligned} \tag{8.3.100}$$

On noting the generalized Hölder's inequality and (8.3.78), (8.3.79) we have

$$|I_{2,1}| \equiv |\int_0^T (\Lambda_\varepsilon(E_\varepsilon^+)\nabla S_\varepsilon^+, \nabla(\pi^h - I)\eta)dt|$$

$$\leq \int_0^T \|\Lambda_\varepsilon(E_\varepsilon^+)\|_\infty \ |S_\varepsilon^+|_1 \ |(\pi^h - I)\eta|_1 dt$$

$$\leq \|\Lambda_\varepsilon(E_\varepsilon^+)\|_{L^\infty(\Omega_T))} \ \|S_\varepsilon^+\|_{L^2(0,T,H^1(\Omega))} \ \|(\pi^h - I)\eta\|_{L^2(0,T,H^1(\Omega))}$$

$$\leq C\,\|(\pi^h - I)\eta\|_{L^2(0,T,H^1(\Omega))} \to 0 \quad \text{as} \quad h \to 0. \tag{8.3.101}$$

Similarly to the treatment of the term $I_{2,1}$, we have from the generalized Hölder's inequality, (8.3.78), (8.3.79) and (2.2.15) that

$$|I_{2,2}| \equiv |\int_0^T ([\Lambda_\varepsilon(E_\varepsilon^+) - \phi(E^+)\mathcal{I}]\nabla S_\varepsilon^+, \nabla(\eta - \widetilde{\eta}))dt|$$

$$\leq \|\Lambda_\varepsilon(E_\varepsilon^+) - \phi(E^+)\mathcal{I}\|_{L^2(\Omega_T))} \ \|S_\varepsilon^+\|_{L^2(0,T,H^1(\Omega))} \ \|\eta - \widetilde{\eta}\|_{L^\infty(0,T,W^{1,\infty}(\Omega))}$$

$$\leq \|\Lambda_\varepsilon(E_\varepsilon^+) - \phi(E^+)\mathcal{I}\|_{L^2(\Omega_T))} \ \|S_\varepsilon^+\|_{L^2(0,T,H^1(\Omega))} \ \|\eta - \widetilde{\eta}\|_{L^\infty(0,T,W^{1,\infty}(\Omega))}$$

$$\leq C\ \|\eta - \widetilde{\eta}\|_{L^\infty(0,T,W^{1,\infty}(\Omega))}. \tag{8.3.102}$$

We also have that

$$|I_{2,3}| \equiv |\int_0^T ([\Lambda_\varepsilon(E_\varepsilon^+) - \phi(E^+)\mathcal{I}]\nabla S_\varepsilon^+, \nabla\widetilde{\eta})dt|$$

$$\leq \|\Lambda_\varepsilon(E_\varepsilon^+) - \phi(E^+)\mathcal{I}\|_{L^2(\Omega_T)} \ \|S_\varepsilon^+\|_{L^2(0,T,H^1(\Omega))} \ \|\nabla\widetilde{\eta}\|_{L^\infty(\Omega_T)}$$

$$\leq C\|\Lambda_\varepsilon(E_\varepsilon^+) - \phi(E^+)\mathcal{I}\|_{L^2(\Omega_T))} \ \|\widetilde{\eta}\|_{L^\infty(0,T;W^{1,\infty}(\Omega))}$$

$$\leq C\|\Lambda_\varepsilon(E_\varepsilon^+) - \phi(E^+)\mathcal{I}\|_{L^2(0,T,L^s(\Omega))} \ \to 0 \quad \text{as} \quad h \to 0. \tag{8.3.103}$$

As the function $\phi(s)$ is bounded, we obtain from (8.3.69) for all $\eta \in L^\infty(0,T;W^{1,\infty}(\Omega))$ that

$$I_{2,4} \equiv \int_0^T (\phi(E^+)\nabla S_\varepsilon^+, \nabla\eta)dt \to \int_0^T (\phi(E^+)\nabla S, \nabla\eta)dt \quad \text{as} \quad h \to 0. \tag{8.3.104}$$

Combining (8.3.100)-(8.3.104) and noting the denseness of the space $W^{1,\infty}(0,T;W^{1,\infty}(\Omega))$ in $L^\infty(0,T;W^{1,\infty}(\Omega))$, yields for all $\eta \in L^\infty(0,T;W^{1,\infty}(\Omega))$ that

$$\int_0^T (\Lambda_\varepsilon(E_\varepsilon^+)\nabla S_\varepsilon^+, \nabla\pi^h\eta)dt \to \int_0^T (\phi(E^+)\nabla S, \nabla\eta)dt \quad \text{as} \quad h \to 0. \tag{8.3.105}$$

Now, we deduce from (8.3.64)-(8.3.65), (8.3.96),(8.3.98), (8.3.99) and (8.3.105) that the functions {E, S} satisfy (8.3.89)-(8.3.90), as well as the results of Theorem 8.3.2. This completes the existence proof.

Chapter 9

Existence and uniqueness for the Keller-Segel Model

In this chapter we show that the solutions can be bounded, independent of M. Based on the analysis in this chapters, the idea is to show the existence of weak solutions to the model (Q), that demands passing to the limits, $\Delta t \to 0^+$ and $M \to \infty$. Then we link the time step Δt to the cutoff parameter $M > 1$ by demanding that $\Delta t = o(M^{-1})$, as $M \to \infty$, so that the only parameter in the problem $(\mathrm{Q}_M^{\Delta t})$ is the cutoff parameter. In Section 9.1 and by using special energy estimates, we show that the solutions can be bounded, independent of M. Then, we use these $M-$independent bounds on the relative entropy to derive $M-$independent bounds on the time-derivatives. In Section 9.2, compactness arguments was used to study the convergence of the finite element approximate problem and the existence of a non-negative weak solution for (Q) was concluded. Finally, the error estimate was introduce in Section 9.3.

9.1 M-independent bounds on the derivatives

We are now ready to embark on the derivation of the required bounds, uniform in the cut-off parameter M, on norms of E^+ and S^+. The appropriate choice of test function in (8.3.89) and (8.3.90) for this purpose is $\eta = \chi_{[0,t]}(\mathcal{F}^M)'(E^+)$ and $\eta = \chi_{[0,t]}S^+$ with $t = t_n$, $n \in \{1, ..., N\}$, and $\chi_{[0,t]}$ denoting the characteristic function of the interval $[0,t]$. While Theorem 8.3.1 guarantees that E^+ is nonnegative *a.e.* on $\Omega\times[0,T]$, there is unfortunately no reason why E^+ should be strictly positive on $\Omega\times[0,T]$, and therefore the expression $(\mathcal{F}^M)'(E^+)$ may in general be undefined; the same is true of $(\mathcal{F}^M)''(E^+)$ which also appears in the algebraic manipulations. In the following theorem, we circumvent this problem by working with $(\mathcal{F}^M)'(E^+ + \epsilon)$ instead of $(\mathcal{F}^M)'(E^+)$, where $\epsilon > 0$. Since E^+ is known to be nonnegative from Theorem 8.3.1, $(\mathcal{F}^M)'(E^+ + \epsilon)$ and $(\mathcal{F}^M)''(E^+ + \epsilon)$ are well-defined. After deriving the relevant bounds, which will involve $\mathcal{F}^M(E^+ + \epsilon)$ only, we shall pass to the limit $\epsilon \to 0^+$, noting that, unlike $(\mathcal{F}^M)'(E^+ + \epsilon)$ and $(\mathcal{F}^M)''(E^+ + \epsilon)$, the functions $\mathcal{F}^M(E^+ + \epsilon)$ is well-defined for any nonnegative E^+.

Theorem 9.1.1 Let $M = \max\sup E^0$ if $E^0 \in L^\infty(\Omega)$, then, the solutions $\{E^\pm, S^\pm\}$ satisfy the following bounds

$$\begin{aligned}
&\int_\Omega (\mathcal{F}^M)(E^n)dx + \frac{\alpha}{2\delta}\int_\Omega (S^n)^2 dx + 2\int_0^t\int_\Omega |\nabla\sqrt{E^+}|^2 dxdt \\
&+\frac{1}{2M\Delta t}\int_0^t\int_\Omega (E^+ - E^-)^2 dxdt + \frac{\alpha}{2\Delta t\delta}\int_0^t\int_\Omega (S^+ - S^-)^2 dxdt \\
&+\frac{1}{2\delta}\int_0^t\int_\Omega |S^+|^2 dxdt + \frac{1}{4\delta}\int_0^t\int_\Omega |\nabla S^+|^2 dxdt \\
&\leq B_1(E^0, S^0).
\end{aligned} \tag{9.1.1}$$

where $B_1(E^0,S^0) = \int_\Omega \mathcal{F}(E^0)dxdt + \frac{\alpha}{2\delta}\int_\Omega (S^0)^2 dxdt + C$.

Proof: We now take any $\epsilon > 0$ and $\epsilon < \min\{1, 1/\delta\}$ to be fixed, whereby $0 < \epsilon < 1 < M$, and we choose

$$\eta = \chi_{[0,t]}(\mathcal{F}^M)'(E^+ + \epsilon) \text{ and } \eta = \frac{1}{\delta}\chi_{[0,t]}S^+ \quad \text{with } t = t_n,\ n \in \{1, ..., N\},$$

as test function in (8.3.89) and (8.3.90), respectively, to get

$$\begin{aligned}
&\int_0^T \big[\langle\frac{\partial E}{\partial t}, \chi_{[0,t]}(\mathcal{F}^M)'(E^+ + \epsilon)\rangle + (\nabla E^+, \nabla\chi_{[0,t]}(\mathcal{F}^M)'(E^+ + \epsilon))\big]dt \\
&= \int_0^T (\phi(E^+)\nabla S^+, \nabla\chi_{[0,t]}(\mathcal{F}^M)'(E^+ + \epsilon))dt,
\end{aligned} \tag{9.1.2}$$

$$\int_0^T \big[\frac{\alpha}{\delta}\langle\frac{\partial S}{\partial t}, \chi_{[0,t]}S^+\rangle + \frac{1}{\delta}(S^+, \chi_{[0,t]}S^+) + \frac{1}{\delta}(\nabla S^+, \nabla\chi_{[0,t]}S^+)$$

$$+(\nabla E^+, \nabla \chi_{[0,t]} S^+)\big] dt = \frac{\mu}{\delta} \int_0^T (E^+, \chi_{[0,t]} S^+) dt. \quad (9.1.3)$$

We now analyze each term individually. Clearly $\mathcal{F}^M(E^+ + \epsilon)$ is twice continuously differentiable on the interval $(-\epsilon, \infty)$ for any $\epsilon > 0$. Thus, using Taylor theorem for $s \in [0, \infty)$ and $c \in [0, \infty)$,

$$(s-c)(\mathcal{F}^M)'(s+\epsilon) = \mathcal{F}^M(s+\epsilon) - \mathcal{F}^M(c+\epsilon) + \frac{1}{2}(s-c)^2(\mathcal{F}^M)''(\theta s + (1-\theta)c + \epsilon),$$

with $\theta \in (0,1)$. Hence, on noting that $t \in [0,T] \to E^+(.,t)$ is piecewise linear relative to the partition $\{0 = t_0, t_1, ..., t_N = T\}$ of the interval $[0,T]$,

$$\hat{T}_1 := \int_0^T \int_\Omega \frac{\partial E}{\partial t} \chi_{[0,t]} (\mathcal{F}^M)'(E^+ + \epsilon) dx dt = \int_0^t \int_\Omega \frac{\partial E}{\partial t} (\mathcal{F}^M)'(E^+ + \epsilon) dx dt$$

$$= \frac{1}{\Delta t} \int_0^t \int_\Omega (E^+ - E^-)(\mathcal{F}^M)'(E^+ + \epsilon) dx dt$$

$$= \frac{1}{\Delta t} \int_0^t \int_\Omega (\mathcal{F}^M)(E^+ + \epsilon) dx dt - \frac{1}{\Delta t} \int_0^t \int_\Omega (\mathcal{F}^M)(E^- + \epsilon) dx dt$$

$$+ \frac{1}{2\Delta t} \int_0^t \int_\Omega (E^+ - E^-)^2 (\mathcal{F}^M)''(\theta E^+ + (1-\theta)E^- + \epsilon) dx dt.$$

Noting from (4.1.3) that $(\mathcal{F}^M)''(s+\epsilon) \geq 1/M$, this then implies, with $t = t_n$, $n \in \{1, ..., N\}$, that

$$\hat{T}_1 \geq \frac{1}{\Delta t} \int_0^t \int_\Omega (\mathcal{F}^M)(E^+ + \epsilon) dx dt - \frac{1}{\Delta t} \int_0^t \int_\Omega (\mathcal{F}^M)(E^- + \epsilon) dx dt$$

$$+ \frac{1}{2M\Delta t} \int_0^t \int_\Omega (E^+ - E^-)^2 dx dt$$

$$= \int_\Omega (\mathcal{F}^M)(E^n + \epsilon) dx - \int_\Omega (\mathcal{F}^M)(E^0 + \epsilon) dx + \frac{1}{2M\Delta t} \int_0^t \int_\Omega (E^+ - E^-)^2 dx dt$$

$$= \int_\Omega (\mathcal{F}^M)(E^n + \epsilon) dx - \int_\Omega (\mathcal{F}^M)(\phi(E^0) + \epsilon) dx + \frac{1}{2M\Delta t} \int_0^t \int_\Omega (E^+ - E^-)^2 dx dt$$

$$\geq \int_\Omega (\mathcal{F}^M)(E^n + \epsilon) dx - \int_\Omega \mathcal{F}(E^0 + \epsilon) dx + \frac{1}{2M\Delta t} \int_0^t \int_\Omega (E^+ - E^-)^2 dx dt - C\epsilon. \quad (9.1.4)$$

We use in the second step the simple fact that if there exists $M > 0$ such that $0 \leq E^0 \leq M$, then $\phi(E^0) = E^0$. Then, in the last inequality, we use the results of Lemma 4.2.1. Now, using the fact that $\phi(s) \leq s$, $\forall s$, we can deal with the second term in (9.1.2) as follows:

$$\hat{T}_2 := \int_0^T \int_\Omega \nabla E^+ \nabla \chi_{[0,t]} (\mathcal{F}^M)'(E^+ + \epsilon) dx dt = \int_0^t \int_\Omega |\nabla E^+|^2 (\mathcal{F}^M)''(E^+ + \epsilon) dx dt$$

$$= \int_0^t \int_\Omega \frac{|\nabla E^+|^2}{\phi(E^+ + \epsilon)} dxdt \geq \int_0^t \int_\Omega \frac{|\nabla E^+|^2}{E^+ + \epsilon} dxdt = 4 \int_0^t \int_\Omega |\nabla \sqrt{E^+ + \epsilon}|^2 dxdt. \tag{9.1.5}$$

Next, we consider the third term in (9.1.2), using Cauchy-Schwarz and Young inequalities, the Lipschitz continuity of ϕ and the fact that $\phi(s+\epsilon) \geq \epsilon$ we have

$$\hat{T}_3 := \int_0^T \int_\Omega \phi(E^+)\nabla S^+ \ \nabla \chi_{[0,t]} (\mathcal{F}^M)'(E^+ + \epsilon) dxdt = \int_0^t \int_\Omega \frac{\phi(E^+)}{\phi(E^+ + \epsilon)} \nabla E^+ \ \nabla S^+ \ dxdt$$

$$= \int_0^t \int_\Omega \nabla E^+ \ \nabla S^+ \, dxdt + \int_0^t \int_\Omega \frac{\phi(E^+) - \phi(E^+ + \epsilon)}{\phi(E^+ + \epsilon)} \nabla E^+ \ \nabla S^+ \, dxdt$$

$$\leq \int_0^t \int_\Omega \nabla E^+ \ \nabla S^+ \, dxdt + \delta \int_0^t \int_\Omega \frac{(\phi(E^+) - \phi(E^+ + \epsilon))^2}{\phi^2(E^+ + \epsilon)} |\nabla E^+|_1^2 \, dxdt$$

$$+ \frac{1}{4\delta} \int_0^t \int_\Omega |\nabla S^+|^2 dxdt$$

$$\leq \int_0^t \int_\Omega \nabla E^+ \ \nabla S^+ \, dxdt + \epsilon\delta \int_0^t \int_\Omega \frac{|\nabla E^+|_1^2}{\phi(E^+ + \epsilon)} \, dxdt + \frac{1}{4\delta} \int_0^t \int_\Omega |\nabla S^+|^2 dxdt \tag{9.1.6}$$

Moreover,

$$\hat{T}_4 := \frac{\alpha}{\delta} \int_0^T \int_\Omega \frac{\partial S}{\partial t} \chi_{[0,t]} S^+ dxdt = \frac{\alpha}{\Delta t \delta} \int_0^t \int_\Omega (S^+ - S^-) S^+ dxdt$$

$$= \frac{\alpha}{2\Delta t\delta} \int_0^t \int_\Omega (S^+)^2 dxdt - \frac{\alpha}{2\Delta t \delta} \int_0^t \int_\Omega (S^-)^2 dxdt + \frac{\alpha}{2\Delta t\delta} \int_0^t \int_\Omega (S^+ - S^-)^2 dxdt.$$

$$= \frac{\alpha}{2\delta} \int_\Omega (S^n)^2 dx - \frac{\alpha}{2\delta} \int_\Omega (S^0)^2 dx + \frac{\alpha}{2\Delta t\delta} \int_0^t \int_\Omega (S^+ - S^-)^2 dxdt. \tag{9.1.7}$$

Now, substituting the results of (9.1.4)-(9.1.7) in (9.1.2) and (9.1.3), then summing the final results, we have

$$\int_\Omega (\mathcal{F}^M)(E^n + \epsilon) dx + \frac{\alpha}{2\delta} \int_\Omega (S^n)^2 dx$$

$$+ \frac{1}{2M\Delta t} \int_0^t \int_\Omega (E^+ - E^-)^2 dxdt + \frac{\alpha}{2\Delta t\delta} \int_0^t \int_\Omega (S^+ - S^-)^2 dxdt$$

$$+ 4(1 - \delta\epsilon) \int_0^t \int_\Omega |\nabla \sqrt{E^+ + \epsilon}|^2 + \frac{1}{\delta} \int_0^t \int_\Omega |S^+|^2 dxdt + \frac{3}{4\delta} \int_0^t \int_\Omega |\nabla S^+|^2 dxdt$$

$$\leq \frac{\mu}{\delta} \int_0^t \int_\Omega E^+ S^+ dxdt + \int_\Omega (\mathcal{F}^M)(E^0 + \epsilon) dx + \frac{\alpha}{2\delta} \int_\Omega (S^0)^2 dx + C\epsilon. \tag{9.1.8}$$

We estimate the first term in right-hand side of (9.1.8) using Hölder's inequality, the Sobolev embedding theorem, the Gagliardo-Nirenberg inequality with $\varpi = d/12$ and Young's inequality for $p_1 = 1/\varpi, p_2 = 2/(1 - 2\varpi)$, and $p_3 = 2$ imply that

$$\hat{T}_8 := \frac{\mu}{\delta} \int_\Omega E^+ S^+ dx \leq \frac{\mu}{\delta} C \ \|E^+\|_{L^{6/5}(\Omega)} \ \|S^+\|_{L^6(\Omega)} \leq \frac{\mu}{\delta} C \ \|\sqrt{E^+}\|^2_{L^{12/5}(\Omega)} \ \|S^+\|_{H^1(\Omega)}$$

$$
\begin{aligned}
&\leq \frac{\mu}{\delta} C \ \|\sqrt{E^+}\|_{L^2(\Omega)}^{2(1-\varpi)} \ \|\sqrt{E^+}\|_{H^1(\Omega)}^{2\varpi} \ \|S^+\|_{H^1(\Omega)} \\
&\leq \frac{\mu}{\delta} C \ \|E^+\|_{L^1(\Omega)}^{1-\varpi} \ \|\sqrt{E^+}\|_{H^1(\Omega)}^{2\varpi} \ \|S^+\|_{H^1(\Omega)} \\
&\leq C(\mu,\delta)\|E^+\|_{L^1(\Omega)}^{2(1-\varpi)/(1-2\varpi)} + 2\|\sqrt{E^+}\|_{H^1(\Omega)}^2 + \frac{1}{2\delta}\|S^+\|_{H^1(\Omega)}^2 \\
&\leq C(\mu,\delta)\|E^+\|_{L^1(\Omega)}^{2(1-\varpi)/(1-2\varpi)} + 2\|\nabla\sqrt{E^+}\|_0^2 + 2\|\sqrt{E^+}\|_0^2 + \frac{1}{2\delta}\|S^+\|_0^2 + \frac{1}{2\delta}|S^+|_1^2 \\
&\leq C(\mu,\delta)\|E^+\|_{L^1(\Omega)}^{2(1-\varpi)/(1-2\varpi)} + 2\|E^+\|_{L^1(\Omega)} + 2\|\nabla\sqrt{E^+}\|_0^2 + \frac{1}{2\delta}\|S^+\|_0^2 + \frac{1}{2\delta}|S^+|_1^2.
\end{aligned} \tag{9.1.9}
$$

Moreover,

$$
\|E^+\|_{L^1(\Omega)} = (E^+, 1) = (e^0, 1) \leq C. \tag{9.1.10}
$$

Substituting (9.1.9) in (9.1.8) and noting (9.1.10), we have

$$
\begin{aligned}
&\int_\Omega (\mathcal{F}^M)(E^n+\epsilon)dx + \frac{\alpha}{2\delta}\int_\Omega (S^n)^2 dx \\
&+\frac{1}{2M\Delta t}\int_0^t\int_\Omega (E^+ - E^-)^2 dxdt + \frac{\alpha}{2\Delta t\delta}\int_0^t\int_\Omega (S^+ - S^-)^2 dxdt \\
&+4(1-\delta\epsilon)\int_0^t\int_\Omega |\nabla\sqrt{E^+ + \epsilon}|^2 dxdt + \frac{1}{2\delta}\int_0^t\int_\Omega |S^+|^2 dxdt + \frac{1}{4\delta}\int_0^t\int_\Omega |\nabla S^+|^2 dxdt \\
&\leq \int_\Omega \mathcal{F}(E^0+\epsilon)dx + \frac{\alpha}{2\delta}\int_\Omega (S^0)^2 dx + 2\int_0^t\int_\Omega |\nabla\sqrt{E^+}|^2 dxdt + C\epsilon.
\end{aligned} \tag{9.1.11}
$$

We shall tidy up the bound (9.1.11) by passing to the limit $\epsilon \to 0^+$. Concerning the ϵ-dependent term on the right-hand side, Lebesgue's dominated convergence theorem implies that

$$
\lim_{\epsilon\to 0^+}\int_\Omega \mathcal{F}(E^0+\epsilon)dx = \int_\Omega \mathcal{F}(E^0)dx.
$$

We can easily pass to the limit on the left-hand side of (9.1.11). By applying Fatou's lemma to the first and fifth terms on the left-hand side of (9.1.11) we get, for $t = t_n, n \in \{1, ..., N\}$, that

$$
\liminf_{\epsilon\to 0^+}\int_\Omega \mathcal{F}^M(E^n+\epsilon)dxdt \geq \int_\Omega \mathcal{F}^M(E^n)dxdt,
$$

and

$$
\liminf_{\epsilon\to 0^+}\int_0^t\int_\Omega |\nabla\sqrt{E^+ + \epsilon}|^2 dxdt \geq \int_0^t\int_\Omega |\nabla\sqrt{E^+}|^2 dxdt.
$$

Thus, after passage to the limit $\epsilon \to 0^+$, we have after a small rearrangement, for all $t = t_n, n \in \{1, ..., N\}$, that

$$\int_\Omega (\mathcal{F}^M)(E^n)dx + \frac{\alpha}{2\delta}\int_\Omega (S^n)^2 dx + \frac{1}{2M\Delta t}\int_0^t \int_\Omega (E^+ - E^-)^2 dxdt$$
$$+\frac{\alpha}{2\Delta t\delta}\int_0^t \int_\Omega (S^+ - S^-)^2 dxdt + 2\int_0^t \int_\Omega |\nabla\sqrt{E^+}|^2 + \frac{1}{2\delta}\int_0^t \int_\Omega |S^+|^2 dxdt$$
$$+\frac{1}{4\delta}\int_0^t \int_\Omega |\nabla S^+|^2 dxdt \leq \int_\Omega (\mathcal{F})(E^0)dx + \frac{\alpha}{2\delta}\int_\Omega (S^0)^2 dx + C. \tag{9.1.12}$$

□

Remark: The denominator in the prefactor of the third integral motivates us to link Δt to M so that $\Delta t M = o(1)$ as $\Delta t \to 0$ (or, equivalently, $\Delta t = o(M^{-1})$ as $M \to \infty$), in order to drive the integral multiplied by the prefactor to 0 in the limit of $M \to \infty$, once the product of the two has been bounded above by a constant, independent of M.

Lemma 9.1.2 The following bounds hold:

$$\|E^+\|_{L^2(0,T;W^{1,1}(\Omega))} + \|E^+\|_{L^{4/3}(0,T;W^{1,4/3}(\Omega))} \leq C, \tag{9.1.13}$$

$$\|E^+\|_{L^2(\Omega_T)} \leq C, \qquad d = 1\,, 2, \tag{9.1.14}$$

where $C > 0$ is independent of M and Δt.

Proof: Using the Cauchy-Schwarz inequality and (9.1.10), we have

$$\|E^+\|_{L^2(0,T;W^{1,1}(\Omega))} = \|\nabla E^+\|_{L^2(0,T;L^1(\Omega))} + C = 4\int_0^T \left\|\sqrt{E^+}\,\nabla\sqrt{E^+}\right\|^2_{L^1(\Omega)} dt + C$$
$$\leq 4\int_0^T \left\|\sqrt{E^+}\right\|^2_{L^2(\Omega)} \left\|\nabla\sqrt{E^+}\right\|^2_{L^2(\Omega)} dt + C$$
$$\leq 4\left\|E^+\right\|_{L^\infty(0,T;L^1(\Omega))} \left\|\nabla\sqrt{E^+}\right\|^2_{L^2(0,T;L^2(\Omega))} + C \leq C,$$

using (9.1.1). This shows the first estimate. Notice that this bound implies, because of the embedding $W^{1,1}(\Omega) \hookrightarrow L^2(\Omega)$ for $d = 2$, that E^+ is bounded in $L^2(\Omega_T)$, where $\Omega_T = \Omega \times (0, T)$. Then the second bound follows from

$$\|E^+\|_{L^{4/3}(0,T;W^{1,4/3}(\Omega))} = \|\nabla E^+\|_{L^{4/3}(\Omega_T)} + C$$
$$\leq 2\left\|\sqrt{E^+}\;\nabla\sqrt{E^+}\right\|_{L^{4/3}(\Omega_T)} + C$$

$$\leq 2\big\|\sqrt{E^+}\big\|_{L^4(\Omega_T)} \big\|\nabla \sqrt{E^+}\big\|_{L^2(\Omega_T)} + C \leq 2\big\|E^+\big\|^2_{L^2(\Omega_T)} \big\|\nabla \sqrt{E^+}\big\|_{L^2(\Omega_T)} + C \leq C,$$

which finishes the proof. □

Lemma 9.1.3 The following bounds on the time-derivatives hold:

$$\left\|\frac{\partial E}{\partial t}\right\|^2_{L^1(0,T;(H^{2+\varphi}(\Omega))')} \leq C,$$

and

$$\left\|\frac{\partial S}{\partial t}\right\|^2_{L^{4/3}(0,T;(W^{1,4}(\Omega))')} \leq C.$$

Proof: We begin by bounding the time-derivative of E using (8.3.89), we shall then bound the time derivative of S in a similar manner. Let $\varphi > 0$ and $\eta \in L^\infty(0,T;H^{2+\varphi}(\Omega))$. By Sobolev embedding, it holds that $\eta \in L^\infty(0,T;W^{1,\infty}(\Omega))$. Then, by using (8.3.89), (9.1.1) and Hölders inequality,

$$\begin{aligned}
|\int_0^T \int_\Omega \frac{\partial E}{\partial t}\eta \, d\,x d\,t| &\leq |\int_0^T \int_\Omega \nabla E^+ \cdot \nabla\eta \, d\,x d\,t| + |\int_0^T \int_\Omega \phi(E^+)\nabla S^+ \cdot \nabla\eta \, d\,x d\,t| \\
&\leq \|\nabla E^+\|_{L^{4/3}(\Omega_T)}\|\nabla\eta\|_{L^4(\Omega_T)} + \|E^+\|_{L^2(\Omega_T)}\|\nabla S^+\|_{L^2(\Omega_T)}\|\nabla\eta\|_{L^\infty(\Omega_T)} \\
&\leq B(E^0,S^0)\|\eta\|_{L^\infty(0,T;W^{1,\infty}(\Omega))} \\
&\leq B(E^0,S^0)\|\eta\|_{L^\infty(0,T;H^{2+\varphi}(\Omega))}.
\end{aligned}$$

Thus, we deduce that

$$\left\|\frac{\partial E}{\partial t}\right\|^2_{L^1(0,T;(H^{2+\varphi}(\Omega))')} \leq C. \tag{9.1.15}$$

In a similar way, let $\eta \in L^4(0,T;W^{1,4}(\Omega))$, then using (8.3.90), (9.1.1) and (9.1.13), we have

$$\begin{aligned}
\left|\int_0^T \int_\Omega \frac{\partial S}{\partial t}\eta \, d\,x d\,t\right| &\leq C\Big[\left|\int_0^T \int_\Omega S^+\eta \, d\,x d\,t\right| + \left|\int_0^T \int_\Omega \nabla S^+ \cdot \nabla\eta \, d\,x d\,t\right| \\
&\quad + \left|\int_0^T \int_\Omega \nabla E^+ \cdot \nabla\eta \, d\,x d\,t\right| + \left|\int_0^T \int_\Omega E^+\eta \, d\,x d\,t\right|\Big] \\
&\leq C\big[\|S^+\|_{L^2(\Omega_T)}\|\eta\|_{L^2(\Omega_T)} + \|\nabla S^+\|_{L^2(\Omega_T)}\|\nabla\eta\|_{L^2(\Omega_T)} \\
&\quad + \|\nabla E^+\|_{L^{4/3}(\Omega_T)}\|\nabla\eta\|_{L^4(\Omega_T)} + \|E^+\|_{L^2(\Omega_T)}\|\eta\|_{L^2(\Omega_T)}\big] \\
&\leq CB(E^0,S^0)\|\eta\|_{L^4(0,T;W^{1,4}(\Omega))}.
\end{aligned}$$

Then, we have

$$\left\|\frac{\partial S}{\partial t}\right\|^2_{L^{4/3}(0,T;(W^{1,4}(\Omega))')} \leq C. \tag{9.1.16}$$

□

9.2 Passage to the limit $M \to \infty$

We shall assume that

$$\Delta t = o(M^{-1}) \quad \text{as} \quad M \to \infty. \tag{9.2.17}$$

Requiring, for example, that $0 < \Delta t \leq C_0/(M \log M), M > 1$, with an arbitrary (but fixed) constant C_0 will suffice to ensure that (9.1.1) holds. The sequences

$$\{E^+\}_{M>1}, \qquad \{S^+\}_{M>1},$$

as well as all sequences of spatial and temporal derivatives of the entries of these two sequences, will thus be, indirectly, indexed by M alone, although for reasons of consistency with our previous notation we shall not introduce new, compressed, notation with Δt omitted from the superscripts. Instead, whenever $M \to \infty$, it will be understood that Δt tends to 0 according to (9.2.17).

On combining (9.1.15) and (9.1.16) with (9.1.1) we arrive at the following bound, which represents the starting point for the convergence analysis that will be developed in the next subsection:

$$\int_\Omega (\mathcal{F}^M)(E^n)dxdt + \frac{\alpha}{2\delta}\int_\Omega (S^n)^2 dxdt + 2\int_0^t\int_\Omega |\nabla\sqrt{E^+}|^2 + \frac{1}{2\delta}\int_0^t\int_\Omega |S^+|^2 dxdt$$
$$+\frac{1}{2M\Delta t}\int_0^t\int_\Omega (E^+ - E^-)^2 dxdt + \frac{\alpha}{2\Delta t\delta}\int_0^t\int_\Omega (S^+ - S^-)^2 dxdt$$
$$+\frac{1}{4\delta}\int_0^t\int_\Omega |\nabla S^+|^2 dxdt + \left\|\frac{\partial E}{\partial t}\right\|^2_{L^1(0,T;(H^{2+\varphi}(\Omega))')} + \left\|\frac{\partial S}{\partial t}\right\|^2_{L^{4/3}(0,T;(W^{1,4}(\Omega))')} \leq C, \tag{9.2.18}$$

where C denotes a generic positive constant independent of M and Δt.

Lemma 9.2.1 Let $\mathcal{E}^\pm = \min\{E^\pm, M\}$. Hence $\mathcal{E}^\pm \to \mathcal{E} = \min\{e, M\}$ *a.e.* then for sufficiently small $\Delta t > 0$, the following bounds hold:

$$\int_\Omega |E^\pm - \mathcal{E}^\pm| dx \leq \frac{1}{\ln M}\left[\int_{E^\pm \geq M} \mathcal{F}(E^\pm)dx + C\right], \tag{9.2.19}$$

$$\int_\Omega |\mathcal{E} - e| dx \leq \frac{1}{\ln M}\left[\int_{e \geq M} \mathcal{F}(e)dx + C\right]. \tag{9.2.20}$$

Proof: It follows from the definition of $\phi(e)$ and by testing (8.1.1) with $\chi \equiv 1$ gives that

$$0 \leq \int_{e \geq M} M dxdt \leq \int_{\Omega} \phi(e) dxdt \leq \int_{\Omega} e dxdt = \int_{\Omega} e^0 dxdt. \tag{9.2.21}$$

and similarly

$$0 \leq \int_{E^{\pm} \geq M} M dxdt \leq \int_{\Omega} E^0 dxdt. \tag{9.2.22}$$

Let us now recall the logarithmic Young's inequality (see Appendix A.1.21):

$$r\, s \leq r \ln r - r + e^s \qquad \forall\ r,\ s \in \mathbb{R}^{\geq 0}. \tag{9.2.23}$$

Applying (9.2.23) with $r = e - M$ and $s = \ln M$ and then with $r = E^{\pm} - M$ and $s = \ln M$, we have for $e \geq M$ and $E^{\pm} \geq M$ that

$$\begin{aligned} \ln M(e - M) &\leq \mathcal{F}(e - M) + M, \\ \ln M(E^{\pm} - M) &\leq \mathcal{F}(E^{\pm} - M) + M. \end{aligned} \tag{9.2.24}$$

The bound $(9.2.24)_2$ and (9.2.22) then imply

$$\begin{aligned} \int_{\Omega} |E^{\pm} - \mathcal{E}^{\pm}| dx &= \int_{E^{\pm} \geq M} (E^{\pm} - M) dx \\ &\leq \frac{1}{\ln M} \left[\int_{E^{\pm} \geq M} \mathcal{F}(E^{\pm} - M) dx + \int_{E^{\pm} \geq M} M dx \right] \\ &\leq \frac{1}{\ln M} \left[\int_{E^{\pm} \geq M} \mathcal{F}(E^{\pm} - M) dx + C \right], \end{aligned} \tag{9.2.25}$$

and similarly, using the bound $(9.2.24)_1$ and (9.2.21) we have

$$\int_{\Omega} |e - \mathcal{E}| dx \leq \frac{1}{\ln M} \left[\int_{e \geq M} \mathcal{F}(e - M) dx + C \right]. \tag{9.2.26}$$

□

In the next lemma, we prove the strong convergence of a sequence of functions bounded in certain Sobolev spaces.

Lemma 9.2.2 Let $\Omega \subset \mathbb{R}^d (d \geq 1)$ be a bounded domain with $\partial\Omega \in C^{0,1}$, $T > 0$. Furthermore, let $\{E^{\pm}\}$ be a sequence of nonnegative functions satisfying

$$\|\mathcal{F}(E^{\pm})\|_{L^\infty(0,T;L^1(\Omega))} + \|\sqrt{E^{\pm}}\|_{L^2(0,T;H^1(\Omega))} + \|\partial_t E\|^2_{L^1(0,T;(H^s(\Omega))')} \leq C, \tag{9.2.27}$$

for some $C > 0$ independent of Δt. Then, up to a subsequence, as $\Delta t \to 0$, $E^{\pm} \to e$ strongly in $L^2(0, T; L^{d/(d-1)}(\Omega))$.

The above uniform estimates are typical for solutions $E^{\pm}$ of nonlinear diffusion equations for which $\int_\Omega \mathcal{F}(E^{\pm})dx$ is an entropy with $\int_\Omega |\nabla\sqrt{E^{\pm}}|^2 dx$ as the corresponding entropy production. Notice that the estimate implies that $\nabla E^{\pm} = 2\sqrt{E^{\pm}}\nabla\sqrt{E^{\pm}}$ is uniformly bounded in $L^2(0,T;L^1(\Omega))$. Hence, since the embedding $W^{1,1}(\Omega)) \hookrightarrow L^p(\Omega)$ is compact for all $p < d/(d-1)$, we conclude from the Aubin lemma that there exists a subsequence of $\{E^{\pm}\}$, which is not relabelled, such that $E^{\pm} \to e$ strongly in $L^2(0,T;L^p(\Omega))$ as $\Delta t \to 0$. The additional estimate for $\mathcal{F}(E^{\pm})$ in $L^1(\Omega)$ allows us to extend this convergence result to $p = d/(d-1)$.

Proof: It holds that $E^{\pm} \to e$ *a.e.* First, we claim that this convergence and the bound for $\mathcal{F}(E^{\pm})$ imply that $\sqrt{E^{\pm}} \to \sqrt{e}$ strongly in $L^\infty(0,T;L^2(\Omega))$ as $\Delta t \to 0$. Indeed, by the Fatou lemma,

$$\sup_{(0,T)}\int_\Omega \mathcal{F}(e)dx = \sup_{(0,T)}\int_\Omega \lim_{\Delta t\to 0}\mathcal{F}(E^{\pm})dx \le \liminf_{\Delta t\to 0}\sup_{(0,T)}\int_\Omega \mathcal{F}(E^{\pm})dx \le C.$$

Note that $|\mathcal{E}^{\pm}-\mathcal{E}| \to 0$, $a.e$, and that $|\mathcal{E}^{\pm}|, |\mathcal{E}| \le M$, then the dominated convergence theorem yields that

$$\int_\Omega |\mathcal{E}^{\pm} - \mathcal{E}|dx \to 0 \ \text{ as } \ \Delta t \to 0,$$

so for Δt sufficiently small

$$\sup\int_\Omega |\mathcal{E}^{\pm} - \mathcal{E}|dx \le \frac{1}{\ln M}. \tag{9.2.28}$$

On noting that $\mathcal{F}(e)$ is non-negative and monotonically increasing on $[1,\infty)$, and that $\mathcal{F}(e) \in [0,1]$ for $e \in [0,1]$, then by using the bound (9.1.1), we deduce that

$$\begin{aligned}
&\int_{e\ge M}\mathcal{F}(e-M)dxdt\\
&= \int_{e\in[M,M+1)}\mathcal{F}(e-M)dxdt + \int_{e\ge M+1}\mathcal{F}(e-M)dxdt\\
&\le \int_{e\in[M,M+1)} dxdt + \int_{e\ge M+1}\mathcal{F}(e)dxdt\\
&\le 1 + \int_\Omega \mathcal{F}(e)dxdt \le C,
\end{aligned} \tag{9.2.29}$$

and similarly, we have

$$\int_{E^{\pm}\ge M}\mathcal{F}(E^{\pm}-M)dxdt \le 1 + \int_\Omega \mathcal{F}(E^{\pm})dxdt \le C. \tag{9.2.30}$$

Then, on noting the bounds (9.2.25), (9.2.26), (9.2.29) and (9.2.30), we arrive:

$$\sup_{(0,T)} \int_\Omega |E^\pm - e|dx \le \sup_{(0,T)} \int_\Omega |E^\pm - \mathcal{E}^\pm|dx + \sup_{(0,T)} \int_\Omega |\mathcal{E}^\pm - \mathcal{E}|dx + \sup_{(0,T)} \int_\Omega |\mathcal{E} - e|dx$$

$$\le \frac{1}{\ln M}\left[\int_{E^\pm \ge M} \mathcal{F}(E^\pm - M)dx + C\right] + \frac{1}{\ln M} + \frac{1}{\ln M}\left[\int_{e \ge M} \mathcal{F}(e - M)dx + C\right]$$

$$\le \frac{C}{\ln M}.$$

This shows that as $M \to \infty$ then $E^\pm \to e$ strongly in $L^\infty(0,T;L^1(\Omega))$. Consequently, since $(x-y)^2 \le |x^2 - y^2|$ for $x, y \ge 0$, then we have $\sqrt{E^\pm} \to \sqrt{e}$ strongly in $L^\infty(0,T;L^2(\Omega))$.

Next, we apply the Gagliardo-Nirenberg inequality

$$\|\sqrt{E^\pm} - \sqrt{e}\|^4_{L^4(0,T;L^{2d/(d-1)}(\Omega))} \le C_1 \int_0^T \|\sqrt{E^\pm} - \sqrt{e}\|^2_{H^1(\Omega)} \|\sqrt{E^\pm} - \sqrt{e}\|^2_{L^2(\Omega)} dt$$

$$\le C_2\left(\|\sqrt{E^\pm}\|^2_{L^2(0,T;H^1(\Omega))} + \|\sqrt{e}\|^2_{L^2(0,T;H^1(\Omega))}\right) \times \|\sqrt{E^\pm} - \sqrt{e}\|^2_{L^\infty(0,T;L^2(\Omega))}$$

$$\to 0 \qquad \text{as} \qquad \Delta t \to 0.$$

Hence, $\sqrt{E^\pm} \to \sqrt{e}$ strongly in $L^4(0,T;L^{2d/(d-1)}(\Omega))$. Now, since

$$\|uv\|^2_{L^2(0,T;L^p(\Omega))} \le \|u\|^2_{L^4(0,T;L^{2p}(\Omega))} \|v\|^2_{L^4(0,T;L^{2p}(\Omega))},$$

then, by using the above fact we have

$$\|E^\pm - e\|^2_{L^2(0,T;L^{d/(d-1)}(\Omega))} \le \|\sqrt{E^\pm} + \sqrt{e}\|^2_{L^4(0,T;L^{2d/(d-1)}(\Omega))} \times \|\sqrt{E^\pm} - \sqrt{e}\|^2_{L^4(0,T;L^{2d/(d-1)}(\Omega))}$$

$$\le \left[\|\sqrt{E^\pm}\|^2_{L^4(0,T;L^{2d/(d-1)}(\Omega))} + \|\sqrt{e}\|^2_{L^4(0,T;L^{2d/(d-1)}(\Omega))}\right] \times \|\sqrt{E^\pm} - \sqrt{e}\|^2_{L^4(0,T;L^{2d/(d-1)}(\Omega))}$$

$$\to 0 \qquad \text{as} \qquad \Delta t \to 0.$$

Hence, $E^\pm \to e$ strongly in $L^2(0,T;L^{d/(d-1)}(\Omega))$. □

Theorem 9.2.3 There exists a subsequence of $\{E^\pm, S^\pm\}_{M>1}$, (not indicated) with $\Delta t = o(M^{-1})$, and a pair of functions $\{e, s\}$ such that

$$E, E^\pm \rightharpoonup e \qquad \text{in} \quad L^{4/3}(0,T;W^{1,4/3}(\Omega)), \tag{9.2.31}$$

$$\frac{\partial E}{\partial t} \rightharpoonup \frac{\partial e}{\partial t} \qquad \text{in} \quad L^1(0,T;(H^{2+\varphi}(\Omega))'), \tag{9.2.32}$$

$$E, E^{\pm} \to e \quad \text{in} \quad L^2(0,T;L^p(\Omega)),\ p \le d/(d-1), \tag{9.2.33}$$

$$\phi(E^{\pm}) \to e \quad \text{in} \quad L^2(0,T;L^p(\Omega)),\ p \le d/(d-1), \tag{9.2.34}$$

$$S, S^{\pm} \rightharpoonup s, \quad \text{in } L^2(0,T;H^1(\Omega)), \tag{9.2.35}$$

$$S, S^{\pm} \rightharpoonup^* s \quad \text{in } L^{\infty}(0,T;L^2(\Omega)), \tag{9.2.36}$$

$$\frac{\partial S}{\partial t} \rightharpoonup \frac{\partial s}{\partial t} \quad \text{in } L^{4/3}(0,T;(W^{1,4}(\Omega))'), \tag{9.2.37}$$

$$S, S^{\pm} \to s \quad \text{in} \quad L^2(0,T;L^q(\Omega)), \quad q < \infty, \tag{9.2.38}$$

$$E^{\pm}\nabla S^{\pm} \to e\nabla s \quad \text{in} \quad L^1(0,T;L^1(\Omega)), \quad d = 1,2, \tag{9.2.39}$$

Proof: The proof of (9.2.31), (9.2.32), (9.2.35), (9.2.36) and (9.2.37) can achieved using a sequential compactness argument and noting the bounds in (9.2.18). Note that (9.2.33) was demonstrated in Lemma 9.2.2 for all $p \le d/(d-1)$. Taking into account (9.2.35), (9.2.37) and Aubins lemma provides the existence of subsequences of $S^{\pm}$, which are not relabeled, such that, as $\Delta t \to 0$, the convergence result (9.2.38) holds, where, we have used the compactness of the embeddings $H^1(\Omega) \hookrightarrow L^q(\Omega)$ for all $1 \le q < \infty$ in two-dimensional domains.

From the Lipschitz continuity of ϕ, we obtain for any $p \le d/(d-1)$ that

$$\begin{aligned}\|\phi(E^{\pm}) - e\|_{L^2(0,T;L^p(\Omega))} &\le \|e - \phi(e)\|_{L^2(0,T;L^p(\Omega))} + \|\phi(e) - \phi(E^{\pm})\|_{L^2(0,T;L^p(\Omega))} \\ &\le \|e - \phi(e)\|_{L^2(0,T;L^p(\Omega))} + \|e - E^{\pm}\|_{L^2(0,T;L^p(\Omega))}.\end{aligned} \tag{9.2.40}$$

The first term on the right-hand side of (9.2.40) converges to zero as $M \to \infty$ on noting that $\phi(e)$ converges to e almost everywhere on $\Omega \times [0,T]$ and applying Lebesgue's dominated convergence theorem see Appendix A.1.20. The second term converges to 0 on noting (9.2.33). That yields the desired result (9.2.34).

Unfortunately, the above convergence results do not allow us to pass to the limit in the term $(E^{\pm}\nabla S^{\pm})$. However, we are able to exploit the boundedness of $\mathcal{F}(E^{\pm})$ in $L^1(\Omega)$. Indeed, Lemma 9.2.2 shows that, up to a subsequence,

$$E^{\pm} \to e \quad \text{in} \quad L^2(0,T;L^2(\Omega)), \quad d = 1,2. \tag{9.2.41}$$

Hence, we find that

$$\int_0^t \int_{\Omega} |E^{\pm}\nabla S^{\pm} - e\nabla s| dxdt$$

$$\leq \int_0^t \int_\Omega |(E^\pm - e)\nabla S^\pm| dxdt + \int_0^t \int_\Omega |(\nabla s - \nabla S^\pm) e| dxdt$$
$$\to 0 \qquad \text{as} \qquad \Delta t \to 0. \tag{9.2.42}$$

Hence, we have

$$E^\pm \nabla S^\pm \to e\nabla s \qquad \text{in} \;\; L^1(0,T;L^1(\Omega)), \quad d \leq 2. \tag{9.2.43}$$

□

Theorem 9.2.4 The functions $\{e, s\}$ are a global weak solution to problem (Q), in the sense that

$$\int_0^T \big[\langle \frac{\partial e}{\partial t}, \eta\rangle_{H^{2+\varphi}(\Omega)} + (\nabla e, \nabla \eta)\big] dt = \int_0^T (e\nabla s, \nabla \eta) dt, \quad \eta \in L^4(\Omega_T) \cap L^\infty(0,T;W^{1,\infty}(\Omega)) \tag{9.2.44}$$

$$\int_0^T \big[\alpha\langle \frac{\partial s}{\partial t}, \eta\rangle_{W^{1,4}(\Omega)} + (s,\eta) + (\nabla s, \nabla \eta) + \delta(\nabla e, \nabla \eta)\big] dt = \mu \int_0^T (e,\eta) dt, \quad \eta \in L^4(\Omega_T). \tag{9.2.45}$$

Proof:

We shall now study the convergence of each term in (8.3.89) and (8.3.90) separately. By using (9.2.32) and (9.2.37) we immediately have that

$$\int_0^T \int_\Omega \frac{\partial E}{\partial t} \eta \, d\,x\,d\,t = \int_0^T \langle \frac{\partial E}{\partial t}, \eta\rangle_{H^{2+\varphi}(\Omega)} \, d\,t \to \int_0^T \langle \frac{\partial e}{\partial t}, \eta\rangle_{H^{2+\varphi}(\Omega)}, \tag{9.2.46}$$

$$\int_0^T \int_\Omega \frac{\partial S}{\partial t} \eta \, d\,x\,d\,t = \int_0^T \langle \frac{\partial S}{\partial t}, \eta\rangle_{W^{1,4}(\Omega)} \, d\,t \to \int_0^T \langle \frac{\partial s}{\partial t}, \eta\rangle_{W^{1,4}(\Omega)}, \tag{9.2.47}$$

as $M \to \infty$ (and $\Delta t \to 0_+$), for $\eta \in L^2(0,T;H^1(\Omega))$, as required. Moreover:

$$\int_0^T \int_\Omega \nabla E^+ \, \nabla \eta d\,x\,d\,t \to \int_0^T \int_\Omega \nabla e \, \nabla \eta d\,x\,d\,t, \tag{9.2.48}$$

$$\int_0^T \int_\Omega \nabla S^+ \, \nabla \eta d\,x\,d\,t \to \int_0^T \int_\Omega \nabla s \, \nabla \eta d\,x\,d\,t, \tag{9.2.49}$$

$$\int_0^T \int_\Omega E^+ \, \eta d\,x\,d\,t \to \int_0^T \int_\Omega e \, \eta d\,x\,d\,t, \tag{9.2.50}$$

$$\int_0^T \int_\Omega S^+ \, \eta d\,x\,d\,t \to \int_0^T \int_\Omega s \, \eta d\,x\,d\,t. \tag{9.2.51}$$

The third term in (9.2.44) will be dealt with by decomposing it into two further terms, the first of which tends to 0, while the second converges to the expected limiting value. We proceed as follows:

$$\int_0^T \int_\Omega \phi(E^+)\, \nabla S^+\, \nabla \eta d\, x\, d\, t$$

$$= \int_0^T \int_\Omega (\phi(E^+) - E^+)\, \nabla S^+\, \nabla \eta d\, x\, d\, t + \int_0^T \int_\Omega E^+\, \nabla S^+\, \nabla \eta d\, x\, d\, t$$
$$=: V_1 + V_2. \tag{9.2.52}$$

We shall show that V_1 converges to 0 and that V_2 converges to the expected limit.

$$|V_1| \leq \int_0^T \int_\Omega |\phi(E^+) - E^+|\, |\nabla S^+|\, |\nabla \eta| d\, x\, d\, t$$
$$\leq \|\phi(E^+) - E^+\|_{L^2(\Omega_T)}\, \|S^+\|_{L^2(0,T;H^1(\Omega))}\, \|\eta\|_{L^\infty(0,T;W^{1,\infty}(\Omega))}.$$

The norm of the difference of the bound on V_1 is known to converge to 0 as $M \to \infty$ (and $\Delta t \to 0_+$), by (9.2.34). This then implies that the term V_1 converges to 0 as $M \to \infty$ (and $\Delta t \to 0_+$).

Concerning the term V_2, we have that

$$V_2 = \int_0^T \int_\Omega E^+\, \nabla S^+\, \nabla \eta d\, x\, d\, t \to \int_0^T \int_\Omega e\, \nabla s\, \nabla \eta d\, x\, d\, t, \tag{9.2.53}$$

as $M \to \infty$ (and $\Delta t \to 0_+$). □

9.2.1 Uniqueness of a weak solution

In this section, in order that we are able to prove uniqueness of a solution, we have to assume that $\|e\|_{L^\infty(\Omega_T)} + \|s\|_{L^\infty(0,T;H^1(\Omega))} \leq C$ holds. We note that we are unable to prove the uniqueness without such a bound.

Theorem 9.2.5 Assume that $\|e\|_{L^\infty(\Omega_T)} + \|s\|_{L^\infty(0,T;H^1(\Omega))} \leq C$, then for δ sufficiently small, there exists a unique solution to (9.2.44)-(9.2.45).

Proof: Assume that there are two weak solutions $\{e_1, s_1\}$ and $\{e_2, s_2\}$ to the system (9.2.44)-(9.2.45). Let the solutions $\{e_1, s_1\}$ and $\{e_2, s_2\}$ satisfy

$$\|e_i\|_{L^\infty(\Omega_T)} + \|s_i\|_{L^\infty(0,T;H^1(\Omega))} \leq C, \quad i = 1, 2, \tag{9.2.54}$$

and

$$e_1(\cdot,0) = e_2(\cdot,0) = e^0(\cdot) \quad \text{and} \quad s_1(\cdot,0) = s_2(\cdot,0) = s^0(\cdot) \quad \text{in} \quad L^2(\Omega). \tag{9.2.55}$$

Setting $\boldsymbol{e} = e_1 - e_2$, $\boldsymbol{s} = s_1 - s_2$ and testing (9.2.44) with $\eta \equiv \boldsymbol{e} \in L^2(0,T;H^1(\Omega))$ and (9.2.45) with $\eta \equiv \frac{1}{\delta}\boldsymbol{s} \in L^2(0,T;H^1(\Omega))$ leads to after subtracting the weak forms

$$\frac{1}{2}\|\boldsymbol{e}(T)\|_0^2 + \|\nabla\boldsymbol{e}\|^2_{L^2(\Omega_T)} = \frac{1}{2}\|\boldsymbol{e}(0)\|_0^2 + \int_0^T (e_2\nabla s_2 - e_1\nabla s_1, \nabla\boldsymbol{e})dt, \tag{9.2.56}$$

$$\frac{\alpha}{2\delta}\|\boldsymbol{s}(T)\|_0^2 + \frac{1}{\delta}\|\boldsymbol{s}\|^2_{L^2(\Omega_T)} + \frac{1}{\delta}\|\nabla\boldsymbol{s}\|^2_{L^2(\Omega_T)} + \int_0^T (\nabla\boldsymbol{e},\nabla\boldsymbol{s})dt = \frac{\alpha}{2\delta}\|\boldsymbol{s}(0)\|_0^2 + \frac{\mu}{\delta}\int_0^T (\boldsymbol{e},\boldsymbol{s})dt. \tag{9.2.57}$$

Adding (9.2.56) and (9.2.57), noting (9.2.54) and employing Hölder's inequality yields that

$$\begin{aligned}
&\frac{1}{2}\big(\|\boldsymbol{e}(T)\|_0^2 + \frac{1}{\delta}\|\boldsymbol{s}(T)\|_0^2\big) + \|\nabla\boldsymbol{e}\|^2_{L^2(\Omega_T)} + \frac{1}{\delta}\|\boldsymbol{s}\|^2_{L^2(\Omega_T)} + \frac{1}{\delta}\|\nabla\boldsymbol{s}\|^2_{L^2(\Omega_T)} \\
&= \frac{\mu}{\delta}\int_0^T (\boldsymbol{e},\boldsymbol{s})dt - \int_0^T (\nabla\boldsymbol{e},\nabla\boldsymbol{s}) + \int_0^T (e_2\nabla s_2 - e_1\nabla s_1,\nabla\boldsymbol{e})dt \\
&= \frac{\mu}{\delta}\int_0^T (\boldsymbol{e},\boldsymbol{s})dt - \int_0^T (\nabla\boldsymbol{e},\nabla\boldsymbol{s}) - \int_0^T (e_1\nabla\boldsymbol{s},\nabla\boldsymbol{e})dt - \int_0^T (\boldsymbol{e}\nabla s_2,\nabla\boldsymbol{e})dt \\
&\le \frac{\mu}{\delta}\int_0^T \|\boldsymbol{e}\|_0\|\boldsymbol{s}\|_0 dt + C\int_0^T |\boldsymbol{e}|_1|\boldsymbol{s}|_1 dt + \int_0^T \|\boldsymbol{e}\|_{0,\infty}|s_2|_1|\boldsymbol{e}|_1 dt \\
&\le \frac{\mu}{\delta}\int_0^T \|\boldsymbol{e}\|_0\|\boldsymbol{s}\|_0 dt + C\int_0^T |\boldsymbol{e}|_1|\boldsymbol{s}|_1 dt.
\end{aligned} \tag{9.2.58}$$

We easily obtain from the Young's inequality that

$$\frac{\mu}{\delta}\int_0^T \|\boldsymbol{e}\|_0\|\boldsymbol{s}\|_0 dt \le \frac{\mu^2}{4\delta}\|\boldsymbol{e}\|^2_{L^2(\Omega_T)} + \frac{1}{\delta}\|\boldsymbol{s}\|^2_{L^2(\Omega_T)}, \tag{9.2.59}$$

$$C\int_0^T |\boldsymbol{e}|_1|\boldsymbol{s}|_1 dt \le C\delta\|\nabla\boldsymbol{e}\|^2_{L^2(\Omega_T)} + \frac{1}{\delta}\|\nabla\boldsymbol{s}\|^2_{L^2(\Omega_T)}. \tag{9.2.60}$$

Putting (9.2.59) and (9.2.60) in (9.2.58) leads to

$$\frac{1}{2}\big(\|\boldsymbol{e}(T)\|_0^2 + \frac{\alpha}{\delta}\|\boldsymbol{s}(T)\|_0^2\big) + (1 - C\delta)\|\nabla\boldsymbol{e}\|^2_{L^2(\Omega_T)} \le \frac{\mu^2}{4\delta}\|\boldsymbol{e}\|^2_{L^2(\Omega_T)}. \tag{9.2.61}$$

As $C\delta \le 1$, then we arrive to the following inequality

$$\|\boldsymbol{e}(T)\|_0^2 + \frac{1}{\delta}\|\boldsymbol{s}(T)\|_0^2 \le C\|\boldsymbol{e}\|^2_{L^2(\Omega_T)}. \tag{9.2.62}$$

Applying the integral version of Grönwall's lemma, see Appendix A.1.5, leads to

$$\|\boldsymbol{e}(T)\|_0^2 + \frac{1}{\delta}\|\boldsymbol{s}(T)\|_0^2 \le 0. \tag{9.2.63}$$

Thus, we conclude $e_1 = e_2$ and $s_1 = s_2$ as required. □

9.3 An error estimate

In this section we study the error estimate between the weak solution of (Q) and the fully discrete approximation defined by (8.2.15) - (8.2.16). Additionally to the uniqueness requirements, the derivation of an error estimate requires extra regularity on the time derivatives of the approximate solutions that we have been unable to prove. The details are given in the following theorem.

Theorem 9.3.1 Let all the assumptions of Theorem 8.2.4 hold. If $\delta < \frac{4}{\hat{\kappa}^2}$ and

$$\|\frac{\partial E_\varepsilon}{\partial t}\|_{L^2(\Omega_T)} + \|\frac{\partial S_\varepsilon}{\partial t}\|_{L^2(\Omega_T)} + \|S_\varepsilon\|_{L^\infty(0,T;H^1(\Omega))}$$

$$+\|e\|_{L^\infty(0,T;H^1(\Omega))} + \|E^+\|_{L^2(0,T;H^1(\Omega))} + \|e\|_{L^2(0,T;H^1(\Omega))} \leq C, \tag{9.3.64}$$

where $\hat{\kappa} = \|e-1\|_{L^\infty(\Omega_T)}$ and let $\|e\|_{L^\infty(\Omega_T)} \leq M$ and $e^0, s^0 \in H^1(\Omega)$, then the solution $\{E_\varepsilon, S_\varepsilon\}$ of $(\mathrm{Q}^{h,\Delta t}_{M,\varepsilon}), h, \Delta t \leq 1$, satisfies the following error bound:

$$\|e - E_\varepsilon\|^2_{L^\infty(0,T;L^2(\Omega))} + \|s - S_\varepsilon\|^2_{L^\infty(0,T;L^2(\Omega))}$$

$$\leq C\big(h + \Delta t + \varepsilon^2 + \big\|\nabla(I-\pi^h)e\big\|_{L^2(\Omega_T)} + \big\|\nabla(I-\pi^h)s\big\|_{L^2(\Omega_T)}\big). \tag{9.3.65}$$

Furthermore, if $e, s \in L^2(0,T;H^2(\Omega))$ then

$$\|e - E_\varepsilon\|^2_{L^\infty(0,T;L^2(\Omega))} + \|s - S_\varepsilon\|^2_{L^\infty(0,T;L^2(\Omega))} \leq C\big(h + \Delta t + \varepsilon^2\big). \tag{9.3.66}$$

Proof: We first mention that $\pi^h e$ and $\pi^h s$ are well defined since $e(\cdot,t), s(\cdot,t) \in H^1(\Omega)$ for *a.e.* $t \in (0,T)$ and the Sobolev embedding result $H^1(\Omega) \hookrightarrow C(\overline{\Omega})$ holds in one space dimension. Noting this, we set

$$\mathfrak{e}^A_y = y - \pi^h y, \quad \mathfrak{e}^{(\pm)}_{y,\varepsilon} = y - Y^{(\pm)}_\varepsilon, \quad \mathfrak{E}^{(\pm)}_{y,\varepsilon} = \pi^h y - Y^{(\pm)}_\varepsilon, \tag{9.3.67}$$

where $y \equiv e$ and s, $Y^{(\pm)}_\varepsilon \equiv E^{(\pm)}_\varepsilon$ and $S^{(\pm)}_\varepsilon$, respectively.

On subtracting (8.3.64) and (8.3.65) from (9.2.44) and (9.2.45) respectively, it follows for *a.e.* $t \in (0,T)$ and for all $\chi \in S^h$ that

$$\big(\frac{\partial \mathfrak{e}_{e,\varepsilon}}{\partial t}, \chi\big) + (\nabla \mathfrak{e}^+_{e,\varepsilon}, \nabla\chi) = (e\nabla s, \nabla\chi) - (\Lambda_\varepsilon(E^+_\varepsilon)\nabla S^+_\varepsilon, \nabla\chi) + \Big\{\big(\frac{\partial E_\varepsilon}{\partial t}, \chi\big)^h - \big(\frac{\partial E_\varepsilon}{\partial t}, \chi\big)\Big\}, \tag{9.3.68}$$

$$\alpha\big(\frac{\partial \mathfrak{e}_{s,\varepsilon}}{\partial t}, \chi\big) + (\mathfrak{e}^+_{s,\varepsilon}, \chi) + (\nabla \mathfrak{e}^+_{s,\varepsilon}, \nabla\chi) + \delta(\nabla \mathfrak{e}^+_{e,\varepsilon}, \nabla\chi) = \mu\,(\mathfrak{e}^+_{e,\varepsilon}, \chi)$$

$$+\alpha\Big\{\big(\frac{\partial S_\varepsilon}{\partial t},\chi\big)^h-\big(\frac{\partial S_\varepsilon}{\partial t},\chi\big)\Big\}+\mu\Big\{(E^+_\varepsilon,\chi)-(E^+_\varepsilon,\chi)^h\Big\}. \quad (9.3.69)$$

Hence, choosing $\chi\equiv\mathfrak{E}^+_{e,\varepsilon}\in S^h$ in (9.3.68) and $\chi\equiv\frac{1}{\delta}\mathfrak{E}^+_{s,\varepsilon}\in S^h$ in (9.3.69) and summing the resulting equations yields that

$$\frac{1}{2}\frac{d}{dt}\|\mathfrak{e}_{e,\varepsilon}\|_0^2+\frac{\alpha}{2\delta}\frac{d}{dt}\|\mathfrak{e}_{s,\varepsilon}\|_0^2+\frac{1}{\delta}\|\mathfrak{e}^+_{s,\varepsilon}\|_0^2+|e^+_{\mathfrak{e},\varepsilon}|_1^2+\frac{1}{\delta}|\mathfrak{e}^+_{s,\varepsilon}|_1^2$$
$$=\Big[\Big(\frac{\partial\mathfrak{e}_{e,\varepsilon}}{\partial t},\mathfrak{e}^A_e\Big)+\frac{\alpha}{\delta}\Big(\frac{\partial\mathfrak{e}_{s,\varepsilon}}{\partial t},\mathfrak{e}^A_s\Big)\Big]$$
$$+\Big[\Big(\frac{\partial\mathfrak{e}_{e,\varepsilon}}{\partial t},E^+_\varepsilon-E_\varepsilon\Big)+\frac{\alpha}{\delta}\Big(\frac{\partial\mathfrak{e}_{s,\varepsilon}}{\partial t},S^+_\varepsilon-S_\varepsilon\Big)\Big]$$
$$+\Big[\big(\nabla\mathfrak{e}^+_{e,\varepsilon},\nabla\mathfrak{e}^A_e\big)+\frac{1}{\delta}\big(\nabla\mathfrak{e}^+_{s,\varepsilon},\nabla\mathfrak{e}^A_s\big)+\frac{1}{\delta}\big(\mathfrak{e}^+_{s,\varepsilon},\mathfrak{e}^A_s\big)\Big]$$
$$+\Big[\Big\{\big(\frac{\partial E_\varepsilon}{\partial t},\mathfrak{E}^+_{e,\varepsilon}\big)^h-\big(\frac{\partial E_\varepsilon}{\partial t},\mathfrak{E}^+_{e,\varepsilon}\big)\Big\}+\frac{\alpha}{\delta}\Big\{\big(\frac{\partial S_\varepsilon}{\partial t},\mathfrak{E}^+_{s,\varepsilon}\big)^h-\big(\frac{\partial S_\varepsilon}{\partial t},\mathfrak{E}^+_{s,\varepsilon}\big)\Big\}$$
$$+\frac{1}{\delta}\Big\{(S^+_\varepsilon,\mathfrak{E}^+_{s,\varepsilon})^h-(S^+_\varepsilon,\mathfrak{E}^+_{s,\varepsilon})\Big\}\Big]+\Big[\big(\nabla\mathfrak{e}^+_{e,\varepsilon},\nabla\mathfrak{e}^A_s\big)-\big(e\nabla\mathfrak{e}^+_{s,\varepsilon},\nabla\mathfrak{e}^A_e\big)\Big]$$
$$+\Big[\big((e-1)\nabla\mathfrak{e}^+_{s,\varepsilon},\nabla\mathfrak{e}^+_{e,\varepsilon}\big)\Big]+\Big[\Big([e-\Lambda_\varepsilon(E^+_\varepsilon)]\nabla S^+_\varepsilon,\nabla\mathfrak{E}^+_{e,\varepsilon}\Big)\Big]$$
$$+\Big[\frac{\mu}{\delta}\,(\mathfrak{e}^+_{e,\varepsilon},\mathfrak{E}^+_{s,\varepsilon})\Big]+\Big[\frac{\mu}{\delta}\Big\{(E^+_\varepsilon,\mathfrak{E}^+_{s,\varepsilon})-(E^+_\varepsilon,\mathfrak{E}^+_{s,\varepsilon})^h\Big\}\Big]$$
$$=\sum_{i=1}^{9}\big[I_i\big], \quad (9.3.70)$$

where we have noticed from (9.3.67) that

$$\mathfrak{E}^{(\pm)}_{y,\varepsilon}=\mathfrak{e}^{(\pm)}_{y,\varepsilon}-\mathfrak{e}^A_y=\mathfrak{e}_{y,\varepsilon}-\mathfrak{e}^A_y+Y_\varepsilon-Y^{(\pm)}_\varepsilon.$$

We now bound each term on the right hand side of (9.3.70) separately.

Using the Cauchy-Schwarz inequality gives that

$$I_1=\Big(\frac{\partial\mathfrak{e}_{e,\varepsilon}}{\partial t},\mathfrak{e}^A_e\Big)+\frac{\alpha}{\delta}\Big(\frac{\partial\mathfrak{e}_{s,\varepsilon}}{\partial t},\mathfrak{e}^A_s\Big)$$
$$\le C\Big(\|\frac{\partial\mathfrak{e}_{e,\varepsilon}}{\partial t}\|_0\|\mathfrak{e}^A_e\|_0+\|\frac{\partial\mathfrak{e}_{s,\varepsilon}}{\partial t}\|_0\|\mathfrak{e}^A_s\|_0):=\tilde{I}_1\Big), \quad (9.3.71)$$

$$I_2=\Big(\frac{\partial\mathfrak{e}_{e,\varepsilon}}{\partial t},E^+_\varepsilon-E_\varepsilon\Big)+\frac{\alpha}{\delta}\Big(\frac{\partial\mathfrak{e}_{s,\varepsilon}}{\partial t},S^+_\varepsilon-S_\varepsilon\Big)$$
$$\le C\Big(\|\frac{\partial\mathfrak{e}_{e,\varepsilon}}{\partial t}\|_0\|E^+_\varepsilon-E_\varepsilon\|_0+\|\frac{\partial\mathfrak{e}_{s,\varepsilon}}{\partial t}\|_0\|S^+_\varepsilon-S_\varepsilon\|_0\Big):=\tilde{I}_2, \quad (9.3.72)$$

$$I_3 = \big(\nabla \mathfrak{e}_{e,\varepsilon}^+, \nabla \mathfrak{e}_e^A\big) + \frac{1}{\delta}\big(\nabla \mathfrak{e}_{s,\varepsilon}^+, \nabla \mathfrak{e}_s^A\big) + \frac{1}{\delta}\big(\mathfrak{e}_{s,\varepsilon}^+, \mathfrak{e}_s^A\big)$$
$$\leq C\left(|\mathfrak{e}_{e,\varepsilon}^+|_1|\mathfrak{e}_e^A|_1 + |\mathfrak{e}_{s,\varepsilon}^+|_1|\mathfrak{e}_s^A|_1 + \|\mathfrak{e}_{s,\varepsilon}^+\|_0\|\mathfrak{e}_s^A\|_0\right) := \tilde{I}_3. \tag{9.3.73}$$

With the aid of (2.4.59), we have that

$$I_4 = \left\{\big(\frac{\partial E_\varepsilon}{\partial t}, \mathfrak{E}_{e,\varepsilon}^+\big)^h - \big(\frac{\partial E_\varepsilon}{\partial t}, \mathfrak{E}_{e,\varepsilon}^+\big)\right\} + \frac{\alpha}{\delta}\left\{\big(\frac{\partial S_\varepsilon}{\partial t}, \mathfrak{E}_{s,\varepsilon}^+\big)^h - \big(\frac{\partial S_\varepsilon}{\partial t}, \mathfrak{E}_{s,\varepsilon}^+\big)\right\}$$
$$+\frac{1}{\delta}\left\{(S_\varepsilon^+, \mathfrak{E}_{s,\varepsilon}^+)^h - (S_\varepsilon^+, \mathfrak{E}_{s,\varepsilon}^+)\right\}$$
$$\leq C\,h\left(\|\frac{\partial E_\varepsilon}{\partial t}\|_0|\mathfrak{E}_{e,\varepsilon}^+|_1 + \|\frac{\partial S_\varepsilon}{\partial t}\|_0|\mathfrak{E}_{s,\varepsilon}^+|_1 + \|S_\varepsilon^+\|_0|\mathfrak{E}_{s,\varepsilon}^+|_1\right) := \tilde{I}_4. \tag{9.3.74}$$

Noting the Cauchy-Schwarz inequality leads to

$$I_5 = \big(\nabla \mathfrak{e}_{e,\varepsilon}^+, \nabla \mathfrak{e}_s^A\big) - \big(e\nabla \mathfrak{e}_{s,\varepsilon}^+, \nabla \mathfrak{e}_e^A\big)$$
$$\leq C(|\mathfrak{e}_{e,\varepsilon}^+|_1|\mathfrak{e}_s^A|_1 + |\mathfrak{e}_{s,\varepsilon}^+|_1|\mathfrak{e}_e^A|_1) := \tilde{I}_5. \tag{9.3.75}$$

We also obtain from Young's inequality that

$$I_6 = \big((e-1)\nabla \mathfrak{e}_{s,\varepsilon}^+, \nabla \mathfrak{e}_{e,\varepsilon}^+\big) \leq \hat{\kappa}|\mathfrak{e}_{e,\varepsilon}^+|_1|\mathfrak{e}_{s,\varepsilon}^+|_1 \leq \frac{\delta\hat{\kappa}^2}{4}|\mathfrak{e}_{e,\varepsilon}^+|_1^2 + \frac{1}{\delta}|\mathfrak{e}_{s,\varepsilon}^+|_1^2 := \tilde{I}_6. \tag{9.3.76}$$

It follows from the Hölder's inequality, the last bound in (9.3.64), (2.4.69), (2.4.55), the Lipschitz continuity of ϕ_ε and (9.3.67) that

$$I_7 = \left([e - \Lambda_\varepsilon(E_\varepsilon^+)]\nabla S_\varepsilon^+, \nabla \mathfrak{E}_{e,\varepsilon}^+\right)$$
$$\leq |S_\varepsilon^+|_1\|\Lambda_\varepsilon(E_\varepsilon^+) - e\|_{0,\infty}|\mathfrak{E}_{e,\varepsilon}^+|_1$$
$$\leq C\|\Lambda_\varepsilon(E_\varepsilon^+) - e\|_{0,\infty}|\mathfrak{E}_{e,\varepsilon}^+|_1$$
$$\leq C\Big(\|\Lambda_\varepsilon(E_\varepsilon^+) - \phi_\varepsilon(E_\varepsilon^+)\|_{0,\infty} + \|\phi_\varepsilon(E_\varepsilon^+) - \phi_\varepsilon(e)\|_{0,\infty}$$
$$+\|\phi_\varepsilon(e) - \phi(e)\|_{0,\infty} + \|\phi(e) - e\|_{0,\infty}\Big)|\mathfrak{E}_{e,\varepsilon}^+|_1$$
$$\leq C\Big(h^{\frac{1}{2}}|E_\varepsilon^+|_1 + \|\mathfrak{e}_{e,\varepsilon}^+\|_{0,\infty} + \varepsilon\Big)|\mathfrak{E}_{e,\varepsilon}^+|_1$$
$$\leq C\Big(h^{\frac{1}{2}}\|E_\varepsilon^+\|_1 + \|\mathfrak{e}_{e,\varepsilon}^+\|_{0,\infty} + \varepsilon\Big)\big(|\mathfrak{e}_{e,\varepsilon}^+|_1 + |\mathfrak{e}_e^A|_1\big)$$
$$= \left[C\Big(h^{\frac{1}{2}}\|E_\varepsilon^+\|_1 + \varepsilon\Big)|\mathfrak{e}_{e,\varepsilon}^+|_1\right] + \left[C\|\mathfrak{e}_{e,\varepsilon}^+\|_{0,\infty}|\mathfrak{e}_{e,\varepsilon}^+|_1\right] + \left[C\Big(h^{\frac{1}{2}}\|E_\varepsilon^+\|_1 + \|\mathfrak{e}_{e,\varepsilon}^+\|_{0,\infty} + \varepsilon\Big)|\mathfrak{e}_e^A|_1\right]$$

$$:= \Big[I_{7,1} \Big] + \Big[I_{7,2} \Big] + \Big[I_{7,3} \Big]. \tag{9.3.77}$$

But, the Young's inequality gives, on making the assumption $\delta < \frac{4}{\hat{\kappa}^2}$, that

$$I_{7,1} \leq C\big(h \, \|E_\varepsilon^+\|_1^2 + \varepsilon^2\big) + \frac{4-\delta\hat{\kappa}^2}{8}\big|\mathfrak{e}_{e,\varepsilon}^+\big|_1^2. \tag{9.3.78}$$

We obtain from (2.1.4) and Young's inequality that

$$\begin{aligned}
I_{7,2} &= C\|\mathfrak{e}_{e,\varepsilon}^+\|_{0,\infty}|\mathfrak{e}_{e,\varepsilon}^+|_1 \\
&\leq C\|\mathfrak{e}_{e,\varepsilon}^+\|_0^{\frac{1}{2}} \ \|\mathfrak{e}_{e,\varepsilon}^+\|_1^{\frac{1}{2}} \ |\mathfrak{e}_{e,\varepsilon}^+|_1 \\
&\leq C\Bigg(\|\mathfrak{e}_{e,\varepsilon}^+\|_0 \ |\mathfrak{e}_{e,\varepsilon}^+|_1 + \|\mathfrak{e}_{e,\varepsilon}^+\|_0^{\frac{1}{2}} \ |\mathfrak{e}_{e,\varepsilon}^+|_1^{\frac{3}{2}} \Bigg) \\
&\leq C\|\mathfrak{e}_{e,\varepsilon}^+\|_0^2 + \frac{4-\delta\hat{\kappa}^2}{8}|\mathfrak{e}_{e,\varepsilon}^+|_1^2 \\
&\leq C\|\mathfrak{e}_{e,\varepsilon}\|_0^2 + C\|E_\varepsilon^+ - E_\varepsilon\|_0^2 + \frac{4-\delta\hat{\kappa}^2}{8}|\mathfrak{e}_{e,\varepsilon}^+|_1^2.
\end{aligned} \tag{9.3.79}$$

Noting the Cauchy-Schwarz inequality and Young's inequality leads to

$$\begin{aligned}
I_8 &= \frac{\mu}{\delta}\,(\mathfrak{e}_{e,\varepsilon}^+, \mathfrak{E}_{s,\varepsilon}^+) \leq \frac{\mu}{\delta}\,\|\mathfrak{e}_{e,\varepsilon}^+\|_0\|\mathfrak{E}_{s,\varepsilon}^+\|_0 \\
&\leq C\big(\|\mathfrak{e}_{e,\varepsilon}^+\|_0^2 + \|\mathfrak{E}_{s,\varepsilon}^+\|_0^2\big) \\
&\leq C\|\mathfrak{e}_{e,\varepsilon}\|_0^2 + C\|\mathfrak{e}_{s,\varepsilon}\|_0^2 + \tilde{I}_8,
\end{aligned} \tag{9.3.80}$$

where $\tilde{I}_8 := C\left(\|E_\varepsilon^+ - E_\varepsilon\|_0^2 + \|S_\varepsilon^+ - S_\varepsilon\|_0^2 + \|\mathfrak{e}_s^A\|_0^2\right)$. Finally, we use (2.4.59) and Young's inequality to obtain that

$$\begin{aligned}
I_9 &= \frac{\mu}{\delta}\Big\{ (E_\varepsilon^+, \mathfrak{E}_{s,\varepsilon}^+) - (E_\varepsilon^+, \mathfrak{E}_{s,\varepsilon}^+)^h \Big\} \\
&\leq C\,h\|E_\varepsilon^+\|_1\|\mathfrak{E}_{s,\varepsilon}^+\|_0 := \tilde{I}_9.
\end{aligned} \tag{9.3.81}$$

Now, combining (9.3.70)-(9.3.81) yields that

$$\frac{d}{dt}\Bigg(\|\mathfrak{e}_{e,\varepsilon}\|_0^2 + \frac{\alpha}{\delta}\|\mathfrak{e}_{s,\varepsilon}\|_0^2\Bigg) \leq C\Bigg(\|\mathfrak{e}_{e,\varepsilon}\|_0^2 + \frac{\alpha}{\delta}\|\mathfrak{e}_{s,\varepsilon}\|_0^2\Bigg) + \sum_{i=1}^{9}\tilde{I}_i, \tag{9.3.82}$$

where

$$\tilde{I}_6 := 0,$$

$$\tilde{I}_7 := I_{7,3} + C\big(h\,\|E_\varepsilon^+\|_1^2 + \varepsilon^2 + \|E_\varepsilon^+ - E_\varepsilon\|_0^2\big).$$

Applying the Grönwall lemma to (9.3.82) leads to for *a.e.* $t \in (0,T)$

$$\|\mathfrak{e}_{e,\varepsilon}(t)\|_0^2+\frac{\alpha}{\delta}\|\mathfrak{e}_{s,\varepsilon}(t)\|_0^2 \le e^{CT}\left(\|\mathfrak{e}_{e,\varepsilon}(0)\|_0^2+\frac{\alpha}{\delta}\|\mathfrak{e}_{s,\varepsilon}(0)\|_0^2\right)+e^{CT}\int_0^T\sum_{i=1}^{9}\tilde{I}_i dt. \quad (9.3.83)$$

To bound the right hand side of (9.3.83), the assumption $e^0, s^0 \in H^1(\Omega)$ and (2.4.56) that

$$\|\mathfrak{e}_{e,\varepsilon}(0)\|_0^2 \le \|e^0 - E_\varepsilon^0\|_0^2 \le Ch^2|e^0|_1^2 \le Ch^2, \quad (9.3.84)$$

$$\|\mathfrak{e}_{s,\varepsilon}(0)\|_0^2 \le \|s^0 - S_\varepsilon^0\|_0^2 \le Ch^2|s^0|_1^2 \le Ch^2. \quad (9.3.85)$$

We also use the estimate (2.4.56) to find that

$$\|\mathfrak{e}_e^A\|_0^2 = \|(I-\pi^h)e\|_0^2 \le Ch^2|e|_1^2, \quad (9.3.86)$$

$$\|\mathfrak{e}_s^A\|_0^2 = \|(I-\pi^h)s\|_0^2 \le Ch^2|s|_1^2. \quad (9.3.87)$$

Similarly to (8.3.80), we have from (9.3.64) that

$$\begin{aligned}&\|E_\varepsilon^\pm - E_\varepsilon\|_{L^2(\Omega_T)}^2 + \|S_\varepsilon^\pm - S_\varepsilon\|_{L^2(\Omega_T)}^2 \\ &\le (\Delta t)^2\|\frac{\partial E_\varepsilon}{\partial t}\|_{L^2(\Omega_T)}^2 + (\Delta t)^2\|\frac{\partial S_\varepsilon}{\partial t}\|_{L^2(\Omega_T)}^2 \le C(\Delta t)^2. \quad (9.3.88)\end{aligned}$$

On noting (9.3.67), (9.2.35), (9.3.64), and (2.4.56), we deduce that

$$\begin{aligned}&\left\|\mathfrak{E}_{e,\varepsilon}^+\right\|_{L^2(0,T;H^1(\Omega))} + \left\|\mathfrak{E}_{s,\varepsilon}^+\right\|_{L^2(0,T;H^1(\Omega))} \\ &\le \left\|\mathfrak{e}_{e,\varepsilon}^+\right\|_{L^2(0,T;H^1(\Omega))} + \left\|\mathfrak{e}_{s,\varepsilon}^+\right\|_{L^2(0,T;H^1(\Omega))} + \left\|\mathfrak{e}_e^A\right\|_{L^2(0,T;H^1(\Omega))} + \left\|\mathfrak{e}_s^A\right\|_{L^2(0,T;H^1(\Omega))}. \quad (9.3.89)\end{aligned}$$

Now, using Hölder's inequality, (9.3.64), (9.3.86), (9.3.87), (9.3.88) and (9.3.89), we can obtain the following estimates:

$$\begin{aligned}\int_0^T \tilde{I}_1 &\le C\,h\left(\|\frac{\partial \mathfrak{e}_{e,\varepsilon}}{\partial t}\|_{L^2(\Omega_T)}\|e\|_{L^2(0,T;H^1(\Omega))} + \|\frac{\partial \mathfrak{e}_{s,\varepsilon}}{\partial t}\|_{L^2(\Omega_T)}\|s\|_{L^2(0,T;H^1(\Omega))}\right) \\ &\le C\,h, \quad (9.3.90)\end{aligned}$$

$$\begin{aligned}\int_0^T \tilde{I}_2 &\le C\left(\|\frac{\partial \mathfrak{e}_{e,\varepsilon}}{\partial t}\|_{L^2(\Omega_T)}\|E_\varepsilon^+ - E_\varepsilon\|_{L^2(\Omega_T)} + \|\frac{\partial \mathfrak{e}_{s,\varepsilon}}{\partial t}\|_{L^2(\Omega_T)}\|S_\varepsilon^+ - S_\varepsilon\|_{L^2(\Omega_T)}\right) \\ &\le C\,\Delta t, \quad (9.3.91)\end{aligned}$$

$$\int_0^T \tilde{I}_3 \le C\left(\|\mathfrak{e}_{e,\varepsilon}^+\|_{L^2(0,T;H^1(\Omega))}\|\nabla \mathfrak{e}_e^A\|_{L^2(\Omega_T)}\right.$$

$$+\|\mathfrak{e}^+_{s,\varepsilon}\|_{L^2(0,T;H^1(\Omega))}\|\nabla\mathfrak{e}^A_s\|_{L^2(\Omega_T)} + Ch\|\mathfrak{e}^+_{s,\varepsilon}\|_{L^2(\Omega_T)}\|s\|_{L^2(0,T;H^1(\Omega))}\Big)$$

$$\leq C\left(\|\nabla\mathfrak{e}^A_e\|_{L^2(\Omega_T)} + \|\nabla\mathfrak{e}^A_s\|_{L^2(\Omega_T)} + h\right), \tag{9.3.92}$$

$$\int_0^T \tilde{I}_4 \leq C\,h\left(\|\frac{\partial E_\varepsilon}{\partial t}\|_{L^2(\Omega_T)}\|\mathfrak{E}^+_{e,\varepsilon}\|_{L^2(0,T;H^1(\Omega))}\right.$$

$$\left.+\|\frac{\partial S_\varepsilon}{\partial t}\|_{L^2(\Omega_T)}\|\mathfrak{E}^+_{s,\varepsilon}\|_{L^2(0,T;H^1(\Omega))} + \|S^+_\varepsilon\|_{L^2(\Omega_T)}\|\mathfrak{E}^+_{s,\varepsilon}\|_{L^2(0,T;H^1(\Omega))}\right)$$

$$\leq C\,h, \tag{9.3.93}$$

$$\int_0^T \tilde{I}_5 \leq C\Big(\|\mathfrak{e}^+_{e,\varepsilon}\|_{L^2(0,T;H^1(\Omega))}\|\nabla\mathfrak{e}^A_s\|_{L^2(\Omega_T)} + \|\mathfrak{e}^+_{s,\varepsilon}\|_{L^2(0,T;H^1(\Omega))}\|\nabla\mathfrak{e}^A_e\|_{L^2(\Omega_T)}\Big)$$

$$\leq C\left(\|\nabla\mathfrak{e}^A_s\|_{L^2(\Omega_T)} + \|\nabla\mathfrak{e}^A_e\|_{L^2(\Omega_T)}\right), \tag{9.3.94}$$

$$\int_0^T \tilde{I}_7 \leq C\left(h^{\frac{1}{2}}\|E^+_\varepsilon\|_{L^2(0,T;H^1(\Omega))} + \|\mathfrak{e}^+_{e,\varepsilon}\|_{L^2(0,T;L^\infty(\Omega))} + \varepsilon\right)\|\nabla\mathfrak{e}^A_e\|_{L^2(\Omega_T)}$$

$$+C\left(\|E^+_\varepsilon - E_\varepsilon\|^2_{L^2(\Omega_T)} + h\|E^+_\varepsilon\|^2_{L^2(0,T;H^1(\Omega))} + \varepsilon^2\right)$$

$$\leq C\left((\Delta t)^2 + h + \varepsilon^2 + \|\nabla\mathfrak{e}^A_e\|_{L^2(\Omega_T)}\right), \tag{9.3.95}$$

$$\int_0^T \tilde{I}_8 \leq C\left(\|E^+_\varepsilon - E_\varepsilon\|^2_{L^2(\Omega_T)} + \|S^+_\varepsilon - S_\varepsilon\|^2_{L^2(\Omega_T)} + Ch^2\|s\|^2_{L^2(0,T;H^1(\Omega))}\right)$$

$$\leq C\left((\Delta t)^2 + h^2\right), \tag{9.3.96}$$

$$\int_0^T \tilde{I}_9 \leq C\,h\|E^+_\varepsilon\|_{L^2(0,T;H^1(\Omega))}\|\mathfrak{E}^+_{s,\varepsilon}\|_{L^2(\Omega_T)} \leq C\,h. \tag{9.3.97}$$

Combining (9.3.83), (9.3.84) and (9.3.90)-(9.3.98) yields for $h, \Delta t \leq 1$, and for $a.e.\ t \in (0,T)$ that

$$\|\mathfrak{e}_{e,\varepsilon}\|^2_0 + \|\mathfrak{e}_{s,\varepsilon}\|^2_0 \leq C\big(h + h^2 + \Delta t + (\Delta t)^2 + \varepsilon^2 + \|\nabla\mathfrak{e}^A_e\|_{L^2(\Omega_T)} + \|\nabla\mathfrak{e}^A_s\|_{L^2(\Omega_T)}\big)$$

$$\leq C\big(h + \Delta t + \varepsilon^2 + \|\nabla\mathfrak{e}^A_e\|_{L^2(\Omega_T)} + \|\nabla\mathfrak{e}^A_s\|_{L^2(\Omega_T)}\big). \tag{9.3.98}$$

This gives the estimate (9.3.66).

If $e, s \in L^2(0,T;H^2(\Omega))$, the result (9.3.66) follows immediately from (9.3.65) on noting the following estimate (see Theorem 3.1.6 in Ciarlet [39]):

$$|(I - \pi^h)\eta|_1 \leq C\,h\,|\eta|_2.$$

□

Chapter 10

The Keller-Segel Model: Numerical experiments

This chapter is devoted to the discussion of some numerical experiments for the model (Q). We introduce an iterative approach to solve our fully discrete finite element approximation to problem (Q). We then establish and discuss some numerical solutions for different choices of the parameters α, δ, μ. We also introduce a modified iterative scheme to obtain the numerical solutions. In addition, we obtain and discuss some other numerical results. All programs were written in Matlab to generate the numerical results and to plot the graphs.

We could find no two-dimensional examples to compare our computations with for $\delta > 0$. However, in the case that $\delta = 0$ the continuous and numerical solution blow up, see references. It should be noted that our entropy bound is not valid in this case but that the numerical approximation still works up to the point of blow-up. We include this simulation to demonstrate the robustness of the approximation. We performed the same experiment with other values of $\delta > 0$ and found that blow up did not occur.

We first introduce the following practical algorithm to solve the nonlinear algebraic system arising from the approximate problem $(\mathrm{Q}_{M,\varepsilon}^{h\,\Delta t,k})$ at each time level:

Given $\{E_\varepsilon^{n,0}, S_\varepsilon^{n,0}\} \in S^h \times S^h$ for $k \geq 1$ find $\{E_\varepsilon^{n,k}, S_\varepsilon^{n,k}\} \in S^h \times S^h$ such that for all $\chi \in S^h$

$$\Big(\frac{E_\varepsilon^{n,k}-E_\varepsilon^{n-1}}{\Delta t},\chi\Big)^h+(\rho\nabla E_\varepsilon^{n,k}-\varrho\Lambda_\varepsilon(E_\varepsilon^{n,k-1})\nabla S_\varepsilon^{n,k},\nabla\chi)=0, \tag{10.0.1}$$

$$\alpha\Big(\frac{S_\varepsilon^{n,k}-S_\varepsilon^{n-1}}{\Delta t},\chi\Big)^h+(S_\varepsilon^{n,k},\chi)^h+(\nabla S_\varepsilon^{n,k},\nabla\chi)+\delta(\nabla E_\varepsilon^{n,k},\nabla\chi)=\mu(E_\varepsilon^{n,k},\chi)^h, \tag{10.0.2}$$

where the coefficients $\rho, \varrho > 0$ have been added to compare with experiments elsewhere in the literature. We note that it is easy to prove all of the results that we have proved, i.e., all of the previous results hold with this modified model. We start with $E_\varepsilon^0 \equiv \pi^h e^0$ and $S_\varepsilon^0 \equiv \pi^h s^0$ and we set, for $n \geq 1$, $E_\varepsilon^{n,0} \equiv E_\varepsilon^{n-1}$ and $S_\varepsilon^{n,0} \equiv S_\varepsilon^{n-1}$. As the system (10.0.1)-(10.0.2) is linear, existence of $\{E_\varepsilon^{n,k}, S_\varepsilon^{n,k}\}$ follows from uniqueness. The standard method to solve the system (10.0.1)-(10.0.2) at each iteration is by testing the equations (10.0.1) and (10.0.2) with $\varphi_j, j = 0, ..., J$, to obtain a $(2J+2)\times(2J+2)$ linear system, in terms of the nodal values of $E_\varepsilon^{n,k}$ and $S_\varepsilon^{n,k}$, which can be solved using linear programming. For our numerical results, we set $TOL = 10^{-6}$ and adopt the stopping criteria

$$|E_\varepsilon^{n,k}-E_\varepsilon^{n,k-1}|_{0,\infty} < TOL \quad \text{and} \quad |S_\varepsilon^{n,k}-S_\varepsilon^{n,k-1}|_{0,\infty} < TOL, \tag{10.0.3}$$

i.e. for k satisfying (10.0.3) we set $E_\varepsilon^n \equiv E_\varepsilon^{n,k}$ and $S_\varepsilon^n \equiv S_\varepsilon^{n,k}$.

Although, we have been unable to prove convergence of $\{E^{n,k}, S^{n,k}\}_{k=1}^\infty$ to $\{E^n, S^n\}$ for n fixed, good convergence properties have been observed in practice. We found that the iterative method always converged well (only a few steps were required to fulfill the stopping criteria at each time level).

As already mentioned, the system is square so proving uniqueness is equivalent to existence. If we attempt to adopt the existence argument in the Schauder fixed point theorem, then unfortunately we are left with the extra term

$$-\Delta t\bigg(\big[\Lambda_\varepsilon(E_\varepsilon^{n,k})\big]^{-1}\Lambda_\varepsilon(E_\varepsilon^{n,k-1})\nabla S_\varepsilon^{n,k},\nabla\chi\bigg),$$

which we are unable to deal with. Next we prove uniqueness directly, which depends on Δt being sufficiently small. In practice, if we found that the iteration did not convergence, then our strategy would to be reduce Δt by a factor of $1/2$ and to repeat the experiment.

Theorem 10.0.2 Let $\{E_\varepsilon^{n,k}, S_\varepsilon^{n,k}\}$ be a solution of the problem $(\mathrm{Q}_{M,\varepsilon}^{h\,\Delta t,k})$ such that

$$\max_{n,k} \|S_\varepsilon^{n,k}\|_0^2 \leq C_b,$$

where C_b is a positive constant independent of the parameters $h, \Delta t$ and ε. Then, for sufficiently small Δt, the solution $\{E_\varepsilon^{n,k}, S_\varepsilon^{n,k}\}, n = 1, ..., N$ is unique.

Proof: Assume there are two solutions $\{E_{\varepsilon,1}^{n,k}, S_{\varepsilon,1}^{n,k}\}$ and $\{E_{\varepsilon,2}^{n,k}, S_{\varepsilon,2}^{n,k}\}$ to the problem $(Q_{M,\varepsilon}^{h,\Delta t,k})$ such that

$$\max_{n,k} \left\{\|S_{\varepsilon,1}^{n,k}\|_0^2, \|S_{\varepsilon,2}^{n,k}\|_0^2\right\} \leq C_b. \tag{10.0.4}$$

Now, setting $\mathcal{E}_\varepsilon^{n,k} = E_{\varepsilon,1}^{n,k} - E_{\varepsilon,2}^{n,k}$ and $\mathcal{S}_\varepsilon^{n,k} = S_{\varepsilon,1}^{n,k} - S_{\varepsilon,2}^{n,k}$, and subtracting the fully discrete approximations yields for all $\chi \in S^h$ that

$$\frac{1}{\Delta t}\big(\mathcal{E}_\varepsilon^{n,k}, \chi\big)^h + (\nabla \mathcal{E}_\varepsilon^{n,k}, \nabla\chi) = (\Lambda_\varepsilon(E_\varepsilon^{n,k-1})\nabla \mathcal{S}_\varepsilon^{n,k}, \nabla\chi), \tag{10.0.5}$$

$$\frac{\alpha}{\Delta t}\big(\mathcal{S}_\varepsilon^{n,k}, \chi\big)^h + (\mathcal{S}_\varepsilon^{n,k}, \chi)^h + (\nabla \mathcal{S}_\varepsilon^{n,k}, \nabla\chi) + \delta(\nabla \mathcal{E}_\varepsilon^{n,k}, \nabla\chi) = \mu(\mathcal{E}_\varepsilon^{n,k}, \chi)^h. \tag{10.0.6}$$

Choosing $\chi \equiv \mathcal{E}_\varepsilon^{n,k}$ in (10.0.5) and $\chi \equiv \frac{1}{\delta}\mathcal{S}_\varepsilon^{n,k}$ in (10.0.6) and adding the resulting equations yields, on using the Hölder's inequality, (2.4.68), (2.4.69), (2.4.54) and (10.0.4), that

$$\begin{aligned}
&\frac{1}{\Delta t}|\mathcal{E}_\varepsilon^{n,k}|_h^2 + |\mathcal{E}_\varepsilon^{n,k}|_1^2 + \frac{\alpha}{\delta\Delta t}|\mathcal{S}_\varepsilon^{n,k}|_h^2 + \frac{1}{\delta}|\mathcal{S}_\varepsilon^{n,k}|_h^2 + \frac{1}{\delta}|\mathcal{S}_\varepsilon^{n,k}|_1^2 \\
&= (\Lambda_\varepsilon(E_\varepsilon^{n,k-1})\nabla \mathcal{S}_\varepsilon^{n,k}, \nabla \mathcal{E}_\varepsilon^{n,k}) + \frac{\mu}{\delta}(\mathcal{E}_\varepsilon^{n,k}, \mathcal{S}_\varepsilon^{n,k})^h - (\nabla \mathcal{E}_\varepsilon^{n,k}, \nabla \mathcal{S}_\varepsilon^{n,k}) \\
&= \big([\Lambda_\varepsilon(E_{\varepsilon,1}^{n,k-1}) - 1]\nabla \mathcal{E}_\varepsilon^{n,k}, \nabla \mathcal{S}_\varepsilon^{n,k}\big) + \frac{\mu}{\delta}(\mathcal{E}_\varepsilon^{n,k}, \mathcal{S}_\varepsilon^{n,k})^h \\
&\leq C_1|\mathcal{E}_\varepsilon^{n,k}|_1\ |\mathcal{S}_\varepsilon^{n,k}|_1 + \frac{\mu}{\delta}|\mathcal{E}_\varepsilon^{n,k}|_h\ |\mathcal{S}_\varepsilon^{n,k}|_h, := I_1 + I_2,
\end{aligned} \tag{10.0.7}$$

where

$$I_1 = C_1|\mathcal{E}_\varepsilon^{n,k}|_1\ |\mathcal{S}_\varepsilon^{n,k}|_1,$$

$$I_2 = \frac{\mu}{\delta}|\mathcal{E}_\varepsilon^{n,k}|_h\ |\mathcal{S}_\varepsilon^{n,k}|_h,$$

and C_1 is a positive constant, independent of the parameters $h, \Delta t$ and ε, that is generated from applying (2.4.54).

It follows from the Young's inequality, (2.4.54) and (2.4.55) that

$$I_1 \leq |\mathcal{E}_\varepsilon^{n,k}|_1^2 + \frac{C_1^2}{4}|\mathcal{S}_\varepsilon^{n,k}|_1^2 \leq |\mathcal{E}_\varepsilon^{n,k}|_1^2 + \frac{C_2^2C_1^2}{4h^2}|\mathcal{S}_\varepsilon^{n,k}|_0^2 = |\mathcal{E}_\varepsilon^{n,k}|_1^2 + a_1|\mathcal{S}_\varepsilon^{n,k}|_h^2, \tag{10.0.8}$$

$$I_2 \leq a_2|\mathcal{E}_\varepsilon^{n,k}|_h^2 + \frac{1}{\delta}|\mathcal{S}_\varepsilon^{n,k}|_h^2, \tag{10.0.9}$$

where C_2 is the positive constant, independent of $h, \Delta t$ and ε, generated from applying (2.4.55), $a_1 = \frac{C_2^2 C_1^2}{4h^2}$ and $a_2 = \frac{\mu^2}{4\delta}$. Combining (10.0.7) and (10.0.8)-(10.0.9) yields on noting the equivalence (2.4.46) that

$$\big(\frac{1}{\Delta t} - a_2\big)|\mathcal{E}_\varepsilon^{n,k}|_h^2 + \big(\frac{\alpha}{\delta \Delta t} - a_1\big)|\mathcal{S}_\varepsilon^{n,k}|_h^2 \leq 0. \tag{10.0.10}$$

Now, we set

$$\tau = \min\{\frac{\alpha}{\delta a_1}, \frac{1}{a_2}\}.$$

On noting (10.0.10), we obtain for any $\Delta t \in (0, \tau)$ that

$$|\mathcal{E}_\varepsilon^{n,k}|_h^2 + |\mathcal{S}_\varepsilon^{n,k}|_h^2 \leq 0.$$

We thus conclude $E_{\varepsilon,1}^{n,k} = E_{\varepsilon,2}^{n,k}$ and $S_{\varepsilon,1}^{n,k} = S_{\varepsilon,2}^{n,k}$, as required. □

10.1 Numerical results

10.1.1 1D numerics

We now present some numerical results in one space dimension. Unless otherwise specified, in all experiments we consider a uniform partitioning of $\Omega = (0, 1)$ into 100 subintervals, i.e. $J = 100$ and $h = 1/100$, and choose $\Delta t = 0.001$, $n \geq 1$, and $\varepsilon = 10^{-9}$. In the first part of our experiments, we considered the initial data $e^0(x) = 1$, and $s^0(x) = 1 + 0.1e^{-10x^2}$, which was also considered in [60], with $\alpha = 1$, $\delta = 0$, $\mu = 1$, $\varrho = 5$ and $\rho = 0.1$.

In Figure 10.1 we plot numerical simulations of the Keller-Segel model. The cell density and chemical concentration are plotted at distinct times, showing the growth of the solution as cells accumulate into a sharp boundary peak. After $t = 1$, the figures do not change significantly.

We note that the steady-state solution of (Q) in space and time, denoted by $\{e_c, s_c\}$, is determined by the following equations

$$[e_x - (es_x)]_x = 0, \qquad s_{xx} + \delta e_{xx} + \mu e - s = 0.$$

In the next experiments, we considered the same initial data of the first experiment, with $\alpha = 1$ and $\rho = 0.1$. For $\delta = 0$, $\mu = 0$ and $\varrho = 5$, if $\{e_c, s_c\}$ is a constant steady state solution, then it is easy to show that $(e_c(x), 1) = \frac{1}{|\Omega|}(e^0, 1)$ and $s_c(x) = 0$, and this behaviour has been shown in Figure 10.2. In the second experiment we choose the same parameters of first one but with $e^0(x) = 2$. The solutions corresponding to $e^0(x) = 2$ are plotted in Figure 10.3 at several times, and the results show the same behaviour of Figure 10.2.

Thirdly, we choose $\delta = 0.1$ and $\varrho = 0$, then if $\{e_c, s_c\}$ is a constant steady state solution, then we have $e_c = \frac{1}{|\Omega|}(e^0, 1)$ and $s_c = \mu e_c$. Firstly, we choose $e^0(x) = 1$, and the solutions corresponding to this experiment have been shown at many time levels in Figures 10.4 for $\mu = 0.5$. We repeated the experiment with $e^0(x) = 1$ and $\mu = 0, 3$ and $e^0(x) = 2$ and $\mu = 0, 0.5, 3$, and we found $\{e_c, s_c\}$ always satisfied $s_c - \mu e_c = 0$, within tolerance. Finally, in Figure 10.5 we plot the term $\rho e_x - \varrho(e s_x)$, for $e^0(x) = 1$, $s^0(x) = 1 + 0.1e^{-10x^2}$, $\alpha = 1$, $\delta = 0$, $\mu = 1$, $\varrho = 1$ and $\rho = 0.1$. It is very clear in Figure 10.5 that the term $e_x - (e s_x)$ has a constant value in the steady state solutions.

10.1.2 2D numerics

In this section, we demonstrate the performance of the proposed finite element scheme in two dimensions for the Keller-Segel model. We take the computational domain to be a square uniform mesh $= [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$ with $\alpha = 1$, $\delta = 0$, $\mu = 1$, $\varrho = 1$, $\rho = 1$. The space step is $h = 1/J$ in both x and y directions where $J + 1$ is the number of the nodes in each direction. Then, we apply a right-angled triangulation on Ω in which each subsquare is bisected by its north-east diagonal. We first consider the initial-boundary value problem for the Keller-Segel system with the radially symmetric bell-shaped initial data,

$$e^0(x, y, 0) = 1000e^{-100(x^2+y^2)}, \qquad s^0(x, y, 0) = 500e^{-50(x^2+y^2)}. \tag{10.1.11}$$

According to the results in [59], both $e-$ and $s-$components of the solution are expected to blow up at the origin in finite time. This situation is especially challenging since capturing blow up solutions with shrinking support is extremely hard [59]. We first apply the finite element method to the initial-boundary value problem (10.0.1)-(10.0.2). The computed cell densities at times $T = 10^{-6}, 5\times10^{-6}, 4.4\times10^{-5}, 6\times10^{-5}$

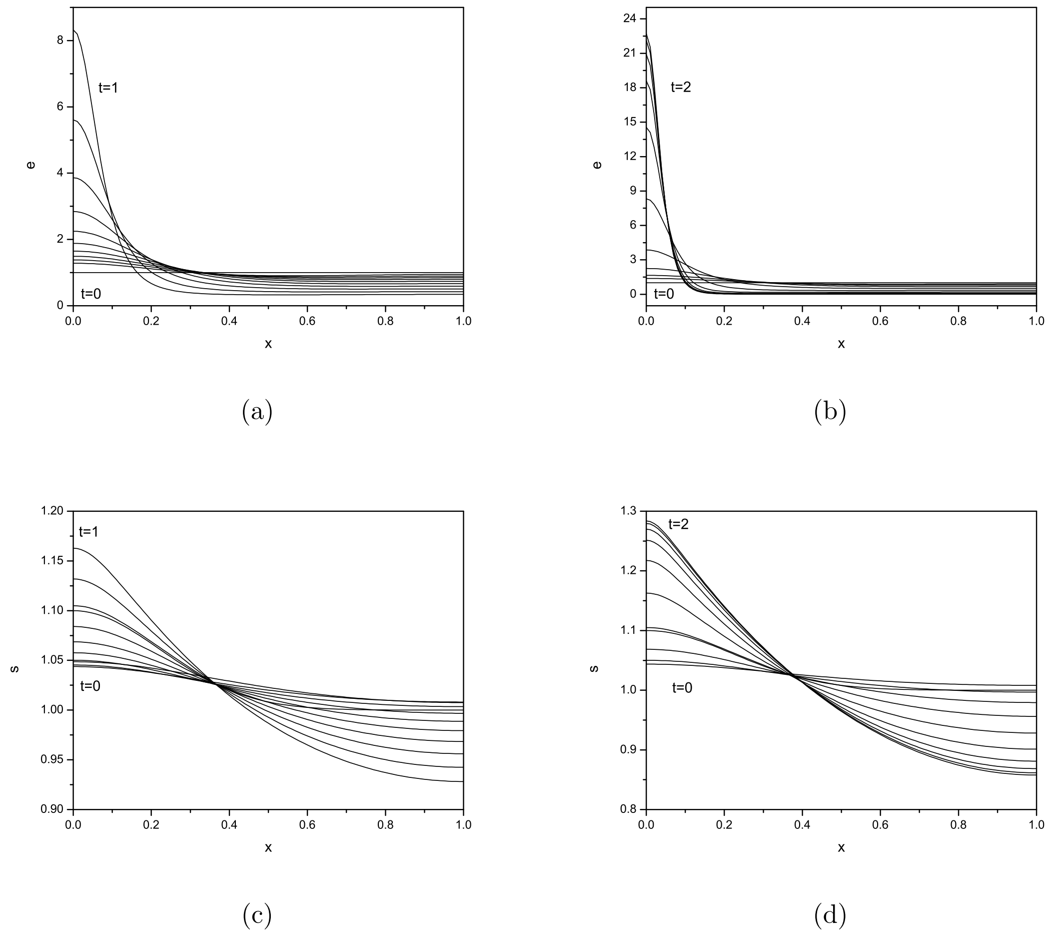
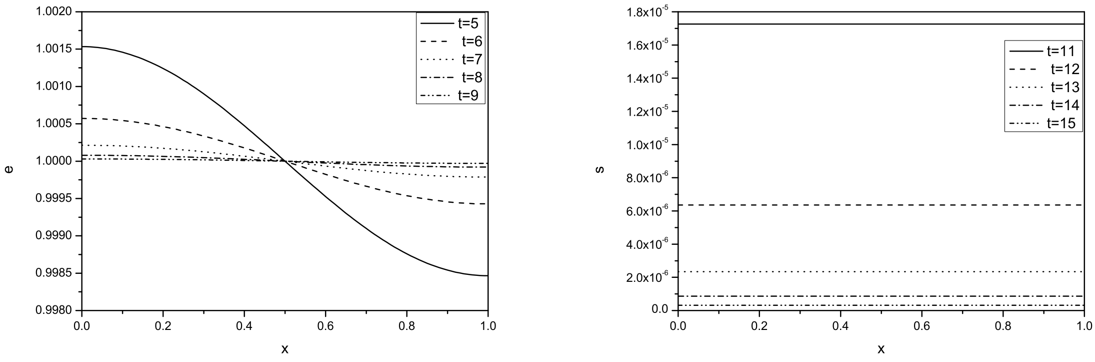
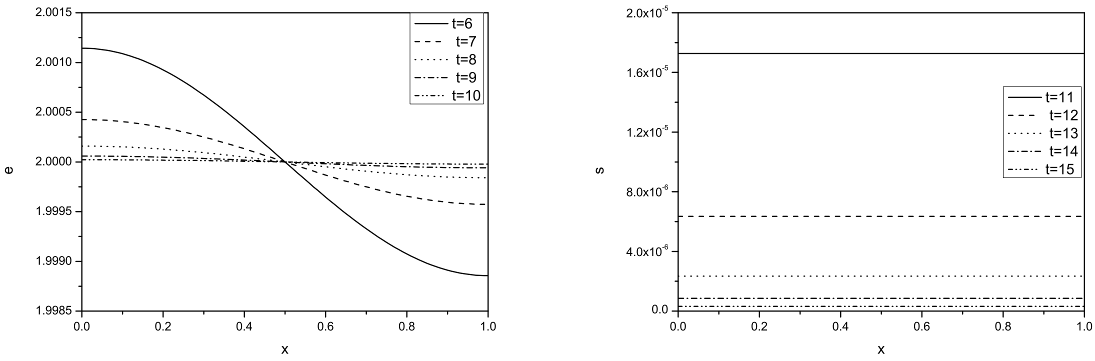


Figure 10.1: The cell density $e(\mathbf{x}, t)$ and the concentration of the chemical signal $s(\mathbf{x}, t)$ versus position with $\Delta t = 0.001$, $\alpha = 1$, $\delta = 0$, $\mu = 1$, $\varrho = 5$ and $\rho = 0.1$. The initial data are $e^0(x) = 1$, $s^0(x) = 1 + 0.1e^{-10x^2}$. In (a) and (c) we plot $e\,\&\,s$ for $t = 0, 0.1, ..., 1$, respectively, while in (b) and (d) we plot $e\,\&\,s$ for $t = 0, 0.2, ..., 2$, respectively.



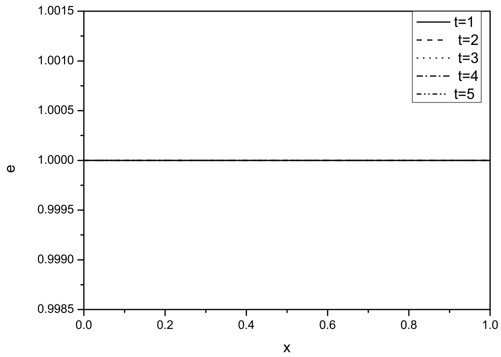
(a) (b)

Figure 10.2: The cell density $e(\mathbf{x}, t)$ and the concentration of the chemical signal $s(\mathbf{x}, t)$ versus position with $\Delta t = 0.001$, $\alpha = 1$, $\delta = 0$, $\mu = 0$, $\varrho = 5$ and $\rho = 0.1$. The initial data are $e^0(x) = 1, \;\; s^0(x) = 1 + 0.1e^{-10x^2}$.

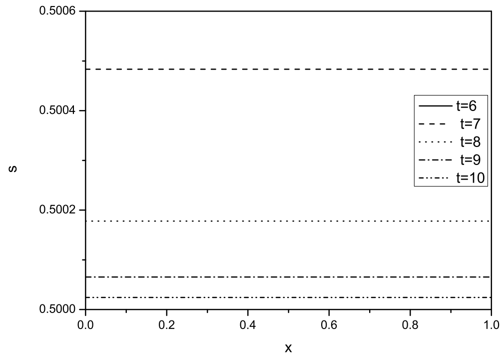


(a) (b)

Figure 10.3: The cell density $e(\mathbf{x}, t)$ and the concentration of the chemical signal $s(\mathbf{x}, t)$ versus position with $\Delta t = 0.001$, $\alpha = 1$, $\delta = 0$, $\mu = 0$, $\varrho = 5$ and $\rho = 0.1$. The initial data are $e^0(x) = 2, \;\; s^0(x) = 1 + 0.1e^{-10x^2}$.



(a)



(b)

Figure 10.4: The cell density $e(\mathbf{x}, t)$ and the concentration of the chemical signal $s(\mathbf{x}, t)$ versus position with $\Delta t = 0.001$, $\alpha = 1$, $\delta = 0.1$, $\mu = 0.5$, $\varrho = 0$ and $\rho = 0.1$. The initial data are $e^0(x) = 1, \quad s^0(x) = 1 + 0.1e^{-10x^2}$.

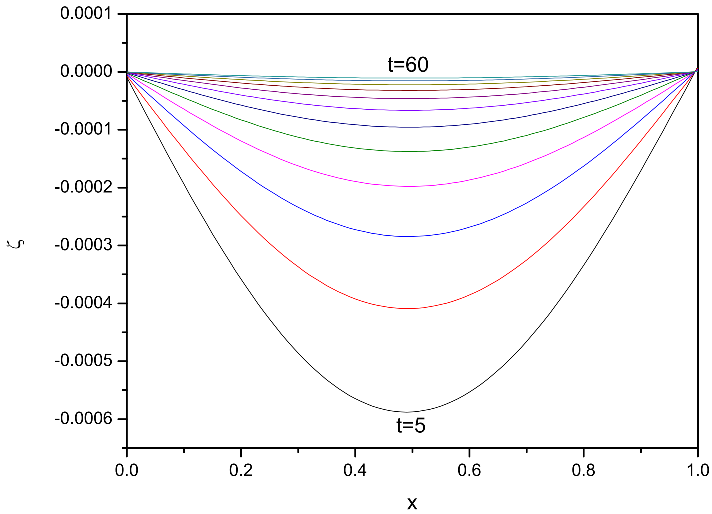
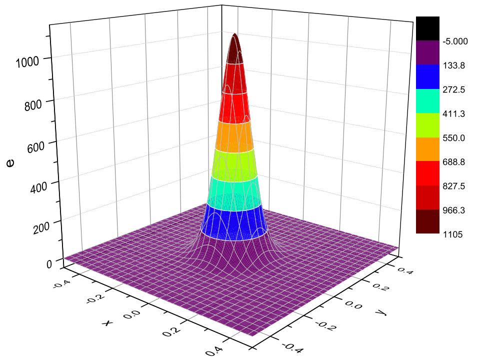


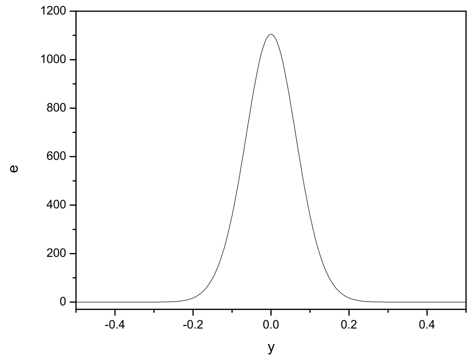
Figure 10.5: The term $\rho e_x - \varrho(es_x)$ versus position with $\Delta t = 0.001$, $\alpha = 1$, $\delta = 0$, $\mu = 1$, $\varrho = 1$ and $\rho = 0.1$. The initial data are $e^0(x) = 1, \quad s^0(x) = 1 + 0.1e^{-10x^2}$. $\zeta = DE^n_\varepsilon - E^n_\varepsilon DS^n_\varepsilon$, where $Dy = (y_{i+1} - y_i)/h, i = 0, ..., J$. In this Figure, we plot for $t = 5, 10, ..., 60$, respectively.

and 10^{-4} are plotted in Figures 10.6, 10.7, 10.8, 10.9 and 10.10, respectively, with $\Delta t = 10^{-7}$. The method performs reasonably well, where we only show the plots with $J = 200$ and $J = 400$ as with other finer grid spacing the plots were quantitatively similar.

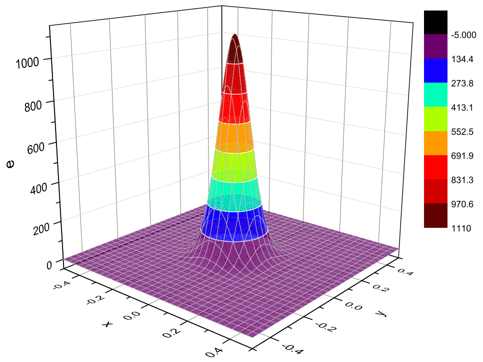
In the results in [38], negative densities appear in numerical solutions which refer to the severe numerical instabilities. We observe a lack of negative values of e or any other numerical instabilities in these experiments, and a high resolution of the solution blowing up. Numerical convergence of the finite element method is verified by running the same test on a finer grid with $h = 0.005$, where we observed that the coarse and the fine grid solutions were in very good agreement at small times $T = 10^{-6}, 5\times10^{-6}$. However, they are quite different at a larger time $T = 4.4\times10^{-5}$, and especially at $T = 6\times10^{-5}$. Therefore, we further refine the grid on the uniform grid with $h = 0.0025$. It seems that these is agreement in the computed solutions at $T = 4.4\times10^{-5}$, but beyond that time there is a difference that keeps increasing (as the grid is refined) at $T = 6\times10^{-5}$. A more precise interpretation of the obtained results would require a knowledge of the blowup time (which is not available). Based on the presented numerical results, we conjecture that the blowup time is most likely $T > 6\times10^{-5}$.



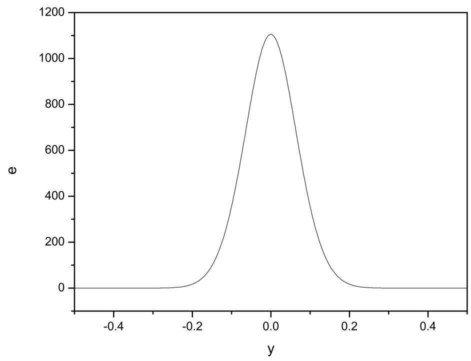
(a)



(b)



(c)



(d)

Figure 10.6: The cell density $e(\mathbf{x}, t)$ at $T = 10^{-6}$ (left) and its one-dimensional (1D) slice along $x = 0$ (right), with $\Delta t = 10^{-7}$. In (a) and (b), $h = 0.005$, in (c) and (d), $h = 0.0025$.

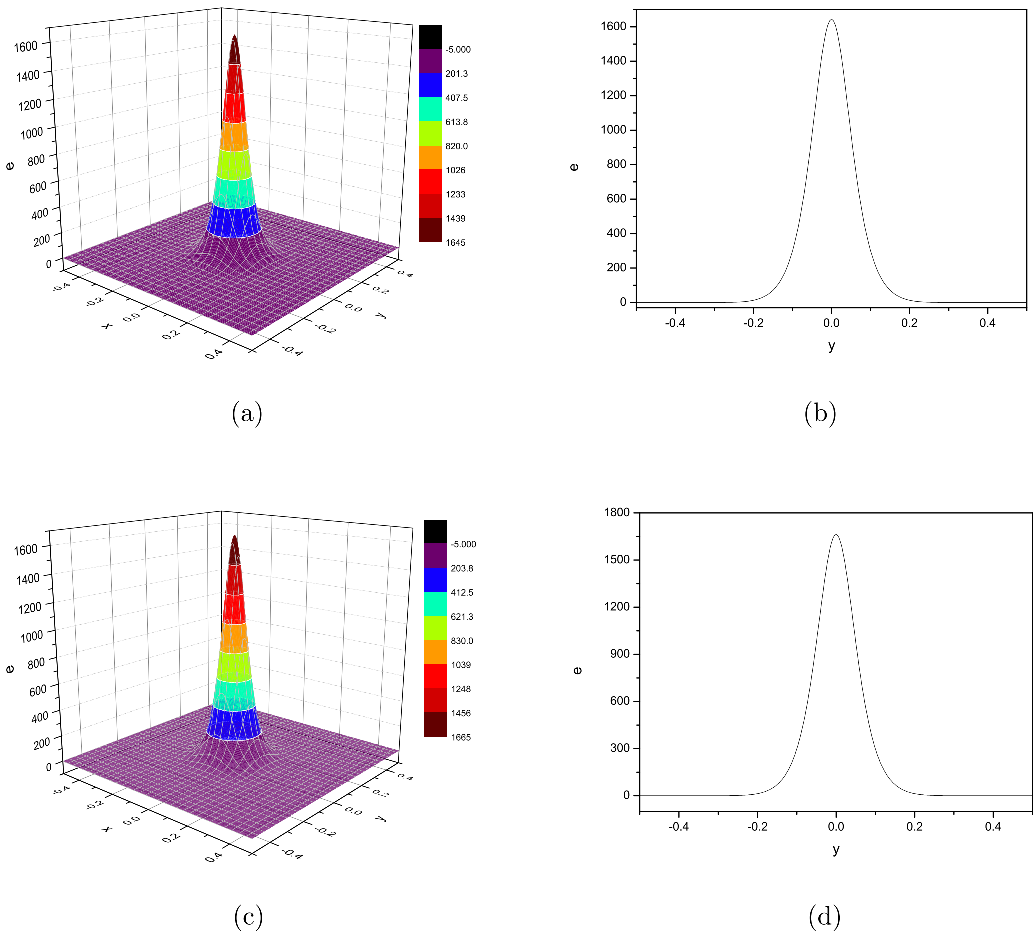
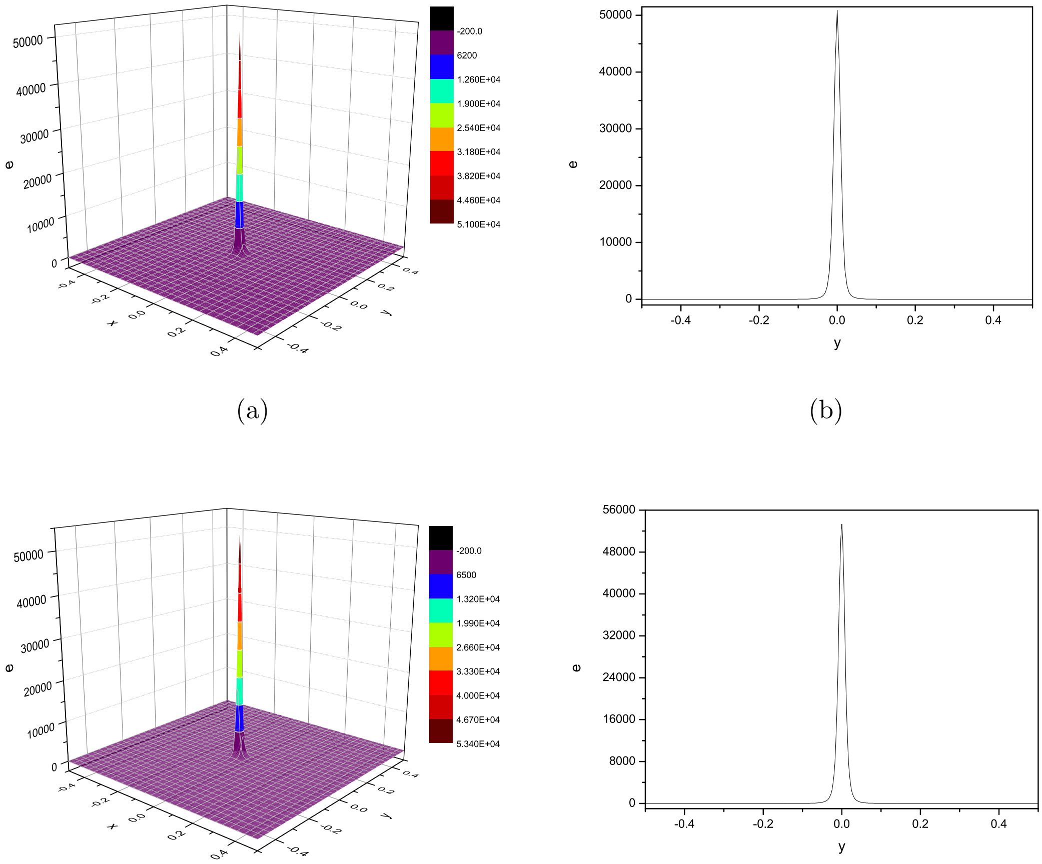


Figure 10.7: The cell density $e(\mathbf{x}, t)$ at $T = 5 \times 10^{-6}$ (left) and its one-dimensional (1D) slice along $x = 0$ (right), with $\Delta t = 10^{-7}$. In (a) and (b), $h = 0.005$, in (c) and (d), $h = 0.0025$.



(a) (b)

(c) (d)

Figure 10.8: The cell density $e(\mathbf{x}, t)$ at $T = 4.4 \times 10^{-5}$ (left) and its one-dimensional (1D) slice along $x = 0$ (right), with $\Delta t = 10^{-7}$. In (a) and (b), $h = 0.005$, in (c) and (d), $h = 0.0025$.

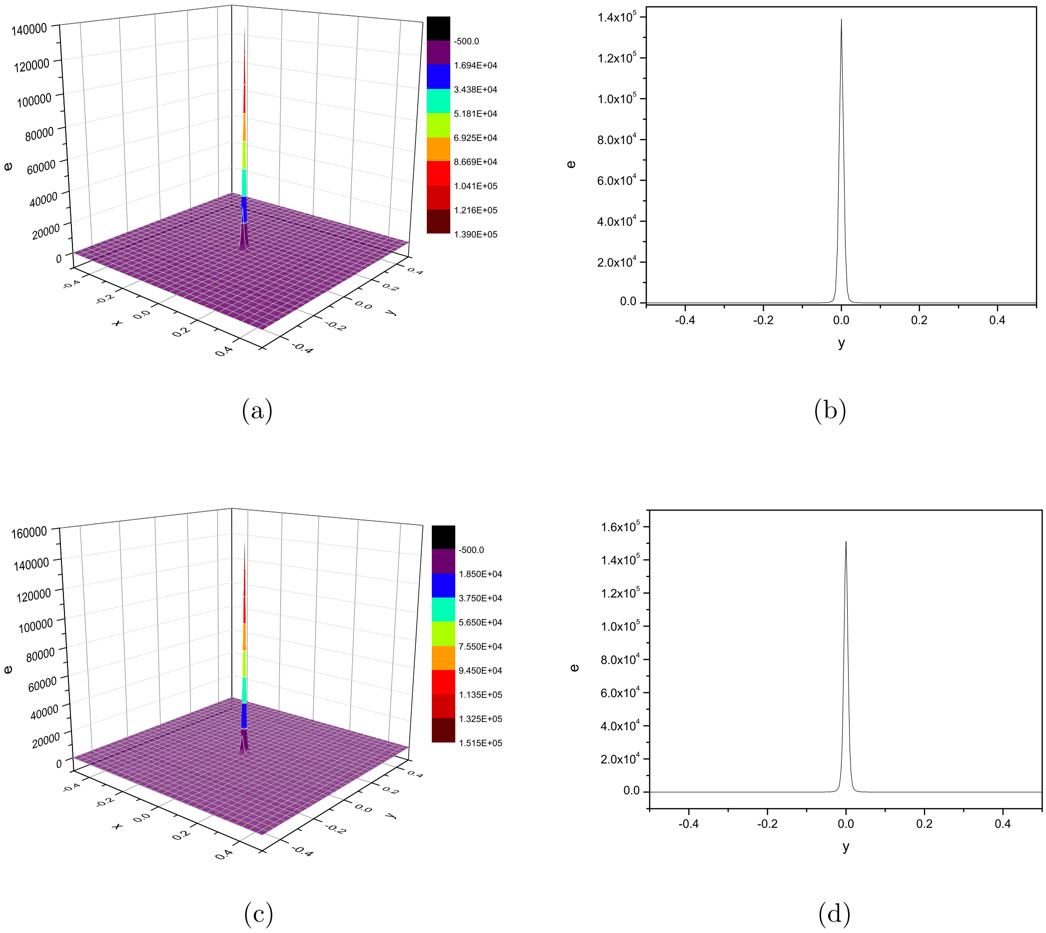
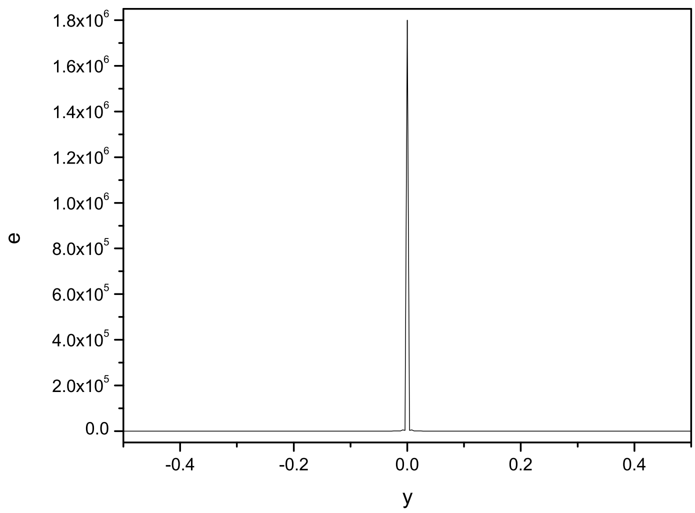
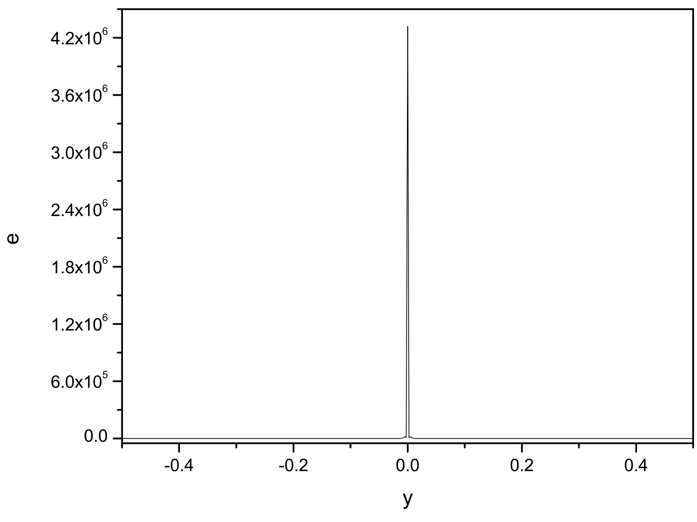


Figure 10.9: The cell density $e(\mathbf{x}, t)$ at $T = 6 \times 10^{-5}$ (left) and its one-dimensional (1D) slice along $x = 0$ (right), with $\Delta t = 10^{-7}$. In (a) and (b), $h = 0.005$, in (c) and (d), $h = 0.0025$.



(a)



(b)

Figure 10.10: The one-dimensional (1D) slice along $x = 0$ of cell density $e(\mathbf{x}, t)$ at $T = 10^{-4}$ and (a) $h = 0.005$, (b) $h = 0.0025$.

Chapter 11

Conclusions

We studied three cross diffusion systems using the finite element method. The first system, (P), is a population model which represents the movement of multi interacting cell populations in $d \leq 3$ space dimensions. The second system, (W), models mechanical tumor-growth. Finally, a Keller-Segel model (Q) with an additional cross-diffusion term in the equation for the chemical signal is analyzed. In the first chapter of the thesis we introduced the models (P), (W) and (Q) and defined the research objectives. Our study of the model (P) was executed in the following four chapters. Also, the model (W) was studied in Chapter 6 and the rest of the thesis was devoted to the study of the model (Q).

It is important to note that the cut-off function $\phi(s)$ and the entropy function F are closely related, viz. $\phi(s) = \min(1/(\mathcal{F}^M)''(s), M)$, see (2.3.22), and this connection plays a crucial role in our argument. Due to the fact that $(\mathcal{F}^M)''(s)$ is unbounded at $s = 0$, the strictly convex entropy function $\mathcal{F}^M$ is replaced by a strictly convex regularization F_ε whose second derivative is bounded above by $1/\varepsilon$ and bounded below by $1/M$, $\varepsilon \in (0, 1)$, $M > 1$, at the same time the cut-off function ϕ is replaced by a strictly positive cut-off function ϕ_ε defined by $\phi_\varepsilon(s) = 1/F''_\varepsilon(s)$.

In Chapter 2, we make a significant step towards showing the existence of a global in-time weak solution of the problem (P). Our approach in proving existence is based on the idea of defining an entropy inequality that leads us to obtain energy estimates. Firstly, we introduce a truncated alternative problem to (P). Then, we introduce a regularized problem of the problem (P). Next, we derive a well defined entropy inequality of the regularized problem. Also, A practical fully discrete finite element approximation of the regularized problem is proposed then we present some necessary lemmata. Finally, the existence of the approximate solutions are discussed by using a fixed point theorem.

In Chapter 3, we prove the existence of a global weak solution to the system $(\mathrm{P}_M^{\Delta t})$ by analysing the convergence of the fully discrete approximate problem $(\mathrm{P}_{M,\varepsilon}^{h,\Delta t})$. A discrete analogue of the entropy inequality is derived and some stability bounds on the approximate solution are shown. Then we prove the existence of non-negative functions $\{U_i\}_{i=1}^m$ bounded in various time-dependent spaces using classical sequential compactness arguments. Finally, we prove that the functions $\{U_i\}_{i=1}^m$ represent a global weak solution of the system $(\mathrm{P}_M^{\Delta t})$ via passage to the limit $\varepsilon, h \to 0$ of the approximate system.

In Chapter 4, to show the existence of weak solutions to the model (P), that demands passing to the limits, $\Delta t \to 0^+$ and $M \to \infty$. Then we link the time step Δt to the cutoff parameter $M > 1$ by demanding that $\Delta t = o(M^{-1})$, as $M \to \infty$, so that the only parameter in the problem $(\mathrm{P}_M^{\Delta t})$ is the cutoff parameter. By using special energy estimates, we show that the solutions can be bounded, independent of M. We then use these $M-$independent bounds on the relative entropy to derive $M-$independent bounds on the time-derivatives. By using sequential compactness arguments, the convergence of the finite element approximate problem has been studied and existence of a non-negative weak solution for (P) was concluded. We also might be able to find the error estimate by adapting the ideas in Barrett and Blowey [7]. We leave this for future investigation. A regularity result stronger than we obtained is required to complete the analysis of problem $(\mathrm{P}_{\Delta t}^{M})$. However, in order to proceed with the convergence analysis we adopted an alternative technique where we assumed that $U_i^{\pm}(\mathbf{x}, t) \in L^\infty(\Omega_T)$.

At the end of our study, in Chapter 5, an algorithm for computing the numerical solutions of the population model (P) was given. Simulations in one and two space dimensions were performed using the implicit scheme. Numerically, there are remaining issues that can be investigated such as existence, uniqueness and error bounds. We were unable to numerically verify the fully discrete error bound for (P) because no exact solution is known. However, experimental work that can be done in this direction is by comparing the computed solution on a coarse mesh with that on a fine mesh.

In Chapter 6 we introduce a fully discrete finite element approximation for the cross-diffusion Tumor-growth model (W). We proved the existence and some stability estimates of the fully discrete approximation. An algorithm for computing the numerical solutions of model (W) and simulations in one space dimension were performed using the implicit scheme in Chapter 7.

In Chapter 8 and 9, the Keller-Segel model (1.2.1)-(1.2.3) is considered. The mathematical analysis used in proving the existence results for (P) was adapted to show that there exists at least one global weak solution of the Keller-Segel model (Q). A regularized fully discrete finite element approximation of the problem (Q) was studied. Existence and uniqueness of the approximations were established. A technical replacement of s by $\phi(s)$ was the key to our study of the system where we considered a truncated alternative problem to (Q). The singular nature of (Q) in $\mathbb{R}^{\geq 0}$ has been treated by employing an appropriate regularization procedure. A well defined entropy inequality of the regularized problem has been derived. A fully discrete finite element approximation to (Q) has been introduced. The existence of the fully discrete solutions has been shown for a sufficiently small time discretization parameter. An analogous discrete entropy inequality has been obtained and some stability bounds on the approximations have been established. Some uniqueness results of approximate and weak solutions have been discussed. An error bound between the fully discrete and weak solutions of (Q) has been proved.

Our mathematical analysis of the Keller-Segel model was for $d = 1, 2$ and 3. However, we use the compactness of the embeddings $H^1(\Omega) \hookrightarrow L^q(\Omega)$ for all $1 \leq q < \infty$ in two-dimensional domains and thus the result in (9.2.38) holds for $d \leq 2$. Moreover, the continuous embedding $H^1(\Omega) \hookrightarrow L^\infty(\Omega)$ holds only for $d = 1$, thus our uniqueness and error bound analysis of (Q) is not valid for multi-dimensional spaces; see and (9.3.79).

Finally, a practical algorithm for computing the numerical solutions of (Q) was given at the beginning of Chapter 10, where simulations in one and two space dimensions were performed. We then performed numerical experiments in two space dimensions demonstrating the blow up behaviour of the numerical solution.

Additional regularity, more than we have been able to prove, was required to complete the uniqueness proof and error bound analysis for problem (Q). Unfortunately, we have been unable to prove the regularity requirement which was essential to establish these results. However, it might be possible and this is left open for future investigation. With regard to the problem (P), an idea for obtaining uniqueness results is to mimic the uniqueness study presented for the model (Q). In this direction, and due to the structure of the model (P), it is more difficult and the issues faced are: analytic; regularity requirements; other technical obstacles. This is also left as an open problem for future work.

The mathematical work in this thesis can be used to analyse other cross diffusion systems. For example, following similar arguments used for (P), one can improve the analysis presented in [35] and [8]. One could also try to adapt the techniques employed in this thesis to study the cross diffusion models in [69] and [57].

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Appendix A

Basic and Auxiliary Results

A.1 Definitions and Auxiliary Results

Theorem A.1.1 (**Schauder's theorem**) Let B be a normed space and let K be a non-empty convex compact set of B. If $f : K \to K$ is a continuous function then f has at least one fixed point (see [5] page 215).

Theorem A.1.2 (**Green's formula**, Rodrigues [85], p.76)
Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain with outward unit normal ν. If $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$, then

$$\int_\Omega \nabla u \cdot \nabla v \; dx = \int_{\partial\Omega} \frac{\partial v}{\partial \nu} \; u \; ds - \int_\Omega u \; \Delta v \; dx. \tag{A.1.1}$$

Theorem A.1.3 (**Lax-Milgram**, see, e.g., [87] page 20 and [49] page 83)
Let V be a Hilbert space. Let a be a bounded bilinear form on $V \times V$ and let $f \in V'$ (i.e. f is a bounded linear functional on V). If a is a coercive, i.e.,

$$\exists \alpha > 0, \quad \forall u \in V, \quad a(u,u) \geq \alpha \|u\|_V^2.$$

Then, there exists a unique $u \in V$ such that

$$a(u,v) = f(v) \equiv \langle f, v \rangle_{V \times V'} \quad \forall v \in V.$$

In addition,

$$\|u\|_V \leq \frac{1}{\alpha} \|f\|_{V'}.$$

Theorem A.1.4 (generalized Lax-Milgram) Let V and W be reflexive Banach spaces. Further let $a(\cdot,\cdot) : V \times W \longrightarrow R$ be a continuous bilinear form such that

$$\sup_{v\in V} a(v,w) \geq 0 \quad \forall w \in W,$$

$$\inf_{0\neq v\in V} \sup_{0\neq w\in W} \frac{a(v,w)}{\|v\|_V \|w\|_W} \geq \alpha,$$

where α is a positive constant. Then for every $F \in W'$ there exists a unique $u \in V$ such that

$$a(u,w) = F(w) \quad \forall w \in W.$$

Furthermore, the following a priori estimate holds:

$$\|u\|_V \leq \frac{1}{\alpha}\|F\|_{W'}.$$

For a proof and applications of the theorem, see for example [87] and [49].

Theorem A.1.5 (The **Grönwall lemma** in its integral and differential form, see e.g. [48]).

We start with the integral form:

Let β be a non-negative constant and let $u(t) \in L^\infty(0,T)$ and $v(t) \in L^1(0,T)$ be non-negative functions such that for *a.e.* $t \in (0,T)$

$$u(t) \leq \beta + \int_0^t u(s)\, v(s)\, ds.$$

Then for *a.e.* $t \in (0,T)$

$$u(t) \leq \beta \exp(\int_0^t v(s)\, ds). \tag{A.1.2}$$

We now state the differential form:

Let $f(t) \in W^{1,1}(0,T)$ and $g(t), h(t), w(t) \in L^1(0,T)$ be non-negative functions such that for *a.e.* $t \in (0,T)$

$$f'(t) + g(t) \leq h(t) + f(t)w(t).$$

Then for *a.e.* $t \in (0,T)$

$$f(t) + \int_0^t g(s)\, ds \leq e^{\Lambda(t)} f(0) + e^{\Lambda(t)} \int_0^t h(s)\, ds, \tag{A.1.3}$$

where $\Lambda(t) = \int_0^t w(s)\, ds$.

Theorem A.1.6 (Sobolev spaces results) Let m be a non-negative integer and let $1 \leq p \leq \infty$. The Sobolev spaces $W^{m,p}(\Omega)$ equipped with the associated norms satisfy the following:

- $W^{m,p}(\Omega)$ is a Banach space (see [83], page 206).
- $W^{m,p}(\Omega)$ is separable if $p \leq \infty$ (see [83], page 206).
- $W^{m,p}(\Omega)$ is reflexive if $1 \leq p \leq \infty$ (see [2], page 47).

Theorem A.1.7 (Sobolev embedding results) Suppose that Ω is a bounded domain. For non-negative integers m and k such that $m \geq k$, we have

$$W^{m,q}(\Omega) \hookrightarrow W^{m,p}(\Omega),$$

whenever $1 \leq p \leq q \leq \infty$ (see, e.g., [27] page 32). If the domain Ω has a Lipschitz boundary, there are more subtle relations among the Sobolev spaces. For instance, there are cases when $k < m$ and $p > q$ and the above embedding is satisfied. In this direction, we refer to the Sobolev embedding theorems in [2], [39] and [5].

Theorem A.1.8 (Time-Dependent spaces results) Let X be a Banach space and let $1 \leq p \leq \infty$. The Sobolev spaces $L^p(0,T;X)$ satisfy the following:

- $L^p(0,T;X)$ is a Banach space (see [70], page 114-116).
- $L^p(0,T;X)$, $(p \leq \infty)$ is separable $\Leftrightarrow$ X is separable (see [70], page 118).
- $L^p(0,T;X)$, $(1 \leq p \leq \infty)$ is reflexive $\Leftrightarrow$ X is reflexive (see [70], page 125).

Theorem A.1.9 (Time-Dependent spaces: embedding results) Let X, Y be Banach spaces with X continuously embedded in Y. Then

$$L^q(0,T;X) \hookrightarrow L^p(0,T;X), \quad 1 \leq p \leq q \leq \infty.$$

(See, for example, [71] page 132).

Theorem A.1.10 (Density results)

- Let Ω be an open bounded domain in $\mathbb{R}^d$ with a Lipschitz boundary $\partial\Omega$. Let m be a non-negative integer and $1 \leq p \leq \infty$. Then $C^\infty(\overline{\Omega})$ is dense in $W^{m,p}(\Omega)$, (see, e.g., [87] page 346).

- Let X be a Banach space and $1 \leq p \leq \infty$. Then $C^\infty([0,T];X)$ is dense in $L^p(0,T;X)$, (see [70], page 118).

Definition A.1.11 (strong convergence) Le V be a normed vector space. Then $x_n \in V$ converges strongly to $x \in V$, written $x_n \to x$, if and only if

$$\|x_n - x\|_V \to 0.$$

Definition A.1.12 (Weak convergence) Let X be a Banach space. Then $x_n \in X$ converges weakly to $x \in X$, written $x_n \rightharpoonup x$, if and only if

$$\langle f, x_n \rangle \to \langle f, x \rangle \quad \forall f \in X',$$

where we use $\langle \cdot, \cdot \rangle$ to denote the duality pairing between X and X'.

Definition A.1.13 (Weak-star convergence) Let X be a Banach space. Then $f_n \in X'$ converges weakly-star to $f \in X'$, written $f_n \rightharpoonup f$, if and only if

$$\langle f_n, x \rangle \to \langle f, x \rangle \quad \forall x \in X.$$

Theorem A.1.14 (Some results of weak and weak-star convergence) Let X be Banach space and X' its dual. Then

- $x_n \to x$ in X implies $x_n \rightharpoonup x$ in X.

- $x_n \rightharpoonup x$ in X implies $\|x\|_X$ is bounded and $\|x\|_X \leq \liminf \|x_n\|_X$.

- $f_n \overset{*}{\rightharpoonup} f$ in X' implies $\|f\|_{X'}$ is bounded and $\|f\|_{X'} \leq \liminf \|f_n\|_{X'}$.

- Weak (weak-star) convergence has a unique limit.

The proof of the above results can be found, for example, in [84] page 102-105.

Theorem A.1.15 (Weak compactness) Let X be a reflexive Banach space, $\{x_n\}$ a bounded sequence in X. Then it is possible to extract from $\{x_n\}$ a subsequence which converges weakly in X(see [43], page 289).

Theorem A.1.16 (Weak-star compactness) Let X be a separable Banach space and X' its dual. Then from every bounded sequence in X', it is possible to extract a subsequence which is weakly-star convergent in X' (see [43], page 291).

Theorem A.1.17 (Convergence) If a sequence $u_n \to u$ in $L^p(\Omega)$, $(1 \leq p \leq \infty)$, then there is a subsequence that converges pointwise to u almost everywhere in Ω, (see, e.g., [84] page 27).

Theorem A.1.18 (Gilfand Triple) Let W be a Banach space continuously and densely embedded in the Hilbert space H. Then

$$W \hookrightarrow H \equiv H' \hookrightarrow W', \quad H' \quad \text{is dense in} \quad W',$$

and we can write

$$\langle f, w \rangle_{W' \times W} = (f, w)_H, \quad \forall f \in H, \;\; w \in W.$$

(See [71], page 103-105).

Theorem A.1.19 (Lions-Aubin Theorem) Let X_0, X , X_1 be three Banach spaces such that

$$X_0 \overset{c}{\hookrightarrow} X \hookrightarrow X_1,$$

where X_0 and X_1 are reflexive. Let T be finite and $1 < p_0, p_1 < \infty$, then the space

$$W = \left\{ v : v \in L^{p_0}(0, T; X), \;\; \frac{dv}{dt} \in L^{p_1}(0, T; X) \right\},$$

with the norm

$$\|v\|_W := \|v\|_{L^{p_0}(0,T;X)} + \|v\|_{L^{p_1}(0,T;X)},$$

is a Banach space and the injection W into $L^{p_0}(0, T; X)$ is compact. (See Temam [91], p.271).

Theorem A.1.20 (Lebesgue dominated convergence theorem). Suppose $f_n : R \to [-\infty, \infty]$ are (Lebesgue) measurable functions such that the pointwise limit $f(x) = \lim_{n\to\infty} f_n(x)$ exists. Assume there is an integrable $g : R \to [0, \infty]$ with $|f_n| \leq g(x)$ for each $x \in \mathbb{R}$. Then f is integrable as is f_n for each n, and

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n(x)dx = \int_{\mathbb{R}} \lim_{n\to\infty} f_n(x)dx = \int_{\mathbb{R}} f(x)dx.$$

Theorem A.1.21 (logarithmic Youngs inequality). Let r, $s \in \mathbb{R}_{\geq 0}$ then we have

$$r\,s \leq r \log r - r + F(s).$$

Proof: Consider $G(r) := \sup_{s\in\mathbb{R}}(r\,s - F(s))$ then using analysis we note that the argument of the supremum attains a maximum at $s = \ln r$ and the argument also tends to $-\infty$ as $x \to \pm\infty$ so that in fact $G(r) := r\,\ln r - r$. However, from the definition of $G(r)$, $\forall s \in \mathbb{R}$

$$G(r) \geq r\,s - F(s),$$

that is

$$r\,s \leq r\,\ln r - r - F(s).$$

□

Theorem A.1.22 (Fatou's Lemma). If f_n is a sequence of nonnegative measurable functions, then

$$\int \liminf_{n\to\infty} f_n\,d\mu \leq \liminf_{n\to\infty} \int f_n\,d\mu.$$