Thermodynamics of Accelerating Black Holes

Three years with the C-metric

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A thesis presented for the degree of
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Abstract: We address a long-standing problem of describing the thermodynamics of an accelerating black hole. We derive a standard first law of black hole thermodynamics, with the usual identification of entropy proportional to the area of the event horizon — even though the event horizon contains a conical singularity. We show how to generalise this result, formulating thermodynamics for black holes with varying conical deficits. We derive a new potential for the varying tension defects: the thermodynamic length, both for accelerating and static black holes. We discuss possible physical processes in which the tension of a string ending on a black hole might vary, and also map out the thermodynamic phase space of accelerating black holes and explore their critical phenomena. We then revisit the critical limit in which asymptotically-AdS black holes develop maximal conical deficits, first for a stationary rotating black hole, and then for an accelerated black hole, by taking various upper bounds for the parameters in the spacetimes presented. We explore the thermodynamics of these geometries and evaluate the reverse isoperimetric inequality, and argue that the ultra-spinning black hole only violates this condition when it is nonaccelerating. Finally, we return to some of our earlier findings and adjust them in light of new results; a new expression for the mass is obtained by computing the dual stress-energy tensor for the spacetime and finding that it corresponds to a relativistic fluid with a nontrivial viscous shear tensor. We compare the holographic computation with the method of conformal completion showing it yields the same result for the mass.

Declaration

The work in this thesis is based on research carried out in the Department of Mathematical Sciences at Durham University. The results presented have in part already appeared in the following publications:

- M. Appels, R. Gregory and D. Kubiznak, *Thermodynamics of Accelerating Black Holes*, *Phys. Rev. Lett.* **117** (2016) 131303, [1604.08812]
- M. Appels, R. Gregory and D. Kubiznak, *Black Hole Thermodynamics with Conical Defects*, *JHEP* **05** (2017) 116, [1702.00490]
- A. Anabalon, M. Appels, R. Gregory, D. Kubiznak, R. B. Mann and A. Övgün, *Holographic Thermodynamics of Accelerating Black Holes*, 1805.02687

Any section reproduced from these articles has been done so with the full consent and permission from the relevant collaborators. No part of this thesis has been submitted elsewhere for any degree or qualification.

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Dedicated to

Mum & Dad,
Grandpa Ian & Granny Liz,
Grandpa Bob & Granny Anne.

Contents

Preface

The 1970s brought in a new era for research into black holes by drawing parallels between the classical theory of thermodynamics and these gravitational solutions. This newly found application for a century old machinery opened many avenues for this research. Indeed, black holes have been discovered to display a wide and diverse variety of phenomena, much of which were made evident from studies into thermodynamics. Even more recently, black hole thermodynamics are often utilised by string theorists seeking to deepen their understanding of quantum gravity via the AdS/CFT correspondence.

In this thesis we seek to ascertain the validity of this framework when applied to exotic solutions discovered — coincidentally, also around the 1970s — to describe accelerating black holes. The work was initially motivated by the discovery that rotating black holes in asymptotically anti-de Sitter space, in a special limit, exhibit extraordinary thermodynamic behaviour. These solutions are unique in particular as they possess two severe conical defects at each pole. Conical defects are an inherent feature to accelerating black holes, hence the desire to investigate the thermodynamics of accelerating black holes in a similar limit. To do so, we needed to first develop a rigid framework for thermodynamics of these black holes.

As we will demonstrate, we have been able to propose a consistent set of thermodynamic relations and quantities for static nonrotating charged accelerating black holes. The situation is somewhat more problematic with the inclusion of rotation and while we are unable to provide a full picture, we are able to form a sufficient picture to investigate the aforementioned limit[1].

The outline is as follows. We will begin by reviewing aspects of black hole thermodynamics in chapter 1, covering some of the important more historical discoveries as well as more recent developments that we have used elsewhere. In chapter 2, we similarly review the C-metric, which describes a uniformly accelerating black hole, its history, as well as certain derivations leading to the metrics we use further. We also

[1] A method for obtaining the thermodynamics of rotating accelerating black holes was discovered in the later stages of production of this thesis [4]. These new results are summarised in the concluding chapter.

outline certain constraints that must be taken into consideration when working with these solutions. In chapter 3, we develop the thermodynamical framework necessary to admit varying conical deficits, and include the tensions of cosmic strings as new extensive variables of our ensemble. This introduces a new potential, the thermodynamic length. In chapter 4, we extend this formulation to include accelerating black holes and consider independently varying conical deficits at the north and south poles. With this framework in place, we explore the thermodynamic phase structure of the solution. We then attempt to generalise this work to include rotating solutions and investigate the thermodynamic properties of critical accelerating solutions, which possess the aforementioned severe conical defects. Finally, in chapter 5, we revisit some of our original conclusions and alter some of our results in light of new calculations that use different techniques.

Conventions

Throughout this thesis we have used the mostly plus convention for metric signatures. These metrics are expressed as the line element $ds^2 = g_{ab}dx^a dx^b$ in most cases. Unless stated otherwise, we are working in 3+1 spacetime dimensions.

Quantities are expressed in Planck units such that $c = G = 4\pi\varepsilon_0 = k_{\mathrm{B}} = 1$, where c is the speed of light in a vacuum, G is Newton's gravitational constant in four dimensions, ε_0 is the permittivity of free space and k_{B} is Boltzmann's constant, and $\hbar = \ell_p^2$, Planck's constant, is the only dimensionful fundamental constant. This implies that physical quantities of length, time, mass and charge have the same dimensions, and temperatures have dimensions L^{-1}.

When a coordinate transformation is used, or any general transformation or relabelling, it will be represented in either one of the following manners. In going from the coordinates/parameters $\{x^i\}$ to the coordinates/parameters $\{y^i\}$, a simple relation will be provided as either

$$x^i = x^i(\{y^i\}) \qquad \text{or} \qquad y^i = y^i(\{x^i\}),$$

where clarity will be the determining factor. Alternatively, it will be more desirable to preserve some particular symbols on either ends of the transformation. when this is the case, the set of coordinates/parameters $\{x^i\}$ is to be replaced, as they appear in any expression, with, for example, a primed set $\{x'^i\}$, for which a relation such as those above will be provided. The replacement procedure is represented by an arrow $\rightarrow$, the direction of which indicates which variable is being replaced (the tail).

This is then followed by the relation to the original variable. Explicitly,

$$x^i \to x' e^i = x'^i(\{x^i\}) \qquad \text{or} \qquad x^i \to x'^i(\{x^i\}),$$

where we have introduced a commonly used shorthand in the second expression signifying simply that "$\to$" is to be interpreted as meaning "$\to x'^i =$".

Chapter 1

Black hole thermodynamics/mechanics

1.1 The laws of black hole mechanics

In 1971, Stephen Hawking discovered that there exists an upper bound on the amount of energy that can be released through gravitational radiation — or any other form of energy release, for that matter — upon the collision of two black holes [5, 6]. This result relies primarily on a proof that through any given physical process, such as a collision or a capture of sorts, the total event horizon area should never decrease, thereby constraining how much energy may be extracted. Now known as Hawking's *black hole area theorem*, the result of this proof is commonly written as:

$$\mathcal{A}_3 \geqslant \mathcal{A}_1 + \mathcal{A}_2 \tag{1.1}$$

This theorem has an important role in the history of black hole thermodynamics as it is responsible for much of the reasoning that was applied to this analogy, in that it is clearly reminiscent of the second law of thermodynamics, and we shall explore these parallels in greater depth below. Along with the area theorem, in [6], Hawking also famously proves that under gravitational collapse, a body of matter will not only form a black hole, but that the event horizon of the black hole formed will have spherical topology, be stationary and axisymmetric.

At a summer school in 1972, Bardeen, Carter and Hawking (BCH) pursued the aforementioned analogy between the macroscopic properties of black holes and thermodynamical systems, work which culminated in the formalisation of *the four laws of black hole mechanics* [7], a homage to their statistical counterparts. Let us now summarise these laws as such:

- The *zeroth law* states that the surface gravity κ of a stationary black hole is constant over its event horizon [7–11].
- The *first law* expresses conservation of energy during physical processes through changes in the properties of the black hole such as its mass M, area $\mathcal{A}$, angular momentum J or charge Q with the following relation:

$$\delta M = \frac{\kappa}{8\pi}\delta\mathcal{A} + \Omega\delta J + \Phi\delta Q, \tag{1.2}$$

  where Ω is the angular velocity of the black hole at its event horizon and Φ is the electrostatic potential. This relation was proved using a variational principle by BCH in the uncharged case, and Carter published the proof for the charged case in the conference proceedings for the same 1972 summer school [9] (repub. in [11]). We review this work in section 1.2.
- The *second law* is the area theorem itself. Following the current theme, we re-express eq. (1.1) simply as:

$$\delta\mathcal{A} \geqslant 0. \tag{1.3}$$

- Finally, the *third law* states that it is impossible through any finite sequence of physical processes to reduce the surface gravity κ to zero. At the time, this law was only conjectured and argued for using logical arguments. It had actually been shown that processes leading to such a configuration did exist if one allowed for infinite divisibility of matter and infinite time [12, 13]. Werner Israel proved the third law a decade later [14].

The analogy between black hole mechanics and classical thermodynamics is complete once some form of identification is made between the surface gravity κ and the temperature T, and between the horizon area $\mathcal{A}$ and the entropy S, both up to some factor determined such that $T\delta S = \kappa\delta\mathcal{A}/8\pi$. The authors of [7] were initially reluctant to make this identification and emphasized that while the similarities existed, these quantities should not lead to the interpretation of the black hole as having either temperature or entropy, and understandably so — classically, the effective temperature of a black hole is absolute zero, as it is (or at least was thought to be) unable to emit any radiation. In fact, they point out that black holes transcend the second law in that one might in theory be able to add entropy to a black hole without changing its final state by much. The concept of horizon area as entropy was not, however, a new one.

Little over half a year prior to *The four laws of black hole mechanics* being received for publication, and only a couple of months before the aforementioned summer school during which much of this work was conceived (but a year after Hawking's initial paper on the area theorem was published), Jacob Bekenstein wrote a letter addressing this specific apparent violation of the second law of thermodynamics [15]. In the letter, he proposes a generalised form of the second law of thermodynamics in which the quantity that is observed to never decrease is given by the sum of common entropy, the entropy of the spacetime outside the black hole and a new *black hole entropy* proportional to the horizon area, i.e. $S_{\mathrm{bh}} = \eta\mathcal{A}$, with η being, at the time, an undetermined proportionality constant. This choice of having entropy proportional to the horizon area was actually motivated by Hawking's work on the area theorem [5] as well as similar work by Christodoulou and Ruffini around the same time [12, 16] on reversible and irreversible processes for black holes. In further works [17, 18], Bekenstein attempts to establish the proportionality factor on heuristic grounds and proposes a value for $\eta = \log(2)/8\pi$. In fact, by considering a differential formula akin to the first law (1.2), Bekenstein was actually able to postulate a possible expression for the temperature, by considering the conjugate quantity to entropy; however he too warns against interpreting this as a true temperature in the thermal sense. That black holes could not seemingly have a nonzero temperature proved to be the obstacle preventing Bekenstein's work on black hole entropy from catching on in other circles early on [19].

In the early 1970s, a classical mechanism for stimulated emission from a black hole had been discovered, known as superradiance. It describes the amplification of waves in a special *superradiant regime*, incident on a generic Kerr-Newman black hole. The easiest way to see this effect is by considering a wave packet of frequency ω, axial quantum number m and charge e, incident on a black hole. The first law (1.2) should hold throughout the capture of this packet and one expects differentials in the ratios of $m : \omega$ and $e : \omega$. It therefore follows that

$$\left(1 - \Omega\frac{m}{\omega} - \Phi\frac{e}{\omega}\right)\delta M = \frac{\kappa}{8\pi}\delta\mathcal{A}. \tag{1.4}$$

Imposing the second law (1.3) allows for $\delta M \leqslant 0$ for wave packets that satisfy

$$\omega \leqslant \Omega m + \Phi e. \tag{1.5}$$

In other words, we expect wave packets incident in this superradiant regime to be scattered off the black hole with a larger amplitude.

This effect can be traced back to works by Yakov Zel'dovich [20, 21], however it was also separately pointed out by Misner [22], apparently unaware of Zel'dovich's work. Other notable contributions to the understanding of these amplifications were made by Press and Teukolsky [23], Starobinskii [24] and by Bekenstein [25]. While most of the work cited above concerns itself with the amplification of superradiant incident waves through stimulated emission, Zel'dovich also suggested that, taking into account quantum mechanical effects, one could in principle also expect spontaneous emission in superradiant modes as well. Writing in his doctoral thesis, Don Page describes how, oblivious to Zel'dovich's work, he and Larry Ford had at the time independently discovered this effect, eventually going on to have discussions with Feynman, Thorne, Press and Teukolsky, before being made aware of the above [26] (along with a few amusing anecdotes from the time in [27]).

What is important to note here is that while it was universally accepted that a black hole could not radiate, a black hole emitting in superradiant modes would technically not violate the second law of black hole mechanics, by definition. It should come, therefore, as no surprise that when Hawking eventually heard about this effect, as it made its way across research circles, his interest was piqued and, after having discussions with Zel'dovich and Starobinskii while in Moscow, began working on a field theory calculation that might help in validifying this quantum phenomenon [27].

In 1974, Hawking made the groundbreaking discovery that not only did black holes radiate in these superradiant modes, but that emission from black holes in fact covered as much of the spectrum it could [28, 29]. Hawking himself later wrote, in retrospect, about how he was initially embarrassed by the result and therefore attempted to introduce various cutoffs in an effort to suppress these additional modes [19]. He eventually accepted the result, citing the fact that the radiation was identical to thermal radiation from a body with temperature $\kappa/2\pi$ as the smoking gun. Another reason Hawking ended up believing in his result was that it made Bekenstein's theory of black hole entropy consistent. This result has since been verified by several other means [29–39] and is generally accepted as a correct result, despite our lack, still to this day, of a consistent picture of quantum gravity.

Hawking temperature, in a sense, was the final piece of the puzzle and the reason why black hole mechanics became black hole *thermodynamics*. With the expression Hawking found, we are now able to rewrite the first law as

$$\delta M = T_{\mathrm{H}}\delta S_{\mathrm{BH}} + \Omega\delta J + \Phi\delta Q, \tag{1.6}$$

where the temperature of the black hole is known as the *Hawking temperature*

$$T_{\mathrm{H}} = \frac{\kappa}{2\pi}, \tag{1.7}$$

and the entropy of the black hole is known as the *Bekenstein-Hawking entropy*

$$S_{\mathrm{BH}} = \frac{\mathcal{A}}{4}. \tag{1.8}$$

Similarly, we promote the second law to the aforementioned generalised second law, with the black hole entropy given by a quarter the horizon area.

1.2 The first law

When Roy Kerr first presented the metric for a rotating black hole in 1963 [40], most commonly written in Boyer-Lindquist[1] coordinates as

$$ds^2 = -\frac{f(r)}{\Sigma}(dt - a\sin^2\theta d\phi)^2 + \frac{\Sigma}{f(r)}dr^2 + \Sigma r^2 d\theta^2 + \frac{\sin^2\theta}{\Sigma r^2}(adt - (r^2+a^2)d\phi)^2,$$
$$f(r) = 1 - \frac{2m}{r} + \frac{a^2+e^2}{r^2}, \qquad \Sigma = 1 + \frac{a^2}{r^2}\cos^2\theta, \tag{1.9}$$

along with the gauge potential

$$B = -\frac{e}{\Sigma r}(dt - a\sin^2\theta d\phi), \tag{1.10}$$

he provided an interpretation for the solution parameters m and a by comparing its Taylor expansion to a previously known approximation of a spinning particle, and concluded that to an observer at infinity this rotating black hole would be equivalent to a particle of mass $M = m$ and angular momentum $J = ma$. The charge term in $f(r)$ was later added by Newman *et al.* [42] and a similar line of reasoning leads one to conclude that a Kerr-Newman black hole with nonzero charge parameter e has an electric charge $Q = e$.

In 1969, Penrose provided a simple mechanism through which one could envision extracting the rotational energy of a Kerr black hole [43], and in 1970, Christodoulou [12, 16] (with Ruffini, in 1971) showed using this picture that a black hole's mass could be decomposed using an irreducible mass M_{ir}, which represents the mass of the remaining black hole when all its rotational and electromagnetic energy is stripped, according to the following formula:

$$M = \sqrt{\left(M_{\text{ir}} + \frac{Q^2}{4M_{\text{ir}}}\right)^2 + \frac{J^2}{4M_{\text{ir}}^2}}. \tag{1.11}$$

Christodoulou and Ruffini point out that this is equivalent in principle to Hawking's area theorem and one can indeed obtain eq. (1.11) by treating M_{ir} as the mass of a Schwarzschild black hole with equal area to a generic Kerr-Newman black hole. One then deduces $M_{\text{ir}}^2 = \mathcal{A}/16\pi$. An appealing aspect of this definition is the resemblance

[1] Boyer and Lindquist were responsible for the maximal extension of the Kerr metric [41]. Boyer-Lindquist coordinates are coordinates best suited for describing ellipsoids and are related to cartesian coordinates through $x = \sqrt{r^2+a^2}\sin\theta\cos\phi$, $y = \sqrt{r^2+a^2}\sin\theta\sin\phi$ and $z = r\cos\theta$.

it bears to the energy of a relativistic particle, when expressed in terms of its rest mass and momentum.

In 1972, Larry Smarr pointed out, using this relation, that if one expressed mass as a function $M = M(\mathcal{A}, J, Q)$ of area, angular momentum and charge, the exact differential

$$dM = \mathcal{T} d\mathcal{A} + \Omega dJ + \Phi dQ \tag{1.12}$$

could be used to obtain the invariant quantities $\mathcal{T} = (\partial M/\partial \mathcal{A})_{J,Q} = \kappa/8\pi$, referred to at the time as the effective surface tension, $\Omega = (\partial M/\partial J)_{\mathcal{A},Q}$, the angular velocity of the black hole, and $\Phi = (\partial M/\partial Q)_{\mathcal{A},J}$, the electromagnetic potential. Smarr then simply observed that with these quantities, eq. (1.11) could be rewritten nicely as

$$M = 2\mathcal{T}\mathcal{A} + 2\Omega J + \Phi Q. \tag{1.13}$$

In fact, this result follows immediately from Euler's theorem on homogeneous functions, given that $M(\lambda\mathcal{A}, \lambda J, \lambda^{\frac{1}{2}} Q) = \lambda^{\frac{1}{2}} M(\mathcal{A}, J, Q)$.

It is in fact quite common to use the exact differential in eq. (1.12) to establish the first law itself. This was the method Bekenstein used in [17] to hypothesise a black hole temperature, and this will also be how we will ultimately establish a first law for accelerating black holes, the object of this thesis.

We began this section by pointing out that the origin of the interpretations of mass and angular momentum as $M = m$ and $J = ma$ was a term-by-term comparison with what might be expected in some low-energy limit. While this was a practical approach, a formal method for identifying such conserved charges had in fact already been developed by Arthur Komar in 1958 [44]. A conserved current can be formed by contracting a killing vector k with the Ricci tensor. The corresponding conserved quantity is obtained by integrating this current over a spacelike hypersurface $\mathcal{S}$ normal to the killing vector. Using the fact that a killing vector satisfies

$$\nabla_a \nabla^a k^b = -R^b{}_a k^a, \tag{1.14}$$

a *Komar integral* associated to a given killing vector is derived as

$$E_k \sim \int_{\partial\mathcal{S}} \nabla^a k^b \; d\Sigma_{ab} = \int_{\partial\mathcal{S}} *\mathbf{d}k, \tag{1.15}$$

where a normalisation is needed, which can be determined at a later stage, and $d\Sigma_{ab}$ is the volume element on $\partial\mathcal{S}$, to give a conserved charge E_k. One finds that for $k = k_t$, the timelike killing vector, and for $k = k_\phi$, the rotational killing vector, this integral, when evaluated at infinity $\partial\mathcal{S}_\infty$ and normalised, yields $M = m$ and $J = ma$ respectively, for the Kerr-Newman metric. One can construct a spacelike surface $\mathcal{S}$ which ends only on the boundary and extends through the horizons to the singularity.

Bardeen, Carter and Hawking were able to prove Smarr's relation, for the uncharged case in a more general setting in [7] and Carter provided the proof for the charged case in [9, 11]. Both methods use this as an intermediary step to deriving the first law, showing how the aforementioned exact differential expression (1.12) is in fact fully self-consistent. We will now briefly review this calculation, omitting technical steps which can all be found in the original works.

Let us consider the Komar integrals for both killing vectors, evaluated over a surface $\mathcal{S}$ which now extends from the boundary $\partial\mathcal{S}_\infty$ and the event horizon $\partial\mathcal{B}$. Integrating both sides of eq. (1.14) will allow us to express mass and angular momentum as

$$\begin{aligned} M &= \frac{1}{4\pi}\int_{\mathcal{S}} R^a{}_b k_t^b \, d\Sigma_a + M_{\mathrm{H}}, & M_{\mathrm{H}} &\equiv -\frac{1}{4\pi}\int_{\partial\mathcal{B}} \nabla^a k_t^b \, d\Sigma_{ab}, \\ J &= -\frac{1}{8\pi}\int_{\mathcal{S}} R^a{}_b k_\phi^b \, d\Sigma_a + J_{\mathrm{H}}, & J_{\mathrm{H}} &\equiv \frac{1}{8\pi}\int_{\partial\mathcal{B}} \nabla^a k_\phi^b \, d\Sigma_{ab}, \end{aligned} \tag{1.16}$$

where M_{H} and J_{H} are the corresponding boundary integrals evaluated at the horizon. For vacuum metrics, $M_{\mathrm{H}} = M = m$ and $J_{\mathrm{H}} = J = ma$. The null generator of the horizon can be defined as $l^a = k_t^a + \Omega_{\mathrm{H}} k_\phi^a$, where Ω_{H} is a scalar quantity which is obtained from the requirement that l be orthogonal to the rotational killing vector k_ϕ. It can further be shown that Ω_{H}, given by

$$\Omega_{\mathrm{H}} = -\frac{g_{t\phi}}{g_{\phi\phi}}, \tag{1.17}$$

is constant over the horizon and represents its angular velocity. Making use of the fact that the surface gravity κ can actually be defined in terms of this vector as $\kappa = n_b l^a \nabla_a l^b$, where n is the unit normal to the horizon, one eliminates M_{H} from eq. (1.16), establishing

$$\frac{1}{2}M = \int_{\mathcal{S}} \left(T^a_{\ b} - \frac{1}{2} T \delta^a_b \right) k_t^b \, d\Sigma_a + \Omega_{\mathrm{H}} J_{\mathrm{H}} + \frac{\kappa \mathcal{A}}{8\pi}, \tag{1.18}$$

which reduces to the Smarr relation in the absence of charge and any external matter. A further decomposition can be made by splitting the stress-energy tensor $T^{ab} = T^{ab}_{\mathrm{M}} + T^{ab}_{\mathrm{F}}$ into its matter and electromagnetic parts, which in turn allows us to express the total angular momentum $J = J_{\mathrm{M}} + J_{\mathrm{F}} + J_{\mathrm{H}}$ in a similar fashion. Finally, one can define the electric charge by integrating the electromagnetic current $j^a = \nabla_b F^{ab}/4\pi$ over the same spacelike surface, and, in analogy to the mass and the angular momentum formulae in eq. (1.16) we write it as

$$Q = -\int_{\mathcal{S}} j^a \, d\Sigma_a + Q_{\mathrm{H}}, \qquad Q_{\mathrm{H}} \equiv -\frac{1}{4\pi}\int_{\partial\mathcal{B}} F^{ab} \, d\Sigma_{ab}. \tag{1.19}$$

It then follows that eq. (1.18) can be rearranged into

$$\begin{aligned}\frac{1}{2}M = \int_{\mathcal{S}} \left(T^a_{\mathrm{M}\ b} - \frac{1}{2}T_{\mathrm{M}}\delta^a_b\right) k^b_t \, d\Sigma_a - \Omega_{\mathrm{H}} J_{\mathrm{M}} - \frac{1}{2}\int_{\mathcal{S}} l^c A_c j^a \, d\Sigma_a + \int_{\mathcal{S}} A_b j^{[b} l^{a]} \, d\Sigma_a \\ + \Omega_{\mathrm{H}} J + \frac{\kappa\mathcal{A}}{8\pi} + \frac{1}{2}\Phi_{\mathrm{H}} Q_{\mathrm{H}},\end{aligned} \tag{1.20}$$

and the Smarr relation is now recovered when the external matter fields and source currents in the first line are switched off.

The formula in eq. (1.20) describes the total mass M of the system, which is a time-conserved quantity. It is possible, however, to look at two neighbouring configurations with slightly different M, and consider the corresponding variation δM, similar to the way in which one varies the action. It is by performing this variation that one obtains the first law in its most general form for a rotating charged black hole surrounded by electric and matter fields:

$$\begin{aligned}\delta M = \int \Omega \delta d J_{\mathrm{M}} + \int \overline{\Theta} \delta d S + \int \overline{\mu}^{(i)} \delta d N_{(i)} + \int \Phi_{\mathcal{S}} \delta d Q \\ + \Phi_{\mathrm{H}} \delta Q_{\mathrm{H}} + \frac{\kappa}{8\pi}\delta\mathcal{A} + \Omega_{\mathrm{H}}(\delta J_{\mathrm{H}} + \delta J_{\mathrm{F}}),\end{aligned} \tag{1.21}$$

where Ω is the angular velocity of the fluid, $\overline{\Theta}$ is its "effective" or "red-shifted" temperature, $\overline{\mu}^{(i)}$ the effective chemical potential corresponding to each type of particle, $\Phi_{\mathcal{S}}$ denotes the electromagnetic potential accross the surface $\mathcal{S}$, and finally the notation "$\int \delta d$" signifies a change in fluid angular momentum (J_{M}), entropy (S) or particle number ($N_{(i)}$) crossing the surface $\mathcal{S}$. This elegant result reduces to the relation initially presented in eq. (1.2) when all external fields and sources are turned off.

This derivation shows why the first law holds, at least for asymptotically flat rotating black holes, from first principles. Starting with the definition of mass as a Komar integral, we are able to show why variations of this quantity are related to all the other variations present in the first law in the way that they are. Previously, it had been known that one could express variations in the mass in this way, as the exact differential (1.12), however this should be seen as more of an observation based on final expressions obtained using independent definitions for mass, area, surface gravity and other quantities. This elegant derivation shows not merely how all of these expressions are related, but how their *definitions* are related.

1.3 The Euclidean approach to black hole thermodynamics

By the end of 1975, Hawking had already devised a way, with Jim Hartle, to reproduce his black hole radiation calculation using the path-integral formulation of quantum field theory [35]. In this approach, the central object is the partition function, which has the form

$$Z = \int \mathcal{D}g\mathcal{D}\Phi \; e^{iS[g,\Phi]}, \tag{1.22}$$

where we are integrating over the space of all possible metrics, including those which are topologically distinct spacetimes such as black hole solutions, and all field configurations. It was then shown, by Gibbons and Hawking in 1976 [45], that, given a properly regularised gravitational action, it was possible to recover all the thermodynamic behaviour of black hole mechanics from the partition function, in analogy to regular euclidean field theory.

The action in eq. (1.22) for a generic gravitational solution is given by

$$S = \frac{1}{16\pi}\int_Y d^4x\sqrt{-g}R + \frac{1}{8\pi}\int_{\partial Y} d^3x\sqrt{-h}K + S_{\mathrm{C}} + S_{\mathrm{M}}. \tag{1.23}$$

It is computed within a region of spacetime Y and is made up of the Einstein-Hilbert, Gibbons-Hawking, counter and matter terms respectively. R is the Ricci scalar for the bulk metric g_{ab}, K is the extrinsic curvature of the boundary ∂Y and h_{ab} is its induced three-dimensional metric. The counterterm S_{C} is determined such that S vanishes in flat space.

An issue that arises when computing the action of a black hole metric is that singularities must be avoided. While it is known that crafty coordinate choices allow one to patch over horizons, curvature singularities are intrinsic to the geometry and may not be removed. It is possible, however, to construct a smooth patch which avoids the singularity all together, by complexifying the timelike coordinate. The subsequent spacetime will only be smooth provided the new coordinate $\tau = it$ is made to have a periodicity $\beta = 2\pi/\kappa$, where κ is the surface gravity of the horizon. Let us illustrate this with the Schwarzschild metric,

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2, \qquad f(r) = 1 - \frac{2m}{r}. \tag{1.24}$$

In the vicinity of the horizon $r_{\mathrm{h}} = 2m$, the Euclidean Schwarzschild metric will take the form given by

$$ds^2 = f'(r_{\mathrm{h}})(r - r_{\mathrm{h}})d\tau^2 + \frac{dr^2}{f'(r_{\mathrm{h}})(r - r_{\mathrm{h}})} + r_{\mathrm{h}}^2 d\Omega^2. \tag{1.25}$$

Introducing temporary coordinates $\rho^2 = 4(r - r_{\mathrm{h}})/f'(r_{\mathrm{h}})$ and $\varphi = f'(r_{\mathrm{h}})\tau/2$, it becomes clear that with $\beta = f'(r_{\mathrm{h}})/4\pi = 8\pi m$, these coordinates describe a spacetime of topology $S^1 \times S^2$. More precisely, each point in the $\tau - r$ subspace corresponds to a 2-sphere of corresponding radius, and the subspace itself has an S^1 symmetry centred around the point $r = r_{\mathrm{h}}$. One could then compute the action on a region bounded by the surface $r = r_{\mathrm{b}}$.

In Euclidean field theory, the periodicity of imaginary time ends up corresponding to the temperature of the system through $T = \beta^{-1}$. This is then used to express the partition function of such a field theory. In the grand canonical ensemble, such a system will have thermodynamically conserved quantities Q_i, and respective conjugate potentials P_i. The partition function can be expressed as

$$Z = \operatorname{Tr} e^{-\beta\left(H - \sum_i P_i Q_i\right)} = \operatorname{Tr} e^{-\beta F}, \tag{1.26}$$

where F is the grand canonical free energy potential, or "grand potential" for short — in the current text, however, this quantity will loosely be referred to as the *free energy potential*, and will be defined according to the ensemble at hand.

Returning to the gravitational partition function, it follows from the path-integral approach that the integral in eq. (1.22) will receive its most dominant contributions from the on-shell metric, that which satisfies Einstein's equations. This allows us to use the following approximation for the partition function, using the euclidean action,

$$Z \approx e^{-S_{\mathrm{E}}[g,\Phi]}. \tag{1.27}$$

In analogy with the nongravitational situation, we derive the free energy from the partition function using eq. (1.26), and write

$$F = -\frac{\log Z}{\beta} = \frac{S_{\mathrm{E}}}{\beta}. \tag{1.28}$$

Finally, one recovers an expression for the mass of the black hole by reverse Legendre transforming the free energy:

$$\begin{aligned} M &= F + TS + \sum_i P_i Q_i \\ &= 2TS + 2\Omega J + \Phi Q, \end{aligned} \tag{1.29}$$

where the final line was obtained by explicitly computing the action for the Kerr-Newman black hole. Historically, this was used to affirm the expression for entropy using the Smarr formula from eq. (1.13), however it could equally be used to verify the converse.

1.4 Incorporating Λ

The story of Einstein reportedly calling his inclusion of a cosmological constant Λ the "biggest blunder of his career" is an oft-repeated one (see [46]). On the contrary, over the past two decades most gravitational research includes a cosmological constant. Of most relevance to astronomers was the discovery that our universe seems to be described by a cosmology with a small-but-nonzero cosmological constant (e.g. ref. [47]). In the world of theoretical physics, arguably one of the most important discoveries, by Maldacena in 1997 [48], was of the conjectured *ADS/CFT correspondence*, or more generally, the *gauge/gravity correspondence* between strongly coupled nongravitational field theories and weakly coupled gravitational theories with negative Λ of one dimension higher.

The sign of the cosmological constant separates gravitational theories into two camps: (asymptotically) de Sitter (dS) theories have $\Lambda > 0$ and (asymptotically) anti-de Sitter (AdS) theories have $\Lambda < 0$. A classical gravitational field theory with $\Lambda \neq 0$ then satisfies the following Einstein field equations (EFEs):

$$R_{ab} - \frac{1}{2} g_{ab} R + g_{ab}\Lambda = 8\pi T_{ab}. \tag{1.30}$$

Finally, in order to make a connection with dimensionality and scale, the cosmological constant of a D-dimenional theory is often parametrised in terms of the real-valued (A)dS length scale ℓ according to

$$\Lambda = \pm \frac{(D-1)(D-2)}{2\ell^2} \overset{D=4}{=} \pm \frac{3}{\ell^2}. \tag{1.31}$$

1.4.1 Thermodynamics of the Kerr-AdS black hole

A natural starting point for introducing a cosmological constant is to review how this fits into the thermodynamic framework for studying black holes presented above. The thermodynamics of black holes in the context of a cosmological constant were actually first discussed in a paper by Gibbons and Hawking in 1976 [49], in which they consider hawking radiation and black hole temperature in a de Sitter ($\Lambda > 0$) background, a delicate topic since de Sitter space also has a cosmological horizon. The original motivation of the paper was in fact to better define the hawking radiation associated to cosmological horizons. In 1982, Hawking and Page presented a thermodynamic description of the Schwarzschild-AdS spacetime [50]. By computing the action using a regularisation scheme similar to the method presented in section 1.3, adapted to an asymptotically-AdS spacetime, Hawking and Page derive expressions for the temperature and entropy of the black hole in this background and in fact used these quantities to show that there existed a phase transition for black holes in an anti-de Sitter background, which we will review below.

While the Kerr-AdS metric has been known since the late 1960s [51], its thermodynamics were only discussed near the turn of the century, when interest in asymptotically-AdS solutions boomed due to the aforementioned AdS/CFT conjecture [52–55]. The metric, in Boyer-Lindquist coordinates, is given by

$$\begin{aligned} ds^2 = -\frac{f(r)}{\Sigma}\Big(dt - a\sin^2\theta\frac{d\phi}{\Xi}\Big)^2 + \frac{\Sigma}{f(r)}dr^2 + \frac{\Sigma r^2}{g(\theta)}d\theta^2 \\ + \frac{g(\theta)\sin^2\theta}{\Sigma r^2}\Big(adt - (r^2+a^2)\frac{d\phi}{\Xi}\Big)^2, \end{aligned} \tag{1.32a}$$

where

$$\begin{aligned} f(r) = \Big(1+\frac{a^2}{r^2}\Big)\Big(1+\frac{r^2}{\ell^2}\Big) - \frac{2m}{r} + \frac{e^2}{r^2}, \qquad g(\theta) = 1 - \frac{a^2}{\ell^2}\cos^2\theta, \\ \Sigma = 1 + \frac{a^2}{r^2}\cos^2\theta, \qquad \Xi = 1 - \frac{a^2}{\ell^2}. \end{aligned} \tag{1.32b}$$

The corresponding gauge potential is given by

$$B = -\frac{e}{\Sigma r}\Big(dt - a\sin^2\theta\frac{d\phi}{\Xi}\Big). \tag{1.33}$$

Hawking, Hunter and Taylor-Robinson [52] were the first to present a set of thermodynamic quantities for the uncharged rotating black hole in AdS. By computing the thermodynamics of rotating bulk spacetimes such as the Kerr-AdS metric, they were able to reconcile the bulk partition function with the partition function of a scalar field coupled to a three-dimensional[2] rotating Einstein universe, which forms the boundary of Kerr-AdS.

Computing the temperature from the metric (1.32) in its Euclidean form (for rotation one must simultaneously take $\tau = it, \alpha = -ia$), and its entropy as a quarter

[2]They also perform this computation in a number of other dimensions; we single out this case for its relevance.

the horizon area, they find:

$$\begin{aligned} T &= \frac{r_+^2 f'(r_+)}{4\pi(r_+^2+a^2)} = \frac{r_+}{4\pi(r_+^2+a^2)}\left[1+\frac{3r_+^2}{\ell^2}+\frac{a^2}{\ell^2}-\frac{a^2}{r_+^2}\right], \\ S &= \frac{\pi(r_+^2+a^2)}{\Xi}, \end{aligned} \tag{1.34}$$

where r_+ denotes the location of the outer event horizon. They compute the mass M and angular momentum J using the timelike and angular killing vectors with corresponding Komar integrals, noting that a background $m = 0$ subtraction is needed for regularisation (see Magnon [56]), and find:

$$M = \frac{m}{\Xi}, \qquad J = \frac{am}{\Xi^2}. \tag{1.35}$$

Finally, the thermodynamic conjugate to the angular momentum, the angular velocity Ω is given as

$$\Omega_{\mathrm{H}} = -\frac{g_{t\phi}}{g_{\phi\phi}}\bigg|_{r_+} = \frac{a\Xi}{r_+^2+a^2}, \tag{1.36}$$

and with all of these quantities, they were able to show that the partition function for this spacetime corresponded to the partition of a scalar field theory coupled to a three-dimensional Einstein universe rotating with angular velocity $\Omega_{\mathrm{H}} + a/\ell^2$.

The authors of [52] did not, however, discuss the first law of thermodynamics, and one can check that eq. (1.6) does not in fact hold with eqs. (1.34) to (1.36). The following year, wanting to present a thorough thermodynamic description of Kerr-Newman-AdS black holes, Caldarelli, Cognola and Klemm [53] addressed this issue. When one computes the mass using a Komar integral, there lies an ambiguity in the normalisation of the timelike killing vector used. In [52], the mass is computed with the killing vector ∂_t, however the authors of [53] compute it using the killing vector ∂_t/Ξ, while the angular momentum is still computed with ∂_ϕ. This leads to the following expressions for the mass and angular momentum:

$$M = \frac{m}{\Xi^2} \qquad J = \frac{am}{\Xi^2}. \tag{1.37}$$

The justification for this choice of normalisation is that the resulting conserved quantities above agree with expressions obtained using a Hamiltonian approach that had been presented by Henneaux and Teitelboim [57] in 1985 for the uncharged case and by Kostelecky and Perry [58] a decade later for the charged case. They also show that the same expressions can be obtained using the Brown-York method [59].

Caldarelli *et al.* then derived a Christodoulou-Ruffini-like formula for the mass, expressing it as a function $M = M(S, J, Q)$. They then reverse-engineered the first law to verify the quantities conjugate to the entropy, angular momentum and charge by computing the relevant partial derivatives. The temperature is found to agree with the charged version of eq. (1.34), and the electrostatic potential is found to agree with

$$\Phi = B_\mu \chi^\mu \Big|_{r_+}^{\infty} = \frac{e r_+}{r_+^2 + a^2}, \tag{1.38}$$

where B is the electric gauge potential from eq. (1.33) and $\chi = \partial_t + \Omega_{\mathrm{H}} \partial_\phi$ is the null generator of the horizon. Most significantly, however, they found that $\Omega = (\partial M / \partial J)_{S,Q} \neq \Omega_{\mathrm{H}}$, but rather that

$$\Omega = \frac{a(1 + r_+^2/\ell^2)}{r_+^2 + a^2} = \Omega_{\mathrm{H}} + \frac{a}{\ell^2}, \tag{1.39}$$

This happens to coincide precisely with the difference between the angular velocity at the horizon (1.36) and the angular velocity at infinity. It therefore makes sense that the physical quantity relevant to thermodynamics be the agnostic quantity $\Omega = \Omega_{\mathrm{H}} - \Omega_\infty$ measuring the angular velocity of the black hole relative to the boundary.

Finally, we would expect to be able to establish the Euler scaling relation that we encountered earlier as Smarr's formula in eq. (1.13). The statement as it stands does not hold, despite the new quantities satisfying the first law. Instead, the authors of [53] show that one may treat the cosmological constant Λ itself as a thermodynamical variable, complete with a conjugate quantity $\Theta = (\partial M / \partial \Lambda)_{S,J,Q}$, and we have the first law

$$\delta M = T \delta S + \Omega \delta J + \Phi \delta Q + \Theta \delta \Lambda. \tag{1.40}$$

This also allows us to rewrite the Christodoulou-Ruffini formula that Caldarelli *et al.* had derived as

$$\frac{1}{2} M = TS + \Omega J + \frac{1}{2} \Phi Q - \Theta \Lambda, \tag{1.41}$$

which agrees with mass being a homogeneous function of S, J, Q and now Λ, after Euler's theorem is applied. We leave the discussion of Λ as a varying quantity to the next section.

Finally, both [52, 53] find that the relation between their respective free energy potentials agree with the euclidean action according to eq. (1.28):

$$F = M - TS - \Omega J - \Phi Q = \frac{S_{\mathrm{E}}}{\beta}. \tag{1.42}$$

The fact that this relation is satisfied when using either eqs. (1.35) and (1.36), or eqs. (1.37) and (1.39) is merely a consequence of the relation

$$\frac{m}{\Xi^2} + \Omega_\infty J = \frac{m}{\Xi}.$$

Silva [54] then showed that the entropy obtained from this thermodynamic prescription agreed with the Cardy entropy [60, 61] by computing central charges of a sub-algebra deduced from the metric (1.32); Gibbons, Perry and Pope [55] extend this description to higher dimensional black holes, clarifying similar subtleties as the one presented above. They also point out that the mass as expressed in eq. (1.37) agrees with masses computed according to Abbott-Deser [62] and Ashtekar-Das-Magnon [63, 64].

1.4.2 Pressure and volume of black hole spacetimes

While the entire discussion concerning having a dynamical cosmological constant may be somewhat controversial to some, the argument can be made that whether physical mechanisms allowing for Λ to change exist or not does not preclude us from considering neighbouring configurations in parametric (incl. Λ) space. Nonetheless, the idea of doing so was initially proposed in the mid 1980s, where, in a series of papers [57, 65–69], Brown, Henneaux and Teitelboim introduce the cosmological constant as an integration constant from a theory which has a 3-form gauge potential coupled to the gravitational field. In this theory, there is a bubble radiation process which reduces the cosmological constant. In [70], it is argued that a change in the cosmological constant in the bulk is equivalent, via the AdS/CFT correspondence, to a change in the number of colors in the nonabelian field theory on the boundary. Finally, in recent work, Gregory, Kastor and Traschen [71, 72] studied the thermodynamics of a black hole system in a background cosmology undergoing slow-roll inflation and confirmed that it changes according to a first law with variable Λ.

The scaling relation (1.41) does, however, suggest that the cosmological constant should be treated as a thermodynamical variable. After this research by Brown *et al.* and the work above by Caldarelli *et al.* [53], this concept was revisited a few times in [73–76], leading to a 2009 paper by Kastor, Ray and Traschen [70] in which some of the computations from the previous section were put on much firmer mathematical ground, and a new thermodynamic interpretation was presented, in an aim to reconnect with traditional statistical mechanics.

In order to prove the Smarr relation, the authors of [70] were able to provide a derivation similar to the one we presented in section 1.2, making use of the killing potential, first introduced in [77, 78] to construct properly defined Komar integrals in asymptotically $\Lambda \neq 0$ geometries. The outcome of this derivation is that they obtained an explicit definition of the conjugate quantity $\Theta = (\partial M/\partial\Lambda)_{S,J,Q}$ that we introduced in eq. (1.41). The cosmological constant can be interpreted as a pressure

$$P = -\frac{\Lambda}{8\pi} \tag{1.43}$$

exerted on the spacetime, and in classical thermodynamics, the conjugate of a pressure is the system's volume. Indeed, Θ has units of volume and we introduce the following quantity:

$$V = -8\pi\Theta, \tag{1.44}$$

known as the *thermodynamic volume* of a black hole. The definition obtained in terms of killing potentials that was found in [70] reveals that this quantity (1.44) may be interpreted as the volume excluded from the full spacetime by the event horizon.

A consequence of the interpretation of the cosmological constant as pressure is that the first law for an uncharged nonrotating black hole now takes the form:

$$\delta M = T\delta S + V\delta P. \tag{1.45}$$

Classical thermodynamics teaches us that the mass of the black hole should therefore not be thought of as the internal energy, but as the *enthalpy* [70]. The enthalpy is related to the internal energy via the Legendre transform $U(S,V) = H(S,P) - PV$. It turns out that because H is linear with respect to P in black hole systems, this transformation is generally noninvertible (see [79] for a recent more formal treatment and discussion of the relation between the enthalpy and internal energy in black hole thermodynamics). Dolan [80, 81] has since investigated the thermodynamical consequences of this interpretation in terms of pressure, volume and enthalpy. In particular, he notes that the heat capacities, whose sign determines local thermodynamic stability, must be computed at constant pressure rather than constant volume, and the free energy which is computed from the action coincides with the Gibbs free energy

$$\begin{aligned} G(T,\Omega,\Phi,P) &= U(S,J,Q,V) - TS - \Omega J - \Phi Q + PV \\ &= H(S,J,Q,P) - TS - \Omega J - \Phi Q \\ &= M - TS - \Omega J - \Phi Q = \frac{S_{\mathrm{E}}}{\beta}. \end{aligned} \tag{1.46}$$

More recently, Cvetič *et al.* [82] computed the thermodynamic volume in a number of asymptotically-AdS black hole spacetimes, comparing it to the "naïve" geometric volume, as they refer to it, given by the integral

$$V' = \int_{r_0}^{r_+} dr \int d\Omega \sqrt{-g}, \tag{1.47}$$

from the singularity at r_0 to the outer horizon at r_+. In general, they find that these quantities agree for static geometries, however differ by a term proportional to the angular momentum, when it is nonzero. This led them to conjecture the *reverse isoperimetric inequality*.

The traditional isoperimetric inequality is the general statement of Euclidean geometry which says that the D-dimensional volume enclosed within a $(D-1)$-dimensional closed surface is maximised when that surface is spherical, or, if V' is the volume enclosed and $\mathcal{A}$ is the area of the surface that encloses it, then

$$\left(\frac{DV'}{\omega_{D-1}}\right)^{\frac{1}{D}} \leqslant \left(\frac{\mathcal{A}}{\omega_{D-1}}\right)^{\frac{1}{D-1}}, \tag{1.48}$$

where ω_{D-1} is the area of a unit D-dimensional sphere. Cvetič *et al.* examined the applicability of this relation to black hole spacetimes using the geometric volume V' and found the relation to be satisfied for uncharged black holes, however charged black holes were found to violate this statement. When the volume they used was the thermodynamic volume V, they found that *all* black hole spacetimes violated the inequality, with the Schwarzschild-AdS geometry saturating the bound. This led to the conjecture that all black holes satisfy the reverse isoperimetric inequality, which is usually re-expressed as the ratio (increasing the number of dimensions by one relative to eq. (1.48) to accommodate for time)

$$\mathcal{R} = \left(\frac{(D-1)V}{\omega_{D-2}}\right)^{\frac{1}{D-1}} \left(\frac{\omega_{D-2}}{\mathcal{A}}\right)^{\frac{1}{D-2}} \geqslant 1, \tag{1.49}$$

which is greater than unity when the conjecture is valid.

Physically, this conjecture is often interpreted to state that for a given thermodynamic volume, the black hole entropy is maximised in a Schwarzschild geometry. The inequality has been shown to hold for a plethora of asymptotically AdS and dS geometries [82, 83], though notable exceptions have also been found [84–86].

1.4.3 Thermodynamic stability and phase transitions

The laws of thermodynamics require the existence of a state of equilibrium. In flat space, one could envisage a system in a box which contains a black hole at equilibrium with a thermal bath of radiation. If this radiation bath is held at a fixed temperature, the equilibrium would be unstable: if a small excess of mass fell into the black hole, its temperature would decrease and would not be able to sustain the rate of radiation necessary to recover equilibrium and the bath would eventually collapse into the black hole; if it were to radiate too much at equilibrium, its temperature would rise and the rate of radiation would exceed the rate of absoption and the black hole would evaporate. In de Sitter (dS) space, the presence of a second, cosmological, horizon implies that equilibrium may only be truly achieved in the special Nariai limit [87, 88] of asymptotically dS black holes where the two horizons coincide, or, if the black hole is charged, in the extremal limit [89, 90].

On the other hand, the negative curvature inherent to anti-de Sitter (AdS) geometries allows the entire spacetime to act as the box we used above. Only massless states are able to escape to infinity and one can have boundary conditions such than the incoming and outgoing states cancel to maintain a thermal bath. In 1982, Hawking and Page [50] studied the physical implications of such a system and made several observations.

Parametrising the cosmological constant as $\Lambda = -3/\ell^2$, the Schwarzschild-AdS line element is

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega_{\text{II}}^2, \qquad f(r) = 1 - \frac{2M}{r} + \frac{r^2}{\ell^2}, \tag{1.50}$$

where $d\Omega_{\text{II}}^2$ is the volume element on the two-sphere, and M is the mass of the black hole. If we then identify the temperature as the period of imaginary time required to regularise the Euclidean section and the entropy as a quarter of the area, we find

$$T = \frac{f'(r_{\text{h}})}{4\pi} = \frac{1}{4\pi r_+}\left(1 + \frac{3r_{\text{h}}^2}{\ell^2}\right), \qquad S = \pi r_{\text{h}}^2, \tag{1.51}$$

where r_{h} is the location of the event horizon and we have used $f(r_{\text{h}}) = 0$ to cast the mass in terms of this parameter, though it should be noted that the mass monotonically increases with respect to r_{h}.

Local stability in thermodynamics requires positivity of the specific heat, which is given by

$$C_P = T\left(\frac{\partial S}{\partial T}\right)_P = -\frac{2\pi r_{\text{h}}^2(1 + 3r_{\text{h}}^2/\ell^2)}{1 - 3r_{\text{h}}^2/\ell^2}. \tag{1.52}$$

We notice therefore that black holes larger than $r_{\text{h}} = \ell/\sqrt{3}$ may be in a locally stable equilibrium, whereas for smaller black holes, equilibrium is unstable. This stable equilibrium is achieved for larger black holes as gaining mass increases temperature, thereby providing a higher rate of radiation for the horizon which may then lead back to equilibrium. This turning point actually corresponds to a minimal black hole temperature $T_{\text{c}} = \sqrt{3}/2\pi\ell$.

Another criteria for stability of a thermodynamical system is that its configuration be that of lowest free energy for the fixed intensive quantities. If more than one configuration is possible, then those of higher free energy will have a tendency to decay to the state of lowest free energy. The previous section tells us that the quantity we are interested in is the Gibbs free energy potential given by

$$G = M - TS = \frac{r_{\mathrm{h}}}{4}\left(1 - \frac{r_{\mathrm{h}}^2}{\ell^2}\right), \tag{1.53}$$

for the Schwarzschild-AdS black hole. There is, however, another possible thermal configuration that we may consider; vacuum AdS filled with a thermal bath has negligible free energy, however is not able to sustain itself against gravitational collapse beyond some temperature $T_{\mathrm{u}} \sim s^{-\frac{1}{4}}\ell^{-\frac{1}{2}}$ where s is the effective number of spin states of the radiation [50].

Figure 1.1 illustrates the situation we have described. For temperatures lower than T_{c}, the only possible configuration is vacuum radiation. Above this temperature, there are two branches corresponding to the larger stable black holes (lower branch) and the smaller unstable black holes (upper branch). We see that for the lower branch, there is a temperature $T_{\mathrm{HP}} = 1/\pi\ell$ above which large black holes on the stable branch have negative free energy, lower than the corresponding radiation in pure AdS. There is therefore a first order phase transition at T_{HP}, known as the *Hawking-Page phase transition.* Between $T_{\mathrm{c}} < T < T_{\mathrm{HP}}$, there are three configurations possible, however the vacuum radiation is still the most thermodynamically favoured and one would expect black holes at this temperature to decay to radiation. For comparison, an asymptotically flat black hole always has positive free energy and is therefore never stable against decay to radiation. Via the AdS/CFT correspondence, it has been shown that in the dual boundary conformal field theory, this phase transition may be interpreted as a confinement/deconfinement phase transition [91].

This analysis may be extended to charged black holes. This was done by Chamblin *et al.* [92] in the early days of holography. The inclusion of charge allows us to consider either the canonical ensemble, where we fix charge itself or the grand canonical ensemble, where we fix the electrostatic potential Φ and allow charge to vary. In the latter case, the free energy potential we will be interested is given by

$$G' = M - TS - \Phi Q. \tag{1.54}$$

In the canonical ensemble we may no longer compare black hole solutions at fixed charge to pure AdS as the vacuum on-shell field equations do not allow for a charged radiation bath in AdS. Referring to fig. 1.2a, we observe two different regimes. For $Q < Q_{\mathrm{c}}$, there are three branches; a first branch of stable black holes with $M < M_1$,

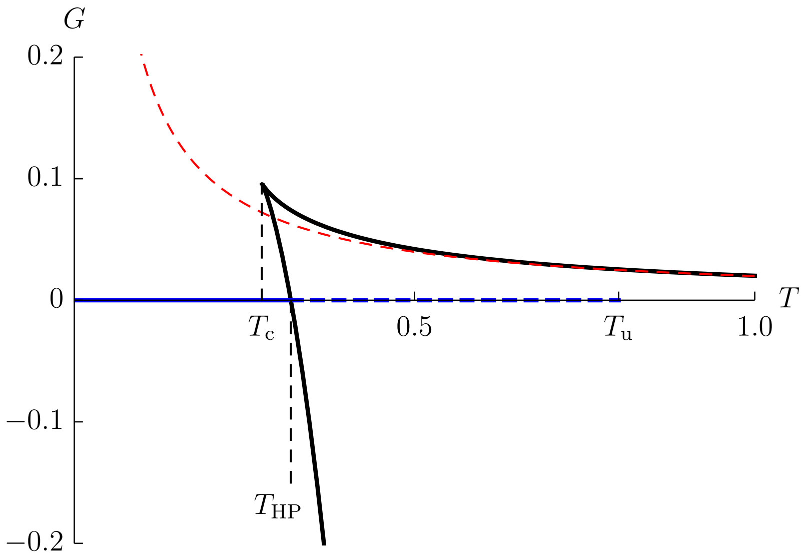


Figure 1.1: *The Hawking-Page phase transition.* The black line represents the Gibbs free energy of Schwarzschild-AdS black holes at different temperatures. The critical temperature T_c separates the upper and lower branches of this plot, corresponding to unstable smaller black holes and stable larger black holes respectively. In blue, we have vacuum radiation, which is unstable for temperatures higher than T_u (which has been arbitrarily chosen here). The red dashed line corresponds to asymptotically flat black holes and is plotted for comparison. These curves have been produced for a geometry which has $\ell = 1$ in planck units.

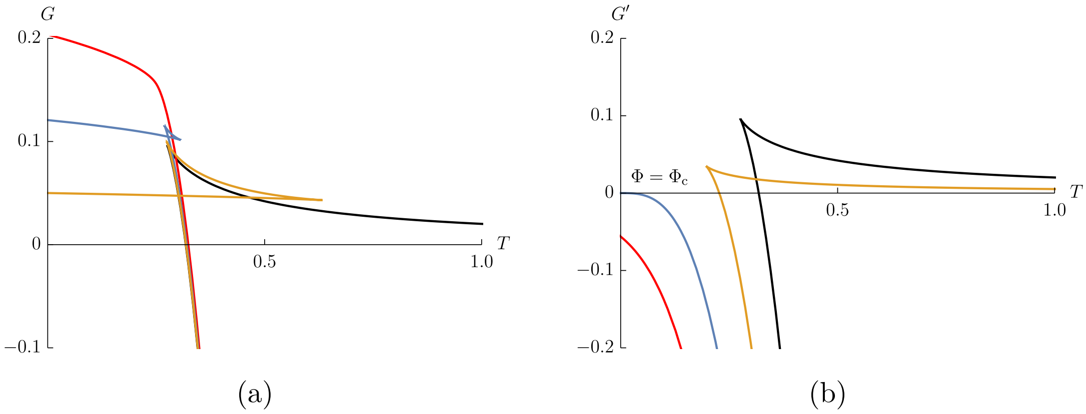


Figure 1.2: The free energy diagrams for the canonical (left) and grand canonical (right) ensembles are reproduced. For the canonical ensemble, three curves at different charges ($e = 0.05\ell$, $e = 0.12\ell$ and $e = 0.2\ell$) are displayed over the uncharged case in black. The two intermdiary curves display the characteristic "swallow tail" behaviour. For the grand canonical ensemble, we have again reproduced three curves at fixed potential ($\Phi = 0.7$, $\Phi = 1.0$ and $\Phi = 1.2$) over the black uncharged plot. These figures have all been reproduced for $\ell = 1$.

a second intermediary branch of unstable black holes with $M_1 < M < M_2$, and a third branch $M > M_2$. In the free energy diagram, the first and third branches intersect for some temperature T_* at which there is a first order phase transition between large and small black holes. These curves display what has been referred to as a "swallow tail" behaviour, referring to the region enclosed by the three branches. As the charge increases towards Q_c, this region shrinks and beyond it, in the second regime, the intersection disappears and the phase transition becomes continuous. It has been noted that this characteristic is very reminiscent of the liquid/gas phase transition of a Van-der-Waals fluid.

In the grand canonical ensemble we may still compare our solutions to a background of vacuum radiation in AdS with a correspondingly fixed electrostatic potential. From fig. 1.2b, we see there is a critical value for this potential which we denote Φ_c. Configurations with a potential smaller than this are similar to the uncharged scenario, with the existence of a Hawking-Page-like phase transition for the lower, stable, of two branches, and no black hole solutions existing below a certain temperature. For larger potentials, the free energy is always negative and the black hole solution is always preferred.

In this chapter, we have seen how the framework of black hole thermodynamics is undoubtedly a fascinating one. The ability to breathe life back into as old a subject as thermodynamics by applying it to modern and exotic solutions of gravitational physics is certainly exciting. The rich phase structure that black holes possess, presented in the last section, is a powerful example of the utility of thermodynamics.

Chapter 2

Acceleration and the C-metric

2.1 Origins of the C-metric

The first nontrivial solution to the Einstein field equations of General Relativity appeared in January 1916, little over a month after their initial publication in late November 1915 [93, 94]. It was Karl Schwarzschild who initially discovered the spacetime and presented it as the most general solution to describe a spherically-symmetric solution to the vacuum field equations [95–97]. This solution went largely misunderstood for the next few decades as various attempts were made to comprehend the singularities the metric contains. It wasn't until 1958 that David Finkelstein, using a set of coordinates earlier discovered by Arthur Eddington that smoothly patched over the coordinate singularity, provided the interpretation of this singularity as a surface which could only be traversed in one direction [98]. The spacetime eventually went on to become known as the Schwarzschild black hole, however it is easy to forget how much time elapsed between the discovery of the solution and for a widely accepted interpretation to be presented.

Within the first few years of the field equations being known, other solutions were found. These were mostly mathematical solutions whose physical meanings were unknown, much like the Schwarzschild metric. Of particular interest to this thesis is a metric belonging to a larger class of solutions discovered by Levi-Civita in 1918 [99].

This metric would then be rediscovered in the early 1960s, in a wave of research presumably inspired by the aforementioned understanding of black holes, by Newman and Tamburino [100], Robinson and Trautman [101] and then again by Ehlers and Kundt [102]. In particular, Ehlers and Kundt explicitly take the initial body of work laid out by Levi-Civita, cited above, and, in their own words, "follow his line of thought" and "simplified and completed his derivations". They continue on to present an invariant classification of degenerate static vacuum fields. Without delving into the details, it is this classification of "A", "B" and "C"-type metrics which gives its name, to this day, to the *C-metric*, the axisymmetric vacuum solution describing an accelerating black hole.

At that point in time, the C-metric was not yet fully understood, but it was seen as a metric that bridges the gap between the Schwarzschild metric and its charged counterpart (Reissner-Nordström), and the Weyl and Robinson-Trautman solutions [103].

Eventually, in 1970, Kinnersley and Walker (KW) picked up the C-metric and finally provided the interpretation of this solution as an *accelerated* black hole [103]. Let us now review how KW obtained their version of the C-metric. The Levi-Civita solution is given in the form

$$ds^2 = \frac{1}{(x+y)^2}\left(-F(y)dt^2 + \frac{dy^2}{F(y)} + \frac{dx^2}{G(x)} + G(x)dz^2\right) \tag{2.1}$$

where

$$G(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \qquad \text{and} \qquad F(y) = -G(-y), \tag{2.2}$$

with $a_0 \dots a_3$ some set of parametrising constants. The metric admits a conformal transformation given by

$$x \to \eta\beta x + \varepsilon, \quad y \to \eta\beta y - \varepsilon, \quad t \to \eta t \quad \text{and} \quad z \to \eta z, \tag{2.3}$$

with η, β and ε serving as real-valued parameters for the transformation and as long as $G(x) \to \beta G(x)$, which imposes the following relations between the coefficients of G before (denoted by a tilde $\tilde{}$) and after the transformation:

$$\beta a_0 = \tilde{a}_0 + \tilde{a}_1\varepsilon + \tilde{a}_2\varepsilon^2 + \tilde{a}_3\varepsilon^3, \tag{2.4a}$$

$$a_1 = \eta\tilde{a}_1 + 2\eta\tilde{a}_2\varepsilon + 3\eta\tilde{a}_3\varepsilon^2, \tag{2.4b}$$

$$a_2 = \eta^2\beta\tilde{a}_2 + 3\eta^2\beta\tilde{a}_3\varepsilon, \tag{2.4c}$$

$$a_3 = \eta^3\beta^2\tilde{a}_3. \tag{2.4d}$$

This symmetry allows us to remove two degrees of freedom from the initial solution, and it is this freedom of parametrisation which has led to some confusion as to its interpretation. The final setup that Kinnersley and Walker present involves setting $a_1 = 0$, $a_0 = -a_2 = 1$ and then labelling $\beta = A^2$ and $a_3 = -2mA$. The final form (including a charged term, introduced later) of the C-metric, as written in [103] is therefore given by

$$ds^2 = \frac{1}{A^2(x+y)^2}\left(-F(y)dt^2 + \frac{dy^2}{F(y)} + \frac{dx^2}{G(x)} + G(x)dz^2\right) \tag{2.5}$$

where

$$G(x) = 1 - x^2 - 2mAx^3 - e^2A^2x^4 \qquad \text{and} \qquad F(y) = -G(-y). \tag{2.6}$$

This paper was published around the time Kinnersley published his doctoral thesis, in which he also shows that the charged C-metric is simply obtained by adding in the quartic term above, along with the gauge potential $B = -eAydt$. For now, the parameters m and A can just be interpreted as the two remaining degrees of freedom for this solution, however it will become apparent that these can be thought of as (being related to) the mass and the acceleration of a massive particle/black hole respectively. The Schwarzschild and Reissner-Nordström spacetimes are then recovered by first changing $y = 1/Ar$ and $t \to At$, taking the limit as $A \to 0$ and then identifying $x = \cos\theta$ and $z = \phi$.

In order to understand the accelerating nature of this solution, it will be useful to first briefly review Rindler coordinates, which are simply a set of coordinates well-suited to uniformly accelerating trajectories. To see what this means, let us consider the path of an uniformly accelerating object in flat 1+1-dimensional space. The accelerations as measured locally by the object, α, and as measured in a static lab frame, a, are related by

$$\alpha = \gamma^3 a, \tag{2.7}$$

where $\gamma = (1-v^2)^{-1/2}$ is the usual Lorentz factor, with v the instantaneous velocity of the object as measured by the static observer. Solving this equation for its position, denoted by x, reveals that the path taken by an accelerating observer satisfies

$$-t^2 + x^2 = \frac{1}{\alpha^2}. \tag{2.8}$$

This is known as hyperbolic motion, and we have the following parametrisation:

$$\begin{aligned} x &= \frac{1}{\alpha}\cosh\alpha\tau, \\ t &= \frac{1}{\alpha}\sinh\alpha\tau. \end{aligned} \tag{2.9}$$

We may use this parametrisation to obtain a set of coordinates which is centred on the idea of accelerated objects, in other words, coordinates for which paths at constant coordinate are these hyperbolic trajectories. Such a transformation is given by:

$$x = \frac{\xi}{\alpha}\cosh\alpha\tau, \qquad t = \frac{\xi}{\alpha}\sinh\alpha\tau. \tag{2.10}$$

This transformation gives rise to what is known as the *Rindler metric*,

$$ds^2 = -\xi^2 d\tau^2 + \frac{d\xi^2}{\alpha^2}. \tag{2.11}$$

These coordinates cover the region $t^2 < x^2$. This is due to the presence of an *acceleration horizon* at $t = \pm x$, or $\xi = 0$. The analytic continuation over the horizon is simply done by reverting back to the original cartesian coordinates.

Let us now return to the C-metric and its interpretation. To see the accelerating nature of the metric, we will work from its flat-space limit, which we obtain from eq. (2.5) by setting $m = e = 0$. The metric is now

$$ds^2 = \frac{1}{A^2(x+y)^2}\left(-(y^2-1)A^2dt^2 + \frac{dy^2}{y^2-1} + \frac{dx^2}{1-x^2} + (1-x^2)dz^2\right). \tag{2.12}$$

Using the following nontrivial coordinate transformation:

$$\xi = \frac{\sqrt{y^2-1}}{x+y}, \qquad \rho = \frac{1}{A}\frac{\sqrt{1-x^2}}{x+y}, \qquad t = A\tau, \qquad z = \varphi, \tag{2.13}$$

we are able to recover the Rindler metric in 3+1-dimensional cylindrical coordinates:

$$ds^2 = -\xi^2 d\tau^2 + \frac{d\xi^2}{A^2} + d\rho^2 + \rho^2 d\varphi^2. \tag{2.14}$$

This reinforces the interpretation of A as the acceleration parameter. The acceleration horizon, $\xi = 0$ is therefore located, in the original coordinates, at $y = 1$. As mentioned above, it is also possible to write the C-metric in a pseudo-spherical coordinate system with a radial coordinate r, related to y through the substitution $y = 1/Ar$, and we find that the acceleration horizon is located at $r = 1/A$. Intuitively, therefore, it makes sense that the acceleration horizon be located farther away with vanishing acceleration. Similarly, we also have that the origin of this spherical system, $r = 0$, which corresponds to $\xi = 1$, follows the trajectory (2.9) that our initial accelerating object did.

2.2 The Plebański-Demiański metric

The Plebański-Demiański (PD) family of solutions [104] was published in 1976, and was described as a new class of stationary and axisymmetric solutions to the Einstein-Maxwell-Λ field equations. In their paper, PD show how, through a series of different coordinate transformations, one could obtain from their metric most of the known black hole solutions, including, as we shall see below, the accelerating C-metric as well as the rotating Kerr metric. Additionally, the solution also accounted for electromagnetic fields, a cosmological constant Λ and even contains a parameter which in certain limits can be identified as the NUT charge. In the sections below, we will demonstrate how one recovers the C-metric from this solution, how the Kerr metric is also obtained from this parent solution, and finally, we shall present modern modifications to the metric that have vastly simplified calculations involving the generalised (rotating, charged) C-metric. Before that, however, let us give the Plebański-Demiański metric[1]:

$$\begin{aligned} ds^2 = \frac{1}{(p+q)^2}\Bigg(&-\frac{Q(q)}{1+(pq)^2}(d\tau + p^2 d\sigma)^2 + \frac{1+(pq)^2}{Q(q)}dq^2 \\ &+\frac{1+(pq)^2}{P(p)}dp^2 + \frac{P(p)}{1+(pq)^2}(d\sigma - q^2 d\tau)^2\Bigg), \end{aligned} \tag{2.15}$$

where

$$\begin{aligned} Q &= -\frac{\Lambda}{6} + g^2 - \gamma - 2nq + \epsilon q^2 - 2mq^3 + \left(-\frac{\Lambda}{6} + e^2 + \gamma\right) q^4, \\ P &= -\frac{\Lambda}{6} - g^2 + \gamma - 2np - \epsilon p^2 - 2mp^3 + \left(-\frac{\Lambda}{6} - e^2 - \gamma\right) p^4, \end{aligned} \tag{2.16}$$

where m, n, e, g, ϵ and γ are real parameters of the solution. Additionally, it is worth drawing attention to the presence of a cosmological constant term, as this term can be carried through the derivation of the C-metric, yielding an accelerating black hole with a nonzero Λ. While unfamiliar, the naming choice for these coordinates facilitates an agnostic treatment of this solution. To simplify the derivations below, we perform the parametric shift $\gamma \to \gamma + g^2 + \frac{\Lambda}{6}$. The metric functions are now given by

$$\begin{aligned} Q &= -\frac{\Lambda}{3} - \gamma - 2nq + \epsilon q^2 - 2mq^3 + \left(\gamma + e^2 + g^2\right) q^4, \\ P &= \gamma - 2np - \epsilon p^2 - 2mp^3 - \left(\gamma + e^2 + g^2 + \frac{\Lambda}{3}\right) p^4. \end{aligned} \tag{2.17}$$

Since the work presented in this thesis focusses on asymptotically AdS spacetimes, we shall write

$$\Lambda = -\frac{3}{\ell^2}.$$

[1]Compared to the original text, we have taken p, q and σ to have opposite signs, for consistency.

2.2.1 From Plebański-Demiański to the C-metric

In order to show how the Plebański-Demiański metric contracts to the C-metric, we must first perform the following rescaling of our coordinates:

$$p \to \sqrt{aA}p, \qquad q \to \sqrt{aA}q, \qquad \tau \to \sqrt{\frac{a}{A^3}}\tau, \qquad \sigma \to \sqrt{\frac{a}{A^3}}\sigma, \tag{2.18}$$

where we have introduced new nonzero parameters a and A in anticipation of what follows. To compensate for this rescaling, we may simplify the metric with the following parameter/function rescalings [104, 105]:

$$\begin{gathered} m \to \left(\frac{A}{a}\right)^{\frac{3}{2}} m, \qquad n \to \left(\frac{A}{a}\right)^{\frac{1}{2}} n, \qquad e \to \frac{A}{a}e, \qquad g \to \frac{A}{a}g, \\ \epsilon \to \frac{A}{a}\epsilon, \qquad \gamma \to A^2\gamma, \qquad P \to A^2 P, \qquad Q \to A^2 Q. \end{gathered} \tag{2.19}$$

The resulting metric is given by

$$\begin{aligned} ds^2 = \frac{1}{A^2(p+q)^2} \Bigg(& -\frac{Q(q)}{1+(aApq)^2}(d\tau + aAp^2 d\sigma)^2 + \frac{1+(aApq)^2}{Q(q)}dq^2 \\ & + \frac{1+(aApq)^2}{P(p)}dp^2 + \frac{P(p)}{1+(aApq)^2}(d\sigma - aAq^2 d\tau)^2 \Bigg), \end{aligned} \tag{2.20}$$

with

$$\begin{aligned} Q &= \frac{1}{A^2\ell^2} - \gamma - \frac{2nq}{A} + \epsilon q^2 - 2mAq^3 + A^2\left(\gamma a^2 + e^2 + g^2\right) q^4, \\ P &= \gamma - \frac{2np}{A} - \epsilon p^2 - 2mAp^3 + \left(\frac{a^2}{\ell^2} - A^2(\gamma a^2 + e^2 + g^2)\right) p^4. \end{aligned} \tag{2.21}$$

Now, according to [104], the curvature invariants of this spacetime do not depend on the parameters ϵ, n and γ. This tells us that these parameters are (locally) simple gauge choices, and we may fix them without affecting the geometry. To recover the C-metric, we first take the limit $a \to 0$, after which we are free to set $\epsilon = \gamma = 1$ and $n = 0$. The C-metric, as written in eq. (2.5), is then simply obtained by identifying $\tau = t$, $q = y$, $p = x$ and $\sigma = z$. The metric (2.20) is actually known as the spinning C-metric (SC-metric), as further work [105–108] showed that while not analytically pleasant, this metric displays characteristics of a rotating black hole while preserving its acceleratng nature.

2.2.2 From Plebański-Demiański to the Kerr-Newman metric

The rotating nature of the metric (2.20) becomes apparent when we switch to Boyer-Lindquist-type, or pseudo-spherical, coordinates. The following transformation:

$$\tau \to A\left(\tau - \frac{a\sigma}{\Xi}\right), \qquad q \to \frac{1}{Ar}, \qquad \sigma \to \frac{\sigma}{\Xi} \tag{2.22}$$

$$\text{with} \quad Q \to \frac{Q}{A^2 r^4}, \quad \text{and} \quad \Xi = 1 - \frac{a^2}{\ell^2} + \mathcal{O}(A), \tag{2.23}$$

where we have included the possibility for a term in Ξ which may depend on acceleration but must vanish in its absence, produces the following metric:

$$\begin{aligned} ds^2 = \frac{1}{(1+Arp)^2}\Bigg(&-\frac{Q(r)}{r^2+a^2p^2}\left(d\tau - a(1-p^2)\frac{d\sigma}{\Xi}\right)^2 + \frac{r^2+a^2p^2}{Q(r)}dr^2 \\ &+\frac{r^2+a^2p^2}{P(p)}dp^2 + \frac{P(p)}{r^2+a^2p^2}\left(ad\tau - (r^2+a^2)\frac{d\sigma}{\Xi}\right)^2\Bigg), \end{aligned} \tag{2.24}$$

with

$$\begin{aligned} Q &= (\gamma a^2 + e^2 + g^2) - 2mr + \epsilon r^2 - 2nr^3 + r^4\left(\frac{1}{\ell^2} - A^2\gamma\right), \\ P &= \gamma - \frac{2np}{A} - \epsilon p^2 - 2mAp^3 + \left(\frac{a^2}{\ell^2} - A^2(\gamma a^2 + e^2 + g^2)\right)p^4. \end{aligned} \tag{2.25}$$

The physical parameters of this solution are m, n, e and g, with γ and ϵ absent from curvature invariants. Unfortunately, this metric does not present a convenient way of writing $p = p(\theta)$ unless we set $A = 0$. The reason for this is that ordinarily, one would write $p = \cos\theta$, however for terms to cancel out neatly, we would expect to be able to factorise P such that a $\sin^2\theta$ piece could be pulled out. For $A = 0$, we are able to do this, provided we are able to set $n = 0$ and $\gamma = 1$ as before, and this time $\epsilon = 1 + a^2/\ell^2$, which gives the Kerr-Newman-AdS metric. In fact, in this limit, the parameter n was identified in [104] as the NUT charge.

2.2.3 The factorised C-metric

The issue that prevented us in the previous section from expressing the C-metric in pseudo-spherical coordinates was that we are unable to factorise the metric functions for the transformation to be convincing. This was an ugly, but well-known, symptom in C-metric calculations that results from the high order polynomials which make up its components. Most computations with this metric will generically revolve around the coordinate ranges, which are dictated by the metric functions and their root configuration. An immediate example is the range of the azimuthal coordinate —

covered in section 2.3.3 — which, if a regularity condition at one of the poles is imposed, depends on one of the roots of P ($\equiv G$).

Without rotation

Inspired by [109], in 2003, Hong and Teo (HT) realised that they could use the symmetry in eq. (2.3) and eq. (2.4) to re-express the metric in a way that simplified calculations [110]. HT realised that by re-tuning this set of function coefficients — those that Kinnersley and Walker picked to provide the original interpretation — they could express these polynomials in a factorised form with simple roots. In the uncharged case, and using the same notation, this was achieved by setting $a_0 = -a_2 = 1$ as before and $a_1 = -a_3 = 2mA$. The C-metric in Hong-Teo form is then still given by eq. (2.5), however the metric functions now factorise nicely:

$$G(x) = (1 - x^2)(1 + 2mAx) \qquad \text{and} \qquad F(y) = -G(-y). \tag{2.26}$$

It is important to stress that the constants m and A used here are not the same as those in the KW form of the metric. The relation between the two metrics and their parameters can be found by simply applying the coordinate transformation (2.3) between the two parameter spaces. For the original metric, we had $\tilde{a}_0 = -\tilde{a}_2 = 1$, $\tilde{a}_1 = 0$ and $\tilde{a}_3 = -2\tilde{m}\tilde{A}$. To preserve the metric (2.5), we see that the parameter β of the transformation must be $\beta = A^2/\tilde{A}^2$. Equations (2.4a) to (2.4c) provide an initial set of relations:

$$\beta = 1 - \varepsilon^2 - 2\tilde{m}\tilde{A} = \frac{A^2}{\tilde{A}^2}, \tag{2.27a}$$

$$mA = -\eta\varepsilon(1 + 3\tilde{m}\tilde{A}\varepsilon), \tag{2.27b}$$

$$\eta = \beta^{-\frac{1}{2}}(1 + 6\tilde{m}\tilde{A}\varepsilon)^{-\frac{1}{2}}, \tag{2.27c}$$

and eq. (2.4d) imposes the following condition on ε:

$$\tilde{m}\tilde{A} + \varepsilon + 8\tilde{m}\tilde{A}\varepsilon^2 + 16\tilde{m}^2\tilde{A}^2\varepsilon^3 = 0. \tag{2.27d}$$

As we show in section 2.3.2, there is an upper bound on mA for this spacetime. In the Kinnersley-Walker parametrisation of the C-metric, $0 \leqslant \tilde{m}\tilde{A} \leqslant 1/(3\sqrt{3})$, and in this regime, the above constraint (2.27d) has only one solution for ε, which can in turn be used to fully determine the remainder of the transformation. As it is unpleasantly apparent, there is a sense in which the complicated root structure of the KW parametrisation has been shifted and swept into the parameters themselves.

With rotation

Following on from their work on the nonrotating C-metric, a year later, Hong and Teo published a similarly factorised version of the C-metric which contained a rotation parameter [111]. Unlike the spinning C-metric, however, due to the factorised nature of this new metric, not only is it more pleasant to work with, but it may also be fully written in Boyer-Lindquist-type coordinates, in such a way that either the nonrotating (charged, AdS) C-metric or the Kerr-Newman-AdS metric may be obtained simply by turning off their respective parameters.

In order to obtain such a metric, the starting point is different to the nonrotating case. Rather than search for a convenient coordinate transformation, HT utilised a top-down approach starting from the Plebański-Demiański metric (2.15). In fact, the derivation requires making the same rescaling as we did earlier, therefore we will pick up from eq. (2.20), the metric, and eq. (2.21), its functions. For convenience, the latter are given by

$$\begin{aligned} Q &= \frac{1}{A^2\ell^2} - \gamma - \frac{2nq}{A} + \epsilon q^2 - 2mAq^3 + A^2\left(\gamma a^2 + e^2 + g^2\right) q^4, \\ P &= \gamma - \frac{2np}{A} - \epsilon p^2 - 2mAp^3 + \left(\frac{a^2}{\ell^2} - A^2(\gamma a^2 + e^2 + g^2)\right) p^4. \end{aligned} \tag{2.28}$$

The gauge freedom in this solution is such that we are free to pick γ and ϵ without affecting the physical geometry of the spacetime. Indeed, earlier this was used to recover the nonrotating C-metric in the KW parametrisation. Additionally, the parameter n can be related to the NUT charge l [112, 113]. The factorised metric is obtained by the requirement that the NUT charge vanish, which sets $n = -mA^2$. We then use the aforementioned gauge freedom to set

$$\gamma = 1, \qquad \epsilon = 1 + \frac{a^2}{\ell^2} - A^2(a^2 + e^2 + g^2).$$

With these choices, we have the following factorised metric functions, all the while preserving the form of the rescaled Plebański-Demiański metric (2.20):

$$\begin{aligned} Q &= \frac{1}{A^2\ell^2}(1 + a^2A^2q^2) + (q^2 - 1)(1 - 2mAq + A^2(a^2 + e^2 + g^2)q^2), \\ P &= (1 - p^2)\left(1 + 2mAp + \left(A^2(a^2 + e^2 + g^2) - \frac{a^2}{\ell^2}\right)p^2\right). \end{aligned} \tag{2.29}$$

The nonrotating C-metric that HT first presented is recovered here for $\Lambda = a = 0$. The factorised nature of this solution allows us to write it in Boyer-Lindquist coordinates with the following similar transformation to the one we used to derive the Kerr-AdS metric (2.23):

$$\tau \to A\left(t - \frac{a\phi}{K}\right), \qquad q \to \frac{1}{Ar}, \qquad p \to \cos\theta, \qquad \sigma \to \frac{\phi}{K}. \tag{2.30}$$

The resulting metric is what we shall henceforth refer to as the *generalised C-metric*, which we will use in future chapters. It is given by

$$\begin{aligned} ds^2 = \frac{1}{\Omega^2}\bigg\{ -\frac{f(r)}{\Sigma}\left[dt - a\sin^2\theta\frac{d\phi}{K}\right]^2 + \frac{\Sigma}{f(r)}dr^2 + \frac{\Sigma r^2}{g(\theta)}d\theta^2 \\ + \frac{g(\theta)\sin^2\theta}{\Sigma r^2}\left[adt - (r^2+a^2)\frac{d\phi}{K}\right]^2\bigg\}, \end{aligned} \tag{2.31a}$$

and the metric functions are

$$\begin{aligned} f(r) &= (1 - A^2r^2)\left[1 - \frac{2m}{r} + \frac{a^2+e^2+g^2}{r^2}\right] + \frac{r^2+a^2}{\ell^2}, \\ g(\theta) &= 1 + 2mA\cos\theta + \left[A^2(a^2+e^2+g^2) - \frac{a^2}{\ell^2}\right]\cos^2\theta, \\ \Sigma &= 1 + \frac{a^2}{r^2}\cos^2\theta, \qquad \Omega = 1 + Ar\cos\theta. \end{aligned} \tag{2.31b}$$

The parameter K that we have introduced allows us to define ϕ such that it has a 2π-periodicity. We will discuss its role extensively in further sections; let it therefore simply be said for now that it affects the distribution of conical defects in the spacetime.

2.3 Reviewing the C-metric and its features

So far, we have seen how the C-metric and its interpretation were developed and better understood. We will now dive in a little deeper and present some subtle aspects of the solution, such as horizon structure, coordinate and parameter ranges as well as clarifying this idea of having a conical defect along one axis. We will begin with the nonrotating form for simplicity, before covering the generalised C-metric.

2.3.1 The nonrotating C-metric

We begin with the asymptotically AdS form of the C-metric. In asymptotically flat space, the C-metric describes a configuration of two black holes accelerating in opposite directions. Each black hole has unequal conical deficits extending from the north and south poles of each event horizon to either the boundary or an acceleration horizon that separates the two. The introduction of a cosmological constant, as we shall see, alters the picture if it is large enough. In that case, the solution describes only one such black hole without an acceleration horizon, and both deficits extend out to the boundary. Although the C-metric is well-known among relativists, there are features of the specific form we will be using that are worth highlighting, discussing how they depend on the parameters of the solution.

For convenience we rewrite the nonrotating charged AdS C-metric [110, 111]:

$$ds^2 = \frac{1}{A^2(x+y)^2}\left(-F(y)dt^2 + \frac{dy^2}{F(y)} + \frac{dx^2}{G(x)} + G(x)dz^2\right), \tag{2.32a}$$

and its metric functions

$$\begin{aligned} F(y) &= \frac{1}{A^2\ell^2} + (y^2-1)(1-2mAy+e^2A^2y^2), \\ G(x) &= (1-x^2)(1+2mAx+e^2A^2x^2). \end{aligned} \tag{2.32b}$$

The factorised metric functions allow us to write this metric also in pseudo-spherical coordinates given by

$$t \to At, \qquad y \to \frac{1}{Ar}, \qquad x \to \cos\theta, \qquad z \to \frac{\phi}{K}. \tag{2.33}$$

The resulting metric is:

$$ds^2 = \frac{1}{\Omega^2}\left[f(r)dt^2 - \frac{dr^2}{f(r)} - r^2\left(\frac{d\theta^2}{g(\theta)} + g(\theta)\sin^2\theta\frac{d\phi^2}{K^2}\right)\right]. \tag{2.34}$$

The conformal factor

$$\Omega = 1 + Ar\cos\theta \tag{2.35}$$

sets the location of the boundary at $r_{\text{bd}} = -1/A\cos\theta$. The other metric functions are given as

$$\begin{aligned} f(r) &= (1-A^2r^2)\left(1-\frac{2m}{r}+\frac{e^2}{r^2}\right) + \frac{r^2}{\ell^2}, \\ g(\theta) &= 1 + 2mA\cos\theta + e^2A^2\cos^2\theta. \end{aligned} \tag{2.36}$$

The remaining parameters, e, m, $A \geqslant 0$ are related to the charge, mass and acceleration of the black hole. In the following sections we will discuss these coordinates and their ranges and review various constraints we must impose on these parameters. We will then also briefly discuss the existence of conical defects in the spacetime before tackling the rotating solution.

2.3.2 Coordinate ranges and parametric restrictions

While easier to interpret physically, Boyer-Lindquist coordinates are not the best suited for the parametric analysis we will present. This is due to the location of the boundary which is shifted from its usual location at $r = \infty$. The interpretation is that for certain values of θ the boundary is closer than infinity, and that for others it is in fact *beyond* infinity, in the sense that we must glue together the regions $0 < r < \infty$ and $-\infty < r < r_{\mathrm{bd}}$. Rather, it is more appropriate to be using the (t, y, x, z) coordinate system.

The premise we will be basing our analysis on is that we will require the signature of the metric be preserved over the coordinate ranges allowed. Since the metric depends on neither t nor z, it follows that we simply require these two coordinates be real. We have also encountered the fact that z behaves like an angular coordinate, and therefore should have some periodicity Δz attributed to it. This will be the subject of the next section. The Kretschmann scalar

$$R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta} = \frac{24}{\ell^4} + (x+y)^6\left(48A^6m^2 + \mathcal{O}(e^2)\right) \tag{2.37}$$

reveals curvature singularities at $|x|, |y| \to \infty$. We have also seen that the Hong-Teo [110] factorisation allows us to map $x = \cos\theta$. It therefore makes sense that $x \in [-1, 1]$, and we must require that $G(x)$ be positive over this range. This condition will be satisfied provided

$$e^2A^2 > \begin{cases} 2mA - 1 & \text{if} \quad mA \leqslant 1, \\ m^2A^2 & \text{if} \quad mA > 1. \end{cases} \tag{2.38}$$

In the absence of charge, this reduces to the simple requirement that $2mA \leqslant 1$. It is for this reason that one may view this condition as an upper bound on acceleration, however this is not so for larger values of charge e. The set of parameters excluded by eq. (2.38) is reproduced as blue hatching in fig. 2.1. By this condition, configurations with larger values of A would need to be charged.

We have also seen that y behaves like an inverse radial coordinate, and the zeros of $F(y)$ therefore correspond to horizons. From the metric (2.32a), the boundary is located at $y_{\mathrm{bd}} = -x$. This distinguishes two possible regions for y, however, for the positive parameter configurations we are restricting ourselves to, the region $y < -x$ will always have a naked singularity at $y \to -\infty$. With charge, generic configurations will always have 0, 2, or 4 distinct horizons. Physically, we have a pair of inner and outer horizons similar to those that characterise the regular Reissner-Nordström (RN) solution which typically approach one another and vanish with larger charge. We also have an acceleration horizon inherent to these accelerating spacetimes. When

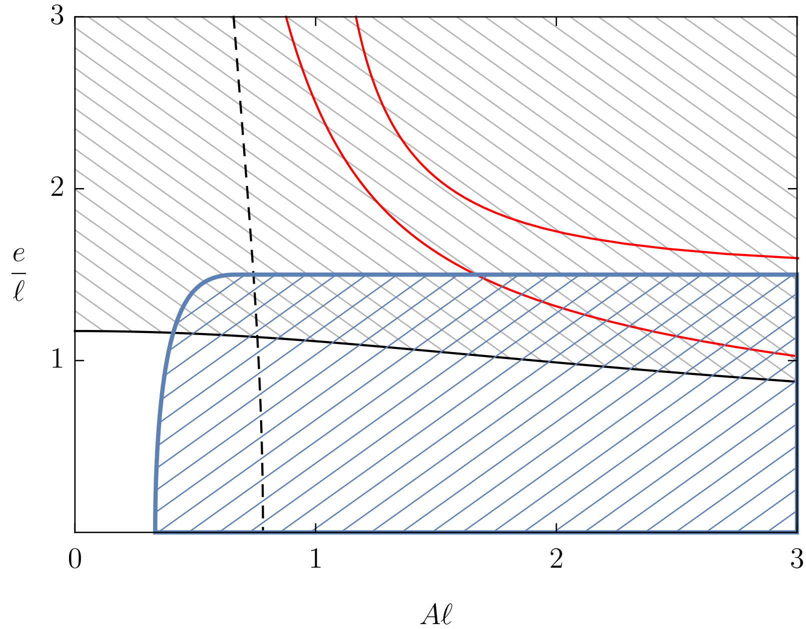


Figure 2.1: Parametric space for the nonrotating C-metric at $m = 1.5\ell$. The blue hatched region corresponds to solutions excluded by eq. (2.38). The region above the solid black line is excluded via cosmic censorship. The dashed black line corresponds to A_{crit}, and separates solutions which have an acceleration horizon to the left, and those that don't to the right. The red lines delimit parametric regions corresponding to spacetimes with additional horizons that intersect the boundary.

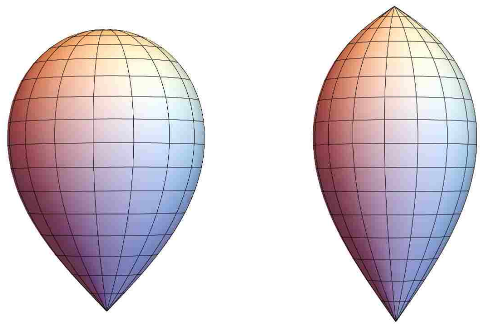
an acceleration horizon is present, there is a second "outer" acceleration horizon, and both of these intersect with the boundary. Pairs of horizons divide the spacetime into regions which share the same signature; a region (i) within the innermost RN horizon which is excluded through cosmic censorship, a region (ii) between the outer RN horizon and the regular acceleration horizon, which we are most interested in, and finally a region (iii) beyond the outermost acceleration horizon which is of little interest to the work we shall be presenting. In fig. 2.1, each line (other than the blue line) delimits configurations with different numbers of horizons. First, the dashed line corresponds to values of A beyond which there is an acceleration horizon, the solid black line corresponds to the extremal limit above which we have a naked singularity in region (ii). The space between the red lines also has four horizons; another pair of inner and outer horizons forms in region (ii), however they too intersect the boundary. This pair further subdivides region (ii), however neither region is of interest as in one we have a curvature singularity visible to the boundary and the other is similar to region (iii). At the uppermost red line in fig. 2.1, the new intermediary region is "absorbed" by the two horizons that bound it, and we are left with two horizons. These nuances were explored in [114], where many of these horizons were sorted into two categories of horizons: black *funnels* — these are the bulk duals of black holes strongly coupled to a field theory plasma, and black *droplets*, which are dual to weakly coupled black holes.

2.3.3 The conical defect

A conical deficit $\delta = 2\pi - \Delta\varphi$ (here φ is a generic azimuthal coordinate) is associated to the presence of a cosmic string with tension $\mu = \delta/8\pi$. One may introduce such defects to familiar spacetimes. For example, one may write the Schwarzschild metric, only this time with $g_{\phi\phi}^{\text{Sch}} = r^2 \sin^2\theta K^{-2}$. For $K > 1$, the result is a black hole with a string running through its core [115]. The deficit along both the $\theta = 0$ and $\theta = \pi$ axes is the same, and the tension of the string is $\mu = \frac{1}{4}(1 - K^{-1})$. The C-metric has unequal deficits, and the resulting string tension imbalance is what physically drives the acceleration.

The conical defect inherent to the C-metric is controlled through the periodicity of z. For simplicity, we pick a new coordinate $\phi = Kz$ such that its periodicity $\Delta\phi = 2\pi$, and the choice of K replaces the choice of Δz. We choose to do this (a) for familiarity and (b) so that this apparent degree of freedom is explicit in computations. The angular part of the metric in eq. (2.34) near the poles is

$$ds^2 \sim \frac{1}{\Omega^2}\frac{r^2}{g(\theta_\pm)}\left[d\rho^2 + \frac{g^2(\theta_\pm)\rho^2}{K^2}d\phi^2\right]. \tag{2.39}$$



(a) (b)

Figure 2.2: Embeddings of the nonrotating C-metric in $\mathbb{E}^3$, for the black hole with $A = 0.01\ell$, $m = 9\ell$ and (a) $K = \Xi$, (b) $K = 1.2\Xi$.

where $\theta = \theta_\pm \pm \rho$ with $\theta_+ = 0$ and $\theta_- = \pi$. The conical deficits along each axis are given by

$$\delta_\pm = 2\pi\left[1 - \frac{g(\theta_\pm)}{K}\right] = 2\pi\left[1 - \frac{1 \pm 2mA + e^2A^2}{K}\right]. \tag{2.40}$$

The tensions of cosmic strings connecting the event horizon to the boundary (or acceleration horizon, if there is one) are related to the deficits, and given by $\mu_\pm = \delta_\pm/8\pi$. It is now evident how our choice of K will impact the geometry of the spacetime; specifically, along with A, it will regulate the distribution of tensions along either axis. It is also worth mentioning that a negative deficit (corresponding to an excess) is possible, however this would be sourced by a negative energy object. We can remove one of the deficits for the C-metric by defining $K = \Xi, \Xi'$ where

$$\Xi, \Xi' = 1 \pm 2mA + e^2A^2. \tag{2.41}$$

Since having $K = \Xi'$ induces an excess at the other pole, it is generally the custom to have $K = \Xi$, regularising the north pole, and only having a string at the south pole. In fig. 2.2 we illustrate how changing K affects an embedding of the horizon in $\mathbb{E}^3$.

2.3.4 The rotating C-metric

It will be of interest to extend the observations made in the previous section to the rotating C-metric. The approaches used in this section will be very similar to what is presented above, and the results obtained below follow accordingly. The metric we will be using is the solution we derived earlier from the Plebański-Demiański metric (2.31),

$$ds^2 = \frac{1}{\Omega^2}\left\{ -\frac{f(r)}{\Sigma}\left[dt - a\sin^2\theta\frac{d\phi}{K}\right]^2 + \frac{\Sigma}{f(r)}dr^2 + \frac{\Sigma r^2}{g(\theta)}d\theta^2 + \frac{g(\theta)\sin^2\theta}{\Sigma r^2}\left[adt - (r^2+a^2)\frac{d\phi}{K}\right]^2\right\}, \tag{2.42a}$$

The conformal factor Ω^{-2} is still given by eq. (2.35), and we have the following metric functions:

$$\begin{aligned} f(r) &= (1 - A^2r^2)\left[1 - \frac{2m}{r} + \frac{a^2+e^2}{r^2}\right] + \frac{r^2+a^2}{\ell^2}, \\ g(\theta) &= 1 + 2mA\cos\theta + \left[A^2(a^2+e^2) - \frac{a^2}{\ell^2}\right]\cos^2\theta, \\ \Sigma &= 1 + \frac{a^2}{r^2}\cos^2\theta. \end{aligned} \tag{2.42b}$$

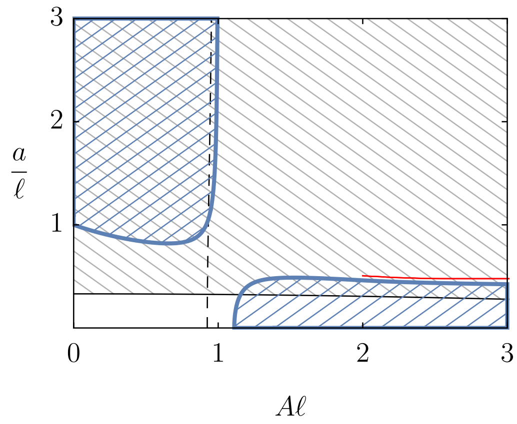
It is worth noting that this form of the rotating C-metric, with $A = 0$, reduces to the familiar Kerr-AdS metric written in Boyer-Lindquist coordinates — assuming K is picked so as to regularise the poles — which allows us to identify a as being a rotation parameter.

With the inclusion of rotation, we have the following parametric restrictions:

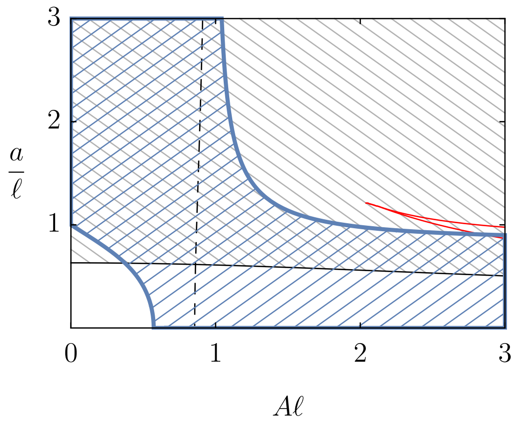
$$A^2(a^2+e^2) - \frac{a^2}{\ell^2} > \begin{cases} 2mA - 1 & \text{if} \quad mA \leqslant 1 \\ m^2A^2 & \text{if} \quad mA > 1 \end{cases} \tag{2.43}$$

once again, obtained by requiring that the metric function $g(\theta)$ be positive over $0 \leqslant \theta < \pi$. We reproduce different parametric spaces in fig. 2.3 and indicate once again with blue hatching the regions excluded by eq. (2.43). As previously stated, we see that for smaller parameter values, this condition acts as an upper bound on the rotation and/or the acceleration. We have also included lines delimiting regions which correspond to solutions with different numbers of horizons. As in the charged nonrotating case, horizons come in pairs, and a generic configuration will have 0, 2 or 4. The meaning of each of these lines is the same as earlier, with the dashed line indicating an acceleration horizon and the solid black line indicating extremal black holes, with grey hatching indicating configurations ruled out by cosmic censorship. For completion, we have also traced the red lines which correspond to further horizon pairs coming in, however these do not concern us. It is worth noting that there are configurations similar to fig. 2.3a in which the upper bound on rotation is due to eq. (2.43) and not censorship. Interestingly, for larger mass and acceleration, these two conditions combine to exclude all possibilities, as in fig. 2.3f.

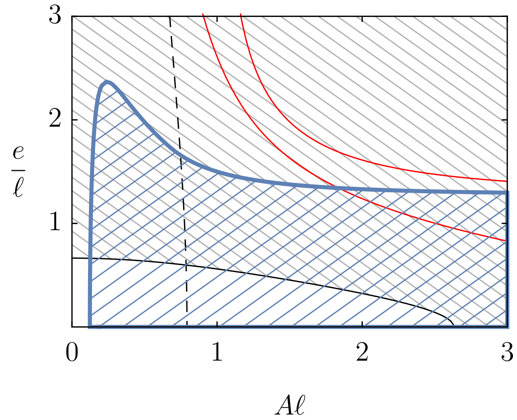
Finally, we may also determine the conical deficits by looking at the region near



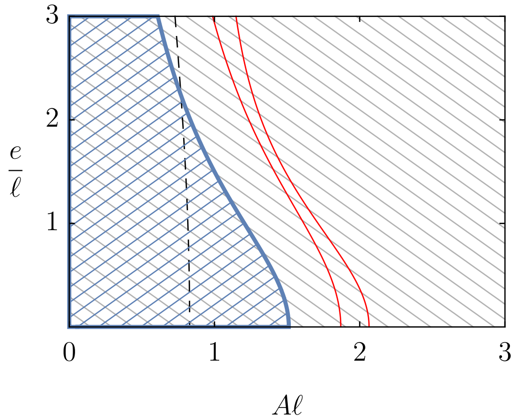
(a) $m = 0.5\ell$, $e = 0.3\ell$.



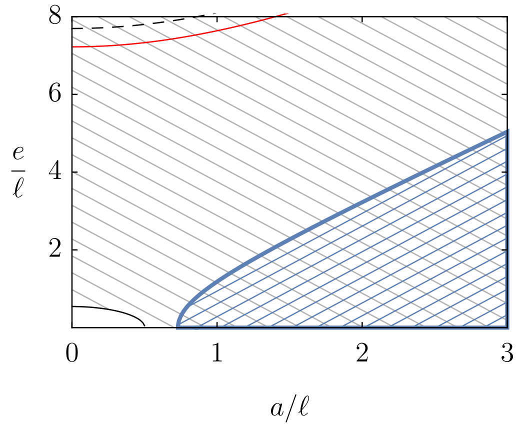
(b) $m = 0.9\ell$, $e = 0.3\ell$.



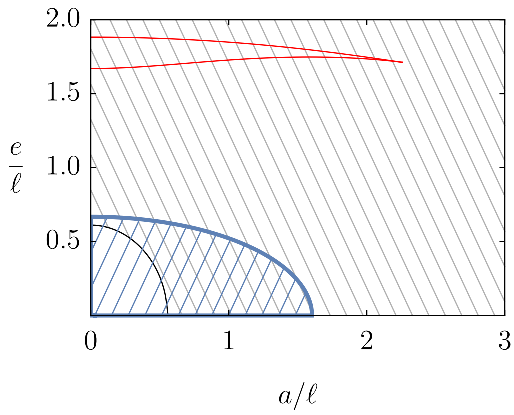
(c) $m = 1.5\ell$, $a = 0.8\ell$.



(d) $m = 1.5\ell$, $a = 2.0\ell$.



(e) $m = 0.6\ell$, $A = 0.5\ell^{-1}$.



(f) $m = 0.7\ell$, $A = 1.1\ell^{-1}$.

Figure 2.3: Allowed parametric regions for the C-metric. The regions marked out with blue hatching correspond to those forbidden by eq. (2.43), and those marked out in grey are excluded by cosmic censorship. The dashed lines correspond to acceleration horizons coming in and the red lines outline regions where an extra pair of boundary-intersecting horizons are formed.

each pole. The string tensions for the rotating C-metric are given by

$$\mu_\pm = \frac{\delta_\pm}{8\pi} = \frac{1}{4}\left[1 - \frac{g(\theta_\pm)}{K}\right] = \frac{1}{4}\left[1 - \frac{1 \pm 2mA + A^2(a^2+e^2) - a^2/\ell^2}{K}\right], \tag{2.44}$$

and the expressions needed to regularise either pole are

$$\Xi, \Xi' = 1 \pm 2mA + \left[A^2(a^2+e^2) - \frac{a^2}{\ell^2}\right]. \tag{2.45}$$

This brings our analysis of the C-metric to an end. See Dias and Lemos for analyses of the coordinate ranges/parameter restrictions on the C-metric in AdS space [116], dS space [117] and in special limiting cases [118]; Krtouš [119] for an analysis of the maximal extension of the charged AdS C-metric; Hubeny, Rangamani and Marolf [114] for a complete overview of the various spacetime solutions contained in the nonrotating uncharged AdS C-metric for different choices of coordinate ranges and different parameter values; and Chen, Ng and Teo [120, 121] for a recent exhaustive analysis of the parametric space of the rotating C-metric.

Chapter 3

Thermodynamics of black holes with conical defects

As we have seen so far, the space of solutions which appear to obey the laws of thermodynamics is rich. Black hole solutions with various conserved charges, may it be electromagnetic, rotational or NUT, in universes with or without a cosmological constant and in any number of dimensions have all been shown to obey the first law once the proper charges and thermodynamic potentials are correctly identified. Any thermodynamic description requires the existence of an equilibrium. In asymptotically flat space, equilibrium is achieved by placing the black hole in a (big) box along with some nondescript fluid surrounding it. In AdS space, the inherent negative curvature of the background geometry allows for the entire spacetime to act as the box containing the equilibrium. From here, one might question whether such a thermodynamic description might exist for the accelerating solutions we introduced in the previous chapter. At a glance, one might be tempted to point at the acceleration as directly preventing the existence of an equilibrium; after all, it must be driven by some external force. This is a valid concern, however, as we will show, we can recover the concept of equilibrium for slowly accelerating black holes in AdS space, by a similar thought process as that for the nonaccelerating solution.

We have shown, in the previous chapter, how acceleration is driven by the existence of a conical deficit corresponding to the influence of a cosmic string attached to the horizon. It will therefore be necessary for us to first investigate how one might formulate thermodynamics of spacetimes in the presence of conical defects, and to simplify that task we restrict ourselves, for now, to nonaccelerating spacetimes.

3.1 Thermodynamics with conical deficits

Let us commence with the basic geometry which describes a static black hole with a cosmic string running through its core: a spacetime first studied by Aryal, Ford and Vilenkin (AFV) [115]. AFV considered a conical deficit through a Schwarzschild black hole:

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\theta^2 + r^2 \sin^2\theta \left(\frac{d\phi}{K}\right)^2 \tag{3.1}$$

where $f(r) = 1 - 2m/r$. They considered a first law of thermodynamics to argue that the entropy of the black hole remained at one quarter of its area, now containing a factor of K: $S = \pi r_+^2/K$. The thermodynamics of a black hole with a string was also considered in greater thoroughness by Martinez and York [122], although the *tension* of the cosmic string, (see section 2.3.3) was held fixed. The only context in which a *varying* tension was considered was in [123], where the varying tension was produced by the capture of a moving cosmic string by a black hole, and it was argued that in the collision of a black hole and cosmic string, the black hole would retain a portion of the string thus increasing its mass.

We revisit this static system first, as a means of exploring the impact of varying tension on black hole thermodynamics. We will consider a charged black hole represented by the metric (3.1), with

$$f(r) = 1 - \frac{2m}{r} + \frac{e^2}{r^2} + \frac{r^2}{\ell^2}\,, \qquad \text{and} \quad B = -\frac{e}{r}dt\,. \tag{3.2}$$

The parameters m and e are related to the black hole's mass and charge respectively, B is the Maxwell potential, and we allow for a negative cosmological constant via $\ell = \sqrt{-3/\Lambda}$.

In order to treat varying tension we leave the parameter K in eq. (3.1) unspecified. As has already been explained, this parameter would typically simply be unity (or a function of rotation in the Kerr-AdS case), however, by keeping K explicitly in the metric we can study conical defects through a well-behaved system in a straightforward manner.

Examining the geometry near $\theta_+ = 0$ and $\theta_- = \pi$ reveals how the parameter K relates to the conical defect. Near the poles, the metric becomes

$$ds^2_{\text{II}} = r^2 \left[d\vartheta^2 + \frac{\vartheta^2}{K^2} d\phi^2 \right], \tag{3.3}$$

on surfaces of constant t and r, where $\vartheta = \pm(\theta - \theta_\pm)$ is the 'distance' to either pole. If $K \neq 1$, there will be a conical defect along the axis of revolution, which corresponds to a cosmic string of tension

$$\mu = \frac{\delta}{8\pi} = \frac{1}{4}\left[1 - \frac{1}{K}\right], \tag{3.4}$$

where δ is the conical deficit. The interpretation of tension is justified by analysing the equations of motion for an actual cosmic string vortex in the presence of a black hole [124], where (3.1) was obtained as the asymptotic form of the metric outside the string core. A tensionless string corresponds to a regular pole, $K = 1$, and in this metric, the tension along either polar axis is equal, allowing simultaneous regularisation of the two poles. The static black hole is inertial, as the deficits balance each other out. This exercise provides insight into the role K plays within a metric. Different values for this parameter determine the severity of an overall defect running through the black hole.

Now let us consider the temperature and entropy (as defined after eq. (3.1)) of the black hole. We compute T by demanding regularity of the Euclidean section of the black hole [45], giving

$$T = \frac{f'(r_+)}{4\pi} = \frac{1}{2\pi r_+^2}\left[m - \frac{e^2}{r_+} + \frac{r_+^3}{\ell^2}\right] \tag{3.5}$$

thus

$$2TS = \frac{m}{K} - \frac{e}{r_+}\left(\frac{e}{K}\right) + 2\left(\frac{3}{8\pi\ell^2}\right)\left(\frac{4\pi}{3}r_+^3\right) = M - \Phi Q + 2PV \tag{3.6}$$

gives a Smarr formula [125] for the black hole, where $M = m/K$ is the mass of the black hole, $Q = (4\pi)^{-1}\int \star dB = e/K$ is the charge on the black hole, $\Phi = e/r_+$ the potential at the horizon, and $P = 3/8\pi\ell^2$, $V = 4\pi r_+^3/3$ the thermodynamic pressure and volume respectively [66, 70, 81].

Now let us consider the effect of changing the parameters of the black hole a small amount; the location of the horizon of the black hole will also shift so that $f + \delta f = 0$ at $r_+ + \delta r_+$:

$$0 = f'(r_+)\delta r_+ - \frac{2\delta m}{r_+} + \frac{2e\delta e}{r_+^2} - 2r_+^2\frac{\delta\ell}{\ell^3} \tag{3.7}$$

However, we can now replace the variation of the parameters m, e, ℓ with the variation of the corresponding thermodynamic charges M, Q, P, and the variation of r_+ with that of entropy, with the important proviso that we must allow for the variation of tension through K. Thus $\delta m = K\delta M + M\delta K$ etc. and $\delta K = 4K^2\delta\mu$ from eq. (3.4). After some rearrangement, eq. (3.7) gives our first law of thermodynamics with varying tension:

$$\delta M = T\delta S + V\delta P + \Phi\delta Q - 2\lambda\delta\mu \tag{3.8}$$

where

$$\lambda = (r_+ - m) \tag{3.9}$$

is a *thermodynamic length* conjugate to the string tension.

This is an important ingredient to this formulation — that string tension (in this case equal along each axis) could be thought of as analogous to a thermodynamic charge that therefore has a corresponding thermodynamic potential. Rather than write a single $\lambda\delta\mu$ term, instead we write two such terms, referring to the deficits emerging from each pole. Although these are obviously equal in this case, one might envision situations where this is not the case. Indeed, our experience with accelerating solutions is that these are spacetimes where the axial configuration exhibits just that; attributing a $\lambda\delta\mu$ term per pole is therefore justified.

3.1.1 A concrete example — capture of a cosmic string

Let us observe the following example to verify the first law in action: the capture, and subsequent escape, of a cosmic string by a black hole. This example was first proposed in [123] in the case of a charged vacuum black hole. The idea is that the string is moving and gets briefly captured by the black hole. In the capture process, the internal energy of the black hole should remain fixed: the physical intuition is that if a cosmic string were to pass through a spherical shell of matter, energy conservation would demand that the spherical shell still have the same total energy throughout the process, thus either it would become denser, or its radius would increase. Of course, in the case of the spherical shell, the cosmic string would simply transit through, leaving the system. For the black hole however, we will see this is not the case, and we have the interpretation of a segment of string having been captured by the black hole, with the black hole increasing its mass accordingly. This process was considered in the probe limit in [126, 127]. We therefore consider the asymptotically flat Reissner-Nordström (RN) metric which has the following structure function:

$$f(r) = 1 - \frac{2m}{r} + \frac{e^2}{r^2} \tag{3.10}$$

with the charge of the black hole being defined via $Q = e/K$, and the electric potential being $\Phi = e/r_+$.

Let us suppose that the string is light, or $\mu \ll 1$, then in the first stage where the black hole captures the string, fixing M and Q implies $\delta m = 4m\delta\mu$ and $\delta e = 4e\delta\mu$ (to first order in μ). Thus

$$T\delta S = \frac{r_+ - r_-}{4\pi r_+^2}\left[2\pi r_+\delta r_+ - 4\pi r_+^2\delta\mu\right] = (r_+ - r_-)\delta\mu = 2\lambda\delta\mu \qquad (3.11)$$

as required. Interestingly, because the internal energy has been fixed, the event horizon has to move outwards to compensate for the conical deficit. Since the entropy contains just one factor of K, but two of r_+, the net effect is an increase of entropy, indicating this is an irreversible thermodynamic process, the one interesting exception being an extremal black hole.

In the second step, the string pulls off the black hole, so $\delta\mu = -\mu$, and since the string is uncharged, δQ must remain zero, and e returns to its original value. However, since entropy cannot decrease, M must increase

$$\delta M = T\delta S + 2(r_+ - m)\mu = \frac{(r_+ - r_-)\delta r_+}{2r_+} + 2(r_+ - r_-)\mu \qquad (3.12)$$

In [123], it was supposed that m did not change, leading to an increase in M of $4m\mu$, which was then stated as being the mass of the string behind the event horizon, however this is in fact only true for the uncharged black hole. Instead, it seems more physically accurate to suppose that r_+ does not decrease, as otherwise the local geodesic congruence defining the event horizon would appear to be contracting in contradiction to the area theorem. In this case, $\delta M = 2(r_+ - r_-)\mu$, or the length of cosmic string trapped between the inner and outer horizons. Even if one allows the local horizon radius to shrink while maintaining constant entropy, $\delta M = (r_+ - r_-)\mu$: half the former amount, but still an increase of mass due to the capture of a length of cosmic string.

3.2 Thermodynamic length for the rotating black hole

From here, we can look into extending this property to rotating spacetimes and see how this affects the corresponding thermodynamic expressions. However, before proceeding into detail, it will be useful to remind ourselves of some of the subtleties introduced when discussing the thermodynamics of a rotating black hole in asymptotically AdS space, initially discussed by Hawking, Hunter and Taylor-Robinson (HHT) in [52], which we covered in section 1.4. These subtleties were pointed out in [53, 55], where it was shown that with a nonzero cosmological constant, the boundary is actually rotating with angular velocity

$$\Omega_\infty = \lim_{r\to\infty} -\frac{g_{t\phi}}{g_{\phi\phi}} = aK\frac{r^2/\ell^2}{a^2r^2\sin^2\theta/\ell^2 - r^2g(\theta)} = -\frac{a}{\ell^2}\frac{K}{\Xi}, \qquad \Xi = 1 - \frac{a^2}{\ell^2} \tag{3.13}$$

implying that the angular velocity ought to be re-normalised and that $\Omega = \Omega_{\mathrm{H}} - \Omega_\infty$ is the true total angular velocity. The mass was then found to be given by $M = m/\Xi^2$ as opposed to m/Ξ, the expression originally given by HHT, which is obtained using the Komar method when a normalisation of the timelike killing vector is omitted. Similarly, the expression for thermodynamic volume,

$$V = \frac{4\pi}{3K}\left(r_+(r_+^2 + a^2) + ma^2\right) \tag{3.14}$$

contains a second, rotation-dependent term which may also be viewed as a re-normalising shift.

Employing similar ideas and viewing thermodynamic potentials as having extra regularising terms, we can actually show that these corrections to the thermodynamic mass and angular velocity are required to satisfy the first law while simultaneously obtaining these expressions for arbitrary and potentially varying string tensions.

The first step is to vary $f(r_+) = 0$ to establish an initial thermodynamic relation. Identifying S as a quarter of the horizon area and T as the temperature given by the Euclideanisation procedure,

$$S = \frac{\pi}{K}(r_+^2 + a^2) \qquad T = \frac{1}{2\pi(r_+^2 + a^2)}\left[m - \frac{a^2}{r_+} + \frac{r_+^3}{\ell^2}\right], \tag{3.15}$$

will lead to the following statement:

$$\delta\left(\frac{m}{K}\right) = T\delta S + V_0\delta P + \Omega_0\delta J - 2r_+\delta\mu + \frac{m\delta K}{2K^2}, \tag{3.16}$$

where we have written V_0 and Ω_0 in anticipation of correction terms, however it is worth noting that $\Omega_0 = \Omega_H$ is the angular velocity at the horizon and that V_0, the first term in eq. (3.14), satisfies a *reduced* Smarr relation given by $m/K = 2(TS - PV_0 + \Omega_0 J)$. These quantities are given by the following relations:

$$\Omega_0 = \frac{aK}{r_+^2 + a^2}, \qquad J = \frac{ma}{K^2}, \qquad V_0 = \frac{4\pi}{3K}r_+(r_+^2 + a^2), \qquad P = \frac{3}{8\pi\ell^2}, \tag{3.17}$$

where the expression for the angular momentum J is obtained unambiguously via the Komar method using background ($m = 0$) subtraction, as per [56].

While eq. (3.16) looks like a first law, it is necessary to remember that K parametrises the tension and can therefore not appear as a standalone term. The aim of this derivation is precisely to find such a first law. Let us now introduce a function

$\gamma = \gamma(a, \ell)$ in our expression for the mass. We know that γ will need to depend on a and ℓ from the relation between K and μ. Using this ansatz for the mass and then perturbing it, we have

$$
\begin{aligned}
M &= \frac{m}{K}\gamma(a, \ell), && (3.18)\\
\delta M &= \delta\left(\frac{m}{K}\right) + \frac{m}{K}(\gamma_a \delta a + \gamma_\ell \delta \ell) \\
&= (\gamma - a\gamma_a)\delta\left(\frac{m}{K}\right) + \gamma_a K \delta J + \frac{m a \gamma_a}{K^2}\delta K + \frac{m}{K}\gamma_\ell \delta \ell. && (3.19)
\end{aligned}
$$

Now, if we rewrite eq. (3.16) using $\Omega = \Omega_0 + \Omega_1$, $V = V_0 + V_1$ as

$$
\begin{aligned}
&T\delta S + V\delta P + \Omega \delta J - 2\lambda \delta \mu \\
&\qquad = \delta\left(\frac{m}{K}\right) + \Omega_1 \delta J - \frac{3V_1}{4\pi \ell^3}\delta \ell + 2(r_+ - \lambda)\delta\mu - \frac{m}{2K^2}\delta K, \qquad (3.20)
\end{aligned}
$$

we may require that eqs. (3.19) and (3.20) be equal to find constraints on γ. Using eq. (4.49) and $\mu = \delta/8\pi$ to express δK in terms of $\delta\mu$, we can infer a differential equation that γ ought to satisfy,

$$
\left(1 + \frac{a^2}{\ell^2}\right)\gamma - \left(1 - \frac{a^2}{\ell^2}\right)a\gamma_a - 1 = 0 \qquad (3.21)
$$

as well as the following expressions for the correction terms, defining them in terms of γ:

$$
\begin{aligned}
V_1 &= -\frac{4\pi}{3K}\frac{m\ell^2}{1 + a^2/\ell^2}\left(\left(1 + \frac{a^2}{\ell^2}\right)\ell\gamma_\ell + 2\frac{a^2}{\ell^2}a\gamma_a + \frac{a^2}{\ell^2}\right), \\
\Omega_1 &= K\gamma_a\left(1 - 2\frac{a^2}{\ell^2}\frac{1}{1 + a^2/\ell^2}\right) - \frac{aK}{\ell^2}\frac{1}{1 + a^2/\ell^2}, \\
\lambda &= r_+ - \frac{m}{1 + a^2/\ell^2}\left(2a\gamma_a + 1\right). \qquad (3.22)
\end{aligned}
$$

We also require that the Smarr relation, which follows from the scaling properties of the system, also be satisfied. Inserting the above expressions into

$$
M = 2(TS - PV + \Omega J), \qquad (3.23)
$$

we obtain another differential equation for γ,

$$
\left(1 + \frac{a^2}{\ell^2}\right)(\gamma - \ell\gamma_\ell) - 2a\gamma_a - 1 = 0. \qquad (3.24)
$$

It is then straightforward to solve eqs. (3.21) and (3.24) and one obtains

$$\begin{aligned}
\gamma &= \frac{1}{\Xi}\left(1+\frac{a}{\ell}\zeta\right), \qquad & \lambda &= r_+ - \frac{m}{\Xi^2}\left(1+\frac{a^2}{\ell^2}+\frac{2a}{\ell}\zeta\right), \\
\Omega_1 &= \frac{a}{\ell^2}\frac{K}{\Xi}\left(1+\frac{\ell}{a}\zeta\right), \qquad & V_1 &= \frac{4\pi}{3}\frac{ma^2}{K\Xi}\left(1+\frac{\ell}{a}\zeta\right),
\end{aligned} \tag{3.25}$$

where ζ is an integration constant. We can fix it by identifying $\Omega_1 = -\Omega_\infty$ provided $\zeta = 0$, which also assures that the angular velocity of the boundary vanishes for $a = 0$. Similarly, we obtain the correct expression for thermodynamic volume if $\zeta = 0$. Finally with $\zeta \neq 0$, γ would supposedly be sensitive to the direction of rotation as given by the sign of a. This apparent asymmetry therefore requires the constant to vanish.

Finally, one can repeat this derivation with the inclusion of charge $Q = e/K$ and introduce a correction term to the potential $\Phi = \Phi_0 + \Phi_1$. This leads to a similar expression for γ, with ζ now a function of charge

$$\gamma = \frac{1}{\Xi}\left[1+\frac{a}{\ell}\zeta\left(\frac{e^2}{\ell^2}\frac{1}{\Xi^2}\right)\right]. \tag{3.26}$$

Eliminating the integration function through a similar line of reasoning as above leads to the correction terms already written above, with the additional proviso that the correction to the gauge potential $\Phi_1 = 0$.

3.3 The thermodynamic length

To recap, we have shown that, allowing for a varying conical deficit in black hole spacetimes, the first law of thermodynamics becomes

$$\delta M = T\delta S + V\delta P + \Phi\delta Q + \Omega\delta J - 2\lambda\delta\mu, \tag{3.27}$$

where the relevant thermodynamical variables are given in (4.38). In order to accommodate varying tension, we have to define a *thermodynamic length*,

$$\lambda = r_+ - \frac{m}{\Xi^2}\left(1+\frac{a^2}{\ell^2}\right) \tag{3.28}$$

for each conical deficit emerging from each pole. Surprisingly perhaps, this thermodynamic length is not simply the geometric length r_+ of the string from pole to singularity. Instead, the mass-dependent adjustment emphasises this is a potential, rather than just an internal energy term that might more appropriately be placed on the left hand side of the equation. Interestingly perhaps, in the absence of a

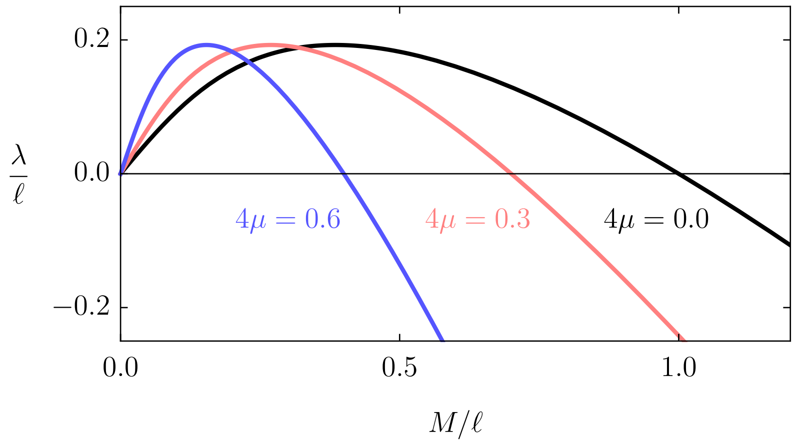


Figure 3.1: Plot of the thermodynamic length for uncharged non-rotating black holes in asymptotically AdS space. As the deficit is increased, the effect on the length of adding mass is amplified.

cosmological constant, the thermodynamic length can actually be interpreted as half the distance between the inner and outer horizons, $\lambda = \frac{r_+ - r_-}{2}$, though this is not the case in AdS.

It is interesting to compare this mass-dependent shift of the thermodynamic length to the correction of the thermodynamic volume for a rotating black hole [82, 128]:

$$V = \frac{4\pi}{3K}\left(r_+(r_+^2 + a^2) + ma^2\right) \tag{3.29}$$

In this case, the first term is the expected geometric volume of the interior of the black hole, the second term being a rotation-dependent correction. It is with this appropriately shifted thermodynamic volume, that the black hole always satisfies the reverse isoperimetric inequality [82].

Notice that the correction term for this thermodynamic volume is always positive, whereas the correction term for thermodynamic length is actually negative. This means that for large enough mass, the thermodynamic length itself becomes negative, as shown in fig. 3.1 for an uncharged black hole. The picture for a charged black hole is similar, although the critical value of M for which λ becomes negative is larger.

From fig. 3.1, we see that the thermodynamic length becomes negative for 'large' black holes, i.e. those for which the thermodynamic mass is of similar order (or higher) than the AdS scale. Setting this in the context of the 'cosmic string' capture process considered in section 3.1 for the vacuum black hole, this would mean that the thermodynamic mass must increase during a capture, as entropy cannot decrease. This seems at first counter to the notion that the string itself does not carry 'ADM' mass, however, the heuristic argument of section 3.1 relies somewhat on the notion that a cosmic string and black hole can be sufficiently separated so that one can consider their thermodynamical (and other) properties independently. For large black holes in AdS this is manifestly not the case.

Chapter 4

Thermodynamics of accelerating black holes

As we have already stated, our goal is to establish whether a thermodynamic interpretation for accelerating black holes may be constructed. Accelerating black holes have always presented a problem in this respect, partly due to the existence of an external driving force which might indicate an inability to attain any kind of equilibrium, but there is also an algebraic obstacle; the existence of a conical deficit and an acceleration parameter for which we have no prior thermodynamic interpretation. Add to that the fact that the boundary is displaced from $r = \infty$ (in Boyer-Lindquist coordinates) resulting in awkward asymptotics to deal with and the existence of an acceleration horizon with its own temperature and it becomes clearer why such a formulation did not exist.

There is at least one of these aspects about which we might feel more confident. The previous chapter discussed how one could include the tension of cosmic strings attached to black hole horizons as a thermodynamic variable, introducing the concept of the thermodynamic length, the conjugate to the tension. Each of the configurations we have dealt with up until now had equal deficits between the north and south poles, however, as we reviewed in section 2.3.3, the C-metric represents an accelerated black hole driven by an imbalance in the cosmic string tensions at each of the poles. What must then be addressed is whether this construction, of black hole thermodynamics with varying conical deficits, may be extended to allow for independent variations in the tensions at the north and south poles. As we shall see below, this is possible, and it will allow us to simultaneously address one of the other potential issues that we brought up at the start of this chapter: whereas in the previous chapter we showed that the parameter K is linked to the overall deficit in the spacetime and that its variations could be re-expressed as variations in the string tension μ, we now have another variable A, representing acceleration, whose variations, together with those for K, may be re-cast as independent variations in the north and south string tensions, $\mu_\pm$.

We will first begin to develop this framework with the simplest case, the uncharged nonrotating accelerating black hole and use the insight gained from the previous chapter to choose to fix the tensions. This will be physically motivated in its own right, and allows us to make initial assertions as to the necessary conditions to formulate consistent thermodynamics. We will then extend this to include the aforementioned generalisation of independently varying conical deficits, at least for charged nonrotating accelerating black holes, deriving the thermodynamic lengths for these solutions along the way.

4.1 Thermodynamics of the C-metric

4.1.1 Establishing a first law

For a general black hole spacetime containing conical defects, any disparity in the sizes of the deficits produces an overall force in the direction of the largest conical *deficit*, and the geometry is described by the C-metric [103], introduced in chapter 2. Typically, C-metrics have both black hole and acceleration horizons. In order to have any chance at constructing a thermodynamic description, we would like to be able to eliminate the acceleration horizon which has its own temperature, different to that of the black hole's. This is possible in asymptotically anti-de Sitter space and for that reason we shall be studying accelerating black holes in negative cosmological constant universes. More specifically, we must restrict ourselves to the so-called *slowly accelerating C-metric* [129], for which the acceleration is small enough that the negative curvature prevents the existence of an acceleration horizon. With this geometry, we may consider the entire spacetime, black hole and string(s) combined, as forming a thermodynamic equilibrium. In the absence of an acceleration horizon, the spacetime can be interpreted as having a black hole maintained a finite distance from the centre of AdS by the cosmic string. We will revisit this shortly, for now let us rewrite the metric here, for convenience. We have

$$ds^2 = \frac{1}{\Omega^2}\left[-f(r)dt^2 + \frac{dr^2}{f(r)} + r^2\left(\frac{d\theta^2}{g(\theta)} + g(\theta)\sin^2\theta\frac{d\phi^2}{K^2}\right)\right], \tag{4.1}$$

and

$$f(r) = (1 - A^2r^2)\Big(1 - \frac{2m}{r}\Big) + \frac{r^2}{\ell^2}, \tag{4.2}$$

$$g(\theta) = 1 + 2mA\cos\theta,$$

$$\Omega = 1 + Ar\cos\theta. \tag{4.3}$$

The conical deficits this spacetime exhibits correspond to cosmic strings with tensions given by (2.40)

$$\mu_\pm = \frac{1}{4}\bigg[1 - \frac{g(\theta_\pm)}{K}\bigg] = \frac{1}{4}\bigg[1 - \frac{1 \pm 2mA}{K}\bigg]. \tag{4.4}$$

Further details concerning the spacetime and its subtleties can be found by referring back to section 2.3.

We have already seen how the parameter K is related to the conical deficits in chapters 2 and 3, however we would like to give some interpretation of the parameter A, which was identified as the acceleration in section 2.1 for the asymptotically flat solution by studying the weak field limit $m = 0$ and exposing it as a reparametrisation of Rindler spacetime. In the presence of a cosmological constant it may be more helpful to view it as the acceleration required to maintain the black hole some distance away from the centre of AdS, as we alluded to previously. Setting $m = 0$ and $K = 1$ to eliminate the conical deficit in eq. (4.1) gives

$$ds^2 = \frac{1}{\Omega^2}\left[-\Big(1 + \frac{r^2}{\ell^2}(1 - A^2\ell^2)\Big)dt^2 + \frac{dr^2}{1 + \frac{r^2}{\ell^2}(1 - A^2\ell^2)} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right]. \tag{4.5}$$

This spacetime no longer has a conical singularity and is locally pure AdS, however in these coordinates the boundary of AdS is not at $r = \infty$, but at $r = -1/(A\cos\theta)$. For θ in the southern hemisphere, this occurs at *finite* r, but in the northern hemisphere $r = \infty$ actually lies within the AdS spacetime (see fig. 4.1). To transform to global AdS coordinates $\{R, \Theta\}$, one takes [129]

$$1 + \frac{R^2}{\ell^2} = \frac{1 + (1 - A^2\ell^2)r^2/\ell^2}{(1 - A^2\ell^2)\Omega^2}, \qquad R\sin\Theta = \frac{r\sin\theta}{\Omega}, \tag{4.6}$$

resulting in the metric for anti-de Sitter space in global coordinates:

$$ds^2_{\text{AdS}} = -(1 - A^2\ell^2)\Big(1 + \frac{R^2}{\ell^2}\Big)dt^2 + \frac{dR^2}{1 + \frac{R^2}{\ell^2}} + R^2\Big(d\Theta^2 + \sin^2\Theta\frac{d\phi^2}{K^2}\Big). \tag{4.7}$$

The boundary, $R \to \infty$ now clearly corresponds to $\Omega \to 0$, and the origin of Rindler coordinates corresponds to $R_0 = A\ell^2/\sqrt{1 - A^2\ell^2}$, in other words, the Rindler

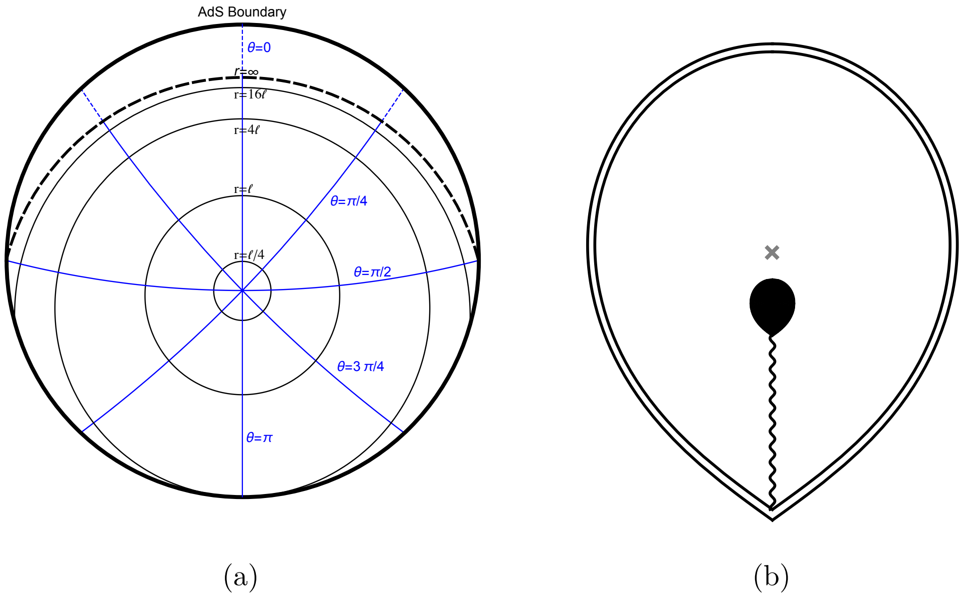


Figure 4.1: (a) The slowly accelerating Rindler spacetime shown here with $A\ell = 1/4$, and $\ell = 1$ for simplicity. The spatial sections of AdS have been compactified to a Poincaré disc, with the constant r Rindler coordinate indicated in black and constant θ in blue. The origin of the Rindler coordinates is clearly visible as being displaced from the centre of the disc, with the limit of the r-coordinate being the thick dashed black line. (b) The black hole distorts the Poincaré disc with a conical deficit, and is displaced from the origin of AdS. The spacetime is again static, and a cross section is shown.

coordinates represent those of an observer displaced from the origin of AdS.

Let us now define an important thermodynamic quantity. We suspect m to be related to the mass of the black hole, however the computation required to obtain such an expression is rather tricky. The awkward asymptotics do not lend themselves well to a Komar approach. Instead, we used the method of conformal completion [63, 64, 130]. This takes the electric part of the Weyl tensor projected along the timelike conformal Killing vector, and integrates over a sphere at conformal infinity. The calculation gives[1]

$$M = \frac{m}{K}, \tag{4.8}$$

thus m gives the mass of the black hole. Note that unlike the rapidly accelerating

[1]More details regarding this computation are provided in section 5.1.1. Though it should be noted that the result stated here was later discovered to be incorrect. In particular, this is related to an issue with the somewhat ambiguous scaling of the timelike killing vector which affects the computation by an overall multiplying factor. Chapter 5 addresses the issue, however these details were discovered in the later stages of the production of this thesis. The results stated in this chapter hold despite this inconsistency.

black hole, this is a genuine ADM-style mass, and not a "rearrangement" of dipoles as discussed in [131], where a boost mass was introduced for the C-metric.

Meanwhile, we identify the entropy with a quarter of the horizon area

$$S = \frac{\mathcal{A}}{4} = \frac{\pi r_+^2}{K(1 - A^2 r_+^2)}\,, \tag{4.9}$$

and calculate the temperature via the usual Euclidean method (see section 1.3) to obtain

$$T = \frac{f'(r_+)}{4\pi} = \frac{1}{2\pi r_+}\left(\frac{m}{r_+}\left(1 + A^2 r_+^2\right) + \frac{r_+^2}{\ell^2} - A^2 r_+^2\right). \tag{4.10}$$

We now identify P with the pressure associated to the cosmological constant according to $P = \frac{3}{8\pi\ell^2}$, which allows us to rewrite the temperature as

$$TS = \frac{M}{2} + P\frac{4\pi r_+^3}{3K(1 - A^2 r_+^2)^2}, \tag{4.11}$$

which is nothing other than the Smarr relation $M = 2(TS - PV)$ provided we identify the black hole thermodynamic volume as

$$V = \frac{4\pi r_+^3}{3K(1 - A^2 r_+^2)^2}. \tag{4.12}$$

So far, this is a rewriting of a relation for the temperature, having identified standard thermodynamic variables or charges for the solution. Now let us consider the first law by considering a variation due to some physical process. Typically, one derives the first law by observing the change in horizon radius during a physical process. The horizon radius is given by a root of $f(r_+) = 0$, and thus depends on m, A and ℓ. The specific form of this algebraic root is not vital, what matters is how the mass varies in terms of the change in horizon area, thermodynamic volume, and charge.

Originally, it was reasoned that during this process, the conical deficits (or lack thereof) could not change, as these corresponded to the physical objects causing the acceleration. Of course, as we have already shown by now, one could envisage physical processes that would alter conical defects on a black hole horizon, nonetheless it is simplest to restrict ourselves to the scenario where all tensions are held fixed. Thus we must consider a variation of m, A and K that preserves the cosmic string tensions, and it turns out that it is precisely through this physical restriction that we are able to derive a first law.

To obtain the first law, we typically consider a perturbation of the equation that determines the location of the event horizon of the black hole: $f(r_+) = 0$. If we allow our parameters to vary, this will typically result in a perturbation also of r_+, hence we can write

$$\frac{\partial f}{\partial r_+}\delta r_+ + \frac{\partial f}{\partial m}\delta m + \frac{\partial f}{\partial A}\delta A + \frac{\partial f}{\partial \ell}\delta \ell = 0 \tag{4.13}$$

where everything is evaluated at $f(r_+, m, A, \ell) = 0$. Clearly we can replace δm and $\delta \ell$ by variations of the thermodynamic parameters M and P, and δr_+ is expressible in terms of δS, δK and δA using the variation of eq. (4.9),

$$\delta S = \frac{2\pi r_+ \delta r_+}{K(1 - A^2 r_+^2)^2} + \frac{2\pi r_+^4 A \delta A}{K(1 - A^2 r_+^2)^2} - \frac{\pi r_+^2}{(1 - A^2 r_+^2)} \frac{\delta K}{K^2}. \tag{4.14}$$

Finally, we replace $(\partial f/\partial r_+)_{m,A,\ell} = 4\pi T$, and use $f(r_+) = 0$ to simplify the terms multiplying δA to obtain:

$$(1 - A^2 r_+^2)(T\delta S + V\delta P) - \delta M - \frac{m A r_+^2}{K}\delta A + \left(\frac{r_+^2}{\ell^2} - (1 + A^2 r_+^2)\right)\frac{r_+ \delta K}{4K^2} = 0. \tag{4.15}$$

At the moment, it seems as if we have extra thermodynamic contributions, however, we now use the physical input from the cosmic string that the conical deficits on each axis must not change. To achieve this, we must require that both $\delta\mu_\pm = 0$. A quick look at the linear combinations

$$\mu_+ + \mu_- = \frac{1}{2}\left[1 - \frac{1}{K}\right] \qquad \text{and} \qquad \mu_+ - \mu_- = -\frac{mA}{K} \tag{4.16}$$

reveals that this is achieved by the conditions $\delta K = 0$ and $\delta(mA) = 0$, or $m\delta A = -A\delta m$. Replacing δA in eq. (4.15) and rearranging, finally, indeed gives the first law:

$$\delta M = T\delta S + V\delta P. \tag{4.17}$$

Thus, this first pass at a thermodynamic construction for accelerating black holes is indeed promising. Despite a few intuitive barriers, we have succeeded in establishing a first law for this black hole solution, suggesting that it ought to display similar thermal behaviour to other known solutions. This was the result we first presented in [1]. We will carry out a survey of its thermodynamic features further on, however we must now generalise this result to include electric charge as well as investigate whether it is feasible to allow the string tensions to vary independently; this formed the core of our following publication [2].

4.1.2 Thermodynamics of the charged C-metric

Having shown that the C-metric appears to obey the laws of black hole thermodynamics, at least in the uncharged case when the cosmic string tensions are held fixed, we will now perform a similar analysis to verify this is the case for the charged black hole, while this time including terms corresponding to each deficit. While at first this might seem unnecessary, an example of a process which would involve a change in the string tensions could be the collision of two accelerating black holes[2]. Keeping the metric as in eq. (4.1), now with $f(r)$ defined to be (2.36)

$$f(r) = (1 - A^2r^2)\left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right) + \frac{r^2}{\ell^2}, \tag{4.18}$$

we need the gauge potential $B = -\frac{e}{r}dt$ to satisfy the Einstein-Maxwell equations. The charge of the black hole is obtained by integrating the electromagnetic field strength tensor:

$$Q = \frac{1}{4\pi}\int_{S^2} \star dB = \frac{e}{K}. \tag{4.19}$$

We start by finding the temperature and entropy of the black hole, using the conventional relations

$$\begin{aligned} T &= \frac{f'(r_+)}{4\pi} = \frac{1}{2\pi r_+}\left(\frac{m}{r_+}\left(1 + A^2r_+^2\right) + \frac{r_+^2}{\ell^2} - A^2r_+^2 - \frac{e^2}{r_+^2}\right), \\ S &= \frac{\mathcal{A}}{4} = \frac{\pi r_+^2}{K(1 - A^2r_+^2)}. \end{aligned} \tag{4.20}$$

Checking the Smarr relation, we compute

$$2TS = \frac{m}{K} - \frac{e^2}{Kr_+^2} + \frac{r_+^3}{K\ell^2(1 - A^2r_+^2)^2}. \tag{4.21}$$

With the charge of the black hole (4.19), $Q = e/K$, the electric potential given by $\Phi_{\mathrm{H}} = e/r_+$ and defining

$$V = \frac{4\pi r_+^3}{3K(1 - A^2r_+^2)^2} \tag{4.22}$$

as the thermodynamic volume, as it was in the previous section, we obtain

$$\frac{m}{K} = 2TS + Q\Phi_H - 2PV. \tag{4.23}$$

Although it is tempting to identify $M = m/K$, this would be to ignore the asymptot-

[2] An unfortunate caveat of our model is that the two black holes would need to be accelerating along the same axis.

ics of the spacetime. As mentioned in earlier chapters (see sections 1.4 and 3.2), the experience of the rotating AdS black hole is that thermodynamic potentials should be normalised at infinity [53, 55], and in the case of rotation, expressing this solution in ordinary Boyer-Lindquist coordinates results in a spacetime that has a rotating boundary. Subtracting off this rotation leads to an extra renormalisation of the thermodynamic mass, a correct Smarr formula and correct first law.

Here, however, we cannot simply perform a similar electromagnetic gauge transformation. Our electrostatic potential no longer vanishes at infinity, and our boundary has an electric flux from pole to pole given by

$$F = eA\sin\theta\, dt\wedge d\theta, \tag{4.24}$$

which, incidentally, also prevents us from carrying out an Ashtekar-Das [64] computation of the mass.

We obviously cannot subtract this charge, as that would be a physical change, but it does lead us to suspect that there may be a renormalization of electrostatic potential and thermodynamic mass. We will show how to do this shortly, but first consider just the uncharged black hole, and consider variations in the position of the horizon as in eq. (4.15):

$$\delta f(r_+) = f'_+\delta r_+ - 2\frac{\delta m}{r_+}(1-A^2r_+^2) - 2A\delta A r_+(r_+-2m) - 2\frac{r_+^2}{\ell^3}\delta\ell = 0 \tag{4.25}$$

The procedure is similar to the previous section, but we now have more algebra involved in the variation of the thermodynamic parameters. For example, in relating δr_+ to δS, we had (4.14):

$$\delta S = \frac{2\pi r_+\delta r_+}{K(1-A^2r_+^2)^2} + \frac{2\pi r_+^4 A\delta A}{K(1-A^2r_+^2)^2} - \frac{\pi r_+^2}{(1-A^2r_+^2)}\frac{\delta K}{K^2} \tag{4.26}$$

where, now, our expressions for the tensions eq. (4.4) give

$$\frac{\delta K}{K^2} = 2\left(\delta\mu_+ + \delta\mu_-\right), \qquad \frac{m}{K}\delta A = -\left[\delta\mu_+ - \delta\mu_- + A\delta\Big(\frac{m}{K}\Big)\right], \tag{4.27}$$

which we can substitute back into the variation of entropy.

As we are dealing with the uncharged black hole, we define $M = m/K$, and after some algebra one gets

$$\delta M = V\delta P + T\delta S - \delta\mu_+\left[\frac{r_+}{1+Ar_+} - m\right] - \delta\mu_-\left[\frac{r_+}{1-Ar_+} - m\right]. \tag{4.28}$$

Thus, the accelerating black hole has the same thermodynamic first law as the nonaccelerating black hole, but now with a thermodynamic length for the piece of string attaching at each pole:

$$\lambda_\pm = \frac{r_+}{1 \pm A r_+} - m \tag{4.29}$$

This obviously agrees with eq. (3.9) for the string threading the black hole, where r_+ has now been replaced by $r_+/\Omega(r_+, \theta_\pm)$ at each pole.

Now let us consider the addition of charge. Following the same procedure of varying the horizon as before leads to the relation

$$\delta\!\left(\frac{m}{K}\right) = T\delta S + V\delta P + \Phi_{\mathrm{H}}\delta Q - \frac{r_+\delta\mu_+}{1 + Ar_+} - \frac{r_+\delta\mu_-}{1 - Ar_+} + \frac{m\delta K}{2K^2} \tag{4.30}$$

where now our expressions for the tensions lead to

$$\begin{aligned} \frac{m}{K}\delta A &= -\delta\mu_+ + \delta\mu_- - A\delta\left(\frac{m}{K}\right) \\ [1 - e^2A^2]\frac{\delta K}{2K^2} &= A^2 e\delta Q - \frac{e^2A^2}{m}\delta\left(\frac{m}{K}\right) + \delta\mu_+\left[1 - \frac{Ae^2}{m}\right] + \delta\mu_-\left[1 + \frac{Ae^2}{m}\right] \end{aligned} \tag{4.31}$$

Keeping an open mind, we define our thermodynamic mass and electrostatic potential as:

$$M = \frac{m}{K}\gamma(A, e), \qquad \Phi = \Phi_0 + \Phi_1, \tag{4.32}$$

where $\Phi_0 = \Phi_{\mathrm{H}}$ and Φ_1 is a correction, re-zeroing the potential, analogous to the correction of the angular potential of the Kerr-AdS black hole, but without the corresponding interpretation of being the value of the original potential at infinity. It also follows from eqs. (4.30) and (4.31) that we can assume γ to be independent of ℓ. The method here is to seek a consistent set of thermodynamic relations while maintaining the temperature and entropy as defined earlier by the surface gravity and area respectively.

Next, we compare

$$\begin{aligned} \delta M &= \gamma\delta\left(\frac{m}{K}\right) + \frac{m}{K}(\gamma_e\delta e + \gamma_A\delta A) \\ &= \gamma\delta\left(\frac{m}{K}\right) + m\gamma_e\delta Q + me\gamma_e\frac{\delta K}{K^2} + \gamma_A\frac{m}{K}\delta A \end{aligned} \tag{4.33}$$

to

$$\begin{aligned} T\delta S + V\delta P + \Phi\delta Q - \lambda_+\delta\mu_+ - \lambda_-\delta\mu_- &= \delta\left(\frac{m}{K}\right) - \Phi_0\delta Q - \frac{m\delta K}{2K^2} \\ &\quad + \left[\frac{r_+}{1 + Ar_+} - \lambda_+\right]\delta\mu_+ + \left[\frac{r_+}{1 - Ar_+} - \lambda_-\right]\delta\mu_-, \end{aligned} \tag{4.34}$$

where $\lambda_\pm$ are to be determined. After some algebra, we obtain

$$\begin{aligned}
&\delta M - T\delta S - V\delta P - \Phi\delta Q + \lambda_+\delta\mu_+ + \lambda_-\delta\mu_- \\
&\quad = \Big[(1-e^2A^2)\gamma - 2e^3A^2\gamma_e - A(1-e^2A^2)\gamma_A - 1\Big]\frac{\delta\,(m/K)}{(1-e^2A^2)} \\
&\quad + \Big[m(1+e^2A^2)\gamma_e + mA^2e + (1-e^2A^2)\Phi_0\Big]\frac{\delta Q}{(1-e^2A^2)} \\
&\quad + \left[\lambda_+ - \frac{r_+}{1+Ar_+} - \gamma_A + \frac{(2e\gamma_e+1)}{1-e^2A^2}\Big(m - e^2A\Big)\right]\delta\mu_+ \\
&\quad + \left[\lambda_- - \frac{r_+}{1-Ar_+} + \gamma_A + \frac{(2e\gamma_e+1)}{1-e^2A^2}\Big(m + e^2A\Big)\right]\delta\mu_-
\end{aligned} \tag{4.35}$$

for our first law to hold, clearly the right-hand side of this equation must vanish, leading to a constraint for γ:

$$(1-e^2A^2)\gamma - 2e^3A^2\gamma_e - A(1-e^2A^2)\gamma_A = 1. \tag{4.36}$$

This equation is solvable, and we obtain

$$\gamma = \frac{1}{1+e^2A^2} + A\ell\zeta\left(\frac{\ell}{e} + A^2e\ell\right), \tag{4.37}$$

where requiring γ be unity in the absence of charge eliminates the integrating function $\zeta(x)$. This specifies our thermodynamic mass, and we determine Φ_1 and $\lambda_\pm$ from eq. (4.35):

$$\begin{aligned}
M &= \frac{m}{K(1+e^2A^2)} \\
\Phi_1 &= -\frac{meA^2}{1+e^2A^2} \\
\lambda_\pm &= \frac{r_+}{1\pm Ar_+} - \frac{m(1-e^2A^2)}{(1+e^2A^2)^2} \mp \frac{e^2A}{(1+e^2A^2)}
\end{aligned} \tag{4.38}$$

This is a rather unusual set of relations, derived in [2], which gives consistent thermodynamics. The offset of the electrostatic potential depends on mass, and the thermodynamic mass depends on charge. We view this as a consequence of the fact that for the accelerating black hole, the electric potential cannot be gauged away at infinity — there is a polar electric field at the AdS boundary, thus mass and charge are inextricably intertwined. These results will be revisited in chapter 5, in light of new studies on the subject.

Finally, we have that

$$\frac{m}{K} = M + e^2A^2M = M + \Phi_1 Q, \tag{4.39}$$

which ensures, along with eq. (4.23), that the Smarr relation will indeed be satisfied with these new quantities.

It is also interesting to compare these results for varying tension to some of our early work [1], which formed the basis of the previous introductory section where K and $\mu_\pm$ were held fixed but in the presence of charge. With these assumptions, both the quantities eA and mA were fixed, however, we did not alter the thermodynamic mass from m/K, nor the electrostatic potential from $\Phi_{\rm H}$. The two sets of results are consistent, since, as we have already pointed out,

$$\Phi_1 Q = \frac{me^2A^2}{K(1+e^2A^2)} = \frac{m}{K}\left[1 - \frac{1}{1+e^2A^2}\right] = \frac{m}{K} - M.$$

The correction to the electrostatic potential therefore balances the shift in thermodynamic mass in both the Smarr formula, and indeed the first law with the assumptions made in [1] since eA was required to be fixed. However, it is worth revisiting these assumptions in the light of our work here on varying tension.

First, notice that our charged C-metric has parameters: m, relating to the mass of the black hole, e to its charge, A to its acceleration, and K, that relates to an overall conical deficit. K is the one parameter that has no immediately obvious physical interpretation, indeed seems more like a parameter which fixes the periodicity of the azimuthal coordinate, thus fixing K was natural. However, now armed with our better understanding of the metric and its thermodynamics, we see that in fixing the tensions of the deficits, we are fixing two physical quantities, thus we should only find that *two* combinations of the solution parameters are fixed. Therefore, we should not fix K a priori, but instead just the combinations of parameters that fix the tensions:

$$\begin{aligned} 2(\mu_+ + \mu_-) &= 1 - \frac{1+e^2A^2}{K} \\ \mu_+ - \mu_- &= -\frac{mA}{K} \end{aligned} \tag{4.40}$$

From these expressions, we see that if charge vanishes, then indeed fixing tensions fixes K and the combination mA, but if charge does not vanish, then we can no longer conclude that $\delta K = 0$. Instead

$$\frac{\delta K}{K} = \frac{\delta(mA)}{mA} = 2\frac{eA\delta(eA)}{1+e^2A^2} \tag{4.41}$$

i.e. we have *two* constraints on the variation of our parameters. Thus, for example if we throw a small mass m_0 into the black hole, we expect $\delta M = m_0$, $\delta Q = \delta P = 0$.

Using the expression for M and the tensions we then find:

$$\frac{\delta K}{K^2} = -2\frac{e^2A^2}{m}\delta M \qquad \delta A = -(1-e^2A^2)\frac{AK}{m}\delta M$$
$$\delta m = (1-3e^2A^2)K\delta M \qquad \delta e = -2\frac{e^3A^2K}{m}\delta M \tag{4.42}$$

indicating that the acceleration of the black hole drops, as expected.

For the accelerating black hole, we want to compare the volume dependence on r_+ to the area dependence via the isoperimetric ratio as a consistency check, introduced in section 1.4.2,

$$\mathcal{R} = \left(\frac{3V}{\omega_2}\right)^{\frac{1}{3}}\left(\frac{\omega_2}{\mathcal{A}}\right)^{\frac{1}{2}}, \tag{4.43}$$

where V is the thermodynamic volume, $\mathcal{A}$ is the horizon area, and $\omega_2 = 4\pi/K$ is the area of a unit 'sphere'. Using eq. (4.22) for V and eq. (4.20) for $\mathcal{A}$, we find

$$\mathcal{R} = \frac{1}{(1-A^2r_+^2)^{1/6}} \geqslant 1. \tag{4.44}$$

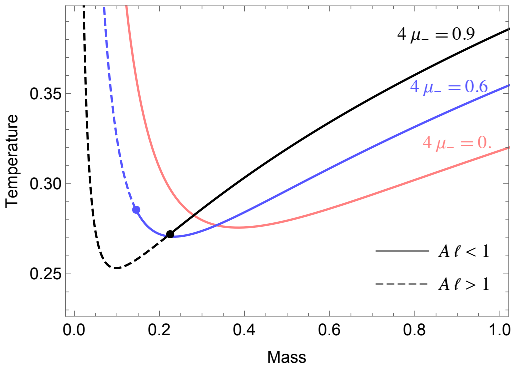
Thus these slowly accelerating black holes do indeed satisfy the reverse isoperimetric inequality.

From here, it would be most interesting to proceed onto rotating black holes, nicely tying the bow on this study of thermodynamics. Alas, the computations involved have proved too tall an order[3]. We shall, however, revisit the topic further in the chapter as we now take a turn to explore what these findings may teach us about accelerating black holes.

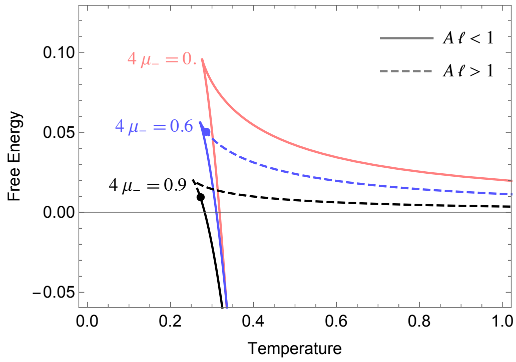
4.2 Critical behaviour of accelerating black holes

Given that we are working in anti-de Sitter spacetime, we can ask whether there is something analogous to a Hawking-Page phase transition [50] for our accelerating black holes, although it is difficult to see how one could actually have a phase transition between a system with a conical deficit along one polar axis only, and a presumably totally regular radiation bath. However, recall, from section 1.4.3, that a black hole in AdS behaves similarly to a black hole in a reflecting box, with the negative curvature of the AdS providing the qualitative reflection. For small black holes, the effect of the negative curvature is subdominant to the local curvature

[3] A method for obtaining the thermodynamics of rotating accelerating black holes was discovered in the later stages of production of this thesis [4]. These new results are summarised in the concluding chapter.



(a)



(b)

Figure 4.2: (a) A plot of temperature as a function of mass (in units of ℓ) for the uncharged black hole. The slowly accelerating regime is shown as a solid line, and the inferred local horizon temperature for $A\ell > 1$ is shown dashed. Note how for larger string tension (hence greater acceleration) the region of positive specific heat increases. (b) A similar plot, but now showing the Gibbs free energy as a function of temperature.

of the black hole, and the black hole has negative specific heat, as in the vacuum Schwarzschild case. For black holes larger than the AdS radius, the vacuum curvature dominates, and the black hole has positive specific heat; in particular, there is a minimum temperature for a black hole in AdS. Below this temperature, only a radiation bath can be a solution to the Einstein equations at finite T. Plotting the Gibbs free energy as a function of temperature shows both the allowed states, as well as the preferred one for a given temperature. At very low T, the only allowed state is a radiation bath. Above a critical temperature $T_c = \sqrt{3}/2\pi\ell$, one can have either a radiation bath, or a black hole (that may be either 'small' or 'large'). However for $T > 1/\pi\ell$, the large black hole is not only thermodynamically stable (in the sense of positive specific heat) but thermodynamically preferred, and a radiation bath will spontaneously transition into a large black hole.

First consider the situation where our accelerating black hole is uncharged.[4] Fixing the tension of the string, we can plot the temperature of our black hole as a function of its mass, M, as shown in figure 4.2. This figure shows how increasing acceleration actually makes a black hole of given mass *more* thermodynamically stable in the sense of positive specific heat. Figure 4.2 also shows the corresponding Gibbs free energy, indicating the would-be Hawking-Page transition occurs at lower temperatures as acceleration increases.

[4]In all explicit examples and figures in this section we take the $\theta = 0$ axis to be regular ($\mu_+ = 0$). This is for simplicity, including a nonzero north pole tension does not alter the essential physics of what we present here.

At first sight, this is rather curious, as a naive examination of the uncharged C-metric shows that the Newtonian potential, $f(r)$ has the cosmological constant ameliorated by the acceleration: $f(r) = r^2(1/\ell^2 - A^2) \simeq r^2/\ell_{\text{eff}}^2$. Given that one often imagines that it is the black hole radius relative to the confining 'box' of AdS that is causing the thermodynamic stability of the large black holes, this looks rather confusing: increasing acceleration appears to counteract the AdS length scale. However, this intuition is too naive: the relevant effect is the spacetime curvature in the vicinity of the horizon, and whether the black hole or the cosmological constant is dominant (larger black holes having smaller tidal forces). Computing the Kretschmann scalar at the event horizon demonstrates that indeed, increasing acceleration for a given mass lowers the local tidal forces due to the black hole. In fact, it is easy to compute the "Hawking-Page" transition temperature, assuming the radiation bath to have zero Gibbs energy from the expressions for TS and M in terms of r_+, A and ℓ. A brief calculation gives

$$\begin{aligned} T_{\text{HP}}(r_+, \ell, A) &= \frac{1}{4\pi r_+}\left[\frac{3r_+^2}{\ell^2(1 - A^2 r_+^2)} + 1\right] \\ &\simeq \frac{1}{2\pi\ell}\left(1 - \frac{3}{2}A^2\ell^2 + \mathcal{O}(A^4\ell^4)\right) . \end{aligned} \tag{4.45}$$

While the acceleration parameter A is not a thermodynamic charge, instead being related to the tension via M, nonetheless, the general picture is that increasing tension increases acceleration, thereby decreasing the temperature at which the "Hawking-Page" transition occurs.

Now consider adding a charge to the black hole, for which we might now expect a richer phase structure, possibly with critical phenomena analogous to the isolated charged AdS black hole [92, 132, 133]. The critical phenomena occur due to the three possible phases of black hole behaviour for varying mass. In the presence of charge, there is now a lower limit on the mass parameter of the black hole, set by the extremal limit where the temperature vanishes. Increasing the mass of the black hole moves it away from extremality, thus increasing temperature, rendering the specific heat positive near this lower limit. For large mass black holes, we are also in a positive specific heat regime where the local vacuum curvature is dominant in the near horizon geometry. Depending on the size of the charge relative to the vacuum energy, there can be an additional negative specific heat regime where the black hole is small enough that its local curvature is dominant, but is far enough from extremality that the usual Schwarzschild negative specific heat type of behaviour pervades. Given that for uncharged accelerating black holes, increasing tension lowers the critical temperature at which the transition to positive specific heat occurs, we expect this "swallow tail" behaviour to be mitigated for charged accelerating black

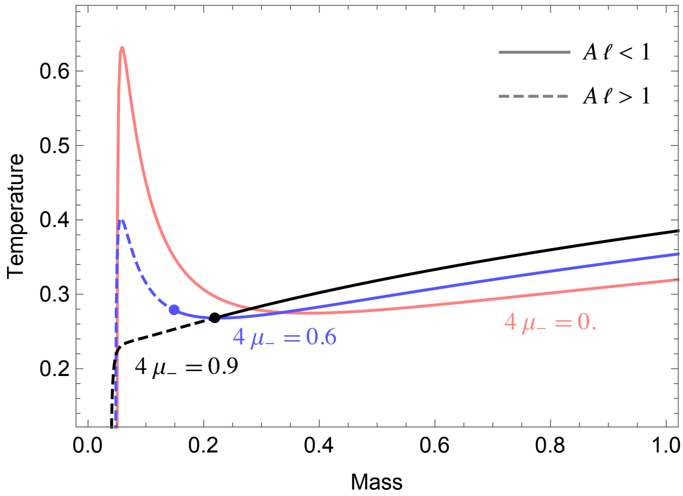


Figure 4.3: A plot of temperature as a function of mass for the charged black hole, with fixed $Q = 0.05\ell$, and varying tension as labelled. As before, the slowly accelerating regime is shown as a solid line, and $A > A^*$ is shown dashed.

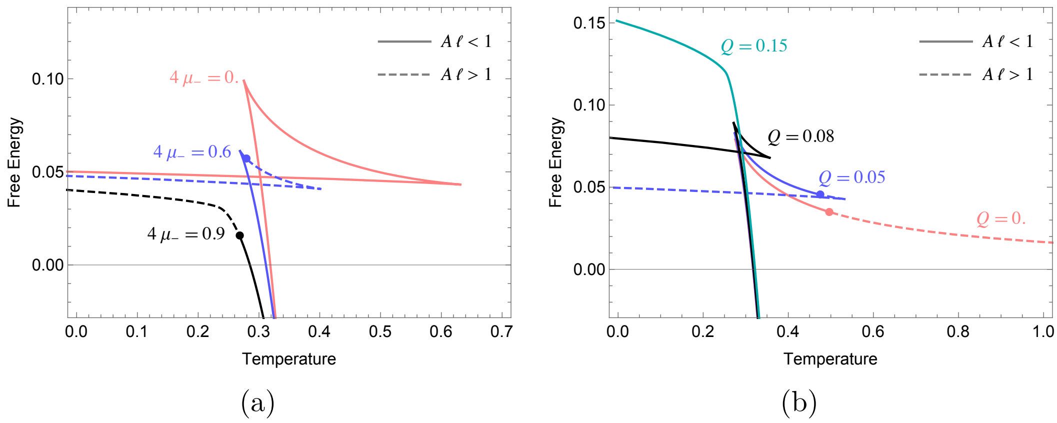
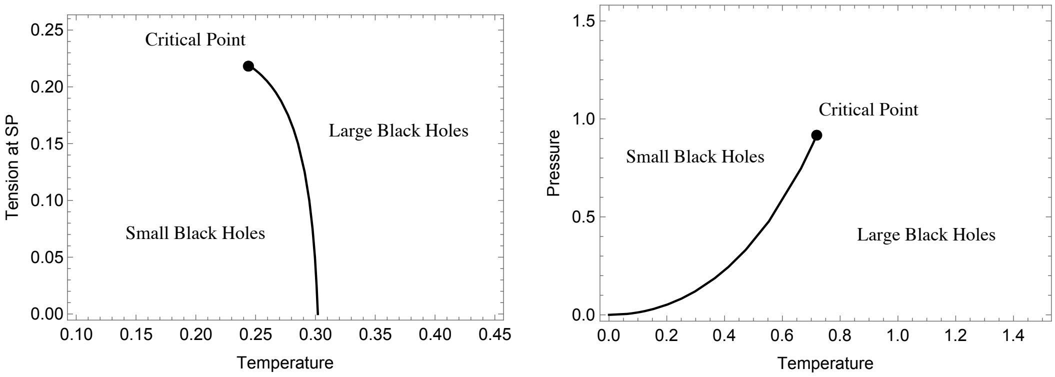


Figure 4.4: A plot of the free energy as a function of temperature for varying tension with $Q = 0.05\ell$ on the left, and varying charge with $4\mu_- = 0.3$ on the right.

holes in the canonical ensemble, and indeed this is what is observed.

We first explore the accelerating black hole in the canonical ensemble, i.e. where the charge, Q, of the black hole is fixed, but we allow M and μ_- to vary. In fig. 4.3, we give a representative plot of temperature as a function of black hole mass for $Q = 0.05\ell$ to illustrate how increasing tension gradually removes the negative specific heat phase of the black hole.

Figure 4.4 shows the variation of the free energy $F = M - TS$ with temperature for varying tension and charge. As tension is increased, the swallow tail becomes smaller, and eventually disappears, analogous to the situation where the charge is gradually increased, shown on the right in fig. 4.4. The free energy plot tells us that at low temperatures, we have the near extremal black hole, however as the mass of the black



(a) (b)

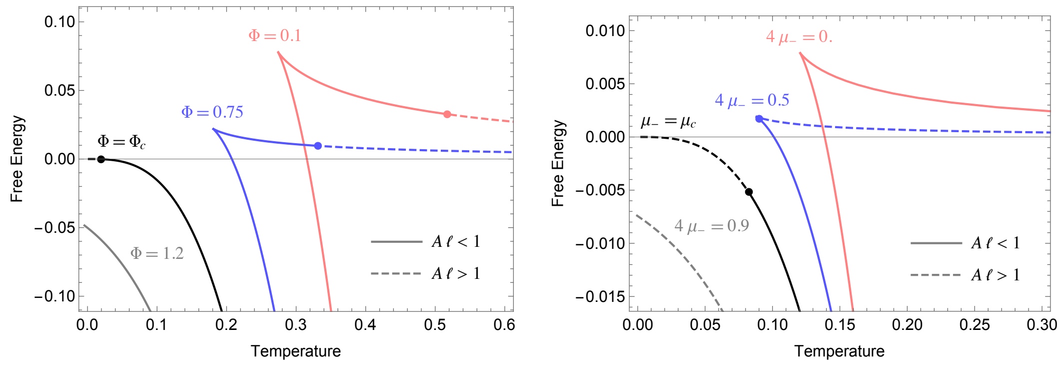
Figure 4.5: The coexistence line for the charged black hole shown for varying tension and cosmological constant with the black hole charge is fixed at $Q = 0.05$. (a) $\ell = 1$, and the value of tension at the critical point is $\mu_c = 0.219$. (b) $4\mu_- = 0.3$, and the critical value of the AdS radius is $\ell_c = 0.36$.

hole increases there is a critical value at which there is a spontaneous transition to a larger black hole with positive specific heat. The existence of this transition relies on the presence of the intermediate region of negative specific heat for the charged black hole. For large enough tension (or charge relative to ℓ), there is a critical point at which this intermediate regime disappears, and the phase transition along with it. Figure 4.5 shows the "Van der Waals"-like behaviour of this coexistence curve for varying tension (in analogy to the varying potential plots of [92]), and cosmological constant (in analogy to [134]).

Finally, for completeness, we consider the thermodynamics of the accelerating charged black hole in the grand canonical ensemble, where we now allow charge to vary. The Gibbs potential is now $G = M - TS - Q\Phi$, with

$$\Phi = \Phi_H + \Phi_1 = \frac{e}{r_+} - \frac{meA^2}{1 + e^2A^2} \tag{4.46}$$

kept fixed. The interesting feature of fixed potential, as noted in [92] for an isolated RNAdS black hole, is that there is a critical value of Φ delineating two qualitatively different behaviours of the black hole. For small fixed potentials, the charged AdS black hole can never approach extremality. This can be seen by noting that $f = f' = 0$ at extremality, where $f(r)$ is the RNAdS black hole potential. Solving these algebraic equations, and substituting $\Phi_{RN} = e/r_+$, one finds the constraint $3r_+^2/\ell^2 = \Phi_{RN}^2 - 1$, thus for $|\Phi_{RN}| < 1$ there is no possibility of extremality. In our case, for the charged accelerating black hole, the algebraic relations for extremality at fixed potential are considerably more complicated partly due to the extra acceleration parameter, but mostly because of the complicated expression for Φ (4.46). However,



(a) (b)

Figure 4.6: The Gibbs potential in the grand canonical ensemble as a function of temperature, on the left with $4\mu_- = 0.3$ for varying potential as labelled, and on the right with $\Phi = 0.9$ and varying tension as labelled in the plot.

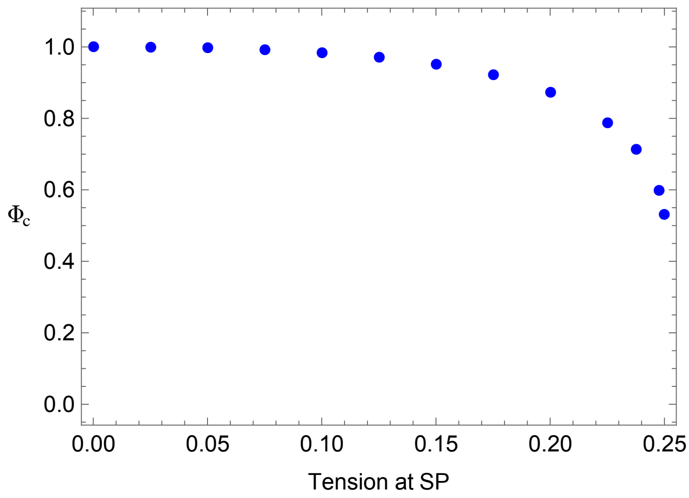


Figure 4.7: A plot of the critical value of $\Phi_c(\mu_-)$ at which a black hole is always preferred for all temperatures as a function of the tension.

the same principle applies, and we also observe a similar phase transition from small to large Φ, where the critical value of Φ is now tension dependent. Figure 4.6 demonstrates this behaviour showing the analogous plot to [92] with acceleration for fixed μ_-, and also how the behaviour depends on μ_- at fixed Φ, illustrating how increasing μ_- improves the thermodynamic viability of the black hole. Figure 4.7 shows how the critical value of the potential, where only positive specific heat black holes are allowed, varies with tension.

4.3 Critical black holes with teardrop-shaped horizons

Another area of interest recently has been to search for and discover spacetimes which do not satisfy the reverse isoperimetric inequality. A little context is needed. In exploring possible black hole solutions in four-dimensional Fayet-Iliopoulos gauged supergravities, Gnecchi *et al.* briefly presented a noncompact black hole horizon with a finite area [135]. It was later clarified in a letter by Klemm that this solution can be interpreted as the *ultra-spinning* limit of the Kerr-AdS solution, where the rotation parameter is taken to be critically large [136]. This limit only becomes sensible if one admits the existence of conical defects running along the main axis of revolution, which in turn become maximal in this limit. The result is a horizon which could be described as roughly spherical near its equator, with sharp conical deficits at each pole that extrude to the boundary.

From here, in a series of papers, Hennigar *et al.* [84, 85, 137], explored the thermodynamic implications of having such an extraordinary spacetime. In particular, they sought to verify the reverse isoperimetric inequality conjecture, which we first introduced back in section 1.4.2, in the context of these solutions. If the nondiverging area of a noncompact horizon was the initial "first of its kind" for this solution, these papers established the second such instance. The ultra-spinning black hole was the first solution found to violate this conjecture, leading the authors to impose more stringent conditions under which the bound might be valid.

Let us now seek to determine the uniqueness of this latter discovery. A curious feature of the ultra-spinning spacetime is that it is seemingly isolated from regularly-spinning black holes by any physical process. It is interesting therefore to ponder whether it truly is a special case, or whether this violation is present in further extensions of this solution by introducing acceleration.

The C-metric is similar in form to Kerr-AdS, but is differentiated by a nonremovable conical defect and a boundary offset from the usual $r \to \infty$, if one treats r as a generic radial coordinate centred on the black hole. The characteristic feature of the ultra-spinning black hole is the pair of maximal deficits at each pole. The accelerated solution has by default one deficit greater than the other, which means that we may only have one such maximal defect. The term "ultra-spinning" to designate this class of solutions in the context of acceleration is misleading, for unlike inertial black holes, this state may be reached by maximising — more appropriately, as will be explained in further sections, extremising — not only rotation but either acceleration or even charge as well. The term *critical*, for lack of an original word, will therefore be used to designate any black hole solution which exhibits either a single or a pair of 2π-deficit(s).

4.3.1 Critical black hole geometries

Let us first consider the nonaccelerating ultra-spinning black hole. We begin by considering the rotating black hole in asymptotically AdS space, described by the Kerr-AdS metric, which was given in eq. (1.32). In Boyer-Lindquist coordinates, it is,

$$ds^2 = -\frac{f(r)}{\Sigma}\left[dt - a\sin^2\theta\frac{d\phi}{K}\right]^2 + \frac{\Sigma}{f(r)}dr^2 + \frac{\Sigma r^2}{g(\theta)}d\theta^2 + \frac{g(\theta)\sin^2\theta}{\Sigma r^2}\left[adt - (r^2+a^2)\frac{d\phi}{K}\right]^2, \tag{4.47}$$

where

$$\begin{aligned} f(r) &= 1 - \frac{2m}{r} + \frac{a^2}{r^2} + \frac{r^2+a^2}{\ell^2}, \\ g(\theta) &= 1 - \frac{a^2}{\ell^2}\cos^2\theta, \\ \Sigma(r,\theta) &= 1 + \frac{a^2}{r^2}\cos^2\theta, \end{aligned} \tag{4.48}$$

and K is a parameter which we choose to leave unspecified. The parameters m and a correspond to the mass and rotation of the spacetime and $\ell = \sqrt{-3/\Lambda}$ is the AdS length scale. In chapter 1, we introduced the Kerr-AdS metric with $K = \Xi = 1 - a^2/\ell^2$. This actually ensures the poles are regular; indeed, one may check that with K undefined, this spacetime has a conical deficit given by

$$\delta = 2\pi\left[1 - \frac{1-a^2/\ell^2}{K}\right]. \tag{4.49}$$

The ultra-spinning limit is obtained by taking the limit in which $a \to \ell$. From the expression above, it is clear that in this limit the deficit along the $\theta = 0$ and $\theta = \pi$ axes is maximal (2π). The $\phi\phi$-component of the traditional metric, with regular poles, diverges, and the workaround presented in [84] amounts to having $K \neq \Xi$. Part of the reasoning behind naming this spacetime as ultra-spinning is that the angular velocity evaluated on the boundary of the spacetime also diverges, despite the adjusted metric. Figure 4.8 shows an embedding of this spacetime.

We now revisit accelerating black holes and the C-metric. We are now interested in the generalised C-metric (2.42) which includes a rotation parameter a. Rewriting

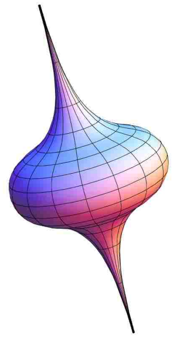


Figure 4.8: A θ–ϕ slice of the ultra-spinning black hole spacetime for $r = r_+$, the outer horizon.

it for convenience, we have

$$\begin{aligned} ds^2 = \frac{1}{\Omega^2}\bigg\{ &-\frac{f(r)}{\Sigma}\left[dt - a\sin^2\theta\frac{d\phi}{K}\right]^2 + \frac{\Sigma}{f(r)}dr^2 + \frac{\Sigma r^2}{g(\theta)}d\theta^2 \\ &+ \frac{g(\theta)\sin^2\theta}{\Sigma r^2}\left[adt - (r^2+a^2)\frac{d\phi}{K}\right]^2\bigg\}, \\ F = dB, \qquad &B = -\frac{e}{\Sigma r}\left[dt - a\sin^2\theta\frac{d\phi}{K}\right]. \end{aligned} \tag{4.50}$$

where

$$\begin{aligned} f(r) &= (1 - A^2r^2)\left[1 - \frac{2m}{r} + \frac{a^2+e^2}{r^2}\right] + \frac{r^2+a^2}{\ell^2}, \\ g(\theta) &= 1 + 2mA\cos\theta + \left[A^2(a^2+e^2) - \frac{a^2}{\ell^2}\right]\cos^2\theta, \\ \Sigma &= 1 + \frac{a^2}{r^2}\cos^2\theta, \qquad \Omega = 1 + Ar\cos\theta. \end{aligned} \tag{4.51}$$

Parametric restrictions exist for this solution and were given in section 2.3, along with the following explicit expressions for the string tensions:

$$\mu_\pm = \frac{\delta_\pm}{8\pi} = \frac{1}{4}\left[1 - \frac{g(\theta_\pm)}{K}\right] = \frac{1}{4}\left[1 - \frac{1 \pm 2mA + A^2(a^2+e^2) - a^2/\ell^2}{K}\right], \tag{4.52}$$

from which we see that setting $K = \Xi \equiv 1 + 2mA + A^2(a^2 + e^2 - a^2/\ell^2)$ removes the conical deficit at the north pole, leaving a positive defect at the south pole. One might also envisage removing the defect at the south pole by setting $K = \Xi' \equiv 1 - 2mA + A^2(a^2 + e^2 - a^2/\ell^2)$ which would leave an excess at the north pole. Generally, however, it is preferable to avoid conical excesses as these would have to be sourced by negative energy objects.

The C-metric provides a mechanism through which we can construct a black hole with strings of unequal tension at either pole, through various choices of the

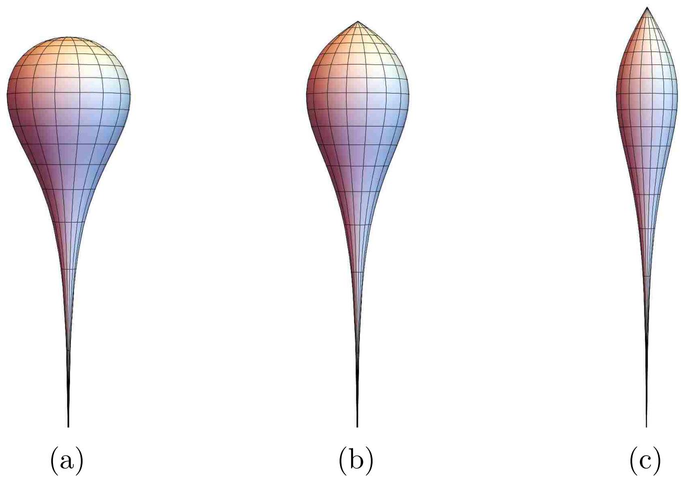


Figure 4.9: Embeddings in $\mathbb{E}^3$ of the ultra-spinning C-metric for $m = 9\ell$, $A = 0.04\ell^{-1}$ and (a) $K = \Xi$, (b) $K = 1.2\Xi$, (c) $K = 2\Xi$.

parameters A and K. The term critical, in this section, is used to describe a black hole where at least one of the tensions is maximal, as in the ultra-spinning black hole. This occurs when the deficit is taken to its upper limit, 2π. While for Kerr-AdS, this corresponds to an upper bound on rotation, for the generalised C-metric, it actually corresponds to a set of bounds, upper and sometimes lower, for the parameters of not only rotation, but charge and acceleration too.

The critical limit for the C-metric is defined as the parametric limit required for $\delta_- \to 2\pi$, since $\delta_- \geqslant \delta_+$. We read off, from our definition of the conical deficits (4.52), that this occurs when $g(\theta_-) = 0$. We have already determined this in section 2.3.4, when investigating parametric restrictions on this metric. The case we are interested in actually corresponds, as long as $mA < 1$, to the solid blue lines in fig. 2.3, described by the relation

$$a^2 = \ell^2 \frac{1 - 2mA + e^2A^2}{1 - A^2\ell^2}, \tag{4.53}$$

the condition for criticality.

This relation, combined with eq. (2.43) provides more stringent conditions in parametric space for which critical C-metrics exist. Indeed, requiring that the right-hand side of eq. (4.53) be positive allows us to draw the following criteria. We have two possibilities, either

$$A < \frac{1}{\ell}, \quad \text{and} \quad 2mA < \min\{1 + e^2A^2, 2\}, \tag{4.54}$$

or

$$A > \frac{1}{\ell}, \quad e^2A^2 < 1, \quad \text{and} \quad 1 + e^2A^2 < 2mA < 2. \tag{4.55}$$

For this limit to be sensible physically, we must ensure that the singularity remain shielded by an event horizon. While for the general C-metric we are limited to numerical techniques in determining when a horizon is formed — this was displayed in fig. 2.3 — we are able to analytically determine expressions for the horizons in the absence of charge[5].

We are interested in determining the relationship between the parameters A, ℓ and m (with a given by eq. (4.53)) for which the function $f(r)$ exhibits a double root. The adjacent sections in parameter space will then correspond to a naked singularity and one which has (at least) a horizon. Let r_{dr} denote the location of the extremal horizon. We then use $f(r_{\mathrm{dr}}) = 0$ to find $m = m(r_{\mathrm{dr}})$ when the spacetime is extremal, and use it to factorise

$$f(r)\Big|_{m=m(r_{\mathrm{dr}})} = \frac{r - r_{\mathrm{dr}}}{r^2}\overline{f}(r). \tag{4.56}$$

Since r_{dr} is a double root of $f(r)$, we also have $\overline{f}(r_{\mathrm{dr}}) = 0$, which yields a constraint on the parameters,

$$(1 + Ar_{\mathrm{dr}})\left[\left(A^2 - \frac{1}{\ell^2}\right)r_{\mathrm{dr}}^2 - 1\right]\left[\left(A^2 - \frac{1}{\ell^2}\right)r_{\mathrm{dr}}^3 - \left(A^2 - \frac{1}{\ell^2}\right)r_{\mathrm{dr}}^2 + 3Ar_{\mathrm{dr}} - 1\right] = 0. \tag{4.57}$$

This has a couple of possible solutions. The first two factors give

$$\begin{aligned} r_{\mathrm{dr}} &= -\frac{1}{A} \qquad m = \frac{1}{2A^3\ell^2}, \\ r_{\mathrm{dr}} &= \pm\frac{\ell}{\sqrt{A^2\ell^2 - 1}} \qquad m = 0 \end{aligned} \tag{4.58}$$

with the corresponding value of m also given. The solutions to the cubic equation in the third factor may be parametrised using hyperbolic or trigonometric functions. To do so, we treat $0 \leqslant A\ell \leqslant 1$ and $A\ell > 1$ separately as follows:

- for $0 \leqslant A\ell \leqslant 1$, we write $A\ell = \sin 3\chi$. The three real solutions are then

  $$r_{\mathrm{dr}} = \frac{\ell}{\tan\bar{\chi}\cos 3\bar{\chi}}, \qquad m = \frac{\ell}{2\sin^3\bar{\chi}}\frac{1 - 4\sin^2\bar{\chi}}{8\sin^2\bar{\chi} - 5}, \tag{4.59}$$

  where $\bar{\chi} = \chi + 2\pi n/3$, and $n \in \{-1, 0, 1\}$.

- for $A\ell > 1$, we write $A\ell = \cosh 3\eta$. There is one real solution and it reads

  $$r_{\mathrm{dr}} = \frac{\ell}{\coth\eta\sinh 3\eta}, \qquad m = \frac{\ell}{2\cosh^3\eta}\frac{4\cosh^2\eta - 1}{8\cosh^2\eta - 5}. \tag{4.60}$$

[5] Again, we do not expect charge to dramatically alter the observations laid out in this section, however it does mute our ability to perform this analysis analytically.

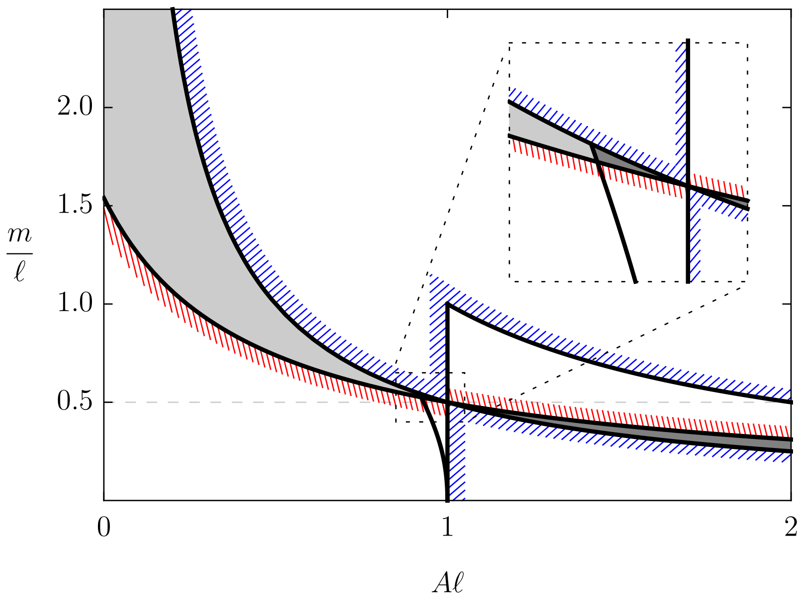


Figure 4.10: Parametric space for the accelerated ultra-spinning black hole. Blue hatched regions are excluded by virtue of the conditions set out in eq. (4.54) and eq. (4.55). Red hatched regions correspond to naked singularities and are therefore also excluded. The remaining space is separated into two regions, one which describes slowly accelerating black holes without an acceleration horizon (lighter shade), and another for spacetimes with an acceleration horizon (darker shade).

The mapped output of these solutions is displayed in fig. 4.10, which displays the parametric regions which result in spacetimes containing either no horizons, an outer event horizon, an acceleration horizon, or both. More importantly, this confirms that it is therefore indeed possible to reach this limit while avoiding a naked singularity.

4.3.2 Thermodynamics of the critical C-metric

We are interested in determining whether these critical black holes violate the reverse isoperimetric inequality. As alluded to in the introduction to this section, regular ultra-spinning black holes seem to have more entropy than they ought to by this upper bound. In order to consider this for the critical C-metric, we will need a thermodynamic description, which we have already provided in the absence of rotation.

The super-entropic black hole

Let us return to the nonaccelerating solution (4.47) to review the ultra-spinning black hole and why it is super-entropic. Consider now the $a \to \ell$ limit. As mentioned above, this causes the original, smooth, Kerr-AdS metric to diverge, and if precautionary measures are taken to allow for conical defects, it has the effect of freezing out the tension and decoupling K as a physical parameter. This can be seen from the expression for conical deficits (4.49): in this limit, K is no longer linked to the deficit, and, in turn, it is no longer linked to the tension. With the thermodynamic quantities derived in chapter 3, we see that some of these, such as the mass $M = m/K\Xi$, also diverge, through their dependence on the function $\gamma \sim \Xi^{-1} = (1 - a^2/\ell^2)^{-1}$ which was introduced as a function multiplying the "naive" mass $M' = m/K$ to satisfy the first law. In particular, the angular velocity of the boundary diverges hence the 'ultra-spinning' label attributed to this limit. That γ diverges poses the main problem for resolving the first law in this limit.

This decoupling of K from μ, or rather, that, combined with the fact that μ is now seemingly constant, does however hold interesting implications for the primitive first law in eq. (3.16),

$$\delta\left(\frac{m}{K}\right) = T\delta S + V_0\delta P + \Omega_0\delta J - 2r_+\delta\mu + \frac{m\delta K}{2K^2}, \tag{4.61}$$

which, itself, does not diverge. The $\delta\mu$ piece obviously vanishes, and we are free to set $\delta K = 0$ too, since it is purely a gauge choice at this point, it no longer has physical significance. This leaves a functioning first law for the ultra-spinning black hole, with thermodynamic potentials, given in eqs. (3.15) and (3.17),

$$\begin{gathered} S = \frac{\pi}{K}(r_+^2 + a^2) \qquad T = \frac{1}{2\pi(r_+^2+a^2)}\left[m - \frac{a^2}{r_+} + \frac{r_+^3}{\ell^2}\right], \qquad \Omega_0 = \frac{aK}{r_+^2+a^2}, \\ J = \frac{ma}{K^2}, \qquad V_0 = \frac{4\pi}{3K}r_+(r_+^2+a^2), \qquad P = \frac{3}{8\pi\ell^2}, \end{gathered} \tag{4.62}$$

(with $a = \ell$) distinct from the regular Kerr-AdS black hole. These can be related to those in [84, 85] by recognising that $K = 2\pi/\bar{\mu}$ (where the bar allows for distinction from what we define as tension, and $\bar{\mu}$ is the periodicity of their redefined azimuthal coordinate).

Having obtained expressions for the volume and horizon area of the black hole, we may now discuss the reverse isoperimetric conjecture. We find that

$$\mathcal{R} = \sqrt[6]{\frac{r_+^2}{r_+^2+\ell^2}} \leqslant 1, \tag{4.63}$$

and the conjecture is violated. The existence of a super-entropic black hole would naively imply the existence of other near-ultra-spinning super-entropic black holes however the physical discontinuity between this black hole and regular Kerr-AdS spacetimes prevents this train of logic. In [84], new more stringent conditions on the validity of this inequality were proposed, stating that the inequality ought to be conjectured to only hold for compact horizons. We can verify these new additional conditions by exploring whether the reverse isoperimetric inequality is violated for noncompact accelerating black hole horizons.

The critical nonrotating C-metric

We return now to the nonrotating C-metric discussed in the previous chapter. We explored some of the conditions one must impose unto this solution in section 2.3.2. In particular, we had the condition (2.38):

$$e^2A^2 > \begin{cases} 2mA - 1 & \text{if} \quad mA \leqslant 1, \\ m^2A^2 & \text{if} \quad mA > 1, \end{cases} \tag{4.64}$$

which ensures that the metric signature is preserved for the entire range of $\theta \in [0, \pi]$. The conical deficits along each axis $\theta_\pm = 0, \pi$ were also given (2.40) as

$$\delta_\pm = 2\pi \left[1 - \frac{g(\theta_\pm)}{K} \right] = 2\pi \left[1 - \frac{1 \pm 2mA + e^2A^2}{K} \right], \tag{4.65}$$

and we see that a maximal deficit is obtained along the $\theta = \pi$ axis provided that $1 - 2mA + e^2A^2 = 0$. This also prevents a maximal deficit from occurring at $\theta = 0$ as $\delta_+ < \delta_-$. This corresponds to saturating the bound (4.64) for $mA \leqslant 1$. Saturating the bound for $mA > 1$ is a different type of limit in which a double root is introduced for $g(\theta)$ over $0 < \theta < \pi$, resulting in two separate positive regions over this domain, all the while preserving the signature. Once again, we see that the 2π-deficit is clearly independent of K.

This limit has already been mentioned in [138, 139]. Unlike the nonaccelerating case however, we still have the possibility of a defect at $\theta = 0$. The tension of the string running from the north pole is $\mu_+ = (K - 2)/4K$, and K does in fact still have a physical role in the spacetime. All this means is that this fact, combined with the absence of any divergences other than those at the horizons, implies that we do not need to treat this spacetime any different to its noncritical counterpart.

Using the thermodynamic quantities we established in section 4.1.2, and turning off the electric charge, which adds little qualitatively, we know therefore that the first law of black hole thermodynamics,

$$\delta M = T\delta S + V\delta P - \lambda_+\delta\mu_+ - \lambda_-\delta\mu_-, \tag{4.66}$$

is satisfied with the following quantities,

$$M = \frac{m}{K}, \qquad T = \frac{1}{2\pi r_+^2}\left[m(1 - A^2 r_+^2) + \frac{r_+^3}{\ell^2(1 - A^2 r_+^2)}\right], \qquad S = \frac{\mathcal{A}}{4} = \frac{\pi r_+^2}{K(1 - A^2 r_+^2)},$$
$$V = \frac{4\pi r_+^3}{3K(1 - A^2 r_+^2)^2}, \qquad P = \frac{3}{8\pi \ell^2}, \qquad \lambda_\pm = \frac{r_+}{1 \pm A r_+} - m, \tag{4.67}$$

From here we can safely take the critical limit $2mA = 1$. The last term in eq. (4.66) vanishes and none of the quantities above diverge. Finally, seeing as μ_+ is still related to K and we therefore need not treat this particular limit any differently, we expect the isoperimetric conjecture to be obeyed. The isoperimetric ratio is

$$\mathcal{R} = \frac{1}{\sqrt[6]{1 - A^2 r_+^2}} = \frac{1}{\sqrt[6]{1 - r_+^2/4m^2}} \geqslant 1. \tag{4.68}$$

Therefore, despite the noncompact horizon, its entropy does fall below the bound imposed unto it by the isoperimetric inequality, and cannot be considered as "super-entropic".

4.3.3 Thermodynamics for the rotating C-metric

We have already alluded to the fact that unfortunately, the thermodynamics of rotating accelerated black holes are not well understood, despite recent attempts [140][6]. Nonetheless, by taking a perturbative approach, one may hope to glance at these thermodynamics by taking an approach similar to what we have already been doing in previous sections, that is, to assume the first law be upheld, introduce correction terms to potentials for which this would not be unreasonable and then solve the ensuing equations. Using this line of analysis, we will attempt to make sufficient headway so as to draw certain conclusions concerning the critical limit.

Once again, we seek to determine an expression for the mass of the black hole as well as other thermodynamic quantities by demanding that the first law,

$$\delta M = T\delta S + \Omega \delta J + \Phi \delta Q + V \delta P - \lambda_+ \delta\mu_+ - \lambda_- \delta\mu_-, \tag{4.69}$$

hold. To do so in an appropriate fashion, we will impose that entropy be a quarter of the outer horizon area, and that the temperature be that obtained by regularising

[6] A method for obtaining the thermodynamics of rotating accelerating black holes was discovered in the later stages of production of this thesis [4]. These new results are summarised in the concluding chapter.

the Euclidean form of this metric. To clarify, we write out all expressions explicitly,

$$T = \frac{f'(r_+)}{4\pi} = \frac{1}{2\pi(r_+^2 + a^2)} \left(m(1 + A^2 r_+^2) - \frac{a^2 + e^2}{r_+} + r_+^3 \left(\frac{1}{\ell^2} - A^2 \right) \right),$$
$$S = \frac{\mathcal{A}}{4} = \frac{\pi(r_+^2 + a^2)}{K(1 - A^2 r_+^2)}. \tag{4.70}$$

We will also preserve the following forms for the charge Q, angular momentum J, pressure P,

$$Q = \frac{e}{K}, \qquad J = \frac{ma}{K^2}, \qquad P = -\frac{\Lambda}{8\pi} = \frac{3}{8\pi\ell^2}, \tag{4.71}$$

and complete the set of extensive variables by including the tensions $\mu_\pm = \delta_\pm/8\pi$ as defined by eq. (4.52). We will be decomposing the angular velocity Ω, electrical potential Φ and thermodynamic volume V into two terms

$$X = X_0 + X_1 \tag{4.72}$$

in order to separate reasonably[7] well-defined quantities, denoted by X_0, and correction terms, X_1, whose existence is required to satisfy the first law, and whose explicit form left to be determined in what follows. As a reminder, our use of correction terms is justified by considering the precedent laid out by the thermodynamics of Kerr-AdS [53, 55] which we covered in sections 1.4 and 3.1. We have, to begin with [53],

$$\Omega_0 = -\frac{g_{t\phi}}{g_{\phi\phi}}\bigg|_{r=r_+} = \frac{aK}{r_+^2 + a^2}, \qquad \Phi_0 = \frac{e r_+^2}{r_+^2 + a^2}. \tag{4.73}$$

The thermodynamic volume is usually determined through the Smarr/scaling Euler relation. We will take V_0 to be that which satisfies the following "reduced" Smarr relation given as

$$\frac{m}{K} = 2(TS - PV_0 + \Omega_0 J) + \Phi_0 Q. \tag{4.74}$$

A straightforward re-arrangement of eq. (4.70) reveals

$$V_0 = \frac{4\pi r_+ (r_+^2 + a^2)}{3K(1 - A^2 r_+^2)^2}. \tag{4.75}$$

Finally, we introduce a function $\gamma(A, a, e, l)$ which we will be using to determine an

[7] By reasonably, we are referring to conventional (see, for example [55]) ways of deriving the quantities in question, but applying them to the current geometry. These are all well-motivated for more standard black holes and therefore are *reasonable* choices.

explicit form for the mass,

$$M = \frac{m}{K}\gamma(A, a, e, l), \tag{4.76}$$

as well as explicit expressions for the correction terms introduced above.

To establish the first law, one usually begins by considering perturbations of $f(r_+) = 0$. The result can usually simply then be massaged directly into the first law itself. Proceeding this way, making use of all the definitions given above, the closest one can get to an expression which resembles the first law is the full form of the expression we have been using throughout the previous sections as the initial starting point of this derivation. In fact, the procedure remains much the same, however we will proceed in full, as this is a crucial part in this write-up. One eventually obtains

$$\delta\frac{m}{K} = T\delta S + \Omega_0\delta J + \Phi_0\delta Q + V_0\delta P - \frac{r_+}{1+Ar_+}\delta\mu_+ - \frac{r_+}{1-Ar_+}\delta\mu_- + \frac{m\delta K}{2K^2}, \tag{4.77}$$

which generalises eqs. (3.16) and (4.30). One then combines eq. (4.69) with eq. (4.77), from which it follows that

$$\delta M = \delta\frac{m}{K} + \Omega_1\delta J + \Phi_1\delta Q + V_1\delta P - \left(\lambda_\pm - \frac{r_+}{1\pm Ar_-}\right)\delta\mu_\pm - \frac{m}{2K^2}\delta K. \tag{4.78}$$

We will be comparing this to what a variation of M as defined in eq. (4.76) yields. In order to be able to do so term-by-term, we must ensure that all the variations are independent. In particular, using eq. (4.52), as well as variations of eq. (4.71), we can re-express δK and, for good measure, δA as

$$\begin{aligned}\delta K &= -\frac{2K^2}{m\Delta}(1+a^2A^2-\Delta)\delta\frac{m}{K} - \frac{2aK}{m\ell^2\Delta}(1-A^2\ell^2)\delta J + \frac{2eA^2K^2}{\Delta}\delta Q - \frac{8\pi a^2K}{3\Delta}\delta P\\ &\qquad + \frac{2K^2}{m\Delta}\Big(m \mp A(a^2+e^2)\Big)\delta\mu_\pm,\\ \delta A &= -\frac{AK}{m}\delta\frac{m}{K} - \frac{K}{m}\delta\mu_+ + \frac{K}{m}\delta\mu_-,\end{aligned} \tag{4.79}$$

where $\Delta \equiv 1 - e^2A^2 + \frac{a^2}{\ell^2}(1-A^2\ell^2)$. It immediately follows, from substituting these into eq. (4.78), that

$$\begin{aligned}\delta M &= \delta\frac{m}{K}\frac{1+a^2A^2}{\Delta} + \delta J\left(\Omega_1 + \frac{aK}{\ell^2\Delta}(1-A^2\ell^2)\right) + \delta Q\left(\Phi_1 - \frac{meA^2}{\Delta}\right)\\ &\quad + \delta P\left(V_1 + \frac{4\pi ma^2}{3K\Delta}\right) + \delta\mu_\pm\left(\frac{r_+}{1\pm Ar_+} - \lambda_\pm - \frac{1}{\Delta}\Big(m\mp A(a^2+e^2)\Big)\right).\end{aligned} \tag{4.80}$$

To perform the term-by-term comparison, we must write out perturbations of M in terms of the same quantities as those above. For this we can use variations Q

and J (4.71) as well as those for K and A outlined above in eq. (4.79) to eventually obtain

$$\begin{aligned}
\delta M = \delta\frac{m}{K}\Big(\gamma - A\gamma_A - \frac{a\gamma_a}{\Delta}(2+2A^2a^2-\Delta) + \frac{2e}{\Delta}\gamma_e(1+a^2A^2-\Delta)\Big)\\
+\frac{K}{\Delta}\delta J\left((2-2e^2A^2-\Delta)\gamma_a - \frac{2ea}{\ell^2}(1-A^2\ell^2)\gamma_e\right)\\
+\frac{m}{\Delta}\delta Q\left(\Delta\gamma_e - 2eaA^2\gamma_a\right)\\
-\frac{4\pi m\delta P}{3K\Delta}\left(\ell^3\Delta\gamma_\ell + 2a^2e\gamma_e + 2a^3\gamma_a\right)\\
-\delta\mu_\pm\left(\frac{2}{\Delta}\left(m \mp A(a^2+e^2)\right)(a\gamma_a + e\gamma_e) \mp \gamma_A\right).
\end{aligned} \tag{4.81}$$

Now that we have obtained two expressions for the variation of M, one from its explicit definition, and one through its relation with the first law, we can require these be equal to determine what functional form γ must have to obtain a set of thermodynamic quantities for this geometry that obey the first law. Doing so yields a set of differential equations which can be separated out into five equations that determine correction terms from γ

$$\begin{aligned}
\Omega_1 &= \frac{K}{\Delta}(2-2e^2A^2-\Delta)\gamma_a + \frac{2aeK}{\ell^2\Delta}(1-A^2\ell^2)\gamma_e - \frac{aK}{\ell^2\Delta}(1-A^2\ell^2),\\
\Phi_1 &= \frac{m}{\Delta}\Big(2eaA^2\gamma_a + (\Delta+2e^2A^2)\gamma_e + eA^2\Big),\\
V_1 &= -\frac{4\pi m}{3\Delta K}(\Delta l^3\gamma_\ell + 2a^3\gamma_a + 2a^2e\gamma_e + a^2),\\
\lambda_\pm &= \frac{r_+}{1\pm Ar_+} \pm \gamma_A - \frac{1}{\Delta}(2a\gamma_a + 2e\gamma_e + 1)\Big(m \mp A(a^2+e^2)\Big),
\end{aligned} \tag{4.82}$$

and a differential equation for γ,

$$\Delta(\gamma - A\gamma_A) - 2e\gamma_e(1+a^2A^2-\Delta)) - a\gamma_a(2+2a^2A^2-\Delta) - 1 - a^2A^2 = 0. \tag{4.83}$$

We also need these quantities to satisfy the Smarr relation

$$M = 2(TS - PV + \Omega J) + \Phi Q. \tag{4.84}$$

We can use eqs. (4.71) to (4.76) together to reduce the Smarr relation above into another differential equation for γ,

$$\Delta(\gamma - \ell\gamma_\ell) - e\gamma_e(2+2a^2A^2-\Delta) - 2a\gamma_a(1+a^2A^2) - 1 - a^2A^2 = 0. \tag{4.85}$$

With these two differential equations, we can discuss solving them. Subtracting one from the other provides a first hint at the form of γ, which is so familiar it could have been guessed at. We infer

$$a\gamma_a + e\gamma_e + \ell\gamma_\ell - A\gamma_A = 0 \qquad \Longleftrightarrow \qquad \gamma(A,a,e,\ell) = \phi\Big(\frac{a}{\ell},\frac{e}{\ell},A\ell\Big). \tag{4.86}$$

We re-write eq. (4.83) in terms of new unitless parameters $x = a^2/\ell^2$, $y = e^2/\ell^2$ and $z = A^2\ell^2$,

$$\begin{aligned}\phi(1+x-z(x+y)) - 2x\phi_x(1-x+z(3x+y)) - 4y\phi_y(-x+z(2x+y))\\ - 2z\phi_z(1+x-z(x+y)) - 1 - zx = 0.\end{aligned} \tag{4.87}$$

This equation has exact solutions for $x = 0$ and $z = 0$, either the nonrotating case or the nonaccelerating case, as have been outlined in previous sections. It is possible to solve this equation perturbatively with respect to either variable at least up to order x^2 or z^2, however for brevity, we shall only keep next-to-leading order terms, and set integration constants to 0. For example, writing $\phi = \phi^{(0)} + z\phi^{(1)} + \mathcal{O}(z^2)$, we may use the zeroth order solution to solve the equation at first order, given by

$$\begin{aligned}x + \phi^{(0)}(x+y) + 2x\phi^{(0)}_x(3x+y) + 4y\phi^{(0)}_y(2x+y) + \phi^{(1)}(1+x) + 2x\phi^{(1)}_x(1-x) - 4xy\phi^{(1)}_y\\ = \frac{(1+x)(x^2+2x+y^2)}{(1-x)^2} + \phi^{(1)}(1+x) + 2x\phi^{(1)}_x(1-x) - 4xy\phi^{(1)}_y = 0.\end{aligned} \tag{4.88}$$

To second order in A (first order in z), γ is given by

$$\begin{aligned}\gamma^{(A\ell\ll 1)} &= \frac{1}{1-a^2/\ell^2}\\ &\quad - \frac{A^2\ell^2}{4}\left(\frac{1+4e^2/\ell^2+3a^4/\ell^4}{(1-a^2/\ell^2)^2} + \frac{\ell}{2a}\left(1-\frac{a^2}{\ell^2}\right)\log\frac{1-a/\ell}{1+a/\ell}\right) + \mathcal{O}(A^4\ell^4).\end{aligned} \tag{4.89}$$

Alternatively, it is also possible to consider perturbations around $x = 0$, for small rotation parameter. Performing much the same as above, we find

$$\gamma^{(a\ll\ell)} = \frac{1}{1+e^2A^2} + \frac{a^2}{4e^2}\left(\frac{1+4e^2/\ell^2+3A^4e^4}{(1+e^2A^2)^2} - \Big(1+e^2A^2\Big)\frac{\arctan(Ae)}{Ae}\right) + \mathcal{O}\Big(\frac{a^4}{\ell^4}\Big). \tag{4.90}$$

Obtaining these solutions means that we are now able to write down, admittedly at low order, expressions for the correction terms and the mass in the small acceleration and/or small rotation limit using eqs. (4.76) and (4.82) with the appropriate expansion. For convenience, we re-write the mass

$$M = \frac{m}{K}\gamma, \tag{4.91}$$

noting that we do recover known solutions both in the absence of rotation and the absence of acceleration.

Let us now return to the critical limit introduced above and examine how this limit affects the thermodynamics of the system using the results from the previously. While the perturbative techniques used constrain us in parameter space, the initial set-0 nature of the ultra-spinning case hints that either this behaviour extends continuously to other critical limits away from $A = 0$, or it truly is set-0 and remains disconnected from other geometries in physical parameter space. As explained above, a characteristic feature of the ultra-spinning limit is that certain thermodynamical quantities for the KNAdS solution blow up. This is a direct consequence of the fact that γ, in the absence of acceleration, diverges for $a \to \ell$. However, the presence of an acceleration parameter shifts this limit to what is presented in eq. (4.53). γ, in the critical limit and for small acceleration, expands to

$$\gamma^{(A\ell \ll 1)}\Big|_{\text{us}} = \frac{1}{2Am} - \frac{A\ell^4}{8m^3}\left(1 + \frac{e^2}{\ell^2}\right)^2 - \frac{A\ell^2}{4m} + \mathcal{O}(A^2\ell^2). \tag{4.92}$$

This is clearly finite for $A \neq 0$. We therefore expect similar behaviour for the remaining potentials. The correction to the angular velocity, for example, expands to

$$\Omega_1^{(A\ell \ll 1)}\Big|_{\text{us}} = \frac{-K}{2\ell} + \frac{K}{2mA\ell} - \frac{AK\ell^3}{8m^3}\left(1 + \frac{e^2}{\ell^2}\right)^2 - \frac{AK}{4m\ell}(m^2 - e^2) + \mathcal{O}(A^2\ell^2), \tag{4.93}$$

which is well-behaved, even in this critical limit. It therefore seems reasonable to infer that the set of thermodynamic quantities must not be redefined as they were for the ultra-spinning black hole and, as a consequence, the reverse isoperimetric inequality is expected to be upheld and critical black hole solutions that accelerate would not, then, be super-entropic in general.

Chapter 5

Holographic thermodynamics of accelerating black holes

The importance of black holes in advancing our understanding of physics cannot be underestimated. They provide a setting for testing our most fundamental ideas about gravity under extreme conditions and offer us insight into the underlying microscopic degrees of freedom that may be associated with quantum gravity. The subject of black hole thermodynamics [17, 18, 29] has proven to be an invaluable tool to this end, and broad classes of black holes have been shown to exhibit a rich and varied range of thermodynamic behaviour, particularly in anti-de Sitter spacetime [141].

The work we have presented so far ambitiously tackles the thermodynamics of accelerated black holes, a topic which had remained largely misunderstood. We were able to make progress by identifying the tension as the extensive property which is added with conical defects. Therefore, surprisingly, even though these black holes are not isolated by virtue of the cosmic strings' presence, it is possible to derive sensible-looking thermodynamics, although recent studies have apparently conflicting results [1, 2, 140, 142], in particular regarding our earlier definitions for the mass and temperature. Since this work failed to convince members of the community, we felt the need to re-examine our work from first principles. Eventually this led to the publication of [3], which forms the basis of this chapter.

5.1 Mass of an accelerated black hole

We consider here the interpretation of an accelerating black hole in anti-de Sitter (AdS) spacetime, with a focus on a holographic interpretation of the thermodynamics. We resolve conflicting issues that exist in the literature, obtain a distinct set of thermodynamic variables that are now consistent with the gravitational action, and agree with both the conformal and holographic methods for computing conserved charges. To this end, we focus our attention to black holes with no acceleration horizon [129], so that there is no ambiguity as to which horizon temperature should be considered, or as to whether there is an equilibrium thermodynamics for the system. In addition, as we discuss, the holographic computation and interpretation are also unambiguous and straightforward. We also comment on the cases when the acceleration horizons appear and provide a novel interpretation of the boundary geometry.

Recall that the metric used to describe an accelerating black hole in AdS is the AdS C-metric, given by (2.34)

$$ds^2 = \frac{1}{\Omega^2}\left[-f(r)dt^2 + \frac{dr^2}{f(r)} + r^2\left(\frac{d\theta^2}{g(\theta)} + g(\theta)\sin^2\theta\frac{d\phi^2}{K^2}\right)\right], \tag{5.1}$$

where

$$\begin{aligned}\Omega = 1 + Ar\cos\theta, \qquad g(\theta) = 1 + 2mA\cos\theta,\\ f(r) = (1 - A^2r^2)\left(1 - \frac{2m}{r}\right) + \frac{r^2}{\ell^2}.\end{aligned} \tag{5.2}$$

The parameters A and ℓ characterise the acceleration and cosmological constant respectively, m is tied to the mass and K controls the overall conical deficit at both poles (and, in a sense, A controls the disparity of the defect at each pole). We also introduced conditions on these parameters in section 2.3.2, thus we require $2mA < 1$ to preserve the metric signature. The absence of an acceleration horizon yields the constraint $f(-1/A\cos\theta) > 0$, in turn constraining the parameter space (m, ℓ) to the white region bounded by the blue and red lines in figure 5.1. It is straightforward to show via a transformation [110] on the coordinates $(x = \cos\theta, y = -1/Ar)$ that the latter bound is equivalent to the absence of black droplets [114]. Looking at eq. (5.2), we see that such an acceleration horizon is indeed present for large values of A. At $A\ell = 1$, the horizon is located precisely at $r = \infty$, intersecting the boundary at $\theta = \pi/2$. For smaller values of A, the horizon progresses beyond $r = \infty$ into largely negative values of r until A reaches the aforementioned red line, at which point the acceleration horizon, intersecting the boundary at some value $0 < \theta < \pi/2$, vanishes. Finally, the tensions at the poles are given by (2.40)

$$\mu_\pm = \frac{\delta_\pm}{8\pi} = \frac{1}{4}\left(1 - \frac{g(\theta_\pm)}{K}\right) = \frac{1}{4}\left(1 - \frac{1 \pm 2mA}{K}\right), \tag{5.3}$$

where $\theta_+ = 0$ and $\theta_- = \pi$ denote the poles.

An intriguing fact that we came across when studying this metric reveals itself when one examines the geometry obtained by setting $m = 0$. As discussed in [2, 142]

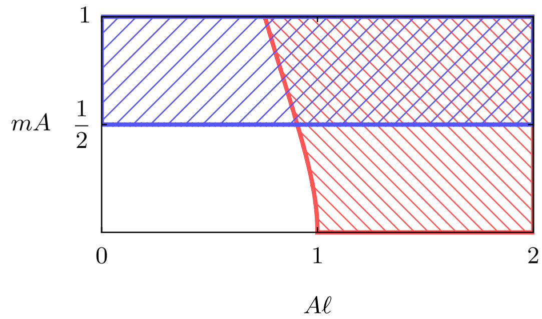


Figure 5.1: Parameter space for the AdS C-metric. The blue and red lines denote the boundaries in the parameter space $(mA, A\ell)$ for which the holographic computation is valid. The hatched red region is where acceleration horizons are present and the hatched blue region is where the metric signature is not preserved, leaving the white region as the physical parameter space.

as well as earlier in chapter 4, setting $m = 0$ removes the black hole horizon, and leaves pure AdS spacetime in Rindler-type coordinates. Performing the coordinate transformation [129]:

$$1 + \frac{R^2}{\ell^2} = \frac{1 + (1 - A^2\ell^2)r^2/\ell^2}{(1 - A^2\ell^2)\Omega^2}\,, \qquad R\sin\vartheta = \frac{r\sin\theta}{\Omega}\,, \tag{5.4}$$

recovers AdS in global coordinates:

$$ds^2_{\mathrm{AdS}} = -\left(1 + \frac{R^2}{\ell^2}\right)\alpha^2 dt^2 + \frac{dR^2}{1 + \frac{R^2}{\ell^2}} + R^2\left(d\vartheta^2 + \sin^2\vartheta\frac{d\phi^2}{K^2}\right), \tag{5.5}$$

however, note that the time coordinate is not the expected AdS time, but is rescaled by a factor of $\alpha = \sqrt{1 - A^2\ell^2}$. Conventionally, we choose the normalisation of our time coordinate so that it corresponds to the "time" of an asymptotic observer. While this is potentially a slightly slippery concept in AdS, taken together with the spherical asymptotic spatial coordinates, this scaling suggests that the correct time coordinate is not in fact t, but rather $\tau = \alpha t$, giving a rescaling of the time-coordinate in (5.1). As we will see, this will inevitably have consequences for the thermodynamics of our spacetime, as the mass and temperature are both sensitive to the definition of the time coordinate. To emphasise the point we have made, let us re-write the metric that we ought to use henceforth, now with rescaled time, as

$$ds^2 = \frac{1}{\Omega^2}\left[-\frac{f(r)}{1 - A^2\ell^2}d\tau^2 + \frac{dr^2}{f(r)} + r^2\left(\frac{d\theta^2}{g(\theta)} + g(\theta)\sin^2\theta\frac{d\phi^2}{K^2}\right)\right], \tag{5.6}$$

where $f(r)$, $g(\theta)$ and Ω retain their earlier definitions.

We now turn to correctly identifying the black hole mass, often the biggest challenge in studying thermodynamics of black holes with nontrivial asymptotics. In what follows, we will provide two independent arguments, beginning with the Ashtekar-Das method [64, 130] applied to the metric (5.6). However, although consistency of thermodynamical relations is a common method of deriving thermodynamics (used for example in [140]), we do not consider this sufficient, hence return to our theme of holography, computing the holographic stress tensor of the boundary theory, thereby confirming our result. As an ancillary argument, we finally check consistency with a computation of the free energy.

5.1.1 The Ashtekar-Das mass

The first argument uses the Ashtekar-Das definition of conformal mass [64, 130], which extracts the mass via conformal regularisation of the AdS C-metric near the boundary. The idea is to perform a conformal transformation on (5.6), $\bar{g}_{\mu\nu} = \bar{\Omega}^2 g_{\mu\nu}$, to remove the divergence near the boundary, which allows us to compute and extract the electric part of the Weyl tensor of the conformal metric

$$\mathcal{E}^{\nu}{}_{\mu} = \frac{\ell^2}{\bar{\Omega}} N^{\alpha} N^{\beta} \bar{C}^{\nu}{}_{\alpha\mu\beta}, \tag{5.7}$$

composed from the Weyl tensor itself, $\bar{C}^{\mu}{}_{\alpha\nu\beta}$, and the normal to the boundary, $N_{\mu} = \partial_{\mu}\bar{\Omega}$. When contracted with a Killing vector, this forms a conserved current, providing us with a novel way of obtaining a conserved charge.

Even though the conformal completion is not unique, the charge thus obtained is independent of the choice of conformal completion. We pick $\bar{\Omega} = \ell\Omega r^{-1}$, which provides a smooth conformal completion in the limit $A = 0$. This allows us to write the conformal metric at the boundary as

$$ds^2 = \ell^2 \left(-\frac{A^2 F(-x)}{1 - A^2\ell^2} d\tau^2 + \frac{dx^2}{A^2\ell^2 F(-x) G(x)} + G(x)\frac{d\phi^2}{K^2} \right), \tag{5.8}$$

where we have written $x = \cos\theta$ and F and G are the Hong-Teo [110] metric functions we introduced in chapter 2, explicitly given by (2.32b)

$$G(x) = \frac{1}{A^2\ell^2} - F(-x) = (1 - x^2)(1 + 2mAx). \tag{5.9}$$

The spacelike surface element tangent to $\bar{\Omega} = 0$ needed to integrate out the current is then obtained by computing the determinant of this metric for a surface at constant τ and multiplying it with a timelike unit normal, yielding

$$d\bar{S}_{\mu} = \delta^{\tau}_{\mu} \frac{\ell^2 dx d\phi}{\alpha K}, \tag{5.10}$$

The nonvanishing components of the normal vector are given by $N_r = -\ell/r^2$ and

$N_x = A\ell$, meaning that the only relevant components of the Weyl tensor are

$$\bar{C}_{r\tau r\tau} = -\frac{2m\ell\bar{\Omega}}{r^4(1-A^2\ell^2)}, \qquad \bar{C}_{x\tau x\tau} = \frac{m\ell f(r)\bar{\Omega}}{(1-A^2\ell^2)r^2G(x)}, \tag{5.11}$$

provided we pick a suitable Killing vector for the mass, $k = \partial_\tau$. The mass is then obtained by performing the appropriately normalised integral at the boundary

$$Q(k) = \frac{\ell}{8\pi}\lim_{\bar{\Omega}\to 0}\oint \mathcal{E}^\nu{}_\mu k_\nu d\bar{S}^\mu, \tag{5.12}$$

finally leading to

$$M = Q(\partial_\tau) = \int_{-1}^{1}\frac{m}{4K\alpha}\Big(2-3A^2\ell^2G(x)\Big)\,dx = \frac{m}{K}\sqrt{1-A^2\ell^2}, \tag{5.13}$$

where we used the fact that $x^2 f(\frac{-1}{Ax}) = F(-x)$.

This is the first result which contradicts both our earlier findings from section 4.3 and results found elsewhere [1, 2, 140]. The absence of acceleration horizons ensures that M vanishes in the limit $A\ell \to 1$ only for $m = 0$ and is positive otherwise.

5.1.2 Holographic derivation of the mass

We now turn to another method for deriving the thermodynamic mass, by computing the holographic stress tensor. This provides an alternate and completely independent method of computation, and will reveal the dual interpretation of this system. The idea here is to perform a Fefferman-Graham expansion of the metric [143], identifying the fall-off of subleading terms in the metric at the boundary. These are then used to compute the dual stress-energy tensor that can be integrated to give the mass of the system.

The action, including boundary counterterms [144–146], is

$$\begin{aligned} I[g] =& \frac{1}{16\pi}\int_{\mathcal{M}} d^4x\sqrt{-g}\left[R+\frac{6}{\ell^2}\right] + \frac{1}{8\pi}\int_{\partial\mathcal{M}} d^3x\sqrt{-h}\mathcal{K} \\ & - \frac{1}{8\pi}\int_{\partial\mathcal{M}} d^3x\sqrt{-h}\left[\frac{2}{\ell}+\frac{\ell}{2}\mathcal{R}\left(h\right)\right], \end{aligned} \tag{5.14}$$

where $\mathcal{K}_{ab} = \nabla_a n_b$, with n the unit normal to the surface $\partial\mathcal{M}$, is the extrinsic curvature of the boundary metric, evaluated asymptotically in an appropriate coordinate system, defined presently, and $\mathcal{K} = \mathcal{K}_{ab}h^{ab}$ is its trace on the boundary. h_{ab} is the intrinsic metric on $\partial\mathcal{M}$, and $\mathcal{R}$ its Ricci curvature. Varying the action gives the energy-momentum tensor:

$$8\pi\mathcal{T}_{ab} = \ell\mathcal{G}_{ab}\left(h\right) - \frac{2}{\ell}h_{ab} - \mathcal{K}_{ab} + h_{ab}\mathcal{K}\,. \tag{5.15}$$

To compute these terms requires new coordinates near the boundary of AdS, typically parametrised by Fefferman-Graham coordinates, in which

$$ds^2 = \frac{\ell^2}{\rho^2}d\rho^2 + \rho^2\left(\gamma^{(0)}_{ab} + \frac{1}{\rho^2}\gamma^{(2)}_{ab} + ...\right)dx^a dx^b, \tag{5.16}$$

placing the boundary now at $\rho = \infty$. Although often one identifies a ρ coordinate globally, due to the complexity of (5.1), we instead perform an asymptotic expansion for the coordinate transformation, writing

$$\frac{1}{Ar} = -\xi - \sum X_n(\xi)\,\rho^{-n}, \qquad \cos\theta = \xi + \sum Y_n(\xi)\,\rho^{-n}, \tag{5.17}$$

determining the functions X_n and Y_n by requiring the metric be of the form in eq. (5.16) up to $\mathcal{O}(\rho^{-3})$; we find this to be achievable by truncating the expansion at $n = 4$. Requiring there be no cross-terms $g_{\rho\xi}$ and then solving order by order, allows us to fix X_1, X_2, X_3 and X_4 in terms of Y_n without having to solve differential equations. With these functions we also find that $g_{\rho\rho} = \ell^2/\rho^2$ is automatically satisfied at leading order and we need only therefore eliminate the two subleading terms. This is most easily achieved from here by fixing Y_2, Y_3 and Y_4, which can be done once again first at next-to-leading order for Y_2, at next order for Y_3 and again similarly at next order for Y_4. With only seven of the eight functions determined, we find that $\gamma^{(1)}_{ab} = 0$ as required and we are left with one functional degree of freedom in Y_1 which we parametrise as

$$Y_1(\xi) = \frac{A^2\ell^3}{\omega(\xi)\alpha}G(\xi)\sqrt{F(-\xi)} \tag{5.18}$$

in order to elucidate the conformal degree of freedom in the boundary metric, ω, and where F and G are as in eq. (5.9). The boundary metric is then given, in these coordinates, by:

$$ds^2_{(0)} = -\frac{\omega^2 d\tau^2}{\ell^2} + \frac{\omega^2\alpha^2 d\xi^2}{A^4\ell^4 F(-\xi)^2 G(\xi)} + \frac{\omega^2\alpha^2 G(\xi)}{A^2\ell^2 F(-\xi)}\frac{d\phi^2}{K^2}, \tag{5.19}$$

which agrees with eq. (5.8) up to an overall factor when $\rho \to \infty$, which $\omega(\xi)$ allows for. Note that the transformation (5.17) is valid in general only when $F(-\xi) > 0$, but this is precisely the constraint that acceleration horizons are absent.

The expectation value of the energy momentum of the CFT_3 can then be calculated, yielding a relativistic fluid with a nontrivial viscous-shear tensor

$$\langle \mathcal{T}_{ab}\rangle = \lim_{\rho\to\infty}\frac{\rho}{\ell}\mathcal{T}_{ab} = \frac{3}{2}\rho_{\mathrm{E}} U_a U_b + \frac{\rho_{\mathrm{E}}}{2}\ell^2\gamma^{(0)}_{ab} + \pi_{ab}\,, \tag{5.20}$$

with $U = \omega^{-1}\partial_\tau$, and boundary indices are raised and lowered with $\ell^2\gamma^{(0)}_{ab}$. The energy density is

$$\rho_{\mathrm{E}} = \frac{mA^3\ell}{8\pi\alpha^3\omega^3}F(-\xi)^{3/2}(2 - 3A^2\ell^2G(\xi))\,, \tag{5.21}$$

yielding the mass

$$M = \int \langle T_{ab}\rangle\, k^a_\tau dS^b = \int \rho_{\mathrm{E}}\ell^3\sqrt{-\gamma^{(0)}}\; dxd\phi = \frac{\alpha m}{K}\,, \tag{5.22}$$

where $k_\tau = \partial_\tau$. This result agrees with eq. (5.13) from the Ashtekar-Das method in the previous section. Note that this calculation is independent of the conformal frame (the choice of ω).

Finally, for completeness, we include the shear tensor is

$$\pi^x_x = \frac{3mA^5\ell^3}{16\pi\alpha^3\omega^3}G(\xi)F(-\xi)^{3/2} = -\pi^\phi_\phi\,, \tag{5.23}$$

with all other components vanishing. The equation of state is that of a thermal gas of massless particles and the dual fluid is anisotropic, as expected from the strongly distorted boundary.

5.2 Thermodynamics

The agreement from two indendent methods for determining the black hole mass hints strongly that our suggested rescaling of time by a factor of $\alpha = \sqrt{1 - A^2\ell^2}$ is indeed correct, and the time coordinate we ought to use when determining thermodynamics is then in fact τ rather than the original coordinate t which we employed in previous chapters. Of most relevance, we expect the temperature to be directly affected, and indeed, computing the temperature associated with the black hole (also the temperature of the boundary field theory) via the Euclidean method, we find:

$$T = \frac{f'(r_+)}{4\pi\alpha} = \frac{1}{2\pi\alpha r_+}\left(\frac{m}{r_+}\left(1 + A^2r_+^2\right) + \frac{r_+^2}{\ell^2} - A^2r_+^2\right), \tag{5.24}$$

which is indeed α-shifted relative to our definition of temperature in chapter 4. It is worth pausing to reflect on this result. In some of our past work [1, 2, 142], which formed the basis of the previous chapter, it appeared to be a natural approach to use the standard time coordinate appearing in the AdS C-metric to derive this temperature, as the blackening factor of the metric was in its canonical form; however, as pointed out in [55], normalising the time and timelike Killing vector is key to obtaining the correct thermodynamics, although the method of obtaining this correct normalisation was less transparent.

Here, having uncovered this suggestive result, we now proceed carefully with considering thermodynamics of the accelerating black hole. As usual, we will take the entropy to be one quarter of the horizon area and the pressure defined as:

$$S = \frac{\mathcal{A}}{4} = \frac{\pi r_+^2}{K(1 - A^2 r_+^2)}, \qquad P = -\frac{3}{8\pi \ell^2}. \tag{5.25}$$

To verify the first law, a simple approach is to recycle a relation we have used previously (4.30), which is obtained by varying the metric function $f(r_+)$, yielding:

$$\delta\left(\frac{m}{K}\right) = \bar{T}\delta S + \bar{V}\delta P - \frac{r_+}{1 \pm A r_+}\delta\mu_\pm + \frac{m}{2K^2}\delta K, \tag{5.26}$$

with $\bar{T}$ and $\bar{V}$ corresponding to our previous definitions of temperature and volume given in eqs. (4.10) and (4.12) as

$$\begin{aligned} \bar{T} &= \frac{1}{2\pi r_+}\left(\frac{m}{r_+}\left(1 + A^2 r_+^2\right) + \frac{r_+^2}{\ell^2} - A^2 r_+^2\right), \\ \bar{V} &= \frac{4\pi r_+^3}{3K(1 - A^2 r_+^2)^2}. \end{aligned} \tag{5.27}$$

The variation of the definition in eq. (5.13) for the mass can be written as

$$\begin{aligned} \delta M &= \alpha\delta\left(\frac{m}{K}\right) + \frac{m}{K}\delta\alpha \\ &= \alpha\delta\left(\frac{m}{K}\right) - \frac{A\ell^2}{\alpha}\frac{m}{K}\delta A - \frac{4\pi}{3}\frac{mA^2\ell^4}{K\alpha}\delta P. \end{aligned} \tag{5.28}$$

The variations of A and K were also given earlier as (4.27):

$$\frac{\delta K}{K^2} = 2(\delta\mu_+ + \delta\mu_-), \qquad \frac{m}{K}\delta A = -A\delta\left(\frac{m}{K}\right) + \delta\mu_- - \delta\mu_+, \tag{5.29}$$

which implies that eq. (5.28) can be re-expressed as

$$\begin{aligned} \delta M &= \frac{1}{\alpha}\delta\left(\frac{m}{K}\right) - \frac{4\pi}{3}\frac{mA^2\ell^4}{K\alpha}\delta P - \frac{A\ell^2}{\alpha}(\delta\mu_- - \delta\mu_+) \\ &= \frac{\bar{T}}{\alpha}\delta S + \left(\frac{\bar{V}}{\alpha} - \frac{4\pi}{3}\frac{mA^2\ell^4}{K\alpha}\right)\delta P - \lambda_\pm\delta\mu_\pm, \end{aligned} \tag{5.30}$$

provided the thermodynamic length is now defined as

$$\lambda_\pm = \frac{1}{\alpha}\left(\frac{r_+}{1 \pm A r_+} - m \mp A\ell^2\right). \tag{5.31}$$

Equation (5.30) constitutes the first law, and we may observe that our definition for the temperature in eq. (5.24) is in agreement. The first law suggests that the

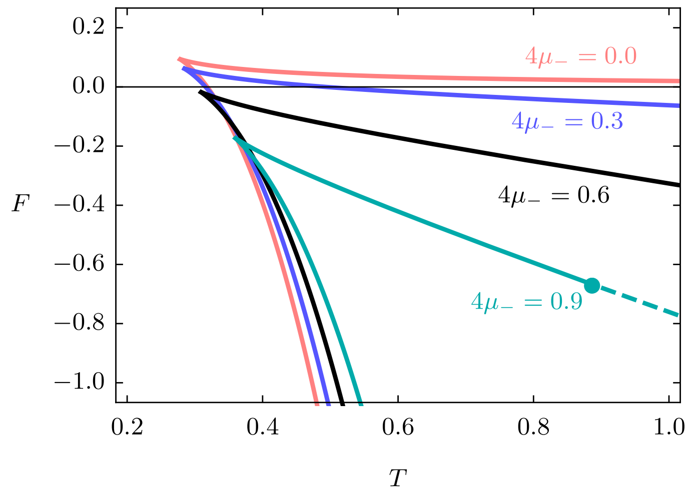


Figure 5.2: *Free energy.* The red curve is the Schwarzschild-AdS case, illustrating the well-known Hawking-Page transition, situated at a temperature given by the intersection of the red curve with $F = 0$. We do not know of any such interpretation for all other curves with $\mu_- \neq 0$. The upper parts of these curves do not continue to arbitrarily large M but terminate at the boundary given in figure 5.1; this is visible in the above plot only for $4\mu_- = 0.9$.

thermodynamic volume is given by

$$V = \frac{\bar{V}}{\alpha} - \frac{4\pi}{3}\frac{mA^2\ell^4}{K\alpha} = \frac{4}{3}\frac{\pi}{K\alpha}\left[\frac{r_+^3}{(1-A^2r_+^2)^2} + mA^2\ell^4\right]. \tag{5.32}$$

This statement is easily verified by the Smarr relation [125], $M = 2TS - 2PV$, which can be shown to hold with the quantities defined as above.

Finally, let us return to the computation of the action (5.33). We find

$$I = \frac{\beta}{2\alpha K}\left(m - 2mA^2\ell^2 - \frac{r_+^3}{\ell^2(1-A^2r_+^2)^2}\right), \tag{5.33}$$

using the time coordinate τ. Some simple algebra then yields the expected result $F = I/\beta = M - TS$ for the free energy, which we plot in figure 5.2.

Although similar in form, the behaviour of the free energy no longer indicates the presence of a standard Hawking-Page transition [50]. As the string tension is fixed for the curves in the plot, no transition to pure radiation (with zero tension) is possible. One may, however, speculate that a transition to a different type of spacetime (for example that of the expanding spherical wave with an attached semi-infinite string of given tension, similar to [147]) may still be possible — such an investigation, however, remains to be carried out.

We can also verify the reverse isoperimetric inequality, discussed in section 1.4, or that the weighted ratio of volume to area $\mathcal{R} = \left(\frac{3V}{\omega_2}\right)^{\frac{1}{3}} \left(\frac{\omega_2}{\mathcal{A}}\right)^{\frac{1}{2}}$ (recall $\omega_2 = 4\pi/K$ here) is greater than unity. Using eq. (5.25) and eq. (5.32), $\mathcal{R}$ may rather unpleasantly be written as

$$\mathcal{R}^6 = 1 + \frac{A^2\ell^6}{4r_+^4\alpha^2(1 - A^2r_+^2)} \left(\frac{4r_+^2}{\ell^2} + A^2\ell^2(1 - A^2r_+^2)^2\right) \left((1 - A^2r_+^2) + \frac{r_+^2}{\ell^2}\right)^2, \tag{5.34}$$

which is greater than unity, provided that $Ar_+ < 1$. This is guaranteed by the geometry; from $f(r_+) = 0$, $Ar_+ < 1$ if and only if $2m > r_+$ which is always the case in the presence of the cosmological constant.

Our full and consistent description of the thermodynamics of an accelerating black hole reconciles discrepancies and conflicts that have appeared in previous investigations of this system [1, 2, 140] which were covered in previous chapters. For example, while a consistent set of thermodynamic variables for charged accelerating black holes was obtained in [1, 2, 142] and in chapter 4, the resultant free energy was not consistent with the action. Alternate expressions for mass and temperature have been posited [140], with the tension of one deficit held fixed to zero. The other tension, while allowed to vary, was not included in the first law, which was derived by assuming integrability of a scaling of mass and temperature. However no physical interpretation was given either for this scaling or for why the energy content of the tension was thermodynamically irrelevant. Furthermore, the vacuum accelerating black hole has an acceleration horizon, akin to a Rindler horizon, and the full structure of the spacetime is that of two accelerating black holes in two Rindler regions. Whether one should be considering a single thermodynamic mass and first law with an additional horizon and black hole, or whether, as suggested in [131], this should be considered as a single system with a mass dipole is an open question.

Note that our computation is independent of the conformal frame, hence we can compare to investigations of holographic C-metrics with an acceleration horizon. For example, by choosing $\omega^2 = A^2\ell^2F(-\xi)\alpha^{-2}$, we recover the form of the boundary metric employed in [114], and our coordinate transformation (5.17) is now valid throughout $\xi \in [-1, 1]$. However, if the condition $F(-\xi) > 0$ is violated, then a black droplet/black funnel is present, and we no longer have an equilibrium temperature for the system in general. The boundary geometry corresponds to a black hole in a spatially compact universe, and so there is no spatial asymptotic region as pointed out in [114]. However, with the full conformal degree of freedom present in our expression, we can easily remedy this shortcoming by, for example, multiplying the ω above by $\frac{1}{\sqrt{1-\xi}}$, giving an $AdS_2 \times S^1$ asymptotic region at $\xi = 1$ with the AdS_2

and S^1 radius being equal. If we multiply by $\frac{1}{\sqrt{1-\xi^2}}$ then there are actually two $AdS_2 \times S^1$ asymptotic regions at $\xi = \pm 1$ and $\gamma_{ab}^{(0)}$ yields the geometry of a wormhole when there are no horizons at the boundary. The $AdS_2 \times S^1$ asymptotic geometry is supersymmetric and to our knowledge has been unnoticed so far in the literature.

Chapter 6

Concluding remarks and outlook

We began in chapter 1 by reviewing some of the remarkable discoveries [5–7, 15, 17, 18, 25, 28, 29, 39] concerning black holes that lead to the inception of a new field of research in black hole thermodynamics. We saw how certain mathematical properties of black holes were indeed very reminiscent of behaviour observed in the study of classical thermodynamics, leading to the establishing of the *four laws of black hole mechanics* [7]. The most important revelation in this process was undoubtedly Hawking's discovery that black holes emit spontaneous radiation and could therefore be considered thermal objects after all [28, 29]. This, combined with Bekenstein's assertion that the area of the event horizon of a black hole directly corresponds to its entropy [15, 17, 18] form the basis of what we now call black hole thermodynamics.

In chapter 2, we introduced the accelerating black hole and its geometrical description via the *C-metric*. Most notably, the C-metric is an interesting geometry as it possesses a conical defect which is interpreted as a cosmic string with positive tension attached to the corresponding pole.

One of our main goals was to adapt the thermodynamic framework to encorporate accelerating black holes. To that end, in chapter 3, we showed how to allow for a varying conical deficit in black hole spacetimes, and found the relevant thermodynamical variables to describe the system. We introduced the thermodynamic length as the conjugate potential to the tension in the first law. This length consists of a direct geometrical part which can be interpreted loosely as the radius of the black hole and a mass-dependent shift. Having identified tension as the correct extensive variable introduced with conical defects, we then move on to acceleration.

In chapter 4, we derive the first law for accelerating black holes, adding to the arsenal of solutions already shown to display thermodynamic characteristics. By considering only slowly accelerating asymptotically AdS solutions, we are able to form an equilibrium in the absence of horizons other than the event horizon. We derive the thermodynamic length for charged accelerating black holes as well as other thermodynamic potentials needed for the first law. Using this description, we explore the thermodynamic phases of accelerating black holes and find that they exhibit similar behaviour to their nonaccelerating AdS cousins, however, it does seem as though the impact of acceleration is to improve the thermodynamic stability of the black holes.

It is interesting to note that the first law indicates that if the tension of a defect is fixed, then there is no contribution to the variation of M coming from tension, yet, if the black hole increases its mass and hence its horizon radius, the horizon will now have consumed a portion of the string along each pole. This does not appear in the thermodynamic relation. This reinforces the interpretation of M as the *enthalpy* of the black hole [70]. Although the black hole increases its internal energy by swallowing some cosmic string, it has also displaced the exact same amount of energy from the environment, resulting in no net overall gain in the total energy of the thermodynamic system (other than the mass that was added to the black hole in the first place).

We also employ this thermodynamic description to explore black hole solutions with maximal deficits. The ultra-spinning black hole of [84, 135, 136] belongs to this class of black holes, as do the “bottle-shaped” black holes mentioned in [114, 120]. It was discovered that ultra-spinning black holes violate the reverse isoperimetric inequality, implying they exceed the maximal amount of entropy as allowed by a bound determined with respect to their thermodynamic volume. We find that ultra-spinning accelerating black holes are not “super-entropic” in this sense, although we were only able to do so by making use of an incomplete thermodynamic description of rotating accelerating black holes, obtained perturbatively in the neighbourhoods of either vanishing acceleration or vanishing rotation.

Finally, in chapter 5, we address some issues that were raised in our original models. We improve upon our thermodynamic definitions and verify the mass of the black hole using a holographic approach. We also found that the dual stress energy tensor for the accelerating black hole corresponds to a relativistic fluid with a nontrivial viscous shear tensor proportional to the acceleration parameter. Given that the acceleration parameter also determines the conical deficit, the source of this anisotropy is clearly due to the impact of the deficit of the fluid. It would be interesting to compare this to the weak coupling calculation of stress tensors in the presence of conical deficits [148].

A caveat of this description is that we have been unable to incorporate charge in any way; in particular, forming a holographic description of a charged accelerating black hole is expected to be a challenge due to the asymptotic structure of the gauge field. In this sense then, a full description of the thermodynamics of charged accelerated black holes is still lacking, however, our hope is that some of the ideas presented here bring us that much closer to its discovery.

6.1 Thermodynamics of the rotating accelerating black hole

During the final stages of the production of this thesis, some of the authors of [3] have succeeded in obtaining a thermodynamic description of the accelerating black hole with rotation [4]. The relevance to the work at hand is clear; we therefore deem it necessary to include these most recent results, appending some final comments. Although the work is not publicly available at the time of writing, the mass of a rotating accelerating black hole, described by the generalised C-metric given in eq. (2.31), was successfully obtained by computing the holographic stress tensor of this geometry following approach as that presented in chapter 5. The resulting mass is given as

$$M = \frac{m}{K}\frac{\sqrt{1-A^2\ell^2}(1+a^2A^2)}{1-\frac{a^2}{\ell^2}(1-A^2\ell^2)}. \tag{6.1}$$

By reworking the derivation presented in section 4.3.3 to account for the rescaling $t = \tau/\alpha = \tau/\sqrt{1-A^2\ell^2}$ of time, which we introduced in chapter 5, we can show how this mass satisfies the first law along with other thermodynamic potentials we provide below. First, an expression for the variation of $m\alpha/K$ can be obtained simply by modifying eq. (4.77) which describes the variation of m/K:

$$\begin{aligned}
\delta\frac{m\alpha}{K} &= \frac{\bar{T}}{\alpha}\delta S + \frac{\bar{\Omega}_0}{\alpha}\delta J + \left(\frac{\bar{V}_0}{\alpha} + \frac{4\pi m A^2\ell^4}{3K\alpha}\right)\delta P - \left(\frac{r_+}{\alpha(1\pm Ar_+)} \mp \frac{A\ell^2}{\alpha}\right)\delta\mu_\pm + \frac{m\delta K}{2\alpha K^2} \\
&= T\delta S + \Omega_0\delta J + V_0\delta P - \left(\frac{r_+}{\alpha(1\pm Ar_+)} \mp \frac{A\ell^2}{\alpha}\right)\delta\mu_\pm + \frac{m\delta K}{2\alpha K^2},
\end{aligned} \tag{6.2}$$

where the quantities

$$\begin{aligned}
S &= \frac{\mathcal{A}}{4} = \frac{\pi(r_+^2+a^2)}{K(1-A^2r_+^2)}, \qquad J = \frac{ma}{K^2}, \qquad P = -\frac{\Lambda}{8\pi} = \frac{3}{8\pi\ell^2}, \\
\bar{T} &= \frac{f'(r_+)}{4\pi} = \frac{1}{2\pi(r_+^2+a^2)}\left(m(1+A^2r_+^2) - \frac{a^2}{r_+} + r_+^3\left(\frac{1}{\ell^2} - A^2\right)\right), \\
\bar{\Omega}_0 &= -\frac{g_{t\phi}}{g_{\phi\phi}}\bigg|_{r=r_+} = \frac{aK}{r_+^2+a^2}, \qquad \bar{V}_0 = \frac{4\pi r_+(r_+^2+a^2)}{3K(1-A^2r_+^2)^2},
\end{aligned} \tag{6.3}$$

are those which we derived in chapter 4 and satisfy

$$\frac{m}{K} = 2\bar{T}S - 2P\bar{V}_0 + 2\bar{\Omega}_0 J. \tag{6.4}$$

In turn, the relation above can also be rewritten as

$$\frac{m\alpha}{K} = 2TS - 2PV_0 + 2\Omega_0 J. \tag{6.5}$$

From here, we may introduce the function $\gamma = \gamma(A, a, \ell)$, and computing the variation of $m\alpha\gamma/K$, both directly and then indirectly using eq. (6.2), as we did in chapter 4, allows us to derive correction terms from γ as follows:

$$\begin{aligned}
V_1 &= \frac{4\pi m}{3K\alpha\Delta_+}\left(\Delta_+(A^2\ell^4\gamma - \alpha^2\ell^3\gamma_\ell) - 2\alpha^2 a^3\gamma_a - a^2 - \Delta_+ A^2\ell^4\right), \\
\Omega_1 &= \frac{\alpha K}{\Delta_+}\left(-\frac{a}{\ell^2} + \Delta_-\gamma_a\right), \\
\lambda_\pm &= \frac{r_+}{\alpha(1 \pm Ar_+)} - \frac{m \mp a^2 A}{\alpha\Delta_+} \mp \frac{A\ell^2\gamma}{\alpha} - \frac{2\alpha(m \mp a^2 A)a\gamma_a}{\Delta_+} \pm \alpha\gamma_A,
\end{aligned} \tag{6.6}$$

where

$$\Delta_\pm = 1 \pm \frac{a^2\alpha^2}{\ell^2}. \tag{6.7}$$

In conjunction with the Smarr relation, we obtain a pair of differential equations, which, together, allow us to specify that $\gamma = \gamma(a^2/\ell^2, a^2A^2)$. Replacing $\gamma = (1 + a^2A^2)/\Delta_-$, as provided by [4], solves the remaining equation, confirming that the first law is now satisfied.

Thus, the first law for a rotating accelerating black hole,

$$\delta M = T\delta S + \Omega\delta J + V\delta P - \lambda_\pm\delta\mu_\pm, \tag{6.8}$$

is satisfied with the following quantities:

$$\begin{aligned}
T &= \frac{1}{2\alpha\pi(r_+^2 + a^2)}\left(m(1 + A^2r_+^2) - \frac{a^2}{r_+} + r_+^3\left(\frac{1}{\ell^2} - A^2\right)\right), \\
\Omega &= \frac{aK}{\alpha(r_+^2 + a^2)} + \frac{a}{\ell^2}\frac{K}{\alpha\Delta_-}, \\
V &= \frac{4\pi}{3K\alpha}\left(\frac{r_+(r_+^2 + a^2)}{(1 - A^2r_+^2)^2} + mA^2\ell^4 + \frac{ma^2\alpha^2}{\Delta_-}\right), \\
\lambda_\pm &= \frac{r_+}{\alpha(1 \pm Ar_+)} - m\frac{\Delta_+}{\alpha\Delta_-^2} \mp \frac{A\ell^2}{\alpha},
\end{aligned} \tag{6.9}$$

along with the previously defined expressions for the entropy S, angular momentum J, pressure P, tension $\mu_\pm$ and finally the mass, given above, M.

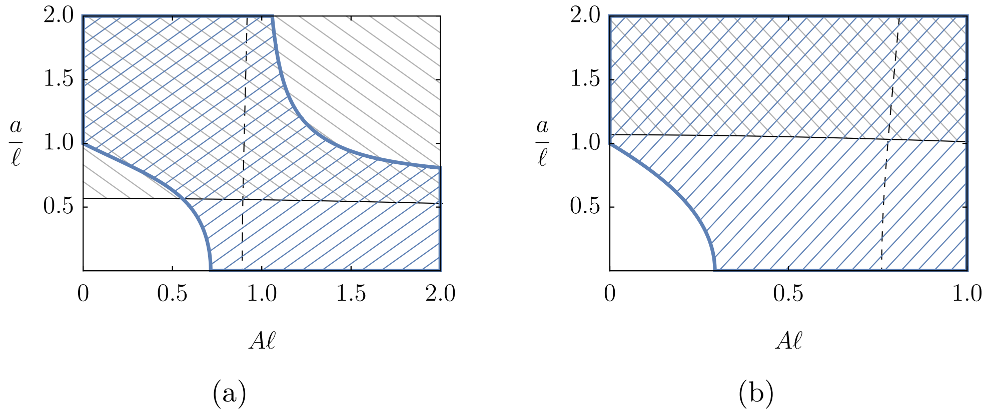


Figure 6.1: Parametric restrictions that apply to the rotating accelerating black hole. The dashed lines correspond to acceleration horizons and the solid black lines to the extremal limit. The blue lines correspond to the critical limit in which one of the poles has a maximal conical deficit. In (a) $m = 0.7\ell$, and in (b) $m = 1.7\ell$.

Equipped with these relations, we may now finally answer the question which led to much of the work presented here. Is the critical rotating accelerating black hole super-entropic? In other words, we must establish whether the critical rotating C-metric violates the reverse isoperimetric inequality. As a reminder, in the critical limit for this geometry (for $mA < 1$ only),

$$\frac{a^2}{\ell^2}(1 - A^2\ell^2) = 1 - 2mA. \tag{6.10}$$

For small mass, the conditions that there be no acceleration horizon and that there be an event horizon shielding the singularity prevent this limit from being reached. For larger mass, the parametric restrictions that apply are displayed in fig. 6.1, and the blue lines directly correspond to the critical limit in each case. The expression given by the reverse isoperimetric ratio is cumbersome, we therefore plot the quantity $\mathcal{R}^6 - 1$, which is positive when the inequality holds, in fig. 6.2. As one may observe, the reverse isoperimetric inequality is satisfied in both cases, provided the parametric restrictions mentioned above are imposed. Our earlier conjecture, based on perturbative arguments, that critical accelerated rotating black holes are not super-entropic holds, and while we have not provided a mathematical proof that this is always the case, fig. 6.2 strongly suggests that this statement is valid in general.

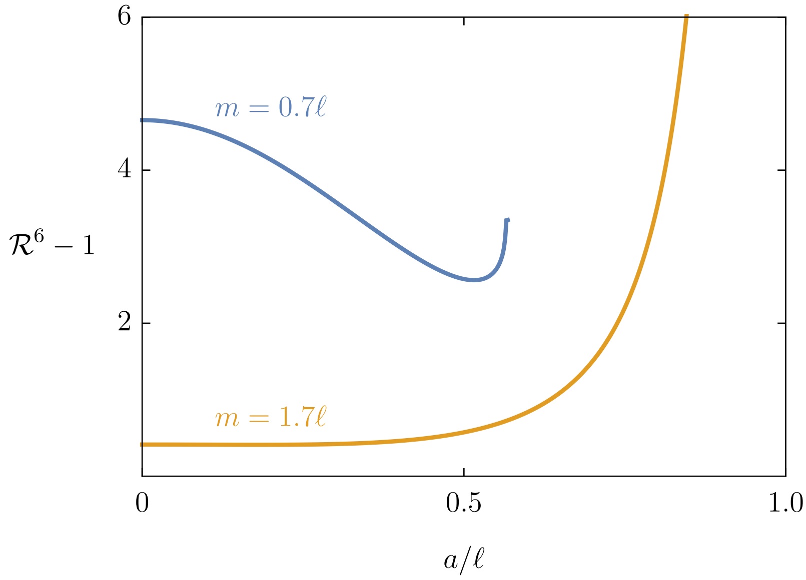


Figure 6.2: The reverse isoperimetric inequality states that the quantity $\mathcal{R}^6 - 1 > 0$ for all black holes. This is a plot of this quantity with respect to the rotation parameter a at different values of the mass parameter m. The represented geometries have parametric restrictions which have been summarised in fig. 6.1.

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