## *Aspects of hydrodynamics in AdS/CMT*

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# Aspects of hydrodynamics in AdS-CMT

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A Thesis presented for the degree of
Doctor of Philosophy


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*Dedicated to*

Dad, Mum and my loving family, particularly granddad Blackie, who passed away during the production of this thesis.

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Condensed matter theory is the study of systems at finite density. In this thesis we will attempt to argue that gauge-gravity dualities can give deep and meaningful insights into the behaviour of strongly coupled condensed matter systems. The first three chapters will be a review of material already available in literature. Chapter 1 will introduce holography and the AdS-CFT correspondence. Particularly, in this chapter, the technique for the extraction of diffusion constants for charge and shear stress-energy-momentum fluctuations in a field theory with a holographic dual will be demonstrated. Chapter 2 will summarise relevant literature on the relativistic fluid-gravity correspondence. In the first half of the chapter it will be shown how to calculate the transport coefficients and Navier-Stokes equations for a suitable thermal field theory. The second half of chapter 2 will then be dedicated to extracting the transport coefficients for a strongly coupled field theory dual to a Reissner-Nordstrøm AdS spacetime. In chapter 3 a scaling of the metric and gauge field found in chapter 2 will be taken such that the boundary field theory admits Galilean, as opposed to relativistic, symmetry. Consequently, the governing hydrodynamic equations will be the non-relativistic, incompressible Navier-Stokes. Chapters 4 and 5 represent novel work. In chapter 4 the transport coefficients for a particular strongly coupled thermal field theory with underlying Schrödinger symmetry will be extracted from a charged, asymptotically Schrödinger spacetime. The governing hydrodynamic

equations will be compressible with non-relativistic symmetry as opposed to those found via the scaling limit of chapter 3. In chapter 5 we show how knowledge of the transport coefficients of a thermal field theory can be used as a test-bed for numerical methods to explore beyond the hydrodynamic (long wavelength and low frequency) regime. With this in mind we consider Reissner-Nordstrøm $AdS_4$ and determine the two point correlators at arbitrary frequency and momentum. Finally in chapter 6 we summarise the work discussed in this thesis and speculate about further applications of hydrodynamic techniques to strongly coupled condensed matter theories.

# Declaration

The work in this thesis is based on research carried out at the Centre for Particle Theory, the Department of Mathematical Sciences, England. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all my own work unless referenced to the contrary in the text. The work is based substantially on the following papers [1–3] where parts have been reproduced with the permission of the respective authors.

# Acknowledgements

I would like to thank my supervisor, Mukund Rangamani, for his guidance over the past three years. I would additionally like to acknowledge Ruth Gregory for her pastoral support and Simon Ross and Veronika Hubeny for their helpful insights into the problems discussed in this thesis. Of special note are the postdocs and PhD students at Durham, including Pau Figueras, Paolo Benincasa, Joan Camps, Ben Withers, Carlos Ballon-Bayona and Simon Gentle whose explanations and discussions often helped clarify difficult concepts. On a more personal note I would like to thank my family for their forbearance during a difficult time in my life. Their support helped me through the more arduous periods during the production of this thesis.

# Contents

# List of Figures

# Chapter 1

# The AdS-CMT programme

Condensed matter physics is the study of large collections of interacting particles which includes solids, liquids, gases and more exotic states of matter such as plasmas. On a theoretical level several outstanding problems continue to puzzle condensed matter theorists among which are the mechanism of high temperature superconductivity and the fundamental description of strange metals. It may be that the resolution to these questions lies in developing an understanding of how strong coupling affects the physics of materials. However systems at strong coupling continue to be one of the least tamed areas of modern physics.

The calculational problems faced at strong coupling are due to the breakdown of perturbation theory. Usually we posit that the current state of our physical system, for example a collection of interacting particles, is not too different from a system without interactions. Perturbation theory then adds small corrections to the observables of these "zero-interaction" systems to calculate the physical observables in the interacting system. In many cases of interest however perturbation theory breaks down as the interactions are strong and thus corrections are expected to be large.

Similar issues with the nature of strong coupling have been encountered in high energy particle physics. The necessity for a method to describe strongly coupled phenomena, such as the quark-gluon plasma (QGP), led many people in this context to consider more exotic toy models in the hopes that generic behaviour could be extracted. Historically however it had even proven difficult to write down a useful

description of the simplest such models at strong coupling. For instance a "simple" theory that will be important to us is planar $\mathcal{N} = 4$ SYM with a large 't Hooft coupling. However, it is for this theory in particular that a fruitful approach originating in string theory, called the AdS-CFT correspondence, was discovered. In this thesis we shall attempt to demonstrate how generalisations to this correspondence can also be applied to condensed matter physics.

To date the most extensive use of this conjecture has been in the study of non-abelian gauge theories since early examples of the AdS-CFT correspondence related $\mathcal{N} = 4$ SYM with many colours (large $N$) and large 't Hooft coupling $\lambda$ to classical gravity on $\mathrm{AdS}_5 \times S^5$. However AdS-CFT is only one precise realisation from string theory of what is believed to be a far larger class of correspondences: the gauge-gravity dualities. These highly speculative dualities are all holographic in nature meaning they relate a theory in $(d + 1)$ dimensions (the "bulk" theory) to one in $d$ dimensions (the "boundary" theory). For most dualities based on the original correspondence, the dictionary states that boundary fall-offs for solutions to classical field equations living in a class of $(d+1)$-dimensional spacetimes correspond to sources and expectation values for some operators living in dual, strongly coupled, conformal field theories[1]. Thus we can translate the hard problem of calculating expectation values in generic strongly coupled field theories into the much easier problem of solving classical field equations in a higher dimensional spacetime. One such example is the now classic set of results on extracting the real-time correlators of some strongly coupled systems from gravity [5, 6].

It is natural to ask how these dualities may be of use in condensed matter physics. By tuning the physics of the bulk theory, the boundary theory can be made to mimic the qualitative features of many condensed matter theories and it is what we can learn from these mimic theories that primarily interests us here. Key examples are the superfliud phase transition [7–9] and a Fermi surface [10–12]. See [13], [14] and [15] for reviews.

[1]Later in this thesis we shall encounter an example where the spacetime is not AdS but instead a Schrödinger spacetime. This is an example of this larger class of gauge-gravity dualities.

The logical process by which we use gauge-gravity dualities to examine condensed matter systems is quite straightforward - we begin by isolating the essential features of the condensed matter system. For example, a low temperature superconductor at the perturbative level consists of a collection of electrons where, at sufficiently low temperatures, electron pairs form bound states under phonon exchange called Cooper pairs [16, 17]. These Cooper pairs being bosonic condense under suitable conditions breaking a global $U(1)$ symmetry. This spontaneous breaking of the $U(1)$ symmetry has also been described in terms of an effective Ginzburg-Landau model [18]. The essential features here are a scalar which gains a non-zero expectation value and breaks a $U(1)$ symmetry. This is the motivation for studying phase transitions in Einstein-Maxwell-scalar theories where, when certain parameters saturate a bound in the system, the scalar becomes tachyonic and condenses [8]. Due to the coupling between the scalar and gauge field this condensation provides the gauge field with a mass and thus breaks the global $U(1)$ in the boundary field theory. Considering such systems allows us to ask questions about superconductivity when the microscopic nature of the theory is strongly coupled.

The kinds of questions that we can answer by conducting these holographic studies include:

- Is it possible to have a particular condensed matter phenomena, such as superconductivity, when the fundamental theory governing the system is strongly coupled? As an example, the answer for a superconductor turns out to be yes as has been shown in [7–9].

- Are there features shared by large classes of systems at strong coupling? The answer is yes, one example being the shear viscosity to entropy ratio. The large size of the class of theories which share this bound gives us hope that we might be able to observe such features in real world physics.

It is the study of these questions that constitutes the current AdS-CMT programme with a hope to constructing more and more realistic models of actual condensed matter systems. However, it should be noted that at present we do not have sufficient control over the AdS-CMT correspondence to construct gravitational duals to an

arbitrary condensed matter theory and thus, for now, we are stuck with somewhat loose models. Indeed, the complexity of the holographic dual necessary to provide an exact description of a particular condensed matter system will almost certainly be too complicated to be of practical use.

One useful technique yielding interesting results when applied to condensed matter systems is hydrodynamics. Hydrodynamics is the framework describing how perturbed, interacting, thermal systems approach global equilibrium at long times. Notably this area has been the subject of intense study and, while there have been significant achievements such as those of Navier, Stokes and Kolmogorov, there remain many phenomena, like turbulence, that still lack a full theoretical description. Importantly, an ability to completely describe the generic behaviour of fluids would lead to a deeper understanding of a vast range of models. This is because a typical feature of interacting field theories with a long-wavelength expansion is a sector well described by the hydrodynamic regime.

For a fluid description whose microscopic origin is strongly coupled we cannot apply the usual perturbative methods to calculate transport coefficients. As such, these fluids represent an interesting theoretical challenge. Moreover, beyond purely theoretical considerations, there are several practical applications where controllable models of such fluids would be useful. One example concerns the dynamics of the QGP whose transport coefficients can be determined indirectly from experimental data [19–21]. The interest in applying gauge-gravity dualities to this system comes from the fact that the correspondence describes a large class of ideal fluids and the QGP seems to be approximately ideal[2]

This thesis is principally concerned with the fluid-gravity sector of the gauge-gravity dualities where it was discovered that the restriction of certain large $N$ gauge theories to a long-wavelength regime is dual to a simplified gravitational description [29]. This programme of studying duals to the fluid description of strongly coupled conformal field theories began with the seminal works of [30,31] where calculations of

[2]A few key papers in the application of AdS-CFT to QCD include [22–27]. Using the general lessons learned from the application of AdS-CFT has already led to a qualitative improvement in understanding the properties of this state of matter like its low viscosity to entropy ratio [28].

the graviton retarded Green's functions were made at the linearised level of gravity. A significant achievement in [32] was a procedure to extend the previous calculations to the full non-linear equations. Again we remind ourselves that the results obtained by these methods can give only qualitative predictions about nature because, as yet, no observed phenomenon is known to have as an underlying description a large $N$ gauge field. However, the fluid-gravity correspondence is a rich area still producing new and significant results and in the conclusion to this thesis we shall put the work presented here into the context of more recent work.

In what follows we shall begin by giving a brief introduction to the generalities of AdS-CFT. This will comprise the next section. In particular we shall indicate how to find expectation values for the boundary operators that will be of interest to us - the stress-energy-momentum (SEM) tensor and charge currents. This discussion will be of necessity cursory and where necessary the interested reader will be referred to literature. However, in the subsequent sections we shall perform in detail two calculations to make explicit how the AdS-CFT dictionary is used in practice. Namely we shall calculate the boundary diffusion constants corresponding to turning on a probe gauge field and linearised shear gravitational perturbations in a Schwarzschild-AdS bulk dual. This will additionally allow us to introduce valuable notation which we shall make use of throughout the thesis. Finally we end the chapter by demonstrating how one may go about showing universality of the quantities we may be interested in calculating. In particular we shall argue that the shear viscosity to entropy ratio of all field theories dual to spatially isotropic, two derivative gravity models is fixed.

In the second chapter we shall discuss charged relativistic fluids being clear about assumptions such as spatial isotropy, parity invariance and scale invariance. This chapter is split into two pieces. In the first and largest part we will detail the nature and process of determining effective, relativistic hydrodynamic descriptions - as until recently this area lacked a standardised description. We shall write down the effective theory governing charged fluids in the absence of background electric and magnetic fields to first order in a derivative expansion. Additionally we shall comment upon the extension to higher orders in derivatives and theories with non-

trivial boundary field strengths but not calculate them as they will not be relevant to the rest of the thesis. The second part of this chapter describes in short the same process as seen from the dual gravitational theory, in particular, how to write down the dual description of the hydrodynamic modes of a strongly coupled field theory. As there are many excellent papers available, which we shall reference, we shall restrict ourselves to a single example skipping details that are explored elsewhere. Where relevant we will attempt to provide a comparison to the linearised analysis in this introduction.

In the next two chapters we discuss incompressible and compressible non-relativistic fluids. These will each have different underlying symmetries; a discussion of which can be found in appendix A. In both cases we shall calculate holographic duals to certain boundary theories with these hydrodynamic modes. The discussion of charged compressible non-relativistic fluids is based on [1] and demonstrates not only that gauge field anomalies may be important for non-relativistic fluids but that a conjecture about the Prandtl number given in [33] is false in the presence of charge.

Having started by discussing a linearised analysis in the introduction we shall come full circle and discuss the linearised perturbation analysis of shear modes around a Reissner-Nordstrøm $AdS_4$ black hole. This will prove slightly more interesting than the linearised examples discussed in the introduction as the presence of a background charge in the gravity theory leads to the coupling of gauge and gravitational modes. We shall show how the analytic hydrodynamic analysis we have formulated in previous chapters can provide a solid basis from which we can explore more exotic areas of the AdS-CFT correspondence outside the long wavelength, low frequency regime.

## 1.1 Holography

Gravity is an inescapable feature of nature. We are often saved from having to consider it as part of our fundamental theories because the local curvature scale in our day to day lives is much larger than the Planck length $\ell_{\mathrm{P}}$. However, when we

consider very high energies it becomes necessary to understand the quantum nature of gravity. A fundamental feature of a quantum theory of gravity appears to be holography - the first example of which is the Bekenstein-Hawking result that the entropy associated with a black hole is proportional to its area in Planck units. This is one of several results that guides us into thinking that gravity in $d+1$ dimensions must be in a correspondence with a field theory in fewer dimensions. In this section we shall outline the basics of the AdS-CFT dictionary as a precise realisation of holography but to keep readability often refer the reader to extensive reviews in literature [13, 29, 34–43].

The first and most natural question to ask is: if holography is a fundamental aspect of nature, in which field theories are the gravitational degrees of freedom most obvious? It is clear at least that gravitational degrees of freedom have not been seen in the weak coupling regimes of quantum field theories. This leads us to expect that if gauge-gravity duality holds fundamentally the only place where it can be obvious must be at strong coupling.

The most precise collection of holographic correspondences come from compactifying Type IIB string theory on $\mathrm{AdS}_5 \times M^5$ where $M^5$ is an Einstein-Sasaki manifold. Such compactifications preserve varying amounts of supersymmetry from the original string theory leading us to believe our gravitational setup is stable. Additionally each compactification leads to different conserved charges in the dual field theory. The approach of considering duals given by string compactifications is called the "top-down" approach and suggests a relationship between partition functions of the form

$$\left\langle \exp\left(-i\int d^d x \phi_{(0)}\hat{O}\right)\right\rangle_{QFT} = Z_{\mathrm{string}}[\phi_{(0)}] \tag{1.1.1}$$

where $Z_{\mathrm{string}}[\phi_{(0)}]$ is the Type IIB string action with the "bulk" fields taking boundary values $\phi_{(0)}$ and $\left\langle \exp\left(-i\int d^d x \phi_{(0)}\hat{O}\right)\right\rangle_{QFT}$ is the generating functional for correlators of $\hat{O}$ in the dual quantum field theory.

The numerous examples of "top-down" holographic duals guides us into conjecturing that any asymptotically, locally AdS space is dual to some strongly coupled field theory. This justifies the more conjectural "bottom-up" approach where we look for some asymptotically, locally AdS spacetime with bulk fields that yield the

desired matter content in the dual field theory. Thus many of the most interesting features of the duality are contained in the nature of the asymptotic geometry of AdS.

We shall now lay out some of the generic features of the dictionaries. Consider an asymptotically, locally AdS spacetime $\mathcal{M}$ with a boundary $\mathcal{B}$ and a metric $g$. The asymptotic form of the metric near the boundary of AdS can be written locally as[3]

$$\begin{aligned} g &= g_{MN} dx^M \otimes dx^N \\ &\overset{z\to 0}{\sim} \ell^2 \left( \frac{dz^2 + \hat{h}_{\mu\nu}(z, x^\mu) dx^\mu dx^\nu}{z^2} \right) \end{aligned} \tag{1.1.2}$$

where $\mathcal{B}$ is given by $z = 0$. As displayed above we use capital Latin letters for bulk indices while Greek letters refer to boundary coordinate indices. This metric can extended across the boundary to a metric $\bar{g}$ on the closed manifold $\bar{\mathcal{M}} = \mathcal{M} \cup \mathcal{B}$ via a defining function $f$ such that $\bar{g} = f^2 g$. The defining function has a simple zero at $z = 0$ and is strictly positive on $\mathcal{M}$ but is otherwise arbitrary. We call the pullback of the metric

$$\gamma = \lim_{\epsilon \to 0} f^2 \, g|_{z=\epsilon} \tag{1.1.3}$$

onto the boundary the "boundary metric". As any defining function $f$ with the specified properties will do we see that $f$ actually specifies an element of a class of boundary metrics [44]. It can be shown that bulk diffeomorphisms induce diffeomorphisms of the boundary and, in pure AdS space, the symmetry algebra generated by the bulk diffeomorphisms furnishes a representation of the conformal group at the boundary.

Similarly for any bulk matter field on $\mathcal{M}$, which we shall denote as $\Phi$, we can define its extension to the closed manifold. If we consider the linear response of the boundary theory to perturbations of the bulk fields we can see that, for probe matter fields governed by second order differential equations, there are two unique

[3]For this section only it is convenient to work with the coordinate $z$. In subsequent sections we shall make the replacement $z = \frac{\ell^2}{r}$ where $\ell$ is the AdS length scale.

fall-offs in $z$ as $z$ tends to zero

$$z^{\Delta_-}, z^{\Delta_+} \ . \tag{1.1.4}$$

Our notation is such that $\Delta_- \leq \Delta_+$ and the bulk field on the manifold $\bar{\mathcal{M}}$ is given by

$$\bar{\Phi} = f^{-\Delta_-}\Phi \ . \tag{1.1.5}$$

When the bulk conserved charges like the energy

$$E = -\int_{\Sigma_t} d^d x \sqrt{g|_{\Sigma_t}} u^M T^{\text{matter}}_{MN} \xi^N \tag{1.1.6}$$

are finite, where $\Sigma_t$ is a spacelike surface, $u^M$ is the velocity of a time-like observer and $\xi^N$ is a time-like Killing vector field, the solution $\Phi$ is called normalisable [34]. When they are not it is called non-normalisable and the leading coefficient, denoted $\phi_{(0)}$, is interpreted as a source in the boundary field theory

$$\phi_{(0)} = \lim_{z\to 0} f^{-\Delta_-}\Phi. \tag{1.1.7}$$

We see that $\phi_{(0)}$ is a scalar density of weight $\Delta_-$. We interpret this in the dual field theory as a deformation of the field theory Lagrangian

$$\mathcal{L} \to \mathcal{L} + \int d^d x \sqrt{-\gamma} \phi_{(0)} \hat{O} \tag{1.1.8}$$

where $\langle O \rangle$ can be read off as the leading coefficient of $z^{\Delta_+}$ in the bulk field $\Phi$. $\hat{O}$ is interpreted as an operator in the field theory. In this way we see that the deformation is irrelevant if $\Delta_+ > d$, marginal if $\Delta_+ = d$ and relevant if $\Delta_+ < d$.

Of particular importance in what follows are the boundary stress-energy-momentum (SEM) tensor ($T^{\mu\nu}$) and charge current ($J^\mu_I$ where $I$ labels a collection of Abelian charges). The boundary SEM tensor can be extracted in two ways - firstly we can transform our metric into Fefferman-Graham form and read off the subleading piece. Here we prefer a second method where we calculate the bulk Brown-York tensor ($T_{\text{BY}}$) on some hypersurface of constant $z = \epsilon$. We then take $\epsilon \to 0$ and read off the value of the Brown-York SEM tensor at the boundary. The asymptotic Brown-York SEM tensor will be naively divergent at the boundary so we

will need to subtract appropriate counterterms (a procedure for determining these counterterms can be found in [45, 46]). Either way we note that we can write

$$\begin{aligned}(T_{\rm BY})_{MN} &= K_{MN} - \Pi_{MN}K \ , \\ F_{MN} &= \frac{1}{2}\left(J_M n_N - J_N n_M\right) + \tilde{F}_{MN}\end{aligned}$$

where $n_M$ is the unit one-form annihilating vectors lying in constant $z$ surfaces, $\Pi_{MN} = g_{MN} - n_M n_N$ is the radial projector and $K_{MN} = \Pi_{(M|}{}^P \nabla_P n_{|N)}$ and $K$ are the extrinsic curvature of a constant $z$ slice and its trace respectively. Without loss of generality it is natural to choose our regulating function to be $f = \frac{z}{\ell}$ [44] which implicitly picks a conformal frame in the boundary. In terms of the bulk metric and bulk field strength the boundary metric and boundary field strength are then

$$\gamma_{\mu\nu} = \lim_{z\to 0}\left(\frac{z}{\ell}\right)^2 \hat{h}_{\mu\nu} \ , \tag{1.1.9}$$

$$\tilde{F}_{\mu\nu} = \lim_{z\to 0} F_{\mu\nu} \ , \tag{1.1.10}$$

while the boundary currents are

$$T_{\mu\nu} = -\frac{1}{\kappa_{d+1}^2}\lim_{z\to 0}\left(\frac{z}{\ell}\right)^{-(d-2)}\left(K_{\mu\nu} - \Pi_{\mu\nu}K - \text{counterterms}\right) \ , \tag{1.1.11}$$

$$J^\mu = -\frac{g_F^2}{\kappa_{d+1}^2}\lim_{z\to 0}\left(\frac{z}{\ell}\right)^{-d} n^M F_M{}^\mu \ , \tag{1.1.12}$$

where $g_F$ is the gauge coupling, $\kappa_{d+1} = \sqrt{8\pi G_{d+1}}$ and we have identified the overall constants by varying the on-shell actions with respect to the source[4]. Strictly the currents on the left hand side of Eqs. (1.1.11) and (1.1.12) should be denoted $\left\langle \hat{T}_{\mu\nu} \right\rangle$ and $\left\langle \hat{J}^\mu \right\rangle$ where $\hat{T}_{\mu\nu}$ and $\hat{J}^\mu$ are the SEM tensor and current operators in the boundary field theory. This notation is cumbersome however and we instead use the above notation. It should be noted that the number and form of the counterterms are dimension dependent [45, 46].

As we stated at the beginning of this section a black hole spacetime has particular thermodynamic properties. Importantly non-extremal black holes have a temperature. The next obvious question to ask is - what is the effect of placing a

[4]For further details on the normalisation of our bulk fields see Eq. (2.2.61).

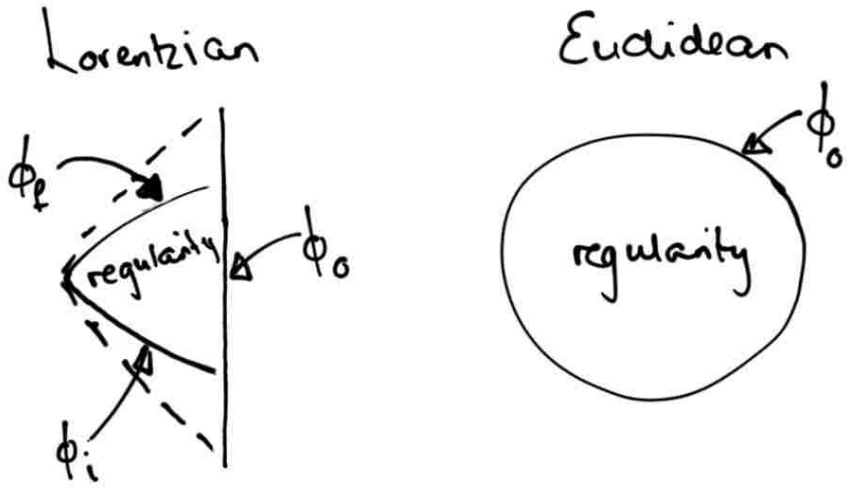


Figure 1.1: An illustration of fixing boundary conditions in Euclidean (right) and Lorentzian (left) AdS-CFT respectively.

black hole in an asymptotically, locally AdS spacetime according to the dual boundary field theory? It is clear that, if there is a one-to-one correspondence between the bulk gravitational theory and the boundary field theory, when the bulk theory has periodicity in imaginary time this should be shared by the boundary field theory[5]. Hence we interpret placing a black hole in the spacetime as heating up the boundary field theory.

Finally, we come to the issue of boundary conditions at asymptotic times which are important in Lorentzian theories for defining correlators. In the above we have often implicitly worked in the Euclideanised version of the bulk theory. In such Euclideanised theories it is generally possible to prove that the asymptotic value of

[5]Analytically continuing a field theory to imaginary time $\tau$ a thermal correlator in the canonical ensemble is defined in terms of the corresponding density matrix, $\exp\left(-\beta\hat{H}\right)$, by

$$\left\langle \hat{O}(\tau,\mathbf{x})\hat{O}(0,\mathbf{0})\right\rangle_{\beta} = \frac{\mathrm{Tr}\left(\exp\left(-\beta\hat{H}\right)\hat{O}(\tau,\mathbf{x})\hat{O}(0,\mathbf{0})\right)}{\mathrm{Tr}\left(\exp\left(-\beta\hat{H}\right)\right)} \tag{1.1.13}$$

where the left hand side is the expectation value of the two point function in the canonical ensemble, $\beta$ the inverse temperature, $\hat{H}$ the Hamiltonian and we have assumed Euclidean time ordering. Using the time evolution operator and applying cyclicity of trace indicates that any field operators are to identified when their time arguments differ by multiples of $\beta$.

the field and regularity in the interior is enough to uniquely specify the classical solution to the bulk field equations. This ensures that there is only one correlator of interest with Euclidean time ordering. Analytic continuation of the Euclidean correlator gives a Feynman correlator. However, in Lorentzian space, there is more than one interesting correlator. This ambiguity is tied to the fact that there exist many normalisable solutions which are regular in Lorentzian AdS and these can always be added arbitrarily to any solution in the bulk without affecting our boundary conditions. A way to get at the other correlators is to specify boundary conditions on initial and final time slices in the bulk (see Fig. (1.1) and [47]). The procedure of [47] requires us to attach pieces of Euclidean AdS on these initial and final slices, which corresponds to the Keldysh time-contour-ordering formalism in the boundary, where the renormalisation terms are known. In [48] it was proven that the commonly used technique for calculating retarded correlators by demanding infalling conditions on a black hole horizon [5] is justified by this "piecewise AdS-CFT" procedure.

This introduction to AdS-CFT has been very brief. There are many excellent reviews available [13, 35–43, 49]. We shall introduce wherever necessary in the rest of this thesis additional pieces of information available in these reviews however, in the next section, we shall show how the AdS-CFT correspondence can be used to extract boundary physics from linearised fluctuations of gauge fields and gravity with an eye to relating to later results.

## 1.2 Linearised perturbations

Consider the following Schwarzschild-AdS spacetime

$$\begin{aligned} ds^2 &= \frac{r^2}{\ell^2}\left(-f(r)dt^2 + d\mathbf{x}_{d-1}^2\right) + \frac{\ell^2 dr^2}{r^2} \\ f(r) &= 1 - \left(\frac{r_+}{r}\right)^d \\ F &= 0 \end{aligned} \tag{1.2.14}$$

where $F$ is the field strength of a $U(1)$ gauge field and our boundary metric at asymptotically large $r$ is the Minkowski metric[6]. The position of the black hole

[6]Note the change of radial coordinate from $z$ to $r$ mentioned previously.

horizon is at $r = r_+$.

In and of themselves the linearised perturbations of space-times are interesting as studying the spectrum of quasinormal modes can indicate whether an instability exists for the spacetime to evolve to a different configuration. In the context of AdS-CFT this has a dual meaning where such a change in the bulk theory corresponds to a phase transition of the boundary field theory. In particular, the Schwarzschild-AdS spacetimes are interesting for us because they describe a thermal boundary field theory and we can use them to compare and contrast with the charged spacetimes in later chapters.

In this section we shall show how to use the AdS-CFT correspondence to work out the boundary source and expectation values for some operators corresponding to gauge and gravitational perturbations of Eq. (1.2.14). Again, this will prove useful later for comparison purposes as well as giving a feel for how to use the correspondence. In chapter 5 we shall return to these calculations in the context of Reissner-Nordstrøm $AdS_4$ where the gauge and gravitational modes become coupled.

It is necessary to lay out some formalism with which we shall be consistent throughout the thesis. We make the following convention choice for our Fourier transforms

$$\begin{aligned} \phi\left(r, x^{\mu}\right) &= \int \frac{d^{d}k}{(2\pi)^{d}} \phi\left(r, \mathbf{k}_{\mu}\right) \exp\left(i k_{\mu} x^{\mu}\right) \\ k_{\mu} &= \left(-\omega, \mathbf{k}\right) \ . \end{aligned}$$

After performing the transformation we remind ourselves that generically a bulk field $\Phi$ at large $r$ looks like

$$\Phi\left(r; \omega, \mathbf{k}\right) = \frac{\phi_0\left(\omega, \mathbf{k}\right)}{r^{\Delta_+}} + \ldots + \frac{\left\langle \hat{O}\left(\omega, \mathbf{k}\right)\right\rangle}{r^{\Delta_-}} + \ldots \ .$$

A priori there is no relationship between the source and expectation value and we need to specify some boundary condition in the bulk to relate them. For retarded Green's functions in linear response where the amplitude of $\phi_0$ is assumed to be sufficiently small we solve the bulk equations of motion with infalling conditions [48, 50] on the future horizon and extract the retarded Green's function from

$$G_R\left(\omega, \mathbf{k}\right) = \frac{\left\langle \hat{O}\left[\phi_0\left(\omega, \mathbf{k}\right)\right]\right\rangle}{\phi_0\left(\omega, \mathbf{k}\right)} \ .$$

Continuing $G_R$ to complex $\omega$ we remind the reader that it is solutions to the unforced equations which govern how the system returns to equilibrium. This corresponds to finding $\omega(\mathbf{k})$ such that $\phi_0(\omega(\mathbf{k}), \mathbf{k}) \equiv 0$. If $\omega \in \mathbb{R}$ then the bulk solution is called a normal mode. However, as is more generally the case, if $\omega \in \mathbb{C}$ then the solution is called a quasinormal mode. For $\omega(\mathbf{k})$ a quasinormal mode if $\Im\omega(\mathbf{k}) > 0$ in our sign convention then there exists an exponentially growing mode in the spectrum of perturbations and thus the bulk is linearly unstable. This is indicative of the aforementioned phase transition.

### 1.2.1 Probe gauge fields in Schwarzschild-AdS

The bulk action we wish to consider is

$$\mathcal{S} = -\frac{g_F^2}{4\kappa_{d+1}^2}\int d^{d+1}x\sqrt{-g}F_{MN}F^{MN} \tag{1.2.15}$$

where we have introduced a normalisation $g_F^2$ to our action to better compare with results obtained from Eq. (2.2.61) in chapter 2. We shall perform a small amplitude perturbation of the boundary by turning on a source for the bulk gauge field. Shift $F_{MN} \to F_{MN} + \epsilon f_{MN}$ where $F_{MN} \equiv 0$ in our background (Eq. (1.2.14)). The resultant action which is second order in $\epsilon$ is

$$\epsilon^2\mathcal{S}^{(2)} = -\epsilon^2\frac{g_F^2}{4\kappa_{d+1}^2}\int d^{d+1}x\sqrt{-g}f_{MN}f^{MN}\ . \tag{1.2.16}$$

Note that the piece which is first order in $\epsilon$ is simply the variation of the background gauge field strength $F_{MN}$. Demanding that the background field equations are solved minimizes this term and, because of our choice of background, sets $F_{MN} \equiv 0$. The variation of $\mathcal{S}^{(2)}$ gives:

$$\begin{aligned}\delta\mathcal{S}^{(2)} &= -\frac{g_F^2}{\kappa_{d+1}^2}\int_{\partial\mathcal{M}} d^dx\left(\sqrt{-g}f^{MN}n_M\right)\delta a_N \\ &\quad +\frac{g_F^2}{\kappa_{d+1}^2}\int_{\mathcal{M}} d^{d+1}x\sqrt{-g}\left[\frac{1}{\sqrt{-g}}\partial_M\left(\sqrt{-g}f^{MN}\right)\right]\delta a_N\ .\end{aligned}$$

Modulo the subtleties of real time AdS-CFT already mentioned - to make this variation zero on-shell we shall specify that the gauge field perturbation be infalling on the future black hole horizon and have the value $a_\mu(t,x)$ at the boundary.

We shall partially fix the gauge freedom of the field and Fourier transform giving

$$a_\mu(r;x^\mu) = \int \frac{d^dk}{(2\pi)^d} a_\mu(r;k_\mu)\exp(ik_\mu x^\mu) \ , \tag{1.2.17}$$

$$a_r(r;x^\mu) = 0 \tag{1.2.18}$$

with $k_\mu = (-\omega, \mathbf{k})$. Maintaining this gauge choice will leave the residual gauge transformations

$$a_\mu(r;x^\mu) \to a_\mu(r;x^\mu) + \partial_\mu\lambda(x^\mu)$$

or in Fourier space

$$a_\mu(r;x^\mu) \to a_\mu(r;k_\mu) + k_\mu\lambda(k_\mu)$$

where we have decomposed the gauge transformation into Fourier components too. The resultant equations of motion are

$$\begin{aligned}
\partial_r^2 a_t(r;k_\mu) &+ \left(\frac{d-1}{r}\right)\partial_r a_t(r;k_\mu) + \frac{\ell^4}{r^4 f(r)}\left[\omega k^i a_i(r;k_\mu) + \mathbf{k}^2 a_t(r;k_\mu)\right] = 0 \ , \\
\partial_r^2 a_i(r;k_\mu) &+ \left[\left(\frac{d-1}{r}\right) + \frac{f'(r)}{f(r)}\right]\partial_r a_i(r;k_\mu) - \frac{\ell^4}{r^4 f(r)}\left[\mathbf{k}^2 a_i(r;k_\mu) - k_i k^j a_j(r;k_\mu)\right] \\
&+ \frac{\ell^4}{r^4 f(r)^2}\left[\omega^2 a_i(r;k_\mu) + \omega k_i a_t(r;k_\mu)\right] = 0
\end{aligned}$$

and

$$0 = \omega\partial_r a_t(r;k_\mu) + k^i f(r)\partial_r a_i(r;k_\mu)$$

which comes from variation with respect to $a_r$.

Now assume that the fields depend only on $r$, $t$ and $k^x = k$. It is clear this is a consistent ansatz for our equations which reduce to:

$$\begin{aligned}
0 &= \partial_r^2 a_t(r;\omega,k) + \left(\frac{d-1}{r}\right)\partial_r a_t(r;\omega,k) \\
&\quad + \frac{\ell^4}{r^4 f(r)}\left[\omega k a_x(r;\omega,k) + k^2 a_t(r;\omega,k)\right] \ , \\
0 &= \partial_r^2 a_x(r;\omega,k) + \left[\left(\frac{d-1}{r}\right) + \frac{f'(r)}{f(r)}\right]\partial_r a_x(r;\omega,k) \\
&\quad + \frac{\ell^4}{r^4 f(r)^2}\left[\omega^2 a_x(r;\omega,k) + \omega k a_t(r;\omega,k)\right] \ .
\end{aligned}$$

The above equations will produce interesting long lived modes when we consider

long wavelengths and low frequencies. Additionally we must consider

$$\begin{aligned} 0 &= \omega\partial_r a_t\left(r;\omega,k\right) + kf(r)\partial_r a_x\left(r;\omega,k\right) \\ 0 &= \partial_r^2 a_i\left(r;\omega,k\right) + \left[\left(\frac{d-1}{r}\right) + \frac{f'(r)}{f(r)}\right]\partial_r a_i\left(r;\omega,k\right) \\ &\quad + \frac{\ell^4}{r^4 f(r)^2}\left[\omega^2 - f(r)k^2\right] a_i\left(r;\omega,k\right) \end{aligned}$$

to check overall consistency. The first of these latter two equations is actually a gauge constraint coming from demanding that $\mathcal{S}^{(2)}$ be gauge invariant under residual gauge transformations.

Now consider the field $E$ given by the gauge invariant object

$$E\left(r;\omega,k\right) = \omega a_x\left(r;\omega,k\right) + k a_t\left(r;\omega,k\right) \ .$$

Writing the equations of motion for our field components in terms of $E$ the resultant equations of motion are

$$\begin{aligned} 0 &= \partial_r^2 E\left(r;\omega,k\right) + \left[\left(\frac{d-1}{r}\right) + \frac{f'(r)}{f(r)} + \frac{k^2 f'(r)}{(\omega^2 - k^2 f(r))}\right]\partial_r E\left(r;\omega,k\right) \\ &\quad + \frac{\ell^4}{r^4 f(r)^2}\left(\omega^2 - k^2 f(r)\right) E\left(r;\omega,k\right) \ , \qquad (1.2.19) \\ 0 &= \partial_r^2 a_i\left(r;\omega,k\right) + \left[\left(\frac{d-1}{r}\right) + \frac{f'(r)}{f(r)}\right]\partial_r a_i\left(r;\omega,k\right) \\ &\quad + \frac{\ell^4}{r^4 f(r)^2}\left[\omega^2 - f(r)k^2\right] a_i\left(r;\omega,k\right) \qquad (1.2.20) \end{aligned}$$

and the gauge constraint

$$0 = \omega\partial_r a_t\left(r;\omega,k\right) + kf(r)\partial_r a_x\left(r;\omega,k\right) \ . \qquad (1.2.21)$$

Eqs. (1.2.19), (1.2.20) and (1.2.21) are still difficult to solve analytically for arbitrary $k$ and $\omega$. There exists a tractable regime however when $\omega$ and $k$ are small (long wavelengths and low frequencies) which was first examined in detail in [30,31]. This, as we shall see more explicitly in chapter 2, is exactly where we expect to see hydrodynamic behaviour in the field theory. Notice that the only difference between our equations of motion for $E$ and $a_i$ is due to a modification of the damping term in the equation of motion for $E$. We shall now attempt to solve the equations of motion in this "hydrodynamic" regime.

### Small $\omega$

In the limit of small $\omega$ we find that Eq. (1.2.20) becomes

$$\begin{aligned} 0 &= \partial_r^2 a_i(r;\omega,k) + \left[\left(\frac{d-1}{r}\right) + \frac{f'(r)}{f(r)}\right]\partial_r a_i(r;\omega,k) \\ \Rightarrow a_i(r;\omega,k) &= a_i^{(0)} + (d-2)\langle J_i\rangle \int_r^\infty \frac{dr}{\ell}\left(\frac{\ell}{r}\right)^{d-1}\frac{1}{f(r)}\,. \end{aligned}$$

For the case of a purely thermal background we have

$$a_i(r;\omega,k) = a_i^{(0)} + (d-2)\langle J_i\rangle \;{}_2F_1\left[1, 1-\frac{2}{d}, 2-\frac{2}{d}, \left(\frac{r_0}{r}\right)^d\right]\,. \quad (1.2.22)$$

In the general case, to make this result finite, as $r \to r_+$ we require that $\langle J_i\rangle \equiv 0$.

Similarly, for the $E$ field, our equation of motion is

$$0 = \partial_r^2 E(r;\omega,k) + \left[\left(\frac{d-1}{r}\right) + \frac{f'(r)}{f(r)}\frac{\omega^2}{\omega^2 - k^2 f(r)}\right]\partial_r E(r;\omega,k)\,.$$

This is a first order equation in $d_r E$ so the solution is easy to find

$$E = E^{(0)} + \frac{(d-2)\langle J_E\rangle}{\left(\frac{\omega}{T}\right)^2 - \left(\frac{k}{T}\right)^2}\int_r^\infty \frac{dr}{\ell}\left(\frac{\ell}{r}\right)^{d-1}\frac{\left(\frac{\omega}{T}\right)^2 - \left(\frac{k}{T}\right)^2 f(r)}{f(r)} \quad (1.2.23)$$

where we have defined

$$E \overset{r\to\infty}{\sim} E^{(0)} + \langle J_E\rangle\left(\frac{\ell}{r}\right)^{d-2} + \ldots$$

The integral is logarithmically divergent in the near horizon except when $\frac{\omega}{T} << \frac{k}{T}$.

### Near region

We would like to impose that the fields are infalling as we expect that freely falling observers should see nothing special at the black hole horizon. This implies that the gauge field must solve the differential equation

$$\partial_r A_\mu = \frac{\ell^2}{r^2 f(r)}\partial_t A_\mu$$

near the horizon. This equation actually relates field strengths at the horizon and is unsurprisingly the condition imposed when considering the membrane paradigm.

In Fourier space we have

$$\begin{aligned}
\partial_r A_\mu &= -i\frac{\omega\ell^2}{r^2 f(r)}A_\mu \\
\Rightarrow A_\mu &= \exp\left(-i\omega\ell^2\int dr\frac{1}{r^2 f(r)}\right)F_\mu(r) \\
&= \tilde{F}_\mu(r)\begin{cases} (r-r_0)^{-\frac{i\omega\ell^2}{r_0^2 f'(r_0)^2}}, & \text{non}-\text{extremal} \\ \exp\left(+\frac{2i\omega\ell^2}{r_0^2 f''(r_0)(r-r_0)}\right), & \text{extremal} \\ \exp\left(\frac{i\omega\ell^2}{r}\right), & \text{vacuum} \end{cases}
\end{aligned}$$

where $F_\mu(r)$ and $\tilde{F}_\mu(r)$ are regular functions in $r$ as $r \to r_+$. Here we shall only look at the non-extremal case. Now let

$$F_p = (E, a_i)\exp\left(i\omega\ell^2\int dr\frac{1}{r^2 f(r)}\right)$$

with $p = 0, 1, \ldots, d-1$ and note that $F_p$ is regular as $r \to r_+$. Our equations of motion become

$$\begin{aligned}
0 &= d_r^2 F_p(r;\omega,k) + \left[-\frac{2i\omega\ell^2}{r^2 f(r)} + \left(\frac{d-1}{r}\right) + \frac{f'(r)}{f(r)} + \frac{k^2 f'(r)}{(\omega^2 - k^2 f(r))}\delta_{0p}\right] d_r F_p(r;\omega,k) \\
&\quad + \left[-\frac{i\omega\ell^2}{r^2 f(r)}\left(\frac{d-3}{r} + \frac{k^2 f'(r)}{(\omega^2 - k^2 f(r))}\delta_{0p}\right) - \frac{k^2\ell^4}{r^4 f(r)}\right] F_p(r;\omega,k) \ .
\end{aligned}$$

Solving our near horizon equation in a power series in $r - r_+$ will supply two integration constants. These will be used to fix the unspecified coefficients in Eq. (1.2.22) and Eq. (1.2.23). We shall search only for regular solutions to our equations of motion which implies that $F_p$ has a power series expansion of the form:

$$F_p(r) = c_p + \tilde{c}_p(r - r_0) + O^2(r - r_0) \ . \tag{1.2.24}$$

Substituting into the equation of motion we find:

$$\tilde{c}_p = \frac{i\omega\ell^2\left(\frac{d-3}{r_+} + \frac{k^2 f'(r_+)}{\omega^2}\delta_{0p}\right) + \frac{k^2\ell^4}{{r_+}^2}}{{r_+}^2 f'(r_+) - 2i\omega\ell^2}c_p \ . \tag{1.2.25}$$

In the limit that $\omega << k$ this becomes

$$\tilde{c}_p = i\frac{\ell^2}{{r_+}^2}\frac{k^2}{\omega}\delta_{0p}c_p \tag{1.2.26}$$

where, to obtain this final result, we have shifted $c_p \to \frac{\omega}{T}c_p$. This latter replacement is in part justified by the fact that when $T$ becomes large the near horizon conditions become asymptotically close to the trivial solution leaving an unperturbed ground state.

### Matching

We now match our hydrodynamic and near horizon solutions. Let's begin with the $a_i$ components. The small $r$ expansion of the hydrodynamic limit of $a_i$ is

$$a_i\left(r;\omega,k\right) \;=\; a_i^{(0)}\ .$$

Matching to the near horizon expansion we see that $a_i^{(0)} \equiv c_i$ and $\tilde{c}_i \equiv 0$. Thus the gauge field in the transverse directions is identically zero when $\frac{\omega}{T}$ is taken to be small.

Up until this point we have not had to place a restriction on the trajectories $\omega(k)$ in the complex $\omega$ plane other than $\frac{\omega(k)}{k} \to 0$ as $k \to 0$. Even once we have fixed the coefficients $E^{(0)}$ and $\langle J_E\rangle$ using the near horizon expansion

$$\begin{aligned}\langle J_E\rangle &= i\left(\frac{r_+}{\ell}\right)^{d-3}\left[\frac{\ell T}{(d-2)}\right]\left(\frac{k}{T}\right)^2 c_E \\ E^{(0)} &= c_E\left[\frac{\omega}{T}+i\left(\frac{T\ell^2}{(d-2)r_+}\right)\left(\frac{k}{T}\right)^2\right]\end{aligned} \tag{1.2.27}$$

all the trajectories noted above are permissable. However, if we require a quasinormal mode then we must set the source term to zero. For a non-trivial solution we cannot set $c_E \equiv 0$ which means we are forced to pick only one of the $\omega(k)$ trajectories which corresponds to a pole in the retarded Green's function. The pole's equation of motion in the complex frequency plane is

$$\frac{\omega}{T} \;=\; -i\left(\frac{d}{4(d-2)\pi}\right)\left(\frac{k}{T}\right)^2+\ldots \tag{1.2.28}$$

to the current order in $\frac{k}{T}$ where we used $r_+ = \frac{4\pi T\ell^2}{d}$. This gives the same result that was originally found in [30] and subsequently for various other decoupling limits in [51]. It becomes asymptotically close to the origin as $T \to \infty$ and thus is parametrically close to thermal equilibrium (where $E^{(0)} = \omega = k = 0$). Moreover the residue of the pole is given simply by evaluating $\langle J_E\rangle$ in Eq. (1.2.27) for $\frac{k}{T}$ small.

## 1.2.2 Fluctuations of a boosted black brane

As another illustration of extracting boundary expectation values for operators in the dual field theory from linearised perturbations we consider gravitational perturbations about a boosted black brane. This will also give us an opportunity to

introduce notation that will be useful to us later. Before we look at our specific geometry of interest however we discuss some of the generalities of solving the linearised Einstein equations in asymptotically, locally AdS spaces. Now, given our background metric $g^{(0)}$, a generic perturbation $h$ looks like

$$g = g^{(0)}_{MN} dx^M dx^N + \epsilon h_{MN} dx^M dx^N \ .$$

Intuitively it is clear that to first order in $\epsilon$ we should be able to write the equation that determines $h$ as an operator defined in terms of $g^{(0)}$ and its derivatives acting linearly upon $h$. In the next section we shall do this for the example of perturbations about the boosted black brane using the symmetry of the background to break up our perturbation into tensors with respect to spatial symmetries.

There are some useful geometric expressions which we shall use below and it is interesting to see them in a generic form. Firstly, the inverse bulk metric to first order in $\epsilon$ is

$$g^{-1} \ = \ \left[ g^{PQ}_{(0)} - \epsilon g^{PK}_{(0)} h_{KL} g^{LQ}_{(0)} \right] \partial_P \otimes \partial_Q \ .$$

The perturbation to the Ricci tensor is

$$\begin{aligned} R^{(1)}_{MP} \ &= \ \frac{1}{2}\nabla^2 h_{MP} - \frac{1}{2}\left[ \nabla_M \nabla_L h^L{}_P + \nabla_P \nabla_L h_M{}^L \right] + \frac{1}{2}\nabla_M \nabla_P h \\ &\quad + \frac{1}{2}\left[ R_M{}^R[g] h_{RP} + R_P{}^R[g] h_{RM} \right] - R_{MNPR}[g] h^{NR} \end{aligned}$$

while the perturbation to the Ricci scalar is

$$R^{(1)} \ = \ \nabla^2 h - \nabla_M \nabla_L h^{LM} \ .$$

For our asymptotically AdS spacetimes the Einstein equation takes the form

$$R_{MN} + \frac{d}{\ell^2} g_{MN} \ = \ s_{MN} \tag{1.2.29}$$

where $s_{MN} = T_{MN} - \frac{1}{d+1} T g_{MN}$ is the part of the Einstein equations that vanishes in the absence of matter. Expanding in $\epsilon$ we have

$$\left( R^{(0)}_{MN} \pm d g^{(0)}_{MN} \right) + \epsilon \left( R^{(1)}_{MN} \pm d h_{MN} \right) \ = \ s^{(0)}_{MN} + \epsilon s^{(1)}_{MN} \ .$$

The order $\epsilon^0$ piece is satisfied by our background metric and we can use it to rearrange the order $\epsilon$ piece to

$$\begin{aligned} & \frac{1}{2}\nabla^2 h_{MP} - \frac{1}{2}\left[\nabla_M \nabla_L h^L{}_P + \nabla_P \nabla_L h_M{}^L\right] + \frac{1}{2}\nabla_M \nabla_P h - R_{MNPR}\left[g_{(0)}\right] h^{NR} \\ = \; & s^{(1)}_{MP} - \frac{1}{2}\left[\left(s^{(0)}\right)_M{}^R h_{RP} + \left(s^{(0)}\right)_P{}^R h_{RM}\right] \; . \end{aligned} \tag{1.2.30}$$

For later use define $\tilde{E}^{(1)}_{MN} dx^M \otimes dx^N$ to be equal to the left hand side of this equation.

### Definitions and gauge choices

Having discussed the generalities of linearised gravity in AdS let's consider the problem we are particularly interested in which is perturbations around a boosted black brane geometry. These have metrics of the form

$$\begin{aligned} ds^2 &= -2u^{(0)}_\mu dx^\mu dr + r^2\left(1 - f(r)\right) u^{(0)}_\mu u^{(0)}_\nu dx^\mu dx^\nu + r^2 \eta_{\mu\nu} dx^\mu dx^\nu \\ &\quad + \epsilon h_{MN} dx^M dx^N \end{aligned} \tag{1.2.31}$$

with $h$ being our perturbation and $u^\mu_{(0)}$ some constant time-like vector which we shall allow to be general up to being unit normalised. We have set $\ell = 1$ in this section for brevity. The "background" part of this metric can be conveniently written in two ways

$$\begin{aligned} ds^2 &= -2u^{(0)}_\mu dx^\mu dr + r^2\left(1 - f(r)\right) u^{(0)}_\mu u^{(0)}_\nu dx^\mu dx^\nu + r^2 \eta_{\mu\nu} dx^\mu dx^\nu \\ &= -2u^{(0)}_\mu dx^\mu dr - r^2 f(r) u^{(0)}_\mu u^{(0)}_\nu dx^\mu dx^\nu + r^2 \Pi_{\mu\nu} dx^\mu dx^\nu \end{aligned} \tag{1.2.32}$$

where the object $\Pi_{\mu\nu}$ is called the spatial projector. It is defined to be

$$\Pi_{\mu\nu} = \gamma_{\mu\nu} + u^{(0)}_\mu u^{(0)}_\nu \tag{1.2.33}$$

where $\gamma$ is the boundary metric which in the current problem is the Minkowski metric $\eta$. We make the choice that raising and lowering of Greek indices is done with the boundary metric $\gamma$.

For posterity we state two further background quantities which will be useful for us to know in the following. Firstly the inverse background metric $g^{-1}_{(0)}$ has the form

$$g^{-1}_{(0)} = r^2 f(r) \partial_r \otimes \partial_r + 2u^\mu_{(0)} \partial_\mu \otimes \partial_r + \frac{1}{r^2}\left(\Pi^{-1}\right)^{\mu\nu} \partial_\mu \otimes \partial_\nu$$

where $(\Pi^{-1})^{\mu\nu}$ is defined so that $\delta^{\mu}{}_{\nu} = (\Pi^{-1})^{\mu\sigma}\,\Pi_{\sigma\nu}$. Secondly the Riemann tensor[7] of the background is

$$\begin{aligned} R_{\mu\nu\alpha\beta} &= r^4 f(r)\left[\Pi_{\alpha\nu}\Pi_{\beta\mu} - \Pi_{\alpha\mu}\Pi_{\beta\nu}\right] \\ &\quad -\frac{1}{2}r^3 f(r)\partial_r\left(r^2 f(r)\right)\times \\ &\quad \left[u^{(0)}_{\mu}\Pi_{\nu\alpha}u^{(0)}_{\beta} - u^{(0)}_{\nu}\Pi_{\mu\alpha}u^{(0)}_{\beta} + u^{(0)}_{\nu}\Pi_{\mu\beta}u^{(0)}_{\alpha} - u^{(0)}_{\mu}\Pi_{\nu\beta}u^{(0)}_{\alpha}\right] \\ R_{r\mu\alpha\nu} &= \frac{1}{2}r\partial_r\left(r^2 f(r)\right)\left[u^{(0)}_{\alpha}\Pi_{\nu\mu} - u^{(0)}_{\nu}\Pi_{\alpha\mu}\right] \\ R_{r\mu r\nu} &= \frac{1}{2}u^{(0)}_{\nu}u^{(0)}_{\mu}\partial_r^2\left(r^2 f(r)\right)\ . \end{aligned}$$

For now we shall consider general perturbations $h_{MN}$ which can potentially change the boundary metric. In the end though we will be looking for perturbations to our background bulk metric that are normalisable and describe a VEV deformation of the SEM tensor. However before we can progress we must deal with the issue of gauge invariance so that when we find solutions for our perturbations we know that they are truly different and not just gauge transformations of each other. To make our lives easier we shall break general invariance and make the following gauge choices:

$$\begin{aligned} g_{rr} &= 0, \\ g_{r\mu} &= \left(1+\epsilon\alpha(r,x)\right)u^{(0)}_{\mu}, \\ \mathrm{Tr}\left(g^{-1}_{(0)}g_{(1)}\right) &= 0\ . \end{aligned} \tag{1.2.34}$$

The first choice kills $h_{rr}$ while the second choice relates $h_{\mu r}$ to $u^{(0)}_{\mu}$ up to a constant of proportionality. The third condition tells us that $\alpha u^{\mu}_{(0)}u^{(0)}_{\mu} + \frac{1}{r^2}h_{\mu\nu}\Pi^{\mu\nu} = 0$. The perturbed metric therefore takes the form:

$$ds^2 = -2\left(1+\epsilon\mathrm{Tr}\left(\hat{h}\Pi\right)\right)u^{(0)}_{\mu}dx^{\mu}dr + \left(-r^2 f(r)u^{(0)}_{\mu}u^{(0)}_{\nu} + r^2\Pi_{\mu\nu} + \epsilon h_{\mu\nu}\right)dx^{\mu}dx^{\nu}\ .$$

[7]The conventions used in this thesis for the Riemann tensor are

$$R_{\mu\nu\alpha}{}^{\beta} = \Gamma^{\rho}_{\alpha\mu}\Gamma^{\beta}_{\nu\rho} - \Gamma^{\rho}_{\alpha\nu}\Gamma^{\beta}_{\mu\rho} - \partial_{\mu}\Gamma^{\beta}_{\nu\alpha} + \partial_{\nu}\Gamma^{\beta}_{\mu\alpha}$$

and $R_{\mu\nu} = R_{\mu\beta\nu}{}^{\beta}$, $R = R_{\mu\nu}g^{\mu\nu}$.

Using the spatial projector we can write

$$\begin{aligned} ds^2 &= -2\left(1+\frac{\epsilon}{r^2}h^{(1)}\right)u^{(0)}_\mu dx^\mu dr \\ &\quad + \left(-r^2 f(r) u^{(0)}_\mu u^{(0)}_\nu + r^2 \Pi_{\mu\nu}\right) dx^\mu \otimes dx^\nu \\ &\quad + \epsilon\left(h^{(0)}u^{(0)}_\mu u^{(0)}_\nu + 2h_{(\mu}u^{(0)}_{\nu)} + h_{\langle\mu\nu\rangle} + \frac{h^{(1)}}{d-1}\Pi_{\mu\nu}\right) dx^\mu \otimes dx^\nu \ , \end{aligned} \tag{1.2.35}$$

where all displayed indices not on a $u^\mu_{(0)}$ are entirely transverse, $h^{(1)} = h_{\mu\nu}\Pi^{\mu\nu}$, $h_\mu = -u^\nu_{(0)}h_{\nu\rho}\Pi^\rho{}_\mu$ and $h^{(0)} = h_{\mu\nu}u^\mu_{(0)}u^\nu_{(0)}$ and the angular brackets of a two-tensor $t_{\mu\nu}$ impose the following relation between its components

$$t_{\mu\nu} = \Pi_\mu{}^\alpha \Pi_\nu{}^\beta \left(t_{(\alpha\beta)} - \frac{1}{d-1}\gamma_{\alpha\beta}t_\lambda{}^\lambda\right) \ . \tag{1.2.36}$$

We could now carry all the computations through and compute the equations of motion governing the perturbation (Eq. (1.2.30)). As discussed in [52,53] it is always possible to break down the equations of motion into decoupled equations between scalar, vector and tensor parts of the metric with respect to the background $SO(d-1)$ spatial symmetry of our black brane. However as an example in this section we shall look only at the transverse, symmetric, traceless tensor piece of the metric which is contained within $h_{\langle\mu\nu\rangle}$.

### The tensor sector equation of motion

As we mentioned above perturbations which are symmetric, traceless, two-tensors with respect to spatial rotations in the $x^i$ directions are contained within $h_{\langle\mu\nu\rangle}$. The

linearised Einstein equation given by acting on $h_{\langle\mu\nu\rangle}$ is

$$\begin{aligned}
\tilde{E}^{(1)}_{MN}dx^M\otimes dx^N &= \frac{1}{2}\Big[r^2 f(r)\partial_r^2 h_{\langle\mu\nu\rangle} + 2u^\alpha_{(0)}\partial_r\partial_\alpha h_{\langle\mu\nu\rangle} + \frac{1}{r^2}\left(\Pi^{-1}\right)^{\alpha\beta}\partial_\alpha\partial_\beta h_{\langle\mu\nu\rangle} \\
&\quad + r^2\partial_r f\partial_r h_{\langle\mu\nu\rangle} + (d-3)rf\partial_r h_{\langle\mu\nu\rangle} + \frac{d-5}{r}u^\sigma_{(0)}\partial_\sigma h_{\langle\mu\nu\rangle} \\
&\quad - h_{\langle\mu\nu\rangle}\left(2r\partial_r f(r) + 2(d-1)f\right) + 2h_{\langle\mu\nu\rangle}\Big]\,dx^\mu\otimes dx^\nu \\
&\quad - \left[\frac{1}{r^3}\left(\Pi^{-1}\right)^{\mu\alpha}\partial_\alpha h_{\langle\mu\nu\rangle}\right]dr\otimes dx^\nu \\
&\quad - \left[\frac{1}{r^3}\left(\Pi^{-1}\right)^{\alpha\nu}\partial_\alpha h_{\langle\mu\nu\rangle}\right]dx^\mu\otimes dr \\
&\quad - \frac{1}{2}\left[\frac{1}{r^2}\left(\Pi^{-1}\right)^{\sigma\rho}\partial_\mu\partial_\rho h_{\langle\sigma\nu\rangle} + \frac{1}{r^2}\left(\Pi^{-1}\right)^{\sigma\rho}\partial_\nu\partial_\rho h_{\langle\sigma\mu\rangle}\right]dx^\mu\otimes dx^\nu \\
&\quad - \frac{1}{2}\left[\frac{1}{r^2}\left(\Pi^{-1}\right)^{\sigma\rho}\partial_r\partial_\rho h_{\langle\sigma\nu\rangle} - \frac{4}{r^3}\left(\Pi^{-1}\right)^{\sigma\alpha}\partial_\alpha h_{\langle\sigma\nu\rangle}\right] \\
&\quad \times\left(dr\otimes dx^\nu + dx^\nu\otimes dr\right) + \ldots
\end{aligned}$$

where the ellipses indicate terms coming from other pieces of the perturbation. We notice that $h_{\langle\mu\nu\rangle}$ in itself does not decouple from the other components of the perturbation $h^{(0)}$, $h^{(1)}$, $h_\mu$, $u^{(1)}_\mu$. However, as suggested in [52,53], we can decompose $h_{\langle\mu\nu\rangle}$ further. Let $x^i$ be our boundary spatial coordinates and align a time coordinate $v$ with $u^\mu_{(0)}$. $h_{\langle\mu\nu\rangle}$ can be written in terms of vector and tensor harmonics

$$h_{\langle ij\rangle}(r,v,x) = h^T(r,v)\mathbb{T}_{jk} + h^V(r,v)\left(\partial_i\mathbb{V}_j(x) + \partial_j\mathbb{V}_i(x)\right) \tag{1.2.37}$$

which satisfy the following conditions

$$\begin{aligned}
\delta^{ij}\partial_i\partial_j\mathbb{T}_{kl} + \mathbf{k}^2\mathbb{T}_{kl} &= 0\,, \\
\delta^{ij}\partial_i\mathbb{T}_{jk} &= 0\,, \\
\delta^{ij}\partial_i\partial_j\mathbb{V}_k + \mathbf{k}^2\mathbb{V}_k &= 0\,, \\
\delta^{ij}\partial_i\mathbb{V}_j &= 0
\end{aligned} \tag{1.2.38}$$

with $\left(\Pi^{-1}\right)^{ij} = \delta^{ij}$ in our coordinate system. It is clear our Einstein equation then reduces to

$$\begin{aligned}
\tilde{E}^{(1)}_{MN}dx^M\otimes dx^N &= \frac{1}{2}\Big[r^2 f(r)\partial_r^2 h_T + 2\partial_r\partial_v h_T - \frac{\mathbf{k}^2}{r^2}h_T \\
&\quad + r^2\partial_r f\partial_r h_T + (d-3)rf\partial_r h_T + \frac{d-5}{r}\partial_v h_T \\
&\quad - h_T\left(2r\partial_r f(r) + 2(d-1)f\right) + 2h_T\Big]\,\mathbb{T}_{ij}dx^i\otimes dx^j + \ldots
\end{aligned}$$

where we have written out only the part of the $\tilde{E}^{(1)}$ proportional to $\mathbb{T}_{ij}$. We see that $h^T$ completely decouples from the other components of the metric perturbation as it is only enters in the term multiplying $\mathbb{T}_{ij}$.

We are now in a position to solve the resultant equation for the tensor part of the metric perturbation. We shall make a simplifying ansatz by only turning on momentum in the $z$ direction and performing a Fourier decomposition with respect to this direction

$$h_{xy}(r,v,z) = r^2 \int \frac{d\omega dk_z}{(2\pi)^2} h_{xy}(r;\omega,k_z) \exp\left(-i\omega v + ik_z z\right) . \tag{1.2.39}$$

We have introduced an $r^2$ for convenience. The the equation of motion becomes

$$\begin{aligned} &\partial_r^2 h_{xy} + \left(\frac{\partial_r f}{f} + \frac{d+1}{r}\right)\partial_r h_{xy} \\ = \; &\frac{1}{r^4 f}\left[k_z^2 h_{xy} + i\omega\left(2r^2 d_r h_{xy} + (d-1) r h_{xy}\right)\right] \end{aligned} \tag{1.2.40}$$

which is the equation of motion for a massless scalar in our choice of coordinates.

We could now attempt to solve Eq. (1.2.40) in the low frequency and momentum regime using exactly the same method as discussed above for the gauge field. For brevity we shall not do this and simply quote the important results, namely, while there is no pole in this sector there is a non-zero diffusion rate that can be calculated from the Green's function as done in [30] for $d = 4$. This diffusion rate is interpreted as the shear viscosity of the boundary fluid and its value is

$$\eta = \frac{1}{2\kappa_{d+1}^2}\left(\frac{r_+}{\ell}\right)^{d-1} . \tag{1.2.41}$$

As we shall see later this is associated with a quadratically dispersing mode - the shear pole.

Finally, before leaving the linearised analysis to look at universality in the next section, we should make an important observation. All we have shown so far is that there are quadratically dispersing modes in a linearised, long wavelength and low frequency analysis of the gauge and shear gravitational perturbations of Schwarzschild-AdS. Hydrodynamics however is a non-linear theory. Furthermore there seems to be no special reason to choose our timelike vector field to satisfy the non-linear Navier-Stokes equations. In the the next chapter we shall demonstrate

that it is possible to solve the Einstein equations non-linearly. Moreover we shall see that satisfying the relativistic Navier-Stokes equations is intimately linked with the resultant perturbed spacetime being regular.

## 1.3 Universality

It is a very natural question to ask how generic our results, Eqs. (1.2.28) and (1.2.41), for transport coefficients are. One known result is that in spatially isotropic fluids dual to two-derivative gravity models the shear viscosity to entropy ratio is given by the universal value

$$\frac{\eta}{s} = \frac{1}{4\pi} \, .$$

We shall outline how to prove this result as it allows us to introduce formalism that will be useful in chapter 5. It will also allow us to link the membrane paradigm to the strongly coupled field theory living at the boundary - an approach which was successfully pushed in [54] and returned to in [3].

As we have already shown in a gravity theory where the background is spatially isotropic it is possible to decompose metric perturbations into representations of the spatial isometry algebra. In particular, we found that the $h_{xy}$ component always obeyed equations of motion coming from an action of the form

$$\mathcal{S} = -\frac{1}{4\kappa_{d+1}^2} \int_{r>r_0} d^{d+1}x \sqrt{-g} \, (\partial\phi)^2 \tag{1.3.42}$$

where $\phi = h_x^y$ and we shall work in the coordinates displayed in Eq. (1.2.14) rather than the Eddington-Finkelstein coordinates of Eq. (1.2.31) for simplicity. We shall find it useful to employ the momentum canonically conjugate to $\phi$ with respect to the $r$ foliation of the spacetime which is denoted $\Pi$ and has the form

$$\Pi = -\frac{1}{4\kappa_{d+1}^2} \sqrt{-g} g^{rr} \partial_r \phi \, . \tag{1.3.43}$$

The equations obeyed by the field $\phi$ can be written in Hamiltonian form with respect to the $r$-foliation. The Hamiltonian equation relating $\Pi$ to derivatives of the $\phi$ comes from Eq. (1.3.43). Additionally, from the equations of motion for the field $\phi$ we

have

$$\begin{aligned} \frac{1}{\sqrt{-g}}\partial_r\left(\sqrt{-g}g^{rr}\partial_r\phi\right) &= g^{\mu\nu}k_\mu k_\nu\phi \\ \Rightarrow \partial_r\Pi &= -\frac{\sqrt{-g}}{4\kappa_{d+1}^2}g^{\mu\nu}k_\mu k_\nu\phi\,. \end{aligned} \tag{1.3.44}$$

The benefit of this split becomes clear once we impose infalling conditions at the black hole horizon on our fields. The retarded Green's function at the boundary, up to renormalisation, then has the form

$$G_R\left(k_\mu\right) = -\lim_{r\to\infty}\frac{\Pi\left(r,k_\mu\right)}{\phi\left(r,k_\mu\right)}\,. \tag{1.3.45}$$

This suggests we define a "Green's function" at each value of $r$ [54] by removing the limit. As the low frequency and momentum limit of the Green's functions defines a transport coefficient by

$$\chi = -\lim_{\omega\to 0}\lim_{\mathbf{k}\to\mathbf{0}}\frac{1}{\omega}\Im G_R\left(k_\mu\right) \tag{1.3.46}$$

we can then connect transport coefficients in the membrane paradigm to transport coefficients at the boundary via the flow of $G_R$ in $r$.

Now consider the limit where $k_\mu \to 0$ with $\omega\phi$ and $\Pi$ held fixed. The resultant equations of motion from Eq. (1.3.43) and Eq. (1.3.44) are

$$\partial_r\phi = 0 \tag{1.3.47}$$

$$\partial_r\Pi = 0\,. \tag{1.3.48}$$

This is just the low frequency and long wavelength limit and we see that our equations of motion have become trivial. This means that the boundary values of the fields are fixed by their near horizon values. In particular, as the shear viscosity of the membrane is fixed, whenever the background metric has spatial isotropy in its $r$-foliation, the shear viscosity to entropy ratio in the boundary is also fixed to be $\frac{1}{4\pi}$ [54]. The same cannot be said for our diffusion constant for the gauge field, Eq. (1.2.28), which explicitly has dimension dependent parameters.

We have shown that under certain, quite general, assumptions the shear viscosity to entropy ratio of every field theory dual to thermal, asymptotically, locally AdS spaces is fixed. It was initially conjectured in [28] that $\frac{1}{4\pi}$ may be a lower bound

to the viscosity of all relativistic quantum field theories at finite temperature. It is important to point out here that there are now several examples for violations of this bound. One such example can be found in Gauss-Bonnet gravity, where it has been shown that the reason the bound is violated is essentially due to the fact that when the Gauss-Bonnet coupling, $\lambda_{GB}$, is greater than zero then gravity is more strongly coupled than in AdS space [55–58]. Similarly, it has recently been shown in theories that explicitly break spatial isotropy in the ground state, components of the shear viscosity tensor can dip below the viscosity bound [59].

# Chapter 2

# Relativistic fluids

In this chapter we shall review work already available in literature. As we have previously indicated it is possible to calculate hydrodynamic-like quantities, using the AdS-CFT correspondence, by considering perturbations of a particular black hole spacetime. However we have yet to show that these constitute true hydrodynamics. In this chapter we shall demonstrate that these long-lived modes truly are governed by the relativistic Navier-Stokes equations. Understanding these modes is important for the applications to real world physics indicated in the introduction.

While there are many interesting phenomena that can be understood by considering uncharged fluids at progressively higher orders in a derivative expansion, for brevity, we shall truncate to first order in derivatives and refer where necessary to literature. The procedure for continuing to higher orders will however be outlined. An extension that we shall make is to add a $U(1)$ charge to the fluid as this has previously led to new conceptual insights. One such discovery is the necessity for parity violating terms in the relativistic fluid expansion, when the fluid is placed in a charged background [60], even at first order.

The subsequent chapter is split into two pieces. In the first we show how it is possible to write down an effective description of long wavelength, low frequency modes for any suitable field theory. As mentioned above, for the purposes of explicit computation, we shall truncate to first order in derivatives and refer the reader to the relevant literature for higher orders [32, 61–63].

Additionally, while we have seen that spatial isotropy in the holographic dual is

a powerful method to simplify the resultant Einstein equations (Sec. 1.2.2), many of the areas that are currently interesting in fluid-gravity are exactly those that break such isotropy [64–69]. Thus we shall not assume spatial isotropy in deriving the effective hydrodynamic theory until we come to the end of the section where we shall wish to compare to known holographic duals.

The second part of this chapter indicates the process by which it is possible to find a holographic dual to the field theories discussed above at strong coupling. We shall only discuss the subsequent Einstein-Maxwell equations briefly as there are already good discussions of the topic in literature [32, 61–63]. What will be novel is our explicit connection of this calculation to the linearised analysis in the previous chapter.

## 2.1 Dynamics of charged fluids

In this section we shall discuss how to formulate an effective hydrodynamic theory from a given field theory. We begin by considering the global thermodynamics (Sec. 2.1.1) of a charged material at a large non-zero temperature which describes the state of the system at very late times when any initial fluctuations have completely dissipated. We then describe the idea of local thermodynamics which applies at late times when fluctuations of long wavelength and small frequency are not negligible in Sec. 2.1.2. This will afford us the opportunity to fix some ambiguities in our local concepts. In the subsequent section (Sec. 2.1.3) we shall then discuss how to construct hydrodynamics as a description of the approach to global equilibrium from regions of local equilibrium.

In Sec. 2.1.4 we lay out the generalities of the hydrodynamic derivative expansion at first order in derivatives. We shall not assume that the system is spatially isotropic which will make the notation a little more unwieldly however, when we then do consider spatially isotropic systems, it will be explicit how strong the simplification of our resultant effective theory is. In Sec. 2.1.5 we consider how charge current anomalies appear in our effective theory. In particular, when such anomalies exist, we must be careful in our assumptions about parity.

As an illustration of the simplicity of the process of determining the transport coefficients in spatially isotropic theories we shall then derive in Secs. 2.1.6 and 2.1.7 the number and form of the transport coefficients needed to completely describe uncharged and charged fluids at first order in derivatives respectively. Finally we shall discuss, although not calculate, how one may proceed to second order in derivatives, consider non-zero electric and magnetic fields or force the fluid by summarising the literature.

### 2.1.1 Global thermodynamics

Assume we are given a charged system with a non-zero temperature. Let $\mathcal{B}_d$ be the $d$-dimensional manifold with metric $\gamma_{\mu\nu}$ on which this system lives. To connect to traditional hydrodynamics we shall choose pressure $P$, temperature $T$ and charge $Q^I$ to be our typical state variables. The thermodynamic potential for such a system will have the general form:

$$G = G\left(P, T, Q^I\right) . \tag{2.1.1}$$

From the first law of thermodynamics the differential of the internal energy of our system is

$$\mathrm{d}E = T\mathrm{d}S - P\mathrm{d}V + \mu_{qI}\mathrm{d}Q^I \tag{2.1.2}$$

and therefore we choose the differential of $G$ to be given by:

$$\mathrm{d}G = \mathrm{d}E - \mathrm{d}\left(ST - PV\right) \tag{2.1.3}$$

$$= V\mathrm{d}P - S\mathrm{d}T + \mu_{qI}\mathrm{d}Q^I . \tag{2.1.4}$$

The result of minimizing the potential is the elimination of one of our state variables (typically we shall choose to eliminate $P$) encoded in an equation of state which relates them. In what follows we shall often assume an underlying conformality to our theory which will supply us with just such an equation of state.

We assume extensivity of the thermodynamic potential and use it to extract a $V$ from the definition of $G$ to give us

$$g = g\left(P, T, q^I\right) \tag{2.1.5}$$

where $q^I$ is the charge density which is more appropriate for infinite volume systems. For a system with scale invariance, under a Weyl transformation $\phi(x)$ of the metric $\gamma$, our state variables transform as:

$$g \rightarrow \exp(-d\phi)\, g\ , \tag{2.1.6}$$

$$P \rightarrow \exp(-d\phi)\, P\ , \tag{2.1.7}$$

$$T \rightarrow \exp(-\phi)\, T\ , \tag{2.1.8}$$

$$q^I \rightarrow \exp(-(d-1)\phi)\, q^I\ . \tag{2.1.9}$$

We can use this scale invariance to pick a conformal frame on the manifold with some reference temperature $T_0$. If we start in a system where the temperature is $T$ consider making a scaling transformation

$$\phi = \ln\left(\frac{T_0}{T}\right)\ . \tag{2.1.10}$$

The energy density in a system with temperature $T$ is related to that in the system with temperature $T_0$ by our scaling transformation in the following manner

$$\begin{aligned} \varepsilon\left(T, q^I\right) &= \left(\frac{T}{T_0}\right)^d \varepsilon\left(T_0, q_0^I\right) \\ &= \left(\frac{T}{T_0}\right)^d \varepsilon\left(T\left(\frac{T_0}{T}\right), q^I\left(\frac{T_0}{T}\right)^{d-1}\right)\ . \end{aligned} \tag{2.1.11}$$

where we have assumed we are at a minimum of the thermodynamic potential $G$ to eliminate $P$ as a variable. Our reference temperature $T_0$ is constant so we can drop the functional dependence on it. As such we can redefine our energy density so that

$$\varepsilon\left(T, q^I\right) = T^d h\left(\frac{q_I}{T^{d-1}}\right) \tag{2.1.12}$$

where $h$ is some function depending on the particular system we are examining. This process also applies to other state variables such as the entropy or pressure.

The scaling of our thermodynamic variables with respect to a conformal transformation also allows us to apply a version of Euler's homogeneous function theorem. Scaling our reference temperature it can be shown that

$$g = dP - sT + q^I \mu_{qI}\ . \tag{2.1.13}$$

Using our definition for $g$ in terms of the energy density, Eq. (2.1.3), we find:

$$\varepsilon\left(T, q^I\right) = (d-1)P\left(T, q^I\right) \tag{2.1.14}$$

which we shall henceforth take as our equation of state. In the following, unless we explicitly state so, we shall not assume scale invariance as it is a powerful tool that is only necessary in certain circumstances.

### 2.1.2 Local thermodynamics and fluid variables

Consider disturbing our system of global equilibrium discussed above. In the case of kinetic theory in classical mechanics we expect at large temperatures random molecular motion will quickly dissipate the disturbance locally. Hence in the neighbourhood of a point the system satisfies the equation of state Eq. (2.1.14). However if the disturbance has changed the system on scales greater than the mean free path then different regions will settle into different equilibria as illustrated in Fig. 2.1. This cannot be a stable situation as, for example, particles will flow from regions of greater concentration to regions of lower concentration (Fick's law).

This process of dissipating disturbances is ubiquitous and does not rely upon kinetic theory - rather all that is required is a microscopic means to reach local equilibrium in "good time". We would like to describe the process of moving from regions of local equilibrium to global equilibrium. What is clear is that because this "hydrodynamic phenomenon" has some universal character the formalism cannot depend on the detailed microscopic nature of the field theory although, as relaxation rates depend on the system considered, there must be model dependent parameters we can tune. This should indicate to us that the description can only be in terms of objects that are shared by our theories of interest (like the SEM tensor) and obey the same equations in these systems (conservation of the SEM tensor).

Additionally we expect then that when we have a hydrodynamic description of the system these conserved currents must be built from the fluid velocity and thermodynamic state variables which occur in all our models of interest and define patches of local equilibrium. The expression for these conserved currents in terms of

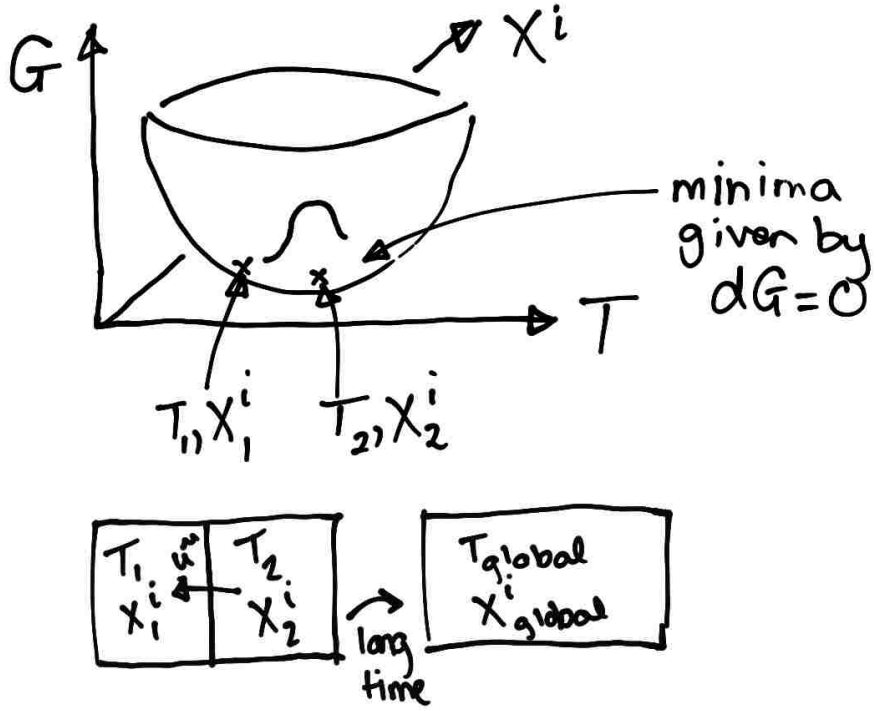


Figure 2.1: An illustration of the concept of local thermal equilibrium. Displayed is some generic potential $G$. The axis $X^i$ is representative of all variables other than $T$. When the system is in global equilibrium it sits at the bottom of our generic potential. We then choose to disturb the system which, after a small time, creates regions of local equilibrium. These regions touch each other as depicted underneath the potential. The situation cannot be stable as, for example, Fick's law tells us that solute will diffuse from a place of high concentration to a region of low concentration. This process is illustrated below the graph of the potential and is the phenomenon we wish to describe.

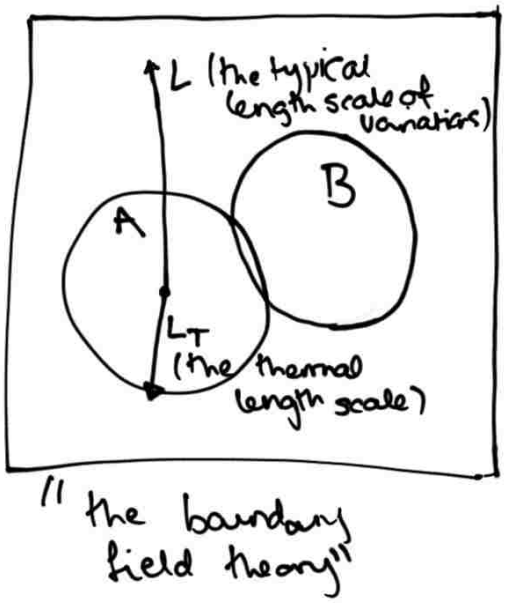


Figure 2.2: An illustration of the concept of local thermal equilibrium where we have displayed the typical scale of fluctuations $L$ and the thermal length scale $L_T$.

state variables, their derivatives and fluid velocity derivatives is called a constitutive relation. The model dependent parameters are the coefficients of the derivative terms and are called transport coefficients.

It is natural to ask when this decomposition via local equilibria is likely to be valid. We shall assume that, in our system in the neighbourhood of any point, we are able to draw an open patch where it can be treated as being in equilibrium (see Fig. 2.2). The typical size of these patches we shall denote $L_T$ while the typical size of fluctuations in our theory we shall denote $L$. We can expect our concepts of local equilibrium to be well defined whenever $\frac{L_T}{L} \ll 1$ - so that we do not disturb local equilibrium. Any thermal patch is in contact with other nearby patches and, because from the neighbourhood of one point to the neighbourhood of another point the thermodynamic state variables are varying, there will be a flow of quantities like heat and charge between these patches of local equilibrium. Hydrodynamics therefore is fundamentally a derivative expansion with higher order derivatives being suppressed compared to those occurring lower down in the expansion.

### Frame choices

We begin by assuming the existence of conserved currents

$$T_{\mu\nu}$$
$$J^{\mu}_{I}$$

which represent stress-energy-momentum and free charge flow between regions of local equilibrium respectively. We shall need a notion of time given that the process of reaching equilibrium is manifestly one that breaks time symmetry due to the production of entropy. Thus we also assume the existence of a unit normalised time-like vector field which has the following property

$$T_{\mu\nu}(x)u^{\nu}(x) = -\varepsilon\left(T(x), q^{I}(x)\right)u_{\mu}(x) \tag{2.1.15}$$

where $\varepsilon(x)$ is the local energy density in terms of the local temperature and charge density fields which we assume we are given. Defining our time coordinate by the integral curves of $u^{\mu}$ in Eq. (2.1.15) puts us in the "Landau frame". In this frame, when we follow a fluid lump, it has constant energy. Note that the eigenvector equation for the SEM tensor fixes an ambiguity in what we mean by fluid velocity using our expression for energy density.

We have already defined the spatial projector $\Pi_{\mu\nu}$ in Eq. (1.2.33) and with this definition it immediately follows that

$$T_{\mu\nu} = \varepsilon u_{\mu}u_{\nu} + \Pi^{\alpha}_{\mu}\Pi^{\beta}_{\nu}T_{\alpha\beta} \tag{2.1.16}$$

where we have made use of the eigenvalue equation of the SEM tensor. We note that the SEM tensor has decomposed into a piece entirely parallel to the fluid velocity (the first term) and a piece entirely transverse to it (the last term).

For the charge current we have

$$J^{\mu}_{I} = q_{I}u^{\mu} + \Pi^{\mu}_{\alpha}J^{\alpha}_{I} \tag{2.1.17}$$

where the charge current has split into a piece parallel to $u^{\mu}$ (whose coefficient is $q^{I}$) and a piece orthogonal to it. As we have a charge current we could alternatively have specified our fluid velocity $u^{\mu}$ by aligning it with the charge current - this is called the Eckart frame.

As we mentioned above - we expect the process of reaching global equilibrium to increase entropy. This will allow us to restrict the type of processes that can sensibly occur. As such we define an entropy current which has the form:

$$J_s{}^{\mu} = s\left(T, q^I\right) u^{\mu} + \Pi^{\mu}_{\nu} J_s{}^{\nu} \ .$$

An alternative approach to using the entropy current was recently developed in literature [70–73]. At present this "generating functional" method is not as powerful as requiring positive divergence of the entropy current as it only constrains a smaller subset of transport coefficients.

#### The currents at zeroth order

We would like to finish this section by specifying the entirely transverse parts of the SEM tensor and charge current that occur in Eqs. (2.1.16) and (2.1.17). If we assume spatial isotropy and scale invariance we can write:

$$\begin{aligned} \varepsilon &= \Pi^{\mu\nu} T_{\mu\nu} \\ &= (d-1)P \ , \end{aligned} \tag{2.1.18}$$

where we have used that the trace of the stress tensor is zero (up to anomalies which occur at a higher order in derivatives) in a field theory which is fundamentally conformal. Additionally the transverse piece of the charge current must be zero when we have a spatially isotropic theory as it would otherwise pick out a special direction. Finally therefore the constitutive relations are

$$T_{\mu\nu} = \varepsilon u_{\mu} u_{\nu} + P\Pi_{\mu\nu}, \tag{2.1.19}$$

$$J^{\mu}_{I} = q_I u^{\mu} \tag{2.1.20}$$

with the assumptions we have made.

### 2.1.3 Conservation, symmetries and ideal fluid dynamics

As we have previously indicated, hydrodynamics is a derivative expansion, with progressively higher order derivatives being suppressed. Thus we should think of Eqs. (2.1.19) and (2.1.20) as being only the first terms in the expansion and, in

the next section, we would like to consider the first order corrections. As such we shall now discuss the equations of motion obeyed by the fluid velocity, namely SEM tensor and charge current conservation, and show in what way positive divergence of the entropy current can be used to constrain such terms.

The parts of the currents we have so far at zeroth order in derivatives are sometimes called "ideal". The corrections are called dissipative and the total currents will have the form

$$\begin{aligned} T_{\mu\nu} &= T^{(\text{ideal})}_{\mu\nu} + \tau_{\mu\nu} \\ J^{\mu}_{I} &= \left(J^{(\text{ideal})}_{I}\right)^{\mu} + \xi^{\mu}_{I} \, . \end{aligned}$$

with $\tau_{\mu\nu}$ and $\xi^{\mu}_{I}$ being at least order one in derivatives. The resultant currents must satisfy the conservation equations which for us are the following

$$\nabla^{\mu} T_{\mu\nu} = J^{\mu}_{I} F^{I}_{\nu\mu} \tag{2.1.21}$$

$$\nabla_{\mu} J^{\mu}_{I} = C^{\text{anomalies}}_{I} \tag{2.1.22}$$

where $\nabla_{\mu}$ is the covariant derivative with respect to the metric of the manifold on which our fluid lives and $C^{\text{anomalies}}_{I}$ is the anomalous part of the charge current conservation[1]. Additionally, as mentioned above, we impose a constraint on our derivative expansion, namely,

$$\nabla_{\mu} \left\langle J_{s}{}^{\mu} \right\rangle \geq 0 \, . \tag{2.1.23}$$

Strictly this constraint is not necessary from the perspective of an effective field theory but seems physically sensible. We can see that this will affect objects that can appear in our constitutive relations in the following way. From the first law of thermodynamics for a constant comoving volume we have

$$u^{\mu} \nabla_{\mu} s = \frac{u^{\mu}}{T} \left[ \nabla_{\mu} \varepsilon - \mu_{qI} \nabla_{\mu} q^{I} \right] \, .$$

For now we shall assume our microscopic theory is anomaly free such that the Landau

[1]Be aware that we have assumed our electric and magnetic fields are order one in derivatives and thus do not appear in the system's governing thermodynamic relation.

frame condition allows us to write

$$\begin{aligned} u_\mu \nabla^\mu \varepsilon &= -\varepsilon \nabla^\mu u_\mu - \tau_{\mu\nu}\nabla^\mu u^\nu - E^I_\mu J^\mu_I \ , \\ u^\mu \nabla_\mu q_I &= -q_I \nabla_\mu u^\mu - \nabla_\mu \xi^\mu_I \end{aligned}$$

where we have defined

$$E^\nu_I = u_\mu F^{\mu\nu}_I \ . \tag{2.1.24}$$

We shall return to the case of anomalous theories shortly as the consequences are quite interesting and deserve attention on their own. Combining the above two pieces of information implies that

$$\begin{aligned} 0 &\leq \nabla_\mu s^\mu \\ &\leq -\frac{\tau_{\mu\nu}}{T}\nabla^\mu u^\nu - \xi^I_\mu \left( \nabla^\mu \left( \frac{\mu_{qI}}{T} \right) + E^\mu_I \right) \\ &\quad + \nabla_\mu \left\{ \xi_{\mathrm{s}}{}^\mu + \xi^\mu_I \frac{\mu_q{}^I}{T} \right\} \ . \end{aligned} \tag{2.1.25}$$

We can define the correction to the entropy current, $\xi_{\mathrm{s}}{}^\mu$, entirely in terms of the correction to the charge current

$$\xi_{\mathrm{s}}{}^\mu = -\xi^\mu_I \frac{\mu_q{}^I}{T} \tag{2.1.26}$$

and we are left with

$$0 \leq -\frac{\tau_{\mu\nu}}{T}\nabla^\mu u^\nu - \xi^\mu_I \left( \nabla_\mu \left( \frac{\mu_{qI}}{T} \right) + E^I_\mu \right) \ .$$

It is clear that we must pick $\tau_{\mu\nu}$ and $\xi^\mu_I$ to make each of the remaining terms a positive square. We shall see shortly that this places restrictions on the transport coefficients.

### 2.1.4 Viscous fluid dynamics at first order

We now construct the objects $\tau_{\mu\nu}$ and $\xi^\mu_I$ at first order in derivatives. Potentially we could have derivative corrections in $T$, $q_I$ and $u^\mu$. The direction $u^\mu$ will always be special so it makes sense to decompose our covariant derivative into pieces transverse

and parallel to the fluid velocity[2]:

$$\nabla_\mu = -u_\mu u^\sigma \nabla_\sigma + \Pi^\sigma_\mu \nabla_\sigma \ .$$

Additionally coefficients multiplying terms involving $\nabla u$ will have projectors in them to account for the fact that

$$\begin{aligned} u^\mu u_\mu &= -1 \ , \\ \Rightarrow u^\mu \nabla_\nu u_\mu &= 0 \ . \end{aligned}$$

With these considerations in mind the most general linear combination of single derivative objects we can give has the rather ugly expression

$$\begin{aligned} \tau_{\mu\nu} &= \left(c^{(1)}\right)^{\sigma_1\sigma_2}_{\mu\nu} \nabla_{\sigma_3} u_{\sigma_4} - \left(c^{(1)}\right)^{\sigma_2}_{\mu\nu} u^{\sigma_1} \nabla_{\sigma_1} u_{\sigma_2} \\ &\quad + \left(c^{(2)}_I\right)^{\sigma_1}_{\mu\nu} \nabla_{\sigma_1} q^I - \left(c^{(2)}_I\right)_{\mu\nu} u^{\sigma_1} \nabla_{\sigma_3} q^I \\ &\quad + \left(c^{(3)}\right)^{\sigma_1}_{\mu\nu} \nabla_{\sigma_1} T - \left(c^{(3)}\right)_{\mu\nu} u^{\sigma_1} \nabla_{\sigma_1} T \ , \\ \xi^\mu_I &= \left(c^{(4)}_I\right)^{\mu\sigma_1\sigma_2} \nabla_{\sigma_1} u_{\sigma_2} - \left(c^{(4)}_I\right)^{\mu\sigma_2} u^{\sigma_1} \nabla_{\sigma_1} u_{\sigma_2} \\ &\quad + \left(c^{(5)}_{IJ}\right)^{\mu\sigma_1} \nabla_{\sigma_1} q^J - \left(c^{(5)}_{IJ}\right)^{\mu} u^{\sigma_1} \nabla_{\sigma_1} q^J \\ &\quad + \left(c^{(6)}_I\right)^{\mu\sigma_1} \nabla_{\sigma_1} T - \left(c^{(6)}_I\right)^{\mu} u^{\sigma_1} \nabla_{\sigma_1} T \end{aligned}$$

where we assume all indices not displayed on a $u$ are transverse.

We note that the number of operators we have allowed in our SEM tensor and charge current corrections generally over-determines our system. At first order in derivatives there are $(d+1)$ (one parallel and $d$ transverse) constraints from ideal SEM tensor and charge current conservation. This will allow us to eliminate $d+2$ operators from the dissipative part of our constitutive relation. Without loss of generality at first order we shall use the constraints from the zeroth order conservation equations to remove $\nabla_\mu T$ and $u^\mu \nabla_\mu q_I$ in the SEM tensor and $u^\mu \nabla_\mu T$

[2]If we were to consider only conformal field theories we could replace $\nabla_\mu$ with the Weyl covariant derivative $D_\mu$. This requires the introduction of a Weyl connection $\mathcal{A}_\mu$ and, by choice of this connection, the derivative can be made transverse. This greatly simplifies the following decomposition but requires extra formalism which we shall not introduce. See [74] for further details.

and $u^\mu \nabla_\mu u^\nu$ in the charge current. Our first order constitutive relations become

$$\tau_{\mu\nu} = \left(c^{(1)}\right)^{\sigma_1\sigma_2}_{\mu\nu} \nabla_{\sigma_3} u_{\sigma_4} - \left(c^{(1)}\right)^{\sigma_2}_{\mu\nu} u^{\sigma_1} \nabla_{\sigma_1} u_{\sigma_2} \,, \tag{2.1.27}$$

$$\begin{aligned} \xi^\mu_I = & \left(c^{(4)}_I\right)^{\mu\sigma_1\sigma_2} \nabla_{\sigma_1} u_{\sigma_2} \\ & + \left(c^{(5)}_{IJ}\right)^{\mu\sigma_1} \nabla_{\sigma_1} q^J - \left(c^{(5)}_{IJ}\right)^{\mu} u^{\sigma_1} \nabla_{\sigma_1} q^J + \left(c^{(6)}_I\right)^{\mu\sigma_1} \nabla_{\sigma_1} T \end{aligned} \tag{2.1.28}$$

where we have redefined our constants as necessary. When we later make use of scale invariance we shall be able to reduce the number of operators again by one - but as this is such a powerful tool we shall hold it in reserve until we absolutely have to use it. In a later section we shall make several simplifying assumptions that will constrain the above expansion and allow us to extract some physics from it. Before doing that however we should discuss how charge current anomalies can affect our hydrodynamic description through parity violation in the next section.

### 2.1.5 Broken parity and inexact symmetries in fluid dynamics

In the above we took the attitude that if an operator could be added to our SEM tensor and charge current then it should be written down. However, historically, a more restrictive approach was taken to deriving the constitutive relations leading to missing terms compared to our expansion. In particular, in non-relativistic physics it is often assumed that the theory we are interested in is fundamentally parity invariant. This discrete symmetry relates components of $c$s (Eqs. (2.1.28) and (2.1.27)) in our expansion. From an effective field theory point of view, as the coefficients of terms are arbitrary, we can simply set to zero anything that is not parity invariant. This truncates the class of field theories our hydrodynamics describes to those which are parity invariant.

From a non-relativistic point of view this truncated theory contains all real world fluids that have ever been encountered. This begs the question - should we assume our relativistic theories are parity invariant too and indeed it is consistent to truncate? Surprisingly the answer is no (due to anomalies) and we shall demonstrate in later chapters that broken parity invariance also has previously unpredicted con-

sequences for certain non-relativistic fluids. For now let's review the ideas behind the calculation of [60].

We begin by describing the general anomaly structure relevant to our setup. For the moment let's consider an arbitrary gauge algebra (and not just the collection of $U(1)$ charges we shall often truncate to). We can extract generators of these charges from our field strength and write the field strength in form notation as

$$F = \frac{1}{2}F^I_{\mu\nu}T_I dx^\mu \wedge dx^\nu \,, \tag{2.1.29}$$

where $T^I$ are the generators of our algebra. Using the Hodge dual operator $*$ the current conservation equation can, in absence of anomalies be written,

$$\mathrm{d}{*}J_I = 0 \,.$$

On the grounds of gauge invariance we expect that in the presence of anomalies the charge conservation equation in even dimensions becomes

$$\mathrm{d}{*}J_I = c_I{}^{J\ldots K}\mathrm{Tr}\left(F_J \wedge \ldots \wedge F_K\right) \tag{2.1.30}$$

where $c_I{}^{J\ldots K}$ is a constant dependent on the algebra.

Let's now examine the entropy current. Unlike in the previous case, where there were no anomalies, the Landau frame condition allows us to write

$$u^\mu \nabla_\mu q_I = -q_I \nabla_\mu u^\mu - \nabla_\mu \xi^\mu_I - C^{\text{anomalies}}_I \tag{2.1.31}$$

where the last term is an anomaly dependent correction coming from Eq. (2.1.30). Positive divergence of the entropy current then implies that

$$\begin{aligned} 0 &\leq \nabla_\mu s^\mu \\ &\leq -\frac{\tau_{\mu\nu}}{T}\nabla^\mu u^\nu - \xi^I_\mu\left(\nabla^\mu\left(\frac{\mu_{qI}}{T}\right) + E^I_\mu\right) + \frac{\mu_{qI}}{T}C^{\text{anomalies}}_I \\ &\quad + \nabla_\mu\left\{\xi^\mu_s + \xi^\mu_I\frac{\mu_q{}^I}{T}\right\} \,. \end{aligned}$$

The correction to the entropy current is defined as before, Eq. (2.1.26), and we are left with

$$0 \leq -\frac{\tau_{\mu\nu}}{T}\nabla^\mu u^\nu - \xi^I_\mu\left(\nabla^\mu\left(\frac{\mu_{qI}}{T}\right) + E^I_\mu\right) + \frac{\mu_{qI}}{T}C^{\text{anomalies}}_I \,.$$

As mentioned above when there are no anomalies we can pick $\tau_{\mu\nu}$ and $\xi_I^\mu$ to make each of the remaining terms a positive square. However, when we do have anomalies, the last term can have either sign and for suitable choices of the electric and magnetic fields can overwhelm the first two terms.

To see the resolution to this problem note that under a parity transformation we have

$$\begin{aligned} \Pi_\mu^\alpha \nabla_\alpha u_\nu &\rightarrow \Pi_\mu^\alpha \nabla_\alpha u_\nu \ , \\ E_\mu^I &\rightarrow -E_\mu^I \end{aligned}$$

which in turn implies that

$$\begin{aligned} \tau_{\mu\nu} &\rightarrow \tau_{\mu\nu} \ , \\ \xi_\mu^I &\rightarrow -\xi_\mu^I \end{aligned}$$

to maintain positivity of the divergence of the entropy current in the absence of anomalies. In the case where there are anomalies the fact that the last term can overwhelm the first two, and changes sign under parity flips, indicates that something is missing from our derivative expansion if both $\tau_{\mu\nu}$ and $\xi_\mu^I$ have the definite parity properties that they had in the anomaly-free case.

We shall return to this issue later when we consider charged, spatially isotropic hydrodynamics at first order in derivatives. It is sufficient now for us to simply remember that we not make parity assumptions when simplifying our coefficients $c$. As such we should make note of the parity violating tensor structure that exists in spatially isotropic systems. In $d$ dimensions there exists an $d$-index $\epsilon$ symbol. We can decompose it in terms of our spatial projector and velocities as:

$$\epsilon_{\mu_1 \ldots \mu_d} = -d! u_{[\mu_1} \Sigma_{\mu_2 \ldots \mu_d]} \ . \tag{2.1.32}$$

The object $\Sigma$ is entirely transverse and totally antisymmetric and we shall later find that it enters our derivative expansion in such a way as to deal with the above anomaly problem.

### 2.1.6 The universal sector at first order

In this subsection we shall consider uncharged fluids, without assumptions about the manifold other than that it is weakly curved, in the absence of background fields that may break $SO(d-1)$ invariance at zeroth or first order. This is clearly the simplest truncation of our general expansion.

#### The equations of fluid dynamics

In the case that there is no charge we are left only with the stress tensor as the charge current vanishes. Our general derivative correction reduces to:

$$\tau_{\mu\nu} = \left(c^{(1)}\right)^{\sigma_1\sigma_2}_{\mu\nu} \nabla_{\sigma_3} u_{\sigma_4} - \left(c^{(1)}\right)^{\sigma_2}_{\mu\nu} u^{\sigma_1} \nabla_{\sigma_1} u_{\sigma_2} \,. \tag{2.1.33}$$

At first order $c^{(1)}$ is locally a constant in the derivative expansion. We note that even if the boundary metric is curved, on the condition it is weakly curved with respect to a formal derivative expansion parameter, $\epsilon$, we can pick a local coordinate system at each point in which Lorentz symmetry is manifest up to second order in derivatives

$$\gamma_{\mu\nu}(x) = \eta_{\mu\nu} - \epsilon^2 \frac{1}{3} R_{\beta\mu\alpha\nu}(x_0) x^\alpha x^\beta + \dots \,.$$

As the $c$s are constants at this order and we have spatial isotropy the current corrections can only be formed from the generalised Kronecker $\delta$s (which contain $\epsilon^{\mu_1\cdots\mu_d}$) contracted with spatial projectors. This is because these are the only non-trivial numerical tensors to have the same value in all frames. It then makes sense to decompose our conserved current corrections into a symmetric traceless and a trace part using the angular brackets defined earlier (see Eq. (1.2.36))

$$\tau_{\mu\nu} = \tau_{\langle\mu\nu\rangle} + \frac{1}{d-1} \Pi_{\mu\nu} \tau^\lambda_\lambda$$

where

$$\tau_{\langle\mu\nu\rangle} = \left(c^{(1)}\right)^{\sigma_1\sigma_2}_{\langle\mu\nu\rangle} \nabla_{\sigma_1} u_{\sigma_2} - \left(c^{(1)}\right)^{\sigma_2}_{\langle\mu\nu\rangle} u^{\sigma_1} \nabla_{\sigma_1} u_{\sigma_2} \,.$$

For $d \geq 3$ the only objects we can construct that transform correctly under rotations and are locally constants are

$$\begin{aligned}\left(c^{(1)}\right)^{\sigma_1\sigma_2}_{\langle\mu\nu\rangle} &= -2\eta\delta^{\sigma_3}_{\langle\mu}\delta^{\sigma_4}_{\nu\rangle}\Pi^{\sigma_1}_{\sigma_3}\Pi^{\sigma_2}_{\sigma_4} \\ &= -2\eta\Pi^{\sigma_1}_{\langle\mu}\Pi^{\sigma_2}_{\nu\rangle} \ , \\ \left(c^{(1)}\right)^{\sigma_2}_{\langle\mu\nu\rangle} &= 0 \ ,\end{aligned}$$

where $\eta$ is a transport coefficient called the shear viscosity. As for the trace part we simply write down the most general linear combination of completely contracted derivative corrections

$$\tau^{\lambda}_{\lambda} = (d-1)\left[-\zeta\nabla_{\lambda}u^{\lambda}\right] \ , \tag{2.1.34}$$

where $\zeta$ is the bulk viscosity and we have extracted a dimension dependent coefficient coming from the trace of the spatial projector. We can write the generic derivative of a fluid velocity as

$$\nabla_{\mu}u_{\nu} = -a_{\mu}u_{\nu} + \sigma_{\mu\nu} + \omega_{\mu\nu} + \frac{1}{d-1}\theta\Pi_{\mu\nu} \tag{2.1.35}$$

where

$$\begin{aligned}\theta = \nabla_{\mu}u^{\mu} \quad &, \quad a^{\mu} = u^{\nu}\nabla_{\nu}u^{\mu} \ , \\ \sigma_{\mu\nu} &= \nabla_{\langle\mu}u_{\nu\rangle} \ , \\ \omega_{\mu\nu} &= \Pi_{\mu}{}^{\alpha}\Pi_{\mu}{}^{\beta}\nabla_{[\alpha}u_{\beta]} \ .\end{aligned}$$

This finally leaves the derivative correction to the SEM tensor as

$$\tau_{\mu\nu} = -2\eta\sigma - \zeta\Pi_{\mu\nu}\theta \ . \tag{2.1.36}$$

The form of this SEM tensor correction is generic to the hydrodynamics of all spatially isotropic field theories where the energy density depends only on the temperature. The precise values of $\eta$ and $\zeta$ are our model dependent parameters mentioned above and depend on the specific nature of the fluid.

Now that we have the constitutive relation for the SEM tensor we plug it into the SEM tensor conservation equations and project parallel and perpendicular to

the fluid motion respectively. The resultant equations of motion are at most second order in derivatives and have the form:

$$\begin{aligned} u^\mu \nabla_\mu \varepsilon &= -(\varepsilon + P)\,\theta + 2\eta\sigma^2 + \zeta\theta^2 \\ \nabla_\alpha^\perp P &= -(\varepsilon + P)\,a_\alpha + 2\eta\nabla_\perp^\nu \sigma_{\alpha\nu} - 2\eta\sigma^2 u_\alpha + 2\sigma_\alpha{}^\nu \nabla_\nu^\perp \eta \\ &\quad + \zeta\nabla_\alpha^\perp\theta + \zeta a_\alpha\theta + \theta\nabla_\alpha^\perp\zeta \end{aligned}$$

While it appears that the second equation is not transverse due to the presence of $u_\alpha$ when acting on this equation with $u^\alpha$ we find

$$0 = u^\alpha \nabla^\nu \sigma_{\alpha\nu} + \sigma^2 \tag{2.1.37}$$

which can be shown to be true using Leibnitz and transverseness of $\sigma_{\alpha\nu}$.

### Entropy currents

To second order in derivatives, using Eq. (2.1.36), it can be readily shown that

$$\nabla_\mu J_s{}^\mu = \frac{2\eta}{T}\sigma^2 + \frac{\zeta}{T}\theta^2$$

where the dissipative correction to the entropy current is zero. The constraints on our transport coefficients are now obvious, $\eta, \zeta \geq 0$.

### Propagation of disturbances

In chapter 1 we interpreted the linearised dispersion relations we found for perturbations of a boosted black brane as hydrodynamic modes. We now seek to demonstrate that linearised disturbances of the effective description we have written down in this section do indeed contain the quadratically dispersing mode found in chapter 1. For a background metric with a flat time direction

$$ds^2 = -dt^2 + \gamma_{ij}(x)dx^i \otimes dx^j$$

we can calculate the speed of sound for our system by linearising

$$\begin{aligned} u_\mu dx^\mu &= -dt + \epsilon^2 v_i(\epsilon x)dx^i \\ (\varepsilon, P, \eta, \zeta) &= \left(\varepsilon_{(0)}, P_{(0)}, \eta_{(0)}, \zeta_{(0)}\right) + \epsilon^2\left(\delta\varepsilon(\epsilon x), \delta P(\epsilon x), \delta\eta(\epsilon x), \delta\zeta(\epsilon x)\right) \\ (\theta, a, \sigma, \omega) &= \epsilon^3 \Bigg( \nabla_i v^i(\epsilon x), \partial_t v^i(\epsilon x)dx^i, \nabla_{(i}v_{j)}(\epsilon x)dx^i \otimes dx^j - \frac{\nabla_i v^i(\epsilon x)}{d-1}\gamma, \\ &\qquad \nabla_{[i}v_{j]}(\epsilon x)dx^i \otimes dx^j \Big)\ , \end{aligned} \tag{2.1.38}$$

where local variations of the thermodynamic quantities are second order in derivatives as local thermal equilibrium holds. This additionally implies that variations in the velocity must also be second order in $\epsilon$. The resulting equations to order $\epsilon^4$ are:

$$\begin{aligned} 0 &= \epsilon^3 \left[\partial_t \delta\varepsilon(\epsilon x) + \left(\varepsilon_{(0)} + P_{(0)}\right)\theta\right] + \mathcal{O}^5(\epsilon) && (2.1.39) \\ 0 &= \epsilon^3 \left[\nabla_i \delta P(\epsilon x) + \left(\varepsilon_{(0)} + P_{(0)}\right)\partial_t v_i(\epsilon x)\right] \\ &\quad + \epsilon^4 \left[-\eta_{(0)}\left(\nabla^j \nabla_i v_j(\epsilon x) + \nabla^2 v_i(\epsilon x)\right) + \frac{2}{d-1}\eta_{(0)}\nabla_i\theta - \zeta_{(0)}\nabla_i\theta\right] \\ &\quad + \mathcal{O}^5(\epsilon)\,. && (2.1.40) \end{aligned}$$

After some tedious manipulation the equation of motion for the pressure fluctuation becomes

$$\begin{aligned} 0 &= \epsilon^4 \left[\nabla^2 \delta P(\epsilon x) - \left(\frac{\delta\varepsilon_{(0)}}{\delta P_{(0)}}\right)\partial_t^2 \delta P(\epsilon x)\right] \\ &\quad + \epsilon^5 \left[\frac{2(d-2)}{d-1}\eta_{(0)} + \zeta_{(0)}\right]\left(\frac{\delta\varepsilon_{(0)}}{\delta P_{(0)}}\right)\frac{\partial_t \nabla^2 \delta P(\epsilon x)}{\left(\varepsilon_{(0)} + P_{(0)}\right)} \\ &\quad + \mathcal{O}^6(\epsilon) \end{aligned}$$

where we have assumed a flat spatial metric. We can replace the covariant derivatives with partials and perform a Fourier expansion

$$\delta P\left(\epsilon t, \epsilon \mathbf{x}\right) = \int \frac{d^d k}{(2\pi)^d} \delta P\left(\omega, \mathbf{k}_\parallel\right) \exp\left[i\epsilon\left(-\omega t + \mathbf{k}\cdot\mathbf{x}\right)\right]$$

where $\mathbf{k}_\parallel \cdot \mathbf{v} = \|\mathbf{v}\| \left\|\mathbf{k}_\parallel\right\|$. The fact that only wave-vectors parallel to the fluid velocity give variations of the pressure follows from Eq. (2.1.39) where for non-zero energy density variations (and thus non-zero pressure variations) the expansion $\theta$ must be non-zero. For a mode whose wave-vector is entirely perpendicular to the fluid velocity, when moving to Fourier space, the expansion is zero. The equation of motion for our pressure wave becomes

$$\begin{aligned} 0 &= \epsilon^4 \left[-\mathbf{k}_\parallel^2 \delta P + \left(\frac{\delta\varepsilon_{(0)}}{\delta P_{(0)}}\right)\omega^2 \delta P\right] \\ &\quad + i\epsilon^5 \left[\frac{2(d-2)}{d-1}\eta_{(0)} + \zeta_{(0)}\right]\left(\frac{\delta\varepsilon_{(0)}}{\delta P_{(0)}}\right)\frac{\omega \mathbf{k}_\parallel^2 \delta P}{\left(\varepsilon_{(0)} + P_{(0)}\right)}\,. \end{aligned}$$

Solving this equation in the limit of small $k$ we find the result of [75] namely

$$\begin{aligned} \omega &= \sqrt{\frac{\delta P_{(0)}}{\delta\varepsilon_{(0)}}}\left\|\mathbf{k}_\parallel\right\| - \frac{i\epsilon}{2\left(\varepsilon_{(0)} + P_{(0)}\right)}\left[\frac{2(d-2)}{d-1}\eta_{(0)} + \zeta_{(0)}\right]\left\|\mathbf{k}_\parallel\right\|^2 \\ &\quad + \mathcal{O}^3\left(k_\parallel\right) && (2.1.41) \end{aligned}$$

and we see that the limit of small $\mathbf{k}_{\parallel}$ is concomitant with the limit of small $\omega$. This is a linear dispersion relation which, given our intuition about hydrodynamics, we can interpret as a sound mode. It is a simultaneous fluctuation of $T^{tt}$, $T^{ii}$ and the longitudinal component $T^{ti}$. There is a second type of fluctuation given by considering fluctuations of the transverse component of $T^{ti}$ only. As noted above we will need to set variations of the pressure and energy density to zero. The equations of motion for such fluctuations are

$$\begin{aligned} 0 &= \epsilon^3\theta + \mathcal{O}^5(\epsilon)\,. \\ 0 &= \epsilon^3\left[\left(\varepsilon_{(0)} + P_{(0)}\right)\partial_t v_i(\epsilon x)\right] + \epsilon^4\left[-\eta_{(0)}\nabla^2 v_i(\epsilon x)\right] + \mathcal{O}^5(\epsilon)\,. \end{aligned}$$

where we have used the first equation to simplify away terms in the second. Taking a flat spatial metric, moving to momentum space once more and solving for the dispersion relation we find

$$\omega = -i\epsilon\left(\frac{\eta_{(0)}}{\varepsilon_{(0)} + P_{(0)}}\right)\|\mathbf{k}_{\perp}\|^2 + \mathcal{O}^3\left(k_{\perp}\right) \tag{2.1.42}$$

with $\mathbf{k}_{\perp}\cdot\mathbf{v} = 0$ from $\theta = 0$. On choosing $d = 4$ both Eqs. (2.1.41) and (2.1.42) match the values given in [30]. Using $\eta_{(0)}/s_{(0)} = 1/4\pi$ for a two derivative gravity model (Eq. (1.2.41)) we have a prediction for the shear viscosity of chargeless fluids dual to such models namely

$$\eta_{(0)} = \frac{1}{4\pi}\left(\varepsilon_{(0)} + P_{(0)}\right) \tag{2.1.43}$$

which we shall demonstrate to be the case in the second half of this chapter.

### Conformal fluids

Let our boundary manifold be a conformal manifold and consider the derivative of the fluid velocity under a Weyl transformation. We find that

$$\tilde{\nabla}_{\mu}\tilde{u}_{\nu} = e^{\phi}\left[\nabla_{\mu}u_{\nu} + \left(\partial_{\mu}\phi\right)u_{\nu} - \left(u_{\nu}\partial_{\mu}\phi + u_{\mu}\partial_{\nu}\phi - \gamma_{\mu\nu}u^{\alpha}\partial_{\alpha}\phi\right)\right]$$

thus it does not transform as a weighted tensor under a scaling transformation. Given that the boundary is a conformal manifold this suggests that the derivative as written is not a natural object in this manifold. This places constraints on transport

coefficients such that the combination of derivatives appearing in our SEM tensor transform correctly under a scaling.

Consider first the expansion $\theta$ which transforms as

$$\tilde{\nabla}_\lambda \tilde{u}^\lambda = e^{-\phi}\left[\nabla_\mu u^\mu + (d-1)u^\mu \partial_\mu \phi\right] .$$

In a conformal theory $T^\lambda_\lambda \equiv 0$ so our bulk viscosity must be zero. Additionally we have

$$\tilde{\nabla}_{\langle\mu}\tilde{u}_{\nu\rangle} = e^{-\phi}\nabla_{\langle\mu}u_{\nu\rangle} . \tag{2.1.44}$$

So we can have a non-zero shear viscosity under the condition that, when one performs a scaling, $\eta \to e^{-(d+1)\phi}\eta$. Finally therefore, for a fluid which has an underlying conformal symmetry in its microscopics, our dissipative correction at first order in derivatives is

$$\tau_{\mu\nu} = -2\eta\sigma_{\mu\nu} . \tag{2.1.45}$$

### 2.1.7 Charged fluid dynamics at first order

In this subsection we shall consider fluids in the absence of electric and magnetic fields but with a non-zero charge. This is a natural step to take given our discussion in the previous subsection and these fluids and their duals will be important in later chapters.

#### The equations of fluid dynamics

In the case that there are no $E$ or $B$ fields there can be no polarisation or magnetisation so our general derivative correction reduces to:

$$\begin{aligned}
\tau_{\mu\nu} &= \left(c^{(1)}\right)^{\sigma_1\sigma_2}_{\mu\nu}\nabla_{\sigma_3}u_{\sigma_4} + \left(c^{(1)}\right)^{\sigma_2}_{\mu\nu}u^{\sigma_1}\nabla_{\sigma_1}u_{\sigma_2} + \left(c^{(2)}_I\right)^{\sigma_1}_{\mu\nu}\nabla_{\sigma_1}q^I \\
\xi^\mu_I &= \left(c^{(4)}_I\right)^{\mu\sigma_1\sigma_2}\nabla_{\sigma_1}u_{\sigma_2} + \left(c^{(5)}_{IJ}\right)^{\mu\sigma_1}\nabla_{\sigma_1}q^J + \left(c^{(5)}_{IJ}\right)^{\mu}u^{\sigma_1}\nabla_{\sigma_1}q^J + \left(c^{(6)}_I\right)^{\mu\sigma_1}\nabla_{\sigma_1}T
\end{aligned}$$

Again, at first order all the $c$'s are locally constants in the derivative expansion. As they are constants and we have spatial isotropy ($SO(d-1)$ invariance) they can

only be formed from the numerical relative tensors $\delta^{\mu}_{\nu}$ and $\epsilon^{\mu_1\cdots\mu_d}$. Hence:

$$\begin{aligned}
\left(c^{(1)}\right)^{\sigma_1\sigma_2}_{\langle\mu\nu\rangle} &= 2\eta\Pi^{\sigma_1}_{\langle\mu}\Pi^{\sigma_2}_{\nu\rangle} \\
\left(c^{(4)}_I\right)^{\mu\sigma_1\sigma_2} &= \begin{cases} 0, & d=3 \\ -\mho_I\Sigma^{\mu\sigma_1\sigma_2}, & d=4 \\ 0, & d>4 \end{cases} \\
\left(c^{(5)}_{IJ}\right)^{\mu\sigma_1} &= \begin{cases} -\kappa_{qIJ}\Pi^{\mu\sigma_1} - \tilde{\kappa}_{qIJ}\Sigma^{\mu\sigma_1}, & d=3 \\ -\kappa_{qIJ}\Pi^{\mu\sigma_1}, & d>3 \end{cases} \\
\left(c^{(6)}_I\right)^{\mu\sigma_1} &= \begin{cases} -\gamma_I\Pi^{\mu\sigma_1} - \tilde{\gamma}_I\Sigma^{\mu\sigma_1}, & d=3 \\ -\gamma_I\Pi^{\mu\sigma_1}, & d>3 \end{cases}
\end{aligned}$$

with any non-displayed coefficients being identically zero.

As before to determine $\tau^{\lambda}_{\lambda}$ we simply take the most general linear combination of scalars that we can produce from objects containing a single derivative in our permitted operators. Our conserved SEM and charge current corrections then become

$$\begin{aligned}
\tau_{\mu\nu} &= -2\eta\nabla_{\langle\mu}u_{\nu\rangle} + \zeta\theta\Pi_{\mu\nu} \\
J^{\mu}_I &= q_I u^{\mu} - \kappa_{qIJ}\Pi^{\mu\sigma}\nabla_{\sigma}q^J - \gamma_I\Pi^{\mu\sigma}\nabla_{\sigma}T \\
&\quad - \begin{cases} \tilde{\kappa}_{qIJ}\Sigma^{\mu\sigma_1}\nabla_{\sigma_1}q^J + \tilde{\gamma}_I\Sigma^{\mu\sigma_1}\nabla_{\sigma_1}T, & d=3 \\ \mho_I\Sigma^{\mu\sigma_1\sigma_2}\omega_{\sigma_1\sigma_2}, & d=4 \\ 0, & d>4 \end{cases}
\end{aligned}$$

Just as before the equations of motion obeyed by the velocity fields can be obtained by plugging these constitutive relations into their respective conservation equations.These corrections have been derived many times in the literature. For a derivation from a slightly different perspective see [29].

### Entropy and heat currents

In the absence of anomalies the divergence of the entropy current to second order in derivatives was given by Eq. (2.1.25). As the form of the SEM tensor has not changed from the uncharged case the only new piece we need to determine comes from the dissipative term in the charge current. The divergence of the entropy

current is

$$\nabla_\mu J_s{}^\mu = \frac{2\eta}{T}\sigma^2 + \frac{\zeta}{T}\theta^2 - \xi^I_\mu \nabla^\mu \left(\frac{\mu_{qI}(q,T)}{T}\right) . \tag{2.1.46}$$

The simplest way to ensure that this divergence is positive definite is to require that

$$\xi^\mu_I = -D_{IJ}\Pi^{\mu\nu}\nabla_\nu \left(\frac{\mu_{qI}(q,T)}{T}\right) , \tag{2.1.47}$$

where $D_{IJ}$ is a positive definite matrix. The transport coefficients $\kappa_{IJ}$ and $\gamma_I$ are then related to $D_{IJ}$ by

$$\kappa_{IJ} = D_{IK}\left(\frac{\partial \mu_q{}^K}{\partial q^J}\right)_T , \tag{2.1.48}$$

$$\gamma_I = D_{IJ}\left(\left(\frac{\partial \mu_q{}^J}{\partial T}\right)_q - \frac{\mu_q{}^J}{T}\right) . \tag{2.1.49}$$

For our scale invariant fluids we shall find that these relationships are indeed obeyed. Finally therefore the divergence of the entropy current in the absence of anomalies has the form

$$\nabla_\mu J_s{}^\mu = \frac{2\eta}{T}\sigma^2 + \frac{\zeta}{T}\theta^2 + D^{IJ}\nabla_\perp\left(\frac{\mu_{qI}}{T}\right)\cdot\nabla_\perp\left(\frac{\mu_{qJ}}{T}\right) \tag{2.1.50}$$

which is manifestly positive definite.

Now we turn to the issue of charge anomalies. Above we included parity violating terms in our expansion of the charge current in anticipation of the fact that they may be needed to ensure the entropy current divergence is positive definite. Traditional approaches to determining the corrections to $J^\mu$ such as Israel-Stewart theory [76,77] typically assume parity invariance and thus set the coefficients of parity violating terms to zero. As was first shown in [60], and we have argued above, consideration of anomalies leads to their presence. For the particular case of $d = 4$ as shown in [60] if we add to the parity invariant expression for the charge current correction

$$\gamma_I \Pi^{\mu\sigma}\nabla_\sigma T + \kappa_{qIJ}\Pi^{\mu\sigma}\nabla_\sigma q^J \tag{2.1.51}$$

a parity violating piece then this piece must have exactly the form

$$\mho_I \Sigma^{\mu\sigma_1\sigma_2}\nabla_{\sigma_1} u_{\sigma_2} , \tag{2.1.52}$$

where $\mathfrak{V}_I$ is completely dictated by anomalies. It is relatively straightforward to see this by applying to the fluid an electromagnetic field which is everywhere order one in derivatives. Repeating the above process while allowing for electric and magnetic fields to appear leads to corrections to the dissipative terms. The first order corrections are

$$\xi_{\mathrm{s}}{}^{\mu} = -\frac{\mu_q{}^I}{T}\xi_I^{\mu} + D_B^I B_I^{\mu} + D_{\omega}\Sigma^{\mu\nu\sigma}\omega_{\nu\sigma} \,, \tag{2.1.53}$$

$$\xi_I^{\mu} = -D_{IJ}\Pi^{\mu\nu}\left[\nabla_{\nu}\left(\frac{\mu_q{}^J}{T}\right) + \frac{1}{T}E_{\nu}^J\right] - \mathfrak{V}_I\Sigma^{\mu\nu\sigma}\omega_{\nu\sigma} - \tilde{\mathfrak{V}}_I{}^J B_J^{\mu} \tag{2.1.54}$$

with $D_B^I$, $D_{\omega}$, $\mathfrak{V}_I$ and $\tilde{\mathfrak{V}}_I{}^J$ as yet undetermined and we have defined

$$B_I^{\mu} = -\Sigma^{\mu}{}_{\sigma_1\sigma_2}F_I^{\sigma_1\sigma_2} \,.$$

The new dissipative coefficients take account of an electromagnetic field at first order in derivatives and possible parity violating corrections to the entropy and charge currents. The charge conservation equation is now:

$$\partial_{\mu}J_I^{\mu} = C_{IJK}E_{\mu}^J\left(B^K\right)^{\mu} \,. \tag{2.1.55}$$

Hence the positive definite divergence of the entropy current implies

$$\begin{aligned} 0 \leq & -\left[\frac{D_I^B}{\varepsilon+P}q^I\delta^{JK} - \frac{1}{T}\tilde{\mathfrak{V}}^{JK} + C^{IJK}\frac{\mu_{qI}}{T}\right]E_J^{\mu}\left(B_K\right)_{\mu} \\ & -\left[\frac{2D_{\omega}}{\varepsilon+P}q^I - \frac{1}{T}\mathfrak{V}^I - 2D_B^I\right]\left(E_I\right)_{\mu}\Sigma^{\mu\sigma_1\sigma_2}\omega_{\sigma_1\sigma_2} \\ & +B_I^{\mu}\left[\nabla_{\mu}D_B^I - \frac{D_B^I}{\varepsilon+P}\nabla_{\mu}^{\perp}P + \nabla_{\mu}\left(\frac{\mu_q{}^J}{T}\right)\tilde{\mathfrak{V}}_J{}^I\right] \\ & +\Sigma^{\mu\nu\sigma}\omega_{\nu\sigma}\left[\nabla_{\mu}D_{\omega} - \frac{2D_{\omega}}{\varepsilon+P}\nabla_{\mu}^{\perp}P + \nabla_{\mu}\left(\frac{\mu_{qI}}{T}\right)\mathfrak{V}^I\right] \,. \end{aligned}$$

Attempting to satisfy the strict equalities for all the terms which are not squares completely fixes expressions for $D_{\omega}$, $D_B$, $\mathfrak{V}^I$ and $\tilde{\mathfrak{V}}_J^I$ in terms of the thermodynamic variables and the anomaly coefficients $C^{IJK}$ as demonstrated in [60].

### Propagation of disturbances

We consider the additional variations

$$\begin{aligned} \left(q^I, \gamma_I, \kappa_{qI}, \mathfrak{V}_I\right) = & \left(q^I_{(0)}, \gamma_I^{(0)}, \kappa_{qI}{}^{(0)}, \mathfrak{V}_I^{(0)}\right) \\ & +\epsilon^2\left(\delta q^I(\epsilon x), \delta\gamma_I(\epsilon x), \delta\kappa_{qI}(\epsilon x), \delta\mathfrak{V}_I(\epsilon x)\right) \end{aligned} \tag{2.1.56}$$

on top of those displayed in Eq. (2.1.38). As the constitutive relation for the SEM tensor has precisely the form it had in the chargeless case the changes to the equations of motion for a linearised disturbance are minimal and simply take into account the dependence of the energy density upon the charge. Choosing $\delta\epsilon = \delta P = 0$ we get the same shear mode that we had before Eq. (2.1.42).

Obtaining the equations of motion for the sound modes is tedious but straightforwardly follows from the method used in the uncharged case although we must be careful because charge and pressure fluctuations couple. As such we shall not do it here. However, to compare to charge diffusion as considered in chapter 1 we should look at charge variations in the absence anomalies and background charge. We need to solve the charge conservation equation

$$0 \;=\; \epsilon^3\left(-i\omega\delta q_I\right)+\epsilon^4\left(+\kappa_{qIJ}\mathbf{k}_\perp^2\delta q^J\right)+\mathcal{O}^6(\epsilon)\;, \tag{2.1.57}$$

where we have set pressure variations to zero. For a single $U(1)$ charge there will be one new dispersive mode where

$$\omega \;=\; -i\epsilon\kappa_q\mathbf{k}_\perp^2+\mathcal{O}^3\left(\mathbf{k}_\perp\right)\;. \tag{2.1.58}$$

This has exactly the form for charge diffusion found from the linearised perturbation analysis of the bulk gauge field in Sec. 1.2.1 with $\kappa_q = \left(\frac{d}{4(d-2)\pi T}\right)$. When there are multiple charges even in this simplified regime there will be a different decay channel corresponding to each charge with $\kappa_{qIJ}$ allowing them to mix.

### Conformality

So far we have not used conformality. Just as before scale invariance implies that $T^\lambda_\lambda \equiv 0$ and thus $\zeta \equiv 0$. Additionally under a Weyl transformation

$$\begin{aligned}
\tilde{\tilde{\kappa}}_{qIJ} = e^{\phi}\tilde{\kappa}_{qIJ},\ \tilde{\tilde{\gamma}}_I = \tilde{\gamma}_{IJ} \quad &, \quad d=3\\
\tilde{\mathfrak{V}}_I = e^{2\phi}\mathfrak{V}_I \quad &, \quad d=4\\
\tilde{\kappa}_{qIJ} = e^{(d+1)\phi}\kappa_{qIJ},\ \tilde{\gamma}_I = e^{3\phi}\gamma_I \quad &, \quad d\geq 3
\end{aligned}$$

and for all dimensions considered

$$(d-1)\kappa_{qIJ}q^J+\gamma_I T=0 \tag{2.1.59}$$

which is consistent with what we found from positivity of the entropy current divergence. For $d = 3$ there is an additional relationship of the form

$$2\tilde{\kappa}_{q_{IJ}} q^J + \tilde{\gamma}_I T = 0 \, . \tag{2.1.60}$$

### 2.1.8 Non-zero $E$ and $B$ fields, higher derivatives and forced fluids

In this thesis we shall not have cause to investigate spacetimes dual to field theories with non-zero electric $E$ and magnetic $B$ fields at zeroth order. As such we shall not dwell on constructing their effective field theories. Nonetheless it would be remiss of us not to remark on how one might go about writing them down. The first thing to note is that in the above we have used $SO(d-1)$ symmetry extensively to simplify the resultant expressions. As such when we have $E$ and $B$ fields, because they pick out spatial directions, our expressions will contain more terms.

The case of non-zero $B$ fields were considered in [65,78,79]. For $B$ fields that scale appropriately in the wavenumber the effect upon the hydrodynamics is mild and the fluid velocity and dissipative coefficients receive $B$ dependent corrections [78]. However, when $B$ is finite in the large $T$ limit its effect on the dispersion relations is drastic [65, 79] with the loss of a sound mode and the introduction of sub-diffusive modes. The case of non-zero electric fields has also been considered in literature and the interested reader is referred to [68].

As for higher order derivative terms the process by which one can proceed should now be clear. For example, at two derivatives, the most general combination of objects that contain two derivatives is written down. If the theory is spatially isotropic and conformal then these can be reduced further. For the uncharged case see [32] and for the case of a charged fluid in absence of electric and magnetic fields see [62].

One subtlety occurs if our background manifold is curved. At second order terms dependent on the background curvature can appear in our expansion. Unfortunately there is insufficient space to discuss the extension of the above procedure

to these cases here. It seems reasonably clear however that we must add terms to our expansions that are constructed from curvature invariants of the boundary. For an uncharged fluid the procedure is discussed in detail in [63]. To the author's knowledge the extension to a charged fluid living on a curved background has not been done in literature.

## 2.2 A fluid dual to asymptotically, local AdS spaces

We begin this section by first describing the thermodynamics of a charged black hole in AdS. This will be the holographic dual to a strongly coupled thermal field theory upon which we shall build our hydrodynamics. We shall then show how by a change of coordinates it can be broken down into the same gauge choices we employed in our discussion of linearised gravitational perturbations. Noting the similarity between the linearised and non-linear calculations we shall then lay out the general structure used in the hydrodynamic analysis of these spacetimes. Much of this work is based on [32, 62].

### 2.2.1 Background and thermodynamics

For the moment let's consider a bottom-up approach to finding the duals to charged, high temperature ($T \gg \mu$) strongly coupled boundary field theories. As mentioned in chapter 1 we isolate the features that are of interest to us - namely the existence of a conserved charge and a non-zero temperature. Thus we look for a bulk gravitational theory that contains a black hole and a $U(1)$ gauge field. A simple action which contains the gravitational and gauge field content necessary for such a bulk setup would be

$$\mathcal{S} = \frac{1}{2\kappa_{d+1}^2} \int d^{d+1}x \sqrt{-g} \left( R_{(d+1)} - 2\Lambda - g_F^2 F^2 \right) \tag{2.2.61}$$

where $g_F$ is the gauge coupling, $\Lambda < 0$ and $F = \mathrm{d}A$ is a $U(1)$ field strength. We have restricted ourselves to a single $U(1)$ charge as it is straightforward to add further $U(1)$ charges but doing so does not provide further theoretical insight. For a well-defined variational principle we should additionally include terms that ensure the

boundary values of the fields remain fixed such as a Gibbons-Hawking term. We note that in consistent truncations from string theory there can be various other fields which must take non-zero values for any uplifts to the full ten-dimensional theory to be valid. One example that will be important for us later is a Chern-Simons term but for now we shall assume this to be zero.

The equations of motion coming from this action are

$$\begin{aligned} R_{MN} + d g_{MN} &= g_F^2 \left( 2F_{MP}F_N^P - \frac{1}{d-1} g_{MN} F^2 \right) , \\ \mathrm{d}{*F} &= 0 \end{aligned}$$

and one solution to this set of equations does indeed describe an electric Reissner-Nordstrøm $\mathrm{AdS}_{d+1}$ black hole with planar horizon

$$ds^2 = \frac{r^2}{\ell^2}\left(-f(r)dt^2 + d\mathbf{x}_{d-2}^2\right) + \frac{\ell^2}{r^2 f(r)} dr^2 , \tag{2.2.62}$$

$$A = \mu_q \left( 1 - \frac{{r_+}^{d-2}}{r^{d-2}} \right) dt , \tag{2.2.63}$$

where

$$f(r) = 1 - \frac{m}{r^d} + \frac{Q^2}{r^{2(d-1)}} , \tag{2.2.64}$$

$m$ is the mass of our black hole and $Q$ its charge which is related to the chemical potential $\mu_q$ by

$$\mu_q = \sqrt{\frac{d-1}{2(d-2)}} \frac{Q}{g_F {r_+}^{d-2}} . \tag{2.2.65}$$

There exist three values of $r$ where the metric (Eq. (2.2.62)) has a singularity, the true singularity at $r = 0$ and the inner and outer black-hole horizons $r_-$ and $r_+$ respectively (where $r_- < r_+$).

We should check that the charged black holes of Eq. (2.2.62) and Eq. (2.2.63) have the correct boundary interpretation. The temperature of these black holes can be calculated by Euclideanising the metric and ensuring the resultant manifold has the topology of a circle as opposed to a cone in Euclideanised time. It is

$$T = \frac{dr_+}{4\pi\ell^2} \left( 1 - \frac{(d-2)Q^2}{d{r_+}^{2d-2}} \right) . \tag{2.2.66}$$

For the background of Eqs. (2.2.62) and (2.2.63) there is no electromagnetic field strength in the boundary, $\tilde{F}_{\mu\nu} \equiv 0$, but there is a non-zero charge current

$$J^t = \frac{2(d-2)\mu_q g_F^2 {r_+}^{d-2}}{\kappa_{d+1}^2 \ell^{d-1}} \tag{2.2.67}$$

which we identify with the boundary charge density $q$. Additionally the boundary SEM tensor has non-zero components

$$T_{tt} = \frac{(d-1)m}{2\kappa_{d+1}^2 \ell^{d+1}} , \tag{2.2.68}$$

$$T_{ij} = \frac{m}{2\kappa_{d+1}^2 \ell^{d+1}} \delta_{ij} , \tag{2.2.69}$$

from which we can identify our energy density and pressure

$$\varepsilon = \frac{(d-1)m}{2\kappa_{d+1}^2 \ell^{d+1}} , \tag{2.2.70}$$

$$P = \frac{m}{2\kappa_{d+1}^2 \ell^{d+1}} . \tag{2.2.71}$$

Note that $\varepsilon = (d-1)P$ as expected for a theory with underlying conformal invariance. The entropy density is also easy to calculate using Bekenstein and Hawking's result and it is given by

$$s = \frac{2\pi}{\kappa_{d+1}^2} \left(\frac{r_+}{\ell}\right)^{d-1} . \tag{2.2.72}$$

As an additional check it is straightforward to show that the first law of thermodynamics, for constant spatial volume, is satisfied. Hence these gravitational duals correspond to boundary field theories with all the ingredients necessary for studying the charged hydrodynamics of strongly coupled field theories.

Before moving on we should justify whether these bottom-up configurations are sensible to consider. We will have at least some evidence in the affirmative if we are able to find an example of a charged black hole from a string theory embedding. We shall now attempt to do this. Consider the following consistent truncation of Type IIB string theory

$$\mathcal{S}_{10} = \frac{1}{2\kappa_{10}^2} \int \left[ (*_{10}1) e^{-2\Phi} \left( R^{(10)} + 4(\partial\Phi)^2 \right) - \frac{1}{4} F_{(5)} \wedge *_{10} F_{(5)} \right] , \tag{2.2.73}$$

where we have chosen the string frame. As usual, self-duality of the RR 5-form must be imposed after variation to complete the specification of the equations of

motion. From [80] consider the following ansatz for solutions to the equations of motion coming from this action

$$\begin{aligned} ds_{10}^2 &= g_{\mu\nu}(x)dx^\mu \otimes dx^\nu \\ &\quad + \left(d\psi + \mathcal{A}_{(1)} - \frac{2}{\sqrt{3}}A_Q\right)^2 + d\Sigma_4^2 \,, \qquad (2.2.74) \\ F_{(5)} &= 2(1+*_{10})\left(d\psi + \mathcal{A}_{(1)} - \frac{2}{\sqrt{3}}A_Q\right) \wedge J_{(2)} \wedge J_{(2)} \\ &\quad -\frac{2}{\sqrt{3}}(1+*_{10})*_5 F_Q \wedge J_{(2)} \,, \qquad (2.2.75) \\ F_Q &= \mathrm{d}A_Q \,, \qquad (2.2.76) \end{aligned}$$

where $*_5$ and $*_{10}$ are the five- and ten-dimensional Hodge stars defined in appendix B and the unusual factors of $-\frac{2}{\sqrt{3}}$ are present so that our $A_Q$ normalisation matches that of [62]. The final two terms of the ten-dimensional metric are the five-dimensional metric on the unit $S^5$ given by a $U(1)$ fibration over $\mathbb{CP}^2$ with a gravi-photon turned on. The one-form $\mathcal{A}_{(1)}$ and the two form $J_{(2)}$ are called the Kähler potential and form respectively and are defined by the following equations:

$$\begin{aligned} J_{(2)} &= \frac{1}{2}\mathrm{d}\mathcal{A}_{(1)} \,, \\ \mathrm{Vol}(\mathbb{CP}^2) &= \frac{1}{2}J_{(2)} \wedge J_{(2)} \,. \end{aligned}$$

Using the five-form Bianchi identify $dF_{(5)} = 0$ we find our choice for the decomposition of the RR 5-form implies that $F_Q$ satisfies the following equation of motion

$$\mathrm{d}*_5 F_Q - 4\kappa_{CS} F_Q \wedge F_Q = 0 \,,$$

where $\kappa_{CS} = -\frac{1}{2\sqrt{3}}$ is the Chern-Simon's coupling of the gauge field $A_Q$. This follows from an action of the form

$$\int \left[F_Q \wedge *_5 F_Q - \frac{8}{3}\kappa_{CS} A_Q \wedge F_Q \wedge F_Q\right]$$

which, upon compactification, we shall use to substitute for the $F_{(5)}$ terms of Eq. (2.2.73). There is one caveat in this replacement and it is that the cosmological constant in the five-dimensional theory receives contributions from the volume form

pieces of Eq. (2.2.75). The correct cosmological constant can be determined by examining the equations of motion.

Also, we now generalise to arbitrary $\kappa_{CS}$ which will clarify the role of the Chern-Simon's term in generating the parity violating coefficients. Compactifying $S^5$ we can set the resultant scalar field associated with the metric, $\sigma$, to be equal to the dilaton $\Phi$. Further, we find $\Phi = 0$ is a consistent solution to the equations of motion and hence our action (2.2.73) reduces to:

$$\mathcal{S}_5 = \frac{1}{2\kappa_5^2}\int\left[\text{vol}_M\left(R^{(5)}+12\right) - 2F_Q\wedge *_5 F_Q + \frac{16\kappa_{CS}}{3}A_Q\wedge F_Q\wedge F_Q\right] \quad (2.2.77)$$

in the Einstein frame. This matches the Einstein-Maxwell action of Eq. (2.2.61) in five dimensions when we introduce a Chern-Simons term and set $g_F = 1$. Hence all the results above apply to the Reissner-Nordstrøm $AdS_5$ solutions that solve the equations of motion from this action and we have found one example of a charged black hole of the type we are interested in that can embedded in string theory.

Finally we note a useful coordinate transformation that can be applied to Eqs. (2.2.62) and (2.2.63) which will be essential in what follows. We note that the form of the metric Eq. (2.2.62) is not manifestly regular on the future horizon where we normally impose infalling conditions on our perturbations. Moving to Eddington-Finkelstein coordinates our five-dimensional charged black hole metric and gauge field become

$$ds^2 = 2dvdr + \frac{r^2}{\ell^2}\left(-f(r)dv^2 + d\mathbf{x}_{d-2}^2\right)\ , \quad (2.2.78)$$

$$A = \mu_q\left(1-\frac{{r_+}^{d-2}}{r^{d-2}}\right)dv\ , \quad (2.2.79)$$

where we have made a gauge transformation in the gauge field to remove a $dr$ piece and set $\ell = 1$. This metric is now manifestly regular on the future horizon. Finally we can use a boost of the boundary coordinates, which does not change the regularity properties of the metric, to write

$$ds^2 = -2u_\mu dx^\mu dr + \frac{r^2}{\ell^2}\left(-f(r)u_\mu u_\nu dx^\mu dx^\nu + d\mathbf{x}_{d-2}^2\right)\ , \quad (2.2.80)$$

$$A = -\mu_q\left(1-\frac{{r_+}^{d-2}}{r^{d-2}}\right)u_\mu dx^\mu\ , \quad (2.2.81)$$

where we choose for $u_\mu$ to be unit normalised. These boosted black brane metrics are a $(d+2)$ parameter family ($d$ from unit normalised, timelike $u^\mu$, one from $T$ and

one from $Q$) that we shall now investigate perturbations of. The fact that this is a $(d+2)$ parameter family is essential as this is exactly the number of independent charged Navier-Stokes equations.

### 2.2.2 A derivative expansion of the metric

We now consider promoting our thermodynamic constants and velocity field to functions of position in the boundary coordinates i.e. $T = T(x)$ and $u_\mu = u_\mu(x)$. We should note that non-zero temperatures and velocities will break the $SO(4,2)$ symmetry in the boundary theory and thus we can loosely identify these promoted fields as Goldstone bosons. The fluid metric with promoted parameters,

$$\begin{aligned} ds^2 &= -2u_\mu(x)dx^\mu dr - r^2 f(m(x),Q(x),r)u_\mu(x)u_\nu(x)dx^\mu dx^\nu \\ &\quad + r^2\Pi_{\mu\nu}(x)dx^\mu dx^\nu \ , \end{aligned} \tag{2.2.82}$$

and other bulk fields generically do not satisfy the supplied bulk equations of motion. To get a better understanding of what is going on let's look at the metric about some point $x_{(0)}$ and construct Riemann normal coordinates $\epsilon x^\mu$ where $\epsilon$ is a formal derivative counting parameter so that in the neighbourhood of this point the boundary metric $\gamma_{\mu\nu}(x_{(0)})$ is locally flat

$$\gamma_{\mu\nu} = \eta_{\mu\nu} - \frac{1}{3}\epsilon^2 R_{\beta\mu\alpha\nu}(x_{(0)})\left(x^\alpha - x^\alpha_{(0)}\right)\left(x^\beta - x^\beta_{(0)}\right) + \dots \ .$$

We note that we still have the freedom to make local Lorentz boosts as the leading term by which we defined this coordinate transformation is invariant under such transformations. Our velocity and thermodynamic fields (we shall later find consistency conditions on exactly which velocity fields we can choose) can be expanded as

$$u^\mu(x) = u^\mu(x_{(0)}) + \epsilon\left(x - x_{(0)}\right)^\nu \partial_\nu u^\mu(x_{(0)}) + \mathcal{O}^2(\epsilon) \ , \tag{2.2.83}$$

$$T(x) = T(x_{(0)}) + \epsilon\left(x - x_{(0)}\right)^\nu \partial_\nu T^\mu(x_{(0)}) + \mathcal{O}^2(\epsilon) \ , \tag{2.2.84}$$

and similarly for the other relevant thermodynamic quantities. Our metric expanded to first order in derivatives about the point $x_{(0)}$ is

$$\begin{aligned} ds^2 &= -2\left(u_\mu^{(0)} + \epsilon\left(x-x_{(0)}\right)^\nu \partial_\nu u_\mu^{(0)}\right) dx^\mu \otimes dr \\ &\quad + \Big(-r^2 f(r) u_\mu^{(0)} u_\nu^{(0)} + r^2 \Pi_{\mu\nu} + 2\epsilon r^2 \left(1-f^{(0)}\right)\left(x-x_{(0)}\right)^\alpha u_{(\mu|}^{(0)} \partial_\alpha u_{|\nu)}^{(0)} \\ &\quad - r^2 \epsilon u_\mu^{(0)} u_\nu^{(0)} \left(x-x_{(0)}\right)^\alpha \left(\frac{\delta f^{(0)}}{\delta m^{(0)}} \partial_\alpha m^{(0)} + \frac{\delta f^{(0)}}{\delta Q^{(0)}} \partial_\alpha Q^{(0)}\right)\Big) dx^\mu \otimes dx^\nu \end{aligned}$$

where we have defined

$$f^{(0)} = f\left(m^{(0)}, Q^{(0)}, r\right) . \tag{2.2.85}$$

We can see that this metric has exactly the same form as Eq. (1.2.35) where we identify

$$u_\mu^{(1)} = \left(x-x_{(0)}\right)^\nu \partial_\nu u_\mu^{(0)} , \tag{2.2.86}$$

$$h^{(0)} = -r^2\left(x-x_{(0)}\right)^\alpha \left(\frac{\delta f^{(0)}}{\delta m^{(0)}} \partial_\alpha m^{(0)} + \frac{\delta f^{(0)}}{\delta Q^{(0)}} \partial_\alpha Q^{(0)}\right) , \tag{2.2.87}$$

$$h^{(1)} = 0 , \tag{2.2.88}$$

$$h_\mu = r^2\left(1-f^{(0)}\right)\left(x-x_{(0)}\right)^\alpha \partial_\alpha u_\mu^{(0)} , \tag{2.2.89}$$

$$h_{\langle\mu\nu\rangle} = 0 . \tag{2.2.90}$$

We note that while the boundary metric is still flat after promotion the SEM tensor at each point $x_{(0)}$ is deformed from the ground state given by setting $u_\mu dx^\mu = -dv$ and $(m = m_{(0)}, Q = Q_{(0)})$. The Einstein equations are not satisfied and as it stands this is not sensible because it is the falloffs to **solutions** of the classical equations of motion in the bulk that correspond to sources and expectation values in the boundary. This suggests that we must add to our metric (2.2.82) an explicit forcing term $r^2 \tilde{h}_{MN} dx^M \otimes dx^N$ which is first order in derivatives so that the bulk

equations of motion are solved. The quantities appearing in our metric become

$$u^{(1)}_\mu = \left(x - x_{(0)}\right)^\nu \partial_\nu u^{(0)}_\mu + \tilde{h}^{(1)} u^{(0)}_\mu \ , \tag{2.2.91}$$

$$h^{(0)} = -r^2 \left(x - x_{(0)}\right)^\alpha \left( \frac{\delta f^{(0)}}{\delta m^{(0)}} \partial_\alpha m^{(0)} + \frac{\delta f^{(0)}}{\delta Q^{(0)}} \partial_\alpha Q^{(0)} \right) + r^2 \tilde{h}^{(0)} \ , \tag{2.2.92}$$

$$h^{(1)} = r^2 \tilde{h}^{(1)} \ , \tag{2.2.93}$$

$$h_\mu = r^2 \left(1 - f^{(0)}\right) \left(x - x_{(0)}\right)^\alpha \partial_\alpha u^{(0)}_\mu + r^2 \tilde{h}_\mu \ , \tag{2.2.94}$$

$$h_{\langle \mu\nu \rangle} = r^2 \tilde{h}_{\langle \mu\nu \rangle} \ , \tag{2.2.95}$$

and we can interpret this metric as a perturbation of the boundary SEM tensor by some forcing $r^2 \tilde{h}_{MN} dx^M \otimes dx^N$. We have employed the gauge choices of Eq. (1.2.34).

Demanding regularity in the interior of the spacetime (and infalling conditions on the future horizon) will fix the deformation of the SEM tensor in terms of the the forcing. We can then search for $r^2 \tilde{h}_{MN} dx^M \otimes dx^N$ such that the bulk equations of motion are satisfied for a vanishing deformation of the boundary metric. An analogous process occurs in the linearised analysis where we search for solutions to the bulk equations in the complex $\omega$ plane attempting to find a quasi-normal mode which is asymptotically close to the ground state when $T$ is large. The ground state $u^\mu(x) = u^\mu_{(0)}$, etc. is of course a solution but we are more interested in nearby solutions which correspond to the system relaxing to global equilibrium.

There exists an additional set of tricks available to us to make the subsequent analysis simpler. First, using a local Lorentz boost with velocity parameter $\bar{v}_i = \frac{\epsilon^2 u^t - 1}{(1 - \epsilon^2 u_i u^i)} u_i$ and $v^i = \epsilon u^i$ in parallel with the scaling

$$\exp\left( - \ln \frac{T(x_{(0)})}{T(x)} \right) \ , \tag{2.2.96}$$

we can bring the velocity and temperature fields respectively into a canonical form about $x_{(0)}$

$$u^\mu = \left( \frac{1}{\sqrt{1 + \epsilon^2 \mathbf{v}^2}}, \frac{\epsilon v^i(x)}{\sqrt{1 + \epsilon^2 \mathbf{v}^2}} \right) \ , \tag{2.2.97}$$

and $T(x_{(0)}) = 1$. To write the fluid velocity in this canonical form we have had to scale the spatial velocity drastically from its finite value $u^i(x)$ to some infinitesimal departure from zero $\epsilon v^i(x)$. This is the origin of some singular behaviour in our

Lorentz boost as $\epsilon \to 0$. The key point to emphasize is that spatial velocity is finite about each point on the boundary but by making a "large" Lorentz boost in regions about any point we can bring it into the canonical (and small) form above. Locally the bulk metric looks like

$$\begin{aligned} ds^2 &= 2dvdr + \epsilon x^\nu \partial_\nu v_i dx^i dr \\ &\quad + \Big(-r^2 f(r) dv^2 + r^2 \delta_{ij} dx^i \otimes dx^j - 2\epsilon r^2 \left(1 - f^{(0)}\right) x^\alpha \partial_\alpha v_i dv \otimes dx^i \\ &\qquad - r^2 \epsilon x^\alpha \left( \frac{\delta f^{(0)}}{\delta m^{(0)}} \partial_\alpha m^{(0)} + \frac{\delta f^{(0)}}{\delta Q^{(0)}} \partial_\alpha Q^{(0)} \right) \Big) dv^2 \\ &\quad + \epsilon r^2 \left( \tilde{h}^{(0)}(r) dv^2 - 2\tilde{h}_i(r) dv \otimes dx^i + \tilde{h}_{\langle ij \rangle}(r) dx^i \otimes dx^j + \frac{\tilde{h}^{(1)}(r)}{d-1} \delta_{ij} dx^i \otimes dx^j \right) , \end{aligned} \tag{2.2.98}$$

where we have shifted our coordinates so that $x_{(0)} = 0$.

A similar process can be applied to the gauge field whose generic, local form at first order in $\epsilon$ is

$$\begin{aligned} A_M dx^M &= \mu_q \left( 1 - \frac{{r_+}^{d-2}}{r^{d-2}} \right) dv \\ &\quad + \epsilon \left[ \frac{\sqrt{3}}{2r^2} w^{(1)} dv + \left( g^{(1)} - \frac{\sqrt{3} Q_0}{2r^2} j^{(1)} \right) dx^i \right] , \end{aligned} \tag{2.2.99}$$

where we have maintained the gauge choice

$$A_r \equiv 0 \tag{2.2.100}$$

and chosen our notation for the gauge field to match [62].

Before turning to actually solving these equations, which we shall do for Reissner-Nordstrøm $AdS_5$ in particular, we introduce an additional piece of terminology. In addition to the scalar, vector and tensor decomposition above we can further separate the Einstein-Maxwell equations in each of these sectors into two groups; constraint equations[3], which are obtained by contracting the Einstein and Maxwell equations with the vector dual to the one-form $(dr)^M$, and dynamical equations.

[3]To call the projected equations constraints is a slight abuse of terminology since we are not dealing with an initial value problem here.

The constraint equations depend on the particular nature of the fluid we are considering in that, as they correspond to covariant conservation of SEM tensor and charge currents, we need to solve the hydrodynamic equations for temperature and charge profiles when given a velocity distribution. However, the dynamical equations can be completely solved by choosing the correct dependence of the undetermined functions in Eq. (2.2.98) and Eq. (2.2.99) on $m$, $Q$ and $r$.

### 2.2.3 First order metric corrections for Reissner-Nordstrøm $AdS_5$

We now turn to solving the resultant Einstein-Maxwell equations sector by sector at first order in derivatives for the particular example of a fluid dual to a non-extremal Reissner-Nordstrøm $AdS_5$ black hole with Chern-Simon's terms. Hence we are attempting to solve order by order the equations of motion coming from the action given in Eq. (2.2.77). We begin with the tensor sector so as to compare to the linearised calculation in the introduction. The tensor sector does not include a constraint equation so we shall also look at the scalar sector in detail to see how these appear. Finally we shall briefly comment on the vector sector and refer the reader to literature for details.

#### Tensor sector

We begin by studying the tensor sector as it connects nicely to our previous discussion of linearised gravity. Again, the tensor sector perturbation is built from two pieces, one coming from the original metric expanded to order $\epsilon$ and a $x^\mu$ independent piece that we must add to fix the Einstein equations. From our discussion of linearised gravity we have

$$\begin{aligned}\tilde{E}^{(1)}_{MN}dx^M\otimes dx^N &= \frac{1}{2}\Big[r^2 f(r)\partial_r^2\left(r^2\tilde{h}_{\langle ij\rangle}(r)\right)+r^2\partial_r f\partial_r\left(r^2\tilde{h}_{\langle ij\rangle}\right)\\ &\quad +(d-3)rf\partial_r\left(r^2\tilde{h}_{\langle ij\rangle}\right)-r^2\tilde{h}_{\langle ij\rangle}(r)\left(2r\partial_r f(r)+2(d-1)f\right)\\ &\quad +2r^2\tilde{h}_{\langle ij\rangle}(r)\Big]\,dx^i\otimes dx^j+\ldots\end{aligned}$$

where ellipses indicate terms coming from other pieces of the perturbation. What is clear is that the operator governing $\tilde{h}_{\langle ij\rangle}$ is precisely

$$\partial_r^2 + \left(\frac{\partial_r f}{f} + \frac{d+1}{r}\right)\partial_r \ . \tag{2.2.101}$$

Similarly, there will be universal vector and scalar operators details of which can be found in [32]. However to follow through with the full decomposition of [52, 53] on the objects displayed in Eq. (2.2.93) would fail to yield the correct results. Essentially this is due to the fact that we wish to add a correction to the expanded spacetime Eq. (2.2.90). Henceforth the terms tensor, vector and scalar sector refer to the decomposition of $\tilde{h}_{MN}$. Hence there will be mixing between what [52, 53] would term the tensor, vector and scalar pieces of the metric and, as we shall see, the equation of motion for $\tilde{h}_{\langle ij\rangle}$ becomes sourced.

Now we work specifically in Reissner-Nordstrøm $AdS_5$. The gauge field does not supply any tensor sector equations and there are no constraint equations. Our Einstein equation evaluates to

$$\partial_r^2\tilde{h}_{ij} + \left(\frac{\partial_r f}{f} + \frac{5}{r}\right)\partial_r\tilde{h}_{ij} \ = \ -\frac{6}{r^3 f(r)}\sigma_{ij}^{(0)} \ , \tag{2.2.102}$$

This is a first order equation in $d_r\tilde{h}_{ij}$ which can be simply integrated to give

$$\tilde{h}_{ij} = \frac{2}{r_+}F_1(m,Q,r)\sigma_{ij} \ , \tag{2.2.103}$$

where

$$F_1(m,Q,r) \ = \ \int_{\frac{r}{r_+}}^{\infty} dx \frac{x\left(x^2+x+1\right)}{(x+1)\left(x^4+x^2-\frac{Q^2}{{r_+}^6}\right)} \ .$$

### Scalar sector

Next we turn to the scalar sector as it supplies a nice illustration of the various roles of constraint and dynamical equations and how it is possible to generically solve them. The three unknown functions we have to contend with are

$$\tilde{h}^{(0)}(r), \tilde{h}^{(1)}, w^{(1)}(r) \ . \tag{2.2.104}$$

Among the Einstein equations there are four independent $SO(3)$ scalar pieces coming from $vv$, $vr$, $rr$ and trace over the $ii$ components. Two linear combinations of these supply constraint equations

$$g^{rr}E_{vr} + g^{rv}E_{vv} = 0\,, \tag{2.2.105}$$

$$g^{rr}E_{rr} + g^{rv}E_{vr} = 0. \tag{2.2.106}$$

The first of these reduces to

$$\partial_\mu T^{\mu v}_{(0)} = 0 \tag{2.2.107}$$

while the second relates $\tilde{h}^{(0)}$, $\tilde{h}^{(1)}$, $w^{(1)}$ according to

$$\begin{aligned} d_r\tilde{h}^{(0)} + \frac{4}{r}\tilde{h}^{(0)} &= \frac{2\partial_i v_i^{(0)}}{r^2} + \left[\left(1 - \frac{m_0}{3r^4}\right) d_r\tilde{h}^{(1)} + \frac{4}{r}\tilde{h}^{(1)}\right] \\ &\quad -\frac{2q_0}{r^6}\left[d_r w^{(1)} - \frac{2}{r}w^{(1)}\right]. \end{aligned} \tag{2.2.108}$$

We see that for the Einstein equations up to order one in derivatives to be satisfied the fluid velocity field must satisfy the Navier-Stokes to order one in derivatives Eq. (2.2.107). As Eq. (2.2.107) is order one in derivatives it only depends on the zeroth order SEM tensor. This is a structure that is also seen at higher orders in the derivative expansion, namely, for the Einstein equations to be satisfied at order $k$ the fluid velocities in terms of which the metric is written must satisfy the Navier-Stokes equations coming from the order $k-1$ constitutive relation.

Of the remaining two Einstein equations any one can be chosen to supply a dynamical equation; we will later choose the $E_{rr}$ equation and check that the solution satisfies all the others. Similarly for the Maxwell field there is one unknown function

$$w^{(1)}(r). \tag{2.2.109}$$

There are two scalar sector Maxwell equations with one being related to the other by a constraint

$$g^{rr}M_r + g^{rv}M_v = 0 \tag{2.2.110}$$

which in turn reduces to

$$\partial_\mu J^\mu_{(0)} = 0\,. \tag{2.2.111}$$

This leaves us one to solve for, for which we choose $M_r$.

As for the dynamical equations we need to solve

$$d_r^2\tilde{h}^{(1)} + \frac{5}{r}d_r\tilde{h}^{(1)} = 0\,, \tag{2.2.112}$$

$$d_r^2 w^{(1)}(r) - \frac{1}{r}d_r w^{(1)}(r) - \frac{2q_0}{r}\tilde{h}^{(1)} = 0\,. \tag{2.2.113}$$

Solving the first equation tells us that

$$\tilde{h}^{(1)} = c_{\tilde{h}^{(1)}} + \frac{1}{r^4}\tilde{c}_{\tilde{h}^{(1)}}\,. \tag{2.2.114}$$

The first term represents a non-normalisable mode and must be set to zero. The second piece represents a VEV deformation of the boundary SEM tensor. Integrating the second equation gives

$$w^{(1)}(r) = c_{w^{(1)}} + \tilde{c}_{w^{(1)}}r^2 - \frac{q_0}{3r^4}\tilde{c}_{\tilde{h}^{(1)}}\,. \tag{2.2.115}$$

Again we see the presence of a non-normalisable mode which requires us to set $\tilde{c}_{w^{(1)}} = 0$.

Finally only $\tilde{h}^{(0)}$ is left to determine. We can see that the constraint equation above relates it to $\tilde{h}^{(1)}$ and $w^{(1)}$. In particular:

$$\tilde{h}^{(0)} = \frac{1}{r^4}\left[\frac{2}{3}\partial_i v_i^{(0)} r^3 + c_{\tilde{h}^{(0)}} - \frac{2q_0}{r^2}c_{w^{(1)}} + \frac{1}{3}\left(-\frac{m_0}{r^4} + \frac{2q_0^2}{r^6}\right)\tilde{c}_{\tilde{h}^{(1)}}\right]\,. \tag{2.2.116}$$

We see that the constants $c_{w^{(1)}}$ and $c_{\tilde{h}^{(0)}}$ can be absorbed into redefinitions of the mass and charge and thus we can ignore them as they do not represent the kind of VEV deformations we are interested in. Similarly the constant $\tilde{c}_{\tilde{h}^{(1)}}$ can be absorbed into a redefinition of $r$ and thus redefines the "renormalisation scale" in the bulk theory. Hence only the first term contributes to the definition of $\tilde{h}^{(0)}$. To summarise the scalar sector correction to the bulk metric is given by

$$ds^2 = \ldots + \epsilon r^2\left(\frac{2}{3}\frac{\partial_i v^i_{(0)}}{r}dv^2\right) \tag{2.2.117}$$

with no scalar correction to the gauge field at this order in $\epsilon$.

### Vector sector

A similar process can be applied in the vector sector where there is a single constraint equation given by

$$g^{rr}E_{ri} + g^{rv}E_{vi} = 0 \tag{2.2.118}$$

which reduces to demanding that

$$\partial_\mu T^{\mu i}_{(0)} = 0 \, . \tag{2.2.119}$$

Once again we see that calling equations such as Eq. (2.2.118) constraint equations makes sense because satisfying the above equation forces first derivatives of the velocity fields to be related to local thermodynamic fields. Indeed if we could generically solve the relativistic Navier-Stokes equation for some initial data then we could investigate interesting duals to exotic fluid flows. Here however it is only necessary to solve the equations ultralocally about a point on the boundary. As the process of solving the constraint and dynamical equations for the vector sector is very similar to the above we shall not go through it here. Instead we simply state the full metric in the next section Eq. (2.2.120). For further details the reader is referred to [32].

### 2.2.4 The metric, SEM tensor and charge current to first order

A relativistic and gauge independent expression for the charged black brane with $AdS_5$ boundary found in [62] to first order in derivatives is:

$$\begin{aligned} ds_5^2 &= -2u_\mu dx^\mu dr - r^2 f(m,Q,r) u_\mu u_\nu dx^\mu dx^\nu + r^2 \Pi_{\mu\nu} dx^\mu dx^\nu \\ &\quad -2r u_\mu \left(u^\lambda \nabla_\lambda u_\nu\right) dx^\mu dx^\nu + \frac{2}{3} r \left(\nabla_\lambda u^\lambda\right) u_\mu u_\nu dx^\mu dx^\nu \\ &\quad +2\frac{r^2}{r_+} F_1(m,Q,r)\sigma_{\mu\nu} dx^\mu dx^\nu - \frac{2\sqrt{3}\kappa_{CS} Q^3}{m r^4} u_\mu l_\nu dx^\mu dx^\nu \\ &\quad -12Q\frac{r^2}{{r_+}^7} F_2(m,Q,r) u_\mu \left(\Pi^\lambda_\nu \nabla_\lambda + 3u^\lambda \nabla_\lambda u_\nu\right) Q dx^\mu dx^\nu \, , \end{aligned} \tag{2.2.120}$$

$$\begin{aligned} A_Q &= \frac{\sqrt{3}Q}{2r^2} u_\mu dx^\mu + \frac{3\kappa_{CS} Q^2}{m r^2} l_\mu dx^\mu \\ &\quad +\frac{\sqrt{3} r^5}{2{r_+}^7} \left[\frac{\partial}{\partial r} F_2(m,Q,r)\right] \left(\Pi^\lambda_\mu \nabla_\lambda + 3u^\lambda \nabla_\lambda u_\mu\right) Q dx^\mu \, , \end{aligned} \tag{2.2.121}$$

where the first line of each definition is zeroth order in fluid derivatives, the subsequent lines first order and:

$$\begin{aligned} f(m,Q,r_+) &= 0\ , \\ F_1(m,Q,r) &= \int_{\frac{r}{r_+}}^{\infty} dx \frac{x\left(x^2+x+1\right)}{(x+1)\left(x^4+x^2-\frac{Q^2}{{r_+}^6}\right)}\ , \\ F_2(m,Q,r) &= \frac{1}{3}\left(1-\frac{m}{r^4}+\frac{Q^2}{r^6}\right)\int_{\frac{r}{r_+}}^{\infty} dx \frac{1}{\left(1-\frac{m}{x^4}+\frac{Q^2}{x^6}\right)^2}\left(\frac{1}{x^8}-\frac{3}{4x^7}\left(1+\frac{{r_+}^4}{m}\right)\right)\ . \end{aligned}$$

These results were obtained in [32] by perturbatively solving the equations of motion from the action of Eq. (2.2.77) with trivial dilaton field. This solution can readily be uplifted to a 10-dimensional solution of Type IIB by adding a suitably deformed $S^5$ term to the metric and folding an $A_Q$ term into the standard expression for the Ramond-Ramond five form [80].

It is not too difficult to read off the SEM tensor and charge current for the above metric and gauge field. The form of the SEM tensor and charge currents are

$$T_{\mu\nu} = P\left(\eta_{\mu\nu}+4u_\mu u_\nu\right)-2\eta\sigma_{\mu\nu}\ , \tag{2.2.122}$$

$$J^\mu = qu^\mu-\kappa_q\left(\Pi^{\mu\nu}\nabla_\nu q+3qu^\nu\nabla_\nu u^\mu\right)-\mho l^\mu \tag{2.2.123}$$

as expected from our discussion of hydrodynamics. As we have a specific model the coefficients, in terms of $G_5$ which can be related to the central charge of the field theory, can be determined. They have the following expressions

$$\begin{aligned} &\varepsilon = 3P\ , \qquad P=\tfrac{m}{16\pi G_5}\ , \qquad \eta=\tfrac{{r_+}^3}{16\pi G_5}\ , \\ &q=\tfrac{\sqrt{3}Q}{4\pi G_5}\ , \qquad \kappa_q=\left(\tfrac{{r_+}^4+m}{4mr_+}\right)\ , \qquad \gamma=-\tfrac{3q}{4\varepsilon}\kappa_q\ , \qquad \mho=-\tfrac{\kappa_{CS}}{2P}q^2\ , \end{aligned}$$

where we have applied the SEM conservation equation at zeroth order in derivatives,

$$u^\nu\nabla_\nu u^\mu+\frac{\Pi^{\mu\nu}\nabla_\nu P}{\varepsilon+P}=0\ ,$$

to the penultimate term of Eq. (2.2.123) to write it in terms of $\varepsilon$. Thus, $\gamma$ displayed above is the coefficient of the derivative of the energy density $\varepsilon$. To convert to a derivative of $T$, which was our choice of variable at the beginning of the chapter, one need only substitute in the expression for the energy density in terms of $T$ and $q$. Note that our parity violation coefficient $\mho$ is indeed determined by the Chern-Simon's parameter $\kappa_{CS}$, the charge $q$ and the fluid pressure $P$.

# Chapter 3

# Incompressible non-relativistic fluids

In this chapter we shall discuss a scaling of the relativistic metrics of chapter 2 which allows our boundary fluid to realise Galilean symmetry. This process is related to that for obtaining the Galilean conformal algebra from the conformal algebra as discussed in appendix A. We shall find that the resultant non-relativistic fluids satisfy an incompressibility condition. The consideration of a fluid dual to a charged black hole, while novel, represents a minor change to [61] and their results can readily be obtained by setting this charge to zero.

## 3.1 Relaxation of the field theory from local to global equilibrium

### 3.1.1 Thermodynamics

The essential starting point for constructing our hydrodynamic theory in the previous chapter was the existence of a global thermodynamic ground state for a perturbed system to decay to. So we should understand how the scaling of appendix A, which we shall now refer to as BMW scaling, affects the thermodynamics. The relativistic ground state was specified by

$$G = G\left(P, T, q^I\right) . \tag{3.1.1}$$

When we consider local thermodynamics each of these thermodynamic state variables became a function of coordinates. The Galilean algebra achieved by a contraction of the Poincaré algebra in appendix A was represented by functions which had an anisotropic scaling in space and time

$$f(\epsilon x) \;\rightarrow\; f\left({\epsilon_{\rm BMW}}^r x^i, {\epsilon_{\rm BMW}}^{r+1} t\right) \;, \tag{3.1.2}$$

where $\epsilon_{\rm BMW}$ is the non-relativistic scaling parameter. In what follows we shall choose $r = 1$. We would like to argue how the thermodynamic variables should decompose under this scaling (see appendix A for a general discussion of the scaling process). The essential physics is contained in the example of an uncharged, free, relativistic point particle whose energy is given by

$$E = \sqrt{m_{(0)}^2 + \mathbf{p}^2} \;. \tag{3.1.3}$$

Under the scaling $\mathbf{p} = \epsilon_{\rm BMW}\mathbf{p}_*$ we find

$$E \;\sim\; m_{(0)} + {\epsilon_{\rm BMW}}^2 \frac{\mathbf{p}_*^2}{2m_{(0)}} + \mathcal{O}^4(\epsilon_{\rm BMW}) \;. \tag{3.1.4}$$

If we were considering kinetic theory we would use this to argue that the total energy of our fluid should split into a "zero-point"-like piece $m_{(0)}$ and a non-relativistic energy[1]. More generally this suggests to us that the thermodynamic variables of our fluid, other than charge and those related to electromagnetic effects, should scale as

$$P \;=\; P_{(0)} + {\epsilon_{\rm BMW}}^2 {P_{\rm nr}}^* \;, \tag{3.1.5}$$

$$T \;=\; T_{(0)} + {\epsilon_{\rm BMW}}^2 \delta T_* \;, \tag{3.1.6}$$

where $P_{(0)}$ and $T_{(0)}$ are the relativistic temperature and pressure respectively and henceforth the script $^*$ will indicate that the arguments of the function scale anisotropically in space and time (Eq. (3.1.2)). This decomposition will receive further justification in the next section where we demonstrate that it leads to the incompressible

[1]Note that if the relativistic dispersion relation is gapless, such as for the hydrodynamic sound mode of a chargeless relativistic fluid, there is no zero point energy to contend with and in our example the mode would disperse linearly.

Navier-Stokes equations. We shall find that when we attempt to generate these non-relativistic fluid equations from their relativistic counterparts we will have to normalise the non-relativistic pressure, charge, etc. by a factor $\rho_{(0)}$ defined by

$$\rho_{(0)} = \varepsilon_{(0)} + P_{(0)} .$$

This will later be interpreted as a mass density. We could of course do this immediately in Eq. (3.1.6) converting the "raw" thermodynamic quantity, ${P_{\rm nr}}^*$ into its properly normalised counterpart ${P_{\rm nr}}^* \to \rho_{(0)}{P_{\rm nr}}^*$ as was done in [61]. However it will be enlightening to see explicitly where this normalisation turns up in the resultant contraction and therefore we shall save doing this until later.

The fact that we need to additionally scale the charge is a little more difficult to see from our particle motion argument. However, imagine that we had the relativistic Gibbs free energy and, using the scalings above, we wanted to construct the corresponding non-relativistic Gibbs' free energy. We would expand our relativistic relations in powers of $\epsilon_{\rm BMW}$. The piece which is order 0 in $\epsilon_{\rm BMW}$ as above we would interpret as something like a "zero point" free energy which should affect our dynamics only through overall rescalings of our raw non-relativistic variables. If the charge did not decompose in the same way as the pressure and the temperature then the resultant free energy would describe an uncharged system. So we also decompose

$$q^I = q^I_{(0)} + {\epsilon_{\rm BMW}}^2 {q_{\rm nr}}^* . \tag{3.1.7}$$

Finally, for scale invariant theories the relativistic equation of state in absence of an external electromagnetic field was

$$\varepsilon = (d-1)P . \tag{3.1.8}$$

Given an equation of state for the fluid, just like for the free energy, we can then work out the equation of state obeyed by its non-relativistic limit. On general grounds these must be the same expression up to introduction of $\rho_{(0)}$ in suitable places. For

a fundamentally scale invariant theory in particular, we find

$$\begin{aligned}\left(\varepsilon_{(0)}+{\epsilon_{\rm BMW}}^2\rho_{(0)}\varepsilon_{\rm nr}\right) &= (d-1)\left(P_{(0)}+{\epsilon_{\rm BMW}}^2\rho_{(0)}{P_{\rm nr}}^*\right)\\ \Rightarrow {\varepsilon_{\rm nr}}^* &= (d-1)\,{P_{\rm nr}}^*\end{aligned}$$

as expected.

### 3.1.2 Fluid variables and fluid dynamics

The equations of motion coming from covariant conservation of the relativistic SEM tensor to first order in derivatives are

$$\begin{aligned}0 &= u^\mu\nabla_\mu\varepsilon+(\varepsilon+P)\,\theta-2\eta\sigma^2-\zeta\theta^2\\ 0 &= \nabla^\perp_\alpha P+(\varepsilon+P)\,a_\alpha-2\eta\nabla^\nu_\perp\sigma_{\alpha\nu}+2\eta\sigma^2u_\alpha-2{\sigma_\alpha}^\nu\nabla^\perp_\nu\eta\,,\\ &\quad -\zeta\nabla^\perp_\alpha\theta-\zeta a_\alpha\theta-\theta\nabla^\perp_\alpha\zeta\end{aligned}$$

where for simplicity we have assumed our manifold is flat. As discussed above we need our thermodynamic variables to undergo the decomposition of Eqs. (3.1.6) and (3.1.7). Given that the transport coefficients are functions of our chosen thermodynamic variables they must also undergo such a decomposition so that

$$\eta \;\rightarrow\; \eta_{(0)}+{\epsilon_{\rm BMW}}^2\eta_{\rm nr}\,, \tag{3.1.9}$$

$$\zeta \;\rightarrow\; \zeta_{(0)}+{\epsilon_{\rm BMW}}^2\zeta_{\rm nr}\,. \tag{3.1.10}$$

Finally, our relativistic velocity field decomposes as

$$\begin{aligned}u^\mu &= \frac{1}{\sqrt{1-{\epsilon_{\rm BMW}}^2\delta_{ij}v^i_*v^j_*}}\left(1,\epsilon_{\rm BMW}v^i_*\right)\\ &= \left(1+\frac{1}{2}{\epsilon_{\rm BMW}}^2\delta_{ij}v^i_*v^j_*,\epsilon_{\rm BMW}v^i_*\right)+\mathcal{O}^3(\epsilon_{\rm BMW})\,.\end{aligned} \tag{3.1.11}$$

These transformations are qualitatively similar to those in the linearised perturbation analysis of Eq. (2.1.40). In fact we can modify those equations by replacing $\epsilon$ with $\epsilon_{\rm BMW}$ and adding to any time derivative an extra $\epsilon_{\rm BMW}$. Care must also be taken over the fact that $u^t$ has an additional order ${\epsilon_{\rm BMW}}^2$ piece and $v^i_*$ is only order one in $\epsilon_{\rm BMW}$ which requires us to reintroduce $v^j_*\partial_j v^i_*$ which dropped out of the

linearised analysis. The result is that

$$0 = {\epsilon_{\rm BMW}}^2 [\theta^*] + \mathcal{O}^4(\epsilon_{\rm BMW}) \tag{3.1.12}$$

$$0 = {\epsilon_{\rm BMW}}^3 \left[ \left(\varepsilon_{(0)} + P_{(0)}\right) \left(\partial_t v_i^* + v_*^j \partial_j v_*^i\right) + \nabla_i {P_{\rm nr}}^* - \eta_{(0)} \left(\nabla^j \nabla_i v_j^* + \nabla^2 v_i^*\right)\right] + \mathcal{O}^4(\epsilon_{\rm BMW}) \tag{3.1.13}$$

where the first equation comes from the parallel part of the relativistic equation and demands incompressibility of the fluid while the second equation comes from the transverse part and is the non-relativistic Navier-Stokes equation. We have used Eq. (3.1.12) to simplify terms in Eq. (3.1.13). We can redefine our non-relativistic pressure and viscosities to have overall factors of $\varepsilon_{(0)} + P_{(0)}$ which brings Eq. (3.1.13) into the more pleasant form

$$0 = \partial_t v_i^* + v_*^j \partial_j v_*^i + \nabla_i {P_{\rm nr}}^* - \nu \left(\nabla^j \nabla_i v_j^* + \nabla^2 v_i^*\right) \tag{3.1.14}$$

where we have interpreted $\nu = \frac{\eta_{(0)}}{\varepsilon_{(0)} + P_{(0)}}$ to be the kinematic viscosity. Similarly for the charge conservation equations we find

$$0 = {\epsilon_{\rm BMW}}^3 \left(\partial_t {q_{\rm nr}}^I\right) + \mathcal{O}^4(\epsilon_{\rm BMW}) \tag{3.1.15}$$

and we simply end up with conservation of charge to this order. These equations have interesting symmetry properties and the interested reader is referred to [61] and [81] for further details.

## 3.2 Dual to asymptotically, locally AdS spaces

Given the duality between the strongly coupled fluids and bulk spacetimes of chapter 2 it is clear that it is possible to take the BMW limit of the bulk metric dual to the relativistic fluid of chapter 2. This will provide us with a bulk spacetime whose dual is a non-relativistic fluid with underlying Galilean conformal symmetry. For relating to the Schrödinger fluid in the next chapter it will be sufficient for us to consider the metric and gauge field at first order for the Reissner-Nordstrøm $AdS_5$ black hole given by Eqs. (2.2.120) and (2.2.121).

### 3.2.1 First order metric corrections for Reissner-Nordstrøm $AdS_5$

We wish to apply the scaling transformations, Eqs. (3.1.6), (3.1.7) and (3.1.11) to the bulk metric and gauge fields of Eqs. (2.2.120) and (2.2.121) which we state here for convenience

$$\begin{aligned} ds^2 &= -2u_\mu dx^\mu dr - r^2 f(m,Q,r) u_\mu u_\nu dx^\mu dx^\nu + r^2 \Pi_{\mu\nu} dx^\mu dx^\nu \\ &\quad -2r u_\mu \left(u^\lambda \nabla_\lambda u_\nu\right) dx^\mu dx^\nu + \frac{2}{3} r \left(\nabla_\lambda u^\lambda\right) u_\mu u_\nu dx^\mu dx^\nu \\ &\quad +2\frac{r^2}{r_+} F_1(m,Q,r)\eta_{\mu\nu} dx^\mu dx^\nu - \frac{2\sqrt{3}\kappa_{CS} Q^3}{m r^4} u_\mu l_\nu dx^\mu dx^\nu \\ &\quad -12Q\frac{r^2}{{r_+}^7} F_2(m,Q,r) u_\mu \left(\Pi^\lambda_\nu \nabla_\lambda + 3u^\lambda \nabla_\lambda u_\nu\right) Q dx^\mu dx^\nu \ , \\ A_Q &= \frac{\sqrt{3}Q}{2r^2} u_\mu dx^\mu + \frac{3\kappa_{CS} Q^2}{m r^2} l_\mu dx^\mu \\ &\quad +\frac{\sqrt{3} r^5}{2{r_+}^7} \left[\frac{\partial}{\partial r} F_2(m,Q,r)\right] \left(\Pi^\lambda_\mu \nabla_\lambda + 3u^\lambda \nabla_\lambda u_\mu\right) Q dx^\mu \ . \end{aligned}$$

Define the following quantities

$$\begin{aligned} m &= m^{(0)} + {\epsilon_{\rm BMW}}^2 m^*_{nr} \ , \\ Q &= Q^{(0)} + {\epsilon_{\rm BMW}}^2 Q^*_{nr} \ , \end{aligned}$$

which follow when $m$ and $Q$ are expressed in terms of $T$ and $q$. The resultant metric and gauge field are

$$\begin{aligned}
ds^2_{(0)} &= 2dtdr + r^2\left(1 - f(m^{(0)}, Q^{(0)}, r)\right)dt^2 + r^2\delta_{ij}dx^i dx^j \ , \\
ds^2_{(1)} &= -2v^*_i dx^i dr - 2r^2\left(1 - f(m^{(0)}, Q^{(0)}, r)\right)v^*_i dx^i dt \ , \\
ds^2_{(2)} &= \delta_{ij}v^i_* v^j_* dtdr + r^2\left(1 - f(m^{(0)}, Q^{(0)}, r)\right)\delta_{ij}v^i_* v^j_* dt^2 \\
&\quad -r^2\left(\frac{\delta f(m^{(0)}, Q^{(0)}, r)}{\delta m^{(0)}}m^*_{nr} + \frac{\delta f(m^{(0)}, Q^{(0)}, r)}{\delta Q^{(0)}}Q^*_{nr}\right)dt^2 \\
&\quad +r^2\left(1 - f(m^{(0)}, Q^{(0)}, r)\right)v^*_i v^*_j dx^i dx^j \\
&\quad +2\frac{r^2}{r_+}F_1\left(m^{(0)}, Q^{(0)}, r\right)\nabla_{(i}v_{j)}dx^i dx^j \\
&\quad -\frac{2\sqrt{3}\kappa_{CS}Q^3_{(0)}}{m_{(0)}r^4}\epsilon_i{}^{jk}\nabla_j v^*_k dtdx^i \ , \\
A^Q_{(0)} &= -\frac{\sqrt{3}}{2r^2}Q^{(0)}dt \ , \\
A^Q_{(1)} &= \frac{\sqrt{3}Q}{2r^2}v^*_i dx^i \ , \\
A^Q_{(2)} &= -\frac{\sqrt{3}}{4r^2}Q^{(0)}\delta_{ij}v^i_* v^j_* - \frac{\sqrt{3}}{2r^2}Q^*_{nr}dt + \frac{3\kappa_{CS}Q^2_{(0)}}{m^{(0)}r^2}\epsilon_i{}^{jk}\nabla_j v^*_k dx^i \ .
\end{aligned}$$

where the subscript refers to the order in $\epsilon_{\rm BMW}$ and we have applied incompressibility as a constraint on the fluid velocity. We should make a few comments on this metric and gauge field. The part of the metric which is zeroth order in $\epsilon_{\rm BMW}$ is clearly the usual metric of $AdS_{d+1}$. The order $\epsilon_{\rm BMW}{}^2$ piece of both the metric and the gauge field contain terms quadratic in spatial velocities leading to the correct terms at the boundary to form the material derivative of the incompressible Navier Stokes equations. The presence of these quadratic terms is due to the fact that we scaled length scale fluctuations to be large at the same rate that we took spatial velocities to be small [61]. Had we simply scaled the spatial velocity amplitudes the resultant metrics would be dual to the linearised relativistic Navier-Stokes equations.

We have thus isolated a metric which corresponds to the desired boundary fluid. There is no need to extract the SEM tensor and charge currents again because all we have done is select particular solutions to the relativistic fluid equations of motion. Thus our previously calculated SEM tensor and charge current (Eqs. (2.2.122) and (2.2.123)) still apply and will decompose exactly as described above. The only new

transport coefficient is $\nu$ which is simply a ratio of relativistic transport coefficients. In the absence of charge the kinematic viscosity has the form

$$\nu = \frac{1}{4\pi T} \tag{3.2.16}$$

where we have used the integrated form of the first law of thermodynamics in the absence of charge $\varepsilon + P = sT$. When charge is present it has the form

$$\nu = \frac{1}{4\pi \left(T + \frac{\mu_q q}{s}\right)} \tag{3.2.17}$$

and thus, unlike the relativistic shear viscosity, lacks a universal character.

### 3.2.2 Beyond first order

It is clear that given any dual to a relativistic fluid we should be able to go through the BMW process to obtain an incompressible fluid with underlying Galilean symmetry. In literature higher orders in $\epsilon_{\text{BMW}}$ have been considered as have been forced fluids where the boundary metric on which the original relativistic fluid lives is curved. At zeroth and first order in $\epsilon_{\text{BMW}}$ the curvature invariants of the boundary are locally zero and so the form of the metric corrections is unchanged from that displayed above. However at second order it is possible to write down Weyl covariant curvature objects and thus the second order corrections to our metrics are different.

For a discussion of forced and higher order non-relativistic fluids with underlying Galilean conformal invariance see [61] and subsequently [3] which corrects a small but important error in [61]. In [3] it is also shown how the membrane paradigm, whose gravitational fluctutations can exactly be described by a non-relativistic, incompressible fluid, can be embedded as an object in the boundary field theory. The degrees of freedom associated with the membrane correspond to imposing a Newton-Cartan structure, associated with non-relativistic gravity, on the boundary as opposed to the usual Lorentzian structure of relativistic theories.

# Chapter 4

# Compressible non-relativistic fluids

Another example of a non-relativistic limit of strongly-coupled field theories are the duals to Schrödinger spacetimes. The interest in the hydrodynamics of these theories in part lies in the fact that it has proven difficult to find generalisations of the AdS-CFT correspondence to other asymptotics. The Galilean limit discussed in the previous section only represents a restriction of the relativistic correspondence to some subsector of solutions. Schrödinger spacetimes however represent a truly different class of asymptotics where the dual field theory satisfies the Schrödinger algebra [82–84] (see appendix A). In the light-cone coordinate system the bulk metric of the zero-temperature ground-state of these spaces has the form

$$ds^2 = r^2 \left( 2dx^+ dx^- - \beta^2 r^2 \left(dx^+\right)^2 + d\mathbf{x}^2 \right) + \frac{dr^2}{r^2} \, ,$$

where $\beta$ is some number. For the case above $\beta$ can be scaled out by a boost but it has non-trivial effects outside of the ground state. Based on the suggestions of [85, 86] it later proved possible to embed these spacetimes in string theory [87–89] although there as yet remains issues of interpretation of boundary quantities [90]. For other works on Schrödinger spacetimes and non-relativistic CFTs see [91–118].

In this chapter we seek to extend the fluid-gravity correspondence to discuss fluids with Schrödinger symmetry and perhaps shed some light on outstanding problems in (compressible) non-relativistic fluid mechanics. The case of an uncharged

fluid has already been considered in [33]. Our focus in this section will be towards understanding charged fluids with Schrödinger symmetry. In [119,120] a $U(1)$-charged, asymptotically Schrödinger spacetime at finite temperature has been discussed and some linear transport coefficients calculated. To obtain these results a truncation of the 10-dimensional effective Type IIB string action was isolated. Specifically we begin with a stack of D3-branes rotating in $S^5$ [121]. The Kaluza-Klein reduction of this solution is associated with a Reissner-Nordstrøm $AdS_5$ black hole. By applying a Null Melvin Twist [122, 123], or alternately a TsT transformation [124], to the 10-dimensional metric we induce Schrödinger symmetry on the boundary spacetime. In this section we will extend the analysis of these charged, thermal systems to hydrodynamics at first order calculating all the relevant transport coefficients.

A fluid with Schrödinger symmetry whose gravitational dual was $(d+1)$-dimensional would occupy $d-2$ spatial dimensions. As discussed in the previous section, an alternate approach to achieving non-relativistic symmetry is given by the "Galilean conformal algebra" which produces fluids moving in $d-1$ spatial dimensions. As noted in chapter 4 this approach preserves the form of the equation of state from the relativistic theory to the non-relativistic theory but projects out sound waves whose dispersion relations take the form $\omega \propto k$ giving an incompressible fluid [61].

Our aim in this chapter is to detail the first order corrections to a charged fluid with Schrödinger symmetry. The general form of these corrections can be obtained by reducing conformal, relativistic currents (Eqs. (2.2.122) and (2.2.123)) to their non-relativistic counterparts via light-cone reduction. We then consider a five-dimensional, asymptotically Schrödinger charged black hole spacetime and construct its fluid dual. The corrections up to first order in boundary derivatives of velocity to the metric, dilaton, gauge field and massive vector field are calculated. Finally, using the holographic dictionary in Schrödinger spacetimes, we shall calculate the boundary values of the stress tensor and gauge field, obtained from our charged black brane, to determine the dependence on charge and temperature profiles of the non-relativistic transport coefficients.

This chapter is organised as follows. In the next section a map is constructed that links the transport coefficients of the relativistic charge current to their non-relativistic counterparts. This is an extension of previous work, notably, we seek to preserve the mappings of [33] between relativistic stress-energy-momentum (SEM) tensor transport coefficients and their non-relativistic counterparts. In the subsequent section we shall construct an action principle for the charged, asymptotically Schrödinger black hole and determine its field content. For this calculation we closely follow the work of [119]. Having obtained the asymptotically AdS metric and gauge field to first order in chapter 2 we take the TsT of this solution to find the corresponding metric and fields with Schrödinger symmetry. Finally we calculate the boundary values of these fields and determine the non-relativistic transport coefficients.

## 4.1 Relaxation of the field theory from local to global equilibrium

### 4.1.1 Generalities

In this section we shall formulate the basics of compressible non-relativistic hydrodynamics with a conserved particle number. The defining thermodynamic potential is again the Gibbs potential

$$G = G\left(P(x), T(x), N(x), Q^I(x)\right) \tag{4.1.1}$$

where $N(x)$ is a new variable compared to the relativistic case which corresponds to a conserved particle number. As noted before the thermodynamic coefficients do not indicate how quantities flow between different patches of local equilibrium and as such we need to supplement our knowledge of the extensive variables with the local fluid velocity $v^i(x)$ and mass density $\rho(x)$. We contrast this with the case of the BMW scaling of the relativistic solution where there is no conserved particle number and thus no mass density. The hydrodynamic regime is now characterised

by four conservation equations

$$\partial_+\rho + \partial_i\left(\rho v^i\right) = 0\,, \tag{4.1.2}$$

$$\partial_+\left(\rho v^i\right) + \partial_j\Pi^{ij} = 0\,, \tag{4.1.3}$$

$$\partial_+\left(\varepsilon_{\rm nr} + \frac{1}{2}\rho v^2\right) + \partial_i {J_\epsilon}^i = 0\,, \tag{4.1.4}$$

$$\partial_+ q_{\rm nr} + \partial_i {J_{\rm nr}}^i = 0\,, \tag{4.1.5}$$

where we have used + to denote the time coordinate to match our later interpretation of $x^+$ under light-cone reduction. These equations are the continuity, momentum conservation, energy conservation and charge conservation equations respectively of the fluid.

To zeroth order in derivatives of velocity and temperature we can expand the undetermined tensor objects, $\Pi_{ij}$, ${J_\epsilon}^i$ and ${J_{\rm nr}}^i$ as

$$\Pi^{ij} = \rho v^i v^j + P_{\rm nr}\delta^{ij}\,, \tag{4.1.6}$$

$${J_\epsilon}^i = \left(\varepsilon_{\rm nr} + P_{\rm nr} + \frac{1}{2}\rho v^2\right) v^i\,, \tag{4.1.7}$$

$${J_{\rm nr}}^i = q_{\rm nr} v^i\,, \tag{4.1.8}$$

where $\varepsilon_{\rm nr}$ and $P_{\rm nr}$ are the fluid's energy density and pressure. We shall call Eqs. (4.1.6), (4.1.7) and (4.1.8) the stress tensor, energy density current and charge density current at zeroth order.

Following the work of previous chapters and [29] we consider adding terms to our spatial stress tensor and currents with single derivatives of velocity. Our undetermined tensor quantities in Eqs. (4.1.3) and (4.1.4) take the form

$$\Pi^{ij} = \rho v^i v^j + P_{\rm nr}\delta^{ij} - \eta_{\rm nr}\sigma^{ij} - \zeta_{\rm nr}\theta\delta^{ij}\,, \tag{4.1.9}$$

$$\begin{aligned} {J_\epsilon}^i = {} & \left(\varepsilon_{\rm nr} + P_{\rm nr} + \frac{1}{2}\rho v^2\right) v^i - \eta_{\rm nr}\sigma^{ij}v_j - \kappa_T\delta^{ij}\partial_j T \\ & -\varpi\delta^{ij}\partial_j \ln\left[\frac{r_+\left(T,\frac{\mu}{T},\frac{\mu_q}{T}\right)\left[{r_+}^4\left(T,\frac{\mu}{T},\frac{\mu_q}{T}\right) - \frac{8}{3}\frac{{\mu_q}^2}{\mu}{r_+}^2\left(T,\frac{\mu}{T},\frac{\mu_q}{T}\right)\right]^{\frac{1}{4}}}{\left({r_+}^2\left(T,\frac{\mu}{T},\frac{\mu_q}{T}\right) + \frac{4}{3}\frac{{\mu_q}^2}{\mu}\right)}\right]\,, \end{aligned} \tag{4.1.10}$$

where we have assumed a flat background. The last term of Eq. (4.1.10) can be decomposed into a sum of differentials of our chosen thermodynamic variables but

as we will be extracting only the single coefficient $\varpi$ from our gravity dual we have chosen to display this more compact expression which indicates the relationship of the new dissipative terms to objects relevant to the gravity dual. The stress tensor $\Pi_{ij}$ is unchanged from the uncharged case of [33] but the energy current $J_\epsilon{}^i$ has received an additional correction which vanishes when the charge is set to zero. In these expressions we have used the following definitions:

$$\begin{aligned} \theta &= \partial_i v^i \,, \\ \sigma^{ij} &= \left(\partial^i v^j + \partial^j v^i - \frac{2\delta^{ij}}{d-2}\theta\right) \,, \\ r_+(x,y,z) &= \frac{\pi}{2\sqrt{2}}\left(-\frac{x}{y}\right)^{\frac{1}{2}}\left[1+\sqrt{1+\frac{32}{3}\frac{z^2}{\pi^2}}\right] \,. \end{aligned} \tag{4.1.11}$$

All but the last coefficient of Eq. (4.1.10) have standard physical interpretations. We shall call the new coefficient, $\varpi$, the contribution to the energy current from charge.

As regards determining the first order corrections to the current vector we note that at zeroth order the conservation equations Eqs. (4.1.2)-(4.1.5) can be written as:

$$\begin{aligned} \partial_+\rho + v^i\partial_i\rho + \rho\partial_i v^i &= 0 \,, \\ \partial_+ v^i + v^j\partial_j v^i + \frac{1}{\rho}\partial^i P_{\rm nr} &= 0 \,, \\ \partial_+\varepsilon_{\rm nr} + \partial_i\left(\varepsilon_{\rm nr} v^i\right) + P_{\rm nr}\partial_j v^j &= 0 \,, \\ \partial_+ q_{\rm nr} + v^i\partial_i q_{\rm nr} + q_{\rm nr}\partial_i v^i &= 0 \,, \end{aligned}$$

where the equation of state for the fluid relates $\varepsilon_{\rm nr}$ and $P_{\rm nr}$. Hence, if we obtain a complete solution to the above equations they allow us to replace time derivatives of the variables $\varepsilon_{\rm nr}$, $\rho$, $q_{\rm nr}$ and $v^i$ for spatial derivatives at first order making an error in our final results that, overall, is second order in derivatives and can therefore be ignored. We can thus write our non-relativistic current vector as:

$$\begin{aligned} J_{\rm nr}{}^i &= q_{\rm nr}v^i - \kappa_{\rm nr}\delta^{ij}\partial_j q_{\rm nr} - \gamma_{\rm nr}{}^{ij}\partial_j\varepsilon_{\rm nr} - \digamma_{\rm nr}{}^{ij}\partial_j\rho \\ &\quad -\mho_{\rm nr}\left[\epsilon^{ij}v^k\left(\partial_j v_k - \partial_k v_j\right) + v^i\epsilon^{jk}\partial_j v_k\right] \,. \end{aligned} \tag{4.1.12}$$

Here $\kappa_{\rm nr}$ is the non-relativistic diffusion constant and $\mho_{\rm nr}$ the parity violation coefficient. We shall call the tensor objects $\gamma_{\rm nr}{}^{ij}$ and $\digamma_{\rm nr}{}^{ij}$ the contributions of energy

density and mass density to the charge current respectively.

### 4.1.2 Light-cone reduction of charged relativistic fluids

One way of obtaining the form of the first order corrections to Eqs. (4.1.6), (4.1.7) and (4.1.8) is to light-cone reduce the SEM tensor and charge current of a relativistic fluid. In particular we shall consider a conformal, relativistic fluid and then the light-cone reduction will lead to a hydrodynamic system with Schrödinger symmetry [33]. We summarise the previous work of [33] on the uncharged fluid before determining a map between relativistic charge coefficients and their non-relativistic counterparts.

As was demonstrated in [33] at first-order in derivatives of fluid velocity there exists a map between the relativistic SEM tensor variables, $(u^\mu, \varepsilon, P, \eta)$, and non-relativistic $(v^i, \rho, \varepsilon_{\rm nr}, P_{\rm nr}, \eta_{\rm nr})$ variables. We shall seek to maintain these relations and augment them with our charge variable maps. In particular we can use them in our reduction of the charge current. Summarising the results of [33] we begin by assuming that our fluid lives on a Minkowskian background with metric

$$ds^2 = 2dx^+dx^- + d\mathbf{x}^2 \tag{4.1.13}$$

and that the relativistic hydrodynamic variables and velocities depend only trivially on the $x^-$ direction. This suggests we make the following identifications

$$\begin{aligned} T^{++} &= \rho \,, \\ T^{+i} &= \rho v^i \,, \\ T^{+-} &= -\left(\varepsilon_{\rm nr} + \frac{1}{2}\rho v^2\right) \,, \\ T^{-i} &= -J_\epsilon^{\ i} \,, \\ T^{ij} &= \Pi^{ij} \,, \end{aligned} \tag{4.1.14}$$

which come from comparing SEM tensor conservation equations in our choice of coordinates and Eqs. (4.1.2)-(4.1.4). The $T^{++}$ component of the SEM tensor implies the following identification between variables

$$\rho = (\varepsilon + P)\left(u^+\right)^2 \tag{4.1.15}$$

while the $T^{+i}$ component indicates that

$$v^i = \frac{u^i}{u^+} - \frac{\eta}{\rho}\left(\partial_i u^+ - \frac{u^+}{(\varepsilon+P)}\partial_i P\right) .$$

The other quantities of importance we list for completeness:

$$\Pi^{ij} = \rho v^i v^j + P\delta^{ij} - \eta u^+ \sigma^{ij} , \tag{4.1.16}$$

$$P_{\rm nr} = P , \tag{4.1.17}$$

$$\eta_{\rm nr} = \eta u^+ , \tag{4.1.18}$$

$$\varepsilon_{\rm nr} = \frac{1}{2}(\varepsilon - P) , \tag{4.1.19}$$

$$J_\epsilon{}^i = \left(\varepsilon_{\rm nr} + P_{\rm nr} + \frac{1}{2}\rho v^2\right) v^i - \eta_{\rm nr}\sigma^{ij}v_j - \frac{2\eta_{\rm nr}P_{\rm nr}}{\rho}\delta^{ij}\left[\frac{3}{2}\frac{\partial_j\varepsilon_{\rm nr}}{\varepsilon_{\rm nr}} - \frac{\partial_j\rho}{\rho}\right] . \tag{4.1.20}$$

We have used conformality of the relativistic theory to set the bulk viscosity of the parent theory to zero. For a conformal relativistic fluid in $d$ spacetime dimensions the equation of state is supplied by tracelessness of the SEM tensor and implies $\varepsilon = (d-1)P$ and therefore, using the above maps, the non-relativistic fluid satisfies $\varepsilon_{\rm nr} = \frac{d-2}{2}P_{\rm nr}$.

In [33] the final two terms of Eq. (4.1.20) are eliminated in favour of $\partial_i T$ and the resultant coefficient is interpreted as the thermal conductivity $\kappa_T$. This can be done because the equation of state for an uncharged fluid is given by

$$\varepsilon_{\rm nr} = \alpha\left(\frac{T^2}{\mu}\right)^{\frac{d}{2}} ,$$

where $\alpha$ is a constant as detailed in [103]. However for a charged fluid there is an additional scale in the problem, $\mu_q$, and therefore our equation of state takes the more generic form $P_{\rm nr} = T^{\frac{d}{2}} g\left(\frac{\mu}{T}, \frac{\mu_q}{T}\right)$ which prevents us from eliminating $\partial_i\varepsilon_{\rm nr}$ and $\partial_i\rho$ completely. We shall return to the interpretation of these terms later.

Consider now the charge density current

$$J^\mu = qu^\mu - \kappa_q\Pi^{\mu\sigma}\nabla_\sigma q - \gamma\Pi^{\mu\sigma}\nabla_\sigma\varepsilon - \mho_I\Sigma^{\mu\sigma_1\sigma_2}\omega_{\sigma_1\sigma_2} .$$

The variables displayed are not the standard variables of chapter 2 however we wish to match the notation of [1]. To convert to our standard variables we need only plug

in the expression for the energy density in terms of charge density and temperature. What is clear is that it satisfies the conservation equation

$$\partial_+ J^+ + \partial_i J^i = 0$$

and using the maps Eqs. (4.1.15)-(4.1.20) we find that $J^+$ has the form

$$J^+ = qu^+ + \mathfrak{V}(u^+)^2 \partial^j v^k \epsilon_{jk} \,, \tag{4.1.21}$$

where we have used the scaling relation $(d-1)\kappa_q q + d\gamma\varepsilon = 0$ to annihilate the term proportional to $\theta$ and set $\varepsilon_{+-ij} = -\varepsilon_{ij}$. It is a satisfying occurrence that the only correction to the identification $J^+ = q_{\rm nr}$ at first order is a piece which accounts for the anomalies in the relativistic theory. Indeed, in the holographic model we shall construct in future sections, if the Chern-Simon's coupling in our action is set to zero, then $q_{\rm nr}$ is just a scaling by $u^+$ of $q$. Reducing the spatial part of the current leads to:

$$\begin{aligned} J^i &= J^+ v^i - \frac{\kappa_q}{u^+}\partial^i J^+ - \mathfrak{V}(u^+)^2 \left[\epsilon^{ij} v^k \left(\partial_j v_k - \partial_k v_j\right) + v^i \epsilon^{jk} \partial_j v_k\right] \\ &\quad - \left(\frac{d-2}{d}\right)\left[J^+\left(\frac{(d+2)}{d-2}\frac{\eta u^+}{\rho} - \frac{\kappa_q}{u^+}\right)\delta^{ij} + \mathfrak{V}\epsilon^{ij}\right]\left(\frac{\partial_j \varepsilon}{2\varepsilon}\right) \\ &\quad + \left[\left(\frac{\kappa_q}{u^+} + \frac{\eta u^+}{\rho}\right)\delta^{ij} + \mathfrak{V}\epsilon^{ij}\right]\left(\frac{\partial_j \rho}{2\rho}\right) \,. \end{aligned}$$

Additionally, in light of the above expansion, it seems reasonable to define $\kappa_{\rm nr} = \frac{\kappa_q}{u^+}$ and $\mathfrak{V}_{\rm nr} = \mathfrak{V}(u^+)^2$. The remaining terms, proportional to $\partial_i P$ and $\partial_i \rho$, merit further consideration for the same reasons as the final terms of Eq. (4.1.20) and we shall return to them later. However, using the equation of state to find $\varepsilon$ in terms of $\varepsilon_{\rm nr}$ the spatial part of the current can be rewritten as

$$\begin{aligned} J^i &= q_{\rm nr} v^i - \kappa_{\rm nr}\partial^i q_{\rm nr} - \mathfrak{V}_{\rm nr}\left[\epsilon^{ij} v^k \left(\partial_j v_k - \partial_k v_j\right) + v^i \epsilon^{jk}\partial_j v_k\right] \\ &\quad - \left(\frac{d-2}{d}\right)\left[q_{\rm nr}\left(\frac{(d+2)}{d-2}\frac{\eta_{\rm nr}}{\rho} - \kappa_{\rm nr}\right)\delta^{ij} + \frac{2d}{d-2}\frac{\mathfrak{V}_{\rm nr}\varepsilon_{\rm nr}}{\rho}\epsilon^{ij}\right]\left(\frac{\partial_j \varepsilon_{\rm nr}}{2\varepsilon_{\rm nr}}\right) \\ &\quad + \left[q_{\rm nr}\left(\kappa_{\rm nr} + \frac{\eta_{\rm nr}}{\rho}\right)\delta^{ij} + \frac{2\,(d+2)}{d}\frac{\mathfrak{V}_{\rm nr}\varepsilon_{\rm nr}}{\rho}\epsilon^{ij}\right]\left(\frac{\partial_j \rho}{2\rho}\right) \end{aligned}$$

which represents the maximal possible reduction into our chosen non-relativistic operators. This matches Eq. (4.1.12) if we identify the coefficient of $\partial_j \varepsilon_{\rm nr}$ with ${\gamma_{\rm nr}}^{ij}$ and that of $\partial_j \rho$ with $-{\digamma_{\rm nr}}^{ij}$.

So far we have only related the non-relativistic and relativistic variables. A priori we have no knowledge of the functional form of the relativistic charge coefficients in terms of local charge and temperature profiles. Hence we also do not know the functional forms of the non-relativistic coefficients in terms of the corresponding non-relativistic charge and temperature. In the next section we specialise our discussion to one particular fluid; a charged, Schrödinger invariant fluid which has as its dual a charged black hole with asymptotically Schrödinger boundary. The holographic dictionary in Schrödinger space-times will then readily allow us to calculate these dependencies; an otherwise difficult problem for a general fluid.

## 4.2 A fluid dual to Schrödinger spacetimes

### 4.2.1 Background metric

As our focus is now on Schrödinger fluids with gravity duals we can apply the holographic dictionary in Schrödinger space-times [45, 87, 88] to our investigation. This provides a map from an asymptotically Schrödinger space-time to a boundary field theory with Schrödinger symmetry and hence allows us to determine expectation values of the spatial stress tensor, energy current and charge current at zeroth order. In particular, we begin with an uplift for Reissner-Nordstrøm AdS which we can TsT transform to give the bulk fields boundary Schrödinger symmetry. To calculate the temperature and charge profiles corresponding to our dual black hole we will need to construct a suitable action yielding these bulk fields. We lean heavily on the formalism of [119] and find a 10-dimensional action from an effective description of Type IIB string theory whose Kaluza-Klein reduction leads to a five dimensional theory with the correct field content. It turns out that this reduced action contains only four fundamental fields; the gravitational field, the dilaton, an $R$-charged gauge field and a massive vector that is now standard fare in Schrödinger spacetimes [85,87–89]. We complete this section by calculating the thermodynamics of our bulk spacetime, in particular, the specific heat at constant pressure, particle number and charge.

We would now like to induce Schrödinger symmetry on the boundary of the spacetime given by solving the equations of motion coming from Eq. (2.2.77) for

a charged black hole. To do this we can apply one of a pair of solution generating techniques at the level of the equations of motion; the Null-Melvin twist [122, 123] or TsT [124]. They both take a Reissner-Nordstrøm $AdS_5 \times \chi^5$ manifold where $\chi^5$ is Sasaki-Einstein and yield an asymptotically Schrödinger charged black hole with a deformed $\chi^5$. The former, NMT, begins by boosting our solution along one of the spatial isometry directions, say $y$ by a rapidity of $\gamma$. Two T-dualisations along $y$ are then performed with a twist of the one-form $d\psi \to \upsilon d\psi$ sandwiched between them. We then boost the resultant fields by $-\gamma$. Finally $\upsilon$ is scaled to zero and $\gamma \to \infty$ while keeping $\beta = \frac{1}{2}\, upsilone^{\gamma}$ constant. The latter technique, TsT, involves a twist in the $x^- = \frac{1}{2\beta}(y-t)$ direction by $\alpha d\psi$ between two T-dualisations along the $\psi$ direction. This second technique can be applied to any spacetime with a $U(1) \times U(1)$ isometry. Moreover, when one of the $U(1)$ isometries is null it can be shown that the two techniques coincide [33].

Applying a TsT along the Hopf direction $\psi$ as discussed in appendix B, where we also give the heavier details of our notation, leads to the following fields

$$\begin{aligned}
(ds^2_{10})'' &= \frac{r^2}{k}\left[-\beta^2 r^2 f(m,Q,r)(dt+dy)^2 - f(m,Q,r)dt^2 + dy^2 + k d\mathbf{x}^2\right] \\
&\quad + \frac{dr^2}{r^2 f(m,Q,r)} + \frac{\left(d\psi + \mathcal{A}_{(1)} - \frac{2}{\sqrt{3}}A_Q\right)^2}{k} + d\Sigma_4^2 \,, \qquad (4.2.22)\\
B''_{(2)} &= \frac{\beta r^2}{k}\,(dy + f(m,Q,r)dt) \wedge \left(d\psi + \mathcal{A}_{(1)} - \frac{2}{\sqrt{3}}A_Q\right) \,, \qquad (4.2.23)\\
F''_{(3)} &= -\frac{2Q\beta}{r^3} J_{(2)} \wedge dr \,, \qquad (4.2.24)\\
F''_{(5)} &= F_{(5)} + B''_{(2)} \wedge F''_{(3)} \,, \qquad (4.2.25)\\
\exp(2\Phi'') &= \frac{1}{k} \,, \qquad (4.2.26)
\end{aligned}$$

where we have set $\alpha = 1$ so that boost and twist parameters of NMT and TsT respectively coincide. We have defined

$$k = 1 + \beta^2 r^2 (1 - f(m,Q,r)) \qquad (4.2.27)$$

in correspondence with the notation of [87] and note that our results are in agreement with [119]. The operation $*''_{10}$ is the Hodge dual in the Melvinised spacetime and its definition is also detailed in the appendix.

The self-duality of the 5-form $F''_{(5)}$ allows us to determine a relation between $*_5 F_Q$ in the un-Melvinised spacetime and the quantities $\Phi''$, $f$, $A_M$ and $F_Q$ in the Melvinised spacetime[1]. Hence we can write

$$\begin{aligned}\mathcal{S}_5 &= \frac{1}{16\pi G_5}\int\Big[\mathrm{vol}_{M''}e^{-2\Phi}\left(R^{(5)}+16-4e^{2\Phi}\right)-4e^{\Phi}F\wedge *''_5F-\frac{1}{2}e^{-3\Phi}F_M\wedge *''_5F_M\\ &\quad-4e^{-\Phi}A_M\wedge *''_5A_M-\frac{2}{3}e^{-\Phi}\left(A_M\wedge F_Q\right)\wedge *''_5\left(A_M\wedge F_Q\right)\\ &\quad-4e^{-\Phi}\left(\frac{1}{\sqrt{3}}F_Q+F\wedge A_M\right)\wedge *''_5\left(\frac{1}{\sqrt{3}}F_Q+F\wedge A_M\right)\\ &\quad-\frac{2}{3}e^{\Phi}F_Q\wedge *''_5F_Q+\frac{16\kappa_{CS}}{3}A_Q\wedge F_Q\wedge F_Q\Big]\ , \end{aligned}\tag{4.2.28}$$

where, as was discovered in [119], the equation of motion for the $F$ field is completely algebraic

$$F=e^{-2\Phi}*''_5\left(A_M\wedge *''_5\left(\frac{1}{\sqrt{3}}F_Q+F\wedge A_M\right)\right)$$

and it is therefore only an auxiliary field which is merely present to simplify our action.

Considering Eq. (4.2.28) we can see that the Kaluza-Klein reduction of the fields Eqs. (4.2.22)-(4.2.26) in the string frame is

$$\begin{aligned}(ds_5^2)'' &= \frac{r^2}{k}\left[-\beta^2r^2f(m,Q,r)(dt+dy)^2-f(m,Q,r)dt^2+dy^2+kd\mathbf{x}^2\right]\\ &\quad+\frac{dr^2}{r^2f(m,Q,r)}\ , \end{aligned}\tag{4.2.29}$$

$$A_Q = \frac{\sqrt{3}}{2}\frac{Q}{r^2}dt\ , \tag{4.2.30}$$

$$A_M = \frac{\beta r^2}{k}\left(dy+f(m,Q,r)dt\right)\ , \tag{4.2.31}$$

where the massless one form and non-trivial dilaton field come from the metric while the massive vector field originates in the NS-NS two form $B_{(2)}$. We note that the existence of a charged, massless one-form whose boundary value shall be interpreted as sourcing the charge of our fluid and a massive vector field which lacks a corresponding conserved current on the boundary.

With the full 5-dimensional zeroth order metric Eq. (4.2.29) available we can now determine the thermodynamics of our fluid. Given $m$ and $Q$, our metric has

[1] See appendices for further details.

a horizon whenever $f(m,Q,r) = 0$ and hence will have an associated Hawking temperature $T$. As before we take $r_+$ to be the location of the outermost horizon which turns out to be Killing as our spacetime is stationary. The temperature can be found by determining the surface gravity, $\kappa$, from the following formula

$$\kappa^2 = -\frac{1}{2}\left(\nabla_\mu \xi_\nu\right)\left(\nabla^\mu \xi^\nu\right)\Big|_{r=r_+} , \tag{4.2.32}$$

where $\xi^a$ is the null generator associated with the horizon. We are working in the Einstein frame attained from Eq. (4.2.29) by conformally rescaling the metric with the dilaton

$$ds_E^2 = e^{-\frac{2}{3}\Phi''}(ds_5^2)'' . \tag{4.2.33}$$

For Eq. (4.2.33) the (Killing) vector generating the horizon is proportional to $(\partial_t)^a$. To determine the constant of proportionality we note that in lightcone coordinates, $x^+$ and $x^-$, $(\partial_+)^a$ is the generator of boundary time translations and hence if we set its coefficient to be unit we find:

$$\begin{aligned} \xi^a &= \frac{1}{\beta}(\partial_t)^a \\ &= (\partial_+)^a - \frac{1}{2\beta^2}(\partial_-)^a \ . \end{aligned} \tag{4.2.34}$$

Substituting this result into Eq. (4.2.32) the temperature, $T = \frac{\kappa}{2\pi}$, is:

$$T = \frac{r_+}{2\pi\beta}\left(2 - \frac{Q^2}{{r_+}^6}\right) . \tag{4.2.35}$$

In the limit that $Q \to 0$ this coincides with the result of [33]. Of note, the temperature of the relativistic precursor theory is given in terms of the non-relativistic temperature Eq. (4.2.35) by $\beta T$.

The entropy associated with Eq. (4.2.29) can be calculated using the Bekenstein-Hawking formula

$$S = \frac{A}{4G_5} ,$$

where $A$ is the area of the event horizon at $r = r_+$. The solution turns out to be independent of $\beta$ which is to do with the fact that our metric was generated by a series of boosts and dualities from Eq. (2.2.74) as discussed in [123]. For our solution, taking $V_3$ to be volume of the horizon, we find the entropy in the boundary

is given by

$$S = \frac{{r_+}^3}{4G_5} V_3 \tag{4.2.36}$$

whose form in terms of $r_+$ is unchanged from the uncharged case discussed in [33].

In Eq. (4.2.34) we have a generator of time translation on the boundary $(\partial_+)^a$ and an additional generator $(\partial_-)^a$. The former corresponds to the Hamiltonian $\hat{H}$ in the dual field theory while the latter should be interpreted as the particle number generator $\hat{N}$. Hence the coefficient of $(\partial_-)^a$ in the null generator $\xi^a$ is the particle number chemical potential:

$$\mu = -\frac{1}{2\beta^2} \ .$$

However, this is not the only chemical potential in the thermodynamics of our solution as the charge can also vary. This charge chemical potential can be found from the asymptotic values of the "boundary time" component of the gauge field and is given by:

$$\begin{aligned} \mu_q &= A_+(r_+) - A_+(\infty) \\ &= \frac{\sqrt{3}Q}{4\beta {r_+}^2} \\ &= \frac{\sqrt{6}}{4} \frac{Q\,(-\mu)^{\frac{1}{2}}}{{r_+}^2} \ . \end{aligned} \tag{4.2.37}$$

Note that we have explicitly defined both the chemical potentials as non-relativistic quantities in this chapter.

We now have sufficient information to specify our density matrix and can therefore obtain the thermodynamic potential of our ensemble. The existence of two chemical potentials implies that the density matrix has the following form:

$$\hat{\rho} = \exp\left[ -\left( \frac{\hat{H} - \mu\hat{\partial}_- - \mu_q \hat{J}^+}{T} \right) \right] \ .$$

The trace of the density matrix gives us the partition function whence the thermodynamic potential is determined by:

$$\tilde{G}(V_2, T, \mu, \mu_q) = -T \ln \Xi(T, \mu, \mu_q) \ , \tag{4.2.38}$$

where $\Xi = \mathrm{tr}\,(\hat{\rho})$ and $V_2$ is the two-dimensional spatial volume of the Schrödinger theory. This free energy represents the work done by the system when the chemical

potential difference between two neighbouring regions of equilibrium changes in an isothermal and isochoric process. The Gibbs potential Eq. (4.1.1) can be obtained from Eq. (4.2.38) by a Legendre transform. Before we do this however we need to obtain the charge and particle number from $\tilde{G}$.

We can obtain the relativistic energy $E$ by computing the ADM mass of Eq. (4.2.22). Using the relativistic equation of state $\varepsilon = 3P$ and Eq. (4.1.17) gives us the non-relativistic pressure $P_{\rm nr}$. Finally, using the integrated form of the first law of thermodynamics and the definition of the free energy[2] implies

$$\begin{aligned} \tilde{G} &= E - TS - \mu N - \mu_q J^+ && (4.2.39)\\ &= -P_{\rm nr}(T,\mu,\mu_q)V_2 \\ &= -\frac{V_2 \Delta x^-}{16\pi G_5}\left( {r_+}^4 - \frac{8}{3}\frac{{\mu_q}^2}{\mu}{r_+}^2 \right) && (4.2.40) \end{aligned}$$

where $\Delta x^-$ has been introduced to characterise the period of the compactified $x^-$-direction [33]. Using the thermodynamic relations supplied by

$$d\tilde{G} = \underbrace{\left(\frac{\partial G}{\partial V}\right)_{T,\mu,\mu_q}}_{-P_{\rm nr}(x)} dV + \underbrace{\left(\frac{\partial G}{\partial T}\right)_{V,\mu,\mu_q}}_{-S(x)} dT + \underbrace{\left(\frac{\partial G}{\partial \mu}\right)_{V,T,\mu_q}}_{-N(x)} d\mu + \underbrace{\left(\frac{\partial G}{\partial \mu_q}\right)_{V,T,\mu}}_{-J^+(x)} d\mu_q$$

we find the following additional quantities

$$N = -\tfrac{2P_{\rm nr}V_2}{\mu} \qquad J^+ = \tfrac{\sqrt{3}\beta Q}{2\pi G_5}\Delta x^- V_2 \qquad S = \tfrac{\beta {r_+}^3}{4G_5}\Delta x^- V_2 \ .$$

In particular the entropy, at constant $t$, matches that obtained from Eq. (4.2.36) as $\beta \Delta x^- V_2 = V_3$. Neither $J^+$ nor $\mu_q$ attain their relativistic values on setting $\beta = 1$ however the combination $\mu_q J^+$ does. We also note that the energy density calculated by inverting Eq. (4.2.39) and dividing by the spatial volume $V_2$ satisfies the equation of state for a non-relativistic fluid

$$\begin{aligned} \varepsilon &= \frac{\Delta x^-}{16\pi G_5}\left( {r_+}^4 - \frac{8}{3}\frac{{\mu_q}^2}{\mu}{r_+}^2 \right) \\ &= P_{\rm nr} \end{aligned}$$

[2] Our formula for the free energy does not appear to agree with the on-shell action of [119]. An important consistency check of the on-shell action is that one recover the correct entropy i.e.; $S = -\left(T\frac{\partial}{\partial T} + 1\right)\frac{\tilde{G}}{T}$. It is not clear to us that the expression in [119] passes this check. On the contrary, because we are working in the grand canonical ensemble and have derived Eq. (4.2.40) by calculating the pressure from the ADM mass of the black hole, $\tilde{G}$ definitely passes this test.

which acts as a check of our thermodynamics.

The Gibbs potential is given by the Legendre transform

$$\begin{aligned} G(P_{\rm nr}, T, N, J^+) &= \tilde{G} + P_{\rm nr} V_2 + \mu N + \mu_q J^+ \\ &= \mu N + \mu_q J^+ \end{aligned}$$

where all the quantities on the right-hand side are functions of $P$, $T$, $N$ and $J^+$. Using Eq. (4.2.40) we find

$$G = -\frac{\Delta x^- V_2}{8\pi G_5}\left[{r_+}^4 + \frac{16}{3}\frac{{\mu_q}^2}{\mu}{r_+}^2\right] . \tag{4.2.41}$$

The specific heat at constant pressure, particle number and charge is then given by

$$\begin{aligned} c_{P_{\rm nr},N,J^+} &= \frac{\pi^2 T}{{r_+}^2\left(T, \frac{\mu}{T}, \frac{\mu_q}{T}\right)} \times \\ &\quad \left[1 - \frac{4}{3}\frac{{\mu_q}^2}{\mu {r_+}^2\left(T, \frac{\mu}{T}, \frac{\mu_q}{T}\right)} + \frac{64}{9}\frac{{\mu_q}^4}{\mu^2 {r_+}^4\left(T, \frac{\mu}{T}, \frac{\mu_q}{T}\right)}\right]^{-1} , \end{aligned} \tag{4.2.42}$$

where $r_+\left(T, \frac{\mu}{T}, \frac{\mu_q}{T}\right)$ is given by Eq. (4.1.11). In the limit that the charge goes to zero this approaches the result for the heat capacity given by [33]. We note that thermodynamic quantities like Eq. (4.2.41) and Eq. (4.2.42) will also apply to the first-order Schrödinger fluid of the next section.

### 4.2.2 First order metric corrections

With the well-established relativistic fluid-gravity derivative expansion, given in chapter 2, as a skeleton we can now consider moving beyond zeroth order for a fluid with Schrödinger symmetry. In particular we shall find the derivative expansion corrections to Eq. (4.2.29) at first order. We begin by noting that Eq. (4.2.29) is not regular across the future horizon due to a coordinate singularity. One of the nice properties of the relativistic fluid gravity metric was its regularity at all points other than $r = 0$. This can be remedied by translating the $x^+$ and $x^-$ coordinates in the following manner:

$$\begin{aligned} dx^+ &\to dx^+ - \frac{\beta}{r^2 f(m,Q,r)} dr , \\ dx^- &\to dx^- + \frac{1}{2\beta r^2 f(m,Q,r)} dr . \end{aligned}$$

Our metric, gauge field and massive vector field correspondingly become

$$\begin{aligned}(ds_5^2)'' &= \frac{r^2}{k}\left[\left(\frac{1-f(m,Q,r)}{4\beta^2}-r^2 f\right)\left(dx^+\right)^2+\beta^2\left(1-f(m,Q,r)\right)\left(dx^-\right)^2\right.\\ &\quad +(1+f(m,Q,r))dx^+dx^- +\left(\frac{1}{\beta r^2}+2\beta\right)dx^+dr\\ &\quad \left.-\left(\frac{2\beta}{r^2}\right)dx^-dr+kd\mathbf{x}^2\right]-\beta^2\frac{dr^2}{k}\ , \qquad (4.2.43)\\ A_Q &= \frac{\sqrt{3}}{2}\frac{Q}{r^2}\left[\frac{dx^+}{2\beta}-\beta dx^-\right]\ , \qquad (4.2.44)\\ A_M &= \frac{\beta r^2}{k}\left[\left(1+f(m,Q,r)\right)\frac{dx^+}{2\beta}+\left(1-f(m,Q,r)\right)\beta dx^- -\frac{dr}{r^2}\right]\ , \qquad (4.2.45)\end{aligned}$$

where we have used a gauge choice to remove a $dr$ term from $A_Q$. It should be noted that because of the dominant scaling of the $x^+$ term in Eq. (4.2.43) we lack a well-defined boundary metric for the asymptotically Schrödinger charged black hole.

Our boundary theory has Galilean symmetry and by boosting Eq. (4.2.43) we obtain a class of solutions with the same thermodynamics but non-zero velocity. The boosted solutions will be classified by five constants $\beta$, $v^i$, $Q$ and $m$ however these contain no new physics. Instead, to obtain results other than the ideal fluid, we need to promote the parameters $m$, $Q$, $\beta$ and $v^i$ to functions of the boundary coordinates and find first order fluid corrections to our fields. A local Galilean boost has the form:

$$\begin{aligned}\mathbf{x} &\to \mathbf{x}+\mathbf{v}(x)x^+\ ,\\ x^- &\to x^-+\mathbf{v}(x)\cdot\mathbf{x}+\frac{1}{2}\mathbf{v}^2(x)x^+\ .\end{aligned}$$

Consider the vicinity of a point $x^\mu=0$ and use global Galilean invariance to set the velocity at this point to zero. Performing our local boost on Eqs. (4.2.43)-(4.2.45), to one derivative in velocity, we find the following additional terms:

$$\begin{aligned}g^{(1)} &= \frac{2\beta^2r^2}{k}\left[(1-f(m,Q,r))dx^-\mathbf{x}+\frac{1}{2}(1+f(m,Q,r))dx^+\mathbf{x}\right.\\ &\quad \left.+\frac{1}{\beta r^2}\mathbf{x}dr+\frac{kx^+}{\beta^2}d\mathbf{x}\right]\cdot d\mathbf{v}\ ,\\ A_Q^{(1)} &= -\frac{\sqrt{3}}{2}\frac{Q}{r^2}\beta\mathbf{x}\cdot d\mathbf{v}\ ,\\ A_M^{(1)} &= +\frac{\beta^2r^2}{k}(1-f(m,Q,r))\mathbf{x}\cdot d\mathbf{v}\ .\end{aligned}$$

Again, generically, the metric, gauge field and massive vector field with the above terms will not satisfy the equations of motion from Eq. (4.2.28) and as such we need to find suitable corrections. Our precursor fields Eqs. (4.2.29)-(4.2.31) have $SO(2)$ spatial invariance and thus we can parameterise our corrections with respect to this symmetry much as we did with the $SO(3)$ symmetry in the relativistic case. However we face two additional complications for the asymptotically Schrödinger spacetime. Firstly, we not only have new equations of motion for the massive vector field and dilaton to satisfy but our equations of motion have become more complex. Secondly, the $SO(2)$ symmetry is not as helpful in the Schrödinger case as the $SO(3)$ was in the relativistic case. For example, in the Schrödinger case there are two possible vector sectors in the metric coming from $dx^+dx^i$ and $dx^-dx^i$ compared to one, $dvdx^i$, for a relativistic fluid.

It is clear then that solving the equations of motion from Eq. (4.2.28) to first order in derivatives would be a cumbersome process. Instead, as previously mentioned, we can perform a TsT transformation of Eqs. (2.2.120) and (2.2.121) to obtain the $U(1)$ charged, Schrödinger fluid at first order. As we previously assumed trivial $x^-$ dependence in our hydrodynamic variables in order to light-cone reduce, we can drop all $x^-$ dependence in the metric coefficients. This ties in nicely with the fact that to use the TsT solution generating technique detailed in the appendix we require $x^-$ and $\psi$ to be isometry directions of the metric. In particular notice that $x^-$ is a null isometry direction in the boundary theory and so the TsT with a twist along this direction will coincide with an NMT of our fields.

Using the identities from [33], in particular the zeroth order current conservation equations, our results indicate that the dilaton now has the form

$$\begin{aligned} e^{-2\Phi''} &= k \\ &= 1+\beta^2 r^2\left[1-f(m,Q,r)\right]+\frac{2\sqrt{3}\beta^3 Q^3}{mr^4}\kappa_{CS}\epsilon^{ij}\partial_i v_j\ , \end{aligned} \tag{4.2.46}$$

where $i,j$ run over the spatial directions $\{x,z\}$ and we have taken $u^+=\beta$. The full

metric at first order is given by:

$$\begin{aligned}(ds_5^2)'' &= -2u_\mu dx^\mu dr - r^2 f(m,Q,r) u_\mu u_\nu dx^\mu dx^\nu + r^2 \Pi_{\mu\nu} dx^\mu dx^\nu \\ &\quad -2r u_\mu \left(u^\lambda \nabla_\lambda u_\nu\right) dx^\mu dx^\nu + \frac{2}{3} r \left(\nabla_\lambda u^\lambda\right) u_\mu u_\nu dx^\mu dx^\nu \\ &\quad +2\frac{r^2}{r_+} F_1(m,Q,r)\eta_{\mu\nu} dx^\mu dx^\nu - \frac{2\sqrt{3}\kappa_{CS} Q^3}{m r^4} u_\mu l_\nu dx^\mu dx^\nu \\ &\quad -12Q\frac{r^2}{{r_+}^7} F_2(m,Q,r) u_\mu \left(\Pi^\lambda_\nu \nabla_\lambda + 3u^\lambda \nabla_\lambda u_\nu\right) Q dx^\mu dx^\nu \\ &\quad -k \left(A_M\right)_\mu \left(A_M\right)_\nu dx^\mu dx^\nu \ . \end{aligned} \tag{4.2.47}$$

The massive vector field, in the light-cone coordinate system, has the form

$$\begin{aligned} A_M &= \frac{1}{k}\Bigg[\left(\beta^2 r^2 \left(1 - f(m,Q,r)\right) + \frac{2\sqrt{3}}{m r^4}\kappa_{CS} Q^3 \beta^3 \varepsilon_{jk}\partial^j v^k\right) dx^- + \left(r^2 dx^+ - \beta dr\right) \\ &\quad + \left(\beta r^2 \left(1 - f(m,Q,r)\right) + \frac{\sqrt{3}\kappa_{CS} Q^3 \beta^2}{m r^4}\varepsilon_{jk}\partial^j v^k\right) u_\alpha dx^\alpha \\ &\quad + \Bigg(2\frac{r^2}{r_+} F_1 \eta_{-\alpha} - \frac{\sqrt{3}\kappa_{CS} Q^3 \beta}{m r^4} l_\alpha - 6\beta Q \frac{r^2}{{r_+}^7} F_2 \left(\nabla_\alpha Q - \frac{3}{2} Q \frac{\nabla_\alpha P_{\rm nr}}{\varepsilon_{\rm nr} + P_{\rm nr}}\right) \\ &\quad + \frac{\beta r}{2}\frac{\nabla_\alpha P_{\rm nr}}{\varepsilon_{\rm nr} + P_{\rm nr}}\Bigg) dx^\alpha\Bigg] \ , \end{aligned}$$

where $\alpha \in \{+, x, z\}$ and $A_Q$ is unchanged from Eq. (2.2.121) by the TsT. Note that all but the last line of the metric Eq. (4.2.47) occurs in Eq. (2.2.120) so that all the deformation comes from the vector field $A_M$. This makes sense in light of the fact that the massive vector field is the only additional dynamical field between the Melvinised and un-Melvinised solutions.

### 4.2.3 Transport coefficients

Now that the bulk metric and vector fields are known we can return to the fluid side of the fluid-gravity correspondence. To calculate the hydrodynamic and charge coefficients we need to determine the boundary value of the conserved currents associated with the bulk metric and the gauge field. However, we face a particular problem in Schrödinger spacetimes due to the slow asymptotic fall-off of the modes. We follow [88], [90] and [33] and interpret the SEM tensor of the asymptotically AdS theory prior to the TsT transformation as a tensor complex (collection of fields) in

the non-relativistic theory. This complex is given exactly by the identifications in Eq. (4.1.14).

To convert our relativistic results into their non-relativistic counterparts we need to fix the normalisation of our boundary velocity $u^\mu$. Fortunately we have already isolated a suitable choice in Eq. (4.2.34) when we fixed the coefficient of $(\partial_+)^a$ in the horizon null generator, $\xi^a$, to be unit. Hence we take:

$$\begin{aligned} u^+ &= \beta \,, \\ u^i &= \beta v^i - \frac{\beta\eta}{\rho}\delta^{ij}\left(\partial_j\beta - \frac{\beta}{(\varepsilon+P)}\partial_j P\right) \,. \end{aligned} \tag{4.2.48}$$

Using the maps Eqs. (4.1.16)-(4.1.20), (4.1.21) and (4.1.22) it is now possible to determine all the non-relativistic quantities in terms of $m$, $Q$, $\beta$ and $v^i$. Modulo the subtlety involving the thermal conductivity which we shall discuss next the zeroth order coefficients are:

$$\begin{aligned} \varepsilon_{\rm nr} = P_{\rm nr} \,, \qquad P_{\rm nr} = \tfrac{m}{16\pi G_5}\Delta x^- \,, \qquad \rho = -\tfrac{2P_{\rm nr}}{\mu} \,, \\ q_{\rm nr} = \frac{\sqrt{3}\beta Q}{4\pi G_5}\left[1 + \frac{\sqrt{3}\beta Q\kappa_{CS}}{m}\varepsilon_{jk}\partial^j v^k\right]\Delta x^- \,. \end{aligned} \tag{4.2.49}$$

The apparent disparity for $q_{\rm nr}$ with the thermodynamic result is due to the previous decomposition of the combination $\mu_q J^+$ into $\mu_q$ and $J^+$ which shuffled a factor of two between them. At first order we also have

$$\begin{aligned} \eta_{\rm nr} = \tfrac{\beta {r_+}^3}{16\pi G_5}\Delta x^- \,, \qquad \kappa_{\rm nr} = \left(\tfrac{{r_+}^4+m}{4m\beta r_+}\right)\Delta x^- \,, \qquad \mho_{\rm nr} = -\tfrac{\kappa_{CS}}{2P_{\rm nr}}{q_{\rm nr}}^2 \,, \\ {\gamma_{\rm nr}}^{ij} = \left(\frac{1}{4\varepsilon_{\rm nr}}\right)\left[q_{\rm nr}\left(3\frac{\eta_{\rm nr}}{\rho} - \kappa_{nr}\right)\delta^{ij} + 4\frac{\mho_{\rm nr}\varepsilon_{\rm nr}}{\rho}\epsilon^{ij}\right]\Delta x^- \,, \\ {\mathcal{F}_{\rm nr}}^{ij} = -\left(\frac{1}{2\rho}\right)\left[q_{\rm nr}\left(\kappa_{nr} + \frac{\eta_{\rm nr}}{\rho}\right)\delta^{ij} + 4\frac{\mho_{\rm nr}\varepsilon_{\rm nr}}{\rho}\epsilon^{ij}\right]\Delta x^- \,. \end{aligned} \tag{4.2.50}$$

where the $\Delta x^-$ factors were introduced to ensure the above quantities are volume densities with respect to the two-dimensional spatial volume $V_2$. The parity violating term $\mho_{\rm nr}$ already multiplies an object that is first order in fluid derivatives, see Eq. (4.1.22). Hence when expressing it in terms of $q_{\rm nr}$ we have dropped any additional velocity derivatives. Similarly for replacing $\gamma$ in ${\gamma_{\rm nr}}^{ij}$.

We would now like to extract the thermal conductivity. Re-expressing the final two terms of Eq. (4.1.20) as the differential of a logarithm and using Eq. (4.2.35)

and Eq. (4.2.49) we can write:

$$\begin{aligned} & -\frac{2\eta_{\rm nr}P_{\rm nr}}{\rho}\delta^{ij}\partial_j \ln\left(\frac{{P_{\rm nr}}^{\frac{3}{2}}}{\rho}\right) \\ = & -\frac{4\eta_{\rm nr}P_{\rm nr}}{\rho T}\delta^{ij}\partial_j T \\ & -\frac{2\eta_{\rm nr}}{\rho}\delta^{ij}\partial_j\left[\frac{1}{2}\ln\left(1+\frac{Q^2}{{r_+}^6}\right)-2\ln\left(1-\frac{Q^2}{2{r_+}^6}\right)\right] . \end{aligned} \tag{4.2.51}$$

If we interpret the thermal conductivity as the coefficient of the term with no explicit charge dependence then $\kappa_T$ has the same functional dependence on $\eta_{\rm nr}$, $P_{\rm nr}$, $\rho$ and $T$ as in the uncharged case of [33]:

$$\kappa_T = \frac{4\eta_{\rm nr}P_{\rm nr}}{\rho T} . \tag{4.2.52}$$

As promised there is a new term which vanishes if the local charge density is set to zero. On substituting for $Q$ in terms of $\mu_q$ using Eq. (4.2.37) and rearranging we obtain the final term of Eq. (4.1.10) with $\varpi = \kappa_T$.

We would now like to calculate the Prandtl number for the fluid. We first note that the kinematic viscosity, which compares the importance of viscous to inertial forces, is defined by

$$\nu = \frac{\eta_{\rm nr}}{\rho} ,$$

where $\rho$ is representative of inertia. The thermal diffusivity is defined by

$$\chi = \frac{\kappa_T}{\rho c_{P_{\rm nr},N,J^+}} ,$$

where $\kappa_T$ measures heat flow from a region of local equilibrium while $\rho c_{P_{\rm nr},N,J^+}$ measures the ability of the region to adjust its temperature to match its surroundings. Thus when $\chi$ is large the region in question quickly responds to the temperature of neighbouring regions and equilibriates its temperature.

The Prandtl number is given by the ratio of kinematic viscosity to thermal diffusivity

$$\Pr = \frac{\nu}{\chi}$$

and thus represents the relative importance of viscous effects and heat conduction in reaching steady state flow. Using Eqs. (4.2.42), (4.2.49), (4.2.50) and (4.2.52) we

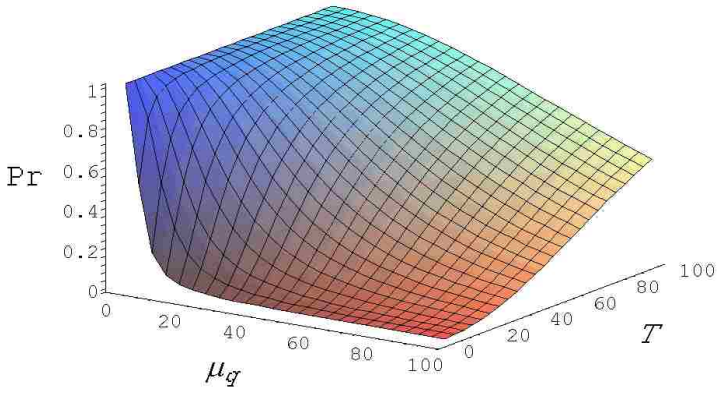


Figure 4.1: A graph of the Prandtl number against $\mu_q$ and $T$. We note that the Prandtl number clearly tends to one when the charge vanishes.

find this number to be

$$\mathrm{Pr} = \frac{\pi^2 T^2}{2}\left[\frac{4}{3}{\mu_q}^2 - {\mu r_+}^2\left(T, \frac{\mu}{T}, \frac{\mu_q}{T}\right) - \frac{64}{9}\frac{{\mu_q}^4}{{\mu r_+}^2\left(T, \frac{\mu}{T}, \frac{\mu_q}{T}\right)}\right]^{-1},$$

where $r_+\left(T, \frac{\mu}{T}, \frac{\mu_q}{T}\right)$ is given by Eq. (4.1.11). Disappointingly this indicates that the fluid does not achieve a universal value in the presence of a conserved electric charge unlike the uncharged case of [33] where it is identically one. Of note is the fact that Pr is independent of the particle number chemical potential and compactification radius $\Delta x^-$. Figure (4.1) is a diagram giving an indication of the dependence of Pr on charge chemical potential and temperature.

In the above we have found the transport coefficients of a class of non-relativistic fluids which are compressible and thus support sound modes. Given some fluctuation of these fluids the behaviour of the Prandtl number with charge chemical potential implies that increases in this chemical potential lead to heat conduction becoming more important in dissipating fluctuations than viscous effects. This number is bounded above in our example by the value of one which indicates viscous effects can never be more significant than heat conduction. However this should not be conjectured to be a bound on all fluids, like the viscosity to entropy ratio was [28], as it is simple to find counterexamples. For example, water has a Prandtl number of approximately 7 at room temperature.

# Chapter 5

# Beyond hydrodynamics

So far we have focused exclusively on using the AdS-CMT correspondence in a regime where the effective field theory governing the boundary physics is hydrodynamics. In this chapter we shall show how to go beyond the small $\frac{\omega}{T}$ and $\frac{k}{T}$ regime to explore more exotic aspects of our condensed matter models and compute the retarded Green's function for arbitrary $\omega$ and $k$. Moreover we shall demonstrate how to use hydrodynamics as a consistency check in these limits.

We shall choose to play with a toy-model whose dual contains gravity and a $U(1)$ gauge field. We shall extract the full retarded, boundary correlator corresponding to linearised perturbations of this spacetime something that, at the time of publication of [2], had not be done before. Indeed many previous studies of retarded correlators had restricted themsleves to calculating just the locations of poles in retarded Green's functions or the spectral function.

The quasinormal modes are found by computing the quasinormal frequencies of the bulk theory [5, 125]. Quasinormal modes typically are exponentially decaying sourceless solutions to the classical equations of the bulk theory (see [126], [127]). They have important physical consequences, for example, if a quasinormal mode has a positive imaginary part this suggests a linear instability of the bulk theory and thus a phase transition in the boundary theory as discussed in chapter 1. For the model we shall consider the quasinormal frequencies for shear-type electromagnetic and gravitational perturbations of a Reissner-Nordstrøm $AdS_4$ black hole were computed in [4] and no such instabilites were found.

Here we review [2] and demonstrate how to compute the retarded Green's functions in full for conserved currents in the shear channel, considering arbitrary frequencies and momenta. We will work exclusively at non-zero temperature by studying a non-extremal black hole to explore the detailed structure of these thermal correlators. We show that their poles have varying residues and that this information can be used to assess the dominance of these poles beyond the hydrodynamic regime. After analysing the properties of these correlators for large and small values of the boundary theory parameters, we examine the curious and intricate motion of their poles at intermediate parameter values.

## 5.1 Background and method

Consider the action of Eq. (2.2.61) in four spacetime dimensions with $g_F = \ell^2$ where we remind the reader that $\ell$ is the AdS length scale. We consider gauge and gravitational fluctuations about a Reissner-Nordstom-AdS$_4$ background

$$\begin{aligned} g_{\mu\nu} &\rightarrow g_{\mu\nu} + h_{\mu\nu} \,, \\ A_\mu &\rightarrow A_\mu + a_\mu \,, \end{aligned}$$

and will extract the retarded Green's functions governing the corresponding shear SEM tensor and charge current perturbations in the boundary field theory. Linear response relates the one-point functions and sources in the boundary field theory by

$$\left\langle \hat{T}_{\mu\nu} \right\rangle = G_{\mu\nu,\rho\sigma} h^{\rho\sigma}_{(0)} + G_{\mu\nu,\rho} A^{\rho}_{(0)} \,, \tag{5.1.1}$$

$$\left\langle \hat{J}_{\mu} \right\rangle = G_{\mu,\rho\sigma} h^{\rho\sigma}_{(0)} + G_{\mu,\rho} A^{\rho}_{(0)} \,, \tag{5.1.2}$$

where $h^{\rho\sigma}_{(0)}$ and $A^{\rho}_{(0)}$ are source terms for gravitational and gauge fluctuations respectively and in these lines only indices are raised and lowered with respect to the boundary metric. As the gravitational and gauge perturbations are coupled we must be careful to realise that not all of the Green's functions are independent due to various Ward identities relating them [128–130]. The work of [52] demonstrates these coupled gauge and gravitational fluctuations can be consistently decoupled in terms of appropriate variables. After using rotation invariance in the $(x, y)$ plane to

set the momentum in the $y$ direction to zero, we can focus on correlators of $\hat{J}_y, \hat{T}_{xy}$ and $\hat{T}_{yt}$. A little work indicates that a set of gauge invariant variables are

$$\begin{aligned} X &= \frac{k}{\mu_q} h^y{}_t(r) + \frac{\omega}{\mu_q} h^x{}_y(r) \,, \\ Y &= a_y(r) \,, \end{aligned}$$

where indices are raised and lowered with respect to the bulk background metric which shall henceforth be our convention. The pair of master fields, denoted $\Phi_\pm$, in which the linearised equations decouple are defined in terms of these gauge invariant fluctuations by

$$\begin{aligned} \Phi_\pm &= \mu_q \frac{\left(\frac{k}{\mu_q}\right) f(r) r^3 {r_+}^2}{\left(\frac{\omega}{\mu_q}\right)^2 - f(r)\left(\frac{k}{\mu_q}\right)^2} X'(r) \\ &\quad - \frac{2Q^2 r_+}{r}\left[\frac{2\left(\frac{k}{\mu_q}\right)^2 f(r)}{\left(\frac{\omega}{\mu_q}\right)^2 - f(r)\left(\frac{k}{\mu_q}\right)^2} + \frac{r}{r_+} g_\pm\left(\frac{k}{\mu_q}\right)\right] Y(r) \end{aligned}$$

with

$$g_\pm(x) = \frac{3}{4}\left(1 + \frac{1}{Q^2}\right)\left(1 \pm \sqrt{1 + \frac{16}{9}x^2\left(1 + \frac{1}{Q^2}\right)^{-2}}\right) \,. \tag{5.1.3}$$

The resultant decoupled equations are too difficult to solve analytically so we shall apply numerics to extract the desired correlators. Our coordinate range in $r$ however is infinite which is not ideal for the numerical calculations we wish to employ. As such we choose to compactify this coordinate via the transformation $z = \frac{r_+}{r}$ such that $z = 0$ is the boundary and $z = 1$ is the horizon. In terms of this new coordinate the master field equations are

$$z^2 f(f\Phi'_\pm)' + \left[-zff' + Q^2 z^2(\mathfrak{w}^2 - \mathfrak{q}^2 f) - 2Q^2 g_\pm(\mathfrak{q}) z^3 f\right]\Phi_\pm = 0, \tag{5.1.4}$$

where a prime denotes differentiation with respect to $z$. The dimensionless parameters $\mathfrak{w}$ and $\mathfrak{q}$ are the frequency and $x$ momentum normalised with respect to the chemical potential:

$$\mathfrak{w} = \frac{\omega}{\mu_q}, \qquad \mathfrak{q} = \frac{k}{\mu_q} \,.$$

We shall solve Eq. (5.1.4) numerically with infalling boundary conditions at the horizon and read-off the asymptotic fall-offs of $\Phi_\pm(z)$ at the $AdS_4$ boundary which have the form:

$$\Phi_\pm \overset{z\to 0}{\sim} \hat{\Phi}_\pm \left(1 + \hat{\Pi}_\pm z + \ldots\right) . \tag{5.1.5}$$

There are certain technical issues for doing this calculation numerically which we review in appendix C. In terms of the $\Pi_\pm$ the Green's functions displayed in Eqs. (5.1.1) and (5.1.2) are:

$$G_{yt,yt} = \frac{2\mathfrak{q}^2 Q^2(g_-\hat{\Pi}_+ - g_+\hat{\Pi}_-)}{3(g_+ - g_-)} \tag{5.1.6}$$

$$G_{xy,xy} = \frac{2\mathfrak{w}^2 Q^2(g_-\hat{\Pi}_+ - g_+\hat{\Pi}_-)}{3(g_+ - g_-)} \tag{5.1.7}$$

$$G_{xy,yt} = -\frac{\mathfrak{q}\mathfrak{w} Q^2(g_-\hat{\Pi}_+ - g_+\hat{\Pi}_-)}{3(g_+ - g_-)} \tag{5.1.8}$$

$$G_{yt,xy} = -\frac{\mathfrak{q}\mathfrak{w} Q^2(g_-\hat{\Pi}_+ - g_+\hat{\Pi}_-)}{3(g_+ - g_-)} \tag{5.1.9}$$

$$G_{xy,y} = \frac{2\mathfrak{q}\mathfrak{w} Q^2(\hat{\Pi}_+ - \hat{\Pi}_-)}{3\mu_q(g_+ - g_-)} \tag{5.1.10}$$

$$G_{y,xy} = \frac{2\mathfrak{q}\mathfrak{w}(\hat{\Pi}_+ - \hat{\Pi}_-)}{\mu_q(g_+ - g_-)} \tag{5.1.11}$$

$$G_{yt,y} = -\frac{2\mathfrak{q}^2 Q^2(\hat{\Pi}_+ - \hat{\Pi}_-)}{3\mu_q(g_+ - g_-)} \tag{5.1.12}$$

$$G_{y,yt} = -\frac{2\mathfrak{q}^2(\hat{\Pi}_+ - \hat{\Pi}_-)}{\mu_q(g_+ - g_-)} \tag{5.1.13}$$

$$G_{y,y} = -\frac{8(g_+\hat{\Pi}_+ - g_-\hat{\Pi}_-)}{{\mu_q}^2(g_+ - g_-)}. \tag{5.1.14}$$

The normalisation of our two-point operators is arbitrary (as we can pick our operator basis at whim) and hence we have chosen to normalise all correlators with respect to the displayed definition of $G_{y,y}$. At zero temperature these expressions reduce to those of [4]. For the rest of this chapter we shall set $\ell = 1$ and absorb factors of $r_+$ into $(t, x, y)$. One important upshot of this is that $\mu_q = Q$.

## 5.2 The retarded Green's function

To ensure that the retarded Green's functions for $\Phi_\pm$ have been calculated correctly, we must check that the locations of their poles match the quasinormal spectrum for the appropriate bulk fluctuations [5, 125]. In Fig. 5.1 we demonstrate that this is indeed the case for our results. The quasinormal frequencies were computed using the determinant method pioneered in [131] and explained in detail in [132]. We find precise agreement with the quasinormal frequency plots for non-zero temperature shown in [4]. Before continuing we point out the following terminology - we shall refer to poles which are entirely imaginary as "on-axis". As none of our modes are normal modes there will be no confusion over which axis the term "on-axis" refers to. Modes with a real part shall be called "off-axis".

As an example of the results contained in this section consider Fig. 5.2 which displays part of the $G_{xy,y}$ correlator, given in Eq. (5.1.10).

### 5.2.1 Matching to hydrodynamics

Let's begin our discussion of the retarded correlator by considering how to reproduce the hydrodynamic results. This will give us an important cross check of our numerics and justify using them to search deeper into the complex frequency plane. Essentially we shall reproduce the analytical expressions for $\hat{\Pi}_\pm$ found in [133] with an eye to matching our notation.

As discussed above the matrix of corrrelators can be constructed from $\hat{\Pi}_\pm$. We can perform a linearised perturbation analysis in small $\mathfrak{w}$ and $\mathfrak{q}$ as in [30]. Parity considerations will ensure that there is no linear piece in $\mathfrak{q}$ in our dispersion relation[1] but as we shall see there will be an order $\mathfrak{w}$ piece. To begin, let's extract the infalling piece of the field and define two functions $F(z)$ and $G(z)$

$$\Phi_\pm(z) = (z-1)^{-i\mathfrak{w}\mu_q/4\pi T} F_\pm(z) G_\pm(z). \tag{5.2.15}$$

[1] To see that the dispersion relation cannot depend on $\mathfrak{q}$ but only powers of $\mathfrak{q}^2$ it is sufficient to examine the equations of motion, Eq. (5.1.4), and the boundary conditions to see that $\mathfrak{q} \to -\mathfrak{q}$ is a symmetry.

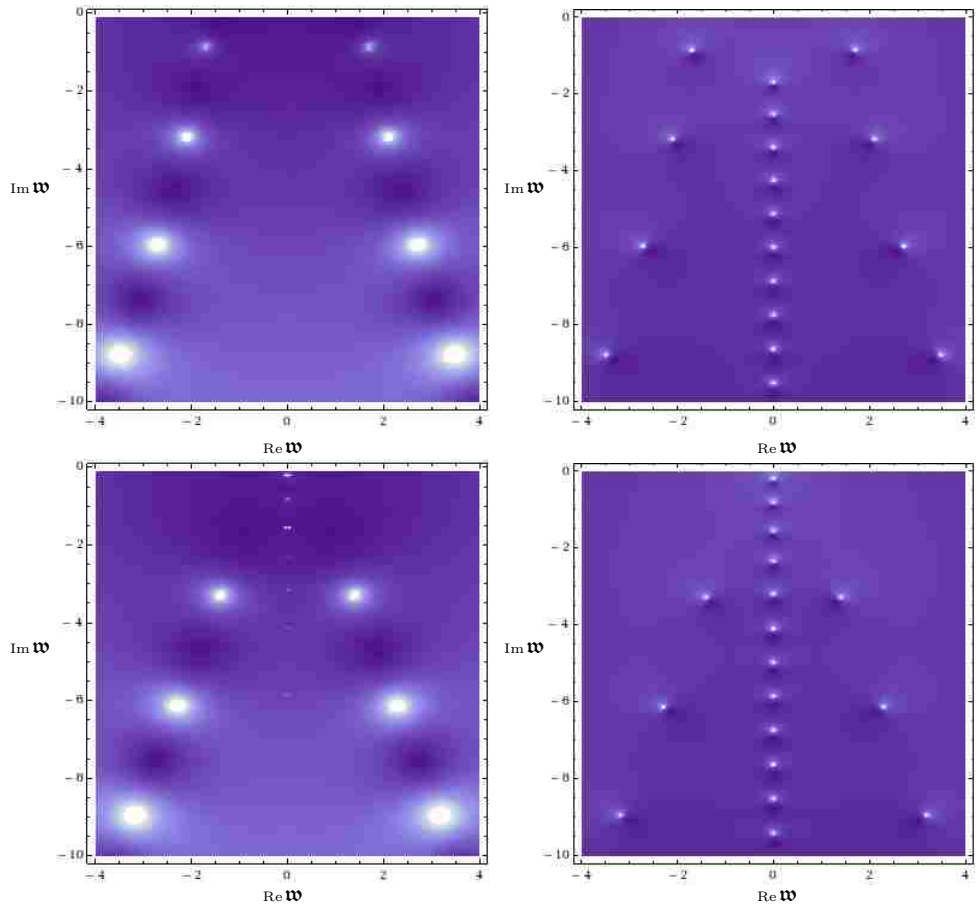


Figure 5.1: A comparison between density plots of $|\hat{\Pi}_{\pm}|$ on the complex $\mathfrak{w}$ plane (left panels) and the quasinormal frequencies for $\Phi_{\pm}$. The top row is for $\Phi_{+}$ and the bottom row is for $\Phi_{-}$. All plots have $\mathfrak{q} = 1$ and $T/\mu_q = 0.09$. As we discuss later, the on-axis modes are weaker but have been tested thoroughly against the quasinormal spectrum in a finer plot.

As the equation of motion for the master fields is non-trivial in the $\mathfrak{w}, \mathfrak{q} \to 0$ limit, to make our analysis simpler, we shall pick $G$ to satisfy the resultant equation of motion

$$zfG'' + zf'G' - (f' + 2Q^2 g_{\pm}(0) z^2)G = 0 \tag{5.2.16}$$

which has solutions of the form

$$G(z) = a(z+b), \quad \text{where} \quad b = \begin{cases} -\frac{3(1+Q^2)}{4Q^2}, & \Phi_{+}, \\ 0, & \Phi_{-}. \end{cases} \tag{5.2.17}$$

Here, $a$ is an overall scaling constant we can specify by a choice of $FG$ at the horizon.

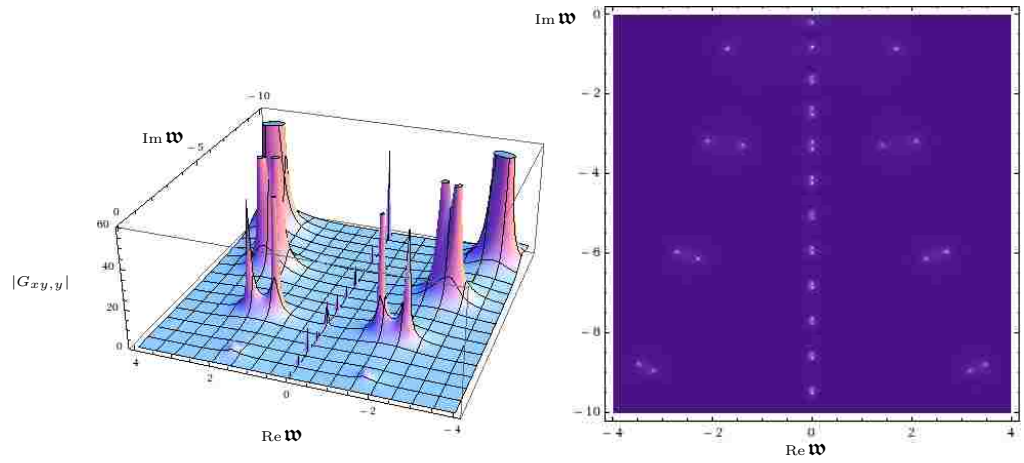


Figure 5.2: A comparison between a surface plot of $|G_{xy,y}|$ on the complex $\mathfrak{w}$ plane (left) and the appropriate quasinormal frequencies. Both plots have $\mathfrak{q} = 1$ and $\frac{T}{\mu_q} = 0.09$.

The remaining equation for $F$ is

$$F'' + \left(\frac{2\alpha}{z-1} + \frac{f'}{f} + \frac{2G'}{G}\right) F' + \left[\frac{\alpha}{z-1}\left(\frac{\alpha-1}{z-1} + \frac{f'}{f} + \frac{2G'}{G}\right) + \frac{Q^2}{f^2}(\mathfrak{w}^2 - \mathfrak{q}^2 f) - 2Q^2 \tilde{g}_\pm(\mathfrak{q})\frac{z}{f}\right] F = 0, \tag{5.2.18}$$

where we have defined

$$\alpha(\mathfrak{w}) = \frac{-i\mathfrak{w}\mu_q}{4\pi T} \quad \text{and} \quad \tilde{g}_\pm(\mathfrak{q}) = g_\pm(\mathfrak{q}) - g_\pm(0). \tag{5.2.19}$$

We now attempt to find a series expansion of $F$ in small $\mathfrak{w}$ and $\mathfrak{q}$

$$F(z) = F_{(0,0)} + \mathfrak{w} F_{(1,0)}(z) + \mathfrak{q}^2 F_{(0,2)}(z) + O(\mathfrak{w}^2, \mathfrak{w}\mathfrak{q}^2), \tag{5.2.20}$$

where $F_{(p,q)}$ is the part of $F$ multiplying $\mathfrak{w}^p \mathfrak{q}^q$ matching the notation in chapter 1. Subsequently we will be able to expand Eq. (5.2.18) when these parameters are small. The infalling boundary condition at $z = 1$ has removed the irregularity of our equations of motion at the horizon and as a consequence $F$, and all its coefficients in the $\mathfrak{w}$, $\mathfrak{q}$ expansion, must be regular there.

Let us argue what pieces of the solution $F(z)$ we will need to calculate $\hat{\Pi}_\pm$. This term comes in at subleading order as $z \to 0$ in $\Phi_\pm$ as such we first note that we

shall only have to determine the leading and subleading pieces of these masterfields. Consider $\hat{\Phi}_+$ and expand to order $z$

$$\begin{aligned}\Phi_\pm &= (-1)^\alpha(1-\alpha z)a\left[F_{(0,0)}+\mathfrak{w}(F_{(1,0)}+F'_{(1,0)}(0)z)\right.\\ &\qquad\left.+\mathfrak{q}^2(F_{(2,0)}(0)+F'_{(2,0)}(0)z)\right](z+b)\\ &\quad +O(z^2,\mathfrak{w}^2,\mathfrak{w}\mathfrak{q}^2)\end{aligned}$$

Each lower order solution provides a source term in the equation of motion at the next order. Beginning at lowest order in $\mathfrak{w}$ we must solve

$$F''_{(1,0)}+\left(\frac{f'}{f}+\frac{2G'}{G}\right)F'_{(1,0)}+\frac{\alpha F_{(0,0)}}{\mathfrak{w}(z-1)}\left(-\frac{1}{z-1}+\frac{f'}{f}+\frac{2G'}{G}\right)=0, \tag{5.2.21}$$

Extracting an integrating factor for the $F_{(1,0)}$ terms we can integrate the source $F_{(0,0)}$ once to find

$$F'_{(1,0)}=-\frac{\alpha F_{(0,0)}}{\mathfrak{w}}\left(\frac{1}{z-1}+\frac{c_1a^2}{fG^2}\right). \tag{5.2.22}$$

This solution diverges as $O(z-1)^{-1}$ when $z$ tends to the horizon which requires that we choose the integration constant $c_1$ to judiciously to remove the divergence. The resultant choices are

$$c_1=\begin{cases}-\frac{(3-Q^2)^3}{16Q^4}, & \Phi_+\\ (3-Q^2), & \Phi-\end{cases}. \tag{5.2.23}$$

The factor of $a^2$ has been removed because $F$ cannot depend on $a$.

The first term in $\mathfrak{q}$ is $O(\mathfrak{q}^2)$ and $F_{(0,2)}$ satisfies

$$F''_{(0,2)}+\left(\frac{f'}{f}+\frac{2G'}{G}\right)F'_{(0,2)}-\frac{Q^2F_{(0,0)}}{f}(1+2\tilde{g}_\pm(1))=0\,. \tag{5.2.24}$$

Again, extracting an integrating factor for $F_{(0,2)}$ and integrating the source against this factor once yields

$$F'_{(0,2)}=\frac{Q^2F_{(0,0)}}{f(z+b)^2}\left[c_2+b\left(bz+z^2+2\tilde{g}_\pm(1)\left(\frac{bz^2}{2}+\frac{2z^3}{3}\right)\right)+\frac{z^3}{3}+\frac{\tilde{g}_\pm(1)z^4}{2}\right]. \tag{5.2.25}$$

This solution is also naively divergent and we must pick $c_2$ appropriately

$$c_2=\begin{cases}-\frac{27+63Q^2+29Q^4+9Q^6}{48Q^4(1+Q^2)}, & \Phi_+\\ -\frac{1}{3(1+Q^2)}, & \Phi_-.\end{cases} \tag{5.2.26}$$

We now have sufficient information to calculate $\Pi_\pm$. Substituting for $b$ we obtain

$$\hat{\Pi}_+ = -\frac{4Q^2}{3(1+Q^2)} + i\mathfrak{w}\left(\frac{Q}{3-Q^2} - \frac{iF'_{(1,0)}(0)}{F_{(0,0)}}\right) + \mathfrak{q}^2\frac{F'_{(0,2)}(0)}{F_{(0,0)}} + O(\mathfrak{w}^2, \mathfrak{w}\mathfrak{q}^2) \ . \quad (5.2.27)$$

Substituting for $F'_{(1,0)}(0)$ and $F'_{(0,2)}(0)$ we obtain the final result

$$\hat{\Pi}_+ = -\frac{4Q^2}{3(1+Q^2)} + i\mathfrak{w}\frac{Q\left(3-Q^2\right)^2}{9\left(1+Q^2\right)^2} - \mathfrak{q}^2\frac{Q^2\left(27+63Q^2+29Q^4+9Q^6\right)}{27\left(1+Q^2\right)^3} + O(\mathfrak{w}^2, \mathfrak{w}\mathfrak{q}^2) \ . \quad (5.2.28)$$

The case of $\hat{\Pi}_-$ is slightly different because the $z$-dependence of $F_{(1,0)}$ and $F_{(0,2)}$ is different. We find

$$F_{(1,0)} = -\frac{\alpha F_0}{\mathfrak{w}}\left(\frac{Q^2-3}{z} + d_1\right) + O(z) \ , \quad (5.2.29)$$

$$F_{(0,2)} = Q^2 F_0\left(\frac{1}{3(1+Q^2)z} + d_2\right) + O(z) \ , \quad (5.2.30)$$

where $d_1$ and $d_2$ are constants. Using $b = 0$ we obtain

$$\Phi_- = (-1)^\alpha a F_0\left[\alpha(3-Q^2) + \frac{\mathfrak{q}^2Q^2}{3(1+Q^2)} + (1 - \alpha d_1 + \mathfrak{q}^2Q^2d_2)z\right] + O(z^2, \mathfrak{w}^2, \mathfrak{w}\mathfrak{q}^2) \quad (5.2.31)$$

and thus

$$\hat{\Pi}_- = \frac{1 - \alpha d_1 + \mathfrak{q}^2Q^2d_2}{\alpha(3-Q^2) + \frac{\mathfrak{q}^2Q^2}{3(1+Q^2)}} + O(\mathfrak{w}^2, \mathfrak{w}\mathfrak{q}^2) \ . \quad (5.2.32)$$

Fluctuations transverse to the direction of momentum flow, as in the shear channel we are studying, excite diffusive modes. For our system we examine Eq. (5.2.32) to find

$$D = \frac{\ell^2}{3(1+Q^2)r_+} \ . \quad (5.2.33)$$

As we have seen previously, conformality implies that

$$D = \frac{\eta}{\varepsilon + P} \ , P = \frac{\varepsilon}{2} \ , \quad (5.2.34)$$

We can invert the relation for the diffusion constant to find $\eta$. Using the thermodynamics of our black hole

$$\varepsilon = \frac{{r_+}^3}{\kappa_4^2\ell^4}(1+Q^2) \ ,$$
$$s = \frac{2\pi}{\kappa_4^2}\left(\frac{r_+}{\ell}\right)^2 \ , \quad (5.2.35)$$

we obtain

$$\frac{\eta}{s} = \frac{1}{4\pi}. \tag{5.2.36}$$

Which we expected on general grounds for a theory dual to a two-derivative gravity theory [134] assuming spatial isotropy (see chapter 1).

We can calculate the diffusion constant numerically by studying the motion of the lowest quasinormal pole of $\hat{\Pi}_-$ as a function of $\mathfrak{q}$. As expected the analytical result Eq. (5.2.33) agrees with the numerical one when $T \gg \mu_q$ (see Fig. 5.3). This provides some vindication of our numerics.

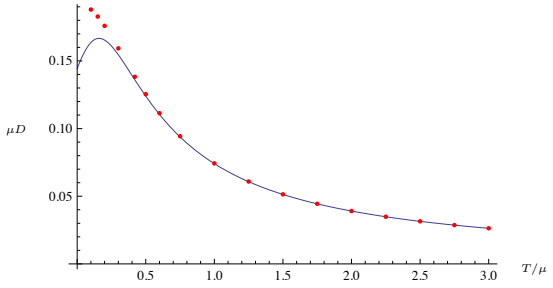


Figure 5.3: Plot of $\mu_q D$ as a function of $\frac{T}{\mu_q}$. The solid curve is the analytical result from Eq. (5.2.33) and the points were extracted from the motion of the lowest quasinormal frequency.

Let's demonstrate the break-down of the effective hydrodynamic theory as $\mathfrak{w}$ becomes too large by examining $\Pi_+$. It is clear from Fig. 5.4 when the hydrodynamics begins to break down. As $\mathfrak{q} \to 0$ we get the correct contact term for $\hat{\Pi}_+$ as in Eq. (5.2.28) and the results are in good agreement over a small range in $\mathfrak{w}$, but they soon deviate. Similarly, as we should expect, the agreement persists over a larger range of $\mathfrak{w}$ as the value $\frac{T}{\mu_q}$ is raised.

### 5.2.2 Residues

Now that we have thoroughly checked the hydrodynamic regime we can go further into the complex $\mathfrak{w}$ plane. In Fig. 5.5 we show surface plots of $\hat{\Pi}_\pm$ on the complex $\mathfrak{w}$ plane. We note the presence of a symmetry in the graph

$$\overline{\hat{\Pi}_\pm(\mathfrak{w}, \mathfrak{q})} = \hat{\Pi}_\pm(-\bar{\mathfrak{w}}, \mathfrak{q}) \ . \tag{5.2.37}$$

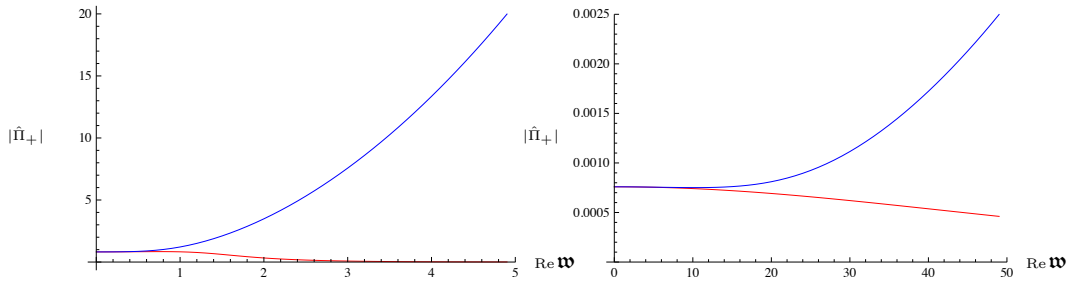


Figure 5.4: A comparison between numerical (red or lower branch) and analytical (blue or upper branch) results for $|\hat{\Pi}_+|$. The left plot is at $\frac{T}{\mu_q} = 0.09$, the right plot is at $\frac{T}{\mu_q} = 10$, and both have $\mathfrak{q} = 10^{-6}$. Note that we have subtracted off the large $\mathfrak{w}$ behaviour from all results, as discussed in Section 5.2.

This is to be expected as it essentially states that the phase velocity is unchanged under a parity transformation. It can be seen at the level of the equations of motion $\mathfrak{w} \leftrightarrow -\bar{\mathfrak{w}}$ and the infalling boundary conditions.

The magnitude of the pole residues in our plots is indicated by the size of the respective peaks. We shall now show how these residues are essential in determining the short to mid time behaviour of the system (as opposed to the hydrodynamic pole which is the only one relevant at late times). Let's assume that the poles of $\hat{\Pi}_\pm$ are simple. The $\Pi_\pm$ as objects from general considerations can be assumed to be meromorphic. Hence they have the form

$$\hat{\Pi}_\pm(\mathfrak{w}, \mathfrak{q}) = \sum_i \frac{R_i\left(\mathfrak{q}; \frac{T}{\mu_q}\right)}{\mathfrak{w} - \mathfrak{w}_i\left(\mathfrak{q}; \frac{T}{\mu_q}\right)} + \text{analytic pieces}, \tag{5.2.38}$$

where the locations of the poles $\mathfrak{w}_i$ in the complex $\mathfrak{w}$ plane have real part $\tilde{\mathfrak{w}}_i(\mathfrak{q}; \frac{T}{\mu_q})$ and imaginary part $-\Gamma_i(\mathfrak{q}; \frac{T}{\mu_q})$. After a Fourier transform it is straightforward to see that

$$R_i\left(\mathfrak{q}; \frac{T}{\mu_q}\right) e^{-\mu_q \Gamma_i(\mathfrak{q}; \frac{T}{\mu_q})(t-t')}$$

determines the contribution of a given pole to $\hat{\Pi}_\pm$. In the ultra-short time regime $t - t' \approx 0$ we can assume that the exponential is negligible and the residues entirely dictate how each pole contributes to the overall Green's function. In the short-mid time regimes $0 \lesssim t - t' \lesssim 1$ there is a delicate interplay between the strong suppression given by the exponential and the residue. When $t - t' > 1$ the exponent

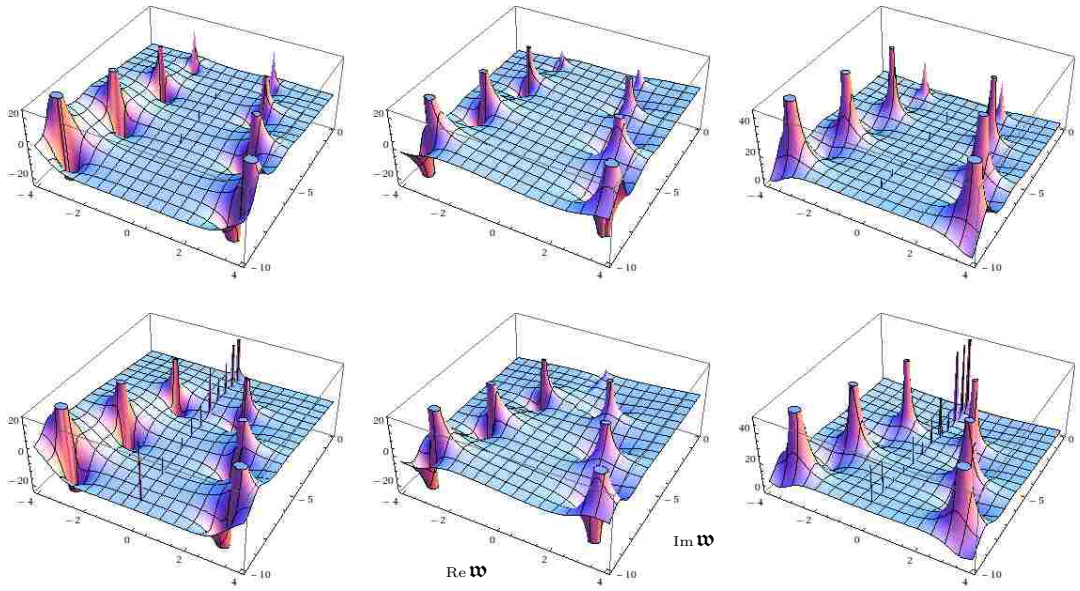


Figure 5.5: Surface plots of $\hat{\Pi}_+$ (top row) and $\hat{\Pi}_-$ (bottom row) on the complex $\mathfrak{w}$ plane at $\mathfrak{q} = 1$, $\frac{T}{\mu_q} = 0.09$. We show (from left to right) the real part, imaginary part and absolute value of each. All plots have the same orientation on the plane, as indicated.

damps away all but the leading pole in the expansion and we enter the hydrodynamic regime.

### 5.2.3 Behaviour at large $\mathfrak{w}$ and $\mathfrak{q}$

A straightforward exercise is to perform a WKB analysis of the equation of motion to determine the large $\mathfrak{w}$ and $\mathfrak{q}$ behaviour. This yields

$$\hat{\Pi} = \pm iQ\sqrt{\mathfrak{w}^2 - \mathfrak{q}^2} \tag{5.2.39}$$

for both $\hat{\Pi}_\pm$ with $\mathfrak{w}$ real. So for $\mathfrak{w} \gg \mathfrak{q}$ or $\mathfrak{q} \gg \mathfrak{w}$ the growth of $|\hat{\Pi}_\pm|$ is linear in the larger parameter, which we demonstrate in Fig. 5.6. A feature to note in our figures is that when $\mathfrak{w} = \pm\mathfrak{q}$ and $\mathfrak{q}$ is large the Greens function is zero.

The WKB approximation is essentially an effect of the vacuum by which we mean once the black hole is removed this large $\Re\mathfrak{w}$ and $\mathfrak{q}$ behaviour remains. To isolate the effect of adding the black hole to the bulk theory we should subtract off this asymptotic behaviour. This will introduce a branch cut into our correlators

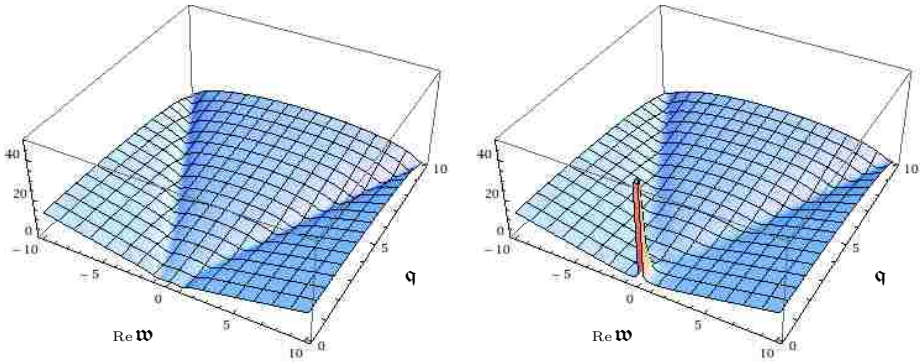


Figure 5.6: Surface plots of $|\hat{\Pi}_+|$ (left) and $|\hat{\Pi}_-|$ (right) on the $(\Re\mathfrak{w}, \mathfrak{q})$ plane at $T/\mu_q = 0.09$ to demonstrate the large $\mathfrak{w}$ and $\mathfrak{q}$ scaling. Note that we avoid $\mathfrak{q} = 0$: in that case the symmetry is enhanced and so the master fields are different, as discussed in [4].

that we should be a little wary of but makes no fundamental difference other than clearing up our plots.

### 5.2.4 Spectral function

The spectral function of a thermal field theory gives the number density of states in the ensemble with a particular real $\mathfrak{w}$ and $\mathfrak{q}$ and is typically defined, up to sign, by

$$\rho(\mathfrak{w}, \mathfrak{q}) = -\Im G_R(\mathfrak{w}, \mathfrak{q}) \tag{5.2.40}$$

for real $\mathfrak{w}$. As it is such an important quantity we shall now seek to extract it from our numerics.

The features on the $\Re\mathfrak{w}$ axis can be explained by studying the general form Eq. (5.2.38). Making use of the symmetry Eq. (5.2.37) we consider a straightforward re-writing of this expression, with dependence on $\mathfrak{q}$ and $\frac{T}{\mu_q}$ suppressed:

$$\hat{\Pi}_\pm(\mathfrak{w}) = \sum_i \left[ \frac{a_i + ib_i}{\mathfrak{w} - |\tilde{\mathfrak{w}}_i| + i\Gamma_i} - \frac{a_i - ib_i}{\mathfrak{w} + |\tilde{\mathfrak{w}}_i| + i\Gamma_i} \right] + \sum_j \frac{ic_j}{\mathfrak{w} + i\Gamma_j} + \text{analytic.} \tag{5.2.41}$$

Here, $i$ runs over one half of the off-axis poles and $j$ runs over the on-axis poles.

Taking real and imaginary parts of the above form we obtain

$$\begin{aligned}\Re\hat{\Pi}_\pm &= \sum_i \left[\frac{a_i\mathfrak{w} - (a_i|\tilde{\mathfrak{w}}_i| - b_i\Gamma_i)}{(\mathfrak{w} - |\tilde{\mathfrak{w}}_i|)^2 + \Gamma_i^2} - \frac{a_i\mathfrak{w} + (a_i|\tilde{\mathfrak{w}}_i| - b_i\Gamma_i)}{(\mathfrak{w} + |\tilde{\mathfrak{w}}_i|)^2 + \Gamma_i^2}\right] \\ &\quad + \sum_j \frac{c_j\Gamma_j}{\mathfrak{w}^2 + \Gamma_j^2} + \text{analytic} \qquad (5.2.42) \\ \Im\hat{\Pi}_\pm &= \sum_i \left[\frac{b_i\mathfrak{w} - (a_i\Gamma_i + b_i|\tilde{\mathfrak{w}}_i|)}{(\mathfrak{w} - |\tilde{\mathfrak{w}}_i|)^2 + \Gamma_i^2} + \frac{b_i\mathfrak{w} + (a_i\Gamma_i + b_i|\tilde{\mathfrak{w}}_i|)}{(\mathfrak{w} + |\tilde{\mathfrak{w}}_i|)^2 + \Gamma_i^2}\right] \\ &\quad + \sum_j \frac{c_j\mathfrak{w}}{\mathfrak{w}^2 + \Gamma_j^2} + \text{analytic}. \qquad (5.2.43)\end{aligned}$$

In Fig. 5.7 we have taken a slice of the $(\Re\mathfrak{w}, \mathfrak{q})$ plane at $\mathfrak{q} = 10^{-6}$. The expressions above take their largest values approximately when $\mathfrak{w} = 0$ or $\pm|\tilde{\mathfrak{w}}_i|$, which is indeed what we find in Fig. 5.7. In effect, the presence of poles lower down in the complex $\mathfrak{w}$ plane is 'projected' onto the retarded Green's functions at real $\mathfrak{w}$.

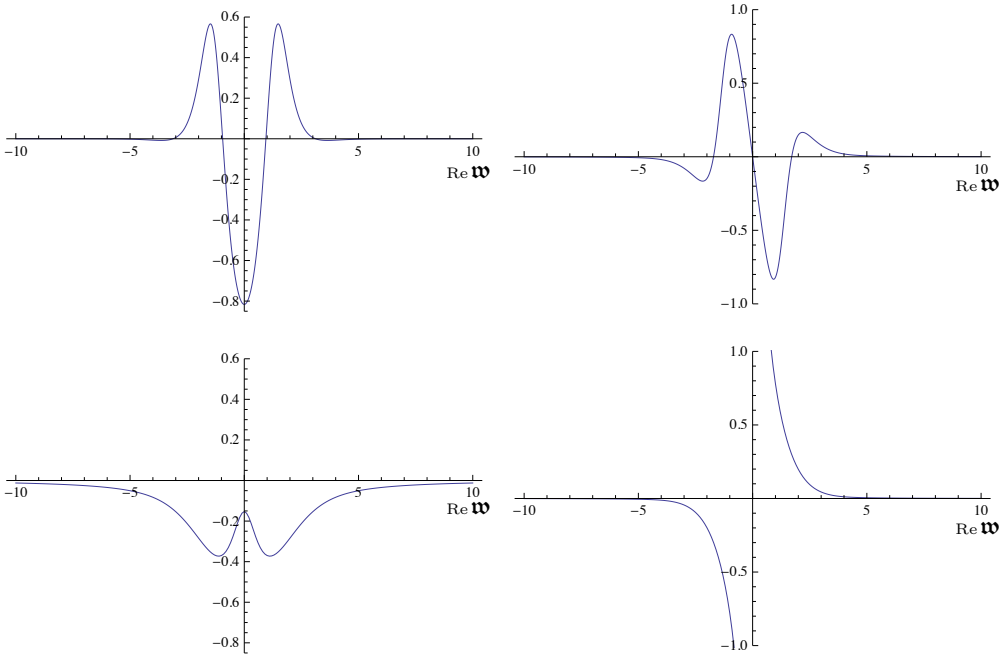


Figure 5.7: Slices through the $(\Re\mathfrak{w}, \mathfrak{q})$ plane of $\hat{\Pi}_+$ (top row) and $\hat{\Pi}_-$ (bottom row) at $\mathfrak{q} = 10^{-6}$ and $\frac{T}{\mu_q} = 0.09$. The left and right panels show the real and imaginary parts, respectively.

Note that in Fig. 5.7 we have subtracted the large $\mathfrak{w}$ and $\mathfrak{q}$ behaviour discussed previously in order to reveal the peaks that occur away from $\mathfrak{w} = 0$. Our numerics confirm that the modified retarded Green's functions tend to zero as $\Re\mathfrak{w}$ increases.

We have chosen to subtract off the positive branch.

By considering $\mathfrak{q} \ll 1$ we are working in the long-wavelength regime. If instead we consider $\mathfrak{q} \gg 1$, for which the spatial perturbations are much smaller than the scales in the boundary theory, the poles on the $\Im\mathfrak{w}$ axis move much further down the complex $\mathfrak{w}$ plane and their effect on the spectral function diminishes.

### 5.2.5 Pole motion

As noted in [4], the lowest quasinormal frequency for $\Phi_-$ approaches zero as $\mathfrak{q} \to 0$. We found the same behaviour for the lowest pole in $\hat{\Pi}_-$; as such, this can be identified as the hydrodynamic pole. The behaviour of the poles as $\mathfrak{q}$ and $\frac{T}{\mu_q}$ are varied is particularly rich and, to our knowledge, has not been studied in detail before.

The trajectories of the on-axis poles of $\hat{\Pi}_-$ at fixed $\frac{T}{\mu_q}$ are shown in Fig. 5.8. As $\mathfrak{q}$ is increased, all these poles move down the $\Im\mathfrak{w}$ axis. The on-axis poles of $\hat{\Pi}_+$ have very simple trajectories and so we focus on the behaviour of $\hat{\Pi}_-$.

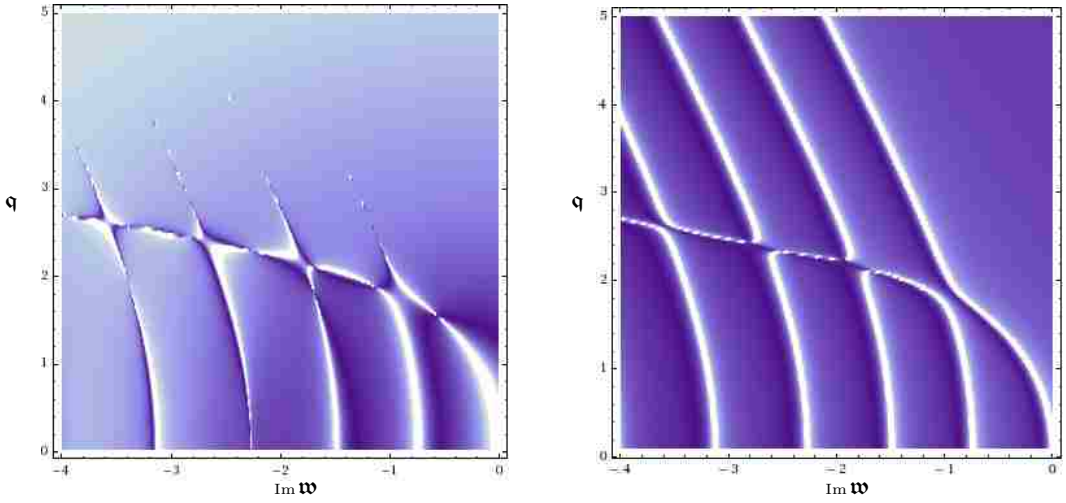


Figure 5.8: Trajectories on the $(\Im\mathfrak{w}, \mathfrak{q})$ plane of the on-axis poles of $\hat{\Pi}_-$ (left) and the motion of the associated quasinormal frequencies. We fix $\frac{T}{\mu_q} = 0.09$ and consider $0.1 \leq \mathfrak{q} \leq 5$.

Firstly we note the apparent continuity of the white line characterising the hydrodynamic pole. The physical implication of this behaviour is that in the hydrodynamic limit, where this is the only relevant pole, the dispersion relation of the

quasi-particle corresponding to this pole is a smooth function of $\mathfrak{q}$. This is given by

$$\mathfrak{w} = -i\mu_q D\mathfrak{q}^2 \tag{5.2.44}$$

with diffusion constant $D$. In the left of Fig. 5.8 we can see there are values of $\mathfrak{q}$ for which a given on-axis mode disappears, as indicated by a gap in the associated trajectory (for example at $\mathfrak{q} \approx 1.6$ and $\mathfrak{w} \approx -0.6i$). If Eq. (5.2.38) holds, then we must conclude that the residue vanishes at these $(\mathfrak{w}, \mathfrak{q})$. With this in mind we see that, for finely tuned $\mathfrak{q}$, it may be possible for a previously sub-leading pole to give the dominant contribution to $\hat{\Pi}_-$. Note that whilst the quasinormal plots can show this continuity, full knowledge of the correlators is required to determine which pole dominates.

Also, we observe that the trajectories of all on-axis poles become less pronounced as $\mathfrak{q}$ is increased, which indicates that the residues of the poles become smaller. Furthermore, there are no poles at small $|\mathfrak{w}|$ for sufficiently large $\mathfrak{q}$.

The 'interactions' between the poles of $\hat{\Pi}_-$ are quite intricate as $\frac{T}{\mu_q}$ and $\mathfrak{q}$ are varied. We observe two types of behaviour: 'repulsion' and and 'clover-leaf crossing'. Suppose we fix $\frac{T}{\mu_q}$ and increase $\mathfrak{q}$. All poles on the $\Im\mathfrak{w}$ axis move downwards but the hydrodynamic pole moves down much more rapidly for the same change in $\mathfrak{q}$, as shown in Fig. 5.8. It slows down as it approaches the next pole and the latter speeds up but the two never touch, just as for two like magnetic poles being brought together. As this second pole then moves down the axis and approaches the third, the two coalesce into a single object, split and move off-axis, coalesce a second time then split and move on-axis once more in a clover-leaf pattern.

Now suppose we fix $\mathfrak{q}$ at some larger value and increase $\frac{T}{\mu_q}$. Again, all the poles move downwards as $\frac{T}{\mu_q}$ is increased, except there is always one pole which moves upwards. An example of this is shown in Fig. 5.9 and a sample crossing is depicted in Fig. 5.10. We also see several repulsions, so it appears that a different pole moves upwards each time.

We have so far not been able to tag the poles in a clover-leaf crossing and track their motion independently. The fact that the poles coalesce follows from the symmetry Eq. (5.2.37): the poles must be symmetric about the $\Im\mathfrak{w}$ axis, so they cannot move round each other but must cross. However, we do not know if this

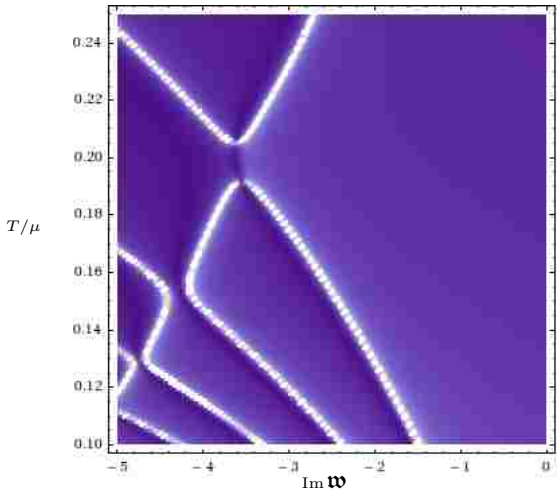


Figure 5.9: Trajectories on the $(\Im\mathfrak{w}, \frac{T}{\mu_q})$ plane of the on-axis quasinormal frequencies of $\Phi_-$. We fix $\mathfrak{q} = 3$ and consider $0.1 \leq \frac{T}{\mu_q} \leq 0.25$.

forms a sum of simple poles or a double pole. If the latter occurs then Eq. (5.2.38) must be modified, so it would be interesting to map the behaviour of $\hat{\Pi}_-$, or even just the hydrodynamic pole, over a larger region of $(\mathfrak{q}, \frac{T}{\mu_q})$ space.

From the gravity perspective, we have shown that certain quasinormal modes switch from purely decaying to partially oscillatory behaviour as the parameters are varied. For example, for the central plot of Fig. 5.10 there exist two modes degenerate in energy which move in opposite directions. Now, those modes closest to the $\Re\mathfrak{w}$ axis dominate the late-time response of the black hole. By studying this crossover we are exploring the intermediate response for perturbations of various $\mathfrak{q}$ of black holes with different $\frac{T}{\mu_q}$. From the field theory perspective, the dispersion relation for the excitations associated with these poles develops a real part at this crossover. This indicates a purely decaying mode becomes propagating.

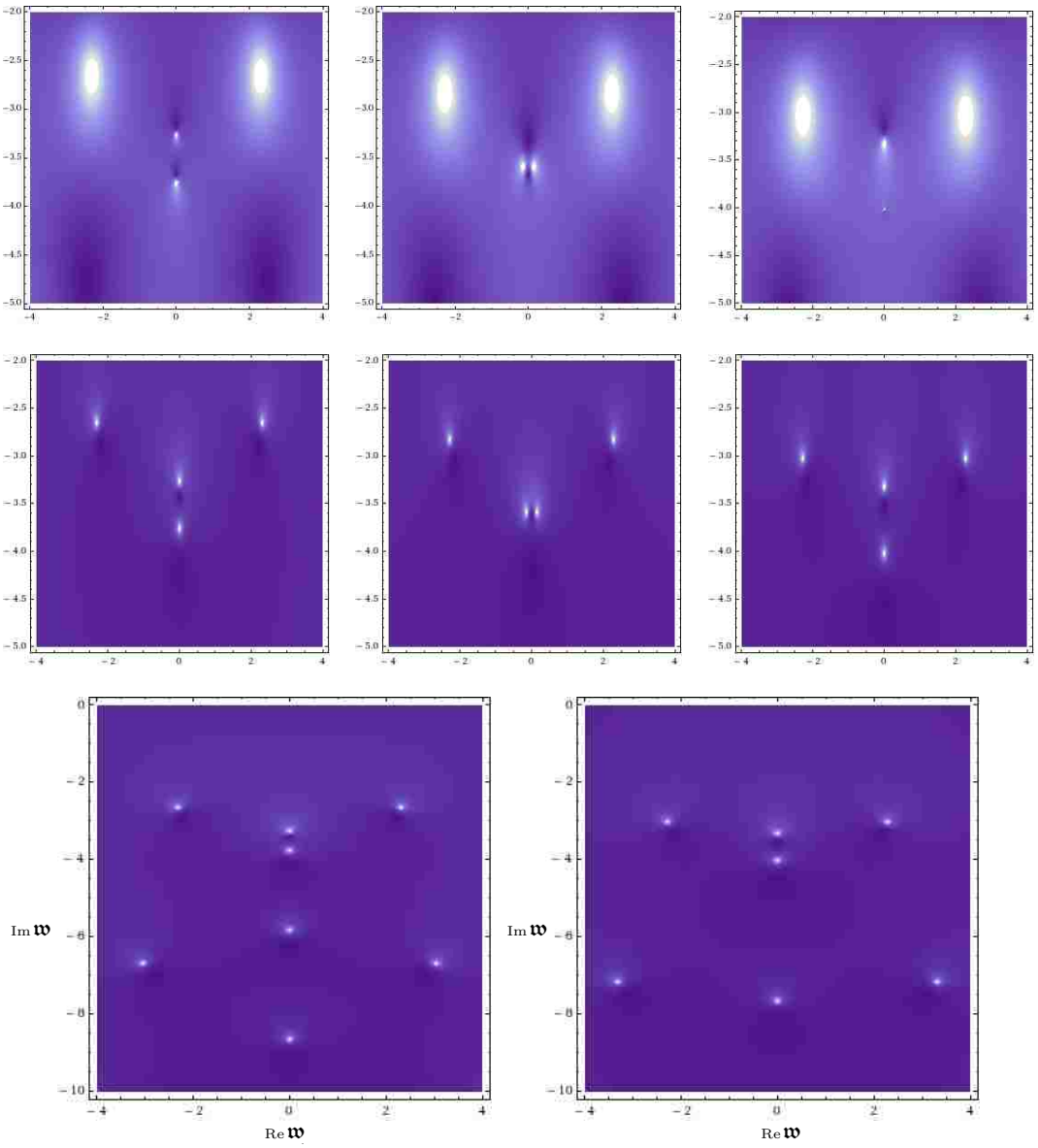


Figure 5.10: Density plots of $|\hat{\Pi}_-|$ on the complex $\mathfrak{w}$ plane at fixed $\mathfrak{q} = 3$ are shown in the upper row. The values of $\frac{T}{\mu_q}$ are (from left to right) 0.185, 0.2 and 0.215. These plots have been cropped to include only the poles which move off-axis and the leading poles. For comparison, the quasinormal frequencies at the same values of parameters are given in the second row. The final row contains two quasinormal frequency plots, one at $\frac{T}{\mu_q} = 0.185$ and one at $\frac{T}{\mu_q} = 0.215$. These show a larger region of the complex $\mathfrak{w}$ plane, including the origin, for orientation.

# Chapter 6

# Conclusion

In this thesis we have attempted to argue that gauge-gravity duality can give us deep and meaningful insights into the behaviour of strongly coupled condensed matter systems. Our focus has been on the application of hydrodynamics to condensed matter physics in a regime where the condensed matter theory is strongly coupled.

In the introduction we laid out the general nature of the AdS-CFT correspondence. The results stated are well known and represent a summary of the literature. The following provide a collection of useful reviews for the interested reader: [13, 35–43, 49]. A calculation of particular interest that we repeated was to demonstrate that the low frequency and momentum dispersion relations of charge and shear stress-energy-momentum fluctuations about Schwarzschild-AdS were reminiscent of hydrodynamics. These calculations gave us the opportunity to introduce much of the notation we later used in the thesis as well as motivating chapter 2. We finished the chapter by summarising arguments stating that for the diffusion constant associated with the gravitational shear mode there was an element of universality. In particular in any thermal field theory dual to a two-derivative gravitational model with spatial isotropy the shear viscosity to entropy ratio is conjectured to be bounded below by $\frac{1}{4\pi}$.

Subsequently, in chapter 2, we demonstrated that the linearised modes found in the introduction genuinely did correspond to hydrodynamic modes in the boundary. This chapter is mostly review of the available literature and is included to provide the necessary background to understand the later chapters. Within we calculated to

first order in derivatives the charged and uncharged constitutive relations for fluid dynamics. In the second half of this chapter we then demonstrated how to calculate the gravitational dual to a boundary field theory with a SEM tensor and charge current satisfying the relativistic Navier-Stokes equations. Thus we showed explicitly the existence of a gravitational dual to the hydrodynamics of certain strongly coupled field theories.

There continues to be much interest in the study of relativistic hydrodynamics dual to black hole spacetimes. Extensions to [32] have included the non-relativistic fluids discussed in this thesis, magnetohydrodynamics [65, 78, 79], forced fluids [63] and even "anisotropic hydrodynamics" [64, 69] which may be of use in describing the quark-gluon plasma. On a purely theoretical level there has been recent interest in generating functionals for the transport coefficients given by considering fundamental symmetries of the theory [70–73]. Moreover, the general fluid-gravity procedure has been extended from its origins as an AdS correspondence to other classes of spacetimes including the study of black-folds [135–139] and zero cosmological constant spaces [140–143]. All these extensions and the continuing output of recent work indicate that this will be an area which continues to produce fascinating fundamental results for some time to come.

While relativistic fluids have nice symmetry properties, our day-to-day experience of fluids generally consists of those whose average molecular velocity is small enough to ignore relativistic effects. As we are attempting to argue that the gauge-gravity correspondence can be a useful tool in condensed matter theory we should attempt to search for gravitational duals with Galilean symmetry. Additionally fluids encountered in the real world can often be assumed to be incompressible. This led us in chapter 3 to find a scaling limit of the results in chapter 2 such that our resultant fluid had an underlying Galilean symmetry. The process by which this was done also had the effect of scaling away sound mode thus making the fluid incompressible. We then showed in the second half of this chapter how the same limit can be enacted on the bulk holographic dual. Again this chapter is mostly review based on [61] which has been included so that we can contrast it with the Schrödinger fluid in the following chapter.

Subsequent to [61] there have been several related studies investigating the area of gauge-gravity dualities in spacetimes whose boundary dual has Galilean conformal symmetry. Most recently the incompressible fluid limit has reappeared in the discussion of the membrane paradigm and holography in Rindler space [140–142, 144]. Subsequently it was shown in [3] how the gravitational Dirichlet problem in AdS is related to the membrane paradigm. In particular it was demonstrated that as the Dirichlet surface approaches the horizon, to retain sensible dynamics, a BMW-like limit has to be applied to the hypersurface fluid. This connection has yet to be completely understood with a much greater understanding of non-relativistic holography required to fully appreciate the significance of this statement. Examples of studies into the nature of non-relativistic holography for the Galilean conformal symmetry group include [81, 145–151].

In chapter 4 we then demonstrated how to obtain the hydrodynamic derivative expansions of a fluid with Schrödinger invariance at first order from a parent charged, conformal, relativistic theory. Specifically we have shown how to generalise the maps of [33] to the case of an $U(1)$ charged fluid at first order. The resultant fluid was compressible which stands in contrast to the fluid of chapter 3.

We then focused upon the hydrodynamic limit of a particular three dimensional, non-relativistic conformal field theory and its dual solution which is a five-dimensional asymptotically Schrödinger, charged black hole. Using the TsT technique on an Reissner-Nordstrøm $AdS_5$ precursor we constructed an action whose equations of motion had the desired black hole as a solution and isolated the thermodynamics. Although in principle we could then have computed the first order corrections to our asymptotically Schrödinger charged black hole using the fluid-derivative procedure we noted that it would be particularly cumbersome to do so and hence opted instead to use the TsT technique. Thus we arrived at expressions for the metric, gauge field, massive vector field and dilaton to first order in derivatives.

With the corrected fields to hand in principle it was possible to calculate the asymptotic values of the metric and gauge field directly to determine their corresponding conserved boundary currents. While this is simple to do for the gauge

field ambiguities in asymptotic fall off of the metric necessitated that we interpret the boundary SEM tensor in the precursor asymptotically AdS theory as a tensor complex of the Schrödinger invariant theory [88]. With these identifications it was relatively simple to apply the holographic dictionary in Schrödinger space-times to compute the boundary coefficients and with a little work obtain the Prandtl number. An important result discovered here was that the universal value of one for the Prandtl number of an uncharged fluid no longer holds when there is an additional non-zero charge. This suggests that it may be interesting to understand the consequences of a scaling where the non-relativistic charge and particle number were related as this would naturally be interpreted as the charge being carried by the fluid particles. We leave this for future work. Moreover, the work represented in this chapter provides an important extension to the literature as, for example, it demonstrates that it is possible to still find terms due to anomalies even in non-relativistic fluids - a development that had not previously been foreseen in [60].

Although our study in chapter 4 concentrated on a fluid occupying two spatial dimensions, the derivative expansions (4.1.9), (4.1.10) and (4.1.12) apply in any dimension with the caveat that the relativistic one-derivative parity violating term only exists in four dimensions. Similarly the generalisations to multiple $U(1)$ charges or indeed different internal symmetries seems clear and we can determine the hydrodynamic coefficients if we can find a suitable dual black hole with the required asymptotics.

The gauge-gravity dictionary in Schrödinger spacetimes is still not well understood due to the somewhat bizarre asymptotics of these geometries. Much work in this area now focuses on better understanding the nature of the correspondence for which [1] (chapter 4) may provide valuable intuition. A non-exhaustive list of areas in the literature where this work may be useful include embedding Schrödinger geometries in string and M-theory [33, 88, 119, 120, 152–154], the generic nature of the gauge-gravity correspondence in Schrödinger spacetimes [83, 84, 90, 155] and applications to condensed matter systems [85, 156, 157]. Additionally the work represented by this chapter can be used as evidence for conjectures such as those in [158].

In the final chapter we attempted to demonstrate the limitations and usefulness of hydrodynamic analysis in a particular field theory which is assumed to be dual to a non-extremal Reissner-Nordstrøm $AdS_4$ black hole. We computed numerically the full retarded Green's functions for conserved currents in the shear channel of this theory for non-trivial density matrix. Our results are consistent with those of [4], but we have gone beyond that treatment to show many interesting features. In particular, the linear response of the boundary theory exhibits rather peculiar behaviour beyond the hydrodynamic regime which we would like to understand in more detail.

An immediate application of the method discussed in this chapter is to study the sound channel correlators of this boundary field theory. This case was not considered in the original paper [159] but, subsequent to [2] on which this chapter is based, it was considered in [160]. A further extension which to the author's knowledge has not yet been done is to compute the full retarded Green's functions for conserved currents in the shear channel at zero temperature. We expect qualitatively different behaviour at zero temperature. It has been argued in many examples (see [132], for instance) that the double zero in $f(r)$ at the horizon of extremal Reissner-Nordstrøm $AdS_4$ leads to a branch cut in the appropriate retarded Green's function(s) of the boundary theory. This was first seen explicitly at small frequencies in [12]. Whilst numerical results were reported in [159] for the sound channel at zero temperature, only the small frequency behaviour was reported in [4] for the shear channel, based on the analytical methods developed in [12]. We would like to go beyond this regime, for which numerical methods are required. However, the double zero in $f(r)$ shows up as an irregular singular point in the master field equations, leading to difficulties in the numerical computation of the retarded Green's functions by our methods. It would be interesting to adapt our methods to tackle zero temperature for the shear channel.

To summarise; the gauge-gravity correspondence is an important tool by which we can learn general lessons about the physics of strongly coupled condensed matter systems. While many basic questions have been resolved there remain many exciting problems even within the context of the hydrodynamics to be answered. Moreover,

we have demonstrated that for thermal field theories, because the hydrodynamic regime of fluctuations often has tractable equations, calculation of hydrodynamic modes can be used as a test-bed for methods to explore beyond the low frequency and momentum regime.

# Appendix A

# Non-relativistic symmetry algebras

In this chapter we shall review the properties of non-relativistic symmetry algebras. In the first section we shall look at the symmetry transformations of Galilean relativity realised as coordinate transformations and use these to imply the corresponding Galilei algebra. We shall then demonstrate how the Galilei algebra can be realised as a parametric contraction of the Poincaré algebra. This process corresponds to our usual notion of expected non-relativistic symmetry when considering small spatial velocities. Then we shall show how the Galilei algebra can also be obtained by light-cone reduction of the Poincaré algebra and compare the two approaches.

In the second section we shall briefly discuss a simple extension to the Galilei algebra - the Milne and Coriolis algebras which allow for time dependent rotations and translations. We shall show how the Galilei algebra occurs as a subalgebra of the Milne and Coriolis algebras.

In the third section we shall look at the Galilean conformal algebra. In the first section we demonstrated how parametric contraction of the Poincaré algebra led to the Galilei algebra. Given that the conformal algebra is an extension of the Poincaré algebra to contain scaling transformations we might ask whether there is an analogue of the Galilei algebra which contains scaling transformations. Taking the parametric contraction of the Poincaré algebra shall yield one such extension called the Galilean conformal algebra. This non-relativistic algebra can be thought

of as describing gapped modes.

As the above discussion hints there exists more than one way to extend the Galilei algebra to contain scaling transformations. In section four we shall discuss the Schrödinger algebra which is given by performing a light cone reduction of the conformal algebra. We shall see it has quantatively different physics to the Galilean conformal algebra. In particular, unlike the previous algebra, it can describe gapped modes.

## A.1 The Galilei algebra

In this section we shall derive the Galilei algebra from coordinate isometries. Much of the material will probably be familiar to the reader however it serves as a good starting point to discuss more complicated algebras. Of particular use later on is the process by which the Galilei algebra can be produced from the Poincaré algebra. We shall discuss two of these methods. The first, parametric contraction, will be useful for considering fluids holographically dual to some spacetime with small spatial velocities. The second, light-cone reduction, will be interesting as it will allows us to interpret boundary field theories dual to certain Schrödinger spacetimes in terms of those dual to certain AdS spacetimes.

The Galilei algebra consists of transformations corresponding to the following coordinate isometries:

| Coordinate | Transformed coordinate | Interpretation |
|---|---|---|
| $t$ | $t + a$ | time reparameterisation |
| $x^i$ | $x^i + a^i$ | spatial translations |
| $x^i$ | $x^i + v^i t$ | boosts |
| $x^i$ | $R\left(x^i\right)$ | rotations |

where $R$ is the rotation matrix. For infinitesimal transformations we can write:

| Coordinate | Transformed coordinate | Interpretation |
|---|---|---|
| $t$ | $[1 + (ia)(-i\partial_t)]\, t$ | time reparameterisation |
| $x^i$ | $[1 + (ia^j)(-i\partial_j)]\, x^i$ | spatial translations |
| $x^i$ | $[1 + (iv^j)(-it\partial_j)]\, x^i$ | boosts |
| $x^i$ | $[1 + (i\theta^{ij})(i(x_i\partial_j - x_j\partial_i))]\, x^k$ | rotations |

We shall define:

$$\hat{H} = -i\partial_t \,, \qquad \hat{P}_i = -i\partial_i \,,$$
$$\hat{B}_i = -it\partial_i \,, \quad \hat{J}_{ij} = i\left(x_i\partial_j - x_j\partial_i\right) \,,$$

where $\hat{J}_{ij}$ is the generator of rotations in the two-plane spanned by $x^i$ and $x^j$. In the following we shall demonstrate that these operators form a closed Lie algebra spanned by elements of the form

$$\hat{X} = a\hat{H} + b^i\hat{P}_i + v^i\hat{B}_i + \theta^{ij}\hat{J}_{ij}$$

where $a$ is a coordinate scalar, $b^i$, $v^i$ coordinate vectors and $\theta^{ij}$ an antisymmetric coordinate two-tensor that parameterise infinitesimal transformations.

The commutation relations obeyed by these generators can be simply written down:

$$\begin{aligned}
\left[\hat{H}, \hat{B}_i\right] &= i\hat{P}_i \,, \\
\left[\hat{P}_i, \hat{J}_{jk}\right] &= i\left[\delta_{ij}\hat{P}_k - \delta_{ik}\hat{P}_j\right] \,, \\
\left[\hat{B}_i, \hat{J}_{jk}\right] &= i\left[\delta_{ij}\hat{B}_k - \delta_{ik}\hat{B}_j\right] \,, \\
\left[\hat{J}_{ij}, \hat{J}_{kl}\right] &= i\left(\delta_{il}\hat{J}_{jk} - \delta_{ik}\hat{J}_{jl} - \delta_{jl}\hat{J}_{ik} + \delta_{jk}\hat{J}_{il}\right) \,,
\end{aligned} \tag{1.1.1}$$

where all non-displayed commutation relations are zero. There exists a central extension to the above algebra which we shall discuss next.

We are familiar with the Schrödinger equation from non-relativistic quantum mechanics

$$\left(\frac{\hbar^2}{2m}\nabla^2 - i\hbar\partial_t\right)\psi = V(x)\psi$$

where we have reintroduced $\hbar$ this one time only and $V(x)$ is an arbitrary potential. It can be rephrased in terms of a Schrödinger operator which is defined to be the

total energy minus the kinetic energy

$$\hat{S} = \hat{H} - \frac{1}{2m}\hat{\mathbf{P}}^2$$

such that $\hat{S}\psi = V(x)\psi$. In a system where there is no potential energy, $\hat{S}\psi = 0$, we would like the Schrödinger operator to be a Casimir of the algebra. The action of any of the symmetry generators of the Galilei algebra then commute with $\hat{S}$ and map solutions of the equation to each other. Unfortunately, looking at our Galilean algebra as written, it is not. In particular, under boosts:

$$\begin{aligned}\left[\hat{S}, \hat{B}_i\right] &= \left[\hat{H} - \frac{1}{2m}\hat{\mathbf{P}}^2, \hat{B}_i\right] \\ &= i\hat{P}_i\end{aligned}$$

This leads us to looking at ways to "correct the algebra" such that the Schrödinger operator is a Casimir. As we don't want energy levels to be affected by boosts this implies that the only thing that can change above is the commutation relations between the translation and boost generators. Let's modify this relationship via the introduction of a central charge such that

$$\left[\hat{P}_i, \hat{B}_j\right] = i\alpha\delta_{ij}$$

then

$$\left[\hat{S}, \hat{B}_i\right] = i\hat{P}_i - \frac{i}{2m}\left(2\alpha\hat{P}_i\right)$$

which suggests if we choose $\alpha$ to be the mass of the particle then the Schrödinger operator is a Casimir of the algebra. The resultant algebra is called the Bargmann algebra.

### A.1.1 As a parametric contraction of the Poincaré algebra

As a prelude to more complicated parametric contractions we shall now demonstrate how to obtain the Galilei algebra from the Poincaré algebra. Given some function $f(t, x)$ we would like to scale its space and time arguments as

$$\begin{aligned} t &\rightarrow \epsilon^r t\,, \\ x^i &\rightarrow \epsilon^{r+1} x^i\,, \end{aligned}$$

where $r$ is some exponent. The process of finding the algebra corresponding to the symmetry implied by the Poincaré algebra on these functions is called Wigner-İnönü contraction. It is most simply illustrated by considering the Lorentz group in $(1+1)$ dimensions whose action upon coordinates is given by

$$\Lambda(v) = \begin{pmatrix} \gamma & \gamma\frac{v}{c} \\ \gamma\frac{v}{c} & \gamma \end{pmatrix} \tag{1.1.2}$$

where $\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ and we have restored $c$ for the rest of this paragraph. Setting $\frac{v}{c} = \epsilon v_*$ where $v_*$ is our non-relativistic velocity we can attempt to take $\epsilon \to 0$ but, as the group stands, this results only in the trivial group. Instead we make a transformation

$$U = \begin{pmatrix} \epsilon^{-r} & 0 \\ 0 & \epsilon^{-(r+1)} \end{pmatrix} \tag{1.1.3}$$

which is singular in the desired limit. It's effect on the Lorentz transformations however is to make them non-trivial in this limit and we find

$$\begin{aligned} \Lambda(\epsilon v_*) &= \begin{pmatrix} \gamma & \gamma\epsilon^2 v_* \\ \gamma v_* & \gamma \end{pmatrix} \\ &\xrightarrow{\epsilon\to 0} \begin{pmatrix} 1 & 0 \\ v_* & 1 \end{pmatrix} \end{aligned} \tag{1.1.4}$$

which is clearly a symmetry transformation of the Galilean group in $(1+1)$ dimensions. This process of Wigner-İnönü contraction is in fact more general than we have described and applies in general to groups with a non-trivial subgroup that is fixed under the contraction.

Given the transformations of the group we are now in a position to infer the transformations of the generators. The generators of the Poincaré algebra are $\left\{\hat{P}_\mu, \hat{M}_{\mu\nu}\right\}$ and they satisfy the commutation relations

$$\begin{aligned} \left[\hat{P}_\mu, \hat{P}_\nu\right] &= 0\,, \\ \left[\hat{P}_\mu, \hat{M}_{\nu\sigma}\right] &= i\left(\eta_{\mu\nu}\hat{P}_\sigma - \eta_{\mu\sigma}\hat{P}_\nu\right)\,, \\ \left[\hat{M}_{\mu\nu}, \hat{M}_{\sigma\rho}\right] &= i\left(\eta_{\mu\rho}\hat{M}_{\nu\sigma} - \eta_{\mu\sigma}\hat{M}_{\nu\rho} - \eta_{\nu\rho}\hat{M}_{\mu\sigma} + \eta_{\nu\sigma}\hat{M}_{\mu\rho}\right)\,, \end{aligned}$$

and their coordinate representations are

$$\begin{aligned} \hat{P}_\mu &= -i\partial_\mu \ , \\ \hat{M}_{\mu\nu} &= i\left(x_\mu\partial_\nu - x_\nu\partial_\mu\right) \ . \end{aligned}$$

The operators themselves have coordinate dependence and so they break up under scaling. The leading pieces of their coordinate representations are

$$\begin{aligned} \hat{P}_t &\rightarrow -i\partial_t = \hat{H} \ , \\ \hat{P}_i &\rightarrow -i\partial_i = \hat{P}_i \ , \\ \hat{M}_{it} &\rightarrow -it\partial_i = \hat{B}_i \ , \\ \hat{M}_{ij} &\rightarrow i\left(x_i\partial_j - x_j\partial_i\right) = \hat{J}_{ij} \ . \end{aligned}$$

As expected time and space become distinct. The commutation relations are those of the Galilei algebra (Eq. 1.1.1) without a central extension to the Bargmann algebra.

### A.1.2 As a light-cone reduction of the Poincaré algebra

Similarly, before tackling more complicated light-cone reductions, we shall demonstrate how to obtain the Galilei algebra from the Poincaré algebra. Given some function $f(x^+, x^-, x^i)$ on some spacetime with a metric of the form

$$ds^2 = -2dx^+dx^- + \left(dx^i\right)^2 \tag{1.1.5}$$

where

$$x^\pm = \frac{t \pm x^1}{2}$$

we ask that our physical objects be independent of the coordinate $x^-$. We require that the action of symmetry generators in our resultant algebra preserves this independence. Thus we will need to remove from the Poincaré algebra any generators that introduce non-trivial dependence on $x^-$. This restricted algebra will turn out to be the Galilean algebra in one fewer dimensions.

When we attempt to restrict the Poincaré algebra it is clear that $\hat{P}_- = -i\partial_-$ will naturally drop out from our algebra as this generates translations along the

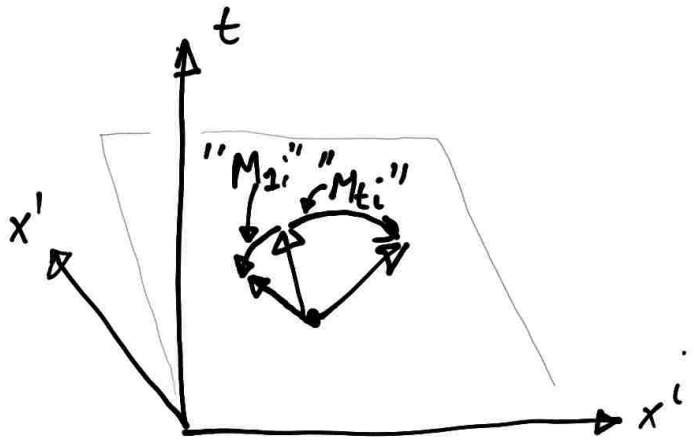


Figure A.1: A figure demonstrating why only certain combinations of boosts and rotations may be retained under light-cone reduction. The sheet displayed is a surface of constant $x^-$. A function which is independent of $x^-$ will have the same $x^+$ and $x^i$ dependent profile for each sheet at a given $x^-$. Drawn on the sheet is the potential tangent vector of some particle. The figure attempts to illustrate that a $M_{ti}$ boost or a $M_{1i}$ boost will take the particle off this sheet meaning it necessarily has dependence on $x^-$. As we do not want this to happen it must be the case that either: neither boost is in our resultant algebra or a combination of the two is. It is this latter case which we find in the text.

$x^-$ direction. What is perhaps less clear is that only certain linear combinations of boosts can remain. We attempt to illustrate this in Fig. A.1. The operators that survive are

$$\begin{aligned}\hat{P}_+ &= -i\partial_+ \,, \\ \hat{P}_i &= -i\partial_i \,, \\ \hat{M}_{i-} &= i\left(x_i\partial_- + x^+\partial_i\right) \,, \\ \hat{M}_{ij} &= i\left(x_i\partial_j - x_j\partial_i\right) \,.\end{aligned}$$

The resultant commutation relations are:

$$\begin{aligned}\left[\hat{P}_+, \hat{M}_{i-}\right] &= i\hat{P}_i \,, \\ \left[\hat{P}_i, \hat{M}_{jk}\right] &= i\left(\delta_{ij}\hat{P}_k - \delta_{ik}\hat{P}_j\right) \,, \\ \left[\hat{M}_{i-}, \hat{M}_{jk}\right] &= i\left(\delta_{ij}\hat{B}_k - \delta_{ik}\hat{B}_j\right) \,, \\ \left[\hat{M}_{ij}, \hat{M}_{kl}\right] &= i\left(\delta_{il}\hat{M}_{jk} - \delta_{ik}\hat{M}_{jl} - \delta_{jl}\hat{M}_{ik} + \delta_{jk}\hat{M}_{il}\right) \,,\end{aligned}$$

where again un-displayed commutation relations are zero. When we identify $\hat{P}_+ = \hat{H}$, $\hat{M}_{i-} = \hat{B}_i$ and $\hat{M}_{ij} = \hat{J}_{ij}$ we get the Galilei algebra (Eq. 1.1.1) in one fewer dimensions than we started with.

## A.2 The Milne and Coriolis algebras

The Galilei transformations noted above are not the most general non-relativistic isometries we could make. In particular we could consider the time dependent, but spatially independent, translations and rotations

$$\begin{aligned}x^i &\rightarrow x^i + a^i(t) \,, \\ x^i &\rightarrow R_{ij}(t)x^i \,.\end{aligned}$$

The corresponding infinitesimal transformations are

| Coordinate | Transformed coordinate | Interpretation |
|---|---|---|
| $t$ | $\left[1 + (ia)\left(-i\partial_t\right)\right] t$ | time reparameterisation |
| $x^i$ | $\left[1 + \sum_{k=-\infty}^{\infty}\left(ia^j_{(k)}\right)\left(-it^k\partial_j\right)\right] x^i$ | spatial translations |
| $x^i$ | $\left[1 + \sum_{k=-\infty}^{\infty}\left(i\theta^{ij}_{(k)}\right)\left(it^k\left(x_i\partial_j - x_j\partial_i\right)\right)\right] x^l$ | rotations |

The usual translations and boosts are simply the $t^0$ and $t^1$ components of the time dependent spatial translation respectively. We shall define:

$$\hat{H} = -i\partial_t \ , \qquad \hat{M}_i^{(k)} = -it^{k+1}\partial_i \ ,$$
$$\hat{J}_{ij}^{(k)} = it^k \left(x_i\partial_j - x_j\partial_i\right) \ .$$

The commutation relations that these generators obey are:

$$\begin{aligned}
\left[\hat{H}, \hat{M}_i^{(k)}\right] &= -i(k+1)\hat{M}_i^{(k-1)} \ , \\
\left[\hat{M}_i^{(m)}, \hat{J}_{jk}^{(n)}\right] &= i\left(\delta_{ij}\hat{M}_k^{(m+n)} - \delta_{ik}\hat{M}_j^{(m+n)}\right) \ , \\
\left[\hat{J}_{ij}^{(m)}, \hat{J}_{kl}^{(n)}\right] &= i\left[\delta_{jk}\hat{J}_{il}^{(n+m)} + \delta_{jl}\hat{J}_{ki}^{(n+m)} + \delta_{ik}\hat{J}_{lj}^{(n)} + \delta_{il}\hat{J}_{jk}^{(n)}\right] \ ,
\end{aligned}$$

where undisplayed commutation relations are zero. We additionally note that the Milne algebra, where the rotation generators are truncated to $\hat{J}_{ij}^{(0)}$ is a consistent limit of the Coriolis algebra which makes our rotations time independent.

## A.3 The Galilean conformal algebra

As discussed above a way to achieve a non-relativistic analogue of the conformal algebra is via parametric contraction [81]. The resulting symmetry algebra, called the Galilean conformal algebra (GCA), will contain as a subalgebra the Galilei algebra in addition to a dilatation operator and analogues of the time and space components of special conformal transformations. The GCA can be viewed as a subalgebra of an infinite symmetry algebra on which we will make some passing comments.

### A.3.1 As a parametric contraction of the conformal algebra

The conformal algebra is an extension of the Poincaré algebra to include scaling transformations. Hence to the usual Poincaré algebra generators we add the following new generators

$$\begin{aligned}
\hat{D} &= -ix\cdot\partial \ , \\
\hat{K}_\mu &= -i\left(2x_\mu x\cdot\partial - x\cdot x\partial_\mu\right) \ ,
\end{aligned}$$

which satisfy the additional commutation relations

$$\begin{aligned}
\left[\hat{D}, \hat{P}_\mu\right] &= i\hat{P}_\mu \ , \\
\left[\hat{D}, \hat{K}_\mu\right] &= -i\hat{K}_\mu \ , \\
\left[\hat{K}_\mu, \hat{P}_\nu\right] &= i\left[2\eta_{\mu\nu}\hat{D} - 2\hat{M}_{\mu\nu}\right] \ , \\
\left[\hat{K}_\mu, \hat{M}_{\nu\sigma}\right] &= i\left[\eta_{\mu\nu}\hat{K}_\sigma - \eta_{\mu\sigma}\hat{K}_\nu\right] \ ,
\end{aligned}$$

where undisplayed commutation relations are zero. Just as before we perform a parametric contraction. The coordinate representations of the new non-relativistic operators are

$$\begin{aligned}
\hat{D} &= -i\left[t\partial_t + x^i\partial_i\right] \ , \\
\hat{K} &= -i\left(2tx^i\partial_i + t^2\partial_t\right) \ , \\
\hat{K}_i &= it^2\partial_i \ .
\end{aligned}$$

Notice that the coordinate representation of the generator of dilatations has exactly the same form as in the relativistic algebra. The commutation relations become:

$$\begin{aligned}
\left[\hat{D}, \hat{H}\right] &= i\hat{H} \ , \\
\left[\hat{D}, \hat{P}_i\right] &= -i\hat{P}_i \ , \\
\left[\hat{D}, \hat{K}_t\right] &= i\hat{K}_t \ , \\
\left[\hat{D}, \hat{K}_i\right] &= i\hat{K}_i \ , \\
\left[\hat{K}_i, \hat{H}\right] &= 2i\hat{B}_i \ , \\
\left[\hat{K}_t, \hat{H}\right] &= -2i\hat{D} \ , \\
\left[\hat{K}_t, \hat{P}_j\right] &= -2i\hat{B}_j \ , \\
\left[\hat{K}_t, \hat{B}_i\right] &= i\hat{K}_i \ ,
\end{aligned}$$

where undisplayed commutation relations are zero. This is called the Galilean conformal algebra. In passing we note that the Galilean conformal algebra does not admit the Galilei central extension and thus intuitively represents "gapless" theories.

### A.3.2 The infinite Galilean conformal algebra

The Galilean conformal group has an extension to an infinite symmetry algebra. We shall not dwell on this but it is nonetheless interesting to see and may yet have

some future applications in holography [81]. First we alter our notation and redefine the following generators:

$$\hat{L}^{(-1)} = \hat{H} \ , \quad \hat{L}^{(0)} = \hat{D} \ , \quad \hat{L}^{(+1)} = \hat{K} \ ,$$
$$\hat{M}_i^{(-1)} = \hat{P}_i \ , \quad \hat{M}_i^{(0)} = \hat{B}_i \ , \quad \hat{M}_i^{(+1)} = \hat{K}_i \ .$$

Examining the resultant commutation relations suggests an obvious extension. Define new generators:

$$\begin{aligned} \hat{L}^{(n)} &= -i\left((n+1)t^n x^i \partial_i + t^{n+1}\partial_t\right) \\ \hat{M}_i^{(n)} &= -it^{n+1}\partial_i \\ \hat{J}_{ij}^{(n)} &= it^n\left(x_i\partial_j - x_j\partial_i\right) \end{aligned}$$

We can see that when $n = 0, \pm 1$ the coordinate representation of the above generators matches that of the finite Galilean conformal algebra. The algebra obeyed by these operators is

$$\begin{aligned} \left[\hat{L}^{(n)}, \hat{L}^{(m)}\right] &= i(n-m)\hat{L}^{(n+m)} \ , \\ \left[\hat{L}^{(n)}, \hat{M}_j^{(m)}\right] &= i(n-m)M_j^{(n+m)} \ , \\ \left[\hat{L}^{(n)}, \hat{J}_{ij}^{(m)}\right] &= -imJ_{ij}^{(m)} \ . \end{aligned}$$

We would like to interpret these new generators. First notice that if we restrict $\hat{L}^{(n)} \in \left\{\hat{L}^{(-1)}, \hat{L}^{(0)}, \hat{L}^{(1)}\right\}$ then the subalgebra we find is the Coriolis algebra where $\hat{M}^{(n)}$ generate time dependent translations (and thus boosts) while $\hat{J}^{(n)}$ generate time dependent rotations. The objects $\hat{L}^{(n)}, |n| \geq 1$ have an interesting interpretation as reparameterisations of absolute time [81].

## A.4 The Schrödinger algebra

As discussed above a way to achieve a non-relativistic analogue of the conformal algebra is via light-cone reduction. The resulting symmetry algebra, called the Schrödinger algebra, will contain as a subalgebra the Galilei algebra in addition to a dilatation operator and an analogue of the time component of special conformal transformations. It will not contain an analogue of the spatial component of the special conformal transformations. The Schrödinger algebra will have a central extension.

### A.4.1 As a contraction of the conformal algebra

Given that we have already shown how to find the Galilei algebra from the Poincaré algebra the process of light cone reducing the conformal algebra is quite easy. As before we need to drop $\hat{P}_-$ and $\hat{M}_{+i}$ but, for our algebra to be consistent, we must also drop any of the new conformal operators that introduce these via commutation relations too. The only commutation relation from our conformal algebra that could potentially be a problem is:

$$\left[\hat{K}_\mu, \hat{P}_\nu\right] = i\left[2\eta_{\mu\nu}\hat{D} - 2\hat{M}_{\mu\nu}\right] .$$

Clearly if $\mu = +$ and $\nu = i$ or $\mu = i$ and $\nu = +$ we get the "forbidden" operators $\hat{M}_{+i}$. To prevent this we remove $\hat{K}_+$ and $\hat{K}_i$ from the algebra. The remaining new operators are the dilatation operator and $\hat{K}_-$ which have the coordinate expressions

$$\begin{aligned}\hat{D} &= -i\left(2x^+\partial_+ + x^i\partial_i\right) ,\\ \hat{K}_- &= ix^+\left(x^+\partial_+ + x^i\partial_i\right) ,\end{aligned}$$

where we have dropped an overall factor of two from the definition of $\hat{K}_-$. The commutation relations are not quite given by making a coordinate choice, $(+,-,i)$, in the relativistic commutation relations because time now scales differently in the dilatation operator compared to the relativistic case. Thus we should take extra care with commutation relations involving $D$. Working with their coordinate representations it is possible to show that the commutation relations are

$$\begin{aligned}\left[\hat{D}, \hat{P}_+\right] &= 2i\hat{P}_+ ,\\ \left[\hat{D}, \hat{P}_i\right] &= iP_i ,\\ \left[\hat{D}, \hat{M}_{i-}\right] &= i\hat{M}_{i-} ,\\ \left[\hat{D}, \hat{K}_-\right] &= -2i\hat{K}_- ,\\ \left[\hat{K}_-, \hat{P}_+\right] &= i\hat{D} ,\\ \left[\hat{K}_-, \hat{P}_i\right] &= -i\hat{M}_{-i} ,\end{aligned} \tag{1.4.6}$$

where we must remember that we have dropped a 2 from $\hat{K}_-$. Again undisplayed commutation relations are zero.

There are some passing remarks to make about the Schrödinger algebra. Firstly, notice that the effect of the dilatation operator on coordinates is to set

$$\begin{aligned} x^i &\rightarrow \lambda x^i \,, \\ x^+ &\rightarrow \lambda^2 x^+ \,. \end{aligned}$$

The anisotropic scaling of space and time is common in condensed matter systems and the exponent of $\lambda$ in the time rescaling is called the "dynamical" exponent. The Schrödinger algebra is similar to the more common Lifshitz algebra which contains the scaling

$$\begin{aligned} x^i &\rightarrow \lambda x^i \,, \\ t &\rightarrow \lambda^z t \,, \end{aligned}$$

but no boosts. Finally we remark that the Schrödinger algebra is also compatible with the Galilean central extension unlike the Galilean conformal algebra.

# Appendix B

# Mathematical conventions

## B.1 Hodge duals

Following [119] we note that the ten-dimensional Hodge dual on Reissner-Nordstrøm $AdS_5 \times S^5$ can be restricted to the 5-dimensional Reissner-Nordstrøm $AdS_5$ manifold, 1 dimensional fibration coordinate and $\mathbb{CP}^2$ in the following manner

$$*_{10} = (-1)^{(5-n_5)n_4+(5-n_5)n_1+(1-n_1)n_4} *_5 *_1 *_4$$

where $n_5$, $n_1$ and $n_4$ are the number of indices in each part. In particular:

$$\begin{aligned}
*_{10}(1) &= \frac{1}{2} e^{\Phi} V_{RNAdS} \wedge \left(d\psi + \mathcal{A}_{(1)}\right) \wedge J_{(2)} \wedge J_{(2)} \ , \\
*_5(1) &= V_{RNAdS} \ , \\
*_1(1) &= e^{\Phi} \left( d\psi + \mathcal{A}_{(1)} - \frac{2}{\sqrt{3}} A_Q \right) \ , \\
*_4(1) &= \frac{1}{2} J_{(2)} \wedge J_{(2)} \ , \\
*_4(J_{(2)}) &= J_{(2)} \ ,
\end{aligned}$$

where $V_{S^5} = *_1(1) \wedge *_4(1)$ when $\Phi = 0$.

After Melvinisation the Hodge dual on the asymptotically Schrödinger charged black brane spacetime, denoted $Schr_5$, is not equal to that on Reissner-Nordstrøm $AdS_5$ and we must determine its effect upon our volume forms and gauge fields. It can be shown that objects whose terms all contain $dx^-$ pick up a factor of $e^{\Phi}$ when acted on by the Melvinised Hodge dual. After Melvinisation the following important

objects Hodge dualise in the manner shown:

$$\begin{aligned} V_{RNAdS} &= e^{-\Phi} V_{\text{Schr}_5} \ , \\ *_5 F_Q &= -2e^{-\Phi} *_5'' \left( \frac{1}{\sqrt{3}} F_Q + F \wedge A_M \right) \ . \end{aligned}$$

The transformation of $*_5 F_Q$ was determined by considering the fact that $B''_{(2)} \wedge F''_{(3)}$ is precisely the quantity that needs to be added to make $F_{(5)}$ self-dual with respect to the Melvinised metric, see [119].

## B.2 TsT transformation

First note that generically we can write the 10-dimensional metric and five-form as:

$$\begin{aligned} ds_{10}^2 &= g_{--} \left(dx^-\right)^2 + 2 g_{-\alpha} dx^- dx^\alpha + g_{\alpha\beta} dx^\alpha dx^\beta + \left(d\psi + \mathcal{A}_{(1)} + A_Q\right)^2 + d\Sigma_4^2 \ , \\ F_{(5)} &= d\psi \wedge \left(dx^- \wedge A_{(3)} + B_{(4)}\right) + dx^- \wedge C_{(4)} + D_{(5)} \ , \end{aligned}$$

where $\alpha, \beta$ belong to $\{r, +, x, z\}$. As the TsT only ever performs algebraic operations on the $\psi$ and $x^-$ isometry directions we only need to keep track of these terms.

We shall need to T-dualise our solution twice so it makes sense to define a standard form for the relevant fields (as in [119]). In particular, we isolate all the dependence on $\psi$ in our fields and write them in the following manner:

$$\begin{aligned} ds_{10}^2 &= g_{\psi\psi} \left(d\psi + g_{(\psi)}\right)^2 + \dots \ , \\ B_{(2)} &= \left(d\psi + \frac{1}{2} g_{(\psi)}\right) \wedge B_{(\psi)} + \dots \ , \\ F_{(p)} &= \left(d\psi + g_{(\psi)}\right) \wedge F_{(p)\psi} + F_{(p)\not\psi} \ , \end{aligned}$$

where $\not\psi$ indicates the piece of the field with no $\psi$ components. We also choose to denote the dilaton by $\Phi_0$. The T-dualisation of these objects then yields:

$$\begin{aligned} \left(ds_{10}^2\right)' &= \frac{1}{g_{\psi\psi}} \left(d\psi - B_{(\psi)}\right)^2 + \dots \ , \\ B'_{(2)} &= \left(d\psi - \frac{1}{2} B_{(\psi)}\right) \wedge \left(-g_{(\psi)}\right) + \dots \ , \\ F'_{(p)} &= \left(d\psi - B_{(\psi)}\right) \wedge \left(F_{(p-1)\not\psi}\right) + F_{(p+1)\psi} \ , \\ e^{\Phi'} &= \frac{e^{\Phi_0}}{g_{\psi\psi}} \ . \end{aligned}$$

We shall denote T-dualised quantities with a $'$.

As was stated in the main body of the thesis the TsT transformation is formed from the following sequence of operations:

1. T-dualise along the $\psi$ direction,
2. twist along $x^-$ sending it to $x^- + \alpha\psi$ where $\alpha$ is a constant,
3. and finally T-dualise along the $\psi$ direction.

Applying these operations to the fields of Eq. (2.2.1) we obtain:

$$\begin{aligned}
\left(ds^2_{10}\right)'' &= ds^2_5 + \frac{\left(d\psi + \mathcal{A}_{(1)} - \frac{2}{\sqrt{3}}A_Q\right)^2}{k} + d\Sigma^2_4 - \frac{\alpha^2}{k}\left(g_{--}dx^- + g_{-\alpha}dx^\alpha\right) \ , \\
B''_{(2)} &= \frac{\alpha}{k}\left(g_{--}dx^- + g_{-\alpha}dx^\alpha\right) \wedge \left(d\psi + \mathcal{A}_{(1)} - \frac{2}{\sqrt{3}}A_Q\right) \ , \\
F''_{(3)} &= \alpha A_{(3)} \ , \\
F''_{(5)} &= F_{(5)} + B''_{(2)} \wedge F''_{(3)} \ , \\
F''_{(7)} &= B''_{(2)} \wedge F''_{(5)} \ , \\
e^{2\Phi''} &= \frac{e^{2\Phi_0}}{k} \ ,
\end{aligned}$$

where $k = 1 + \alpha^2 g_{--}$ and $ds^2_5$ is the original five dimensional metric. From the above formulae we can readily identify $A_M$ to be $\frac{\alpha}{k}g_{-\mu}dx^\mu$.

# Appendix C

# Numerical method for extracting correlators

In this appendix we give details of the method used to extract the retarded Green's functions for $\Phi_\pm$ numerically. As outlined in Section 5.1, we must first solve Eq. (5.1.4). This is a second-order, linear ODE with a regular singular point at $z = 1$ (the horizon). We choose the following ansatz, where $\Phi$ denotes one of the $\Phi_\pm$:

$$\Phi(z) = (z-1)^{-i\mathfrak{w}\mu_q/4\pi T}\phi(z). \tag{3.0.1}$$

The first factor imposes the infalling boundary condition at $z = 1$. A unique solution is then specified by $\phi(1)$, which we are free to choose because the equation is linear. We expand $\phi$ about $z = 1$ up to some order, $N$, to generate the initial condition for a Runge-Kutta algorithm at some $z = 1 - \epsilon$. To extract $\hat{\Pi}_\pm(\mathfrak{w}, \mathfrak{q})$, we integrate out to the boundary and match this numerical solution to the asymptotic expansion given in Eq. (5.1.5). The matching can be performed using a root-finding method, for example.

The main numerical issue comes from the initial condition. Naively, we would like to choose a small $\epsilon$ so that the series expansion of $\phi$ is accurate with only a few terms. However, we found that the pattern of poles was completely washed out below a certain line in the complex $\mathfrak{w}$ plane. This numerical instability appears if $\epsilon$ is small for $\Im\mathfrak{w}$ large and negative (and/or for $\frac{T}{\mu_q}$ small). To see why, note that the

ansatz Eq. (3.0.1) for $\Phi$ at $z = 1 - \epsilon$ contains the factor

$$\epsilon^{-i\mathfrak{w}\mu_q/4\pi T}.$$

This factor becomes very large in these regimes if $\epsilon$ is small, leading to round-off errors.

Thus, a larger $\epsilon$ must be chosen in these regimes. As a consequence, a sufficiently large $N$ must be chosen to offset the error from starting the integration further from $z = 1$. The values of $\epsilon, N$ can be constrained by ensuring that the locations of the poles match the quasinormal spectrum for the appropriate bulk fluctuations, as stated in Section 5.2.

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