Counting and Averaging Problems in Graph Theory

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Abstract

Paul Günther (1966), proved the following result: Given a continuous function f on a compact surface M of constant curvature -1 and its periodic lift $\tilde{f}$ to the universal covering, the hyperbolic plane, then the averages of the lift $\tilde{f}$ over increasing spheres converge to the average of the function f over the surface M.

Heinz Huber (1956) considered the following problem on the hyperbolic plane $\mathbb{H}$: Consider a strictly hyperbolic subgroup of automorphisms on $\mathbb{H}$ with compact quotient, and choose a conjugacy class in this group. Count the number of vertices inside an increasing ball, which are images of a fixed point $x \in \mathbb{H}$ under automorphisms in the chosen conjugacy class, and describe the asymptotic behaviour of this number as the size of the ball goes to infinity.

In this thesis, we use a well-known analogy between the hyperbolic plane and the regular tree to solve the above problems, and some related ones, on a tree. We deal mainly with regular trees, however some results incorporate more general graphs.

Declaration

The work in this thesis is based on research carried out at the Pure Maths Group, the Department of Mathematical Sciences, Durham University, England. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all my own work unless referenced to the contrary in the text.

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Introduction

One of the things that makes mathematics such a beautiful and interesting subject is the fact that all its different areas seem to be linked in some way. In this thesis I will exploit one of these links to produce new results in the field of graphs from older results on Riemann surfaces.

This thesis contains results I have proved while studying for my PhD. Chapters 2 and 3 contain results on radial averages of functions on regular and semi-regular graphs, whereas chapter 6 deals with a similar problem on non-regular graphs, although the result is far less detailed here. I summarised the main results in an article [21]. In chapter 4 we will discuss a lattice point problem on the regular tree, where we count the vertices linked by some conjugacy class in a group of automorphisms of the tree (see also [22]).

Both the radial average and the lattice point problem were motivated by results on Riemann surfaces, in particular, the radial average result for vertices on a regular graph is a discrete analogue of a result by Paul Günther [28], whereas the lattice point problem was inspired by an article by Heinz Huber [34]. The generalisations of theorem 2.4 in chapters 2 and 3 are natural extensions of the discrete radial average result.

Before stating these results, however, we will cover some basic graph theory, definitions and notation in chapter 1. After discussing my own results in chapters 2, 3 and 4, we will deal with the Selberg and Ahumada trace formulas in chapter 5. In chapter 6 we then discuss a generalisation of part of the radial average results in chapters 2 and 3, and suggest where this research may lead in the future.

Chapter 1

Preliminary Material

1.1 Basic Definitions

Let us start with some basic definitions and notation. See also for example [8], [19], [6] or any basic book on graph theory.

Definition 1.1 *A* graph G *is made up of a non-empty set* $V = V(G)$ *of* vertices*, and a set* $E = E(G)$ *of unordered pairs of elements in* V *called* edges*. We write* $e = \{v, w\}$ *for the edge* e *defined by the vertices* v *and* w*, and we say this edge connects the two vertices, and the vertices are* adjacent*. The vertices* v *and* w *are said to be* incident *with the edge* e*. Two edges are said to be adjacent if they are both incident to a vertex* v*.*

When we draw a graph, we represent vertices by points and edges by lines, in such a way that the edge connecting two vertices is represented by a line joining the corresponding points. Note that both the vertex and edge set may be infinitely large in the definition above. If both sets have finite size we say that the graph is *finite*. An edge can connect a vertex to itself, in which case it is called a *loop*, and there may be *multiple edges* connecting the same two vertices. If both loops and multiple edges are not allowed, then the graph is called *simple*.

Definition 1.2 *The* degree $d(v)$ *of a vertex* $v \in V$ *is defined as the number of edges incident with* v*.*

A loop contributes twice to the degree of the vertex, as this edge is incident with the vertex at both its endpoints. A graph in which every vertex has the same degree is called a *regular* graph. If all vertices have finite degree, then the graph is called *locally finite*. Note that a graph can be infinite, but locally finite. A nice example of a regular finite graph is the Petersen graph in figure 1.1. Note that it is regular of degree 3. Vertex u is adjacent to v but not to w, and edges a and e are adjacent as they are both incident to vertex v.

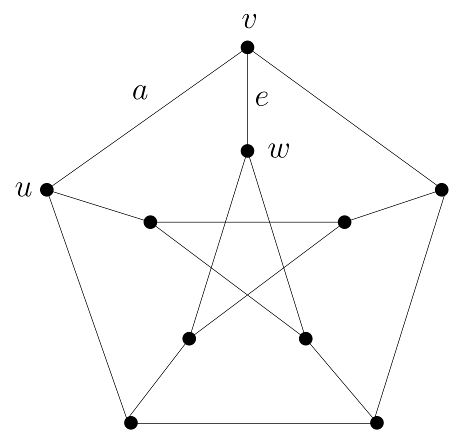

Figure 1.1: The Petersen graph

Any edge $e = \{v, w\}$ can be given a direction, say, from v to w, to give a directed edge denoted $\overrightarrow{\{v, w\}}$. In this case, v is the *origin* of e and w the *terminus*. In the case of a loop, the notation $\overrightarrow{\{v, v\}}$ is not sufficient, and we have to specify the direction separately.

We define a *path* on a graph as a sequence of adjacent edges, each usually with a given direction, and we write down a path in terms of the vertices defining each edge. In figure 1.1, for example, if we let $\overrightarrow{a} = \overrightarrow{\{u, v\}}$ and $\overrightarrow{e} = \overrightarrow{\{v, w\}}$, then we write the path consisting of $\overrightarrow{a}$ and $\overrightarrow{e}$ as $\overrightarrow{\{u, v, w\}}$. A path has *backtracking* if it has two consecutive edges which are the same except that they have opposite directions. A path is *closed* if it ends at the same vertex as it started. A non-empty closed path in which no vertex or edge is visited more than once is a *cycle*.

Definition 1.3 *A graph G is* connected *if we can find a path in G from any vertex $v \in V$ to any other vertex $w \in V$.*

Even if a graph is not connected, we can still find non-empty subsets C_i of V where we can find a path from any vertex in C_i to every other. If we choose these C_i to be maximal, then the sets C_i are called the *connected components* of G, and each vertex of G belongs to exactly one connected component.

Definition 1.4 *A connected graph G which contains no cycles is called a* tree. *A graph in which every connected component is a tree is called a* forest.

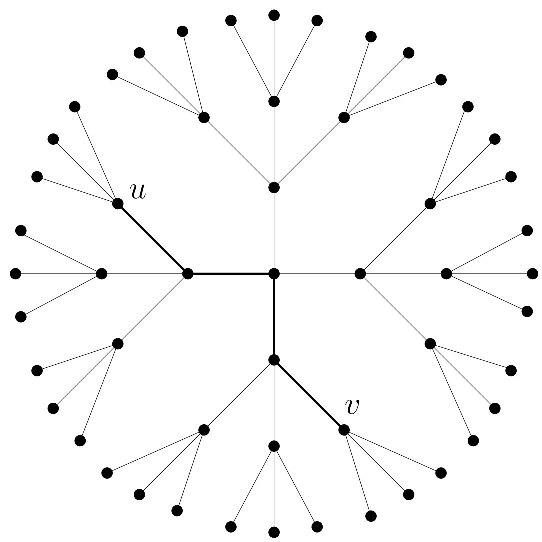

Figure 1.2: A finite tree with the path from u to v in bold.

Note that a regular tree has to be infinite, and that the regular tree of a given degree is unique.

Besides determining whether or not a graph is connected, we have another use for paths on graphs. We can use them to define a distance function on the set of vertices of G.

Definition 1.5 *The* distance $d(v, w)$ *between two vertices $v, w \in V$ is defined as the number of edges in a shortest path connecting them.*

We will use the *combinatorial* distance, where each edge is defined to have length one. It is possible to define other distance functions by assigning a length to each edge, but we will not consider this here. See also figure 1.2, where the length of the shortest path connecting u and v in the tree is 4, so $d(u, v) = 4$. Note that in the

Petersen graph (see figure 1.1) the distance between any pair of non-adjacent edges is 2.

We now give a definition of a special type of graph, which we will need in the next chapter.

Definition 1.6 *A graph G is called* bipartite *if we can assign one of two colours to each vertex in such a way that no edge connects two vertices of the same colour.*

Proposition 1.7 *A graph G is bipartite iff every non-backtracking closed path in G has even length.*

PROOF Clearly a graph is bipartite iff each of its connected components is bipartite, so we assume G is connected. Let γ be a non-backtracking closed path on G defined by $\overrightarrow{\{v_0, v_1, \ldots, v_n, v_0\}}$. Suppose v_0 is coloured black, then v_1 must be coloured white as these two vertices are joined by an edge. Continuing in this manner, we find the vertices along the path must have alternating colours. Looking at the end of the path, we find that as v_0 is black, v_n must be white, hence n must be odd and the path must have even length.

Conversely, assume every non-backtracking closed path in G has even length. Let T be a spanning tree of G, that is, a sub-graph of G which is a tree and has the same vertex set as G. This tree, and in fact any tree, is clearly bipartite, as we can easily colour the vertices alternating black and white without having an edge connecting two vertices of the same colour. We are left to check that the edges in $G - T$ do not connect two vertices of the same colour. Suppose $e = \{x, y\}$ is such an edge, and suppose it does connect two vertices of the same colour, then the non-backtracking closed path made up of the path from x to y in the tree and the edge e will have odd length, which contradicts our assumption. Hence there can be no such edge in G, and thus G is bipartite. $\square$

An example of a bipartite graph is given in figure 1.3, where we have coloured the vertices black and white as by the definition. It is the *complete bipartite graph* $K_{3,3}$ with a set of white vertices of size 3 and a set of black vertices of size 3, where each white vertex is adjacent to all black vertices and vice versa.

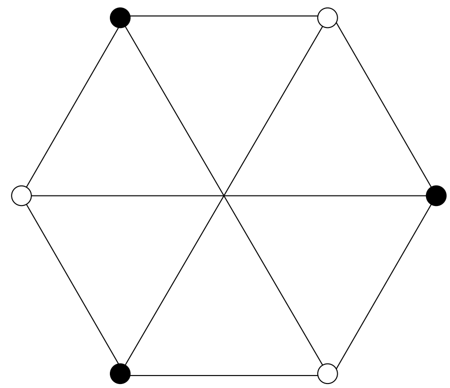

Figure 1.3: The complete bipartite graph $K_{3,3}$.

There are a number of matrices that can be associated to a graph, however for the time being we shall only need one of them, namely the *adjacency matrix*, which we first use in section 1.3.

Definition 1.8 *The* adjacency matrix A_G *of a graph* G *with* n *vertices is an* $n \times n$ *matrix with rows and columns labelled by the vertices of* G*. When* $i \neq j$*, the* i, j^{th} *entry* $a_{i,j}$ *is the number of edges connecting vertex* i *and vertex* j*. The entry* $a_{i,i}$ *on the diagonal is twice the number of loops at vertex* i*.*

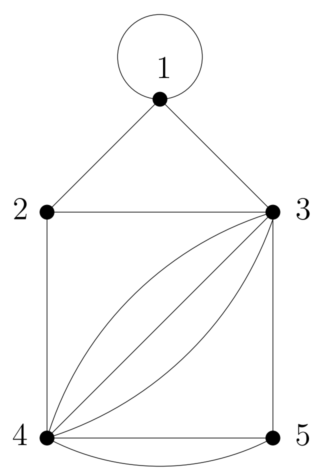

Figure 1.4: A finite graph G.

As an example, we give the adjacency matrix of the graph G given in figure 1.4:

$$A_G = \begin{pmatrix} 2 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 3 & 1 \\ 0 & 1 & 3 & 0 & 2 \\ 0 & 0 & 1 & 2 & 0 \end{pmatrix}$$

Note that the matrix is symmetric, and that the sum of the entries on row (or column) i equals the degree of vertex i. Incidentally, this is why loops are counted twice.

1.2 Covers

Let G be a finite graph with vertex set V and edge set E. We define an *n-fold covering space* G' of G in the topological sense as a covering of the graph viewed as a cell complex. Geometrically this means the following. Let V' be the vertex set of G', which consists of n copies of each vertex in V. Define a surjective map $\pi : G' \to G$ which takes each $v' \in V'$ back to the original vertex in G that it is a copy of. Let E' be the edge set of G', which consists of n copies of each edge in E with the following conditions.

1. Let $e' = \{v', w'\} \in E'$, with $\pi(v') = v$ and $\pi(w') = w$. Then $\{v, w\}$ must be an edge in E.

2. The degree of each vertex must be preserved, that is, if $\pi(v') = v$ then $d(v') = d(v)$ for all $v' \in V'$.

Note that generally there are many possible covering spaces, as for most $e' = \{v', w'\}$ we have a choice of which set $\{v', w'\}$ of pre-images of the vertices v and w we use. The map $\pi : G' \to G$ is called the *covering map* defining the covering space G' of G. However, rather than referring to a covering map defining a covering space, we will refer to the two simultaneously as a covering of G.

We can generalise the concept of covering spaces to an infinite covering of a graph called the *universal cover*. There are (at least) two ways to consider this.

In topology, the universal cover is defined as a simply connected cover. In graph theoretical terms, this means the cover is connected and has no cycles, hence this is a tree. The universal cover is said to cover all connected covers of the graph G.

Unfortunately this is difficult to visualise, but there is a second, more obvious way to construct the universal cover $\widetilde{G}$ of a graph G. Let the vertex set of $\widetilde{G}$ be the set of all non-backtracking paths in G starting at a fixed base point $v_0 \in V$. Two vertices in $\widetilde{G}$ are adjacent if the paths differ by one edge at the end, in such a way that one path can be obtained from the other by removing the last edge of the path (see e.g. [3]). All trees obtained in this way by choosing a different base point are the same up to isomorphism. The universal cover of any finite connected graph is a locally finite tree and, in particular, the universal cover of any regular finite connected graph of degree d is the regular tree of the same degree.

For the universal cover, we have the covering map $\pi : \widetilde{G} \to G$ in the same way as for the n-fold covering. We can use this map and its inverse to compare the graph and its universal cover. For example, by the definition of $\pi(v)$, if we take the pre-image $\pi^{-1}(v)$ of a vertex $v \in G$ we obtain the set of all vertices in $\widetilde{G}$ that are copies of v in the sense of the n-fold covering. This is called the *fibre* of v, and will be useful later on.

There is another way of comparing a regular tree and a finite regular graph of the same degree, which we will first use in chapter 4. Let X be a regular tree of degree $q + 1 \geq 3$. *Automorphisms* of the tree are maps $A : X \to X$ that preserve edge relations (see also [24]). There are three types of non-trivial automorphisms of a regular tree, namely rotations, inversions and translations. Rotations fix one vertex and rotate all others around it, inversions fix one edge but reverse its orientation, and translations fix no vertices and no edges. We can take a subset Γ of the set of translations on X such that Γ is a group, and define the *quotient* X/Γ, which will be a graph. If we then further restrict our choice of Γ, we can find a group Γ of translations on X so that the quotient X/Γ is a finite, simple graph G. This graph will also be regular of degree $q+1$, and clearly the universal cover of G as described above is the tree X.

Let $\tilde{v}_0 \in \pi^{-1}(v)$ be some vertex in the fibre of v. Note that

$$\pi^{-1}(v) = \{\tilde{v} \in V(X) : \pi(\tilde{v}) = v\} = \Gamma\tilde{v}_0$$

that is, the fibre of v is the orbit of $\tilde{v}_0$ by Γ. This holds for any choice of $\tilde{v}_0 \in \pi^{-1}(v)$. Look for example at the graph G in figure 1.5 and its universal cover in figure 1.6,

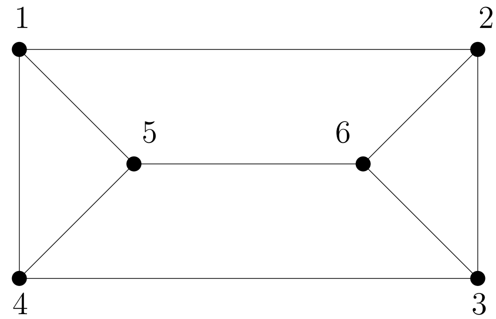

Figure 1.5: A regular graph G with labelled vertices

and choose $v = 1$ and $\tilde{v}_0$ to be the vertex at the centre of the picture. Then the set of all vertices labelled $\tilde{1}$ is mapped to 1 by the projection map π, so this set is the fibre of 1, and it is also the orbit of $\tilde{v}_0$ by Γ.

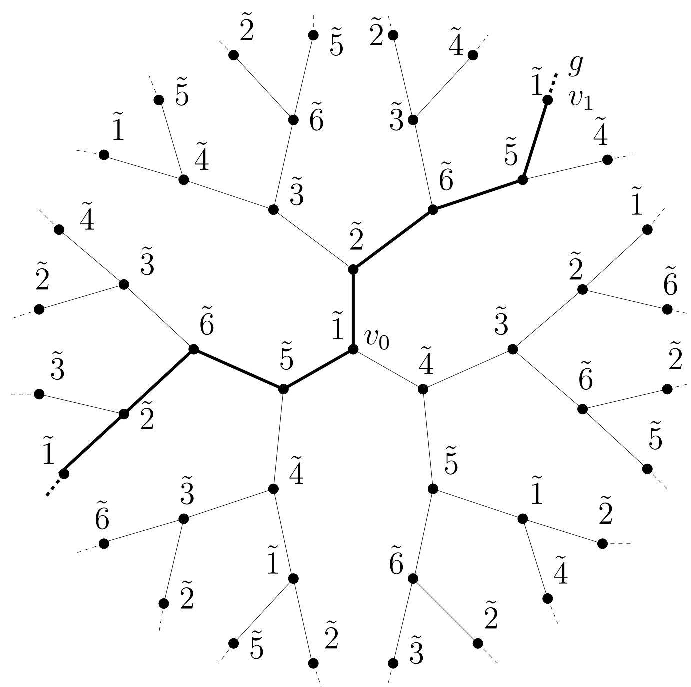

Figure 1.6: Part of the universal cover of G, with labelled vertices.

Recall we defined translations on X to be automorphisms that fix no vertices and no edges. Define a *geodesic* on X to be a bi-infinite, non-backtracking path. By bi-infinite we mean infinite in both directions. We find (see for example [24]) that translations fix a geodesic g in X, in the sense that the vertices of g are mapped by the automorphism to other vertices on g. Now let the geodesic g be defined by a translation $\gamma \in \Gamma$, where Γ is chosen so that $G = X/\Gamma$ is a simple finite graph, and let $d(v_0, \gamma(v_0)) = m$ for some $v_0 \in g$. As γ is an automorphism, this distance m will be the same for any choice of $v_0 \in g$. Then the path $\pi(g)$ obtained on G by applying the projection map will be a finite closed path of length m. This length m will be called the *displacement length* (see also chapter 4, definition 4.5).

The fundamental group $\pi_1(G, v)$ of a finite graph G is defined in terms of closed paths starting and ending at a given vertex v. Given a fixed vertex $v \in G$ and a fixed vertex $\tilde{v}_0$ in the fibre of v, there is a one-to-one correspondence between closed paths starting (and hence terminating) at v and geodesics through $\tilde{v}_0$. It is obtained by applying π^{-1} to the closed path, taking $\pi^{-1}(v) = \tilde{v}_0$, and pre-images of the other vertices of g so that we obtain a non-backtracking path in X. Now extend this path infinitely in both directions in the natural way to obtain a geodesic. Letting the length of the closed path equal the displacement length of the translation, this gives an isomorphism between $\pi_1(G, v)$ and Γ. For example, the closed path p_1 in G through vertices $1, 2, 6, 5, 1$ corresponds to a translation γ_1 along the infinite geodesic g in figure 1.6 which takes v_0 to v_1, so the element in $\pi_1(G, v)$ corresponding to this closed loop corresponds to $\gamma_1 \in \Gamma$. See also section 4.2 for more definitions regarding translations on X.

1.3 Functions and Operators on Graphs

1.3.1 Finite Graphs

It is easy to define a real or complex valued function on the vertices of a finite graph G by assigning a value to each vertex. Later (in chapter 3) we will also define functions on the edges of G, but for now we concentrate on functions defined on the vertices. A nice way to describe these functions is by listing the values as entries in a vector. This will be useful when we apply linear operators to the given functions. Operators are maps which we can apply to one or more functions to produce other functions, and linear operators must also satisfy two conditions, namely additivity and homogeneity of degree 1. For a general linear operator T acting on functions $f(x)$ and $g(x)$, this means that

$$T(f(x) + g(x)) = T(f(x)) + T(g(x)), \text{ and}$$

$$T(af(x)) = aT(f(x))$$

for any scalar a. Now as a linear operator on a finite graph is finite dimensional, we can represent it by a matrix. To apply the operator to a function, all we have to do is multiply the vector representing the relevant function by this matrix.

The operator we will use is called the *Laplace operator* or Laplacian, which is defined by the following equation

$$\mathcal{L}_G f(v) = \frac{1}{d(v)} \sum_{v \sim w} f(w) \tag{1.1}$$

where by $w \sim v$ we mean w is adjacent to v, and we sum over all vertices w adjacent to v. In fact, this is one of several different definitions of the Laplace operator, which we will discuss in section 1.5. For a $(q+1)$-regular finite graph, we can represent our Laplacian as a matrix in terms of the adjacency matrix A_G of G, namely

$$\mathcal{L}_G = \frac{1}{q+1} A_G. \tag{1.2}$$

Associated to the Laplacian on any graph G is a special type of function which will be very important to us in later chapters.

Definition 1.9 *An* eigenfunction *of the Laplacian $\mathcal{L}_G$ is a function φ such that*

$$\mathcal{L}_G \varphi = \lambda \varphi.$$

The scalar λ is called the eigenvalue *associated to φ.*

We will discuss these functions in some more detail in section 1.4, but for the time being note that when we write $\mathcal{L}_G$ for a finite graph G in terms of a matrix, we can find its eigenvectors and eigenvalues in the usual way. Writing the eigenfunctions of G in vector form then gives us exactly the eigenvectors associated to the matrix of $\mathcal{L}_G$.

1.3.2 Infinite Graphs

We can also define functions and operators on infinite graphs, although we are now no longer able to represent them by finite vectors and matrices. We can still apply the Laplacian to functions on infinite graphs, for example the universal cover $\widetilde{G}$ of G, via equation (1.1). We can also use the covering map π as defined in section 1.2 to define a function $\tilde{f}$ on $\widetilde{G}$ from a function f on the finite graph G as follows:

$$\tilde{f} = f \circ \pi \tag{1.3}$$

This is called the *lift* of f from G to $\widetilde{G}$. Take for example the eigenfunctions on G as defined in definition 1.9.

Lemma 1.10 *Lifting an eigenfunction $\varphi(x)$ on G to $\widetilde{G}$ gives an eigenfunction $\widetilde{\varphi}(x)$ of the Laplacian $\mathcal{L}_{\widetilde{G}}$. The eigenfunction $\widetilde{\varphi}(x)$ has the same eigenvalue λ as $\varphi(x)$.*

Proof Due to the construction of the lift, the set of vertices adjacent to any $v \in V(G)$ is in one-to-one correspondence with the set of vertices adjacent to any $\tilde{v} \in V(\widetilde{G})$ such that $\pi(\tilde{v}) = v$, so the sum in the Laplacian is taken over the same function values in each case:

$$\mathcal{L}_G\varphi(v) = \frac{1}{d(v)}\sum_{v\sim w}\varphi(w) = \frac{1}{d(\tilde{v})}\sum_{\tilde{v}\sim w}\widetilde{\varphi}(w) = \mathcal{L}_{\widetilde{G}}\widetilde{\varphi}(\tilde{v}) \tag{1.4}$$

This equation also shows that the eigenvalue is preserved, that is, if we denote the eigenvalue of $\mathcal{L}_{\widetilde{G}}$ by $\widetilde{\lambda}$, then $\widetilde{\lambda} = \lambda$. □

We finish this section with one final remark, which will be important in chapter 4. Let f be a function on the finite graph G defined as the quotient $G = X/\Gamma$ for a regular tree X and a group of translations Γ on X. Lift the function to $X = \widetilde{G}$ by equation (1.3). Then, by its construction, the function $\tilde{f}$ is Γ-*periodic* (or Γ-invariant) on X, that is, the function takes the same value on all vertices in the orbit Γx of any vertex $x \in X$.

1.4 The Graph Spectrum

A useful tool often used to describe the characteristics of a graph is that of the *graph spectrum*. The spectrum of a graph is defined to be the spectrum of its adjacency matrix, that is, the set of eigenvalues associated to this matrix. Now recall that we defined the Laplacian operator $\mathcal{L}_G$ in such a way that, for a regular graph, it can be represented by a constant $(\frac{1}{q+1})$ multiple of the adjacency matrix. This means that for each property of the spectrum of the graph, we can easily find an equivalent property of the spectrum of the Laplacian. We state some of these properties here for future reference.

The adjacency matrix is, by construction, real and symmetric, hence the Laplacian is a real and symmetric operator. We know from elementary matrix theory that a real symmetric matrix has only real eigenvalues.

Proposition 1.11 *Let G be a regular graph. Then all the eigenvalues λ of the Laplacian on G satisfy $|\lambda| \leq 1$, and in particular we know $\lambda_0 = 1$ is an eigenvalue.*

Proof We follow the proof as given in [18, proposition 1.1.2] for the eigenvalues of the adjacency operator.

Let $\varphi_0 \equiv 1$ be the constant function on G. Then clearly φ_0 is an eigenfunction of the Laplacian on G with eigenvalue $\lambda_0 = 1$. Now let φ be any eigenfunction of $\mathcal{L}_G$ with eigenvalue λ, and suppose the vertex v satisfies $|\varphi(v)| = \max_{w \in V} |\varphi(w)|$. Assume that $\varphi(v) > 0$ (if it is not then work with $-\varphi$ to obtain this condition). Then

$$|\lambda|\varphi(v) = |\lambda\varphi(v)| = |\mathcal{L}_G\varphi(v)| = \left|\frac{1}{q+1}\sum_{v\sim w}\varphi(w)\right| \leq \frac{1}{q+1}(q+1)\varphi(v) = \varphi(v).$$

Dividing the resulting equation $|\lambda|\varphi(v) \leq \varphi(v)$ by $\varphi(v)$ we obtain $|\lambda| \leq 1$ as required. □

Proposition 1.12 *Let G be a connected regular graph. Then the eigenvalue $\lambda_0 = 1$ is a simple eigenvalue, i.e. an eigenvalue with multiplicity one.*

Proof Again, we use ideas from [18].

Suppose G is connected, and let φ be an eigenfunction associated to $\lambda_0 = 1$. Let $v \in V$ satisfy $|\varphi(v)| = \max_{w \in V} |\varphi(w)|$, and suppose $\varphi(v) > 0$ as in the proof of proposition 1.11. Now

$$\varphi(v) = \mathcal{L}_G\varphi(v) = \frac{1}{q+1}\sum_{v\sim w}\varphi(w) \leq \frac{1}{q+1}(q+1)\varphi(v) = \varphi(v).$$

For this to hold, we need $\varphi(w) = \varphi(v)$ for all w adjacent to v. Applying the same principle to all such w and continuing in the same manner, we find that $\varphi(w) = \varphi(v)$ for all $w \in V$. As G is connected, this implies that the constant function obtained is the only possible eigenfunction associated to $\lambda_0 = 1$ up to multiplication by a constant, so λ_0 is a simple eigenvalue. □

Proposition 1.13 *The Laplacian on a connected regular graph has eigenvalue* -1 *iff the graph is bipartite. In fact, the graph is bipartite iff the spectrum of the Laplacian is symmetric around* 0.

PROOF Again we follow a proof in [18, proposition 1.1.4].

Suppose G is bipartite, with associated bipartition of the vertex set V into two disjoint sets A, B so that $V = A \cup B$ and every edge in G connects one vertex in A to one in B. Suppose φ is an eigenfunction on G with eigenvalue λ. Define a new function

$$\psi(x) = \begin{cases} \varphi(x) & \text{if } x \in A \\ -\varphi(x) & \text{if } x \in B. \end{cases}$$

Then it is easy to see that $\psi(x)$ is also an eigenfunction of G, with associated eigenvalue $-\lambda$. Hence the spectrum of $\mathcal{L}_G$ is symmetrical, and by proposition 1.11 we find that the Laplacian has an eigenvalue equal to -1.

Now suppose the Laplacian has an eigenvalue equal to -1 with associated eigenfunction φ. Let $v \in V$ satisfying $|\varphi(v)| = \max_{w \in V} |\varphi(w)|$, and as in the proof of proposition 1.11 we assume that $\varphi(v) > 0$. We find that

$$\varphi(v) = -1 \cdot \mathcal{L}_G \varphi(v) = \frac{1}{q+1} \sum_{v \sim w} (-\varphi(w)).$$

We know that $|\varphi(w)| \leq \varphi(v)$ for all $w \in V$, so for the above equation to hold we must have $\varphi(w) = -\varphi(v)$ for all w in the sum. Similarly, for a vertex u adjacent to any such w we find that $\varphi(u) = -\varphi(w) = \varphi(v)$. We can use this to define two sets in V. Let A be the set of vertices w for which $\varphi(w) = \varphi(v) > 0$, and let B be the set of vertices w for which $\varphi(w) = -\varphi(v) < 0$. By the way we constructed these sets, and as G is connected, $V = A \cup B$ and $A \cap B = \emptyset$. Every edge in G now connects a vertex in A to a vertex in B, hence G is bipartite. □

Note that for any finite regular graph G, A_G is a real symmetric matrix, hence so is the matrix for $\mathcal{L}_G$. This means the Laplacian is a self-adjoint operator, and its eigenfunctions φ_i form a *basis* for all functions on V. Now take the lift to $\widetilde{G}$ of this set of $|V(G)| = N$ eigenfunctions, and call the resulting functions $\widetilde{\varphi}_i$, where $i = 0, \ldots, N-1$. Recall that the lift of a function from $G = \widetilde{G}/\Gamma$ to $\widetilde{G}$ is Γ-periodic on $\widetilde{G}$. Clearly the set of functions $\{\widetilde{\varphi}_i\}_{i=0}^{N-1}$ forms a basis for all such Γ-periodic functions on $\widetilde{G}$.

Definition 1.14 *Let G be a finite connected regular graph, and let*

$$1 = \lambda_0 > \lambda_1 \geq \ldots \geq \lambda_{N-1}$$

be the eigenvalues of $\mathcal{L}_G$ in non-increasing order. Then the spectral gap *is defined to be*

$$\min\{|1-\lambda_1|, |-1-\lambda_{N-1}|\}.$$

We will see in later chapters that some convergence properties are better for graphs with a larger spectral gap. Graphs with a large spectral gap at least equal to a certain value are defined as follows.

Definition 1.15 *A finite $(q+1)$-regular connected graph G is called* Ramanujan *if all eigenvalues λ other than 1 and -1 satisfy*

$$|\lambda| \leq \frac{2\sqrt{q}}{q+1},$$

so a Ramanujan graph has a spectral gap of at least

$$1 - \frac{2\sqrt{q}}{q+1} = \frac{(\sqrt{q}-1)^2}{q+1}.$$

1.5 Analogy Between Regular Trees and the Hyperbolic Plane

The starting point of this thesis is the remarkable analogy between the hyperbolic disc and the regular tree. Many authors have already taken advantage of this analogy, see for example Cartier [13], Figà-Talamanca and Nebbia [24], or Sunada [55], where we find the illustration in figure 1.7.

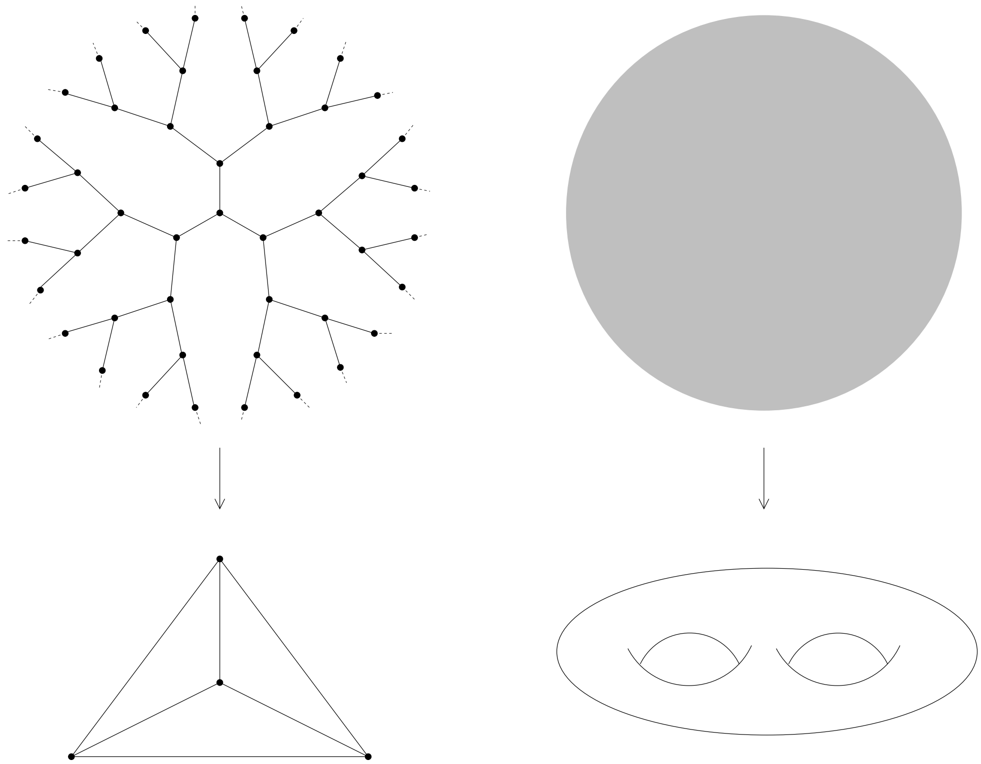

Figure 1.7: Analogy between the tree and the hyperbolic disc, from [55, p 73].

The analogy arises when we carry out harmonic analysis on either space. For example, when we consider Fuchsian groups (discrete subgroups of the group of isometries, see for example [37]) on the hyperbolic disc, we can compare these to discrete subgroups of isometries on the tree. In particular, we find similar types of isometries on both the hyperbolic plane $\mathbb{H}$ and the tree X. A hyperbolic isometry on $\mathbb{H}$, for example, fixes two points on the boundary, and fixes the geodesic connecting these two points as a set. The points on the geodesic are translated along the geodesic by a fixed hyperbolic distance, which is called the displacement length. Similarly, we have a type of isometry on X which fixes a geodesic γ as a set but translates the vertices on γ along it by a fixed distance, again called the displacement length. We defined these isometries in section 1.2 to be translations.

Now take a strictly hyperbolic discrete subgroup Γ of isometries on $\mathbb{H}$, and note that there is a fundamental domain $\mathfrak{F}$ associated to Γ on $\mathbb{H}$. The *fundamental domain* of a group Γ acting on a topological space is a domain which contains exactly one point from the orbit of the group action of each point in the space, except perhaps on the boundary of the domain, where we may have more than one point from an orbit. So $\mathfrak{F}$ contains exactly one representative of the orbit Γx of each point $x \in \mathbb{H}$ (with exception of the boundary). Exactly the same holds for a group of translations on T. In both settings we can take the quotient of the space by the group to obtain, in certain cases, a finite surface or graph, as pictured in figure 1.7. The arrow in the pictures denotes the projection map π.

The Laplace operator as defined on hyperbolic surfaces has several analogues on the tree, one of which we defined in section 1.3.1. Two other, more common ways of defining the Laplacian on a graph are

- the combinatorial Laplacian $\mathcal{L}_{\text{comb}}\ f(x) = \sum_{x \sim y} \big(f(x) - f(y)\big)$
- the canonical Laplacian $\mathcal{L}_{\text{can}}\ f(x) = \frac{1}{d(x)} \sum_{x \sim y} \big(f(x) - f(y)\big)$.

Note however that for a regular graph the two operators defined above are just constant multiples of each other, and in the case of a finite graph both can be represented by matrices. Letting A_G be the adjacency matrix of the graph G, and D_G the matrix with the degree of vertex i on the i^{th} diagonal entry and zeros everywhere else, the combinatorial Laplacian is represented by $D_G - A_G$ and the canonical Laplacian by $I - D_G^{-1} A_G$, where I is the identity matrix. Our Laplacian defined in equation (1.1) can be obtained from the canonical Laplacian via

$$\mathcal{L}_G f(x) = f(x) - \mathcal{L}_{\text{can}} f(x)$$

and is hence represented in matrix form for finite graphs by $D_G^{-1} A_G$, which for a $(q+1)$-regular graph is just $\frac{1}{q+1} A_G$, as noted in equation (1.2).

Many other objects can be defined both on the hyperbolic plane and the tree, or on a surface and a finite graph, for example horocycles (see section 2.6.2), the ideal boundary (see for example [24] for the case of trees), the Radon transform (see for example [14] for the tree case), and as we will see in chapter 5 there is a discrete analogue of the Selberg Trace Formula as defined by Selberg [50]. Hence it is only natural to look for more results for graphs and trees that are analogues of results on surfaces and hyperbolic space.

Chapter 2

Radial Averages for Vertices

2.1 Motivation

The motivation for the research problem described in this chapter was work by Paul Günther on spherical averages in hyperbolic space. Using the well-known analogy between hyperbolic space and regular graphs described in section 1.5, we want to look at the equivalent problem in the discrete case. The results in the continuous case can be found in [28], and we shall outline the basic ideas from his work in this section.

Günther works in n dimensions, but we just state the two-dimensional case as it is the one most like our case. Consider the hyperbolic plane modelled by the Poincaré disc $\mathbb{D}$, and consider a polygon $\mathcal{F}$ in $\mathbb{D}$ with $4g$ sides. The interior angles and side-lengths of $\mathcal{F}$ have to satisfy certain conditions, which we state after the following observation. Identifying the sides of $\mathcal{F}$ in an appropriate way results in a surface M of genus g with constant curvature $K = -1$. Hence the lengths of the sides which are identified must be equal, and the interior angles at the corner points which are identified must add up to 2π.

Let Γ be the group generated by the side-identifications on $\mathbb{D}$ used to produce the surface from the polygon. This group Γ is a subgroup of $PSU(1,1)$, and images $\gamma\mathcal{F}$ of $\mathcal{F}$ under $\gamma \in \Gamma$ form a tessellation of $\mathbb{D}$. Let $\pi : \mathbb{D} \to M$ be the projection map, which is one-to-one from our polygon onto the surface, except for points on the boundary of $\mathcal{F}$. The pre-image of a point in M under this map consists of one point in each $\gamma\mathcal{F}$, again with the exception of points on the boundary of $\mathcal{F}$. We find that $M = \mathbb{D}/\Gamma$, and note that $\mathbb{D}$ is the *universal covering* of the surface M in the topological sense.

Now let $f : M \to \mathbb{R}$ be a continuous function on the surface, and let $\tilde{f}$ be its lift to the Poincaré disc, that is $\tilde{f} = f \circ \pi : \mathbb{D} \to \mathbb{R}$. Note that $\tilde{f}$ is Γ-periodic on $\mathbb{D}$, i.e. it is the same on each copy of $\mathcal{F}$. Choose a point $p_0 \in \mathcal{F}$. Take a (two-dimensional) sphere $S_r(p_0)$ in $\mathbb{D}$ centred at p_0, and of hyperbolic radius r large enough so that the sphere incorporates several distinct copies $\gamma\mathcal{F}$ of $\mathcal{F}$. We then define the *spherical mean* by the following formula:

$$M_r(f) = \frac{1}{\ell(S_r(p_0))} \int_{S_r(p_0)} \tilde{f} d\sigma$$

Here $\ell(S_r(p_0))$ is the length of the circle, and σ is a length element along it (the arc-length).

Günther studied the behaviour of the spherical mean as $r \to \infty$. His theorem states that as $r \to \infty$ we get

$$M_r(f) \longrightarrow \frac{1}{\mathrm{vol}(M)} \int_M f d\mathrm{vol}.$$

The problem outlined above is referred to as the *equidistribution of increasing spheres*. In this chapter, we prove similar results for functions on the vertices of regular graphs with, in addition, a convergence rate. However, rather than just looking at spheres, we also consider averages over more general sets, namely spherical arcs, which in turn imply results for spheres, tubes, horocycles, balls and sectors.

2.2 Result for Regular Graphs

Let G be a finite regular graph as defined in section 1.1, with degree $d(v) \geq 3$, universal cover $\widetilde{G}$ (which is a regular tree of the same degree), and projection map $\pi : \widetilde{G} \to G$ as defined in section 1.2. Take any function $f : V \to \mathbb{R}$ and lift it to the universal cover via $\tilde{f} = f \circ \pi$. We define sets of vertices on the tree $\widetilde{G}$ called *spheres* and *arcs* as follows.

Definition 2.1 *A* sphere *of radius n on the tree $\widetilde{G}$ centred at the vertex v_0 is defined as*

$$S_n(v_0) = \{v \in V(X) : d(v, v_0) = n\}. \quad (2.1)$$

Note that $d(v, w) \in \mathbb{N}$ for all $v, w \in X$, so this definition only makes sense for integer radius.

Definition 2.2 *Let $a = \overrightarrow{\{w', w\}}$ be the directed edge in the tree $\widetilde{G}$. We define the (spherical)* arc *of radius $n+1$ on $\widetilde{G}$ based at the vertex w' in the direction of w as*

$$A_{n+1}(a) = S_{n+1}(w') \cap S_n(w) \quad (2.2)$$

for $n + 1 \geq 1$. We set $A_0(a) = \{w'\}$.

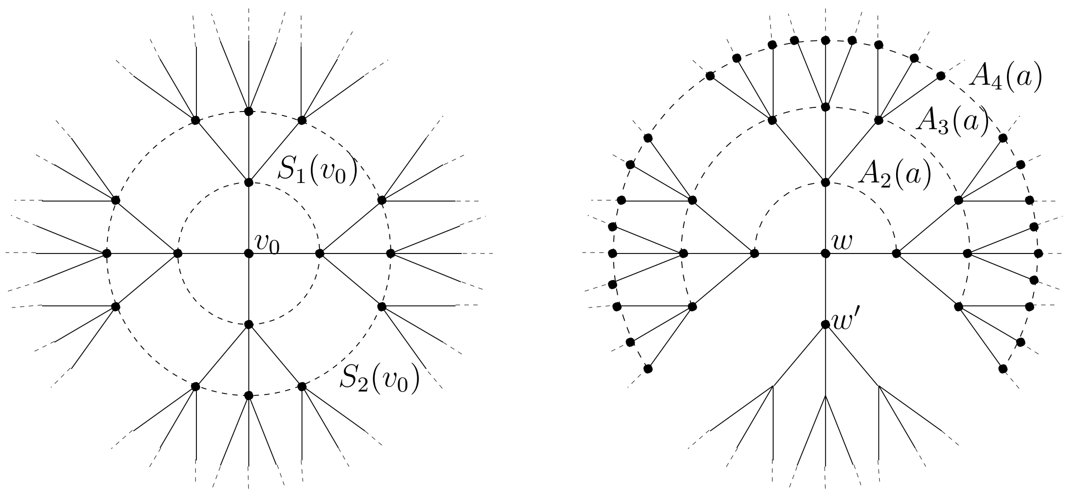

Figure 2.1: Spheres and arcs on the regular tree of degree 4.

Note that the arc is a section of the sphere, and we can make up a sphere of radius n centred at w' by taking for example the union of the set of arcs

$$\{A_n(a_i) : i = 1, \ldots, d(w')\}$$

for $a_i = \overrightarrow{\{w', w_i\}}$ where $\{w_i\}$ runs through all vertices adjacent to w'. We can now define our more general discrete version of Günther's spherical mean.

Definition 2.3 *The* arc average *of the function $f : V \to \mathbb{R}$ is defined as*

$$M_{n,a}(f) = \frac{1}{|A_n(a)|} \sum_{x \in A_n(a)} \tilde{f}(x).$$

The main purpose of this chapter is to study the asymptotic behaviour of this average as $n \to \infty$ for regular graphs. We then use the result for arcs to prove similar results on other level sets in $\widetilde{G}$. To this end, we use the analogy between hyperbolic surfaces and regular graphs described in section 1.5. The problem studied by Günther [28] for spheres in the hyperbolic plane as described in section 2.1 is analogous to the case of spheres on the universal cover of the regular graph. We shall discuss spheres later in section 2.6, as we derive it from a more general result for spherical arcs, which we now state.

Theorem 2.4 *Let G be a finite, non-bipartite, regular connected graph of degree $d(v) = q + 1 \geq 3$ and $f : V \to \mathbb{R}$ a function on its vertices. Then we have for any directed edge a in G*

$$\left| M_{n,a}(f) - \frac{1}{|V|} \sum_{v \in V} f(v) \right| \leq C_G ||f||_2 \beta_{max}^n.$$

Here C_G is a constant depending on G but independent of a, and $\beta_{max} \in [q^{-1/2}, 1)$.

Obviously, this implies that

$$\lim_{n \to \infty} M_{n,a}(f) = \frac{1}{|V|} \sum_{v \in V} f(v) \tag{2.3}$$

for any directed edge a. We call the right hand side of equation (2.3) the *graph average* of the function. The norm $||f||_2$ comes from the inner product $\langle f, g \rangle = \sum_{v \in V} f(v)g(v)$, so $||f||_2 = \sqrt{\langle f, f \rangle}$. We exclude bipartite graphs in this theorem because spheres of even and odd radii have to be treated separately in this case - see section 2.5 for details.

We shall see in the proof that the convergence rate $\beta_{\max}$ depends on the Fourier coefficients of f, and the spectral gap. Recall from section 1.4 that Ramanujan graphs have a large spectral gap (see e.g. [18] or [40]). These graphs have either $\beta_{\max} = q^{-1/2}$ or $\beta_{\max} = q^{-1/2+\varepsilon}$ for arbitrarily small $\varepsilon > 0$, giving the best convergence rate for a general function. For more details see the proof in section 2.4.2.

2.3 Background

Before we turn to the proof of the above theorem, let us briefly explain how radial averages are related to non-backtracking random walks (NBRW), a subject of active current research.

2.3.1 Random Walks

A *random walk* is a path (sometimes also called a walk) on a graph where each edge is independent of all previous edges, or more precisely, if the i^{th} edge is $e_i = \overrightarrow{\{v,w\}}$, then the $(i+1)^{\text{th}}$ edge can be any edge originating from w with equal probability. Random walks are a special case of Markov chains, which are random processes where every next state depends only on the present and not on other past states, i.e. they satisfy the Markov property. A *non-backtracking* random walk on a graph is almost a random walk, except that the walk is not allowed to backtrack, so if the i^{th} edge is $e_i = \overrightarrow{\{v,w\}}$, then the $(i+1)^{\text{th}}$ edge may with equal probability be any edge incident with w *except* $\overrightarrow{\{w,v\}}$.

We can use our spherical average result (corollary 2.7 in section 2.6) to find out with what probability a NBRW on G starting at v_0 will end up at a vertex $v \in V$ after r steps. To do this, we use the characteristic function δ_v on V, defined by

$$\delta_v(w) = \begin{cases} 1 & \text{if } w = v \\ 0 & \text{if } w \neq v. \end{cases}$$

We lift δ_v to $\widetilde{G}$, and take the average of this lifted function $\tilde{\delta}_v$ over the sphere of radius r to get the required probability. The probabilities we obtain here coincide with those found by other authors, for example Alon et al. [2] or Ortner and Woess [45], and the convergence rate we obtain coincides with the mixing rate obtained in [2], which measures how fast the probability distribution of NBRW converges to the stationary distribution, our graph average. Our spherical average result, however, is a corollary of the main theorem of this chapter, which concerns arc averages. The random walk equivalent of our main result would be a NBRW *with prescribed first step*, which to the best of our knowledge has not been studied.

Another difference between our result and that of NBRW is that we allow *any* real function on G rather than just δ_v. We will show in section 2.4.2 that we can improve the general convergence rate if we know the Fourier coefficients of the function we are dealing with. Finally, we will also discuss results for tubes, horocycles, balls and sectors in section 2.6. None of these results have an obvious NBRW equivalent.

2.3.2 Cogrowth

NBRW have been studied in the context of cogrowth on graphs, hence we briefly discuss this. Cogrowth was first introduced in the context of groups and their Cayley graphs, and was studied in the early 1980s by, amongst others, Grigorchuk [27], Cohen [16] and Woess [61]. In the 1990s the application of cogrowth was extended to arbitrary graphs, see for example Northshield [44] or Bartholdi [4].

We define the growth of the tree $\widetilde{G}$ by

$$\mathrm{gr}(\widetilde{G}) = \limsup_{r\to\infty} \left|S_r(\tilde{v})\right|^{1/r},$$

and the cogrowth of the graph G by

$$\mathrm{cogr}(G) = \limsup_{r\to\infty} \left|S_r(\tilde{v}) \cap \pi^{-1}(v)\right|^{1/r},$$

both of which are independent of $\tilde{v} \in \widetilde{V}$, where $\pi(\tilde{v}) = v \in V$. Then the cogrowth constant is $\eta = \frac{\ln \mathrm{cogr}(G)}{\ln \mathrm{gr}(\widetilde{G})}$.

More recently, Ortner and Woess [45] generalised the definition of cogrowth and used it to study NBRW. They set

$$\mathrm{cog}_r^{\nu}(v, w) = \nu_{\tilde{v},r}(\pi^{-1}(v))$$

where $\nu = (\nu_{\tilde{v},r})_{\tilde{v}\in\widetilde{V},r\geq 0}$ is a sequence of probability measures concentrated on the sphere $S_r(\tilde{v})$, subject to some regularity conditions. Choosing particular measures one obtains cogrowth or NBRW probabilities, and both notions coincide in the case of a regular graph.

2.4 Proof of the Theorem

We now discuss the proof of theorem 2.4, which concerns functions on the vertices of a finite connected non-bipartite regular graph G of degree $d(v) = q + 1 \geq 3$. Let $f : V \to \mathbb{R}$ be such a function, and recall that in section 1.3.1 we defined the Laplacian of f at $v \in V$ as

$$\mathcal{L}_G f(v) = \frac{1}{d(v)} \sum_{v \sim w} f(w)$$

and showed that it is a real symmetric operator with eigenvalues λ satisfying $-1 \leq \lambda \leq 1$. The eigenvalue -1 occurs iff G is bipartite (see proposition 1.13), and we have excluded this case from the theorem precisely due to this eigenvalue. The simple eigenvalue 1 is associated to the constant eigenfunction, so for all non-constant eigenfunctions we now have $|\lambda| < 1$. First we will show that the arc average converges to the graph average, and in the next section we will use the proof to calculate the convergence rate.

2.4.1 Convergence of Eigenfunctions

First we prove the convergence result for a basis of functions on G. As any function can be written as a linear combination of basis functions, this is sufficient to prove that any function converges to the graph average. We choose the orthonormal basis of eigenfunctions φ_i of the Laplacian with corresponding eigenvalues λ_i. Let φ_0 be the constant eigenfunction, and note that here the arc average $M_{n,a}(\varphi_0)$ clearly equals the graph average for all n. The φ_i are orthogonal, so $\langle \varphi_i, \varphi_0 \rangle = 0$ and hence the graph average $\sum_{v \in V} \varphi_i(v) = 0$ for $i \neq 0$. To prove convergence, we aim to show that $M_{n,a}(\varphi_i) \to 0$ for $i = 1, 2, \ldots, |V| - 1$.

For each eigenfunction $\varphi_i \neq \varphi_0$ on G let $\widetilde{\varphi}_i = \varphi_i \circ \pi$ be its lift onto the universal cover $\widetilde{G}$, where it is an eigenfunction of $\mathcal{L}_{\widetilde{G}}$ with the same eigenvalue λ_i by lemma 1.10. Recall we defined the arc average of a function in definition 2.3. For the eigenfunctions $\widetilde{\varphi}_i$, let

$$F_i(v) = M_{d(v,w'),a}(\varphi_i) = \frac{1}{|A_{d(v,w')}(a)|} \sum_{y \in A_{d(v,w')}(a)} \widetilde{\varphi}_i(y) \tag{2.4}$$

where $a = \overrightarrow{\{w', w\}}$. The function $F_i(v)$ depends only on $d(w', v) = n$, hence we shall denote it $F_i(n)$ for all $v \in A_n(a)$.

Lemma 2.5 *We can interchange the order of the Laplacian and the radial average as operators acting on functions on the tree, and obtain the same result.*

PROOF Applying the Laplacian first and then the radial average, we obtain

$$\begin{aligned} M_{n,a}(\mathcal{L}_{\widetilde{G}}(\widetilde{\varphi}_i(v))) =& \frac{1}{|A_n(a)|} \sum_{z \in A_n(a)} \left(\frac{1}{q+1} \sum_{v \sim y} f(z) \right) \\ =& \frac{1}{(q+1)|A_n(a)|} \Big(\sum_{y \in A_{n+1}(a)} f(y) + q \cdot \sum_{y \in A_{n-1}(a)} f(y) \Big) \end{aligned} \tag{2.5}$$

where $a = \overrightarrow{\{w', w\}}$, and n is chosen such that $v \in A_n(a)$. Applying the radial average first and then the Laplacian gives

$$\begin{aligned} \mathcal{L}_{\widetilde{G}} M_{n,a}(\widetilde{\varphi}_i(v)) &= \frac{1}{q+1} \sum_{z \sim y} \left(\frac{1}{|A_n(a)|} \sum_{y \in A_n(a)} f(z) \right) \\ &= \frac{1}{q+1} \Big(q \cdot \frac{1}{|A_{n+1}(a)|} \sum_{y \in A_{n+1}(a)} f(y) + \frac{1}{|A_{n-1}(a)|} \sum_{y \in A_{n-1}(a)} f(y) \Big) \\ &= \frac{1}{q+1} \Big(q \cdot \frac{1}{q|A_n(a)|} \sum_{y \in A_{n+1}(a)} f(y) + \frac{1}{\frac{1}{q}|A_n(a)|} \sum_{y \in A_{n-1}(a)} f(y) \Big) \\ &= \frac{1}{(q+1)|A_n(a)|} \Big(\sum_{y \in A_{n+1}(a)} f(y) + q \cdot \sum_{y \in A_{n-1}(a)} f(y) \Big) \end{aligned} \tag{2.6}$$

where again $v \in A_n(a)$ and we note that $|A_{n+1}(a)| = q|A_n(a)| \ \forall \ n \geq 1$. Now note that equations (2.5) and (2.6) are equal to complete the proof. □

Using the lemma above and lemma 1.10 we obtain a recursion relation for $F_i(n)$ namely

$$F_i(n+1) - \frac{q+1}{q} \lambda_i F_i(n) + \frac{1}{q} F_i(n-1) = 0 \ \forall \ n \geq 1. \tag{2.7}$$

Note $F_i(0) = \varphi_i(w')$ and $F_i(1) = \varphi_i(w)$ give the initial conditions. We assume $F_i(n) = \alpha_i^n$ to solve the recursion relation, and obtain two solutions $\alpha_i^{\pm}$ which depend on $D_i = (q+1)^2 \lambda_i^2 - 4q$, and are distinct iff $D_i \neq 0$. The $\alpha_i^{\pm}$ are given by

$$\alpha_i^{\pm} = \frac{q+1}{2q} \lambda_i \pm \frac{1}{2q} \sqrt{D_i}. \tag{2.8}$$

The general solution for $D_i \neq 0$ is then $F_i(n) = u_i^+ (\alpha_i^+)^n + u_i^- (\alpha_i^-)^n$ where

$$u_i^{\pm} = \widetilde{\varphi}_i(w') \frac{\sqrt{D} \pm \big((q+1)\lambda - 2\sqrt{D} \big)}{2\sqrt{D}} \mp \widetilde{\varphi}_i(w) \frac{q}{\sqrt{D}}. \tag{2.9}$$

For $D_i = 0$, $\alpha_i^{\pm} = \alpha_i = \pm\frac{1}{\sqrt{q}}$ and the general solution is $F_i(n) = u_i(\alpha_i)^n + v_i n(\alpha_i)^n$ for constants $u_i = \widetilde{\varphi}_i(w')$ and

$$v_i = \pm\sqrt{q}\widetilde{\varphi}_i(w) - \widetilde{\varphi}_i(w'). \tag{2.10}$$

It now just remains to check that $|\alpha_i^{\pm}| < 1$ for $|\lambda| < 1$, and $|n(\alpha_i)^n| = |nq^{n/2}| \to 0$ to obtain $\lim_{n\to\infty} F_i(n) = 0$ for $i \neq 0$ as required.

2.4.2 The Convergence Rate

For the calculation of the convergence rate we distinguish three cases:

Case 1 $D_i < 0$ ($|\lambda_i| < \frac{2\sqrt{q}}{q+1}$): We find $|\alpha_i^{\pm}| = \frac{1}{\sqrt{q}}$ and

$$|F_i(n)| \leq (|u_i^+| + |u_i^-|)\big(\frac{1}{\sqrt{q}}\big)^n \leq C_i q^{-n/2}$$

for some constant $C_i > 0$ which depends on u_i^+ and u_i^-, i.e. the initial conditions given by $\widetilde{\varphi}_i(w)$ and $\widetilde{\varphi}_i(w')$. Now there are only finitely many values of $u_i^{\pm}$, as there are only finitely many choices of a. Therefore we can choose C_i large enough so that it is independent of a.

Case 2 $D_i = 0$ ($|\lambda_i| = \frac{2\sqrt{q}}{q+1}$): Here we have

$$|F_i(n)| \leq \big(|u_i| + |v_i|n\big)\big(\frac{1}{\sqrt{q}}\big)^n \leq C_i' \cdot (n+1) \cdot q^{-n/2}$$

for some $C_i' > 0$. Choosing $\beta_i = q^{-1/2+\varepsilon}$ for arbitrary $\varepsilon > 0$ and adjusting the constant $C_i(\varepsilon)$ appropriately, we obtain

$$|F_i(n)| \leq C_i(\varepsilon)\beta_i^n$$

for $C_i(\varepsilon) > 0$ independent of a.

Case 3 $D_i > 0$ ($\frac{2\sqrt{q}}{q+1} < |\lambda_i| < 1$): We find $\alpha_i^{\pm}$ are both real and $|\alpha_i^{\pm}| < 1$. Let $\beta_i = \max\{|\alpha_i^+|, |\alpha_i^-|\}$, which we can find explicitly as follows. Suppose $\frac{2\sqrt{q}}{q+1} < \lambda_i < 1$, then

$$|\alpha_i^{\pm}| = \frac{q+1}{2q}|\lambda_i| \pm \frac{\sqrt{D_i}}{2q}.$$

Now suppose $-1 < \lambda_i < \frac{-2\sqrt{q}}{q+1}$, then

$$|\alpha_i^{\pm}| = \frac{q+1}{2q}|\lambda_i| \mp \frac{\sqrt{D_i}}{2q}.$$

In either case the maximum of the two is $\frac{q+1}{2q}|\lambda_i| + \frac{\sqrt{D_i}}{2q}$, so $\beta_i = \frac{q+1}{2q}|\lambda_i| + \frac{\sqrt{D_i}}{2q}$. Then using $\frac{2\sqrt{q}}{q+1} < |\lambda_i| < 1$ and $0 < |D_i| \leq (q-1)^2$ we find $\frac{1}{\sqrt{q}} < \beta_i < 1$, hence we have

$$|F_i(n)| \leq C_i \beta_i^n$$

for some $C_i > 0$ independent of a.

A general function $f : V \to \mathbb{R}$ can be written as $f = \sum_{i=0}^{|V|-1} a_i \varphi_i$ and we obtain

$$\Big|M_{r,a}(f) - \frac{1}{|V|}\sum_{v\in V} f(v)\Big| \leq \Big|\sum_{i=1}^{|V|-1} a_i F_i(r)\Big| \leq \left(\sum_{i=1}^{|V|-1} |a_i| C_i\right) \beta_{\max}^r. \tag{2.11}$$

Here $\beta_{\max} = \max_{i=1,\ldots|V|-1}\{\beta_i\}$ is the convergence rate obtained from the eigenvalue $\lambda_i \neq 1$ of largest modulus, so the larger the spectral gap of G, the smaller $\beta_{\max}$. If we know the Fourier coefficients a_i of f then we can improve $\beta_{\max}$ by taking the maximum β_i over $i = 1, \ldots, |V| - 1$ such that $a_i \neq 0$. When $a_i = 0$ for the largest eigenvalue λ_i not equal to 1, this gives us a smaller $\beta_{\max}$.

Applying Cauchy-Schwarz to equation (2.11), we obtain

$$\Big|M_{r,a}(f) - \frac{1}{|V|}\sum_{v\in V} f(v)\Big| \leq C_G \sqrt{\sum_{i=1}^{|V|-1} |a_i|^2}\ \beta_{\max}^r \leq C_G\ ||f||_2\ \beta_{\max}^r$$

where $C_G = \sqrt{|V|-1} \cdot \max_i C_i$. Note that this convergence is independent of a, and that for Ramanujan graphs we obtain $\beta_{\max} = q^{-1/2}$ (or $q^{-1/2+\varepsilon}$ if $|\lambda_{\max}| = \frac{2\sqrt{q}}{q+1}$) as all their eigenvalues give $D \leq 0$.

It turns out that the general $\beta_{\max}$ (for unknown Fourier coefficients a_i) is exactly the mixing rate for NBRW found in [2]. Recall from section 2.3.1 that their result corresponds to taking an average over a vertex *sphere* S_r rather than an arc. If we know that one or more Fourier coefficients of f vanish, we can get a value of $\beta_{\max}$ smaller than this mixing rate.

2.5 Bipartite Graphs

In this section we shall briefly revisit bipartite graphs, before extending theorem 2.4 in the next section to increasing subsets of $\widetilde{G}$ other than arcs.

Recall in definition 1.6 we defined a bipartite graph as one whose vertices can be coloured, using just two distinct colours, so that no two adjacent vertices have the same colour. Let G be a $(q+1)$-regular bipartite graph with N vertices, where $V = P \cup Q$ is the corresponding partition into two sets of non-adjacent vertices of the same colour. Suppose $v_0 \in P$. Let $f : V \to \mathbb{C}$ be a function on the vertices of G. We can still write f in terms of eigenfunctions of the Laplacian, but to investigate the convergence of its arc average we have to deal with the eigenvalue -1. The other eigenvalues are dealt with as in theorem 2.4.

In proposition 1.13 we showed that the spectrum of a bipartite graph is symmetric, so if we label the eigenvalues such that $\lambda_0 > \lambda_1 \geq \ldots \geq \lambda_N$, we have $\lambda_i = -\lambda_{N-i}$ for all $i = 0, \ldots, N$. Let $\varphi_i(x)$ be an eigenfunction of the Laplacian on G with eigenvalue λ_i, then

$$\varphi_{N-i}(x) = \begin{cases} \varphi_i(x) & \text{if } x \in P \\ -\varphi_i(x) & \text{if } x \in Q \end{cases} \tag{2.12}$$

is an eigenfunction with eigenvalue $\lambda_{N-i} = -\lambda_i$. Now use this and the fact that $\sum_{v \in V} \varphi_i(v) = 0$ for $i \neq 0$ to find, for all $i \neq 0, N$,

$$\sum_{v \in V} \varphi_i(v) = \sum_{v \in V} \varphi_{N-i}(v) \tag{2.13}$$

$$\sum_{v \in P} \varphi_i(v) = \sum_{v \in P} \varphi_{N-i}(v). \tag{2.14}$$

Subtracting equation (2.14) from equation (2.13) we obtain

$$\sum_{v \in Q} \varphi_i(v) = \sum_{v \in Q} \varphi_{N-i}(v) \tag{2.15}$$

but using equation (2.12) we find

$$\sum_{v \in Q} \varphi_i(v) = -\sum_{v \in Q} \varphi_{N-i}(v). \tag{2.16}$$

Equations (2.15) and (2.16) imply

$$\sum_{x \in Q} \varphi_i(x) = 0 \tag{2.17}$$

for all $i \neq 0, N$, and similarly $\sum_{x \in P} \varphi_i(x) = 0$. We also find

$$\sum_{x \in P} \varphi_0(x) = \sum_{x \in P} \varphi_N(x) \quad \text{and} \quad \sum_{x \in P} \varphi_0(x) = -\sum_{x \in Q} \varphi_N(x). \tag{2.18}$$

We will now show that the average of a function over arcs of increasing even radius approaches the average over P, and the average over arcs of odd radius approaches the average over Q.

Proposition 2.6 *Let G be a $(q+1)$-regular bipartite graph as above. Then for an arc $A_r(a)$ on $\widetilde{G}$ based at v_0 with even r*

$$\Big| M_{r,a}(f) - \frac{1}{|P|} \sum_{v \in P} f(v) \Big| \leq C_G ||f||_2 \beta_{max}^r$$

and with odd r

$$\Big| M_{r,a}(f) - \frac{1}{|Q|} \sum_{v \in Q} f(v) \Big| \leq C_G ||f||_2 \beta_{max}^r.$$

Proof The result clearly holds for $\varphi_0(x)$. Equation (2.12) shows that $\varphi_N(x)$ is equal to a constant K on arcs of even radius and equal to $-K$ on arcs of odd radius, and equation (2.18) guarantees that in either case the constant is equal to the required average. The method of proof from the non-bipartite case and equation (2.17) above imply the result for $\varphi_i(x)$ with $i \neq 0, N$. Writing a general function f in terms of $\{\varphi_i(x)\}$ as before then gives the result. $\square$

2.6 Applications

We finish this chapter by giving applications of theorem 2.4 to different radial averages of functions on regular graphs.

2.6.1 Spheres and Tubes

Corollary 2.7 *Let G, f, C_G and β_{max} be as in theorem 2.4. Then for a sphere $S_r(v_0)$ on $\widetilde{G}$ we have*

$$\Big| \frac{1}{|S_r(v_0)|} \sum_{v \in S_r(v_0)} \tilde{f}(v) - \frac{1}{|V|} \sum_{v \in V} f(v) \Big| \leq C_G ||f||_2 \beta_{max}^r.$$

It is easy to see that a sphere of radius $r > 0$ is the disjoint union of $q+1$ arcs of the same radius, all with $w' = v_0$. Hence the result follows from theorem 2.4.

Definition 2.8 *Let Z be a subset of the vertices and edges of $\widetilde{G}$ so that Z is a finite connected graph. We then define the* tube $\mathcal{T}_r(Z)$ *of radius r around Z in $\widetilde{G}$ by*

$$\mathcal{T}_r(Z) = \{v \in \widetilde{V} : \min_{x \in V(Z)} d(v,x) = r\}. \tag{2.19}$$

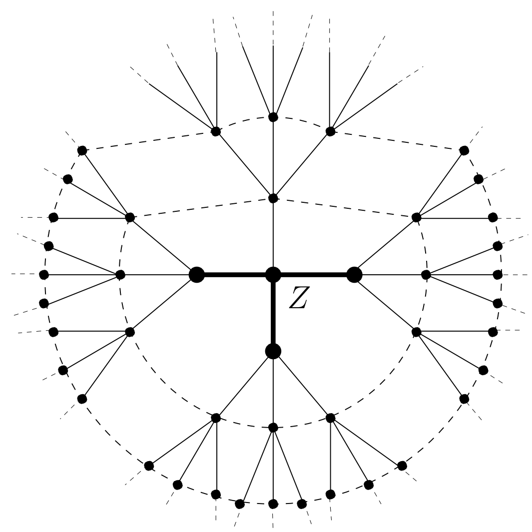

Figure 2.2: Tubes on the regular tree $\widetilde{G}$ of degree 4.

See figure 2.2 for an example of tubes. We can construct the tube from arcs using the edges in $\widetilde{G}$ that connect a vertex in Z to a vertex not in Z. These edges make up the *boundary* ∂Z of Z. Give each of these edges a direction away from Z, so that their origin is in Z and their terminus is not. The collection of arcs given by $\{A_r(a_i) : a_i \in \partial Z \text{ directed away from } Z\}$ is then the same as the tube $\mathcal{T}_r(Z)$. From this fact, the following result immediately follows.

Corollary 2.9 *Let G, f, C_G and β_{max} be as in theorem 2.4. Then for tubes on $\widetilde{G}$ we have*

$$\left| \frac{1}{|\mathcal{T}_r(Z)|} \sum_{v \in \mathcal{T}_r(Z)} \tilde{f}(v) - \frac{1}{|V|} \sum_{v \in V} f(v) \right| \leq C ||f||_2 \beta_{max}^r.$$

2.6.2 Horocycles

Next, we consider increasing subsets of horocycles on $\widetilde{G}$ to find a discrete analogue of a result by Furstenberg [25] on the "unique ergodicity of the horocycle flow" (see also [7, chapter IV]). Horocycles are sometimes also called horospheres, and have the following intuitive definition. One way of defining increasing circles on a plane is by taking a fixed centre and increasing the radius. Alternatively, we can define increasing circles by fixing a point on the circle, and letting the centre move along a geodesic on the surface in such a way that the distance between the fixed point and the centre increases. This is illustrated in figure 2.3 below. The limit of these circles as the radius goes to infinity is then defined to be the horocycle defined by the geodesic and the fixed point, which in figure 2.3 are the dashed line and the black dot.

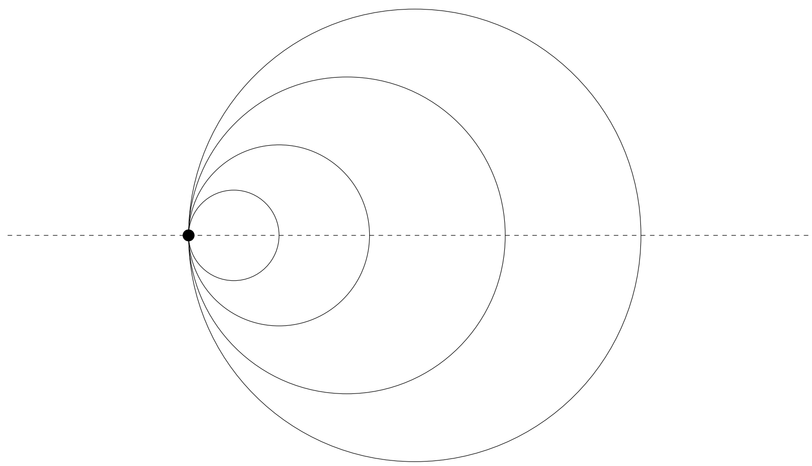

Figure 2.3: Horocycles on the Euclidean plane.

Horocycles on trees were first introduced by Cartier in [12]. As in the case of surfaces, we use a geometrically motivated definition of horocycles, in fact we define them as level sets of Busemann functions, which we will explain below.

We defined a geodesic γ on the tree $\widetilde{G}$ as a bi-infinite non-backtracking path, denoted by its vertices $\ldots, v_{-1}, v_0, v_1, \ldots \in \widetilde{V}$, where v_i is adjacent to v_{i+1} and $v_i \neq v_{i+2}\ \forall\ i \in \mathbb{Z}$. Recall $d(v, w)$ is the combinatorial distance between vertices v and w, and define the Busemann function

$$b_{\gamma,v_k}(w) = \lim_{n\to\infty} d(w, v_{k+n}) - n.$$

For $k \in \mathbb{Z}$ we then define the horocycle $H_k = b^{-1}_{\gamma,v_0}(k)$ (see also figure 2.4). For explanation and another illustration of horocycles, see also [24, Chapter I, Section 9].

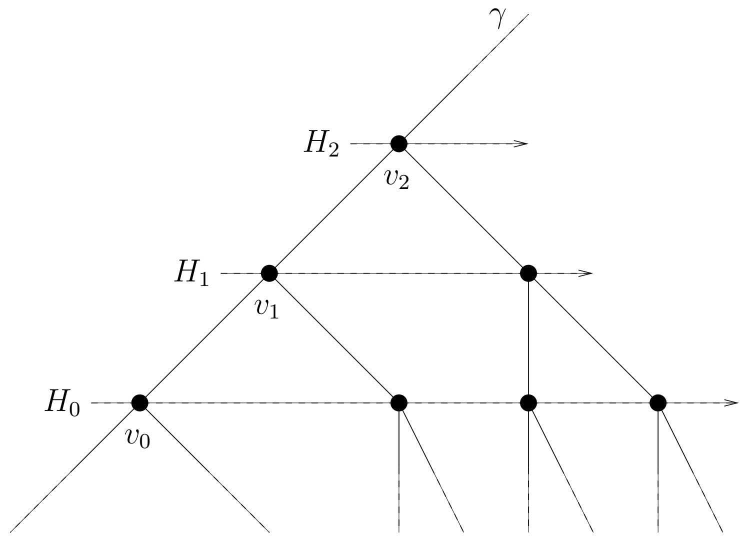

Figure 2.4: Horocycles on the regular tree of degree 4.

Rather than full horocycles, we will consider subsets of the horocycle H_0 defined by

$$\mathcal{H}_{\gamma,r}(v_0) = H_0 \cap S_r(v_r)$$

for which we obtain the following theorem.

Theorem 2.10 *Let G, f, C_G and β_{max} be as in theorem 2.4. Then*

$$\left|\frac{1}{|\mathcal{H}_{\gamma,r}(v_0)|}\sum_{v\in\mathcal{H}_{\gamma,r}(v_0)}\tilde{f}(v) - \frac{1}{|V|}\sum_{v\in V} f(v)\right| \leq C_G||f||_2\beta^r_{max}.$$

Proof Note that we can view the subset of the horocycle as an arc

$$\mathcal{H}_{\gamma,r}(v_0) = A_{r+1}\left(\overrightarrow{\{v_{r+1}, v_r\}}\right)$$

where v_i are vertices on the geodesic defining H_k, and $\overrightarrow{\{v_{r+1}, v_r\}}$ is the directed edge from v_{r+1} to v_r. As $r \to \infty$ we have a set of increasing circular arcs, where the origin of the arc changes at each step. But the convergence for arcs in theorem 2.4 is independent of the origin of the arc, so the subsets can be viewed just as increasing circular arcs, and the theorem follows. □

2.6.3 Balls and Sectors

We finish this chapter by considering averages of functions over two more types of increasing subsets of $\widetilde{G}$, namely *balls* and *sectors*. These are defined as follows.

Definition 2.11 *A closed* ball $B_R(x_0)$ *of radius* R *based at* $x_0 \in V(X)$ *is defined as*

$$B_R(x_0) = \bigcup_{r=0}^{R} S_r(x_0). \tag{2.20}$$

Definition 2.12 *A* sector $K_R(a)$ *in the direction of* $a = \overrightarrow{\{w', w\}}$ *of radius* R *is defined as*

$$K_R(a) = \bigcup_{r=0}^{R} A_r(a). \tag{2.21}$$

Theorem 2.13 *Let* G, f, C_G *and* $\beta = \beta_{max}$ *be as in theorem 2.4. Define*

$$C'_G = 2\frac{q\beta - \beta}{q\beta - 1}C_G.$$

Then for a sector $K_R(a)$ *of radius* $R \geq 2$ *on* $\widetilde{G}$ *we have*

$$\Big|\frac{1}{|K_R(a)|}\sum_{x\in K_R(a)} \tilde{f}(x) - \frac{1}{|V|}\sum_{v\in V} f(v)\Big| \leq C'_G ||f||_2 \beta^R.$$

Proof We proved in theorem 2.4 that

$$\Big|\frac{1}{|A_r(a)|}\sum_{x\in A_r(a)} \tilde{f}(x) - \frac{1}{|V|}\sum_{v\in V} f(v)\Big| \leq C_G ||f||_2 \beta^r. \tag{2.22}$$

We rearrange this equation and write it as

$$\sum_{x\in A_r(a)} \tilde{f}(x) = \frac{|A_r(a)|}{|V|}\sum_{v\in V} f(v) + |A_r(a)|\varepsilon_r C_G ||f||_2 \beta^r \tag{2.23}$$

for some $|\varepsilon_r| \leq 1$. Using equation (2.21) and the fact that $\sum_{r=0}^{R} |A_r(a)| = |K_R(a)|$, we can take the sum of both sides of this equation from $r = 0$ to R to obtain

$$\sum_{x\in K_R(a)} \tilde{f}(x) = \frac{|K_R(a)|}{|V|}\sum_{v\in V} f(v) + \sum_{r=0}^{R} |A_r(a)|\varepsilon_r C_G ||f||_2 \beta^r \tag{2.24}$$

which implies

$$\frac{1}{|K_R(a)|}\sum_{x\in K_R(a)} \tilde{f}(x) = \frac{1}{|V|}\sum_{v\in V} f(v) + C_G ||f||_2 \frac{\sum_{r=0}^{R} |A_r(a)|\varepsilon_r \beta^r}{\sum_{r=0}^{R} |A_r(a)|}. \tag{2.25}$$

To finish the proof, we just need to show that

$$\frac{\sum_{r=0}^{R} |A_r(a)|\varepsilon_r \beta^r}{\sum_{r=0}^{R} |A_r(a)|} \leq 2\frac{q\beta - \beta}{q\beta - 1}\beta^R \tag{2.26}$$

which goes to zero as $R \to \infty$ since $|\beta| < 1$. Let E_R denote the left hand side of equation (2.26). Note $|A_r(a)| = q^{r-1}$ for $r \geq 1$, and $|A_0(a)| = 1$. Using the fact that $|\varepsilon_r| \leq 1$, we now obtain

$$|E_R| \leq \frac{1 + \sum_{r=1}^{R} q^{r-1}\beta^r}{1 + \sum_{r=1}^{R} q^{r-1}} = \frac{\frac{q-1}{q} + \sum_{r=0}^{R} q^{r-1}\beta^r}{\frac{q-1}{q} + \sum_{r=0}^{R} q^{r-1}} \tag{2.27}$$

$$\leq \frac{q - 1 + \sum_{r=0}^{R} q^r\beta^r}{\sum_{r=0}^{R} q^r} = \frac{q - 1 + \frac{1-(q\beta)^{R+1}}{1-q\beta}}{\frac{1-q^{R+1}}{1-q}} \tag{2.28}$$

where the last equality is due to the geometric series formula $\sum_{r=0}^{R} x^r = \frac{1-x^{R+1}}{1-x}$ for $|x| \neq 1$. We rearrange to get

$$|E_R| \leq \frac{q-1}{q\beta - 1} \cdot \frac{(q-1)(q\beta - 1) + (q\beta)^{R+1} - 1}{q^{R+1} - 1} \tag{2.29}$$

where for $R \geq 2$ we find

$$|E_R| \leq 2\frac{q-1}{q\beta - 1} \cdot \frac{(q\beta)^{R+1} - 1}{q^{R+1} - 1}. \tag{2.30}$$

To simplify this last expression, consider the following inequalities for three positive numbers a, b, c, where $b > c$. It is easy to show that

$$\frac{a-c}{b-c} > \frac{a}{b} \iff a > b \tag{2.31}$$

$$\frac{a-c}{b-c} \leq \frac{a}{b} \iff a \leq b. \tag{2.32}$$

In our case, let $a = (q\beta)^{R+1}$, $b = q^{R+1}$ and $c = 1$. Clearly $\frac{a}{b} = \beta^{R+1} < 1$ so $a < b$ and $\frac{a-c}{b-c} < \beta^{R+1}$, i.e.

$$|E_R| \leq 2\frac{q-1}{q\beta - 1} \cdot \frac{(q\beta)^{R+1} - 1}{q^{R+1} - 1} < 2\frac{q-1}{q\beta - 1}\beta^{R+1} = 2\frac{q\beta - \beta}{q\beta - 1}\beta^R. \tag{2.33}$$

Since $|\beta| < 1$ this goes to zero as R goes to infinity, and also gives us the required error estimate for finite R. □

Corollary 2.14 *Let G, f, C_G and $\beta = \beta_{max}$ be as in theorem 2.4. Then for a ball $B_R(x_0)$ of radius $R \geq 2$ on $\widetilde{G}$ we have*

$$\left| \frac{1}{|B_R(x_0)|} \sum_{x \in B_R(x_0)} \tilde{f}(x) - \frac{1}{|V|} \sum_{v \in V} f(v) \right| \leq \frac{q\beta - \beta}{q\beta - 1} C_G ||f||_2 \beta^R.$$

Proof Following the method of proof in theorem 2.13 above we get

$$\frac{1}{|B_R(x_0)|}\sum_{x\in B_R(x_0)}\tilde{f}(x)=\frac{1}{|V|}\sum_{v\in V}f(v)+C_G||f||_2\frac{\sum_{r=0}^{R}|S_r(x_0)|\varepsilon_r\beta^r}{\sum_{r=0}^{R}|S_r(x_0)|}. \tag{2.34}$$

We just need to show that

$$E_R=\frac{\sum_{r=0}^{R}|S_r(x_0)|\varepsilon_r\beta^r}{\sum_{r=0}^{R}|S_r(x_0)|} \tag{2.35}$$

with $|\varepsilon_r|\leq 1$ satisfies $|E_R|\leq\frac{q\beta-\beta}{q\beta-1}\beta^R$. Note that $|S_0(x_0)|=1$ and $|S_r(x_0)|=\frac{q+1}{q}q^r$ for $r\geq 1$ to obtain

$$|E_R|\leq\frac{\frac{-1}{q+1}+\sum_{r=0}^{R}(q\beta)^r}{\frac{-1}{q+1}+\sum_{r=0}^{R}q^r}<\frac{\sum_{r=0}^{R}(q\beta)^r}{\sum_{r=0}^{R}q^r} \tag{2.36}$$

using the inequalities for $a,b,c>0$ in equations (2.31) and (2.32), where $a=\sum_{r=0}^{R}(q\beta)^r$, $b=\sum_{r=0}^{R}q^r$ and $c=\frac{1}{q+1}$. Now use the geometric series used in the proof of theorem 2.13 to obtain

$$|E_R|\leq\frac{\frac{(q\beta)^{R+1}-1}{q\beta-1}}{\frac{q^{R+1}-1}{q-1}}=\frac{q-1}{q\beta-1}\cdot\frac{(q\beta)^{R+1}-1}{q^{R+1}-1}. \tag{2.37}$$

We simplify the second fraction using the inequalities in equations (2.32) and (2.31) again, this time for $a=(q\beta)^{R+1}$, $b=q^{R+1}$ and $c=1$ to obtain

$$|E_R|\leq\frac{q-1}{q\beta-1}\frac{(q\beta)^{R+1}}{q^{R+1}}=\frac{q-1}{q\beta-1}\beta^{R+1}=\frac{q\beta-\beta}{q\beta-1}\beta^R \tag{2.38}$$

as required. □

Chapter 3

Radial Averages for Edges

3.1 Motivation

In chapter 2 we dealt with functions that were defined on the vertices of a graph G. In this chapter we discuss similar results for functions defined on the edges of a simple connected graph G. An *edge function* $f : E \to \mathbb{R}$ on the graph G is defined by assigning a value to each edge of the graph. We are motivated to do this by the fact that all definitions used in the vertex case can easily be modified to deal with edges, due to the existence of a *line graph*.

Definition 3.1 *The vertex set of the* line graph $L(G)$ *of* G *is defined to be the edge set of* G. *Two vertices are adjacent in* $L(G)$ *if the corresponding edges in* G *are adjacent.*

Hence an edge function on G is just a function on the vertices of $L(G)$. We will find, however, that the edge spheres we define in the next section do not coincide with vertex spheres on the line graph due to the following. Recall that an edge e on a simple graph G is uniquely defined by two vertices v, w, its endpoints, and an edge a is adjacent to e if it meets e in either of its endpoints (see also figure 3.1). This means that all edges incident to the vertex v are mutually adjacent, but they are not necessarily adjacent to edges incident to w. In fact, as G is simple, e is the only edge adjacent to both v and w. We obtain two sets E_1, E_2 of mutually adjacent edges with common endpoints v and w respectively, with $E_1 \cap E_2 = \{e\}$.

If we now look at the vertices on $L(G)$ corresponding to E_1 and E_2, we find they

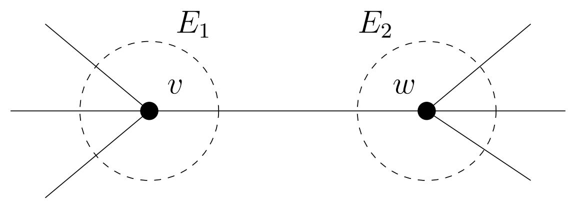

Figure 3.1: Two sets E_1, E_2 of mutually adjacent edges.

are all mutually adjacent. The discrepancy between the spheres is caused by this fact, and it means that the results we obtain in this chapter can not be obtained by applying theorem 2.4 to the line graph of G.

3.2 Definitions for Edge Functions

To work with functions on the edges rather than on the vertices of a graph, we need to redefine some notions introduced for vertices in section 1.1 in terms of edges. When possible, we use the line graph of G to obtain these definitions. As mentioned in section 3.1, we shall only deal with simple graphs in this chapter, i.e. a graph is not allowed to have loops or multiple edges.

Definition 3.2 *The* edge degree $d'(e)$ *of an edge* $e \in E(G)$ *is defined as the number of edges adjacent to it.*

The degree of an edge is clearly just the degree of the corresponding vertex in the line graph. A regular graph of (vertex) degree $q+1$ has constant edge degree $d'(e) = 2q$, and hence is also regular in the sense of edges. There is however another type of graph with constant edge degree, but not constant vertex degree, which we now define.

Definition 3.3 *A* semi-regular *graph is a bipartite graph in which all the vertices of one colour have the same degree, say* $q+1$*, and all the vertices of the other colour have the same degree, say* $p+1$*, which may be different from* $q+1$*.*

Definition 3.4 *The* length *of a path defined in terms of the directed edges* $\overrightarrow{e_0}$, $\overrightarrow{e_1}, \ldots, \overrightarrow{e_n}$ *is equal to* n*, which is the length of the equivalent path in the line graph defined by the vertices corresponding to* $e_0, \ldots, e_n$.

Definition 3.5 *The* edge distance $d'(a, e)$ *between two edges* $a, e \in E(G)$ *is defined as the length of a shortest path connecting them, where we define the length of a path as in definition 3.4.*

This is the same as the distance between the vertices a and e in the line graph $L(G)$ as defined in definition 1.5.

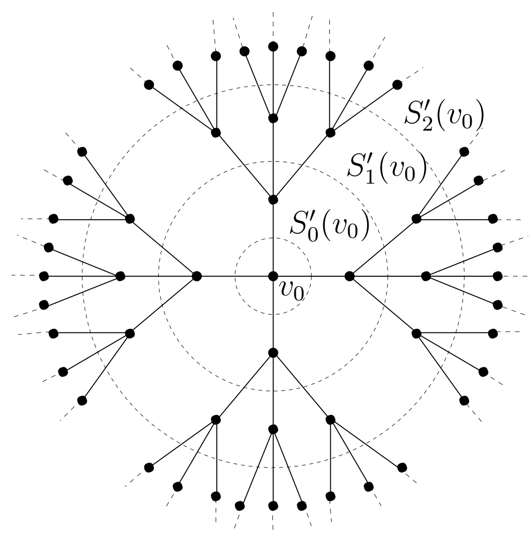

Figure 3.2: Edge spheres on the regular tree of degree 4.

Next we need to define level sets for edges. We work analogously to the vertex case. See also figure 3.2.

Definition 3.6 *An* edge sphere $S_r'(v_0)$ *of radius* r *centred on a vertex* v_0 *in a tree* $\widetilde{G}$ *is defined by*

$$S_r'(v_0) = \big\{ e = \{x, y\} \in E(X) : \min\{d(x, v_0), d(y, v_0)\} = r \big\}. \tag{3.1}$$

We centre the sphere on a vertex rather than an edge as this makes our definition of arcs easier. We will see in section 3.6.1 that a sphere centred on an edge is just a tube around that edge, and the same results will apply, so our choice is irrelevant.

Now we use the definition of edge spheres above to define a spherical edge arc (see also figure 3.3).

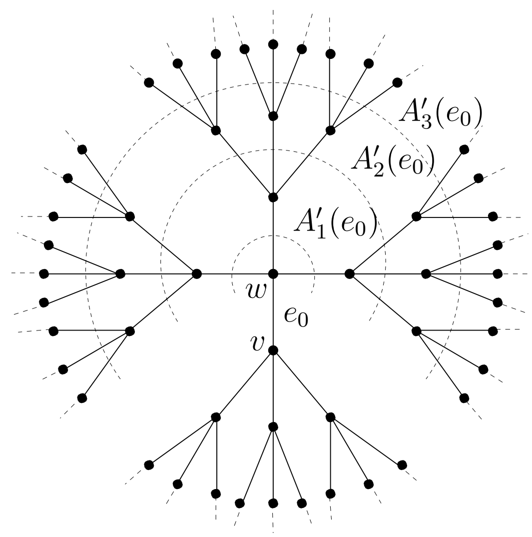

Figure 3.3: Edge arcs on the regular tree of degree 4.

Definition 3.7 *Let* $e_0 = \overrightarrow{\{v, w\}}$ *be a directed edge in* $\widetilde{G}$. *We define the* edge arc *of radius* $r + 1 \geq 1$ *emanating from* e_0 *by*

$$A'_{r+1}(e_0) = S'_{r+1}(v) \cap S'_r(w) \tag{3.2}$$

and we set $A'_0(e_0) = \{e_0\}$.

Again, this is a section of a sphere, and we can make up a sphere centred on the vertex v by taking the union of the arcs $A'_r(\overrightarrow{\{v, w_i\}})$ for all vertices w_i adjacent to v.

Definition 3.8 *The edge adjacency matrix* A'_G *of a graph* G *with* $|E| = m$ *edges is an* $m \times m$ *matrix with rows and columns labelled by the edges of* G. *The* i, j^{th} *entry* $a_{i,j}$ *is one if edges* i *and* j *are adjacent, and zero if they are not.*

Note that an edge cannot be adjacent to itself as this gives a loop, and we have excluded such non-simple graphs from our consideration in this chapter. This means the entries on the diagonal of A'_G are all zero.

Recall that in section 1.2 we defined the universal cover $\widetilde{G}$ of a graph G, and the covering map $\pi : \widetilde{G} \to G$ in terms of vertices of G. Observe that both notions are easily extended to edges, as edges are defined in terms of adjacent vertices. Therefore we can also lift edge functions f to functions $\tilde{f}$ on the edges of the universal cover of G.

The *arc average* of the function $f : E \to \mathbb{R}$ on the edges is now defined as

$$M_{r,a}(f) = \frac{1}{|A'_r(a)|} \sum_{e \in A'_r(a)} \tilde{f}(e).$$

The main purpose of this chapter is to find analogous results to theorem 2.4 in chapter 2 for functions on the edges of regular and semi-regular graphs. Again we apply these results for arcs to find results for spheres and tubes. Horocycles, however, are only defined on the vertices of $\widetilde{G}$, so we cannot find an edge equivalent of theorem 2.10.

3.3 Results for Regular and Semi-regular Graphs

Using the definitions in the previous section, we can now define an analogue of theorem 2.4 for functions on the edges of a regular graph G of degree $q+1$.

Theorem 3.9 *Let G be a finite regular connected simple graph with $d'(e) = 2q \geq 4$ and let $f : E \to \mathbb{R}$ be a function on its edges. Then we have for any directed edge a in G*

$$\left| M_{r,a}(f) - \frac{1}{|E|} \sum_{e \in E} f(e) \right| \leq C_G ||f||_2 \beta_{max}^r.$$

Here C_G is a constant depending on G but independent of a, and $\beta_{max} \in \{\frac{1}{q}\} \cup [q^{-1/2}, 1)$.

The norm here comes from the inner product $\langle f, g \rangle = \sum_{e \in E} f(e)g(e)$, so $||f||_2 = \sqrt{\langle f, f \rangle}$. The precise value of $\beta_{\max}$ is related to the spectrum of the edge Laplacian (see section 3.4) and the Fourier coefficients of f. Again, we see that $M_{r,a}(f)$ converges to the graph average, which is defined for edge functions as $\frac{1}{|E|} \sum_{e \in E} f(e)$.

We reiterate that it is important to note that it is not possible to use theorem 2.4 to prove theorem 3.9 (and 3.10 below) by looking at the corresponding line graph $L(G)$. This is due to the fact that vertex arcs on $\widetilde{L(G)}$ and edge arcs on $\widetilde{G}$ do not coincide, because $\widetilde{L(G)} \neq L(\widetilde{G})$ due to the discrepancy in the adjacency relation described in section 3.1.

Now we state the theorem for *semi-regular* graphs analogous to theorem 2.4. Let G be semi-regular with vertex degrees $p+1$ and $q+1$, and hence constant edge degree $p+q$.

Theorem 3.10 *Let G be a finite connected semi-regular simple graph with edge degree $p+q$, where $p,q \geq 2$, and let $f : E \to \mathbb{R}$ be a function on its edges. Then we have for any directed edge a in G*

$$\left| M_{r,a}(f) - \frac{1}{|E|}\sum_{e\in E} f(e) \right| \leq C_G ||f||_2 \beta_{max}^r.$$

Here C_G is a constant depending on G but independent of a, and $\beta_{max} \in \{(pq)^{-1/2}\} \cup [(pq)^{-1/4}, 1)$.

Note that theorem 3.10 only deals with bipartite graphs, whereas in theorem 3.9 the graph may be either bipartite or not, so it is not a special case of theorem 3.10. We need $p,q \geq 2$ in this theorem, as $p = 1$ can give a non-converging function on the graph. Take for example $K_{2,3}$, call the two vertices of degree three x and y, and define a function $g : E \to \{-1, 1\}$ such that $g(e) = 1$ if x is an endpoint of e, and -1 otherwise (see figure 3.4). Clearly the arc average of g takes the values ± 1 in a recurring pattern and never converges.

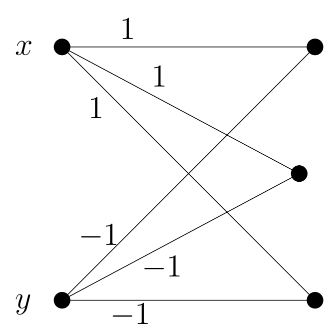

Figure 3.4: The graph $K_{2,3}$ with values annotated for the function g.

3.4 Proof of the Regular Case

Theorem 3.9 concerns functions on the edges of a regular graph G, and the method of proof follows that of the vertex case in section 2.4 apart from a small deviation towards the end. Recall that we no longer allow the graph to have loops or multiple edges, and require $d'(e) = 2q \geq 4\ \forall\ e \in E$. Let $f : E \to \mathbb{R}$ be a function on the edges of a graph G. Then the (edge) Laplacian of f at $e \in E$ is defined as

$$\mathcal{L}'_G f(e) = \frac{1}{d'(e)} \sum_{d'(e,a)=1} f(a).$$

Note that this is equivalent to the vertex Laplacian on the line graph $L(G)$ of G. We find that here the range of eigenvalues of the edge Laplacian is smaller, namely $-1/q \leq \lambda_i \leq 1$, as the eigenvalues of the adjacency matrix of a line graph satisfy $\mu \geq -2$, see [20] or [33].

The recursion relation, which we obtain using the same methods as in the previous chapter, in this case looks as follows

$$F_i(n+1) + \frac{q-1-2\lambda_i q}{q} F_i(n) + \frac{1}{q} F_i(n-1) = 0 \text{ for } n \geq 1. \tag{3.3}$$

Here the initial conditions are $F_i(0) = \varphi_i(a)$, where $a = \overrightarrow{\{v,w\}}$ is the directed edge from which the arc emanates, and

$$qF_i(1) + F_i(0) = \sum_{w \sim v_j} \varphi_i(\{w, v_j\}) \tag{3.4}$$

where the sum runs through all vertices v_j adjacent to w. Using this we want to show that $\lim_{n\to\infty} F_i(n) = 0$ for $-\frac{1}{q} \leq \lambda_i < 1$ with the appropriate convergence rate. For $D_i = (q-1-2\lambda_i q)^2 - 4q \neq 0$ we find again that $F_i(n) = u_i^+(\alpha_i^+)^n + u_i^-(\alpha_i^-)^n$, where this time

$$\alpha_i^\pm = \lambda_i - \frac{q-1}{2q} \pm \frac{1}{2q}\sqrt{D_i}, \tag{3.5}$$

$$u_i^\pm = \pm\frac{q}{\sqrt{D_i}} F_i(1) + \frac{\sqrt{D_i} \pm (q-1-2\lambda_i q)}{2\sqrt{D_i}} F_i(0). \tag{3.6}$$

For $D_i = 0$ we have $\lambda_i = \frac{q-1}{2q} \pm \frac{1}{\sqrt{q}}$ so $F_i(n) = u_i\alpha_i^n + v_i n\alpha_i^n$ where $\alpha_i = \pm\frac{1}{\sqrt{q}}$,

$$u_i = F_i(0), \tag{3.7}$$

$$v_i = -F_i(0) \pm \sqrt{q} F_i(1). \tag{3.8}$$

When $D_i \leq 0$ the proof now follows that of the vertex case, and we obtain $|\alpha_i^+| = |\alpha_i^-| = \beta_i = \frac{1}{\sqrt{q}}$ for the convergence rate.

Note that $D_i > 0$ for $\lambda_i \in [-1/q, m_1) \cup (m_2, 1] = I$, where $m_1 = \frac{q-1}{2q} - q^{-1/2}$ and $m_2 = \frac{q-1}{2q} + q^{-1/2}$. Define two functions

$$\alpha^\pm(\lambda) = \lambda - \frac{q-1}{2q} \pm \frac{1}{2q}\sqrt{D(\lambda)} \tag{3.9}$$

where $D(\lambda) = (q-1-2\lambda q)^2 - 4q$. We calculate that

$$\begin{aligned}\frac{\partial}{\partial\lambda}\alpha^{\pm}(\lambda) =& 1 \pm \frac{1}{2q}\cdot\frac{1}{2}\cdot\left((q-1-2\lambda q)^2-4q\right)^{-1/2}\cdot 2\cdot(q-1-2\lambda q)\cdot(-2q) \\ =& 1 \pm \frac{2\lambda q - q + 1}{\sqrt{(q-1-2\lambda q)^2-4q}} \end{aligned} \tag{3.10}$$

where the fraction in equation 3.10 clearly has absolute value greater than one. This means that the functions $\alpha^{\pm}(\lambda)$ are both monotone on $[-1/q, m_1)$ and $(m_2, 1]$, because $\frac{\partial}{\partial\lambda}\alpha^{\pm}(\lambda)$ does not change sign anywhere. Calculating $|\alpha^{\pm}(\lambda)|$ for boundary values of I gives $|\alpha^{\pm}(\lambda)| < 1 \;\forall\; \lambda \in I$ with two exceptions. These are $\alpha^{+}(1) = 1$, which corresponds to the constant function, where the arc average always equals the graph average, and $|\alpha^{-}(-1/q)| = 1$, which we will investigate below. For $\lambda_i \in (-1/q, m_1)\cup(m_2, 1)$, the convergence rate is again easily calculated from $\frac{1}{\sqrt{q}} < |\alpha_i| < 1$.

To check what happens if $\lambda = \frac{-1}{q}$, we use the following lemma from [20, Theorem 3], which we restate in our notation:

Lemma 3.11 *Let f be any eigenfunction of the (edge) Laplacian with eigenvalue $\frac{-1}{q}$. Then*

$$\sum_{v_0\sim w_j} f(\{v_0, w_j\}) = 0 \tag{3.11}$$

for all $v_0 \in V$.

This means that $F_i(0) + qF_i(1) = 0$ in equation (3.4) when $\lambda_i = \frac{-1}{q}$. Use this and the recursion relation in equation (3.3) to obtain $F_i(n) = (-1/q)^n F_i(0)$ which clearly converges to zero as $n \to \infty$ with $\beta_i = 1/q$. Using the expression of a function f in terms of its Fourier coefficients as before, this completes the proof of the fact that the arc average of functions on the edges of G converges to the graph average. To find the convergence rates for a function with given Fourier coefficients, we work completely analogously to the vertex case in theorem 2.4. Note that the only functions which can have $\beta_{\max} = \frac{1}{q}$ are eigenfunctions of the Laplacian with eigenvalue $-1/q$.

3.5 Proof of the Semi-regular Case

We now prove theorem 3.10. Here we deal with functions on the edges of a simple semi-regular graph with edge degree $p+q$, where we require that $p, q \geq 2$. As for theorem 3.9, we reduce the problem to the radialisation of non-constant eigenfunctions of the edge Laplacian, which now has eigenvalues $\frac{-2}{p+q} \leq \lambda \leq 1$. Recall that the arc is defined as emanating from an edge $a = \overrightarrow{\{v, w\}}$. We assume the vertex v has degree $p+1$ (see also figure 3.5).

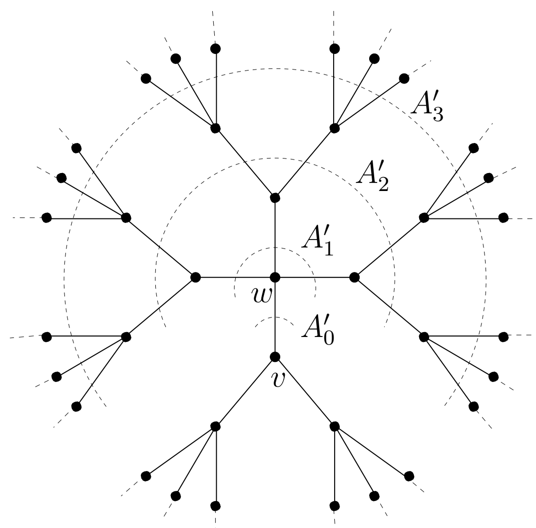

Figure 3.5: Arcs on the semi-regular tree with $p+1=3$ and $q+1=4$.

3.5.1 Recursion

Because G is semi-regular, there is a more complicated recursion formula for the radialised eigenfunction $F_i(n)$ with eigenvalue λ_i on the edges of the universal cover $\widetilde{G}$. Using the Laplacian on $\widetilde{G}$, given by $\mathcal{L}_{\widetilde{G}} f(e) = \frac{1}{p+q} \sum_{d'(e,a)=1} f(a)$, we find for positive integers k

$$\lambda_i F_i(2k) = \frac{1}{p+q}\Big(qF_i(2k+1) + (p-1)F_i(2k) + F_i(2k-1)\Big)$$

and

$$\lambda_i F_i(2k-1) = \frac{1}{p+q}\Big(pF_i(2k) + (q-1)F_i(2k-1) + F_i(2k-2)\Big).$$

Rearranging the expressions and then combining the two equations, we obtain

$$\begin{pmatrix} F_i(2k+1) \\ F_i(2k) \end{pmatrix} = A_i \cdot \begin{pmatrix} F_i(2k-1) \\ F_i(2k-2) \end{pmatrix} \qquad \text{for } k \geq 1,\ k \in \mathbb{N}, \text{ where}$$

$$A_i = \begin{pmatrix} \frac{\left(p-1-\lambda_i(p+q)\right)\left(q-1-\lambda_i(p+q)\right)-p}{pq} & \frac{p-1-\lambda_i(p+q)}{pq} \\ -\frac{q-1-\lambda_i(p+q)}{p} & -\frac{1}{p} \end{pmatrix}.$$

Hence $\binom{F_i(2k+1)}{F_i(2k)} = A_i^k \cdot \binom{F_i(1)}{F_i(0)}$. The convergence properties of the arc average are now determined by the eigenvalues of the matrix A_i. Define

$$t_\pm(\lambda) = \frac{\left(p-1-\lambda(p+q)\right)\left(q-1-\lambda(p+q)\right)-p-q \pm \sqrt{D(\lambda)}}{2pq}$$

$$\text{where} \quad D(\lambda) = \Big(\left(p-1-\lambda(p+q)\right)\left(q-1-\lambda(p+q)\right)-p-q\Big)^2 - 4pq.$$

Then the eigenvalues of A_i are $t_\pm(\lambda_i)$ for all λ_i that occur. Convergence of the arc average can only fail if we have λ_i such that $|t_+(\lambda_i)| \geq 1$ or $|t_-(\lambda_i)| \geq 1$ (by formulas (3.13) and (3.14) below). Therefore we investigate $|t_\pm(\lambda_i)|$ for all possible λ_i.

3.5.2 Convergence

We distinguish the cases $D(\lambda) \leq 0$ and $D(\lambda) > 0$. For $D(\lambda) \leq 0$ we have $|t_\pm(\lambda_i)| = \frac{1}{\sqrt{pq}}$ which means the arc average converges for the corresponding eigenfunctions. We look at the various regions of λ for which $D(\lambda) > 0$ separately.

Note $D(\lambda) = 0$ for

$$\lambda = m_{\pm\pm} = \frac{p+q-2 \pm \sqrt{(p-q)^2 + 4(\sqrt{p} \pm \sqrt{q})^2}}{2(p+q)}. \tag{3.12}$$

Deduce that $D(\lambda) > 0$ in the intervals $I_1 = [\frac{-2}{p+q}, m_{-+})$, $I_2 = (m_{--}, m_{+-})$ and $I_3 = (m_{++}, 1]$, where the subscripts $+$ and $-$ refer to the choices of $\pm$ in $m_{\pm\pm}$ in order of appearance. Solving $\frac{\partial}{\partial\lambda} t_\pm(\lambda) = 0$ gives $\lambda = m' = \frac{p+q-2}{2(p+q)}$ as the only solution for $\lambda \in I_1 \cup I_2 \cup I_3$. Therefore $t_\pm(\lambda)$ is monotone on I_1 and I_3, so $t_\pm(\lambda)$ has possible maxima and minima only at the endpoints of the intervals. On I_2, maxima and minima can only occur at m' and at the endpoints.

There exist values of p and q so that $|t_\pm(m')| > 1$, so in theory we could have $|t_\pm(\lambda_i)| > 1$, and we need to investigate whether such λ_i can occur. There is a

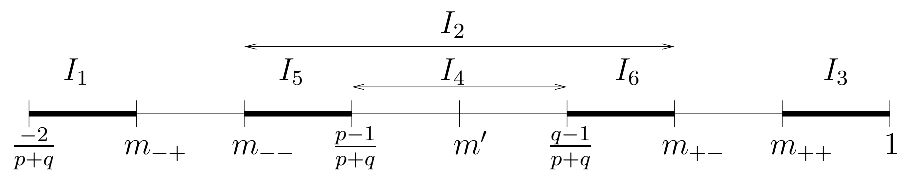

Figure 3.6: Values and intervals of λ which are used in the proof.

useful lemma (lemma 3.12) which we shall prove in section 3.5.4, which states that for $p < q$ the edge Laplacian has no eigenvalues λ in the interval $I_4 = (\frac{p-1}{p+q}, \frac{q-1}{p+q})$ (if $p > q$ just switch the roles of p and q here). This means that we do not need $|t_\pm(\lambda)| < 1$ for all $\lambda \in I_2$, just for $I_2 - I_4 = I_5 \cup I_6$ where $I_5 = (m_{--}, \frac{p-1}{p+q}]$ and $I_6 = [\frac{q-1}{p+q}, m_{+-})$. As $m' \in I_4$, the functions $t_\pm(\lambda)$ are monotone on I_5 and I_6, so maxima and minima can only occur at the endpoints of the intervals. Hence to check $|t_\pm(\lambda)|$ for $D(\lambda) > 0$ for all values λ_i which may occur, we now just have to check $t_\pm(\lambda)$ at the boundary values of each of the intervals I_1, I_5, I_6 and I_3. See also figure 3.6. Note that if $p = q$, we have $\frac{p-1}{p+q} = \frac{q-1}{p+q}$ so $I_4 = \emptyset$. Using $D(m_{\pm\pm}) = 0$ we find

$$\begin{aligned}
\Big|t_\pm(m_{\pm\pm})\Big| &= \frac{1}{\sqrt{pq}} < 1,\\
\Big|t_+\Big(\frac{p-1}{p+q}\Big)\Big| = \Big|t_+\Big(\frac{q-1}{p+q}\Big)\Big| &= \frac{1}{p} < 1,\\
\Big|t_-\Big(\frac{p-1}{p+q}\Big)\Big| = \Big|t_-\Big(\frac{q-1}{p+q}\Big)\Big| &= \frac{1}{q} < 1,\\
\Big|t_-(1)\Big| = \Big|t_-\Big(\frac{-2}{p+q}\Big)\Big| &= \frac{1}{pq} < 1,
\end{aligned}$$

so $|t_-(\lambda_i)| < 1$ for all eigenvalues λ_i that occur. Finally, $t_+(1) = t_+(\frac{-2}{p+q}) = 1$, and $|t_+(\lambda_i)| < 1$ for all λ_i except these two values. The eigenvalue $\lambda_i = 1$ corresponds to the constant eigenfunction which, as before, is equal to the radial average. When $\lambda_i = \frac{-2}{p+q}$, we use theorem 3 in [20] again to find $F_i(1) = -\frac{1}{q}F_i(0)$, $F_i(2) = -\frac{1}{p}F_i(1)$, $F_i(n+1) = -\frac{1}{q}F_i(n)$ for $n > 0$ even, and $F_i(n+1) = -\frac{1}{p}F_i(n)$ for $n > 1$ odd. This means $|F_i(n)| \leq C_i(\frac{1}{\sqrt{pq}})^n$ for some constant C_i, and $F_i(n)$ converges to zero as $n \to \infty$ as required. Hence we have shown that the arc average of a function on the edges of a semi-regular graph converges to the graph average. In the next section we will investigate the convergence rate.

3.5.3 Convergence rate

First we assume that $D(\lambda) \neq 0$, and let $\underline{u}_i^+$, $\underline{u}_i^-$ be a basis of unit eigenvectors of A_i corresponding to the eigenvalues $t_+(\lambda_i)$, $t_-(\lambda_i)$ respectively. Writing the initial vector $\binom{F_i(1)}{F_i(0)} = a_i^+ \underline{u}_i^+ + a_i^- \underline{u}_i^-$ we find

$$\begin{pmatrix} F_i(2k+1) \\ F_i(2k) \end{pmatrix} = A_i^k \begin{pmatrix} F_i(1) \\ F_i(0) \end{pmatrix} = a_i^+ \big(t_+(\lambda_i)\big)^k \underline{u}_i^+ + a_i^- \big(t_-(\lambda_i)\big)^k \underline{u}_i^- \quad \text{for } k \in \mathbb{N}. \tag{3.13}$$

For $D(\lambda) < 0$ we now use the fact that $|t_+(\lambda_i)| = |t_-(\lambda_i)| = \frac{1}{\sqrt{pq}}$ to find

$$|F_i(2k+j)| \leq B_j \big(\frac{1}{\sqrt{pq}}\big)^k \qquad \text{for } j = 0, 1$$

with suitable constants B_0, B_1 both depending only on $F_i(0)$ and $F_i(1)$, hence

$$|F_i(n)| \leq C_i (pq)^{-\frac{n}{4}}$$

for some $C_i > 0$ depending on $F_i(0)$ and $F_i(1)$.

When $D(\lambda) > 0$ the convergence will depend on the eigenvalue of A_i with largest absolute value. Letting $\beta_i = \max\{|t_+(\lambda_i)|, |t_-(\lambda_i)|\}$ and using the same methods as before we find

$$|F_i(n)| \leq C_i \beta_i^{\frac{n}{2}}$$

for some $C_i > 0$ depending on $F_i(0)$ and $F_i(1)$, and $\frac{1}{\sqrt{pq}} < \beta_i < 1$.

Now in the case that $D(\lambda) = 0$, A_i has an eigenvalue $t = \frac{1}{\sqrt{pq}}$ (or $\frac{-1}{\sqrt{pq}}$) of algebraic multiplicity two. Note that in this case we can find a Jordan base of unit length vectors $\underline{u}_i$, $\underline{v}_i$ such that $A_i u_i = t u_i$ and $a_i v_i = u_i + t v_i$. Then we can find constants a_i, b_i such that $\binom{F_i(1)}{F_i(0)} = a_i \underline{u}_i + b_i \underline{v}_i$, and derive

$$\begin{pmatrix} F_i(2k+1) \\ F_i(2k) \end{pmatrix} = A_i^k \begin{pmatrix} F_i(1) \\ F_i(0) \end{pmatrix} = (a_i t^k + b_i k t^{k-1}) \underline{u}_i + b_i t^k \underline{v}_i. \tag{3.14}$$

We obtain

$$|F(2k+j)| \leq B_j'(k+1)|t|^k \qquad \text{for } j = 0, 1$$

with constants B_0', B_1' depending only on $F_i(0)$ and $F_i(1)$. This implies that

$$|F_i(n)| \leq C_i' \cdot (1+n) \cdot (pq)^{-\frac{n}{4}} \leq C_i \beta_i^{\frac{n}{2}}$$

for $C_i' > 0$ depending on $F_i(0)$ and $F_i(1)$, $\beta_i = (pq)^{-\frac{1}{2}+\varepsilon}$ for arbitrarily chosen $\varepsilon > 0$, and appropriately adjusted C_i.

As with the previous two theorems, we write $f = \sum_{i=0}^{|V|-1} c_i\varphi_i$ and use the largest value of β_i to find

$$\left| M_{r,a}(f) - \frac{1}{|E|}\sum_{e\in E} f(e)\right| \leq C_G||f||_2\beta_{\max}^r$$

where as before $C_G > 0$ large enough to provide independence of the directed edge a in G.

3.5.4 Proof of the Lemma

To complete the proof of theorem 3.10, it remains to prove the following lemma.

Lemma 3.12 *Let G be a semi-regular graph as in theorem 3.10, and $p < q$. Then the edge Laplacian has no eigenvalues λ such that*

$$\frac{p-1}{p+q} < \lambda < \frac{q-1}{p+q}.$$

PROOF Let G be a semi-regular graph with n_1 vertices of degree $p+1$ and n_2 vertices of degree $q+1$, where $n_1 \geq n_2$ and all vertices of the same degree are mutually non-adjacent. Then [17, Theorem 1.3.18] gives the following relation between the characteristic polynomials $P_G(x)$ and $P_{L(G)}(x)$ of G and its line graph $L(G)$ respectively:

$$P_{L(G)}(x) = (x+2)^m\sqrt{\left(\frac{-\alpha_1(x)}{\alpha_2(x)}\right)^{n_1-n_2} P_G\left(\sqrt{\alpha_1(x)\alpha_2(x)}\right)P_G\left(-\sqrt{\alpha_1(x)\alpha_2(x)}\right)}$$

where $m = |E| - |V|$, $\alpha_1 = x - p + 1$ and $\alpha_2 = x - q + 1$. Recall $P_{L(G)}(\mu) = 0$ for eigenvalues μ of the edge adjacency matrix $A_{L(G)}$, and as $\mathcal{L}'_{L(G)} = \frac{1}{p+q}A_{L(G)}$ we have

$$\lambda = \frac{\mu}{p+q}$$

so $\mu \in [-2, p+q]$ by [20]. Using the above formula for $P_{L(G)}$, we find its roots can only be $\mu = -2$, $\mu = p-1$, or μ such that $\pm\sqrt{\alpha_1(\mu)\alpha_2(\mu)}$ is an eigenvalue of the original graph. Note that G has only real eigenvalues. However since $\sqrt{\alpha_1(\mu)\alpha_2(\mu)}$ is *purely imaginary* for $p-1 < \mu < q-1$, $L(G)$ cannot have eigenvalues in this region. □

3.6 Applications

To finish this chapter, we give applications of theorems 3.9 and 3.10 to different radial averages of functions on regular and semi-regular graphs.

3.6.1 Spheres and Tubes

Corollary 3.13 *Let G, f, C_G and β_{max} be as in theorem 3.9. Then for an edge sphere $S'_r(v_0)$ on $\widetilde{G}$ we have*

$$\Big|\frac{1}{|S'_r(v_0)|}\sum_{e\in S'_r(v_0)}\tilde{f}(e) - \frac{1}{|E|}\sum_{e\in E} f(e)\Big| \leq C_G||f||_2\beta^r_{max}.$$

As noted in section 3.2, it is easy to see that an edge sphere is made up of $q+1$ edge arcs emanating from its centre v. Hence theorem 3.9 immediately implies this result. In the same way, we obtain the following theorem for semi-regular graphs.

Corollary 3.14 *Let G, f, C_G and β_{max} be as in theorem 3.10. Then for an edge sphere $S'_r(v_0)$ on $\widetilde{G}$ we have*

$$\Big|\frac{1}{|S'_r(v_0)|}\sum_{e\in S'_r(v_0)}\tilde{f}(e) - \frac{1}{|E|}\sum_{e\in E} f(e)\Big| \leq C_G||f||_2\beta^r_{max}.$$

Recall that in section 2.6 we defined (vertex) tubes around a connected graph in $\widetilde{G}$. We now do the same for edges.

Definition 3.15 *Let Z be a subset of the vertices and edges of $\widetilde{G}$ so that Z is a connected graph. We then define the* edge tube *of radius r around Z in $\widetilde{G}$ by*

$$\mathcal{T}'_r(Z) = \{e \in \widetilde{E} : \min_{a\in E(Z)} d'(e,a) = r\}. \tag{3.15}$$

We note, as in the vertex case, that edge tubes can be made up of several edge arcs, and we use this fact to deduce the following corollaries from theorems 3.9 and 3.10.

Corollary 3.16 *Let G, f, C_G and β_{max} be as in theorem 3.9. Then for edge tubes*

$$\Big|\frac{1}{|\mathcal{T}'_r(X)|}\sum_{e\in \mathcal{T}'_r(X)}\tilde{f}(e) - \frac{1}{|E|}\sum_{e\in E} f(e)\Big| \leq C_G||f||_2\beta^r_{max}.$$

Corollary 3.17 *Let G, f, C_G and β_{max} be as in theorem 3.10. Then for edge tubes*

$$\left|\frac{1}{|\mathcal{T}_r'(X)|}\sum_{e\in\mathcal{T}_r'(X)}\tilde{f}(e)-\frac{1}{|E|}\sum_{e\in E}f(e)\right|\leq C_G||f||_2\beta_{max}^r.$$

3.6.2 Balls and Sectors

We finish this chapter by giving, as in the vertex case, an application of theorems 3.9 and 3.10 to edge balls and sectors.

Definition 3.18 *A closed* edge ball $B_R'(v)$ *of radius R based at $v \in V(\widetilde{G})$ is defined as*

$$B_R'(v)=\bigcup_{r=0}^{R}S_r'(v). \tag{3.16}$$

Definition 3.19 *An* edge sector $K_R'(e)$ *in the direction of $e=\overrightarrow{\{v,w\}}$ of radius R is defined as*

$$K_R'(e)=\bigcup_{r=0}^{R}A_r'(e). \tag{3.17}$$

Again, we can construct the ball by taking the union of the appropriate sectors. Denote $a_i=\overrightarrow{\{v,w_i\}}$ for every w_i adjacent to v, numbered from 1 to $d(v)$. Then

$$B_R'(v)=\bigcup_{i=1}^{d(v)}K_R'(a_i). \tag{3.18}$$

Note that this is a disjoint union, so we will be able to deduce results for balls directly from results for sectors.

Theorem 3.20 *Let G, f, C_G and β_{max} be as in theorem 3.9. Define*

$$C_G'=\frac{q\beta-\beta}{q\beta-1}C_G. \tag{3.19}$$

Then for an edge sector $K_R'(a)$ of radius R on $\widetilde{G}$ we have

$$\left|\frac{1}{|K_R'(a)|}\sum_{e\in K_R'(a)}\tilde{f}(e)-\frac{1}{|E|}\sum_{e\in E}f(e)\right|\leq C_G'||f||_2\beta^R. \tag{3.20}$$

Proof We proceed in a similar way to the proof of the vertex case in 2.13. In theorem 3.9 we proved that

$$\left|\frac{1}{|A_r'(a)|}\sum_{e\in A_r'(a)}\tilde{f}(e)-\frac{1}{|E|}\sum_{e\in E}f(e)\right|\leq C_G||f||_2\beta^r \tag{3.21}$$

which we can rearrange to

$$\sum_{e\in A'_r(a)} \tilde{f}(e) \le \frac{|A'_r(a)|}{|E|}\sum_{e\in E} f(e) + |A'_r(a)|\varepsilon_r C_G ||f||_2 \beta^r \tag{3.22}$$

for some $|\varepsilon_r| \le 1$. Using equation (3.17) and $\sum_{r=0}^{R} |A'_r(a)| = |K'_R(a)|$ we take the sum of both sides from $r = 0$ to R and rearrange to obtain

$$\frac{1}{|K'_R(a)|}\sum_{e\in K'_R(a)} \tilde{f}(e) \le \frac{1}{|E|}\sum_{e\in E} f(e) + C_G||f||_2 \frac{\sum_{r=0}^{R} |A'_r(a)|\varepsilon_r\beta^r}{\sum_{r=0}^{R}|A'_r(a)|}. \tag{3.23}$$

We are left to show that

$$E_R = \frac{\sum_{r=0}^{R} |A'_r(a)|\varepsilon_r\beta^r}{\sum_{r=0}^{R}|A'_r(a)|} \tag{3.24}$$

goes to zero as $R \to \infty$. Use $|A'_r(a)| = q^r$, $|\varepsilon_r| \le 1$ and the geometric series formula $\sum_{r=0}^{R} x^r = \frac{1-x^{R+1}}{1-x}$ to obtain

$$|E_R| \le \frac{\sum_{r=0}^{R} q^r\beta^r}{\sum_{r=0}^{R} q^r} = \frac{\frac{1-(q\beta)^{R+1}}{1-q\beta}}{\frac{1-q^{R+1}}{1-q}} = \frac{1-q}{1-q\beta}\cdot\frac{1-(q\beta)^{R+1}}{1-q^{R+1}}. \tag{3.25}$$

Now use the same trick from equations 2.31 and 2.32 to obtain

$$|E_R| < \frac{1-q}{1-q\beta}\beta^{R+1} = \frac{q\beta-\beta}{q\beta-1}\beta^R \tag{3.26}$$

which goes to zero as R goes to infinity, and gives us the required convergence rate.

□

Again, we use the fact that a ball is made up of several sectors to obtain:

Corollary 3.21 *Let G, f, C_G and β_{max} be as in theorem 3.9. Define*

$$C'_G = \frac{q\beta-\beta}{q\beta-1}C_G \tag{3.27}$$

as in theorem 3.20. Then for an edge ball $B'_R(v)$ of radius R on $\widetilde{G}$ we have

$$\left|\frac{1}{|B'_R(v)|}\sum_{x\in B'_R(v)} \tilde{f}(e) - \frac{1}{|E|}\sum_{e\in E} f(e)\right| \le C'_G||f||_2\beta^R. \tag{3.28}$$

We do the same for semi-regular graphs.

Theorem 3.22 *Let G, f, C_G and β_{max} be as in theorem 3.10. Define*

$$C'_G = \frac{pq\beta^2-\beta^2}{pq\beta^2-1}\left(p\sqrt{pq}+pq\right)C_G. \tag{3.29}$$

Then for an edge sector $K'_R(e)$ of radius $R \geq 1$ on $\widetilde{G}$ we have

$$\Big|\frac{1}{|K'_R(a)|}\sum_{x\in K'_R(a)}\tilde{f}(e)-\frac{1}{|E|}\sum_{e\in E}f(e)\Big|\leq C'_G||f||_2\beta^R. \tag{3.30}$$

Proof The proof is identical to that of theorem 3.20 up to the calculation of E_R. In this case we have $|A'_r(a)| = (pq)^{r/2}$ for even r and $|A'_r(a)| = q(pq)^{r/2}$ for odd r. This means that

$$\sum_{r=0}^{R}|A'_r(a)| = \sum_{r=0}^{[R/2]}(pq)^r + \sum_{r=0}^{[(R-1)/2]} q(pq)^r \tag{3.31}$$

where $[x]$ is the Gauss bracket, which means that we take the integer part of this rational number, so for example $[2] = 2$ and $[5/2] = 2$. Using this we now have

$$|E_R| \leq \frac{\sum_{r=0}^{R}|A'_r(a)|\beta^r}{\sum_{r=0}^{R}|A'_r(a)|} = \frac{\sum_{r=0}^{[R/2]}(pq)^r\beta^{2r}+\sum_{r=0}^{[(R-1)/2]}q(pq)^r\beta^{2r+1}}{\sum_{r=0}^{[R/2]}(pq)^r+\sum_{r=0}^{[(R-1)/2]}q(pq)^r}. \tag{3.32}$$

Apply the geometric series formula quoted in the proof of theorem 3.20 to each of the finite sums in this equation to get

$$|E_R| \leq \frac{\frac{(pq\beta^2)^{[R/2]+1}-1}{pq\beta^2-1}+q\beta\frac{(pq\beta^2)^{[(R-1)/2]+1}-1}{pq\beta^2-1}}{\frac{(pq)^{[R/2]+1}-1}{pq-1}+q\frac{(pq)^{[(R-1)/2]+1}-1}{pq-1}} \tag{3.33}$$

which we rearrange and write as

$$|E_R| \leq \frac{pq-1}{pq\beta^2-1}\cdot\frac{(pq\beta^2)^{[R/2]+1}-1+q\beta(pq\beta^2)^{[(R-1)/2]+1}-q\beta}{(pq)^{[R/2]+1}-1+q(pq)^{[(R-1)/2]+1}-q} \tag{3.34}$$

$$\leq \frac{pq-1}{pq\beta^2-1}\cdot\frac{(pq\beta^2)^{R/2+1}-1+q\beta(pq\beta^2)^{(R-1)/2+1}-q\beta}{(pq)^{(R-1)/2}-1+q(pq)^{R/2-1}-q} \tag{3.35}$$

$$= \frac{pq-1}{pq\beta^2-1}\cdot\frac{\big(pq\beta^2+q\beta\sqrt{pq\beta^2}\big)(pq\beta^2)^{R/2}-1-q\beta}{(\sqrt{pq}+q)(pq)^{(R-1)/2}-1-q}. \tag{3.36}$$

Note that for $R \geq 1$ we have $q+1 < \sqrt{pq}(pq)^{(R-1)/2}$ which implies from equation (3.36) that

$$|E_R| \leq \frac{pq-1}{pq\beta^2-1}\cdot\frac{\big(pq\beta^2+q\beta\sqrt{pq\beta^2}\big)(pq\beta^2)^{R/2}-1-q\beta}{(\sqrt{pq}+q)(pq)^{(R-1)/2}-\sqrt{pq}(pq)^{(R-1)/2}} \tag{3.37}$$

$$\leq \frac{pq-1}{pq\beta^2-1}\cdot\frac{\big(pq\beta^2+q\beta\sqrt{pq\beta^2}\big)(pq\beta^2)^{R/2}}{q(pq)^{(R-1)/2}} \tag{3.38}$$

$$= \frac{pq-1}{pq\beta^2-1}\cdot\big(p\beta^2+\beta\sqrt{pq\beta^2}\big)\sqrt{pq\beta^2}\frac{(pq\beta^2)^{(R-1)/2}}{(pq)^{(R-1)/2}} \tag{3.39}$$

$$= \frac{pq-1}{pq\beta^2-1}\cdot\big(p\beta^2+\beta\sqrt{pq\beta^2}\big)\cdot\sqrt{pq}\cdot\beta^R \tag{3.40}$$

$$= \frac{pq-1}{pq\beta^2-1}\cdot\big(p\sqrt{pq}+pq\big)\cdot\beta^{R+2} \tag{3.41}$$

which gives the result. □

Corollary 3.23 *Let G, f, C'_G and β_{max} be as in theorem 3.22. Then for an edge ball $B'_R(v)$ of radius R on $\widetilde{G}$ we have*

$$\Big|\frac{1}{|B'_R(v)|}\sum_{e\in B'_R(v)}\tilde{f}(e)-\frac{1}{|E|}\sum_{e\in E}f(e)\Big|\leq C'_G||f||_2\beta^R. \tag{3.42}$$

This follows from theorem 3.22 in the same way as corollary 3.21 follows from theorem 3.20, using equation (3.18).

Chapter 4

Lattice Point Problems

4.1 Motivation

In 1956 Heinz Huber [34] considered the following problem on the hyperbolic plane $\mathbb{H}$. Consider a strictly hyperbolic subgroup Γ of automorphisms on $\mathbb{H}$. We can then identify points of $\mathbb{H}$ which are equivalent with respect to Γ to obtain an orientable Riemannian manifold of constant curvature -1, which we call $\mathbb{H}/\Gamma = M$. Furthermore, if we choose Γ such that M is compact, we can define the genus p of the group Γ to be the topological genus of the closed manifold M. Now take a non-trivial class $\mathcal{K}$ of conjugate elements in Γ. Let $\rho(z_1, z_2)$ be the usual hyperbolic distance between two points $z_1, z_2 \in \mathbb{H}$. Now count

$$N_{\mathcal{K}}(z,t) = \#\{T \in \mathcal{K} : \rho(z, Tz) \leq t\},$$

which is the number of points inside a ball of radius t, that are the image of z under an element $T \in \mathcal{K}$. Huber describes the asymptotic behaviour of this number in the following theorem.

Theorem 4.1 *Let Γ be a hyperbolic group of automorphisms on $\mathbb{H}$ so that the manifold $M = \mathbb{H}/\Gamma$ is compact with genus p. Define $N_{\mathcal{K}}(z,t)$ as above. Then*

$$N_{\mathcal{K}}(z,t) \sim \frac{1}{4\pi(p-1)} \cdot \frac{1}{\nu(\mathcal{K})} \cdot \frac{\mu(\mathcal{K})}{\sinh(\mu(\mathcal{K}/2))} \cdot e^{t/2},$$

where $\nu(\mathcal{K})$ and $\mu(\mathcal{K})$ are constants defined by $\mathcal{K}$. By the notation $N_{\mathcal{K}}(z,t) \sim Ce^{t/2}$ we mean that $N_{\mathcal{K}}(z,t)e^{-t/2} \to C$ as n goes to infinity.

The term $\mu(T)$ is defined as the displacement length of a translation T, and as $\mu(V^{-1}TV) = \mu(T)$ for all $T, V \in \Gamma$ this is constant for all $T \in \mathcal{K}$. Hence we can use $\mu(T) = \mu(\mathcal{K})$ as a constant in the theorem. The term $\nu(T)$ is defined as the multiplicity of a translation T. We can write any element $T \in \Gamma$ as a power $T = P^k$ of some unique primitive element P of Γ, where $k > 0$. This number k is defined to be the multiplicity $\nu(T)$ of T. Again, this number is constant for all $T \in \mathcal{K}$ so we can use $\nu(\mathcal{K})$ as a constant in the theorem. We will explain both constants in more detail in the next section.

In this chapter, we will use the analogy between the hyperbolic plane and the regular tree described in section 1.5 to prove a similar result on the regular tree. We will see that this result (theorem 4.10) can be proved using similar methods to those used by Huber. In particular, Huber uses a Dirichlet series

$$G_{\mathcal{K}}(z, s) = \sum_{T \in \mathcal{K}} (\cosh(\rho(z, Tz)) - 1)^{-s}$$

to which we find a discrete analogue. He uses a Tauberian theorem in the final stages of his proof, and although we cannot use the exact same theorem, we can use a different Tauberian theorem to finish our proof (see section 4.3.5).

4.2 Main Theorem

4.2.1 Definitions

Let X be a regular tree of degree $q + 1 \geq 3$ with vertex set $V(X)$. Let $T : X \to X$ be a non-trivial hyperbolic automorphism or *translation* on X, which we defined in section 1.2 as an element in $\mathrm{Aut}(X)$ with no fixed vertices or edges (see for example [24, Chapter 1]). Let Γ be a group of translations in $\mathrm{Aut}(X)$ such that $G = X/\Gamma$ is a *finite, simple, non-bipartite*, $(q + 1)$-regular graph. We will treat the bipartite case separately in section 4.4. Let $\mathcal{K}$ be the conjugacy class of a non-trivial element $T_0 \in \Gamma$.

Definition 4.2 *For $t \in \mathbb{R}$, $x \in V(X)$, we define*

$$N_{\mathcal{K}}(x, t) = \#\{T \in \mathcal{K} : d(x, Tx) \leq t\}. \tag{4.1}$$

This is the number of elements in the conjugacy class that map a given vertex x to a vertex at distance at most t away from x.

If we think of the set $\{Tx : T \in \mathcal{K}\}$ as a set of points making up an infinite lattice on X, then $N_{\mathcal{K}}(x,t)$ counts the number of *lattice points* in a ball of radius t centred at x. In this chapter we study the limiting behaviour of the counting function in equation (4.1) for increasing t. To do this we define a function $G_{\mathcal{K}} : V(X) \times \mathbb{C} \to \mathbb{C}$ for fixed $\mathcal{K}$ as follows:

$$G_{\mathcal{K}}(x,s) = \sum_{T \in \mathcal{K}} q^{-d(x,Tx)s} \tag{4.2}$$

Lemma 4.3 *The function $G_{\mathcal{K}}(x,s)$ as defined above* (4.2) *is absolutely convergent for $Re(s) > 1$.*

Proof Note

$$\sum_{T \in \mathcal{K}} q^{-d(x,Tx)s} < \sum_{y \in V(X)} q^{-d(x,y)s}$$

as $\{Tx : T \in \mathcal{K}\} \subset V(X)$. We can rewrite the sum on the right hand side in terms of spheres $S_n(x) = \{y \in V(X) : d(x,y) = n\}$ as $\sum_{n=0}^{\infty} |S_n(x)| q^{-ns}$. Observe that $|S_n(x)| = (q+1)q^{n-1}$ for $n \geq 1$ and use this to find

$$\sum_{n=0}^{\infty} |S_n(x)| q^{-ns} \leq \sum_{n=0}^{\infty} 2q^n q^{-ns} = 2\sum_{n=0}^{\infty} q^{n(1-s)}.$$

This sum converges for $\mathrm{Re}(s) > 1$ by the geometric series formula. □

In fact even more is true. In due course (see equation (4.40)) we will see that $G_{\mathcal{K}}$ has a meromorphic extension to $\mathbb{C}$, which is holomorphic for $\mathrm{Re}(s) > \frac{1}{2}$. For the time being, however, we shall use the above lemma to allow us to manipulate the function $G_{\mathcal{K}}(x,s)$ into something we can explicitly calculate. This will be a crucial part of the proof of the main theorem below.

Definition 4.4 *The* axis $\mathfrak{a}(T)$ *of a non-trivial translation T on $V(X)$ is the unique geodesic which is mapped to itself by T.*

Definition 4.5 *The* displacement length $\mu(T)$ *of T is given by*

$$\mu(T) = \min_{x \in V(X)} d(x, Tx).$$

Note that $\mu(T) \in \mathbb{N}$ and $\mu(T) \geq 1$ for non-trivial T. It is easy to see that the minimum is attained exactly for those vertices that lie in $\mathfrak{a}(T)$, as the vertices in $\mathfrak{a}(T)$ are shifted along the axis by $\mu(T)$ vertices under the action of T. Let $\delta(x, T)$ be the distance from a point x to the axis of a translation T, that is

$$\delta(x, T) = \min_{y \in \mathfrak{a}(T)} d(x, y). \tag{4.3}$$

We then observe

$$d(x, Tx) = \mu(T) + 2\delta(x, T). \tag{4.4}$$

Definition 4.6 *An element $P \in \Gamma$ is called* primitive *if $\nexists\ Q \in \Gamma$ and $n > 1$ such that $P = Q^n$. For every non-trivial $T \in \Gamma$, we can find a unique primitive $P \in \Gamma$ so that we can write $T = P^k$ for some $k \geq 1$, and we call this the* standard representation *of T. Now write $k = \nu(T)$ and call it the* multiplicity *of T.*

Clearly $\mathfrak{a}(T) = \mathfrak{a}(P)$, and $\mu(T) = \mu(P) \cdot \nu(T)$.

Lemma 4.7 *For any $P, V \in \Gamma$, P is primitive iff $V^{-1}PV$ is primitive.*

Proof Let P be primitive in Γ, and let $P^* = V^{-1}PV$. Suppose P^* is not primitive, so we can write $P^* = Q^k$ for some $k > 1$. Then $P = VP^*V^{-1} = VQ^kV^{-1} = (VQV^{-1})^k$, which contradicts the assumption that P is primitive. Hence P is primitive iff P^* is primitive. □

Lemma 4.8 *$\mu(T)$ is invariant under conjugation in Γ.*

Proof Use the fact that $V \in \Gamma$ is an isometry, a distance preserving map, to obtain

$$\mu(V^{-1}TV) = \min_{x \in V(X)} d(x, V^{-1}TVx) = \min_{x \in V(X)} d(Vx, VV^{-1}TVx) \tag{4.5}$$

$$= \min_{y = Vx \in V(X)} d(y, Ty) = \min_{y \in V(X)} d(y, Ty) = \mu(T). \tag{4.6}$$

This means that for any $T \in \mathcal{K}$ we can write $\mu(T) = \mu(\mathcal{K})$. □

Lemma 4.9 *$\nu(T)$ is invariant under conjugation in Γ.*

Proof Let $T = P^{\nu(T)}$. Then

$$V^{-1}TV = V^{-1}P^{\nu(T)}V = \left(V^{-1}PV\right)^{\nu(T)} = (P^*)^{\nu(T)}, \tag{4.7}$$

where we know P^* is primitive by lemma 4.7. This holds for all $V \in \Gamma$, hence the multiplicity of an element is preserved under conjugation, which means we can write $\nu(T) = \nu(\mathcal{K})$ for all $T \in \mathcal{K}$. □

4.2.2 Statement of the Theorem

We can now state our **main theorem** of this chapter.

Theorem 4.10 *Let $N_{\mathcal{K}}(x, n)$ be defined as in equation* (4.1) *for positive integers n. Then as $n \to \infty$, where $n - \mu(\mathcal{K})$ is even,*

$$N_{\mathcal{K}}(x, n) \sim q^{\frac{n-\mu(\mathcal{K})}{2}} \frac{\mu(\mathcal{K})}{\nu(\mathcal{K})|G|}$$

with $\mu(\mathcal{K})$ and $\nu(\mathcal{K})$ as defined above.

We need $n - \mu(\mathcal{K})$ to be an even integer in the theorem for the following reason. Equation (4.4) implies that we can rewrite the counting function as

$$N_{\mathcal{K}}(x, n) = \#\{T \in \mathcal{K} : \delta(x, T) \leq \frac{n - \mu(\mathcal{K})}{2}\}. \tag{4.8}$$

Note that $\delta(x, T) \in \mathbb{N}$, so the counting function can only change at integer values of $\frac{n-\mu(\mathcal{K})}{2}$, that is, when $n - \mu(\mathcal{K})$ is even.

4.3 Proof of Main Theorem

4.3.1 Background

We noted before that we are dealing here with a type of problem that counts the number of lattice points on a graph inside an increasing ball. This type of problem is often solved using a discrete version of Selberg's Trace Formula on a regular tree (see [50] and for the discrete version [1], [10], [57] or [59]). We will discuss the trace formula and its applications in chapter 5. In this particular case, however, we are unable to use the trace formula for reasons outlined below.

The trace formula is obtained using a Γ-invariant function $G(x, y)$ (a so-called *point-pair invariant*) such that

$$G(\gamma x, y) = G(x, \gamma y) = G(x, y) \ \forall \ \gamma \in \Gamma \tag{4.9}$$

(see [50] or [57]). A natural choice for us to use would be

$$G_{\mathcal{K}}(x, y, s) = \sum_{T \in \mathcal{K}} q^{-d(x,Ty)s} \tag{4.10}$$

but one easily checks that this function does not satisfy equation (4.9) for arbitrary x, y. Instead, we follow in our proof the method of Huber [34] using the function defined in equation (4.2), which satisfies $G_{\mathcal{K}}(x, s) = G_{\mathcal{K}}(\gamma x, s)$ for all $\gamma \in \Gamma$.

4.3.2 Manipulation of $G_{\mathcal{K}}(x, s)$

Recall from equation (1.1) in section 1.3.1 that we defined the Laplacian on a $(q+1)$-regular graph G by

$$\mathcal{L}_G f(v) = \frac{1}{q+1} \sum_{v \sim w} f(w).$$

We also saw that if $|V(G)| = N + 1$ we have an orthonormal basis $\{\varphi_i\}_{i=0}^N$ of eigenfunctions of the Laplacian on G with eigenvalues $1 = \lambda_0 > \lambda_1 \geq \ldots \geq \lambda_N \geq -1$. As before, lift the eigenfunctions to the universal cover X to obtain a basis $\{\tilde{\varphi}_i = \varphi_i \circ \pi\}_{i=0}^N$ of eigenfunctions of the Laplacian for functions on X which are Γ-invariant.

We defined in equation (4.2) a function $G_{\mathcal{K}}(x, s)$ on X which satisfies $G_{\mathcal{K}}(x, s) = G_{\mathcal{K}}(\gamma x, s)$ for all $\gamma \in \Gamma$, so it is Γ-invariant. Therefore we can view it as a function on $V(G) \times \mathbb{C}$, where G is the quotient X/Γ. We call this function $g_{\mathcal{K}}(x, s)$, and write $g_{\mathcal{K}}(x, s) = \sum_{i=0}^N F_i(s)\varphi_i(x)$, with Fourier coefficients $F_i(s)$ given by

$$F_i(s) = \sum_{x \in V(G)} g_{\mathcal{K}}(x, s)\varphi_i(x).$$

Now 'lift' $g_{\mathcal{K}}(x, s)$ back up to X and obtain

$$G_{\mathcal{K}}(x, s) = \sum_{i=0}^{N} F_i(s)\widetilde{\varphi}_i(x) \tag{4.11}$$

where

$$F_i(s) = \sum_{x \in \mathfrak{F}} G_{\mathcal{K}}(x, s)\widetilde{\varphi}_i(x) \tag{4.12}$$

for a fundamental domain $\mathfrak{F} \subset V(X)$ of Γ on $V(X)$. Note that there is a canonical one-to-one correspondence between $\mathfrak{F}$ and $V(G)$.

Choose and fix a translation $T^* \in \mathcal{K}$ with the standard representation $T^* = P^{\nu(\mathcal{K})}$ for some primitive P. Let $H = \langle P \rangle$ be the subgroup of Γ generated by P. Then we can write Γ as a disjoint union of right cosets of H, i.e.

$$\Gamma = \bigcup_{n=1}^{\infty} HA_n \tag{4.13}$$

with a fixed set $\{A_n\}_{n=1}^{\infty} \subset \Gamma$. The elements $A_n^{-1}T^*A_n = T_n$ are pairwise disjoint and run through all of $\mathcal{K}$ as $n = 1, 2, \ldots, \infty$, and we define

$$\mathfrak{F}^* = \bigcup_{n=1}^{\infty} A_n(\mathfrak{F}) \tag{4.14}$$

where $A_n(\mathfrak{F}) = \{A_n x : x \in \mathfrak{F}\}$.

Recall our definition of a fundamental domain in section 1.5. We now make this definition more specific. For a group Γ acting on a tree X, a fundamental domain $\mathfrak{F}$ must satisfy the following two conditions:

- $\mathfrak{F} \cap g\mathfrak{F} = \emptyset$ for any $g \in \Gamma$ which is not the identity element
- $\displaystyle\bigcup_{g\in\Gamma} g\mathfrak{F} = X$

Lemma 4.11 $\mathfrak{F}^*$ *is a fundamental domain of the cyclic group* H*.*

Proof $\mathfrak{F}$ is a fundamental domain for the group Γ, which means it satisfies the two conditions above. We need to show the same conditions hold for the domain $\mathfrak{F}^* = \bigcup_{n=1}^{\infty} A_n\mathfrak{F}$ and the group $H = \langle P \rangle$.

To show $\mathfrak{F}^* \cap h\mathfrak{F}^* = \emptyset$, that is,

$$\left(\bigcup_{n=1}^{\infty} A_n\mathfrak{F}\right) \cap \left(\bigcup_{n=1}^{\infty} hA_n\mathfrak{F}\right) = \emptyset \tag{4.15}$$

for all $h \in H$ not equal to the identity, it is sufficient to prove $A_n\mathfrak{F} \cap hA_m\mathfrak{F} = \emptyset$, which is equivalent to $\mathfrak{F} \cap A_n^{-1}hA_m\mathfrak{F} = \emptyset$. $\mathfrak{F}$ is a fundamental domain for Γ, so for this to hold we need $A_n^{-1}hA_m \neq e$ where e is the identity element in Γ. Now $A_n^{-1}hA_m = e$ iff $h = A_nA_m^{-1} \in H$, but this implies $A_n \in HA_m$ so by definition we must have $m = n$ and hence $h = e$, which we excluded. This means equation 4.15, and hence the first condition, is satisfied.

For the second condition, we need to show $\cup_{h\in H} h\mathfrak{F}^* = X$. Note that

$$\bigcup_{h\in H} h\mathfrak{F}^* = \bigcup_{h\in H}\bigcup_{n=1}^{\infty} hA_n\mathfrak{F} = \bigcup_{g\in\Gamma} g\mathfrak{F} = X$$

using equation (4.13) and the fact that $\mathfrak{F}$ is a fundamental domain for Γ. This shows the second condition is also satisfied, hence $\mathfrak{F}^*$ is a fundamental domain of H. □

Lemma 4.12 *The Fourier coefficients $F_i(s)$ are given by*

$$F_i(s) = q^{-s\mu(\mathcal{K})}\sum_{x\in\mathfrak{F}^*} q^{-2s\delta(x,P)}\widetilde{\varphi}_i(x).$$

Proof Let $k = \nu(\mathcal{K})$. Use equations (4.2) and (4.12), and the definition of T_n above to get

$$\begin{aligned} F_i(s) = \sum_{x\in\mathfrak{F}} G_{\mathcal{K}}(x,s)\widetilde{\varphi}_i(x) &= \sum_{x\in\mathfrak{F}}\sum_{T\in\mathcal{K}} q^{-d(x,Tx)s}\widetilde{\varphi}_i(x) \\ &= \sum_{x\in\mathfrak{F}}\sum_{n=1}^{\infty} q^{-d(x,T_nx)s}\widetilde{\varphi}_i(x) \\ &= \sum_{x\in\mathfrak{F}}\sum_{n=1}^{\infty} q^{-d(x,A_n^{-1}P^kA_nx)s}\widetilde{\varphi}_i(x). \end{aligned} \tag{4.16}$$

Now use the fact that the distance function $d(x,y)$ is an isometry, and change the order of summation to obtain

$$F_i(s) = \sum_{x\in\mathfrak{F}}\sum_{n=1}^{\infty} q^{-d(A_nx,P^kA_nx)s}\widetilde{\varphi}_i(x) = \sum_{n=1}^{\infty}\sum_{x\in A_n(\mathfrak{F})} q^{-d(x,P^kx)s}\widetilde{\varphi}_i(x). \tag{4.17}$$

From equation (4.14) we then obtain

$$F_i(s) = \sum_{x\in\mathfrak{F}^*} q^{-d(x,P^kx)s}\widetilde{\varphi}_i(x) = q^{-sk\mu(P)}\sum_{x\in\mathfrak{F}^*} q^{-2s\delta(x,P)}\widetilde{\varphi}_i(x), \tag{4.18}$$

where the last equality is due to equation (4.4). Use $k\mu(P) = \mu(\mathcal{K})$ to obtain the final result. □

4.3.3 Transfer the Functions onto the Quotient Graph

Note that $\delta(x,P) = \delta(P^nx,P)$ for any integer n, as the axes of P and P^n coincide, and $\widetilde{\varphi}_i(x) = \widetilde{\varphi}_i(P^nx)$ as $P^n\in\Gamma$ and $\widetilde{\varphi}_i(x)$ is Γ-invariant. This means the terms in the sum in equation (4.18) are invariant under $H = \langle P \rangle$. Hence we can replace $\mathfrak{F}^*$ in the sum of (4.18) by *any* fundamental domain of H.

Take a segment of $\mathfrak{a}(P)$ of length $\mu(P)$ and all branches emanating from the vertices in this segment away from the axis. This means we exclude the two branches that emanate from the vertices at the ends of the segment in the direction of the axis. The vertices in the set we have just defined clearly form a fundamental domain of H, which we call $\mathfrak{F}_P$. Using the fact we can interchange fundamental domains of H shown above, we now sum over the vertices in $\mathfrak{F}_P$ instead of $\mathfrak{F}^*$ in equation (4.18).

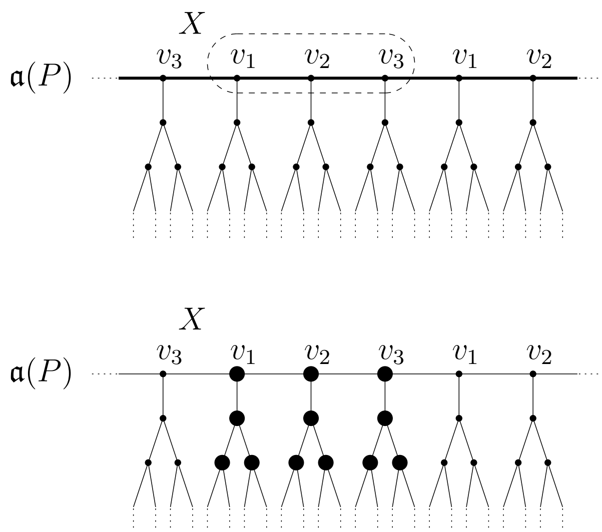

Figure 4.1: A segment of $\mathfrak{a}(P)$, and the resulting fundamental domain $\mathfrak{F}_P$.

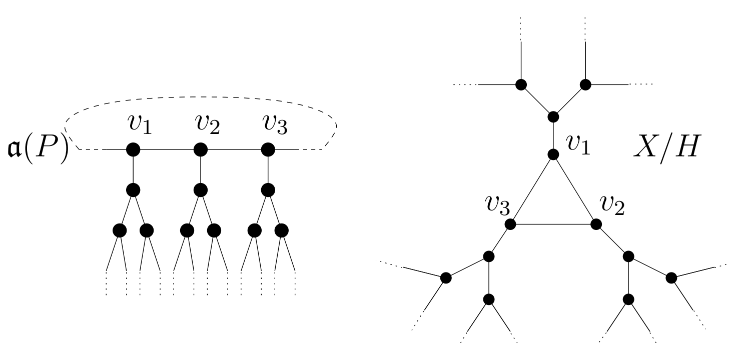

Figure 4.2: How to deduce the structure of X/H from $\mathfrak{F}_P$.

The structure of the quotient graph X/H can easily be deduced from $\mathfrak{F}_P$. Look for example at figure 4.1 and 4.2, where $\mu(P) = 3$. The vertices in $\mathfrak{F}_P$ are bold in figure 4.1, and in figure 4.2 we show how to 'glue' the edges together to obtain X/H.

We now transfer the functions $\widetilde{\varphi}_i$ from $\mathfrak{F}_P$ to functions on the vertices of X/H, and to do this we use the obvious one-to-one correspondence between the vertex sets $\mathfrak{F}_P$ and $V(X/H)$. Note that the edge relations are preserved, and call the new function $\overline{\varphi_i} : V(X/H) \to \mathbb{R}$ for $i = 0, \ldots, N$.

Lemma 4.13 *The functions $\overline{\varphi_i}$ are eigenfunctions of the Laplacian on X/H.*

PROOF For each $v \in V(X/H)$, the set of vertices at distance 1 from v is preserved when we glue together $\mathfrak{F}_P$ to obtain X/H. Hence we can use the same argument as in the proof of lemma 1.10 to say

$$\mathcal{L}_{\mathfrak{F}} f(v) = \frac{1}{d(v)} \sum_{\substack{d(v,w)=1 \\ v,w \in \mathfrak{F}}} f(w) = \frac{1}{d(v)} \sum_{\substack{d(v,w)=1 \\ v,w \in X/H}} \overline{f}(w) = \mathcal{L}_{X/H} \overline{f}(v) \tag{4.19}$$

Again, we see that the eigenvalue λ_i is preserved. □

Using lemma 4.13 and equation (4.18) we now obtain

$$F_i(s) = q^{-s\mu(\mathcal{K})} \sum_{x \in V(X/H)} q^{-2s\delta'(x,P)} \overline{\varphi_i}(x) \tag{4.20}$$

where $\delta'(x, P)$ is the distance from the vertex x to the central circuit in X/H, which is exactly equal to $\delta(x, P)$ on X. For example, the triangle in figure 4.2 is the central circuit.

4.3.4 Explicit Calculation of $G_{\mathcal{K}}(x, s)$

Definition 4.14 *We define* levels *in X/H as follows:*

$$L_n = L_n(X/H) = \{x \in V(X/H) : \delta'(x, P) = n\} \quad \textit{for } n \geq 0 \tag{4.21}$$

The *radial average* of a function $f : V(X/H) \to \mathbb{R}$ with respect to these levels is

$$\frac{1}{|L_n|} \sum_{x \in L_n} \overline{f}(x) \tag{4.22}$$

Lemma 4.15 *The radial average of $\overline{\varphi_i}(x)$ gives an eigenfunction of the Laplacian on X/H with eigenvalue λ_i, which we call $\Phi_i(x)$.*

Proof Use

$$\Phi_i(x) = \frac{1}{|L_{\delta'(x,P)}|} \sum_{y \in L_{\delta'(x,P)}} \overline{\varphi_i}(y)$$

and the definition of the Laplacian in equation (1.1), observing $d(x) = q+1$, to obtain

$$\begin{aligned}
\mathcal{L}_{X/H}\Phi_i(x) =& \frac{1}{q+1} \sum_{x \sim y} \Phi_i(y) \\
=& \frac{1}{q+1} \sum_{x \sim y} \Big(\frac{1}{|L_{\delta'(x,P)}|} \sum_{w \in L_{\delta'(x,P)}} \overline{\varphi_i}(w) \Big) \\
=& \frac{1}{q+1} \frac{1}{|L_{\delta'(x,P)}|} \Big(\sum_{w \in L_{\delta'(x,P)-1}} \overline{\varphi_i}(x) + \sum_{z \in L_{\delta'(x,P)+1}} \overline{\varphi_i}(z) \Big) \\
=& \frac{1}{|L_{\delta'(x,P)}|} \sum_{w \in L_{\delta'(x,P)}} \Big(\frac{1}{q+1} \sum_{y \sim w} \overline{\varphi_i}(y) \Big) \\
=& \frac{1}{|L_{\delta'(x,P)}|} \sum_{w \in L_{\delta'(x,P)}} \lambda_i \overline{\varphi_i}(x) = \lambda_i \Phi_i(x).
\end{aligned}$$

□

Clearly $\Phi_i(x) = \Phi_i(y)$ whenever $\delta'(x,P) = \delta'(y,P) = n$, so we shall write $\Phi_i(n)$ from now on, where $n \in \mathbb{Z}^{\geq 0}$.

Note that

$$V(X/H) = \bigcup_{n=0}^{\infty} L_n, \text{ and} \tag{4.23}$$

$$\sum_{x \in L_n} \overline{\varphi_i}(x) = |L_n| \Phi_i(n). \tag{4.24}$$

Use these facts and equation (4.20) to obtain

$$F_i(s) = q^{-s\mu(\mathcal{K})} \sum_{n=0}^{\infty} |L_n| q^{-2sn} \Phi_i(n). \tag{4.25}$$

Lemma 4.16 *If $\lambda_i \neq \pm\frac{2\sqrt{q}}{q+1}$, there are constants $\alpha_i^{\pm}$, $u_i^{\pm}$ (see equations* (4.30) *and* (4.32) *below) depending only on φ_i, λ_i and q such that*

$$F_i(s) = q^{-s\mu(\mathcal{K})} \frac{\mu(\mathcal{K})}{\nu(\mathcal{K})} \frac{q-1}{q} \Big(\frac{u_i^+ + u_i^-}{q-1} + \frac{u_i^+}{1-\alpha_i^+ q^{1-2s}} + \frac{u_i^-}{1-\alpha_i^- q^{1-2s}} \Big). \tag{4.26}$$

In the case that $\lambda_i = \pm\frac{2\sqrt{q}}{q+1}$, we obtain

$$\begin{aligned}
F_i(s) = \frac{1}{q}\; q^{-s\mu(\mathcal{K})} \frac{\mu(\mathcal{K})}{\nu(\mathcal{K})} \Phi_i(0) + q^{-s\mu(\mathcal{K})} \frac{\mu(\mathcal{K})}{\nu(\mathcal{K})} \frac{q-1}{q} \frac{\Phi_i(0)}{1 \mp q^{1/2-2s}} \\
+ q^{-s\mu(\mathcal{K})} \frac{\mu(\mathcal{K})}{\nu(\mathcal{K})} \frac{q-1}{q} \Phi_i(0) \Big(\frac{\sqrt{q}-1}{\sqrt{q}+1} \Big)^{\pm 1} \frac{\pm q^{1/2-2s}}{(1 \mp q^{1/2-2s})^2}.
\end{aligned} \tag{4.27}$$

Proof Assume first that $\lambda_i \neq \pm\frac{2\sqrt{q}}{q+1}$. By lemma 4.15 we can use $\mathcal{L}_{X/H}\Phi_i(n) = \lambda_i\Phi_i(n)$ to find the recursion relation

$$\Phi_i(n+1) - \frac{q+1}{q}\lambda_i\Phi_i(n) + \frac{1}{q}\Phi_i(n-1) = 0 \quad \text{for } n \geq 1 \tag{4.28}$$

and initial conditions $\Phi_i(0)$ and $\Phi_i(1)$ determined by

$$(q+1)\lambda_i\Phi_i(0) = (q-1)\Phi_i(1) + 2\Phi_i(0). \tag{4.29}$$

Assume $\Phi_i(n) = \alpha_i^n$ and use equation (4.28) to find two possibilities for α_i, namely

$$\alpha_i^{\pm} = \frac{q+1}{2q}\lambda_i \pm \frac{\sqrt{(q+1)^2{\lambda_i}^2 - 4q}}{2q}. \tag{4.30}$$

This means we have a general solution

$$\Phi_i(n) = u_i^+(\alpha_i^+)^n + u_i^-(\alpha_i^-)^n \tag{4.31}$$

for constants $u_i^{\pm}$, which we can determine from equation (4.29) and the general solution for $n = 0, 1$ to be

$$u_i^{\pm} = \left(\frac{1}{2} \pm \frac{(q+1)^2\lambda_i - 4q}{2(q-1)\sqrt{(q+1)^2{\lambda_i}^2 - 4q}}\right) \cdot \Phi_i(0). \tag{4.32}$$

Observe that $\alpha_i^+ \neq \alpha_i^-$ as the square root is non-zero due to the exclusion of $\lambda_i = \pm\frac{2\sqrt{q}}{q+1}$. Use equation (4.31) in equation (4.25) to get

$$F_i(s) = q^{-s\mu(\mathcal{K})}\sum_{n=0}^{\infty}|L_n|q^{-2sn}\big(u_i^+(\alpha_i^+)^n + u_i^-(\alpha_i^-)^n\big). \tag{4.33}$$

Note that $|L_0| = \mu(P) = \frac{\mu(\mathcal{K})}{\nu(\mathcal{K})}$, and $|L_n| = q^{n-1}(q-1)\frac{\mu(\mathcal{K})}{\nu(\mathcal{K})}$ for $n \geq 1$, and use this to obtain

$$F_i(s) = q^{-s\mu(\mathcal{K})}\frac{\mu(\mathcal{K})}{\nu(\mathcal{K})}\big(u_i^+ + u_i^-\big) \\ + q^{-s\mu(\mathcal{K})}\frac{\mu(\mathcal{K})}{\nu(\mathcal{K})}\frac{q-1}{q}\sum_{n=1}^{\infty}q^n q^{-2sn}\big(u_i^+(\alpha_i^+)^n + u_i^-(\alpha_i^-)^n\big) \tag{4.34}$$

$$= \frac{1}{q}q^{-s\mu(\mathcal{K})}\frac{\mu(\mathcal{K})}{\nu(\mathcal{K})}\big(u_i^+ + u_i^-\big) \\ + q^{-s\mu(\mathcal{K})}\frac{\mu(\mathcal{K})}{\nu(\mathcal{K})}\frac{q-1}{q}\sum_{n=0}^{\infty}q^{(1-2s)n}\big(u_i^+(\alpha_i^+)^n + u_i^-(\alpha_i^-)^n\big). \tag{4.35}$$

We now split the infinite sum into two parts, to each of which we apply the geometric series formula $\sum_{r=0}^{\infty} x^r = \frac{1}{1-x}$ $(|x| < 1)$ which results in

$$F_i(s) = \frac{1}{q} q^{-s\mu(\mathcal{K})} \frac{\mu(\mathcal{K})}{\nu(\mathcal{K})} \left(u_i^+ + u_i^-\right) + q^{-s\mu(\mathcal{K})} \frac{\mu(\mathcal{K})}{\nu(\mathcal{K})} \frac{q-1}{q} \sum_{n=0}^{\infty} u_i^+ (q^{(1-2s)} \alpha_i^+)^n + q^{-s\mu(\mathcal{K})} \frac{\mu(\mathcal{K})}{\nu(\mathcal{K})} \frac{q-1}{q} \sum_{n=0}^{\infty} u_i^- (q^{(1-2s)} \alpha_i^-)^n \tag{4.36}$$

$$= \frac{1}{q} q^{-s\mu(\mathcal{K})} \frac{\mu(\mathcal{K})}{\nu(\mathcal{K})} \left(u_i^+ + u_i^-\right) + q^{-s\mu(\mathcal{K})} \frac{\mu(\mathcal{K})}{\nu(\mathcal{K})} \frac{q-1}{q} \frac{u_i^+}{1-\alpha_i^+ q^{1-2s}} + q^{-s\mu(\mathcal{K})} \frac{\mu(\mathcal{K})}{\nu(\mathcal{K})} \frac{q-1}{q} \frac{u_i^-}{1-\alpha_i^- q^{1-2s}}. \tag{4.37}$$

Observe that for the infinite sums to converge, we need $|\alpha_i^{\pm} q^{1-2s}| < 1$. For $|\lambda_i| < 1$ it is easy to check that $|\alpha_i^{\pm}| < 1$. Hence there is a real number $\sigma_0 < \frac{1}{2}$ so that the sums obtained from equation (4.25) converge for $\mathrm{Re}(s) > \sigma_0$. As G is non-bipartite, the eigenvalue $\lambda = -1$ does not occur, however for the eigenvalue $\lambda_0 = 1$ we have $\alpha_0^+ = 1$ and the infinite sum for $F_0(s)$ will only converge for $\mathrm{Re}(s) > \frac{1}{2}$.

Now let $\lambda_i = \pm\frac{2\sqrt{q}}{q+1}$. The square root in equation (4.30) is zero iff we have $\lambda_i = \pm\frac{2\sqrt{q}}{q+1}$, which implies $\alpha_i^+ = \alpha_i^- = \alpha_i = \pm\frac{1}{\sqrt{q}}$. In this case

$$\Phi_i(n) = u_i \alpha_i^n + v_i n \alpha_i^n$$

for some constants u_i and v_i. We calculate the constants using the initial conditions and obtain

$$\Phi_i(n) = \left(1 + n\left(\frac{\sqrt{q}-1}{\sqrt{q}+1}\right)^{\pm 1}\right) \Phi_i(0) \alpha_i^n \quad \text{for} \quad \lambda_i = \pm\frac{2\sqrt{q}}{q+1}. \tag{4.38}$$

Using this in equation (4.25) we obtain

$$F_i(s) = q^{-s\mu(\mathcal{K})} \sum_{n=0}^{\infty} |L_n| q^{-2sn} \left(1 + n\left(\frac{\sqrt{q}-1}{\sqrt{q}+1}\right)^{\pm 1}\right) \Phi_i(0) \alpha_i^n$$

$$= \frac{1}{q} q^{-s\mu(\mathcal{K})} \frac{\mu(\mathcal{K})}{\nu(\mathcal{K})} \Phi_i(0) + q^{-s\mu(\mathcal{K})} \frac{\mu(\mathcal{K})}{\nu(\mathcal{K})} \frac{q-1}{q} \sum_{n=0}^{\infty} q^n q^{-2sn} \Phi_i(0) \Big(\pm\frac{1}{\sqrt{q}}\Big)^n + q^{-s\mu(\mathcal{K})} \frac{\mu(\mathcal{K})}{\nu(\mathcal{K})} \frac{q-1}{q} \sum_{n=0}^{\infty} q^n q^{-2sn} n \left(\frac{\sqrt{q}-1}{\sqrt{q}+1}\right)^{\pm 1} \Phi_i(0) \Big(\pm\frac{1}{\sqrt{q}}\Big)^n. \tag{4.39}$$

Grouping together the terms in each sum with a power n, and using the geometric series formula we used above and the series $\sum_{i=1}^{\infty} i x^i = \frac{x}{(1-x)^2}$, we obtain the required expression from (4.25). For the convergence of the two infinite sums obtained here we require $|q^{1/2-2s}| < 1$ which implies $\mathrm{Re}(s) > \frac{1}{4}$, which is consistent with the general case above. □

Calculating α_0^- and $u_0^\pm$ using $\widetilde{\varphi_0}(x) = \frac{1}{\sqrt{|G|}}$ $\forall\ x \in V(G)$ we obtain the explicit formula

$$G_{\mathcal{K}}(x,s) = \frac{\mu(\mathcal{K})}{\nu(\mathcal{K})|G|q^{s\mu(\mathcal{K})+1}}\big(1 + \frac{q-1}{1-q^{1-2s}}\big) + \sum_{i=1}^{N} F_i(s)\widetilde{\varphi}_i(x). \tag{4.40}$$

This is a meromorphic extension (which is holomorphic for $\mathrm{Re}(s) > \frac{1}{2}$) to the complex plane of $G_{\mathcal{K}}(x,s)$, which was defined in equation (4.2) by

$$G_{\mathcal{K}}(x,s) = \sum_{T\in\mathcal{K}} q^{-d(x,Tx)s}.$$

4.3.5 Tauberian Theorem

To finish the proof of theorem 4.10, we use a refined version of the Tauberian theorem by Wiener-Ikehara from [26] (see also [39, chapter III theorem 5.4]), which is a refinement of the Tauberian theorem in [60]. This theorem requires a function $f(s)$ which converges for $\mathrm{Re}(s) > 1$ and has a simple pole at $s = 1$. We choose $f(s) = G_{\mathcal{K}}(x, \frac{s}{2})$, which from equation (4.40) has a pole at $s = 1$. The residue of $f(s)$ at $s = 1$ is

$$\mathrm{Res}(f(s),1) = \lim_{s\to 1}(s-1)G_{\mathcal{K}}(x,\frac{s}{2}) = \frac{\mu(\mathcal{K})(q-1)}{\nu(\mathcal{K})|G|q^{(\mu(\mathcal{K})/2)+1}}\,\frac{1}{\ln q} := A. \tag{4.41}$$

That means $f(s) = \frac{A}{s-1} + g(s)$ for some function $g(s)$ which is analytic for $\mathrm{Re}(s) > 1$. Check that $g(s)$ is analytic for $\mathrm{Re}(s) \searrow 1$ when $|\mathrm{Im}(s)| < \frac{2\pi}{\ln q}$. Indeed, $g(s)$ has poles wherever $f(s)$ does, except we have removed the pole at $s = 1$. As $f(s)$ has poles on the line $l = \{s : \mathrm{Re}(s) = 1\}$ at $s = 1 + ki\frac{2\pi m}{\ln q}$ for any $m \in \mathbb{Z}$, $g(s)$ has no poles on l for $|\mathrm{Im}(s)| < \frac{2\pi}{\ln q}$. Note also $A > 0$.

Recall $N_{\mathcal{K}}(x,t) = \#\{T \in \mathcal{K} : d(x,Tx) \le t\}$ from definition 4.2. For fixed $x \in V(X)$, let

$$S(t) = N_{\mathcal{K}}(x,2t) = \#\{T \in \mathcal{K} : d(x,Tx) \le 2t\}.$$

This is a non-decreasing step-function, which vanishes for $t < 0$.

We now have all the ingredients we need to apply the Tauberian theorem, which in our notation reads as follows.

Theorem 4.17 *Let $S(t)$ vanish for $t < 0$, be non-decreasing, continuous from the right and such that*

$$f(s) = \int_0^\infty q^{-st} dS(t), \quad s = \sigma_1 + i\sigma_2$$

exists for $Re(s) = \sigma_1 > 1$. Suppose that for some number $\rho > 0$, there is a constant $A \geq 0$ such that the analytic function

$$g(s) = f(s) - \frac{A}{s-1}, \quad s = \sigma_1 + i\sigma_2, \quad \sigma_1 > 1$$

converges to a boundary function $g(1 + i\sigma_2)$ in $L^1(-\rho < \sigma_2 < \rho)$ as $\sigma_1 \searrow 1$. Let τ be the supremum of all possible numbers ρ. Then

$$\frac{2\pi/\tau}{e^{2\pi/\tau} - 1} A \leq \liminf_{t\to\infty} q^{-t}S(t) \leq \limsup_{t\to\infty} q^{-t}S(t) \leq \frac{2\pi/\tau}{1 - e^{-2\pi/\tau}} A.$$

In our case $\tau = 2\pi/\ln q$. Using this and equation (4.41) we obtain

$$\frac{1}{q}\frac{\mu(\mathcal{K})}{\nu(\mathcal{K})|G|q^{\mu(\mathcal{K})/2}} \leq \liminf_{t\to\infty}\ q^{-t}S(t) \leq \limsup_{t\to\infty}\ q^{-t}S(t) \leq \frac{\mu(\mathcal{K})}{\nu(\mathcal{K})|G|q^{\mu(\mathcal{K})/2}}. \quad (4.42)$$

These estimates no longer depend on the choice of x.

We noticed that as a consequence of (4.4) when $\mu(\mathcal{K})$ is even, $S(t)$ will jump only at integer values of t (the case where $\mu(\mathcal{K})$ is odd works similarly, except jumps occur only when $t + \frac{1}{2}$ is an integer). In this (even) case, writing $m = [t]$ or equivalently

$$t = m + \varepsilon \text{ with } m \in \mathbb{Z} \text{ and } \varepsilon \in [0, 1) \quad (4.43)$$

we have

$$S(t) = S(m)\ \forall\ \varepsilon \in [0, 1) \quad (4.44)$$

Letting

$$a = \frac{\mu(\mathcal{K})}{\nu(\mathcal{K})|G|q^{\mu(\mathcal{K})/2}} \quad (4.45)$$

we obtain the following estimates for the lim inf and lim sup respectively:

$$\frac{a}{q} \leq \liminf_{m\to\infty, m\in\mathbb{N}} q^{-(m+\varepsilon)}S(m) = q^{-\varepsilon} \liminf_{m\to\infty, m\in\mathbb{N}} q^{-m}S(m) \leq a \quad (4.46)$$

$$\frac{a}{q} \leq \limsup_{m\to\infty, m\in\mathbb{N}} q^{-(m+\varepsilon)}S(m) = q^{-\varepsilon} \limsup_{m\to\infty, m\in\mathbb{N}} q^{-m}S(m) \leq a \quad (4.47)$$

As this must hold for all $\varepsilon \in [0,1)$, we obtain

$$\frac{a}{q} \leq q^{-1} \liminf_{m\to\infty, m\in\mathbb{N}} S(m) \leq \liminf_{m\to\infty, m\in\mathbb{N}} S(m) \leq a$$

which implies

$$\liminf_{m\to\infty, m\in\mathbb{N}} S(m) = a$$

Similarly we obtain

$$\limsup_{m\to\infty, m\in\mathbb{N}} S(m) = a$$

and hence

$$\lim_{m\to\infty, m\in\mathbb{N}} S(m) = a$$

This means that for large integers $n = 2m$ we have an approximation of $S(m)$ and hence $N_{\mathcal{K}}(x,n)$ as follows

$$S(m) \sim q^{m-\frac{\mu(\mathcal{K})}{2}} \frac{\mu(\mathcal{K})}{\nu(\mathcal{K})|G|} \qquad N_{\mathcal{K}}(x,n) = S(\frac{n}{2}) \sim q^{\frac{n-\mu(\mathcal{K})}{2}} \frac{\mu(\mathcal{K})}{\nu(\mathcal{K})|G|}$$

where from the definition of m we require $n - \mu(\mathcal{K})$ to be even (because when $\mu(\mathcal{K})$ is odd, we take $m \in \frac{1}{2} + \mathbb{Z}$ in equation (4.43)). Using equation (4.44) it is clear that for any real t such that $\mu(\mathcal{K}) + 2r \leq t < \mu(\mathcal{K}) + 2r + 2$ for a non-negative integer r, we have $N_{\mathcal{K}}(x,t) = N_{\mathcal{K}}(x, \mu(\mathcal{K}) + 2r)$. This completes the proof of the main theorem 4.10 of this chapter.

4.4 Bipartite Graphs

We finish this chapter by discussing why we required that G was non-bipartite. Most of the proof above can be used to show a weaker result, but theorem 4.10 will not hold for bipartite G.

The method of proof works for the bipartite case up to lemma 4.16, where we have to consider the effects on $G_{\mathcal{K}}(x,s)$ of $\lambda_N = -1$, the eigenvalue of the Laplacian which occurs exactly when G is bipartite. We can easily calculate that

$$\begin{aligned}
\alpha_N^+ =& \frac{-1}{q}, \\
\alpha_N^- =& -1, \\
u_N^+ =& \frac{-4q}{(q-1)^2}\ \Phi_N(0), \\
u_N^- =& \frac{(q+1)^2}{(q-1)^2}\ \Phi_N(0),
\end{aligned}$$

and obtain

$$\begin{aligned}
F_N(s) = & \frac{1}{q}\ q^{-s\mu(\mathcal{K})}\ \frac{\mu(\mathcal{K})}{\nu(\mathcal{K})}\ \left(u_i^+ + u_i^-\right) \\
& + q^{-s\mu(\mathcal{K})}\ \frac{\mu(\mathcal{K})}{\nu(\mathcal{K})}\ \frac{q-1}{q}\ \frac{-4q}{(q-1)^2}\ \frac{1}{1+q^{-2s}} \\
& + q^{-s\mu(\mathcal{K})}\ \frac{\mu(\mathcal{K})}{\nu(\mathcal{K})}\ \frac{q-1}{q}\ \frac{(q+1)^2}{(q-1)^2}\ \frac{1}{1+q^{1-2s}}.
\end{aligned}$$

Due to $\alpha_N^- = -1$ (the last line of the equation above), the series for $F_N(s)$ requires $\mathrm{Re}(s) > \frac{1}{2}$ to converge, and we obtain

$$\begin{aligned}
G_{\mathcal{K}}(x,s) = & \frac{\mu(\mathcal{K})}{\nu(\mathcal{K})|G|q^{s\mu(\mathcal{K})+1}}\left(1+\frac{q-1}{1-q^{1-2s}}\right) \\
& + \frac{\mu(\mathcal{K})}{\nu(\mathcal{K})|G|q^{s\mu(\mathcal{K})+1}}\left(1+\frac{q-1}{1+q^{1-2s}}\right)\ +\ \mathbb{F}(x,s)
\end{aligned}$$

for some function $\mathbb{F}(x,s)$ which is analytic for $\mathrm{Re}(s) \geq \frac{1}{2}$.

As before, the residue of $f(s) = G_{\mathcal{K}}(x, \frac{s}{2})$ at $s = 1$ equals A (from equation (4.41)), but now the function $g(s) = f(s) - \frac{A}{s-1}$ only converges to a boundary function for $|\mathrm{Im}(s)| < \frac{\pi}{\ln q} = \tau$. The Tauberian theorem can still be applied, but only shows

$$\begin{aligned}
\frac{1}{q^2}\ \frac{2q}{q+1}\ \frac{\mu(\mathcal{K})}{\nu(\mathcal{K})|G|q^{\mu(\mathcal{K})/2}} \ \leq & \ \liminf_{t\to\infty}\ q^{-t}S(t) \\
\leq & \ \limsup_{t\to\infty}\ q^{-t}S(t)\ \leq\ \frac{2q}{q+1}\ \frac{\mu(\mathcal{K})}{\nu(\mathcal{K})|G|q^{\mu(\mathcal{K})/2}}.
\end{aligned}$$

The difference between this estimate and that in equation (4.42) is that the left and right estimates differ by a factor q^2 instead of the factor q obtained in the non-bipartite case. This means it is no longer possible to deduce a precise limit from the estimates. All we can say here is

$$\frac{1}{q^2}\ \frac{2q}{q+1}\ \frac{\mu(\mathcal{K})}{\nu(\mathcal{K})|G|q^{\mu(\mathcal{K})/2}} \ \leq\ \liminf_{t\to\infty}\ q^{-t/2}\ N_{\mathcal{K}}(x,t)$$
$$\leq\ \limsup_{t\to\infty}\ q^{-t/2}\ N_{\mathcal{K}}(x,t)\ \leq\ \frac{2q}{q+1}\ \frac{\mu(\mathcal{K})}{\nu(\mathcal{K})|G|q^{\mu(\mathcal{K})/2}}.$$

Note that $S(t)$ can jump at any integer value t as $\mu(\mathcal{K})$ is always even for a bipartite graph. This does not, however, help us find a better estimate for $N_{\mathcal{K}}(x,t)$. To solve this we would have to investigate the possibility of proving this theorem in a different way, but this is beyond the scope of this thesis.

Chapter 5

Trace Formulas and Applications

We start this chapter with a short survey of counting problems we have come across in the literature which are relevant to our research. Some of the proofs involve zeta functions, namely the Riemann zeta function and its geometric analogue the Selberg zeta function. Later proofs involve the Selberg Trace Formula, which is connected to the Selberg zeta function, and the discrete analogue of the Selberg Trace Formula, namely Ahumada's Trace Formula. Both these trace formulas will be discussed in later sections (sections 5.3 and 5.2 respectively). In the final two sections of this chapter we will discuss an application of the trace formulas and other related problems.

5.1 Counting Problems

Historically, the Prime Number Theorem is the first counting problem relevant to our research. The setting may be far removed from that of counting paths on a graph, but it turns out there are striking similarities. The Prime Number Theorem states that the number of primes $\Pi(N)$ less than or equal to the positive integer N can be approximated by $N/\log N$, with relative error approaching zero as N goes to infinity. It was first conjectured by A.–M. Legendre in 1796, and also studied by C. F. Gauss, although he did not publish his results. Hadamard [29] and de la Vallée–Poussin [58] subsequently proved the Prime Number Theorem independently, both in 1896. In 1949 Selberg [49] and Erdös [23] both gave proofs for the result using only elementary methods. Improvements were also made on the original method of proof, for example by Newman [43] in 1980. What all but the elementary proofs have in common is that they rely on the Riemann zeta function [48] or, more specifically, they rely on the distribution of its zeros.

A geometric analogue to the Riemann zeta function is the Selberg zeta function, which is connected to the Selberg Trace Formula [50]. It can be used to calculate the number of primitive oriented closed geodesics on a hyperbolic surface. This is sometimes referred to as the Prime Number Theorem for compact Riemann surfaces, taking primitive geodesics to be the 'primes' here. Simply stated, a geodesic is primitive if it is not a power of another. The theorem, which we discuss in section 5.5.2, states that if $\Pi(x)$ is the number of primitive oriented closed geodesics of length at most $\ln x$, then $\Pi(x)$ approaches $x/\ln x$ as $x \to \infty$. It was originally proved by Huber [35], and later by Hejhal [32] using Selberg's work [50]. Hejhal notes that Selberg wrote in [51, p179], without proof, that the zeta function could be used to obtain information about the relevant counting function. Both this Prime Number Theorem and the original Prime Number Theorem are explained very well in a book by Buser [11, Chapter 9], to which we also refer the interested reader.

Finally, the trace formula most relevant to our work is Ahumada's Trace Formula for the regular tree [1], which is the discrete analogue of Selberg's Trace Formula. We shall derive it ourselves in section 5.2 following a method of proof similar to that in chapter 4. See also for example [10], [57] or [59]. Our method is most similar to that of Terras and Wallace [57], the only real difference being the notation. In the same article, the authors use this discrete trace formula to prove a discrete version of the Prime Number Theorem on the regular tree, which we will state in section 5.5.2.

5.2 Derivation of the Ahumada Trace Formula

5.2.1 Geometric Derivation

Let $K : [0, \infty) \to \mathbb{C}$ be a function which is 'sufficiently decreasing' (we will define this condition shortly). We define a function $k : G \times G \to \mathbb{C}$ on the vertices of a finite, simple, regular, connected graph G with universal cover $\widetilde{G}$ as follows

$$k(x,y) = \sum_{\gamma \in \Gamma} K(d(\tilde{x}, \gamma\tilde{y})) \tag{5.1}$$

where $\tilde{x}, \tilde{y}$ are some chosen fixed preimages of x, y under the projection map π. The condition 'sufficiently decreasing' is then defined as the condition that the infinite sum in equation (5.1) must converge.

Proposition 5.1 *The definition of $k(x,y)$ is independent of the choice of preimages $\tilde{x}, \tilde{y}$ of x, y.*

Proof Note that $k(x,y) = k(\eta x, y)$ for all $\eta \in \Gamma$, as we sum over all $\gamma \in \Gamma$ in equation (5.1), and the distance function $d(x,y)$ is invariant under Γ. Similarly, $k(x,y) = k(x, \eta y)$ for all $\eta \in \Gamma$, and we see that in fact $k(x,y) = k(\eta_1 x, \eta_2 y)$ for all $\eta_1, \eta_2 \in \Gamma$. Now as we choose $\tilde{x} \in \pi^{-1}(x) = \{\gamma\tilde{x} : \gamma \in \Gamma\}$ and $\tilde{y} \in \pi^{-1}(y) = \{\gamma\tilde{y} : \gamma \in \Gamma\}$, the result follows. □

Let $\mathfrak{F} = \mathfrak{F}(\Gamma)$ be a fundamental domain of Γ on $\widetilde{G}$. Look at $\sum_{x\in G} k(x,x)$, which we can write as

$$\sum_{x \in G} k(x,x) = \sum_{\gamma \in \Gamma} \sum_{\tilde{x} \in \mathfrak{F}} K(d(\tilde{x}, \gamma\tilde{x})) \tag{5.2}$$

$$= |\mathfrak{F}| \cdot K(0) + \sum_{\gamma \in \Gamma'} \sum_{\tilde{x} \in \mathfrak{F}} K(d(\tilde{x}, \gamma\tilde{x})) \tag{5.3}$$

where Γ' is Γ minus the identity element e. We want to simplify equation (5.3) by writing it as a sum over conjugacy classes in Γ. Note that $\{e\}$ is the trivial conjugacy class, so the sum over elements in Γ' is the sum over all elements in non-trivial conjugacy classes. Every element in Γ' belongs to exactly one equivalence class $[\tau]$ in the set Π of all non-trivial conjugacy classes in Γ, so we can rewrite the sum in equation (5.3) as

$$\sum_{x \in G} k(x,x) = |\mathfrak{F}| \cdot K(0) + \sum_{[\tau] \in \Pi} \sum_{\gamma \in [\tau]} \sum_{\tilde{x} \in \mathfrak{F}} K(d(\tilde{x}, \gamma\tilde{x})). \tag{5.4}$$

In the next section we will simplify the sum in equation (5.4) from an algebraic to a geometric form. To do this we will need the following lemma.

Lemma 5.2 *There is a one-to-one correspondence between elements $[\gamma] \in \Pi$ and oriented closed paths on G without backtracking, which arises from the projection map π.*

Proof Recall from lemma 4.8 that $\mu(\gamma)$ is invariant under conjugation, so $\mu(\gamma') = \mu(\gamma)$ for all $\gamma' \in [\gamma]$ for any $[\gamma] \in \Pi$. It is easy to show, however, that the axis $\mathfrak{a}(\gamma)$ does vary under conjugation, namely by $\mathfrak{a}(\eta^{-1}\gamma\eta) = \eta^{-1}\mathfrak{a}(\gamma)$. The set of axes of all translations in $[\gamma]$ is thus given by $\Gamma\mathfrak{a}(\gamma)$. When this set is projected onto $G = \widetilde{G}/\Gamma$ using the projection map $\pi : \widetilde{G} \to G$ defined in section 1.2, each axis is clearly mapped to the same oriented closed path $\overline{\gamma}$, where the orientation comes from the direction of the translation γ along the axis.

Conversely, suppose we have an oriented closed path $\overline{\gamma}$ in G. We can choose any point $x \in V(G)$ on this path to be its base-point. Once we have chosen $\tilde{x} \in \pi^{-1}(x)$, there is a unique lift of $\overline{\gamma}$ to $\widetilde{G}$. Extending this lift by letting $\overline{\gamma}$ be repeated infinitely many times in each direction gives us a geodesic in $\widetilde{G}$, which is the axis $\mathfrak{a}(\gamma)$ of some translation γ in $\widetilde{G}$ in the direction given by the orientation of $\overline{\gamma}$, and with translation length $\mu(\gamma)$ equal to the length of the closed path $\overline{\gamma}$. The set of translations obtained by choosing all possible $\tilde{x} \in \pi^{-1}(x)$ is $\Gamma\mathfrak{a}(\gamma)$, which is exactly the set of axes of the conjugacy class $[\gamma]$. Note that the choices of the base-point x of $\overline{\gamma}$ and its lift $\tilde{x}$ along the axis $\mathfrak{a}(\gamma)$ do not influence the set of geodesics obtained, so we truly have a one-to-one correspondence between $[\gamma]$ and $\overline{\gamma}$. □

This lemma will allow us to replace the sum over all conjugacy classes in Π with a sum over all oriented closed paths on G without backtracking.

5.2.2 Simplifying the Sum

To simplify the sum in equation (5.4) further, we now find a different way to write $[\tau]$. Let $\Gamma_\tau = \{\eta \in \Gamma : \eta^{-1}\tau\eta = \tau\}$ be the centraliser of τ. Note that we can write Γ as a disjoint union of cosets of Γ_τ, that is,

$$\Gamma = \bigcup_{i=0}^{\infty} \Gamma_\tau \eta_i, \tag{5.5}$$

where $H = \{\eta_i : i = 0, \ldots, \infty\}$ is a set consisting of one fixed representative for each coset $\Gamma_\tau\eta_i$. Clearly the choice of representatives is irrelevant. We use this to prove the following lemma.

Lemma 5.3 *The conjugacy class* $[\tau] = \{\eta^{-1}\tau\eta : \eta \in \Gamma\}$ *can be written as* $\{\eta_i^{-1}\tau\eta_i : \eta_i \in H\}$ *with* H *defined as above. This gives a one-to-one correspondence between elements in* $[\tau]$ *and elements in* $\Gamma_\tau\backslash\Gamma$*, in fact the map* $M : H \to [\tau]$ *defined by* $\eta_i \mapsto \eta_i^{-1}\tau\eta_i$ *is a bijection.*

Proof Suppose we have $\eta_i, \eta_j \in H$ such that $\eta_i^{-1}\tau\eta_i = \eta_j^{-1}\tau\eta_j$. This equation is equivalent to $\eta_j\eta_i^{-1}\tau\eta_i\eta_j^{-1} = \tau$, which is true iff $\eta_i\eta_j^{-1} \in \Gamma_\tau$. Now $\eta_i\eta_j^{-1} \in \Gamma_\tau$ iff $\Gamma_\tau\eta_i\eta_j^{-1} = \Gamma_\tau$, which is equivalent to $\Gamma_\tau\eta_i = \Gamma_\tau\eta_j$, so η_i and η_j are in the same coset. Hence $\eta^{-1}\tau\eta = \eta_i^{-1}\tau\eta_i$ for all $\eta \in \Gamma_\tau\eta_i$, and as we know from equation (5.5) that $\Gamma = \bigcup_{i=0}^{\infty}\Gamma_\tau\eta_i$, this means $[\tau] = \{\eta^{-1}\tau\eta : \eta \in \Gamma\} = \{\eta_i^{-1}\tau\eta_i : \eta_i \in H\}$. The one-to-one correspondence between elements $\eta_i^{-1}\tau\eta_i$ of $[\tau]$ and elements $\Gamma_\tau\eta_i$ of $\Gamma_\tau\backslash\Gamma$ thus occurs via the η_i. The map M is clearly a bijection, as we have shown above that $\eta_i^{-1}\tau\eta_i = \eta_j^{-1}\tau\eta_j$ iff $i = j$. □

Lemma 5.4 *We can rewrite the centraliser of* τ *as* $\Gamma_\tau = \{\tau_0^n : n \in \mathbb{Z}\}$*, where* τ_0 *is the primitive element in* Γ *such that* $\tau = \tau_0^p$ *for some integer* p*.*

Proof Clearly $\tau_0^n \in \Gamma_\tau$ for $n \in \mathbb{Z}$, as $\tau = \tau_0^p$ and powers of τ_0 commute. Conversely, suppose $\eta \in \Gamma_\tau$ is not of the form τ_0^n. By the definition of Γ_τ, we know $\eta^{-1}\tau\eta = \tau$. Let the axis of τ be $\mathfrak{a}(\tau)$, then we know that $\mathfrak{a}(\eta^{-1}\tau\eta) = \eta^{-1}\mathfrak{a}(\tau)$. As $\eta^{-1}\tau\eta = \tau$ we find $\eta^{-1}\mathfrak{a}(\tau) = \mathfrak{a}(\tau)$, so η^{-1} fixes $\mathfrak{a}(\tau)$ as a set and hence $\mathfrak{a}(\eta^{-1}) = \mathfrak{a}(\eta) = \mathfrak{a}(\tau)$. The only way we could now have $\eta \neq \tau_0^n$ is if the displacement lengths $\mu(\eta)$ and $\mu(\tau_0)$ were incompatible, that is, $\mu(\eta) \neq m\mu(\tau_0)$ for any integer m. If this were the case, however, we could produce a word in τ_0 and η with displacement length less than $\mu(\tau_0)$, which contradicts the fact that τ_0 is primitive. Hence we must have $\mu(\eta) = m\mu(\tau_0)$ for some integer $m = \nu(\eta)$, so $\eta = \tau_0^m$ as required. □

This means we can rewrite the sum in equation (5.4) as

$$\sum_{x\in G} k(x,x) = |\mathfrak{F}| \cdot K(0) + \sum_{[\tau]\in\Pi}\sum_{\eta\in H}\sum_{\tilde{x}\in\mathfrak{F}} K(d(\tilde{x}, \eta^{-1}\tau\eta\tilde{x})) \tag{5.6}$$

$$= |\mathfrak{F}| \cdot K(0) + \sum_{[\tau]\in\Pi}\sum_{\eta\in H}\sum_{\tilde{x}\in\mathfrak{F}} K(d(\eta\tilde{x}, \tau\eta\tilde{x})) \tag{5.7}$$

since $d(x, y)$ is invariant under Γ. To simplify this further, we need the following lemma.

Lemma 5.5 *The disjoint union*

$$\mathfrak{F}(\Gamma_\tau) = \bigcup_{\eta \in H} \eta \mathfrak{F}(\Gamma) \tag{5.8}$$

is a fundamental domain of Γ_τ.

Proof Recall from lemma 5.3 that $[\tau] = \{\eta_i^{-1} \tau \eta_i : \eta_i \in H\}$ where $\eta_i^{-1}\tau\eta_i = \eta_j^{-1}\tau\eta_j$ iff $i = j$. Recall also from chapter 4 the two conditions for a fundamental domain, which we restate here. $\mathfrak{F}$ is a fundamental domain for the group Γ, so

- $\mathfrak{F} \cap g\mathfrak{F} = \emptyset \quad \forall g \in \Gamma$ not equal to the identity
- $\bigcup_{g \in \Gamma} g\mathfrak{F} = \widetilde{G}$

We need to show that $\mathfrak{F}(\Gamma_\tau) \cap \gamma\mathfrak{F}(\Gamma_\tau) = \emptyset \;\forall\; \gamma \in \Gamma_\tau$ not equal to the identity, that is,

$$\mathfrak{F}(\Gamma_\tau) \cap \gamma\mathfrak{F}(\Gamma_\tau) = \Big(\bigcup_{\eta \in H} \eta\mathfrak{F}(\Gamma)\Big) \cap \Big(\bigcup_{\eta \in H} \gamma\eta\mathfrak{F}(\Gamma)\Big) = \emptyset.$$

It is sufficient to prove $\eta_i\mathfrak{F}(\Gamma) \cap \gamma\eta_j\mathfrak{F}(\Gamma) = \emptyset$ for $i \neq j$, which is equivalent to

$$\mathfrak{F}(\Gamma) \cap \eta_i^{-1}\gamma\eta_j\mathfrak{F}(\Gamma) = \emptyset.$$

Now as $\mathfrak{F}(\Gamma)$ is a fundamental domain, this would require $\eta_i^{-1}\gamma\eta_j$ to be equal to the identity element, that is, $\gamma\eta_j = \eta_i$ for some $\gamma \in \Gamma_\tau$. This implies $\Gamma_\tau\gamma\eta_j = \Gamma_\tau\eta_i$, and as $\gamma \in \Gamma_\tau$ we find $\Gamma_\tau\gamma = \Gamma_\tau$, so we now have $\Gamma_\tau\eta_j = \Gamma_\tau\eta_i$ which implies $i = j$ by lemma 5.3. This gives a contradiction, hence $\mathfrak{F}(\Gamma_\tau) \cap \gamma\mathfrak{F}(\Gamma_\tau) = \emptyset$ as required.

For the second condition, we find that

$$\bigcup_{\gamma \in \Gamma_\tau} \gamma\mathfrak{F}(\Gamma_\tau) = \bigcup_{\gamma \in \Gamma_\tau} \gamma \bigcup_{\eta \in H} \eta\mathfrak{F}(\Gamma) = \bigcup_{\gamma \in \Gamma_\tau} \bigcup_{\eta \in H} \gamma\eta\mathfrak{F}(\Gamma) = \widetilde{G}.$$

The double union over Γ_τ and H gives us $\gamma\eta \in \Gamma$ by equation (5.5), so the second condition also holds, which completes the proof. $\square$

This means we can rewrite equation (5.7) as follows

$$\sum_{x\in G} k(x,x) = |\mathfrak{F}| \cdot K(0) + \sum_{[\tau]\in\Pi} \sum_{\eta\in H} \sum_{\tilde{y}\in\eta\mathfrak{F}(\Gamma)} K(d(\tilde{y},\tau\tilde{y})) \tag{5.9}$$

$$= |\mathfrak{F}| \cdot K(0) + \sum_{[\tau]\in\Pi} \sum_{\tilde{x}\in\mathfrak{F}(\Gamma_\tau)} K(d(\tilde{x},\tau\tilde{x})) \tag{5.10}$$

where the last sum is over the fundamental domain $\mathfrak{F}(\Gamma_\tau)$ of Γ_τ as defined in equation (5.8). We would now like to be able to take the sum over a fundamental domain of Γ_τ of our choice, but we need to check whether this is possible, that is, we need to check that $K(d(\tilde{x},\tau\tilde{x}))$ is invariant under Γ_τ. Recall that $d(x,y)$ is invariant under Γ, that is

$$d(x,y) = d(\gamma x, \gamma y) \text{ for all } \gamma \in \Gamma. \tag{5.11}$$

Now Γ_τ is a subset of Γ, so (5.11) also holds for all $\gamma \in \Gamma_\tau$, so $K(d(\tilde{x},\tau\tilde{x}))$ is invariant under Γ_τ as required. Hence we can use any fundamental domain of Γ_τ in the sum in equation (5.10).

Recall from lemma 5.4 that we can write $\Gamma_\tau = \{\tau_0^n : n \in \mathbb{Z}\}$, so Γ_τ is generated by the element τ_0. This means there is a fundamental domain of Γ_τ which consists of a segment of $\mathfrak{a}(\tau_0)$ of length $\mu(\tau_0)$, together with all branches emanating from it except those which contain vertices which belong to the axis. Note the similarity to the fundamental domain chosen in section 4.3.3, figure 4.1. We call this fundamental domain $\mathfrak{F}(\tau_0)$, and use this in equation (5.10) to obtain

$$\sum_{x\in G} k(x,x) = |\mathfrak{F}| \cdot K(0) + \sum_{[\tau]\in\Pi} \sum_{x\in\mathfrak{F}(\tau_0)} K(d(x,\tau x)). \tag{5.12}$$

Just like in chapter 4, we can view the fundamental domain $\mathfrak{F}(\tau_0)$ as a union of levels in the quotient graph $\widetilde{G}/\Gamma_\tau$, defined by

$$L_n = \{x \in \widetilde{G}/\Gamma_\tau : \delta'(x,\tau) = n\} \tag{5.13}$$

where $\delta'(x,\tau)$ is the distance from a vertex x to the central circuit in $\widetilde{G}/\Gamma_\tau$. Using this in equation (5.12) we obtain

$$\sum_{x\in G} k(x,x) = |\mathfrak{F}| \cdot K(0) + \sum_{[\tau]\in\Pi} \sum_{j=0}^{\infty} \sum_{x\in L_j} K(d(x,\tau x)) \tag{5.14}$$

$$= |\mathfrak{F}| \cdot K(0) + \sum_{[\tau]\in\Pi} \sum_{j=0}^{\infty} |L_j| \; K(\mu(\tau)+2j) \tag{5.15}$$

where the last equality is due to $d(x, \tau x) = \mu(\tau) + 2\delta(x, \tau) = \mu(\tau) + 2\delta'(x, \tau)$, which is similar to equation (4.4) in chapter 4.

Recall from lemma 5.2 that we have a one-to-one correspondence between non-trivial conjugacy classes $[\tau] \in \Pi$ and oriented closed paths on G without backtracking. In fact, in the proof of the lemma, we showed how to determine the conjugacy class from the path and vice versa. Recall that the displacement length $\mu(\tau)$ of the translation in $[\tau]$ is equal to the length of the geodesic γ obtained on G, which we can denote $\mu(\gamma)$. This means we can write the sum over conjugacy classes in equation (5.15) as a sum over the set $\mathcal{C}$ of all oriented closed non-backtracking paths on G

$$\sum_{x \in G} k(x, x) = |\mathfrak{F}| \cdot K(0) + \sum_{\gamma \in \mathcal{C}} \sum_{j=0}^{\infty} |L_j| \; K(\mu(\gamma) + 2j). \tag{5.16}$$

Note that every oriented closed non-backtracking path γ can be written as a positive power of a primitive oriented closed non-backtracking path γ_0, so $\gamma = \gamma_0^n$ for some positive integer n. Recall that a primitive path γ_0 in G is one that can be written as γ_0^n only when $n = 1$. We use the observation above for the final simplification of equation (5.16). Let $\mathcal{P}$ be the set of all primitive oriented closed non-backtracking paths on G, then

$$\sum_{x \in G} k(x, x) = |\mathfrak{F}| \cdot K(0) + \sum_{\gamma \in \mathcal{P}} \sum_{n=1}^{\infty} \sum_{j=0}^{\infty} |L_j| \; K(n\mu(\gamma) + 2j). \tag{5.17}$$

At the end of section 5.2.3 we will use the equation above to define the discrete version of Selberg's trace formula, first discovered by Ahumada [1]. It can be calculated as the trace of an operator, which we explain in the following section.

5.2.3 The Trace of an Operator

The formula in equation (5.17) can be calculated as the trace of an integral operator $I_k : C(G, \mathbb{C}) \to C(G, \mathbb{C})$ defined by

$$(I_k f)(x) = \sum_{y \in G} k(x, y) f(y) \tag{5.18}$$

where $k(x, y)$ is as defined in equation (5.1). I_k is a linear operator on the space $C(G, \mathbb{C})$ of complex valued functions $f(x)$ on the vertices of the graph G. This space is a vector space with an inner product, defined for $f, g \in C(G, \mathbb{C})$ by

$$(f, g) = \sum_{x \in G} f(x)\overline{g(x)}. \tag{5.19}$$

Definition 5.6 *For a linear operator $A : V \to V$ on a vector space V, the trace of A is defined by*

$$tr(A) = \sum_{i=1}^{n} (Av_i, v_i) \tag{5.20}$$

for some orthonormal basis $\{v_i\}$ of V.

Note that the operator A is represented by a matrix. It is a well-known fact from linear algebra that the trace of a matrix is independent of the basis chosen for the vector space. This means that we can choose any (orthonormal) basis of $C(G, \mathbb{C})$ to calculate the trace of I_k.

Proposition 5.7 *Let δ_x be the characteristic function of a vertex $x \in G$. The set $\{\delta_x : x \in V(G)\}$ forms the standard basis of functions on $V(G)$. Using this basis to calculate the trace of I_k gives*

$$tr(I_k) = \sum_{x \in G} k(x, x). \tag{5.21}$$

Proof Using the definitions of the inner product on $C(G, \mathbb{C})$ in equation (5.19) and of the operator I_k in equation (5.18), and the fact that $\delta_x(y) = 0$ when $x \neq y$, and $\delta_x(x) = 1$ we find

$$\begin{aligned} \mathrm{tr}(I_k) &= \sum_{x \in G} (I_k\delta_x, \delta_x) = \sum_{x \in G} \sum_{y \in G} (I_k\delta_x)(y)\overline{\delta_x(y)} \\ &= \sum_{x \in G} (I_k\delta_x)(x) = \sum_{x \in G} \sum_{y \in G} k(x, y)\delta_x(y) = \sum_{x \in G} k(x, x) \end{aligned} \tag{5.22}$$

as required. □

Definition 5.8 *Let φ_i be an eigenfunction of the Laplacian on G with eigenvalue λ_i, and let the functions $K(z)$ and $k(x, y)$ be defined as in equation (5.1). We then define the* spectral function *associated to $k(x, y)$ as*

$$h(\lambda_j) = \sum_{n=0}^{\infty} |S_n(\tilde{x})| K(n) w_j(n) \tag{5.23}$$

where $S_n(\tilde{x})$ *is the vertex sphere of radius* n *based at* $\tilde{x}$*, and* $w_j(n) = w_j(d(\tilde{x}, \tilde{y}))$ *is the* spherical eigenfunction *on* $\widetilde{G}$ *based at* $\tilde{x}$*, with eigenvalue* λ_j*. This means that* w_j *is invariant under rotation, and* $w_j(0) = 1$ *(see for example [57, p505] or [24, p41]).*

Proposition 5.9 *Let* $\{\varphi_i\}$ *be the orthonormal basis of (normalised) eigenfunctions of the Laplacian on* G*, with corresponding eigenvalues* λ_i*. Calculating the trace of* I_k *using this basis gives*

$$tr(I_k) = \sum_{j=1}^{N} h(\lambda_j) \tag{5.24}$$

where $h(\lambda_j)$ *is the* spectral function *associated to* $k(x, y)$ *defined above in equation (5.23).*

Proof For any function $f : V(G) \to \mathbb{C}$ on the vertices of G, let $\tilde{f}$ be its Γ-periodic lift to the tree $\widetilde{G}$ as defined in 1.3.2. Let x be a vertex in G, and $\tilde{x}$ a lift of x on $\widetilde{G}$. Applying the operator I_k to the function f using the definitions of I_k in equation (5.18) and of $k(x, y)$ in (5.1) gives

$$\begin{aligned}(I_k f)(x) = \sum_{y \in G} k(x, y) f(y) &= \sum_{\tilde{y} \in \mathfrak{F}} \sum_{\gamma \in \Gamma} K(d(\tilde{x}, \gamma\tilde{y}))\tilde{f}(\tilde{y}) \\ &= \sum_{\tilde{y} \in \widetilde{G}} K(d(\tilde{x}, \tilde{y}))\tilde{f}(\tilde{y})\end{aligned} \tag{5.25}$$

where $\mathfrak{F}$ is a fundamental domain of Γ in $\widetilde{G}$ as usual.

Define the radialisation of $\tilde{f}(x)$ with respect to $\tilde{x}$ as

$$\widetilde{f}_x^{\#}(\tilde{y}) = \frac{1}{|S_{d(\tilde{x},\tilde{y})}(\tilde{x})|} \sum_{d(\tilde{x},y)=d(\tilde{x},\tilde{y})} \tilde{f}(y). \tag{5.26}$$

We can use this to rewrite equation (5.25) as follows

$$\begin{aligned}(I_k f)(x) &= \sum_{n=0}^{\infty} \sum_{\tilde{y} \in S_n(\tilde{x})} K(d(\tilde{x}, \tilde{y}))\tilde{f}(\tilde{y}) = \sum_{n=0}^{\infty} K(n) \sum_{\tilde{y} \in S_n(\tilde{x})} \tilde{f}(\tilde{y}) \\ &= \sum_{n=0}^{\infty} K(n) \sum_{y \in S_n(\tilde{x})} \widetilde{f}_x^{\#}(y) = \sum_{n=0}^{\infty} K(n)|S_n(\tilde{x})|\widetilde{f}_x^{\#}(y_n)\end{aligned} \tag{5.27}$$

where $y_n \in S_n(\tilde{x})$. Now we use this equation to apply I_k to an eigenfunction $\varphi_i(x)$ of the Laplacian on G. Note that $\widetilde{\varphi}_{ix}^{\#}(y) = \varphi_i(x) \cdot w_i(y)$, where $w_i(y)$ is the spherical eigenfunction defined in definition 5.8. We obtain

$$\begin{aligned}(I_k\varphi_i)(x) =& \sum_{n=0}^{\infty} K(n)|S_n(\tilde{x})|\widetilde{\varphi}_{ix}^{\#}(y_n) \\ =& \varphi_i(x)\sum_{n=0}^{\infty}|S_n(\tilde{x})|K(n)w_i(n) \\ =& h(\lambda_i)\cdot\varphi_i(x) \end{aligned} \tag{5.28}$$

for the spectral function $h(\lambda_i)$ as defined in equation (5.23). Note that $h(\lambda_i)$ does not depend on x, so $\varphi_i(x)$ is an eigenfunction of the operator I_k with eigenvalue $h(\lambda_i)$. This means it is easy to calculate

$$\mathrm{tr}(I_k) = \sum_{j=1}^{N}(I_k\varphi_j, \varphi_j) = \sum_{j=1}^{N} h(\lambda_j)(\varphi_j, \varphi_j) = \sum_{j=1}^{N} h(\lambda_j) \tag{5.29}$$

as the eigenfunctions $\{\varphi_j\}$ are orthonormal. □

Combining propositions 5.7 and 5.9 we obtain the following identity

$$\sum_{x\in G} k(x,x) = \sum_{j=1}^{N} h(\lambda_j). \tag{5.30}$$

This equation is sometimes called the *pre-trace formula.* Combining it with equation (5.17) gives

$$\sum_{j=1}^{N} h(\lambda_j) = |\mathfrak{F}|\cdot K(0) + \sum_{\gamma\in\mathcal{P}}\sum_{n=1}^{\infty}\sum_{j=0}^{\infty}\ |L_j|\ K(n\mu(\gamma)+2j) \tag{5.31}$$

which is the discrete equivalent of Selberg's trace formula, also called *Ahumada's trace formula.* Equations (5.30) and (5.31) can be found for example in an article by Terras and Wallace [57], where they are derived using spherical and horocycle transforms.

Before we discuss an application of this trace formula, we discuss its analogue in the setting of a Riemannian space, which is the Selberg Trace Formula. Chronologically speaking, the Selberg Trace Formula was the earlier of the two trace formulas discussed in this chapter, in fact it was the inspiration for Ahumada's discrete analogue in [1].

5.3 The Selberg Trace Formula

We now briefly discuss the trace formula defined by Selberg [50] in 1956, and compare it to the discrete analogue defined by Ahumada [1] and derived in the previous section. In the most general case, Selberg works with a Riemannian space S and a locally compact group Γ_0 of isometries on S. An element in Γ_0 is called m, and Γ_0 must act transitively, i.e.

$$\forall\ x, y \in S \text{ there exists } m \in \Gamma_0 \text{ such that } x = my. \tag{5.32}$$

He looks at linear operators on functions on S, such that the operator is invariant under Γ_0, so that it commutes with isometries $m \in \Gamma_0$. One such operator is the integral operator of the form

$$\int_S K(x,y)f(y)dy \tag{5.33}$$

which is invariant under Γ_0 iff the relation

$$K(mx, my) = K(x,y) \tag{5.34}$$

holds for all $x, y \in S$ and $m \in \Gamma_0$. This is referred to as *point-pair invariance*. Note that this corresponds to our function $K(d(\tilde{x}, \tilde{y}))$ in section 5.2.1, which is invariant under all automorphisms of the tree as they preserve the distance function $d(x, y)$.

Now let Γ be a discrete subgroup of Γ_0 which acts properly discontinuously on S. In the special case where S is the hyperbolic plane, this gives a manifold $M = \mathbb{H}/\Gamma$, which corresponds to the graph given in our case by the quotient $G = \widetilde{G}/\Gamma$. The isometries we call translations are called hyperbolic isometries here, and we note that Selberg does not rule out non-hyperbolic isometries in the general case like we did. In the special case of $S = \mathbb{H}$, however, certain restrictions on the fundamental domain $\mathfrak{F}$ of Γ in S rule out the possibility of parabolic isometries, and only allow a finite number of elliptic isometries.

We then define

$$k(x,y) = \sum_{\gamma \in \Gamma} K(x, \gamma y) \tag{5.35}$$

under some assumptions that ensure this function $k(x, y)$ converges and is uniformly bounded. The equivalent equation in the discrete case is given in equation (5.1).

Selberg proceeds to give two ways to calculate the trace of the operator in equation (5.33).

Let $S = \mathbb{H}$, and expand $k(x, y)$ in terms of eigenfunctions $F_i(x)$

$$k(x,y) = \sum_{\gamma \in \Gamma} K(x, \gamma y) = \sum_i h(\lambda_i) F_i(x) \overline{F_i'}(y) \tag{5.36}$$

where $\overline{F_i'}$ is the conjugate transpose of F_i. We then find the trace is equal to the following two expressions

$$\int_{\mathfrak{F}} \operatorname{tr}(k(x,x))dx = \sum_i h(\lambda_i) \tag{5.37}$$

$$= \sum_{\gamma \in \Gamma} \int_{\mathfrak{F}} K(x, \gamma x) dx \tag{5.38}$$

which, when calculated, gives the *Selberg Trace Formula*

$$\begin{aligned} \sum_i h(\lambda_i) =& \frac{A(\mathfrak{F})\nu}{2\pi} \int_{-\infty}^{\infty} r \tanh(\pi r) h(r) dr \\ &+ 2 \sum_{\{P\}_\Gamma} \sum_{k=1}^{\infty} \frac{\log N\{P\}}{N\{P\}^{k/2} - (N\{P\})^{-k/2}} g(k \log N\{P\}) \end{aligned} \tag{5.39}$$

with the following definitions for constants and functions used. $A(\mathfrak{F})$ is the area of the fundamental domain $\mathfrak{F}$, and ν is the dimension of S. $\{P\}_\Gamma$ is the set of conjugacy classes of primitive hyperbolic elements in Γ, and $N\{P\}$ is the *norm* of the element P, that is, $N\{P\} = e^{\mu(P)}$ where $\mu(P)$ is the displacement length. Define the function

$$K(t) = K\Big(d(z, z')\Big) = K\Big(\frac{|z - z'|^2}{yy'}\Big) \text{ where } z = x + iy \text{ for real } x, y$$

and use this to define an integral

$$Q(w) = \int_w^{\infty} \frac{K(t)}{\sqrt{t - w}} dt.$$

This in turn helps us define the function

$$g(u) = Q\Big(2(\cosh u - 1)\Big) = \int_{2(\cosh u - 1)}^{\infty} \frac{K(t)}{\sqrt{t - 2(\cosh u - 1)}} dt$$

which is used in equation (5.39). Finally, we define a function

$$h(r) = \int_{-\infty}^{\infty} e^{iru} g(u) du = \int_{-\infty}^{\infty} e^{iru} \int_{2(\cosh u - 1)}^{\infty} \frac{K(t)}{\sqrt{t - 2(\cosh u - 1)}} dt du.$$

This completes our short description of the Selberg Trace Formula. For details of its derivation and application we refer the interested reader to Selberg's article [50], or alternatively [32] or [42]. There are many more references available on this subject, but these are too many to mention.

5.4 An Application

We now show how the trace formula we derived in equation (5.31) can be applied to a counting problem on a regular graph G. We follow methods similar to Brooks [10] to obtain a result also stated in Terras and Wallace [57].

Define $N(\ell)$ to be the number of closed paths of length ℓ on G without back-tracking but possibly having 'tails'. Here, two paths consisting of the same sequence of vertices are counted as 'different' if they have a different starting point. A path has a *tail* if, when its vertices are given by $v_0, v_1, \ldots, v_{n-1}, v_n = v_0$, we have $v_i = v_{n-i}$ for $i = 1, \ldots, k$ for some $0 < k < n/2$. Recall that non-backtracking means that $v_i \neq v_{i+2}$ for all $i = 0, \ldots, n-2$. Define

$$K_\ell(n) = \begin{cases} 1 & \text{if } n = \ell \\ 0 & \text{otherwise} \end{cases} \tag{5.40}$$

for $\ell \geq 0$. Using equation (5.1), this gives us a function

$$k_\ell(x, y) = \sum_{\gamma \in \Gamma} K_\ell\big(d(\tilde{x}, \gamma\tilde{y})\big). \tag{5.41}$$

Observe then that $N(\ell) = \sum_{x \in G} k_\ell(x, x)$.

We now use the pre-trace formula (5.30). For the functions $K_\ell(n)$ and $k_\ell(x, y)$ defined above, recalling equation (5.23), we find for $\ell > 0$

$$\begin{aligned}
\sum_{x \in G} k_\ell(x, x) = \sum_{j=1}^{N} h(\lambda_j) &= \sum_{j=1}^{N} \sum_{n=1}^{\infty} |S_n(\tilde{x})| K_\ell(n) w_j(n) \\
&= \sum_{j=1}^{N} |S_\ell(\tilde{x})| w_j(\ell) \qquad (5.42) \\
&= (q+1) q^{\ell-1} \sum_{j=1}^{N} w_j(\ell) \qquad (5.43)
\end{aligned}$$

using $|S_\ell(\tilde{x})| = (q+1)q^{\ell-1}$ for $\ell > 0$. For $\ell = 0$, we use $|S_0(\tilde{x})| = 1$ in equation (5.42) to find

$$\sum_{x \in G} k_0(x,x) = \sum_{j=1}^{N} w_j(0) = N. \tag{5.44}$$

For $\ell > 0$ we can now use a nice property of the $w_j(n)$ to obtain an approximation for $N(\ell)$ as $\ell \to \infty$. Using an argument identical to that used in the proof of theorem 2.4 on page 25, we obtain a solution for $w_j(n)$ in terms of

$$\alpha_j^{\pm} = \frac{q+1}{2q}\lambda_i \pm \frac{1}{2q}\sqrt{(q+1)^2{\lambda_i}^2 - 4q}$$

namely $w_j(n) = u_j^+(\alpha_j^+)^n + u_j^-(\alpha_j^-)^n$. Here $w_j(0) = 1$ and $w_j(1) = \mathcal{L}_G w_j(0) = \lambda_j w_j(0) = \lambda_j$ give us the constants $u_j^{\pm}$. Hence $|\alpha_j^{\pm}| < 1$ gives us $w_j(n) \to 0$ as $n \to \infty$ for w_j not equal to the constant function $w_0 = 1$ (with eigenvalue $\lambda_0 = 1$) or, for bipartite G, $w_N = \pm 1$. This means for non-bipartite G that, as $\ell \to \infty$, $\sum_{j=1}^{N} w_j(\ell) \to w_0(\ell) = 1$, and combining this with equation (5.43) we obtain

$$N(\ell) = \sum_{x \in G} k_\ell(x,x) \to \frac{q+1}{q}\, q^\ell \qquad \text{as } \ell \to \infty. \tag{5.45}$$

For bipartite G, we use

$$w_N(\ell) = \begin{cases} 1 & \text{when } \ell \text{ is even} \\ -1 & \text{when } \ell \text{ is odd} \end{cases}$$

to obtain

$$N(\ell) \to \frac{2(q+1)}{q} q^\ell \qquad \text{for even } \ell, \text{ as } \ell \to \infty,$$

noticing that as all closed non-backtracking paths have even length, $N(\ell) = 0$ for all odd ℓ.

Recall that in $N(\ell)$ we defined paths as different if they had different starting points. Notice, however, that two paths (without tails) can be identical apart from their starting points, and that we count such paths ℓ times in $N(\ell)$. To avoid this, we will solve a similar counting problem in the next section, where we count the number of paths with a fixed starting point. This follows from something we call the *full lattice point problem.*

5.5 Related Problems

5.5.1 The Full Lattice Point Problem

In this section, we discuss some problems related to the counting problem in section 5.4, for graphs as well as surfaces. We start with the full lattice point problem, and note that its proof does not require the trace formula in either setting. In the case of surfaces, Patterson [46] proved the following result (see also [11, Chapter 9]). Let Γ be a discrete group of orientation preserving isometries on $\mathbb{H}$ so that the quotient $\mathbb{H}/\Gamma = M$ is a compact Riemann surface. For any two points $z, w \in \mathbb{H}$, define a counting function

$$N_\Gamma(t; z, w) = \#\{T \in \Gamma : \mathrm{dist}(z, Tw) \leq t\}, \tag{5.46}$$

where $\mathrm{dist}(x, y)$ is the hyperbolic distance defined in the usual way. Then as $t \to \infty$,

$$N_\Gamma(t; z, w) \sim \frac{\pi}{\mathrm{Area}(M)}\ e^t = \frac{1}{4(g-1)}\ e^t, \tag{5.47}$$

where g is the genus of the surface M.

The analogous result for graphs is easily obtained from corollary 2.14. Let Γ be a group of translations on the regular tree $\widetilde{G}$ so that the graph $G = \widetilde{G}/\Gamma$ is finite, and define a counting function

$$N_\Gamma(t; z, w) = \#\{T \in \Gamma : d(z, Tw) \leq t\}. \tag{5.48}$$

Note that we can interpret this as

$$\#\{T \in \Gamma : d(z, Tw) \leq t\} = \sum_{y \in B_t(z)} \tilde{\delta}_w(y) \tag{5.49}$$

where $\tilde{\delta}_w(y)$ is the following function. Let $\delta_x(y)$ be the characteristic function of the vertex x in G, and choose x to be the image of w under the projection map π. Then $\tilde{\delta}_w(y)$ is the lift to the tree of the function $\delta_{\pi(w)}(y)$. Using this function in the corollary gives

$$\left| \frac{1}{|B_t(z)|} N_\Gamma(t; z, w) - \frac{1}{|V|} \right| \leq C'_G \beta^t \tag{5.50}$$

which we rewrite in a similar fashion to that in the proof of corollary 2.14 as

$$N_\Gamma(t; z, w) = \frac{|B_t(z)|}{|V|} + |B_t(z)| C'_G \beta^t \varepsilon_t \tag{5.51}$$

for some $|\varepsilon_t| \leq 1$. Recall

$$|B_t(z)| = \sum_{r=0}^{t} |S_r(z)| = 1 + \sum_{r=1}^{t} (q+1)q^{r-1} = 1 + \frac{q+1}{q-1}(q^t - 1) \tag{5.52}$$

using the geometric series formula in the last step. We obtain

$$N_\Gamma(t; z, w) = \frac{q+1}{q-1}\frac{1}{|V|}q^t + \frac{q+1}{q-1}C'_G \varepsilon_t (q\beta)^t + \text{constant.} \tag{5.53}$$

Now $|\beta| < 1$ so as $t \to \infty$ we can approximate $N_\Gamma(t; z, w)$ by the leading term, that is

$$N_\Gamma(t; z, w) \sim \frac{q+1}{q-1}\frac{1}{|V|}q^t. \tag{5.54}$$

Note that in theorem 4.10 we counted a subset of the points counted in the problem above, as we only counted vertices that are images of the base-point under elements in a chosen conjugacy class in Γ. Recall that this problem was also not solved using the trace formula, nor is its counterpart on a surface (see [34]).

Going back to the full lattice point problem on a graph, we see that $N_\Gamma(t; z, z)$ gives us some information about the number of closed paths on G of length t based at z. More precisely, it is the number of closed non-backtracking paths on G of length t based at z possibly having tails. This is still not the most natural thing to count, as we would prefer not to specify a base point. In the next section we will discuss the Prime Number Theorem, which counts the number of *primitive* oriented closed geodesics on a surface or, in the discrete case, on a tree. Here we do not specify a base-point, and we do not allow backtracking or tails in the discrete case.

5.5.2 Prime Number Theorem

As we already stated in section 5.1, the Prime Number Theorem for compact Riemann surfaces counts the number of primitive oriented closed geodesics on a surface. Following the notation used by Buser [11, Chapter 9], let $\mathscr{C}(M)$ be the set of all oriented closed geodesics on a compact Riemann surface M (defined by $M = \mathbb{H}/\Gamma$), and let $\mathscr{P}(M)$ be the set of all primitive oriented closed geodesics. Recall that a geodesic is primitive if it is not the m-fold iterate, with $m \geq 2$, of another closed geodesic on M. Let $\ell(\gamma)$ be the length of the geodesic γ on M. Define counting functions as follows

$$\Phi(t) = \#\{\gamma \in \mathscr{C}(M) : \ell(\gamma) \leq t\} \tag{5.55}$$

$$\Pi(t) = \#\{\gamma \in \mathscr{P}(M) : \ell(\gamma) \leq t\}. \tag{5.56}$$

Now define the norm $N(\gamma)$ of the geodesic by $\ell(\gamma) = \ln N(\gamma)$ and define counting functions

$$\phi(x) = \#\{\gamma \in \mathscr{C}(M) : N(\gamma) \leq x\} \tag{5.57}$$

$$\pi(x) = \#\{\gamma \in \mathscr{P}(M) : N(\gamma) \leq x\}. \tag{5.58}$$

The theorem then reads as follows.

Theorem 5.10 (PNT for Surfaces) *Defining $\pi(x)$ and $\phi(x)$ as above, we have for any compact Riemann surface of genus $g \geq 2$*

$$\pi(x) \sim x/\log x$$

$$\phi(x) \sim x/\log x$$

as $x \to \infty$.

For a proof, see for example [11, chapter 9] or [32].

The corresponding result for graphs can be obtained using the Ihara zeta function. This function, and its application to an analogue of the Prime Number Theorem on graphs, has been discussed in various papers, and we should mention [1], [5], [30], [36], [53], [54], [56] and [59] in this context. We found the account by Terras and Wallace [57] useful, as it fits in with earlier work in this chapter. The graph analogue of the Prime Number Theorem is the statement that for a finite non-bipartite regular graph, defining

$$\pi_G(n) = \#\{\text{primitive closed non-backtracking paths without tails of length} = n\}$$

we have

$$\pi_G(n) \sim \frac{q^n}{n} \qquad \text{as } n \to \infty. \tag{5.59}$$

For bipartite graphs, we find for even n (see for example [55, p 71])

$$\pi_G(n) \sim 2\frac{q^n}{n} \qquad \text{as } n \to \infty.$$

Note that in a bipartite graph there are no closed paths of odd length, by proposition 1.7.

5.5.3 Homological Constraints

One final problem we should mention related to that of counting paths on graphs or surfaces deals with homological constraints. That is to say, rather than counting all paths, we just count those in a certain homology class. For a formal definition of homology, see for example Hatcher [31]. Before we state the relevant results, let us briefly review the concept of homology classes on graphs. This is best done in terms of an example.

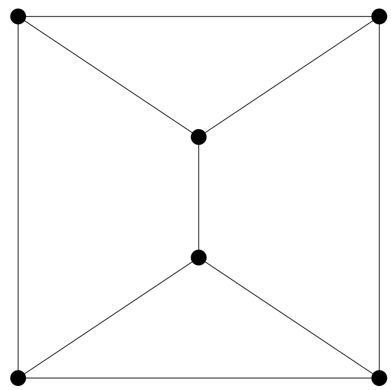

Figure 5.1: The graph G.

Let G be the graph in figure 5.1, and suppose we have some closed, non-backtracking path γ on G for which we want to determine the homology class. Find a spanning tree of G (see figure 5.2), and give arbitrary directions to each of the edges not in this spanning tree. Now collapse the spanning tree to a single vertex using a deformation retract (also called a 'contraction' in graph theory. See for example [8, p24] or [19, p18]). We are left with a set of loops at a single vertex, which is called a bouquet. The original path γ now has a corresponding path γ' on the bouquet. This path γ' will traverse the various loops in a certain direction and in a certain order. Now count the number of times each loop is traversed, counting $+1$ if the direction is the same and -1 if the direction is opposite to the given direction of the edge. We obtain four integers corresponding to the four loops $c_1, \ldots, c_4$. Now suppose we have some other closed non-backtracking path γ_1 on G, and we do the same for this path using the same spanning tree and directions chosen earlier. Then γ and γ_1 are in the same homology class if we obtain the same four integers corresponding to $c_1, \ldots, c_4$ for both graphs.

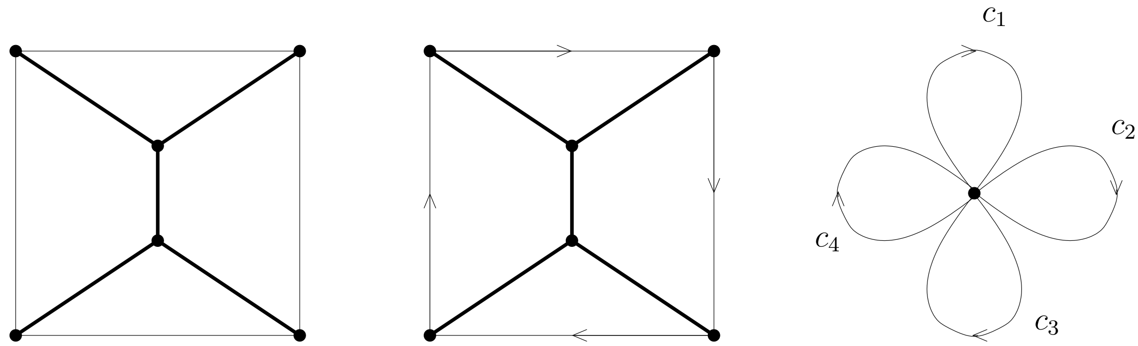

Figure 5.2: Construction of the bouquet from G.

It turns out that the homology classes obtained are independent of the spanning tree chosen, however the reasons for this are beyond the scope of this thesis. We refer the interested reader to Hatcher's book [31] for more details on this subject.

For graphs we then define the following counting function

$$\pi(n,[g]) = \{p : |p| = n, [p] = [g]\},$$

where $|p|$ is the length of the path p, and $[g]$ denotes the homology class of the path g on the $(q+1)$-regular graph G. Sunada (see [55]) found that as $n \to \infty$ for non-bipartite graphs

$$\pi(n,[g]) \sim \kappa(G)^{1/2}\Big(\frac{|V|(q-1)}{4\pi}\Big)^{b_1/2}\frac{q^n}{n^{b_1/2+1}},$$

where b_1 is the first Betti number of G (see [31, p 130]) and $\kappa(G)$ is the tree number, which is defined as the number of spanning trees of G. For non-bipartite graphs, he finds for even n that as $n \to \infty$

$$\pi(n,[g]) \sim 2\kappa(G)^{1/2}\Big(\frac{|V|(q-1)}{4\pi}\Big)^{b_1/2}\frac{q^n}{n^{b_1/2+1}}.$$

Some earlier results can also be found in [47, p411]

In the case of a compact Riemann surface M of genus G, Katsuda and Sunada [38] defined $\pi(x,\alpha)$ to be the number of prime closed geodesics in a homology class $\alpha \in H_1(M,\mathbb{Z})$ with length less than x, and found that when $x \to \infty$,

$$\pi(x,\alpha) \sim (g-1)^g \frac{e^x}{x^{g+1}}.$$

For more general results and further references, see also chapter 11 in Sharp's contribution to [41].

Chapter 6

Radial Averages for Non-Regular Graphs

6.1 Setting

In this final chapter we return to the problem discussed in chapters 2 and 3 regarding radial averages of functions on graphs. The graphs considered in these chapters were either regular or semi-regular, so we will discuss here what we know about the more general case.

Let G be a simple, connecte, finite, non-bipartite graph, and $\widetilde{G}$ its universal cover, which is an infinite tree. Recall the definitions for vertex and edge spheres (equation (2.1) and definition 3.6) and vertex and edge arcs (equation (2.2) and definition 3.7) from chapters 2 and 3. Note that all these definitions still hold if the tree is non-regular.

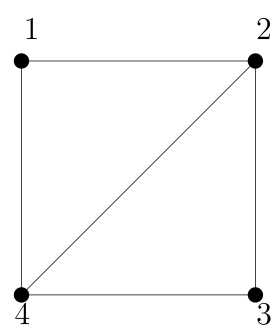

Figure 6.1: The graph H.

Throughout this chapter we will use a simple example to illustrate our definitions and claims. For ease of computation, we choose a (non-regular and non-semi-regular) graph H with a small number of edges, illustrated in figure 6.1. We will see in section 6.3 why it is the number of edges that determines the complexity of our calculations. See also figure 6.2 for an illustration of spheres on the universal cover of H.

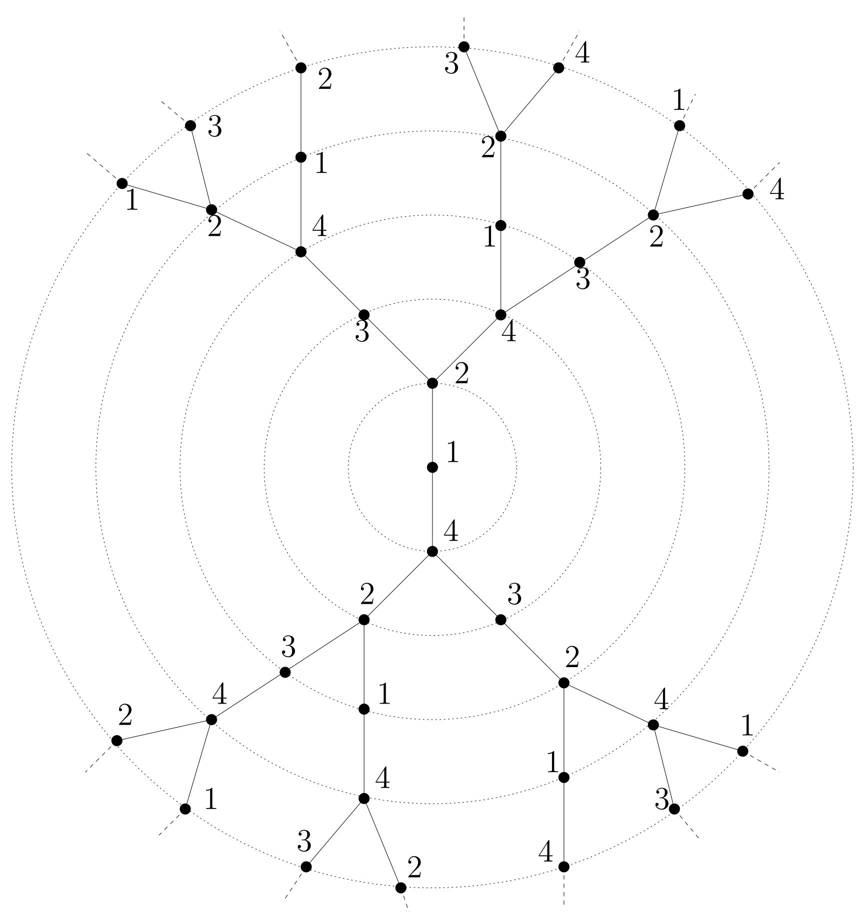

Figure 6.2: Vertex spheres on the universal cover of H.

6.2 Comparison with NBRW

Before we go into more detail, let us compare the radial average setting to that of non-backtracking random walks (NBRW) as introduced in section 2.3. Recall that authors working with NBRW are concerned with the probability distribution of the possible endpoints of a non-backtracking path of length n starting at some given vertex. We used the characteristic function δ_x to find out how many times a particular vertex showed up in a sphere of radius n as a proportion of the total number of vertices in the sphere. In the regular case this gave the same answer as for NBRW (see for example [2] and [45]).

Now in the non-regular case, we find that NBRW probabilities and the distribution of possible endpoints in the sphere are different. Take for example our graph H and compare the distribution of vertices in the sphere of radius 5 with the corresponding NBRW probabilities (see figure 6.3).

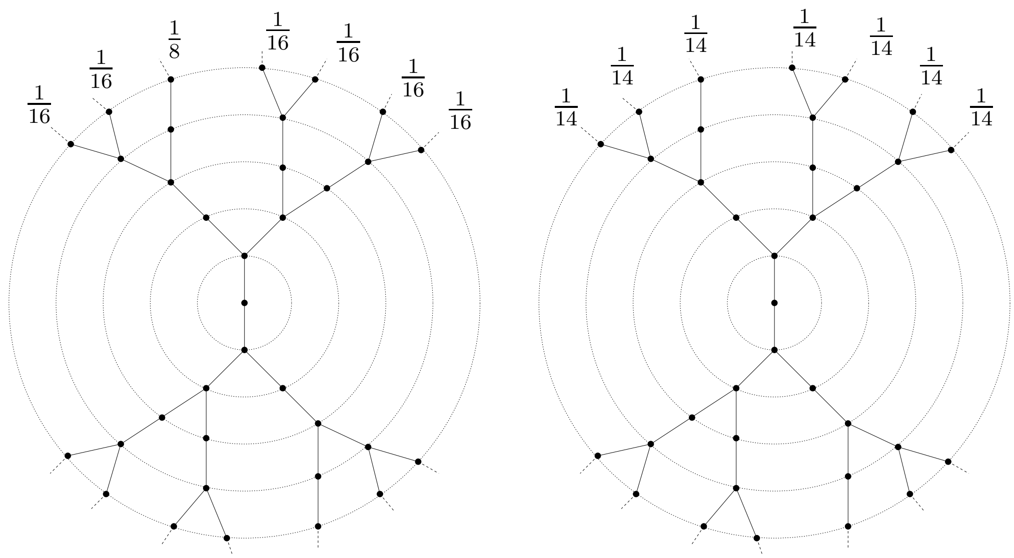

Figure 6.3: NBRW probabilities and weights for the spherical average on H.

In the NBRW case, the probability given by each vertex v depends on how many times we have had to make a choice of direction along the path from 1 to v. For spherical averages, however, each endpoint is given the same weight when taking the average. This leads to different answers for the two problems, so we see in the case of a non-regular graph that radial averages and NBRW are truly different concepts.

6.3 The Oriented Edge Adjacency Matrix

To investigate the arc average of functions defined on the vertices or edges of non-regular graphs, we have to define a new matrix A_O associated to the graph G, which we call the *oriented edge adjacency matrix*. Let $\overline{E}$ be the set of oriented edges of G, where for each unoriented edge $e \in E$ we have two oriented edges $\overrightarrow{e}, \overrightarrow{e}' \in \overline{E}$

with opposite directions. Labelling these edges $\overrightarrow{e_1}, \ldots \overrightarrow{e_m}$, A_O is defined as an $\overline{E} \times \overline{E}$ matrix with entries $a_{i,j}$ given by the following rule:

- $a_{i,j} = 1$ if $\overrightarrow{e_j}$ followed by $\overrightarrow{e_i}$ forms a non-backtracking path
- $a_{i,j} = 0$ otherwise.

Note that two oriented edges may form a backtracking path, but we do not count these. As an example we give A_O for our graph H,

$$A_O = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

where the rows and columns are both labelled by the oriented edges in the following order:

$$\overrightarrow{\{1,2\}}, \overrightarrow{\{2,1\}}, \overrightarrow{\{1,4\}}, \overrightarrow{\{4,1\}}, \overrightarrow{\{2,3\}}, \overrightarrow{\{3,2\}}, \overrightarrow{\{2,4\}}, \overrightarrow{\{4,2\}}, \overrightarrow{\{3,4\}}, \overrightarrow{\{4,3\}}.$$

Note that A_O is not symmetric, unlike the vertex and edge adjacency matrices in definitions 1.8 and 3.8 respectively. This means its eigenvalues and eigenvectors do not always have the nice properties we exploited in the proofs of theorems 2.4 and 3.9. We do know, however, that A_O is *non-negative*, as it has only two possible entries (0 and 1). If the graph G is non-bipartite, we also know that A_O is primitive, as G is connected. A matrix M is *primitive* if there is some integer N such that M^N has all entries greater than zero. This means we can apply the Perron-Frobenius Theorem to A_O, which reads as follows (see for example [9, p 197-198] and [52, p 3-4]):

Theorem 6.1 (Perron-Frobenius) *Let A be a non-negative primitive $r \times r$ matrix. Then there exists a real eigenvalue λ_1 with algebraic and geometric multiplicity 1 such that $\lambda_1 > 0$, and $\lambda_1 > |\lambda_j|$ for all other eigenvalues λ_j of A. Also, the left eigenvector u_1 and the right eigenvector v_1 can be chosen such that $u_1^T v_1 = 1$. We can choose u_1 and v_1 to be strictly positive.*

Let $\lambda_2, \ldots, \lambda_r$ be the other eigenvalues of A ordered such that

$$\lambda_1 > |\lambda_2| \geq \ldots \geq |\lambda_r|$$

and if $|\lambda_2| = |\lambda_j|$ for some $j \geq 3$, then $m_2 \geq m_j$ for all λ_j with $|\lambda_j| = |\lambda_2|$, where m_j is the algebraic multiplicity of λ_j. Then

$$A^n = \lambda_1^n v_1 u_1^T + \Theta(n^{m_2-1}|\lambda_2|^n)$$

where $\Theta(f(n))$ represents a function of n such that there exist $\alpha, \beta \in \mathbb{R}$, $0 < \alpha \leq \beta < \infty$, such that $\alpha f(n) \leq \Theta(f(n)) \leq \beta f(n)$ for sufficiently large n.

Recall that a right eigenvector v of the matrix A with eigenvalue λ satisfies $Av = \lambda v$, whereas a left eigenvector u with eigenvalue λ satisfies $uA = \lambda u$. We will see in the next section how this theorem helps us deduce the proportion of each vertex (or indeed edge) in a radial average.

6.4 Results for Non-Regular Graphs

We can deduce the number of paths of length $n+1$ starting with oriented edge e_j and ending with e_i from the entries of A_O^n: it is just the ij entry of this matrix. Now rather than the actual number of paths appearing, we would like to know the proportion of paths in a certain arc ending at a specific directed edge in comparison to the total number of paths in this arc, so that we can find the radial average over this arc of a given function on the edges of G. To do this we have to normalise the entries of A_O^n, but without a simple way to calculate the total number of vertices or edges in a radial set this is not as easy as in the regular case.

Recall the definition of an edge arc $A'_n(\overrightarrow{e})$ in 3.7.

Definition 6.2 *Let e_i be an undirected edge in $\widetilde{G}$. The* proportion *with which the lifts of this edge occur in the edge arc $A'_n(e)$ is denoted $p_n(e_i)$. Hence the arc average at radius n of a function f on the edges of G is given by*

$$M_{n,e}(f) = \frac{1}{|A'_n(e)|} \sum_{e_i \in A'_n(e)} \widetilde{f}(e_i) = \sum_{e_i \in G} p_n(e_i) f(e_i). \tag{6.1}$$

Similarly, we can define the proportion with which a directed edge occurs in $\widetilde{G}$. Note that given an undirected edge e_i, and the same edge with each of the two given orientations $\overrightarrow{e_i}$, $\overrightarrow{e_i}'$, we have

$$p_n(e_i) = p_n(\overrightarrow{e_i}) + p_n(\overrightarrow{e_i}'). \tag{6.2}$$

To find the proportion with which a vertex occurs in a vertex arc of radius n we note that $|A'_n(e)| = |A_{n+1}(e)|$ and

$$p_{n+1}(v) = \sum_{e_j \in \overline{E} \,:\, t(e_j) = v} p_n(e_j), \tag{6.3}$$

where $t(e_j) = v$ means v is the terminus of e_j, that is, $e_j = \overrightarrow{\{w, v\}}$ for some vertex w.

Theorem 6.3 *Label the directed edges of a graph G as $\overline{E} = \{e_1, e_2, \ldots, e_m\}$, and define the vector v_1 as in theorem 6.1. Then as $n \to \infty$ we find that the proportions p_n with which the lifts of each edge occur in $A'_n(e)$ satisfy*

$$\begin{pmatrix} p_n(e_1) \\ p_n(e_2) \\ \vdots \\ p_n(e_m) \end{pmatrix} \to \frac{1}{\sum_{i=1}^m v_{1,i}} \cdot v_1 \tag{6.4}$$

where $v_{1,i}$ is the i^{th} entry of the vector v_1. This result is independent of the directed edge e determining the arc defining p_n.

Proof We noted that A_O is primitive and non-negative for any simple, finite, non-bipartite graph, so we can apply theorem 6.1 to it. Dividing by λ_1^n, we find that

$$\frac{A_O^n}{\lambda_1^n} \to v_1 u_1^T \tag{6.5}$$

when $n \to \infty$. To see in what proportion the endpoints of paths starting with the directed edge e_i occur, we now need to calculate the following. Let w_i be the i^{th} standard basis vector of an $\overline{E} \times \overline{E}$ matrix, which has a 1 in the i^{th} entry and zeros everywhere else. Then $p_n(e_i)$ is proportional to the vector $A_O^n w_i$. When n is large, we can use equation (6.5) to approximate the distribution, as it implies

$$\frac{A_O^n}{\lambda_1^n} w_i \to v_1 u_1^T w_i = u_{1,i} v_1$$

where $u_{1,i}$ is the i^{th} entry of u_1 obtained from $u_1^T \cdot w_i$. Note that u_1 is strictly positive, so $u_{1,i}$ is never zero, hence this is true for any choice of edge e determining the arc. To obtain $\{p_n(e_i)\}_{i=1}^m$ we now divide $u_{1,i} v_1$ by the sum of its entries, and obtain equation (6.4) above. Note that this is now independent of the choice of e_i, as we divide by $u_{1,i}$ in the last step. □

In the example H given earlier, we have largest eigenvalue

$$\lambda = \frac{1}{3}(27 + 3\sqrt{78})^{1/3} + \frac{1}{(27 + 3\sqrt{78})^{1/3}}$$

and corresponding left eigenvector

$$u = \frac{1}{8\lambda + 2(\lambda^2 - 1)^2}[\lambda,\ 1,\ \lambda,\ 1,\ 1,\ \lambda,\ \lambda^2 - 1,\ \lambda^2 - 1,\ \lambda,\ 1]^T$$

and right eigenvector

$$v = [1,\ \lambda,\ 1,\ \lambda,\ \lambda,\ 1,\ \lambda^2 - 1,\ \lambda^2 - 1,\ 1,\ \lambda]^T \tag{6.6}$$

where we have normalised the first vector to ensure $u^T v = 1$. The proportions p_n of directed edges then approximate $\frac{1}{2(\lambda+1)^2} \cdot v$ as $n \to \infty$. Notice that only three different terms show up, namely 1, λ and $\lambda^2 - 1$. Comparing the weights with the edges of the graph, we find that the weight 1 holds for edges from a vertex of degree 3 to one of degree 2, weight λ holds for the inverses of such edges, and weight $\lambda^2 - 1$ holds for edges joining two vertices of degree 3. This is most likely due to the symmetry in H, and does not necessarily indicate that the proportions of edges in an arc depend only on the degrees of the two vertices determining the edge.

Now to obtain arc averages for functions on the vertices or edges of a graph G, we use equation (6.1). In the case of vertex functions, we find

$$M_{n,e}(f) \to \sum_{v \in V(G)} p_n(v) f(v) \tag{6.7}$$

as $n \to \infty$, using equation (6.3) to find $p_n(v)$. In the case of edge functions, we use equation (6.2) to obtain

$$M_{n,e}(f) \to \sum_{e \in E(G)} p_n(e) f(e) \tag{6.8}$$

as $n \to \infty$.

6.4.1 Corollaries

As in the regular case, we now also state results for averages over spheres and tubes. Recall that the sphere is just a special case of a tube, so we just state the latter here.

Corollary 6.4 *With the same assumptions as in theorem 6.3, we find the proportions $P_n(e_i)$ with which the lifts of the directed edges e_i occur in any tube $\mathcal{T}_n$ of radius n satisfy*

$$\begin{pmatrix} P_n(e_1) \\ P_n(e_2) \\ \vdots \\ P_n(e_m) \end{pmatrix} \to \frac{1}{\sum_{i=1}^{m} v_{1,i}} \cdot v_1 \tag{6.9}$$

as $n \to \infty$.

Proof The proof follows that of theorem 6.3, except that we use more than one w_i. Recall that a tube is made up of several arcs. Let z be the vector whose i^{th} entry z_i equals the number of times $A_n(\widetilde{e_i})$ appears in the tube, so z is the entry wise sum of the relevant w_i, possibly with some repetitions. We find

$$\frac{A_O^n}{\lambda_1^n} z \to v_1 u_1^T z = \Big(\sum_{i=1}^{m} z_i u_{1,i} \Big) v_1$$

where the sum is a positive constant depending only on the choice of tube. Now to find the proportions $P_n(e_i)$, we normalise the right hand side of this equation, which means we divide by the constant, so the choice of tube is irrelevant. □

Using $P_{n+1}(v) = \sum_{e_j \in \overline{E}\,:\,t(e_j)=v} P_n(e_j)$ we find

$$\frac{1}{|\mathcal{T}_n|} \sum_{v \in \mathcal{T}_n} \widetilde{f}(v) \to \sum_{v \in V(G)} P_n(v) f(v) \tag{6.10}$$

as $n \to \infty$, and using $P_n(e) = P_n(\overrightarrow{e}) + P_n(\overrightarrow{e}')$ we find

$$\frac{1}{|\mathcal{T}'_n|} \sum_{e \in \mathcal{T}'_n} \widetilde{f}(e) \to \sum_{e \in V(G)} P_n(e) f(e) \tag{6.11}$$

as $n \to \infty$.

6.5 Conclusion

We finish this chapter with some short comments on theorem 6.3, in particular regarding its relation to the results in chapters 2 and 3. Clearly theorem 6.3 also holds for regular graphs, where in addition we know that all entries in v_1 are the same. This means that in the limit all directed edges appear the same number of times in A'_n, which is in line with our findings in earlier chapters. Theorem 6.3 does not give us such a nice convergence rate, nor does it say anything about bipartite graphs. It does, however, rely on external theory, namely the Perron-Frobenius Theorem, which our proof of the regular cases in chapters 2 and 3 did not. The non-regular equivalent of these results would require further research, and is a problem worth investigating in the future. It is, however, outside the scope of this thesis.

Bibliography

[1] G. Ahumada: *Fonctions priodiques et formule des traces de Selberg sur les arbres*, C. R. Acad. Sci. Paris Sr. I Math. 305 no. 16, (1987), 709–712

[2] N. Alon, I. Benjamin, E. Lubetzky, S. Sodin: *Non-Backtracking random walks mix faster*, Commun. Contemp. Math. **9** no. 4, (2007), 585–603

[3] O. Angel, J. Friedman, S. Hoory: *The Non-Backtracking Spectrum of the Universal cover of a Graph*, http://arxiv.org/abs/0712.0192v1, (2007)

[4] L. Bartholdi: *Counting Paths in Graphs*, Enseign. Math. (2) **45** no. 1-2, (1999), 83–131

[5] H. Bass: *The Ihara-Selberg zeta function of a tree lattice*, Internat. J. Math. 3 no. 6, (1992), 717–797

[6] L. W. Beineke, R. J. Wilson: *Topics in Algebraic Graph Theory*, Cambridge University Press, Cambridge, (2004)

[7] M.B. Bekka, M. Mayer: *Ergodic theory and topological dynamics of group actions on homogeneous spaces*, London Mathematical Society Lecture Note Series, **269** Cambridge University Press, Cambridge, (2000)

[8] B. Bollobás: *Modern Graph Theory*, Graduate Texts in Mathematics 184, Springer, (1998)

[9] P. Brémaud: *Markov Chains; Gibbs Fields, Monte Carlo Simulation, and Queues*, Springer, New York, (1999)

[10] R. Brooks: *The spectral geometry of k-regular graphs*, J. Anal. Math. 57, (1991), 120–151

[11] P. Buser: *Geometry and Spectra of Compact Riemann Surfaces*, Progress in Mathematics 106, Birkhäuser Boston, (1992)

[12] P. Cartier: *Fonctions harmoniques sur un arbre*, Symposia Mathematica, Vol. IX (Convegno di Calcolo delle Probabilit, INDAM, Rome, 1971), Academic Press, London, (1972), 203–270

[13] P. Cartier: *Harmonic Analysis on Trees*, Harmonic analysis on homogeneous spaces (Proc. Sympos. Pure Math., Vol. XXVI, Williams Coll., Williamstown, Mass., 1972), 419–424 Amer. Math. Soc., Providence, R.I., 1973

[14] E. Casadio Tarabusi, J. M. Cohen, M. A. Picardello: *The horocyclic Radon transform on Non-Homogeneous Trees*, Israel Journal of Mathematics, 78, (1992), 363-380

[15] F. R. K. Chung: *Spectral Graph Theory*, CBMS Regional Conference Series in Mathematics, 92. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, (1997)

[16] J. M. Cohen: *Cogrowth and Amenability of Discrete Groups*, J. Funct. Anal. **48** no. 3, (1982), 301–309

[17] D. Cvetković, P. Rowlinson, S. Simić: *Eigenspaces of Graphs*, Encyclopedia of Mathematics and its Applications **66**, Cambridge University Press, Cambridge, (1997)

[18] G. Davidoff, P. Sarnak, A. Valette: *Elementary Number Theory, Group Theory, and Ramanujan Graphs*, London Mathematical Society Student Texts **55**, Cambridge University Press, Cambridge, (2003)

[19] R. Diestel: *Graph Theory*, Graduate Texts in Mathematics 173, third edition, Springer-Verlag, Berlin, (2006)

[20] M. Doob: *A geometric Interpretation of the least Eigenvalue of a Line Graph*, Proc. Second Chapel Hill Conf. on Combinatorial Mathematics and its Applications (Univ. North Carolina, Chapel Hill, N.C., 1970), Univ. North Carolina, Chapel Hill, N.C., (1970), 126–135

[21] F. Douma: *Radial Averages on Regular and Semiregular Graphs*, arXiv: 0909.5573v1 [math.CO]

[22] F. Douma: *A Lattice Point Problem on the Regular Graph*, arXiv: 1002.0932v1 [math.CO]

[23] P. Erdös: *On a new method in elementary number theory which leads to an elementary proof of the prime number theorem*, Proc. Nat. Acad. Sci. U. S. A. 35, (1949), 374–384

[24] A. Figà-Talamanca, Claudio Nebbia: *Harmonic analysis and representation theory for groups acting on homogeneous trees*, LMS Lecture Note Series 162. Cambridge University Press, Cambridge, (1991)

[25] H. Furstenberg: *The Unique Ergodicity of the Horocycle Flow*, Recent advances in topological dynamics (Proc. Conf., Yale Univ., New Haven, Conn., 1972; in honor of Gustav Arnold Hedlund), Lecture Notes in Math., Vol. **318**, Springer, Berlin, (1973), 95–115

[26] S. W. Graham, Jeffrey D. Vaaler: *A class of extremal functions for the Fourier transform* Trans. Amer. Math. Soc. 265, no. 1, (1981), 283–302

[27] R.I. Grigorchuk: *Symmetrical Random Walks on Discrete Groups*, Multicomponent Random Systems, Advances in Probability and Related Topics **6**, Dekker, New York, (1980), 285–325

[28] P. Günther: *Sphärische Mittelwerte in kompakten harmonischen Riemannchen Mannigfaltigkeiten*, Math. Ann. **165**, (1966), 281–296

[29] J. Hadamard: *Sur la distribution des zéros de la fonction $\zeta(s)$ et ses consquences arithmtiques* Bull. Soc. Math. France 24 (1896), 199–220

[30] K. Hashimoto: *Zeta functions of finite graphs and representations of p-adic groups*, Automorphic forms and geometry of arithmetic varieties, Adv. Stud. Pure Math., 15, Academic Press, Boston, MA, 211–280, (1989)

[31] A. Hatcher: *Algebraic Topology*, Cambridge University Press, Cambridge, (2002)

[32] D. A. Hejhal: *The Selberg trace formula for* $\mathrm{PSL}(2, R)$, Vol. I. Lecture Notes in Mathematics, Vol. 548. Springer-Verlag, Berlin-New York, (1976); Vol. 2. Lecture Notes in Mathematics, 1001. Springer-Verlag, Berlin, (1983)

[33] A. J. Hoffman, D. K. Ray-Chaudhuri: *On the Line Graph of a Symmetric Balanced Incomplete Block Design*, Transactions of the American Mathematical Society, Vol. 116 (Apr., 1965), 238–252

[34] H. Huber: *Über eine neue Klasse automorpher Funktionen und ein Gitterpunktproblem in der hyperbolischen Ebene. I.* Comment. Math. Helv. 30 (1956), 20–62, (1955)

[35] H. Huber: *Zur analytischen Theorie hyperbolischen Raumfomen und Bewegungsgruppen*, Math. Ann., 138, (1959), 1–26

[36] Y. Ihara: *On discrete subgroups of the two by two projective linear group over p-adic fields*, J. Math. Soc. Japan, vol 18, (1966), 219–235

[37] S. Katok: *Fuchsian Groups*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, (1992)

[38] A. Katsuda, T. Sunada: *Homology and closed geodesics in a compact Riemann surface*, American Journal of Mathematics, vol 110, no 1, (1988), 145–155

[39] J. Korevaar: *Tauberian Theory, A century of developments*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 329. Springer-Verlag, Berlin, (2004)

[40] A. Lubotzky, R. Phillips, P. Sarnak: *Ramanujan Graphs*, Combinatorica **8** no. 3, (1988), 261–277

[41] G. A. Margulis: *On Some Aspects of the Theory of Anosov Systems*, Springer Monographs in Mathematics, Springer-Verlag, Berlin, (2004)

[42] H. P. McKean: *Selberg's Trace Formula as Applied to a Compact Riemann Surface*, Communications on Pure and Applied Mathematics, vol XXV, 225–246, (1972)

[43] D. J. Newman: *Simple Analytic Proof of the Prime Number Theorem*, The American Mathematical Monthly, Vol. 87, No. 9, (1980), 693–696

[44] S. Northshield: *Cogrowth of Arbitrary Graphs*, Random walks and geometry, Walter de Gruyter GmbH & Co. KG, Berlin, (2004), 501–513

[45] R. Ortner and W. Woess: *Non-Backtracking Random Walks and Cogrowth of Graphs*, Canad. J. Math. **59**, no.4, (2007) 828–844

[46] S. J. Patterson: *A lattice-point problem in hyperbolic space*, Mathematika 22, no. 1, (1975), 81–88

[47] Mark Pollicott, Richard Sharp: *Rates of recurrence for* $\mathbb{Z}^q$ *and* $\mathbb{R}^q$ *extensions of subshifts of finite type*, J. London Math. Soc., 49, (1994), 401–416

[48] B. Riemann: *Über die Anzahl der Priemzahlen unter einer gegebenen Grösse*, Monatsber. Berlin. Akad., (1859), 671–680. Also in: *Bernard Riemann's gesammelte mathematische Werke und wissenschaftliche Nachlass*, 2nd edition, Teubner, Leipzig, (1892) 145–155, reprinted Dover, New York, (1953)

[49] A. Selberg: *An elementary proof of the prime-number theorem*, Ann. of Math. (2) 50, (1949), 305–313

[50] A. Selberg: *Harmonic analysis and discontinuous groups in weakly symmetric Riemannian spaces with applications to Dirichlet series* J. Indian Math. Soc. (N.S.), 20, (1956), 47–87

[51] A. Selberg: *Discontinuous groups and harmonic analysis*, Proc. International Math. Congr. Stockholm, (1962)

[52] E. Seneta: *Non-negative Matrices and Markov Chains*, 2nd edition, Springer-Verlag, New York, (1981)

[53] H. M. Stark and A. A. Terras: *Zeta functions of finite graphs and coverings*, Adv. Math. 121, no. 1, (1996), 124–165

[54] H. M. Stark and A. A. Terras: *Zeta functions of finite graphs and coverings II*, Adv. Math. 154, no. 1, (2000), 132–195

[55] T. Sunada: *Discrete geometric analysis*, Analysis on graphs and its applications, Proc. Sympos. Pure Math., 77, Amer. Math. Soc., Providence, RI, (2008), 51–83

[56] T. Sunada: *Fundamental Groups and Laplacians*, Geometry and analysis on Manifolds (Katata/Kyoto, 1987), Lecture Notes in Mathematics vol. 1339, Springer, Berlin, (1988), 248–277

[57] A. Terras, D. Wallace: *Selberg's trace formula on the k-regular tree and applications*, Int. J. Math. Math. Sci., no. 8, (2003), 501–526

[58] C. de la Vallée Poussin, Annales de la Société scientifique de Bruxelles, vol. 20, part 2, (1896)

[59] A. B. Venkov, A. M. Nikitin: *The Selberg trace formula, Ramanujan graphs and some problems in mathematical physics*, Algebra i Analiz 5, (1993), no. 3, 1–76; translation in St. Petersburg Math. J., 5, no. 3, (1994), 419–484

[60] N. Wiener: *Tauberian theorems* Ann. of Math. (2), 33, no. 1, (1932)

[61] W. Woess: *Cogrowth of Groups and Simple Random Walks*, Arch. Math. (Basel), **41**, (4), (1983), 363–370