On generalised Deligne–Lusztig constructions

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$$\begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ * & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ * & 1 \end{bmatrix} \times \begin{bmatrix} * & 0 \\ 0 & * \end{bmatrix} \times \begin{bmatrix} 1 & * \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} * & * \\ * & * \end{bmatrix}$$

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Abstract

This thesis is on the representations of connected reductive groups over finite quotients of a complete discrete valuation ring. Several aspects of higher Deligne–Lusztig representations are studied.

First we discuss some properties analogous to the finite field case; for example, we show that the higher Deligne–Lusztig inductions are compatible with the Harish-Chandra inductions.

We then introduce certain subvarieties of higher Deligne–Lusztig varieties, by taking preimages of lower level groups along reduction maps; their constructions are motivated by efforts on computing the representation dimensions. In special cases we show that their cohomologies are closely related to the higher Deligne–Lusztig representations.

Then we turn to our main results. We show that, at even levels the higher Deligne–Lusztig representations of general linear groups coincide with certain explicitly induced representations; thus in this case we solved a problem raised by Lusztig. The generalisation of this result for a general reductive group is completed jointly with Stasinski; we also present this generalisation. Some discussions on the relations between this result and the invariant characters of finite Lie algebras are also presented.

In the even level case, we give a construction of generic character sheaves on reductive groups over rings, which are certain complexes whose associated functions are higher Deligne–Lusztig characters; they are accompanied with induction and restriction functors. By assuming some properties concerning perverse sheaves, we show that the induction and restriction functors are transitive and admit a Frobenius reciprocity.

Declaration and copyright

The work in this thesis is based on research carried out in the Department of Mathematical Sciences, Durham University, United Kingdom. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all my own work unless referenced to the contrary in the text.

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About seventy years ago, John von Neumann remarked that, "if people do not believe that mathematics is simple, it is only because they do not realize how complicated life is". Without yet having specific opinions on whether mathematics is simple, I want to thank my families for their continued support, which helps me to deal with the complicated life.

Contents

Chapter 1

Introduction

1.1 Background

In 1976 Deligne and Lusztig published a seminal work [DL76] on the representation theory of reductive groups over a finite field. Their methods are geometric and are based on the ℓ-adic cohomology theory. Subsequently, Lusztig [Lus84a] established the Jordan decomposition of representations for these groups, which leads to the classification of representations of connected reductive groups over a finite field.

In his Corvallis paper [Lus79] in 1979, Lusztig proposed two generalisations of [DL76] for reductive groups over a local field and over a local ring (or more precisely, the ring of integers in the local field), respectively.

In the local field case, Boyarchenko stated in [Boy12] three problems related to the constructions in [Lus79]: The first one is to formulate Lusztig's proposed construction in an explicit manner; the second one is to compute the representations of Lusztig explicitly in certain sense; the third problem is to generalise Lusztig's idea beyond the unramified extension case. Boyarchenko gave a solution to the first problem in the division algebra case, and then studied the second problem in a general setting called Deligne–Lusztig patterns, which comes from his joint work with Weinstein [BW16] on the local Langlands correspondence.

In the local ring case, Lusztig proved in [Lus04] several fundamental results for his constructions in the case that the reductive groups are over a finite local ring of the form $\mathbb{F}_q[[\pi]]/(\pi^r)$, and Stasinski [Sta09] extended this work to reductive groups over an arbitrary finite local ring, based on the techniques introduced by Greenberg in [Gre61] and [Gre63]. While the irreducibility and the orthogonality of these representations have been obtained since then, several basic properties of these representations themselves, such as their dimensions and signs, were unknown until now (for even levels); besides, the case of the smallest group $\mathrm{SL}_2(\mathbb{F}_q[[\pi]]/\pi^2)$ was figured out by Lusztig himself in [Lus04] by analysing the fibres of the reduction map.

Meanwhile, in the case of GL_n and SL_n, Hill found an analogue of Lusztig's Jordan decomposition for these groups over local rings, and in Hill's Jordan decomposition one

series is called nilpotent representations; see e.g. [Hil94] and [Hil95]. In order to recover the nilpotent representations, Stasinski [Sta11] constructed the extended Deligne–Lusztig varieties for tamely ramified maximal tori of GL_n and SL_n and thereby found all nilpotent representations for $\mathrm{SL}_2(\mathbb{F}_q[[\pi]]/\pi^2)$ (it is still an open question whether the extended Deligne–Lusztig varieties give all nilpotent representations in general).

Nowadays, both the classical and the generalised Deligne–Lusztig theories have deep connections with the Langlands program (see [Lus09], [Yos10], [BW16], [Cha16], and [Iva16]), and in particular, the representations of reductive groups over local rings, besides the interest in their own right, are related to the inertial Langlands correspondence (see [BM02] and [CEG+16]). Though, in this thesis we do not enter into these aspects.

In this thesis, we call the generalised Deligne–Lusztig constructions for reductive groups $\mathbb{G}$ over a complete discrete valuation ring $\mathcal{O}$, developed in [Lus79], [Lus04], and [Sta09], the higher Deligne–Lusztig theory. Due to the smoothness of $\mathbb{G}$ and the topology of $\mathcal{O}$, constructing the smooth representations of $\mathbb{G}(\mathcal{O})$ is equivalent to constructing the representations of $\mathbb{G}(\mathcal{O}/\pi^r)$, where π is a uniformiser and r runs over all positive integers. The higher Deligne–Lusztig theory can then be viewed as a geometric approach to the representation theory of $\mathbb{G}(\mathcal{O}/\pi^r)$, unified for all $r \geq 1$.

Meanwhile, in the case $r \geq 2$, under some restrictions on the group $\mathbb{G}$, Gérardin [Gér75] found an algebraic approach (i.e. to use extensions of representations, ordinary induction, and Weil representations) to construct irreducible representations of $\mathbb{G}(\mathcal{O}/\pi^r)$, following Shintani's earlier work [Shi68].

Both the higher Deligne–Lusztig representations and Gérardin's representations rely on the same set of parameters (at least in the generic case). With this observation, Lusztig [Lus04] raised the problem of whether the higher Deligne–Lusztig representations coincide with the purely algebraically constructed representations given by Gérardin (for $r \geq 2$). This suggests a beautiful bridge between the algebraic methods and the geometric methods in representation theory. Besides, an affirmative answer to this problem will give several important consequences. Indeed, as already happened in the case of classical Deligne–Lusztig theory, the quantitative properties (like dimensions, or more generally, characters) of the geometrically constructed representations are hard; usually problems concerning computing the Frobenius trace of étale cohomologies are involved. On the other hand, if one can show that these geometrically constructed representations are isomorphic to certain explicitly induced representations, then their characters (which are regarded as very difficult objects) can be computed explicitly via characters of much simpler subgroups, and thereby many quantitative properties can be obtained. One of the main aims of this thesis is to answer Lusztig's question.

In 1985, based on the works around Weil conjectures and cohomology theory for singular spaces (see e.g. [Del80] and [BBD82]), Lusztig established a geometric theory of characters for reductive groups over an algebraically closed field, called character sheaves; see [Lus86]. Besides their inherent interests, this abstract machinery admits very concrete applications.

For example, many works on determining the rationality properties and the character tables of a reductive group over a finite field heavily rely on the theory of character sheaves; see e.g. [Lus92], [Sho95], [Gec03], and [Bon06].

In 2006, Lusztig [Lus06] proposed a generalisation of principal series character sheaves for reductive groups over a local ring of the form $\mathbb{F}_q[[\pi]]$ (or equivalently, $\mathbb{F}_q[[\pi]]/\pi^r$ for r runs over positive integers), and conjectured that the complexes involved are simple perverse (e.g. this implies the induction functors are transitive). Some special cases of this conjecture were established by Lusztig himself in [Lus15].

Beyond the principal series case and the function field case, motivated by our algebraisation theorems, we find there is a natural way to geometrise the characters of higher Deligne–Lusztig representations, at even levels (i.e. r is even). We regard this geometric theory as a possible (unramified) character sheaf theory for reductive groups over finite local rings, as there are several similarities between our constructions and Lusztig's character sheaf theory for reductive groups over a finite field, like the transitivity and Frobenius reciprocity of the induction and restriction functors. Moreover, as can be seen in Lusztig's work [Lus15], in the principal series case with $\mathcal{O} = \mathbb{F}_q[[\pi]]$, under the assumptions that $\mathrm{char}(\mathbb{F}_q)$ is big enough and $r \leq 4$, the complexes we constructed coincide (up to shifts) with Lusztig's generalised principal series character sheaves in [Lus06] (the assumption concerning r is removed later in [Kim16]). One advantage of our construction is that it works for any series, rather than just the principal series, or its twisted forms (which are needed in the case $r = 1$; see [Lus86]). As in the principal series case in [Lus06], we expect our complexes are also simple perverse. We remark that in the function field case, Fan [Fan12] also considered a construction of character sheaves (at level 2), based on a method very different from ours.

1.2 Structure and results

Here we present an overview of the contents of each chapter.

In Chapter 2 we give a quick review of classical Deligne–Lusztig theory for reductive groups over a finite field. As nowadays, besides the original articles, the material can be found in several other places (e.g. [Sri79], [Car93], [DM91], and [Gec13]), our exposition will omit most proofs, and instead, we focus on explaining ideas and explicit examples. We start with background material on linear algebraic groups, and then turn to Deligne–Lusztig theory. Some basics on ℓ-adic cohomology theory will be introduced first, then the bimodule induction approach to the Deligne–Lusztig (virtual) representations, as presented in [DM91], will be given. On the other hand, following [Car93], we give a down to earth discussion of Deligne–Lusztig varieties; some simple computations in the special case of SL_2, which seems not easily to be found in the literature, are presented. The modular aspect is not touched in this thesis; there is an introduction [Bon11] with a focus on the case of SL_2.

In the beginning of Chapter 3, we first give a brief overview of the works [Lus79], [Lus04], and [Sta09], which provided the framework of higher Deligne–Lusztig theory. Then we

investigate some basic properties of these generalised constructions, in a manner similar to the ones in the classical case over a finite field, like the compatibility with the Harish-Chandra induction. In the second half of this chapter we discuss some considerations around the dimensions of higher Deligne–Lusztig representations. These considerations motivate the notion of essential parts (see Definition 3.4.1), which are certain varieties whose cohomologies seems closely related to the higher Deligne–Lusztig representations.

In Chapter 4, among other things, we present our main results, which gives an affirmative answer to a question raised by Lusztig at even levels (see Question 4.3.1): First, let $\mathbb{G} = \mathrm{GL}_n$ and r even, then

$$R^{\theta}_{T,U} \cong \mathrm{Ind}^{G^F}_{(TU^{\pm})^F}\widetilde{\theta}$$

for $\theta \in \widehat{T^F}$ regular and in general position. Here $R^{\theta}_{T,U}$ denotes a higher Deligne–Lusztig representation (see Definition 3.1.6) and $\mathrm{Ind}^{G^F}_{(TU^{\pm})^F}\widetilde{\theta}$ is an induced representation first considered by Gérardin (see Proposition 4.1.6); they are both parametrised by pairs (T, θ), where T is a commutative algebraic group associated to a maximal torus and θ is a character of its subgroup of rational points. For more details, see Theorem 4.3.2 and Corollary 4.3.6. The generalisation of this result for an arbitrary reductive group $\mathbb{G}$ is achieved in the joint work [CS16], and will be our second main result (the statement is the same as the above isomorphism). We present this generalisation as Theorem 4.3.9 and Corollary 4.3.10. In this chapter we start with a review of Clifford theory as well as some preliminary algebraic constructions, which are required in the proofs of the main results. Then we first consider the principal series case; in this case one can get the above isomorphism via a very simple algebraic argument; however, in this case we also give a geometric argument, which shares a similar philosophy with the general case (but it is much simpler than the general case). Then we present the complete proofs of the main results, first for GL_n and then for an arbitrary (connected) reductive group. Finally, we turn to invariant characters of finite reductive Lie algebras; we discuss their relations with generic Deligne–Lusztig representations via an argument analogous to standard arguments in Deligne–Lusztig theory.

In Chapter 5 we discuss a geometrisation of generic characters of reductive groups over local rings (at even levels); they are certain complexes living in the bounded derived category of constructible $\overline{\mathbb{Q}}_\ell$-sheaves, and their constructions are motivated by the main results in Chapter 4. The characteristic functions of the these complexes are the characters of the generic Deligne–Lusztig representations (see Proposition 5.3.4). Under certain assumptions we show that the associated induction and restriction functors are transitive (see Proposition 5.3.10 and Proposition 5.3.13) and they admit a Frobenius reciprocity (Proposition 5.3.14). Partial results in this chapter are presented in [Che16].

1.3 Conventions and prerequisites

Throughout this thesis, when there is no confusion, we use the conjugation notation $a^b := b^{-1}ab =: {}^{b^{-1}}a$ for a, b in a group. For a map f from a set S, and $s \in S$, we may write ${}^f s$ for

$f(s)$. By a reductive group we always mean a connected reductive group.

For a scheme X over another scheme S, we may write X/S; when S is the spectrum of a ring R, we also write X/R. In this thesis by a variety we mean a reduced scheme of finite type over a field. All varieties in this thesis are quasi-projective (hence separated). When the variety is over an algebraically closed field, we may ignore its non-closed points. For a quasi-projective variety X over $k = \overline{\mathbb{F}}_q$, there is the bounded derived category of constructible $\overline{\mathbb{Q}}_\ell$-sheaves, constructed by Deligne [Del80]; we denote it by $\mathcal{D}(X)$. Let $\mathcal{D}(X)^{\leq 0} \subseteq \mathcal{D}(X)$ be the full subcategory consisting of the objects $K \in \mathcal{D}(X)$ satisfying: The support of $\mathcal{H}^i(K)$ has dimension $\leq -i$ for any $i \in \mathbb{Z}$ (so, in particular, $\mathcal{H}^i(K) = 0$ for $i > 0$, and the support of $\mathcal{H}^0(K)$ is a finite set). Meanwhile, let $\mathcal{D}^{\geq 0}(X)$ be the full subcategory (of $\mathcal{D}(X)$) whose objects are the K such that $\mathbb{D}_X(K) \in \mathcal{D}^{\leq 0}(X)$, where $\mathbb{D}_X$ denotes the Verdier dual functor on $\mathcal{D}(X)$. The category of perverse sheaves on X is the full subcategory $\mathcal{M}(X) := \mathcal{D}^{\leq 0}(X) \cap \mathcal{D}^{\geq 0}(X)$. Note that $\mathbb{D}_X$ preserves $\mathcal{M}(X)$. Some basic properties of derived categories and perverse sheaves are nicely summarised in [Lus86] and [BD10].

In this thesis we are only concerned with representations over $\overline{\mathbb{Q}}_\ell$, where ℓ is some fixed rational prime.

We assume basic knowledges of algebraic geometry and algebraic groups, which can be found in e.g. [Har77], [Liu06], and [Spr09], [MT11], respectively. Some knowledge of reductive group schemes over rings (e.g. in [DG70]) is helpful for understanding certain arguments. Basics of étale cohomology can be found in e.g. [Del77], [FK88], and [Mil13]; collections of some properties of étale cohomology used in Deligne–Lusztig theory, originally proved in [DL76], can be found in [Car93] and [DM91]. For derived category of constructible $\overline{\mathbb{Q}}_\ell$-sheaves presented in [Del80] and [BBD82], one standard book in English is [KW01]; nice summaries can be found in [Lus86] and [BD10]. Concerning representation theory and number theory, the material in [Ser77] and [Neu99] is more than enough.

Chapter 2

Classical Deligne–Lusztig theory

This chapter aims to give a quick introduction to the Deligne–Lusztig representations of connected reductive groups over a finite field. It originates from the first year PhD report of progression of the author. We start with basics on algebraic groups, then turn to the fundamental constructions and results.

2.1 Algebraic groups

2.1.1 Linear groups and rational structures

Let k be an algebraically closed field. Affine algebraic groups over k are exactly the linear algebraic groups over k; see e.g. [Spr09, Theorem 2.3.7]. In the following we review some basic notions concerning closed subgroups of a connected affine algebraic group over k.

Definition 2.1.1. Given a connected algebraic group over k, the maximal closed connected solvable subgroups are called *Borel subgroups*, and a closed subgroup containing some Borel subgroup is called a *parabolic subgroup*.

It is known that any two Borel subgroups in a connected algebraic group are conjugate to each other; see e.g. [Spr09, Theorem 6.2.7].

Example 2.1.2. In GL_n over k, the parabolic subgroups have geometric natures in term of flags: An ascending chain of vector spaces

$$0 = W_0 \subsetneq \cdots \subsetneq W_m = k^n$$

is called a flag. It is known that the stabilisers (in GL_n) of flags are exactly the parabolic subgroups, and, when $m = n$, the stabiliser is a Borel subgroup. When $W_i = k^i$ (for $i \in (0, m) \cap \mathbb{Z}$) and the inclusions are the natural ones, the corresponding parabolic subgroup is called standard; they consist of block upper triangular matrices. See e.g. [AB95] for details.

This example leads to the Lie–Kolchin theorem:

Theorem 2.1.3. *Suppose H is a connected solvable closed subgroup of GL_n over k, then H is conjugate to a subgroup of the group of upper triangular matrices.*

This theorem follows from the above example and the fact that Borel subgroups are conjugate to each other.

Definition 2.1.4. A *torus* of a connected algebraic group over k is a closed subgroup isomorphic to a product of multiplicative groups $\mathbb{G}_m \times \cdots \times \mathbb{G}_m$.

Every torus is contained in a maximal torus, and any maximal torus is contained in a Borel subgroup. Moreover, any two maximal tori are conjugate; for the proofs of these facts see e.g. [Spr09, Chapter 6].

Now we turn to rational structures.

Definition 2.1.5. An $\mathbb{F}_q$*-rational structure* on a variety V over $\overline{\mathbb{F}}_q$ is an $\overline{\mathbb{F}}_q$-isomorphism $V \simeq \mathrm{Spec}\ \overline{\mathbb{F}}_q \times_{\mathrm{Spec}\ \mathbb{F}_q} V_0$ for some variety V_0 over $\mathbb{F}_q$. When this is the case, we say V is $\mathbb{F}_q$-rational, or defined over $\mathbb{F}_q$.

Definition 2.1.6. In the above definition, by tensoring the morphism induced from the Frobenius morphism on the structural sheaf of V_0 with the identity automorphism of $\overline{\mathbb{F}}_q$, we get an $\overline{\mathbb{F}}_q$-endomorphism on V; we call it a *geometric Frobenius* of V. Meanwhile, the Frobenius in $\mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$ induces an $\mathbb{F}_q$-automorphism of V by tensoring with the identity automorphism of V_0; we call the resulting automorphism an *arithmetic Frobenius*.

The notions geometric Frobenius, arithmetic Frobenius, and rational structures are equivalent for affine or projective varieties; see [DM91, Page 35] for details.

Remark 2.1.7. When there is no confusion, we use the following convention: We drop the "geometric" in the term "geometric Frobenius"; a morphism between ($\mathbb{F}_q$-) rational varieties is said to be *rational* if it commutes with the (geometric) Frobenius; when a (geometric) Frobenius F is given, we may use the terms "F-rational" or "F-stable" to indicate "rational".

For an algebraic group with geometric Frobenius F, we are interested in the finite group consisting of the F-stable points. An algebraic group over $\overline{\mathbb{F}}_q$ may admit different rational structures over $\mathbb{F}_q$.

Example 2.1.8. Consider the group $\mathrm{GL}_n/\overline{\mathbb{F}}_q$; it admits a natural rational structure

$$\mathrm{GL}_n/\overline{\mathbb{F}}_q \simeq \mathrm{Spec}\ \overline{\mathbb{F}}_q \times_{\mathrm{Spec}\ \mathbb{F}_q} \mathrm{GL}_n/\mathbb{F}_q.$$

Meanwhile, GL_n admits another important rational structure, i.e. unitary group structure: If we denote by Fr the geometric Frobenius associated to the natural rational structure on GL_n over $\mathbb{F}_q$, then the group

$$U_n(\mathbb{F}_q) := \mathrm{GL}_n(\overline{\mathbb{F}}_q)^{F'} = \{x \in \mathrm{GL}_n(\mathbb{F}_{q^2}) \mid \mathrm{Fr}(x)^t = x^{-1}\},$$

where t means taking transpose, is also the set of rational points of an $\mathbb{F}_q$-rational structure; here $F' \colon x \mapsto \mathrm{Fr}(x^{-1})^t$ is the geometric Frobenius endomorphism. This follows from the fact that $F'^2 = \mathrm{Fr}^2$; see [DM91, Proposition 3.3(i)].

From now on, whenever we talk about rational structures on an algebraic group over $\overline{\mathbb{F}}_q$, we assume the group operations are rational. In particular, this is the case if the group is the base change of a group scheme over the finite field involved.

Definition 2.1.9. Let F be an endomorphism of a connected algebraic group G/k. We call

$$L : G \longrightarrow G, \quad g \mapsto g^{-1} \cdot {}^F g$$

the *Lang map* associated to F.

Theorem 2.1.10. *Along with the above notation, if F is surjective with finitely many fixed points, then the Lang map is surjective.*

This is the so-called Lang–Steinberg theorem; it is fundamental for algebraic groups over a finite field. For example, it implies that, if F is a geometric Frobenius on $G/\overline{\mathbb{F}}_q$, then for any F-rational closed connected subgroup H, one has $(G/H)^F = G^F/H^F$. See e.g. [DM91, Chapter 3] for more details.

The proof of this theorem can be found in [Ste68, 10]; one can also find an outline in [DM91, 3.10]. We refer to [DM91, P38-44] for applications of the Lang–Steinberg theorem.

2.1.2 Reductive groups

The fundamental objects in our concerns are connected reductive groups, e.g. GL_n, PGL_n, and SO_n. These are affine algebraic groups with nice combinatorial properties. When they are defined over an algebraic closure of a finite field, the subgroups consisting of points over the finite field are called finite reductive groups, and are also referred to as finite groups of Lie type.

Definition 2.1.11. Given a connected algebraic group G/k, its *unipotent radical*, denoted by $R_u(G)$, is defined to be the maximal connected normal unipotent closed subgroup. If $R_u(G) = 1$, then we say G is *reductive*.

For the existence and uniqueness of $R_u(G)$, see [DM91, Proposition 0.16].

One has the following decomposition (see e.g. [DM91, Chapter 1]).

Proposition 2.1.12. *Let P be a parabolic subgroup of a connected reductive group G/k, then there exists a closed subgroup M of P such that $P = M \ltimes R_u(P)$. Such a semi-direct product decomposition is called a Levi decomposition, and M is called a Levi subgroup (of P). In particular, if P is a Borel subgroup, then M is a maximal torus.*

Example 2.1.13. In GL_3, if we take $P = \begin{bmatrix} \mathrm{GL}_2 & * \\ 0 & \mathrm{GL}_1 \end{bmatrix}$ to be a standard parabolic subgroup, then the standard Levi decomposition is $P = L \ltimes U = \begin{bmatrix} \mathrm{GL}_2 & 0 \\ 0 & \mathrm{GL}_1 \end{bmatrix} \ltimes \begin{bmatrix} I_2 & * \\ 0 & 1 \end{bmatrix}$.

Natural examples of reductive groups include the general linear groups; one direct way to see they are reductive is to use the Lie–Kolchin theorem (note that an upper triangular matrix will be conjugated to a lower triangular matrix by some element in the Weyl group), or to use the fact that every normal subgroup of a general linear group (except for some very small ones) is either a subgroup of the scalar matrices or contains the special linear subgroup. Here we present a completely elementary proof without referring to these facts; maybe it was known but we couldn't find it in literature.

Proposition 2.1.14. *The general linear group* GL_n/k *is reductive.*

Proof. The idea of the proof is: For any non-identity unipotent element A, by taking conjugation of A we get certain "nice" form J, then we take a conjugation of J by some permutation matrix P to get another matrix J'; we show one can choose the permutation matrix P to make JJ' has an eigenvalue not equal to 1, so there is no non-trivial unipotent normal subgroup of GL_n.

For simplifying computations we follow Stasinski's suggestion to require J to be a Jordan normal form. And for simplifying notation we omit the easier cases $n = 2, 3$ where one can check by hands the same method works; so in the below we assume $n \geq 4$.

As A is not the identity, its (upper) Jordan normal form $J = \mathrm{diag}(J_1, \cdots, J_m)$ has a non-trivial $s \times s$ Jordan block J_i. Let P_i be the $s \times s$ permutation matrix that switch the first row/column and the last row/column, i.e. $P_i = \begin{bmatrix} 0 & 0 & 1 \\ 0 & I_{s-2} & 0 \\ 1 & 0 & 0 \end{bmatrix}$, and put $J_i' := P_i^{-1} J_i P_i$. The entries of $J_i J_i'$ are as follows: The first row is $(1, 1, 1, 0, \cdots, 0)$, the last three rows are $(1, 0, \cdots, 0, 1, 2, 0), (1, 1, 0, \cdots, 0, 1, 1), (0, 1, 0, \cdots, 0, 1)$, and for $j \in [2, s-3]$ the j-th row is $(\cdots, 0, 1, 2, 1, 0, \cdots)$ where the number of the beginning zeros is $j - 1$.

Now, for $w \neq i$ let P_w be the identity matrix with same size as J_w , and consider $P = \mathrm{diag}(P_1, \cdots, P_m)$. We want to show the matrix

$$JP^{-1}JP = \mathrm{diag}(J_1^2, \cdots, J_{i-1}^2, J_i J_i', J_{i+1}^2, \cdots, J_m^2)$$

has an eigenvalue not equals 1. It is sufficient to show $J_i J_i'$ has no eigenvalue equals 1.

Indeed, suppose $J_i J_i'(x_1, \cdots, x_s)^t = (x_1, \cdots, x_s)^t$, then the last row gives $x_2 + x_s = x_s$, the $(s-1)$-th row gives $x_1 + x_2 + x_{s-1} + x_s = x_{s-1}$, the $(s-2)$-th row gives $x_1 + x_{s-2} + 2x_{s-1} = x_{s-2}$, the first row gives $x_1 + x_2 + x_3 = x_1$, and all other rows gives $x_j + 2x_{j+1} + x_{j+2} = x_j$. Hence $(x_1, \cdots, x_s) = 0$, done. □

The above proof also implies:

Proposition 2.1.15. *Let* k *be an algebraically closed field, and let* $n \geq 2$*. Suppose* $A \in \mathrm{GL}_n(k)$ *is unipotent, then we can find in* $\mathrm{GL}_n(k)$ *two conjugates of* A*, denoted by* B *and* C*, such that there are at least two eigenvalues of* BC *not equal to* 1*.*

The connected reductive groups are with BN-pairs (see [DM91, 1.1]), and hence admit the following Bruhat decomposition.

Theorem 2.1.16. *Let G/k be a connected reductive group, T a maximal torus of G with Weyl group $W = N_G(T)/T$, and B a Borel subgroup containing T. Then there is a finite partition*

$$G = \coprod_{w \in W} BwB.$$

into locally closed subvarieties. Here we abuse notation by writing w for a lift in $N_G(T)$.

Here the Borel subgroup B and the normaliser $N_G(T)$ form a BN-pair; see [DM91, Theorem 1.2 and Proposition 1.4] for more details.

2.2 Deligne–Lusztig theory

In this section we denote by X a quasi-projective variety over an algebraic closure $\overline{\mathbb{F}}_q$ of the finite field $\mathbb{F}_q$, and we take ℓ to be a rational prime not equal to $p = \mathrm{char}(\mathbb{F}_q)$.

2.2.1 Étale cohomology and induction

We start with some preparations on Lefschetz numbers.

Let $H_c^i(X, \overline{\mathbb{Q}}_\ell)$ be the i-th compactly supported ℓ-adic cohomology of X. The formal alternating sum $H_c^*(X) := \sum_i (-1)^i H_c^i(X, \overline{\mathbb{Q}}_\ell)$ is a virtual $\overline{\mathbb{Q}}_\ell$-vector space of finite dimension. If g is a finite endomorphism of X, then it induces a (linear) endomorphism on each ℓ-adic cohomology group, and we call

$$\mathcal{L}(g, X) := \mathrm{Tr}(g \mid H_c^*(X)) = \sum_i (-1)^i \mathrm{Tr}(g \mid H_c^i(X, \overline{\mathbb{Q}}_\ell))$$

the *Lefschetz number* of g.

Example 2.2.1. When g is the Frobenius endomorphism on X with respect to some rational structure, we have

$$|X^g| = \mathcal{L}(g, X);$$

this is the Grothendieck fixed point formula.

One consequence of the above formula is that the value of Lefschetz number is independent of the choice of ℓ; see e.g. [DM91, 10.6].

Most basic properties of ℓ-adic cohomology and Lefschetz number that we needed can be found in [Del77] and [DL76], and we will use them as standard facts; in the following we discuss one of them, which is fundamental in Deligne–Lusztig theory:

Theorem 2.2.2. *Let σ be a finite order automorphism on X with $\sigma = su$, where s is a power of σ of order prime to p and u is a power of σ of p-power order (i.e. Jordan decomposition of the automorphism). Then one has*

$$\mathcal{L}(\sigma, X) = \mathcal{L}(u, X^s),$$

which is usually referred to as the Deligne–Lusztig fixed point formula.

Sketch of proof. Here we give a sketch of the arguments, and refer to [DL76] for the detailed proof (see also the very helpful treatment in [Sri79]).

Firstly, put $X_1 = X$, and then define the varieties $X_{i+1} = \{x \in X_i \mid \sigma^i x \neq x\}$ inductively. So we get a partition into locally closed subvarieties $X = \coprod_i X_i^{\sigma^i}$, with σ acts on each $X_i^{\sigma^i}$ freely as a cyclic group. Then by basic properties of Lefschetz number we have

$$\mathcal{L}(\sigma, X) = \sum_i \mathcal{L}(\sigma, X_i^{\sigma^i}) \quad \text{and} \quad \mathcal{L}(u, X^s) = \sum_i \mathcal{L}(u, (X_i^{\sigma^i})^s).$$

Thus it suffices to show

$$\mathcal{L}(\sigma, X_i^{\sigma^i}) = \mathcal{L}(u, (X_i^{\sigma^i})^s)$$

for each i. And more generally, as the finite cyclic group $\langle\sigma\rangle$ acts on $X_i^{\sigma^i}$ freely, it would suffice to show

$$\mathcal{L}(\sigma, Y) = \mathcal{L}(u, Y^s)$$

for any quasi-projective variety $Y/\overline{\mathbb{F}}_q$ on which $\langle\sigma\rangle$ acts freely.

If $s = 1$, then this equality is trivial.

Now assume $s \neq 1$, then $Y^s = \emptyset$ since the action is free, so $\mathcal{L}(u, Y^s) = 0$, and it remains to show $\mathcal{L}(\sigma, Y) = 0$ in this case. As $s \neq 1$, there is a prime $\ell' \neq p$ dividing the order of σ. We can take $\ell = \ell'$ since the Lefschetz number is independent of the choice of ℓ.

In the case $H_c^*(Y)$ is a virtual projective $\mathbb{Z}_\ell$-module, its character value at σ is zero since the order of σ is divisible by ℓ; see [Ser77, P143 (i)] for a proof of this assertion. In the general case, by techniques of derived categories, when one is calculating the Lefschetz number, the module $H_c^*(Y)$ can always be replaced by a virtual projective $\mathbb{Z}_\ell$-module; we refer to [DL76, Proposition 3.5] for this last fact. □

We turn to inductions. Let A_1 and A_2 be two finite groups. For any A_1-module-A_2 $\mathfrak{M}$, i.e. a bimodule with a left $\overline{\mathbb{Q}}_\ell[A_1]$-action and a right $\overline{\mathbb{Q}}_\ell[A_2]$-action, there is an associated functor from the virtual $\overline{\mathbb{Q}}_\ell$-representations of A_2 to that of A_1 given by

$$\mathfrak{N} \mapsto \mathfrak{M} \otimes_{\overline{\mathbb{Q}}_\ell[A_2]} \mathfrak{N},$$

where the tensor product means the tensor product of $\overline{\mathbb{Q}}_\ell$-vector spaces modulo the relations generated by $\langle m \otimes a_2 n - m a_2 \otimes n \rangle$, with $a_2 \in A_2$, $n \in \mathfrak{N}$, and $m \in \mathfrak{M}$. This bimodule approach follows the treatment in [DM91].

Throughout the remaining part of this chapter, we let G be a connected reductive group over $\overline{\mathbb{F}}_q$ with a geometric Frobenius F over $\mathbb{F}_q$, let L be the associated Lang map, and let T be a rational maximal torus. Given a (not necessarily rational) parabolic subgroup P of G, and a rational Levi subgroup M of P, the variety $L^{-1}(FU)$ admits a left G^F-action and a right M^F-action, where U denotes the unipotent radical of P; these two actions make $H_c^*(L^{-1}(FU))$ into a virtual bimodule. The variety $L^{-1}(FU)$ is one of the *Deligne–Lusztig varieties*; we will take a closer look at them in the next subsection.

Definition 2.2.3. The *Lusztig induction* from the virtual $\overline{\mathbb{Q}}_\ell$-representations of M^F to that of G^F is the induction functor given by the G^F-module-M^F $H_c^*(L^{-1}(FU))$. In the case that $P = B$ is a Borel subgroup and $M = T$ is a maximal torus, we also call it a *Deligne–Lusztig induction*; in this case, for an irreducible representation $\theta \in \widehat{T^F} := \mathrm{Hom}(T^F, \overline{\mathbb{Q}}_\ell^\times)$, the G^F-representation $H_c^*(L^{-1}(FU)) \otimes_{\overline{\mathbb{Q}}_\ell[T^F]} \theta$ is denoted by R_T^θ. The representations R_T^θ are called *Deligne–Lusztig representations*.

In the above, if P is rational, then this induction is actually an induction functor with bimodule $\overline{\mathbb{Q}}_\ell[G^F/U^F] \cong H_c^0(L^{-1}(FU))$, which is usually referred to as a Harish-Chandra induction or parabolic induction. The Harish-Chandra inductions provide a classification of $\mathrm{Irr}(G^F)$ based on the notion of cuspidal representations (a class function is called cuspidal if its translation by any rational point of any proper rational parabolic subgroup became zero after integrating over the rational points in the unipotent radical); see [DM91, Chapter 6], as well as the original text of Harish-Chandra [HC70], for more details. The Deligne–Lusztig construction generalises this idea and "induce" representations from any rational maximal torus, quasi-split (i.e. contained in a rational Borel subgroup) or not.

Remark 2.2.4. The G^F-conjugacy classes of rational maximal tori are parametrised by F-conjugacy classes of the Weyl group W of some fixed quasi-split rational maximal torus. In particular, the class of quasi-split maximal tori corresponds to the class containing the trivial element $1 \in W$. This was proved in [DL76, 1.14]; see also [DM91, Chapter 3].

Remark 2.2.5. One sees that the notation R_T^θ does not involve B. Indeed, as a G^F-representation it is independent of the choice of B; see [DL76, Corollary 4.3].

Remark 2.2.6. Note that $R_T^\theta = H_c^*(L^{-1}(FU)) \otimes_{\overline{\mathbb{Q}}_\ell[T^F]} \theta \cong H_c^*(L^{-1}(FU))_\theta$ is the θ-isotypical part of $H_c^*(L^{-1}(FU))$, so for any $g \in G^F$ the character value can be written as

$$R_T^\theta(g) = \sum_{i \in \mathbb{Z}} (-1)^i \cdot \mathrm{Tr}\left(g \mid H_c^i(L^{-1}(FU))_\theta\right).$$

2.2.2 Deligne–Lusztig varieties

Now we discuss the variety $L^{-1}(FU)$ itself, as well as some closely related quotients.

In this subsection we assume T_0 to be a quasi-split rational maximal torus of G and B_0 a rational Borel subgroup containing T_0, and we denote by $W = W(T_0)$ the Weyl group. We want to discuss Deligne–Lusztig varieties; they are defined via the notion of relative positions concerning pairs of Borel subgroups.

Definition 2.2.7. Consider the set of pairs (B_1, B_2) of Borel subgroups of G. Note that G acts on this set by conjugation $g \in G$: $(B_1, B_2) \mapsto ({}^gB_1, {}^gB_2)$. We say (B_1, B_2) is in *relative position* $w \in W$ if it lies in the same G-orbit as $(B_0, {}^{\hat{w}}B_0)$, for some lift $\hat{w} \in N_G(T_0)$ of w.

Definition 2.2.8. Let $X(w)$ be the G^F-set of all Borel subgroups B such that B and FB are in relative position w. Besides $L^{-1}(FU)$, we also call $X(w)$ a *Deligne–Lusztig variety* (see the below remark).

Remark 2.2.9. Actually, $X(w)$ is a locally closed subset of the flag variety G/B_0 of Borel subgroups (indeed, it can be viewed as a transversal intersection between an orbit and a graph), hence it admits a natural reduced variety structure; it is smooth and purely of dimension $\ell(w)$, where $\ell(w)$ denotes the length of the Weyl element w; see [DL76, 1.4] for more details.

Deligne–Lusztig varieties can be described more explicitly; let us do it step by step (the details can be found in [Car93, Chapter 7]):

(i) The group $B_0 \cap {}^{\hat{w}}B_0$ acts on $L^{-1}(\hat{w}B_0)$ by right multiplication, and $g \mapsto {}^gB_0$ is a surjective morphism from $L^{-1}(\hat{w}B_0)$ to $X(w)$, with fibres being the orbits of $B_0 \cap {}^{\hat{w}}B_0$; see [Car93, 7.7.6].

(ii) Let $B_0 = U_0T_0$ be the Levi decomposition of B_0, and let $T_0(w)$ be the twisted rational maximal torus of T_0 of type $w \in W$. We have $T_0(w)^F = \{t \in T_0 \mid wF(t)w^{-1} = t\}$; see [DM91, 3.23 and 3.24].

(iii) By (i) one can see $(U_0 \cap {}^{\hat{w}}U_0)T_0(w)^F$ acts on $L^{-1}(\hat{w}U_0)$ by right multiplication, and $g \mapsto {}^gB_0$ is a surjective morphism from $L^{-1}(\hat{w}U_0)$ to $X(w)$ with fibres being orbits of $(U_0 \cap {}^{\hat{w}}U_0)T_0(w)^F$; see [Car93, 7.7.7].

(iv) Assume $\hat{w} = x^{-1}F(x) \in N_G(T_0)$ for $x \in G$. Take $T = {}^xT_0$; note that T is F-stable. Let $B = {}^xB_0$, and let $B = UT$ be the Levi decomposition.

(v) The varieties $L^{-1}(FU)$ and $L^{-1}(\hat{w}U_0)$ are isomorphic by $g \mapsto gx$; together with (iii) one can deduce $L^{-1}(FU)/(U \cap FU)T^F \cong X(w)$; see [DL76, 1.11].

(vi) From the above we see $\tilde{X}(w) := L^{-1}(FU)/U \cap FU$ is a G^F-equivariant T^F-torsor over $X(w)$ (see also [DL76, 1.8]). By abuse of notation, we call $X(w)$, $\tilde{X}(w)$, and $L^{-1}(FU)$ Deligne–Lusztig varieties at w.

Indeed, since $U \cap FU$ is an affine space (see e.g. [DM91, 0.33]), by homotopy considerations (see e.g. [DM91, 10.12]) we have the isomorphisms of $\overline{\mathbb{Q}}_\ell[G^F]$-modules (recall that R_T^θ is independent of B)

$$R_T^\theta = H_c^*(L^{-1}(FU))_\theta \cong H_c^*(\tilde{X}(w))_\theta.$$

Remark 2.2.10. The Deligne–Lusztig varieties $\tilde{X}(1)$ and $X(1)$ are exactly the ones used to give the Harish-Chandra induction, because the involved rational maximal tori are quasi-split. These varieties are of dimension zero, with cardinalities $|\tilde{X}(1)| = |G^F/B_0^F| \cdot |T_0^F|$ and $|X(1)| = |G^F/B_0^F|$ respectively.

For $\mathrm{SL}_2(\mathbb{F}_q)$, prior to the story of Deligne–Lusztig theory, Drinfeld found that the compactly supported ℓ-adic cohomology of the affine curve $xy^q - yx^q = 1$ can be used to produce the discrete series representations (i.e. cuspidal representations) of $\mathrm{SL}_2(\mathbb{F}_q)$. This curve is called the Drinfeld curve, and is actually a Deligne–Lusztig variety associated to SL_2. Let us illustrate this point below:

Firstly, as $W(T_0) \cong S_2$, where T_0 is the group of all diagonal matrices, we need to consider two Weyl elements. For the trivial Weyl element, by the above remark we see $X(1)$ is a set of $|G^F/B_0^F| = q+1$ points, and $\tilde{X}(1)$ is a set of $|G^F/B_0^F| \cdot |T_0^F| = q^2 - 1$ points.

Now denote by w the non-trivial Weyl element; we fix a lift $\hat{w} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ in G. In order to compute $\tilde{X}(w)$, first note that $U_0 \cap {}^{\hat{w}}U_0 = 1$ (here U_0 is the group of upper triangular matrices with diagonal entries $= 1$), so $\tilde{X}(w) \cong L^{-1}(\hat{w}U_0)$. Suppose

$$L\colon \begin{bmatrix} x & y \\ z & w \end{bmatrix} \mapsto \hat{w} \cdot \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & a \end{bmatrix},$$

where L is the Lang map. Then we see the points in $\tilde{X}(w)$ are the points ($\in \mathbb{A}^4$) satisfying

$$xw - yz = 1, \quad wx^q = yz^q, \quad wy^q - yw^q = -1, \quad xz^q - zx^q = 1,$$

which is equivalent to (first times $wy^q - yw^q = -1$ with x^q)

$$y = x^q, \quad w = z^q, \quad xw - yz = 1.$$

Thus these points form the curve $xz^q - zx^q = 1$ (on $\mathbb{A}^2$), i.e. the Drinfeld curve. The group $\mathrm{SL}_2(\mathbb{F}_q)$ acts on the curve via left matrix multiplication by writing the points as

$$\tilde{X}(w) = \left\{ \begin{bmatrix} x & x^q \\ z & z^q \end{bmatrix} \right\}_{x,z \in \overline{\mathbb{F}}_q;\ xz^q - zx^q = 1}.$$

Meanwhile, $T_0(w)^F$ acts on this curve by right matrix multiplication, and

$$T_0(w)^F = \left\{ \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} \middle| \begin{bmatrix} a & 0 \\ 0 & a^{-1} \end{bmatrix} = \begin{bmatrix} a^{-q} & 0 \\ 0 & a^q \end{bmatrix} \right\} \cong \mathbb{Z}/(q+1).$$

So $X(w)$ is a quotient of $\tilde{X}(w)$ by the group of $(q+1)$-th roots of unity. A detailed direct topological computation of the cohomology groups of these varieties can be found in [Che13]; on the other hand, the cohomology groups can also be evaluated by looking at the representations of $\mathrm{SL}_2(\mathbb{F}_q)$ (see [Lus78]).

2.2.3 Main properties and applications

First, like the case of ordinary induction, there is a Mackey intertwining formula for the Lusztig inductions, at least when one of the involved Levi subgroups is a torus; see [DM91, Chapter 11]. An immediate consequence is the following orthogonality relation.

Theorem 2.2.11. *Let T and T' be two rational maximal tori of G, and take $\theta \in \mathrm{Irr}(T^F)$ and $\theta' \in \mathrm{Irr}(T'^F)$, then one has*

$$\langle R_T^\theta, R_{T'}^{\theta'} \rangle_{G^F} = \frac{1}{|T^F|} \#\{g \in G^F \mid {}^gT = T',\ {}^g\theta = \theta'\}.$$

Proof. See [DM91, 11.15]. □

The original proof of the above orthogonality relation in [DL76, 6.9] uses Green functions, which are defined to be the restrictions of the characters R_T^1 to unipotent elements, for various rational T. These functions (denote them by $Q_{T,G}$) are very helpful for studying Deligne–Lusztig characters, because they reduce the problem on evaluating Deligne–Lusztig characters to unipotent elements of smaller reductive groups in the following sense:

Theorem 2.2.12. *Let $g = su \in G^F$ with s semisimple and u unipotent (Jordan decomposition), then*

$$R_T^\theta(g) = \frac{1}{C_G^\circ(s)^F} \sum_{\substack{x \in G^F \\ s^x \in T^F}} \theta(s^x) Q_{^xT, C_G^\circ(s)}(u),$$

where $C_G(s)$ is the centraliser of s in G.

Proof. See e.g. [Car93, Theorem 7.2.8]. □

The orthogonality also implies the following irreducibility, which confirms Macdonald's conjecture that one can associate an irreducible representation of G^F for each pair (T, θ).

Theorem 2.2.13. *Take $\theta \in \mathrm{Irr}(T^F)$. If θ is in general position, i.e. no non-trivial element in $W(T)^F$ fixes θ, then one of the G^F-representations $\pm R_T^\theta$ is irreducible.*

Besides the above irreducibility in general positions, actually all irreducible representations of G^F can be realised in Deligne–Lusztig theory:

Theorem 2.2.14. *Given any $\rho \in \mathrm{Irr}(G^F)$, one can find a pair (T, θ) such that $\langle \rho, R_T^\theta \rangle_{G^F} \neq 0$.*

One way to prove this theorem is to prove that the regular character Reg_G can be written as a linear combination of Deligne–Lusztig characters.

Definition 2.2.15. A *uniform function* on G^F is a class function on G^F such that it is a ($\overline{\mathbb{Q}}_\ell$-) linear combination of Deligne–Lusztig characters.

Theorem 2.2.16. *The regular character Reg_G, the trivial character 1_G, and the Steinberg character St_G (see e.g. [DM91, 9.1]) are uniform functions.*

Proof. This can be proved by using duality functors; see e.g. [DM91, 12.13 and 12.14]. □

By the orthogonality relation, if T and T' are not G^F-conjugate, then $\langle R_T^\theta, R_{T'}^{\theta'}\rangle_{G^F} = 0$. However, this does not mean they have no irreducible constituents in common, since they are virtual modules. The way to fix this problem is to introduce the notion of geometric conjugacy classes, which provides a classification of irreducible representations of G^F in the context of Deligne–Lusztig theory.

Definition 2.2.17. For a rational maximal torus T and a positive integer n, the *norm map* is defined to be the morphism:

$$N_{F^n/F}\colon T \longrightarrow T; \quad \tau \mapsto \tau \cdot {}^F\tau \cdot {}^{F^2}\tau \cdots {}^{F^{n-1}}\tau.$$

Note that the norm maps are transitive.

Definition 2.2.18. Let T and T' be two rational maximal tori, and θ (resp. θ') an irreducible character of T^F (resp. T'^F). We say (T,θ) and (T',θ') are *geometrically conjugate* if $(T, \theta \circ N_{F^n/F})$ and $(T', \theta' \circ N_{F^n/F})$ are G^{F^n}-conjugate for some positive integer n.

Theorem 2.2.19. *If (T,θ) and (T',θ') are not geometrically conjugate, then R_T^θ and $R_{T'}^{\theta'}$ have no common irreducible constituent.*

Proof. See [DL76, 6.2 and 6.3]. □

Let G^* be the dual reductive group of G with dual Frobenius F^* (the two pairs (G,F) and (G^*,F^*) satisfy some compatibility conditions; see [DM91, 13.10]), then there is a canonical one-to-one correspondence between the geometric conjugacy classes of G and the F^*-rational conjugacy classes of semisimple elements in G^* (see e.g. [DM91, 13.12]), and in this bijection $\{1\} \subseteq G^*$ corresponds to the pairs of the form $(T,1)$. Under this bijection, we denote by $\mathcal{E}(G,(s))$ the subset of elements in $\mathrm{Irr}(G^F)$ corresponding to the conjugacy class $(s) \subseteq G^*$ containing $s \in G^*$. Thus the above theorem can be rewritten as

$$\mathrm{Irr}(G^F) = \coprod_{(s)} \mathcal{E}(G,(s)),$$

A main result in Lusztig's book [Lus84a], with the assumption that the centre of G is connected (this condition is to ensure the connectedness of the centraliser of every s, and was removed in a later work of Lusztig), is a bijection $\mathcal{E}(G,(s)) = \mathcal{E}(C_{G^*}(s),\{1\})$ with certain nice properties (see e.g. [DM91, 13.23]); this gives a bijection

$$\mathrm{Irr}(G^F) = \coprod_{(s)} \mathcal{E}(C_{G^*}(s),\{1\}), \tag{2.1}$$

where (s) runs over F^*-rational conjugacy classes of semisimple elements in G^*. As the elements in $\mathcal{E}(C_{G^*}(s),\{1\})$ are the unipotent representations of $(C_{G^*}(s))^{F^*}$, one usually refers (2.1) as Lusztig's Jordan decomposition of representations. For more details, see [DM91, Chapter 13] and Lusztig's original book [Lus84a].

Chapter 3

Generalised Deligne–Lusztig constructions

In the previous chapter we have given an introduction to Deligne and Lusztig's work [DL76], and in this chapter we will discuss a generalisation for reductive groups over a finite quotient of a complete discrete valuation ring. This generalisation was introduced in [Lus79], and then developed in [Lus04] for function fields which was generalised for general case in [Sta09].

3.1 Higher Deligne–Lusztig theory

In this chapter, we denote by $\mathcal{O}$ the ring of integers in a non-archimedean local field, π a uniformiser, and $\mathbb{F}_q$ the residue field with characteristic p. Moreover, we denote by $\mathcal{O}^{\mathrm{ur}}$ the ring of integers of the maximal unramified extension of the local field, and denote its residue field by $k = \overline{\mathbb{F}}_q$. We want to study the smooth representations of connected reductive groups over $\mathcal{O}$, by geometric methods via passing to $\mathcal{O}^{\mathrm{ur}}$.

By properties of the profinite topology, every smooth representation of a connected reductive group over $\mathcal{O}$ factors through a representation of this group over a finite quotient $\mathcal{O}_r := \mathcal{O}/\pi^r$, where r is a positive integer. So we focus on the study of representations of reductive groups over $\mathcal{O}_r$, with r runs over all positive integers.

3.1.1 Group schemes and the Greenberg functor

From now on, let $\mathbb{G}$ be a reductive group scheme over $\mathcal{O}_r$ (i.e. $\mathbb{G}$ is a smooth affine group scheme over $\mathcal{O}_r$ with geometric fibres being connected reductive groups in the usual sense; this is the definition used in [DG70, XIX 2.7]), where r is a fixed arbitrary positive integer. We denote by $\mathbf{G}$ the base change of $\mathbb{G}$ to $\mathcal{O}_r^{\mathrm{ur}} := \mathcal{O}^{\mathrm{ur}}/\pi^r$.

Remark 3.1.1. One can view $\mathbf{G}$ as a closed subgroup scheme of some $\mathrm{GL}_n/\mathcal{O}_r^{\mathrm{ur}}$. Indeed, when $r = 1$ this is well-known; see e.g. [Wat79, 3.4 Theorem] for a short Hopf algebra argument. For a general r, note that $\mathcal{O}_r^{\mathrm{ur}}$ is an artinian local ring with algebraically closed

residue field, hence it is a strictly henselian local ring, so by [Sta09, Lemma 2.1] the group $\mathbf{G}$ is split in the sense of [DG70, XXII 1.13]. Now it follows from [Tho87, Theorem 3.1 and Corollary 3.2] that $\mathbf{G}$ is a closed subgroup scheme of some $\mathrm{GL}_n/\mathcal{O}_r^{\mathrm{ur}}$.

Remark 3.1.2. For a smooth affine group scheme $\mathbb{H}$ over $\mathcal{O}_r$, it is known that

$$|\mathbb{H}(\mathcal{O}_r)| = |\mathbb{H}(\mathbb{F}_q)| \cdot q^{(r-1)\cdot \dim \mathbb{H}_k},$$

where $\mathbb{H}_k$ denotes the geometric fibre over k. This follows from the fact that the kernels of reduction maps admit an affine space structure; see [Gre63, 3. Remark].

We describe in the below some properties of Greenberg functors; they were used in [Sta09] to generalise the higher Deligne–Lusztig theory from the case of $\mathcal{O} = \mathbb{F}_q[[\pi]]$ (which is in [Lus04]) to the case of a general $\mathcal{O}$, and we will work in this generalised situation. Here the proofs are omitted; see the original texts [Gre61] and [Gre63], as well as the more modern treatments [Sta12] and [BDA16], for the details and further properties.

Let A be an artinian local ring with perfect residue field $\mathrm{res}(A)$. The Greenberg functor is a functor from the category of schemes of finite type over A to the category of schemes of finite type over $\mathrm{res}(A)$, denoted by $\mathcal{F}\colon X \mapsto \mathcal{F}X$, with the property that $(\mathcal{F}X)(\mathrm{res}(A)) = X(A)$ in a canonical way. This allows us to translate certain questions concerning schemes over A to questions concerning schemes over $\mathrm{res}(A)$. (Actually $\mathcal{F}$ can be defined on larger categories, but for our purpose we can focus on finite type schemes.) The functor $\mathcal{F}$ has many nice properties. For example, it preserves affineness, separatedness, and smoothness, of objects; it preserves both closed immersions and open immersions, and it preserves fibre products. Moreover, $\mathcal{F}$ takes a finite type group scheme over A to a finite type group scheme over $\mathrm{res}(A)$; in this case the set-theoretical identification $(\mathcal{F}X)(\mathrm{res}(A)) = X(A)$ is an isomorphism of abstract groups.

Example 3.1.3. Recall that $W_r(\mathbb{F}_q) \cong \mathcal{O}_F/p^r$ and $W_r(\overline{\mathbb{F}}_q) \cong \mathcal{O}_{F^{\mathrm{ur}}}/p^r$, where F (resp. F^{ur}) is the unramified extension of $\mathbb{Q}_p$ of degree $\log_p q$ (resp. the maximal unramified extension of $\mathbb{Q}_p$). In particular we can view $\mathrm{GL}_n(\mathbb{Z}_p^{\mathrm{ur}}/p^r) = \mathrm{GL}_n(W_r(\overline{\mathbb{F}}_p))$ as the $\overline{\mathbb{F}}_p$-points of an algebraic group over $\overline{\mathbb{F}}_p$; this is how the Greenberg functor works in this special case.

The Frobenius element in $\mathrm{Gal}(k/\mathbb{F}_q)$ extends to an automorphism of $\mathcal{O}_r^{\mathrm{ur}}$, and then (by taking tensor product with $\mathrm{Id}_{\mathbb{G}}$) to $\mathbf{G}$. By the Greenberg functor this "arithmetic Frobenius" on $\mathbf{G}$ gives an $\mathbb{F}_q$-rational structure to the algebraic group $G = G_r := \mathcal{F}\mathbf{G}$; we denote the associated geometric Frobenius by F. One has

$$\mathbb{G}(\mathcal{O}_r) \cong G_r^F \quad \text{and} \quad \mathbf{G}(\mathcal{O}_r^{\mathrm{ur}}) \cong G_r(k)$$

as abstract groups. We denote by $L\colon g \mapsto g^{-1}F(g)$ the Lang map associated to F. For each $i \in \mathbb{Z} \cap [1, r]$, we have the reduction map $\rho_{r,i}\colon G \to G_i$ modulo π^i, which is a surjective morphism between algebraic groups; we denote the kernel subgroup by $G^i = G_r^i$. For convenience we put $G^0 := G$; do not confuse it with the identity component notation G°. Similar notation applies to closed subgroups of G.

From now on, let $\mathbf{T}$ be a maximal torus of $\mathbf{G}$ such that the Greenberg functor image $T = \mathcal{F}\mathbf{T}$ is F-stable. Take $\mathbf{B}$ a Borel subgroup of $\mathbf{G}$ containing $\mathbf{T}$, then one has a Levi decomposition $\mathbf{B} = \mathbf{U}\mathbf{T}$, and $\mathcal{F}$ gives a semi-direct product $B = UT$ of algebraic groups over k, where $B = \mathcal{F}\mathbf{B}$ and $U = \mathcal{F}\mathbf{U}$. We denote by $\Phi = \Phi(\mathbf{G}, \mathbf{T})$ the root system of $\mathbf{T}$; note that $\mathbf{B}$ determines the positive roots Φ^+ and the negative roots Φ^-. The opposite unipotent group to $\mathbf{U}$ is denoted by $\mathbf{U}^-$, and its Greenberg functor image is denoted by U^-. For every root $\alpha \in \Phi$, we write $\mathbf{T}^\alpha$ for the image group scheme of the coroot $\check{\alpha}\colon \mathbb{G}_{m/\mathcal{O}_r^{\mathrm{ur}}} \to \mathbf{T}$ (this is a closed subgroup of $\mathbf{T}$; see [DG70, VIB 1.2]), and write $\mathbf{U}_\alpha$ for the unipotent root subgroup; their Greenberg functor images are denoted in the natural way. For convenience, we write $\mathcal{T}^\alpha$ for $(T^\alpha)^{r-1}$.

Remark 3.1.4. Concerning the notion of Weyl group, one has $W(T_1) := N_{G_1}(T_1)/T_1 \cong N_G(T)/T$: This follows from the fact that $\mathbf{G}$ is split with respect to every maximal torus; see [DG70, XXII 3.4]. So we put $W(T) := W(T_1)$.

Remark 3.1.5. Note that one can always find a $\mathbf{T}$ with T being F-stable: Firstly, T is always a Cartan subgroup, i.e. the centraliser of a maximal torus of G, and every Cartan subgroup of G comes in this way (see [Sta12]). Meanwhile, it is clear that if a maximal torus is rational, then so is its centraliser. Now the existence of such a $\mathbf{T}$ follows from the Lang–Steinberg theorem; see e.g. [DM91, 3.15].

3.1.2 Higher Deligne–Lusztig representations

Along with the above notation, we are interested in the following $\overline{\mathbb{F}}_q$-variety:

$$S_{T,U} = L^{-1}(FU) = \{g \in G \mid g^{-1}F(g) \in FU\}.$$

Note that $G^F \times T^F$ acts on $S_{T,U}$ naturally by $(g,t)\colon x \mapsto gxt$, so $G^F \times T^F$ also acts on the compactly supported ℓ-adic cohomology groups $H_c^i(S_{T,U}, \overline{\mathbb{Q}}_\ell)$ (here $\ell \neq p$).

For any $\theta \in \widehat{T^F} = \mathrm{Hom}(T^F, \overline{\mathbb{Q}}_\ell^\times)$, we denote by $H_c^i(S_{T,U}, \overline{\mathbb{Q}}_\ell)_\theta$ the θ-isotypical part of $H_c^i(S_{T,U}, \overline{\mathbb{Q}}_\ell)$. This $\overline{\mathbb{Q}}_\ell$-linear subspace is a G^F-submodule of $H_c^i(S_{T,U}, \overline{\mathbb{Q}}_\ell)$. We write $H_c^i(-)$ (resp. $H_c^*(-)$) as shorthand for $H_c^i(-, \overline{\mathbb{Q}}_\ell)$ (resp. the formal sum $\sum(-1)^i H_c^i(-, \overline{\mathbb{Q}}_\ell)$).

Definition 3.1.6. The virtual G^F-representation

$$R_{T,U}^\theta := \sum_{i \in \mathbb{Z}} (-1)^i H^i(S_{T,U}, \overline{\mathbb{Q}}_\ell)_\theta.$$

is called a *(higher) Deligne–Lusztig representation* with respect to (T, θ); we also denote its character by the same notation.

By introducing the notion of regularity of characters, one obtains a generalisation of the orthogonality relation and the Macdonald conjecture (theorem). These are the main results proved in [Lus04] and [Sta09].

Definition 3.1.7. A character $\theta \in \widehat{T^F}$ is called *regular* if it is non-trivial on $N_F^{F^a}((\mathcal{T}^\alpha)^{F^a})$ for every root $\alpha \in \Phi$ of $\mathbf{T}$, where $N_F^{F^a}(t) := t \cdot F(t) \cdots F^{a-1}(t)$ is the norm map and a is some positive integer such that $F^a(\mathcal{T}^\alpha) = \mathcal{T}^\alpha$ for $\forall \alpha$.

This definition is independent of the choice of a (see [Lus04, 1.5] and [Sta09, 2.8]). We collect the main results of [Lus04] and [Sta09] below.

Theorem 3.1.8. *Let $\theta \in \widehat{T^F}$ be a regular character, then as a G^F-representation, $R_{T,U}^\theta$ is independent of the choice of $\mathbf{U}$. And if θ is moreover in general position (i.e. no non-trivial element in $W^F(T)$ fixes θ), then $R_{T,U}^\theta$ is irreducible (up to a sign). If (T,θ) and (T',θ') are not G^F-conjugate, then the inner product between $R_{T,U}^\theta$ and $R_{T',U'}^{\theta'}$ is zero.*

Proof. See [Lus04] for the function field case and see [Sta09] for the general case. □

The classical Deligne–Lusztig theory developed in [DL76] is a geometric approach to the representation theory of reductive groups over $\mathcal{O}_1$. Lusztig suggested a generalisation of this construction to $\mathcal{O}_r$ in [Lus79]. In [Lus04] Lusztig proved several fundamental results in the case $\mathcal{O}_r = \mathbb{F}_q[[\pi]]/\pi^r$. By applying the Greenberg functor technique, Stasinski generalised Lusztig's results to a general $\mathcal{O}_r$ in [Sta09]. In the above we have recalled this generalised construction, and in the following we will work in this general framework.

3.2 Some basic properties

In this section we consider some properties of $R_{T,U}^\theta$ along the classical Deligne–Lusztig theory; a nice reference is [Car93].

3.2.1 Character formula and unipotent elements

Proposition 3.2.1. *There is a character formula: For any $g \in G^F$, one has*

$$R_{T,U}^\theta(g) = \frac{1}{|T^F|} \cdot \sum_{t \in T^F} \theta(t^{-1}) \cdot \mathcal{L}((g,t), S_{T,U}).$$

Proof. Since T^F is a finite abelian group, the proof is the same as in the classical case; see e.g. [Car93, 7.2.2 and 7.2.3]. □

Let $u \in G^F$ be a unipotent element, then since $T^F \cong (T_1)^F \times (T^1)^F$ (note that T^F is abelian), the above implies

$$\begin{aligned} R_{T,U}^\theta(u) &= \frac{1}{|T^F|} \sum_{t \in T^F} \theta(t^{-1}) \mathcal{L}((u,t), S_{T,U}) = \frac{1}{|T^F|} \sum_{t' \in (T_1)^F, t'' \in (T^1)^F} \theta(t^{-1}) \mathcal{L}((u, t''t'), S_{T,U}) \\ &= \frac{1}{|T^F|} \sum_{t' \in (T_1)^F, t'' \in (T^1)^F} \theta(t^{-1}) \mathcal{L}((u,t''), (S_{T,U})^{t'}) = \frac{1}{|T^F|} \sum_{t'' \in (T^1)^F} \theta(t'')^{-1} \mathcal{L}((u,t''), S_{T,U}), \end{aligned}$$

where the third equality follows from Deligne and Lusztig's fixed point formula (see 2.2.2). So the value $R^{\theta}_{T,U}(u)$ only depends on $\theta|_{(T^1)^F}$, a character of a commutative unipotent group; this is the generalised version of [Car93, 7.2.9]. In particular, for $\theta \in \widehat{T^F}$ such that $\theta|_{(T^1)^F} = 1$, one has $R^{\theta}_{T,U}(u) = R^{1}_{T,U}(u)$.

3.2.2 Compatibility with parabolic induction

During a summer school lunch in 2015, Eitan Sayag asked the author whether the higher Deligne–Lusztig representations are compatible with parabolic inductions. One way to formulate this question in a more precise form is as follows.

Let $\mathbf{P}$ be a parabolic subgroup of $\mathbf{G}$ containing $\mathbf{B}$, such that its Greenberg functor image P is F-rational; let $\mathbf{M}$ be a Levi subgroup of $\mathbf{P}$ containing $\mathbf{T}$, such that its Greenberg functor image M is F-rational. Then one can consider the G^F-representation $\mathrm{Ind}_{P^F}^{G^F} \widetilde{R}^{\theta}_{T,U\cap M}$ for each $\theta \in \widehat{T^F}$, where $\widetilde{R}^{\theta}_{T,U\cap M}$ is the trivial extension of $R^{\theta}_{T,U\cap M}$ to P^F. A natural question is whether it coincides with $R^{\theta}_{T,U}$.

Note that if this question is positive, then in some sense, the study of higher Deligne–Lusztig representations can be reduced to the studies of the cuspidal case and decompositions of parabolic induced representations. In the $r = 1$ case this was proved as [DL76, 8.2]; an expanded version of the argument can be found in [Car93, 7.4.4]. Actually the argument for the $r = 1$ case also works for a general r; we illustrate this point below.

Proposition 3.2.2. *Along with the above notation, one has*

$$R^{\theta}_{T,U} = \mathrm{Ind}_{P^F}^{G^F} \widetilde{R}^{\theta}_{T,U\cap M}$$

for every $\theta \in \widehat{T^F}$.

Proof. Let P_i, $i = 1, \cdots$ (with $P_1 = P$) be the distinct conjugates of P by elements of G^F. Consider $S_i := \{g \in S_{T,U} \mid {}^gP = P_i\}$ (these are closed subvarieties as they are defined by closed conditions). Note that for any $g \in S_{T,U}$, one has $L(g) \in FU \subseteq FB \subseteq P$, so the Lang–Steinberg theorem implies the existence of $p \in P$ with $L(g) = L(p)$; in particular $gp^{-1} \in G^F$, and hence ${}^gP$ is one of P_i. Therefore we have a finite partition

$$S_{T,U} = \coprod_i S_i.$$

Clearly, the left G^F-action permutes $\{S_i\}_i$, and the right T^F-action preserves each S_i.

As $N_G(P) = P$ (by [DG70, XXII 5.8.5] we get the equality on scheme level, then use [Sta12, 4.15]), the stabiliser of S_1 in G^F is P^F. Then by Proposition 3.2.1 and [Car93, 7.1.7] we see $R^{\theta}_{T,U}$ is the character induced by

$$p \mapsto \frac{1}{|T^F|} \sum_{t\in T^F} \theta(t^{-1})\mathcal{L}((p,t), S_1),$$

a character of P^F.

Now consider $S_P := M \cap L^{-1}(FU)$; the quotient morphism $P \to M$ induces a morphism $S_1 \to S_P$ given by $g \mapsto gU_P$, where U_P is the Greenberg functor image of the unipotent radical of $\mathbf{P}$. Note that this is a surjective morphism with fibres being isomorphic to the affine space U_P, and it induces a $P^F \times T^F$-action from S_1 to S_P, so

$$\frac{1}{|T^F|}\sum_{t\in T^F}\theta(t^{-1})\mathcal{L}((p,t),S_1) = \frac{1}{|T^F|}\sum_{t\in T^F}\theta(t^{-1})\mathcal{L}((p,t),S_P)$$

for every $p \in P^F$. The right hand side is the trivial extension of the character

$$m \mapsto \frac{1}{|T^F|}\sum_{t\in T^F}\theta(t^{-1})\mathcal{L}((m,t),S_P) = R^{\theta}_{T,M\cap U}(m)$$

(of M^F) to P^F, since the left $(U_P)^F$-action on S_P is trivial (note that the P^F-action on S_P comes from the quotient morphism). Therefore $R^{\theta}_{T,U} = \mathrm{Ind}_{P^F}^{G^F}\widetilde{R}^{\theta}_{T,U\cap M}$, as desired. □

Remark 3.2.3. From the above argument we see that the involved $S_{T,U}$ is usually disconnected, as each S_i is a closed subvariety and usually G^F does not normalise P; however, this argument does not apply to the case that T is not contained in any rational P.

3.3 Towards a dimension formula

At even levels (i.e. r is even), the dimension formula of generic higher Deligne–Lusztig representations follows from the algebraisation theorems in the next chapter; see Corollary 4.3.6 and Corollary 4.3.10. However, if $r \neq 1$ is odd, the dimension formula is in general unknown. While the algebraisation for r odd is not yet available, there is another (very different) approach towards a dimension formula the author took in his early stages in the PhD studies; in this section we present some ideas in this different approach, and in the next section we present some constructions motivated by these efforts. Indeed, by studying the fibres of the reduction maps, Lusztig obtained the dimension formula for $\mathrm{SL}_2(\mathbb{F}_q[[\pi]]/\pi^2)$; the works in these two sections can be viewed as attempts to generalise Lusztig's method for general groups, thus also admit their own interests as independent research topics.

3.3.1 A review of the classical case

Let us start with introducing the dimension formula in classical Deligne–Lusztig theory. In this subsection, we make the temporary assumption that $r = 1$; in particular, $\mathbf{G} = G$ is a connected reductive group over k. Let T_0 be a rational maximal torus contained in a rational Borel B_0. We refer to [DM91] for the details.

The *duality functor* for the $\overline{\mathbb{Q}}_\ell$ ($\cong \mathbb{C}$ as a field) -valued class functions on G^F, associated to T_0, is defined to be

$$D_G = \sum (-1)^{r(N)} R_N \circ {}^*R_N,$$

where the sum runs over the rational parabolic subgroups containing B_0, N denotes an arbitrary rational Levi subgroup of the parabolic, $r(N)$ denotes the semisimple $\mathbb{F}_q$-rank (i.e. the dimension of a maximal split subtorus of a maximal quasi-split torus of the quotient of the algebraic group by its radical) of N, and R_N (resp. ${}^*R_N$) denotes the associated Lusztig induction (resp. restriction, i.e. right adjoint) functor; this definition does not depend on the choice of the Borel and the N's. The *Steinberg character* of G^F is defined to be

$$\mathrm{St}_G = D_G(\mathrm{Id}_{G^F}).$$

It is known that St_G is irreducible. The Steinberg character is "orthogonal" to Deligne–Lusztig characters in the sense that it is always trivial on non-trivial unipotent elements while Deligne–Lusztig characters admit a Green function formula (see Theorem 2.2.12). There are three properties of St_G used for establishing the dimension formula:

1. For a non-trivial unipotent element $u \in G^F$, one has $\mathrm{St}_G(u) = 0$.
2. Given a rational maximal torus T, one has
$$^*R_T\mathrm{St}_G = \varepsilon_G\varepsilon_T\mathrm{St}_T = \varepsilon_G\varepsilon_T\mathrm{Id}_T,$$
where $\varepsilon_H := (-1)^{\mathbb{F}_q\text{-rank}(H)}$ for any rational closed subgroup H of G.
3. The dimension of St_G is $|G^F|_p$, the p-part of the cardinality.

Theorem 3.3.1. *One has* $\dim R_T^\theta = R_T^\theta(1) = \varepsilon_G\varepsilon_T|G^F|_{p'}/|T^F|$.

Proof. (This is essentially from [DM91, 12.9].) By the second property listed above we see

$$\langle R_T^\theta, \mathrm{St}_G\rangle_{G^F} = \varepsilon_G\varepsilon_T\delta_{1,\theta}.$$

Taking summation we get

$$\langle \sum_{\theta\in\widehat{T^F}} R_T^\theta, \mathrm{St}_G\rangle_{G^F} = \varepsilon_G\varepsilon_T;$$

note that $\sum_{\theta\in\widehat{T^F}} R_T^\theta$ is afforded by the virtual G^F-module $H_c^*(L^{-1}(FU))$. Thus the Deligne–Lusztig fixed point formula for Jordan decomposition (see Theorem 2.2.2) implies

$$\varepsilon_G\varepsilon_T = \frac{1}{|G^F|}\cdot \sum_{u \text{ unipotent in } G^F} \left(\sum_{\theta\in\widehat{T^F}} R_T^\theta(u)\cdot \mathrm{St}_G(u)\right),$$

since $L^{-1}(FU)$ cannot be fixed by any non-trivial semisimple element. Note that $\dim R_T^\theta$ does not depend on $\theta \in \mathrm{Irr}(T^F)$ by the Green function formula (see Theorem 2.2.12), so the first and the third properties of St_G listed above imply

$$\varepsilon_G\varepsilon_T = \frac{|T^F|}{|G^F|}\dim R_T^\theta\cdot \mathrm{St}_G(1) = \frac{|T^F|}{|G^F|_{p'}}\dim R_T^\theta.$$

This completes the proof. □

Remark 3.3.2. A first idea on establishing an analogue of the above dimension formula for general r is to develop the needed properties to get an analogous proof. The main step is to construct a Steinberg character for higher r such that the three properties involved in the $r = 1$ case have useful analogues. Actually, Lees has considered a generalisation of the Steinberg character for general linear groups for higher r in [Lee78], and Campbell gave another generalisation for general linear groups in [Cam04]. Later, Campbell also considered the general Chevalley group case in [Cam07]. These constructions emphasised different analogous properties of the classical Steinberg character. However, after some computations we found their constructions, as well as some variations, do not work for our purpose, and the reason is roughly that the "unipotent radical" of reductive groups over a finite ring can be very large, which results that the first property listed above is hopeless for these analogues for higher r.

3.3.2 Reduction maps and fibres

One way to study Deligne–Lusztig theory for higher r is to study the reduction maps between higher Deligne–Lusztig varieties, in an inductive manner, and this naturally leads to the study of their fibres.

In this subsection we assume $r > 1$, and let $1 \leq h < r$ be an integer. When different levels are involved, we may denote the (higher) Deligne–Lusztig varieties at level $1 \leq i \leq r$ by $S^i_{T,U} := S_{T_i,U_i}$; note that the reduction map $\rho_{r,i}$ naturally restricts to these varieties

$$\rho_{r,i} \colon S^r_{T,U} \longrightarrow S^i_{T,U}.$$

Note that $\rho^{-1}_{h+1,h}(1) \cap S^{h+1}_{T,U}$ is a commutative closed subgroup of G_{h+1}.

Lemma 3.3.3. *The reduction map $\rho_{h+1,h} \colon S^{h+1}_{T,U} \longrightarrow S^h_{T,U}$ is surjective.*

Proof. There is a commutative diagram

$$\begin{array}{ccc} S^{h+1}_{T,U} & \xrightarrow{L} & F(U_{h+1}) \\ \downarrow{\scriptstyle \rho_{h+1,h}} & & \downarrow{\scriptstyle \rho_{h+1,h}} \\ S^h_{T,U} & \xrightarrow{L} & F(U_h). \end{array}$$

Note that all the arrows are known to be surjective except for the left $\rho_{h+1,h}$ (the surjectivity of the right $\rho_{h+1,h}$ follows from the smoothness of $\mathbf{U}$ and Grothendieck's infinitesimal lifting of formal smoothness; see e.g. [Liu06, Proposition 6.2.15]). Suppose $a \in S^h_{T,U}$, $L(a) = a'$, $\rho_{h+1,h}(b') = a'$, and $L(b) = b'$, then $L(\rho_{h+1,h}(b)) = \rho_{h+1,h}(L(b)) = a' = L(a)$ implies $a = g \cdot \rho_{h+1,h}(b)$ for some $g \in G^F_h$. Suppose $\rho_{h+1,h}(g_1) = g$ for $g_1 \in G^F_{h+1}$ (the existence of such a g_1 follows from the smoothness of $\mathbb{G}$), then $g_1 b \in S^{h+1}_{T,U}$ and $\rho_{h+1,h}(g_1 b) = g \cdot \rho_{h+1,h}(b) = a$. □

From now on, we let $\mathbf{B}_0 \subseteq \mathbf{G}$ be a Borel subgroup and $\mathbf{T}_0 \subseteq \mathbf{B}_0$ a maximal torus, such that their Greenberg functor images B_0 and T_0 are F-stable; we denote the unipotent radical

of $\mathbf{B}_0$ by $\mathbf{U}_0$, and denote its Greenberg functor image by U_0; furthermore, we let $\hat{\lambda} \in G$ be such that $B = \hat{\lambda} B_0 \hat{\lambda}^{-1}$ and $T = \hat{\lambda} T_0 \hat{\lambda}^{-1}$, and we write $\lambda := \rho_{r,1}(\hat{\lambda})$. The existence of such a quasi-split pair follows from the Lang–Steinberg theorem; see [DG70, XXII 5.8.5] and [DM91, 3.12]. Note that $\hat{\lambda}^{-1} F(\hat{\lambda}) = \hat{w} \in N_G(T_0)$ is a lift of some Weyl element $w \in W(T_0)$ (see also Remark 3.1.4), and $S_{T,U}$ is isomorphic to $L^{-1}(\hat{w} U_0)$, via right multiplication by $\hat{\lambda}$.

In case $\mathbb{G}$ is a general or special linear group, we take $\mathbf{B}_0$ and $\mathbf{T}_0$ to be the standard Borel subgroup and the standard maximal torus, respectively, and in this case $\hat{w}$ is taken to be a monomial matrix (for special linear groups, we allow signs) and the lift of the trivial Weyl element is taken to be the identity matrix.

Following Remark 3.1.1, we view $\mathbf{G}$ as a closed subgroup scheme of some GL_n, and denote by M_n the ring of $n \times n$ matrices over $\mathcal{O}^{\mathrm{ur}}/\pi^r$. When we write GL_n or SL_n we automatically assume $n > 1$; the case $n = 1$ is on the one hand trivial and on the other hand special in the sense that there are no roots involved.

We start with a description of fibres.

Lemma 3.3.4. *Regard $G(k)$ as $\mathbf{G}(\mathcal{O}_r^{\mathrm{ur}})$. Pick $\hat{g} \in L^{-1}(\hat{w} U_0)$, and write $\bar{g} := \rho_{r,r-1}(\hat{g})$ and $\hat{g}^{-1} F(\hat{g}) = \hat{w} u \in \hat{w} U_0$. Then on closed points one can identify $\rho_{r,r-1}^{-1}(\bar{g}) \cap L^{-1}(\hat{w} U_0)$ with*

$$\{\hat{g}(I + g_{r-1}) \in G \mid \ g_{r-1} \in \pi^{r-1} \cdot \mathrm{M}_n(\mathcal{O}_r^{\mathrm{ur}}),\ I + ({}^u F(g_{r-1}) - g_{r-1}^{\hat{w}}) \in U_0\}.$$

Proof. Note that for $s \in \mathrm{M}_n(\mathcal{O}^{\mathrm{ur}}/\pi^r)$ one has $(\hat{g} + \pi^{r-1} s)^{-1} = \hat{g}^{-1} - \pi^{r-1} \hat{g}^{-1} s \hat{g}^{-1}$. Therefore

$$\begin{aligned}(\hat{g} + \pi^{r-1} s)^{-1} \cdot F(\hat{g} + \pi^{r-1} s) =& \hat{g}^{-1} F(\hat{g}) + \pi^{r-1} \hat{g}^{-1} F(s) - \pi^{r-1} \hat{g}^{-1} s \hat{g}^{-1} F(\hat{g}) \\ =& \hat{w} u + \pi^{r-1} \hat{w} u F(\hat{g}^{-1} s) - \pi^{r-1} \hat{g}^{-1} s \hat{w} u.\end{aligned}$$

The above is an element of $\hat{w} U_0$ if and only if

$$I + \pi^{r-1} u F(\hat{g}^{-1} s) u^{-1} - \pi^{r-1} \hat{w}^{-1} \hat{g}^{-1} s \hat{w} \in U_0.$$

Denote $\hat{g}^{-1} s$ by g'; note that when s runs over $\mathrm{M}_n(\mathcal{O}^{\mathrm{ur}}/\pi^r)$, so is g'. Thus

$$\begin{aligned}\rho_{r,r-1}^{-1}(\bar{g}) \cap L^{-1}(\hat{w} U_0) =& \{\hat{g} + \pi^{r-1} s \in G(k) \mid s \in \mathrm{M}_n(\mathcal{O}_r^{\mathrm{ur}}),\ L(\hat{g} + \pi^{r-1} s) \in \hat{w} U_0\} \\ =& \{\hat{g} + \pi^{r-1} \hat{g} g' \in G(k) \mid g' \in \mathrm{M}_n(\mathcal{O}_r^{\mathrm{ur}}),\ I + \pi^{r-1}({}^u F(g') - g'^{\hat{w}}) \in U_0\}.\end{aligned}$$

Denote $\pi^{r-1} g'$ by g_{r-1}, then we see

$$\rho_{r,r-1}^{-1}(\bar{g}) \cap L^{-1}(\hat{w} U_0) = \{\hat{g} + \hat{g} g_{r-1} \in G_r \mid g_{r-1} \in \pi^{r-1} \cdot \mathrm{M}_n(\mathcal{O}_r^{\mathrm{ur}}),\ I + ({}^u F(g_{r-1}) - g_{r-1}{}^{\hat{w}}) \in U_0\};$$

the lemma follows. □

Remark 3.3.5. When trying to link cohomology of fibres and cohomology of the base space, one may be interested in the rational points (see e.g. the argument of [DM91, 10.12]). Note that for $\mathcal{O} = \mathbb{F}_q[[\pi]]$, the fibre of $\rho_{r,1}$ between Deligne–Lusztig varieties at any rational point contains a rational point, since $G_r = G^{r-1} \rtimes G_1$ in this case.

It seems $\rho_{r,1}^{-1}(1) \cap L^{-1}(\hat{w}U_0)$ can be decomposed into clopen subvarieties on which $(T_r^1)^F$ acts simply and transitively. We first check the following toy example.

Example 3.3.6. Let $\mathbb{G}$ be GL_2 over $\mathbb{F}_q[[\pi]]/\pi^2$. Suppose $T_0 = T$ (i.e. the Harish-Chandra case). By Lemma 3.3.4 we see

$$\rho_{2,1}^{-1}(1) \cap L^{-1}(\hat{w}U_0) = \left\{ \begin{bmatrix} 1+x\pi & y\pi \\ z\pi & 1+w\pi \end{bmatrix} \;\middle|\; x, z, w \in \mathbb{F}_q,\ y \in \overline{\mathbb{F}}_q \right\}.$$

Note that $t \in (T_2^1)^F$ has form $t = \begin{bmatrix} 1+a\pi & 0 \\ 0 & 1+b\pi \end{bmatrix}$, where $a, b \in \mathbb{F}_q$. So $(T_2^1)^F$ acts (right matrix multiplication) on $\rho_{2,1}^{-1}(1) \cap L^{-1}(\hat{w}U_0)$ by

$$t\colon \begin{bmatrix} 1+x\pi & y\pi \\ z\pi & 1+w\pi \end{bmatrix} \mapsto \begin{bmatrix} 1+(x+a)\pi & y\pi \\ z\pi & 1+(w+b)\pi \end{bmatrix}.$$

In particular, $\rho_{2,1}^{-1}(1) \cap L^{-1}(\hat{w}U_0)$ decomposes into q^2 copies of disjoint unions of q affine lines; these copies are parametrised by $x, w \in \mathbb{F}_q$. So we can write $\rho_{2,1}^{-1}(1) \cap L^{-1}(\hat{w}U_0) = \coprod Y_{x,w}$, where $Y_{z,w}$ is a disjoint union of q affine lines, such that $(T_2^1)^F$ acts on the set $\{Y_{x,w}\}_{x,w}$ simply and transitively. Therefore (see e.g. [DM91, 10.7, 10.8, and 10.12])

$$\mathcal{L}((1,t), \rho_{2,1}^{-1}(1) \cap L^{-1}(\hat{w}U_0)) = q \cdot \mathrm{Reg}_{(T_2^1)^F}(t).$$

By general property of the regular representation of a finite group, this is zero unless $t = 1$.

The decomposition in the above example gives some helpful ideas for the general case. Firstly, by Iwahori decomposition (see the version in [Sta09, Lemma 2.2]) we have

$$\begin{aligned} \rho_{r,1}^{-1}(1) \cap S_{T,U} &= \left\{ g \in G_r^1 \mid L(g) \in FU_r^1 \right\} = \{ u^- tu \in (U^-)_r^1 \cdot T_r^1 \cdot U_r^1 \mid L(u^- tu) \in FU_r^1 \} \\ &= \{ u^- ut \in (U^-)_r^1 \cdot U_r^1 \cdot T_r^1 \mid L(u^- ut) \in FU_r^1 \}. \end{aligned}$$

Denote by Y_t the closed subvariety $\rho_{r,1}^{-1}(1) \cap S_{T,U} \cap (U^-)_r^1 \cdot U_r^1 \cdot t$ for any $t \in T_r^1$; the varieties Y_t and $Y_{t'}$ are disjoint for $t \neq t'$, and $Y_{t'} \cong Y_t$ if $t/t' \in (T_r^1)^F$. Fix a set R of representatives of $T_r^1/(T_r^1)^F$ (a group, because T_r is commutative). Consider $Y^R := \bigcup_{\tilde{t} \in R} Y_{\tilde{t}}$ and its translations $Y^R \cdot t$, where $t \in (T_r^1)^F$. Then we get a set-theoretic finite decomposition of closed points:

$$\rho_{r,1}^{-1}(1) \cap S_{T,U} = \coprod_{t \in (T_r^1)^F} Y^R \cdot t;$$

note that $(T_r^1)^F$ acts simply and transitively on $\{Y^R \cdot t\}_{t \in (T_r^1)^F}$. It would be interesting to know whether these are decompositions into closed subvarieties rather than just subsets (or more precisely, in R, there are only finitely many t such that Y_t non-empty).

There is a similar decomposition for $\rho_{r,r-1}^{-1}(1) \cap S_{T,U}$, which is much easier: Consider $u^- tu \in \rho_{r,r-1}^{-1}(1) \cap S_{T,U}$, then $L(u^- tu) \in FU$ implies $t \in (T^{r-1})^F$ by the uniqueness of Iwahori decomposition and by the fact that G^{r-1} is abelian. In particular, $\rho_{r,r-1}^{-1}(1) \cap S_{T,U}$ is a disjoint union of $(T^{r-1})^F$-copies of an affine space; this is a variant version of the "crucial lemma" in [Lus79].

3.3.3 Two by two special linear groups

Throughout this subsection we assume $\mathcal{O} = \mathbb{F}_q[[\pi]]$, $\mathbb{G} = \mathrm{SL}_2$, and $\mathbf{T}$ non-split. We study the fibres (of Deligne–Lusztig varieties) along $\rho_{r,r-1}$ in this special case.

Put $\hat{w} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$; take $\hat{g} \in L^{-1}(\hat{w}U_0)$ and let $\bar{g} := \rho_{r,r-1}(\hat{g})$; suppose $\rho_{r-1,1}(\bar{g}^{-1}F(\bar{g})) = \hat{w}u' = \hat{w}\begin{bmatrix} 1 & u_0 \\ 0 & 1 \end{bmatrix}$ (so $u_0 \in \overline{\mathbb{F}}_q$). Then Lemma 3.3.4 implies

$$\rho_{r,r-1}^{-1}(\bar{g}) \cap L^{-1}(\hat{w}U_0) = \left\{ \hat{g}(I + g_{r-1}\pi^{r-1}) \;\middle|\; g_{r-1} = \begin{bmatrix} a & -c^q \\ c & -a \end{bmatrix} \in M_2(\overline{\mathbb{F}}_q),\ a = -a^q - u_0 c^q \right\}.$$

So there are two cases:

(i) When $u_0 = 0$, i.e. when $\bar{g} \in \rho_{r-1,1}^{-1}(G_1^F \cdot \lambda)$, the fibre is $\mathbb{A}^1 \times \{a \in \overline{\mathbb{F}}_q \mid a + a^q = 0\}$;

(ii) Otherwise, the fibre is $\mathbb{A}^1$ (consider the variable change $c \mapsto (c-a)/\sqrt[q]{u_0}$).

In case (i), we want to understand how $(T^{r-1})^F$ acts (from the right hand side) on the components of the fibre. Pick $\begin{bmatrix} t_0 & 0 \\ 0 & t_0^{-1} \end{bmatrix} = I + \pi^{r-1}t' \in \hat{\lambda}^{-1}(T^{r-1})^F\hat{\lambda}$; note that $t' = \begin{bmatrix} t_0' & 0 \\ 0 & -t_0' \end{bmatrix} \in M_2(\overline{\mathbb{F}}_q)$ satisfies $t_0' + {t_0'}^q = 0$. Writing the elements in $\rho_{r,1}^{-1}(G_1^F\lambda) \cap L^{-1}(\hat{w}U_0)$ as triples $(\hat{g}, a, c)$, a direct computation shows the $(T^{r-1})^F$-action on $\rho_{r,1}^{-1}(G_1^F\lambda) \cap L^{-1}(\hat{w}U_0)$ is

$$\begin{bmatrix} t_0 & 0 \\ 0 & t_0^{-1} \end{bmatrix} : (\hat{g}, a, c) \mapsto (\hat{g}, a + t_0', c). \tag{3.1}$$

In particular, $(T^{r-1})^F$ acts on a simply transitively.

Suppose $\theta \in \widehat{T^F}$ is regular, then since $(T^{r-1})^F$ acts trivially on $H_c^*(S_{T,U} \setminus \rho_{r,1}^{-1}(G_1^F))_\theta \cong H_c^{*-2}(S_{T,U}^{r-1} \setminus \rho_{r-1,1}^{-1}(G_1^F))_\theta$, we see $H_c^*(S_{T,U})_\theta = H_c^*(\rho_{r,1}^{-1}(G_1^F\lambda) \cap L^{-1}(\hat{w}U_0))_\theta$.

Example 3.3.7. For $r = 2$, Lusztig sketched in [Lus04] a wedge-product argument to show $\dim R_{T,U}^\theta = q^2 - q$ for θ regular. We give an explicit computation here. Firstly, from above the variety $\rho_{r,1}^{-1}(G_1^F\lambda) \cap L^{-1}(\hat{w}U_0) = \rho_{2,1}^{-1}(G_1^F\lambda) \cap L^{-1}(\hat{w}U_0)$ can be viewed as a line bundle over $G_1^F\lambda \times \{a \in \overline{\mathbb{F}}_q \mid a + a^q = 0\}$, thus by basic properties of ℓ-adic cohomology we see

$$H_c^*(\rho_{2,1}^{-1}(G_1^F\lambda) \cap L^{-1}(\hat{w}U_0)) = \overline{\mathbb{Q}}_\ell[G_1^F\lambda \times \{a \in \overline{\mathbb{F}}_q \mid a + a^q = 0\}].$$

A basis of this vector space is $(g\lambda, a)$, where $g \in G_1^F$ and $a \in \overline{\mathbb{F}}_q$ satisfying $a^q + a = 0$. Suppose $\sum c_{g,a}(g\lambda, a)$ is a θ-eigenvector, i.e.

$$\sum_{(g,a)} \theta(t) c_{g,a}(g\lambda, a) = \sum_{(g,a)} c_{g,a}(g\rho_{2,1}(t)\lambda, a + t_0'),\ \forall t \in T^F, \tag{3.2}$$

where t_0' is from the $(T^1)^F$-component of t (conjugated by λ). Clearly, (3.2) is equivalent to

$$c_{g\rho_{2,1}(t),a+t_0'} = \theta(t)^{-1} c_{g,a},\ \forall t \in T^F, \forall (g,a).$$

Therefore

$$H_c^*(\rho_{2,1}^{-1}(G_1^F\lambda) \cap L^{-1}(\hat{w}U_0))_\theta = \mathrm{Span}\left\{ \sum_{t\in T^F} c_{g',0}\theta(t)^{-1}(g'\rho_{2,1}(t)\lambda, t_0') \right\}_{g'},$$

where $g' \in G_1^F$ runs over a set of representatives of the coset G_1^F/T_1^F. In particular, for θ regular one has

$$\dim R_{T,U}^\theta = \dim H_c^*(\rho_{2,1}^{-1}(G_1^F\lambda) \cap L^{-1}(\hat{w}U_0))_\theta = \frac{|G_1^F\lambda \times \{a \mid a + a^q = 0\}|}{|T^F|} = q^2 - q.$$

3.4 Some subvarieties

In this section we introduce a family of subvarieties, called essential parts, of higher Deligne–Lusztig varieties. This construction is motivated by the studies in Subsection 3.3.3.

3.4.1 Essential parts and primitivity

Definition 3.4.1. The variety $E_{T,U,h} := \rho_{r,h}^{-1}(G_h^F \cdot \rho_{r,h}(\hat{\lambda})) \cap L^{-1}(\hat{w}U_0)$ is called the *essential part* of $L^{-1}(\hat{w}U_0)$ at level h. For each $\theta \in \widehat{T^F}$, we denote by $R_{T,U,h}^\theta$ the representation $H^*(E_{T,U,h})_\theta$ of G^F, and call it an essential part representation.

Remark 3.4.2. Note that $E_{T,U,h} \cong \rho_{r,h}^{-1}(G_h^F) \cap S_{T,U}$ by the conjugation by $\hat{\lambda}$.

From the computations in Subsection 3.3.3, we see that:

Proposition 3.4.3. *For $\mathbb{G} = \mathrm{SL}_2$ and $\mathcal{O} = \mathbb{F}_q[[\pi]]$. If $\mathbf{T}$ is non-split and $\theta \in \widehat{T^F}$ is regular. Then $R_{T,U}^\theta = R_{T,U,1}^\theta$.*

A representation space V of G_{r-1}^F can also be acted on by G_r^F via the reduction map; following the terminology in [Sha04], we call a representation of G^F NOT of this type a *primitive representation* of G^F.

Proposition 3.4.4. *For each $i \in \mathbb{Z}$, if the right $(T^{r-1})^F$-action on $H_c^i(E_{T,U,h})$ is not trivial, then $H_c^i(E_{T,U,h})$ is primitive.*

Proof. It suffices to show that the left action of some non-trivial subgroup of $(G^{r-1})^F$ is non-trivial on $H_c^i(E_{T,U,h})$. Write $G_h^F = \{g_j\}_j$. Let E_j be the pre-image of $g_j \in G_h$ in $E_{T,U,h}$. Then $E_{T,U,h}$ is the disjoint union of the varieties E_j. So $H_c^i(E_{T,U,h})$ is the direct sum of $H_c^i(E_j)$ for all j (see e.g. [DM91, 10.7]). Without loss of generality, suppose $g_1 = 1$.

For any $e \in E_1$ and any $t \in (T^{r-1})^F$ we have $t \cdot e = e \cdot t$, so the left $(T^{r-1})^F$-action and the right $(T^{r-1})^F$-action coincide on E_1. Therefore

$$(T^{r-1})^F \backslash E_1 \cong E_1/(T^{r-1})^F$$

as varieties, and hence

$${}^{(T^{r-1})^F}H_c^i(E_1) \cong H_c^i(E_1)^{(T^{r-1})^F}$$

as vector spaces (see e.g. [DM91, 10.10]); here both the left and the right superscripts mean that we are taking the subspace of the vectors fixed by $(T^{r-1})^F$. By our assumption $H_c^i(E_{T,U,h})^{(T^{r-1})^F}$ is a strictly smaller subspace of $H_c^i(E_{T,U,h})$, so ${}^{(T^{r-1})^F}H_c^i(E_1)$ is a strictly smaller subspace of $H_c^i(E_{T,U,h})$, which implies the primitivity. □

We expect that $R^\theta_{T,U,h}$ (for each h) "contains" $R^\theta_{T,U}$ for any regular θ, and that in the case r is even and the torus is anisotropic, they coincide. In the next subsection we present some computations; together with the results in Chapter 4 (see Proposition 4.1.8, Remark 4.1.9, and Corollary 4.3.6) they give support for this expectation.

3.4.2 Small special linear groups

In this subsection we assume $\mathbb{G} = \mathrm{SL}_n$ and $\mathcal{O} = \mathbb{F}_q[[\pi]]$.

Proposition 3.4.5. *Let $n = 2$, $\mathbf{T}$ non-split, and $\hat{w} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. Write $l := [\frac{r+1}{2}]$. Then for $\theta \in \widehat{T^F}$ non-trivial on $(T^{r-1})^F$, the representation $R^\theta_{T,U,l}$ is of dimension $(q-1)q^{2l-1}$.*

Proof. This is similar to what we did in Lemma 3.3.4. For any $g \in G_l^F \rho_{r,l}(\hat{\lambda})$ fix a lift $\hat{g} \in G^F\hat{\lambda}$. Then we have

$$\rho_{r,l}^{-1}(g) \cap L^{-1}(\hat{w}U_0) = \{\hat{g}m \in G \mid \ (\hat{g}m)^{-1}F(\hat{g}m) \in \hat{w}U_0\},$$

where $m = I + g_l\pi^l + g_{l+1}\pi^{l+1} + \cdots + g_{r-1}\pi^{r-1}$ with each $g_i \in \mathrm{M}_2(\overline{\mathbb{F}}_q)$. Note that $m^{-1} = I - g_l\pi^l - g_{l+1}\pi^{l+1} - \cdots - g_{r-1}\pi^{r-1}$. A direct computation shows

$$(\hat{g}m)^{-1}F(\hat{g}m) = \hat{w} + (\hat{w}F(g_l) - g_l\hat{w})\pi^l + \cdots + (\hat{w}F(g_{r-1}) - g_{r-1}\hat{w})\pi^{r-1},$$

and the condition "$(\hat{g}m)^{-1}F(\hat{g}m) \in \hat{w}U_0$" is thus equivalent to

$$I + (F(g_l) - g_l^{\hat{w}})\pi^l + \cdots + (F(g_{r-1}) - g_{r-1}^{\hat{w}})\pi^{r-1} \in U_0. \tag{3.3}$$

Write $g_i = \begin{bmatrix} a_i & b_i \\ c_i & -a_i \end{bmatrix}$, Then (3.3) is equivalent to

$$a_i \in \mathbb{F}_{q^2},\ a_i^q + a_i = 0,\ c_i^q + b_i = 0,\ \forall i.$$

Therefore

$$\begin{aligned}\rho_{r,l}^{-1}(g)\cap L^{-1}(\hat{w}U_0) &\cong \prod_{i=l}^{r-1}\{(a_i,c_i)\mid\ a_i,c_i\in\overline{\mathbb{F}}_q, a_i+a_i^q=0\}\\ &\cong \mathbb{A}^{r-l}\times\prod_{i=l}^{r-1}\{a_i\mid\ a_i\in\overline{\mathbb{F}}_q, a_i+a_i^q=0\}\end{aligned}$$

as varieties. Now the same arguments in Subsection 3.3.3 imply that

$$\dim R^{\theta}_{T,U,l}=\frac{|G_l^F\rho_{r,l}(\hat{\lambda})\times\prod_{i=l}^{r-1}\{a_i\mid a_i+a_i^q=0\}|}{|T^F|}=(q-1)q^{2l-1}.$$

This completes the proof. □

Recall that for $\mathrm{SL}_n(\mathbb{F}_q)$, the conjugacy classes of the Weyl group $W\cong S_n$ are determined by the possible partitions of n. In particular, in the case $n=3$ there are three classes, and hence three different classes of rational maximal tori; the one corresponding to $\mathbb{F}_q^\times\times\mathbb{F}_q^\times\times\mathbb{F}_q^\times$ (provides the principal series), the one corresponding to $\mathbb{F}_{q^2}^\times\times\mathbb{F}_q^\times$ (provides the neither-principal nor-cuspidal series), and the one corresponding to $\mathbb{F}_{q^3}^\times$ (provides the cuspidal series). While the higher (and classical) Deligne–Lusztig representations only depend on the conjugacy classes of Weyl elements, the essential part representations may depend on the Weyl elements themselves.

By the same method in Proposition 3.4.5, one can compute the dimensions of essential part representations explicitly for small groups. For $n=3$ and $r=2$, we list the results below (done with the help of software Wolfram Mathematica):

(1) If $w=1$, then

$$|T^F|=q^2(q-1)^2,$$

and the fibres at $\mathrm{SL}_3(\mathbb{F}_q)$ are isomorphic to

$$\rho_{2,1}^{-1}(1)\cap S_{T,U}\cong(\mathbb{F}_q)^6\times\mathbb{A}^3.$$

Thus

$$|\dim H_c^*(E_{T,U,1})|=\dim H_c^6(E_{T,U,1})=q^9\cdot(q-1)^2\cdot(q+1)\cdot(q^2+q+1),$$

and

$$|\dim R^{\theta}_{T,U,1}|=q^7\cdot(q+1)\cdot(q^2+q+1).$$

(2) If $w\neq 1$, not a Coxeter element nor the longest Weyl element, then

$$|T^F|=q^2(q^2-1),$$

and the fibres at $\mathrm{SL}_3(\mathbb{F}_q)$ are isomorphic to

$$\rho_{2,1}^{-1}(1) \cap S_{T,U} \cong \mathbb{F}_{q^2} \times \{(a,e) \in \mathbb{F}_{q^2} \times \mathbb{F}_q \mid \ a^q + a + e = 0\} \times \mathbb{A}^2.$$

Thus

$$|\dim H_c^*(E_{T,U,1})| = \dim H_c^4(E_{T,U,1}) = q^7 \cdot (q-1)^2 \cdot (q+1) \cdot (q^2+q+1),$$

and

$$|\dim R_{T,U,1}^{\theta}| = q^5 \cdot (q-1) \cdot (q^2+q+1).$$

(3) If w is the longest Weyl element, then

$$|T^F| = q^2(q^2-1),$$

and the fibres at $\mathrm{SL}_3(\mathbb{F}_q)$ are isomorphic to

$$\rho_{2,1}^{-1}(1) \cap S_{T,U} \cong \{(a,e) \in \mathbb{F}_{q^2} \times \mathbb{F}_q \mid \ a^q + a + e = 0\} \times \mathbb{A}^3.$$

Thus

$$|\dim H_c^*(E_{T,U,1})| = \dim H_c^6(E_{T,U,1}) = q^7 \cdot (q-1)^2 \cdot (q+1) \cdot (q^2+q+1),$$

and

$$|\dim R_{T,U,1}^{\theta}| = q^3 \cdot (q-1) \cdot (q^2+q+1).$$

(4) If w is a Coxeter element, then

$$|T^F| = q^2(q^2+q+1),$$

and the fibres at $\mathrm{SL}_3(\mathbb{F}_q)$ are isomorphic to

$$\rho_{2,1}^{-1}(1) \cap S_{T,U} \cong \{a \in \mathbb{F}_{q^3} \mid \ a^{q^2} + a^q + a = 0\} \times \mathbb{A}^3.$$

Thus

$$|\dim H_c^*(E_{T,U,1})| = \dim H_c^6(E_{T,U,1}) = q^5 \cdot (q-1)^2 \cdot (q+1) \cdot (q^2+q+1),$$

and

$$|\dim R_{T,U,1}^{\theta}| = q^3 \cdot (q-1)^2 \cdot (q+1).$$

Remark 3.4.6. We note that all the computations concerning the essential parts in this subsection also work for a general GL_n and SL_n for the Coxeter tori and even r. For example, consider the Coxeter Weyl element

$$w = \begin{bmatrix} 0 & I_{n-1} \\ 1(\text{or} -1) & 0 \end{bmatrix} \in S_n.$$

The condition "$I_n + \pi(F(g) - g^w) \in U_0$" implies every non-$(i,i)$ entry is finally determined by a free variable in the upper triangular part, hence contributes an affine line, and thus in the alternating sum of cohomologies only the diagonal part matters, and $(T^{r/2})^F$ acts on this diagonal part simply and transitively. Similar things are true for the symplectic group Sp_4 with $r = 2$. However, at the current moment we don't know the precise relation between essential parts and Deligne–Lusztig representations for a general reductive group.

Chapter 4

Main results - algebraisation

In this chapter we give an affirmative answer to Lusztig's question (see Question 4.3.1) for $\mathbb{G} = \mathrm{GL}_n$ with r even. The generalisation of this result for an arbitrary reductive group is obtained later in the joint work [CS16]; we also present this work. After then, we discuss some applications to the character theory of reductive Lie algebras over a finite field. Throughout this chapter we assume r is even, and write $r = 2l$ (do not confuse l with the prime ℓ).

4.1 Some algebraic constructions

The main results in this chapter are on links between algebraic methods and geometric methods in representation theory, so we present some algebraic constructions in this section.

4.1.1 Clifford theory

Clifford theory is an efficient algebraic method to construct smooth representations of reductive groups over $\mathcal{O}$. The general idea of using Clifford theory to construct irreducible representations of $\mathrm{GL}_n(\mathcal{O})$ can be traced back to Shintani [Shi68], and was recovered by Hill, who introduced the notion of regular representations (see [Hil95]); very recently Stasinski and Stevens constructed all regular representations of $\mathrm{GL}_n(\mathcal{O})$ (see [SS16]). Let us start with a collection of the fundamentals in Clifford theory (the proofs can be found e.g. in [Isa06]):

Lemma 4.1.1 (Clifford theory)**.** *Let N be a normal subgroup of a finite group G. For any $\sigma \in \mathrm{Irr}(N)$, denote the stabiliser under conjugation by $\mathrm{Stab}_G(\sigma) := \{g \in G \mid \ \sigma^g \cong \sigma\}$. Note that $N \subseteq \mathrm{Stab}_G(\sigma)$ because characters are class functions. Then*

(i) For any $\rho \in \mathrm{Irr}(G)$, one has $\rho|_N = c \cdot \bigoplus_{g \in G/\mathrm{Stab}_G(\sigma)} \sigma^g$ for some $\sigma \in \mathrm{Irr}(N)$ and some positive integer c.

(ii) For $\sigma \in \mathrm{Irr}(N)$, consider $A := \{\sigma' \in \mathrm{Irr}(\mathrm{Stab}_G(\sigma)) \mid \langle \sigma'|_N, \sigma\rangle \neq 0\}$ and $B := \{\rho \in \mathrm{Irr}(G) \mid \langle \rho|_N, \sigma\rangle \neq 0\}$. Then $\sigma' \mapsto \mathrm{Ind}_{\mathrm{Stab}_G(\sigma)}^G \sigma'$ is bijective from A to B.

(iii) Let H be a subgroup of G containing N, and suppose $\sigma \in \mathrm{Irr}(N)$ admits an extension σ'' to H (i.e. $\exists \sigma'' \in \mathrm{Rep}(H)$ with $\sigma''|_N = \sigma$). Then

$$\mathrm{Ind}_N^H \sigma = \bigoplus_{\chi \in \mathrm{Irr}(H/N)} \chi\sigma'',$$

and each $\chi\sigma''$ is irreducible, where $\mathrm{Irr}(H/N)$ denotes the irreducible representations of H that are trivial on N.

(iv) If $\mathrm{Stab}_G(\sigma)/N$ is cyclic, there is an extension of $\sigma \in \mathrm{Irr}(N)$ to $\mathrm{Stab}_G(\sigma)$.

We give a brief description of how Clifford theory works in the case of $\mathbb{G} = \mathrm{GL}_n$ with the standard Frobenius; interested reader should refer to the detailed nice survey [Sta16]. As additive groups we have $\mathcal{O}^{\mathrm{ur}}/\pi^l \cong \pi^l\mathcal{O}^{\mathrm{ur}}/\pi^r$, which induces an isomorphism $\mathrm{M}_n(\mathcal{O}^{\mathrm{ur}}/\pi^l) \cong \mathrm{M}_n(\pi^l\mathcal{O}^{\mathrm{ur}}/\pi^r)$ of additive groups. As $x \mapsto x - I$ gives an isomorphism from G^l to $\mathrm{M}_n(\pi^l\mathcal{O}^{\mathrm{ur}}/\pi^r)$, we see $(G^l)^F \cong \mathrm{M}_n(\mathcal{O}/\pi^l)$. Explicitly, the isomorphism $(G^l)^F \cong \mathrm{M}_n(\mathcal{O}/\pi^l)$ can be viewed as the composite morphism

$$x \mapsto x - I \mapsto a_x\pi^l + b_x\pi^r \mapsto a_x + b_x\pi^l \mapsto x' \in \mathrm{M}_n(\mathcal{O}/\pi^l),$$

where $a_x\pi^l + b_x\pi^r$ is an arbitrary lift of $x - I \in \mathrm{M}_n(\pi^l\mathcal{O}/\pi^r)$ to $\mathrm{M}_n(\mathcal{O})$, and x' is the residue class of $a_x \in \mathrm{M}_n(\mathcal{O})$ modulo π^l. (In literature it is also common to write $\pi^{-l}(x - I)$ for x'.)

Now fix a $\overline{\mathbb{Q}}_\ell$-valued additive character ψ of $\mathcal{O}$ with conductor (π^l), i.e. ψ is trivial on the ideal (π^l) but not trivial on (π^{l-1}). Then we have an exact pairing

$$\mathrm{M}_n(\mathcal{O}/\pi^l) \times (G^l)^F \longrightarrow \overline{\mathbb{Q}}_\ell^\times;\ (\beta, x) \mapsto \psi(\mathrm{Tr}(\beta x')),$$

which induces an isomorphism (depending on ψ) from an "additive" group to a "multiplicative" group.

$$\mathrm{M}_n(\mathcal{O}/\pi^l) \cong \mathrm{Hom}((G^l)^F, \overline{\mathbb{Q}}_\ell^\times).$$

For $\beta \in \mathrm{M}_n(\mathcal{O}/\pi^l)$ we denote by ψ_β the corresponding image in $\mathrm{Hom}((G^l)^F, \overline{\mathbb{Q}}_\ell^\times)$. If ψ_β can be extended to an irreducible character $\widetilde{\psi_\beta}$ on $\mathrm{Stab}_{G^F}(\psi_\beta)$, i.e. $\widetilde{\psi_\beta}|_{(G^l)^F} = \psi_\beta$, then Clifford theory (Lemma 4.1.1) implies $\mathrm{Ind}_{\mathrm{Stab}_{G^F}(\psi_\beta)}^{G^F} \widetilde{\psi_\beta}$ is an irreducible character of G^F.

Remark 4.1.2. A similar description works for an arbitrary reductive group, provided the characteristic of the residue field is big enough and the trace form is replaced by a non-degenerate symmetric bilinear form on $(G^l)^F$.

4.1.2 Arithmetic radicals

In this subsection we introduce a variety which can be used to realising an algebraically constructed representation given by Gérardin (see Remark 4.1.7).

Definition 4.1.3. Consider the commutative unipotent group $U^\pm := (U^-)^l U^l$; it is called the *arithmetic radical* with respect to $\mathbf{T}$.

Lemma 4.1.4. *$U^{\pm}$ is normalised by $N_G(T)$, and it is F-rational.*

Proof. Note that $U^{\pm} = \prod_{\alpha\in\Phi} U_\alpha^l$, where $\Phi = \Phi(\mathbf{G},\mathbf{T})$ is the root system for $\mathbf{T}$ and $U_\alpha = \mathcal{F}(\mathbf{U}_\alpha)$ is the Greenberg functor image of the root subgroup of $\mathbf{U}$ for $\alpha \in \Phi$. For any $v \in W(T)$ we have $\hat{v}U_\alpha\hat{v}^{-1} = U_{v(\alpha)}$, so $\hat{v}\prod_{\alpha\in\Phi} U_\alpha^l\hat{v}^{-1} = \prod_{\alpha\in\Phi} U_{v(\alpha)}^l = \prod_{\alpha\in\Phi} U_\alpha^l$, i.e. $U^{\pm}$ is normalised by $N_G(T)$. On the other hand, note that $FU^{\pm} = FG^l/FT^l$, so the rationality of $U^{\pm}$ follows from that of G^l and of T^l. □

We want to study the variety $L^{-1}(U^{\pm})$, which is an analogue of the Deligne–Lusztig variety $L^{-1}(FU)$.

Remark 4.1.5. Indeed, there is a more general family of analogues, i.e. the varieties $L^{-1}(F(U^i(U^-)^{r-i}))$, where $i \in [0,r]\cap\mathbb{Z}$ (in this family r can also be odd). The higher Deligne–Lusztig variety is then the case with $i = 0$ or r, and the variety $L^{-1}(U^{\pm})$ is the case with r even and $i = r/2$. However, our arguments for the main results in Section 4.3 do not work directly for this general family.

Note that there is a natural $G^F \times T^F$-action on $L^{-1}(U^{\pm})$, so $H_c^*(L^{-1}(U^{\pm}))$ is a (virtual) G^F-module-T^F. Indeed, it is an actual module:

Proposition 4.1.6. *The G^F-representation $H_c^*(L^{-1}(U^{\pm}))_\theta = H_c^*(L^{-1}(FU^{\pm}))_\theta$ is isomorphic to $\mathrm{Ind}_{(TU^{\pm})^F}^{G^F}\widetilde{\theta}$ for $\forall\theta \in \widehat{T^F}$, where $\widetilde{\theta}$ is the trivial lift of θ from T^F to $(TU^{\pm})^F$ ($TU^{\pm}$ is a semi-direct product); this is a primitive representation if $\theta|_{(T^{r-1})^F} \neq 1$ (e.g. this is the case if θ is regular)*

Proof. The argument for the first part is similar to rewriting Deligne–Lusztig representations as parabolically induced representations in the Harish-Chandra case (see e.g. [DM91, Page 81]). Consider the quotient morphism $L^{-1}(U^{\pm}) \to G/U^{\pm}$. Note that $F(gU^{\pm}) = gL(g)U^{\pm} = gU^{\pm}$, so its image is $(G/U^{\pm})^F \cong G^F/(U^{\pm})^F$. As its fibres are isomorphic to an affine space, we see $H_c^*(L^{-1}(U^{\pm})) \cong \overline{\mathbb{Q}}_\ell[G^F/(U^{\pm})^F]$ by basics of ℓ-adic cohomology. Now, $\overline{\mathbb{Q}}_\ell[G^F/(U^{\pm})^F] \otimes_{\overline{\mathbb{Q}}_\ell[T^F]} \theta \cong \overline{\mathbb{Q}}_\ell[G^F] \otimes_{\overline{\mathbb{Q}}_\ell[(TU^{\pm})^F]} \widetilde{\theta}$ as $\overline{\mathbb{Q}}_\ell[G^F]$-modules, so $H_c^*(L^{-1}(U^{\pm}))_\theta \cong \mathrm{Ind}_{(TU^{\pm})^F}^{G^F}\widetilde{\theta}$.

Now the primitivity follows immediately from the Mackey intertwining formula. □

For the below remark, first note that any split reductive group over a non-archimedean local field is the base change of a reductive group over the ring of integers; this is in Tits's Corvallis article; see e.g. https://mathoverflow.net/questions/184540 for a proof.

Remark 4.1.7. Suppose $\mathbb{G}(\mathcal{O}_r)$ is the $\mathcal{O}_r$-points of a split reductive group over $\mathrm{Frac}(\mathcal{O})$, with derived subgroup being simply connected, and suppose T^F is the $\mathcal{O}_r$-points of a "special" maximal torus over $\mathrm{Frac}(\mathcal{O})$ in the sense of [Gér75, 3.3.9]. (In the case $\mathbb{G} = \mathrm{GL}_n$ or SL_n, these conditions are always satisfied; see [Gér75, 3.4.2].) Under these conditions (see [Gér75, 4.1.1]), the representations $\mathrm{Ind}_{(TU^{\pm})^F}^{G^F}\widetilde{\theta}$ were first considered by Gérardin in [Gér75], in which he proved they are irreducible if θ is regular and in general position; see [Gér75, 4.4.1]. We note that in Gérardin's work the notion of regularity of characters is formulated by the concept of conductor along Galois orbits; see [Gér75, 4.2.2 and 4.2.3].

In the case of general and special linear groups, we show that $H_c^*(L^{-1}(U^{\pm}))_\theta$ is always a sub-representation of the essential part representation $R^\theta_{T,U,l}$:

Proposition 4.1.8. *Suppose $\mathbb{G} = \mathrm{GL}_n$ or SL_n. Then $H_c^*(L^{-1}(U^{\pm}))_\theta$ and $R^\theta_{T,U,l}$ are actual representations, and $H_c^*(L^{-1}(U^{\pm}))_\theta$ is a sub-representation of $R^\theta_{T,U,l}$.*

Proof. The map $g \mapsto g\hat{\lambda}$ gives a $G^F \times T^F$-isomorphism $L^{-1}(U^{\pm}) \cong L^{-1}(\hat{w}U_0^{\pm})$, where the right T^F-action on $L^{-1}(\hat{w}U_0^{\pm})$ is twisted by conjugation by $\hat{\lambda}$. We need to compare $H_c^*(L^{-1}(\hat{w}U_0^{\pm}))_\theta$ and $H_c^*(E_{T,U,l})_\theta$. The reduction map $\rho'\colon L^{-1}(\hat{w}U_0^{\pm}) \to G_l^F\rho_{r,l}(\hat{\lambda})$ (the restriction of $\rho_{r,l}$ on $L^{-1}(\hat{w}U_0^{\pm})$) is surjective and with isomorphic fibres. A similar argument of Lemma 3.3.4 shows

$$\rho'^{-1}(\bar{g}\lambda) = \{\hat{g}\hat{\lambda}(I+g_l) \in G_r \mid \ g_l \in \pi^l \mathrm{M}_n(\mathcal{O}_r^{\mathrm{ur}}),\ I + (F(g_l) - g_l^{\hat{w}}) \in U_0^{\pm}\}$$

for every $\bar{g} \in G_l^F$, where $\hat{g} \in G^F$ is a fixed lift of $\bar{g}$. Denote the permutation on $\{1, \cdots, n\}$ corresponding to w by σ, and write $g_l = (g_{i,j})$, then the condition "$I + (F(g_l) - g_l^{\hat{w}}) \in U_0^{\pm}$" is equivalent to "$F(g_{i,i}) = \pm g_{\sigma(i),\sigma(i)}$ for all i", where the sign $\pm$ is determined by the monomial matrix $\hat{w}$. So the fibre of $\bar{g}\rho_{r,l}(\hat{\lambda}) \in G_l^F\rho_{r,l}(\hat{\lambda})$ along ρ' is isomorphic to an affine bundle over the finite set consisting of diagonal matrices $D \in \mathrm{M}_n(\mathcal{O}^{\mathrm{ur}}/\pi^r)$ subject to $I + D \in \hat{\lambda}^{-1}(T^l)^F\hat{\lambda}$. Thus

$$H_c^*(L^{-1}(\hat{w}U_0^{\pm})) \cong \overline{\mathbb{Q}}_\ell[G_l^F\rho_{r,l}(\hat{\lambda}) \times (\hat{\lambda}^{-1}(T^l)^F\hat{\lambda} - I)] \cong \overline{\mathbb{Q}}_\ell[G_l^F \times \pi^{-l}(\hat{\lambda}^{-1}(T^l)^F\hat{\lambda} - I)].$$

A basis of this vector space is $X := \{(\bar{g}, t_l) \in G_l^F \times \pi^{-l}(\hat{\lambda}^{-1}(T^l)^F\hat{\lambda} - I)\}$. For each $\bar{g} \in G_l^F$ let $\hat{g}$ be a fixed lift in G^F, then by tracing back the $G^F \times T^F$-action on the variety one can see how $(g,t) \in G^F \times T^F$ acts on this basis: Write $t = t'(I + t'')$ and $g = g'(I + g'')$, where $g'', t'' \in \pi^l \cdot \mathrm{M}_n(\mathcal{O}^{\mathrm{ur}}/\pi^r)$, such that $g'\hat{g}t'$ is a fixed lift of $\rho_{r,l}(g)\bar{g}\rho_{r,l}(t)$, then on X we have

$$(g,t)\colon (\bar{g}, t_l) \mapsto (\rho_{r,l}(g)\bar{g}\rho_{r,l}(t), d(\pi^{-l}g''^{\bar{g}t'\hat{\lambda}} + \pi^{-l}t''^{\hat{\lambda}} + t_l)),$$

where $d(x)$ is defined to be the diagonal matrix given by the diagonal of a matrix x. In particular, the right T^F-action on the basis set X is free. Thus for each $\theta \in \widehat{T^F}$, since $|X| = |G_l^F \cdot (T^l)^F|$, we see $\dim H_c^*(L^{-1}(\hat{w}U_0^{\pm}))_\theta = |G_l^F|/|T_l^F|$.

Similarly, the fibres of the essential parts along reduction maps are isomorphic to (note that $E_{T,U,l} = L^{-1}(\hat{w}U_0^l)$)

$$\rho_{r,l}^{-1}(\bar{g}\hat{\lambda}) \cap\ E_{T,U,l} = \{\hat{g}\hat{\lambda}(I+g_l) \in G_r \mid \ g_l \in \pi^l \cdot \mathrm{M}_n(\mathcal{O}_r^{\mathrm{ur}}),\ I + (F(g_l) - g_l^{\hat{w}}) \in U_0^l\},$$

which, in the same way as above, can be viewed as an affine bundle over a finite set S containing $\hat{\lambda}^{-1}(T^l)^F\hat{\lambda} - I$, thus $H_c^*(E_{T,U,l}) \cong \overline{\mathbb{Q}}_\ell[G_l^F\hat{\lambda} \times S]$, and then a similar description of the $G^F \times T^F$-action on this space implies $H_c^*(L^{-1}\hat{w}U_0^{\pm})_\theta \subseteq H_c^*(E_{T,U,l})_\theta$ for each θ. □

Remark 4.1.9. Note that, in the above proof, if $S = \hat{\lambda}^{-1}(T^l)^F\hat{\lambda} - I$, then $H_c^*(L^{-1}(U^{\pm}))_\theta \cong R^\theta_{T,U,l}$. For example, this happens if w is the Coxeter element $(1, 2, \cdots, n)$.

4.1.3 Generic characters

In a connected reductive algebraic group, one can talk about regular semisimple elements, which form an open subvariety (see e.g. [Hum95, 2.5]). Here we define a similar notion for characters of rational maximal tori. One motivation for this notion is to get a better understanding of the regularity (of characters; see Definition 3.1.7); on the other hand, Gérardin's results are for split reductive groups with simply connected derived subgroups and certain maximal tori, and we hope to find a condition of characters such that the same result is true for a general reductive group with these characters.

Definition 4.1.10. A character $\theta \in \widehat{T^F}$ is called *generic*, if it is regular, in general position, and satisfies the stabiliser condition $\mathrm{Stab}_{G^F}(\widetilde{\theta}|_{(G^l)^F}) = (TU^{\pm})^F \cdot \mathrm{Stab}_{N_G(T)^F}(\widetilde{\theta}|_{(G^l)^F})$.

This notion was introduced in [CS16], and, though it looks complicated, it is natural: Note that $\mathrm{Stab}_{N_G(T)^F}(\widetilde{\theta}|_{(G^l)^F})$ is a subgroup of $\mathrm{Stab}_{G^F}(\widetilde{\theta}|_{(G^l)^F})$, and $\mathrm{Stab}_{N_G(T)^F}(\widetilde{\theta}|_{(G^l)^F})/T^F \subseteq W(T)^F$. We remark that the genericity is closely related to regularity and being in general position; see [CS16] for a comparison result in the case of Coxeter tori of general linear groups. We also remark that, if θ is regular and in general position, then $\mathrm{Stab}_{N_G(T)^F}(\widetilde{\theta}|_{(G^l)^F})/T^F$ seems is always trivial, unless $\mathrm{char}(\mathbb{F}_q)$ is too small.

Proposition 4.1.11. *If θ is generic, then $\mathrm{Ind}_{(TU^{\pm})^F}^{G^F}\widetilde{\theta}$ is irreducible.*

Proof. (See also [CS16, 4.7].) We have

$$\mathrm{Ind}_{(TU^{\pm})^F}^{G^F}\widetilde{\theta} = \mathrm{Ind}_{(TU^{\pm})^F\cdot\mathrm{Stab}_{N_G(T)^F}(\widetilde{\theta}|_{(G^l)^F})}^{G^F}\mathrm{Ind}_{(TU^{\pm})^F}^{(TU^{\pm})^F\cdot\mathrm{Stab}_{N_G(T)^F}(\widetilde{\theta}|_{(G^l)^F})}\widetilde{\theta},$$

so by Clifford theory (see Lemma 4.1.1) it suffices to show $\mathrm{Ind}_{(TU^{\pm})^F}^{(TU^{\pm})^F\cdot\mathrm{Stab}_{N_G(T)^F}(\widetilde{\theta}|_{(G^l)^F})}\widetilde{\theta}$ is irreducible. This latter irreducibility follows immediately from the Mackey intertwining formula and the assumption that θ is in general position. □

4.2 The Harish-Chandra case

In this section we assume $B = FB$.

We investigate the Harish-Chandra case in this section. The algebraisation problem in this case is rather easy and can be established in a few lines in an algebraic way. But we also consider a geometric argument in a special case, which may serve as a toy example.

4.2.1 Algebraic approach

We want to compare $\mathrm{Ind}_{B^F}^{G^F}\widetilde{\theta}$ and $\mathrm{Ind}_{T^F(U^{\pm})^F}^{G^F}\widetilde{\theta}$, where $\theta \in \widehat{T^F}$ is extended trivially in both sides in an obvious way. By Mackey intertwining formula and Frobenius reciprocity we see

$$\langle \mathrm{Ind}_{B^F}^{G^F}\widetilde{\theta}, \mathrm{Ind}_{T^F(U^{\pm})^F}^{G^F}\widetilde{\theta}\rangle_{G^F} = \sum_{x\in B^F\backslash G^F/T^F(U^{\pm})^F} \langle \widetilde{\theta}|_{B^F\cap{}^x((TU^{\pm})^F)}, \widetilde{\theta}^x|_{B^F\cap{}^x((TU^{\pm})^F)}\rangle_{B^F\cap{}^x((TU^{\pm})^F)},$$

which is not zero (by looking at $x = 1$). Now for θ regular and in general position, we know from Deligne–Lusztig theory that $\mathrm{Ind}_{B^F}^{G^F}\theta$ is irreducible. Since the two representations $\mathrm{Ind}_{B^F}^{G^F}\widetilde{\theta}$ and $\mathrm{Ind}_{T^F(U^\pm)^F}^{G^F}\widetilde{\theta}$ have the same dimensions (note that $|(U^l)^F| = |U_l^F|$; see e.g. [DM91, 10.11]), we get

$$\mathrm{Ind}_{B^F}^{G^F}\widetilde{\theta} \cong \mathrm{Ind}_{T^F(U^\pm)^F}^{G^F}\widetilde{\theta}. \tag{4.1}$$

4.2.2 Geometric approach

The above argument on inner product is algebraic. Now let us do it in a geometric way. In this subsection we assume $\mathcal{O} = \mathbb{F}_q[[\pi]]$, $r = 2$, and $\mathbb{G} = \mathrm{SL}_2$; in the following we show (4.1) by a geometric method in this special case.

Recall that $\mathrm{Ind}_{B^F}^{G^F}\widetilde{\theta} = H_c^*(L^{-1}(U))_\theta$ and $\mathrm{Ind}_{T^F(U^\pm)^F}^{G^F}\widetilde{\theta} = H_c^*(L^{-1}(U^\pm))_\theta$. Similar to classical Deligne–Lusztig theory, by Künneth formula and adjunction we have

$$\langle H_c^*(L^{-1}(U))_\theta, H_c^*(L^{-1}(U^\pm))_\theta\rangle_{G^F} = \dim H_c^*(\Sigma)_{\theta^{-1},\theta},$$

where

$$\Sigma := \{(x, x', y) \in U^\pm \times U \times G \mid \ xF(y) = yx'\}.$$

By Bruhat decomposition we see $G = (U^-)^1TU \sqcup UwTU$ (where $w \in W(T)$ is non-trivial; we do not distinguish it with its lift to G), hence we get a decomposition into locally closed subvarieties $\Sigma = \Sigma_1 \sqcup \Sigma_2$, where Σ_1 is the subvariety with $y \in (U^-)^1TU$. Therefore by [DM91, 10.7] we only need to calculate each $\dim H_c^*(\Sigma_i)_{\theta^{-1},\theta}$.

Consider the locally trivial fibration

$$\hat{\Sigma}_1 = \{(x, x', z, \tau, u) \in U^\pm \times U \times (U^-)^1 \times T \times U \mid xF(z\tau u) = z\tau ux'\}.$$

By the variable changes $xF(z) \mapsto x$ and $x'F(u^{-1}) \mapsto x'$ we can rewrite $\hat{\Sigma}_1$ as $\{(x, x', z, \tau, u) \in U^\pm \times U \times (U^-)^1 \times T \times U \mid xF(\tau) = z\tau ux'\}$. Now the $T^F \times T^F$-action on $\hat{\Sigma}_1$ is compatible with a T_1-action given by $t\colon (x, x', z, \tau, u) \mapsto (txt^{-1}, tx't^{-1}, tzt^{-1}, \tau, tut^{-1})$, thus

$$\dim H_c^*(\Sigma_1)_{\theta^{-1},\theta} = \dim H_c^*(\hat{\Sigma}_1)_{\theta^{-1},\theta} = \dim H_c^*(\hat{\Sigma}_1^{T_1})_{\theta^{-1},\theta}.$$

Since $\hat{\Sigma}_1^{T_1} = \{(1, 1, 1, \tau, 1) \mid \tau \in T, F(\tau) = \tau\}$, we see $H_c^*(\hat{\Sigma}_1^{T_1}) \cong \overline{\mathbb{Q}}_\ell[T^F]$, on which the $T^F \times T^F$-action is $(t, t')\colon \tau \mapsto t\tau t'^{-1}$. Therefore

$$\dim H_c^*(\Sigma_1)_{\theta^{-1},\theta} = \dim \overline{\mathbb{Q}}_\ell[T^F]_{\theta^{-1},\theta} = 1.$$

We now turn to Σ_2. Similarly, consider the locally trivial fibration $\hat{\Sigma}_2 = \{(x, x', u, \tau, u') \in U^\pm \times U \times U \times T \times U \mid xF(uw\tau u') = uw\tau u'x'\}$. By variable change $u'x'F(u^{-1}) \mapsto x'$ we can rewrite $\hat{\Sigma}_2$ as $\{(x, x', u, \tau, u') \in U^\pm \times U \times U \times T \times U \mid xF(uw\tau) = uw\tau x'\}$, on which $T^F \times T^F$ acts by

$$(t, t')\colon (x, x', u, \tau, u') \mapsto (txt'^{-1}, tx't'^{-1}, tut'^{-1}, w^{-1}tw\tau t'^{-1}, tu't'^{-1}).$$

By Bruhat decomposition and the defining equation, in $\hat{\Sigma}_2$ we see $x' \in U^1$, so $\hat{\Sigma}_2 = \{(x, x', u, \tau, u') \in U^{\pm} \times U^1 \times U \times T \times U \mid xF(uw\tau) = uw\tau x'\}$. Now write $x = vv'$, where $v \in (U^-)^1$ and $v' \in U^1$, then

$$xF(uw\tau) = vv'F(u)wF(\tau) = v'F(u) \cdot v[v^{-1}, F(u^{-1})]w \cdot F(\tau),$$

where $[v^{-1}, F(u)^{-1}] := v^{-1}F(u^{-1})vF(u)$ denotes the commutator. By writing

$$v = \begin{bmatrix} 1 & 0 \\ \underline{v} & 1 \end{bmatrix} \quad \text{and} \quad u = \begin{bmatrix} 1 & \underline{u} \\ 0 & 1 \end{bmatrix},$$

where $\underline{v} \in \pi\mathcal{O}^{\mathrm{ur}}/\pi^2$, we see

$$v[v^{-1}, F(u)^{-1}]w = \begin{bmatrix} 1 & -F(\underline{u}^2)\underline{v} \\ 0 & 1 \end{bmatrix} \cdot w \cdot w^{-1}vw \cdot w^{-1} \begin{bmatrix} 1 - F(\underline{u})\underline{v} & 0 \\ 0 & 1 + F(\underline{u})\underline{v} \end{bmatrix} w,$$

and therefore

$$xF(uw\tau) = \left(v'F(u) \begin{bmatrix} 1 & -F(\underline{u}^2)\underline{v} \\ 0 & 1 \end{bmatrix} \right) \cdot w \cdot v^w \cdot \left(\begin{bmatrix} 1 - F(\underline{u})\underline{v} & 0 \\ 0 & 1 + F(\underline{u})\underline{v} \end{bmatrix}^w \cdot F(\tau) \right),$$

which gives "$xF(uwt)$" a decomposition into the form "$UwUT$". In our situation such a decomposition is unique, so by comparing it with $uw\tau x' = uw(\tau x'\tau^{-1})\tau$ we conclude that

$$\hat{\Sigma}_2 = \{(v, v', x', u, \tau, u') \in (U^-)^1 \times U^1 \times U^1 \times U \times T \times U \mid S\},$$

where S is the system of equations

$$u = v'F(u) \begin{bmatrix} 1 & -F(\underline{u}^2)\underline{v} \\ 0 & 1 \end{bmatrix}, \ \tau x'\tau^{-1} = w^{-1}vw, \ \tau = \begin{bmatrix} 1 + F(\underline{u})\underline{v} & 0 \\ 0 & 1 - F(\underline{u})\underline{v} \end{bmatrix} F(\tau).$$

These equations imply the alternating sum of cohomology of $\hat{\Sigma}_2$ is $T^F \times T^F$-equivariant isomorphic (see e.g. [DM91, 10.12]) to that of

$$\tilde{\Sigma}_2 := \{(v, u, \tau) \in U^1 \times U \times T \mid S'\},$$

where S' is the system of equations

$$F(u) = u \mod \pi \quad \text{and} \quad \tau = \begin{bmatrix} 1 + F(\underline{u})\underline{v} & 0 \\ 0 & 1 - F(\underline{u})\underline{v} \end{bmatrix} F(\tau).$$

In order to proceed we write $\underline{u} = u_0 + u_1\pi$, $\underline{v} = v_1\pi$, $\tau' = \tau'_0 + \tau'_1\pi$, $\tau'' = \tau''_0 + \tau''_1\pi$, where $u_0, u_1, v_1, \tau'_0, \tau'_1, \tau''_0, \tau''_1 \in \overline{\mathbb{F}}_q$, and $\tau = \begin{bmatrix} \tau' & 0 \\ 0 & \tau'' \end{bmatrix}$. Then the above equations imply the alternating sum of the cohomology of $\tilde{\Sigma}_2$ is $T^F \times T^F$-equivariant isomorphic to that of

$$\bar{\Sigma}_2 = \{(u_0, \tau'_0, \tau''_0, v_1, \tau'_1, \tau''_1) \in \mathbb{F}_q \times \mathbb{F}_q \times \mathbb{F}_q \times \overline{\mathbb{F}}_q \times \overline{\mathbb{F}}_q \times \overline{\mathbb{F}}_q \mid S''\},$$

where S'' is the system of conditions

$$\tau_1' = (\tau_1')^q + u_0 v_1 \tau_0', \quad \tau_1'' = (\tau_1'')^q - u_0 v_1 \tau_0'', \quad \mathrm{diag}(\tau', \tau'') \in T.$$

For $\mathbb{G} = \mathrm{SL}_2$, we have $\tau' = \tau''^{-1}$, so the above equations imply that v_1 and τ'' are determined by the other variables, hence $\bar{\Sigma}_2$ is a $T^F \times T^F$-equivariant line bundle over $\{(u_0, \tau_0') \in \mathbb{F}_q \times \mathbb{F}_q^*\}$, on which $(T^1)^F \times (T^1)^F$ acts trivially. So for θ regular, we see that $\dim H_c^*(\Sigma_2)_{\theta^{-1},\theta} = \dim H_c^*(\bar{\Sigma}_2)_{\theta^{-1},\theta} = 0$.

Therefore, for θ regular, $H_c^*(\Sigma)_{\theta^{-1},\theta} = H_c^*(\Sigma_1)_{\theta^{-1},\theta}$ is of dimension 1, i.e.

$$\langle H_c^*(L^{-1}(U))_\theta, H_c^*(L^{-1}(U^\pm))_\theta \rangle_{G^F} = 1.$$

Moreover, if θ is in general position, then by the irreducibility of $\mathrm{Ind}_{B^F}^{G^F}\widetilde{\theta}$ and the comparison between the representation dimensions we obtain $\mathrm{Ind}_{B^F}^{G^F}\widetilde{\theta} \cong \mathrm{Ind}_{T^F(U^\pm)^F}^{G^F}\widetilde{\theta}$.

4.3 Algebraisation

4.3.1 General linear groups

The main problem considered in this chapter, raised by Lusztig in [Lus04], is that whether the representations of the form $R_{T,U}^\theta$ and that of the form $\mathrm{Ind}_{(TU^\pm)^F}^{G^F}\widetilde{\theta}$ coincide. Let us state a precise version of this problem:

Question 4.3.1. If $\theta \in \widehat{T^F}$ is regular and in general position, then does one have

$$R_{T,U}^\theta \cong \mathrm{Ind}_{(TU^\pm)^F}^{G^F}\widetilde{\theta},$$

under either Gérardin's conditions on groups (see Remark 4.1.7) or the genericity condition on characters (see Definition 4.1.10)?

The difficult step in answering Question 4.3.1 is to show that the inner product between $R_{T,U}^\theta$ and $\mathrm{Ind}_{(TU^\pm)^F}^{G^F}\widetilde{\theta}$ is one. In this subsection we present the proof for general linear groups, which is the first main result of this chapter. The generalisation of this result to any reductive group is achieved later in the joint work [CS16], and in the next subsection we will present this generalisation as our second main result.

In the remaining of this subsection we assume $\mathbb{G} = \mathrm{GL}_n$. Note that though $W(T)$ is rational, one may not have $W(T) = W(T)^F$ unless $n = 2$ or $T = T_0$. For any $v \in W(T)$, we fix a lift $\hat{v} \in N_G(T)$ in such a way that $\hat{v}' := \hat{v}^{\hat{\lambda}}$ coincide with the monomial matrix lift we had chosen for an element in $W(T_0)$.

Theorem 4.3.2. *Suppose $\mathbb{G} = \mathrm{GL}_n$. If $\theta \in \widehat{T^F}$ is regular and in general position, then*

$$\langle \mathrm{Ind}_{(TU^\pm)^F}^{G^F}\widetilde{\theta}, R_{T,U}^\theta \rangle_{G^F} = 1.$$

Proof. We need to compute the inner product between the alternating sum of the cohomology of $L^{-1}(FU)$ and that of the Lang pre-image $L^{-1}(FU^{\pm})$ of the arithmetic radical. One has

$$\langle H_c^*(L^{-1}(FU^{\pm}))_\theta, R_{T,U}^\theta\rangle_{G^F} = \dim H_c^*(\Sigma)_{\theta^{-1},\theta},$$

where $\Sigma := \{(x, x', y) \in U^{\pm} \times FU \times G \mid xF(y) = yx'\}$; this follows from the $T^F \times T^F$-equivariant isomorphism

$$G^F\backslash L^{-1}(U^{\pm}) \times L^{-1}(FU) \cong \Sigma; \quad (g, g') \mapsto (g^{-1}F(g), g'^{-1}F(g'), g^{-1}g'),$$

the Künneth formula (see e.g. [DM91, 10.9]), and the Hom–tensor adjunction. (To see a morphism as above is an isomorphism rather than just a bijective morphism, it is enough to show the morphism is separable (see [Spr09, 5.5.4]), which results from the étaleness of the Lang morphism; for details see the argument in [Car93, P221 to P222].)

The Bruhat decomposition $G_1 = \coprod_{v\in W(T)} B_1\hat{v}B_1$ of $G_1 = \mathbf{G}(\overline{\mathbb{F}}_q)$ gives the finite stratification $G = \coprod_{v\in W(T)} G_v$, where

$$G_v := (U \cap \hat{v}U^-\hat{v}^{-1})(\hat{v}(U^-)^1\hat{v}^{-1})\hat{v}TU$$

(a variant of this was used in [Lus04, 1.9] and proved in details in [Sta09, 2.3]), and hence a finite stratification of locally closed subvarieties

$$\Sigma = \coprod_{v\in W(T)} \Sigma_v,$$

where $\Sigma_v := \{(x, x', y) \in U^{\pm} \times FU \times G_v \mid xF(y) = yx'\}$. Write

$$\mathcal{Z}_v := (U \cap \hat{v}U^-\hat{v}^{-1}) \times \hat{v}(U^-)^1\hat{v}^{-1},$$

then Σ_v admits the following locally trivial fibration by an affine space ($\cong U \cap \hat{v}(U^-)^1\hat{v}^{-1}$)

$$\hat{\Sigma}_v = \{(x, x', u', u^-, \tau, u) \in U^{\pm} \times FU \times \mathcal{Z}_v \times T \times U \mid xF(u'u^-\hat{v}\tau u) = u'u^-\hat{v}\tau ux'\},$$

on which $T^F \times T^F$ acts as

$$(t, t')\colon (x, x', u', u^-, \tau, u) \mapsto (t^{-1}xt, t'^{-1}x't', t^{-1}u't, t^{-1}u^-t, (t^{\hat{v}})^{-1}\tau t', t'^{-1}ut').$$

By the change of variable $x'F(u)^{-1} \mapsto x'$ we rewrite $\hat{\Sigma}_v$ as

$$\hat{\Sigma}_v = \{(x, x', u', u^-, \tau, u) \in U^{\pm} \times FU \times \mathcal{Z}_v \times T \times U \mid xF(u'u^-\hat{v}\tau) = u'u^-\hat{v}\tau ux'\},$$

on which the $T^F \times T^F$-action does not change.

For $i \in [0, r-1] \cap \mathbb{Z}$, let $\mathcal{Z}_v(i)$ be the pre-image of $(\hat{v}U^-\hat{v}^{-1})^i = \hat{v}(U^-)^i\hat{v}^{-1}$ under the product morphism

$$\mathcal{Z}_v = (U \cap \hat{v}U^-\hat{v}^{-1}) \times \hat{v}(U^-)^1\hat{v}^{-1} \longrightarrow \hat{v}U^-\hat{v}^{-1}.$$

Recall that for $i = 0$ we let $G^0 = G$ for an algebraic group G when a reduction map is involved; don't confuse it with the identity component G°. We also write $\mathcal{Z}_v^*(i) := \mathcal{Z}_v(i) \setminus \mathcal{Z}_v(i+1)$. For each v consider the partition $\hat{\Sigma}_v = \Sigma_v' \sqcup \Sigma_v''$ into locally closed subvarieties:

$$\Sigma_v' := \{(x, x', u', u^-, \tau, u) \in \hat{\Sigma}_v \mid (u', u^-) \in \mathcal{Z}_v \setminus \mathcal{Z}_v(l)\}$$

and

$$\Sigma_v'' := \{(x, x', u', u^-, \tau, u) \in \hat{\Sigma}_v \mid (u', u^-) \in \mathcal{Z}_v(l)\}.$$

Our target is to show:

(a) $\dim H_c^*(\Sigma_v'')_{\theta^{-1},\theta} = 1$ if $v = 1$, and $= 0$ if $v \neq 1$;

(b) $\dim H_c^*(\Sigma_v')_{\theta^{-1},\theta} = 0$ for all v.

We start with the much easier (a):

Lemma 4.3.3. *(a) is true.*

Proof. Consider the defining equation $xF(u'u^-\hat{v}\tau) = u'u^-\hat{v}\tau ux'$ of Σ_v''; note that

$$u'u^- \in \hat{v}(U^-)^l\hat{v}^{-1} \subseteq U^\pm = FU^\pm,$$

so we can apply the changes of variables $(u'u^-)^{-1}x \mapsto x$, and then $xF(u'u^-) \mapsto x$. This allows us to rewrite Σ_v'' as $\bar{\Sigma}_v'' = \{(x, x', u', u^-, \tau, u) \in U^\pm \times FU \times \mathcal{Z}_v(l) \times T \times U \mid xF(\hat{v}\tau) = \hat{v}\tau ux'\}$, on which $T^F \times T^F$ acts in the same way on each component as before.

Take $H = \{(t, t') \in T \times T \mid tF(t^{-1}) = F(\hat{v})t'F(t')^{-1}F(\hat{v}^{-1})\}$; this is an algebraic group, and note that H acts on $\bar{\Sigma}_v''$ in the same way as $T^F \times T^F$. Write T_* for the reductive part of T (it is a torus isomorphic to T_1 since T is abelian), then the identity component $H_* := (H \cap (T_* \times T_*))^\circ$ is a torus acting on $\bar{\Sigma}_v''$, thus by basic properties of ℓ-adic cohomology we see

$$\dim H_c^*(\bar{\Sigma}_v'')_{\theta^{-1},\theta} = \dim H_c^*((\bar{\Sigma}_v'')^{H_*})_{\theta^{-1},\theta}.$$

Note that the Lang–Steinberg theorem implies both the first projection and the second projection of H_* to T_* are surjective, therefore $(x, x', u', u^-, \tau, u) \in (\bar{\Sigma}_v'')^{H_*}$ only if $x = x' = u' = u^- = u = 1$. Thus $(\bar{\Sigma}_v'')^{H_*} = \{(1, 1, 1, 1, \tau, 1) \mid F(\hat{v}\tau) = \hat{v}\tau\}^{H_*}$. The set $(\hat{v}T)^F$ is empty unless $\hat{v}^{-1}F(\hat{v}) \in T$ (in particular, v is F-stable), in which case $\{(1, 1, 1, 1, \tau, 1) \mid F(\hat{v}\tau) = \hat{v}\tau\}$ is indeed stable under the action of H, so it is also stable under the action of the connected group H_*. We only need to concern the non-empty case. As a finite set, $(\hat{v}T)^F$ admits only the trivial action of H_* (note that a continuous map takes a connected component into a connected component), thus $(\bar{\Sigma}_v'')^{H_*} = \{(1, 1, 1, 1, \tau, 1) \mid F(\hat{v}\tau) = \hat{v}\tau\}^{H_*} = (\hat{v}T)^F$. Therefore $H_c^*(\bar{\Sigma}_v'') = \overline{\mathbb{Q}}_\ell[(\hat{v}T)^F]$, on which $T^F \times T^F$ acts via (t, t'): $\hat{v}\tau \mapsto \hat{v}(t^{\hat{v}})^{-1}\tau t'$ (i.e. a permutation representation for both the first and the second T^F). In particular, the irreducible constituents of $H_c^*(\bar{\Sigma}_v'')$ are the $H_c^*(\bar{\Sigma}_v'')_{(\phi^{\hat{v}})^{-1},\phi}$'s, where ϕ runs over $\widehat{T^F}$. Hence $H_c^*(\bar{\Sigma}_v'')_{\theta^{-1},\theta}$ is non-zero if and only if $\theta^{\hat{v}} = \theta$, but since θ is assumed to be in general position, this can happen only if $v = 1$. For $v = 1$, we have $\dim H_c^*(\bar{\Sigma}_1'')_{\theta^{-1},\theta} = 1$ for any θ since $|\widehat{T^F}| = |T^F|$. This proves (a). □

The argument of (b) requires the following homotopy property proved in [DL76].

Lemma 4.3.4. *Let H be a variety over an algebraically closed field k, and Y a separated variety over k. Let $f\colon H\times Y\to Y$ be a morphism such that $(p,f)\colon H\times Y\to H\times Y$ is an automorphism, where p is the left projection of $H\times Y$. Then for h varies in H, the induced endomorphism of $f(h,-)$ on $H_c^i(Y,\overline{\mathbb{Q}}_\ell)$ only depends on the first homotopy group of H (i.e. on the connected component containing h).*

Proof. Indeed, the proof is same with the one presented in [DL76, 6.4, 6.5]; we restate it for convenience, as the assertion stated here is in a more general form. Consider the following cartesian square

$$\begin{array}{ccc} H\times Y & \longrightarrow & Y \\ {\scriptstyle p=p\circ(p,f)}\downarrow & & \downarrow \\ H & \longrightarrow & \mathrm{Spec}\ (k). \end{array}$$

The proper base change theorem implies the sheaf $R^ip_!\mathbb{Z}/n$ on H is the constant sheaf $H_c^i(Y,\mathbb{Z}/n)$; the isomorphism (p,f) induces a sheaf endomorphism on it, and on the stalk $H_c^i(Y,\mathbb{Z}/n)_h=H_c^i(Y,\mathbb{Z}/n)$ the endomorphism is the induced map of $f(h,-)$. However, the stalk endomorphisms of a constant sheaf on a connected variety are constant with respect to the change of stalks, so the induced map of $f(h,-)$ on $H_c^i(Y,\mathbb{Z}/n)$ is constant when h varies in a connected component of H. Now the proof is completed by taking the projective limit with respect to $\mathbb{Z}/n$, where n runs over the powers of ℓ. □

Lemma 4.3.5. *(b) is true.*

Proof. By applying the changes of variables $\hat{v}\tau\hat{v}^{-1}\mapsto\tau$, $\tau^{-1}u^-\tau\mapsto u^-$, and $\tau^{-1}u'\tau\mapsto u'$ we rewrite Σ_v' as

$$\bar{\Sigma}_v'=\{(x,x',u',u^-,\tau,u)\in U^\pm\times FU\times\mathcal{Z}_v\setminus\mathcal{Z}_v(l)\times T\times U\mid xF(\tau u'u^-\hat{v})=\tau u'u^-\hat{v}ux'\},$$

on which $(t,t')\in T^F\times T^F$ acts by taking (x,x',u',u^-,τ,u) to

$$(t^{-1}xt,t'^{-1}x't',(t'^{\hat{v}})^{-1}u'(t')^{\hat{v}},(t'^{\hat{v}})^{-1}u^-(t')^{\hat{v}},t^{-1}\tau(t')^{\hat{v}},t'^{-1}ut').$$

To show $\dim H_c^*(\bar{\Sigma}_v')_{\theta^{-1},\theta}=0$, it suffices to show

$$\dim H_c^*(\bar{\Sigma}_v')_{\theta^{-1}|_{(T^{r-1})^F}}=0,$$

where the subscript group is $(T^{r-1})^F=(T^{r-1})^F\times 1\subseteq T^F\times T^F$. Note that the $(T^{r-1})^F$-action on $\bar{\Sigma}_v'$ is given by

$$t\colon(x,x',u',u^-,\tau,u)\mapsto(x,x',u',u^-,t^{-1}\tau,u).$$

By the changes of variables $\hat{\lambda}^{-1}x\hat{\lambda}\mapsto x$, $F(\hat{\lambda}^{-1})x'F(\hat{\lambda})\mapsto x'$, $\hat{\lambda}^{-1}u'\hat{\lambda}\mapsto u'$, $\hat{\lambda}^{-1}u^-\hat{\lambda}\mapsto u^-$, $\hat{\lambda}^{-1}\tau\hat{\lambda}\mapsto\tau$, and $\hat{\lambda}^{-1}u\hat{\lambda}\mapsto u$, we can rewrite $\bar{\Sigma}_v'$ as

$$\tilde{\Sigma}_v'=\{(x,x',u',u^-,\tau,u)\in U_0^\pm\times U_0\times\mathcal{Z}_0\times T_0\times U_0\mid x\hat{w}F(\tau u'u^-)\hat{v}'=\tau u'u^-\hat{v}'u\hat{w}x'\},$$

where $\mathcal{Z}_0 := (\mathcal{Z}_v \setminus \mathcal{Z}_v(l))^{\hat{\lambda}}$ (for $(u', u^-) \in \mathcal{Z}_v$, we put $(u', u^-)^{\hat{\lambda}} := ((u')^{\hat{\lambda}}, (u^-)^{\hat{\lambda}})$). Note that

$$u'u^- \in (\hat{v}U^-\hat{v}^{-1})^{\hat{\lambda}} = (U_0^-)^{\hat{v}'^{-1}},$$

and the $(T^{r-1})^F$-action on $\tilde{\Sigma}'_v$ becomes

$$t\colon (x, x', u', u^-, \tau, u) \mapsto (x, x', u', u^-, (t^{\hat{\lambda}})^{-1}\tau, u).$$

We have a finite disjoint partition $\mathcal{Z}_0 = \coprod_{i=0}^{l-1} \mathcal{Z}_0(i)$ of locally closed subvarieties, where $\mathcal{Z}_0(i) := (\mathcal{Z}_v^*(i))^{\hat{\lambda}}$. Consider the finite set

$$I := \{(\alpha, \beta) \in \mathbb{Z} \times \mathbb{Z} \mid 1 \leq \alpha \leq n-1, 1 \leq \beta \leq n-\alpha\}.$$

Let us define a total order on I: If $\alpha' < \alpha$, then $(\alpha', \beta') < (\alpha, \beta)$; if $\beta' < \beta$, then $(\alpha, \beta') < (\alpha, \beta)$. Note that the map

$$(\alpha, \beta) \mapsto (\alpha + \beta, \beta)$$

is a bijection from I to the set of positions in strictly lower triangular areas of $n \times n$ matrices. Let $\mathcal{Z}_0^{(\alpha,\beta)}(i) \subseteq \mathcal{Z}_0(i)$ be the locally closed subvariety consisting of elements (u', u^-) such that: The $(\alpha+\beta, \beta)$-entry of $\hat{v}'^{-1}F(u'u^-)\hat{v}'$ is non-zero, and for all $(\alpha', \beta') < (\alpha, \beta)$ the $(\alpha'+\beta', \beta')$-entries of $\hat{v}'^{-1}F(u'u^-)\hat{v}'$ is zero. Then we obtain a finite partition $\mathcal{Z}_0(i) = \coprod_{(\alpha,\beta)\in I} \mathcal{Z}_0^{(\alpha,\beta)}(i)$ into locally closed subvarieties, and hence a partition

$$\tilde{\Sigma}'_v = \coprod_{i,I} \Sigma_v^{i,(\alpha,\beta)},$$

where

$$\Sigma_v^{i,(\alpha,\beta)} := \{(x, x', u', u^-, \tau, u) \in \tilde{\Sigma}'_v \mid (u', u^-) \in \mathcal{Z}_0^{(\alpha,\beta)}(i)\}.$$

Note that each subvariety $\Sigma_v^{i,(\alpha,\beta)}$ inherits the $(T^{r-1})^F$-action

$$t\colon (x, x', u', u^-, \tau, u) \mapsto (x, x', u', u^-, (t^{\hat{\lambda}})^{-1}\tau, u),$$

thus by basic properties of ℓ-adic cohomology, to prove the lemma it suffices to show:

$$H_c^*(\Sigma_v^{i,(\alpha,\beta)})_{\theta^{-1}|_{(T^{r-1})^F}} = 0,$$

for every $i \in \{0, \cdots, l-1\}$ and every $(\alpha, \beta) \in I$.

Let $U_{0,(\alpha,\beta)}$ (resp. $U^-_{0,(\alpha,\beta)}$) be the closed subvariety (of U_0 (resp. U_0^-)) consisting of elements such that all entries are zero, except for the diagonal entries and possibly the $(\beta, \beta+\alpha)$-entry (resp. $(\alpha+\beta, \beta)$-entry). Denote by

$$\eta\colon U_{0,(\alpha,\beta)} \longrightarrow \mathcal{F}\mathbb{G}_a$$

and

$$\eta_-\colon U^-_{0,(\alpha,\beta)} \longrightarrow \mathcal{F}\mathbb{G}_a$$

the natural isomorphisms of algebraic groups; these two morphisms can be extended to the whole U_0 (resp. U_0^-) by compositing the natural projections. Let $\mathbf{T}_0^{(\alpha,\beta)}$ be the root subgroup of $\mathbf{T}_0$ of the form

$$\mathbf{T}_0^{(\alpha,\beta)} := \{\mathrm{diag}(1,\cdots,1,\tilde{\tau},1,\cdots,1,\tilde{\tau}^{-1},1,\cdots,1)\},$$

where $\tilde{\tau}$ is the (β,β)-entry and $\tilde{\tau}^{-1}$ is the $(\alpha+\beta,\alpha+\beta)$-entry; denote by

$$\eta_*\colon \mathcal{F}\mathbf{T}_0^{(\alpha,\beta)} \longrightarrow \mathcal{F}\mathbb{G}_m$$

the inverse automorphism of the coroot. We always view an element in $\mathcal{F}\mathbb{G}_m$ as an element in $\mathcal{F}\mathbb{G}_a$, by the natural open immersion. For any $i \in \{0,1,\cdots,l-1\}$, the Greenberg functor lifts the isomorphism of additive groups $(\pi^i) \cong \mathcal{O}^{\mathrm{ur}}/(\pi^{r-i})\colon \pi^i a + (\pi^r) \mapsto a + (\pi^{r-i})$ to an isomorphism of affine spaces

$$\mu_i\colon (\mathcal{F}\mathbb{G}_a)^i \longrightarrow (\mathcal{F}\mathbb{G}_a)_{r-i}.$$

On the other hand, take a section morphism

$$\mu^i\colon (\mathcal{F}\mathbb{G}_a)_{r-i} \cong \mathcal{F}\mathbb{G}_a/(\mathcal{F}\mathbb{G}_a)^{r-i} \longrightarrow \mathcal{F}\mathbb{G}_a$$

to the quotient morphism such that $\mu^i(0) = 0$ (the existence of μ^i follows from the fact that $\mathcal{F}\mathbb{G}_a$ is an affine space). Write $\mathcal{T}_0^{(\alpha,\beta)} = (\mathcal{F}\mathbf{T}_0^{(\alpha,\beta)})^{r-1}$. Then for any $(\alpha,\beta) \in I$, any $i \in \{0,1,\cdots,l-1\}$, and any $t \in T^{r-1}$ such that $\hat{v}'^{-1}F(t^{\hat{\lambda}})\hat{w}^{-1}(t^{\hat{\lambda}})^{-1}\hat{w}\hat{v}' \in \mathcal{T}_0^{(\alpha,\beta)}$, consider the morphism $g_t\colon U_0 \to U_0$ given by

$$g_t\colon x' \mapsto x' \cdot \left(\eta^{-1} \circ \mu^i \left(\mu_i \left(\eta_* \left(\hat{v}'^{-1}F(t^{\hat{\lambda}})\hat{w}^{-1}(t^{\hat{\lambda}})^{-1}\hat{w}\hat{v}'\right) - 1\right) \cdot (\mu_i \circ \eta_-)\left(\hat{v}'^{-1}F(z)\hat{v}'\right)^{-1}\right)\right),$$

with the parameter $(u',u^-) \in \mathcal{Z}_0^{(\alpha,\beta)}(i)$, where $z := u'u^-$. The multiplication operation "$\cdot$" is by viewing $\mathbb{G}_a$ as a ring scheme. Note that since the $(\alpha+\beta,\beta)$-entry of z is not in $((U_0^-)^{v'^{-1}})^{i+1}$, the inverse of $\mu_i \circ \eta_-\,(\hat{v}'^{-1}F(z)\hat{v}')$ exists, hence this morphism is well-defined. Also note that, for $F(t) = t$ one has $\hat{v}'^{-1}F(t^{\hat{\lambda}})\hat{w}^{-1}(t^{\hat{\lambda}})^{-1}\hat{w}\hat{v}' = 1$, so $g_t(x') = x'$ in this case.

Meanwhile, for any $(\alpha,\beta) \in I$, any $i \in \{0,1,\cdots,l-1\}$, and any $t \in T^{r-1}$ such that

$$\hat{v}'^{-1}F(t^{\hat{\lambda}})\hat{w}^{-1}(t^{\hat{\lambda}})^{-1}\hat{w}\hat{v}' \in \mathcal{T}_0^{(\alpha,\beta)},$$

consider the morphism $f_t\colon U_0^{\pm} \to U_0^{\pm}$ given by

$$f_t\colon x \mapsto x(t^{\hat{\lambda}})^{-1}\hat{w}F(\tau z)\hat{v}'(x')^{-1}g_t(x')\hat{v}'^{-1}F(\tau z)^{-1}F(t^{\hat{\lambda}})\hat{w}^{-1},$$

with the parameters $x' \in U_0$, $\tau \in T_0$, and $(u',u^-) \in \mathcal{Z}_0^{(\alpha,\beta)}(i)$, where $z := u'u^-$. To see this is well-defined one needs to check the right hand side is in $U_0^{\pm}$: The condition

$$x(t^{\hat{\lambda}})^{-1}\hat{w}F(\tau z)\hat{v}'(x')^{-1}g_t(x')\hat{v}'^{-1}F(\tau z)^{-1}F(t^{\hat{\lambda}})\hat{w}^{-1} \in U_0^{\pm}$$

is equivalent to

$$F(z)\hat{v}'(x')^{-1}g_t(x')\hat{v}'^{-1}F(z)^{-1}F(t^{\hat{\lambda}})\hat{w}^{-1}(t^{\hat{\lambda}})^{-1}\hat{w} \in U_0^{\pm},$$

which is equivalent to

$$(\hat{v}'^{-1}F(z)\hat{v}')\cdot(x')^{-1}g_t(x')\cdot(\hat{v}'^{-1}F(z)^{-1}\hat{v}')\cdot(\hat{v}'^{-1}F(t^{\hat{\lambda}})\hat{w}^{-1}(t^{\hat{\lambda}})^{-1}\hat{w}\hat{v}') \in U_0^{\pm}.$$

Since $(x')^{-1}g_t(x')$ is in $G^{r-i-1} \subseteq G^l$, by considering to modulo π^l we see the element $(\hat{v}'^{-1}F(z)\hat{v}')\cdot(x')^{-1}g_t(x')\cdot(\hat{v}'^{-1}F(z)^{-1}\hat{v}')\cdot(\hat{v}'^{-1}F(t^{\hat{\lambda}})\hat{w}^{-1}(t^{\hat{\lambda}})^{-1}\hat{w}\hat{v}')$ is in G^l, so we only need to show the diagonal entries of this element (view it as a matrix) are 1. As $(u', u^-) \in \mathcal{Z}_0^{(\alpha,\beta)}(i)$, we can write the $(\alpha+\beta,\beta)$-entry of $\hat{v}'^{-1}F(z)\hat{v}'$ to be $\pi^i\bar{z}$, where $\bar{z} \in (\mathcal{O}_r^{\mathrm{ur}})^*$, then the $(\alpha+\beta,\beta)$-entry of $\hat{v}'^{-1}F(z)^{-1}\hat{v}'$ is $-\pi^i\bar{z}$, and for all $(\alpha',\beta') < (\alpha,\beta)$ the $(\alpha'+\beta',\beta')$-entries of $\hat{v}'^{-1}F(z)^{-1}\hat{v}'$ are zero. Meanwhile, we write

$$\hat{v}'^{-1}F(t^{\hat{\lambda}})\hat{w}^{-1}(t^{\hat{\lambda}})^{-1}\hat{w}\hat{v}' = \mathrm{diag}(1,\cdots,1,1+\pi^{r-1}\bar{t},1,\cdots,1,1-\pi^{r-1}\bar{t},1,\cdots,1),$$

where $\bar{t} \in \mathcal{O}_r^{\mathrm{ur}}$, and $1+\pi^{r-1}\bar{t}$ is the (β,β)-entry and $1-\pi^{r-1}\bar{t}$ is the $(\alpha+\beta,\alpha+\beta)$-entry. Thus $x'^{-1}g_t(x') \in U_{0,(\alpha,\beta)}$ and its $(\beta,\beta+\alpha)$-entry is $\mu^i(\mu_i(\pi^i\bar{z})^{-1}\cdot\mu_i(\pi^{r-1}\bar{t}))$. With these notations, by direct computations one can write out the diagonal entries of

$$(\hat{v}'^{-1}F(z)\hat{v}')\cdot(x')^{-1}g_t(x')\cdot(\hat{v}'^{-1}F(z)^{-1}\hat{v}')\cdot(\hat{v}'^{-1}F(t^{\hat{\lambda}})\hat{w}^{-1}(t^{\hat{\lambda}})^{-1}\hat{w}\hat{v}')$$

to be

(i) The (β,β)-entry is $1-\pi^i\bar{z}\mu^i(\mu_i(\pi^i\bar{z})^{-1}\cdot\mu_i(\pi^{r-1}\bar{t}))+\pi^{r-1}\bar{t}$;

(ii) The $(\beta+\alpha,\beta+\alpha)$-entry is $1+\pi^i\bar{z}\mu^i(\mu_i(\pi^i\bar{z})^{-1}\cdot\mu_i(\pi^{r-1}\bar{t}))-\pi^{r-1}\bar{t}$;

(iii) Other diagonal entries are 1.

Now it remains to show $\pi^{r-1}\bar{t} = \pi^i\bar{z}\mu^i(\mu_i(\pi^i\bar{z})^{-1}\cdot\mu_i(\pi^{r-1}\bar{t}))$. As elements in cosets we can write $\bar{z} = \tilde{z}+(\pi^r)$ and $\bar{t} = \tilde{t}+(\pi^r)$, where $\tilde{z} \in (\mathcal{O}^{\mathrm{ur}})^*$ and $\tilde{t} \in \mathcal{O}^{\mathrm{ur}}$. Then

$$\begin{aligned}
\pi^i\bar{z}\mu^i(\mu_i(\pi^i\bar{z})^{-1}\cdot\mu_i(\pi^{r-1}\bar{t})) &= \pi^i(\tilde{z}+(\pi^r))\mu^i((\tilde{z}^{-1}+(\pi^{r-i}))\cdot(\pi^{r-i-1}\tilde{t}+(\pi^{r-i})))+(\pi^r)\\
&= \pi^i(\tilde{z}+(\pi^r))\mu^i(\pi^{r-i-1}\tilde{z}^{-1}\tilde{t}+(\pi^{r-i}))+(\pi^r)\\
&= \pi^i(\tilde{z}+(\pi^r))(\pi^{r-i-1}\tilde{z}^{-1}\tilde{t}+\pi^{r-i}s+(\pi^r))+(\pi^r)\\
&= \pi^i(\pi^{r-i-1}\tilde{t}+\pi^{r-i}s\tilde{z}+(\pi^r))+(\pi^r) = \pi^{r-1}\tilde{t}+(\pi^r) = \pi^{r-1}\bar{t},
\end{aligned}$$

for some $s \in \mathcal{O}^{\mathrm{ur}}$ (s depends on μ^i). Therefore f_t is well-defined. Note that, similar to g_t, if $F(t) = t$, then $f_t(x) = x$.

Now consider the closed subgroup

$$H := \{t \in T^{r-1} \mid \hat{v}'^{-1}F(t^{\hat{\lambda}})\hat{w}^{-1}(t^{\hat{\lambda}})^{-1}\hat{w}\hat{v}' \in \mathcal{T}_0^{(\alpha,\beta)}\}$$

of T^{r-1}. For any $t \in H$, the above preparations on f_t and g_t allow us to define the following automorphism on $\Sigma_v^{i,(\alpha,\beta)}$:

$$h_t \colon (x, x', u', u^-, \tau, u) \mapsto (f_t(x), g_t(x'), u', u^-, (t^{\hat{\lambda}})^{-1}\tau, u),$$

where the involved parameters are as presented. To see this is well-defined, one needs to show the right hand side satisfies the defining equation of $\Sigma_v^{i,(\alpha,\beta)}$, and this can be seen by direct computations by expanding the definition of f_t. Note that, in the case $F(t) = t$, the automorphism h_t coincides with the $(T^{r-1})^F$-action. Thus by Lemma 4.3.4, since h_1 is the identity map, the induced endomorphism of h_t on $H_c^*(\Sigma_v^{i,(\alpha,\beta)})$ is the identity map for any t in the identity component H° of H.

The condition $\hat{v}'^{-1}F(t^{\hat{\lambda}})\hat{w}^{-1}(t^{\hat{\lambda}})^{-1}\hat{w}\hat{v}' \in \mathcal{T}_0^{(\alpha,\beta)}$ is equivalent to

$$F(t)t^{-1} \in F(\hat{v}\hat{\lambda}\mathcal{T}_0^{(\alpha,\beta)}\hat{\lambda}^{-1}\hat{v}^{-1}),$$

therefore $H = \{t \in T^{r-1} \mid F(t)t^{-1} \in \mathcal{T}^{(\alpha,\beta)}\}$, where $\mathcal{T}^{(\alpha,\beta)} := F(\hat{v}\hat{\lambda}\mathcal{T}_0^{(\alpha,\beta)}\hat{\lambda}^{-1}\hat{v}^{-1}) \subseteq T$; this is the kernel of $\rho_{r,r-1}$ restricting to the image of some coroot of $\mathbf{T}$. Let $a \geq 1$ be such that $F^a(\mathcal{T}^{(\alpha,\beta)}) = \mathcal{T}^{(\alpha,\beta)}$, then the image of the norm map $N_F^{F^a}(t) = t \cdot F(t) \cdots F^{a-1}(t)$ on $\mathcal{T}^{(\alpha,\beta)}$ is a connected subgroup of H, hence contained in H°. Now, since

$$N_F^{F^a}((\mathcal{T}^{(\alpha,\beta)})^{F^a}) \subseteq (T^{r-1})^F \cap H^\circ,$$

we see

$$H_c^*(\Sigma_v^{i,(\alpha,\beta)})_{\theta^{-1}\circ N_F^{F^a}|_{(\mathcal{T}^{(\alpha,\beta)})^{F^a}}} = H_c^*(\Sigma_v^{i,(\alpha,\beta)})_{\theta^{-1}|_{N_F^{F^a}((\mathcal{T}^{(\alpha,\beta)})^{F^a})}} = 0,$$

by the regularity of θ. Therefore $H_c^*(\Sigma_v^{i,(\alpha,\beta)})_{\theta^{-1}|_{(T^{r-1})^F}} = 0$. The lemma is proved. □

By above, $\dim H_c^*(\Sigma)_{\theta^{-1},\theta} = \dim H_c^*(\Sigma_1'')_{\theta^{-1},\theta} = 1$, so the theorem is proved. □

Now we are able to answer Question 4.3.1 for the general linear groups.

Corollary 4.3.6. *Let $\mathbb{G} = \mathrm{GL}_n$. If $\theta \in \widehat{T^F}$ is regular and in general position, then*

$$R_{T,U}^\theta \cong \mathrm{Ind}_{(TU^\pm)^F}^{G^F}\widetilde{\theta},$$

and they are primitive irreducible representations of dimension $|G_l^F|/|T_l^F|$. In particular, in this case $R_{T,U}^\theta$ is a true representation rather than a virtual representation.

Proof. This follows immediately from Theorem 4.3.2, Remark 4.1.7, and Theorem 3.1.8. □

4.3.2 Arbitrary reductive groups

In this subsection we discuss how to generalise the arguments in the above subsection for a general reductive group $\mathbb{G}$. This generalisation is the main result of the joint work [CS16].

Such a generalisation requires some combinatorial information on roots, which is recorded as a variant of [Lus04, Lemma 1.7] in the below (for groups like GL_n and SL_n this can be done in an ad hoc way as in the above subsection). We start with fixing some notation:

Definition 4.3.7. Recall that Φ^- is the set of negative roots of $\mathbf{T}$ with respect to $\mathbf{B}$.

(1) Suppose Φ^- is equipped with a fixed arbitrary total order. For $z \in U^-$ and $\beta \in \Phi^-$, define $x_\beta^z \in \mathcal{F}\mathbf{U}_\beta$ by the decomposition $z = \prod_{\beta\in\Phi^-} x_\beta^z$, where the product is given according to the following order: If $\mathrm{ht}(\beta) < \mathrm{ht}(\beta')$, then x_β^z is left to $x_{\beta'}^z$; and if $\mathrm{ht}(\beta) = \mathrm{ht}(\beta')$ and $\beta < \beta'$, then x_β^z is left to $x_{\beta'}^z$; here, for $\beta \in \Phi^-$, we denote by $\mathrm{ht}(\beta)$ the largest integer n such that $\beta = \sum_{i=1}^n \beta_i$ for some $\beta_i \in \Phi^-$.

(2) For a fixed $\alpha \in \Phi^+$ and $i \in \{0, \cdots, l-1\}$, denote by $Z^\alpha(i) \subseteq U^-$ the subvariety consisted of all z such that:

i. $x_{-\alpha}^z \neq 1$;
ii. $z \in (U^-)^i \setminus (U^-)^{i+1}$;
iii. $x_\beta^z = 1$ for $\forall \beta \in \Phi^-$ such that $\mathrm{ht}(\beta) < \mathrm{ht}(-\alpha)$;
iv. $x_\beta^z = 1$ for $\forall \beta \in \Phi^-$ such that $\mathrm{ht}(\beta) = \mathrm{ht}(-\alpha)$ and $\beta < -\alpha$.

(3) $\mathcal{T}^\alpha := (\mathcal{F}\mathbf{T}^\alpha)^{r-1}$; this is a 1-dimensional affine space by the Greenberg functor.

Lemma 4.3.8. *Suppose $\alpha \in \Phi^+$ and $i \in \{0, \cdots, l-1\}$. Then for $z \in Z^\alpha(i)$ and $\xi \in U_\alpha^{r-i-1}$, one has*

$$[\xi, z] := \xi z \xi^{-1} z^{-1} = \tau_{\xi,z}\omega_{\xi,z},$$

where $\tau_{\xi,z} \in \mathcal{T}^\alpha$ and $\omega_{\xi,z} \in (U^-)^{r-1}$ are uniquely determined. Moreover,

$$U_\alpha^{r-i-1} \to \mathcal{T}^\alpha; \quad \xi \mapsto \tau_{\xi,z}$$

is a surjective morphism admitting a section Ψ_z^α such that: (i) $\Psi_z^\alpha(1) = 1$; (ii) the map $Z^\alpha(i) \times \mathcal{T}^\alpha \to U_\alpha^{r-i-1}\colon (z, \tau) \mapsto \Psi_z^\alpha(\tau)$ admits a structure of morphism of varieties.

Proof. Write $z = x_{-\alpha}^z z'$, then

$$[\xi, z] = \xi x_{-\alpha}^z z' \xi^{-1} z'^{-1} (x_{-\alpha}^z)^{-1} = [\xi, x_{-\alpha}^z] \cdot {}^{x_{-\alpha}^z}[\xi, z'].$$

We need to determine $[\xi, x_{-\alpha}^z]$ and ${}^{x_{-\alpha}^z}[\xi, z']$ separately.

Following the notation in [DG70, XX] we write $p_\beta\colon \mathbb{G}_{a/\mathcal{O}_r^{\mathrm{ur}}} \cong \mathbf{U}_\beta$ for every $\beta \in \Phi$ (and we use the same notation for the isomorphism induced by p_β via the Greenberg functor), then for some $a \in \mathbb{G}_m(\mathcal{O}_r^{\mathrm{ur}})$ we have

$$p_\alpha(x)p_{-\alpha}(y) = p_{-\alpha}(\frac{y}{1+axy})\check{\alpha}(1+axy)p_\alpha(\frac{x}{1+axy}),$$

for $\forall x, y \in \mathbb{G}_a(\mathcal{O}_r^{\mathrm{ur}})$; see [DG70, XX 2.2]. Let x, y be such that $p_\alpha(x) = \xi$ and $p_{-\alpha}(y) = x_{-\alpha}^z$ (note that $x^2 = 0$), then by applying the above formula to the commutator $[p_\alpha(x), p_{-\alpha}(y)] = p_\alpha(x)p_{-\alpha}(y)p_\alpha(-x)p_{-\alpha}(-y)$ repeatedly we see that

$$[\xi, x_{-\alpha}^z] = \check{\alpha}(1 + axy)p_{-\alpha}(axy^2).$$

Note that $p_{-\alpha}(axy^2) \in U_{-\alpha}^{r-1}$. In the below we will see $\check{\alpha}(1 + axy)$ is the required $\tau_{\xi,z}$.

Now turn to $[\xi, z']$; we want to show $[\xi, z'] \in (U^-)^{r-1}$. Let us do this by induction on the cardinal $\#\{\beta \in \Phi^- \mid x_\beta^{z'} \neq 1\}$. If the cardinal is zero then this is clear. When $\#\{\beta \in \Phi^- \mid x_\beta^{z'} \neq 1\} = 1$, we can put $z' = x_\beta^{z'}$, then by the Chevalley commutator formula (see [Lus04, 1.6 (b)] or [Sta09, Lemma 2.9 (b)]) we have

$$[\xi, z'] \in \prod_{j,j' \geq 1,\ j\beta + j'\alpha \in \Phi} U_{j\beta + j'\alpha}^{r-1}.$$

By basic properties of root system, if $j\beta + j'\alpha \in \Phi^+$ for some j, j', then $\beta + \alpha \in \Phi^+$ (see the arguments in [Lus04, 1.7]), which implies $\mathrm{ht}(-\alpha) > \mathrm{ht}(\beta)$, a contradiction to our assumption on z, so $[\xi, z'] \in (U^-)^{r-1}$ in this case. Suppose $[\xi, z'] \in (U^-)^{r-1}$ for $\#\{\beta \in \Phi^- \mid x_\beta^{z'} \neq 1\} \leq N$, then in the case $\#\{\beta \in \Phi^- \mid x_\beta^{z'} \neq 1\} = N + 1$, we decompose the product $z' = \prod_{\beta \in \Phi^-} x_\beta^{z'} = z_1' z_2'$ in such a way that both $[\xi, z_1']$ and $[\xi, z_2']$ are in $(U^-)^{r-1}$. Note that

$$[\xi, z'] = [\xi, z_1'] \cdot {}^{z_1'}[\xi, z_2'].$$

Since $z_1' \in U^-$, we see ${}^{z_1'}[\xi, z_2'] \in (U^-)^{r-1}$, and therefore $[\xi, z'] \in (U^-)^{r-1}$ for $\#\{\beta \in \Phi^- \mid x_\beta^{z'} \neq 1\} = N + 1$.

As $[\xi, z'] \in (U^-)^{r-1}$ implies ${}^{x_{-\alpha}^z}[\xi, z'] \in (U^-)^{r-1}$, in

$$[\xi, z] = [\xi, x_{-\alpha}^z] \cdot {}^{x_{-\alpha}^z}[\xi, z'] = \check{\alpha}(1 + axy) \cdot p_{-\alpha}(axy^2) \cdot {}^{x_{-\alpha}^z}[\xi, z']$$

we can take $\tau_{\xi,z} = \check{\alpha}(1 + axy) \in \mathcal{T}^\alpha$ and $\omega_{\xi,z} = p_{-\alpha}(axy^2) \cdot {}^{x_{-\alpha}^z}[\xi, z'] \in (U^-)^{r-1}$. They are uniquely determined by Iwahori decomposition.

Now, as $\tau_{\xi,z}$ is defined to be $\check{\alpha}(1 + ap_\alpha^{-1}(\xi)p_{-\alpha}^{-1}(x_{-\alpha}^z))$, the map $\xi \mapsto \tau_{\xi,z}$ is a surjective algebraic group morphism (note that $z \mapsto x_{-\alpha}^z$ is a projection, hence a morphism). The section morphism Ψ_z^α can be defined in the following way: Consider the isomorphism of additive groups $(\pi^i) \cong \mathcal{O}_{r-i}^{\mathrm{ur}}$: $\pi^i a + (\pi^r) \mapsto a + (\pi^{r-i})$; by the Greenberg functor it can be viewed as an isomorphism of affine spaces

$$\mu_i \colon (\mathcal{F}\mathbb{G}_{a/\mathcal{O}_r^{\mathrm{ur}}})^i \longrightarrow (\mathcal{F}\mathbb{G}_{a/\mathcal{O}_r^{\mathrm{ur}}})_{r-i}.$$

Note that such an isomorphism depends on the choice of π. Meanwhile, let

$$\mu^i \colon (\mathcal{F}\mathbb{G}_{a/\mathcal{O}_r^{\mathrm{ur}}})_{r-i} \cong \mathcal{F}\mathbb{G}_{a/\mathcal{O}_r^{\mathrm{ur}}}/(\mathcal{F}\mathbb{G}_{a/\mathcal{O}_r^{\mathrm{ur}}})^{r-i} \longrightarrow \mathcal{F}\mathbb{G}_{a/\mathcal{O}_r^{\mathrm{ur}}}$$

be a section morphism to the quotient morphism such that $\mu^i(0) = 0$ (μ^i exists because $\mathcal{F}\mathbb{G}_{a/\mathcal{O}_r^{\mathrm{ur}}}$ is an affine space). We put

$$\Psi_z^\alpha(\tau) := p_\alpha\left(a^{-1}\cdot\mu^i\left(\mu_i\left(\check{\alpha}^{-1}(\tau)-1\right)\cdot\mu_i\left(p_{-\alpha}^{-1}(x_{-\alpha}^z)\right)^{-1}\right)\right).$$

Here $\check{\alpha}^{-1}$ is defined on $\mathcal{T}^\alpha = (\mathcal{F}\mathbf{T}^\alpha)^{r-1} \cong (\mathcal{F}\mathbb{G}_{m/\mathcal{O}_r^{\mathrm{ur}}})^{r-1}$ as the inverse to $\check{\alpha}$, and we view $\check{\alpha}^{-1}(\tau)$ as an element in $\mathcal{F}\mathbb{G}_{a/\mathcal{O}_r^{\mathrm{ur}}}$ by the natural open immersion $\mathbb{G}_{m/\mathcal{O}_r^{\mathrm{ur}}} \to \mathbb{G}_{a/\mathcal{O}_r^{\mathrm{ur}}}$, so the minus operation $\check{\alpha}^{-1}(\tau) - 1$ is well-defined. On the other hand, by our assumption on z, $\mu_i\left(p_{-\alpha}^{-1}(x_{-\alpha}^z)\right)$ is an element in $\mathcal{F}\mathbb{G}_{m/\mathcal{O}_r^{\mathrm{ur}}}$, so its inverse exits. Moreover, the multiplication operation "$\cdot$" is by viewing $\mathbb{G}_{a/\mathcal{O}_r^{\mathrm{ur}}}$ (resp. $\mathcal{F}\mathbb{G}_{a/\mathcal{O}_r^{\mathrm{ur}}}$) as a ring scheme (resp. k-ring variety). Thus Ψ_z^α is well-defined as a morphism.

Finally, by direct computations one sees the morphism $\tau \mapsto \Psi_z^\alpha(\tau) \mapsto \tau_{\Psi_z^\alpha(\tau),z}$ is the identity map on the k-points $\mathcal{T}^\alpha(k)$ of the 1-dimensional affine space $\mathcal{T}^\alpha \cong \mathbb{A}_k^1$, so it is the identity morphism, hence Ψ_z^α is a section to $\xi \mapsto \tau_{\xi,z}$. The other assertions in the lemma follow from the definition of Ψ_z^α. □

Now we turn to the generalised result itself (see [CS16, Theorem 4.1]):

Theorem 4.3.9. *If $\theta \in \widehat{T}$ regular and in general position, then*

$$\langle H_c^*(L^{-1}(FU^\pm))_\theta, R_{T,U}^\theta\rangle_{G^F} = 1$$

Proof. Similar to the previous subsection, one has

$$\langle H_c^*(L^{-1}(FU^\pm))_\theta, R_{T,U}^\theta\rangle = \dim H_c^*(\Sigma)_{\theta^{-1},\theta},$$

where $\Sigma := \{(x, x', y) \in U^\pm \times FU \times G \mid xF(y) = yx'\}$. And similarly, we only need to show:

(a) $\dim H_c^*(\Sigma_v'')_{\theta^{-1},\theta} = 1$ if $v = 1$, and $= 0$ if $v \neq 1$;

(b) $\dim H_c^*(\Sigma_v')_{\theta^{-1},\theta} = 0$ for all v,

where

$$\Sigma_v' := \{(x, x', u', u^-, \tau, u) \in \hat{\Sigma}_v \mid (u', u^-) \in \mathcal{Z}_v \setminus \mathcal{Z}_v(l)\}$$

and

$$\Sigma_v'' := \{(x, x', u', u^-, \tau, u) \in \hat{\Sigma}_v \mid (u', u^-) \in \mathcal{Z}_v(l)\};$$

here $\mathcal{Z}_v := (U\cap\hat{v}U^-\hat{v}^{-1})\times\hat{v}(U^-)^1\hat{v}^{-1}$, $\mathcal{Z}_v(i)$ the pre-image of $\hat{v}(U^-)^i\hat{v}^{-1}$ under $(U\cap\hat{v}U^-\hat{v}^{-1})\times\hat{v}(U^-)^1\hat{v}^{-1} \to \hat{v}U^-\hat{v}^{-1}$, and $\mathcal{Z}_v^*(i) := \mathcal{Z}_v(i) \setminus \mathcal{Z}_v(i+1)$. This (a) can be proved in the same way as Lemma 4.3.3, so we focus on (b).

The similar argument in Lemma 4.3.5 implies it suffices to show

$$\dim H_c^*(\bar{\Sigma}_v')_{\theta^{-1}|_{(T^{r-1})^F}} = 0$$

for the subgroup $(T^{r-1})^F = (T^{r-1})^F \times 1 \subseteq T^F \times T^F$, where

$$\bar{\Sigma}'_v := \{(x, x', u', u^-, \tau, u) \in U^{\pm} \times FU \times \mathcal{Z}_v \setminus \mathcal{Z}_v(l) \times T \times U \mid xF(\tau u' u^- \hat{v}) = \tau u' u^- \hat{v} u x'\},$$

on which the $(T^{r-1})^F$-action is given by

$$t\colon (x, x', u', u^-, \tau, u) \mapsto (x, x', u', u^-, t^{-1}\tau, u).$$

We fix a total order on the finite set Φ^-. For $\beta \in \Phi^-$, let $F(\beta) \in \Phi$ be the root defined by $F(U)_{F(\beta)} = F(U_\beta)$, then the order on Φ^- produces an order on $F(\Phi^-)$; similarly we can define F from Φ^+, and hence get a bijection on $\Phi = \Phi^- \sqcup \Phi^+ = F(\Phi^-) \sqcup F(\Phi^+)$, and then a bijection on $\{U_\beta\}_{\beta\in\Phi}$; it is clear $F(-\alpha) = -F(\alpha)$ for any $\alpha \in \Phi$. Let $\mathcal{Z}_v^\beta(i) \subseteq \mathcal{Z}_v^*(i)$ be the subvariety consisting of the (u', u^-) such that, in the decomposition $F(z) := F(\hat{v}^{-1}u'u^-\hat{v}) = \prod_{\beta' \in F(\Phi^-)} x_{\beta'}^{F(z)}$ one has: $x_{\beta'}^{F(z)} = 1$ whenever $\mathrm{ht}(\beta') < \mathrm{ht}(F(\beta))$, $x_{\beta'}^{F(z)} = 1$ whenever $\mathrm{ht}(\beta') = \mathrm{ht}(F(\beta))$ and $\beta' < F(\beta)$, and $x_{F(\beta)}^{F(z)} \neq 1$. We then obtain a finite disjoint partition

$$\mathcal{Z}_v \setminus \mathcal{Z}_v(l) = \coprod_{i=0}^{l-1} \coprod_{\beta\in\Phi^-} \mathcal{Z}_v^\beta(i).$$

And hence a partition of $\bar{\Sigma}'_v$ into locally closed subvarieties

$$\bar{\Sigma}'_v = \coprod_{i=0}^{l-1} \coprod_{\beta\in\Phi^-} \Sigma_v^{i,\beta},$$

where

$$\Sigma_v^{i,\beta} := \{(x, x', u', u^-, \tau, u) \in U^{\pm} \times FU \times \mathcal{Z}_v^\beta(i) \times T \times U \mid xF(\tau u' u^- \hat{v}) = \tau u' u^- \hat{v} u x'\}.$$

Each subvariety $\Sigma_v^{i,\beta}$ inherits the $(T^{r-1})^F$-action

$$t\colon (x, x', u', u^-, \tau, u) \mapsto (x, x', u', u^-, t^{-1}\tau, u),$$

so it suffices to show:

$$H_c^*(\Sigma_v^{i,\beta})_{\theta^{-1}|_{(T^{r-1})^F}} = 0,$$

for every $i \in \{0, \cdots, l-1\}$ and every $\beta \in \Phi^-$.

From now on we fix an $\alpha \in \Phi^+$. For any $t \in T^{r-1}$ such that $F(\hat{v})^{-1}F(t)t^{-1}F(\hat{v}) \in \mathcal{T}^{F(\alpha)}$, define $g_t\colon FU \to FU$ by

$$g_t\colon x' \mapsto x' \cdot \Psi_{F(z)}^{F(\alpha)}(F(\hat{v})^{-1}F(t^{-1})tF(\hat{v}))^{-1}$$

with the parameter $(u', u^-) \in \mathcal{Z}_v^{-\alpha}(i)$, where $z := \hat{v}^{-1}u'u^-\hat{v}$. This is well-defined because $F(z)$ satisfies the conditions in Lemma 4.3.8, with respect to $F(U^-)$ and $F(\Phi^-)$. Note that, if $F(t) = t$, then $g_t(x') = x'$.

Moreover, for any $t \in T^{r-1}$ such that $F(\hat{v})^{-1}F(t)t^{-1}F(\hat{v}) \in \mathcal{T}^{F(\alpha)}$, define the morphism $f_t \colon U^{\pm} \to U^{\pm}$ by

$$f_t \colon x \mapsto x \cdot F(\tau)t^{-1}F(\hat{v})F(z)x'^{-1}g_t(x')F(z^{-1})F(\hat{v})^{-1}F(t)F(\tau)^{-1},$$

with the parameters $x' \in FU$, $\tau \in T$, and $(u', u^-) \in \mathcal{Z}_v^{-\alpha}(i)$, where $z := \hat{v}^{-1}u'u^-\hat{v}$. To see this is well-defined one need to check the right hand side is in $U^{\pm}$: By the definition of $\Psi_{F(z)}^{F(\alpha)}$ we have

$$F(z)x'^{-1}g_t(x')F(z^{-1}) = \Psi_{F(z)}^{F(\alpha)}(F(\hat{v})^{-1}F(t^{-1})tF(\hat{v}))^{-1} \cdot F(\hat{v})^{-1}F(t^{-1})tF(\hat{v}) \cdot \omega,$$

for some $\omega \in (U^-)^{r-1}$. Hence

$$(x^{-1}f_t(x))^{F(\tau)} = ({}^{F(\hat{v})}\Psi)^t \cdot ({}^{F(\hat{v})}\omega)^{F(t)} \in \prod_{\beta \in \Phi} U_\beta^{r-i-1} \subseteq U^{\pm},$$

where $\Psi := \Psi_{F(z)}^{F(\alpha)}(F(\hat{v})^{-1}F(t^{-1})tF(\hat{v}))^{-1}$. Thus $x^{-1}f_t(x) \in \prod_{\beta\in\Phi} U_\beta^{r-i-1} \subseteq U^{\pm}$, and f_t is therefore well-defined. Moreover, if $F(t) = t$, then $f_t(x) = x$.

Consider the closed subgroup

$$H := \{t \in T^{r-1} \mid F(\hat{v})^{-1}F(t)t^{-1}F(\hat{v}) \in \mathcal{T}^{F(\alpha)}\}$$

of T^{r-1}. For any $t \in H$, the above preparations on f_t and g_t allow us to define the following automorphism on $\Sigma_v^{i,-\alpha}$:

$$h_t \colon (x, x', u', u^-, \tau, u) \mapsto (f_t(x), g_t(x'), u', u^-, t^{-1}\tau, u),$$

where the involved parameters are as presented. To see this is well-defined, one needs to show the right hand side satisfy the defining equation of $\Sigma_v^{i,-\alpha}$; this can be seen by expanding the definition of f_t. Meanwhile, note that in the case $F(t) = t$, the automorphism h_t coincides with the $(T^{r-1})^F$-action. Thus by Lemma 4.3.4, for any t in the identity component H° of H, the induced map of h_t on $H_c^*(\Sigma_v^{i,-\alpha})$ is the identity map.

Let $a \geq 1$ be such that $F^a(F(\hat{v})\mathcal{T}^{F(\alpha)}F(\hat{v})^{-1}) = F(\hat{v})\mathcal{T}^{F(\alpha)}F(\hat{v})^{-1}$, then the image of the norm map $N_F^{F^a}(t) = t \cdot F(t) \cdots F^{a-1}(t)$ on $F(\hat{v})\mathcal{T}^{F(\alpha)}F(\hat{v})^{-1}$ is a connected subgroup of H, hence contained in H°. Moreover $N_F^{F^a}((F(\hat{v})\mathcal{T}^{F(\alpha)}F(\hat{v})^{-1})^{F^a}) \subseteq (T^{r-1})^F \cap H^\circ$, thus

$$H_c^*(\Sigma_v^{i,-\alpha})_{\theta^{-1}\circ N_F^{F^a}|_{(F(\hat{v})\mathcal{T}^{F(\alpha)}F(\hat{v})^{-1})^{F^a}}} = H_c^*(\Sigma_v^{i,-\alpha})_{\theta^{-1}|_{N_F^{F^a}((F(\hat{v})\mathcal{T}^{F(\alpha)}F(\hat{v})^{-1})^{F^a})}} = 0,$$

as θ is regular. Therefore $H_c^*(\Sigma_v^{i,-\alpha})_{\theta^{-1}|_{(T^{r-1})^F}} = 0$. This completes the proof. □

Now a solution to Question 4.3.1 for a general reductive group is obtained:

Corollary 4.3.10. *Suppose $\theta \in \widehat{T^F}$ is regular and in general position, then*

$$R_{T,U}^{\theta} \cong \mathrm{Ind}_{(TU^{\pm})^F}^{G^F}\widetilde{\theta},$$

if either Gérardin's conditions (see Remark 4.1.7) on groups are satisfied or the genericity condition (see Definition 4.1.10) on θ is satisfied.

Proof. This follows immediately from Theorem 4.3.9, Remark 4.1.7, Proposition 4.1.11, and Theorem 3.1.8. □

4.4 Some remarks on finite Lie algebras

In this section we take $\mathcal{O} = \mathbb{F}_q[[\pi]]$. The group G^{r-1} can be regarded as the additive group of the Lie algebra $\mathfrak{g}$ of G_1, and the adjoint action of G_1^F on $\mathfrak{g}^F$ is the conjugation of G_1^F on $(G^{r-1})^F$. We do not distinguish the groups G^{r-1} and $\mathfrak{g}$. Given a fixed representation of a group, we do not distinguish it from its character or its representation space (so we do not distinguish the terminologies group characters, group modules, and group representations). In the remaining part of this section we assume $r = 2$.

A character (of some virtual representation) of the finite abelian group $\mathfrak{g}^F = (G^1)^F$ is called *invariant* if it is invariant under the adjoint action of G_1^F, and an invariant character is called an irreducible invariant character if it can not be decomposed into the sum of two invariant characters. Note that, if χ is an irreducible character of $(G^1)^F$, then

$$\chi^O := \bigoplus_{s \in G^F/\mathrm{Stab}_{G^F}(\chi)} \chi^s$$

is an invariant character of $(G^1)^F$, and any invariant character containing χ contains χ^O. In particular, this means χ^O is the unique irreducible invariant character containing χ. Invariant characters are related to character sheaves; see [Lus87] and [Let05].

Let θ^1 be a character of $\mathfrak{t}^F = (T^1)^F$, and write its trivial extension to T^F by θ. Since $R_{T,U}^{\theta}$ is also a $\mathfrak{g}^F \cong (G^1)^F$-module, we can regard $R_{\mathfrak{t},\mathfrak{u}}^{\theta^1} := \mathrm{Res}_{\mathfrak{g}^F}^{G^F} R_{T,U}^{\theta}$ as a Deligne–Lusztig theory of invariant characters of $\mathfrak{g}^F$ (for various θ).

Letellier [Let09] compared the above analogue of Deligne–Lusztig representations with a different analogue he considered earlier (in [Let05]), and conjectured that every irreducible invariant character of $\mathfrak{g}^F$ appeared in some $R_{\mathfrak{t},\mathfrak{u}}^{\theta^1}$ (in the sense that the inner product is non-zero). On the other hand, any character of G^F is an invariant character of $(G^1)^F$, thus in order to prove Letellier's conjecture, it suffices to show that, every irreducible character of $(G^1)^F$ is contained in some $R_{\mathfrak{t},\mathfrak{u}}^{\theta^1}$ (in the sense that their inner product is non-zero).

4.4.1 The Harish-Chandra case

In this subsection we focus on the Harish-Chandra case, i.e. we assume $\mathbf{T} = \mathbf{T}_0$ and $\mathbf{B} = \mathbf{B}_0$ (see the notation setting below Lemma 3.3.3). In particular the Deligne–Lusztig representation $R_{\mathfrak{t},\mathfrak{u}}^{\theta^1}$ is the parabolically induced representation $\mathrm{Res}_{\mathfrak{g}^F}^{G^F} \mathrm{Ind}_{B_0^F}^{G^F} \widetilde{\theta}$. Let χ be an irreducible character of $\mathfrak{g}^F$ such that the restriction of χ to $\mathfrak{t}^F$ is $\theta^1 = \theta|_{(T^1)^F}$. Then we have

$$\left\langle \mathrm{Res}_{\mathfrak{g}^F}^{G^F} \mathrm{Ind}_{B_0^F}^{G^F} \widetilde{\theta}, \chi \right\rangle_{\mathfrak{g}^F} = \bigoplus_{s \in B_0^F \backslash G^F / \mathfrak{g}^F} \left\langle \mathrm{Ind}_{({}^sB_0^1)^F}^{\mathfrak{g}^F} \left(\widetilde{\theta}^{s^{-1}}|_{({}^sB_0^1)^F} \right), \chi \right\rangle_{\mathfrak{g}^F}$$

by the Mackey intertwining formula. Note that

$$\left\langle \mathrm{Ind}_{({}^sB_0^1)^F}^{\mathfrak{g}^F} \left(\widetilde{\theta}^{s^{-1}}|_{({}^sB_0^1)^F} \right), \chi \right\rangle_{\mathfrak{g}^F} = \left\langle \left(\widetilde{\theta}^{s^{-1}}|_{({}^sB_0^1)^F} \right), \chi|_{({}^sB_0^1)^F} \right\rangle_{({}^sB_0^1)^F}$$

by Frobenius reciprocity. Therefore χ appears in a Harish-Chandra type Deligne–Lusztig representation if and only if χ is trivial on the rational points of the Lie algebra of the unipotent radical of some conjugate of $\mathbf{B}_0$.

Example 4.4.1. Let us consider the more specific case that $\mathbb{G} = \mathrm{SL}_2$ and $\mathbb{F}_q = \mathbb{F}_p$ is a prime finite field. Let χ be an irreducible character of $\mathfrak{g}^F$; we want to see explicitly when χ can be recovered by some Harish-Chandra theoretic representation $\mathrm{Res}^{G^F}_{\mathfrak{g}^F} \mathrm{Ind}^{G^F}_{B_0^F} \widetilde{\theta}$. First consider this condition: For some $x, y, z, w \in \mathbb{F}_p$ with $xw - zy = 1$, one has that

$$\chi\left(\begin{bmatrix} x & y \\ z & w \end{bmatrix}\begin{bmatrix} 1 & u\pi \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix}^{-1}\right) = \chi\left(\begin{bmatrix} 1 - xzu\pi & x^2u\pi \\ -z^2u\pi & 1 + zxu\pi \end{bmatrix}\right) = 1 \tag{4.2}$$

for any $u \in \mathbb{F}_p$. We want to rewrite this condition. By viewing $\mathfrak{g}^F$ as a product of cyclic groups, we can write

$$\chi\left(\begin{bmatrix} 1 - xzu\pi & x^2u\pi \\ -z^2u\pi & 1 + zxu\pi \end{bmatrix}\right) = \chi_1(-xzu)\chi_2(x^2u)\chi_3(-z^2u),$$

where χ_j are some degree 1 (multiplicative) characters of the cyclic group $\mathbb{Z}/p$. Then, by considering lifts of u from $\mathbb{F}_p = \mathbb{Z}/p$ to $\mathbb{Z}$, we see that the condition (4.2) is equivalent to:

$$(\chi_1(-xz)\chi_2(x^2)\chi_3(-z^2))^u = 1$$

for any $u \in \mathbb{Z}$ for some $x, z \in \mathbb{F}_p$ not both zero. (So we can take $u = 1$.) Write $\chi_j(1) = e^{2\pi i \frac{k_j}{p}}$. Then the condition (4.2) is equivalent to:

$$-k_1xz + k_2x^2 - k_3z^2 \equiv 0 \mod p \tag{4.3}$$

for some $x, z \in \mathbb{Z}$ with p not divides x and z simultaneously. If $p \mid k_2$, then by taking $z = 0$ we see χ can be realised by Harish-Chandra theory by definition. If k_2 is prime to p, then z is prime to p as well; denote x/z by t. So, with the additional assumption that k_2 is prime to p, condition (4.3) can be reinterpreted as:

$$t^2 - \frac{k_1}{k_2}t - \frac{k_3}{k_2} \equiv 0 \mod p$$

for some $t \in \mathbb{Z}$. Therefore, the irreducible invariant character containing χ can be recovered from the Harish-Chandra theory if and only if

1. $p \mid k_2$, or
2. $p \nmid k_2$ and the Legendre symbol $\left(\frac{k_1^2+4k_2k_3}{p}\right) \geq 0$,

in other word, if and only if the Legendre symbol $\left(\frac{k_1^2+4k_2k_3}{p}\right) \geq 0$.

4.4.2 Invariant characters and arithmetic radicals

In this subsection we use a geometric method to investigate the invariant characters of $\mathfrak{g}^F$. This is not the easiest approach and it is unnecessarily complicated, but it has the advantage of being an analogue of the standard arguments in Deligne–Lusztig theory.

Note that $(G^1)^F$ acts on $L^{-1}(U^\pm)$ by both right and left action, so if we denote by $H_c^*(L^{-1}(U^\pm))_\chi$ the subspace of $H_c^*(L^{-1}(U^\pm))$ on which the right action of $(G^1)^F$ is given by an irreducible character χ, then we can consider the left $(G^1)^F$-module $H_c^*(L^{-1}(U^\pm))_\chi$. Let $\theta^1 := \chi|_{(T^1)^F}$, then $H_c^*(L^{-1}(U^\pm))_\chi$ is a $(G^1)^F$-submodule of $H_c^*(L^{-1}(U^\pm))_{\theta^1}$, which admits a decomposition

$$H_c^*(L^{-1}(U^\pm))_{\theta^1} = \bigoplus_{\theta\in\widehat{T^F},\ \theta|_{(T^1)^F}=\theta^1} H_c^*(L^{-1}(U^\pm))_\theta = \bigoplus_{\theta\in\widehat{T^F},\ \theta|_{(T^1)^F}=\theta^1} \mathrm{Ind}_{(TU^\pm)^F}^{G^F}\widetilde{\theta}.$$

So, if χ appears in $H_c^*(L^{-1}(U^\pm))_\chi$, then it must appear in some $\mathrm{Ind}_{(TU^\pm)^F}^{G^F}\widetilde{\theta}$. Meanwhile, $\chi \cong H_c^*((G^1)^F)_\chi$ where the subscript "$_\chi$" is from the right action of $(G^1)^F$ on the commutative algebra $H_c^*((G^1)^F) \cong \overline{\mathbb{Q}}_\ell[(G^1)^F]$.

By $(g, g') \mapsto (g'^{-1}F(g'), g^{-1}g')$ we get an isomorphism

$$(G^1)^F\backslash((G^1)^F \times L^{-1}(U^\pm) \cong \Pi := \{(x, y) \in U^\pm \times G \mid x = y^{-1}F(y)\}$$

(the graph of Lang map on $L^{-1}(U^\pm)$), which is $(G^1)^F \times (G^1)^F$-equivariant (acts from the right hand side) with the action of $(G^1)^F \times (G^1)^F$ on Π given by

$$(G^1)^F \times (G^1)^F \ni (s, s')\colon (x, y) \mapsto (x^{s'}, s^{-1}ys') = (x, s^{-1}ys').$$

Therefore

$$\langle\chi, H_c^*(L^{-1}(U^\pm))_\chi\rangle_{(G^1)^F} = \langle H_c^*((G^1)^F)_\chi, H_c^*(L^{-1}(U^\pm))_\chi\rangle_{(G^1)^F} = \dim H_c^*(\Pi)_{\chi^{-1},\chi},$$

by Künneth formula and (Hom–tensor) adjunction. Note that the reduction map $\rho_{2,1}\colon G_2 \to G_1$ takes Π surjectively to $\{1\}\times G_1^F$, so each $(x, y) \in \Pi$ is of the form (x, gy') for some $g \in G_1^F$ and $y' \in G^1$, hence there is a decomposition into closed subvarieties

$$\Pi = \coprod_{g\in G_1^F} \Pi_g,$$

where $\Pi_g := \{(x, gy) \in U^\pm \times G^1 \mid x = y^{-1}F(y)\}$. So

$$\Pi_g \cong \{(x, y) \in U^\pm \times G^1 \mid x = y^{-1}F(y)\},$$

on which $(G^1)^F \times (G^1)^F$ acts by

$$(G^1)^F \times (G^1)^F \ni (s, s')\colon (x, y) \mapsto (x, g^{-1}g^s s^{-1}ys').$$

Thus

$$H_c^*(\Pi)_{\chi^{-1},\chi} = \bigoplus_{g \in G_1^F} H_c^*(\Pi_g)_{\chi^{-1},\chi}.$$

By the Iwahori decomposition $G^1 = T^1U^{\pm}$ and the Lang–Steinberg theorem we see that $\Pi_g \cong (T^1)^F \times U^{\pm}$, so (see e.g. [DM91, 10.12])

$$H_c^*(\Pi_g) \cong \overline{\mathbb{Q}}_\ell[(T^1)^F],$$

on which $(s, s') \in (G^1)^F \times (G^1)^F$ acts by $t \mapsto g^{-1}g^s s^{-1}s't$ composed with the projection $G^1 = T^1U^{\pm} \to T^1$. In particular, $H_c^*(\Pi)_{\chi^{-1},\chi}$ is non-zero only if $\chi((U^{\pm})^F) = 1$; in other words, χ appears in $\mathrm{Ind}_{(TU^{\pm})^F}^{G^F}\widetilde{\theta}$ for some $\theta \in T^F$ with $\theta|_{(T^1)^F} = \chi|_{(T^1)^F}$ only if χ is trivial on $(U^{\pm})^F$.

On the other hand, for $g = 1$, we see that, if χ is trivial on $(U^{\pm})^F$, then χ appears in $H_c^*(\Pi_1)_{\chi^{-1},\chi} = \overline{\mathbb{Q}}_\ell[(T^1)^F]$, on which $(s, s') \in (G^1)^F \times (G^1)^F$ acts by $t \mapsto s^{-1}s't$ composed with the projection $G^1 = T^1U^{\pm} \to T^1$ (this can be viewed as the regular representation of $(T^1)^F$). Therefore, χ appears in $\mathrm{Ind}_{(TU^{\pm})^F}^{G^F}\widetilde{\theta}$ for some $\theta \in T^F$ with $\theta|_{(T^1)^F} = \chi|_{(T^1)^F}$ if and only if χ is trivial on the rational points of the arithmetic radical $U^{\pm}$.

Remark 4.4.2. This result can be obtained in a much easier way by directly applying the Mackey intertwining formula; see the arguments in [CS16, 5.1].

One has a special case of Letellier's conjecture:

Proposition 4.4.3. *For $\mathbb{G} = \mathrm{GL}_2$ or GL_3, every irreducible invariant character of $\mathfrak{g}^F$ appears in a Deligne–Lusztig character $R_{\mathfrak{t},\mathfrak{u}}^{\theta^1}$ in the sense that their inner product is non-zero.*

Proof. This is proved in [CS16, 5.1]. □

Chapter 5

Towards character sheaves over rings

In this chapter we turn to another aspect of the representation theory of $\mathbb{G}(\mathcal{O}_r)$. We first recall the notion of character sheaves introduced in [Lus86] and [Lus06], and then propose a construction of generic character sheaves on reductive groups over quotients of complete discrete valuation rings. We will define the associated induction and restriction functors, and discuss their transitive properties and the adjunction relation. Recall that $\mathcal{D}(-)$ denotes the bounded derived category of constructible $\overline{\mathbb{Q}}_\ell$-sheaves and $\mathcal{M}(-)$ denotes the subcategory of perverse sheaves; these notation settings can be found in Section 1.3.

5.1 Sheaves on abelian groups

In this section we recall the setting of character sheaves on abelian groups in [Lus06, 5].

Throughout this section let H be a connected commutative (affine) algebraic group over $k = \overline{\mathbb{F}}_q$. A prototype of H in our mind is the Greenberg functor image T of a torus group scheme $\mathbf{T} \cong \mathbb{G}_{m/\mathcal{O}_r^{\mathrm{ur}}} \times \cdots \times \mathbb{G}_{m/\mathcal{O}_r^{\mathrm{ur}}}$ over $\mathcal{O}_r^{\mathrm{ur}}$.

Consider the set of all pairs (F, ψ) for various geometric Frobenius F on H, and $\psi \in \widehat{H^F} = \mathrm{Hom}(H^F, \overline{\mathbb{Q}}_\ell^\times)$; by identifying (F, ψ) with $(F^n, \psi \circ N_{F^n/F})$ for every positive integer n we get an equivalence relation on this set. We denote the equivalence classes by H^*.

Lemma 5.1.1. *H^* is an abelian group with multiplication $(F, \psi) \cdot (F, \psi') := (F, \psi \cdot \psi')$.*

Proof. By the fact that every two Frobenius endomorphisms become the same one after being raised to some powers (see e.g. [DM91, 3.6]), and by the definition of the equivalence relation, we see the multiplication is well-defined. The lemma follows. □

For any $(F, \psi) \in H^*$, we want to associate a local system of rank 1, i.e. a locally constant $\overline{\mathbb{Q}}_\ell$-sheaf of rank 1.

Let $L\colon H \to H$ be the Lang map associated to F, and consider the $\overline{\mathbb{Q}}_\ell$-local system $L_*\overline{\mathbb{Q}}_\ell$

on H. We have a direct sum decomposition

$$L_*\overline{\mathbb{Q}}_\ell = \bigoplus_{\psi \in \widehat{H^F}} E^\psi, \tag{5.1}$$

where E^ψ is a locally constant $\overline{\mathbb{Q}}_\ell$-sheaf of rank 1, whose stalk at $t \in H$ is the 1-dimensional representation of H^F given by ψ:

$$E^\psi_t = \{f\colon L^{-1}(t) \to \overline{\mathbb{Q}}_\ell \mid f(t_1 t_2) = \psi(t_1) f(t_2), \forall t_1 \in H^F, t_2 \in L^{-1}(t)\}.$$

Up to isomorphisms, E^ψ only depends on the equivalence class of (F, ψ); see [Lus06, 5].

Definition 5.1.2. We denote the set of isomorphism classes of local systems on H of the form E^ψ, where $(F, \psi) \in H^*$, by $\mathcal{S}(H)$.

The multiplication of functions on $L^{-1}(t)$ gives an isomorphism $E^\psi_t \otimes E^{\psi'}_t \cong E^{\psi\cdot\psi'}_t$, which induces an isomorphism $E^\psi \otimes E^{\psi'} \cong E^{\psi\cdot\psi'}$. So $\mathcal{S}(H)$ admits a natural group structure. Indeed, this group is isomorphic to H^*; to see it, we first recall the notion of characteristic function: The *characteristic function* $\chi_{E^\psi,\varphi}\colon H^F \to \overline{\mathbb{Q}}_\ell$ of E^ψ, with respect to an isomorphism of locally constant sheaves $\varphi\colon F^*E^\psi \to E^\psi$, is defined by taking $t \in H^F$ to $\mathrm{Tr}(\varphi_t, E^\psi_t)$.

Proposition 5.1.3. *The map $(F, \psi) \mapsto E^\psi$ is an isomorphism of abelian groups $H^* \cong \mathcal{S}(H)$.*

Proof. This is in [Lus06, 5]. This map is by definition a surjective group morphism. To see the surjectivity it suffices to show that, for a fixed F, there is a unique ψ such that the image of (F, ψ) is E^ψ. This can be done by looking at the characteristic functions. Firstly, there is a unique isomorphism of locally constant $\overline{\mathbb{Q}}_\ell$-sheaves $\varphi\colon F^*E^\psi \to E^\psi$, such that at stalks $E^\psi_{F(t)} \to E^\psi_t$ it is $\varphi_t\colon f \mapsto f \circ F$; see [Lus06, 5]. If $t \in H^F$, then for $y \in L^{-1}(t)$ one has $F(y) = ty$, so $f \circ F(y) = \psi(t) f(y)$. Thus ψ appeared as the trace of φ at each stalk, i.e. the characteristic function of E^ψ with respect to φ. □

5.2 The finite field case

In this section we recall the notion of character sheaves on G_1 given by Lusztig in [Lus86]. Throughout this section we assume $r = 1$. In particular, $G = G_1$ is a connected reductive group and $T = T_1$ is a maximal torus in a Borel subgroup $B = B_1$.

Fix a local system $\mathcal{L}$ of rank 1 in $\mathcal{S}(T)$. We can consider its stabiliser $W'_\mathcal{L} := \{w \in W \mid (w^{-1})^*\mathcal{L} = \mathcal{L}\}$ in $W = W(T)$; we will define an object $K^\mathcal{L}_w \in \mathcal{D}(G)$ for $w \in W'_\mathcal{L}$. The character sheaves are defined via the perverse cohomology of $K^\mathcal{L}_w$. To define $K^\mathcal{L}_w$, first consider a diagram

$$G \xleftarrow{\pi_w} Y_w \longleftarrow \dot{Y}_w \longrightarrow T \tag{5.2}$$

for $w \in W$: Here

$$Y_w := \{(g, B') \in G \times G/B \mid (B', {}^gB') \in O(w)\},$$

where $O(w)$ denotes the G-conjugation orbit of $(B, {}^wB) \in G/B \times G/B$, and

$$\dot{Y}_w := \{(g, hU) \in G \times G/U \mid g^h \in BwB\}.$$

The morphism $\dot{Y}_w \to Y_w$ is the principal fibration (with T) defined by $(g, hU) \mapsto (g, {}^hB)$; here T acts on $\dot{Y}_w$ by $t_0 \colon (g, hU) \mapsto (g, ht_0^{-1}U)$. The morphism $\pi_w \colon Y_w \to G$ is the natural left projection, which can be viewed as a twist of the Grothendieck–Springer resolution (the case $w = 1$ is exactly the Grothendieck–Springer resolution, which is proper because Y_1 is closed in $G \times G/B$; see e.g. [Sho88, II.4]). And finally, $\dot{Y}_w \to T$ is the projection of $g^h \in BwB = UwTU$ to the T-component; this morphism is T-equivariant with respect to the action $t_0 \colon t \mapsto t_0^w \cdot t_0^{-1}t$ of T on T. For more details see [Lus86, 2].

Remark 5.2.1. Note that Y_w is an analogue of the Deligne–Lusztig variety $X(w) := \{B' \in G/B \mid (B', FB') \in O(w)\}$ studied in [DL76], in the sense of viewing the Frobenius action as a "conjugation".

For the local system $\mathcal{L} \in \mathcal{S}(T)$, if $w \in W'_{\mathcal{L}}$, then the shift $\dot{\mathcal{L}}[\dim T]$ of the inverse image $\dot{\mathcal{L}}$ on $\dot{Y}_w$ is T-equivariant in the sense of [Lus86, 1.9]. This implies the existence of a unique $\overline{\mathbb{Q}}_\ell$-local system $\widetilde{\mathcal{L}}$ of rank 1 on Y_w such that its inverse image on the T-fibration $\dot{Y}_w$ is $\dot{\mathcal{L}}$; see [Lus10, 8.1.7(c)] or [Lus86, 1.9.3].

Now we can define character sheaves.

Definition 5.2.2. Along with the above notation, for $\mathcal{L} \in \mathcal{S}(T)$ and $w \in W'_{\mathcal{L}}$, put $K_w^{\mathcal{L}} := R(\pi_w)_! \widetilde{\mathcal{L}} \in \mathcal{D}(G)$.

Definition 5.2.3. A *character sheaf* on G is a simple perverse sheaf on G appearing as a constituent of ${}^pH^i(K_w^{\mathcal{L}})$ for some $\mathcal{L} \in \mathcal{S}(T)$, some $w \in W'_{\mathcal{L}}$, and some $i \in \mathbb{Z}$ (here ${}^pH^i(-)$ denotes perverse cohomology; see e.g. [Lus86, 1.4]). The set of isomorphism classes of character sheaves is denoted by $\mathrm{CS}(G)$.

Example 5.2.4. When $G = T$ is a torus, the elements in $\mathrm{CS}(T)$ are all of the form $\mathcal{L}[\dim T]$, where $\mathcal{L} \in \mathcal{S}(T)$ (see the last sentence in [Lus86, 2.10]).

Example 5.2.5. Take $w = 1$, then the fact that $N_G(B) = B$ implies

$$Y_w = \{(g, B') \in G \times G/B \mid g \in B'\} = \{(g, hB) \in G \times G/B \mid g^h \in B\},$$

which gives a diagram $G \leftarrow Y_w \to T$, with $Y_w \to T$ being defined by projecting $g^h \in B$ to T. The construction of character sheaves based on the longer diagram (5.2) can be recovered by this shorter diagram. Indeed, this gives the theory of character sheaves in the principal series case. Actually, since w is 1, we have $\dot{Y}_w = \{(g, hU) \in G \times G/U \mid g^h \in B\}$, so the morphism $\dot{Y}_w \to T$ naturally factors through Y_w, and thus $\widetilde{\mathcal{L}}$ can be defined as the inverse image of $\mathcal{L}$ (according to [Lus86, 2.4], such $\widetilde{\mathcal{L}}$ is unique).

5.3 The local ring case

5.3.1 Character sheaves and its functions

In the remaining part of this thesis we assume $r = 2l$ is even.

The algebraisation of the generic Deligne–Lusztig representations $H_c^*(L^{-1}(FU))_\theta$ at even levels suggests that one can develop a generic character sheaf theory for reductive groups over $\mathcal{O}_r^{\mathrm{ur}}$ with r even, in analogy with Gérardin's constructions. Consider the diagram

$$T \xleftarrow{\ b\ } Z_T \xrightarrow{\ a\ } G, \tag{5.3}$$

where

$$Z_T := \{(g, xTU^{\pm}) \in G \times G/TU^{\pm} \mid g^x \in TU^{\pm}\};$$

here a is the left projection $(g, xTU^{\pm}) \mapsto g$, and b is $(g, xTU^{\pm}) \mapsto \pi_T(g^x)$, where π_T is the projection from $TU^{\pm}$ to T.

Lemma 5.3.1. *The variety Z_T, as well as the morphisms a and b, are F-rational.*

Proof. This follows from the rationality of $U^{\pm}$. □

Lemma 5.3.2. *The variety Z_T is smooth and connected.*

Proof. This can be proved in a similar way to [Lus86, 2.5.2]. Let $\widetilde{Z_T}$ be the base change of $Z_T \subseteq G \times G/TU^{\pm}$ along the surjective flat morphism (the flatness follows from, e.g. [Liu06, 1.2.14] and [Spr09, Page 93])

$$G \times G \longrightarrow G \times G/TU^{\pm}.$$

Then by [GD67, 17.7.7] it suffices to show $\widetilde{Z_T} = \{(g, x) \in G \times G \mid g^x \in TU^{\pm}\} \subseteq G \times G$ is smooth and connected. Applying the change of variables $b = g^x$ we get

$$\widetilde{Z_T} \cong \{(b, x) \in G \times G \mid b \in TU^{\pm}\} = TU^{\pm} \times G,$$

which is smooth and connected. □

Back to the diagram (5.3). Let $E^\theta \in \mathcal{S}(T)$ for some $\theta \in \widehat{T^F}$; then as in the proof of Proposition 5.1.3, there is a natural isomorphism $\varphi\colon F^*E^\theta \cong E^\theta$, which induces an isomorphism $F^*Ra_!(b^*E^\theta) \cong Ra_!(b^*E^\theta)$ (by proper base change; see also [Lus06, 8]). We denote the latter isomorphism again by φ. Let $K^\theta := Ra_!(b^*E^\theta) \in \mathcal{D}(G)$.

Definition 5.3.3. If either θ is generic, or $\mathbb{G}$ satisfies the conditions in Remark 4.1.7 and θ is regular and in general position, then we call K^θ a *generic character sheaf* on $\mathbf{G}$.

The *characteristic function* of the complex K^θ with respect to the isomorphism φ is the $\overline{\mathbb{Q}}_\ell$-valued function

$$\chi_{K^\theta,\varphi}\colon G^F \longrightarrow \overline{\mathbb{Q}}_\ell, \quad g \mapsto \chi_{K^\theta,\varphi}(g) := \sum_{i\in\mathbb{Z}} (-1)^i \cdot \mathrm{Tr}(\varphi_g, \mathcal{H}^i(K^\theta)_g).$$

In the generic case this is the character of the higher Deligne–Lusztig representation $R_{T,U}^\theta$ (thanks to our main results):

Proposition 5.3.4. *Along with the above notation, we have*

$$\chi_{K^\theta,\varphi}(g) = \mathrm{Tr}(g, R^\theta_{T,U})$$

for any $g \in G^F$.

Proof. The argument is standard (see also [Lus86, Proposition 13.4]). First of all, by the proper base change theorem in derived category, the function $\chi_{Ra_!(b^*E^\theta),\varphi}(g)$ can be linked to the φ-trace on ℓ-adic cohomology

$$\begin{aligned}\chi_{Ra_!b^*E^\theta,\varphi}(g) &= \sum_{i\in\mathbb{Z}}(-1)^i \cdot \mathrm{Tr}\left(\varphi_g, \mathcal{H}^i(Ra_!(b^*E^\theta))_g\right) \\ &= \sum_{i\in\mathbb{Z}}(-1)^i \cdot \mathrm{Tr}\left(\varphi, H^i_c(a^{-1}(g), b^*E^\theta)\right).\end{aligned} \tag{5.4}$$

By applying the Grothendieck–Lefschetz trace formula (see [Del77, Rapport-3.2]) we get:

$$\begin{aligned}(5.4) &= \sum_{xTU^\pm\in(G/TU^\pm)^F,\ (g,xTU^\pm)\in Z_T} \mathrm{Tr}\left(\varphi, b^*E^\theta_{(g,xTU^\pm)}\right) \\ &= \sum_{xTU^\pm\in(G/TU^\pm)^F,\ g^x\in TU^\pm} \mathrm{Tr}\left(\varphi, E^\theta_{\pi_T(g^x)}\right).\end{aligned} \tag{5.5}$$

By passing from $G/TU^\pm$ to G we see:

$$\begin{aligned}(5.5) &= \frac{1}{|(TU^\pm)^F|}\cdot \sum_{x\in G^F,\ g^x\in TU^\pm} \mathrm{Tr}\left(\varphi, E^\theta_{\pi_T(g^x)}\right) \\ &= \frac{1}{|(TU^\pm)^F|}\cdot \sum_{x\in G^F,\ g^x\in TU^\pm} \theta\left(\pi_T(g^x)\right).\end{aligned}$$

By the definition of induced characters, this is the character value of $\mathrm{Ind}_{(TU^\pm)^F}^{G^F}\widetilde{\theta}$ at g. Now the assertion follows from Corollary 4.3.10. □

5.3.2 Induction and restriction

In this subsection we define induction functors (from equivariant perverse sheaves on Levi subgroups to the derived category $\mathcal{D}(G)$) and restriction functors (from the derived category $\mathcal{D}(G)$ to that of the Levi subgroups), and show that they are transitive (for the induction functor, we require an assumption concerning perverse sheaves). Our definitions of induction and restriction functors are motivated by 5.3.7 and 5.3.8, the transitivity properties (Proposition 5.3.10 and Proposition 5.3.13), a Frobenius reciprocity (Proposition 5.3.14), and Lusztig's definitions of induction and restriction functors in the finite field case [Lus86].

Fix a parabolic subgroup $\mathbf{P}$ of $\mathbf{G}$, and let $\mathbf{M}$ be a Levi subgroup of $\mathbf{P}$ (when there is no confusion we will say $\mathbf{M}$ is a Levi subgroup of $\mathbf{G}$); denote by P and M the corresponding

Greenberg functor images, respectively. Write $\mathbf{U_P}$ for the unipotent radical of $\mathbf{P}$, and write $\mathbf{U_P^-}$ for the unipotent radical of its opposite parabolic subgroup; their Greenberg functor images are denoted by U_P and U_P^-, respectively. We put $U^{\pm}_{G-M} := (U_P)^l(U_P^-)^l$ and $U^{\pm}_M := M \cap U^{\pm}$. For detailed properties of parabolic subgroups and Levi subgroups (over a general base) we refer to [DG70, XXVI].

For a smooth morphism $f\colon X \to Y$ with connected fibres of dimension d, we use the notation $\widetilde{f} := f^*[d]$; it is known that $\widetilde{f}$ is a fully faithful functor from $\mathcal{M}(Y)$ to $\mathcal{M}(X)$ (see [Lus86, 1.8.3]). For perverse sheaves, we will use the notion of equivariance in [Lus86, 1.9] (roughly, for a variety with a group action, the equivariance of sheaves is the compatibility between two pull-backs, one along group action and one along projection, and for perverse sheaves the definition is simpler; see also the discussion in [Lus84b, 0]).

We start with the induction functor, which requires some technical preparations:

Consider the varieties

$$Z^G_M := \{(g, xMU^{\pm}) \in G \times G/MU^{\pm} \mid g^x \in MU^{\pm}\}$$

and

$$\widetilde{Z^G_M} := \{(g, x) \in G \times G \mid g^x \in MU^{\pm}\} \cong MU^{\pm} \times G;$$

they admit the G-action $y \in G\colon (g, x) \mapsto ({}^y g, yx)$. Consider the natural G-equivariant smooth morphism with connected fibres

$$\pi'_{M,G}\colon \widetilde{Z^G_M} \longrightarrow Z^G_M, \quad (g, x) \mapsto (g, xMU^{\pm}).$$

This is a principal $MU^{\pm}$-fibration of Z^G_M, where the action of $MU^{\pm}$ on $\widetilde{Z^G_M}$ is given by $y' \in MU^{\pm}\colon (g, x) \mapsto (g, xy'^{-1})$. Moreover, note that the quotient $G \to G/MU^{\pm}$ is locally trivial: As G^l is a connected unipotent group, it is special (in the sense of Remark 5.3.6), so it suffices to show $G_l \to G_l/M_l$ is locally trivial, which follows from the fact that the multiplication morphism $\mathbf{U_P^-} \times \mathbf{U_P} \times \mathbf{M} \to \mathbf{G}$ is an open immersion; see [DG70, XXVI 4.3.2] (by looking at the Lie algebras, one can also deduce the open immersion property by using the fact that it is an étale monomorphism; see the argument of [CGP15, 2.1.2]). Therefore $\pi'_{M,G}$ is a locally trivial principal fibration by $MU^{\pm}$. With respect to the trivial G-action on M, there is another G-equivariant smooth morphism with connected fibres

$$\pi_{M,G}\colon \widetilde{Z^G_M} \longrightarrow M, \quad (g, x) \longmapsto \pi_M(g^x),$$

where π_M denotes the projection from $MU^{\pm}$ to M. Note that the action of $MU^{\pm}$ on $\widetilde{Z^G_M}$ induces an action of $MU^{\pm}$ on M through $\pi_{M,G}$, which is compatible with the conjugation action of M on M, so $\pi_{M,G}$ is also an $MU^{\pm}$-equivariant morphism. Now we get a diagram

$$M \xleftarrow{\pi_{M,G}} \widetilde{Z^G_M} \xrightarrow{\pi'_{M,G}} Z^G_M \xrightarrow{\pi''_{M,G}} G, \tag{5.6}$$

where $\pi''_{M,G}$ is the left projection (which is G-equivariant with respect to the conjugation action of G on G itself).

Consider (5.6), if $K \in \mathcal{M}(M)$ is M-equivariant with respect to the conjugation action, then $\widetilde{\pi}_{M,G}K$ is $MU^{\pm}$-equivariant and G-equivariant by the definition of equivariance ([Lus86, 1.9]; note that $\pi_{M,G}$ is $MU^{\pm}$-equivariant). Moreover, since $\pi'_{M,G}$ is a locally trivial principal fibration by $MU^{\pm}$, by [Lus86, 1.8.3] and [Lus86, 1.9.3] there is a unique (up to isomorphisms) perverse sheaf $K_{M,G} \in \mathcal{M}(Z_M^G)$ such that $\widetilde{\pi}_{M,G}K \cong \widetilde{\pi}'_{M,G}K_{M,G}$; this $K_{M,G}$ is G-equivariant, by [Lus86, 1.8.3] and the G-equivariance of $\widetilde{\pi}_{M,G}K$. Now we get a complex $R(\pi''_{M,G})_!K_{M,G} \in \mathcal{D}(G)$; if this complex is perverse, then it is G-equivariant by [Lus86, 1.9.2].

Definition 5.3.5. Given $K \in \mathcal{M}(M)$ equivariant with respect to the conjugation action of M, then along with the above notation, we put $\mathrm{ind}_M^G K := R(\pi''_{M,G})_!K_{M,G} \in \mathcal{D}(G)$.

Remark 5.3.6. An algebraic group is called special if any principal fibration by such a group is locally trivial, and it is known that connected unipotent groups are among them; see [Ser58, 4.1, 4.4] and [Gro60, 6].

Example 5.3.7. If $\mathbf{M} = \mathbf{G}$, then $\mathrm{ind}_G^G K = K$ by the equivariance property.

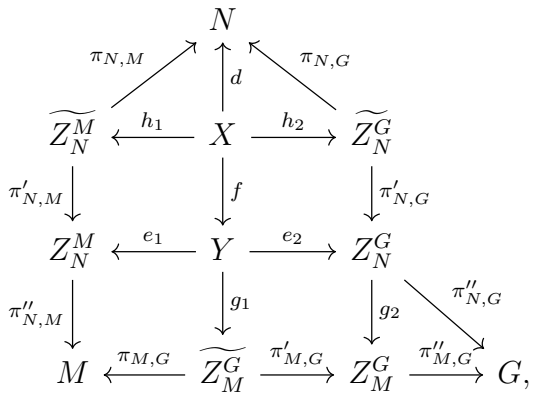
Example 5.3.8. If $\mathbf{M} = \mathbf{T}$ is a maximal torus such that $FT = T$ for the Frobenius F of some rational structure on G, then $Z_T^G = Z_T$, and $\pi_{M,G}$ naturally factors through $b\colon Z_T^G \to T$ (b is the morphism in (5.3)) by $\pi'_{M,G}$, which implies $K_{M,G} \cong \widetilde{b}K$. Thus in this simpler situation, we see that $\mathrm{ind}_T^G E^\theta[\dim T] \cong K^\theta[\dim G]$ for any $\theta \in \widehat{T^F}$.

Remark 5.3.9. In the situation of the above example, we expect that $\mathrm{ind}_T^G E^\theta[\dim T]$ ($\cong K^\theta[\dim G]$) is a simple perverse sheaf, provided K^θ is a generic character sheaf: In the special case that $\mathcal{O} = \mathbb{F}_q[[\pi]]$, $r = 2$ or 4, $\mathrm{char}(\mathbb{F}_q)$ big enough, and T is contained in an F-stable B for some Borel $\mathbf{B}$, this is obtained in Lusztig's work [Lus15].

Proposition 5.3.10. *Let* $\mathbf{N}$ *be a Levi subgroup of* $\mathbf{M}$*; denote by* N *its Greenberg functor image. If* $K \in \mathcal{M}(N)$ *is an* N*-equivariant perverse sheaf such that* $\mathrm{ind}_N^M K$ *is a perverse sheaf (e.g. in the special case mentioned in the above remark,* K *can be* $E^\theta[\dim T]$*), then*

$$\mathrm{ind}_N^G K \cong \mathrm{ind}_M^G \circ \mathrm{ind}_N^M K.$$

Proof. The argument is an analogue of the finite field case in [Lus86, 4.2]. Consider the commutative diagram



where $X := \{(g,x,z) \in G\times G\times MU^{\pm} \mid g^x \in NU^{\pm}\}$, and Y is the quotient of X by the $NU^{\pm}$-action given by $q \in NU^{\pm}$: $(g,x,z) \mapsto (g, xq^{-1}, zq^{-1})$; f denotes the quotient morphism. The other morphisms are as below:

d: $(g,x,z) \mapsto \pi_N(g^x)$, where π_N is the projection from $NU^{\pm}$ to N;

h_1 is $(g,x,z) \mapsto (\pi_M(g^{xz^{-1}}), \pi_M(z))$;

h_2: $(g,x,z) \mapsto (g,x)$;

e_1: $(g,x,z) \mapsto (\pi_M(g^{xz^{-1}}), \pi_M(z)NU^{\pm}_M) \in Z^M_N \subseteq M \times M/NU^{\pm}_M$;

e_2: $(g,x,z) \mapsto (g, xNU^{\pm}) \in Z^G_N = \{(g,x) \in G \times G/NU^{\pm} \mid g^x \in NU^{\pm}\}$;

g_1: $(g,x,z) \mapsto (g, xz^{-1})$;

g_2: $(g, xNU^{\pm}) \mapsto (g, xMU^{\pm})$.

Note that in the above diagram, the two bottom squares are cartesian, and e_i and f are smooth morphisms with connected fibres.

To show $\mathrm{ind}^G_N K \cong \mathrm{ind}^G_M \mathrm{ind}^M_N K$, in other words, to show

$$R(\pi''_{M,G})_! R(g_2)_! K_{N,G} \cong R(\pi''_{M,G})_! (\mathrm{ind}^M_N)_{M,G},$$

it suffices to show

$$R(g_2)_! K_{N,G} \cong (\mathrm{ind}^M_N)_{M,G}.$$

Since $\widetilde{\pi}'_{M,G}$ is fully faithful on perverse sheaves (see [Lus86, 1.8.3]), this assertion can be deduced by showing

$$\widetilde{\pi}'_{M,G} R(g_2)_! K_{N,G} \cong \widetilde{\pi}'_{M,G} (\mathrm{ind}^M_N K)_{M,G}. \tag{5.7}$$

Note that

$$\widetilde{\pi}'_{M,G} (\mathrm{ind}^M_N K)_{M,G} \cong \widetilde{\pi}_{M,G} \mathrm{ind}^M_N K \cong \widetilde{\pi}_{M,G} R(\pi''_{N,M})_! K_{N,M}$$

by the definition of $(\mathrm{ind}^M_N K)_{M,G}$, so (5.7) is equivalent to

$$R(g_1)_! \widetilde{e}_2 K_{N,G} \cong R(g_1)_! \widetilde{e}_1 K_{N,M}$$

by applying the proper base change theorem (on both sides). Thus we only need to show $\widetilde{e}_2 K_{N,G} \cong \widetilde{e}_1 K_{N,M}$, which is equivalent to

$$\widetilde{f}\widetilde{e}_2 K_{N,G} \cong \widetilde{f}\widetilde{e}_1 K_{N,M} \tag{5.8}$$

by the full faithfulness of $\widetilde{f}$ on perverse sheaves ([Lus86, 1.8.3]). By the definitions of $K_{N,M}$ and $K_{N,G}$, (5.8) follows from

$$\widetilde{h}_1 \widetilde{\pi}_{N,M} K = \widetilde{d} K = \widetilde{h}_2 \widetilde{\pi}_{N,G} K.$$

This completes the proof. □

We turn to the restriction functor.

Definition 5.3.11. Consider the diagram

$$M \xleftarrow{\pi_M} MU^{\pm} \xrightarrow{i} G, \tag{5.9}$$

where i is the natural closed immersion and π_M denotes the projection from $MU^{\pm} \cong M \times U^{\pm}_{G-M}$ to M. For any $K \in \mathcal{D}(G)$, we put $\mathrm{res}^G_M K := R(\pi_M)_! i^* K \in \mathcal{D}(M)$.

Example 5.3.12. In the above definition, consider the case that $K = K^\theta$ is a generic character sheaf for $\theta \in \widehat{T^F}$. Suppose M is F-rational and contains T. Then since both π_M and i are F-rational, the isomorphism $\varphi\colon F^*K^\theta \cong K^\theta$ induces an isomorphism $F^*\mathrm{res}^G_M K^\theta \cong \mathrm{res}^G_M K^\theta$, denote which again by φ. For any $s \in M^F$ we have

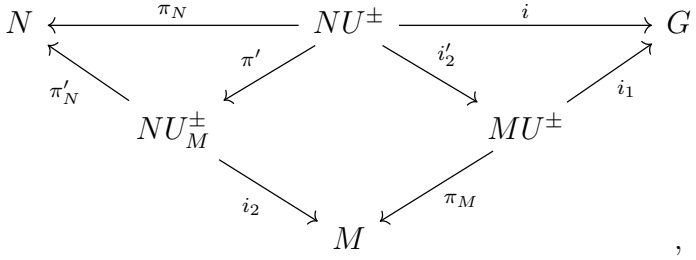
$$\chi_{\mathrm{res}^G_M K^\theta, \varphi}(s) = \sum_{u \in (U^{\pm}_{G-M})^F} \chi_{K^\theta, \varphi}(su) = \sum_{u \in (U^{\pm}_{G-M})^F} \mathrm{Tr}(su, R^\theta_{T,U}),$$

where the first equality follows from standard properties of characteristic functions with respect to proper push-forward and pull-back (see e.g. [KW01, III.12.1]).

Proposition 5.3.13. *Suppose* $\mathbf{N}$ *is a Levi subgroup of* $\mathbf{M}$*, and denote by* N *its Greenberg functor image, then*

$$\mathrm{res}^G_N \cong \mathrm{res}^M_N \circ \mathrm{res}^G_M.$$

Proof. Consider the following commutative diagram



,

Where π' and π'_N are the natural projections, and i_1, i_2, and i'_2 are the natural inclusions. Note that the middle diamond is cartesian, so by proper base change we have

$$\begin{aligned}
\mathrm{res}^G_N &= R(\pi_N)_! i^* \\
&= R(\pi'_N)_! R(\pi')_! (i'_2)^* i_1^* \\
&= R(\pi'_N)_! (i_2)^* R(\pi_M)_! i_1^* \\
&= \mathrm{res}^M_N \circ \mathrm{res}^G_M.
\end{aligned}$$

Thus the transitivity holds. □

5.3.3 Frobenius reciprocity

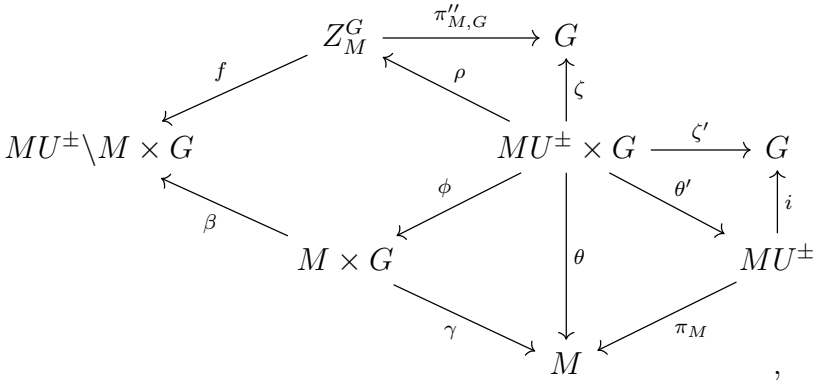
In this subsection we will be concerned with the adjunction between induction and restriction functors. First note that, if K is an equivariant perverse sheaf on M, then by the commutativity between the Verdier duality functor and (proper) pull-backs (see e.g. [BD10, E.4]), its Verdier dual $\mathbb{D}_M K$ is also an equivariant perverse sheaf (with respect to conjugation actions). Throughout this subsection, we ignore Tate twists.

As in the $r = 1$ case, one desires the property $\mathrm{res}^G_M A \in \mathcal{D}^{\leq 0}(M)$; in the $r = 1$ case this property was proved by Lusztig for his character sheaves in [Lus86, 4.4(c)], and we expect it is also true for our $K^\theta[\dim G]$ with K^θ a generic character sheaf.

Proposition 5.3.14. *Let $A \in \mathcal{M}(G)$ (resp. $A_1 \in \mathcal{M}(M)$) be G-equivariant (resp. M-equivariant). If $\mathrm{res}^G_M A \in \mathcal{D}^{\leq 0}(M)$, then*

$$\mathrm{Hom}_{\mathcal{D}(M)}(\mathrm{res}^G_M A, A_1) \cong \mathrm{Hom}_{\mathcal{D}(G)}(A, \mathbb{D}_G \circ \mathrm{ind}^G_M \circ \mathbb{D}_M A_1).$$

Proof. We combine the method in [Lus86, 4.4] with properties of Verdier duality. We will use extensively the notation appearing in the technical preparations of induction functors in Subsection 5.3.2. Consider the commutative diagram



where i and π_M are as in (5.9), $\pi''_{M,G}$ is as in (5.6), and β is the quotient morphism of the $MU^\pm$-action on $M \times G$ given by

$$y \in MU^\pm \colon (g, x) \mapsto (\pi_M(y) g \pi_M(y)^{-1}, xy^{-1});$$

and the other morphisms are as below:

ζ', θ', and γ are left projections;

$\theta \colon (g, x) \mapsto \pi_M(g)$;

$\phi := \pi_M \times \mathrm{id}$;

$\zeta \colon (g, x) \mapsto xgx^{-1}$;

concerning the definitions of f and ρ, first recall that $Z_M^G = \{(g,x) \in G \times G/MU^{\pm} \mid g^x \in MU^{\pm}\}$, we then put

$$f\colon (g, xMU^{\pm}) \mapsto (\pi_M(g^x), x) \mod MU^{\pm};$$

$$\rho\colon (g,x) \mapsto (xgx^{-1}, xMU^{\pm}).$$

Note that in this diagram, by identifying $MU^{\pm} \times G$ with $\widetilde{Z_M^G}$ we see ρ becomes $\pi'_{M,G}$, so (ϕ, γ, π_M, θ') and (f, β, ϕ, ρ) form two cartesian diagrams. Also note that, β is a locally trivial fibration (since the Zariski topologies on $MU^{\pm}\backslash M \times G$ and $M \times G$ are quotient topologies, and ϕ and ρ are locally trivial fibrations by Remark 5.3.6), and f is smooth (by e.g. [GD67, 17.7.7]) with connected fibres isomorphic to $U^{\pm}_{G-M}$.

Firstly, by [Lus86, 1.8.2] we have

$$\mathrm{Hom}_{\mathcal{D}(M)}\left(\mathrm{res}_M^G A, A_1\right) \cong \mathrm{Hom}_{\mathcal{D}(M\times G)}\left(\widetilde{\gamma}\mathrm{res}_M^G A, \widetilde{\gamma}A_1\right). \tag{5.10}$$

Consider the right hand side; by the proper base change theorem

$$\begin{aligned}\widetilde{\gamma}\mathrm{res}_M^G A =&\gamma^* R(\pi_M)_! i^* A[\dim G]\\ =&R\phi_!(\theta')^* i^* A[\dim G]\\ =&R\phi_!(\zeta')^* A[\dim G],\end{aligned}$$

which is actually $R\phi_!(\zeta)^* A[\dim G]$ by the equivariance of A, and then again by proper base change we get

$$\begin{aligned}R\phi_!(\zeta)^* A[\dim G] =&R\phi_!\rho^*(\pi''_{M,G})^* A[\dim G]\\ =&\beta^* Rf_!(\pi''_{M,G})^* A[\dim G]\\ =&\widetilde{\beta}Rf_!(\pi''_{M,G})^* A[\dim U^{\pm}_{G-M}].\end{aligned}$$

On the other hand, since γ is $MU^{\pm}$-equivariant with respect to the conjugation action of $MU^{\pm}$ composed by π_M, we see $\widetilde{\gamma}A_1$ is $MU^{\pm}$-equivariant, thus by [Lus86, 1.9.3] we have $\widetilde{\gamma}A_1 = \widetilde{\beta}A'_1$ for some $A'_1 \in \mathcal{M}(MU^{\pm}\backslash M \times G)$, so (5.10) becomes

$$\mathrm{Hom}_{\mathcal{D}(M)}\left(\mathrm{res}_M^G A, A_1\right) \cong \mathrm{Hom}_{\mathcal{D}(M\times G)}\left(\widetilde{\beta}Rf_!(\pi''_{M,G})^* A[\dim U^{\pm}_{G-M}], \widetilde{\beta}A'_1\right). \tag{5.11}$$

Now, by [Lus86, 1.8.1] and the above computations, the condition $\mathrm{res}_M^G A \in \mathcal{D}^{\leq 0}(M)$ is equivalent to

$$\widetilde{\beta}Rf_!(\pi''_{M,G})^* A[\dim U^{\pm}_{G-M}] \in \mathcal{D}^{\leq 0}(M \times G),$$

thus [Lus86, 1.8.1], [Lus86, 1.8.2], and adjunctions imply

$$\begin{aligned}(5.11) =&\mathrm{Hom}_{\mathcal{D}(MU^{\pm}\backslash M\times G)}\left(Rf_!(\pi''_{M,G})^* A[\dim U^{\pm}_{G-M}], A'_1\right)\\ =&\mathrm{Hom}_{\mathcal{D}(G)}\left(A, R(\pi''_{M,G})_* f^! A'_1[-\dim U^{\pm}_{G-M}]\right).\end{aligned}$$

Note that

$$\begin{aligned} R(\pi''_{M,G})_* f^! A'_1[-\dim U^{\pm}_{G-M}] &= R(\pi''_{M,G})_* f^! (\mathbb{D}_{MU^{\pm}\backslash M\times G} \circ \mathbb{D}_{MU^{\pm}\backslash M\times G}) A'_1[-\dim U^{\pm}_{G-M}] \\ &= \left(\mathbb{D}_G \circ R(\pi''_{M,G})_! f^* \mathbb{D}_{MU^{\pm}\backslash M\times G} A'_1\right)[-\dim U^{\pm}_{G-M}] \\ &= \mathbb{D}_G \left(R(\pi''_{M,G})_! f^* (\mathbb{D}_{MU^{\pm}\backslash M\times G} A'_1)[\dim U^{\pm}_{G-M}]\right) \\ &= \mathbb{D}_G R(\pi''_{M,G})_! \widetilde{f} \mathbb{D}_{MU^{\pm}\backslash M\times G} A'_1 \end{aligned} \tag{5.12}$$

by the Verdier duality (see e.g. [BD10, E.4]).

Meanwhile, since $\widetilde{\gamma} A_1 = \widetilde{\beta} A'_1$, we see

$$\begin{aligned} \widetilde{\gamma}\mathbb{D}_M A_1 &= \gamma^*(\mathbb{D}_M A_1)[\dim G] = \mathbb{D}_{M\times G}(\gamma^! A_1[-\dim G]) = \mathbb{D}_{M\times G}\widetilde{\gamma} A_1 \\ &= \mathbb{D}_{M\times G}\widetilde{\beta} A'_1 = \beta^!(\mathbb{D}_{MU^{\pm}\backslash M\times G} A'_1)[-\dim MU^{\pm}] = \widetilde{\beta}\mathbb{D}_{MU^{\pm}\backslash M\times G} A'_1, \end{aligned}$$

so

$$\widetilde{\rho}(\mathbb{D}_M A_1)_{M,G} \cong \widetilde{\phi}\widetilde{\gamma}\mathbb{D}_M A_1 \cong \widetilde{\phi}\widetilde{\beta}\mathbb{D}_{MU^{\pm}\backslash M\times G} A'_1 \cong \widetilde{\rho}\widetilde{f}\mathbb{D}_{MU^{\pm}\backslash M\times G} A'_1.$$

Thus by the uniqueness of $(\mathbb{D}_M A_1)_{M,G}$ (see the paragraph below (5.6) for the definition of $(-)_{M,G}$ and the uniqueness) we get $\widetilde{f}\mathbb{D}_{MU^{\pm}\backslash L\times G} A'_1 \cong (\mathbb{D}_M A_1)_{M,G}$. Therefore

$$(5.12) = \mathbb{D}_G R(\pi''_{M,G})_! (\mathbb{D}_M A_1)_{M,G} = \mathbb{D}_G \circ \mathrm{ind}_M^G \circ \mathbb{D}_M A_1.$$

This completes the proof. □

5.3.4 Some remarks on Springer theory

In this subsection we assume $\mathcal{O} = \mathbb{F}_q[[\pi]]$ and $r = 2$. In particular $G = G_1 \ltimes G^1$.

Recall that the representations of the symmetric group S_n are classified by the partitions of n; meanwhile, the partitions of n can be interpreted as the unipotent conjugacy classes of $\mathrm{GL}_n(\overline{\mathbb{F}}_q)$, so the representations of S_n are classified by the unipotent conjugacy classes. Springer theory generalises this classification for a general reductive group and its Weyl group; see [Spr78] and [Lus81].

In the following we first state the Springer correspondence (in the form due to Borho and MacPherson [BM81]), a main result of Springer theory, and then explain the involved basic terms by linking them with (5.3), a fundamental diagram for generic character sheaves. In this subsection we mainly refer to [Sho88] for details.

Theorem 5.3.15. *The complex* $\mathrm{IC}(G_1, \mathcal{E})[\#\Phi]|_{(G_1)_{\mathrm{uni}}}$*, where* $(G_1)_{\mathrm{uni}}$ *denotes the variety of unipotent elements and* $\#\Phi$ *is the number of roots, is a semisimple object in* $\mathcal{M}((G_1)_{\mathrm{uni}})$*, and is decomposed as*

$$\mathrm{IC}(G_1, \mathcal{E})[\#\Phi]|_{(G_1)_{\mathrm{uni}}} = \bigoplus_{(C,\mathcal{L})} V_{C,\mathcal{L}} \otimes \mathrm{IC}(\overline{C}, \mathcal{L})[\dim C],$$

where $(C, \mathcal{L})$ runs over the unipotent conjugacy classes C ($\overline{C}$ denotes the closure of C in G_1) and the irreducible local systems $\mathcal{L}$ on C with the property that $\mathcal{L}[\dim C]$ is G-equivariant; here $V_{C,\mathcal{L}}$ is a $\overline{\mathbb{Q}}_\ell$-vector space (multiplicity space). Moreover, the action of $W(T_1)$ on $\mathrm{IC}(G_1, \mathcal{E})[\#\Phi]|_{(G_1)_{\mathrm{uni}}}$ induces an action of $W(T_1)$ on each $V_{C,\mathcal{L}}$, and

$$(C, \mathcal{L}) \mapsto V_{C,\mathcal{L}}$$

is a bijection from the pairs $(C, \mathcal{L})$ such that $V_{C,\mathcal{L}} \neq 0$ to the irreducible representations of $W(T_1)$ over $\overline{\mathbb{Q}}_\ell$; this bijection is called the Springer correspondence.

In the above theorem, $\mathrm{IC}(-,-)$ denotes an intersection cohomology complex on a variety; it is defined with respect to a local system (locally constant $\overline{\mathbb{Q}}_\ell$-sheaf) on a locally closed smooth irreducible subvariety (and every simple perverse sheaf is of this form after shifting); see e.g. [Sho88, 3].

In the above theorem, $\mathcal{E}$ is a local system which can be defined in the following way: First note that

$$Z_T = \{(g, x) \in G \times G/TU^{\pm} \mid g^x \in TU^{\pm}\} \cong G^1 \times \{(g, x) \in G_1 \times G_1/T_1 \mid g^x \in T_1\} \quad (5.13)$$

as varieties, hence by combining (5.3) we get a commutative diagram

$$\begin{array}{ccc} Z_T & \xrightarrow{\quad a \quad} & G \\ \downarrow{\scriptstyle á} & & \downarrow{\scriptstyle \rho_{2,1}} \\ \{(g, x) \in G_1 \times G_1/T_1 \mid g^x \in T_1\} & \xrightarrow{\grave{a}} & G_1, \end{array}$$

where $\rho_{2,1}$ is the reduction map, $á$ is the right projection (from the product variety in (5.13)) and $\grave{a}$ is the natural left projection. In particular, the morphism a in (5.3) appeared as the base change of $\grave{a}$ along a trivial vector bundle.

By definition, the image of $\grave{a}$ is the constructible set $(G_1)_{\mathrm{ss}}$ consisting of semisimple elements (the constructibility follows from Chevalley's theorem [GD67, 1.8.4]); let $(G_1)_{\mathrm{rs}}$ be the subset consisting of regular semisimple elements. Indeed, $(G_1)_{\mathrm{rs}}$ is an open subvariety of G_1 (see e.g. [Hum95, 2.5]). The restriction of $\grave{a}$ on the pre-image of $(G_1)_{\mathrm{rs}}$ is an unramified covering of $(G_1)_{\mathrm{rs}}$ with Galois group $W(T_1)$ (note that $w \in W(T_1)$ acts on Z_T by $(g, xTU^{\pm}) \mapsto (g, xw^{-1}TU^{\pm})$), and the local system $\mathcal{E}$ is defined to be $\mathcal{E} := (\grave{a}|_{\grave{a}^{-1}((G_1)_{rs})})_* \overline{\mathbb{Q}}_\ell$, which admits a natural $W(T_1)$-action; see [Sho88, 4.1].

Since $(G_1)_{\mathrm{rs}}$ is open and dense in G_1, one can consider the intersection cohomology complex $\mathrm{IC}(G_1, \mathcal{E})$, and then the perverse sheaf $\mathrm{IC}(G_1, \mathcal{E})[\dim G_1] \in \mathcal{M}(G_1)$. By viewing $\mathrm{IC}(G_1, -)[\dim G_1]$ as a functor from local systems on $(G_1)_{\mathrm{rs}}$ to perverse sheaves on G_1, we see the group $W(T_1)$ acts on $\mathrm{IC}(G_1, \mathcal{E})[\dim G_1]$. The above theorem is now a restatement of the combination of [Sho88, 4.2] (proved in [Lus81]) and [Sho88, 6.2] (proved in [BM81]). More details can be found in [Sho88].

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