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Discrete analogues of some classical special functions

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DISCRETE ANALOGUES OF SOME CLASSICAL SPECIAL FUNCTIONS

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ABSTRACT

Analogues of special functions on time scales are studied with special focus on the time scale $\mathbb{T} = h\mathbb{Z}$. Functions investigated in particular include complex monomials, new trigonometric functions, Gaussian bell, Hermite and Laguerre polynomials, Bessel functions, and hypergeometric series.

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In loving memory of my grandfather Gary Thomas Wingrove.

4 July 1938 — 9 November 2012

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NOMENCLATURE

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$\approx$	asymptotically equivalent	3
$J_\nu(t,s,\xi,\alpha,\gamma;h)$	discrete Bessel function	67
$J_\nu(t,s;h)$	standard discrete Bessel function	68
$\cos_p(t,s;h)$	discrete cosine function	33
$\Delta_h f(t)$	Δ_h-derivative	18
$\nabla_h f(t)$	∇_h-derivative	18
$e_p(t,s;h)$	discrete exponential	29
$f^{(n)}$	nth derivative of f	3
${}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;t,n,\xi,h)$	discrete hypergeometric series	79
$\gamma(\xi,s)$	incomplete gamma function	13
$\mathscr{L}_h$	discrete Laplace transform	39
$L_n^{(\alpha)}(t,s;h)$	discrete associated Laguerre polynomials	60
$(t-s)_h^n$	discrete monomial centered about s	20
$h_n(t,s;h)$	discrete weighted monomial centered about s	20
$\underline{h_n}(t,s;h)$	falling complexification of h_n	21
$\ominus_{\mu_h}$	regressive minus	28
$\Omega_h(t,s)$	Gaussian bell on $h\mathbb{Z}$	46
$\mathcal{R}_{\mu_h}$ and $\mathcal{R}_{\nu_h}$	set of h-regressive functions	26
$\mathcal{R}^c_{\mu_h}$	regressive constants	28
$a^{\underline{t}}$	falling factorial	21
$a^{\overline{t}}$	rising factorial	16
Υ_h	the Υ_h operator	80
$\sin_p(t,s;h)$	discrete sine	33

1. INTRODUCTION

Continuous calculus, finite difference calculus, and the so-called q-calculus have worked in tandem for hundreds of years, but are typically approached using different notations and intuitions. They perhaps most richly interact within the study of special functions — those functions which we have found it useful to name such as Bessel functions, the gamma function, orthogonal polynomials, and trigonometric functions. Once a function has entered the "list of special functions", its use to solve problems becomes acceptable. In this way, special functions "constitute a common currency" of mathematics [5, page 12], and their use allows us to express solutions of problems in a closed form. The theory of special functions encompasses a huge amount of mathematics, and they have historically proven useful time after time.

Most special functions were encountered while doing other scientific endeavors. For instance, Bessel functions appear in solving Laplace's partial differential equation in cylindrical coordinates. Bessel functions are not generally reducible to simpler functions, so we allow the solution to be expressed in terms of them. Special functions frequently manifest from a differential equation, and their properties are often found by manipulating power series. We have reserved Section 2 to summarize the well-known classical results for which we will find analogues in the sequel. We have included domain colorings of the classical special functions in Figure 2.8.

Time scale calculus is a relatively new branch of mathematics that unifies and extends the similarities between differential calculus, difference calculus, q-calculus, and more. Its investigation was initiated in 1988 by Stefan Hilger in his PhD thesis and has since led to many papers and a few books. One does time scale calculus upon a closed subset of $\mathbb{R}$ (of which there are uncountably many) called a time scale and consequently for each closed subset of $\mathbb{R}$, there is a theory of calculus unique to that time scale. If one proves a theorem in the full generality of time scale calculus, one gains as a consequence a theorem in the theory of differential equations, a theorem in the theory of difference equations, a theorem in the theory of q-calculus, and in fact an analogous theorem for every time scale. This unifying principle can be seen in some applications. For instance, in [32, Example 39], the wealth generated by different interest schemes are compared using the time scale exponential function – with the time scale $\mathbb{T} = \mathbb{R}$ the interest rate is continuous compounding interest, while the time scale $\mathbb{T} = \mathbb{Z}$ may model yearly compounding interest. In this way, time scale calculus is a powerful technique and method of organizing mathematics.

Our primary goal is to find analogues of the classical special functions in the generality offered by time scale calculus. This investigation was formally initiated in [27]. In Section 3 we look at two complexifications of the discrete monomials which we have visualized in Figure 3.1 and Figure 3.2. After this, Section 3 contains a detailed investigation in the particular time scale $\mathbb{T} = h\mathbb{Z}$ with the "obvious" analogues for some more general time scales reserved for Section 4 using the polynomial shift operator as defined by Figure 4.1. Of the functions we have found analogues for are tangent and other unstudied trigonometric functions, Bessel functions, Gaussian bell, Hermite polynomials, Laguerre polynomials, and generalized hypergeometric series. The last item in particular will yield many new special functions that do not appear in this thesis by comparing to the existing literature for hypergeometric series. Our methods unfortunately do not extend to all time scales because of the lack of a critical ingredient: a formula for the inverse Laplace transform for all time scales.

2. CLASSICAL SPECIAL FUNCTIONS

2.1. REAL CALCULUS

The derivative of a function $f\colon \mathbb{R} \to \mathbb{C}$ is defined by

$$f'(t) = \lim_{h \to 0^+} \frac{f(t+h) - f(t)}{h}. \tag{1}$$

For higher-order derivatives we use the notation $f^{(n)}$. If f is infinitely-times differentiable at a point t_0, then in a neighborhood of t_0 the following formula, called a Taylor series, holds [36, Theorem 8.4]

$$f(t) = \sum_{k=0}^{\infty} f^{(k)}(t_0)(t-t_0)^k. \tag{2}$$

The Laplace transform $\mathscr{L}$ is a linear operator on functions $f\colon \mathbb{R} \to \mathbb{C}$ defined by the formula

$$\mathscr{L}\{f\}(z) = \int_0^{\infty} f(\tau) e^{-z\tau} \mathrm{d}\tau. \tag{3}$$

We have included many common Laplace transforms in Table 2.1. The following formula holds [37, Theorem 2.12]:

$$\mathscr{L}\{f^{(n)}\}(z) = z^n \mathscr{L}\{f\}(z) - \sum_{j=0}^{n-1} z^{n-j-1} f^{(j)}(0). \tag{4}$$

We say that two functions f and g are asymptotically equivalent, written $f \approx g$ if $\lim\limits_{x \to \infty} \dfrac{f(x)}{g(x)} = 1$. The convolution integral of two functions f and g is the function

$$(f * g)(t) = \int_0^t f(t-\tau) g(\tau) \mathrm{d}\tau. \tag{5}$$

Table 2.1. Laplace transforms of classical special functions.

Function	Laplace transform	Source
$\frac{t^n}{n!}$	$\frac{1}{z^{n+1}}$	
$e^{\alpha x}$	$\frac{1}{z-\alpha}$	
$\cosh(\alpha t)$	$\frac{z}{z^2-\alpha^2}$	
$\cos(\alpha t)$	$\frac{z}{\alpha^2+z^2}$	
$\sin(\alpha t)$	$\frac{\alpha}{\alpha^2+z^2}$	
$J_0(t)$	$\frac{1}{\sqrt{z^2+1}}$	[39, page 61]
$J_\nu(t)$	$\frac{1}{\sqrt{z^2+1}[\sqrt{z^2+1}+z]^\nu}$	[39, page 61]
$\frac{t^{\frac{\nu}{2}}}{\alpha^{\frac{\nu}{2}}} J_\nu(2\sqrt{\alpha t}); \mathrm{Re}(\nu) > -1$	$\frac{1}{z^{\nu+1} e^{\frac{\alpha}{z}}}$	[39, page 22 (E1.3.1)]

The convolution obeys the so-called convolution theorem [18, Theorem 10.1, page 46]

$$\mathscr{L}\{f * g\}(z) = \mathscr{L}\{f\}(z)\mathscr{L}\{g\}(z). \tag{6}$$

2.2. COMPLEX CALCULUS

The symbol i stands for the imaginary number $i = \sqrt{-1}$. The complex numbers are the elements of the set $\mathbb{C} = \{x + iy \colon x, y \in \mathbb{R}\}$. We represent points in $\mathbb{C}$ on a plane by corresponding $x + iy \in \mathbb{C}$ to the point (x, y) in the plane. This induces the ability to associate to any $x + iy \in \mathbb{C}$ an angle θ (measured from the x-axis) and a length r (measured from $(0, 0)$) called the polar form of $x + iy = re^{i\theta}$. Given $z_0 \in \mathbb{C}$, we define the notation for a disk of radius ε centered at z_0 by

$$D_\varepsilon(z_0) = \{z \in \mathbb{C} \colon |z - z_0| < \varepsilon\}.$$

The product of two series may be computed and is called a Cauchy product:

$$\left(\sum_{k=0}^{\infty} a_k\right)\left(\sum_{k=0}^{\infty} b_k\right) = \sum_{k=0}^{\infty}\sum_{j=0}^{k} a_j b_{k-j}. \tag{7}$$

Let $f \colon \mathbb{C} \to \mathbb{C}$. The complex derivative of f at z_0 is defined [15, page 56] by the formula

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0},$$

provided the limit exists. If $U \subset \mathbb{C}$ and f possesses a complex derivative at all $z \in U$, then we say that f is holomorphic on U. It is known [23, Corollary 3.1.2] that if f is once complex-differentiable at z_0, then it is infinitely-times complex-differentiable at z_0. It is known [23, Theorem 3.3.1] that if f is holomorphic on U, then at each point $z_0 \in U$, there exists an $\varepsilon_{z_0} > 0$ with $D_{\varepsilon_{z_0}}(z_0) \subset U$ such that for all $z \in D_{\varepsilon_{z_0}}(z_0)$,

$$f(z) = \sum_{k=0}^{\infty} \frac{\mathrm{d}f^n}{\mathrm{d}z^n}(z_0)\frac{(z - z_0)^k}{k!}.$$

If $f \colon D_\varepsilon(z_0) \setminus \{z_0\} \to \mathbb{C}$ is holomorphic, then we know [23, Theorem 4.3.2] that there exists a so-called Laurent series expansion for $z \in D_\varepsilon(z_0) \setminus \{z_0\}$

$$f(z) = \sum_{j=-\infty}^{\infty} a_j (z - z_0)^j \tag{8}$$

where for $0 < s < \varepsilon$ and ∂D denoting the boundary of the disk, the coefficients are given by the contour integral

$$a_j = \frac{1}{2\pi i} \oint_{\partial D(z_0,s)} \frac{f(\zeta)}{(\zeta - z_0)^{j+1}} \mathrm{d}\zeta.$$

If $a_j = 0$ for all $j < 0$, then we call z_0 a removable singularity and (8) defines a holomorphic function. If $k > 0$ and $a_j = 0$ for all $-\infty < j < -k$, then we call z_0 a pole of order k of f. If $\inf\{a_j : a_j \neq 0\} = -\infty$, then we call z_0 an essential singularity of f. If f has Laurent series $f(z) = \displaystyle\sum_{k=-\infty}^{\infty} a_k(z - z_0)^k$ in the annulus $D_\varepsilon(z_0) \setminus \{z_0\}$, then we define [15, page 231] the residue of f to be $\operatorname*{Res}_{z=z_0} f(z) = a_{-1}$, i.e., the coefficient of the term $\dfrac{1}{z - z_0}$.

Since both the domain and codomain of a function $f : \mathbb{C} \to \mathbb{C}$ are two-dimensional, it is not immediately clear how to render a graph of such a function. What we will do is associate to each $\theta \in [0, 2\pi)$ a color as in Figure 2.1. For instance, all positive real numbers are "red", all negative real numbers are "light blue", the number i is "green".

To visualize a complex function $f : \mathbb{C} \to \mathbb{C}$, we visualize the mapping in the domain and range through a domain coloring. For example, if $f(z) = z^2$, then we know that $f(i) = -1$. This suggests in the domain coloring of f at the position of i (i.e., at at $(0, 1)$) in the plane matches the color of -1 at $(-1, 0)$ in the plane (light blue) as demonstrated in Figure 2.2. In the sequel we will omit the left image in all pictures because it will always be $\mathbb{C}$. The magnitude of each point is described by the darkness of the image at the point. For example, the function $f(z) = e^{-z^2}$ has small magnitude away from zero on the real line as shown in Figure 2.3. Using domain coloring, we can observe qualitatively interesting properties of complex functions. For example, the periodic nature of e^{iz} manifests itself as repeating bands of color along any vertical line in $\mathbb{C}$ as shown in Figure 2.4. The zeros of a function are represented as black spots, as Figure 2.5 shows. A chosen branch cut of a multi-valued function

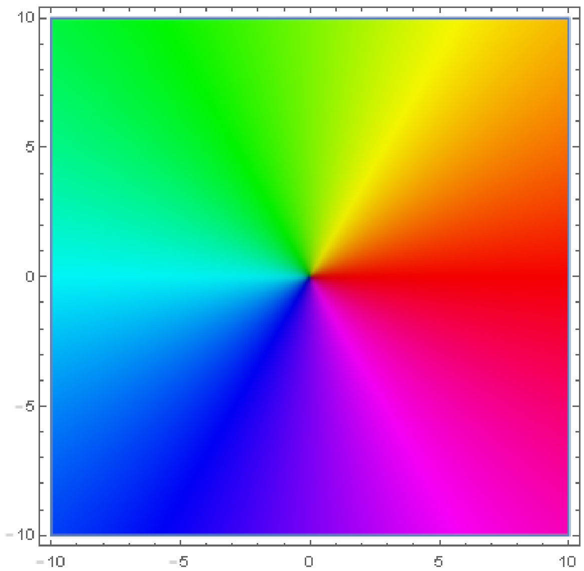


Figure 2.1. Domain coloring of $f(z) = z$.

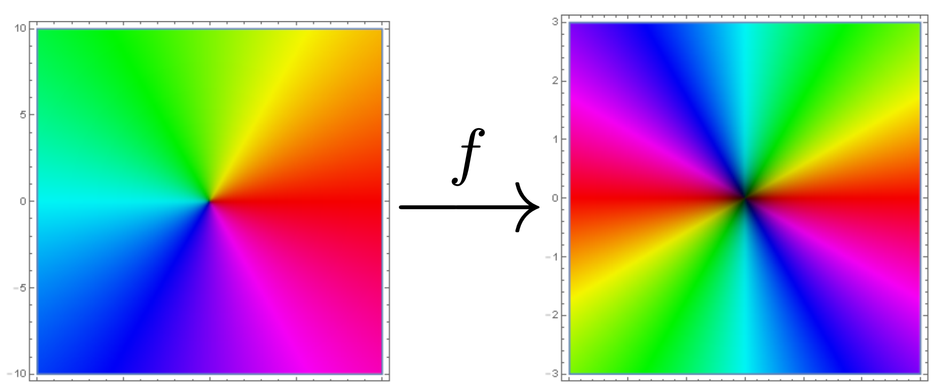


Figure 2.2. Domain coloring of $f(z) = z^2$.

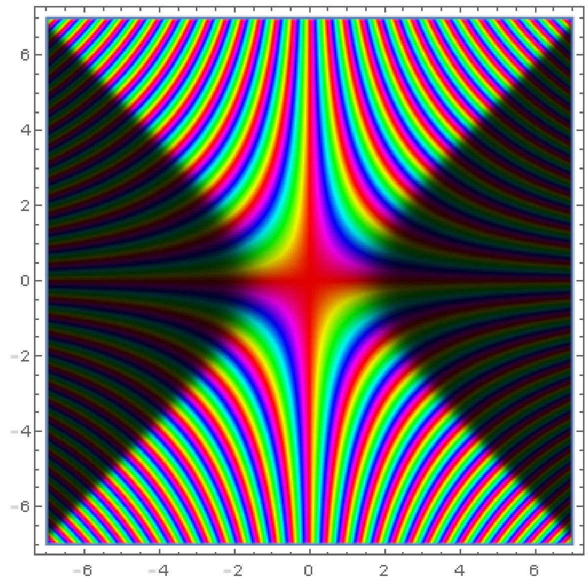


Figure 2.3. Domain coloring of $f(z) = e^{-z^2}$.

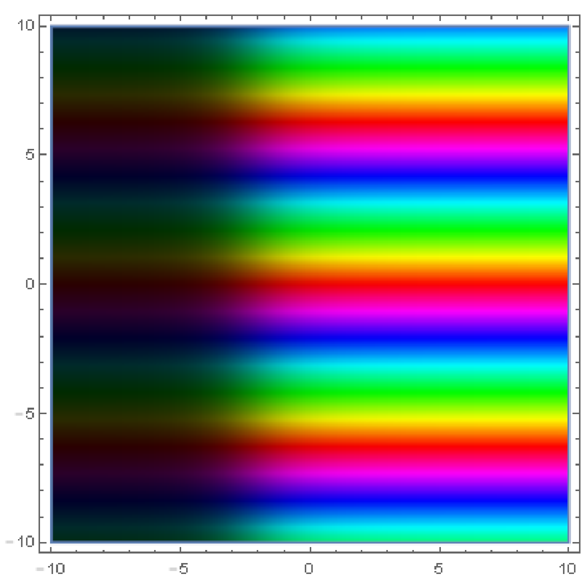


Figure 2.4. Domain coloring of e^z.

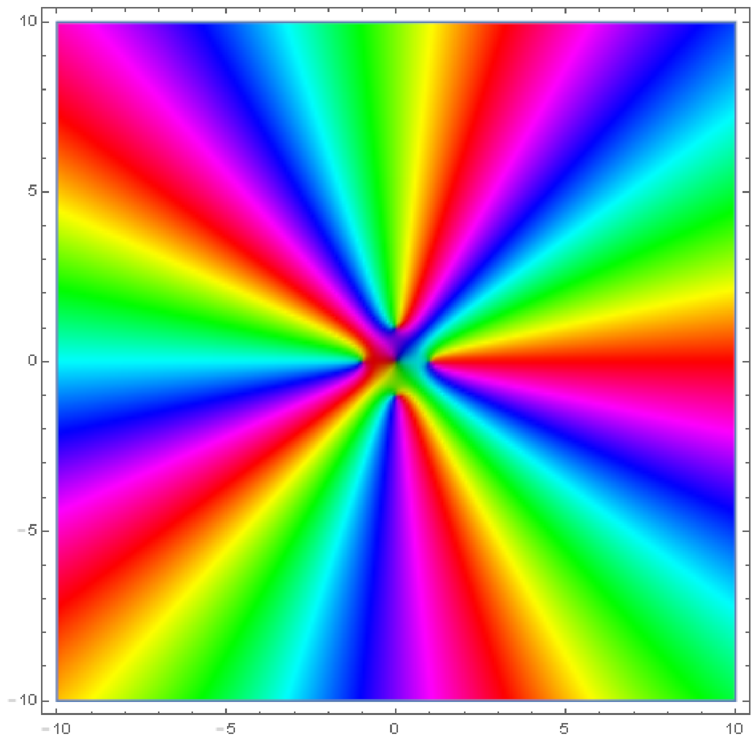


Figure 2.5. Domain coloring of $f(z) = z(z^2 - 1)(z^2 + 1)$.

is clear from the domain coloring Figure 2.6. Suppose f is a holomorphic function with an essential singularity at z. The big Picard theorem [23, Theorem 10.5.6, page 323] states that in any neighborhood z, the function f maps to all values of $\mathbb{C}$ except possibly one. The effect such a singularity has on a domain coloring is an infinite repeating pattern of striped colors converging to a point as shown in Figure 2.7. We have included domain colorings of classical special function in Figure 2.8.

2.3. ELEMENTARY CLASSICAL SPECIAL FUNCTIONS

The function exp is defined to be the solution of the initial value problem

$$y'(t) = y(t), \quad y(0) = 1.$$

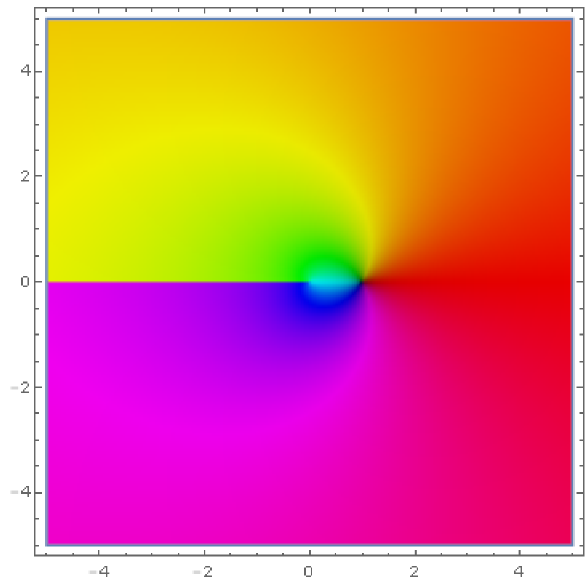


Figure 2.6. Domain coloring of $f(z) = \log(z)$ with branch cut $(-\infty, 0]$.

We will often express $\exp(t)$ as e^t, where $e = \exp(1)$. The following Taylor series is known:

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}. \tag{9}$$

The function sin is defined to be the solution of the initial value problem

$$y''(t) = -y(t), \quad y(0) = 0, \quad y'(0) = 1. \tag{10}$$

The function cos is the solution of the initial value problem

$$y''(t) = -y(t), \quad y(0) = 1, \quad y'(0) = 0. \tag{11}$$

The following formula is called Euler's identity:

$$e^{it} = \cos(t) + i\sin(t),$$

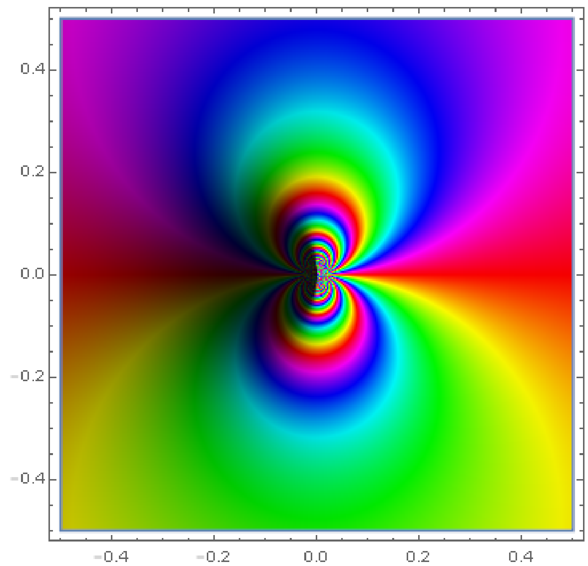


Figure 2.7. Domain coloring of $e^{\frac{1}{z}}$ with essential singularity at $z = 0$.

and consequently we have

$$\cos(t) = \frac{e^{it} + e^{-it}}{2} \tag{12}$$

and

$$\sin(t) = \frac{e^{it} - e^{-it}}{2i}.$$

The following formula is well known:

$$\cos^2(t) + \sin^2(t) = 1. \tag{13}$$

The tangent function is $\tan = \dfrac{\sin}{\cos}$. If we restrict tan to the interval $\left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right)$, then we may define an inverse tangent function $\arctan\colon \mathbb{R} \to \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$. The following complex formulation of arctan is known [1, 4.4.28, page 80]:

$$\arctan t = \frac{i}{2} \log\left(\frac{1 - it}{1 + it}\right). \tag{14}$$

A sequence of orthogonal polynomials is a sequence of polynomials $\{P_n\}_{n=0}^{\infty}$ such that there exists an inner product $\langle \cdot, \cdot \rangle$ such that $\langle P_n, P_m \rangle = 0$ for $m \neq n$. The following theorem is called the three-term-recurrence of a sequence of orthogonal polynomials [28, Theorem 2.2.1].

Theorem 2.3.1. *Let $\langle \cdot, \cdot \rangle$ be an inner product to which the sequence $\{p_n\}_{n=0}^{\infty}$ of polynomials is orthogonal respect to. Then there exist constants $\alpha_n, \beta_n, \gamma_n$ such that the following formula holds for all $n \in \mathbb{N}_0$:*

$$p_{n+1}(t) = (\alpha_n t + \beta_n) p_n(t) + \gamma_n p_{n-1}(t).$$

2.4. GAMMA FUNCTION AND INCOMPLETE GAMMA FUNCTION

The Gamma function Γ is defined for $t > 0$ by the formula

$$\Gamma(t) = \int_0^{\infty} \xi^{t-1} e^{-\xi} \mathrm{d}\xi. \tag{15}$$

Using integration by parts, it can be shown that

$$\Gamma(t+1) = t\Gamma(t). \tag{16}$$

It is known [2, page 7] that $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Combining this with (16) for $k \in \mathbb{N}_0$ yields

$$\Gamma\left(\frac{1}{2} - k\right) = \frac{(-2)^k \sqrt{\pi}}{(2k-1) \cdot (2k-3) \cdot \ldots \cdot 5 \cdot 3 \cdot 1} = \frac{(-1)^k 2^{2k} \sqrt{\pi} k!}{(2k)!}. \tag{17}$$

The formula (16) can be used to extend the domain of Γ to negative values. For instance, $\Gamma\left(-\frac{1}{2}\right) = \frac{\Gamma(-\frac{1}{2}+1)}{-\frac{1}{2}}$. With this formula, we must deduce that Γ has a pole at $t = 0$ and hence also at $t = -1, -2, \ldots$. Sometimes we exploit this fact to see the zeros of the reciprocal gamma function $\frac{1}{\Gamma}$ are at $t = 0, -1, -2, \ldots$. The following formula is known as the Euler reflection formula [2, page 9, Theorem 1.21]:

$$\Gamma(t)\Gamma(1-t) = \frac{\pi}{\sin(t\pi)}. \tag{18}$$

The following formula holds [38, (12)] for any $\alpha \in \mathbb{C}$ and $z \in \mathbb{C} \setminus \{-\alpha, -\alpha-1, \ldots -\beta, -\beta-1, \ldots\}$:

$$\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} \approx z^{\alpha-\beta}. \tag{19}$$

The incomplete gamma function γ is defined by the formula

$$\gamma(\xi, x) = \int_0^x t^{\xi-1} e^{-t} \mathrm{d}t. \tag{20}$$

It is known [21, page 135 (4)] that

$$\gamma(a, x) = e^{-x} \sum_{k=0}^{\infty} \frac{x^{a+k}}{a^{\overline{n+1}}}. \tag{21}$$

We will use the gamma function to define binomial coefficients as studied in detail in [22]:

$$\binom{\alpha}{\beta} = \frac{\Gamma(\alpha+1)}{\Gamma(\beta+1)\Gamma(\alpha-\beta+1)}. \tag{22}$$

The following formula is called the binomial series [1, page 14]:

$$(1+x)^{\alpha} = \sum_{k=0}^{\infty} \binom{\alpha}{k} x^k = \sum_{k=0}^{\infty} \frac{\Gamma(\alpha+1)}{k!\Gamma(\alpha-k+1)} x^k. \tag{23}$$

2.5. BESSEL FUNCTIONS

The Bessel function J_ν is defined by the formula

$$J_\nu(t) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2k+\nu}}{\Gamma(k+\nu+1) 2^{2k+\nu} k!}. \tag{24}$$

This function is a solution to Bessel's differential equation

$$t^2y'' + ty' + (t^2 - \nu^2)y = 0. \tag{25}$$

It is known in general that $y(t) = t^\alpha J_\nu(\xi x^\gamma)$ solves the differential equation

$$t^2y'' + (1 - 2\alpha)ty' + (\xi^2\gamma^2 z^{2\gamma} + (\alpha^2 - \nu^2\gamma^2))y = 0. \tag{26}$$

The following formulas are well known [31, (11), page 39]:

$$\sin(z) = \sqrt{\frac{\pi z}{2}} J_{\frac{1}{2}}(z) \tag{27}$$

and

$$\cos(z) = \sqrt{\frac{\pi z}{2}} J_{-\frac{1}{2}}(z). \tag{28}$$

The following Laplace transform holds:

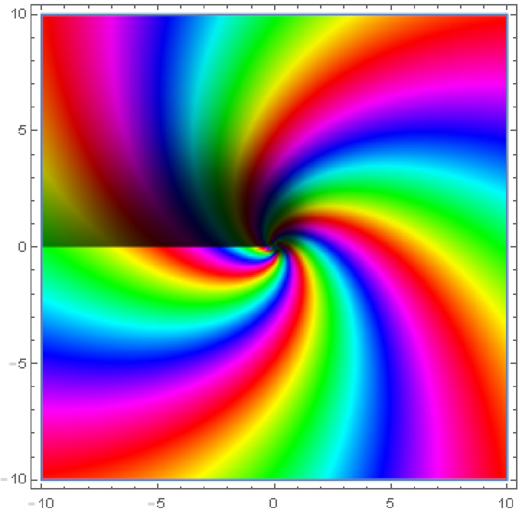
$$\mathscr{L}\{J_\nu(\cdot)\}(z) = \frac{1}{\sqrt{z^2+1}[\sqrt{z^2+1}+z]^{-\nu}}$$

and it is deduced in [39, page 61 (E2.4.3)] as the solution of the differential equation

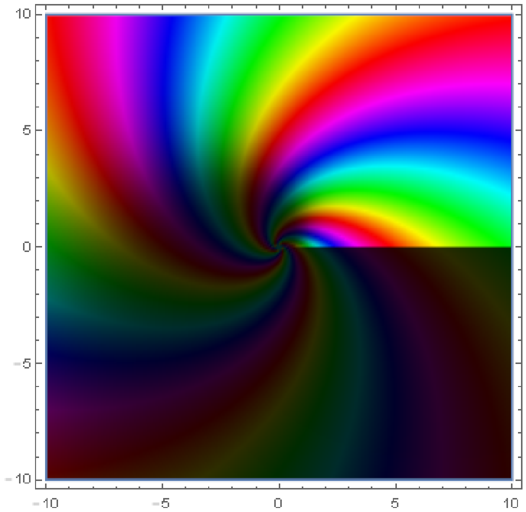
$$(z^2 + 1)y''(z) + 3zy'(z) + (1 - n^2)y(z) = 0 \tag{29}$$

The following Laplace transform holds for $\mathrm{Re}(\nu) > -1$ [39, page 22 (E1.3.1)]:

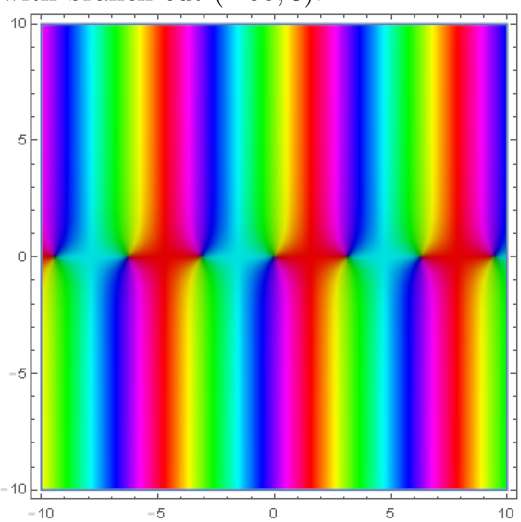
$$\mathscr{L}\left\{t^{\frac{\nu}{2}} J_\nu\left(2\sqrt{\alpha t}\right)\right\}(z) = \frac{\alpha^{\frac{\nu}{2}}}{z^{\nu+1}e^{\frac{\alpha}{z}}}. \tag{30}$$



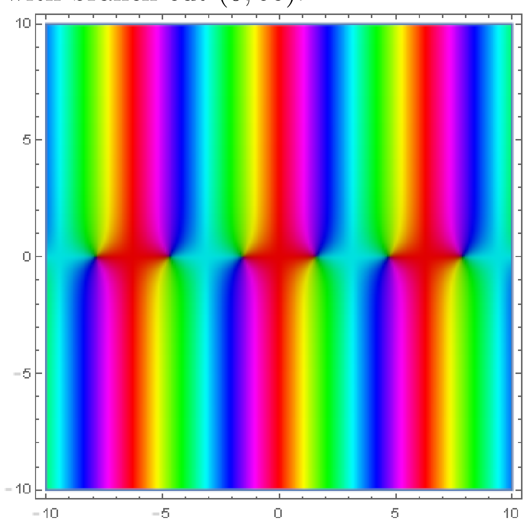
(a) Domain coloring of $\dfrac{z^{4+3i}}{\Gamma(4+3i+1)}$ with branch cut $(-\infty, 0)$.



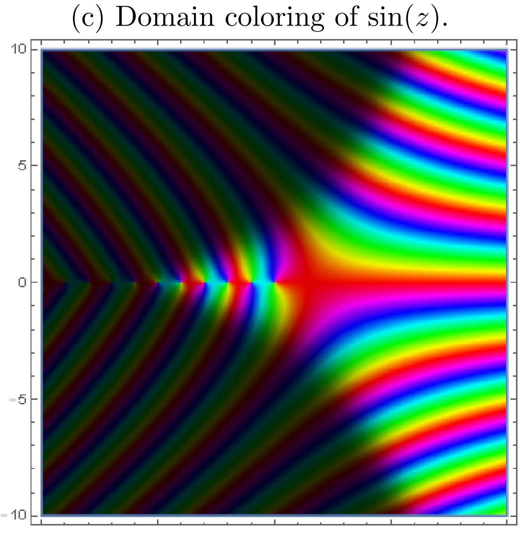
(b) Domain coloring of $\dfrac{z^{4+3i}}{\Gamma(4+3i+1)}$ with branch cut $(0, \infty)$.



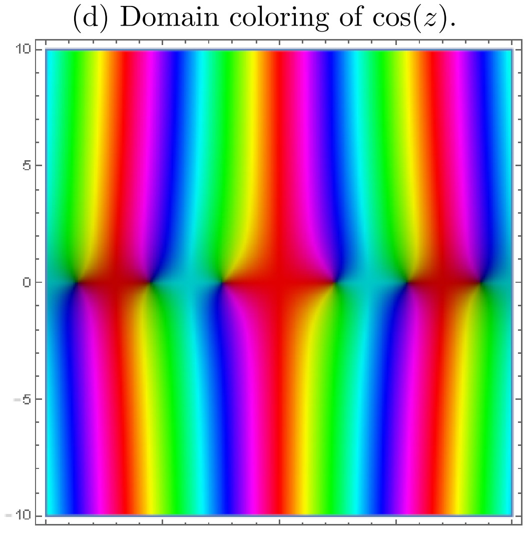
(c) Domain coloring of $\sin(z)$.



(d) Domain coloring of $\cos(z)$.



(e) Domain coloring of $\Gamma(z)$.



(f) Domain coloring of Bessel $J_0(z)$.

Figure 2.8. Domain colorings of classical special functions.

2.6. HYPERGEOMETRIC SERIES

The rising factorial is defined by

$$a^{\overline{t}} = \frac{\Gamma(a+t)}{\Gamma(a)}. \tag{31}$$

We will use the notation $\mathbf{a}$ to refer to the ordered p-tuple $(a_1, \ldots, a_p)$, and we will abuse notation writing $\mathbf{a}^{\overline{k}}$ to refer to the product $\prod_{j=1}^{p} a_j^{\overline{k}}$. We will sometimes write $\mathbf{a}^{\overline{1}}$ as simply $\mathbf{a}$. We will also use $\mathbf{a}+n$ to refer to the tuple $(a_1+n, \ldots, a_p+n)$ and $(\mathbf{a}+n)^{\overline{k}}$ to refer to the product $\prod_{j=1}^{p}(a_j+n)^{\overline{k}}$. The hypergeometric series ${}_pF_q$ is defined by the formula

$${}_pF_q(\mathbf{a};\mathbf{b};t) = \sum_{k=0}^{\infty} \frac{\mathbf{a}^{\overline{k}}}{\mathbf{b}^{\overline{k}}} \frac{t^k}{k!}. \tag{32}$$

We have included many common hypergeometric function representations in Table 2.2. The following formula is known as the negative binomial series [2, page 64]:

$$(1-x)^{-a} = {}_1F_0(a;-;x) = \sum_{k=0}^{\infty} \frac{a^{\overline{k}}}{k!} x^k. \tag{33}$$

Define the operator

$$\theta = z\frac{\mathrm{d}}{\mathrm{d}z}. \tag{34}$$

If $y(t) = {}_pF_q(\vec{a};\vec{b};t)$, then we know [35, page 75 (3)]

$$\left[\theta \prod_{j=1}^{q}(\theta + b_j - 1) - z\prod_{i=1}^{p}(\theta + a_i)\right] y = 0. \tag{35}$$

Table 2.2. Representations of classical special functions as hypergeometric series.

Function	Hypergeometric series representation	Source
$\cosh(az)$	${}_0F_1\left(;\frac{1}{2};-\frac{(az)^2}{4}\right)$	
$\sinh(az)$	$az{}_0F_1\left(;\frac{3}{2};\frac{(az)^2}{4}\right)$	
$\cos(az)$	${}_0F_1\left(;\frac{1}{2};-\frac{(az)^2}{4}\right)$	
$\sin(az)$	$az{}_0F_1\left(;\frac{3}{2};-\frac{(az)^2}{4}\right)$	
e^x	${}_0F_0(;;z)$	[31, page 38]
$L_n^{(\alpha)}(z)$	$\frac{(\alpha+1)^{\overline{n}}}{n!}{}_1F_1(-n;\alpha+1;z)$	[1, page 780, 22.5.24]
$J_\nu(z)$	$\frac{z^\nu}{2^\nu\Gamma(\nu+1)}{}_0F_1\left(;\nu+1;-\frac{z^2}{4}\right)$	[31, page 39]

3. DISCRETE SPECIAL FUNCTIONS

3.1. DISCRETE CALCULUS

There are two discrete derivatives that we will use from the theory of difference calculus. For $f\colon h\mathbb{Z} \to \mathbb{C}$, we define the delta derivative

$$\Delta_h f(t) = \frac{f(t+h) - f(t)}{h}, \tag{36}$$

and we define the nabla derivative

$$\nabla_h f(t) = \frac{f(t) - f(t-h)}{h}. \tag{37}$$

Both the Δ_h and ∇_h derivatives yield the classical derivative (1) in the limit. In other words,

$$\frac{\mathrm{d}}{\mathrm{d}t} f(t) = \lim_{h \to 0^+} \Delta_h f(t) = \lim_{h \to 0^+} \nabla_h f(t).$$

We must be careful with this limit because, for example, $2 \in 2\mathbb{Z}$ but $2 \notin 2.1\mathbb{Z}$, so we understand the limit in the sense of [30, Definition 4.2] which formalizes this notion via "convergence through the $h\mathbb{Z}$s". The product rule for Δ_h is given by

$$\Delta_h[f(t)g(t)] = g(t+h)\Delta_h f(t) + f(t)\Delta_h g(t), \tag{38}$$

and the quotient rule for Δ_h is given by

$$\Delta_h \left(\frac{f(t)}{g(t)} \right) = \frac{g(t)\Delta_h f(t) - f(t)\Delta_h g(t)}{g(t)g(t+h)}. \tag{39}$$

The two discrete analogues of the integral $\displaystyle\int_s^t f(\tau)\mathrm{d}\tau$ are the Δ_h-integral

$$\int_s^t f(\tau)\Delta_h\tau = h\sum_{k=\frac{s}{h}}^{\frac{t}{h}-1} f(kh) \tag{40}$$

and the ∇_h-integral

$$\int_s^t f(\tau)\nabla_h\tau = h\sum_{k=\frac{s}{h}+1}^{\frac{t}{h}} f(kh). \tag{41}$$

These definitions imply the fundamental theorem of Δ_h-calculus,

$$\Delta_h\left[h\sum_{k=\frac{s}{h}}^{\frac{t}{h}-1} f(kh)\right] = f(t), \tag{42}$$

and

$$h\sum_{k=\frac{s}{h}}^{\frac{t}{h}-1} \Delta_h f(kh) = f(t) - f(s),$$

and the fundamental theorem of ∇_h-calculus,

$$\nabla_h\left[h\sum_{k=\frac{s}{h}+1}^{\frac{t}{h}} f(kh)\right] = f(t),$$

and

$$h\sum_{k=\frac{s}{h}+1}^{\frac{t}{h}-1} \nabla_h f(kh) = f(t) - f(s).$$

3.2. DISCRETE POLYNOMIALS AND RELATED FUNCTIONS

We will be using the symbol h in two different ways in this section: h alone refers to a number in $(0,\infty)$, while h_n will refer to discrete polynomials. Define the weighted h_n monomials of $h\mathbb{Z}$ centered about s by to be the functions $h_n\colon h\mathbb{Z}\times h\mathbb{Z}\times$

$\mathbb{R}^+ \to \mathbb{R}$ given by

$$\begin{cases} h_0(t,s;h) & = 1 \\ h_{n+1}(t,s;h) & = h \displaystyle\sum_{k=\frac{s}{h}}^{\frac{t}{h}-1} h_n(kh,s;h). \end{cases}$$

Hence it is clear in lieu of (42) that $\Delta_h h_n(t,s;h) = h_{n-1}(t,s;h)$. It is well known [26, Example 8] that

$$h_n(t,s;h) = \frac{1}{n!} \prod_{k=0}^{n-1} (t-s-kh)\,. \tag{43}$$

For the series methods that follow, it will be useful to define the (unweighted) discrete monomials of $h\mathbb{Z}$ centered about s by the formula

$$(t-s)_h^n = n! h_n(t,s;h). \tag{44}$$

The following "shift lemma" will allow us to emulate series methods from classical special functions theory in discrete calculus.

Lemma 3.2.1. *The following formula holds for $n, m \in \mathbb{N}_0$:*

$$(t-s)_h^n (t-s-hn)_h^m = (t-s)_h^{n+m}.$$

Proof. We compute

$$\begin{aligned} (t-s)_h^n (t-s-hn)_h^m &= \left(\prod_{k=0}^{n-1} (t-s-kh) \right) \left(\prod_{k=0}^{m-1} (t-s-hn-kh) \right) \\ &= \prod_{k=0}^{n+m-1} (t-s-kh) \\ &= (t-s)_h^{n+m}, \end{aligned}$$

as was to be shown. $\square$

Corollary 3.2.1. *If a function f has a series representation $f(t) = \sum_{k=0}^{\infty} a_k (t-s)_h^k$, then*

$$(t-s)_h^{\eta} f(t-\eta h) = \sum_{k=0}^{\infty} a_k (t-s)_h^{k+\eta}.$$

At certain times, we will want to use non-integer values of n in the expression $(t-s)_h^n$. As written, this is impossible because of how the product notation is defined. We will get around this by defining the product using the gamma function (and thus we will have complexifications), however there is not a unique way to do this without imposing more conditions. We will consider two natural complexifications of h_n.

3.2.1. Falling Complexification. We define the falling factorial notation

$$a^{\underline{t}} = \frac{\Gamma(a+1)}{\Gamma(a-t+1)}.$$

We define the falling complexification of $h_n(t,s;h)$ by

$$\underline{h_n}(t,s;h) = \frac{h^n}{\Gamma(n+1)} \left(\frac{t-s}{h}\right)^{\underline{n}} = \frac{h^n}{\Gamma(n+1)} \frac{\Gamma(\frac{t-s}{h}+1)}{\Gamma(\frac{t-s}{h}-n+1)}. \tag{45}$$

Theorem 3.2.1. *If $n \in \mathbb{N}_0$ and $t, s \in h\mathbb{Z}$, then*

$$\underline{h_n}(t,s;h) = h_n(t,s;h).$$

Proof. Recall 43 and calculate

$$\begin{aligned}
\underline{h_n}(t,s;h) &= \frac{h^n}{n!} \frac{\Gamma(\frac{t-s}{h}+1)}{\Gamma(\frac{t-s}{h}-n+1)} \\
&= \frac{1}{n!} \prod_{k=0}^{n-1} (t-s-kh) \\
&= h_n(t,s;h),
\end{aligned}$$

as was to be shown. $\square$

Corollary 3.2.2. *If $n \in \mathbb{N}$ and $t, s \in \mathbb{C}$, then $\underline{h_n}(\cdot, s; h)$ has n zeros at $z_j = s + jh$ for $j \in \{0, 1, \ldots, n-1\}$.*

Corollary 3.2.3. *If $n \in \mathbb{N}_0$, then $\displaystyle\lim_{h \to 0+} \underline{h_n}(t, s; h) = \frac{(t-s)^n}{n!}$.*

Theorem 3.2.2. *The following formulas hold:*

$$\underline{h_0}(t, s; h) = 1 \text{ for all } t, s \in \mathbb{C}$$

and

$$\underline{h_n}(s, s; h) = \frac{h^n \sin(\pi n)}{n\pi} \text{ for all } s \in \mathbb{C}.$$

Proof. Calculate

$$\begin{aligned} \underline{h_0}(t, s; h) &= \frac{h^0}{\Gamma(1+0)} \left(\frac{t-s}{h}\right)^{\underline{0}} \\ &= \frac{\Gamma(\frac{t-s}{h} + 1)}{\Gamma(\frac{t-s}{h} - 0 + 1)} \\ &= 1, \end{aligned}$$

as was to be shown. Now we use (18) to compute

$$\begin{aligned} \underline{h_n}(s, s; h) &= \frac{h^n}{\Gamma(n+1)} (0)^{\underline{n}} \\ &= \frac{h^n}{\Gamma(n+1)} \frac{\Gamma(1)}{\Gamma(0 - n + 1)} \\ &= \frac{h^n}{n\Gamma(n)\Gamma(1-n)} \\ &= \frac{h^n \sin(n\pi)}{n\pi}, \end{aligned}$$

as was to be shown. □

Theorem 3.2.3. *If $n \in \mathbb{Z}$ is negative, then $\underline{h_n}(t, s; h) = 0$ for all $t, s \in \mathbb{C}$.*

Proof. The factor $\dfrac{1}{\Gamma(n+1)}$ equals zero when n is a negative integer. □

Theorem 3.2.4. *If $n \in \mathbb{C} \setminus \mathbb{N}_0$, then $\underline{h_n}(\cdot, s; h)$ has infinitely many zeros at $t = s + (n - m - 1)h$ for $m \in \mathbb{N}_0$. Also $\underline{h_n}(\cdot, s; h)$ has infinitely many poles at $t = s - (m+1)h$ for $m \in \mathbb{N}_0$.*

Proof. From formula (45), we see that $\underline{h_n}$ has zeros whenever $\dfrac{t-s}{h} - n + 1 = -m$, for $m \in \mathbb{N}_0$. Rearrangement of this yields $t = s - (m-n+1)h$. Also from formula (45), we see that $\underline{h_n}$ has poles whenever $\dfrac{t-s}{h} + 1 = -m$, where $m \in \mathbb{N}_0$. Rearrangement of this formula yields $t = s - (m+1)h$. $\square$

From Theorem 3.2.4, we see that as $h \to 0^+$, the poles of $\underline{h_n}$ become more and more dense and cluster along the interval $(-\infty, s)$. Note the similarity between Figure 2.8(a) and Figure 3.1(f).

Theorem 3.2.5. *The following formula holds for* $n \in \mathbb{C} \setminus \mathbb{Z}$*:*

$$\lim_{h \to 0^+} \underline{h_n}(t,s;h) = \frac{(t-s)^n}{\Gamma(n+1)},$$

where the branch cut of $(t-s)^n$ *is taken to be* $(-\infty, s)$.

Proof. Using (19), $\alpha = 1$, $\beta = 1-n$, and $u = \dfrac{t-s}{h}$, we calculate

$$\begin{aligned}
\lim_{h \to 0^+} \underline{h_n}(t,s;h) &= \lim_{h \to 0^+} \frac{h^n}{\Gamma(n+1)} \frac{\Gamma(\frac{t-s}{h}+1)}{\Gamma(\frac{t-s}{h}-n+1)} \\
&= \lim_{u \to \infty} \frac{\left(\frac{t-s}{u}\right)^n}{\Gamma(n+1)} \frac{\Gamma(u+1)}{\Gamma(u-n+1)} \\
&= \frac{(t-s)^n}{\Gamma(n+1)},
\end{aligned}$$

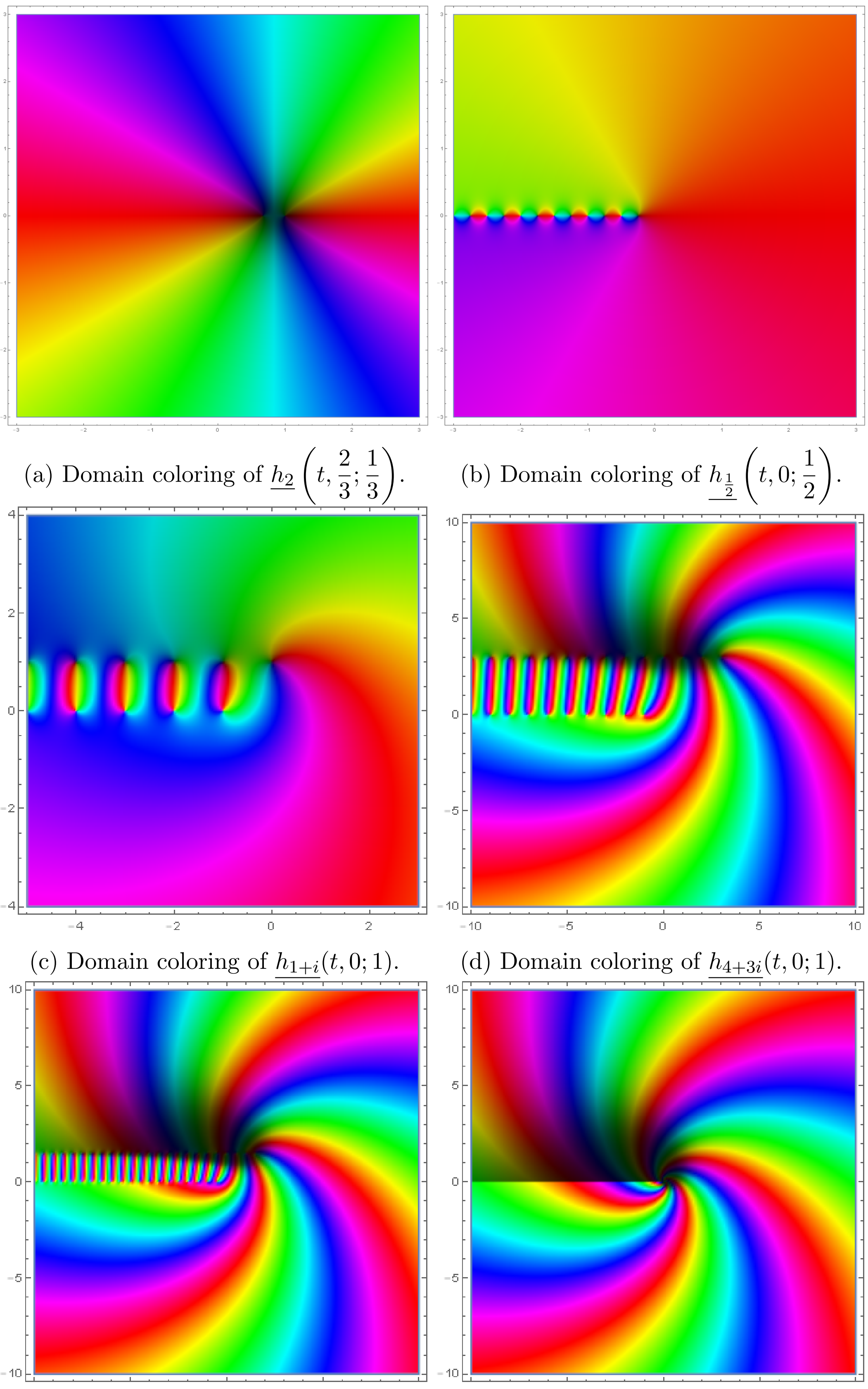
as was to be shown. $\square$

3.2.2. Rising Complexification. Recall the rising factorial (31). The rising complexification of $h_n(t,s;h)$ is defined by

$$\overline{h_n}(t,s,h) = \frac{(-1)^n h^{n-1}}{\Gamma(n+1)} (s-t) \left(\frac{s-t}{h} + 1\right)^{\overline{n-1}} = \frac{(-1)^n h^n}{\Gamma(n+1)} \frac{\Gamma(\frac{s-t}{h}+n)}{\Gamma(\frac{s-t}{h})}. \tag{46}$$

Theorem 3.2.6. *If* $n \in \mathbb{N}_0$ *and* $t, s \in h\mathbb{Z}$*, then*

$$\overline{h_n}(t,s;h) = h_n(t,s;h).$$



(a) Domain coloring of $\underline{h_2}\left(t, \frac{2}{3}; \frac{1}{3}\right)$.

(b) Domain coloring of $\underline{h_{\frac{1}{2}}}\left(t, 0; \frac{1}{2}\right)$.

(c) Domain coloring of $\underline{h_{1+i}}(t, 0; 1)$.

(d) Domain coloring of $\underline{h_{4+3i}}(t, 0; 1)$.

(e) Domain coloring of $\underline{h_{4+3i}}(t, 0; 0.5)$.

(f) Domain coloring of $\underline{h_{4+3i}}(t, 0; 0.01)$.

Figure 3.1. Falling complexification of discrete monomials.

Proof. Observe that
$$\frac{\Gamma(\frac{s-t}{h}+n)}{\Gamma(\frac{s-t}{h})}=\prod_{k=0}^{n-1}\frac{s-t+kh}{h}.$$

Recall (43) and calculate

$$\begin{aligned}\overline{h_n}(t,s;h) &= \frac{(-1)^n h^n}{\Gamma(n+1)}\prod_{k=0}^{n-1}\frac{s-t+kh}{h}\\ &= \frac{1}{n!}\prod_{k=0}^{n-1}(t-s-kh)\\ &= h_n(t,s;h),\end{aligned}$$

as was to be shown. $\square$

Corollary 3.2.4. *If $n\in\mathbb{N}$ and $t,s\in\mathbb{C}$, then $\overline{h_n}(\cdot,s;h)$ has n zeros at $z_j=s+jh$ for $j\in\{0,1,\ldots,n-1\}$.*

Corollary 3.2.5. *If $n\in\mathbb{N}_0$, then $\displaystyle\lim_{h\to 0^+}\overline{h_n}(t,s;h)=\frac{(t-s)^n}{n!}$.*

Theorem 3.2.7. *The following formulas hold:*

$$\overline{h_0}(t,s;h)=1 \text{ for all } t,s\in\mathbb{C}$$

and

$$\overline{h_n}(s,s;h)=0 \text{ for all } s\in\mathbb{C}.$$

Proof. Calculate
$$\overline{h_0}(t,s;h)=\frac{(-1)^0h^0}{\Gamma(1)}\frac{\Gamma(\frac{t-s}{h})}{\Gamma(\frac{t-s}{h})}=1,$$

and since $\dfrac{1}{\Gamma(0)}=0$, we have

$$\overline{h_n}(s,s;h)=\frac{(-1)^nh^n}{\Gamma(n+1)}\frac{\Gamma(n)}{\Gamma(0)}=0,$$

as was to be shown. $\square$

Theorem 3.2.8. *If $n \in \mathbb{Z}$ is negative, then $\overline{h_n}(t, s; h) = 0$ for all $t, s \in \mathbb{C}$.*

Proof. The factor of $\dfrac{1}{\Gamma(n+1)}$ in formula (46) proves this. $\square$

Theorem 3.2.9. *If $n \in \mathbb{C}\backslash\mathbb{N}_0$, then $\overline{h_n}(\cdot, s; h)$ has infinitely many zeros at $t = s+mh$ for $m \in \mathbb{N}_0$. Also $\overline{h_n}(\cdot, s; h)$ has infinitely many poles at $t = s+(m+n)h$ for $m \in \mathbb{N}_0$.*

Proof. The factor of $\dfrac{\Gamma(\frac{s-t}{h}+n)}{\Gamma(\frac{s-t}{h})}$ in (46) shows us that $\overline{h_n}$ has zeros whenever $\dfrac{s-t}{h} = -m$ for $m \in \mathbb{N}_0$. Rearrangement yields $t = s + mh$. We also see that $\overline{h_n}$ has poles whenever $\dfrac{s-t}{h} + n = -m$ for $m \in \mathbb{N}_0$. Rearrangement yields $t = s + (m+n)h$, as was to be shown. $\square$

Note the similarity between Figure 2.8(b) and Figure 3.2(f).

Theorem 3.2.10. *The following formula holds for $n \in \mathbb{C} \setminus \mathbb{Z}$:*

$$\lim_{h \to 0^+} \overline{h_n}(t, s; h) = \frac{(t-s)^n}{\Gamma(n+1)},$$

where the branch cut of $(t-s)^n$ is taken to be (s, ∞).

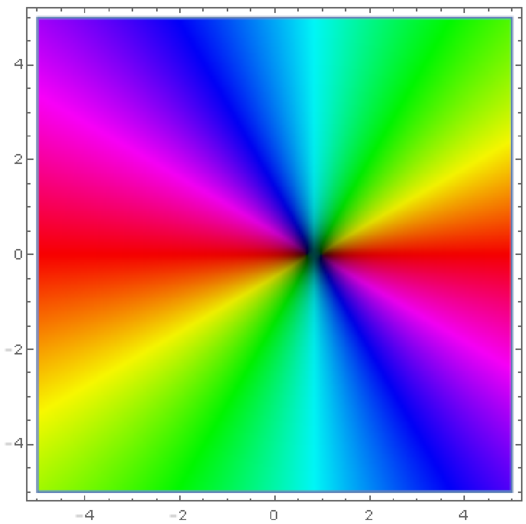
Proof. Using (19), $\alpha = n$, $\beta = 0$, and $u = \dfrac{s-t}{h}$, we calculate

$$\begin{aligned}\lim_{h\to 0^+} \frac{(-1)^n h^n}{\Gamma(n+1)} \frac{\Gamma(\frac{s-t}{h}+n)}{\Gamma(\frac{s-t}{h})} &= \lim_{u\to\infty} \frac{(-1)^n \left(\frac{s-t}{u}\right)^n}{\Gamma(n+1)} u^n \\ &= \frac{(s-t)^n}{\Gamma(n+1)},\end{aligned}$$

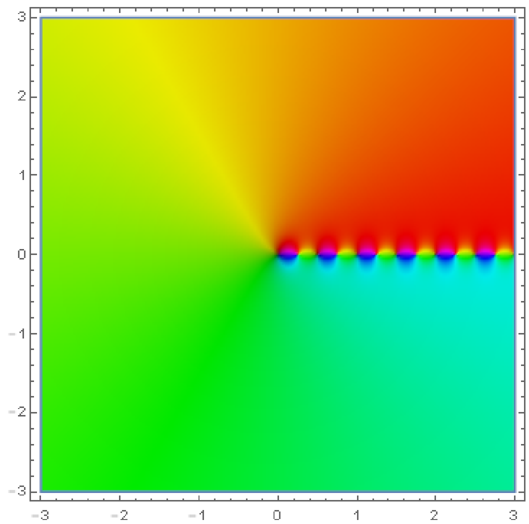
as was to be shown. $\square$

3.3. DISCRETE EXPONENTIAL

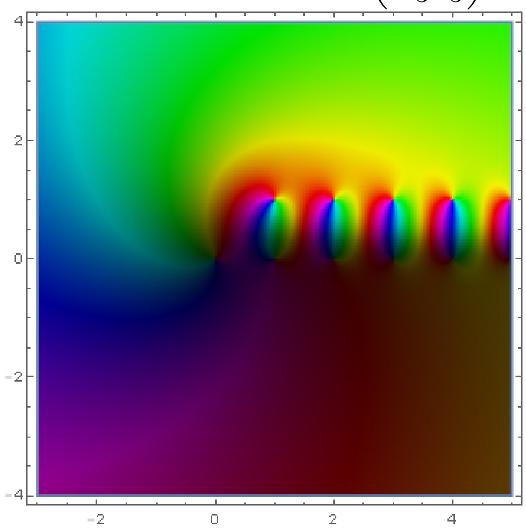
A function $f\colon h\mathbb{Z} \to \mathbb{C}$ is called μ_h-regressive if $1 + hf(t) \neq 0$ for all $t \in h\mathbb{Z}$. A function $f\colon h\mathbb{Z} \to \mathbb{C}$ is called ν_h-regressive if $1 - hf(t) \neq 0$ for all $t \in h\mathbb{Z}$. We denote the set of μ_h-regressive functions by the symbol $\mathcal{R}_{\mu_h}$. If $1 + hf(t) > 0$ for all



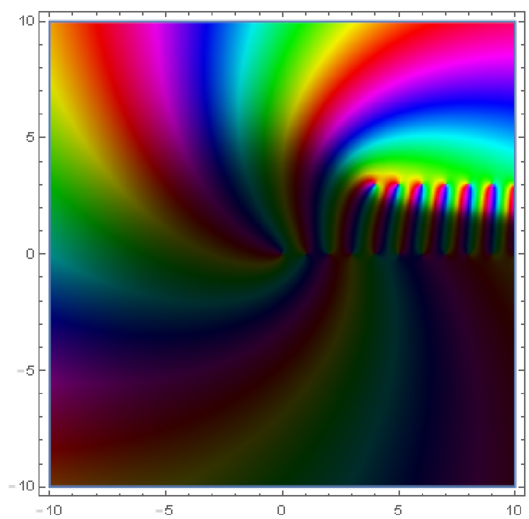
(a) Domain coloring of $\overline{h_2}\left(t, \frac{2}{3}; \frac{1}{3}\right)$.



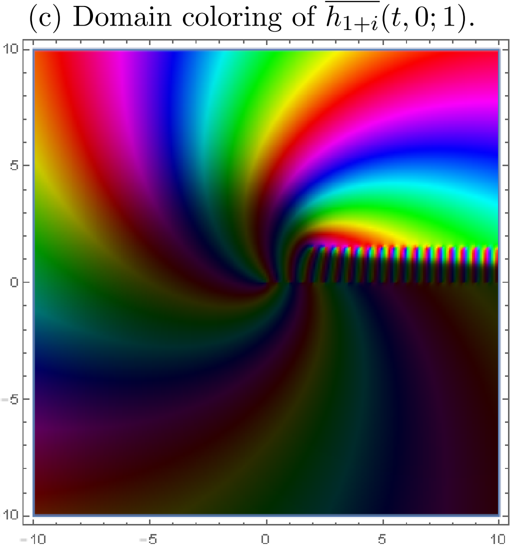
(b) Domain coloring of $\overline{h_{\frac{1}{2}}}\left(t, 0; \frac{1}{2}\right)$.



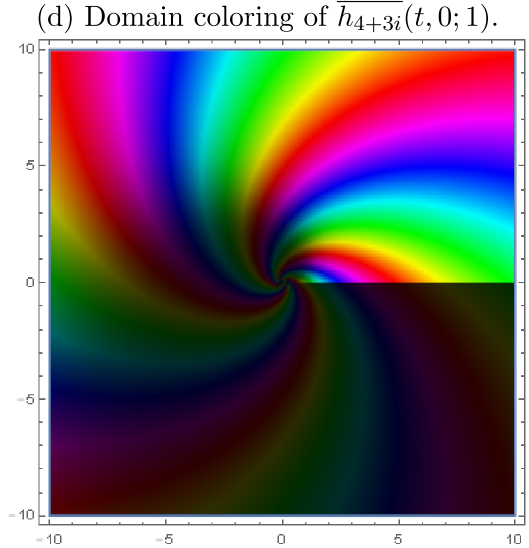
(c) Domain coloring of $\overline{h_{1+i}}(t, 0; 1)$.



(d) Domain coloring of $\overline{h_{4+3i}}(t, 0; 1)$.



(e) Domain coloring of $\overline{h_{4+3i}}(t, 0; 0.5)$.



(f) Domain coloring of $\overline{h_{4+3i}}(t, 0; 0.01)$.

Figure 3.2. Rising complexification of discrete monomials.

$t \in h\mathbb{Z}$, then we say that f is positively μ_h-regressive ($\mathcal{R}^+_{\mu_h}$), and if $1 + hf(t) < 0$ for all $t \in h\mathbb{Z}$, then we say that f is negatively μ_h-regressive ($\mathcal{R}^-_{\mu_h}$). We use the notation $\mathcal{R}^c_{\mu_h}$ to denote the constant functions $f\colon h\mathbb{Z} \to \mathbb{C}$ that are μ_h-regressive. We use similar notation for the ν_h-regressive functions $\mathcal{R}_{\nu_h}$.

There is a natural vector space structure on the set of regressive functions. Define the group addition operation $\oplus_h\colon \mathcal{R}_{\mu_h} \times \mathcal{R}_{\mu_h} \to \mathcal{R}_{\mu_h}$ by

$$(p \oplus_{\mu_h} q)(t) = p(t) + q(t) + hp(t)q(t)$$

and its additive inverse $\ominus_{\mu_h}\colon \mathcal{R}_{\mu_h} \to \mathcal{R}_{\mu_h}$

$$(\ominus_{\mu_h} p)(t) = -\frac{p(t)}{1 + hp(t)}. \tag{47}$$

Define the operation $\odot_{\mu_h}\colon \mathbb{R} \times \mathcal{R}_{\mu_h} \to \mathcal{R}_{\mu_h}$ by

$$(\alpha \odot_{\mu_h} p)(t) = \alpha p(t)\frac{1}{h}\left[(1 + hp(t))^\alpha - 1\right].$$

It is well known that the structure $(\mathcal{R}_{\mu_h}, \oplus_{\mu_h})$ is a group [14, Exercise 1.35] and the structure $(\mathcal{R}, \oplus_{\mu_h}, \odot_{\mu_h})$ is a real vector space [14, Theorem 2.46]. It is known that if $p \in \mathcal{R}^+_{\nu_h}$, then $\hat{e}_p(t, s) > 0$ for all $t, s \in h\mathbb{Z}$ [14, Theorem 3.18 (i)].

Let $p \in \mathcal{R}_{\mu_h}$ and consider the initial value problem

$$\Delta_h y(t) = p(t)y(t), \quad y(s) = 1. \tag{48}$$

Expanding the derivative in this equation, rearranging, and applying the initial condition yields the following solution which we call the discrete Δ_h-exponential function:

$$e_p(t,s;h) = \begin{cases} \displaystyle\prod_{k=\frac{t}{h}}^{\frac{s}{h}-1} \frac{1}{1+hp(hk)}, & t<s \\ 1, & t=s \\ \displaystyle\prod_{k=\frac{s}{h}}^{\frac{t}{h}-1} (1+hp(hk)), & t>s. \end{cases} \tag{49}$$

We may perform the same construction but instead using the ∇_h derivative: let $p \in \mathcal{R}_h$ and consider the initial value problem

$$\nabla_h y(t) = p(t)y(t), \quad y(s) = 1.$$

Rearrangement yields the discrete ∇_h-exponential

$$\hat{e}_p(t,s;h) = \begin{cases} \displaystyle\prod_{k=\frac{t}{h}+1}^{\frac{s}{h}} (1-hp(hk)), & t<s \\ 1, & t=s \\ \displaystyle\prod_{k=\frac{s}{h}+1}^{\frac{t}{h}} \frac{1}{1-hp(hk)}, & t>s. \end{cases} \tag{50}$$

Both of the functions e_p and $\hat{e}_p$ are analogues of the classical exponential function $\exp\left(\int p(\tau)\mathrm{d}\tau\right)$. If $\alpha \in \mathcal{R}^c_{\mu_h}$, then $e_\alpha(t,s;h) = (1+\alpha h)^{\frac{t-s}{h}}$, and if $\beta \in \mathcal{R}^c_{\nu_h}$, then $\hat{e}_\beta(t,s;h) = (1-\alpha h)^{\frac{t-s}{h}}$. If α is a regressive constant, then we know [33, Proposition 6.9] that

$$e_\alpha(t,s;h) = \sum_{k=0}^{\infty} \alpha^k h_k(t,s) = \sum_{k=0}^{\infty} \frac{\alpha^k (t-s)_h^k}{k!}. \tag{51}$$

In lieu of (49), we see that

$$e_\alpha(t,s;h) = (1+\alpha h)^{\frac{t-s}{h}}. \tag{52}$$

We now use (51) to consider the falling complexification of e_α given by

$$\underline{e_\alpha}(t,s;h) = \sum_{k=0}^{\infty} \alpha^k \underline{h_k}(t,s;h).$$

Theorem 3.3.1. *The following formula holds:*

$$\underline{e_\alpha}(t,s;h) = {}_1F_1\left(\frac{s-t}{h}, 1; -\alpha h\right)$$

Proof. Note that for $k \in \mathbb{N}_0$,

$$\left(\frac{s-t}{h}\right)^{\overline{k}} = \prod_{j=0}^{k-1} \frac{s-t+jh}{h} = k!\frac{(-1)^k}{h^k} h_k(t,s;h)$$

and $1^{\overline{k}} = \dfrac{\Gamma(1+k)}{\Gamma(1)} = k!$. Calculate

$$\begin{aligned} {}_1F_1\left(\frac{s-t}{h}, 1; -\alpha h\right) &= \sum_{k=0}^{\infty} \frac{(\frac{s-t}{h})^{\overline{k}}}{1^{\overline{k}}}(-\alpha h)^k \\ &= \sum_{k=0}^{\infty} \alpha^k h_k(t,s;h) \\ &= \underline{e_\alpha}(t,s;h), \end{aligned}$$

as was to be shown. □

3.4. DISCRETE HYPERBOLIC TRIGONOMETRIC FUNCTIONS

Suppose that p is a function such that both p and $-p$ are in $\mathcal{R}_{\mu_h}$. The discrete hyperbolic sine and cosine are defined in [12, Definition 3.17, page 89]

$$\cosh_p(t,s;h) = \frac{e_p(t,s;h) + e_{-p}(t,s;h)}{2} \tag{53}$$

and

$$\sinh_p(t,s;h) = \frac{e_p(t,s;h) - e_{-p}(t,s;h)}{2}. \tag{54}$$

We have included plots of some of these functions in Figure 3.3. We see from (48) that $\cosh_p(s,s;h) = 1$ and $\sinh_p(s,s;h) = 0$. The following Δ_h derivatives follow as well:

$$\Delta_h \cosh_p(t,s;h) = p(t)\sinh_p(t,s;h) \tag{55}$$

and

$$\Delta_h \sinh_p(t,s;h) = p(t)\cosh_p(t,s;h). \tag{56}$$

It is known [12, Theorem 3.21, page 90] that if $\alpha > 0$ is a constant such that both α and $-\alpha$ are in $\mathcal{R}_{\mu_h}$, then the general solution of $\Delta_h^2 y(t) - \alpha^2 y(t) = 0$ is

$$y(t) = c_1 \cosh_\alpha(t,s;h) + c_2 \sinh_\alpha(t,s;h).$$

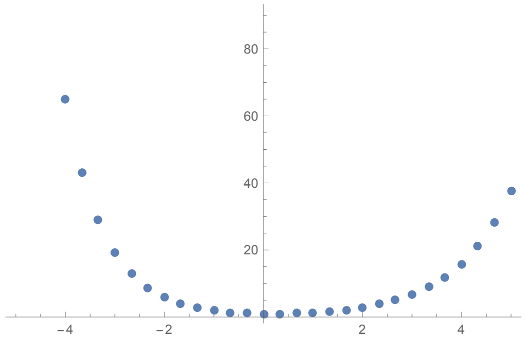
The case of $\alpha = 1$ of the following theorem is pointed out in [25, page 6]; it follows directly from (51).

Theorem 3.4.1. *The following formulas hold:*

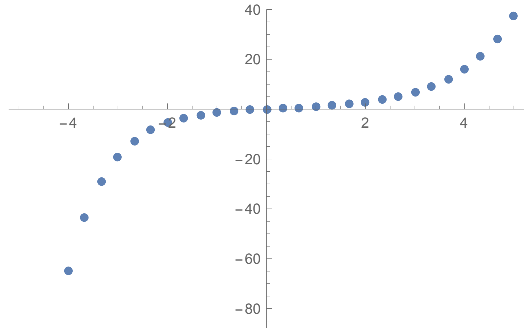
$$\cosh_\alpha(t,s;h) = \sum_{k=0}^{\infty} \frac{\alpha^{2k}}{(2k)!}(t-s)_h^{2k}$$

and

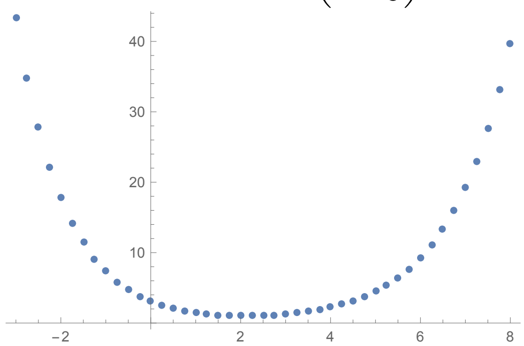
$$\sinh_\alpha(t,s;h) = \sum_{k=0}^{\infty} \frac{\alpha^{2k+1}}{(2k+1)!}(t-s)_h^{2k+1}.$$



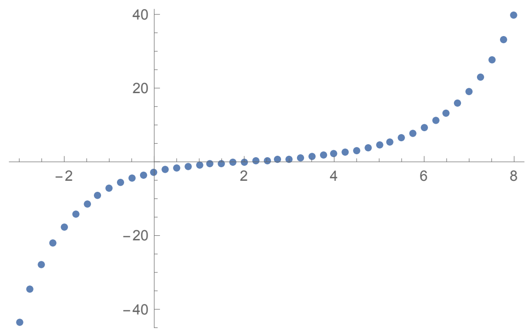
(a) Plot of $\cosh_1\left(t, 0; \dfrac{1}{3}\right)$.



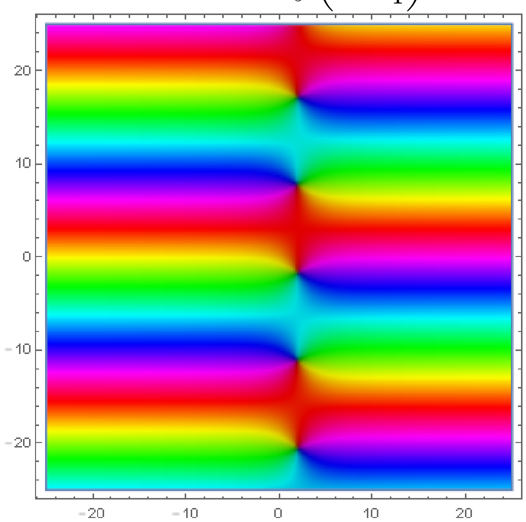
(b) Plot of $\sinh_1\left(t, 0; \dfrac{1}{3}\right)$.



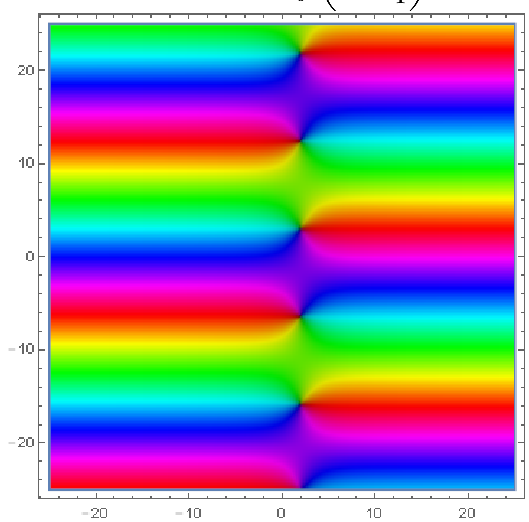
(c) Plot of $\cosh_{\frac{4}{5}}\left(t, 2; \dfrac{1}{4}\right)$.



(d) Plot of $\sinh_{\frac{4}{5}}\left(t, 2; \dfrac{1}{4}\right)$.



(e) Domain coloring of the function $\underline{\cosh_{\frac{1}{3}}}\left(z, 2+3i; \dfrac{1}{10}\right)$.



(f) Domain coloring of the function $\underline{\sinh_{\frac{1}{3}}}\left(z, 2+3i; \dfrac{1}{10}\right)$.

Figure 3.3. Discrete hyperbolic trigonometric functions.

3.5. DISCRETE TRIGONOMETRIC FUNCTIONS

Let $i = \sqrt{-1}$ be the imaginary number and p a function such that both ip and $-ip$ are in $\mathcal{R}_{\mu_h}$. The trigonometric functions are defined in [12, Definition 3.5, page 92] to be

$$\cos_p(t,s;h) = \frac{e_{ip}(t,s;h) + e_{-ip}(t,s;h)}{2} \tag{57}$$

and

$$\sin_p(t,s;h) = \frac{e_{ip}(t,s;h) - e_{-ip}(t,s;h)}{2i}. \tag{58}$$

We have included some plots of these functions in Figure 3.4. From [12, Lemma 3.26], we know

$$\cos_p^2(t,s;h) + \sin_p^2(t,s;h) = e_{hp^2}(t,s;h). \tag{59}$$

We may further define $\tan_p(t,s;h) = \dfrac{\sin_p(t,s;h)}{\cos_p(t,s;h)}$, and similarly we may define the analogues $\sec_\alpha = \dfrac{1}{\cos_\alpha}$, $\csc_\alpha = \dfrac{1}{\sin_\alpha}$ and $\cot_\alpha = \dfrac{1}{\tan_\alpha}$. We have visualized all of these trigonometric functions in Figure 3.5. As a consequence of (59), we see that

$$1 + \tan_p^2(t,s;h) = e_{hp^2}(t,s;h)\sec_p^2(t,s;h)$$

and

$$\cot_p^2(t,s;h) + 1 = e_{hp^2}(t,s;h)\csc_p^2(t,s;h).$$

We see from (48) that $\cos_p(s,s;h) = 1$ and $\sin_p(s,s;h) = 0$. The following delta derivatives follow as well:

$$\Delta_h \cos_p(t,s;h) = -p(t)\sin_p(t,s;h)$$

and

$$\Delta_h \sin_p(t,s;h) = p(t)\cos_p(t,s;h).$$

Theorem 3.5.1. *The following formulas hold:*

$$\Delta_h \tan_p(t,s;h) = p(t)e_{hp^2}(t,s;h)\sec_p(t,s;h)\sec_p(t+h,s;h),$$

$$\Delta_h \sec_p(t,s;h) = p(t)\tan_p(t,s;h)\sec_p(t+h,s;h),$$

$$\Delta_h \csc_p(t,s;h) = -p(t)\cot_p(t,s;h)\csc_p(t+h,s;h),$$

and

$$\Delta_h \cot_p(t,s;h) = -p(t)e_{hp^2}(t,s;h)\csc_p(t,s;h)\csc_p(t+h,s;h).$$

Proof. Using (39) and (59), we see that

$$\begin{aligned}\Delta_h \tan_p(t,s;h) &= \Delta_h\left[\frac{\sin_p(t,s;h)}{\cos_p(t,s;h)}\right]\\ &= \frac{p\cos_p^2(t,s;h)+p\sin_p^2(t,s;h)}{\cos_p(t,s;h)\cos_p(t+h,s;h)}\\ &= pe_{hp^2}(t,s;h)\sec_p(t,s;h)\sec_p(t+h,s;h),\end{aligned}$$

as was to be shown. The other formulas are proven similarly. □

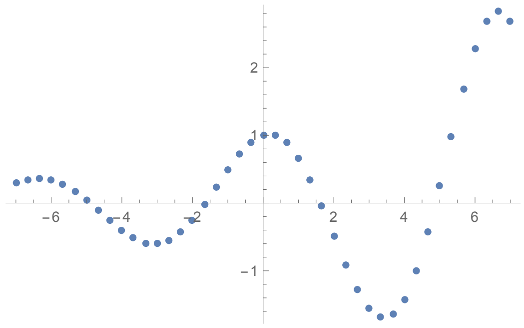
It is known [12, Theorem 3.31, page 93] that if $\alpha > 0$ is a constant such that both α and $-\alpha$ are in $\mathcal{R}_{\mu_h}$, then the general solution of $\Delta_h^2 y(t) + \alpha^2 y(t) = 0$ is

$$y(t) = c_1\cos_\alpha(t,s;h) + c_2\sin_\alpha(t,s;h).$$

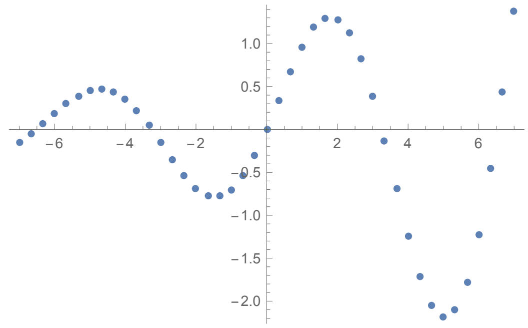
The following theorem follows directly from manipulation of (51).

Theorem 3.5.2. *The following formulas hold:*

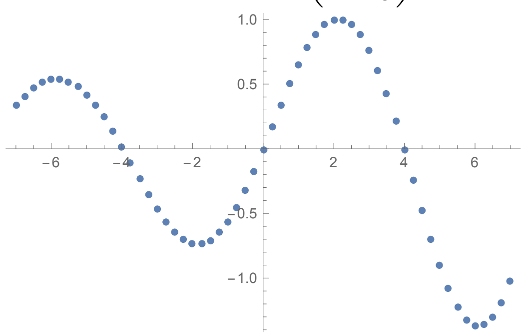
$$\cos_\alpha(t,s;h) = \sum_{k=0}^{\infty}\frac{(-1)^k\alpha^{2k}}{(2k)!}(t-s)_h^{2k}$$



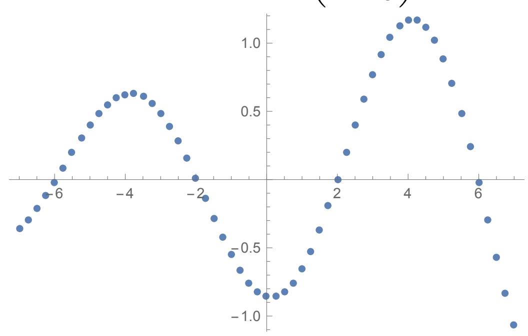
(a) Plot of $\cos_1\left(t, 0; \dfrac{1}{3}\right)$.



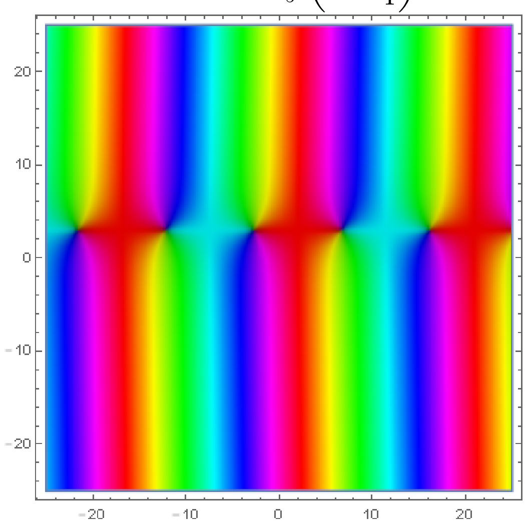
(b) Plot of $\sin_1\left(t, 0; \dfrac{1}{3}\right)$.



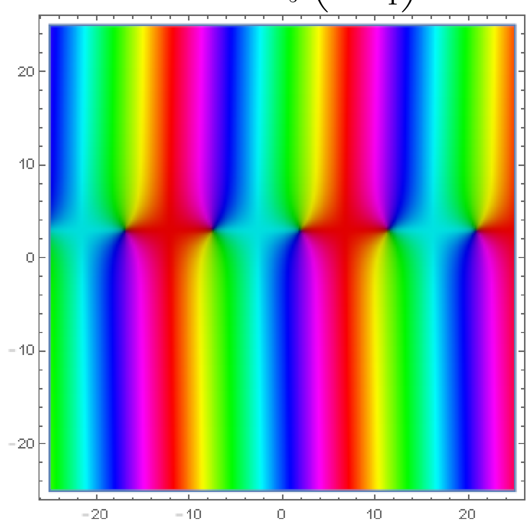
(c) Plot of $\cos_{\frac{4}{5}}\left(t, 2; \dfrac{1}{4}\right)$.



(d) Plot of $\sin_{\frac{4}{5}}\left(t, 2; \dfrac{1}{4}\right)$.



(e) Domain coloring of the function $\underline{\cos_{\frac{1}{3}}}\left(z, 2+3i; \dfrac{1}{10}\right)$.



(f) Domain coloring of the function $\underline{\sin_{\frac{1}{3}}}\left(z, 2+3i; \dfrac{1}{10}\right)$.

Figure 3.4. Discrete sin and cos along with their complexifications.

and

$$\sin_\alpha(t,s;h) = \sum_{k=0}^{\infty} \frac{(-1)^k \alpha^{2k+1}}{(2k+1)!} (t-s)_h^{2k+1}.$$

It is known that our trigonometric functions are unbounded. We now provide a characterization of precisely how it is unbounded by factoring $\cos_\alpha$ and $\sin_\alpha$ into an unbounded part and a bounded part.

Theorem 3.5.3. *The following formulas hold:*

$$\cos_\alpha(t,s;h) = \left(1+\alpha^2h^2\right)^{\frac{t-s}{2h}} \cos\left(\frac{(t-s)\arctan(\alpha h)}{h}\right)$$

and

$$\sin_\alpha(t,s;h) = (1+\alpha^2h^2)^{\frac{t-s}{2h}} \sin\left(\frac{(t-s)\arctan(\alpha h)}{h}\right).$$

Proof. By (57) and (52), we see that

$$\cos_\alpha(t,s;h) = \frac{e_\alpha(t,s;h) + e_{-\alpha}(t,s;h)}{2} = \frac{(1+\alpha h i)^{\frac{t-s}{h}} + (1-\alpha h i)^{\frac{t-s}{h}}}{2}.$$

Using (14), we get

$$\arctan(\alpha h) = \frac{i}{2}\log\left(\frac{1-\alpha h i}{1+\alpha h i}\right).$$

We may calculate

$$\begin{aligned}
\cos\left(\frac{(t-s)\arctan(\alpha h)}{h}\right) &= \frac{\exp\left(i\frac{(t-s)\arctan(\alpha h)}{h}\right) + \exp\left(-i\frac{(t-s)\arctan(\alpha h)}{h}\right)}{2} \\
&= \frac{\left(\frac{1+\alpha h i}{1-\alpha h i}\right)^{\frac{t-s}{2h}} + \left(\frac{1-\alpha h i}{1+\alpha h i}\right)^{\frac{t-s}{2h}}}{2},
\end{aligned}$$

and since $(1+\alpha^2h^2) = (1-\alpha hi)(1+\alpha hi)$, we obtain

$$(1+\alpha^2h^2)^{\frac{t-s}{2h}} \cos\left(\frac{(t-s)\arctan(\alpha h)}{h}\right) = \frac{(1+\alpha hi)^{\frac{t-s}{h}} + (1-\alpha hi)^{\frac{t-s}{h}}}{2},$$

as was to be shown. The formula for $\sin_\alpha$ is proven similarly. □

Theorem 3.5.3 shows us that both the discrete sine and discrete cosine are unbounded because they have an exponential factor. Since they have the same exponential factor, this shows that the discrete tangent is in fact a classical tangent function.

Corollary 3.5.1. *The following formula holds:*

$$\tan_\alpha(t,s;h) = \tan\left(\frac{(t-s)\arctan(\alpha h)}{h}\right).$$

Since $\cot_h$ is the reciprocal of $\tan_h$, it follows that $\cot_h$ is a classical cotangent function.

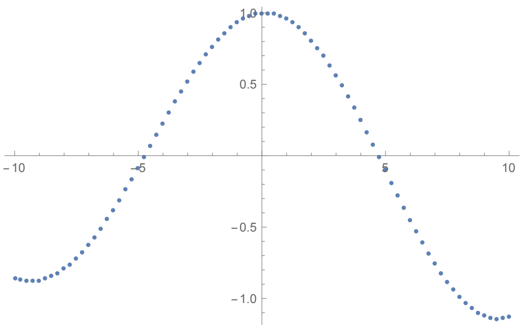
3.6. DISCRETE LAPLACE TRANSFORM

The $\mathcal{Z}$-transform is a well-known analogue of the Laplace transform [29, Definition 3.4] defined for functions $f\colon \mathbb{Z} \to \mathbb{C}$ by the formula

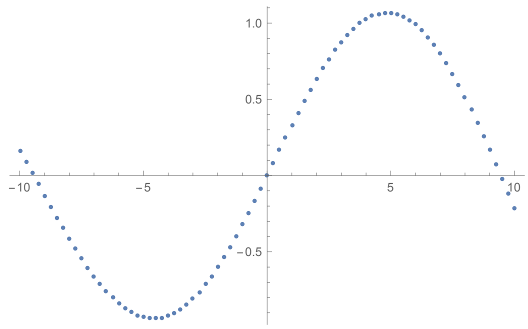
$$\mathcal{Z}\{f\}(z) = \sum_{k=0}^{\infty} \frac{f(k)}{z^k}.$$

The following discrete analogue of (4) is known [19, Exercise 13, page 281]:

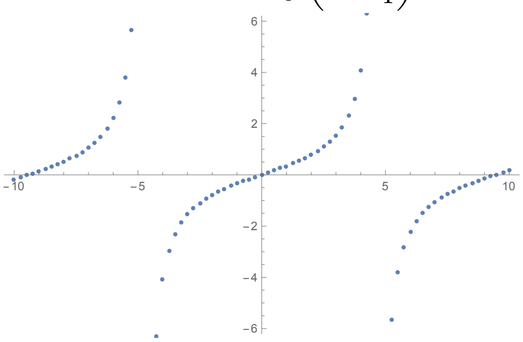
$$\mathcal{Z}\{\Delta_1^n f\}(z) = (z-1)^n \mathcal{Z}\{f\}(z) - z\sum_{j=0}^{n-1}(z-1)^{n-j-1}\Delta_1^j f(0). \tag{60}$$



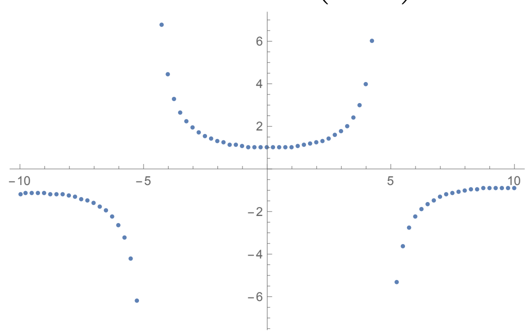
(a) Plot of $\cos_{\frac{1}{3}}\left(t, 0; \frac{1}{4}\right)$.



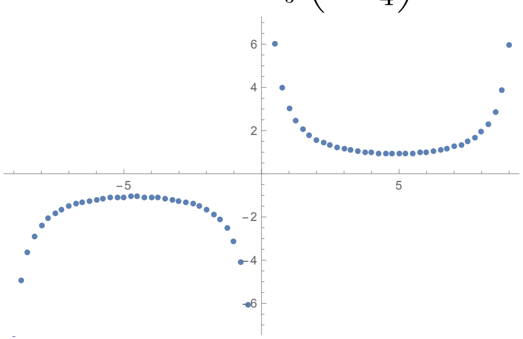
(b) Plot of $\sin_{\frac{1}{3}}\left(t, 0; \frac{1}{4}\right)$.



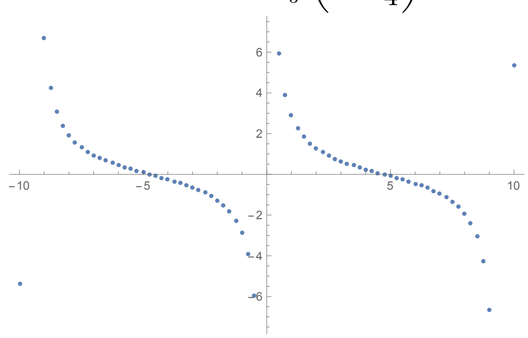
(c) Plot of $\tan_{\frac{1}{3}}\left(t, 0; \frac{1}{4}\right)$.



(d) Plot of $\sec_{\frac{1}{3}}\left(t, 0; \frac{1}{4}\right)$.



(e) Plot of $\csc_{\frac{1}{3}}\left(t, 0; \frac{1}{4}\right)$.



(f) Plot of $\cot_{\frac{1}{3}}\left(t, 0; \frac{1}{4}\right)$.

Figure 3.5. Discrete trigonometric functions.

Using (47), we may calculate $\ominus z = -\dfrac{z}{1+hz}$, and using (49), we see for $t > s$,

$$e_{\ominus z}(t,s) = \prod_{k=\frac{s}{h}}^{\frac{t}{h}-1}\left(1 - \frac{hz}{1+hz}\right) = \prod_{k=\frac{s}{h}}^{\frac{t}{h}-1}\frac{1}{1+hz} = \left(\frac{1}{1+hz}\right)^{\frac{t-s}{h}}.$$

We say a function $f\colon h\mathbb{Z} \to \mathbb{R}$ has exponential order α on $[s,\infty)\cap h\mathbb{Z}$ if $\alpha \in \mathcal{R}_{\mu_h}$ and there exists $K > 0$ such that $|f(t)| \leq K e_\alpha(t,s)$ for all $t \in [s,\infty)\cap h\mathbb{Z}$. An alternative transformation to $\mathscr{Z}$ was introduced in [13] and studied in detail in [9,10] for a function f of exponential order α on $[s,\infty)\cap h\mathbb{Z}$:

$$\begin{aligned}\mathscr{L}_h\{f\}(z;s) &= h\sum_{k=\frac{s}{h}}^{\infty} f(hk)e_{\ominus z}(hk+h,s;h)\\ &= h\sum_{k=\frac{s}{h}}^{\infty}\frac{f(hk)}{(1+hz)^{k+1-\frac{s}{h}}}.\end{aligned} \tag{61}$$

We have included a table of common discrete Laplace transforms in Table 3.1. Being a discrete integral, the discrete Laplace transform is a linear transformation. We have an analogue of (60) from [10, Corollary 6.3] which better resembles (4):

$$\mathscr{L}_h\{\Delta_h^n f\}(z;s) = z^n\mathscr{L}_h\{f\}(z;s) - \sum_{k=0}^{n-1} z^{n-1-k}\Delta_h^k f(s). \tag{62}$$

Complex derivatives of discrete h-Laplace transforms manifest as polynomial factors.

Theorem 3.6.1. *If* $f(t,s) = \sum_{j=0}^{\infty} a_k (t-s)_h^{\underline{k}}$, *then*

$$\frac{\mathrm{d}^n}{\mathrm{d}z^n}\mathscr{L}_h\{f(\cdot,s)\}(z;s) = (-1)^n\mathscr{L}_h\{f_n\}(z;s),$$

where $f_n(t) = (t-s)_h^{\underline{n}} f(t-s-hn)$.

Proof. Using Lemma 3.2.1, the computation

$$\begin{aligned}
(\mathscr{L}_h)^{(n)}(z) &= h\frac{\mathrm{d}^n}{\mathrm{d}z^n}\sum_{k=0}^{\infty}\frac{f(hk)}{(z+1)^{k+1}} \\
&= h(-1)^n\sum_{k=0}^{\infty}\frac{(k+1)(k+2)\dots(k+n)f(hk)}{(z+1)^{k+n+1}} \\
&= h(-1)^n\sum_{k=n}^{\infty}\frac{(k-n+1)(k-n+2)\dots(k+1)kf(h(k-n))}{(z+1)^{k+1}} \\
&= (-1)^n\sum_{k=0}^{\infty}\frac{k_h^n f(hk-hn)}{(z+1)^{k+1}} \\
&= (-1)^n\mathscr{L}_h\{f_n\}(z;s)
\end{aligned}$$

proves the claim. □

The paper [17, Theorem 1.4] proves an inversion theorem to find the inverse discrete Laplace transform $\mathscr{L}_h^{-1}$. If $\int_{c-i\infty}^{c+i\infty}|F(z)||\mathrm{d}z|<\infty$, then we may express f in the form

$$\mathscr{L}_h^{-1}\{F\}(t)=f(t)=\sum_{k=1}^{n}\mathrm{Res}_{z=z_k}e_z(t,0)F(z). \tag{63}$$

Since $h\mathbb{Z}$ is a topologically discrete set, the "almost everywhere" (see [24]) condition present in [17, Theorem 1.5, page 1300] does not apply since all points of $h\mathbb{Z}$ are "right-scattered". Consequently, we have uniqueness of the inverse Laplace transform.

We would like to have a discrete analogue of the convolution integral (5). To achieve this, we first consider the "shifting problem" partial h-difference equation for a function $f\colon h\mathbb{Z}\to\mathbb{C}$. The function $\hat{f}(t,s)$ is the solution of the initial value problem

$$\begin{cases}
\Delta_{h,1}(t,\sigma(s)) = -\Delta_{h,2}u(t,s), & t,s\in h\mathbb{Z}, \quad t\geq s\geq t_0 \\
u(t,t_0)=f(t), & t\in h\mathbb{Z}, \quad t\geq t_0,
\end{cases}$$

where $\Delta_{h,m}f(t_1,\dots,t_n)$ denotes a partial h-difference in the mth argument of f.

When simplified, this partial h-difference equation yields the formula

$$u(t,s) = u(t+h, s+h).$$

Taking the initial condition into account shows that $u(t,s) = \hat{f}(t,s) = f(t-s+t_0)$. The h-convolution is defined in [8, (2.6), page 4] by

$$(f * g)(t,s) = h \sum_{k=\frac{s}{h}}^{\frac{t}{h}-1} f(t-hk+t_0)g(hk). \tag{64}$$

With the h-convolution, we have the convolution theorem [8, Theorem 3.2, page 8]

$$\mathscr{L}_h\{(f*g)(\cdot,s)\}(z;s) = \mathscr{L}_h\{f\}(z;s)\mathscr{L}_h\{g\}(z;s). \tag{65}$$

Table 3.1. Discrete Laplace transforms of discrete special functions.

$f(t)$	$\mathscr{L}_h\{f\}(z;s)$	Source
$h_n(t,s;h)$	$\dfrac{1}{z^{n+1}}$	[9, Table 5]
$e_\alpha(t,s;h)$	$\dfrac{1}{z-\alpha}$	[9, Table 5]
$\cosh_\alpha(t,s;h)$	$\dfrac{z}{z^2-\alpha^2}$	[12, Table 3.2, page 133]
$\sinh_\alpha(t,s;h)$	$\dfrac{\alpha}{z^2-\alpha^2}$	[12, Table 3.2,page 133]
$\cos_\alpha(t,s;h)$	$\dfrac{z}{z^2+\alpha^2}$	[12, Table 3.2, page 133]
$\sin_\alpha(t,s;h)$	$\dfrac{\alpha}{z^2+\alpha^2}$	[12, Table 3.2, page 133]
$J_0(t,1,0,1;h)$	$\dfrac{1}{\sqrt{z^2+1}}$	Theorem 3.10.2
$J_\nu(t,\xi,0,1;h)$	$\dfrac{1}{\sqrt{z^2+1}[\sqrt{z^2+1}+z]^{-\nu}}$	
$J_\nu\left(t,2\sqrt{\alpha},\dfrac{\nu}{2},\dfrac{1}{2};h\right)$	$\dfrac{\alpha^{\frac{\nu}{2}}}{z^{\nu+1}e^{\frac{\alpha}{2}}}$	Theorem 3.10.4

3.7. DISCRETE GAMMA FUNCTION

The gamma function Γ_h is defined in [11] by the formula

$$\Gamma_h(t,s) = h\sum_{k=0}^{\infty}\left(\prod_{j=\frac{s}{h}}^{k-1}\frac{j+t}{j+1}\right)\frac{1}{(1+h)^{k+1}}.$$

We have visualized this function in Figure 3.6. This function has many properties analogous to the properties of the classical gamma function. It diverges as $t \to 0^+$ and as $t \to \infty$. We always know that $\Gamma_h(1;s) = 1$.

Lemma 3.7.1. *The following formula holds:*

$$\Gamma_h(t,h) = \frac{1}{t}\left(1+\frac{1}{h}\right)^{t-1}.$$

Proof. Using (33) with $x = \dfrac{1}{1+h}$ and $a = t$, we see

$$\left(\frac{h}{1+h}\right)^{-t} = \left(1-\frac{1}{1+h}\right)^{-t} = \sum_{k=0}^{\infty}\frac{1}{(1+h)^k}\frac{t^{\overline{k}}}{k!}.$$

Using (16) and (31), we compute

$$\begin{aligned}
\Gamma_h(t,h) &= h\sum_{k=0}^{\infty}\left(\prod_{j=1}^{k-1}\frac{j+t}{j+1}\right)\frac{1}{(1+h)^{k+1}} \\
&= h\sum_{k=0}^{\infty}\frac{\Gamma(t+k)}{k!\Gamma(t+1)(1+h)^{k+1}} \\
&= \frac{h}{1+h}\frac{1}{t}\sum_{k=0}^{\infty}\frac{\Gamma(t+k)}{\Gamma(t)}\frac{1}{k!}\frac{1}{(1+h)^k} \\
&= \frac{h}{1+h}\frac{1}{t}\sum_{k=0}^{\infty}\frac{1}{(1+h)^k}\frac{t^{\overline{k}}}{k!} \\
&= \frac{1}{t}\left(1+\frac{1}{h}\right)^{t-1},
\end{aligned}$$

as was to be shown. □

Theorem 3.7.1. *The following formula holds for $t, s \in h\mathbb{Z}^+$:*

$$\Gamma_h(t,s) = \frac{\left(\frac{s}{h}\right)!}{t(t+1)\dots(t+\frac{s}{h}-1)}\left(1+\frac{1}{h}\right)^{t-1}.$$

Proof. We use Lemma 3.7.1 to compute

$$\begin{aligned}\Gamma_h(t,s) &= h\sum_{k=0}^{\infty}\left(\prod_{j=\frac{s}{h}}^{k-1}\frac{j+t}{j+1}\right)\frac{1}{(1+h)^{k+1}}\\ &= \left(\prod_{j=1}^{\frac{s}{h}-1}\frac{j+1}{j+t}\right)\Gamma_h(t,h)\\ &= \frac{\left(\frac{s}{h}\right)!}{t(t+1)\dots(t+\frac{s}{h}-1)}\left(1+\frac{1}{h}\right)^{t-1},\end{aligned}$$

as was to be shown. □

The complexification $\overline{\Gamma_h}$ follows by simple algebra and is unique by analytic continuation:

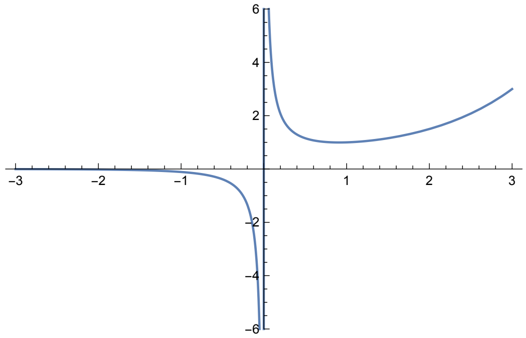
$$\overline{\Gamma}_h(t,s) = \frac{\Gamma\left(\frac{s}{h}+1\right)\left(1+\frac{1}{h}\right)^{t-1}}{t(1+t)^{\overline{s/h-1}}}.$$

3.8. DISCRETE GAUSSIAN BELL

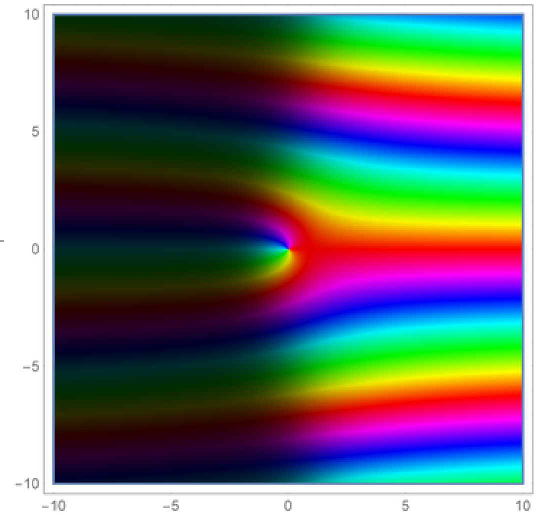
An analogue of the Gaussian bell is defined in [20] by the formula

$$\mathbf{E}_h(t) = \left[(1+h)^{\frac{1}{h}}\right]^{-\frac{t(t-h)}{2}}.$$

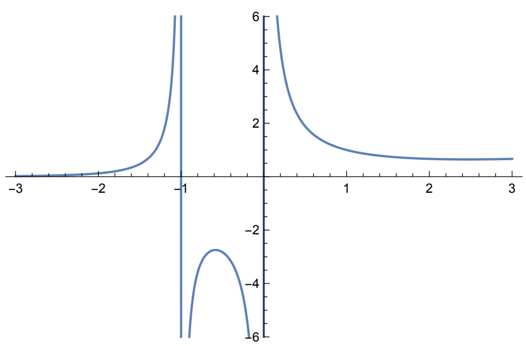
This Gaussian bell has the unfortunate property that its Δ-derivative is proportional to an exponential function: $\Delta\mathbf{E}_h(t) = (2^t-1)\mathbf{E}_h(t)$. We want to use a Gaussian bell



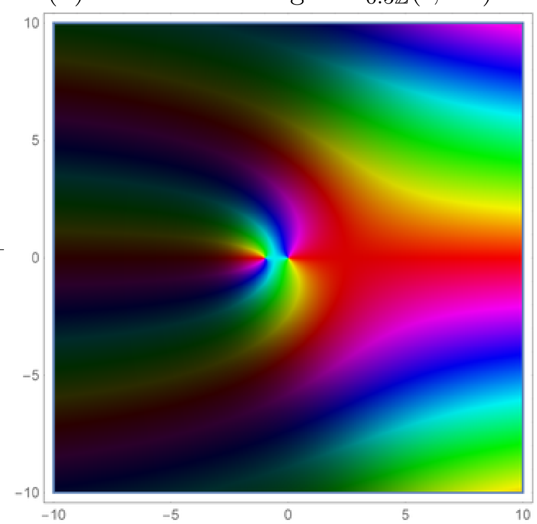
(a) Plot of $\Gamma_{0.5\mathbb{Z}}(t, 0.5)$ on $[-3, 3]$.



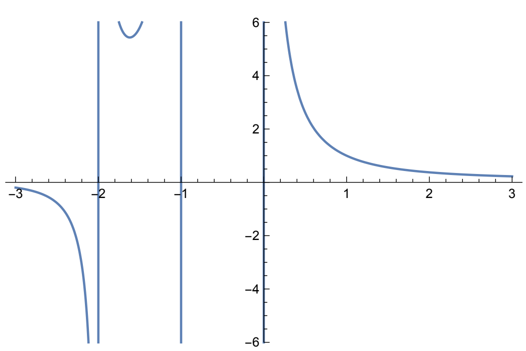
(b) Domain coloring of $\overline{\Gamma}_{0.5\mathbb{Z}}(t, 0.5)$.



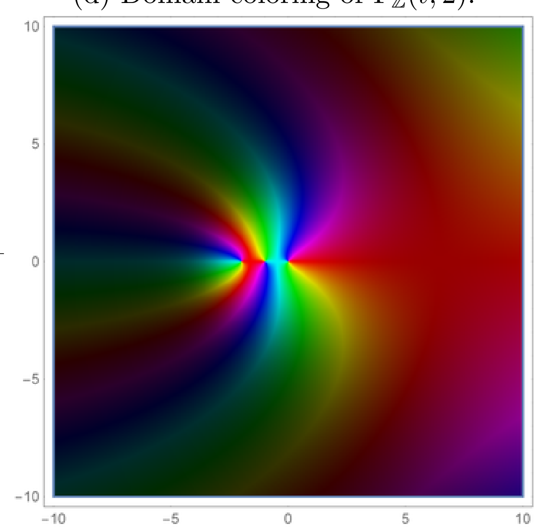
(c) Plot of $\Gamma_{\mathbb{Z}}(t, 2)$ on $[-3, 3]$.



(d) Domain coloring of $\overline{\Gamma}_{\mathbb{Z}}(t, 2)$.



(e) Plot of $\Gamma_{2\mathbb{Z}}(t, 6)$ on $[-3, 3]$.



(f) Domain coloring of $\overline{\Gamma}_{2\mathbb{Z}}(t, 6)$.

Figure 3.6. Plots of discrete gamma functions.

to look at analogues of Hermite polynomials, and so we prefer a Gaussian bell whose difference is proportional to t.

Choose some $\alpha > 0$ and $s \in h\mathbb{Z}$ and define the function $p\colon h\mathbb{Z} \to \mathbb{R}$ by $p(t) = -\alpha(t-s)$. We define a Gaussian bell analogue by considering the split $\Delta_h - \nabla_h$ initial value problem

$$\begin{cases} \Delta_h y(t) = p(t)y(t) = -\alpha(t-s)y(t), & t < s \\ 1, & t = s \\ \nabla_h y(t) = p(t)y(t) = -\alpha(t-s)y(t), & t > s. \end{cases} \tag{66}$$

We now use our knowledge of discrete exponential functions in formulas (49) and (50) to deduce that the solution of (66) is given by

$$\Omega_h(t,s,\alpha) = \left\{ \begin{array}{ll} e_p(t,s;h) = \displaystyle\prod_{k=\frac{t}{h}}^{\frac{s}{h}-1} \frac{1}{1+\alpha h|hk-s|}, & t < s \\ 1, & t = s \\ \hat{e}_p(t,s;h) = \displaystyle\prod_{k=\frac{s}{h}+1}^{\frac{t}{h}} \frac{1}{1+\alpha h|hk-s|}, & t > s \end{array} \right\} = \prod_{k=1}^{\left|\frac{t-s}{h}\right|} \frac{1}{1+\alpha hk} \tag{67}$$

which we call the Gaussian bell on $h\mathbb{Z}$. We say that $h\mathbb{Z}$ is symmetric about $s \in h\mathbb{Z}$ because if $s+\delta \in h\mathbb{Z}$, then $s-\delta \in h\mathbb{Z}$. We define the operation $-_s$ so that if $t = s+\delta_t$, then $-_s t = s - \delta_t$. With this operation, we see that for $t = s + \delta_t$,

$$p(-_s t) = -\alpha(-_s t - s) = -\alpha(s - (\delta_t + s)) = \alpha(t-s) = -p(t).$$

Theorem 3.8.1. *The following formula holds:*

$$\Omega_h(t,s,\alpha) = \Omega_h(-_s t, s, \alpha).$$

Proof. The proof for $t = s$ is obvious from the definition. Suppose that $t > s$. Now compute

$$\begin{aligned}
\Omega_h(t,s,\alpha) &= \hat{e}_p(t,s) \\
&= \prod_{k=\frac{s}{h}+1}^{\frac{t}{h}} \frac{1}{1-hp(hk)} \\
&= \frac{1}{(1-hp(s+h))\dots(1-hp(t))},
\end{aligned}$$

while

$$\begin{aligned}
\Omega_h(-_s t,s,\alpha) &= e_p(-_s t,s) \\
&= \prod_{k=\frac{-_s t}{h}}^{\frac{s}{h}-1} \frac{1}{1+hp(hk)} \\
&= \frac{1}{(1+hp(-_s t))\dots(1+hp(s-h))} \\
&= \frac{1}{(1+hp(s-\delta_t))\dots(1+hp(s-h))} \\
&= \frac{1}{(1-hp(t))\dots(1-hp(s+h))}.
\end{aligned}$$

Hence we observe that $\Omega_h(-_s t,a,\alpha) = \Omega_h(t,s,\alpha)$. The proof for $t < s$ is similar. □

Theorem 3.8.2. *The discrete Gaussian bell is always positive, i.e., for all $t \in h\mathbb{Z}$,*

$$\Omega_h(t,s,\alpha) > 0.$$

Proof. If $t = s$, then $\Omega_h(t,s,\alpha) = 1$. Let $t > s$. Since $p(t) = -\alpha(t-s)$, we see that

$$1 - \nu(t)p(t) = 1 + \alpha\nu(t)(t-s) > 0.$$

This implies that $p \in \mathcal{R}^+_{\nu_h}([s,\infty) \cap h\mathbb{Z}, \mathbb{R})$, and hence by [14, Theorem 3.18 (i)], we have $\Omega_h(t,s,\alpha) > 0$. From this and Theorem 3.8.1, it follows that for all $t < s$,

$\Omega_h(t, s, \alpha) > 0.$ □

Theorem 3.8.3. *The following formula holds:*

$$\Omega_h(t,s,\alpha) = \left(\frac{1}{\alpha h^2}\right)^{\frac{|t-s|}{h}} \frac{1}{\left(1+\frac{1}{\alpha h^2}\right)^{\overline{\frac{|t-s|}{h}}}}.$$

Proof. For $t = s$, the claim is immediately clear. For $t > s$, we compute

$$\begin{aligned}
\left(\frac{1}{\alpha h^2}\right)^{\frac{|t-s|}{h}} \frac{1}{\left(1+\frac{1}{\alpha h^2}\right)^{\overline{\frac{|t-s|}{h}}}} &= \left(\frac{1}{\alpha h^2}\right)^{\frac{|t-s|}{h}} \frac{\Gamma\left(\frac{1}{\alpha h^2}+1\right)}{\Gamma\left(1+\frac{1}{\alpha h^2}+\frac{t-s}{h}\right)} \\
&= \left(\frac{1}{\alpha h^2}\right)^{\frac{|t-s|}{h}} \prod_{k=\frac{s}{h}+1}^{\frac{t}{h}} \frac{1}{\frac{1}{\alpha h^2}+\left(k-\frac{s}{h}\right)} \\
&= \prod_{k=\frac{s}{h}+1}^{\frac{t}{h}} \frac{1}{1+\alpha h(hk-s)} \\
&= \Omega_h(t,s,\alpha),
\end{aligned}$$

as was to be shown. The case for $t < s$ is similar. □

A natural application for the Gaussian bell is as the basis of a discrete normal distribution. To achieve this, we will need to be able to sum over the Gaussian bell. First we define

$$\overline{L}_h(s,\alpha) = h \sum_{k=-\infty}^{\infty} \Omega_h(kh, s, \alpha).$$

Theorem 3.8.4. *The number $\overline{L}_h(s,\alpha)$ exists, and*

$$\overline{L}_h(s,\alpha) = 2he^{\frac{1}{\alpha h^2}} (\alpha h^2)^{\frac{1}{\alpha h^2}} \gamma\left(1+\frac{1}{\alpha h^2}; \frac{1}{\alpha h^2}\right).$$

Proof. From (21), we see

$$\gamma\left(1+\frac{1}{\alpha h^2};\frac{1}{\alpha h^2}\right)=e^{-\frac{1}{\alpha h^2}}\left(\frac{1}{\alpha h^2}\right)^{\frac{1}{\alpha h^2}}\sum_{k=1}^{\infty}\frac{\left(\frac{1}{\alpha h^2}\right)^k}{\left(1+\frac{1}{\alpha h^2}\right)^{\overline{k}}}.$$

and using Theorem 3.8.3, we compute

$$\begin{aligned}\overline{L}_h(s,\alpha)&=2h\sum_{k=1}^{\infty}\Omega_h(hk,s,\alpha)\\&=2h\sum_{k=1}^{\infty}\frac{\left(\frac{1}{\alpha h^2}\right)}{\left(1+\frac{1}{\alpha h^2}\right)^{\overline{\frac{|t-s|}{h}}}}\\&=2he^{\frac{1}{\alpha h^2}}(\alpha h^2)^{\frac{1}{\alpha h^2}}\gamma\left(1+\frac{1}{\alpha h^2};\frac{1}{\alpha h^2}\right),\end{aligned}$$

as was to be shown. □

3.9. ANALOGUES OF ORTHOGONAL POLYNOMIALS

3.9.1. Hermite I.

We define the h-difference equation analogue of the Hermite differential equation of type I by

$$\Delta_h^2 y(t)-(t-s)\Delta_h y(t-h)+ny(t)=0. \tag{68}$$

We define the discrete Hermite polynomials of type I $\mathcal{H}_n$ by

$$\mathcal{H}_n(t,s;h)=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\frac{(-1)^k n!(t-s)_h^{n-2k}}{k!\Gamma(n-2k+1)2^k}. \tag{69}$$

We have included some of these functions in Table 3.2. We justify the names of these functions in the following theorem.

Theorem 3.9.1. *The function* $y(t)=\mathcal{H}_n(t,s;h)$ *solves* (68).

Proof. First assume that $n=2m$ for some $m\in\mathbb{N}$. Then using (16) and the fact that

$\frac{1}{\Gamma}$ has zeros whenever $t \in \{0, -1, -2, \ldots\}$, we see

$$\Delta_h \mathcal{H}_n(t,s;h) = \sum_{k=0}^{m-1} \frac{(-1)^k n!(t-s)_h^{n-2k-1}}{k!\Gamma(n-2k)2^k} = \sum_{k=0}^{m} \frac{(-1)^k n!(t-s)_h^{n-2k-1}}{k!\Gamma(n-2k)2^k},$$

and hence by Lemma 3.2.1,

$$(t-s)\Delta_h \mathcal{H}_n(t-h,s;h) = \sum_{k=0}^{m} \frac{(-1)^k n!(t-s)_h^{n-2k}}{k!\Gamma(n-2k)2^k}.$$

We take a second difference and reindex to find

$$\begin{aligned}
\Delta_h^2 \mathcal{H}_n(t,s;h) &= \sum_{k=0}^{m-1} \frac{(-1)^k n!(t-s)_h^{n-2k-2}}{k!\Gamma(n-2k-1)2^k} \\
&= \sum_{k=1}^{m} \frac{(-1)^{k-1} n!(t-s)_h^{n-2(k-1)-2}}{(k-1)!\Gamma(n-2(k-1)-1)2^{k-1}} \\
&= -2\sum_{k=1}^{m} \frac{k(-1)^k n!(t-s)_h^{n-2k}}{k!\Gamma(n-2k+1)2^k} \\
&= -2\sum_{k=0}^{m} \frac{k(-1)^k n!(t-s)_h^{n-2k}}{k!\Gamma(n-2k+1)2^k}.
\end{aligned}$$

Therefore we compute

$$\begin{aligned}
&\Delta^2 \mathcal{H}_n(t,s;h) - (t-s)\Delta \mathcal{H}_n(t-h,s;h) + n\mathcal{H}_n(t,s;h) \\
&= \sum_{k=0}^{m} \frac{(-1)^k n!(t-s)_h^{n-2k}}{k!2^k} \left[\frac{-2k}{\Gamma(n-2k+1)} - \frac{1}{\Gamma(n-2k)} + \frac{n}{\Gamma(n-2k+1)} \right] \\
&= \sum_{k=0}^{m} \frac{(-1)^k n!(t-s)_h^{n-2k}}{k!\Gamma(n-2k+1)2^k} [-2k - (n-2k) + n] \\
&= 0,
\end{aligned}$$

as was to be shown. The case $n = 2m+1$ is essentially the same. □

Theorem 3.9.2. *The following formula holds:*

$$\Delta_h \mathcal{H}_n(t,s;h) = n\mathcal{H}_{n-1}(t,s;h).$$

Table 3.2. Discrete Hermite polynomials of type I.

n	$\mathcal{H}_n(t;h)$
0	1
1	$t-s$
2	$(t-s)_h^2-1$
3	$(t-s)_h^3-3(t-s)$
4	$(t-s)_h^4-6(t-s)_h^2+3$
5	$(t-s)_h^5-10(t-s)_h^3+15(t-s)$
6	$(t-s)_h^6-15(t-s)_h^4+45(t-s)_h^2-15$
$\vdots$	$\vdots$

Proof. If $n=2m+1$, then $\left\lfloor\frac{n-1}{2}\right\rfloor=m$, and so

$$\begin{aligned}\Delta_h\mathcal{H}_n(t,s;h) &= \sum_{k=0}^{m}\frac{(-1)^k n!(t-s)_h^{n-2k-1}}{k!\Gamma(n-2k)2^k} \\ &= n\sum_{k=0}^{m}\frac{(-1)^k(n-1)!(t-s)^{(n-1)-2k-1}}{k!\Gamma((n-1)-2k)2^k} \\ &= n\mathcal{H}_{n-1}(t,s;h),\end{aligned}$$

as was to be shown. The proof for $n=2m$ is essentially the same. □

Theorem 3.9.3. *The following formula holds:*

$$\mathcal{H}_{n+1}(t,s;h)=t\mathcal{H}_n(t-h,s;h)-n\mathcal{H}_{n-1}(t,s;h).$$

Proof. If $n=2m$, then $\left\lfloor\frac{n+1}{2}\right\rfloor=m$ and $\left\lfloor\frac{n-1}{2}\right\rfloor=m-1$, so we compute

$$\mathcal{H}_{n+1}(t,s;h)=\sum_{k=0}^{m}\frac{(-1)^k(n+1)!(t-s)_h^{(n+1)-2k}}{k!\Gamma((n+1)-2k+1)2^k}$$

$$
\begin{aligned}
&= \sum_{k=0}^{m} \frac{(-1)^k n!(t-s)_h^{n+1-2k}}{k!\Gamma(n-2k+1)2^k} \left(\frac{n+1}{n+1-2k} \right) \\
&= \sum_{k=0}^{m} \frac{(-1)^k n!(t-s)_h^{n+1-2k}}{k!\Gamma(n-2k+1)2^k} \left(1 + \frac{2k}{n+1-2k} \right) \\
&= (t-s)\mathcal{H}_n(t-h,s;h) + n \sum_{k=0}^{m} \frac{k(-1)^k (n-1)!(t-s)_h^{(n+1)-2k}}{k!\Gamma((n+1)-2k+1)2^{k-1}} \\
&= (t-s)\mathcal{H}_n(t-h,s;h) - n \sum_{k=-1}^{m-1} \frac{(-1)^k (n-1)!(t-s)_h^{(n-1)-2k}}{k!\Gamma((n-1)-2k+1)2^k} \\
&= (t-s)\mathcal{H}_n(t-h,s;h) - n\mathcal{H}_{n-1}(t,s;h),
\end{aligned}
$$

as was to be shown. □

Theorem 3.9.4. *There do not exist constants $\alpha_2, \beta_2, \gamma_2$ such that*

$$\mathcal{H}_3(t,0;h) = (\alpha_2 t + \beta_2)\mathcal{H}_2(t,0;h) + \gamma_2 \mathcal{H}_1(t,0;h).$$

Moreover, there does not exist an inner product with respect to which all of the $\mathcal{H}_n$ functions are orthogonal.

Proof. Consider the equation

$$\mathcal{H}_3(t,0;h) = (\alpha_2 t + \beta_2)\mathcal{H}_2(t,0;h) + \gamma_2 \mathcal{H}_1(t,0;h). \tag{70}$$

First note that

$$\mathcal{H}_2(t,0;h) = t_h^2 - 1 = t^2 - th - 1.$$

The right-hand side of this equation becomes

$$
\begin{aligned}
(\alpha_2 + t\beta_2)\mathcal{H}_2(t,0;h) + \gamma_2 \mathcal{H}_1(t,0;h) &= (\alpha_2 + t\beta_2)(t^2 - th - 1) + \gamma_2 t \\
&= t^3\beta_2 + t^2(\alpha_2 - \beta_2 h) + t(\gamma_2 - \beta_2 - \alpha_2 h) - \alpha_2.
\end{aligned}
$$

Since

$$\mathcal{H}_3(t,0;h) = t^3 - 3ht^2 + (2h^2 - 3)t,$$

the equation (70) yields the system of equations

$$\begin{cases} \beta_2 & = 1 \\ \alpha_2 - \beta_2 h & = -3h \\ \gamma_2 - \beta_2 - \alpha_2 h & = 2h^2 - 3 \\ -\alpha_2 & = 0, \end{cases}$$

which is an inconsistent system because $\beta_2 = 1$ and $\alpha_2 = 0$, but $\alpha_2 - \beta_2 h \neq -3h$. By the contrapositive of Theorem 2.3.1, we see that there is no inner product for which the polynomials $\{\mathcal{H}_n\}_{n=0}^{\infty}$ are orthogonal with respect to. $\square$

3.9.2. Hermite II. Because of Theorem 3.9.4, we seek alternative Hermite polynomials that obey an orthogonality property. Define $\phi_h(t,s) = \Omega_h(t,s,1)$. By (66), we know

$$\begin{cases} \Delta_h \phi_h(t,s) = -(t-s)\phi_h(t,s), & t < s \\ \nabla_h \phi_h(t,s) = -(t-s)\phi_h(t,s), & t > s. \end{cases}$$

We now define an analogue of the Hermite polynomials on $h\mathbb{Z}$ by a Rodrigues type. The piecewise defined nature reflects the fact that the nth Δ_h or ∇_h difference requires an argument of the form $\phi_h(t \pm nh)$:

$$H_n(t,s;h) = \begin{cases} (-1)^n \dfrac{\Delta_h^n \phi_h(t,s)}{\phi_h(t,s)}, & t \leq s - nh \\ (-1)^n \dfrac{\nabla_h^n \phi_h(t,s)}{\phi_h(t,s)}, & t \geq s + nh. \end{cases} \tag{71}$$

The piecewise domain of these functions gives us freedom to force orthogonality but we lose uniqueness on $h\mathbb{Z}$ as a consequence.

Theorem 3.9.5. *The following formulas hold:*

$$H_{n+1}(t,s;h) = \begin{cases} (t-s)H_n(t+h,s;h) - \Delta_h H_n(t,s;h), & t \le s-(nh+h) \\ (t-s)H_n(t-h,s;h) - \nabla_h H_n(t,s;h), & t \ge s+nh+h \end{cases}$$

and

$$H_{n+1}(t,s;h) = \begin{cases} -(1+|t-s|)\Delta_h H_n(t,s;h) + (t-s)H_n(t,s;h), & t \le s-nh \\ -(1+(t-s))\nabla_h H_n(t,s;h) + (t-s)H_n(t,s;h), & t \ge s+nh. \end{cases}$$

Proof. For $t \ge s+nh+h$, we compute

$$\begin{aligned} H_{n+1}(t,s;h) &= (-1)^{n+1}\frac{\nabla_h^{n+1}\phi_h(t,s)}{\phi_h(t,s)} \\ &= (-1)^{n+1}\frac{\nabla_h[(-1)^n H_n(t,s;h)\phi_h(t,s)]}{\phi_h(t)} \\ &= -\frac{\nabla_h[H_n(t,s;h)\phi_h(t,s)]}{\phi_h(t,s)} \\ &= -\frac{\nabla_h H_n(t,s;h)\phi_h(t,s) - (t-s)\phi_h(t,s)H_n(t-h,s;h)}{\phi_h(t,s)} \\ &= (t-s)H_n(t-h,s;h) - \nabla_h H_n(t,s;h), \end{aligned}$$

and the proof for $t \le s-(nh+h)$ is similar. Now note that for $t \ge s+nh$, the formula

$$\frac{\phi(t-h,s)}{\phi(t,s)} = \frac{\prod_{k=1}^{\left|\frac{t-s}{h}\right|-1}\frac{1}{1+kh}}{\prod_{k=1}^{\left|\frac{t-s}{h}\right|}\frac{1}{1+kh}} = (1+|t-s|) = 1+t-s$$

holds. To prove the other formula, use the other form of the product rule

$$\begin{aligned} H_{n+1}(t,s;h) &= -\frac{\nabla_h[H_n(t,s;h)\phi_h(t,s)]}{\phi_h(t,s)} \\ &= \frac{-\phi_h(t-h,s)\nabla H_n(t,s;h) + (t-s)H_n(t,s;h)\phi_h(t,s)}{\phi_h(t,s)} \\ &= -(1+t-s)\nabla H_n(t,s;h) + (t-s)H_n(t,s;h), \end{aligned}$$

as was to be shown. The proof for $t \leq s - (nh + h)$ is similar. □

Theorem 3.9.6. *The following formulas hold:*

$$H_{n+1}(t, s; h) = \begin{cases} (t-s)H_n(t, s; h) - nH_{n-1}(t+h, s; h), & t \leq s - nh \\ (t-s)H_n(t, s; h) - nH_{n-1}(t-h, s; h), & t \geq s + nh \end{cases}$$

Proof. Suppose that $t \geq s + (n+1)h$. From $\nabla_h \phi(t, s) = -(t-s)\phi(t, s)$, we compute

$$\nabla_h \phi_h(t, s) + (t-s)\phi_h(t, s) = 0,$$

$$\nabla_h^2 \phi_h(t, s) + (t-s)\nabla_h \phi_h(t, s) + \phi_h(t-h, s) = 0,$$

$$\nabla_h^3 \phi_h(t, s) + (t-s)\nabla_h^2 \phi_h(t, s) + 2\nabla_h \phi_h(t-h, s) = 0,$$

$$\vdots$$

$$\nabla_h^{n+1} \phi_h(t, s) + (t-s)\nabla_h^n \phi_h(t, s) + n\nabla_h^{n-1} \phi_h(t-h, s) = 0. \tag{72}$$

Applying the definition of H_n now yields

$$H_{n+1}(t, s; h) = (t-s)H_n(t, s; h) - nH_{n-1}(t-h, s; h),$$

as was to be shown. The proof for $t < s$ is similar. □

Theorem 3.9.7. *The following formulas hold:*

$$(1+h(t-s))H_{n+1}(t; h) = \begin{cases} (t-s)H_n(t+h, s; h) - nH_{n-1}(t+h, s), t \leq s - (nh+h) \\ (t-s)H_n(t-h, s; h) - nH_{n-1}(t-h, s), t \geq s + nh + h. \end{cases}$$

Proof. We will prove the claim for $t \geq s + nh + h$. From $\nabla_h \phi(t, s) = -(t-s)\phi(t, s)$, we compute

$$\nabla_h \phi_h(t,s) + (t-s)\phi_h(t,s) = 0,$$

$$\nabla_h^2 \phi_h(t,s) + (t-s)\nabla_h \phi_h(t,s) + \phi_h(t-h,s) = 0,$$

$$\nabla_h^3 \phi_h(t,s) + (t-s)\nabla_h^2 \phi_h(t,s) + 2\nabla_h \phi_h(t-h,s) = 0,$$

$$\vdots$$

$$\nabla_h^{n+1} \phi_h(t,s) + (t-s)\nabla_h^n \phi_h(t,s) + n\nabla_h^{n-1} \phi_h(t-h,s) = 0.$$

Using the relationship $f(t) = h\nabla_h f(t) + f(t-h)$, we examine the middle term in the formula

$$\nabla_h^{n+1} \phi_h(t,s) + (t-s)\nabla_h^n \phi_h(t,s) + n\nabla_h^{n-1} \phi_h(t-h,s) = 0 \tag{73}$$

and notice that

$$(t-s)\nabla_h^n \phi_n(t,s) = (t-s)\left[h\nabla_h^{n+1} \phi_h(t,s) + \nabla_h^n \phi_h(t-h,s)\right].$$

Hence (73) becomes

$$(1 + h(t-s))\nabla_h^{n+1} \phi_h(t,s) + (t-s)\nabla_h^n \phi_h(t-h,s) + n\nabla_h^{n-1} \phi_h(t-h,s) = 0.$$

Substitution of the definition of H_n yields

$$(1 + h(t-s))H_{n+1}(t,s;h) = (t-s)H_n(t-h,s;h) - nH_{n-1}(t-h,s),$$

as was to be shown. The proof for $t \leq s - (nh + h)$ is similar. $\square$

We have purposely left the values $|t-s| < hn$ undefined in the definition of the Hermite polynomials so that we can force orthogonality. This is necessary because if we do not impose this restriction, then the resulting polynomials are not orthogonal with respect to $\phi_h(t,s)$. Define the inner product

$$\langle f, g\rangle_s = h \sum_{k=-\infty}^{\infty} f(hk,s;h)g(hk,s;h)\phi(hk,s;h).$$

It is easy to show that if n is even and m is odd, then $\langle H_n, H_m\rangle_s = 0$. So we restrict our attention to even-indexed H_{2n} functions and decide on values for $H_n(t,s;h)$ for $|t-s| < 2n$. We extend first by allowing $H_{2n}(t,s;h)$ to be extended using Theorem 3.9.6 for $t-s \in \{nh+h-s, nh+2h-s, \ldots, 2hn-s\}$. We extend these values to the negative points via the formula $H_{2n}(-t,s;h) = H_{2n}(t,s;h)$. We must pick the remaining values $H_{2n}(s,s;h), H_{2n}(s+h,s;h), \ldots, H_{2n}(s+nh,s;h)$ so that the following n equations hold:

$$\begin{cases} \langle H_{2n}, H_0\rangle_s = 0 \\ \langle H_{2n}, H_2\rangle_s = 0 \\ \vdots \\ \langle H_{2n}, H_{2n-2}\rangle_s = 0. \end{cases} \tag{74}$$

We now define the notations

$$(H_{2n}, f)_\ell = h \sum_{k=\ell}^{n-1} H_{2n}(hk,s;h)g(hk)\phi(hk,s;h),$$

$$\begin{cases} \alpha_0^{(1)}(t,s;h) \equiv 1 & \\ \alpha_m^{(1)}(t,s;h) = H_{2m}(t,s;h) & m = 1,2,\ldots,n-1 \\ \alpha_m^{(N+1)}(t,s;h) = \alpha_m^{(N)}(t,s;h) - \dfrac{\alpha_m^{(N)}(N-1,s;h)}{\alpha_{N-1}^{(N)}(N-1,s;h)} & m = 1,2,\ldots,n-1, \end{cases}$$

$$\begin{cases} \psi_m^{(1)} = 2\displaystyle\sum_{k=n}^{\infty} \frac{H_{2n}(s+hk,s;h)H_{2m}(s+hk,s;h)}{(k+1)!} & m = 0,1,\ldots,n-1 \\ \psi_m^{(N+1)} = \psi_m^{(N)} - \dfrac{\alpha_m^{(N)}(s+Nh,s;h)}{\alpha_{N-1}^{(N)}(s+Nh,s;h)}\psi_{N-1}^{(N)}, & \end{cases}$$

and for $\ell = 1, 2, \ldots, n$,

$$\xi(n-\ell) = \frac{\left[-\psi_{n-\ell}^{(n-\ell+1)} - \sum_{k=1}^{\ell-1} \xi(n-k)\alpha_{n-k-1}^{(n-k)}(s+h(n-k), s; h)\right]}{\alpha_{n-\ell}^{(n-\ell+1)}(s+h(n-\ell), s; h)}.$$

Lemma 3.9.1. *Suppose $H_0, H_2, \ldots, H_{2n-2}$ are known and defined on $h\mathbb{Z}$. The system* (74) *is equivalent to*

$$\begin{cases} H_{2n}(s,s;h) + 2(H_{2n}(\cdot,s;h), \alpha_0^{(1)}(\cdot,s;h))_1 & = -\psi_0^{(1)} \\ H_{2n}(s+h,s;h)\alpha_1^{(2)}(s+h,s;h) + (H_{2n}(\cdot,s;h), \alpha_1^{(2)}(\cdot,s;h))_2 & = -\psi_1^{(2)} \\ \vdots & \\ H_{2n}(s+h(n-1),s;h)\alpha_{n-1}^{(n)}(s+h(n-1),s;h) & = -\psi_{n-1}^{(n)}. \end{cases}$$

Proof. First rewrite (74) as

$$\begin{cases} H_{2n}(s,s;h) + 2(H_{2n}(\cdot,s;h), \alpha_0^{(1)}(\cdot,s;h))_1 = -\psi_0^{(1)} \\ H_{2n}(s,s;h)H_{2m}(s,s;h) + 2(H_{2n}(\cdot,s;h), \alpha_m^{(1)}(\cdot,s;h))_1 = -\psi_m^{(1)}; m = 1, \ldots, n-1. \end{cases}$$

Consider the row operations

$$r_m^{\text{new}} = r_m - \frac{H_{2m}(s,s;h)}{1} r_1 = r_m - \frac{\alpha_m^{(1)}(s,s;h)}{\alpha_0^{(1)}(s,s;h)} r_1$$

applied for $m = 1, \ldots, n-1$. Compute

$$\begin{aligned} 2(H_{2n}(\cdot,s;h), \alpha_m^{(1)}(\cdot,s;h))_1 - \frac{\alpha_m^{(1)}(s,s;h)}{\alpha_0^{(1)}(s,s;h)} 2(H_{2n}(\cdot,s;h), \alpha_0^{(1)}(\cdot,s;h))_1 \\ = 2\left(H_{2n}(\cdot,s;h), \alpha_m^{(1)}(\cdot,s;h) - \frac{\alpha_m^{(1)}(s,s;h)}{\alpha_0^{(1)}(s,s;h)} \alpha_0^{(1)}(\cdot,s;h)\right) \\ = 2(H_{2n}(\cdot,s;h), \alpha_m^{(2)}(\cdot,s;h))_1, \end{aligned}$$

and the right-hand-side becomes $\psi_m^{(2)}$. Therefore the row operation yields

$$\begin{cases} H_{2n}(s,s;h) + 2(H_{2n}(\cdot,s;h),\alpha_0^{(1)}(\cdot,s;h))_1 = -\psi_0^{(1)} \\ H_{2n}(s+h,s;h)\alpha_1^{(2)}(s+h,s;h) + (H_{2n}(\cdot,s;h),\alpha_1^{(2)}(\cdot,s;h))_2 = -\psi_1^{(2)} \\ H_{2n}(s,s;h)\alpha_1^{(2)}(s+h,s;h) + 2(H_{2n}(\cdot,s;h),\alpha_m^{(1)}(\cdot,s;h))_2 = -\psi_m^{(2)} \end{cases}$$

for $m = 2,\ldots,n-1$. Now we apply the row operations

$$r_m^{\text{new}} = r_m - \frac{\alpha_m^{(\eta)}(s+(\eta-1)h,s;h)}{\alpha_1^{(\eta)}(s+(\eta-1)h,s;h)} r_\eta$$

for $\eta = 2,\ldots,n-1$ and each $m = \eta,\ldots,n-1$. Then the result follows. □

Theorem 3.9.8. *If we define*

$$H_{2n}(s+(n-1)h,s;h) = \xi(n-1),$$

$$H_{2n}(s+(n-2)h,s;h) = \xi(n-2),$$

$$\vdots$$

$$H_{2n}(s,s;h) = \xi(0),$$

then the resulting Hermite functions obey (74).

Proof. From Lemma 3.9.1, we have

$$\begin{cases} H_{2n}(s,s;h) + 2(H_{2n}(\cdot,s;h),\alpha_0^{(1)}(\cdot,s;h))_1 & = -\psi_0^{(1)} \\ H_{2n}(s+h,s;h)\alpha_1^{(2)}(s+h,s;h) + (H_{2n}(\cdot,s;h),\alpha_1^{(2)}(\cdot,s;h))_2 & = -\psi_1^{(2)} \\ \vdots & \\ H_{2n}(s+h(n-1),s;h)\alpha_{n-1}^{(n)}(s+h(n-1),s;h) & = -\psi_{n-1}^{(n)}. \end{cases}$$

So starting from the bottom, we see we must choose

$$H_{2n}(s+h(n-1),s;h) = \frac{-\psi_{n-1}^{(n)}}{\alpha_{n-1}^{(n)}(s+h(n-1),s;h)},$$

which equals $\xi(n-\ell)$ with $\ell = 1$. The other terms follow similarly. $\square$

3.9.3. Laguerre. We define the Laguerre h-difference equation by

$$(t-s)\Delta_h^2 y(t-h;h) + (\alpha+1)\Delta_h y(t;h) - (t-s)\Delta y(t-h;h) + ny(t;h) = 0. \quad (75)$$

We define the discrete associated Laguerre polynomials by the formula

$$L_n^{(\alpha)}(t,s;h) = \sum_{k=0}^{n} \frac{(-1)^k (t-s)_h^k}{k!} \binom{n+\alpha}{n-k}. \quad (76)$$

The first few of these functions are given by

$$L_0^{(\alpha)}(t,s;h) = 1,$$

$$L_1^{(\alpha)}(t,s;h) = -(t-s) + (1+\alpha),$$

$$L_2^{(\alpha)}(t,s;h) = \frac{1}{2!}[(t-s)^2 - (t-s)(4+2\alpha+h) + (\alpha^2+3\alpha+2)],$$

$$\vdots$$

We now justify the names of these functions.

Theorem 3.9.9. *The functions $L_n^{(\alpha)}(t,s;h)$ solve* (75)*.*

Proof. Compute

$$\begin{aligned}\Delta_h L_n^{(\alpha)}(t,s;h) &= \sum_{k=1}^{n} \frac{(-1)^k (t-s)_h^{k-1}}{(k-1)!} \binom{n+\alpha}{n-k} \\ &= -\sum_{k=0}^{n-1} \frac{(-1)^k (t-s)_h^k}{k!} \binom{n+\alpha}{n-k-1}\end{aligned}$$

and

$$\begin{aligned}\Delta_h^2 L_n^{(\alpha)}(t,s;h) &= \sum_{k=2}^{n} \frac{(-1)^k (t-s)_h^{k-1}}{(k-1)!}\binom{n+\alpha}{n-k} \\ &= -\sum_{k=1}^{n-1} \frac{(-1)^k (t-s)_h^{k-1}}{(k-1)!}\binom{n+\alpha}{n-k-1}.\end{aligned}$$

Hence by Lemma 3.2.1

$$(t-s)\Delta_h^2 L_n^{(\alpha)}(t-h,s;h) = -(n+\alpha)!\sum_{k=1}^{n-1} \frac{(-1)^k (t-s)_h^k k}{k!(n-k-1)!(\alpha+k+1)}.$$

Now compute

$$\begin{aligned}(\alpha+1)\Delta_h L_n^{(\alpha)}(t,s;h) &= -(\alpha+1)\sum_{k=0}^{n-1} \frac{(-1)^k (t-s)_h^k}{k!}\binom{n+\alpha}{n-k-1} \\ &= -\frac{(\alpha+1)(n+\alpha)!}{(n-1)!(\alpha+1)!} - \sum_{k=1}^{n-1} \frac{(\alpha+1)(n+\alpha)!(-1)^k (t-s)_h^k}{k!(n-k-1)!(\alpha+k+1)!} \\ &= -\frac{(n+\alpha)!}{(n-1)!\alpha!} - \sum_{k=1}^{n-1} \frac{(\alpha+1)(n+\alpha)!(-1)^k (t-s)_h^k}{k!(n-k-1)!(\alpha+k+1)!},\end{aligned}$$

and again using Lemma 3.2.1,

$$\begin{aligned}-(t-s)\Delta_h L_n^{(\alpha)}(t-h,s;h) &= -\sum_{k=1}^{n} \frac{(-1)^k (t-s)_h^k}{k!} k\binom{n+\alpha}{n-k} \\ &= -\frac{(-1)^n (t-s)_h^n}{(n-1)!} - \sum_{k=1}^{n-1} \frac{(n+\alpha)!(-1)^k (t-s)_h^k k}{k!(n-k)!(\alpha+k)!}.\end{aligned}$$

Finally compute

$$\begin{aligned}nL_n^{(\alpha)}(t,s;h) &= n\sum_{k=0}^{n} \frac{(-1)^k (t-s)_h^k}{k!}\binom{n+\alpha}{n-k} \\ &= n\binom{n+\alpha}{n} + \frac{n(-1)^n (t-s)_h^n}{n!} + \sum_{k=1}^{n-1} \frac{n(n+\alpha)!(-1)^k (t-s)_h^k}{k!(n-k)!(\alpha+k)!}\end{aligned}$$

$$= \frac{(n+\alpha)!}{(n-1)!\Gamma(\alpha+1)} + \frac{(-1)^n(t-s)_h^n}{(n-1)!} + \sum_{k=1}^{n-1} \frac{n(n+\alpha)!(-1)^k(t-s)_h^k}{k!(n-k)!(\alpha+k)!}.$$

Combine these formulas together and compute

$$(t-s)\Delta_h^2 L_n^{(\alpha)}(t-h,s;h) + (\alpha+1)\Delta_h L_n(t,s;h) - (t-s)\Delta_h L_n^{(\alpha)}(t-h,s;h)$$
$$+nL_n^{(\alpha)}(t,s;h) = 0,$$

proving the claim. □

Theorem 3.9.10. *The following formula holds:*

$$(n+1)L_n^{(\alpha)}(t,s;h) = (2n+\alpha+1)L_n^{(\alpha)}(t,s;h) - (t-s)L_n^{(\alpha)}(t-h,s;h) - (n+\alpha)L_{n-1}^{(\alpha)}(t,s;h).$$

Proof. Compute

$$\begin{aligned}(2n+\alpha+1)L_n^{(\alpha)}(t,s;h) &= (2n+\alpha+1)\left[\sum_{k=0}^{n} \frac{(-1)^k(t-s)_h^k}{k!}\binom{n+\alpha}{n-k}\right] \\ &= (2n+\alpha+1)\left[\binom{n+\alpha}{n}\right. \\ &\quad \left. + \frac{(-1)^n(t-s)_h^n}{n!} + \sum_{k=1}^{n-1} \frac{(-1)^k(t-s)_h^k}{k!}\binom{n+\alpha}{n-k}\right],\end{aligned}$$

$$\begin{aligned}-tL_n^{(\alpha)}(t-h,s;h) &= -\sum_{k=0}^{n} \frac{(-1)^k(t-s)_h^{k+1}}{k!}\binom{n+\alpha}{n-k} \\ &= \sum_{k=1}^{n+1} \frac{(-1)^k(t-s)_h^k}{k!}k\binom{n+\alpha}{n-k+1} \\ &= \frac{(-1)^{n+1}(t-s)_h^{n+1}}{(n+1)!}(n+1) + \frac{(-1)^n(t-s)_h^n}{n!}n\binom{n+\alpha}{1} \\ &\quad + \sum_{k=1}^{n-1} \frac{(-1)^k(t-s)_h^k}{k!}k\binom{n+\alpha}{n-k+1},\end{aligned}$$

and

$$-(n+\alpha)L_{n-1}^{(\alpha)}(t,s;h) = -(n+\alpha)\sum_{k=0}^{n-1}\frac{(-1)^k(t-s)_h^k}{k!}\binom{n-1+\alpha}{n-1-k}$$
$$= -(n+\alpha)\left[\binom{n-1+\alpha}{n-1} + \sum_{k=1}^{n-1}\frac{(-1)^k(t-s)_h^k}{k!}\binom{n-1+\alpha}{n-1-k}\right].$$

Adding these formulas together yields

$$(2n+\alpha+1)L_n^{(\alpha)}(t,s;h) - (t-s)L_n^{(\alpha)}(t,s;h) - (n+\alpha)L_{n-1}^{(\alpha)}(t,s;h)$$
$$= A_{n,\alpha} + B_{n,\alpha}(t-s)_h^n + C_n(t-s)_h^{n+1}\alpha(t) + \textstyle\sum_{k=1}^{n-1}\frac{(-1)^k(t-s)_h^k}{k!}D_{n,k,\alpha}, \tag{77}$$

where

$$A_{n,\alpha} = (2n+\alpha+1)\binom{n+\alpha}{n} - (n+\alpha)\binom{n-1+\alpha}{n-1},$$

$$B_{n,\alpha} = (2n+\alpha+1)\frac{(-1)^n}{n!} + \frac{(-1)^n}{n!}n(n+\alpha),$$

$$C_n = \frac{(-1)^{n+1}}{(n+1)!}(n+1),$$

and

$$D_{n,k,\alpha} = (2n+\alpha+1)\binom{n+\alpha}{n-k} + k\binom{n+\alpha}{n-k+1} - (n+\alpha)\binom{n-1+\alpha}{n-1-k}.$$

Now examine

$$A_{n,\alpha} = (2n+\alpha+1)\binom{n+\alpha}{n} - (n+\alpha)\binom{n-1+\alpha}{n-1}$$
$$= \frac{(2n+\alpha+1)\Gamma(n+\alpha+1)}{n!\Gamma(\alpha+1)} - \frac{(n+\alpha)\Gamma(n+\alpha)}{(n-1)!\Gamma(\alpha+1)}$$
$$= \frac{\Gamma(n+\alpha+1)}{(n-1)!\Gamma(\alpha+1)}\left[\frac{2n+\alpha+1}{n} - 1\right]$$
$$= \frac{\Gamma(n+\alpha+1)}{(n-1)!\Gamma(\alpha+1)}\frac{n+\alpha+1}{n}$$

$$= (n+1)\binom{n+\alpha+1}{n+1},$$

$$\begin{aligned}
B_{n,\alpha} &= (2n+\alpha+1)\frac{(-1)^n}{n!} + \frac{(-1)^n}{n!}n(n+\alpha) \\
&= \frac{(-1)^n}{n!}(2n+\alpha+1+n^2+n\alpha) \\
&= (n+1)\frac{(-1)^n(t-s)_h^n}{n!}(n+\alpha+1),
\end{aligned}$$

and

$$\begin{aligned}
D_{n,k,\alpha} &= (2n+\alpha+1)\binom{n+\alpha}{n-k} + k\binom{n+\alpha}{n-k+1} - (n+\alpha)\binom{n-1+\alpha}{n-1-k} \\
&= \frac{(2n+\alpha+1)\Gamma(n+\alpha+1)}{(n-k)!\Gamma(\alpha+k+1)} + \frac{k\Gamma(n+\alpha+1)}{(n-k+1)!\Gamma(\alpha+k)} - \frac{\Gamma(n+\alpha+1)\Gamma(n+\alpha)}{(n-1-k)!\Gamma(\alpha+k+1)} \\
&= \frac{\Gamma(n+\alpha+1)}{(n-k-1)!\Gamma(\alpha+k)}\left[\frac{1}{(n-k)(n-k+1)(\alpha+k)}\right] \\
&\times [(2n+\alpha+1)(n-k+1) + k(\alpha+k) - (n-k)(n-k+1)] \\
&= \frac{\Gamma(n+\alpha+1)}{(n-k+1)!\Gamma(\alpha+k+1)} \\
&\times [2n^2 - 2nk + 2n + n\alpha - \alpha k + \alpha + n - k + 1 + \alpha k + k^2 + kn - k^2 + k \\
&- n^2 + kn - n] \\
&= \frac{\Gamma(n+\alpha+1)}{(n-k+1)!\Gamma(\alpha+k+1)}(n^2+2n+\alpha n+\alpha+1) \\
&= \frac{\Gamma(n+\alpha+1)}{(n-k+1)!\Gamma(\alpha+k+1)}(n+1)(n+\alpha+1) \\
&= (n+1)\frac{\Gamma(n+\alpha+2)}{(n-k+1)!\Gamma(\alpha+k+1)}.
\end{aligned}$$

Therefore (77) implies

$$\begin{aligned}
&(2n+\alpha+1)L_n^{(\alpha)}(t,s;h) - tL_n^{(\alpha)}(t-h,s;h) - (n+\alpha)L_{n-1}^{(\alpha)}(t,s;h) \\
&= (n+1)\left[\binom{n+\alpha+1}{n+1} + \frac{(-1)^n(t-s)_h^n}{n!}(n+\alpha+1) + \frac{(-1)^{n+1}(t-s)_h^{n+1}}{(n+1)!}\right. \\
&\qquad\left. + \sum_{k=1}^{n-1}\frac{(-1)^k(t-s)_h^k}{k!}\binom{n+1+\alpha}{n+1-k}\right]
\end{aligned}$$

$$= (n+1)L_{n+1}^{(\alpha)}(t,s;h),$$

as was to be shown. □

Theorem 3.9.11. *The following formula holds:*

$$(t-s)\Delta_h L_n^{(\alpha)}(t-h,s;h) = nL_n^{(\alpha)}(t,s;h) - (n+\alpha)L_{n-1}^{(\alpha)}(t,s;h).$$

Proof. Apply Lemma 3.2.1 to the left-hand side and calculate

$$\begin{aligned}
(t-s)\Delta_h L_n^{(\alpha)}(t-h,s;h) &= (t-s)\Delta_h \left[\sum_{k=0}^{n} \frac{(-1)^k (t-h-s)^k}{k!}\binom{n+\alpha}{n-k}\right] \\
&= (t-s)\left[\sum_{k=1}^{n} \frac{(-1)^k (t-s-h)^{k-1}}{(k-1)!}\binom{n+\alpha}{n-k}\right] \\
&= \sum_{k=1}^{n} \frac{(-1)^k (t-s)_h^k}{(k-1)!}\binom{n+\alpha}{n-k} \\
&= \frac{(-1)^n (t-s)_h^n}{(n-1)!} + \sum_{k=1}^{n-1} \frac{(-1)^k (t-s)_h^k}{(k-1)!}\binom{n+\alpha}{n-k}.
\end{aligned}$$

Now expand to right-hand side to get

$$\begin{aligned}
&n\sum_{k=0}^{n} \frac{(-1)^k (t-s)_h^k}{k!}\binom{n+\alpha}{n-k} - (n+\alpha)\sum_{k=0}^{n-1} \frac{(-1)^k (t-s)_h^k}{k!}\binom{n-1+\alpha}{n-1-k} \\
&= \frac{(-1)^n (t-s)_h^n}{(n-1)!} + \sum_{k=1}^{n-1} \frac{(-1)^k (t-s)_h^k}{k!}\left[n\binom{n+\alpha}{n-k} - (n+\alpha)\binom{n-1+\alpha}{n-1-k}\right] \\
&= \frac{(-1)^n (t-s)_h^n}{(n-1)!} + \sum_{k=1}^{n-1} \frac{(-1)^k (t-s)_h^k}{k!}\left[\frac{\Gamma(n+\alpha+1)k}{(n-k)!\Gamma(\alpha+k+1)}\right] \\
&= \frac{(-1)^n (t-s)_h^n}{(n-1)!} + \sum_{k=1}^{n-1} \frac{(-1)^k (t-s)_h^n}{(k-1)!}\binom{n+\alpha}{n-k},
\end{aligned}$$

proving the claim. □

The following corollary follows from expanding the derivative on the left-hand-side of the formula in Theorem 3.9.11.

Corollary 3.9.1. *The following formula holds:*

$$-(t-s)L_n^{(\alpha)}(t-h,s;h) = (hn-(t-s))L_n^{(\alpha)}(t,s;h) - h(n+\alpha)L_{n-1}^{(\alpha)}(t,s;h).$$

Theorem 3.9.12. *The following formula holds:*

$$\Delta_h L_n^{(\alpha)}(t,s;h) = -L_{n-1}^{(\alpha+1)}(t,s;h).$$

Proof. Compute

$$\begin{aligned}
\Delta_h L_n^{(\alpha)}(t,s;h) &= \sum_{k=1}^{n} \frac{(-1)^k (t-s)_h^{k-1}}{(k-1)!} \binom{n+\alpha}{n-k} \\
&= -\sum_{k=0}^{n-1} \frac{(-1)^k (t-s)_h^{k}}{k!} \binom{n+\alpha}{n-k-1} \\
&= -L_{n-1}^{(\alpha+1)}(t,s;h),
\end{aligned}$$

as was to be shown. □

Theorem 3.9.13 (Three-term recurrence)**.** *The following formula holds:*

$$(n+1)L_{n+1}^{(\alpha)}(t,s;h) = [(2+h)n+\alpha+1-(t-s)]L_n^{(\alpha)}(t,s;h) - (h+1)(n+\alpha)L_{n-1}^{(\alpha)}(t,s;h).$$

Proof. Apply Corollary 3.9.1 to Theorem 3.9.10 and the result is immediate. □

3.10. DISCRETE BESSEL FUNCTIONS

We define the general discrete Bessel function $J_\nu(\cdot, s, \xi, \alpha, \gamma; h)$ by the formula

$$J_\nu(t, s, \xi, \alpha, \gamma; h) = \sum_{k=0}^{\infty} \frac{(-1)^k \xi^{2k+\nu}}{\Gamma(k+\nu+1)2^{2k+\nu}} \frac{(t-s)_h^{\gamma(2k+\nu)+\alpha}}{k!}. \tag{78}$$

We have visualized some of these functions in Figure 3.7. This function is meant to be a discrete analogue of the function $t^\alpha J_\nu(\xi t^\gamma)$ that solves (26). We now justify our claim that this is the analogue of $t^\alpha J_\nu(\xi t^\gamma)$.

Theorem 3.10.1. *Let $\gamma \in \mathbb{Z}$. The function $J_\nu(\cdot, s, \xi, \alpha, \gamma; h)$ solves the h-difference equation*

$$\begin{aligned}(t-s)_h^2 \Delta_h^2 y(t-2h) + (1-2\alpha)(t-s)_h \Delta_h y(t-h) + \xi^2\gamma^2(t-s)_h^{2\gamma} y(t-2h\gamma) \\ + (\alpha^2 - \nu^2\gamma^2) y(t) = 0, \end{aligned} \tag{79}$$

which we call the general discrete Bessel h-difference equation.

Proof. Let $\psi(t) = J_\nu(t, s, \xi, \alpha, \gamma; h)$. Use Lemma 3.2.1 to compute

$$(t-s)_h^2 \Delta_h^2 \psi(t-2h) = \sum_{k=0}^{\infty} \frac{(-1)^k \xi^{2k+\nu} (\gamma(2k+\nu)+\alpha)(\gamma(2k+\nu)+\alpha-1)(t-s)_h^{\gamma(2k+\nu)+\alpha}}{\Gamma(k+\nu+1)2^{2k+\nu}k!},$$

compute

$$(t-s)_h \Delta_h \psi(t-h) = \sum_{k=0}^{\infty} \frac{(-1)^k \xi^{2k+\nu} (\gamma(2k+\nu)+\alpha)(t-s)_h^{\gamma(2k+\nu)+\alpha}}{\Gamma(k+\nu+1)2^{2k+\nu}k!},$$

and finally

$$(t-s)_h^{2\gamma} \psi(t-2h\gamma) = -\sum_{k=1}^{\infty} \frac{4k(k+\nu)(-1)^k \xi^{2k+\nu} (t-s)_h^{\gamma(2k+\nu)+\alpha}}{\Gamma(k+\nu+1)2^{2k+\nu}k!\xi^2}$$

$$= -\sum_{k=0}^{\infty} \frac{4k(k+\nu)(-1)^k \xi^{2k+\nu}(t-s)_h^{\gamma(2k+\nu)+\alpha}}{\Gamma(k+\nu+1)2^{2k+\nu}k!\xi^2}.$$

Plugging these expressions into the left-hand side of the general discrete Bessel h-difference equation yields zero after routine calculations. □

The direct discrete analogue of the classical Bessel function J_ν from (24) is

$$J_\nu(t,s;h) = J_\nu(t,s,1,0,1;h). \tag{80}$$

The following corollary defines the discrete analogue of the Bessel differential equation (25).

Corollary 3.10.1. *The function $J_\nu(t,s;h)$ solves the discrete Bessel h-difference equation*

$$(t-s)_h^2\Delta_h^2 y(t-2h) + (t-s)_h\Delta_h y(t-h) + (t-s)_h^2 y(t-2h) - \nu^2 y(t) = 0. \tag{81}$$

The notation $(t-s)_h^{\gamma(2k+\nu)+\alpha}$ in (78) suggests the exponent must be a positive integer. Otherwise we will appeal to Theorem 3.2.5 and in recognition that the standard branch cut in classical complex analysis is $(-\infty, 0)$ use the falling complexification in such cases (without explicitly noting it). All proofs will be written with this complexification in mind.

First we will show it is easy to derive the Laplace transform of $J_0(t,s;h)$.

Theorem 3.10.2. *The following formula holds:*

$$\mathscr{L}_h\{J_0(\cdot,s;h)\}(z;s) = \frac{1}{\sqrt{z^2+1}}.$$

Proof. From (81), the equation that J_0 solves is

$$(t-s)_h^2\Delta_h^2 J_0(t-2h,s;h)+(t-s)_h\Delta_h J_0(t-h,s;h)+(t-s)_h^2 J_0(t-2h,s;h)=0.$$

Dividing by $(t-s)_h$ yields

$$(t-s-h)_h\Delta_h^2 J_0(t-2h,s;h)+\Delta_h J_0(t-h,s;h)+(t-s-h)_h J_0(t-2h,s;h)=0,$$

and replace t by $t+h$ yielding

$$(t-s)_h\Delta_h^2 J_0(t-h,s;h)+\Delta_h J_0(t,s;h)+(t-s)_h J_0(t-h,s;h)=0.$$

Note by direct computation that $J_0(s,s;h)=1$. We apply Theorem 3.6.1 to see that $\mathscr{L}\{J_0(\cdot,s;h)\}$ obeys

$$-z\mathscr{L}\{J_0\}(z;s)-(z^2+1)\mathscr{L}\{J_0\}'(z;s)=0,$$

a first-order differential equation solvable by separation of variables whose solution is $\mathscr{L}\{J_0\}(z;s)=\dfrac{1}{\sqrt{z^2+1}}$, as was to be shown. □

Theorem 3.10.3. *The following formula holds:*

$$\mathscr{L}_h\{J_\nu\}(z)=\frac{[\sqrt{z^2+1}+z]^{-\nu}}{\sqrt{z^2+1}}.$$

Proof. Applying Theorem 3.6.1 to (81), yields

$$(\mathscr{L}_h\{\Delta^2 J_\nu\})''(z)-(\tilde{\mathscr{Z}}\{\Delta J_\nu\})'(z)+(\mathscr{L}_h\{J_\nu\})''(z)-\nu^2\mathscr{L}_h\{J_\nu\}(z)=0.$$

Now we apply (62) to get

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2}\left[z^2\mathscr{L}_h\{J_\nu\}(z) - zJ_\nu(0) - \Delta J_\nu(0)\right] \quad -\frac{\mathrm{d}}{\mathrm{d}z}\left[z\mathscr{L}_h\{J_\nu\}(z) - J_\nu(0)\right] \\ +\frac{\mathrm{d}^2}{\mathrm{d}z^2}\tilde{\mathscr{Z}}\{J_\nu\}(z) - \nu^2\mathscr{L}_h\{J_\nu\}(z) = 0.$$

We compute the derivatives and simplify to obtain

$$(z^2+1)(\mathscr{L}_h\{J_n\})''(z) + 3z(\mathscr{L}_h\{J_n\})'(z) + (1-n^2)\mathscr{L}_h\{J_n\}(z).$$

This is simply (29) with $y(z) = \mathscr{L}_h\{J_n\}(z)$. Therefore

$$\mathscr{L}_h\{y\}(z) = \frac{[\sqrt{z^2+1}+z]^{-n}}{\sqrt{z^2+1}},$$

as claimed. □

Corollary 3.10.2. *The following formula holds:*

$$(J_0 * J_0)(t,s;h) = \sin_1(t,s;h).$$

Proof. By Theorem 3.10.2 and the convolution theorem (65), we see that

$$\mathscr{L}_h\{J_0 * J_0\}(z;s) = \left(\frac{1}{\sqrt{z^2+1}}\right)^2 = \frac{1}{z^2+1},$$

while Table 3.1 shows that

$$\mathscr{L}_h\{\sin_1(t,s;h)\}(z;s) = \frac{1}{z^2+1},$$

and so the uniqueness of the inverse transform proves the claim. □

We shall derive an analogue of (30). Our proof is inspired by [39, page 22

(E1.3.1)].

Theorem 3.10.4. *The following formula holds:*

$$\mathscr{L}_h \left\{ J_\nu \left(t, s, 2\sqrt{\alpha}, \frac{\nu}{2}, \frac{1}{2}; h \right) \right\} (z; s) = \frac{\alpha^{\frac{\nu}{2}}}{z^{\nu+1} e^{\frac{\alpha}{2}}}.$$

Proof. First note that

$$\begin{aligned} J_\nu \left(t, s, 2\sqrt{\alpha}, \frac{\nu}{2}, \frac{1}{2}; h \right) &= \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k+\nu} (\sqrt{\alpha})^{2k+\nu}}{\Gamma(k+\nu+1) 2^{2k+\nu} k!} (t-s)_h^{\frac{1}{2}(2k+\nu)+\frac{\nu}{2}} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k \alpha^{k+\frac{\nu}{2}}}{\Gamma(k+\nu+1) k!} (t-s)_h^{k+\nu}, \end{aligned}$$

and so since $\mathscr{L}_h\{(t-s)_h^{k+\nu}\}(z;s) = \dfrac{\Gamma(k+\nu+1)}{z^{k+\nu+1}}$, by appealing to (9), we see

$$\mathscr{L}_h \left\{ J_\nu \left(t, s, 2\sqrt{\alpha}, \frac{\nu}{2}, \frac{1}{2}; h \right) \right\} (z; s) = \frac{\alpha^{\frac{\nu}{2}}}{z^{\nu+1}} \sum_{k=0}^{\infty} \left(\frac{-\alpha}{z} \right)^k \frac{1}{k!} = \frac{\alpha^{\frac{\nu}{2}}}{z^{\nu+1} e^{\frac{\alpha}{z}}},$$

as was to be shown. $\square$

Theorem 3.10.5. *For $n \in \mathbb{Z}$, we have $J_{-n}(t,s;h) = (-1)^n J_n(t,s;h)$.*

Proof. Recall that for negative $m \in \mathbb{Z}$, $\dfrac{1}{\Gamma(m+1)} = 0$. Now compute

$$\begin{aligned} J_{-n}(t,s;h) &= \sum_{k=0}^{\infty} \frac{(-1)^k (t-s)_h^{2k-n}}{k! \Gamma(k-n+1) 2^{2k-n}} \\ &= (-1)^n \sum_{k=-n}^{\infty} \frac{(-1)^k (t-s)_h^{2k+n}}{(k+n)! \Gamma(k+1) 2^{2k+n}} \\ &= (-1)^n \sum_{k=0}^{\infty} \frac{(-1)^k (t-s)_h^{2k+n}}{k! \Gamma(k+n+1) 2^{2k+n}} \\ &= (-1)^n J_n(t,s;h), \end{aligned}$$

as was to be shown. $\square$

Theorem 3.10.6. *For all ν,*

$$(t-s)\Delta_h J_\nu(t-h,s;h) = \nu J_\nu(t,s;h) - (t-s)J_{\nu+1}(t-h,s;h).$$

Proof. Using Lemma 3.2.1, we compute

$$\begin{aligned}
(t-s)\Delta_h J_\nu(t-h,s;h) &= \sum_{k=0}^{\infty} \frac{(-1)^k(2k+\nu)(t-s)_h^{2k+\nu}}{k!\Gamma(k+\nu+1)2^{2k+\nu}} \\
&= \nu J_\nu(t,s;h) + (t-s)\sum_{k=0}^{\infty} \frac{(-1)^k(t-s-h)_h^{2k+\nu-1}}{\Gamma(k)\Gamma(k+\nu+1)2^{2k+\nu-1}} \\
&= \nu J_\nu(t,s;h) + (t-s)\sum_{k=1}^{\infty} \frac{(-1)^k(t-s-h)_h^{2k+\nu-1}}{\Gamma(k)\Gamma(k+\nu+1)2^{2k+\nu-1}} \\
&= \nu J_\nu(t,s;h) - (t-s)\sum_{k=0}^{\infty} \frac{(-1)^k(t-s-h)_h^{2k+\nu+1}}{k!\Gamma(k+(\nu+1)+1)2^{2k+\nu+1}} \\
&= \nu J_\nu(t,s;h) - (t-s)J_{\nu+1}(t-h,s;h),
\end{aligned}$$

as was to be shown. $\square$

Theorem 3.10.7. *For all ν,*

$$(t-s)\Delta_h J_\nu(t-h,s;h) = -\nu J_\nu(t,s;h) + (t-s)J_{\nu-1}(t-h,s;h).$$

Proof. Using Lemma 3.2.1, we compute

$$\begin{aligned}
(t-s)\Delta_h J_\nu(t-s-h;h) &= \sum_{k=0}^{\infty} \frac{(-1)^k(2k+\nu)(t-s)_h^{2k+\nu}}{k!\Gamma(k+\nu+1)2^{2k+\nu}} \\
&= -\nu J_\nu(t,s;h) + \sum_{k=0}^{\infty} \frac{(-1)^k(2k+2\nu)(t-s)_h^{2k+\nu}}{k!\Gamma(k+\nu+1)2^{2k+\nu}}
\end{aligned}$$

$$= -\nu J_\nu(t,s;h) + t\sum_{k=0}^{\infty} \frac{(-1)^k (t-h)^{2k+\nu-1}(t-s)_h^{2k+(\nu-1)}}{k!\Gamma(k+(\nu-1)+1)2^{2k+(\nu-1)}}$$

$$= -\nu J_\nu(t,s;h) + (t-s)J_{\nu-1}(t-h,s;h),$$

as was to be shown. □

The formulas in the following corollary come from subtracting and adding the formulas from Theorem 3.10.6 and Theorem 3.10.7.

Corollary 3.10.3. *For all ν, the following formulas hold:*

$$2\nu J_\nu(t,s;h) = tJ_{\nu-1}(t-h,s;h) + tJ_{\nu+1}(t-h,s;h)$$

and

$$2\Delta_h J_\nu(t,s;h) = J_{\nu-1}(t,s;h) - J_{\nu+1}(t,s;h).$$

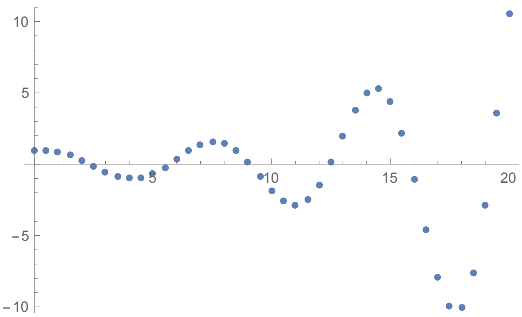
Theorem 3.10.8. *The following formula holds:*

$$\Delta_h \left[(t-s)_h^\nu J_\nu(t-\nu,s;h)\right] = (t-s)_h^\nu J_{\nu-1}(t-\nu,s;h).$$

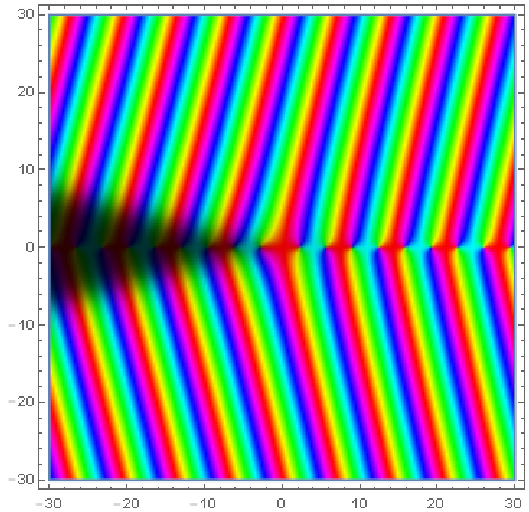
Proof. Using Lemma 3.2.1, we compute

$$\begin{aligned}\Delta_h \left[(t-s)_h^\nu J_\nu(t-\nu,s;h)\right] &= \Delta_h \left[\sum_{k=0}^{\infty} \frac{(-1)^k (t-s)_h^{2k+2\nu}}{k!\Gamma(k+\nu+1)2^{2k+\nu}}\right] \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k (t-s)_h^{2k+2\nu-1}}{k!\Gamma(k+(\nu-1)+1)2^{2k+(\nu-1)}} \\ &= (t-s)_h^\nu J_{\nu-1}(t-\nu,s;h),\end{aligned}$$

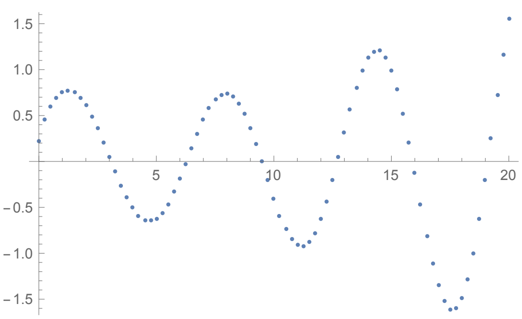
as was to be shown. □



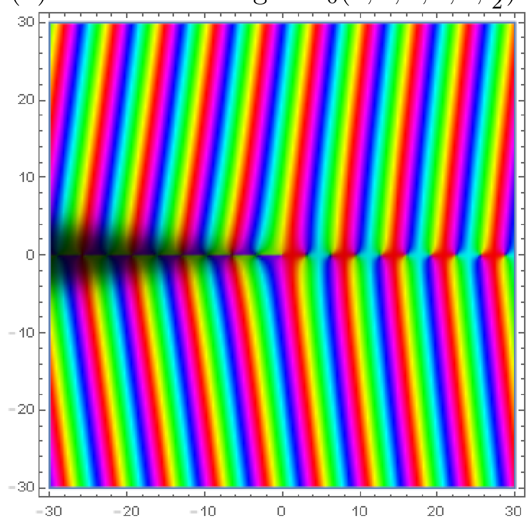
(a) Plot of $J_0(t,0,1,0,1;\frac{1}{2})$.



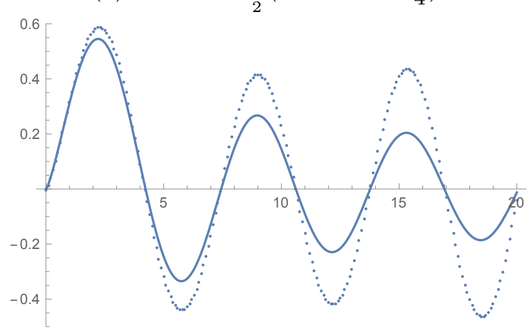
(b) Domain coloring of $J_0(z,0,1,0,1;\frac{1}{2})$.



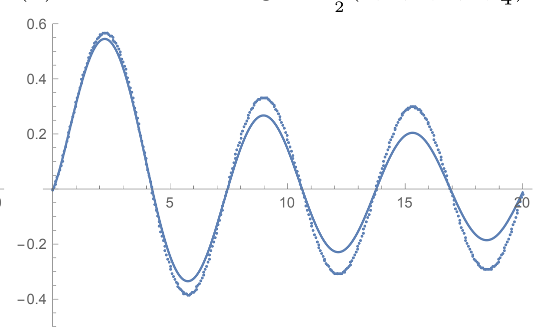
(c) Plot of $J_{\frac{1}{2}}(t,0,2,0,1;\frac{1}{4})$.



(d) Domain coloring of $J_{\frac{1}{2}}(z,0,2,0,1;\frac{1}{4})$.



(e) Plot of $J_{1.3}(t,0,1,0,1;0.1)$ and the classical Bessel $J_{1.3}(t)$.



(f) Plot of $J_{1.3}(t,0,1,0,1;0.05)$ and the classical Bessel $J_{1.3}(t)$.

Figure 3.7. Discrete Bessel functions.

Theorem 3.10.9. *The following formula is equivalent to* (81)*:*

$$[(t+2h-s)_h^2 + h^2(t+2h-s)_h^2]\Delta_h^2 y(t)$$
$$+[(t+2h-s) - h(t+2h-s)_h^2 - \nu^2]\Delta_h y(t+h) + [(t+2h-s)_h^2 - \nu^2]y(t+h) = 0.$$

Proof. First map $t \mapsto t+2h$ to turn (81) into

$$(t+2h-s)_h^2\Delta_h^2 y(t) + (t+2h-s)_h\Delta_h y(t+h) + (t+2h-s) - h^2 y(t) - \nu^2 y(t+2h) = 0.$$

The result follows from substituting in the formulas

$$y(t) = y(t+1) - h\Delta_h y(t+1) + h^2\Delta_h^2 y(t)$$

and

$$y(t+2h) = \Delta_h y(t+1) + y(t+1)$$

and then simplifying. $\square$

A second-order h-difference equation is said to be written in self-adjoint form if it is written as $\Delta(p(t)\Delta y(t)) + q(t)y(t+1) = 0$. The following theorem is called the Leighton–Wintner theorem [12, Theorem 4.64]. It says that if the following condition holds, then y is oscillatory (i.e., equalling zero or changing signs infinitely often): for some $a \in h\mathbb{Z}$,

$$h\sum_{k=\frac{a}{h}}^{\infty} \frac{1}{p(hk)} = h\sum_{k=\frac{a}{h}}^{\infty} q(hk) = \infty. \tag{82}$$

Lemma 3.10.1. *The function $J_0(t,s;1)$ is oscillatory.*

Proof. The difference equation of this function can be put into the form

$$\Delta[(t+1)\Delta y(t)] + (t+1)y(t) = 0,$$

which is self-adjoint form with $p(t) = t+1$ and $q(t) = t+1$. Indeed,

$$\sum_{k=0}^{\infty} \frac{1}{k+1} = \sum_{k=0}^{\infty} k+1 = \infty,$$

satisfying the condition (82), and so by the Leighton-Wintner theorem we may conclude that J_0 is oscillatory. □

Theorem 3.10.10. *For all $n \in \mathbb{Z}$, $J_n(t,s;1)$ is oscillatory.*

Proof. If $n = 0$, then Lemma 3.10.1 guarantees the result. If n is negative, we may use Theorem 3.10.5 to reduce the argument to checking $-n$. Let $n \in \mathbb{N}$. Let y solve the equation in Theorem 3.10.9 so that

$$\Delta^2 y(t) = \frac{t(t+2)+n^2}{2(t+1)(t+2)}\Delta y(t+1) - \frac{(t+1)(t+2)-n^2}{2(t+1)(t+2)}y(t+1). \tag{83}$$

Write $u(t) = v(t)y(t)$. It follows that

$$\Delta u(t) = y(t+1)\Delta v(t) + v(t)\Delta y(t)$$

and

$$\Delta^2 u(t) = y(t+1)\Delta^2 v(t) + (\Delta v(t+1) + \Delta v(t))\Delta y(t+1) + v(t)\Delta^2 y(t).$$

Using (83), we see

$$\begin{aligned}\Delta^2 u(t) &= \left[\frac{\Delta^2 v(t)}{v(t+1)} - \frac{v(t)}{v(t+1)}\frac{(t+1)(t+2)-n^2}{2(t+1)(t+2)}\right] u(t+1) \\ &+ \left[\Delta v(t+1) + \Delta v(t) + v(t)\frac{t(t+2)+n^2}{2(t+1)(t+2)}\right] \Delta y(t+1).\end{aligned} \tag{84}$$

Let $N > 2n^2 - 3$ and let $t \geq N$. Now define the function v to be the solution of the initial value problem

$$\Delta v(t+1) + \Delta v(t) + v(t)\frac{t(t+2)+n^2}{2(t+1)(t+2)} = 0, \quad v(N) = v(N+1) = 1,$$

i.e., for $t \geq N$,

$$v(t+2) = \frac{(t+2)^2 - n^2}{2(t+1)(t+2)} v(t), \quad v(N) = v(N+1) = 1. \tag{85}$$

Consequently this forces (84) to reduce to a self-adjoint difference equation

$$\Delta^2 u(t) + \left[\frac{v(t)}{v(t+1)}\frac{(t+1)(t+2)-n^2}{2(t+1)(t+2)} - \frac{\Delta^2 v(t)}{v(t+1)}\right] u(t+1) = 0,$$

where the functions p and q in the self-adjoint form obey the formulas $p(t) = 1$ and $q(t) = \frac{v(t)}{v(t+1)}\frac{(t+1)(t+2)-n^2}{2(t+1)(t+2)} - \frac{\Delta^2 v(t)}{v(t+1)}$. Using $\Delta^2 v(t) = v(t+2) - 2v(t+1) + v(t)$ and (85), we may compute

$$\begin{aligned}q(t) &= \frac{v(t)}{v(t+1)}\frac{(t+1)(t+2)-n^2}{2(t+1)(t+2)} - \frac{v(t+2) - 2v(t+1) + v(t)}{v(t+1)} \\ &= \frac{v(t)}{v(t+1)}\left[\frac{(t+1)(t+2)-n^2}{2(t+1)(t+2)} - 1\right] - \frac{(t+2)^2-n^2}{2(t+1)(t+2)}\frac{v(t)}{v(t+1)} + 2 \\ &= \frac{v(t)}{v(t+1)}\left[\frac{-(t+2)(t+1) - n^2 - (t+2)^2 + n^2}{2(t+1)(t+2)}\right] + 2 \\ &= 2 - \frac{v(t)}{v(t+1)}\frac{2t+3}{2t+2}.\end{aligned} \tag{86}$$

It follows from (85) that

$$\frac{v(t+2)}{v(t)} = \frac{(t+2)^2 - n^2}{2(t+1)(t+2)}, \quad v(n) = v(n+1) = 1,$$

and $v(t) > 0$ for all $t \geq n$. It is also clear that

$$\lim_{t\to\infty} \frac{v(t+2)}{v(t)} = \frac{1}{2}.$$

We have shown that $u = vy$ has self-adjoint form

$$\Delta(1 \cdot \Delta u(t)) + q(t)u(t+1) = 0.$$

Following (82), it is obvious that $\sum_{k=N}^{\infty} \frac{1}{1} = \infty$, so all we must demonstrate is that $\sum_{k=N}^{\infty} q(k) = \infty$. Assume that $\sum_{k=N}^{\infty} q(k)$ is convergent. Hence $\lim_{t\to\infty} q(t) = 0$. Rearrange (86) to get $\frac{v(t)}{v(t+2)} = (2 - q(t))\frac{2t+2}{2t+3}$. Now notice that

$$\lim_{t\to\infty} \frac{v(t)}{v(t+2)} = \lim_{t\to\infty} \frac{v(t)}{v(t+1)} \frac{v(t+1)}{v(t+2)} = \frac{1}{4}.$$

However,

$$\lim_{t\to\infty} (2 - q(t))\frac{2t+2}{2t+3} = 2,$$

which is a contradiction. Hence $\sum_{k=N}^{\infty} q(k)$ diverges. To complete the proof, we will argue that $q(t)$ is positive for all $t \geq N$ (and hence the sum diverges to ∞). Notice that

$$q(N) = 2 - \left(1 + \frac{1}{2N+2}\right) > 0$$

and the inequality $q(N+1) > 0$ is algebraically equivalent to $N > 2n^2 - 3$, the condition we imposed earlier. Hence $q(N+1) > 0$. Now note for $\ell \in \mathbb{N}$ that

$$q(N+2\ell) - q(N+2(\ell-1)) = \frac{v(N+2(\ell-1))}{2v(N+2(\ell-1)+1)} \frac{8\ell^2+8\ell N+8\ell+2n^2+4N+1}{(N+2\ell-1)[(N+2\ell+1)^2-n^2]} > 0$$

and

$$q(N+2\ell+1) - q(N+2(\ell-1)+1) = \frac{v(N+2\ell-1)(4n^2-1)}{2v(N+2\ell)(N+2\ell)[(N+2\ell+2)^2-n^2]} > 0.$$

This shows that the q function is essentially two interlaced increasing sequences that start at positive values. Hence $q(t)$ is positive for all $t \geq N$, and we may conclude that J_n is oscillatory. $\square$

3.11. DISCRETE HYPERGEOMETRIC SERIES

Following the notation defining the hypergeometric series (32), we define the discrete hypergeometric series by the formula

$$ {}_pF_q(\mathbf{a};\mathbf{b};t,s,n,\xi;h) = \sum_{k=0}^{\infty} \frac{\mathbf{a}^{\overline{k}}}{\mathbf{b}^{\overline{k}}} \xi^k \frac{(t-s)_h^{nk}}{k!}. \tag{87}$$

We have included representations of discrete special functions in terms of this function in Table 3.3.

3.11.1. Elementary Properties and Difference Equations.

Theorem 3.11.1. *The following formula holds:*

$$\Delta_h \left[{}_pF_q(\boldsymbol{a};\boldsymbol{b};t,s,1,\xi;h)\right] = \frac{a_1a_2\dots a_p}{b_1b_2\dots b_q} \xi {}_pF_q(\boldsymbol{a}+1;\boldsymbol{b}+1;t,s,1,\xi;h).$$

Proof. Compute

$$\begin{aligned}\Delta_h\left[{}_pF_q(\mathbf{a};\mathbf{b};t,s,1,\xi;h)\right] &= \sum_{k=1}^{\infty}\frac{\mathbf{a}^{\overline{k}}}{\mathbf{b}^{\overline{k}}}\xi^k\frac{k(t-s)_h^{k-1}}{k!}\\ &= \sum_{k=0}^{\infty}\frac{\mathbf{a}^{\overline{k+1}}}{\mathbf{b}^{\overline{k+1}}}\xi^{k+1}\frac{(t-s)_h^k}{k!}\\ &= \frac{a_1\dots a_p}{b_1\dots b_q}\xi\sum_{k=0}^{\infty}\frac{(\mathbf{a}+1)^{\overline{k}}}{(\mathbf{b}+1)^{\overline{k}}}\xi^k\frac{(t-s)_h^k}{k!},\end{aligned}$$

as was to be shown. □

Similarly, we may prove the following result.

Theorem 3.11.2. *The following formula holds for integers $n\in\mathbb{N}_0$:*

$$\Delta_h^n\left[{}_pF_q(\boldsymbol{a};\boldsymbol{b};t,s,1,\xi;h)\right] = \frac{\boldsymbol{a}^{\overline{n}}}{\boldsymbol{b}^{\overline{n}}}\xi^n{}_pF_q(\boldsymbol{a}+n;\boldsymbol{b}+n;t,s,1,\xi;h).$$

Let $f\colon h\mathbb{Z}\to\mathbb{C}$ be a function and define the function shift operator ϱ_h by $(\varrho_h f)(t)=f(t-h)$. Now define an analogue of (34), Υ_h, by the formula

$$\Upsilon_h=(t-s)_h\varrho_h\Delta_h.$$

Lemma 3.11.1. *The following formula holds:*

$$\Upsilon_h(t-s)_h^k=k(t-s)_h^k.$$

Proof. Compute using Lemma 3.2.1,

$$\begin{aligned}\Upsilon_h(t-s)_h^k &= t\varrho_h\Delta_h(t-s)_h^k\\ &= t\varrho_h\left(k(t-s)_h^{k-1}\right)\end{aligned}$$

$$= kt(t-s-h)_h^{k-1}$$

$$= k(t-s)_h^k,$$

as was to be shown. □

We now present an h-difference equation analogue of the formula (35) that the discrete hypergeometric function satisfies.

Theorem 3.11.3. *Define $y(t) = {}_pF_q(\boldsymbol{a}; \boldsymbol{b}; t, s, n, \xi; h)$. Then y satisfies the equation*

$$\left[\Upsilon_h \prod_{j=1}^{q} \left(\frac{1}{n}\Upsilon_h + b_j - 1\right) - n\xi(t-s)_h^n \varrho_h^n \prod_{i=1}^{p} \left(\frac{1}{n}\Upsilon_h + a_i\right)\right] y = 0.$$

Proof. First compute

$$\begin{aligned}
n\xi(t-s)_h^n \varrho_h^n \left[\prod_{i=1}^{p} \Upsilon_h + a_i\right] y &= n\xi(t-s)_h^n \varrho_h^n \sum_{k=0}^{\infty} \frac{\mathbf{a}^{\overline{k}} \xi^k}{\mathbf{b}^{\overline{k}} k!} \left[\prod_{i=1}^{p} \frac{1}{n}\Upsilon_h + a_i\right] (t-s)_h^{nk} \\
&= n\xi(t-s)_h^n \varrho_h^n \sum_{k=0}^{\infty} \frac{\mathbf{a}^{\overline{k}} \xi^k}{\mathbf{b}^{\overline{k}} k!} \left[\prod_{i=1}^{p} k + a_i\right] (t-s)_h^{nk} \\
&= n\xi \sum_{k=0}^{\infty} \frac{\mathbf{a}^{\overline{k}} \xi^k}{\mathbf{b}^{\overline{k}} k!} \left[\prod_{i=1}^{p} k + a_i\right] (t-s)_h^{nk+n}.
\end{aligned}$$

Now compute

$$\begin{aligned}
\Upsilon_h \left[\prod_{j=1}^{q} \frac{1}{n}\Upsilon_h + b_j - 1\right] y &= \Upsilon_h \sum_{k=0}^{\infty} \frac{\mathbf{a}^{\overline{k}} \xi^k}{\mathbf{b}^{\overline{k}} k!} \left[\prod_{j=1}^{q} \frac{1}{n}\Upsilon_h + b_j - 1\right] (t-s)_h^{nk} \\
&= \Upsilon_h \sum_{k=0}^{\infty} \frac{\mathbf{a}^{\overline{k}} \xi^k}{\mathbf{b}^{\overline{k}} k!} \left[\prod_{j=1}^{q} k + b_j - 1\right] (t-s)_h^{nk} \\
&= \Upsilon_h \sum_{k=0}^{\infty} \frac{\mathbf{a}^{\overline{k}} \xi^k}{\mathbf{b}^{\overline{k-1}} k!} (t-s)_h^{nk} \\
&= n(t-s)_h \varrho_h \sum_{k=1}^{\infty} \frac{\mathbf{a}^{\overline{k}} \xi^k}{\mathbf{b}^{\overline{k-1}} (k-1)!} (t-s)_h^{nk-1} \\
&= n(t-s)_h \varrho_h \sum_{k=0}^{\infty} \frac{\mathbf{a}^{\overline{k+1}} \xi^{k+1}}{\mathbf{b}^{\overline{k}} k!} (t-s)_h^{nk+(n-1)}
\end{aligned}$$

$$= n\xi \sum_{k=0}^{\infty} \frac{\mathbf{a}^{\overline{k}} \xi^k}{\mathbf{b}^{\overline{k}} k!} \left[\prod_{i=1}^{p} k + a_i \right] (t-s)_h^{nk+n},$$

which is the same series as before, which completes the proof. □

Corollary 3.11.1. *Define* $y(t) = {}_pF_q(\boldsymbol{a}; \boldsymbol{b}; t, s, 1, \xi; h)$. *Then* y *satisfies the equation*

$$\left[\Upsilon_h \prod_{j=1}^{q} (\Upsilon_h + b_j - 1) - \xi(t-s)\varrho_h \prod_{i=1}^{p} (\Upsilon_h + a_i) \right] y = 0.$$

3.11.2. Contiguous Relations. Define $\mathbf{a}_j^{\pm}$ to be the p-tuple

$$(a_1, \ldots, a_{j-1}, a_j \pm 1, a_{j+1}, \ldots, a_p)$$

and similarly define $\mathbf{b}_j^{\pm}$ as a q-tuple. We will now derive the so-called contiguous relations for the ${}_pF_q$ function. We adapt the notations and theorems throughout this section from [35, page 81] for the discrete case

$$F = {}_pF_q(\mathbf{a}; \mathbf{b}; t, s, n, \xi; h),$$

$$F(a_j+) = F(\mathbf{a}_j^+; \mathbf{b}; t, s, n, \xi; h) = \sum_{k=0}^{\infty} \frac{\mathbf{a}^{\overline{k}}}{\mathbf{b}^{\overline{k}}} \frac{a_j + k}{a_j} \xi^{nk} \frac{(t-s)_h^{nk}}{k!}, \tag{88}$$

$$F(a_j-) = {}_pF_q(\mathbf{a}_j^-; \mathbf{b}; t, s, n, \xi; h) = \sum_{k=0}^{\infty} \frac{\mathbf{a}^{\overline{k}}}{\mathbf{b}^{\overline{k}}} \frac{a_j - 1}{a_j + k - 1} \frac{(t-s)_h^{nk}}{k!},$$

$$F(b_j+) = {}_pF_q(\mathbf{a}; \mathbf{b}_j^+; t, s, n, \xi; h) = \sum_{k=0}^{\infty} \frac{\mathbf{a}^{\overline{k}}}{\mathbf{b}^{\overline{k}}} \frac{b_j}{b_j + k} \frac{(t-s)_h^{nk}}{k!},$$

and

$$F(b_j-) = {}_pF_q(\mathbf{a}; \mathbf{b}_j^-; t, s, n, \xi; h) = \sum_{k=0}^{\infty} \frac{\mathbf{a}^{\overline{k}}}{\mathbf{b}^{\overline{k}}} \frac{b_j + k - 1}{b_j - 1} \frac{(t-s)_h^{nk}}{k!}.$$

Lemma 3.11.2. *The following recurrences hold:*

$$(\Upsilon_h + a_j)F = a_j F(a_j+), \quad k \in \{1, 2, \ldots, p\},$$

and

$$(\Upsilon_h + b_j - 1)F = (b_j - 1)F(b_j-), \quad k \in \{1, 2, \ldots, q\}.$$

Proof. Using (88), we compute

$$\begin{aligned}(\Upsilon_h + a_j)F &= \Upsilon_h F + a_j F \\ &= \sum_{k=0}^{\infty}(a_j + k)\frac{\mathbf{a}^{\overline{k}}}{\mathbf{b}^{\overline{k}}}\frac{(t-s)_h^k}{k!} \\ &= a_j F(a_j+),\end{aligned}$$

as was to be shown. The rest of the proof is similar. □

Lemma 3.11.3. *The following formulas hold:*

$$(a_1 - a_j)F = a_1 F(a_1+) - a_j F(a_j+), \quad j \in \{2, 3, \ldots, p\}$$

and

$$(a_1 - b_j + 1)F = a_1 F(a_1+) - (b_j - 1)F(b_j-), \quad j \in \{1, 2, \ldots, q\}.$$

Proof. As a consequence of Lemma 3.11.2, we see for $j = 1, 2, \ldots, p$

$$a_j F = a_j F(a_j+) - \Upsilon_h F.$$

Using this formula with $j = 1$ and $j \in \{2, \ldots, p\}$, we get

$$\begin{aligned}(a_1 - a_j)F &= a_1 F - a_j F \\ &= (a_1 F(a_1+) - \Upsilon_h F) - (a_j F(a_j+) - \Upsilon F) \\ &= a_1 F(a_1+) - a_j F(a_j+),\end{aligned}$$

as was to be shown. The other formula follows similarly using the other part of

Lemma 3.11.2. $\square$

Given $\mathbf{a}$ and $\mathbf{b}$, it will be useful to use the following definitions:

$$\prod_{k=1,(j)}^{q} z_k = z_1 z_2 \ldots z_{j-1} z_{j+1} \ldots z_q,$$

$$U_j = \frac{\prod_{k=1}^{p} a_k - b_j}{b_j \prod_{k=1,(j)}^{q} b_k - b_j},$$

$$c_n = \frac{\mathbf{a}^{\overline{n}}}{\mathbf{b}^{\overline{n}}},$$

and

$$S_n = \frac{\mathbf{a}+n}{\mathbf{b}+n}.$$

Note that $c_{n+1} = S_n c_n$.

Theorem 3.11.4. *If $p < q$ and all entries in $\boldsymbol{b}$ are pairwise different, then*

$$a_1 F = a_1 F(a_1+) - (t-s) \sum_{j=1}^{q} U_j F^{\rho}(b_j+).$$

Proof. Consider the partial fraction decomposition [35, page 82]

$$S_n = \frac{\mathbf{a}+n}{\mathbf{b}+n} = \sum_{j=1}^{q} \frac{b_j U_j}{b_j + n}.$$

Now compute

$$\begin{aligned} \Upsilon_h F &= (t-s)\varrho_h \sum_{k=1}^{\infty} c_k \frac{(t-s)_h^{k-1}}{(k-1)!} \\ &= (t-s) \sum_{k=0}^{\infty} c_{k+1} \frac{(t-s-h)_h^k}{k!} \\ &= (t-s) \sum_{k=0}^{\infty} S_k c_k \frac{(t-s-h)_h^k}{k!} \end{aligned}$$

$$= (t-s)\sum_{k=0}^{\infty}\left(\sum_{j=1}^{q}\frac{b_jU_j}{b_j+n}\right)\frac{\mathbf{a}^{\overline{k}}}{\mathbf{b}^{\overline{k}}}\frac{(t-s-h)_h^k}{k!},$$

but since

$$F^{\rho}(b_j+) = \sum_{k=0}^{\infty}\frac{b_j}{b_j+k}\frac{\mathbf{a}^{\overline{k}}}{\mathbf{b}^{\overline{k}}}\frac{(t-h)_h^k}{k!},$$

we see

$$\Upsilon_h F = t\sum_{j=1}^{q}U_jF^{\rho}(b_j+).$$

Using this alongside Lemma 3.11.2 with $j=1$, we compute

$$\begin{aligned} a_1F &= a_1F(a_1+) - \Upsilon_h F \\ &= a_1F(a_1+) - t\sum_{j=1}^{q}U_jF^{\rho}(b_j+), \end{aligned}$$

as was to be shown. □

Theorem 3.11.5. *If $p=q$ and all entries in $\boldsymbol{b}$ are pairwise different, then*

$$\Upsilon_h F = tF^{\rho} + t\sum_{j=1}^{q}U_jF^{\rho}(b_j+).$$

Proof. Let $p=q$. The partial fraction decomposition [35, page 84] yields

$$S_n = 1 + \sum_{k=1}^{q}\frac{b_jU_j}{b_j+n}.$$

Now similarly to before,

$$\begin{aligned} \Upsilon_h F &= t\sum_{k=0}^{\infty}c_{k+1}\frac{(t-h)_h^k}{k!} \\ &= t\sum_{k=0}^{\infty}\left(1+\sum_{k=1}^{q}\frac{b_jU_j}{b_j+n}\right)\frac{\mathbf{a}^{\overline{k}}}{\mathbf{b}^{\overline{k}}}\frac{(t-h)_h^k}{k!}, \end{aligned}$$

and so we get

$$\Upsilon_h F = tF^\rho + t\sum_{j=1}^{q} U_j F^\rho(b_j+),$$

as was to be shown. $\square$

3.11.3. Relations to Discrete Special Functions.

Theorem 3.11.6. *The following formula holds:*

$$e_\alpha(t,s;h) = {}_0F_0\left(;;t,s,1,\alpha;h\right).$$

Proof. Recall (51) and compute

$${}_0F_0\left(;;t,s,1,\alpha;h\right) = \sum_{k=0}^{\infty} \frac{\alpha^k (t-s)_h^k}{k!},$$

as was to be shown. $\square$

Theorem 3.11.7. *The following formula holds:*

$$\cosh_\alpha(t,s;h) = {}_0F_1\left(;\frac{1}{2};t,s,2,\frac{\alpha^2}{4};h\right).$$

Proof. First note that

$$\left(\frac{1}{2}\right)^{\overline{k}} k!2^{2k} = \left(\frac{1}{2}\right)\left(\frac{1}{2}+1\right)\cdots\left(\frac{2k-1}{2}\right)k!2^{2k} = (2k)!.$$

By Theorem 3.4.1, we compute

$${}_0F_1\left(;\frac{1}{2};t,s,2,\frac{\alpha^2}{4};h\right) = \sum_{k=0}^{\infty} \frac{\alpha^{2k}}{2^{2k}} \frac{(t-s)_h^{2k}}{\left(\frac{1}{2}\right)^{\overline{k}} k!}$$

$$= \sum_{k=0}^{\infty} \frac{\alpha^{2k}(t-s)_h^{2k}}{(2k)!},$$

as was to be shown. $\square$

Theorem 3.11.8. *The following formula holds:*

$$\sinh_\alpha(t,s;h) = \alpha t_0 F_1\left(;\frac{3}{2};t-h,s,2,\frac{\alpha^2}{4};h\right).$$

Proof. First note that

$$\left(\frac{3}{2}\right)^{\overline{k}} k! 2^{2k} = \left(\frac{3}{2}\right)\ldots\left(\frac{2k+1}{2}\right) k! 2^{2k} = (2k+1)!.$$

Now using Lemma 3.2.1 and Theorem 3.4.1, we see

$$\begin{aligned}\alpha t_0 F_1\left(;\frac{3}{2};t,2,\frac{\alpha^2}{4};h\right) &= \alpha t \sum_{k=0}^{\infty} \frac{\alpha^{2k}(t-h)_h^{2k}}{2^{2k}(\frac{3}{2})^{\overline{k}} k!} \\ &= \sum_{k=0}^{\infty} \frac{\alpha^{2k+1}(t-s)_h^{2k+1}}{(2k+1)!},\end{aligned}$$

as was to be shown. $\square$

The proof of the following theorem is essentially the same as the proofs of Theorem 3.11.7 and Theorem 3.11.8.

Theorem 3.11.9. *The following formulas hold:*

$$\cos_\alpha(t,s;h) = {}_0F_1\left(;\frac{1}{2};t,s,2,-\frac{\alpha^2}{4};h\right)$$

and

$$\sin_\alpha(t,s;h) = a(t-s)_0 F_1\left(;\frac{3}{2};t-h,s,2,-\frac{\alpha^2}{4};h\right).$$

Theorem 3.11.10. *The following formula holds:*

$$L_n^{(\alpha)}(t,s;h) = \frac{(\alpha+1)^{\overline{n}}}{n!} {}_1F_1(-n;\alpha+1;t,s,1,1;h).$$

Proof. By (76), we have

$$L_n^{(\alpha)}(t,s;h) = \sum_{k=0}^{n} \frac{(-1)^k (t-s)_h^k}{k!} \binom{n+\alpha}{n-k}.$$

Using (22), we see

$$\binom{n+\alpha}{n-k} = \frac{\Gamma(n+\alpha+1)}{(n-k)!\Gamma(\alpha+k+1)}.$$

Note that $(-n)^{\overline{k}} = (-n)(-n+1)\ldots(-n+k-1) = 0$ if $k \geq n+1$. Now compute

$$\begin{aligned}
\frac{(\alpha+1)^{\overline{n}}}{n!} {}_1F_1(-n;\alpha+1;x) &= \frac{(\alpha+1)^{\overline{n}}}{n!} \sum_{k=0}^{\infty} \frac{(-n)^{\overline{k}}(t-s)_h^k}{(\alpha+1)^{\overline{k}} k!} \\
&= (\alpha+1)^{\overline{n}} \sum_{k=0}^{n} \frac{(-1)^k n(n-1)\ldots(n-k+1)(t-s)_h^k}{n!(\alpha+1)^{\overline{k}} k!} \\
&= \frac{\Gamma(\alpha+1+n)}{\Gamma(\alpha+1)} \sum_{k=0}^{n} \frac{(-1)^k (t-s)_h^k \Gamma(\alpha+1)}{(n-k)!\Gamma(\alpha+1+k)k!} \\
&= \sum_{k=0}^{n} \frac{(-1)^k (t-s)_h^k}{k!} \binom{n+\alpha}{n-k} \\
&= L_n^{(\alpha)}(t,s;h),
\end{aligned}$$

as was to be shown. $\square$

Table 3.3. Representations of discrete special functions as discrete hypergeometric series.

Function	Hypergeometric representation	Source
$e_\alpha(t,s;h)$	${}_0F_0\left(;;t,s,1,\alpha;h\right)$	Theorem 3.11.6
$\cosh_\alpha(t,s;h)$	${}_0F_1\left(;\frac{1}{2};t,s,2,\frac{\alpha^2}{4};h\right)$	Theorem 3.11.7
$\sinh_\alpha(t,s;h)$	$\alpha t {}_0F_1\left(;\frac{3}{2};t-h,s,2,\frac{\alpha^2}{4};h\right)$	Theorem 3.11.8
$\cos_\alpha(t,s;h)$	${}_0F_1\left(;\frac{1}{2};t,s,2,-\frac{\alpha^2}{4};h\right)$	Theorem 3.11.9
$\sin_\alpha(t,s;h)$	$\alpha t {}_0F_1\left(;\frac{3}{2};t-h,s,2,-\frac{\alpha^2}{4};h\right)$	Theorem 3.11.9
$\frac{n!}{(\alpha+1)^{\overline{n}}}L_n^{(\alpha)}(t,s;h)$	${}_1F_1(-n;\alpha+1;t,s,1,1;h)$	Theorem 3.11.10
$2^\nu\Gamma(\nu+1)J_\nu(t,s,1,-\nu,1;h)$	${}_0F_1\left(;\nu+1;t,s,2,-\frac{1}{4};h\right)$	Theorem 3.11.11

Theorem 3.11.11. *The following formula holds:*

$$2^\nu\Gamma(\nu+1)J_\nu(t,1,-\nu,1) = {}_0F_1\left(;\nu+1;t,s,2,-\frac{1}{4},h\right).$$

Proof. We proceed via direct calculation using (78):

$$\begin{aligned}
2^\nu\Gamma(\nu+1)J_\nu\left(t,s,1,-\nu,1;\mathbb{T}\right) &= 2^\nu\Gamma(\nu+1)\sum_{k=0}^{\infty}\frac{(-1)^k(t-s)_h^{2k}}{\Gamma(k+\nu+1)2^{2k+\nu}k!} \\
&= \sum_{k=0}^{\infty}\frac{(t-s)_h^{2k}\left(-\frac{1}{4}\right)^k}{(\nu+1)^{\overline{k}}k!} \\
&= {}_0F_1\left(;\nu+1;t,s,2,-\frac{1}{4},h\right),
\end{aligned}$$

as was to be shown. $\square$

Corollary 3.11.2. *The Bessel h-difference equation* (3.10.1) *can be factored as*

$$\left[\Upsilon_h\left(\frac{1}{2}\Upsilon_h+\nu\right)+\frac{1}{2}t_h^2\varrho_h^2\right]y=0.$$

4. EXTENSION TO TIME SCALES

4.1. DEFINITIONS FROM TIME SCALES

Let $X \subseteq \mathbb{R}$. We say that $x \in X$ is a limit point of X if for every $\delta > 0$, the set $(x-\delta, x+\delta) \setminus \{x\}$ is nonempty. We say that a set is a closed set if it contains all of its limit points. A time scale $\mathbb{T}$ is defined to be a closed subset of $\mathbb{R}$. Sometimes if a time scale $\mathbb{T}$ has a left-scattered maximum, it is useful to use the notation $\mathbb{T}^\kappa = \mathbb{T} \setminus \{\max \mathbb{T}\}$.

Let $\mathbb{T}$ be a time scale. A function $f\colon \mathbb{T} \to \mathbb{C}$ is called rd-continuous if it is continuous at all $t \in \mathbb{T}$ such that $\sigma(t) = t$ (i.e., t is "right-dense"). We define the forward jump function $\sigma\colon \mathbb{T} \to \mathbb{T}$ by $\sigma(t) = \inf\{s \in \mathbb{T}\colon s > t\}$, where we interpret $\inf \emptyset = \sup \mathbb{T}$. We define the forward graininess (or "stepsize") function $\mu\colon \mathbb{T} \to \mathbb{R}$ by $\mu(t) = \sigma(t) - t$. Let $f\colon \mathbb{T} \to \mathbb{C}$. We may define the delta derivative f^Δ of f at a point $t \in \mathbb{T}^\kappa$ by

$$f^\Delta(t) = \begin{cases} \dfrac{f(\sigma(t)) - f(t)}{\mu(t)}, & \sigma(t) > 0 \\ f'(t), & \sigma(t) = t. \end{cases}$$

A function $f\colon \mathbb{T} \to \mathbb{C}$ is called rd-continuous if it is continuous at all right-dense points in $\mathbb{T}$ and left-sided limits exist at all left-dense points in $\mathbb{T}$. We write $\mathrm{C_{rd}}$ to denote the set of rd-continuous functions on $\mathbb{T}$. The quotient rule on time scales is

$$\left(\frac{f}{g}\right)^\Delta (t) = \frac{g(t) f^\Delta(t) + f(t) g^\Delta(t)}{g(t) g(\sigma(t))}. \tag{89}$$

Integration on a time scale is often defined as the inverse operation of differentiation [12, Theorem 1.74] in the sense that

$$\int_a^b f^\Delta(\tau) \Delta\tau = f(b) - f(a),$$

but it may also be defined in a Riemann-integration or Lebesgue-integration sense [14, Chapter 5].

With Δ-integration, we define the (weighted) monomials $h_k \colon \mathbb{T} \times \mathbb{T} \to \mathbb{R}$ by

$$\begin{cases} h_0(t,s;\mathbb{T}) = 1 \\ h_{n+1}(t,s;\mathbb{T}) = \displaystyle\int_s^t h_n(\tau,s;\mathbb{T})\Delta\tau. \end{cases} \tag{90}$$

These functions are called "time scale analogues" of the classical (weighted) monomial functions $\dfrac{x^k}{k!}$, and obey the important property that $h_{k+1}^{\Delta}(t,s) = h_k(t,s)$. Just as in (44), we define the (unweighted) monomial functions on a time scale by

$$(t-s)_{\mathbb{T}}^n = n! h_n(t,s;\mathbb{T}),$$

and these functions will be used for our power series.

Given the Δ-derivative, it is natural to look at Δ-differential equations, also known as dynamic equations. Let $p \colon \mathbb{T} \to \mathbb{C}$ be a function such that $1 + p(t)\mu(t) \neq 0$. We say that such p is a regressive function on $\mathbb{T}$ and write $p \in \mathcal{R}_\mu(\mathbb{T})$. If $1+\mu(t)p(t) > 0$, then we say p is positively regressive and write $p \in \mathcal{R}_\mu^+$. We define the operation $\oplus \colon \mathcal{R}_\mu(\mathbb{T}) \times \mathcal{R}_\mu(\mathbb{T}) \to \mathcal{R}_\mu(\mathbb{T})$ by

$$(p \oplus q)(t) = p(t) + q(t) + \mu(t)p(t)q(t).$$

Similarly, the operation $\ominus \colon \mathcal{R}_\mu(\mathbb{T}) \to \mathcal{R}_\mu(\mathbb{T})$ is defined by

$$(\ominus p)(t) = -\frac{p(t)}{1+\mu(t)p(t)}.$$

We also define the operation $\odot\colon \mathbb{R} \times \mathcal{R}_\mu \to \mathcal{R}_\mu$ by the formula

$$(\alpha \odot p)(t) = \begin{cases} \dfrac{(1+\mu(t)p(t))^\alpha - 1}{\mu(t)} & \text{if } \mu(t) > 0 \\ \alpha p(t) & \text{if } \mu(t) = 0. \end{cases}$$

It is well-known that the structure $(\mathcal{R}_\mu(\mathbb{T}), \oplus_{\mu_h})$ is a group [14, Exercise 1.35] and the structure $(\mathcal{R}_\mu(\mathbb{T}), \oplus, \odot)$ is a real vector space [14, Theorem 2.46].

Consider the dynamic initial value problem $y^\Delta(t) = p(t)y(t), \quad y(s) = 1$. The solution of this equation, $e_p\colon \mathbb{T} \times \mathbb{T} \to \mathbb{C}$, is the time scale analogue of (49) and can be defined in terms of a Δ-integral to be [12, page 59]

$$e_p(t,s;\mathbb{T}) = \exp\left(\int_s^t \frac{1}{\mu(\tau)} \mathrm{Log}(1+\mu(\tau)p(\tau))\Delta\tau \right).$$

Let $p\colon \mathbb{T} \to \mathbb{C}$ be regressive and define the time scale trigonometric functions by

$$\cos_p(t,s;\mathbb{T}) = \frac{e_{ip}(t,s;\mathbb{T}) + e_{-ip}(t,s;\mathbb{T})}{2} \tag{91}$$

and

$$\sin_p(t,s;\mathbb{T}) = \frac{e_{ip}(t,s;\mathbb{T}) - e_{-ip}(t,s;\mathbb{T})}{2i}. \tag{92}$$

The following formulas are known [12, Lemma 3.26]:

$$\cos_p^\Delta(\cdot,s;\mathbb{T})(t) = -p(t)\sin_p(t,s;\mathbb{T}), \tag{93}$$

$$\sin_p^\Delta(\cdot,s;\mathbb{T})(t) = p(t)\cos_p(t,s;\mathbb{T}), \tag{94}$$

and

$$\cos_p^2(t,s;\mathbb{T}) + \sin_p^2(t,s;\mathbb{T}) = e_{\mu p^2}(t,s;\mathbb{T}). \tag{95}$$

We now prove an analogue to Theorem 3.5.3.

Theorem 4.1.1. *Let $\mathbb{T}$ be an isolated time scale $\mathbb{T} = \{\ldots, t_{-1}, t_0, t_1, \ldots\}$ and define the bijection $\pi(t_k) = k$. The following formulas hold for regressive constants α:*

$$\cos_\alpha(t, s; \mathbb{T}) = \cos\left(\int_s^t \frac{\arctan(\alpha\mu(\tau))}{\mu(\tau)^2}\Delta\tau\right)\prod_{k=\pi(s)}^{\pi(t)-1}(1+\mu(t_k)^2\alpha^2)^{\frac{1}{2\mu(t_k)}}$$

and

$$\sin_\alpha(t, s; \mathbb{T}) = \sin\left(\int_s^t \frac{\arctan(\alpha\mu(\tau))}{\mu(\tau)^2}\Delta\tau\right)\prod_{k=\pi(s)}^{\pi(t)-1}(1+\mu(t_k)^2\alpha^2)^{\frac{1}{2\mu(t_k)}}.$$

Proof. From (91), we have

$$\cos_\alpha(t, s; \mathbb{T}) = \frac{\displaystyle\prod_{k=\pi(s)}^{\pi(t)-1}(1+\mu(t_k)\alpha i) + \prod_{k=\pi(s)}^{\pi(t)-1}(1-\mu(t_k)\alpha i)}{2}.$$

From (14), we see

$$\frac{\arctan(\alpha\mu(t))}{\mu(t)} = \frac{i}{2\mu(t)}\log\left(\frac{1-\alpha\mu(t)}{1+\alpha\mu(t)}\right) = i\log\left[\left(\frac{1-\alpha\mu(t)}{1+\alpha\mu(t)}\right)^{\frac{1}{2\mu(t)}}\right].$$

First compute using (12)

$$\cos\left(\int_s^t \frac{\arctan(\alpha\mu(\tau))}{\mu(\tau)^2}\Delta\tau\right) = \frac{1}{2}\left[\prod_{k=\pi(s)}^{\pi(t)-1}\left(\frac{1-\mu(t_k)\alpha i}{1+\mu(t_k)\alpha i}\right)^{-\frac{1}{2\mu(t_k)}} + \prod_{k=\pi(s)}^{\pi(t)-1}\left(\frac{1-\mu(t_k)\alpha i}{1+\mu(t_k)\alpha i}\right)^{\frac{1}{2\mu(t_k)}}\right].$$

Therefore since $(1+\mu(t_k)^2\alpha^2) = (1-\mu(t_k)\alpha i)(1+\mu(t_k)\alpha i)$, we see

$$\prod_{k=\pi(s)}^{\pi(t)-1}(1+\mu(t_k)^2\alpha^2)^{\frac{1}{2\mu(t_k)}}\cos\left(\int_s^t \frac{\arctan(\alpha\mu(\tau))}{\mu(\tau)^2}\Delta\tau\right) = \cos_\alpha(t, s; \mathbb{T}),$$

as was to be shown. The proof for $\sin_\alpha$ is similar. $\square$

Define a time scale tangent function by

$$\tan_p(t,s;\mathbb{T}) = \frac{\sin_p(t,s;\mathbb{T})}{\cos_p(t,s;\mathbb{T})}, \tag{96}$$

and similarly define the other time scale trigonometric functions by $\sec_p = \dfrac{1}{\cos_p}$, $\csc_p = \dfrac{1}{\sin_p}$, and $\cot_p = \dfrac{1}{\tan_p}$. These definitions leads to the following analogue of Corollary 3.5.1.

Corollary 4.1.1. *Let $\mathbb{T}$ be an isolated time scale. The following theorem holds:*

$$\tan_\alpha(t,s;\mathbb{T}) = \tan\left(\int_s^t \frac{\arctan(\alpha\mu(\tau))}{\mu(\tau)^2}\Delta\tau\right).$$

From (95), we get the following formulas.

Theorem 4.1.2. *The following formulas hold:*

$$1 + \tan_p^2(t,s;\mathbb{T}) = e_{\mu p^2}(t,s;\mathbb{T})\sec_p^2(t,s;\mathbb{T})$$

and

$$\cos_p^2(t,s;\mathbb{T}) + 1 = e_{\mu p^2}(t,s;\mathbb{T})\csc_p^2(t,s;\mathbb{T}).$$

We also have differentiation formulas for all the new time scale trigonometric functions.

Theorem 4.1.3. *Let $\mathbb{T}$ be an isolated time scale. Then the following formulas hold:*

$$\tan_p^\Delta(t,s;\mathbb{T}) = p(t)e_{\mu p^2}(t,s;\mathbb{T})\sec_p(t,s;\mathbb{T})\sec_p(\sigma(t),s;\mathbb{T}),$$

$$\sec_p^\Delta(t,s;\mathbb{T}) = p(t)\tan_p(t,s;\mathbb{T})\sec_p(\sigma(t),s;\mathbb{T}),$$

$$\csc_p^\Delta(t,s;\mathbb{T}) = -p(t)\cot_p(t,s;\mathbb{T})\csc_p(\sigma(t),s;\mathbb{T}),$$

and

$$\cot_p^{\Delta}(t,s;\mathbb{T}) = -p(t)e_{\mu p^2}(t,s;\mathbb{T})\csc_p(t,s;\mathbb{T})\csc_p(\sigma(t),s;\mathbb{T}).$$

Proof. Using (89), (93), (94), (95), and (96), we see

$$\begin{aligned}\tan_p^{\Delta}(t,s;\mathbb{T}) &= \left[\frac{\sin_p(t,s;\mathbb{T})}{\cos_p(t,s;\mathbb{T})}\right]^{\Delta} \\ &= \frac{p(t)\sin^2(t,s;\mathbb{T}) + p(t)\cos^2(t,s;\mathbb{T})}{\cos(t,s;\mathbb{T})\cos(\sigma(t),s;\mathbb{T})} \\ &= p(t)e_{\mu p^2}(t,s;\mathbb{T})\sec(t,s;\mathbb{T})\sec(\sigma(t),s;\mathbb{T}),\end{aligned}$$

as was to be shown. The other formulas are proven similarly. □

A Laplace transform on time scales was first defined in [27], but was modified and studied in detail in [9, 10, 13, 17]. We have included a list of some Laplace transforms of special functions on time scales in Table 4.1. We now summarize some of the properties of the time scale Laplace transform. Let $s \in \mathbb{T}$. A function $f \in \mathrm{C_{rd}}(\mathbb{T})$ is said to be of exponential order α, where α is a positive regressive constant, provided there exists $K > 0$ with the property that

$$|f(t)| \leq Ke_{\alpha}(t,s) \text{ for all } t \in [s,\infty) \cap \mathbb{T}.$$

Let f be of exponential order α. Then we define its Laplace transform, centered at s, to be

$$\mathscr{L}_{\mathbb{T}}\{f\}(z;s) = \int_s^{\infty} f(\tau)e_{\ominus z}(\sigma(\tau),s)\Delta\tau.$$

We introduce the minimal graininess function $\mu_*\colon \mathbb{T} \to \mathbb{R}$ by

$$\mu_*(s) = \inf\{\mu(\tau)\colon t \in [s,\infty) \cap \mathbb{T}\}.$$

We define the Hilger real part of a complex number by

$$\mathrm{Re}_h(z) = \frac{1}{h}(|1 + hz| - 1)$$

and the sets

$$\mathbb{C}_h = \left\{ z \in \mathbb{C} \colon z \neq -\frac{1}{h} \right\}$$

and

$$\mathbb{C}_h(\lambda) = \{ z \in \mathbb{C}_h \colon \mathrm{Re}_h(z) > \lambda \}.$$

It is known [9, Theorem 5.1] that $\mathscr{L}\{f\}(z;s)$ converges absolutely for $z \in \mathbb{C}_{\mu_*(s)}(\alpha)$ and [9, Theorem 5.3] converges uniformly in $\mathbb{C}_{\mu_*(s)}(\beta)$ for any $\beta > \alpha$. It is also known [9, Theorem 6.1] that if there exists an $M > 0$ such that $|a_k| \leq M\alpha^k$ for all $k \in \mathbb{N}_0$, then

$$\mathscr{L}\left\{ \sum_{k=0}^{\infty} a_k h_k(\cdot, s; \mathbb{T}) \right\}(z;s) = \sum_{k=0}^{\infty} \frac{a_k}{z^{k+1}}.$$

It is also shown in the proof of [9, Theorem 7.1] that

$$\frac{\mathrm{d}}{\mathrm{d}z} e_z(t,s;\mathbb{T}) = m_z(t,s;\mathbb{T}) e_z(t,s;\mathbb{T}), \tag{97}$$

where $m_z(t,s;\mathbb{T}) = \displaystyle\int_s^t \frac{1}{1+\mu(\tau)z} \Delta\tau$. Define the functions (related to the u_n from [9, Corollary 7.2])

$$v_n(z;t,s;\mathbb{T}) = \begin{cases} 1, & n = 0 \\ \dfrac{\mathrm{d}}{\mathrm{d}z} v_{n-1}(z;t,s;\mathbb{T}) + v_{n-1}(z;t,s;\mathbb{T}) m_z(t,s;\mathbb{T}), & n \in \mathbb{N}. \end{cases} \tag{98}$$

The following lemma will allow us to write e_z as a power series involving u_n.

Lemma 4.1.1. *The following formula holds:*

$$\frac{\mathrm{d}^n}{\mathrm{d}z^n} e_z(t,s;\mathbb{T}) = v_n(z;t,s;\mathbb{T}) e_z(t,s;\mathbb{T}).$$

Proof. From (97), we see that the claim holds for $n = 1$. Now assume that the claim holds for $n = N - 1$. Now compute

$$\begin{aligned}\frac{\mathrm{d}^N}{\mathrm{d}z^N} e_z(t,s;\mathbb{T}) &= \frac{\mathrm{d}}{\mathrm{d}z}\left[v_{N-1}(z;t,s;\mathbb{T}) e_z(t,s;\mathbb{T})\right] \\ &= \left[\frac{\mathrm{d}}{\mathrm{d}z} v_{N-1}(z;t,s;\mathbb{T}) + v_{N-1}(z;t,s;\mathbb{T}) m_z(t,s;\mathbb{T})\right] e_z(t,s;\mathbb{T})\end{aligned}$$

and compare to (98), proving the claim. □

The following corollary is immediate from Lemma 4.1.1 and (2).

Corollary 4.1.2. *The following formula holds for any regressive* $z_0 \in \mathbb{C}$*:*

$$e_z(t,s;\mathbb{T}) = e_{z_0}(t,s;\mathbb{T}) \sum_{k=0}^{\infty} v_k(z_0;t,s;\mathbb{T}) \frac{(z-z_0)^k}{k!}.$$

An inverse Laplace transform is known, but only for time scales whose graininess obeys $0 < \mu_{\min} \leq \mu(t) \leq \mu_{\max} < \infty$ (an isolated time scale with bounded graininess). Using such a time scale and $c > 0$, suppose that the complex function F is analytic for $z \in \mathbb{C}_{\mu_*(s)}(\alpha)$ for some $s, \alpha > 0$. If $\oint_{c-i\infty}^{c+i\infty} |F(z)||dz| < \infty$, then

$$f(t) = \mathscr{L}_{\mathbb{T}}^{-1}\{f\}(t;s) = \sum_{i=1}^{n} \operatorname*{Res}_{z=z_i} e_z(t,s) F(z) \tag{99}$$

is a function which obeys $\mathscr{L}_{\mathbb{T}}\{f\}(z;s)$. Of course, inverse Laplace transforms are not necessarily unique on all time scales, but as noted in [17, Theorem 1.5], any two inverse transforms are equal almost everywhere, where almost everywhere is determined in the sense defined in [24]. Consequently, if $\mathbb{T}$ is an isolated time scale, then inverse Laplace transforms are everywhere unique.

When generalizing the Bessel functions, we will want to allow time scale polynomials of arbitrary order. The recursion (90) will not suffice for this. Instead, we will

Table 4.1. Laplace transforms of time scale special functions.

$f(t)$	$\mathscr{L}_{\mathbb{T}}\{f\}(z;s)$	Source
$h_n(t,s;\mathbb{T})$	$\dfrac{1}{z^{n+1}}$	[9, Table 5]
$(t-s)_{\mathbb{T}}^{n}$	$\dfrac{n!}{z^{n+1}}$	Definition
$(t-s)_{\mathbb{T}}^{\nu}$	$\dfrac{\Gamma(\nu+1)}{z^{\nu+1}}$	Definition
$e_\alpha(t,s;\mathbb{T})$	$\dfrac{1}{z-\alpha}$	[9, Table 5]
$\cosh_\alpha(t,s;\mathbb{T})$	$\dfrac{z}{z^2-\alpha^2}$	[12, Table 3.2, page 133]
$\sinh_\alpha(t,s;\mathbb{T})$	$\dfrac{\alpha}{z^2-\alpha^2}$	[12, Table 3.2, page 133]
$\cos_\alpha(t,s;\mathbb{T})$	$\dfrac{z}{z^2+\alpha^2}$	[12, Table 3.2, page 133]
$\sin_\alpha(t,s;\mathbb{T})$	$\dfrac{\alpha}{z^2+\alpha^2}$	[12, Table 3.2, page 133]
$J_0(t,s;\mathbb{T})$	$\dfrac{1}{\sqrt{z^2+1}}$	Theorem 4.4.1
$J_\nu\left(t,s,2\sqrt{\alpha},\dfrac{\nu}{2},\dfrac{1}{2};\mathbb{T}\right)$	$\dfrac{\alpha^{\frac{\nu}{2}}}{z^{\nu+1}e^{\frac{\alpha}{2}}}$	Table 4.2

interpret noninteger subscripts in h_n in a sense similar to that studied in [4, Proposition 3.2], i.e.,

$$h_n(t,s;\mathbb{T}) = \mathscr{L}_{\mathbb{T}}^{-1}\left\{\frac{1}{z^{n+1}}\right\}(t,s).$$

4.2. WHY ANALOGUES ON TIME SCALES ARE DIFFICULT TO FIND

The papers [25,33] are concerned with Taylor series representations of functions on time scales. In [25, Theorem 6.1, page 6], the following formula is derived for any time scale $\mathbb{T}$ with constant graininess μ (i.e., $\mathbb{R}$ or $h\mathbb{Z}$ for some $h > 0$):

$$h_n(t, s; \mathbb{T})h_m(t, s; \mathbb{T}) = \sum_{k=0}^{n} \frac{\mu^{n-k}(m + k)^{(n)}}{k!(n - k)!} h_{m+k}(t, s; \mathbb{T}).$$

This formula gives us the ability to do series methods for dynamic equations on such a $\mathbb{T}$, but the generalization of this formula to all time scales seems difficult to penetrate. The way we chose to get around this difficulty was to use Lemma 3.2.1. The benefit of this formula was that it allowed us to shift a single monomial without introducing other terms, but the downside is that it does this by forcing us to deal with h-difference equations with a delay in the argument. It was always possible to rewrite our delay difference equations to have no delay, but that would have ruined the nice correspondence between the differential equation and the delay h-difference equation analogue.

In fact, we would have no problem defining the Gaussian bell on a time scale, but defining a function in a way analogous to (71) on a general time scale inevitably leads to nondifferentiability because of the product rule. One way to dodge this problem is to define a time scale analogue of $H_n(t, s; h)$ via the Gram–Schmidt orthogonalization, but this just shifts the burden to finding in what way (71) holds on a general time scale.

Our proof of Lemma 3.2.1 relies on the product representation of $h_k(t, s; h)$, and so until we have a similar representation for $h_k(t, s)$ on a general time scale, we cannot use the same trick. Instead we will approach this using inverse Laplace transforms.

We may take any formula defining a special function on $h\mathbb{Z}$ and look at the

$$\begin{tikzcd}
\displaystyle\sum_{k=0}^{\infty} a_k(t-s)_{\mathbb{T}}^{k} \arrow[r, "\mathscr{L}_{\mathbb{T}}"] \arrow[d, "\mathscr{U}_{\mathbb{T}}"'] & \displaystyle\sum_{k=0}^{\infty} \frac{a_k k!}{z^{k+1}} \arrow[d, "-\frac{\mathrm{d}}{\mathrm{d}z}"] \\
\displaystyle\sum_{k=0}^{\infty} a_k(t-s)_{\mathbb{T}}^{k+1} & \displaystyle\sum_{k=0}^{\infty} \frac{a_k(k+1)!}{z^{k+2}} \arrow[l, "\mathscr{L}_{\mathbb{T}}^{-1}"]
\end{tikzcd}$$

Figure 4.1. Diagram for the polynomial shift operator $\mathscr{U}$ on time scales.

consequences of it on a time scale. For example, we could exploit (78) and define for any ν such that $\gamma(2k+\nu)+\alpha$ is an integer,

$$J_\nu(t,s,\xi,\alpha,\gamma;\mathbb{T}) = \sum_{k=0}^{\infty} \frac{(-1)^k \xi^{2k+\nu}}{\Gamma(k+\nu+1)2^{2k+\nu}} \frac{(t-s)_{\mathbb{T}}^{\gamma(2k+\nu)+\alpha}}{k!}.$$

What sorts of theorems could we prove about this function? Could we find a dynamic equation analogue to (3.10.1) for this function? Without a Lemma 3.2.1 analogue, we cannot. It is unclear how any theorem involving a polynomial times J_ν would generalize to an arbitrary time scale, but analogues of other theorems like Theorem 3.10.5 are proven exactly the same.

If we can find an analogue of a polynomial shift operator on time scales, we may recover many of the theorems in the thesis on a general time scale in terms of such an operator.

4.3. POLYNOMIAL SHIFT OPERATOR

Consider the diagram of operators in Figure 4.1 to justify our definition of a polynomial shift on time scales. Hence we define $\mathscr{U}_{\mathbb{T}}$ by the formula

$$\mathscr{U}_{\mathbb{T}}\{f\} := \left(\mathscr{L}_{\mathbb{T}}^{-1} \circ \left[-\frac{\mathrm{d}}{\mathrm{d}z}\right] \circ \mathscr{L}_{\mathbb{T}}\right)\{f\}.$$

The following theorem justifies this definition of $\mathscr{U}_{\mathbb{T}}$ as a time scale analogue of Lemma 3.2.1.

Theorem 4.3.1. *If $\mathbb{T} = h\mathbb{Z}$ and $f(t) = \displaystyle\sum_{k=0}^{\infty} \frac{a_k(t-s)_{h\mathbb{Z}}^k}{k!}$, then*

$$\mathscr{U}_{h\mathbb{Z}}\{f\}(t;s) = (t-s)_{h\mathbb{Z}} f(t-h).$$

Proof. Compute

$$\begin{aligned}
\mathscr{U}_{h\mathbb{Z}}\{f(\cdot;s)\}(t,s) &= \left(\mathscr{L}_{h\mathbb{Z}}^{-1} \circ \left[-\frac{\mathrm{d}}{\mathrm{d}z}\right]\right) h \sum_{k=0}^{\infty} \frac{f(hk)}{(1+hz)^{k-\frac{s}{h}+1}} \\
&= \mathscr{L}_{h\mathbb{Z}}^{-1}\left\{ h \sum_{k=0}^{\infty} \frac{(kh-s+h)f(hk)}{(1+hz)^{k-\frac{s}{h}+2}} \right\}(t;s) \\
&= \mathscr{L}_{h\mathbb{Z}}^{-1}\left\{ h \sum_{k=0}^{\infty} \frac{(kh-s)f(hk-h)}{(1+hz)^{k-\frac{s}{h}+1}} \right\}(t;s) \\
&= (t-s)f(t-h),
\end{aligned}$$

as was to be shown. ☐

The follow theorem is proven by repeated application of Theorem 4.3.1.

Theorem 4.3.2. *The following formula holds:*

$$\mathscr{U}_{h\mathbb{Z}}^{n}\{f(\cdot)\}(t;s) = (t-s)_{h\mathbb{Z}}^{n} f(t-nh).$$

Theorem 4.3.3. *Let $\mathbb{T}$ be a time scale of isolated points with either constant graininess or $0 < \mu_{\min} \leq \mu_{\max} < \infty$. Let $f\colon \mathbb{T} \to \mathbb{C}$ be of exponential order α and write $F(z) = \mathscr{L}\{f\}(z)$. If F has n regressive poles $\{z_1, \ldots, z_n\}$ of orders $\theta_1, \ldots, \theta_n$*

respectively, then

$$\mathscr{U}_{\mathbb{T}}\{f\}(z) = \sum_{i=1}^{n} e_{z_i}(t,s;\mathbb{T}) \sum_{j=0}^{\theta_i - 1} \frac{(\theta_i - j) a_{j-\theta_i}^{(i)} v_{\theta_i - j}(z_i;t,s;\mathbb{T})}{(\theta_i - j)!}.$$

Proof. By (8), such an F obeys a series of the form

$$F(z) = \sum_{k=-\theta_i}^{\infty} a_k^{(i)} (z - z_i)^k,$$

for all z in some annulus $D_{\varepsilon_i}(z_i) \setminus \{z_i\}$. We may compute

$$-\frac{\mathrm{d}}{\mathrm{d}z} F(z) = \sum_{k=-\theta_i}^{\infty} \tilde{a_k}^{(i)} (z - z_i)^{k-1},$$

where

$$\tilde{a_k}^{(i)} = \begin{cases} -k a_k^{(i)} & \text{if } k \neq 0 \\ 0 & \text{if } k = 0. \end{cases}$$

Now Corollary 4.1.2 shows

$$e_z(t,s;\mathbb{T}) = e_{z_i}(t,s;\mathbb{T}) \sum_{k=0}^{\infty} v_k(z_i;t,s;\mathbb{T}) \frac{(z - z_i)^k}{k!},$$

and so we may use (99) and the Cauchy product (7) to compute

$$\begin{aligned}
&\mathscr{U}_{\mathbb{T}}\{f\}(t;s) = \\
&\quad \mathscr{L}_{\mathbb{T}}^{-1} \left\{ \sum_{k=-\theta_i}^{\infty} \tilde{a_k}^{(i)} (\cdot - z_i)^{k-1} \right\} (t;s) \\
&\quad = \sum_{i=1}^{n} \operatorname*{Res}_{z=z_i} e_z(t,s;\mathbb{T}) \sum_{k=-\theta_i}^{\infty} \tilde{a}_k^{(i)} (z - z_i)^{k-1} \\
&\quad = \sum_{i=1}^{n} \operatorname*{Res}_{z=z_i} e_{z_i}(t,s;\mathbb{T}) \left(\sum_{k=-\theta_i}^{\infty} \tilde{a}_k^{(i)} (z - z_i)^{k-1} \right) \left(\sum_{k=0}^{\infty} v_k(z_i;t,s;\mathbb{T}) \frac{(z - z_i)^k}{k!} \right)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{n} \operatorname*{Res}_{z=z_i} e_{z_i}(t,s;\mathbb{T}) \left(\sum_{k=0}^{\infty} \tilde{a}^{(i)}_{k-\theta_i} (z-z_i)^{k-\theta_i-1} \right) \left(\sum_{k=0}^{\infty} v_k(z_i;t,s;\mathbb{T}) \frac{(z-z_i)^k}{k!} \right) \\
&= \sum_{i=1}^{n} \operatorname*{Res}_{z=z_i} e_{z_i}(t,s;\mathbb{T}) \sum_{k=0}^{\infty} \sum_{j=0}^{k} \tilde{a}^{(i)}_{j-\theta_i} (z-z_i)^{j-\theta_i-1} v_{k-j}(z_i;t,s;\mathbb{T}) \frac{(z-z_i)^{k-j}}{(k-j)!} \\
&= \sum_{i=1}^{n} \operatorname*{Res}_{z=z_i} e_{z_i}(t,s;\mathbb{T}) \sum_{k=0}^{\infty} \sum_{j=0}^{k} \tilde{a}^{(i)}_{j-\theta_i} v_{k-j}(z_i;t,s;\mathbb{T}) \frac{(z-z_i)^{k-\theta_i-1}}{(k-j)!}.
\end{aligned}
$$

Notice that the factor $(z-z_i)^{k-\theta_i-1}$ does not depend on j. Therefore, the term $k=\theta_i$ forces $k-\theta_i-1=-1$, i.e., this is the term whose coefficient we need to compute the residue. Thus

$$
\begin{aligned}
\mathscr{U}\{f\}(t;s) &= \sum_{i=1}^{n} e_{z_i}(t,s;\mathbb{T}) \sum_{j=0}^{\theta_i} \frac{\tilde{a}^{(i)}_{j-\theta_i} v_{\theta_i-j}(z_i;t,s;\mathbb{T})}{(\theta_i-j)!} \\
&= \sum_{i=1}^{n} e_{z_i}(t,s;\mathbb{T}) \sum_{j=0}^{\theta_i} \frac{(\theta_i-j) a^{(i)}_{j-\theta_i} v_{\theta_i-j}(z_i;t,s;\mathbb{T})}{(\theta_i-j)!} \\
&= \sum_{i=1}^{n} e_{z_i}(t,s;\mathbb{T}) \sum_{j=0}^{\theta_i-1} \frac{(\theta_i-j) a^{(i)}_{j-\theta_i} v_{\theta_i-j}(z_i;t,s;\mathbb{T})}{(\theta_i-j)!},
\end{aligned}
$$

as was to be shown. □

Corollary 4.3.1. *The following formula holds:*

$$\mathscr{U}_{\mathbb{T}}\{1\}(t;s) = (t-s)_{\mathbb{T}}.$$

Proof. This is evident from Figure 4.1, but we will compute it from Theorem 4.3.3. From Table 4.1, we have $F(z) = \mathscr{L}\{1\}(z;s) = \dfrac{1}{z}$. This is a function with one pole of order 1 at $z=0$. Hence Theorem 4.3.3 shows

$$\mathscr{U}_{\mathbb{T}}\{1\}(t;s) = e_0(t,s;\mathbb{T}) v_1(0;t,s) = (t-s)_{\mathbb{T}},$$

as was to be shown. □

Corollary 4.3.2. *The following formula holds:*

$$\mathscr{U}_{\mathbb{T}}\{(t-s)_{\mathbb{T}}\}(t;s) = (t-s)^2_{\mathbb{T}}.$$

Proof. Table 4.1 shows us that $F(z) = \mathscr{L}_{\mathbb{T}}\{(t-s)_{\mathbb{T}}\}(z;s) = \dfrac{2!}{z^2}$, and we see that F has a pole of order 2 at $z_1 = 0$. Since F is already expressed as a Laurent series, we have $a^{(1)}_{-1} = 0$ and $a^{(1)}_{-2} = 1$. We may also compute

$$\begin{aligned} v_2(z;t,s;\mathbb{T}) &= \frac{\mathrm{d}}{\mathrm{d}z} m_z(t,s) + m_z(t,s)^2 \\ &= -\int_s^t \frac{\mu(\tau)}{1+\mu(\tau)z}\Delta\tau + \left(\int_s^t \frac{1}{1+\mu(\tau)z}\Delta\tau\right)^2, \end{aligned}$$

and so

$$v_2(0;t,s;\mathbb{T}) = -\int_s^t \mu(\tau)\Delta\tau + \left(\int_s^t 1\Delta\tau\right)^2 = (t-s)^2 - \int_s^t \mu(\tau)\Delta\tau.$$

Hence

$$\begin{aligned} \mathscr{U}_{\mathbb{T}}\{h_1(\cdot,s;\mathbb{T})\}(t;s) &= e_0(t,s;\mathbb{T}) \sum_{j=0}^{1} \frac{(2-j)a^{(1)}_{j-2} v_{2-j}(0;t,s)}{(2-j)!} \\ &= \frac{2v_2(0;t,s)}{2!} \\ &= (t-s)^2 - \int_s^t \mu(\tau)\Delta\tau. \end{aligned}$$

It is well known [17, page 1298] [32, pages 21–22] that

$$2!h_2(t,s;\mathbb{T}) = (t-s)^2_{\mathbb{T}} = (t-s)^2 - \int_s^t \mu(\tau)\Delta\tau,$$

and so we have verified

$$\mathscr{U}_{\mathbb{T}}\{(t-s)_{\mathbb{T}}\}(t;s) = (t-s)^2_{\mathbb{T}},$$

as was to be shown. $\square$

Corollary 4.3.3. *The following formula holds for a regressive constant α:*

$$\mathscr{U}_{\mathbb{T}}\{e_\alpha(t,s)\}(t) = m_\alpha(t,s)e_\alpha(t,s).$$

Proof. By Table 4.1, we see that

$$F(z) = \mathscr{L}_{\mathbb{T}}\{e_\alpha(\cdot,s;\mathbb{T})\}(z;s) = \frac{1}{z-\alpha},$$

which has one pole of order 1 at $z_1 = \alpha$. It is already expanded in an appropriate series, and so we see that $a_{-1}^{(1)} = 1$. Therefore

$$\mathscr{U}_{\mathbb{T}}\{e_\alpha(\cdot,t;\mathbb{T})\}(t;s) = e_\alpha(t,s;\mathbb{T})\left[\frac{1\cdot 1\cdot v_1(\alpha;t,s)}{1!}\right] = m_\alpha(t,s)e_\alpha(t,s;\mathbb{T}),$$

as was to be shown. $\square$

4.4. SPECIAL FUNCTIONS ON TIME SCALES USING $\mathscr{U}_{\mathbb{T}}$

In this section, we restrict our attention to time scales $\mathbb{T}$ for which Theorem 4.3.3 holds. To generalize the results to all time scales requires extension of the formula (99) to all time scales. We know that $\mathscr{U}_{\mathbb{T}}$ is a time scale analogue of Lemma 3.2.1. Theorem 4.3.3 tells us how to compute $\mathscr{U}_{\mathbb{T}}$ for any function $f\colon \mathbb{T}\to\mathbb{C}$ of exponential order α. If we allow the use of the $\mathscr{U}_{\mathbb{T}}$ operator in dynamic equations, then we may replicate most of the results obtained in Section 2. Define the time scale Bessel function analogous to (78)

$$J_\nu(t,s,\xi,\alpha,\gamma;\mathbb{T}) = \sum_{k=0}^{\infty} \frac{(-1)^k \xi^{2k+\nu}}{\Gamma(k+\nu+1)2^{2k+\nu}} \frac{(t-s)_{\mathbb{T}}^{\gamma(2k+\nu)+\alpha}}{k!}. \tag{100}$$

Examine the proof of Theorem 3.10, and notice that all steps in the proof involve only using the formula $\Delta(t-s)_{h\mathbb{Z}}^{k} = k(t-s)_{h\mathbb{Z}}^{k-1}$, algebraic manipulation of coefficients, and appeals to Lemma 3.2.1. Each time we apply Lemma 3.2.1 here, we differentiate first, and so to generalize this to time scales, we claim that $J_\nu(\cdot,s,\xi,\alpha,\gamma;\mathbb{T})$ solves the following analogue of (3.10.1) for $\gamma \in \mathbb{Z}$:

$$\mathscr{U}_{\mathbb{T}}^{2}\left\{y^{\Delta^2}\right\}(t;s) + (1-2\alpha)\mathscr{U}\{y^{\Delta}\}(t;s) + \xi^2\gamma^2\mathscr{U}_{\mathbb{T}}^{2\gamma}\{y\}(t;s) + (\alpha^2-\nu^2\gamma^2)y(t) = 0.$$

We will work through the proof that the time scale Bessel functions solve this equation, and then cite the rest of the theorems in this section in the $\mathscr{U}$-notation. Let $\psi(t) = J_\nu(t,s,\xi,\alpha,\gamma;\mathbb{T})$. Calculate

$$\mathscr{U}_{\mathbb{T}}^{2}\{\psi^{\Delta^2}\}(t;s) = \sum_{k=2}^{\infty}\frac{(-1)^k\xi^{2k+\nu}(\gamma(2k+\nu)+\alpha)(\gamma(2k+\nu)+\alpha-1)(t-s)_{\mathbb{T}}^{\gamma(2k+\nu)+\alpha}}{\Gamma(k+\nu+1)2^{2k+\nu}k!},$$

now compute

$$\mathscr{U}_{\mathbb{T}}\{\psi^{\Delta}\}(t;s) = \sum_{k=1}^{\infty}\frac{(-1)^k\xi^{2k+\nu}(\gamma(2k+\nu)+\alpha)(t-s)_{\mathbb{T}}^{\gamma(2k+\nu)+\alpha}}{\Gamma(k+\nu+1)2^{2k+\nu}k!},$$

and finally reindex the series to see

$$\psi(t) = -\frac{1}{\xi^2}\sum_{k=1}^{\infty}\frac{4k(k+\nu)(-1)^k\xi^{2k+\nu}(t-s)_{\mathbb{T}}^{\gamma(2k+\nu)+\alpha-2\gamma}}{\Gamma(k+\nu+1)2^{2k+\nu}k!}$$

and thus

$$\xi^2\mathscr{U}_{\mathbb{T}}^{2\gamma}\{\psi\}(t;s) = -\sum_{k=1}^{\infty}\frac{4k(k+\nu)(-1)^k\xi^{2k+\nu}(t-s)_{\mathbb{T}}^{\gamma(2k+\nu)+\alpha}}{\Gamma(k+\nu+1)2^{2k+\nu}k!}.$$

The proof completes by plugging these expressions into the left-hand side of the time scale Bessel equation and computing

$$(\gamma(2k+\nu)+\alpha)(\gamma(2k+\nu)+\alpha-1)+(1-2\alpha)(\gamma(2k+\nu)+\alpha)-\gamma^2(4k(k+\nu))+\alpha^2-\nu^2\gamma^2=0.$$

We define the time scale analogue of (80) by

$$J_\nu(t,s;\mathbb{T})=J_\nu(t,s,1,0,1;\mathbb{T})=\sum_{k=0}^{\infty}\frac{(-1)^k(t-s)_{\mathbb{T}}^{2k+\nu}}{\Gamma(k+\nu+1)2^{2k+\nu}k!}.$$

We have listed the properties of this function in Table 4.2. We now summarize the time scale analogues of discrete Bessel function results. First we prove an analogue of Theorem 3.10.2, because the proof does not work without a time scale analogue of Theorem 3.6.1.

Theorem 4.4.1. *The following formula holds:*

$$\mathscr{L}_{\mathbb{T}}\{J_0(\cdot,s;\mathbb{T})\}(z;s)=\frac{1}{\sqrt{z^2+1}}.$$

Proof. Notice that

$$\frac{z}{\sqrt{z^2+1}}=\frac{1}{\sqrt{1+\frac{1}{z^2}}}=z\left(1+\frac{1}{z^2}\right)^{-\frac{1}{2}}.$$

By (17) and (23),

$$\begin{aligned}\left(1+\frac{1}{z^2}\right)^{-\frac{1}{2}}&=\sum_{k=0}^{\infty}\binom{-\frac{1}{2}}{k}\frac{1}{z^{2k}}\\&=\sum_{k=0}^{\infty}\frac{\Gamma(-\frac{1}{2}+1)}{k!\Gamma(-\frac{1}{2}-k+1)}\frac{1}{z^{2k}}\end{aligned}$$

$$= \sum_{k=0}^{\infty} \frac{\sqrt{\pi}}{k!\Gamma(\frac{1}{2}-k)} \frac{1}{z^{2k}}$$
$$= \sum_{k=0}^{\infty} \frac{(-1)^k (2k)!}{(k!)^2 2^{2k}} \frac{1}{z^{2k}}.$$

Therefore
$$\frac{1}{\sqrt{z^2+1}} = \sum_{k=0}^{\infty} \frac{(-1)^k (2k)!}{(k!)^2 2^{2k}} \frac{1}{z^{2k+1}}.$$

By Table 4.1, we see $\mathscr{L}_{\mathbb{T}}^{-1}\left\{\frac{1}{z^{2k+1}}\right\}(z;s) = \frac{1}{(2k)!}(t-s)_{\mathbb{T}}^{2k}$, and so

$$\mathscr{L}_{\mathbb{T}}^{-1}\left\{\frac{1}{\sqrt{z^2+1}}\right\}(z;s) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(k!)^2 2^{2k}}(t-s)_{\mathbb{T}}^{2k},$$

which is the series for $J_0(t,s;\mathbb{T})$, as was to be shown. $\square$

Table 4.2. Properties of time scale Bessel functions.

Formula	Analogue of
$\mathscr{L}_{\mathbb{T}}\{J_0(\cdot,s;\mathbb{T})\}(z;s)=\dfrac{1}{\sqrt{z^2+1}}$	Theorem 4.4.1
$\left(J_0 * J_0\right)(t,s;\mathbb{T})=\sin_1(t,s;\mathbb{T})$	Corollary 3.10.2
$\mathscr{L}_{\mathbb{T}}\left\{J_\nu\left(t,s,2\sqrt{\alpha},\dfrac{\nu}{2},\dfrac{1}{2};\mathbb{T}\right)\right\}(z;s)=\dfrac{\alpha^{\frac{\nu}{2}}}{z^{\nu+1}e^{\frac{\alpha}{2}}}$	Theorem 3.10.4
$J_{-n}(t,s;\mathbb{T})=(-1)^n J_n(t,s;\mathbb{T})$	Theorem 3.10.5
$\mathscr{U}_{\mathbb{T}}\left\{J_\nu^\Delta(\cdot,s;\mathbb{T})\right\}(t;s)=\nu J_\nu(t,s;\mathbb{T})-\mathscr{U}_{\mathbb{T}}\left\{J_{\nu+1}(\cdot,s;\mathbb{T})\right\}(t;s)$	Theorem 3.10.6
$\mathscr{U}_{\mathbb{T}}\left\{J_\nu^\Delta(\cdot,s;\mathbb{T})\right\}=-\nu J_\nu(t,s;\mathbb{T})+\mathscr{U}_{\mathbb{T}}\left\{J_{\nu-1}(\cdot,s;\mathbb{T})\right\}$	Theorem 3.10.7
$2\nu J_\nu(t,s;\mathbb{T})=\mathscr{U}_{\mathbb{T}}\left\{J_{\nu-1}(\cdot,s;\mathbb{T})\right\}(t;s)+\mathscr{U}_{\mathbb{T}}\left\{J_{\nu+1}(\cdot,s;\mathbb{T})\right\}(t;s)$	Corollary 3.10.3
$2\Delta_h J_\nu(t,s;\mathbb{T})=J_{\nu-1}(t,s;\mathbb{T})-J_{\nu+1}(t,s;\mathbb{T})$	Corollary 3.10.3
$\Delta_h\left[\mathscr{U}_{\mathbb{T}}^n\left\{J_\nu(\cdot,s;\mathbb{T})\right\}(t;s)\right]=\mathscr{U}_{\mathbb{T}}^n\left\{J_{\nu-1}(\cdot,s;\mathbb{T})\right\}(t;s);n=0,1,2,\ldots$	Theorem 3.10.8

An analogue of Hermite polynomials already exists on time scales [34], but it is different than both analogues in this thesis. We will not attempt to generalize the second Hermite polynomials because they are defined via repeated differentiation which is not guaranteed to exist on an arbitrary time scale. However, we will write out the time scale analogues of the formulas for the analogue of (69). Define the time scale Hermite polynomials (of type I) to be

$$\mathcal{H}_n(t,s;\mathbb{T}) = \sum_{k=0}^{m} \frac{(-1)^k n!(t-s)_{\mathbb{T}}^{n-2k-1}}{k!\Gamma(n-2k)2^k}.$$

We have included the properties of the time scale Hermite polynomials in Table 4.4. We also define the time scale associated Laguerre polynomials by

$$L_n^{(\alpha)}(t,s;\mathbb{T}) = \sum_{k=0}^{n} \frac{(-1)^k (t-s)_{\mathbb{T}}^k}{k!} \binom{n+\alpha}{n-k}.$$

We have included properties of the time scale associated Laguerre polynomials in Table 4.5. Finally we define the generalized hypergeometric series on time scales by the following analogue of (87):

$${}_pF_q(\mathbf{a};\mathbf{b};t,s,n,\xi;\mathbb{T}) = \sum_{k=0}^{\infty} \frac{\mathbf{a}^{\overline{k}}}{\mathbf{b}^{\overline{k}}} \xi^k \frac{(t-s)_{\mathbb{T}}^{nk}}{k!}.$$

We have included the properties of this function in Table 4.3. We define the $\Upsilon_{\mathbb{T}}$ operator by $\Upsilon_{\mathbb{T}} = \mathscr{U}_{\mathbb{T}} \dfrac{\Delta}{\Delta t}$.

Table 4.3. Properties of generalized hypergeometric series on time scales.

Formula	Analogue of
${}_pF_q^{\Delta^n}(\mathbf{a};\mathbf{b};t,s,1,\xi;\mathbb{T}) = \dfrac{\mathbf{a}^{\overline{n}}}{\mathbf{b}^{\overline{n}}}{}_pF_q(\mathbf{a}+n;\mathbf{b}+n;t,s,1,\xi;\mathbb{T})$	Corollary 3.11.2
$e_\alpha(t,s;\mathbb{T}) = {}_0F_0(;;t,s,1,\alpha;\mathbb{T})$	Theorem 3.11.6
$\cosh_\alpha(t,s;\mathbb{T}) = {}_0F_1\left(;\dfrac{1}{2};t,s,2,\dfrac{\alpha^2}{4};\mathbb{T}\right)$	Theorem 3.11.7
$\sinh_\alpha(t,s;\mathbb{T}) = \alpha\mathscr{U}_{\mathbb{T}}\left\{{}_0F_1\left(;\dfrac{3}{2};t,s,2,\dfrac{\alpha^2}{4};\mathbb{T}\right)\right\}$	Theorem 3.11.8
$\cos_\alpha(t,s;\mathbb{T}) = {}_0F_1\left(;\dfrac{1}{2};t,s,2,-\dfrac{\alpha^2}{4};\mathbb{T}\right)$	Theorem 3.11.9
$\sin_\alpha(t,s;\mathbb{T}) = \alpha\mathscr{U}_{\mathbb{T}}\left\{{}_0F_1\left(;\dfrac{3}{2};\cdot,s,2,-\dfrac{\alpha^2}{4};\mathbb{T}\right)\right\}$	Theorem 3.11.9
$L_n^{(\alpha)}(t,s;\mathbb{T}) = \dfrac{(\alpha+1)^{\overline{n}}}{n!}{}_1F_1(-n;\alpha+1;t,s,1,1;\mathbb{T})$	Theorem 3.11.10
$2^\nu\Gamma(\nu+1)J_\nu(t,s,1,-\nu,1;\mathbb{T}) = {}_0F_1\left(;\nu+1;t,s,2,-\dfrac{1}{4};\mathbb{T}\right)$	Theorem 3.11.11

Table 4.4. Properties of $\mathcal{H}_n(t,s;\mathbb{T})$.

Formula	Analogue of
$\Delta\mathcal{H}_n^{\Delta\Delta}(t,s;\mathbb{T}) - \mathscr{U}_{\mathbb{T}}\left\{\mathcal{H}_n^{\Delta}(t,s;\mathbb{T})\right\} + n\mathcal{H}_n(t,s;\mathbb{T}) = 0$	Theorem 3.9.1
$\mathcal{H}_n^{\Delta}(t,s;\mathbb{T}) = n\mathcal{H}_{n-1}(t,s;\mathbb{T})$	Theorem 3.9.2
$\mathcal{H}_{n+1}(t,s;\mathbb{T}) = \mathscr{U}_{\mathbb{T}}\{H_n(\cdot - h,s;\mathbb{T})\}(t;s) - n\mathcal{H}_{n-1}(t,s;\mathbb{T})$	Theorem 3.9.3

Table 4.5. Properties of time scale $L_n^{(\alpha)}(t,s;\mathbb{T})$.

Formula	Analogue of
$\mathscr{U}_{\mathbb{T}}\{L_n^{(\alpha)^{\Delta\Delta}}\} + (\alpha+1)L_n^{(\alpha)^{\Delta}} - \mathscr{U}_{\mathbb{T}}\{y^{\Delta}\} + nL_n^{(\alpha)} = 0$	Theorem 3.9.9
$(n+1)L_n^{(\alpha)} = (2n+\alpha+1)L_n^{(\alpha)} - \mathscr{U}_{\mathbb{T}}\{L_n^{(\alpha)}\}(t;s) - (n+\alpha)L_{n-1}^{(\alpha)}$	Theorem 3.9.10
$\mathscr{U}_{\mathbb{T}}\{L_n^{(\alpha)^{\Delta}}\} = nL_n^{(\alpha)} - (n+\alpha)L_{n-1}^{(\alpha)}$	Theorem 3.9.11
$L_n^{\alpha^{\Delta}} = -L_{n-1}^{(\alpha+1)}$	Theorem 3.9.12

5. CONCLUSION

We have seen many analogues of special functions on time scales, and it is clear that we have barely scratched the surface. We have introduced new natural time scale trigonometric functions and derived some of their properties. We have found a nice formula for the gamma function on $h\mathbb{Z}$. We have shown that the time scale monomials may converge to different branches of the power function depending on which complexification is chosen. We have derived some formulas for an analogue of the Gaussian bell whose Δ-derivative is proportional to a linear function instead of an exponential. We have demonstrated that orthogonality of a set of polynomials is not preserved under finding time scale analogues using Lemma 3.2.1, so perhaps another method of generalization can be developed to preserve orthogonality. We have fleshed out in detail a theory of Bessel functions of the first kind. In the end, we have tied together many of these topics into a theory of an analogue of a hypergeometric series. It is clear that much work needs to be done in expanding these results to a general time scale.

The standard special functions on time scales still have properties we need to discover — for instance, can analogues of Rodrigues-type formulas like (71) exist on a time scale? Are there nice representations of the general time scale gamma function? In what sense can we find a partial dynamic equation analogous to Laplace's equation in cylindrical coordinates that leads to classical Bessel functions? Does the time scale gamma function in [11] obey a Bohr–Mollerup theorem analogue similar to the results in [3]? A major open problem in the theory remains to be the inverse Laplace transform for a general time scale. With one, we may extend $\mathscr{U}_{\mathbb{T}}$ to all time scales and replicate the results in this thesis on all time scales. Can the polynomial shift operator be simplified in terms of standard time scale operations? A major area left alone by this thesis is generating functions — such analogues are difficult to achieve because of the regressive function operations and the lack of an easy-to-use polynomial shift operator. In what way may we find complexifications similar to the discrete rising and falling complexifications? If a time scale contains a limit point, do we obtain any type of uniqueness for complex extensions in neighborhoods of the limit point? How do the results extend to the “time scale complex plane” as defined in the sense of [7]?

BIBLIOGRAPHY

[1] Milton Abramowitz and Irene A. Stegun. *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, volume 55 of *National Bureau of Standards Applied Mathematics Series*. For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C., 1964.

[2] George E. Andrews, Richard Askey, and Ranjan Roy. *Special functions*, volume 71 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1999.

[3] Richard Askey. The q-gamma and q-beta functions. *Applicable Anal.*, 8(2):125–141, 1978/79.

[4] Nuno R. O. Bastos, Dorota Mozyrska, and Delfim F. M. Torres. Fractional derivatives and integrals on time scales via the inverse generalized Laplace transform. *Int. J. Math. Comput.*, 11(J11):1–9, 2011.

[5] Michael Berry. Why are special functions special? *Phys. Today*, 54(4):11–12, apr 2001.

[6] Abigail Bishop, Tom Cuchta, Kathryn Lokken, and Oliver Pechenik. The nilradical and non-nilradical graphs of commutative rings. *Int. J. Algebra*, 2(17-20):981–994, 2008.

[7] Martin Bohner and Gusein Sh. Guseinov. An introduction to complex functions on products of two time scales. *J. Difference Equ. Appl.*, 12(3-4):369–384, 2006.

[8] Martin Bohner and Gusein Sh. Guseinov. The convolution on time scales. *Abstr. Appl. Anal.*, pages Art. ID 58373, 24, 2007.

[9] Martin Bohner, Gusein Sh. Guseinov, and Başak Karpuz. Properties of the Laplace transform on time scales with arbitrary graininess. *Integral Transforms Spec. Funct.*, 22(11):785–800, 2011.

[10] Martin Bohner, Gusein Sh. Guseinov, and Başak Karpuz. Further properties of the Laplace transform on time scales with arbitrary graininess. *Integral Transforms Spec. Funct.*, 24(4):289–301, 2013.

[11] Martin Bohner and Başak Karpuz. The gamma function on time scales. *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.*, 20(4):507–522, 2013.

[12] Martin Bohner and Allan Peterson. *Dynamic equations on time scales*. Birkhäuser Boston, Inc., Boston, MA, 2001. An introduction with applications.

[13] Martin Bohner and Allan Peterson. Laplace transform and Z-transform: unification and extension. *Methods Appl. Anal.*, 9(1):151–157, 2002.

[14] Martin Bohner and Allan Peterson, editors. *Advances in dynamic equations on time scales*. Birkhäuser Boston, Inc., Boston, MA, 2003.

[15] Ruel V. Churchill and James Ward Brown. *Complex variables and applications*. McGraw-Hill Book Co., New York, fourth edition, 1984.

[16] Tom Cuchta. Infinitesimal time scale calculus. *Marshall University Theses, Dissertations and Capstones*. http://mds.marshall.edu/etd/36.

[17] John M. Davis, Ian A. Gravagne, Billy J. Jackson, Robert J. Marks, II, and Alice A. Ramos. The Laplace transform on time scales revisited. *J. Math. Anal. Appl.*, 332(2):1291–1307, 2007.

[18] Gustav Doetsch. *Introduction to the theory and application of the Laplace transformation*. Springer-Verlag, New York-Heidelberg, 1974. Translated from the second German edition by Walter Nader.

[19] Saber Elaydi. *An introduction to difference equations*. Undergraduate Texts in Mathematics. Springer, New York, third edition, 2005.

[20] L. Erbe, A. Peterson, and M. Simon. Square integrability of Gaussian bells on time scales. *Comput. Math. Appl.*, 49(5-6):871–883, 2005.

[21] Arthur Erdélyi, Wilhelm Magnus, Fritz Oberhettinger, and Francesco G. Tricomi. *Higher transcendental functions. Vol. II*. Robert E. Krieger Publishing Co., Inc., Melbourne, Fla., 1981. Based on notes left by Harry Bateman, Reprint of the 1953 original.

[22] David Fowler. The binomial coefficient function. *Amer. Math. Monthly*, 103(1):1–17, 1996.

[23] Robert E. Greene and Steven G. Krantz. *Function theory of one complex variable*, volume 40 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, third edition, 2006.

[24] Gusein Sh. Guseinov. Integration on time scales. *J. Math. Anal. Appl.*, 285(1):107–127, 2003.

[25] B. D. Haile and L. M. Hall. Polynomial and series solutions of dynamic equations on time scales. *Dynam. Systems Appl.*, 12(1-2):149–157, 2003. Special issue: dynamic equations on time scales.

[26] Raegan J. Higgins and Allan Peterson. Cauchy functions and Taylor's formula for time scales $\mathbb{T}$. In *Proceedings of the Sixth International Conference on Difference Equations*, pages 299–308. CRC, Boca Raton, FL, 2004.

[27] Stefan Hilger. Special functions, Laplace and Fourier transform on measure chains. *Dynam. Systems Appl.*, 8(3-4):471–488, 1999.

[28] Mourad E. H. Ismail. *Classical and quantum orthogonal polynomials in one variable*, volume 98 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 2005. With two chapters by Walter Van Assche, With a foreword by Richard A. Askey.

[29] Walter G. Kelley and Allan C. Peterson. *Difference equations*. Harcourt/Academic Press, San Diego, CA, second edition, 2001. An introduction with applications.

[30] Bonita A. Lawrence and Ralph W. Oberste-Vorth. Solutions of dynamic equations with varying time scales. In *Difference equations, special functions and orthogonal polynomials*, pages 452–461. World Sci. Publ., Hackensack, NJ, 2007.

[31] Yudell L. Luke. *The special functions and their approximations, Vol. I.* Mathematics in Science and Engineering, Vol. 53. Academic Press, New York-London, 1969.

[32] Thomas Matthews. *Probability theory on time scales and applications to finance and inequalities*. ProQuest LLC, Ann Arbor, MI, 2011. Thesis (Ph.D.)–Missouri University of Science and Technology.

[33] Dorota Mozyrska and Ewa Pawłuszewicz. Functional series on time scales. *Int. J. Math. Stat.*, 2(S08):94–105, 2008.

[34] Dorota Mozyrska and Ewa Pawłuszewicz. Hermite's equations on time scales. *Appl. Math. Lett.*, 22(8):1217–1219, 2009.

[35] Earl D. Rainville. *Special functions*. Chelsea Publishing Co., Bronx, N.Y., first edition, 1971.

[36] Walter Rudin. *Principles of mathematical analysis*. McGraw-Hill Book Co., New York-Auckland-Düsseldorf, third edition, 1976. International Series in Pure and Applied Mathematics.

[37] Joel L. Schiff. *The Laplace transform*. Undergraduate Texts in Mathematics. Springer-Verlag, New York, 1999. Theory and applications.

[38] F. G. Tricomi and A. Erdélyi. The asymptotic expansion of a ratio of gamma functions. *Pacific J. Math.*, 1:133–142, 1951.

[39] Eric John Watson. *Laplace transforms and applications*, volume 10 of *VNR New Mathematics Library*. Van Nostrand Reinhold Co., New York-London, 1981.

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Tom Cuchta was born in Wheeling, West Virginia on 6 January 1988, and his hometown is Moundsville, West Virginia. He graduated from John Marshall High School in 2006 in Glen Dale, West Virginia. After high school, he attended Marshall University in Huntington, West Virginia and soon began working on the Marshall University Differential Analyzer Team. With this team, he attended the annual Joint Mathematics Meeting in New Orleans, presented a poster at the Council of Undergraduate Research poster session, and he spoke at an MAA conference in 2007. In 2008, he attended the Wabash College REU in Crawfordsville, Indiana funded by NSF Grant DMS-0755260 and published [6]. In 2009 he attended the Joint Mathematics Meeting in Washington, D.C. regarding his REU research. He graduated with degrees in mathematics and applied mathematics in 2009. He stayed at Marshall University for his masters and began studying time scale calculus while working as a graduate teaching assistant. This study culminated in the thesis [16] for which he received his Master of Arts in mathematics in 2011. Afterwards he attended Missouri S&T in Rolla, Missouri and worked as a graduate teaching assistant. While there, he was an active participant in the time scales seminar, analysis seminar, and the two topology seminars. In Spring 2014, he visited Vahrn, Italy to attend the Conference on Partial Differential Equations. On 9 August 2014, he married Brittany Whited. He received his PhD from Missouri S&T in December 2015.