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Bootstrap-based confidence intervals in partially accelerated life testing

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BOOTSTRAP-BASED CONFIDENCE INTERVALS IN PARTIALLY ACCELERATED LIFE TESTING

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ABSTRACT

Accelerated life testing (ALT) is utilized to estimate the underlying failure distribution and related parameters of interest in situations where the components under study are designed for long life and therefore will not yield failure data within a reasonable test period. In ALT, life testing is carried out under two or more higher than normal stress levels, with the resulting acceleration of the failure process yielding a sufficient amount of un-censored life-span data within a practical test duration. Usually one (or more) parameters of the life distribution is linked to the stress level through a suitably selected model based on a well-understood relationship. The estimate of this model is then utilized to determine the life distribution of the components under normal use (design use) conditions. Partially accelerated life testing (PALT) is preferable over accelerated life testing (ALT) in situations where such a model linking the stress to the distribution parameters is unavailable. In this study, parametric and nonparametric bootstrap based methods for obtaining confidence intervals for the parameters of the life distribution as well as a the lower confidence bound for the mean life under normal conditions are developed for both the Weibull and Generalized exponential life distributions under Type I censoring. Monte-Carlo simulation studies are carried out to study the performance of the confidence intervals based on the proposed methods against those of intervals obtained using the traditional delta method. Results show that the bootstrap-based methods performs as well as or better than asymptotic distribution-based methods in most cases.

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1. INTRODUCTION

When products are designed to be highly reliable and therefore have a long life-span, standard life testing, where a sample of units is tested under normal use conditions, will not produce a sufficient number of failures to enable the researcher to obtain good estimates of the parameters of interest. One solution to the problem is to subject the specimens in the sample to higher than normal stress levels. The stress factors can be temperature, humidity, pressure, repetitive flexing at a higher than normal rate, or any other variable that can accelerate the failure process. Since the goal of the study is to estimate the parameters of the underlying life distribution and the expected life-span of the products under normal (design) use conditions, a mathematical model that relate the stress level to one or more parameters of the life distribution has to be estimated and then utilized to extrapolate results obtained at high stress levels to those at the normal level. This model that links stress to the distribution parameter(s), however, must be based on well-understood and/or empirically verified relationship (Meeker and Escobar 1998, p. 495). When such a model is available, an accelerated life test (ALT) can be performed where test specimens are subjected to two or more distinct higher than normal stress levels. The higher stress levels accelerate the failure process, thus yielding a sufficient number of un-censored failure data within a reasonable test period. When a reasonable model that links the stress level to distributional parameter(s) is not available, the partially accelerated life test (PALT) procedure is available as an alternative. In PALT, the test specimens are subjected to a single high stress level as well as stress at the normal level.

Each of these accelerated life test methods can be implemented in two different ways, namely using a constant stress protocol or utilizing a step-stress approach to life testing. In the constant stress procedure, independent samples of specimens are assigned to each of the designated high stress levels, and all specimens in a sample are kept at the assigned stress throughout the experiment. That is, the stress is kept constant within a sample. For example, in PALT, some specimens may experience normal stress throughout the experiment while others are subjected to a higher stress level which is kept constant during the test period. In step-stress method, all specimens are first subjected to one level of stress for a given period of time, and the test specimens that are still functional are subjected to a higher stress level. In this study, the focus will be limited to the constant stress approach so discussions from here on will be on this method only.

In a certain type of constant stress life testing, a sample of product specimens are put to test over a pre-specified test period T and the life spans of the items that failed during this period are recorded. Since not all items on test may fail by time T, the life-span of some specimens are censored. This type of censoring is called Type I censoring. Alternatively, the experimenter can wait until a specific number of items fail and then stop the experiment. For example he/she can wait until 50% of the items fail. In this case we have what is termed as Type II censoring. Since the experimenter sets a specific time at which the experiment will end, the Type I censoring approach is preferable over Type II censoring. The experimenter who conducts a Type II censored experiment will not have a precise idea when the experiment is going to end because the time it takes for a specific percentage of items to fail is a random variable. However, the mathematics of the estimation procedure under Type I censoring can be complicated because the number of failures, R, is a random variable rather than a fixed number as is the case in Type II censoring. The work herein centers on experiments conducted under Type I censoring.

1.1. ACCELERATED LIFE TESTS (CONSTANT STRESS CASE)

In Accelerated Life Tests (ALT), the life-span, X, of a product is assumed to have a distribution (termed the life-distribution) with a probability density function $f(x, \underline{\theta})$, where $\underline{\theta} = (\theta_1, \theta_2, \cdots, \theta_p)'$ is a vector of parameters associated with the distribution such that one or more of the parameters in $\underline{\theta}$ are related to the stress S through a relationship whose functional form is known except for a few parameters. For example, θ_1 may be related to S through the function: $\theta_1 = g(S, \emptyset_0, \emptyset_1) = exp\{\emptyset_0 + \emptyset_1 S\}$. It is assumed that the other parameters in $\underline{\theta}$ are not related to S. To estimate the parameters, two independent samples of specimens of the product are tested, with one sample undergoing stress at an accelerated level S_L and the other sample subjected to an even higher stress level S_H. If $S_D$is the stress level at normal (design) use conditions, then we have the ordering $S_L < S_D < S_H$. From the experimental data, the parameters of the function g are estimated (in the example above we estimate $\emptyset_0$ and $\emptyset_1$). The parameters $\theta_i, i = 2,3, \ldots . p$ are estimated using combined data from both samples because they do not depend on the stress level. Then, using the estimated function, g, the value of θ_1 at the design stress level is estimated by the relationship $\hat{\theta}_1 = g(S_D, \hat{\emptyset}_0, \hat{\emptyset}_1)$ This yields an estimate of the life-distribution at normal use stress level.

1.1.1 A Brief Review of Relevant Literature. There are a large number of publications on ALTs and a relatively smaller but an appreciable number also available for PALTs. For brevity, we will refrain from discussing all of these, but limit the discussion to a select few of these publications. An excellent coverage of Accelerated Life tests is given in Nelson (1990). Other books include Mann, Schafer, and Singapuwalla (1974), Lawless (1982), Viertl (1988), Marvin Rausand and Hsyland (2004) Michelle, Hoang Jr, and David (2006),Guangbin Yang (2007), Tobias and Trindade (2011), and Meeker and Escobar (1998).

One of the more recent publications is Jayawardhana and Samaranayake (2003), that discussed obtaining lower prediction bounds for a future observation from a Weibull population at design (normal use) stress level, using Type II censored accelerated life test data. The scale parameter of the life distribution is assumed to have an inverse power relationship with the stress level. They showed that the method works well when the low and high stresses are reasonably far apart. Alferink and Samaranayake (2011) considered accelerated degradation models and developed confidence intervals for mean life using the Delta method and the bootstrap, assuming lognormal distribution with variance dependent on stress. Another interesting paper is Kamal, et al (2013), who presented a step stress ALT plan that works well. In step stress, the components are first put at a lower stress and the unfailed components are subjected to higher stress after a specific period. More recently, Jayawardhana and Samaranayake (2014), obtained predictive density of a future observation at normal use conditions using ALT method under lognormal life distribution and Type II censoring with non-constant variance.

1.2. PARTIALLY ACCELERATED LIFE TESTS (CONSTANT STRESS CASE)

The main drawback of accelerated life tests is the fact that the functional form of the model that relates stress to the parameters of the life-distribution has to be known. The form of this function can be dependent on the nature of the material the product under study is made of or the construction of the product. For some materials such as electrical insulators, the functional form of g is well known (Nelson, 1990). For some products, especially those constructed of new materials, such a function may not be easily assumed. In many situations, Partially Accelerated Life Tests (PALT) can overcome this problem. In PALT scenario, one set of product specimens are tested at normal use conditions while the other set is tested under high stress conditions. Rather than assume a function that links the model parameter θ_1 with stress, it is assumed that at higher stress, θ_1 takes a new value $\theta_1^* = \beta\theta_1$. That is, the acceleration changes θ_1 through a multiplicative constant. While the mathematics behind estimating both θ_1 and θ_1^* as well as the other parameters of the life distribution is not simple, the PALT methodology avoids the assumption of the linkage function g thus eliminating the chance of using an incorrect functional form. The main drawback of the PALT procedure is that one set of product specimens has to be tested at the normal use stress level thus forcing the experimenter to increase the product test time T in order to ensure that a sufficient number of specimens will fail under normal use conditions. This method, however, is ideal for life testing products such as chemicals, whose usable life-span is moderately long but may not run into many years.

Within the PALT, the literature works mention. Saxena and Zarrin (2013) used the Constant Stress Partially Accelerated Life Test (CSPALT) and assumed Type-I censoring under the Extreme Value Type-III distribution. The Extreme Value Type-III distribution has been recommended as appropriate for high reliability components. The authors used the Maximum Likelihood (ML) method to estimate the parameters of CSPALT model and confidence intervals for the model parameters were constructed. Note that the CSPALT plan is used to minimize the Generalized Asymptotic Variance (GAV) of the ML estimators of the model parameters.

Ismail (2013) derived the maximum likelihood estimators (MLEs) of the parameters of the GE distribution and the acceleration factor when the data are Type-II censored under constant-stress PALT model. The likelihood ratio bounds (LRB) method was used to obtain confidence bounds of the model parameters when the sample size is small. It is also shown that the maximum likelihood estimators are consistent and their asymptotic variances decrease as the sample size increases. The numerical results reported in the paper support the theoretical findings and showed that the estimated approximate confidence intervals for the three parameters are smaller when the sample size is larger.

Abdel-Hamid (2009), considered a constant PALT model when the observed failure times come from Burr(c,k) distribution under progressively Type-II right censoring. The MLEs of the parameters were obtained and their performance was studied through their mean squared errors and relative absolute biases. The paper also showed how to constructed approximate and bootstrap CIs for the parameters. The bootstrap CIs give more accurate results than the approximate intervals for small sample sizes, the Student's-t bootstrap CIs are better than the Percentile bootstrap CIs in the sense of having smaller widths. However, the differences between the lengths of CIs for the two methods decrease with the increase in sample size.

In this study, we develop PALT methodologies for constructing confidence intervals not only for the distribution parameters and the acceleration factor, but also a lower confidence bound for the mean life, under Type I censoring. Three types of confidence intervals and bounds are considered. They are the asymptotic intervals/bounds constructed from the delta-method and those constructed using the parametric bootstrap or the non-parametric bootstrap. The underlying distributions considered are the Weibull and the Generalized Exponential (GE). Methods for obtaining asymptotic or bootstrap-based confidence bounds for the mean life under PALT are not discussed in currently available literature for any type of life distribution, censoring scheme. Also, not available in current literature on PALT are bootstrap-based methods for constructing confidence intervals for distribution parameters of Weibull and

GE distribution and the acceleration factor under Type I censoring. This research aims to fill this gap.

2. BOOTSTRAP-BASED CONFIDENCE INTERVALS IN PARTIALLY ACCELERATED LIFE TESTING UNDER THE WEIBULL DISTRIBUTION

2.1 INTRODUCTION

Products which under normal use conditions last for a long period pose a problem in determining their mean life using standard life tests because only a very small fraction of them will fail under a testing period of reasonable duration. In such situations, practitioners resort to accelerated life tests (ALT). As Nelson (1980) puts it: “Accelerated life testing of a product or material is used to get information quickly on its life distribution.” In an ALT scenario, test units are run under two or more high stress levels to accelerate the failure process conditions yielding failure-time data sooner than under normal (design, field) use conditions. A model is fitted to the accelerated failure times and then extrapolated to estimate the life distribution under normal conditions. Alternatively, a known acceleration factor that adjusts a parameter of the life distribution to account for the higher stress is utilized for this purpose. This is quicker, cheaper, and more practical than testing at design use conditions. When there exists a mathematical model, which specifies the life-stress relationship, or an acceleration factor is known, the ALT is a very suitable approach to quickly obtain information useful for estimating the life distribution under normal use conditions. However, there are some situations in which neither the acceleration factor is known nor do life-stress models exist, or are very hard to assume. In such cases partially accelerated life tests (PALT) provide a better approach.

Under the PALT method, a portion of the test units are placed under the normal use stress conditions and the remaining units are tested under a suitably selected higher than normal stress level. The life distribution under the higher stress level is assumed to be the same as that under normal use, but with the scale parameter multiplied by an acceleration factor. This factor is estimated together with the other distribution parameters. Since there are more failure data from the units that received higher than normal stress level, the combined data provide better estimates of the common parameters.

One drawback of the PALT method is that unlike in the ALT, some units have to be tested under normal use. Thus this method is not suitable for components that are very long lasting. But items such as chemicals that have shelf-lives that are measured in months or a year or two can be tested using this method.

In the following, we develop PALT-based methodologies to obtain confidence bounds for the mean life and confidence intervals for the acceleration factor as well as the distribution parameters when the underlying distribution is Weibull. Type I censoring is also assumed. The methodologies considered are asymptotic methods as well as those relying on the parametric or the non-parametric bootstrap. This research extends the work of Ismail (2013) who assumed Type II censoring and employed only the traditional large sample approach to obtaining prediction intervals. While Ismail's work assumed a Generalized Exponential distribution as the underlying life distribution, we assume the Weibull in this study. The performances of the three methods are compared using a Monte-Carlo simulation study.

2.1.1 A Brief Review of Relevant Literature. Partial accelerated life test (PALT) is the one of methods used for reliability demonstration and prediction of components at normal conditions using data obtained at accelerated condition. It is a type of testing method that enables one to quickly get information over a variety of conditions, and is therefore an important tool for the reliability engineer. A brief outline of previous work on PALT is given below.

Nelson (1990) showed that the stress can be applied in two ways; as constant stress over the test period or in a step-stress fashion. In step-stress partially accelerated life tests (SS-PALT), a test item is first run at normal use conditions and, if it does not fail for a specified time, then it is subjected to a higher than normal stress level for another testing period. The SS-PALT were studied extensively by many authors, for example: Preeti Wanti Srivastava, Mittal (2010), Abdel-Hamid (2009).

However, the constant-stress PALT runs every item at either normal use condition or accelerated use condition only. Thus, we have two samples and units in each sample are run at a constant stress level unique to that sample, the levels being either normal or a pre-determined higher than normal level. Within the literature on PALTs, the following studies are worth mentioning. Saxena and Zarrin (2013) used the constant stress Partially Accelerated Life Test (CSPALT) and assumed Type-I censoring. The underlying life-distribution they incorporated was the Extreme Value Type-III distribution, which has been recommended as appropriate for high reliability components. The Maximum Likelihood (ML) method was employed by the authors to estimate the parameters of CSPALT model and confidence intervals for the model parameters were also constructed.

Ismail (2013) assumed a constant-stress PALT testing scenario under Type-II censoring. In addition to asymptotic confidence bounds, likelihood ratio bounds (LRB) method employed to obtain confidence bounds of the model parameters in small sample situations. The authors showed that the maximum likelihood estimators are consistent and their asymptotic variances decrease as the sample size increases. They also established that the estimated approximate confidence intervals for the three parameters become narrower with increase in sample size. These asymptotic results were confirmed using numerical simulations.

A constant PALT model was developed by Abdel-Hamid (2009), for the case when the underlying life distribution is Burr(c,k). They considered that sample is subjected to progressive Type-II right censoring. The MLEs of the parameters were obtained and their performance with respect to their mean squared errors and relative absolute biases were investigated. The author also constructed approximate and parameters bootstrap-based confidence intervals (CIs) for the parameters. It was shown that the bootstrap CIs gave more accurate results than the approximate intervals for small sample sizes, and that the Student's-t bootstrap CIs have smaller widths than the

Percentile bootstrap CIs. The differences between the lengths of CIs for the two methods, however, decreased with on increase in sample size.

2.1.2 The Weibull Distribution. The proposed PALT method is developed for the case where the underlying life distribution is Weibull. The Weibull probability density function is given by:

$$f\left(x;\alpha,\lambda\right)=\frac{\alpha}{\lambda}\left(\frac{x}{\lambda}\right)^{\alpha-1}e^{-\left(\frac{x}{\lambda}\right)^{\alpha}}, \qquad x>0,\alpha>0,\lambda>0, \tag{1}$$

And the cumulative distribution function is:

$$F\left(x;\alpha,\lambda\right)=1-e^{-\left(\frac{x}{\lambda}\right)^{\alpha}}, \tag{2}$$

where α is the shape parameter and λ the scale parameter.
Note that the Weibull distribution is used extensively in reliability literature because of the different shapes its hazard function can take based on different shape parameter values. The hazard (or the failure rate) function of the Weibull distribution is given by:

$$h\left(x;\alpha,\lambda\right)=\frac{f\left(x\right)}{1-F\left(x\right)}=\frac{\alpha}{\lambda}\left(\frac{x}{\lambda}\right)^{\alpha-1}. \tag{3}$$

2.2 THE PROPOSED PALT METHOD AND BOOTSTRAP INTERVALS

The following assumptions are made regarding the proposed PALT method.

1. The total number of units under test is n .
2. π denotes the proportion of sample units allocated to accelerated condition
3. $n\left(1-\pi\right)=n\overline{\pi}$ units, where $\overline{\pi}=1-\pi$, are allocated to normal (field) use conditions.
4. $n\pi$ units are allocated to the high stress condition (subject to acceleration)

2.2.1 Likelihood Function under Type I Censoring and Asymptotic C.I.s. Under Type I censoring, the censoring time, τ is fixed but the number of failures observed during the test duration τ is a random variable, say R.

Notation

x_i : Observed lifetime of item i tested at the normal (field) use conditions.

y_j : Observed lifetime of item j tested at high stress conditions.

δ_{u_i} : Indicator function denoting the censoring state of ith observation under normal use condition, with $\delta_{u_i} = 1$ if the observation is uncensored.

δ_{a_j} : Indicator function denoting the censoring state of jth observation under high stress condition, with $\delta_{a_j} = 1$ if the observation is uncensored.

n_u : Number of items that failed at normal use condition.

n_a : Number of items that failed at high stress condition.

τ: The censoring time of the life test (for all units).

$x_{(1)} \le \cdots \le x_{(n_u)} \le \tau$: Ordered failure times at normal use condition.

$y_{(1)} \le \cdots \le y_{(n_u)} \le \tau$: Ordered failure times at high stress condition.

β : Denotes the acceleration factor $(\beta > 1)$.

In type I censoring, τ is fixed but the number of failure values observed in time τ is a random variable. The number of items, *R*, failing before time τ is assumed to follow a binomial distribution $R \sim Bin(n, p)$, where $p = F_X(\tau;\alpha,\lambda) = 1 - e^{-\left(\frac{\tau}{\lambda}\right)^{\alpha}}$, under normal use conditions. Under high stress conditions the number of items failing will have a Binomial $Bin(n, p^*)$, distribution where $p^* = F_X(\tau;\alpha,\lambda,\beta) = 1 - e^{-\left(\frac{\beta x}{\lambda}\right)^{\alpha}}$. Then, for observation *i* under normal use conditions, we have,

$$\delta_{u_i} = \begin{cases} 1 & x_i \le \tau \\ 0 & 0/w \end{cases}, \quad i = 1, 2, \cdots, n\overline{\pi}. \tag{4}$$

Similarly, for observation *j* under a high stress condition, we have,

$$\delta_{a_j} = \begin{cases} 1 & y_j \le \tau \\ 0 & 0/w \end{cases}, \quad j = 1, 2, \cdots, n\pi, \tag{5}$$

$$\delta_{u_i} = 1 - \overline{\delta}_{u_i}, \qquad \delta_{a_j} = 1 - \overline{\delta}_{a_j},$$

with

$$\delta_{u_i} \sim Ber(p) \Rightarrow \sum_{i=1}^{n\bar{\pi}} \delta_{u_i} \sim Bin(n\bar{\pi}, p), \tag{6}$$

$$\delta_{a_j} \sim Ber(p) \Rightarrow \sum_{j=1}^{n\pi} \delta_{a_j} \sim Bin(n\pi, p). \tag{7}$$

We also have, under normal use conditions,

$$F_\tau\left(x;\alpha,\lambda \middle| \tau\right) = P\left(X \le x \middle| X \le \tau\right) = \frac{P(X \le x)}{P(X \le \tau)} = \frac{1-e^{-\left(\frac{x}{\lambda}\right)^\alpha}}{1-e^{-\left(\frac{\tau}{\lambda}\right)^\alpha}}$$

$$= \begin{cases} \dfrac{1-e^{-\left(\frac{x}{\lambda}\right)^\alpha}}{1-e^{-\left(\frac{\tau}{\lambda}\right)^\alpha}}, & x \le \tau \\ 1, & x > \tau, \end{cases} \tag{8}$$

and

$$f_\tau\left(x;\alpha,\lambda \middle| \tau\right) = \frac{f_X\left(x;\alpha,\lambda\right)}{F_X\left(\tau;\alpha,\lambda\right)} = \frac{\frac{\alpha}{\lambda}\left(\frac{x}{\lambda}\right)^{\alpha-1} e^{-\left(\frac{x}{\lambda}\right)^\alpha}}{1-e^{-\left(\frac{x}{\lambda}\right)^\alpha}}. \tag{9}$$

Thus, given $R = n_u$, the conditional density of the first r failure times under a normal use condition is equivalent to the joint density of an ordered random sample of size n_u from a truncated Weibull distribution, given by

$$f\left(x_{(1)},\cdots,x_{(n_u)} \middle| R = n_u\right) = n_u! \prod_{i=1}^{n_u} f_\tau\left(x_{(i)};\alpha,\lambda\right) = n_u! \prod_{i=1}^{n_u} \left[\frac{\frac{\alpha}{\lambda}\left(\frac{x_i}{\lambda}\right)^{\alpha-1} e^{-\left(\frac{x_i}{\lambda}\right)^\alpha}}{1-e^{-\left(\frac{\tau}{\lambda}\right)^\alpha}} \right]$$

$$= n_u! \frac{\left(\frac{\alpha}{\lambda}\right)^{n_u} e^{-\lambda \sum_{i=1}^{n_u} \left(\frac{x_i}{\lambda}\right)^\alpha}}{\left(1-e^{-\left(\frac{\tau}{\lambda}\right)^\alpha}\right)^{n_u}} \prod_{i=1}^{n_u} \left(\frac{x_i}{\lambda}\right)^{\alpha-1}. \tag{10}$$

The joint density of obtaining $R = n_u$ ordered observations at the values $x_{(1)},\cdots,x_{(n_u)}$ before time τ may be expressed as

$$f\left(x_{(1)},\cdots,x_{(n_u)}\right)= f\left(x_{(1)},\cdots,x_{(n_u)}\,\middle|\,R=n_u\right) bin(n_u;n\overline{\pi},p)=$$

$$= n_u!\frac{\left(\frac{\alpha}{\lambda}\right)^{n_u} e^{-\sum_{i=1}^{n_u}\left(\frac{x_i}{\lambda}\right)^{\alpha}}}{\left(1-e^{-\left(\frac{\tau}{\lambda}\right)^{\alpha}}\right)^{n_u}}\prod_{i=1}^{n_u}\left(\frac{x_i}{\lambda}\right)^{\alpha-1}\binom{n\overline{\pi}}{n_u}p^{n_u}\left(1-p\right)^{n\overline{\pi}-n_u}$$

$$=\frac{(n\overline{\pi})!}{(n\overline{\pi}-n_u)!}\frac{\left(\frac{\alpha}{\lambda}\right)^{n_u} e^{-\sum_{i=1}^{n_u}\left(\frac{x_i}{\lambda}\right)^{\alpha}}}{\left(1-e^{-\left(\frac{\tau}{\lambda}\right)^{\alpha}}\right)^{n_u}}\left(1-e^{-\left(\frac{\tau}{\lambda}\right)^{\alpha}}\right)^{n_u}\left(e^{-\left(\frac{\tau}{\lambda}\right)^{\alpha}}\right)^{n\overline{\pi}-n_u}\prod_{i=1}^{n_u}\left(\frac{x_i}{\lambda}\right)^{\alpha-1}$$

$$=\frac{(n\overline{\pi})!}{(n\overline{\pi}-n_u)!}\left(\frac{\alpha}{\lambda}\right)^{n_u} e^{-\sum_{i=1}^{n_u}\left(\frac{x_i}{\lambda}\right)^{\alpha}}\left(e^{-\left(\frac{\tau}{\lambda}\right)^{\alpha}}\right)^{n\overline{\pi}-n_u}\prod_{i=1}^{n_u}\left(\frac{x_i}{\lambda}\right)^{\alpha-1}.$$

Therefore, we have

$$f\left(x_{(1)},\cdots,x_{(n_u)}\right)\propto\left(\frac{\alpha}{\lambda}\right)^{n_u}\left(e^{-\left(\frac{\tau}{\lambda}\right)^{\alpha}}\right)^{n\overline{\pi}-n_u} e^{-\sum_{i=1}^{n_u}\left(\frac{x_i}{\lambda}\right)^{\alpha}}\prod_{i=1}^{n_u}\left(\frac{x_i}{\lambda}\right)^{\alpha-1}.$$

In a fashion similar to the argument made about the joint density of observations under normal use conditions, given $R = n_a$ the conditional density of the first r failure times under acceleration is equivalent to the joint density of an ordered random sample of size n_a from a truncated accelerated Weibull distribution. Therefore, for an item tested at accelerated condition, the probability density function is given by

$$f\left(x;\alpha,\lambda,\beta\right)=\frac{\beta\alpha}{\lambda}\left(\frac{\beta x}{\lambda}\right)^{\alpha-1} e^{-\left(\frac{\beta x}{\lambda}\right)^{\alpha}}, \qquad x>0,\ \alpha>0,\ \lambda>0,\ \beta>1,$$

where $Y=\beta^{-1}X$.

$$f\left(y_{(1)},\cdots,y_{(n_a)}\,|R=n_a\right)=n_a\,!\prod_{j=1}^{n_a}f_\tau\left(y_{(i)};\alpha,\lambda,\beta\right)=n_a\,!\prod_{j=1}^{n_a}\left[\frac{\frac{\alpha\beta}{\lambda}\left(\frac{\beta y_j}{\lambda}\right)^{\alpha-1}e^{-\left(\frac{\beta y_j}{\lambda}\right)^{\alpha}}}{1-e^{-\left(\frac{\beta\tau}{\lambda}\right)^{\alpha}}}\right]$$

$$=n_a\,!\frac{\left(\frac{\beta\alpha}{\lambda}\right)^{n_a}e^{-\sum_{i=1}^{n_a}\left(\frac{\beta y_j}{\lambda}\right)^{\alpha}}}{\left(1-e^{-\left(\frac{\beta\tau}{\lambda}\right)^{\alpha}}\right)^{n_a}}\prod_{j=1}^{n_a}\left(\frac{\beta y_j}{\lambda}\right)^{\alpha-1}. \qquad (11)$$

The joint density of obtaining $R=n_a$ ordered observations at the values $Y_{(1)},\cdots,Y_{(n_a)}$ before time, may be expressed as

$$f\left(y_{(1)},\cdots,y_{(n_a)}\right)=f\left(y_{(1)},\cdots,y_{(n_a)}\,|R=n_a\right)bin(n_a;n\pi,p^*)$$

$$=n_a\,!\frac{\left(\frac{\beta\alpha}{\lambda}\right)^{n_a}e^{-\sum_{i=1}^{n_a}\left(\frac{\beta y_j}{\lambda}\right)^{\alpha}}}{\left(1-e^{-\left(\frac{\beta\tau}{\lambda}\right)^{\alpha}}\right)^{n_a}}\prod_{j=1}^{n_a}\left(\frac{\beta y_j}{\lambda}\right)^{\alpha-1}\binom{n\pi}{n_a}\left(p^*\right)^{n_a}\left(1-p^*\right)^{n\pi-n_a}$$

$$=\frac{(n\pi)!}{(n\pi-n_a)!}\frac{\left(\frac{\beta\alpha}{\lambda}\right)^{n_a}e^{-\sum_{i=1}^{n_a}\left(\frac{\beta y_j}{\lambda}\right)^{\alpha}}}{\left(1-e^{-\left(\frac{\beta\tau}{\lambda}\right)^{\alpha}}\right)^{n_a}}\prod_{j=1}^{n_a}\left(\frac{\beta y_j}{\lambda}\right)^{\alpha-1}\left(1-e^{-\left(\frac{\beta\tau}{\lambda}\right)^{\alpha}}\right)^{n_a}\left(e^{-\left(\frac{\beta\tau}{\lambda}\right)^{\alpha}}\right)^{n\pi-n_a}$$

$$=\frac{(n\pi)!}{(n\pi-n_a)!}\left(\frac{\beta\alpha}{\lambda}\right)^{n_a}\left(e^{-\left(\frac{\beta\tau}{\lambda}\right)^{\alpha}}\right)^{n\pi-n_a}e^{-\sum_{i=1}^{n_a}\left(\frac{\beta y_j}{\lambda}\right)^{\alpha}}\prod_{j=1}^{n_a}\left(\frac{\beta y_j}{\lambda}\right)^{\alpha-1},$$

thus,

$$f\left(y_{(1)},\cdots,y_{(n_a)}\right)\propto\left(\frac{\beta\alpha}{\lambda}\right)^{n_a}\left(e^{-\left(\frac{\beta\tau}{\lambda}\right)^{\alpha}}\right)^{n\pi-n_a}e^{-\sum_{i=1}^{n_a}\left(\frac{\beta y_j}{\lambda}\right)^{\alpha}}\prod_{j=1}^{n_a}\left(\frac{\beta y_j}{\lambda}\right)^{\alpha-1},$$

and the total likelihood function for $\left(x_1;\delta_{u_1},\cdots,x_{n\bar{\pi}};\delta_{u_n\bar{\pi}},y_1;\delta_{a_1},\cdots,y_{n\pi};\delta_{a_n\bar{\pi}}\right)$ can be expressed as follows:

$$L = L\left(\alpha,\lambda,\beta \middle| \underline{x}\ ,\underline{y}\right) = L_{u_i}\left(\alpha,\lambda \middle| x_i,\delta_{u_i}\right) L_{u_j}\left(\alpha,\lambda,\beta \middle| y_j,\delta_{u_j}\right)$$

$$= \prod_{i=1}^{n\bar{\pi}} \left[\frac{\alpha}{\lambda}\left(\frac{x_i}{\lambda}\right)^{\alpha-1} e^{-\left(\frac{x_i}{\lambda}\right)^{\alpha}} \right]^{\delta_{u_i}} \left[e^{-\left(\frac{\tau}{\lambda}\right)^{\alpha}} \right]^{\bar{\delta}_{u_i}} \prod_{j=1}^{n\pi} \left[\frac{\beta\alpha}{\lambda}\left(\frac{\beta y_j}{\lambda}\right)^{\alpha-1} e^{-\left(\frac{\beta y_j}{\lambda}\right)^{\alpha}} \right]^{\delta_{u_j}} \left[e^{-\left(\frac{\beta\tau}{\lambda}\right)^{\alpha}} \right]^{\bar{\delta}_{u_j}} \tag{12}$$

$$= \prod_{i=1}^{n_u} \left[\frac{\alpha}{\lambda}\left(\frac{x_i}{\lambda}\right)^{\alpha-1} e^{-\left(\frac{x_i}{\lambda}\right)^{\alpha}} \right] \prod_{i=n_u+1}^{n\bar{\pi}} \left[e^{-\left(\frac{\tau}{\lambda}\right)^{\alpha}} \right] \prod_{j=1}^{n_a} \left[\frac{\beta\alpha}{\lambda}\left(\frac{\beta y_j}{\lambda}\right)^{\alpha-1} e^{-\left(\frac{\beta y_j}{\lambda}\right)^{\alpha}} \right] \prod_{j=n_a+1}^{n\pi} \left[e^{-\left(\frac{\beta\tau}{\lambda}\right)^{\alpha}} \right].$$

The MLE's of the parameters can estimated numerically by minimizing the log likelihood function.

$$\Rightarrow \ln L = \ln L\left(\alpha,\lambda,\beta \middle| \underline{x}\ ,\underline{y}\right) = l$$

$$\Rightarrow l = \sum_{i=1}^{n_u} \ln\alpha - \sum_{i=1}^{n_u} \ln\lambda + (\alpha-1)\sum_{i=1}^{n_u} \ln\left(\frac{x_i}{\lambda}\right) - \sum_{i=1}^{n_u}\left(\frac{x_i}{\lambda}\right)^{\alpha} - \sum_{i=n_u+1}^{n\bar{\pi}}\left(\frac{\tau}{\lambda}\right)^{\alpha} \tag{13}$$

$$+ \sum_{j=1}^{n_a} \ln\alpha - \sum_{j=1}^{n_a} \ln\lambda + \sum_{j=1}^{n_a} \ln\beta + (\alpha-1)\sum_{j=1}^{n_a} \ln\left(\frac{\beta y_j}{\lambda}\right) - \sum_{j=1}^{n_a}\left(\frac{\beta y_j}{\lambda}\right)^{\alpha} - \sum_{j=n_a+1}^{n\pi}\left(\frac{\beta\tau}{\lambda}\right)^{\alpha}.$$

$$\Rightarrow l = (n_u + n_a)(\ln\alpha - \ln\lambda) + n_a \ln\beta + \sum_{i=1}^{n_u}\psi_{7i} + \sum_{j=1}^{n_a}\psi_{8j} - \sum_{i=n_u+1}^{n\bar{\pi}}\left(\frac{\tau}{\lambda}\right)^{\alpha} - \sum_{j=n_a+1}^{n\pi}\left(\frac{\beta\tau}{\lambda}\right)^{\alpha}$$

$$\Rightarrow l = (n_u + n_a)(\ln\alpha - \ln\lambda) + n_a \ln\beta + \sum_{i=1}^{n_u}\psi_{7i} + \sum_{j=1}^{n_a}\psi_{8j}$$

$$-(n\bar{\pi} - n_u)\left(\frac{\tau}{\lambda}\right)^{\alpha} - (n\pi - n_a)\left(\frac{\beta\tau}{\lambda}\right)^{\alpha}.$$

The score equations are obtained by differentiating the log likelihood with respect to the parameters and setting them to zero. These equations are:

$$\Rightarrow \frac{\partial l}{\partial\alpha} = \frac{n_u + n_a}{\alpha} + \sum_{i=1}^{n_u}\left(\ln\left(\frac{x_i}{\lambda}\right) - \psi_{1i}\right) + \sum_{j=1}^{n_a}\left(\ln\left(\frac{\beta y_j}{\lambda}\right) - \psi_{2j}\right) \tag{14}$$

$$-(n\bar{\pi} - n_u)\psi_3 - (n\pi - n_a)\psi_5 = 0,$$

$$\Rightarrow \frac{\partial l}{\partial\lambda} = \frac{\alpha}{\lambda}\left[1 + (n\bar{\pi} - n_u)\left(\frac{\tau}{\lambda}\right)^{\alpha} + (n\pi - n_a)\left(\frac{\beta\tau}{\lambda}\right)^{\alpha} + \sum_{i=1}^{n_u}\left(\frac{x_i}{\lambda}\right)^{\alpha} + \sum_{j=1}^{n_a}\left(\frac{\beta y_j}{\lambda}\right)^{\alpha}\right] = 0, \tag{15}$$

and

$$\Rightarrow \frac{\partial l}{\partial\beta} = \frac{\alpha}{\lambda}\left[1 - (n\pi - n_a)\left(\frac{\beta\tau}{\lambda}\right)^{\alpha} - \sum_{j=1}^{n_a}\left(\frac{\beta y_j}{\lambda}\right)^{\alpha}\right] = 0. \tag{16}$$

Now, we have a system of three nonlinear equations in three unknowns $\alpha, \lambda, and\ \beta$. It is clear that a closed form solution is very difficult to obtain. Therefore, an iterative procedure must be used to find a numerical solution of the above system. Asymptotic confidence intervals for parameter $\underline{\theta} = (\alpha, \lambda, \beta)$ can be obtained using the following convergence in distribution result;

Result 2.1

The MLEs obtained from the above procedure has the asymptotic distribution given by the following convergence result:

$$\sqrt{n}\left((\hat{\alpha}-\alpha),(\hat{\lambda}-\lambda),(\hat{\beta}-\beta) \rightarrow \left(\underline{0}, I^{-1}(\alpha,\lambda,\beta)\right)\right),$$

where the $I = (\alpha, \lambda, \beta)$ is the fisher information matrix given by

$$I(\alpha,\lambda,\beta) = \begin{bmatrix} I_{11}(\alpha) & I_{12}(\alpha\lambda) & I_{13}(\alpha\beta) \\ I_{21}(\lambda\alpha) & I_{22}(\lambda) & I_{23}(\lambda\beta) \\ I_{31}(\beta\alpha) & I_{32}(\beta\lambda) & I_{33}(\beta) \end{bmatrix} = \begin{bmatrix} \frac{\partial^2 l}{\partial\alpha^2} & \frac{\partial^2 l}{\partial\alpha\partial\lambda} & \frac{\partial^2 l}{\partial\alpha\partial\beta} \\ \frac{\partial^2 l}{\partial\lambda\partial\alpha} & \frac{\partial^2 l}{\partial\lambda^2} & \frac{\partial^2 l}{\partial\lambda\partial\beta} \\ \frac{\partial^2 l}{\partial\beta\partial\alpha} & \frac{\partial^2 l}{\partial\beta\partial\lambda} & \frac{\partial^2 l}{\partial\beta^2} \end{bmatrix}.$$

Proof: A proof that shows that the regularity conditions needed for asymptotic normality for Weibull parameters estimates under Type I censoring in the PALT setup is given in the appendix. Note that since the Weibull distribution belongs to the log-location-scale family and the distributions in this family satisfy the regularity conditions needed, the above asymptotic result does hold for MLE estimators of the Weibull parameters (see Escobar and Meeker (2000)), but their results do not consider the case where PALT data are used. Thus, the proof given in the appendix is of importance.

The elements of the 3x3 matrix $I, I_{ij}(\underline{\theta}), \quad i, j = 1,2,3,$ can be approximated by $I_{ij}(\hat{\underline{\theta}}) = I_{ij}(\hat{\alpha}, \hat{\lambda}, \hat{\beta})$,

$$I_{ij}(\hat{\theta}) = \frac{\partial^2 l(\theta)}{\partial\theta_i \partial\theta_j}\Big|_{\theta=\hat{\theta}}.$$

From Eq. (13), we get the following:

$$\frac{\partial^2 l}{\partial \alpha^2} = \frac{n_u + n_a}{\alpha^2} + 2\left(n\bar{\pi} - n_u\right)\psi_3 + 2\left(n\pi - n_a\right)\psi_5 + \sum_{i=1}^{n_u}\psi_{1i} + \sum_{j=1}^{n_a}\psi_{2j}, \tag{17}$$

$$\frac{\partial^2 l}{\partial \lambda^2} = \frac{\alpha}{\lambda^2}\left[n_u + n_a - (\alpha - 1)\left(\left(n\bar{\pi} - n_u\right)\left(\frac{\tau}{\lambda}\right)^{\alpha} + \left(n\pi - n_a\right)\left(\frac{\beta\tau}{\lambda}\right)^{\alpha}\right)\right], \tag{18}$$

$$\frac{\partial^2 l}{\partial \beta^2} = \frac{-\alpha}{\beta^2}\left[1 + (\alpha - 1)\left(n\pi - n_a\right)\left(\frac{\beta\tau}{\lambda}\right)^{\alpha} + (\alpha - 1)\sum_{j=1}^{n_a}\left(\frac{\beta y_j}{\lambda}\right)^{\alpha}\right], \tag{19}$$

$$\frac{\partial^2 l}{\partial \alpha \partial \beta} = n_a \ln\beta + \psi_{7i} + \psi_{8j} - \left(n\bar{\pi} - n_u\right)\left(\frac{\tau}{\lambda}\right)^{\alpha} - \left(n\pi - n_a\right)\left(\frac{\beta\tau}{\lambda}\right)^{\alpha}, \tag{20}$$

$$\frac{\partial^2 l}{\partial \alpha \partial \lambda} = \frac{-1}{\lambda}\left[n_u + n_a - \left(n\bar{\pi} - n_u\right)\psi_4 - \left(n\bar{\pi} - n_u\right)\psi_6 - \sum_{i=1}^{n_u}\psi_{4i} - \sum_{j=1}^{n_a}\psi_{6j}\right], \tag{21}$$

and

$$\frac{\partial^2 l}{\partial \beta \partial \lambda} = \frac{\alpha^2}{\lambda\beta}\left[\left(n\pi - n_a\right)\left(\frac{\beta\tau}{\lambda}\right)^{\alpha} + \sum_{j=1}^{n_a}\left(\frac{\beta y_j}{\lambda}\right)^{\alpha}\right], \tag{22}$$

where

$$\psi_{1i} = \left(\frac{x_i}{\lambda}\right)^{\alpha}\ln\left(\frac{x_i}{\lambda}\right), \qquad \psi_{2j} = \left(\frac{\beta y_j}{\lambda}\right)^{\alpha}\ln\left(\frac{\beta y_j}{\lambda}\right),$$

$$\psi_3 = \left(\frac{\tau}{\lambda}\right)^{\alpha}\ln\left(\frac{\tau}{\lambda}\right), \qquad \psi_4 = \left(\frac{\tau}{\lambda}\right)^{\alpha}\left[\alpha\ln\left(\frac{\tau}{\lambda}\right) + 1\right],$$

$$\psi_{4i} = \left(\frac{x_i}{\lambda}\right)^{\alpha}\left[\alpha\ln\left(\frac{x_i}{\lambda}\right) + 1\right], \qquad \psi_5 = \left(\frac{\beta\tau}{\lambda}\right)^{\alpha}\ln\left(\frac{\beta\tau}{\lambda}\right),$$

$$\psi_6 = \left(\frac{\beta\tau}{\lambda}\right)^{\alpha}\left[\alpha\ln\left(\frac{\beta\tau}{\lambda}\right) + 1\right], \qquad \psi_{6j} = \left(\frac{\beta y_j}{\lambda}\right)^{\alpha}\left[\alpha\ln\left(\frac{\beta y_j}{\lambda}\right) + 1\right],$$

$$\psi_{7i} = (\alpha - 1)\ln\left(\frac{x_i}{\lambda}\right) - \left(\frac{x_i}{\lambda}\right)^{\alpha}, \qquad \psi_{8j} = (\alpha - 1)\ln\left(\frac{\beta y_j}{\lambda}\right) - \left(\frac{\beta y_j}{\lambda}\right)^{\alpha}.$$

Now by employing the standard z-based confidence interval formulations,

$$\hat{\alpha} \pm Z_{\gamma/2}\sqrt{I_{11}^{-1}\left(\hat{\alpha}\right)}, \quad \hat{\lambda} \pm Z_{\gamma/2}\sqrt{I_{22}^{-1}\left(\hat{\lambda}\right)}, \quad \hat{\beta} \pm Z_{\gamma/2}\sqrt{I_{33}^{-1}\left(\hat{\beta}\right)},$$

we obtain the confidence intervals for the parameters based on the asymptotic distribution.

The asymptotic confidence interval for the mean life at normal use conditions is given by

$$\hat{\mu} \pm Z_{\gamma/2}\sqrt{Var\left(\hat{\mu}\right)},$$

where $Var\left(\hat{\mu}\right)$ is obtained using the Delta method.

2.3 THE BOOTSTRAP RESAMPLING METHODS AND THE MONTE-CARLO PROCEDURE

There are two different methods for generating bootstrap sample data. One is the parametric bootstrap, where once the parameters of the underlying distribution are estimated, they are plugged into the assumed distribution and pseudo random numbers then drawn from this estimated distribution to produce the bootstrap sample. The non-parametric bootstrap does not assume a set underlying distribution, but resample from the sample data to produce new samples. The resampling procedure, of course, should be adopted to fit the underlying structure of the problem. For example, in a regression setting, resampling must be done on the residuals of a fitted model rather than from the original data.

In the following, we combine the bootstrap steps with the steps needed to carry out a Monte-Carlo comparison of the proposed methods of building confidence bounds and intervals. The steps for the parametric bootstrap and the non-parametric bootstrap are given separately. Note that the confidence bounds and intervals based on the asymptotic distribution can be computed at each Monte-Carlo simulation sample and does no require bootstrap resampling.

2.3.1 The Proposed Parametric Bootstrap Method and the Monte-Carlo Procedure for Studying its Performance. The Monte-Carlo procedure employed to study the performance of the parametric bootstrap method is described below. The steps for the parametric bootstrap method for obtaining confidence bounds for $\alpha, \lambda,$ and β and lower bounds for the mean life are embedded in this procedure and are given in italics. Note that distributional parameters are varied in the study as follows:

$(\alpha = 1.5, nad\ 2, \lambda = 1, \beta = 1.5, and\ 2)$ with $\mu = \lambda\Gamma\left(1 + \frac{1}{\alpha}\right)$. The censoring time was set at τ=1, and1.5. Note that without loss of generality, scale parameter λ can be set at 1.

(1) For fixed values of n and π, generate a random sample $x_i, i = 1,2, \dots, n\overline{\pi}$ from the Weibull (α, λ) distribution. This would be considered data from the normal use sample. Similarly, generate the data set $y_j, j = 1,2, \dots, n\pi$, representing the sample under the high stress condition, from the Weibull $(\alpha, \beta\lambda)$ distribution.

(2) Use the ML method to estimate the parameters with the same censoring time τ used for both samples. In this study, the nonlinear equations of the maximum likelihood estimates were solved iteratively using the Newton Raphson method.

(3) Employ the resulting estimates of the parameters and acceleration factor to construct asymptotic confidence limits with confidence level at $\gamma = 0.95$. Also, plug-in the MLEs into the Fisher Information matrix to obtain the asymptotic variance and covariance matrix of the estimators and then use them in the delta method to compute the lower bound for mean life.

(4) *Replace the unknown parameters,* α, λ, *in the Weibull distribution for the normal use case with their MLEs,* $\hat{\alpha}, \hat{\lambda}$, *and utilize the estimated distribution to generate a bootstrap sample* $x_i^*, i = 1,2, \dots, n\overline{\pi}$ *of size* $n\overline{\pi}$. Censor the data based on the censoring time τ.

(5) *Similarly replace the unknown parameters,* α, λ, β *in the Weibull distribution for the high stress case with their MLEs,* $\hat{\alpha}, \hat{\lambda}, \hat{\beta}$ *and utilize the estimated distribution to generate a bootstrap sample* $y_j^*, j = 1,2, \dots, n\pi$ *of size* $n\pi$. Censor the data based on the censoring time τ.

(6) Re-estimate the Weibull parameters of the normal use distribution were using the combined bootstrap samples. Denote the bootstrap sample-based MLEs of α, λ, β *and* μ *obtained at bootstrap step k by* $\hat{\alpha}^{*(k)}, \hat{\lambda}^{*(k)}, \hat{\beta}^{*(k)}$ *and* $\hat{\mu}^{*(k)}$ *respectively.*

(7) *Repeat Steps* (4) *to* (6) *1,000 times*. Construct the empirical distributions of the bootstrap estimates $\hat{\alpha}^{*(k)}, \hat{\lambda}^{*(k)}, \hat{\beta}^{*(k)}$ *and* $\hat{\mu}^{*(k)}$, *k*=1, 2, …, 1,000

(8) Use the empirical distributions obtained from bootstrap estimates to construct, confidence interval for α, λ, β *using quantiles at* $\left(\frac{1-\gamma}{2}\right)$*100% and* $1 - \left(\frac{1-\gamma}{2}\right)$*100% of the respective empirical distribution as the lower and upper bounds respectively. Use the* $(1-\gamma)$*100% quantile of the empirical distribution of* $\hat{\mu}^{*(k)}$, *k*=1, 2, …, 1,000, as the lower bound for μ.

(9) Repeat steps (1) through (8) 1,000 times and compute the average number of times each parameter fell within the bound(s). This would yield an estimate of the

expected coverage for each interval. For each parameter except μ, the widths of the two sided interval computed in Steps (3) and (8) are averaged to obtained an estimate of the expected, width.

2.3.2 The Proposed Nonparametric Bootstrap Method and the Monte-Carlo Procedure for Studying its Performance. The Monte-Carlo procedure employed to study the performance of the nonparametric bootstrap method is described below. The steps for the parametric bootstrap method for obtaining confidence bounds for $\boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{and}\ \boldsymbol{\beta}$ and lower binds for the mean life are imbedded in this procedure and given in italics.

(1) For fixed values of *n* and π, generate a random sample $x_i, i = 1,2, \dots, n\overline{\pi}$ from the Weibull (α, λ) distribution. This would be considered data from the normal use sample. Similarly, generate the data set $y_j, j = 1,2, \dots, n\pi$, representing the sample under the high stress condition, from the Weibull $(\alpha, \beta\lambda)$ distribution.

(2) *Obtain a bootstrap resample from each of the two samples generated in Step (1) above, with each bootstrap sample of size* πn *(or* $\overline{\pi}n$ *) obtained by sampling with replacement from the respective sample obtained in (1).*

(3) *New "bootstrap estimates"* $\hat{\alpha}^*, \hat{\lambda}^*, and\ \hat{\beta}^*$ *are computed from the combined bootstrap sample using the ML method as described in Step (2) given in Section 2.3.1. Also estimate the mean life* μ *under normal conditions, accounting for the censoring.*

(4) *Repeat the process given in Steps (2) and (3) 1,000 times and obtain the empirical distributions of* $\hat{\alpha}^*, \hat{\lambda}^*, \widehat{and\ \beta}^*$..

(5) *Using the empirical distributions of the* $\hat{\alpha}^*, \hat{\lambda}^*$*, and* $\hat{\beta}^*$ *obtained from bootstrap estimates, construct confidence interval for* $\alpha, \lambda, and\ \beta$ *using respective quantiles at* $\left(\frac{1-\gamma}{2}\right)$*100% and* $1 - \left(\frac{1-\gamma}{2}\right)$*100%.*

(6) *Using the empirical distributions of the mean* $\hat{\mu}^*$ *obtained from bootstrap estimates, construct the lower confidence bound for* μ *is using quantile at* $(1-\gamma)$*100% .*

(7) Coverage probabilities were computed based on 1,000 simulation runs by repeating Steps (1) – (6).

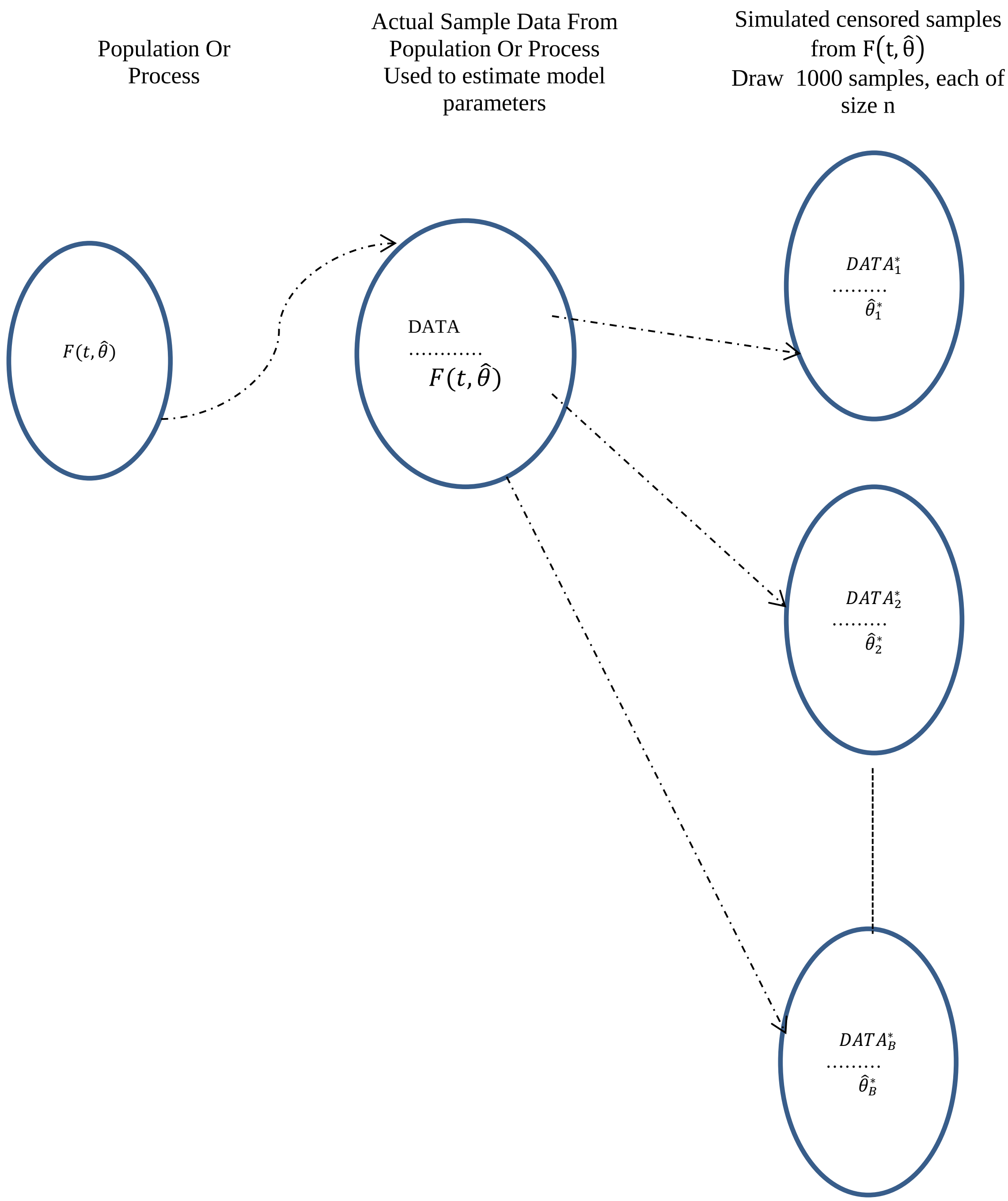

Figure 2.1.Illustrates the parametric bootstrap resampling method

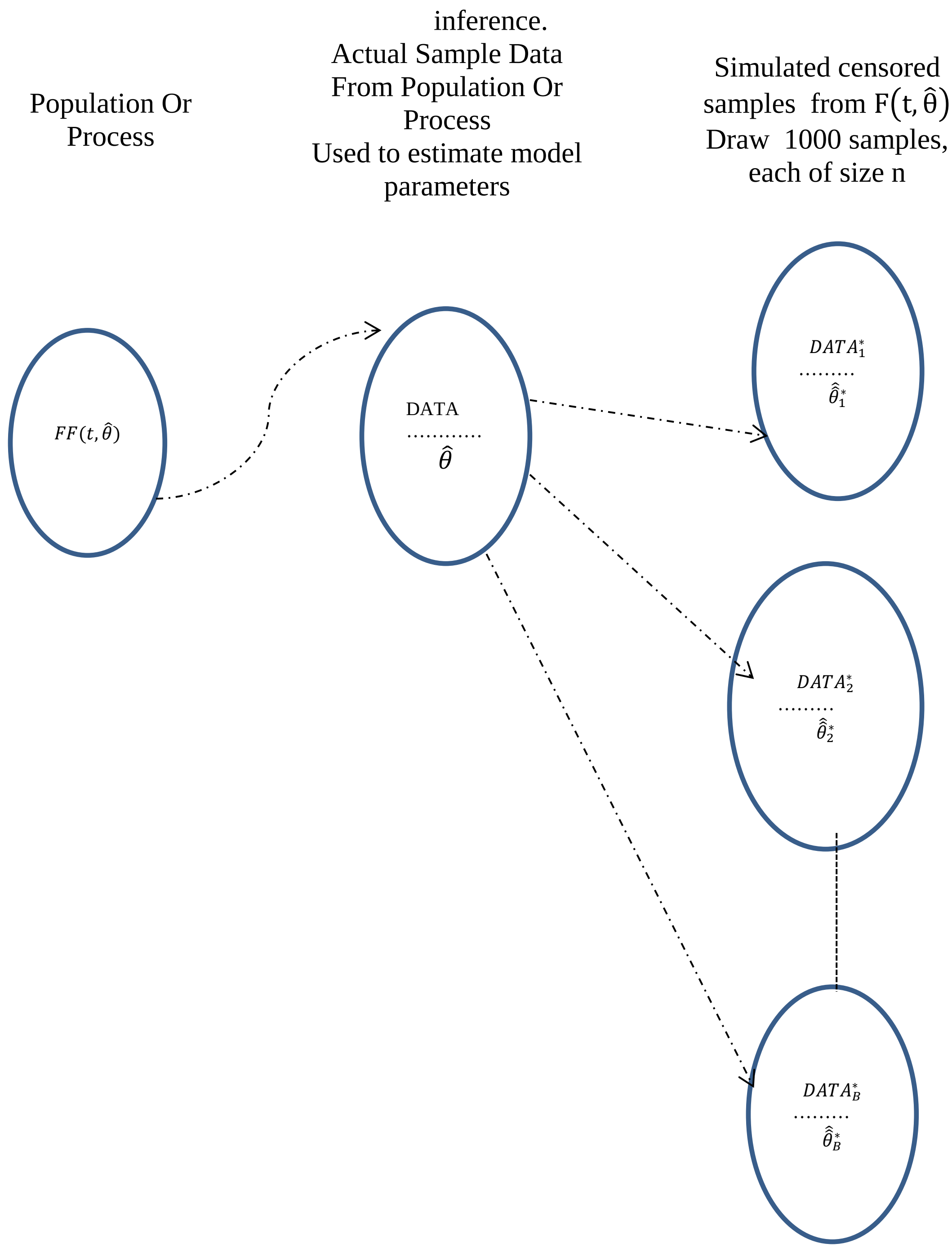

Figure 2.2. Illustrates the nonparametric bootstrap resampling for parametric inference.

2.4 MONTE-CARLO SIMULATION RESULTS AND DISCUSSION

All simulations results reported here are for α = (1.5, and 2) and λ = 1, with the acceleration factor β set at 1.5 and 2.0. The censoring parameter τ was set at values 1, and 1.5

The simulation study was conducted using a computer code written in Matlab, and the simulation results are reported in Table 2.1a to Table 2.31. Tables 2.1a and 2.1b show the results of the maximum likelihood estimation of $(\alpha, \lambda, \beta, and\ \mu)$. The estimated expected values of the MLEs are reasonably close to the true values, even for *n*=30. There is no discernible pattern linking the means of the estimates to changes in the parameter values, at least over the range of parameter values considered in this study. Tables 2.2 to 2.31 show the performance of the asymptotic, parametric bootstrap, and nonparametric bootstrap confidence intervals for $(\alpha, \lambda,\ and\ \beta)$at the 95% confidence level and the performance of the Asymptotic, Parametric Bootstrap, and Nonparametric Bootstrap based 95% confidence bound of mean-life under normal conditions.

Table 2.1a Weibull Parameters, Acceleration Factor, and Type I Censoring

π	τ	α	λ	β	μ	n	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\mu}$
0.5	1.0	1.5	1.0	1.5	0.9027	30	1.576672	1.043488	1.54321	0.949033
						50	1.553463	1.014072	1.565054	0.920318
						75	1.543858	0.983789	1.485776	0.891226
						100	1.537005	1.015341	1.539113	0.918978
	1.5	1.5	1.0	1.5	0.9027	30	1.615358	0.981995	1.516009	0.886497
						50	1.564676	0.986138	1.499799	0.890782
						75	1.538481	1.016195	1.531349	0.919195
						100	1.518111	1.008931	1.522719	0.911857

Table 2.1b Weibull Parameters, Acceleration Factor, and Type I Censoring

π	τ	α	λ	β	μ	n	$\widehat{\alpha}$	$\widehat{\lambda}$	$\widehat{\beta}$	$\hat{\mu}$
0.5	1.0	2.0	1.0	1.5	0.88623	30	2.10566	1.005542	1.526783	0.894765
						50	2.069241	1.015067	1.528142	0.901825
						75	2.03491	1.000411	1.505851	0.887366
						100	2.045635	1.009265	1.516721	0.895606
	1.5	2.0	1.0	1.5	0.88623	30	2.144481	0.996427	1.516204	0.885228
						50	2.082549	1.005506	1.521205	0.892661
						75	2.056499	1.003315	1.51703	0.890216
						100	2.035133	0.995743	1.498669	0.883108
0.667	1.0	2.0	1.0	1.5	0.88623	30	2.097268	1.000108	1.491671	0.889862
						50	2.071283	1.0045	1.505745	0.892569
						75	2.059496	1.009989	1.524947	0.896732
						100	2.032169	1.002487	1.508073	0.889318
	1.5	2.0	1.0	1.5	0.88623	30	2.056806	0.969927	1.431312	0.861615
						50	2.064046	0.992657	1.496456	0.881282
						75	2.031231	0.988086	1.474673	0.877013
						100	2.021136	0.998281	1.501615	0.885341
0.5	1.0	2.0	1.0	2.0	0.88623	30	2.08247	1.007274	2.033726	0.894476
						50	2.060175	1.009152	2.025427	0.930197
						75	2.043235	1.00704	2.02148	0.893325
						100	2.030411	1.001163	2.005237	0.888148
	1.5	2.0	1.0	2.0	0.88623	30	2.088419	1.002052	2.025996	0.889443
						50	2.056056	1.003584	2.012276	0.890461
						75	2.042821	1.00153	2.015177	0.888306
						100	2.036468	1.00281	2.011049	0.889302
0.667	1.0	2.0	1.0	2.0	0.88623	30	2.032135	0.993783	1.97212	0.882527
						50	2.034193	1.004697	1.992329	0.89193
						75	2.042055	1.000766	2.005288	0.887732
						100	2.037142	1.005378	2.025448	0.891691
	1.5	2.0	1.0	2.0	0.88623	30	1.970186	0.971658	1.919988	0.863211
						50	1.980244	0.98249	1.942167	0.872111
						75	2.035332	0.99344	1.990141	0.881122
						100	2.023208	1.001917	2.010083	0.888552

Table 2.2 Coverage of Asymptotic 95% C.I.s

α =1.5, λ =1, β =1.5, μ=0.9027, π=.5, τ=1

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	1.058891	2.297576	1.238686	0.207054	0.967
	λ	0.737031	1.535213	0.798183	0.186898	0.956
	β	1.04402	2.459394	1.415374	0.350738	0.969
	μ	0.711167				0.954
50	α	1.042634	2.174402	1.131768	0.172591	0.966
	λ	0.654653	1.411619	0.756966	0.104981	0.953
	β	1.068087	2.288784	1.220697	0.212695	0.967
	μ	0.658812				0.959
75	α	1.088779	2.131125	1.042346	0.141036	0.965
	λ	0.762073	1.275386	0.513312	0.069603	0.948
	β	1.075537	2.114673	1.039136	0.184005	0.965
	μ	0.711217				0.9554
100	α	1.147291	2.019342	0.872052	0.130529	0.960
	λ	0.78837	1.22566	0.437289	0.041496	0.945
	β	1.183164	2.010407	0.827243	0.105109	0.957
	μ	0.731194				0.952

Table 2.3 Coverage of Parametric Bootstrap 95% C.I.s

α =1.5, λ =1, β =1.5, μ=0.9027, π=.5, τ=1

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	1.131354	2.141941	1.010587	0.135168	0.964
	λ	0.754875	1.532676	0.777801	0.166295	0.954
	β	1.059377	2.344477	1.2851	0.259647	0.968
	μ	0.717816				0.953
50	α	1.085333	2.017417	0.932085	0.108718	0.962
	λ	0.663783	1.426243	0.762459	0.090188	0.953
	β	1.097772	2.200414	1.102642	0.166013	0.966
	μ	0.663387				0.958
75	α	1.143289	1.985733	0.842444	0.112366	0.959
	λ	0.777822	1.262373	0.484551	0.062424	0.947
	β	1.109645	2.089899	0.980254	0.140481	0.964
	μ	0.722811				0.953
100	α	1.210594	1.918238	0.707644	0.091796	0.952
	λ	0.796675	1.219792	0.423118	0.034572	0.945
	β	1.212475	1.915997	0.703522	0.084472	0.952
	μ	0.734488				0.952

Table 2.4 Coverage of Nonparametric Bootstrap 95% C.I.s

α =1.5, λ =1, β =1.5, μ=0.9027, π=.5, τ=1

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	1.094575	2.17102	1.076446	0.172978	0.966
	λ	0.746554	1.786962	1.040408	0.252883	0.965
	β	1.10094	2.802566	1.701625	0.307192	0.971
	μ	0.681494				0.956
50	α	1.194322	2.240323	1.046002	0.124307	0.965
	λ	0.797266	1.480086	0.682821	0.165115	0.951
	β	1.119282	2.155023	1.035741	0.269971	0.965
	μ	0.73212				0.952
75	α	1.125589	2.033443	0.907854	0.130005	0.961
	λ	0.743859	1.489184	0.745326	0.03947	0.953
	β	1.191063	2.113575	0.922512	0.101411	0.962
	μ	0.712294				0.954
100	α	1.272253	1.826239	0.553986	0.095875	0.950
	λ	0.762935	1.290845	0.527909	0.071676	0.949
	β	1.058365	1.852788	0.794424	0.195381	0.956
	μ	0.711974				0.954

Table 2.5 Coverage of Asymptotic 95% C.I.s

α =1.5, λ =1, β =1.5, μ=0.9027, π=.5, τ=1.5

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	1.033085	2.426617	1.393531	0.180204	0.975
	λ	0.678737	1.372683	0.693945	0.073595	0.961
	β	1.005162	2.32126	1.316098	0.206278	0.972
	μ	0.655996				0.959
50	α	1.14241	2.2658	1.12339	0.129082	0.971
	λ	0.746816	1.404307	0.657491	0.074644	0.960
	β	1.029431	2.096434	1.067004	0.100187	0.970
	μ	0.710759				0.955
75	α	1.159242	2.022276	0.863035	0.1017	0.967
	λ	0.745781	1.29498	0.549199	0.109312	0.956
	β	1.125778	2.184085	1.058307	0.090719	0.970
	μ	0.704224				0.955
100	α	1.231624	1.906322	0.674698	0.125054	0.960
	λ	0.822826	1.223822	0.400996	0.076077	0.949
	β	1.204736	1.944556	0.73982	0.078233	0.962
	μ	0.770035				0.952

Table 2.6 Coverage of Parametric Bootstrap 95% C.I.s

α =1.5, λ =1, β =1.5, μ=0.9027, π=.5, τ=1.5

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	1.044993	2.156078	1.111085	0.1244	0.971
	λ	0.774367	1.385444	0.611076	0.052272	0.957
	β	1.065791	2.195218	1.129427	0.126623	0.971
	μ	0.690502				0.957
50	α	1.20184	2.135915	0.934076	0.078184	0.969
	λ	0.739619	1.346937	0.607318	0.068972	0.957
	β	1.188012	2.0868	0.898788	0.061109	0.968
	μ	0.726995				0.953
75	α	1.198535	1.904932	0.706397	0.041833	0.961
	λ	0.788263	1.272586	0.484323	0.126915	0.953
	β	1.075457	1.957574	0.882117	0.093547	0.968
	μ	0.711617				0.955
100	α	1.279874	1.820302	0.540428	0.06384	0.956
	λ	0.85188	1.214498	0.362618	0.063717	0.947
	β	1.286863	1.823956	0.537092	0.060072	0.955
	μ	0.788831				0.951

Table 2.7 Coverage of Nonparametric Bootstrap 95% C.I.s

α =1.5, λ =1, β =1.5, μ=0.9027, π=.5, τ=1.5

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	1.181807	2.249394	1.067588	0.209897	0.970
	λ	0.685653	1.327392	0.641739	0.094274	0.958
	β	1.04436	2.16286	1.1185	0.144993	0.971
	μ	0.677046				0.958
50	α	1.178017	1.970022	0.792005	0.138289	0.964
	λ	0.755765	1.348221	0.592456	0.108979	0.957
	β	1.015456	2.094452	1.078996	0.191347	0.970
	μ	0.70394				0.956
75	α	1.215212	1.968906	0.753694	0.106314	0.963
	λ	0.800374	1.273839	0.473465	0.0537	0.953
	β	1.166294	2.090804	0.92451	0.11965	0.969
	μ	0.713027				0.954
100	α	1.287767	1.806507	0.51874	0.097141	0.954
	λ	0.780335	1.236837	0.456501	0.072061	0.951
	β	1.189513	1.91431	0.724797	0.081611	0.962
	μ	0.731749				0.953

Table 2.8 Coverage of Asymptotic 95% C.I.s

α =2, λ =1, β =1.5, μ=0.88623, π=.5, τ=1

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	1.428128	2.939773	1.511645	0.060395	0.978
	λ	0.770204	1.174486	0.404282	0.031394	0.971
	β	1.108281	1.986048	0.877767	0.066968	0.967
	μ	0.710548				0.973
50	α	1.481771	2.689149	1.207378	0.086403	0.963
	λ	0.804323	1.154149	0.349826	0.01449	0.963
	β	1.163985	1.86008	0.696095	0.035691	0.958
	μ	0.73394				0.966
75	α	1.569334	2.524061	0.954727	0.029612	0.958
	λ	0.832946	1.08939	0.256444	0.016714	0.953
	β	1.215491	1.788695	0.573204	0.043942	0.951
	μ	0.756085				0.957
100	α	1.615016	2.469886	0.85487	0.051646	0.952
	λ	0.828196	1.060944	0.232748	0.011203	0.949
	β	1.219649	1.726205	0.506556	0.02449	0.948
	μ	0.752861				0.958

Table 2.9 Coverage of Parametric Bootstrap 95% C.I.s

α =2, λ =1, β =1.5, μ=0.88623, π=.5, τ=1

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	1.528454	2.785755	1.257301	0.077077	0.971
	λ	0.798505	1.260899	0.462394	0.030082	0.975
	β	1.15374	2.000944	0.847204	0.077483	0.964
	μ	0.736512				0.961
50	α	1.611724	2.595878	0.984154	0.101286	0.958
	λ	0.833037	1.237362	0.404325	0.008988	0.971
	β	1.200454	1.871415	0.670961	0.044053	0.959
	μ	0.757589				0.957
75	α	1.658296	2.478763	0.820467	0.047923	0.956
	λ	0.860875	1.177373	0.316498	0.010782	0.955
	β	1.256715	1.828712	0.571997	0.047047	0.951
	μ	0.779962				0.949
100	α	1.707784	2.424014	0.71623	0.038172	0.949
	λ	0.856417	1.148437	0.29202	0.00501	0.953
	β	1.264424	1.766898	0.502474	0.023945	0.947
	μ	0.775597				0.949

Table 2.10 Coverage of Nonparametric Bootstrap 95% C.I.s

α =2, λ =1, β =1.5, μ=0.88623, π=.5, τ=1

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	1.529158	2.651451	1.122293	0.098632	0.965
	λ	0.783752	1.220464	0.436712	0.027064	0.973
	β	1.125854	1.898379	0.772525	0.056806	0.961
	μ	0.715692				0.973
50	α	1.558791	2.627216	1.068425	0.095078	0.961
	λ	0.795752	1.148941	0.353189	0.037386	0.964
	β	1.151173	1.810551	0.659378	0.056138	0.953
	μ	0.72729				0.967
75	α	1.62623	2.416409	0.790179	0.051491	0.953
	λ	0.819761	1.118779	0.299018	0.02911	0.954
	β	1.251547	1.776176	0.524629	0.035295	0.949
	μ	0.75656				0.957
100	α	1.620668	2.34255	0.721882	0.062642	0.949
	λ	0.841021	1.141333	0.300312	0.025753	0.954
	β	1.232678	1.753693	0.521015	0.039505	0.949
	μ	0.763656				0.951

Table 2.11 Coverage of Asymptotic 95% C.I.s

α =2, λ =1, β =1.5, μ=0.88623, π=.5, τ=1.5

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	1.520019	2.922929	1.40291	0.035159	0.979
	λ	0.756544	1.245536	0.488992	0.02013	0.978
	β	1.09424	2.028082	0.933842	0.039966	0.978
	μ	0.707699				0.969
50	α	1.559902	2.690285	1.130383	0.033658	0.971
	λ	0.802351	1.227825	0.425474	0.022293	0.966
	β	1.16077	1.982451	0.821681	0.02153	0.967
	μ	0.738667				0.967
75	α	1.631716	2.557852	0.926136	0.04647	0.966
	λ	0.831179	1.170676	0.339497	0.004215	0.955
	β	1.216213	1.86435	0.648137	0.014805	0.957
	μ	0.764202				0.959
100	α	1.646833	2.521604	0.874771	0.035506	0.957
	λ	0.850687	1.157054	0.306367	0.000251	0.953
	β	1.243681	1.82187	0.578189	0.014259	0.954
	μ	0.775236				0.957

Table 2.12 Coverage of Parametric Bootstrap 95% C.I.s

$\alpha=2$, $\lambda=1$, $\beta=1.5$, $\mu=0.88623$, $\pi=.5$, $\tau=1.5$

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	1.56777	2.697066	1.129296	0.070908	0.971
	λ	0.781595	1.229183	0.447588	0.031947	0.967
	β	1.145747	1.921492	0.775745	0.062558	0.964
	μ	0.728119				0.967
50	α	1.620587	2.521198	0.900611	0.055174	0.964
	λ	0.833704	1.218953	0.385249	0.031161	0.959
	β	1.217152	1.90776	0.690608	0.045046	0.959
	μ	0.759754				0.962
75	α	1.670304	2.429366	0.759062	0.062678	0.952
	λ	0.85118	1.159789	0.308609	0.014821	0.953
	β	1.258623	1.776935	0.518312	0.023148	0.952
	μ	0.781167				0.955
100	α	1.697873	2.407773	0.7099	0.053035	0.948
	λ	0.870352	1.141185	0.270833	0.006514	0.947
	β	1.293153	1.746251	0.453098	0.028371	0.946
	μ	0.784996				0.955

Table 2.13 Coverage of Nonparametric Bootstrap 95% C.I.s

$\alpha=2$, $\lambda=1$, $\beta=1.5$, $\mu=0.88623$, $\pi=.5$, $\tau=1.5$

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	1.606009	2.720998	1.114989	0.063759	0.970
	λ	0.794537	1.225881	0.431344	0.026063	0.966
	β	1.168365	1.905797	0.737432	0.046977	0.963
	μ	0.735065				0.967
50	α	1.654509	2.564973	0.910464	0.051808	0.965
	λ	0.843868	1.19489	0.351022	0.0103	0.957
	β	1.245075	1.837008	0.591933	0.035625	0.955
	μ	0.76032				0.961
75	α	1.665246	2.443821	0.778575	0.055733	0.953
	λ	0.864866	1.16557	0.300704	0.008555	0.952
	β	1.291665	1.780783	0.489118	0.031462	0.948
	μ	0.785837				0.954
100	α	1.689984	2.363461	0.673477	0.051352	0.943
	λ	0.874408	1.141715	0.267307	0.010536	0.947
	β	1.287561	1.769521	0.48196	0.01726	0.948
	μ	0.796818				0.951

Table 2.14 Coverage of Asymptotic 95% C.I.s

α =2, λ =1, β =1.5, μ=0.88623, π=.667, τ=1

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	1.469722	2.896838	1.427116	0.0385	0.978
	λ	0.738007	1.349913	0.611906	0.020207	0.978
	β	1.093268	2.083072	0.989803	0.057317	0.975
	μ	0.688976				0.967
50	α	1.492544	2.789179	1.296636	0.052213	0.971
	λ	0.771562	1.320843	0.549281	0.036053	0.964
	β	1.136512	2.045376	0.908865	0.042843	0.975
	μ	0.713673				0.963
75	α	1.586767	2.612116	1.025349	0.028957	0.962
	λ	0.826597	1.22184	0.395243	0.026544	0.957
	β	1.201475	1.882793	0.681319	0.039577	0.953
	μ	0.754699				0.957
100	α	1.578048	2.545202	0.967155	0.030645	0.958
	λ	0.826024	1.216026	0.390002	0.020137	0.956
	β	1.217061	1.897053	0.679992	0.037355	0.951
	μ	0.75678				0.955

Table 2.15 Coverage of Parametric Bootstrap 95% C.I.s

α =2, λ =1, β =1.5, μ=0.88623, π=.667, τ=1

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	1.542738	2.695054	1.152316	0.071666	0.967
	λ	0.754464	1.35406	0.599596	0.025528	0.975
	β	1.123545	2.012949	0.889404	0.071674	0.974
	μ	0.694204				0.965
50	α	1.566228	2.637608	1.07138	0.073092	0.963
	λ	0.784254	1.319521	0.535268	0.041696	0.962
	β	1.14788	1.988201	0.840321	0.061033	0.964
	μ	0.718012				0.962
75	α	1.642439	2.537224	0.894786	0.044157	0.954
	λ	0.831346	1.217517	0.386171	0.030114	0.954
	β	1.220708	1.848062	0.627354	0.052174	0.949
	μ	0.759773				0.954
100	α	1.627974	2.437957	0.809983	0.048325	0.951
	λ	0.836019	1.218105	0.382086	0.024322	0.953
	β	1.240228	1.865566	0.625338	0.051281	0.948
	μ	0.762306				0.953

Table 2.16 Coverage of Nonparametric Bootstrap 95% C.I.s

α =2, λ =1, β =1.5, μ=0.88623, π=.667, τ=1

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	1.506926	2.678344	1.171419	0.089682	0.969
	λ	0.747277	1.340167	0.59289	0.052143	0.974
	β	1.101348	1.982783	0.881435	0.061581	0.973
	μ	0.685901				0.969
50	α	1.576145	2.52648	0.950336	0.088759	0.957
	λ	0.771919	1.302267	0.530348	0.063096	0.962
	β	1.158171	1.992152	0.833981	0.054018	0.964
	μ	0.71257				0.963
75	α	1.656934	2.418502	0.761568	0.033334	0.949
	λ	0.825347	1.25867	0.433323	0.02956	0.959
	β	1.224955	1.926176	0.701221	0.030402	0.954
	μ	0.750564				0.959
100	α	1.679754	2.412073	0.732319	0.076219	0.947
	λ	0.831369	1.196916	0.365547	0.029107	0.951
	β	1.235504	1.828671	0.593167	0.041163	0.945
	μ	0.765924				0.951

Table 2.17 Coverage of Asymptotic 95% C.I.s

α =2, λ =1, β =1.5, μ=0.88623, π=.667, τ=1.5

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	1.511932	2.745184	1.233251	0.027192	0.978
	λ	0.71954	1.270625	0.551085	0.021859	0.969
	β	1.052062	1.991453	0.939392	0.031368	0.974
	μ	0.6759				0.969
50	α	1.561383	2.630201	1.068818	0.027701	0.966
	λ	0.746884	1.224853	0.477969	0.028403	0.967
	β	1.094703	1.887362	0.792659	0.035397	0.962
	μ	0.695307				0.967
75	α	1.647156	2.589251	0.942095	0.035601	0.959
	λ	0.821089	1.201453	0.380365	0.018444	0.957
	β	1.207406	1.860281	0.652875	0.015337	0.956
	μ	0.757438				0.955
100	α	1.654901	2.494256	0.839355	0.03133	0.954
	λ	0.821951	1.195714	0.373763	0.019084	0.957
	β	1.218479	1.850327	0.631848	0.016362	0.954
	μ	0.753369				0.957

Table 2.18 Coverage of Parametric Bootstrap 95% C.I.s

α =2, λ =1, β =1.5, μ=0.88623, π=.667, τ=1.5

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	1.558948	2.582157	1.023209	0.056101	0.965
	λ	0.759619	1.252893	0.493274	0.031319	0.967
	β	1.120572	1.905564	0.784991	0.048903	0.962
	μ	0.704306				0.966
50	α	1.628638	2.473594	0.844956	0.044464	0.955
	λ	0.772152	1.208786	0.436634	0.038431	0.961
	β	1.155274	1.799511	0.644237	0.053251	0.955
	μ	0.717311				0.962
75	α	1.707015	2.490781	0.783766	0.050419	0.952
	λ	0.845906	1.189656	0.34375	0.028677	0.954
	β	1.261705	1.799855	0.53815	0.032134	0.951
	μ	0.780221				0.951
100	α	1.689858	2.378117	0.688259	0.052034	0.947
	λ	0.843054	1.185081	0.342027	0.020943	0.954
	β	1.266483	1.791549	0.525066	0.02699	0.949
	μ	0.772328				0.953

Table 2.19 Coverage of Nonparametric Bootstrap 95% C.I.s

α =2, λ =1, β =1.5, μ=0.88623, π=.667, τ=1.5

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	1.601104	2.628303	1.027199	0.040553	0.965
	λ	0.772562	1.257633	0.48507	0.043927	0.967
	β	1.1283	1.898682	0.770382	0.057681	0.961
	μ	0.715079				0.963
50	α	1.646487	2.515288	0.8688	0.077675	0.957
	λ	0.785353	1.204297	0.418944	0.03367	0.959
	β	1.154279	1.85254	0.698261	0.066559	0.957
	μ	0.716794				0.963
75	α	1.675806	2.411131	0.735324	0.052411	0.951
	λ	0.820617	1.184242	0.363625	0.033534	0.956
	β	1.224426	1.771015	0.54659	0.039734	0.951
	μ	0.747658				0.959
100	α	1.687046	2.394657	0.70761	0.050762	0.949
	λ	0.83771	1.162611	0.324901	0.017601	0.951
	β	1.259962	1.755438	0.495477	0.027759	0.947
	μ	0.765106				0.954

Table 2.20 Coverage of Asymptotic 95% C.I.s

α =2, λ =1, β =2, μ=0.88623, π=.5, τ=1

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	1.483977	2.964364	1.480387	0.07662	0.971
	λ	0.760693	1.289926	0.529233	0.02404	0.969
	β	1.452246	2.822865	1.370619	0.059191	0.974
	μ	0.703819				0.969
50	α	1.541615	2.795729	1.254114	0.046893	0.969
	λ	0.815671	1.292058	0.476386	0.034501	0.961
	β	1.529307	2.718522	1.189215	0.039204	0.963
	μ	0.75032				0.963
75	α	1.605842	2.614159	1.008317	0.041621	0.958
	λ	0.835775	1.203938	0.368163	0.032678	0.955
	β	1.620502	2.561225	0.940722	0.032304	0.955
	μ	0.764811				0.959
100	α	1.63589	2.569063	0.933173	0.031881	0.956
	λ	0.852459	1.192569	0.34011	0.011751	0.952
	β	1.645568	2.488622	0.843054	0.03875	0.953
	μ	0.779406				0.954

Table 2.21 Coverage of Parametric Bootstrap 95% C.I.s

α =2, λ =1, β =2, μ=0.88623, π=.5, τ=1

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	1.573883	2.778111	1.204229	0.10896	0.968
	λ	0.772769	1.29223	0.519461	0.028304	0.967
	β	1.494966	2.659774	1.164807	0.074735	0.962
	μ	0.707184				0.969
50	α	1.523706	2.607722	1.084016	0.234415	0.963
	λ	0.785877	1.296655	0.510777	0.126723	0.966
	β	1.550444	2.614866	1.064422	0.109566	0.957
	μ	0.791947				0.952
75	α	1.661268	2.491358	0.830089	0.067737	0.952
	λ	0.838859	1.205743	0.366885	0.038591	0.955
	β	1.663064	2.463082	0.800017	0.050784	0.949
	μ	0.770055				0.956
100	α	1.696021	2.442705	0.746684	0.042784	0.947
	λ	0.856891	1.192207	0.335317	0.018008	0.951
	β	1.691015	2.419798	0.728783	0.05458	0.946
	μ	0.782631				0.953

Table 2.22 Coverage of Nonparametric Bootstrap 95% C.I.s

α =2, λ =1, β =2, μ=0.88623, π=.5, τ=1

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	1.598029	2.799223	1.201194	0.065988	0.967
	λ	0.797661	1.31236	0.514699	0.046478	0.967
	β	1.561397	2.669822	1.108425	0.095103	0.961
	μ	0.738896				0.966
50	α	1.582964	2.624709	1.041745	0.086998	0.959
	λ	0.818457	1.280681	0.462224	0.042453	0.959
	β	1.610014	2.661945	1.051931	0.102445	0.956
	μ	0.759059				0.962
75	α	1.697661	2.482775	0.785114	0.073166	0.951
	λ	0.847026	1.209156	0.36213	0.024926	0.954
	β	1.653746	2.463097	0.80935	0.060862	0.951
	μ	0.769909				0.957
100	α	1.685963	2.455716	0.769753	0.067814	0.949
	λ	0.865267	1.186219	0.320952	0.033396	0.949
	β	1.69385	2.436623	0.742773	0.064615	0.948
	μ	0.778681				0.955

Table 2.23 Coverage of Asymptotic 95% C.I.s

α =2, λ =1, β =2, μ=0.88623, π=.5, τ=1.5

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	1.519411	2.909902	1.390492	0.061313	0.971
	λ	0.774595	1.217566	0.442971	0.024039	0.969
	β	1.518904	2.700644	1.181739	0.041725	0.971
	μ	0.721357				0.969
50	α	1.594864	2.727219	1.132355	0.051139	0.968
	λ	0.81636	1.212945	0.396586	0.023425	0.965
	β	1.555829	2.612693	1.056864	0.054011	0.965
	μ	0.751918				0.963
75	α	1.633685	2.583715	0.95003	0.050887	0.963
	λ	0.837753	1.168198	0.330445	0.015659	0.959
	β	1.628016	2.476788	0.848772	0.033739	0.957
	μ	0.76741				0.961
100	α	1.666533	2.507369	0.840836	0.025476	0.956
	λ	0.853058	1.160286	0.307228	0.017121	0.954
	β	1.644257	2.446682	0.802426	0.018657	0.954
	μ	0.778088				0.956

Table 2.24 Coverage of Parametric Bootstrap 95% C.I.s

$\alpha=2, \lambda=1, \beta=2, \mu=0.88623, \pi=.5, \tau=1.5$

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	1.598659	2.707797	1.109138	0.092883	0.967
	λ	0.802635	1.193767	0.391133	0.029194	0.965
	β	1.632064	2.511551	0.879487	0.056993	0.959
	μ	0.748624				0.965
50	α	1.658097	2.545814	0.887717	0.069414	0.958
	λ	0.841991	1.200204	0.358213	0.032026	0.962
	β	1.647926	2.457365	0.809439	0.080083	0.955
	μ	0.7662				0.962
75	α	1.683376	2.469675	0.786299	0.076359	0.953
	λ	0.862218	1.153746	0.291528	0.020141	0.951
	β	1.69323	2.360561	0.667331	0.05416	0.949
	μ	0.780868				0.954
100	α	1.729485	2.393175	0.66369	0.045071	0.949
	λ	0.87499	1.141151	0.266161	0.023524	0.949
	β	1.720341	2.341989	0.621648	0.042074	0.949
	μ	0.793004				0.951

Table 2.25 Coverage of Nonparametric Bootstrap 95% C.I.s

$\alpha=2, \lambda=1, \beta=2, \mu=0.88623, \pi=.5, \tau=1.5$

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	1.575055	2.729891	1.154836	0.089533	0.969
	λ	0.80811	1.246761	0.438652	0.029353	0.967
	β	1.59222	2.561914	0.969694	0.07735	0.962
	μ	0.739658				0.968
50	α	1.679571	2.574616	0.895045	0.061833	0.959
	λ	0.846712	1.207035	0.360323	0.024383	0.963
	β	1.672726	2.480116	0.80739	0.055024	0.955
	μ	0.771552				0.958
75	α	1.699384	2.426089	0.726705	0.074618	0.951
	λ	0.847472	1.1556	0.308127	0.019242	0.954
	β	1.68427	2.42793	0.74366	0.055335	0.951
	μ	0.780956				0.954
100	α	1.6863	2.41247	0.726169	0.056313	0.951
	λ	0.863844	1.16041	0.296566	0.023083	0.952
	β	1.703262	2.339657	0.636396	0.060274	0.947
	μ	0.784534				0.952

Table 2.26 Coverage of Asymptotic 95% C.I.s

α =2, λ =1, β =2, μ=0.88623, π=.667, τ=1

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	1.501868	2.772073	1.270204	0.038875	0.971
	λ	0.732235	1.290081	0.557847	0.049982	0.967
	β	1.399317	2.77564	1.376323	0.094227	0.971
	μ	0.68061				0.969
50	α	1.537503	2.674505	1.137002	0.056834	0.969
	λ	0.758493	1.270095	0.511602	0.026488	0.963
	β	1.458741	2.702765	1.244024	0.061638	0.962
	μ	0.704273				0.962
75	α	1.634756	2.589415	0.954659	0.027532	0.961
	λ	0.803353	1.233541	0.430188	0.024894	0.954
	β	1.58919	2.556285	0.967094	0.048037	0.954
	μ	0.742476				0.955
100	α	1.635246	2.518069	0.882823	0.036994	0.956
	λ	0.832208	1.204684	0.372476	0.033863	0.951
	β	1.638842	2.516205	0.877364	0.057793	0.951
	μ	0.760237				0.949

Table 2.27 Coverage of Parametric Bootstrap 95% C.I.s

α =2, λ =1, β =2, μ=0.88623, π=.667, τ=1

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	1.584251	2.566311	0.98206	0.07286	0.964
	λ	0.746065	1.292176	0.546111	0.057218	0.965
	β	1.449592	2.649389	1.199797	0.123255	0.959
	μ	0.687468				0.967
50	α	1.616916	2.536815	0.9199	0.07817	0.958
	λ	0.766718	1.266552	0.499834	0.03195	0.961
	β	1.492434	2.609861	1.117427	0.08488	0.956
	μ	0.70979				0.962
75	α	1.706257	2.496656	0.790399	0.046235	0.952
	λ	0.809151	1.239306	0.430156	0.033166	0.954
	β	1.629579	2.480842	0.851263	0.063122	0.948
	μ	0.749622				0.953
100	α	1.683559	2.402323	0.718765	0.043911	0.9948
	λ	0.839364	1.20608	0.366715	0.039321	0.949
	β	1.686397	2.448049	0.761652	0.07715	0.946
	μ	0.764513				0.948

Table 2.28 Coverage of Nonparametric Bootstrap 95% C.I.s

α =2, λ =1, β =2, μ=0.88623, π=.667, τ=1

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	1.525844	2.653593	1.127749	0.072596	0.968
	λ	0.754816	1.329374	0.574558	0.045802	0.969
	β	1.507521	2.7966	1.289079	0.106334	0.965
	μ	0.700187				0.963
50	α	1.566619	2.58238	1.015762	0.06782	0.966
	λ	0.78307	1.257077	0.474007	0.038776	0.959
	β	1.514494	2.601731	1.087237	0.09075	0.955
	μ	0.71844				0.959
75	α	1.672603	2.439149	0.766546	0.057978	0.950
	λ	0.805391	1.23648	0.431089	0.035548	0.955
	β	1.621088	2.548998	0.92791	0.06989	0.953
	μ	0.736908				0.956
100	α	1.694588	2.415594	0.721006	0.048877	0.949
	λ	0.826174	1.186477	0.360303	0.024121	0.948
	β	1.64081	2.412504	0.771694	0.043612	0.947
	μ	0.756104				0.951

Table 2.29 Coverage of Asymptotic 95% C.I.s

α =2, λ =1, β =2, μ=0.88623, π=.667, τ=1.5

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	1.478369	2.624401	1.146033	0.040713	0.971
	λ	0.698611	1.275153	0.576542	0.025833	0.971
	β	1.377872	2.614177	1.236306	0.03808	0.971
	μ	0.662728				0.969
50	α	1.557089	2.526809	0.96972	0.017892	0.969
	λ	0.748954	1.211557	0.462603	0.023503	0.963
	β	1.44547	2.50873	1.06326	0.073951	0.964
	μ	0.698295				0.965
75	α	1.640691	2.544269	0.903578	0.030188	0.963
	λ	0.803109	1.18672	0.383611	0.019288	0.956
	β	1.573525	2.446088	0.872562	0.027582	0.957
	μ	0.739598				0.959
100	α	1.674023	2.503273	0.82925	0.025159	0.958
	λ	0.832987	1.18701	0.354023	0.013232	0.952
	β	1.634495	2.443882	0.809388	0.034866	0.953
	μ	0.762572				0.956

Table 2.30 Coverage of Parametric Bootstrap 95% C.I.s

α =2, λ =1, β =2, μ=0.88623, π=.667, τ=1.5

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	1.549899	2.467697	0.917798	0.071158	0.965
	λ	0.731346	1.250344	0.518998	0.038741	0.967
	β	1.477881	2.474033	0.996152	0.061427	0.963
	μ	0.688011				0.967
50	α	1.632197	2.404597	0.7724	0.043842	0.955
	λ	0.783189	1.189062	0.405872	0.026159	0.958
	β	1.511334	2.386952	0.875618	0.106242	0.957
	μ	0.720073				0.961
75	α	1.690757	2.433836	0.743078	0.040686	0.952
	λ	0.823991	1.174488	0.350496	0.023103	0.952
	β	1.640487	2.346654	0.706166	0.034308	0.949
	μ	0.763419				0.955
100	α	1.723998	2.398318	0.67432	0.033039	0.949
	λ	0.864714	1.171753	0.307039	0.018894	0.949
	β	1.696336	2.349142	0.652807	0.04112	0.946
	μ	0.784101				0.951

Table 2.31 Coverage of Nonparametric Bootstrap 95% C.I.s

α =2, λ =1, β =2, μ=0.88623, π=.667, τ=1.5

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	1.539082	2.472456	0.933374	0.05274	0.966
	λ	0.752717	1.231717	0.479	0.034023	0.965
	β	1.469029	2.482562	1.013533	0.051888	0.964
	μ	0.688344				0.967
50	α	1.621013	2.448673	0.827661	0.08118	0.958
	λ	0.773339	1.248316	0.474977	0.042215	0.964
	β	1.515091	2.502296	0.987205	0.1304	0.963
	μ	0.710522				0.962
75	α	1.70718	2.448783	0.741603	0.072849	0.952
	λ	0.841297	1.188695	0.347398	0.026076	0.951
	β	1.687001	2.396295	0.709293	0.047098	0.949
	μ	0.770576				0.953
100	α	1.732012	2.404005	0.671993	0.029558	0.949
	λ	0.849432	1.190326	0.340894	0.025235	0.951
	β	1.695365	2.353513	0.658147	0.060011	0.946
	μ	0.769226				0.954

The simulations results show that all the three methods provide at least 95% coverage in almost all cases. While the asymptotic confidence intervals for the parameters α and β provide consistently conservative coverage when the sample size is 30 (e.g. 0.967 and 0.969 respectively in Table 2.2) these intervals provide marginally less conservative coverage for larger sample sizes (see Tables 2.2, 2.5, 2.8, 2.11, 2.14, 2.17, 2.20, 2.23, 2.26 and 2.29). The same pattern of less conservative coverage with increasing sample size is seen for the asymptotic confidence intervals for β (see Tables 2.2 through 2.29), except showing a near normal coverage for sample size 30 in Table 2.2. The intervals for α, λ, and β based on the parametric and nonparametric bootstrap show a similar pattern of decreasing coverage with increase in sample size, but in almost all cases the coverage stays near or above the normal value. In general, the coverage probabilities of the confidence intervals for α, λ, and β do not differ by much when compared across the mode of construction.

While the coverage probabilities do not show any distinctive differe3nces between the three methods, inspection of the widths of the confidence intervals for α, λ, and β show some slight differences. In general, the intervals based on the parametric bootstrap are slightly narrower than those based on the asymptotic distribution (see Tables 2.2 and 2.3, 2.5 and 2.6) when $\alpha = 1.5$. This phenomenon disappears when $\alpha = 2$ except for intervals constructed for α (see Tables 2.8 and 2.9 for example). Other than that, one cannot find any discernible pattern that separates the two bootstrap methods as far as interval widths are concerned. When the ratio of normal to accelerated sample sizes changes from 0.5 to 2/3, the widths of the asymptotic intervals for λ and β increases slightly when the censoring time $\tau = 1$ (see Tables 2.8 and 2.14), but such a pattern is not seen when $\tau = 1.5$ (see Tables 2.11 and 2.17). This increase is also seen for parametric and nonparametric bootstrap-based intervals when $\tau = 1$ (Tables 2.12, 2.13, 2.15, and 2.16).

For a practicing chemist or an engineer, estimation of the mean life under normal conditions is even more important than building confidence intervals for the distribution parameters. Manufacturers of products such as chemicals would like to provide customers with information on the shelf-life of the product when stored or used under normal conditions. Usually such assurances are given in terms of a lower confidence bound for the mean life. Therefore, it is of interest to note how the confidence bounds for the mean life performed in the Monte-Carlo study. The confidence bounds for mean life constructed using all three methods show near normal coverage, especially when the sample size is at or above 75. The coverage probability can be conservative for small sample sizes, but this occurs only when the shape parameter $\alpha = 2.$ The expected value of the bounds also do not vary by much across the three methods.

In summary, all three methods produce confidence intervals with reasonable coverages as well as lower confidence bounds for mean life that are comparable and provide coverage ranging from conservative to normal . The relative performance of the asymptotic distribution-based method relative to the bootstrap-based methods is somewhat surprising, even when the sample size is 30. Further studies, with smaller sample sizes and a higher level of censoring may differentiate the bootstrap methods from the asymptotic method.

2.5 CONCLUSIONS AND FUTURE WORK

PALT is a method that is preferable over the ALT procedure when the accelerating factor is unknown or a suitable model that links parameters of the life distribution to the stress level is not available. While PALT is not suitable when the products under test have a very long mean life, it is applicable in situations where the life-span of tested products is only moderately long. This research extended previous work to cover Type I censoring in the Weibull case while at the same time developing bootstrap-based methods for obtaining prediction intervals for distribution parameters and the acceleration factor. In addition, asymptotic distribution based intervals were also considered. More importantly, a method of obtaining lower confidence bounds for the mean life under normal use conditions was also developed. The performance of the three methods was studies using a Monte-Carlo study. Results show that all methods perform reasonable well under all parameter combinations employed in the Monte-Carlo study.

Future work studying the performance of the three approaches under additional parameter combinations and censoring levels is warranted. The performance of the bootstrap methods when estimates other than MLEs, such as the closed form approximations introduced by Englehardt (1975), are used would be of interest. A possible generalization of the proposed procedure is to consider the case where the censoring times are different for the accelerated and normal use samples. Studies on the robustness of the three methods in the presence of outliers or distributional miss-specification may also be valuable. Extending the proposed methodologies to Step-Stress PALT experiments as well as Progressive Step-Stress PALT situations would also be of added value.

3. BOOTSTRAP-BASED CONFIDENCE INTERVALS IN PARTIALLY ACCELERATED LIFE TESTING UNDER THE GENERALIZED EXPONENTIAL DISTRIBUTION

3.1 INTRODUCTION

Accelerated life tests (ALT) are often used to obtain information about the life distribution of products that are designed to last a long time under normal use conditions. This is because, under normal use conditions, only a very small fraction of them will fail during a feasible testing period. Nelson (1980) drives this point home in his statement: “Accelerated life testing of a product or material is used to get information quickly on its life distribution. Test units are run under severe conditions and fail sooner than under usual conditions. ….. This is quicker and cheaper than testing at usual conditions, which is usually impractical because life is so long.” In situations where the acceleration factor is known or one can find a mathematical model describing the life-stress relationship, ALT is provides a quick way to get a sufficient amount of information to estimate the life distribution. However, in situations where neither the acceleration factor is known nor a reasonable life-stress model can be found, partially accelerated life tests (PALT) provide a suitable approach to estimating the life distribution and related parameters.

Under the PALT method, a subset of the test units are placed under the normal use (field use, design use) stress conditions and the remaining units are tested under a suitably selected higher than normal stress level. This provides a statistically viable approach by assuming that life distribution of the units under the higher stress level is the same as that of units under normal use, but with the scale parameter multiplied by an acceleration factor. This factor is estimated together with the other distribution parameters by utilizing the combined data set. Since there is more failure data from the units that received higher than normal stress level, the combined data provide better estimates of the common parameters.

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One drawback of the PALT method is that unlike in the ALT, some units have to be tested under normal use. Thus this method is not suitable for components that are very long lasting. But items such as chemicals that have shelf-lives that are measured in months or a year or two can be tested using this method.

In this paper, we introduce three approaches for the construction confidence intervals for model parameters and lower confidence bounds for the mean life under normal use conditions using Type I censored data from a constant stress PALT when the underlying distribution is Generalized Exponential (GE). The methods introduced are, namely, intervals and bounds based on the asymptotic distribution of the model parameters, the parametric bootstrap, and the nonparametric bootstrap. While results based on the asymptotic distribution is available for the case where the PALT is carried out for GE data, such results are for the Type II censoring scenario. In addition, no bootstrap-based intervals have been developed for cases where the underlying distribution is GE or when the censoring mechanism is Type I.

3.1.1 A Brief Review of Relevant Literature. Compared to the large number of publications on ALTs the publications on PALT is relatively smaller. For brevity, we will focus only on a limit number of these publications. For details on ALT, we refer the reader to the excellent coverage of the topic given in Nelson (1990). Other good references include Mann, Schafer, and Singapuwalla (1974), Lawless (1982), Tobias and Trindade (2011), and Meeker and Escobar (1998).

A relatively recent publication on ALT is Jayawardhana and Samaranayake (2003), which discussed obtaining lower prediction bounds for a future observation from a Weibull population at design (normal use) stress level, using Type II censored accelerated life test data. The authors assumed that the scale parameter of the life distribution have an inverse power relationship with the stress level. They showed that the method works well when the low and high stresses are reasonably far apart. Alferink and Samaranayake (2011) considered accelerated degradation models and developed confidence intervals for mean life using the Delta method and the bootstrap, assuming lognormal distribution with variance dependent on stress. This contrasts with other approaches, which assume that the variance if not affected by increasing stress. Another important publication is Kamal, et al (2013), which presented a step stress ALT plan with good performance. In step stress, the components are first put at a lower stress and the unfailed components are subjected to higher stress after a specific period. More recently, Jayawardhana and Samaranayake (2014), obtained predictive density of a future observation at normal use conditions using ALT method under lognormal life distribution and Type II censoring with non-constant variance.

Among the publications on PALTs, the following are worth mentioning. Saxena and Zarrin (2013) used the Constant Stress Partially Accelerated Life Test (CSPALT) and assumed Type-I censoring under the Extreme Value Type-III distribution. Note that the Extreme Value Type-III distribution has been recommended as appropriate for high reliability components. The authors used the Maximum Likelihood (ML) method to estimate the parameters of CSPALT model and confidence intervals for the model parameters were constructed. Note that the CSPALT plan is used to minimize the Generalized Asymptotic Variance (GAV) of the ML estimators of the model parameters.

Abdel-Hamid (2009), considered a constant PALT model when the observed failure times come from $\mathrm{Burr}(c, k)$ distribution under progressively Type-II right censoring. The MLEs of the parameters were obtained and their performance was studied through their mean squared errors and relative absolute biases. The paper also showed how to construct approximate and bootstrap CIs for the parameters. The bootstrap CIs give more accurate results than the approximate intervals for small sample sizes, and the Student's-t bootstrap CIs are better than the Percentile bootstrap CIs in the sense of having smaller widths. However, the differences between the lengths of CIs for the two methods decrease with increased sample size.

A publication that motivated the work in this dissertation is by Ismail (2013), who derived the maximum likelihood estimators (MLEs) of the parameters of the GE distribution and the acceleration factor when the data are Type-II censored under constant-stress PALT model. The likelihood ratio bounds (LRB) method was used to obtain confidence bounds of the model parameters when the sample size is small. It is also shown that the maximum likelihood estimators are consistent and their variances decrease as the sample size increases. The numerical results reported in the paper support the theoretical findings and showed that the estimated approximate confidence intervals for the three parameters are smaller when the sample size is larger.

3.1.2 The Generalized Exponential Distribution. The proposed PALT method is developed for the case where the underlying life distribution is GE. The generalized exponential distribution has been introduced and studied quite extensively by Gupta and Kundu (1999, 2001a, 2001b), and by Ragab and Ahsanullah (2001). The probability density function and the cumulative distribution function of the generalized exponential distribution function has the forms:

$$f\left(x;\alpha,\lambda\right)=\alpha\lambda e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{\alpha-1} \qquad x>0,\alpha>0,\lambda>0, \tag{1}$$

$$F\left(x;\alpha,\lambda\right)=\left(1-e^{-\lambda x}\right)^{\alpha}, \tag{2}$$

respectively, where α is the shape parameter and λ the scale parameter.

The GE distribution has certain features which are distinct from the Gamma and Weibull distributions (see Gupta and Kundu (1999, 2001)). The GE model can be used as a possible alternative for analyzing skewed datasets. An interesting fact is that both Gamma and GE distributions have the likelihood ratio ordering property while Weibull does not. On the other hand, GE and Weibull distributions have the common feature of having closed form expressions for Cumulative Distribution Function (CDF) and the hazard function. One aspect that makes the GE distribution outperform the Weibull is the fact that the convergence of MLE's of Weibull parameters can be very slow (Bain(1976)) whereas the asymptotic confidence intervals obtained under the GE assumption maintain normal coverage even for small sample sizes (Gupta and Kundu (2001)). Gupta and Kundu (2001) also showed that the hazard function of the GE distribution has proprieties similar to those of the Gamma and Weibull distributions. These properties are summarized in Table 3.1.

Table 3.1 Properties of the Hazard Function[1]

Parameters	**Gamma**	**Weibull**	**GE**
$\boldsymbol{\alpha = 1}$	Constant	Constant	Constant
$\boldsymbol{\alpha > 1}$	Increasing from 0 to λ	Increasing from 0 to ∞	Increasing from 0 to λ
$\boldsymbol{\alpha < 1}$	Decreasing from ∞ to λ	Decreasing from ∞ to 0	Decreasing from ∞ to λ

Table 3.1. Note that the Hazard function of the GE distribution is given by

$$h(x;\alpha,\lambda) = \frac{\alpha\lambda e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{\alpha-1}}{\left(1-e^{-\lambda x}\right)^{\alpha}}.$$

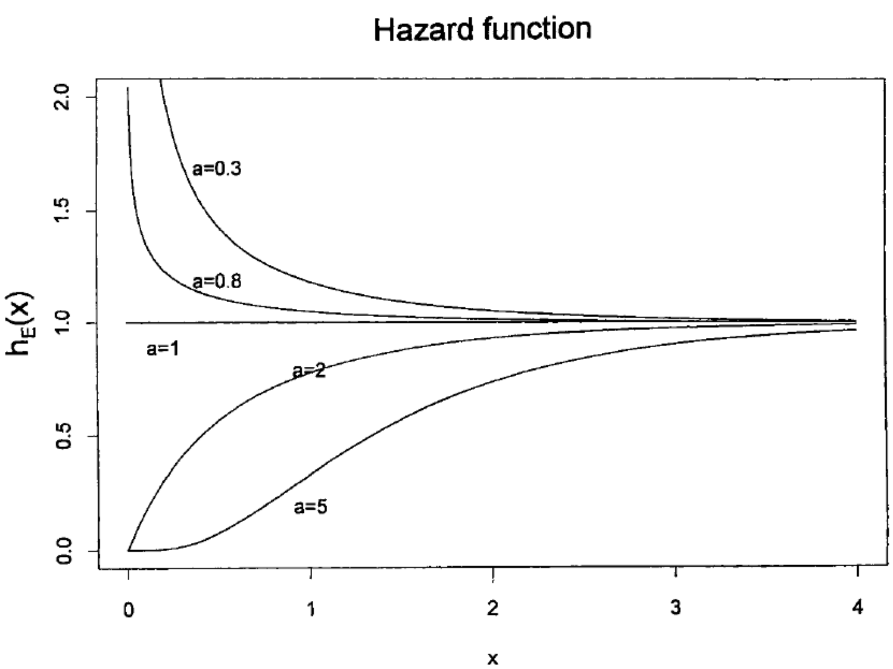

Figure 3.1. Properties of the Hazard Function[2]

3.2 THE PROPOSED PALT METHOD AND BOOTSTRAP INTERVALS

The following assumptions are made regarding the proposed PALT method.

[1] Exponentiated Exponential Family:An Alternative to Gamma and Weibull Distributions(2001)
[2] Exponentiated Exponential Family:An Alternative to Gamma and Weibull Distributions(2001)

1. The total number of units under test is n .
2. π denotes the proportion of sample units allocated to accelerated condition
3. $n(1-\pi) = n\bar{\pi}$of these units are allocated to normal (field) use conditions.
4. $n\pi$ units are allocated to the high stress condition (subject to acceleration)

3.2.1 Likelihood Function under Type I Censoring and Asymptotic C.I.s.

Under Type I censoring, the censoring time, $\boldsymbol{\tau}$, is fixed but the number of failures observed in the time $\boldsymbol{\tau}$ is a random variable, say *R*. We assume that the number of items failing before time $\boldsymbol{\tau}$ follows binomial distribution with parameters $(\boldsymbol{n}, \boldsymbol{p})$ with $\boldsymbol{p} = \boldsymbol{F_X}(\boldsymbol{\tau}; \underline{\boldsymbol{\theta}})$, where $\underline{\boldsymbol{\theta}}$ is the vector of parameters of the GE distribution.

Notation:

x_i: Observed lifetime of item i tested at the normal (field) use condition.

y_j: Observed lifetime of item j tested at high stress condition.

δ_{ui}: Indicator function denoting the censoring state of ith observation under normal . use condition, with $\delta_{ui} = 1$ if the observation is uncensored.

δ_{aj}: Indicator function denoting the censoring state of jth observation under high stress condition, with $\delta_{ai} = 1$ if the observation is uncensored.

n_u: Number of items that failed at normal use condition.

n_a: Number of items that failed at a high stress condition.

τ: The censoring time of the life test (for all units).

$x_{(1)} \leq \cdots \leq x_{(n_u)} \leq \tau$: Ordered failure times at normal use condition.

$y_{(1)} \leq \cdots \leq y_{(n_a)} \leq \tau$: Ordered failure times at high stress condition.

β: Denotes the acceleration factor ($\beta > 1$).

The number of items failing before time τ follows a binomial distribution $R \sim Bin(n, p)$

where

$$p = F_X(\tau; \alpha, \lambda) = \left(1 - e^{-\lambda\tau}\right)^{\alpha}.$$

We let,

$$\delta_{u_i} = \begin{cases} 1 & x_i \leq \tau \\ 0 & 0/w \end{cases}, \quad i = 1, 2, \cdots, n\overline{\pi},$$

(4)

$$\delta_{a_j} = \begin{cases} 1 & y_j \leq \tau \\ 0 & 0/w \end{cases}, \quad j = 1, 2, \cdots, n\pi, \tag{5}$$

and $\delta_{u_i} = 1 - \overline{\delta}_{u_i}$, $\delta_{a_j} = 1 - \overline{\delta}_{a_j}$.

Then,

$$\delta_{u_i} \sim Ber(p) \Rightarrow \sum_{i=1}^{n\overline{\pi}} \delta_{u_i} \sim Bin(n\overline{\pi}, p), \tag{6}$$

$$\delta_{a_j} \sim Ber(p) \Rightarrow \sum_{j=1}^{n\pi} \delta_{a_j} \sim Bin(n\pi, p), \tag{7}$$

where $p^* = F_X(\tau; \alpha, \lambda, \beta) = \left(1 - e^{-\lambda\beta\tau}\right)^{\alpha}$, with

$$F_\tau\left(x;\alpha,\lambda\middle|\tau\right)=P\left(X\le x\middle|X\le\tau\right)=\frac{P(X\le x)}{P(X\le\tau)}=\frac{\left(1-e^{-\lambda x}\right)^{\alpha}}{\left(1-e^{-\lambda\tau}\right)^{\alpha}}$$

$$=\begin{cases}\dfrac{\left(1-e^{-\lambda x}\right)^{\alpha}}{\left(1-e^{-\lambda\tau}\right)^{\alpha}}, & x\le\tau\\ 1, & x>\tau,\end{cases}\tag{8}$$

and

$$f_\tau\left(x;\alpha,\lambda\middle|\tau\right)=\frac{f_X\left(x;\alpha,\lambda\right)}{F_X\left(\tau;\alpha,\lambda\right)}=\frac{\alpha\lambda e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{\alpha-1}}{\left(1-e^{-\lambda\tau}\right)^{\alpha}}.\tag{9}$$

Thus, given $R = n_u$, the conditional density of the first r failure times is equivalent to the joint density of an ordered random sample of size n_u from a truncated GE distribution,

$$f\left(x_{(1)},\cdots,x_{(n_u)}\middle|R=n_u\right)=n_u!\prod_{i=1}^{n_u}f_\tau\left(x_{(i)};\alpha,\lambda\right)=n_u!\prod_{i=1}^{n_u}\left[\frac{\alpha\lambda e^{-\lambda x_i}\left(1-e^{-\lambda x_i}\right)^{\alpha-1}}{\left(1-e^{-\lambda\tau}\right)^{\alpha}}\right]$$

$$=n_u!\frac{(\alpha\lambda)^{n_u}e^{-\lambda\sum_{i=1}^{n_u}x_i}}{\left(1-e^{-\lambda\tau}\right)^{\alpha n_u}}\prod_{i=1}^{n_u}\left[\left(1-e^{-\lambda x_i}\right)^{\alpha-1}\right].\tag{10}$$

The joint density of obtaining $R=n_u$ ordered observations at the values $x_{(1)},\cdots,x_{(n_u)}$ before time, may be expressed as

$$f\left(x_{(1)},\cdots,x_{(n_u)}\right)=f\left(x_{(1)},\cdots,x_{(n_u)}\middle|R=n_u\right)bin\left(n_u;n\overline{\pi},p\right)$$

$$=n_u!\frac{(\alpha\lambda)^{n_u}e^{-\lambda\sum_{i=1}^{n_u}x_i}}{\left(1-e^{-\lambda\tau}\right)^{\alpha n_u}}\prod_{i=1}^{n_u}\left[\left(1-e^{-\lambda x_i}\right)^{\alpha-1}\right]\binom{n\overline{\pi}}{n_u}p^{n_u}\left(1-p\right)^{n\overline{\pi}-n_u}$$

$$=\frac{(n\overline{\pi})!}{(n\overline{\pi}-n_u)!}\frac{(\alpha\lambda)^{n_u}e^{-\lambda\sum_{i=1}^{n_u}x_i}}{\left(1-e^{-\lambda\tau}\right)^{\alpha n_u}}\left(1-e^{-\lambda\tau}\right)^{\alpha n_u}\left[1-\left(1-e^{-\lambda\tau}\right)^{\alpha}\right]^{n\overline{\pi}-n_u}\prod_{i=1}^{n_u}\left[\left(1-e^{-\lambda x_i}\right)^{\alpha-1}\right]$$

$$=\frac{(n\overline{\pi})!}{(n\overline{\pi}-n_u)!}(\alpha\lambda)^{n_u}e^{-\lambda\sum_{i=1}^{n_u}x_i}\left[1-\left(1-e^{-\lambda\tau}\right)^{\alpha}\right]^{n\overline{\pi}-n_u}\prod_{i=1}^{n_u}\left[\left(1-e^{-\lambda x_i}\right)^{\alpha-1}\right].$$

Therefore we can write,

$$f\left(x_{(1)},\cdots,x_{(n_u)}\right)\propto(\alpha\lambda)^{n_u}\left[1-\left(1-e^{-\lambda\tau}\right)^{\alpha}\right]^{n\bar{\pi}-n_u}e^{-\lambda\sum_{i=1}^{n_u}x_i}\prod_{i=1}^{n_u}\left[\left(1-e^{-\lambda x_i}\right)^{\alpha-1}\right].$$

Similarly, given $R=n_a$ the conditional density of the first r failure times is equivalent to the joint density of an ordered random sample of size n_a from a truncated accelerated GE distribution.

For an item tested at accelerated condition, the probability density function is given by

$$f\left(x;\alpha,\lambda\right)=\alpha\beta\lambda e^{-\beta\lambda x}\left(1-e^{-\beta\lambda x}\right)^{\alpha-1}\qquad x>0,\ \alpha>0,\ \lambda>0,\ \beta>1,$$

where $Y=\beta^{-1}X$ and therefore the conditional joint distribution given $R=n_a$ is

$$f\left(y_{(1)},\cdots,y_{(n_u)}\middle|R=n_a\right)=n_a!\prod_{j=1}^{n_a}f_{\tau}\left(x_{(j)};\alpha,\lambda,\beta\right)=n_a!\prod_{j=1}^{n_a}\left[\frac{\alpha\lambda\beta e^{-\lambda\beta y_j}\left(1-e^{-\lambda\beta y_j}\right)^{\alpha-1}}{\left(1-e^{-\lambda\beta\tau}\right)^{\alpha}}\right] \quad (11)$$

$$=n_a!\frac{(\alpha\lambda\beta)^{n_a}e^{-\lambda\beta\sum_{i=1}^{n_u}y_j}}{\left(1-e^{-\lambda\beta\tau}\right)^{\alpha n_a}}\prod_{j=1}^{n_a}\left[\left(1-e^{-\lambda\beta y_j}\right)^{\alpha-1}\right].$$

The joint density of obtaining $R=n_a$ ordered observations at the values $Y_{(1)},\ldots,Y_{(n_a)}$ before time, may be expressed as

$$f\left(y_{(1)},\cdots,y_{(n_a)}\right)=f\left(y_{(1)},\cdots,y_{(n_a)}\middle|R=n_a\right)bin\left(n_a;n\pi,p^*\right)$$

$$=n_u!\frac{(\alpha\lambda\beta)^{n_a}e^{-\lambda\beta\sum_{j=1}^{n_a}y_j}}{\left(1-e^{-\lambda\beta\tau}\right)^{\alpha n_a}}\prod_{j=1}^{n_a}\left[\left(1-e^{-\lambda\beta y_j}\right)^{\alpha-1}\right]\binom{n\pi}{n_a}p*^{n_a}\left(1-p^*\right)^{n\pi-n_a}$$

$$=\frac{(n\pi)!}{(n\pi-n_a)!}\frac{(\alpha\lambda\beta)^{n_a}e^{-\lambda\beta\sum_{j=1}^{n_a}y_j}}{\left(1-e^{-\lambda\beta\tau}\right)^{\alpha n_a}}\left(1-e^{-\lambda\beta\tau}\right)^{\alpha n_a}\left[1-\left(1-e^{-\lambda\beta\tau}\right)^{\alpha}\right]^{n\pi-n_a}\prod_{j=1}^{n_a}\left(1-e^{-\lambda\beta y_j}\right)^{\alpha-1}$$

$$=\frac{(n\pi)!}{(n\pi-n_a)!}(\alpha\lambda\beta)^{n_a}e^{-\lambda\beta\sum_{j=1}^{n_a}y_j}\left[1-\left(1-e^{-\lambda\beta\tau}\right)^{\alpha}\right]^{n\pi-n_a}\prod_{i=1}^{n_a}\left[\left(1-e^{-\lambda\beta y_j}\right)^{\alpha-1}\right].$$

Therefor we can write,

$$f\left(y_{(1)},\cdots,y_{(n_a)}\right)\propto(\alpha\lambda\beta)^{n_a}\,e^{-\lambda\beta\sum_{j=1}^{n_a}y_j}\left[1-\left(1-e^{-\lambda\beta\tau}\right)^{\alpha}\right]^{n\pi-n_a}\prod_{i=1}^{n_a}\left[\left(1-e^{-\lambda\beta y_j}\right)^{\alpha-1}\right],$$

and the total likelihood function for $\left(x_1;\delta_{u_1},\cdots,x_{n\bar{\pi}};\delta_{u_{n\bar{\pi}}},y_1;\delta_{a_1},\cdots,y_{n_{n\pi}};\delta_{a_{n\pi}}\right)$ is given by

$$\begin{aligned}
L\left(\alpha,\lambda,\beta\middle|\underline{x},\underline{y}\right)&=\prod_{i=1}^{n\bar{\pi}}L_{u_i}\left(\alpha,\lambda\middle|x_i,\delta_{u_i}\right)\prod_{j=1}^{n\pi}L_{u_j}\left(\alpha,\lambda,\beta\middle|y_j,\delta_{u_j}\right)\\
&=\prod_{i=1}^{n\bar{\pi}}\left[\alpha\lambda e^{-\lambda x_i}\left(1-e^{-\lambda x_i}\right)^{\alpha-1}\right]^{\delta_{u_i}}\left[1-\left(1-e^{-\lambda\tau}\right)^{\alpha}\right]^{\bar{\delta}_{u_i}}\\
&\quad\times\prod_{j=1}^{n\pi}\left[\alpha\lambda\beta e^{-\lambda\beta y_j}\left(1-e^{-\lambda\beta y_j}\right)^{\alpha-1}\right]^{\delta_{u_j}}\left[1-\left(1-e^{-\lambda\beta\tau}\right)^{\alpha}\right]^{\bar{\delta}_{u_j}} \qquad (12)\\
&=\prod_{i=1}^{n_u}\left[\alpha\lambda e^{-\lambda x_i}\left(1-e^{-\lambda x_i}\right)^{\alpha-1}\right]\prod_{i=1}^{n\bar{\pi}}\left[1-\left(1-e^{-\lambda\tau}\right)^{\alpha}\right]\\
&\quad\times\prod_{j=1}^{n_a}\left[\alpha\lambda\beta e^{-\lambda\beta y_j}\left(1-e^{-\lambda\beta y_j}\right)^{\alpha-1}\right]\prod_{i=1}^{n\pi}\left[1-\left(1-e^{-\lambda\beta\tau}\right)^{\alpha}\right].
\end{aligned}$$

The MLE's of the parameters can estimated numerically by minimizing the log likelihood function.

$$\Rightarrow \ln L\left(\alpha,\lambda,\beta\middle|\underline{x},\underline{y}\right)=l,$$

$$\begin{aligned}
\Rightarrow l&=\sum_{i=1}^{n_u}\ln\alpha+\sum_{i=1}^{n_u}\ln\lambda-\lambda\sum_{i=1}^{n_u}x_i+(\alpha-1)\sum_{i=1}^{n_u}\ln\left(1-e^{-\lambda x_i}\right)\\
&\quad-\sum_{i=n_u+1}^{n\bar{\pi}}\ln\left(1-\left(1-e^{-\lambda\tau}\right)^{\alpha}\right)+\sum_{j=1}^{n_a}\ln\alpha+\sum_{j=1}^{n_a}\ln\lambda+\sum_{j=1}^{n_a}\ln\beta\\
&\quad-\lambda\beta\sum_{j=1}^{n_a}y_j+(\alpha-1)\sum_{j=1}^{n_a}\ln\left(1-e^{-\lambda\beta y_j}\right)-\sum_{j=n_a+1}^{n\pi}\ln\left(1-\left(1-e^{-\lambda\beta\tau}\right)^{\alpha}\right),
\end{aligned}$$

$$\begin{aligned}
\Rightarrow l&=n_u\left(\ln\alpha+\ln\lambda\right)+n_a\left(\ln\alpha+\ln\lambda+\ln\beta\right)-\lambda\left(\sum_{i=1}^{n_u}x_i-\beta\sum_{j=1}^{n_a}y_j\right)\\
&\quad+(\alpha-1)\left[\sum_{i=1}^{n_u}\ln\left(1-e^{-\lambda x_i}\right)+\sum_{j=1}^{n_a}\ln\left(1-e^{-\lambda\beta y_j}\right)\right] \qquad (13)\\
&\quad-\left(n\bar{\pi}-n_u\right)\ln\left(1-\left(1-e^{-\lambda\tau}\right)^{\alpha}\right)-\left(n\pi-n_a\right)\ln\left(1-\left(1-e^{-\lambda\beta\tau}\right)^{\alpha}\right).
\end{aligned}$$

The Score equations become

$$\frac{\partial l}{\partial \alpha}=\frac{n_u+n_a}{\alpha}+\sum_{i=1}^{n_u}\ln\left(1-e^{-\lambda x_i}\right)+\left(n\overline{\pi}-n_u\right)\frac{\left(1-e^{-\lambda\tau}\right)^{\alpha}\ln\left(1-e^{-\lambda\tau}\right)}{\left(1-\left(1-e^{-\lambda\tau}\right)^{\alpha}\right)}$$
$$+\sum_{j=1}^{n_a}\ln\left(1-e^{-\lambda\beta y_j}\right)+\left(n\pi-n_a\right)\frac{\left(1-e^{-\lambda\beta\tau}\right)^{\alpha}\ln\left(1-e^{-\lambda\beta\tau}\right)}{\left(1-\left(1-e^{-\lambda\beta\tau}\right)^{\alpha}\right)}=0, \quad (14)$$

$$\frac{\partial l}{\partial \lambda}=\frac{n_u+n_a}{\lambda}+\sum_{i=1}^{n_u}x_i+\left(\alpha-1\right)\sum_{i=1}^{n_u}\frac{x_i e^{-\lambda x_i}}{\left(1-e^{-\lambda x_i}\right)}+\alpha\left(n\overline{\pi}-n_u\right)\frac{e^{-\lambda\tau}\left(1-e^{-\lambda\tau}\right)^{\alpha-1}}{\left(1-\left(1-e^{-\lambda\tau}\right)^{\alpha}\right)}$$
$$-\beta\sum_{j=1}^{n_a}y_j+\left(\alpha-1\right)\sum_{j=1}^{n_a}\frac{y_i e^{-\lambda\beta y_j}}{\left(1-e^{-\lambda\beta y_j}\right)}+\alpha\left(n\pi-n_a\right)\frac{e^{-\lambda\beta\tau}\left(1-e^{-\lambda\beta\tau}\right)^{\alpha-1}}{\left(1-\left(1-e^{-\lambda\beta\tau}\right)^{\alpha}\right)}=0, \quad (15)$$

$$\frac{\partial l}{\partial \beta}=\frac{n_a}{\beta}-\lambda\sum_{j=1}^{n_a}y_j+\left(\alpha-1\right)\sum_{j=1}^{n_a}\frac{\lambda y_i e^{-\lambda\beta y_j}}{\left(1-e^{-\lambda\beta y_j}\right)}+\alpha\lambda\tau\left(n\pi-n_a\right)\frac{e^{-\lambda\beta\tau}\left(1-e^{-\lambda\beta\tau}\right)^{\alpha-1}}{\left(1-\left(1-e^{-\lambda\beta\tau}\right)^{\alpha}\right)}=0. \quad (16)$$

Now, we have a system of three nonlinear equations in three unknowns $\alpha, \lambda,$ and β. It is clear that a closed form solution is intractable. Therefore, iterative procedure can be used to find a numerical solution of the above system.

The asymptotic confidence intervals for the parameters $\underline{\theta}=\left(\alpha,\lambda,\beta\right)$ can be obtained using following hypothesized convergence in distribution result:

$$\sqrt{n}\left(\left(\hat{\alpha}-\alpha\right),\left(\hat{\lambda}-\lambda\right),\left(\hat{\beta}-\beta\right)\rightarrow\left(\underline{0},I^{-1}\left(\alpha,\lambda,\beta\right)\right)\right),$$

where the $I=\left(\alpha,\lambda,\beta\right)$ is the Fisher information matrix given by

$$I\left(\alpha,\lambda,\beta\right)=\begin{bmatrix} I_{11}\left(\alpha\right) & I_{12}\left(\alpha\lambda\right) & I_{13}\left(\alpha\beta\right) \\ I_{21}\left(\lambda\alpha\right) & I_{22}\left(\lambda\right) & I_{23}\left(\lambda\beta\right) \\ I_{31}\left(\beta\alpha\right) & I_{32}\left(\beta\lambda\right) & I_{33}\left(\beta\right) \end{bmatrix}=\begin{bmatrix} \frac{\partial^2 l}{\partial\alpha^2} & \frac{\partial^2 l}{\partial\alpha\partial\lambda} & \frac{\partial^2 l}{\partial\alpha\partial\beta} \\ \frac{\partial^2 l}{\partial\lambda\partial\alpha} & \frac{\partial^2 l}{\partial\lambda^2} & \frac{\partial^2 l}{\partial\lambda\partial\beta} \\ \frac{\partial^2 l}{\partial\beta\partial\alpha} & \frac{\partial^2 l}{\partial\beta\partial\lambda} & \frac{\partial^2 l}{\partial\beta^2} \end{bmatrix}.$$

Note that Gupta and Kundu (1999), focusing on the three parameter GE distribution (in our case it is assumed that the location parameter is zero), stated the asymptotic normality of the MLEs under the assumption that the shape parameter $\alpha > 2$, and mentioned that further investigation is needed for the case $\alpha \le 2$. They indicate that the regularity conditions can be established using techniques similar to those employed for the gamma and the Weibull families. The above authors, however, studied the behavior of the estimators for $\alpha \le 2$ using Monte-Carlo simulation in Gupta and Kundu (2000), and did not detect any anomalous behavior when $\alpha \le 2$. Also, their results are for Type II censored data and does not consider the PALT scenario. Ismail (2013), however, assumed the above asymptotic result and obtained reasonable confidence intervals for distribution parameters under Type-II censoring. Based on these empirical findings, we will assume that the above result holds in the PALT situations under Type I censoring and also when $\alpha \le 2$. As simulation results given later show, this assumption does not lead to poorly performing confidence intervals.

The elements of the 3x3 matrix $I, I_{ij}\left(\hat{\alpha},\hat{\lambda},\hat{\beta}\right), \quad i, j = 1,2,3,$ can be approximated by $I_{ij}\left(\hat{\alpha},\hat{\lambda},\hat{\beta}\right)$, where

$$I_{ij}\left(\hat{\theta}\right) = \frac{\partial^2 l(\theta)}{\partial\theta_i \partial\theta_j}\Big|_{\theta=\hat{\theta}}.$$

From Eq. (12), we get the following:

$$\frac{\partial^2 l}{\partial\alpha^2} = \frac{n_u + n_a}{\alpha^2} + \frac{(n\bar{\pi} - n_u)\psi_2^\alpha \ln\left(\psi_2^2\right)}{\psi_4^2} + \frac{(n\pi - n_a)\psi_6^\alpha \ln\left(\psi_6^2\right)}{\psi_5^2}, \tag{17}$$

$$\begin{aligned}\frac{\partial^2 l}{\partial\lambda^2} = \frac{n_u + n_a}{\lambda^2} + (\alpha-1)\left\{\sum_{i=1}^{n_u} x_i^2 e^{-\lambda x_i}\right\} - \frac{\tau^2\alpha(n\bar{\pi} - n_u)\psi_2^\alpha\left\{-\alpha + e^{-\lambda\tau} - \psi_2^\alpha e^{-\lambda\tau}\right\}}{\left[-1 + \left(\left(e^{-\lambda\tau} - 1\right)e^{-\lambda\tau}\right)^\alpha\right]^2\left(e^{-\lambda\tau} - 1\right)^2} \\ - \frac{\tau^2\beta^2\alpha(n\pi - n_a)\psi_6^\alpha\left\{-\alpha + e^{-\lambda\beta\tau} - \psi_6^\alpha e^{-\lambda\beta\tau}\right\}}{\left[1 - \psi_6^\alpha e^{-\lambda\beta\tau}\right]^2\psi_6^2} + (\alpha-1)\beta^2\left\{\sum_{i=1}^{n_u} y_j^2 e^{-\lambda\beta y_j}\right\}\end{aligned} \tag{18}$$

$$\frac{\partial^2 l}{\partial \beta^2} = \frac{n_a}{\beta^2} + (\alpha - 1)\lambda^2 \left\{ \sum_{i=1}^{n_u} y_j^2 e^{-\lambda \beta y_j} \right\} - \frac{\tau^2 \beta^2 \alpha (n\pi - n_a) \psi_6^{\alpha} \left\{ -\alpha + e^{-\lambda\beta\tau} - \psi_6^{\alpha} e^{-\lambda\beta\tau} \right\}}{\left[1 - \psi_6^{\alpha} e^{-\lambda\beta\tau} \right]^2 \psi_6^2}$$

(19)

$$\frac{\partial^2 l}{\partial \alpha \partial \beta} = \sum_{i=1}^{n_u} \frac{\lambda y_j^2 e^{-\lambda\beta y_j}}{\psi_{3j}} - \frac{\tau\lambda (n\pi - n_a) \left\{ \alpha \psi_6^{\alpha} \ln(\psi_6) + \psi_6^{\alpha} - \psi_6^{2\alpha} \right\}}{\psi_5^{\alpha} \left(e^{-\lambda\beta\tau} - 1 \right)} \tag{20}$$

$$\frac{\partial^2 l}{\partial \alpha \partial \lambda} = \sum_{i=1}^{n_u} \frac{x_i^2 e^{-\lambda x_i}}{\psi_{1i}} + \sum_{i=1}^{n_u} \frac{\beta y_j^2 e^{-\lambda\beta y_j}}{\psi_{3j}} + \frac{\tau\alpha (n\overline{\pi} - n_u) \psi_2^{\alpha} \left\{ -\alpha \ln(\psi_2) + 1 - \psi_2^{\alpha} \right\}}{\psi_4^{\alpha} \left(e^{-\lambda\tau} - 1 \right)}$$
$$+ \frac{\tau\beta (n\pi - n_a) \psi_6^{\alpha} \left\{ -\alpha \ln\left(1 - e^{-\lambda\beta\tau}\right) + 1 - \psi_6^{\alpha} \right\}}{\psi_5^2 \left(e^{-\lambda\beta\tau} - 1 \right)}, \tag{21}$$

$$\frac{\partial^2 l}{\partial \beta \partial \lambda} = (\alpha - 1) \sum_{i=1}^{n_u} \left[\frac{\beta y_j e^{-\lambda\beta y_j}}{\psi_{3j}} - \frac{\beta y_j^2 e^{-\lambda\beta y_j}}{\psi_{3j}} - \frac{\beta\lambda y_j^2 e^{-2\lambda\beta y_j}}{\psi_{3j}^2} \right]$$
$$+ \frac{\tau\alpha (n\pi - n_a) \left\{ \psi_6^{\alpha} \left(-\alpha\lambda\tau\beta - e^{-\lambda\beta\tau} + 1 + \beta\lambda\tau e^{-\lambda\beta\tau} - \beta\lambda\tau \right) \right\}}{\psi_5 \left(e^{-\lambda\beta\tau} - 1 \right)^2} \tag{22}$$
$$- \frac{\tau^2 \beta (n\pi - n_a) \psi_6^{\alpha} \left\{ 1 - \psi_6^{\alpha} - \alpha \psi_6^{\alpha} \right\}}{\psi_5^2 \left(e^{-\lambda\beta\tau} - 1 \right)^2},$$

where

$$\psi_{1i} = 1 - e^{-\lambda x_i}, \qquad \psi_2 = 1 - e^{-\lambda\tau}, \qquad \psi_{1j} = 1 - e^{-\lambda\beta y_j},$$
$$\psi_4 = 1 - \left(1 - e^{-\lambda\tau}\right)^{\alpha}, \quad \psi_5 = 1 - \left(1 - e^{-\lambda\beta\tau}\right)^{\alpha}, \quad \psi_6 = 1 - e^{-\lambda\beta\tau},$$

and employing the standard z-based confidence interval formulations,

$$\hat{\alpha} \pm Z_{\gamma/2} \sqrt{I_{11}^{-1}(\hat{\alpha})}, \quad \hat{\lambda} \pm Z_{\gamma/2} \sqrt{I_{22}^{-1}(\hat{\lambda})}, \; \hat{\beta} \pm Z_{\gamma/2} \sqrt{I_{33}^{-1}(\hat{\beta})}.$$

The asymptotic confidence interval for the mean life at normal use condition is given by

$$\hat{\mu} \pm Z_{\gamma/2} \sqrt{Var(\hat{\mu})},$$

where $Var(\hat{\mu})$ is obtained using the standard delta method.

3.3 THE BOOTSTRAP SAMPLING METHODS

There are several different methods for generating the needed bootstrap samples data

3.3.1 The Proposed Parametric Bootstrap Method and the Monte-Carlo Procedure. The Monte-Carlo procedure used for the simulation study is given below. The steps for the parametric bootstrap method that can be utilized to obtain confidence bounds for $\boldsymbol{\alpha}, \boldsymbol{\lambda}, \boldsymbol{and}\ \boldsymbol{\beta}$ and lower binds for the mean life is imbedded in this procedure and are given in italics.

Distribution parameters are varied in the study as following $(\alpha = 1.5, and\ 2\ , \lambda = 1\ ,\ \beta = 1.5, \text{and } 2)$ and $\mu = 1/\lambda\,[\psi(\alpha+1) - \psi(1)]$ where $\psi(.)$ Digamma function is presented here. The censoring time was set at τ =1, and 1.5. The n test items were divided into (a) equal sample proportions by setting $\pi = 0.5$, such that 1/2 the items are allocated at accelerated condition and the remaining 1/2 are allocated to the normal use condition and (b) by setting $\pi = 0.667$ such that 1/3 the items are allocated at accelerated condition and the remaining 2/3 are allocated to the normal use condition.

(1) Generate random samples from the GE distribution by using the transformation $x_i = \left(\frac{-1}{\lambda}\right)\ln\left[1 - u_i^{(1/\alpha)}\right],\ i = 1,2, \ldots, n$ where $u_i's$ are random sample from a uniform (0, 1) distribution. Similarly, generate data for the high stress condition by replacing λ with $\beta\lambda$. Employ censoring time τ for both samples.

(2) Employ Maximum likelihood method to estimate the parameters with the same censoring time τ used for both samples. [The nonlinear equations of the maximum likelihood estimates were solved iteratively using Newton Raphson method.]

(3) Use the resulting estimates of the parameters and acceleration factor to construct asymptotic confidence limits with confidence level at $\gamma = 0.95$ and also the asymptotic variance and covariance matrix of the estimators (for use in the delta method based confidence bounds).

(4) Used the estimated parameters $\hat{\alpha}, \hat{\lambda}$, and $\hat{\beta}$ to generate data from the estimated normal use and accelerated GE distribution using the transformation $x_i = \left(-\frac{1}{\hat{\lambda}}\right)\ln\left[1 - u_i^{(1/\hat{\alpha})}\right]$, [$\hat{\lambda}$ is replaced by $\hat{\beta}\hat{\lambda}$ for the accelerated sample.]

(5) Repeat Step (4) to obtain 1,000 bootstrap samples.

(6) *Obtain MLEs of the GE parameters and the acceleration factor using each bootstrap sample and label these* $\hat{\alpha}^*, \hat{\lambda}^*, and\ \hat{\beta}^*$.

(7) *Using the empirical distributions of the estimates* $\hat{\alpha}^*, \hat{\lambda}^*, and\ \hat{\beta}^*$ *obtained from bootstrap estimates, construct confidence interval for* α, λ, β *using respective quantiles at* $\left(\frac{1-\gamma}{2}\right)$*100% and* $1-\left(\frac{1-\gamma}{2}\right)$*100%.*

(8) *Using the empirical distribution of the estimated means* $\hat{\mu}^*$, *obtained from bootstrap samples, construct lower bound confidence bound for* μ *using quantile at* $(1-\gamma)$*100%*

(9) Coverage probabilities were computed based on 1,000 simulation runs by repeating Steps (1) through (7) 1,000 times.

3.3.2 The Proposed Nonparametric Bootstrap Method and the Monte-Carlo Procedure. The Monte-Carlo procedure used for the simulation study is given below. The steps for the nonparametric bootstrap method that can be utilized to obtain confidence bounds for $\boldsymbol{\alpha, \lambda, and\ \beta}$ and lower bounds for the mean life is imbedded in this procedure and are given in *italics*.

(1) Generate random samples from the GE distribution by using the transformation $x_i = \left(\frac{-1}{\lambda}\right) \ln\left[1-u_i^{(1/\alpha)}\right]$, $i = 1,2, \dots, n$ where $u_i's$ are random sample from a uniform (0, 1) distribution. Similarly, generate data for the high stress condition by replacing λ with $\beta\lambda$. Employ censoring time τ for both samples.

(2) *Obtain a bootstrap resample from each of the two samples generated in Step (1) above, with each bootstrap sample of size* πn *(or* $\overline{\pi} n$ *) obtained by sampling with replacement from the respective sample obtained in (1).*

(3) *New "bootstrap estimates"* $\hat{\alpha}^*, \hat{\lambda}^*, and\ \hat{\beta}^*$ *are computed from the combined bootstrap sample using the ML method. Also estimate the mean life* μ *under normal conditions, accounting for the censoring.*

(4) *Repeat the process given in Steps (2) and (3) 1,000 times and obtain the empirical distributions of* $\hat{\alpha}^*, \hat{\lambda}^*, \hat{\beta}^*$, *and* $\hat{\mu}^*$.

(5) *Using the empirical distributions of the* $\hat{\alpha}^*, \hat{\lambda}^*$*, and* $\hat{\beta}^*$ *obtained from bootstrap estimates, confidence interval for* $\alpha, \lambda, and\, \beta$ *is constructed using respective quantiles at* $\left(\frac{1-\gamma}{2}\right)$*100% and* $1 - \left(\frac{1-\gamma}{2}\right)$*100%..*

(6) *Using the empirical distributions of the mean* $\hat{\mu}^*$ *obtained from bootstrap estimates, construct the lower bound confidence interval for using quantile at* $(1-\gamma)$*100%.*

(7) Coverage probabilities were computed based on 1,000 simulation runs obtained by repeating Steps 1 -7.

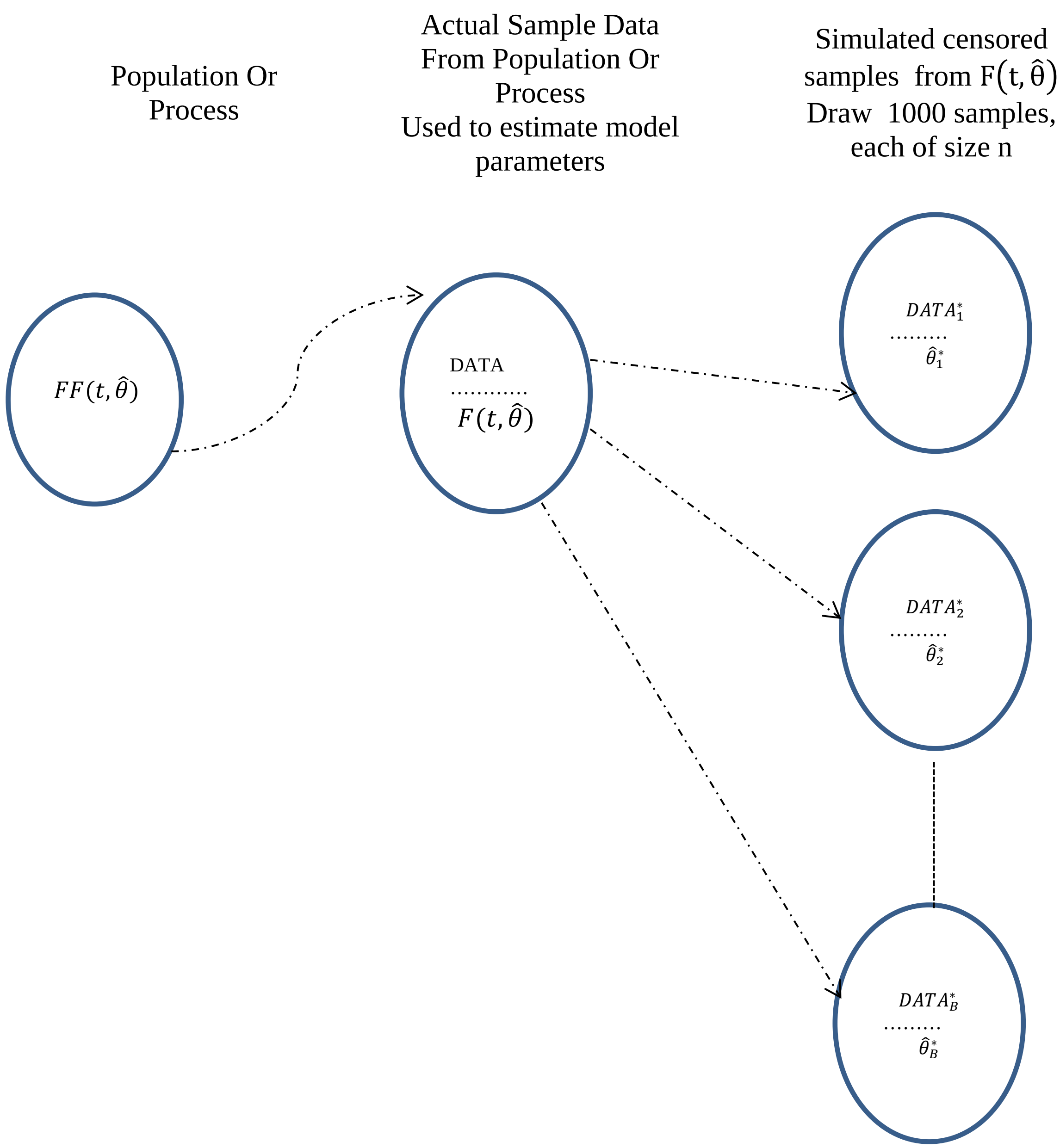

Figure 3.2.Illustrates the parametric bootstrap resampling method

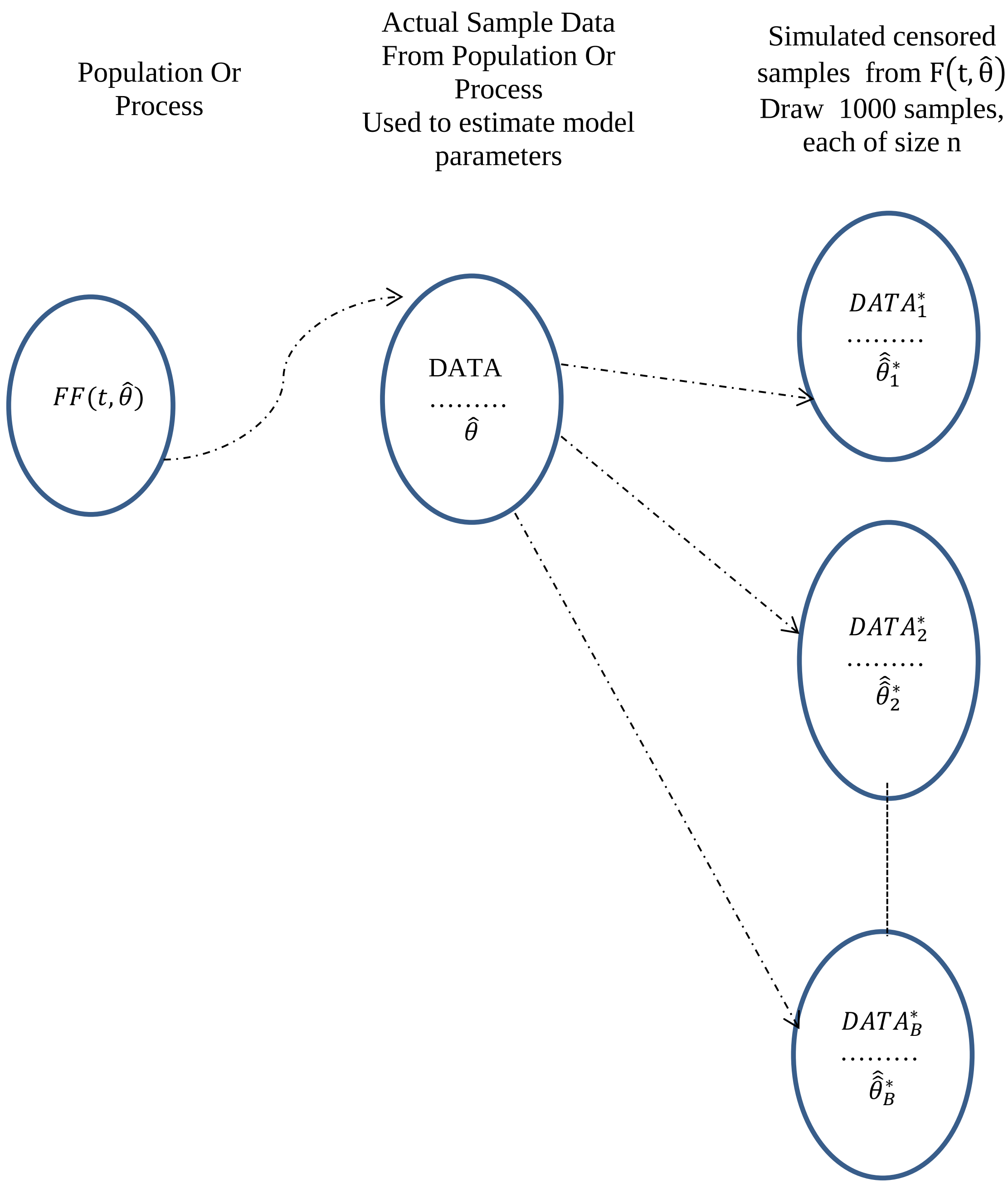

Figure 3.3. Illustrates the nonparametric bootstrap resampling for parametric inference.

3.4 RESULTS AND DISCUSSION

Only select results from the simulation experiments are reported below for brevity. All simulations results reported here are for α = (1.5, and 2) and $\lambda = 1$, with the acceleration factor β is set at 1.5 and 2.0. The censoring parameter τ was set at values 1, and 1.5

By conducting the steps given in Section 3.2 using a computer program written in the Matlab, the simulation results reported in Tables 3.2 to Tables 3.32 are obtained. Tables 3.2 and 3.3 show the maximum likelihood estimates of $(\alpha, \lambda, \beta, \text{and } \mu)$. The estimated expected value of the MLEs for α are close to the true value for when the sample size is 100 but show a slight upwards bias for smaller sample sizes. A similar pattern is observed for estimates of β when the censoring time is 1. The results improve when the censoring time increases to 1.5 or when α increases to 2. Estimates of μ and λ are quite reasonable when the sample size is greater than 30. In general, when the sample size increases the estimates of the parameters approach the true values. Tables 3.4 to 3.32 show the simulation result of (asymptotic, parametric bootstrap, and nonparametric bootstrap) of 95% confidence interval for $(\alpha, \lambda, \text{and } \beta)$ and the lower 95% confidence bound of (asymptotic, parametric bootstrap, and nonparametric bootstrap) for the mean.

Table 3.2a GE Parameters, Acceleration Factor, and Type I Censoring

π	τ	α	λ	β	μ	n	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\mu}$
.5	1	1.5	1	1.5	1.2804	30	1.740536	1.098127	1.625868	1.385353
						50	1.675096	1.082763	1.570141	1.330857
						75	1.634216	1.062605	1.532303	1.314904
						100	1.570929	1.027731	1.546379	1.30996
	1.5	1.5	1	1.5	1.2804	30	1.708948	1.071374	1.57073	1.33327
						50	1.673527	1.073493	1.548163	1.312707
						75	1.622074	1.052761	1.536937	1.305793
						100	1.568747	1.023732	1.534235	1.305156

Table 3.2b GE Parameters, Acceleration Factor, and Type I Censoring

π	τ	α	λ	β	μ	n	$\hat{\alpha}$	$\hat{\lambda}$	$\hat{\beta}$	$\hat{\mu}$
.5	1	2	1	1.5	1.5	30	2.318859	1.095341	1.553225	1.551502
						50	2.24871	1.079897	1.52247	1.528549
						75	2.134716	1.041066	1.529489	1.527969
						100	2.122318	1.038666	1.533429	1.52009
	1.5	2	1	1.5	1.5	30	2.282045	1.095051	1.538302	1.498027
						50	2.179126	1.05482	1.519219	1.509462
						75	2.09156	1.019907	1.527478	1.526625
						100	2.117973	1.039584	1.516357	1.502626
.667	1	2	1	1.5	1.5	30	2.278941	1.080764	1.590074	1.579644
						50	2.20633	1.05488	1.564727	1.561578
						75	2.115322	1.030532	1.525772	1.543245
						100	2.1418	1.036304	1.551029	1.536084
	1.5	2	1	1.5	1.5	30	2.211647	1.062217	1.541497	1.537559
						50	2.175302	1.040722	1.554918	1.54164
						75	2.122398	1.037873	1.521313	1.519886
						100	2.096505	1.024071	1.531034	1.524627
.5	1	2	1	2	1.5	30	2.399464	1.120903	2.124475	1.589719
						50	2.250541	1.06822	2.098928	1.571614
						75	2.172331	1.065452	2.028721	1.511861
						100	2.160057	1.048692	2.034841	1.526222
	1.5	2	1	2	1.5	30	2.385585	1.097882	2.105311	1.553023
						50	2.258429	1.075495	2.035044	1.513903
						75	2.173423	1.061296	2.028121	1.502438
						100	2.111775	1.041332	2.008467	1.499085
.667	1	2	1	2	1.5	30	2.400406	1.136162	2.12515	1.599587
						50	2.238159	1.096672	2.054929	1.544576
						75	2.134428	1.039408	2.051435	1.562317
						100	2.110595	1.037786	2.033213	1.529881
	1.5	2	1	2	1.5	30	2.339819	1.1254	2.008896	1.517782
						50	2.23878	1.095246	1.991892	1.501189
						75	2.147047	1.056632	2.029514	1.516827
						100	2.106217	1.03097	2.0268	1.518701

Table 3.3 Coverage of Asymptotic 95% C.I.s

α =1.5, λ =1, β =1.5, μ= 1.2804, π=.5, τ=1

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	0.886303	3.665721	2.779418	0.738984	0.978
	λ	0.423517	1.949804	1.526287	2.016739	0.965
	β	1.000001	2.749131	1.74913	2.097398	0.973
	μ	0.906484				0.956
50	α	0.955879	2.983422	2.027543	1.469277	0.974
	λ	0.550844	1.806479	1.255635	1.977485	0.960
	β	1.001205	2.310694	1.309489	2.494733	0.961
	μ	0.946696				0.955
75	α	1.013576	2.686825	1.673249	0.764873	0.968
	λ	0.587548	1.653909	1.066361	1.619368	0.954
	β	1.031394	2.206177	1.174782	1.70078	0.957
	μ	1.006098				0.955
100	α	1.103747	2.330024	1.226278	0.547898	0.959
	λ	0.664616	1.473539	0.808923	1.27331	0.949
	β	1.106806	2.099396	0.99259	0.979626	0.952
	μ	1.053292				0.948

Table 3.4 Coverage of Parametric Bootstrap 95% C.I.s

α =1.5, λ =1, β =1.5, μ= 1.2804, π=.5, τ=1

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	1.331254	2.149819	0.818565	0.145201	0.949
	λ	0.391843	1.80441	1.412567	0.049314	0.963
	β	0.743076	2.50866	1.765583	0.115011	0.973
	μ	1.014195				0.954
50	α	1.308252	2.041941	0.733689	0.083439	0.947
	λ	0.439819	1.725707	1.285888	0.04242	0.961
	β	0.828421	2.311862	1.483442	0.105412	0.964
	μ	1.012725				0.954
75	α	1.311399	1.957033	0.645634	0.10657	0.946
	λ	0.473016	1.652194	1.179178	0.050176	0.957
	β	0.897685	2.16692	1.269235	0.068122	0.961
	μ	1.011334				0.954
100	α	1.309674	1.832185	0.522512	0.081731	0.945
	λ	0.54133	1.514132	0.972802	0.038495	0.951
	β	1.022915	2.069844	1.046929	0.03674	0.953
	μ	1.017121				0.953

Table 3.5 Coverage of Nonparametric Bootstrap 95% C.I.s

α =1.5, λ =1, β =1.5, μ= 1.2804, π=.5, τ=1

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	1.252881	2.228192	0.975311	0.173224	0.951
	λ	0.256597	1.939656	1.683059	0.061136	0.969
	β	0.574031	2.677705	2.103674	0.179796	0.974
	μ	1.045158				0.951
50	α	1.238005	2.112187	0.874182	0.132915	0.950
	λ	0.316702	1.848824	1.532122	0.061149	0.965
	β	0.686389	2.453894	1.767505	0.15137	0.973
	μ	1.047231				0.951
75	α	1.249583	2.018849	0.769266	0.127347	0.948
	λ	0.360116	1.765094	1.404978	0.05818	0.962
	β	0.776162	2.288443	1.512281	0.08068	0.964
	μ	1.0460111				0.951
100	α	1.259646	1.882213	0.622567	0.107027	0.946
	λ	0.44819	1.607272	1.159083	0.050324	0.956
	β	0.922677	2.170082	1.247405	0.044463	0.960
	μ	1.047725				0.949

Table 3.6 Coverage of Asymptotic 95% C.I.s

α =1.5, λ =1, β =1.5, μ= 1.2804, π=.5, τ=1.5

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	0.933868	3.407471	2.473602	1.290271	0.984
	λ	0.567705	1.774346	1.206641	1.799216	0.96
	β	1.000349	2.445438	1.445089	5.322325	0.975
	μ	0.970362				0.957
50	α	0.981381	2.82447	1.843089	0.605814	0.980
	λ	0.594318	1.639513	1.045195	1.346216	0.962
	β	1.030054	2.31746	1.287407	1.840845	0.969
	μ	0.984372				0.956
75	α	1.071548	2.518645	1.447097	0.757173	0.975
	λ	0.668835	1.543877	0.875043	2.094424	0.960
	β	1.064394	2.180605	1.116211	2.583192	0.963
	μ	1.01071				0.955
100	α	1.108671	2.288746	1.180075	0.361684	0.964
	λ	0.705641	1.397243	0.691602	0.940337	0.955
	β	1.165785	2.031982	0.866197	1.344257	0.959
	μ	1.065617				0.952

Table 3.7 Coverage of Parametric Bootstrap 95% C.I.s

α =1.5, λ =1, β =1.5, μ= 1.2804, π=.5, τ=1.5

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	1.382336	2.03556	0.653224	0.106312	0.954
	λ	0.505363	1.637384	1.13202	0.049502	0.963
	β	0.694585	2.446875	1.75229	0.046163	0.979
	μ	1.013786				0.955
50	α	1.410145	1.93691	0.526765	0.088477	0.949
	λ	0.558426	1.58856	1.030134	0.054323	0.962
	β	0.834799	2.261527	1.426728	0.050832	0.974
	μ	1.012222				0.955
75	α	1.390069	1.854078	0.464009	0.065731	0.948
	λ	0.552144	1.553378	1.001234	0.04045	0.961
	β	0.855868	2.218006	1.362138	0.069633	0.973
	μ	1.012077				0.955
100	α	1.373397	1.764096	0.390699	0.046292	0.947
	λ	0.588992	1.458472	0.869481	0.02252	0.959
	β	0.933072	2.135398	1.202326	0.040381	0.965
	μ	1.010425				0.955

Table 3.8 Coverage of Nonparametric Bootstrap 95% C.I.s

α =1.5, λ =1, β =1.5, μ= 1.2804, π=.5, τ=1.5

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	1.319793	2.098102	0.778309	0.1715	0.957
	λ	0.396978	1.745769	1.34879	0.073165	0.971
	β	0.526812	2.614647	2.087835	0.068938	0.981
	μ	1.045852				0.954
50	α	1.35971	1.987344	0.627635	0.117548	0.953
	λ	0.459796	1.68719	1.227394	0.056736	0.967
	β	0.698198	2.398128	1.699931	0.066081	0.978
	μ	1.045817				0.954
75	α	1.345643	1.898504	0.552861	0.088861	0.951
	λ	0.456281	1.64924	1.192959	0.047648	0.965
	β	0.72545	2.348423	1.622973	0.102484	0.978
	μ	1.047438				0.953
100	α	1.33599	1.801504	0.465514	0.043429	0.948
	λ	0.505743	1.54172	1.035977	0.030483	0.962
	β	0.817956	2.250514	1.432558	0.080309	0.974
	μ	1.047012				0.953

Table 3.9 Coverage of Asymptotic 95% C.I.s

α =2, λ =1, β =1.5, μ=1.5, π=.5, τ=1

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	1.124812	5.488589	4.363777	0.255789	0.981
	λ	0.45653	2.015863	1.559333	0.069615	0.961
	β	1.000825	2.534655	1.53383	0.20974	0.958
	μ	1.065479				0.957
50	α	1.212398	4.513783	3.301385	2.134716	0.976
	λ	0.522441	1.854566	1.332125	1.041066	0.957
	β	1.001737	2.321992	1.320255	1.529489	0.957
	μ	1.081032				0.956
75	α	1.263107	3.871243	2.608136	0.093859	0.972
	λ	0.574729	1.664997	1.090268	0.058044	0.954
	β	1.104767	2.196196	1.091429	0.087538	0.954
	μ	1.163424				0.954
100	α	1.31801	3.525307	2.207297	0.104948	0.968
	λ	0.604603	1.584901	0.980298	0.041411	0.952
	β	1.110492	2.120666	1.010174	0.112572	0.953
	μ	1.189367				0.953

Table 3.10 Coverage of Parametric Bootstrap 95% C.I.s

α =2, λ =1, β =1.5, μ=1.5, π=.5, τ=1

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	1.694165	2.943554	1.249389	4.30871	0.956
	λ	0.320408	1.870273	1.549865	5.301323	0.961
	β	0.595046	2.511404	1.916358	7.503472	0.967
	μ	1.0316127				0.958
50	α	1.780996	2.716424	0.935428	0.974083	0.953
	λ	0.478922	1.680873	1.201951	1.425656	0.956
	β	0.811726	2.233213	1.421487	1.743399	0.957
	μ	1.0316179				0.958
75	α	1.745389	2.524044	0.778655	0.785926	0.948
	λ	0.494851	1.582482	1.087631	1.278039	0.954
	β	0.957679	2.101298	1.143619	1.105818	0.954
	μ	1.0326788				0.958
100	α	1.744137	2.5005	0.756363	0.85133	0.947
	λ	0.511075	1.571056	1.059981	1.492294	0.954
	β	0.962567	2.104292	1.141725	1.437931	0.954
	μ	1.032988				0.958

Table 3.11 Coverage of Nonparametric Bootstrap 95% C.I.s

α =2, λ =1, β =1.5, μ=1.5, π=.5, τ=1

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	1.251818	4.411799	3.159981	0.351353	0.979
	λ	0.478803	1.943141	1.464338	0.085403	0.958
	β	1.005472	2.691726	1.686254	0.244308	0.964
	μ	1.064006				0.957
50	α	1.288635	3.74735	2.458715	0.241691	0.971
	λ	0.570361	1.633126	1.062765	0.080249	0.954
	β	1.046718	2.272096	1.225378	0.127038	0.956
	μ	1.127359				0.955
75	α	1.367648	3.21649	1.848842	0.109827	0.966
	λ	0.608303	1.53222	0.923917	0.055685	0.951
	β	1.109874	2.164214	1.05434	0.1211	0.954
	μ	1.180841				0.953
100	α	1.421612	3.17387	1.752258	0.127457	0.965
	λ	0.653574	1.464142	0.810568	0.042688	0.949
	β	1.125632	2.122699	0.997067	0.057624	0.952
	μ	1.203878				0.952

Table 3.12 Coverage of Asymptotic 95% C.I.s

α =2, λ =1, β =1.5, μ=1.5, π=.5, τ=1.5

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	1.203329	4.645745	3.442416	0.130657	0.979
	λ	0.582958	1.742203	1.159245	0.072167	0.965
	β	1.032079	2.33018	1.298101	0.039626	0.966
	μ	1.11376				0.956
50	α	1.269906	3.929247	2.659341	0.142574	0.974
	λ	0.636697	1.620818	0.984122	0.056806	0.963
	β	1.060733	2.146322	1.085589	0.060346	0.964
	μ	1.1796				0.955
75	α	1.35047	3.349937	1.999467	0.05589	0.972
	λ	0.653978	1.486139	0.83216	0.024997	0.958
	β	1.098455	2.027858	0.929403	0.038901	0.961
	μ	1.212948				0.952
100	α	1.412011	3.294311	1.882299	0.089669	0.971
	λ	0.708374	1.456289	0.747916	0.029385	0.954
	β	1.151063	2.002362	0.8513	0.045278	0.959
	μ	1.212369				0.952

Table 3.13 Coverage of Parametric Bootstrap 95% C.I.s

α =2, λ =1, β =1.5, μ=1.5, π=.5, τ=1.5

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	1.919609	2.644481	0.724872	0.523763	0.953
	λ	0.662232	1.447409	0.785177	0.952636	0.956
	β	0.977084	2.09952	1.122436	1.199089	0.965
	μ	1.32115				0.951
50	α	1.852541	2.505711	0.653171	1.305673	0.951
	λ	0.703134	1.486967	0.783833	2.348957	0.956
	β	0.984396	2.054043	1.069647	2.643919	0.964
	μ	1.320038				0.951
75	α	1.859323	2.323797	0.464474	0.37725	0.947
	λ	0.731617	1.308196	0.576579	0.685785	0.949
	β	1.126363	1.928592	0.80223	1.217863	0.957
	μ	1.338596				0.949
100	α	1.913367	2.322579	0.409212	0.217596	0.946
	λ	0.794288	1.28488	0.490591	0.46561	0.948
	β	1.188489	1.844225	0.655736	0.695918	0.951
	μ	1.323883				0.951

Table 3.14 Coverage of Nonparametric Bootstrap 95% C.I.s

α =2, λ =1, β =1.5, μ=1.5, π=.5, τ=1.5

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	1.251818	4.411799	3.159981	0.351353	0.978
	λ	0.478803	1.943141	1.464338	0.085403	0.967
	β	1.005472	2.691726	1.686254	0.244308	0.969
	μ	1.064006				0.958
50	α	1.288635	3.74735	2.458715	0.241691	0.973
	λ	0.570361	1.633126	1.062765	0.080249	0.964
	β	1.046718	2.272096	1.225378	0.127038	0.966
	μ	1.127359				0.956
75	α	1.367648	3.21649	1.848842	0.109827	0.971
	λ	0.608303	1.53222	0.923917	0.055685	0.961
	β	1.109874	2.164214	1.05434	0.1211	0.964
	μ	1.180841				0.954
100	α	1.421612	3.17387	1.752258	0.127457	0.970
	λ	0.653574	1.464142	0.810568	0.042688	0.957
	β	1.125632	2.122699	0.997067	0.057624	0.963
	μ	1.203878				0.953

Table 3.15 Coverage of Asymptotic 95% C.I.s

$\alpha = 2$, $\lambda = 1$, $\beta = 1.5$, $\mu = 1.5$, $\pi = .667$, $\tau = 1$

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	1.154164	5.039237	3.885073	0.324596	0.979
	λ	0.43768	1.97847	1.54079	0.092184	0.966
	β	1.001028	2.757582	1.756553	0.117573	0.969
	μ	1.039089				0.958
50	α	1.229204	4.292912	3.063708	0.160972	0.978
	λ	0.489315	1.841252	1.351937	0.089279	0.961
	β	1.014078	2.484245	1.470167	0.08339	0.963
	μ	1.07771				0.956
75	α	1.30664	3.55098	2.244339	0.164655	0.972
	λ	0.552958	1.620801	1.067843	0.042779	0.957
	β	1.064717	2.252631	1.187913	0.142825	0.958
	μ	1.140656				0.954
100	α	1.377051	3.475573	2.098522	0.071369	0.971
	λ	0.603089	1.597809	0.994721	0.041832	0.955
	β	1.11633	2.204177	1.087848	0.071244	0.957
	μ	1.15184				0.953

Table 3.16 Coverage of Parametric Bootstrap 95% C.I.s

$\alpha = 2$, $\lambda = 1$, $\beta = 1.5$, $\mu = 1.5$, $\pi = .667$, $\tau = 1$

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	1.881184	2.676698	0.795514	0.771997	0.951
	λ	0.544195	1.617332	1.073136	1.379114	0.957
	β	0.825775	2.354374	1.528599	1.759849	0.966
	μ	1.316223				0.949
50	α	1.865775	2.546885	0.68111	0.537325	0.948
	λ	0.57059	1.53917	0.96858	1.159222	0.954
	β	0.884242	2.245212	1.36097	1.510346	0.961
	μ	1.319241				0.949
75	α	1.852858	2.377786	0.524928	0.27227	0.947
	λ	0.638843	1.42222	0.783377	0.73819	0.951
	β	0.969072	2.082472	1.113401	1.056698	0.958
	μ	1.324147				0.948
100	α	1.898036	2.385564	0.487527	0.25388	0.946
	λ	0.675433	1.397174	0.721741	0.653684	0.949
	β	1.034355	2.067703	1.033348	0.996751	0.956
	μ	1.328961				0.948

Table 3.17 Coverage of Nonparametric Bootstrap 95% C.I.s

α =2, λ =1, β =1.5, μ=1.5, π=.667, τ=1

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	1.26568	4.129411	2.863731	0.271449	0.977
	λ	0.484636	1.840151	1.355515	0.13168	0.961
	β	1.001162	2.504954	1.503792	0.092099	0.965
	μ	1.058091				0.957
50	α	1.390639	3.947453	2.556814	0.133658	0.975
	λ	0.541281	1.748228	1.206947	0.082081	0.959
	β	1.012852	2.395613	1.382761	0.148703	0.962
	μ	1.077546				0.956
75	α	1.434628	3.28381	1.849182	0.179964	0.970
	λ	0.650661	1.560505	0.909844	0.102515	0.952
	β	1.111393	2.134884	1.023491	0.066214	0.956
	μ	1.161951				0.952
100	α	1.406637	3.110757	1.70412	0.122568	0.968
	λ	0.630473	1.502043	0.87157	0.082322	0.952
	β	1.143815	2.129617	0.985802	0.077929	0.955
	μ	1.184248				0.951

Table 3.18 Coverage of Asymptotic 95% C.I.s

α =2, λ =1, β =1.5, μ=1.5, π=.667, τ=1.5

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	1.232809	4.457629	3.22482	0.181657	0.979
	λ	0.54125	1.733208	1.191958	0.052937	0.956
	β	1.001427	2.382551	1.381125	0.068153	0.968
	μ	1.083422				0.958
50	α	1.260234	3.903778	2.643544	0.07673	0.977
	λ	0.562567	1.635615	1.073047	0.071044	0.963
	β	1.06169	2.281618	1.219928	0.053243	0.965
	μ	1.133477				0.956
75	α	1.337075	3.459852	2.122777	0.084385	0.975
	λ	0.638029	1.539833	0.901804	0.042428	0.957
	β	1.086239	2.087624	1.001385	0.065787	0.961
	μ	1.171499				0.953
100	α	1.399008	3.275216	1.876208	0.072845	0.974
	λ	0.669059	1.449411	0.780353	0.037839	0.955
	β	1.140491	2.050325	0.909834	0.048918	0.957
	μ	1.212545				0.951

Table 3.19 Coverage of Parametric Bootstrap 95% C.I.s

α =2, λ =1, β =1.5, μ=1.5, π=.667, τ=1.5

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	1.905518	2.517776	0.612258	0.226052	0.950
	λ	0.739691	1.384743	0.645052	0.601005	0.951
	β	1.0212	2.061794	1.040595	1.295443	0.962
	μ	1.321983				0.949
50	α	1.911204	2.439399	0.528195	0.237054	0.948
	λ	0.757291	1.324153	0.566862	0.7951	0.949
	β	1.083335	2.026501	0.943166	1.481413	0.959
	μ	1.325385				0.948
75	α	1.915097	2.3297	0.414603	0.185886	0.946
	λ	0.805611	1.270135	0.464524	0.644159	0.947
	β	1.152093	1.890533	0.73844	1.358236	0.954
	μ	1.325509				0.948
100	α	1.9096	2.283411	0.373811	0.108726	0.945
	λ	0.812185	1.235958	0.423774	0.439657	0.946
	β	1.193668	1.868399	0.674732	0.878456	0.952
	μ	1.327446				0.947

Table 3.20 Coverage of Nonparametric Bootstrap 95% C.I.s

α =2, λ =1, β =1.5, μ=1.5, π=.667, τ=1.5

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	1.269203	3.87178	2.602577	0.134755	0.977
	λ	0.547979	1.70635	1.158371	0.0731	0.965
	β	1.004247	2.260736	1.256489	0.097995	0.966
	μ	1.083247				0.958
50	α	1.410204	3.556727	2.146523	0.139628	0.975
	λ	0.642983	1.556018	0.913035	0.064789	0.958
	β	1.090268	2.235839	1.145571	0.072421	0.964
	μ	1.145845				0.955
75	α	1.407773	3.034033	1.62626	0.123123	0.971
	λ	0.662992	1.443012	0.78002	0.046011	0.955
	β	1.127706	2.140591	1.012885	0.059599	0.962
	μ	1.190708				0.952
100	α	1.446856	2.916606	1.46975	0.093843	0.969
	λ	0.69255	1.394713	0.702163	0.045603	0.953
	β	1.14157	1.985018	0.843448	0.066669	0.956
	μ	1.2051				0.951

Table 3.21 Coverage of Asymptotic 95% C.I.s

α =2, λ =1, β =2, μ=1.5, π=.5, τ=1

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	1.180678	5.254714	4.074036	0.982995	0.983
	λ	0.506592	1.988138	1.481546	1.347648	0.963
	β	1.247111	3.310564	2.063453	2.46987	0.972
	μ	1.066164				0.958
50	α	1.254238	4.207019	2.952782	0.779648	0.978
	λ	0.525802	1.756541	1.230739	0.921677	0.957
	β	1.35092	3.189649	1.838729	1.702697	0.967
	μ	1.139075				0.955
75	α	1.345242	3.560474	2.215231	0.641804	0.974
	λ	0.630792	1.641162	1.01037	0.984202	0.955
	β	1.41278	2.846945	1.434165	1.982925	0.962
	μ	1.166284				0.953
100	α	1.423207	3.352306	1.929098	0.740477	0.970
	λ	0.656122	1.526605	0.870483	1.059187	0.952
	β	1.503427	2.710758	1.207332	1.952878	0.956
	μ	1.229383				0.952

Table 3.22 Coverage of Parametric Bootstrap 95% C.I.s

α =2, λ =1, β =2, μ=1.5, π=.5, τ=1

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	1.892279	2.906649	1.01437	0.235433	0.955
	λ	0.557248	1.684558	1.12731	0.066727	0.955
	β	1.152335	3.096615	1.944281	0.160542	0.971
	μ	1.16297				0.953
50	α	1.841297	2.659786	0.818489	0.172752	0.950
	λ	0.620751	1.515689	0.894938	0.04958	0.952
	β	1.31134	2.886517	1.575178	0.101799	0.964
	μ	1.139042				0.955
75	α	1.828085	2.516577	0.688491	0.130191	0.947
	λ	0.661838	1.469066	0.807228	0.041491	0.949
	β	1.331545	2.725896	1.394352	0.07054	0.961
	μ	1.120542				0.956
100	α	1.866915	2.453198	0.586283	0.075365	0.945
	λ	0.72914	1.368244	0.639104	0.024837	0.946
	β	1.483533	2.586148	1.102614	0.048198	0.955
	μ	1.110236				0.956

Table 3.23 Coverage of Nonparametric Bootstrap 95% C.I.s

α =2, λ =1, β =2, μ=1.5, π=.5, τ=1

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	1.795159	3.00377	1.208611	0.361801	0.956
	λ	0.449315	1.792492	1.343178	0.093729	0.959
	β	0.96618	3.28277	2.31659	0.241459	0.975
	μ	1.323013				0.951
50	α	1.762931	2.738152	0.975221	0.2239	0.954
	λ	0.535066	1.601375	1.066309	0.060284	0.955
	β	1.160525	3.037332	1.876807	0.110246	0.969
	μ	1.31011				0.951
75	α	1.762166	2.582496	0.82033	0.180288	0.950
	λ	0.58455	1.546353	0.961803	0.047957	0.954
	β	1.198043	2.859398	1.661355	0.092702	0.965
	μ	1.340352				0.949
100	α	1.810782	2.509332	0.69855	0.0834	0.947
	λ	0.66795	1.429435	0.761486	0.029565	0.948
	β	1.377964	2.691717	1.313753	0.069997	0.958
	μ	1.329761				0.951

Table 3.24 Coverage of Asymptotic 95% C.I.s

α =2, λ =1, β =2, μ=1.5, π=.5, τ=1.5

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	1.11577	5.913243	4.797474	0.580694	0.984
	λ	0.517366	1.982876	1.46551	0.931705	0.970
	β	1.194996	3.683123	2.488127	1.731279	0.975
	μ	1.04588				0.959
50	α	1.339687	3.971925	2.632239	0.310943	0.978
	λ	0.669776	1.636548	0.966772	0.619589	0.961
	β	1.394634	2.880582	1.485948	1.649991	0.967
	μ	1.178838				0.952
75	α	1.413564	3.616671	2.203107	0.372951	0.974
	λ	0.713172	1.538877	0.825706	0.556209	0.957
	β	1.462322	2.740709	1.278387	1.557488	0.964
	μ	1.202326				0.951
100	α	1.467187	3.153735	1.686548	0.109747	0.972
	λ	0.754687	1.399557	0.64487	0.290948	0.952
	β	1.528815	2.568824	1.04001	0.56538	0.961
	μ	1.263944				0.949

Table 3.25 Coverage of Parametric Bootstrap 95% C.I.s

α =2, λ =1, β =2, μ=1.5, π=.5, τ=1.5

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	2.020808	2.750363	0.729555	0.223073	0.955
	λ	0.766721	1.429044	0.662323	0.056178	0.953
	β	1.406384	2.804238	1.397854	0.138132	0.967
	μ	1.166141				0.953
50	α	1.991935	2.524924	0.532989	0.1023	0.949
	λ	0.808887	1.342104	0.533217	0.035152	0.949
	β	1.492788	2.577299	1.084511	0.079342	0.962
	μ	1.136777				0.954
75	α	1.951479	2.395367	0.443888	0.142408	0.948
	λ	0.842167	1.280425	0.438259	0.071572	0.948
	β	1.594817	2.461425	0.866607	0.088798	0.959
	μ	1.12091				0.955
100	α	1.950381	2.273168	0.322786	0.078887	0.945
	λ	0.881478	1.201186	0.319709	0.030296	0.945
	β	1.716333	2.300601	0.584269	0.068257	0.951
	μ	1.105877				0.956

Table 3.26 Coverage of Nonparametric Bootstrap 95% C.I.s

α =2, λ =1, β =2, μ=1.5, π=.5, τ=1.5

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	1.950957	2.820214	0.869257	0.350217	0.959
	λ	0.703307	1.492458	0.789151	0.065922	0.956
	β	1.272547	2.938076	1.665529	0.180675	0.971
	μ	1.311975				0.947
50	α	1.940904	2.575955	0.635051	0.17827	0.952
	λ	0.757834	1.393157	0.635322	0.037498	0.952
	β	1.388952	2.681135	1.292183	0.111022	0.965
	μ	1.321858				0.946
75	α	1.908979	2.437867	0.528888	0.176183	0.949
	λ	0.800206	1.322386	0.522181	0.100848	0.949
	β	1.511844	2.544398	1.032553	0.11485	0.961
	μ	1.333867				0.945
100	α	1.919476	2.304073	0.384596	0.101437	0.947
	λ	0.850867	1.231797	0.380929	0.04092	0.947
	β	1.660392	2.356542	0.69615	0.094245	0.954
	μ	1.332677				0.945

Table 3.27 Coverage of Asymptotic 95% C.I.s

α =2, λ =1, β =2, μ=1.5, π=.667, τ=1

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	1.18527	5.169248	3.983978	0.764236	0.983
	λ	0.485836	2.144273	1.658437	1.246684	0.969
	β	1.197337	3.762314	2.564977	1.838645	0.976
	μ	1.019534				0.961
50	α	1.265367	4.094663	2.829295	0.53232	0.979
	λ	0.533431	1.84381	1.310379	0.761906	0.962
	β	1.301443	3.129461	1.828018	1.449937	0.972
	μ	1.074997				0.957
75	α	1.34338	3.477807	2.134427	0.554648	0.973
	λ	0.594233	1.613714	1.019481	1.09164	0.956
	β	1.404189	2.959669	1.55548	2.097117	0.967
	μ	1.154882				0.953
100	α	1.441018	3.172844	1.731826	0.210842	0.971
	λ	0.66922	1.529088	0.859869	0.425829	0.953
	β	1.449447	2.746254	1.296807	0.910654	0.961
	μ	1.191878				0.951

Table 3.28 Coverage of Parametric Bootstrap 95% C.I.s

α =2, λ =1, β =2, μ=1.5, π=.667, τ=1

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	2.008872	2.79194	0.783068	0.216999	0.951
	λ	0.697514	1.574811	0.877297	0.059573	0.954
	β	1.37921	2.871089	1.491879	0.340091	0.964
	μ	1.163706				0.952
50	α	1.924886	2.551433	0.626548	0.084447	0.946
	λ	0.750658	1.442686	0.692029	0.039608	0.948
	β	1.412114	2.697744	1.28563	0.106689	0.959
	μ	1.13942				0.955
75	α	1.856146	2.41271	0.556563	0.117915	0.945
	λ	0.712056	1.366759	0.654704	0.050946	0.947
	β	1.433622	2.669249	1.235628	0.088742	0.957
	μ	1.118694				0.956
100	α	1.906126	2.315065	0.408939	0.069336	0.941
	λ	0.815369	1.260203	0.444834	0.04016	0.943
	β	1.584649	2.481777	0.897128	0.084253	0.954
	μ	1.108328				0.956

Table 3.29 Coverage of Nonparametric Bootstrap 95% C.I.s

α =2, λ =1, β =2, μ=1.5, π=.667, τ=1

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	1.933897	2.866914	0.933017	0.263416	0.955
	λ	0.613517	1.658807	1.04529	0.077511	0.956
	β	1.236371	3.013929	1.777558	0.431662	0.971
	μ	1.329451				0.948
50	α	1.864897	2.611422	0.746525	0.112463	0.949
	λ	0.6844	1.508944	0.824545	0.049949	0.952
	β	1.289022	2.820836	1.531815	0.144096	0.965
	μ	1.319644				0.949
75	α	1.802858	2.465998	0.663139	0.147549	0.947
	λ	0.649371	1.429444	0.780072	0.061241	0.951
	β	1.315317	2.787554	1.472237	0.120466	0.963
	μ	1.325531				0.948
100	α	1.866972	2.354219	0.487247	0.08616	0.944
	λ	0.772779	1.302794	0.530015	0.055875	0.945
	β	1.498753	2.567672	1.068919	0.12288	0.956
	μ	1.322481				0.948

Table 3.30 Coverage of Asymptotic 95% C.I.s

α =2, λ =1, β =2, μ=1.5, π=.667, τ=1.5

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	1.237376	4.600557	3.36318	0.449652	0.981
	λ	0.608433	1.870604	1.262172	0.932927	0.969
	β	1.221473	3.041747	1.820275	1.995731	0.975
	μ	1.05579				0.959
50	α	1.347492	3.843713	2.496221	0.185363	0.978
	λ	0.656041	1.675728	1.019688	0.571424	0.965
	β	1.346022	2.92537	1.579348	1.208899	0.974
	μ	1.111593				0.955
75	α	1.392674	3.386375	1.993702	0.111297	0.976
	λ	0.681337	1.554317	0.872979	0.47853	0.963
	β	1.402039	2.776832	1.374793	1.352452	0.972
	μ	1.153385				0.952
100	α	1.463351	3.086464	1.623113	0.069089	0.974
	λ	0.724027	1.397377	0.673351	0.277008	0.957
	β	1.571962	2.667108	1.095146	0.701327	0.966
	μ	1.259611				0.949

Table 3.31 Coverage of Parametric Bootstrap 95% C.I.s

α =2, λ =1, β =2, μ=1.5, π=.667, τ=1.5

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	2.024841	2.654797	0.629956	0.174726	0.956
	λ	0.823314	1.427485	0.60417	0.065627	0.955
	β	1.42803	2.589763	1.161733	0.105384	0.968
	μ	1.164098				0.951
50	α	1.996617	2.480943	0.484325	0.121479	0.951
	λ	0.884618	1.305873	0.421255	0.030382	0.949
	β	1.617316	2.441713	0.824397	0.113162	0.959
	μ	1.138449				0.953
75	α	1.945004	2.349089	0.404085	0.128183	0.948
	λ	0.868486	1.244779	0.376293	0.055322	0.947
	β	1.58077	2.403015	0.822246	0.056835	0.959
	μ	1.128913				0.954
100	α	1.945608	2.266826	0.321218	0.041146	0.946
	λ	0.898642	1.163298	0.264656	0.025534	0.944
	β	1.741905	2.311696	0.569791	0.037611	0.954
	μ	1.107509				0.956

Table 3.32 Coverage of Nonparametric Bootstrap 95% C.I.s

α =2, λ =1, β =2, μ=1.5, π=.667, τ=1.5

n	parameter	Lower Bound	Upper Bound	width	SD(W)	Coverage
30	α	1.964526	2.715112	0.750586	0.258982	0.957
	λ	0.765468	1.485331	0.719862	0.073161	0.958
	β	1.3168	2.700993	1.384193	0.143171	0.972
	μ	1.305051				0.947
50	α	1.950246	2.527314	0.577069	0.144537	0.954
	λ	0.844285	1.346206	0.501921	0.037447	0.952
	β	1.538384	2.520645	0.982261	0.149884	0.965
	μ	1.319397				0.946
75	α	1.906316	2.387778	0.481462	0.167801	0.951
	λ	0.832458	1.280807	0.448349	0.071815	0.950
	β	1.502044	2.481741	0.979697	0.081917	0.964
	μ	1.32296				0.946
100	α	1.914853	2.297581	0.382727	0.047641	0.947
	λ	0.873303	1.188637	0.315334	0.027154	0.946
	β	1.68735	2.36625	0.6789	0.043447	0.957
	μ	1.332709				0.945

Looking at the performance of the confidence intervals for the distribution parameters and the acceleration factor, one clear observation that can be made is that the coverage of all three types of intervals are conservative for small sample sizes and that the coverage probabilities drop towards the normal or slightly below normal levels as the sample size increases. When the sample size is 30, the asymptotic intervals for the shape parameter α remains above normal, and in some cases it becomes highly conservative (see Tables 3.3, 3.6 for example). In addition, the width of the interval for α is quite wide relative to the widths of intervals based on the bootstrap methods. While this width decreases with sample size, it remains much wider than the bootstrap intervals for any of the sample sizes considered in this study. Among the bootstrap methods, the parametric intervals for α are consistently narrower than the intervals based on the nonparametric method even though occasionally the coverage dips slightly below the normal level (e.g. Table 3.13, sample size ≥ 75). The latter intervals for α are conservative even for large sample sizes (e.g. Table (3.17), but in other cases they tend to be slightly liberal (e.g. Table 3.5, *n*=100). Overall, for intervals estimation of the shape parameter α, the parametric bootstrap method has good properties that mean the width of the confidence interval is narrowest compared to the intervals based on other methods while at the same time, the coverage does not drop much below the normal level. If slightly liberal intervals for α are a concern, then the recommendation is to use nonparametric intervals when the sample size is fifty or more, and use the parametric bootstrap intervals when the sample size falls below fifty.

Results on the intervals for the scale parameter λ show that, in general, the parametric bootstrap methods yields narrower intervals for small sample sizes 3.15 – 3.17), but there are a few exceptions (e.g. Tables 3.9 – 3.11). In addition, the coverage of the parametric bootstrap-based intervals does not fall below normal for small sample sizes. When the sample size is ≥ 75, however, the asymptotic distribution-based intervals are narrower than those obtained using the other two methods when the sample size is 100 (e.g. Tables 3.3 – 3.5), but this is not always the case (for example see Tables 3.12 and 3.13; Tables 3.27 and 3.28). However, in some instances when this happens, the coverage of the parametric bootstrap intervals is liberal. The nonparametric bootstrap-based intervals tend to provide intervals that are wider than their parametric counterparts but they also tend to be come liberal for large sample sizes (e.g. Table 3.29). Therefore, for intervals estimation of λ, parametric intervals are recommended for sample sizes below 75 and the asymptotic distribution based intervals are recommended for larger sample sizes.

Intervals estimates for the acceleration factor β show conservative coverage for all three types of intervals when the sample size is 30. This conservative coverage decrease as the sample size increases, but never becomes liberal as was the case for other parameters. For sample size 30 with the shape parameter $\alpha = 1.5$, the asymptotic and parametric bootstrap methods provide less conservative coverage than the nonparametric bootstrap-based intervals. When the shape parameter in increased to 2, the parametric bootstrap-based intervals are the narrowest in general when the sample size is 30, while maintaining appropriate coverage (Tables 3.9 through 3.32). When $\alpha = 2,$ the parametric bootstrap-based intervals are narrower than the other two types of intervals for sample sizes 50 and 75, and they maintain coverage at or above the normal level. When $\alpha = 1.5$ and the sample size is 100, the asymptotic distribution-based intervals are narrowest (see Tables 3.3 – 3.8). When the shape parameter is equal to 2 and the sample size is 100, the parametric bootstrap method tends to consistently produce narrower intervals. Overall, the parametric bootstrap-based intervals can be recommended for the acceleration factor.

For constructing 95% lower confidence bounds for the mean life under normal use conditions, no discernible difference is seen between the three methods when $\alpha = 1.5$ (see Tables 3.3 – 3.8). The estimates of the expected value of the lower bound are very close to one another for sample sizes 75 and 100. The asymptotic distribution-based bounds are somewhat lower than the bounds based on the bootstrap methods when the sample size is 30 or 50. When $\alpha = 2$ and the censoring time τ=1, both the asymptotic method and the nonparametric bootstrap-based method provide bounds with slightly above normal coverage with expected values are close to each other while the parametric bootstrap-based bounds display slightly lower expected values with slightly higher than normal coverage (Tables 3.9 – 3.11). When the censoring time increases to 1.5 with α remaining at the value 2, the asymptotic and nonparametric methods provides bounds with very close expected values and coverages which are slightly above normal , but the parametric method yields bounds that are higher with closer to normal coverage (Tables 3.12 – 3.14). The above results were obtained for the case where the size of the normal use and accelerated samples are the same and the acceleration factor is set at 1.5. When the proportion of the sample allocated to the accelerated condition was increased from ½ to 2/3, the same pattern is seen irrespective of the censoring time, but the coverage of the parametric bootstrap-based bounds drops slightly below normal when the acceleration factor remains at 1.5 (Tables 3.15 – 3.17 and 3.18 – 3.20). When the acceleration factor is increased to 2, however, it is the nonparametric bootstrap that yields higher bounds with slightly above normal coverage decreasing to slightly below normal as the sample size increases. (Tables 3.21 – 3.23). From the above results it is apparent that the bounds based on the asymptotic method would suffice if slightly conservative bounds that in some cases are less sharper than other types of bounds are acceptable. However, the bootstrap methods provide sharper bounds in some cases but which of the bootstrap methods perform better depends on the values of the underlying parameters. Since the practitioner will have no idea what the true value of α and the acceleration factor β are, but have control over the censoring time π, it is recommended to use the parametric bounds when using a relatively high censoring time but the sample is divided equally between the normal use and accelerated use. When π is close to 2/3 and one has some an idea that the acceleration factor should be high, the nonparametric bootstrap may be a good choice.

3.5 CONCLUSIONS AND FUTURE WORK

PALT have advantages over the ALT procedure under two scenarios: (1) when the accelerating factor is unknown or (2) a suitable model that links parameters of the life distribution to the stress level is not available. However, a drawback to PALT is that it is not suitable when the products under test have a very long mean life. This is because part of the sample is tested under normal use conditions and components with a very long expected life span may not fail at all during a reasonably chosen test period. It is. However, applicable in situations where the life-span of tested products is only moderately long. This is mostly the case in the chemical industry, where the shelf-life of a specialty chemical may be only a few months to an year long. This research contributes to the area of PALT by generalizing an existing procedure that considers testing products with a generalized exponential distribution. While the previous work considered Type II censoring, the more difficult case of Type I censoring was considered in this paper. In addition, this paper develops two bootstrap-based methods for obtaining confidence intervals for the distribution parameters and the acceleration factor. Moreover, it utilizes the three methods to obtain lower confidence bounds for the mean life of the product under normal use conditions. Monte-Carlo simulation Results show that one or more of the methods perform very well under a wide variety of conditions.

Future work would involve developing a theoretical justification for using the result of asymptotic normality for maximum likelihood estimates derived from a PALT scenario under Type I censoring scheme, while at the same time extending the results for the case where the shape parameter is less than or equal to two. Additional extensions would involve generalizing the test situation to include two censoring times for the two sub-samples, and investigating the behaviour of the proposed procedures under a wider set of distributional and test parameters.

4. CONCLUSION

Partially accelerated life tests (PALT) have advantages over the accelerated life test (ALT) procedure when either (1) the accelerating factor is unknown or (2) a suitable model that links parameters of the life distribution to the stress level is not available. PALT, however, has a drawback in the sense that it is not suitable when the products under test have a very long expected life. If the products have a long life, then the portion of the test sample that is tested under normal use conditions may not produce a few failures at best during a reasonably chosen test period. It is, however, applicable in situations where the life-span of tested products is only moderately long. This is mostly the case in the chemical industry, where the shelf-life of a specialty chemical may be only a few months to a year long.

This research consisted of two main studies. The first study extended a currently available method, for construction confidence intervals for distributional parameters of the underlying Weibull distribution and the acceleration factor, to cover Type I censoring case. It also developed two bootstrap-based methods for obtaining prediction intervals for distribution parameters and the acceleration factor. In addition, asymptotic distribution based intervals were also considered. More importantly, a method of obtaining lower confidence bounds for the mean life under normal use conditions was also developed. The performance of the three methods was studies using a Monte-Carlo study. Results show that all methods perform reasonable well under all parameter combinations employed in the Monte-Carlo study.

The second study contributes to the area of PALT by generalizing an existing procedure that considers testing products with a generalized exponential distribution. While the previous work considered Type II censoring, the more difficult case of Type I censoring was considered in this paper. In addition, this paper develops two bootstrap-based methods for obtaining confidence intervals for the distribution parameters and the acceleration factor. Moreover, it utilizes the three methods to obtain lower confidence bounds for the mean life of the product under normal use conditions. Monte-Carlo simulation Results show that one or more of the methods perform very well under a wide variety of conditions.

APPENDIX

Result 2.1 given in Section 2 can be proved using the Theorem B.41 in Meeker and Escobar (1998) which gives the regularity conditions necessary for the asymptotic normality of the MLEs.

Regularity Conditions for Location-Scale Distributions

When Y [or a transformation of T such as Y=log(T)] is location-scale with pdf, $f_Y\left(y;\theta\right)=\frac{1}{\sigma}\phi\left(\frac{y-\mu}{\sigma}\right), \theta=\left(\mu,\sigma\right),\ -\infty<y<\infty,\ -\infty<\mu<\infty, \sigma>0,$ the “regularity” conditions can be expressed as follows:

- $\phi\left(z\right)>0$ for all $-\infty<z<\infty$
- The following limits hold:

$$\lim_{z\to\pm\infty}\left[z^2\times\frac{\partial\phi\left(z\right)}{\partial z}\right]=0$$

- The second derivative $\partial^2\phi\left(z\right)/\partial z^2$ is continuous.
- The matrix

$$E\left\{-\frac{\partial^2\log\left[\phi\left(z\right)\right]}{\partial\underline{\theta}\partial\underline{\theta}'}\right\},$$

is positive definite and all its elements are finite.

First we show that the log of the Type I censored Weibull variables have a location-scale Family.

Let $X\sim W\left(\lambda,\alpha\right)$, $\quad Y=\ln\left(X\right)$, and $\tau_0=\ln\left(\tau\right)$

Y has a location-scale distribution, namely its cumulative distribution function (cdf) is

$$P\left(Y \le y \middle| Y \le \tau_0\right) = P\left(\ln\left(X\right) \le y \middle| \ln\left(X\right) \le \tau_0\right) = \frac{P(\ln\left(X\right) \le y)}{P(\ln\left(X\right) \le \tau_0)}$$

$$= \frac{P(X \le e^y)}{P(X \le e^{\tau_0})} = \frac{1 - \exp\left\{-\left(\frac{e^y}{\lambda}\right)^\alpha\right\}}{1 - \exp\left\{-\left(\frac{e^{\tau_0}}{\lambda}\right)^\alpha\right\}}$$

$$= \frac{1 - \exp\left\{-e\left(\frac{y - \ln\left(\lambda\right)}{1/\alpha}\right)\right\}}{1 - \exp\left\{-e\left(\frac{\tau_0 - \ln\left(\lambda\right)}{1/\alpha}\right)\right\}} = \frac{1 - \exp\left\{-e\left(\frac{y-u}{b}\right)\right\}}{1 - \exp\left\{-e\left(\frac{\tau_0 - u}{b}\right)\right\}},$$

with location parameter $u = \ln\left(\lambda\right)$ and scale parameter $b = \frac{1}{\alpha}$ since

$$H\left(y \middle| \tau_0\right) = P\left(Y \le y \middle| Y \le \tau_0\right) = P\left(u + bZ \le y \middle| u + bZ \le \tau_0\right) = P\left(Z \le \frac{y-u}{b} \middle| Z \le \frac{\tau_0 - u}{b}\right)$$

$$= \Phi\left(\frac{y-u}{b} \middle| \frac{\tau_0 - u}{b}\right) = \Phi\left(z \middle| \tau^*\right) = \frac{1 - \exp\left\{-e^{\left(\frac{y-u}{b}\right)}\right\}}{1 - \exp\left\{-e^{\left(\frac{\tau_0 - u}{b}\right)}\right\}} = \frac{1 - \exp\left\{-e^z\right\}}{1 - \exp\left\{-e^{\tau^*}\right\}}.$$

where $\Phi\left(z \middle| \tau^*\right)$ is the CDF of a Gumbel with Type I censoring. Therefore,

$$\phi\left(z \middle| \tau^*\right) = \frac{\partial \Phi\left(z\right)}{\partial z} = \frac{\partial}{\partial z}\left(\frac{1 - \exp\left\{-e^{(z)}\right\}}{1 - \exp\left\{-e^{\tau^*}\right\}}\right) = \frac{e^z \exp\left\{-e^{(z)}\right\}}{1 - \exp\left\{-e^{\tau^*}\right\}},$$

$$\frac{\partial \phi\left(z \middle| \tau^*\right)}{\partial z} = \frac{\partial}{\partial z}\left(\frac{e^z \exp\left\{-e^{(z)}\right\}}{1 - \exp\left\{-e^{\tau^*}\right\}}\right) = \frac{-\exp\left\{z - e^{(z)}\right\}\left(e^z - 1\right)}{1 - \exp\left\{-e^{\tau^*}\right\}},$$

where $\phi\left(z \middle| \tau^*\right)$ is the standard Gumbel Distribution with Type I censoring.

We need show $\lim_{z \to \pm\infty}\left[z^2 \times \frac{\partial \phi\left(z \middle| \tau^*\right)}{\partial z}\right] = 0$, which is equivalent to $\lim_{z\to\infty}\left[z^2\times\frac{\partial\phi\left(z\middle|\tau^*\right)}{\partial z}\right]=\lim_{z\to\infty}\left[z^2\times\frac{-\exp\left\{z-e^{(z)}\right\}\left(e^z-1\right)}{1-\exp\left\{-e^{\tau^*}\right\}}\right]=0$,

and therefore,

$$\lim_{z\to-\infty}\left[z^2\times\frac{\partial\phi\left(z\middle|\tau^*\right)}{\partial z}\right]=\lim_{z\to-\infty}\left[z^2\times\frac{-\exp\left\{z-e^{(z)}\right\}\left(e^z-1\right)}{1-\exp\left\{-e^{\tau^*}\right\}}\right]=0.$$

We also need to show that $\frac{\partial^2\phi\left(z\middle|\tau^*\right)}{\partial z^2}$ is continuous. Note that

$$\frac{\partial^2\phi\left(z\middle|\tau^*\right)}{\partial z^2}=\frac{\partial^2}{\partial z^2}\left(\frac{\exp\left\{z-e^{(z)}\right\}}{1-\exp\left\{-e^{\tau^*}\right\}}\right)=\frac{\partial}{\partial z}\left(\frac{-\exp\left\{z-e^{(z)}\right\}\left(e^z-1\right)}{1-\exp\left\{-e^{\tau^*}\right\}}\right)$$

$$=\frac{\exp\left\{z-e^{(z)}\right\}\left(-3e^z+e^{2z}+1\right)}{1-\exp\left\{-e^{\tau^*}\right\}}.$$

The distribution under the acceleration factor β is:

$X\sim W\left(\lambda,\alpha\right)$ $\qquad Y=\beta^{-1}X$. $\quad T=\ln\left(Y\right)=\ln\left(\beta^{-1}X\right)$. Therefore,

Y has a location-scale distribution, and its cumulative distribution function (cdf) is

$$P\left(T\le t\middle|T\le\tau_0\right)=P\left(\ln\left(\beta^{-1}X\right)\le t\middle|\ln\left(\beta^{-1}X\right)\le\tau_0\right)=\frac{P(\ln\left(\beta^{-1}X\right)\le t)}{P(\ln\left(\beta^{-1}X\right)\le\tau_0)}$$

$$=\frac{P(\beta^{-1}X\le e^t)}{P(\beta^{-1}X\le e^{\tau_0})}=\frac{P(X\le\beta e^t)}{P(X\le\beta e^{\tau_0})}=\frac{1-\exp\left\{-\left(\frac{\beta e^t}{\lambda}\right)^\alpha\right\}}{1-\exp\left\{-\left(\frac{\beta e^{\tau_0}}{\lambda}\right)^\alpha\right\}}$$

$$=\frac{1-\exp\left\{-\exp\left(\frac{t-\ln\left(\lambda\right)+\ln\left(\beta\right)}{1/\alpha}\right)\right\}}{1-\exp\left\{-\exp\left(\frac{\tau_0-\ln\left(\lambda\right)+\ln\left(\beta\right)}{1/\alpha}\right)\right\}}=\frac{1-\exp\left\{-\exp\left(\frac{t-u}{b}\right)\right\}}{1-\exp\left\{-\exp\left(\frac{\tau_0-u}{b}\right)\right\}}$$

with location parameter $u=\ln\left(\lambda\right)-\ln\left(\beta\right)$ and scale parameter $b=\frac{1}{\alpha}$.

Now let $T=bZ+b \quad \Rightarrow Z=\frac{T-u}{b}$. Then

$$P\left(T \le t \middle| T \le \tau_0\right) = P\left(u + bZ \le t \middle| u + bZ \le \tau_0\right) = P\left(Z \le \frac{t-u}{b} \middle| Z \le \frac{\tau_0 - u}{b} \right)$$

$$= \Phi\left(\frac{t-u}{b} \middle| \frac{\tau_0 - u}{b} \right) = \Phi\left(z \middle| \tau^*\right) = \frac{1 - \exp\left\{-\exp\left(\frac{t-u}{b}\right)\right\}}{1 - \exp\left\{-\exp\left(\frac{\tau_0 - u}{b}\right)\right\}}$$

$$= \frac{1 - \exp\left\{-\exp\left(z\right)\right\}}{1 - \exp\left\{-\exp\left(\tau^*\right)\right\}},$$

where $\Phi\left(z \middle| \tau^*\right)$ is the CDF of a Gumbel with Type I censoring and $\tau^* = \frac{\tau_0 - u}{b}$.

This implies that $\phi\left(z \middle| \tau^*\right) = \frac{\partial \Phi\left(z\right)}{\partial z} = \frac{\partial}{\partial z}\left(\frac{1 - \exp\left\{-\exp\left(z\right)\right\}}{1 - \exp\left\{-\exp\left(\tau^*\right)\right\}} \right) = \frac{e^z \left[\exp\left\{-\exp\left(z\right)\right\}\right]}{1 - \exp\left\{-\exp\left(\tau^*\right)\right\}}$

and that
$$\frac{\partial \phi\left(z \middle| \tau^*\right)}{\partial z} = \frac{\partial}{\partial z}\left(\frac{e^z \left[\exp\left\{-\exp\left(z\right)\right\}\right]}{1 - \exp\left\{-\exp\left(\tau^*\right)\right\}} \right) = \frac{-\exp\left\{z - e^z\right\}\left[\exp\left(z\right) - 1\right]}{1 - \exp\left\{-\exp\left(\tau^*\right)\right\}},$$

where $\phi\left(z \middle| \tau^*\right)$ is the standard Gumbel Distribution with Type I censoring.

Then we show that the condition

$$\lim_{z \to \pm\infty}\left[z^2 \times \frac{\partial \phi\left(z \middle| \tau^*\right)}{\partial z} \right] = 0$$

holds as follows:
$$\lim_{z \to \infty}\left[z^2 \times \frac{\partial \phi\left(z \middle| \tau^*\right)}{\partial z} \right] = \lim_{z \to \infty}\left[z^2 \times \frac{-e^{z - e^z}\left(e^z - 1\right)}{1 - \exp\left\{-\exp\left(\tau^*\right)\right\}} \right] = 0 \quad \text{and}$$

$$\lim_{z \to -\infty}\left[z^2 \times \frac{\partial \phi\left(z \middle| \tau^*\right)}{\partial z} \right] = \lim_{z \to -\infty}\left[z^2 \times \frac{-e^{z - e^z}\left(e^z - 1\right)}{1 - \exp\left\{-\exp\left(\tau^*\right)\right\}} \right] = 0.$$

Also we need show that $\frac{\partial^2 \phi\left(z \middle| \tau^*\right)}{\partial z^2}$ is continuous. Note that

$$\frac{\partial^2 \phi\left(z \middle| \tau^*\right)}{\partial z^2} = \frac{\partial^2}{\partial z^2}\left(\frac{\exp\left\{z - e^z\right\}}{1 - \exp\left\{-\exp\left(\tau^*\right)\right\}} \right) = \frac{\partial}{\partial z}\left(\frac{-\exp\left\{z - e^z\right\}\left[\exp\left(z\right) - 1\right]}{1 - \exp\left\{-\exp\left(\tau^*\right)\right\}} \right)$$

$$= \frac{\exp\left\{z - e^z\right\}\left[-3e^z + e^{2z} + 1\right]}{1 - \exp\left\{-\exp\left(\tau^*\right)\right\}}.$$

Clearly, the above function is continuous.

In addition, we wish to show that

$$\lim_{z\to-\infty}\left[z^2\times\frac{\partial\phi\left(z\middle|\tau^*\right)}{\partial z}\right]=\lim_{z\to-\infty}\left[z^2\times\frac{-e^{z-e^z}\left(e^z-1\right)}{1-\exp\left\{-\exp\left(\tau^*\right)\right\}}\right]=0\,.$$

In order to show this we examine $\lim_{z\to-\infty}\left[z^2\times -e^{z-e^z}\left(e^z-1\right)\right]$.

Applying the quotient rule, we write $\lim_{z\to-\infty}\left[z^2\times -e^{z-e^z}\left(e^z-1\right)\right]$ as

$$\left[\frac{\lim_{z\to-\infty}-z^2e^z\left(e^z-1\right)}{\lim_{z\to-\infty}e^{e^z}}\right]=\left[\frac{\lim_{z\to-\infty}-z^2e^z\left(e^z-1\right)}{e^{\lim_{z\to-\infty}e^z}}\right]=\left[\frac{\lim_{z\to-\infty}-z^2e^z\left(e^z-1\right)}{e^{e^{-\infty}}}\right]$$

$$=\left[\frac{\lim_{z\to-\infty}-z^2e^z\left(e^z-1\right)}{e^0}\right]=\lim_{z\to-\infty}-z^2e^z\left(e^z-1\right).$$

By the product rule,

$$\lim_{z\to-\infty}z^2e^z\left(1-e^z\right)=\left(\lim_{z\to-\infty}z^2e^z\right)\lim_{z\to-\infty}\left(1-e^z\right)=\left(\lim_{z\to-\infty}z^2e^z\right)\left(1-e^{-\infty}\right)=\left(\lim_{z\to-\infty}z^2e^z\right).$$

To prepare product z^2e^z for solution by L'Hôpital's rule, we write it as $\frac{z^2}{e^{-z}}$.

Applying L'Hôpital's rule, we obtain,

$$\lim_{z\to-\infty}\frac{z^2}{e^{-z}}=\lim_{z\to-\infty}\frac{\frac{d}{dz}z^2}{\frac{d}{dz}e^{-z}}=\lim_{z\to-\infty}\frac{2z}{-e^{-z}}=\lim_{z\to-\infty}-2ze^{-z}=-2\lim_{z\to-\infty}ze^{-z}\,.$$

To prepare the product ze^{-z} for solution by L’Hôpital’s rule, write as $\frac{z}{e^{-z}}$:

Applying L'Hôpital's rule, we obtain,

$$-2\lim_{z\to-\infty}\frac{z}{e^{-z}}=-2\lim_{z\to-\infty}\frac{\frac{d}{dz}z}{\frac{d}{dz}e^{-z}}=-2\lim_{z\to-\infty}\frac{1}{-e^{-z}}=2\lim_{z\to-\infty}e^z=2e^{-\infty}=0\,,$$

which implies $\lim_{z\to-\infty}\left[z^2\times\frac{-e^{z-e^z}\left(e^z-1\right)}{1-\exp\left\{-\exp\left(\tau^*\right)\right\}}\right]=0\,.$ (1)

In order to show that $\lim_{z\to\infty}\left[z^2 \times \frac{\partial\phi\left(z\middle|\tau^*\right)}{\partial z}\right] = \lim_{z\to\infty}\left[z^2 \times \frac{-e^{z-e^z}\left(e^z-1\right)}{1-\exp\left\{-\exp\left(\tau^*\right)\right\}}\right] = 0$, we look at

$$\lim_{z\to\infty}\left[z^2 \times -e^{z-e^z}\left(e^z-1\right)\right] = \lim_{z\to\infty}\left[-z^2e^{2z-e^z} + z^2e^{z-e^z}\right] = \lim_{z\to\infty} z^2e^{z-e^z} - \lim_{z\to\infty} z^2e^{2z-e^z}$$
$$= \lim_{z\to\infty}\frac{z^2e^z}{e^{e^z}} - \lim_{z\to\infty}\frac{z^2e^{2z}}{e^{e^z}}.$$

Applying L'Hôpital's rule, we obtain,

$$\lim_{z\to\infty}\frac{z^2e^z}{e^{e^z}} - \lim_{z\to\infty}\frac{z^2e^{2z}}{e^{e^z}} = \lim_{z\to\infty}\frac{\frac{d}{dz}z^2e^z}{\frac{d}{dz}e^{e^z}} - \lim_{z\to\infty}\frac{\frac{d}{dz}z^2e^{2z}}{\frac{d}{dz}e^{e^z}}$$
$$= \lim_{z\to\infty}\frac{2ze^z + z^2e^z}{e^{e^z+z}} - \lim_{z\to\infty}\frac{2ze^{2z} + 2z^2e^{2z}}{e^{e^z+z}}$$
$$= \lim_{z\to\infty}\frac{2ze^z + z^2e^z}{e^{e^z+z}} - 2\lim_{z\to\infty}\frac{ze^{2z} + z^2e^{2z}}{e^{e^z+z}}$$
$$= \lim_{z\to\infty}\frac{e^z\left(2z+z^2\right)}{e^{e^z+z}} - 2\lim_{z\to\infty}\frac{e^{2z}\left(z+z^2\right)}{e^{e^z+z}}$$
$$= \lim_{z\to\infty}\frac{\left(2z+z^2\right)}{e^{e^z}} - 2\lim_{z\to\infty}\frac{\left(z+z^2\right)}{e^{e^z-z}}.$$

Also, applying L'Hôpital's rule, we obtain,

$$= \lim_{z\to\infty}\frac{\left(2z+z^2\right)}{e^{e^z}} - 2\lim_{z\to\infty}\frac{\left(z+z^2\right)}{e^{e^z-z}} = \lim_{z\to\infty}\frac{\frac{d}{dz}\left(2z+z^2\right)}{\frac{d}{dz}e^{e^z}} - 2\lim_{z\to\infty}\frac{\frac{d}{dz}\left(z+z^2\right)}{\frac{d}{dz}e^{e^z-z}}$$
$$= \lim_{z\to\infty}\frac{\left(2+2z\right)}{e^{z+e^z}} - 2\lim_{z\to\infty}\frac{\left(1+2z\right)}{e^{e^z-z}\left(e^z-1\right)}.$$

Again, applying L'Hôpital's rule, we get that

$$\lim_{z\to\infty}\frac{(2+2z)}{e^{z+e^z}}-2\lim_{z\to\infty}\frac{(1+2z)}{e^{e^z-z}\left(e^z-1\right)}=2\lim_{z\to\infty}\frac{\frac{d}{dz}(1+z)}{\frac{d}{dz}e^{z+e^z}}-2\lim_{z\to\infty}\frac{\frac{d}{dz}(1+2z)}{\frac{d}{dz}e^{e^z-z}\left(e^z-1\right)}$$

$$=2\lim_{z\to\infty}\frac{1}{e^{e^z-z}\left(e^z+1\right)}-2\lim_{z\to\infty}\frac{2}{e^{e^z-z}e^z+\left(e^z-1\right)^2e^{e^z-z}}$$

$$=\frac{2}{e^{e^\infty-\infty}\left(e^\infty+1\right)}-\frac{4}{e^{e^\infty-\infty}e^\infty+\left(e^\infty-1\right)^2e^{e^\infty-\infty}}=\frac{2}{\infty}-\frac{4}{\infty}=0,$$

And therefore $\lim_{z\to\infty}\left[z^2\times\frac{-e^{z-e^z}\left(e^z-1\right)}{1-\exp\left\{-\exp\left(\tau^*\right)\right\}}\right]=0$. (2)

From (1) and (2) we obtain,

$$\lim_{z\to\pm\infty}\left[z^2\times\frac{-e^{z-e^z}\left(e^z-1\right)}{1-\exp\left\{-\exp\left(\tau^*\right)\right\}}\right]=0.$$

We need to show further that $E\left\{-\frac{\partial^2\log\left[\phi(z)\right]}{\partial\underline{\theta}\partial\underline{\theta}'}\right\}$ is positive definite and all its elements are finite.

First we need find ML function of the location-scale Distribution.

In Type I censoring, τ is fixed but the number of failure values observed in time τ is a random variable. The number of items, *R*, failing before time τ is assumed to follow a binomial distribution $R\sim Bin(n,p)$, where

$p=F_Z\left(\tau_0;u,b\right)=1-\exp\left\{-\exp\left(\frac{\tau_0-u}{b}\right)\right\}=1-\exp\left\{-\exp\left(\tau^*\right)\right\}$, under nominal use conditions. Under high stress conditions the number of items failing will have a Binomial $Bin\left(n,p^*\right)$, distribution where

$p^*=F_Z\left(\tau_0;u^*,b\right)=1-\exp\left\{-\exp\left(\frac{\tau_0-u^*}{b}\right)\right\}=1-\exp\left\{-\exp\left(\tau^*\right)\right\}$. Then, for observation *i* under nominal use conditions, we have,

$$\delta_{u_i} = \begin{cases} 1 & x_i \le \tau \\ 0 & 0/w \end{cases}, \quad i = 1, 2, \cdots, n\overline{\pi}.$$

$$\delta_{a_j} = \begin{cases} 1 & y_j \le \tau \\ 0 & 0/w \end{cases}, \quad j = 1, 2, \cdots, n\pi,$$

$$\delta_{u_i} = 1 - \overline{\delta}_{u_i}, \qquad \delta_{a_j} = 1 - \overline{\delta}_{a_j},$$

with

$$\delta_{u_i} \sim Ber(p) \Rightarrow \sum_{i=1}^{n\overline{\pi}} \delta_{u_i} \sim Bin(n\overline{\pi}, p), \qquad \delta_{a_j} \sim Ber(p) \Rightarrow \sum_{j=1}^{n\pi} \delta_{a_j} \sim Bin(n\pi, p).$$

We also have, under nominal use conditions,

$$\Phi\left(\frac{t-u}{b} \middle| \frac{\tau_0 - u}{b}\right) = \begin{cases} \dfrac{1-\exp\left\{-\exp\left(\dfrac{t-u}{b}\right)\right\}}{1-\exp\left\{-\exp\left(\dfrac{\tau_0-u}{b}\right)\right\}}, & t \le \dfrac{\tau_0 - u}{b} \\ 1, & t > \dfrac{\tau_0 - u}{b} \end{cases}$$

$$\phi\left(z \middle| \tau^*\right) = \frac{\partial \Phi(z)}{\partial z} = \frac{\partial}{\partial z}\left(\frac{1-e^{-e^{(z)}}}{1-e^{-e^{\tau^*}}}\right) = \frac{e^z e^{-e^z}}{1-e^{-e^{\tau^*}}},$$

where $\phi\left(z \middle| \tau^*\right)$ is the standard Gumbel distribution.

Also, letting $z = \dfrac{t-u}{b}$ we have,

$$\phi\left(z \middle| \tau_0\right) = \phi\left(\frac{t-u}{b} \middle| \frac{\tau_0 - u}{b}\right) = \frac{\exp\left(\dfrac{t-u}{b}\right)\exp\left\{-\exp\left(\dfrac{t-u}{b}\right)\right\}}{1-\exp\left\{-\exp\left(\dfrac{\tau_0-u}{b}\right)\right\}}.$$

Thus, given $R = n_u$, the conditional density of the first r failure times under the nominal use condition is equivalent to the joint density of an ordered random sample of size n_u from a truncated Weibull distribution, given by

$$\phi_\tau\left(z_{(1)},\cdots,z_{(n_u)}\middle|R=n_u\right)=\phi_\tau\left(t_{(1)},\cdots,t_{(n_u)}\middle|R=n_u\right)=n_u!\prod_{i=1}^{n_u}\phi_\tau\left(z_i\middle|\tau^*\right)=\phi_\tau\left(\frac{t-u}{b}\middle|\frac{\tau_0-u}{b}\right)$$

$$=n_u!\prod_{i=1}^{n_u}\left[\frac{e^{z_i}\exp\left\{-\exp\left(z_i\right)\right\}}{1-\exp\left\{-\exp\left(\frac{\tau_0-u}{b}\right)\right\}}\right]=n_u!\frac{e^{\sum_{i=1}^{n_u}z_i}\exp\left\{-\exp\sum_{i=1}^{n_u}\left(z_i\right)\right\}}{\left[1-\exp\left\{-\exp\left(\frac{\tau_0-u}{b}\right)\right\}\right]^{n_u}}.$$

The joint density of obtaining $R = n_u$ ordered observations at the values $z_{(1)},\cdots,z_{(n_u)}$ before time τ may be expressed as,

$$\phi_\tau\left(z_{(1)},\cdots,z_{(n_u)}\right)=\phi_\tau\left(z_{(1)},\cdots,z_{(n_u)}\middle|R=n_u\right)bin(n_u;n\overline{\pi},p)$$

$$=n_u!\frac{e^{\sum_{i=1}^{n_u}z_i}\exp\left\{-\exp\sum_{i=1}^{n_u}\left(z_i\right)\right\}}{\left[1-\exp\left\{-\exp\left(\frac{\tau-u}{b}\right)\right\}\right]^{n_u}}\binom{n\overline{\pi}}{n_u}p^{n_u}\left(1-p\right)^{n\overline{\pi}-n_u}$$

$$=\frac{(n\overline{\pi})!}{(n\overline{\pi}-n_u)!}\frac{e^{\sum_{i=1}^{n_u}z_i}\exp\left\{-\exp\sum_{i=1}^{n_u}\left(z_i\right)\right\}}{\left[1-\exp\left\{-\exp\left(\tau^*\right)\right\}\right]^{n_u}}\left(1-\exp\left\{-\exp\left(\tau^*\right)\right\}\right)^{n_u}\left(\exp\left\{-\exp\left(\tau^*\right)\right\}\right)^{n\overline{\pi}-n_u}$$

$$=\frac{(n\overline{\pi})!}{(n\overline{\pi}-n_u)!}e^{\sum_{i=1}^{n_u}z_i}\exp\left\{-\exp\sum_{i=1}^{n_u}\left(z_i\right)\right\}\left(\exp\left\{-\exp\left(\tau^*\right)\right\}\right)^{n\overline{\pi}-n_u}.$$

Therefore, we can state that,

$$\phi_\tau\left(z_{(1)},\cdots,z_{(n_u)}\right)=\frac{(n\overline{\pi})!}{(n\overline{\pi}-n_u)!}e^{\sum_{i=1}^{n_u}z_i}\exp\left\{-\exp\sum_{i=1}^{n_u}\left(z_i\right)\right\}\left(\exp\left\{-\exp\left(\tau^*\right)\right\}\right)^{n\overline{\pi}-n_u}$$

$$\phi_\tau\left(z_{(1)},\cdots,z_{(n_u)}\right)=\phi_\tau\left(t_{(1)},\cdots,t_{(n_u)}\right)\propto e^{\sum_{i=1}^{n_u}z_i\left(\frac{t_i-u}{b}\right)}\exp\left\{-\exp\sum_{i=1}^{n_u}\left(\frac{t_i-u}{b}\right)\right\}\left(\exp\left\{-\exp\left(\frac{\tau_0-u}{b}\right)\right\}\right)^{n\overline{\pi}-n_u}.$$

Similarly to the argument made about the joint density of observations under nominal use conditions, given $R = n_a$ the conditional density of the first r failure times under acceleration is equivalent to the joint density of an ordered random sample of size n_a

from a truncated accelerated Weibull distribution. Therefore, for an item tested at accelerated condition, the probability density function is given by

$$\Phi\left(\frac{t'-u'}{b}\middle|\frac{\tau_0-u'}{b}\right)=\begin{cases}\dfrac{1-\exp\left\{-\exp\left(\dfrac{t'-u'}{b}\right)\right\}}{1-\exp\left\{-\exp\left(\dfrac{\tau_0-u'}{b}\right)\right\}}, & t'\le\dfrac{\tau_0-u'}{b}\\ 1, & t'>\dfrac{\tau_0-u'}{b}\end{cases}$$

which yields $\phi\left(z'\middle|\tau^{*\prime}\right)=\dfrac{\partial\Phi(z')}{\partial z'}=\dfrac{\partial}{\partial z'}\left(\dfrac{1-e^{-e^{(z')}}}{1-e^{-e^{\tau^{*\prime}}}}\right)=\dfrac{e^{z'}e^{-e^{z'}}}{1-e^{-e^{\tau^{*\prime}}}}$,

where $\phi\left(z'\middle|\tau^{*\prime}\right)$ is the standard Gumbel distribution.

Also, $z'=\dfrac{t'-u'}{b}$, *and* $\tau^{*\prime}=\dfrac{\tau_0-u'}{b}$, and therefore,

$$\phi\left(z'\middle|\tau^{*\prime}\right)=\phi\left(\frac{t'-u'}{b}\middle|\frac{\tau_0-u'}{b}\right)=\frac{\exp\left(\dfrac{t'-u'}{b}\right)\exp\left\{-\exp\left(\dfrac{t'-u'}{b}\right)\right\}}{1-\exp\left\{-\exp\left(\dfrac{\tau_0-u'}{b}\right)\right\}},$$

The joint density of obtaining $R=n_a$ ordered observations at the values $Y_{(1)},\cdots,Y_{(n_a)}$ before time, may be expressed as,

$$\phi_{\tau_0}\left(z'_{(1)},\cdots,z'_{(n_u)}\right)=\phi_{\tau_0}\left(z'_{(1)},\cdots,z'_{(n_u)}\middle|R=n_u\right)bin(n_u;n\bar{\pi},p)$$

$$=n_u!\frac{e^{\sum_{i=1}^{n_u}z'_i}\exp\left\{-\exp\sum_{i=1}^{n_u}(z'_i)\right\}}{\left[1-\exp\left\{-\exp\left(\dfrac{\tau_0-u'}{b}\right)\right\}\right]^{n_u}}\binom{n\bar{\pi}}{n_u}p^{n_u}(1-p)^{n\bar{\pi}-n_u}$$

$$=\frac{(n\bar{\pi})!}{(n\bar{\pi}-n_u)!}\frac{e^{\sum_{i=1}^{n_u}z'_i}\exp\left\{-\exp\sum_{i=1}^{n_u}(z'_i)\right\}}{\left[1-\exp\left\{-\exp\left(\tau^{*\prime}\right)\right\}\right]^{n_u}}\left(1-\exp\left\{-\exp\left(\tau^{*\prime}\right)\right\}\right)^{n_u}\left(\exp\left\{-\exp\left(\tau^{*\prime}\right)\right\}\right)^{n\bar{\pi}-n_u}$$

$$=\frac{(n\bar{\pi})!}{(n\bar{\pi}-n_u)!}e^{\sum_{i=1}^{n_u}z'_i}\exp\left\{-\exp\sum_{i=1}^{n_u}(z'_i)\right\}\left(\exp\left\{-\exp\left(\tau^{*\prime}\right)\right\}\right)^{n\bar{\pi}-n_u}.$$

Therefore, we can state that

$$\phi_{\tau_0}\left(z'_{(1)},\cdots,z'_{(n_u)}\right)=\frac{(n\bar{\pi})!}{(n\bar{\pi}-n_u)!}e^{\sum_{i=1}^{n_u} z'_i}\exp\left\{-\exp\sum_{i=1}^{n_u}\left(z'_i\right)\right\}\left(\exp\left\{-\exp\left(\tau^{*\prime}\right)\right\}\right)^{n\bar{\pi}-n_u}$$

$$\phi_{\tau_0}\left(z'_{(1)},\cdots,z'_{(n_u)}\right)=\phi_{\tau}\left(t'_{(1)},\cdots,t'_{(n_u)}\right)$$

$$\propto e^{\sum_{i=1}^{n_u} z'_i\left(\frac{t'_i-u'}{b}\right)}\exp\left\{-\exp\sum_{i=1}^{n_u}\left(\frac{t'_i-u'}{b}\right)\right\}\left(\exp\left\{-\exp\left(\frac{\tau_0-u'}{b}\right)\right\}\right)^{n\bar{\pi}-n_u},$$

and the total likelihood function for $\left(t_1,\delta_{u_1},...,t_{n\bar{\pi}},\delta_{u_{n\bar{\pi}}},t'_1,\delta_{a_1},...,t'_{n\pi},\delta_{a_{n\pi}}\right)$ is as follows,

$$L=L\left(\underline{\theta}\middle|\underline{t}\ ,\underline{t}'\right)=L_{u_i}\left(u,b\middle|t_i,\delta_{u_i}\right)L_{u_j}\left(u',b\middle|t'_j,\delta_{u_j}\right)=L_{u_i}\left(u,b\middle|t_i,\right)L_{u_j}\left(u,b,\beta'\middle|t'_j,\delta_{u_j}\right)$$

$$=\prod_{i=1}^{n\bar{\pi}}\left[e^{\left(\frac{t_i-u}{b}\right)}\exp\left\{-\exp\left(\frac{t_i-u}{b}\right)\right\}\right]^{\delta_{u_i}}\left[\left(\exp\left\{-\exp\left(\frac{\tau_0-u}{b}\right)\right\}\right)\right]^{\bar{\delta}_{u_i}}$$

$$\times\prod_{j=1}^{n\pi}\left[e^{\left(\frac{t'_j-u'}{b}\right)}\exp\left\{-\exp\left(\frac{t'_j-u'}{b}\right)\right\}\right]^{\delta_{a_j}}\left[\left(\exp\left\{-\exp\left(\frac{\tau_0-u'}{b}\right)\right\}\right)\right]^{\bar{\delta}_{a_j}}$$

$$L=\prod_{i=1}^{n_u}\left[e^{\left(\frac{t_i-u}{b}\right)}\exp\left\{-\exp\left(\frac{t_i-u}{b}\right)\right\}\right]\prod_{i=n_u+1}^{n\bar{\pi}}\left[\left(\exp\left\{-\exp\left(\frac{\tau_0-u}{b}\right)\right\}\right)\right]$$

$$\times\prod_{j=1}^{n_a}\left[e^{\left(\frac{t'_j-u'}{b}\right)}\exp\left\{-\exp\left(\frac{t'_j-u'}{b}\right)\right\}\right]\prod_{i=n_{a+1}}^{n\pi}\left[\left(\exp\left\{-\exp\left(\frac{\tau_0-u'}{b}\right)\right\}\right)\right]$$

$$=e^{\sum_{i=1}^{n_u}\left(\frac{t_i-u}{b}\right)}\exp\left\{-\sum_{i=1}^{n_u}\exp\left(\frac{t_i-u}{b}\right)\right\}\left[\left(\exp\left\{-\exp\left(\frac{\tau_0-u}{b}\right)\right\}\right)\right]^{n\bar{\pi}-n_u}$$

$$\times e^{\sum_{j=1}^{n_a}\left(\frac{t'_j-u'}{b}\right)}\exp\left\{-\exp\sum_{j=1}^{n_a}\left(\frac{t'_j-u'}{b}\right)\right\}\left[\left(\exp\left\{-\exp\left(\frac{\tau_0-u'}{b}\right)\right\}\right)\right]^{n\pi-n_a}.$$

The MLE's of the parameters can be estimated numerically by minimizing the log likelihood function. The log likelihood is expressed as,

$$\ln L = \ln L\left(\underline{\theta} \middle| \underline{t}\ ,\underline{t'}\right) = l,$$

$$l = \ln\left\{\begin{array}{l} e^{\sum_{i=1}^{n_u}\left(\frac{t_i-u}{b}\right)} \exp\left\{-\sum_{i=1}^{n_u}\exp\left(\frac{t_i-u}{b}\right)\right\}\left[\left(\exp\left\{-\exp\left(\frac{\tau_0-u}{b}\right)\right\}\right)\right]^{n\overline{\pi}-n_u} \\ \times e^{\sum_{j=1}^{n_a}\left(\frac{t'_j-u'}{b}\right)} \exp\left\{-\exp\sum_{j=1}^{n_a}\left(\frac{t'_j-u'}{b}\right)\right\}\left[\left(\exp\left\{-\exp\left(\frac{\tau_0-u'}{b}\right)\right\}\right)\right]^{n\pi-n_a} \end{array}\right\}$$

$$= \left[\sum_{i=1}^{n_u}\left(\frac{t_i-u}{b}\right) - \sum_{i=1}^{n_u}\exp\left(\frac{t_i-u}{b}\right) + \sum_{j=1}^{n_a}\left(\frac{t'_j-u'}{b}\right) - \exp\sum_{j=1}^{n_a}\left(\frac{t'_j-u'}{b}\right)\right]$$

$$\times \ln\left\{\left[\left(\exp\left\{-\exp\left(\frac{\tau_0-u}{b}\right)\right\}\right)\right]^{n\overline{\pi}-n_u}\left[\left(\exp\left\{-\exp\left(\frac{\tau_0-u'}{b}\right)\right\}\right)\right]^{n\pi-n_a}\right\}$$

$$= \left[\sum_{i=1}^{n_u}\left(\frac{t_i-u}{b}\right) - \sum_{i=1}^{n_u}\exp\left(\frac{t_i-u}{b}\right) + \sum_{j=1}^{n_a}\left(\frac{t'_j-u'}{b}\right) - \sum_{j=1}^{n_a}\exp\left(\frac{t'_j-u'}{b}\right)\right]$$

$$-\left(n\overline{\pi}-n_u\right)\exp\left(\frac{\tau_0-u}{b}\right) - \left(n\pi-n_a\right)\exp\left(\frac{\tau_0-u'}{b}\right),$$

where $u' = u + \ln(\beta)$.

The normal equations are obtained by differentiating the log likelihood with respect to the parameters and setting them to zero. The Score equations are:

$$\frac{\partial l}{du} = \frac{-n_u}{b} + \frac{1}{b}\sum_{i=1}^{n_u}\exp\left(\frac{t_i-u}{b}\right) - \frac{n_a}{b} + \frac{1}{b}\exp\sum_{j=1}^{n_a}\left(\frac{t'_j-u'}{b}\right)$$

$$+\frac{\left(n\overline{\pi}-n_u\right)}{b}\exp\left(\frac{\tau_0-u}{b}\right) + \frac{\left(n\pi-n_a\right)}{b}\exp\left(\frac{\tau_0-u'}{b}\right)$$

$$= \frac{-n_u-n_a}{b} + \frac{1}{b}\sum_{i=1}^{n_u}\exp\left(\frac{t_i-u}{b}\right) + \frac{1}{b}\exp\sum_{j=1}^{n_a}\left(\frac{t'_j-u'}{b}\right)$$

$$+\left(\frac{n\overline{\pi}-n_u}{b}\right)\exp\left(\frac{\tau_0-u}{b}\right) + \frac{\left(n\pi-n_a\right)}{b}\exp\left(\frac{\tau_0-u'}{b}\right) = 0,$$

$$\frac{\partial l}{db} = -\sum_{i=1}^{n_u}\left(\frac{t_i-u}{b^2}\right) + \sum_{i=1}^{n_u}\left(\frac{t_i-u}{b^2}\right)\exp\left(\frac{t_i-u}{b}\right) - \sum_{j=1}^{n_a}\left(\frac{t'_j-u'}{b^2}\right) + \sum_{j=1}^{n_a}\left(\frac{t'_j-u'}{b^2}\right)\exp\left(\frac{t'_j-u'}{b}\right)$$

$$+\left(n\overline{\pi}-n_u\right)\left(\frac{\tau_0-u}{b^2}\right)\exp\left(\frac{\tau_0-u}{b}\right) + \left(n\pi-n_a\right)\left(\frac{\tau_0-u'}{b^2}\right)\exp\left(\frac{\tau_0-u'}{b}\right) = 0,$$

and $\dfrac{\partial l}{du'} = -\dfrac{n_a}{b} + \dfrac{1}{b}\exp\sum_{j=1}^{n_a}\left(\dfrac{t_j' - u'}{b}\right) + \dfrac{(n\pi - n_a)}{b}\exp\left(\dfrac{\tau_0 - u'}{b}\right) = 0.$

We also derive the second derivatives:

$$\begin{aligned}\frac{\partial^2 l}{\partial u \partial b} = \frac{n_u + n_a}{b^2} &- \frac{1}{b}\sum_{i=1}^{n_u}\left(\frac{t_i - u}{b^2}\right)\exp\left(\frac{t_i - u}{b}\right) - \frac{1}{b^2}\sum_{i=1}^{n_u}\exp\left(\frac{t_i - u}{b}\right) \\ &- \frac{1}{b}\sum_{j=1}^{n_a}\left(\frac{t_j' - u'}{b^2}\right)\exp\left(\frac{t_j' - u'}{b}\right) - \frac{1}{b^2}\sum_{j=1}^{n_a}\exp\left(\frac{t_j' - u'}{b}\right) \\ &- \frac{(n\overline{\pi} - n_u)}{b}\left(\frac{\tau_0 - u}{b^2}\right)\exp\left(\frac{\tau_0 - u}{b}\right) - \frac{(n\overline{\pi} - n_u)}{b^2}\exp\left(\frac{\tau_0 - u}{b}\right) \\ &- \frac{(n\pi - n_a)}{b}\left(\frac{\tau_0 - u'}{b}\right)\exp\left(\frac{\tau_0 - u'}{b}\right) - \frac{(n\pi - n_a)}{b^2}\exp\left(\frac{\tau_0 - u'}{b}\right),\end{aligned}$$

$$\frac{\partial^2 l}{\partial u \partial u'} = \frac{-n_a}{b^2}\exp\sum_{j=1}^{n_a}\left(\frac{t_j' - u'}{b}\right) - \frac{(n\pi - n_a)}{b^2}\exp\left(\frac{\tau_0 - u'}{b}\right),$$

$$\frac{\partial^2 l}{\partial b \partial u'} = \frac{n_a}{b^2} - 2\sum_{j=1}^{n_a}\left(\frac{t_j' - u'}{b^3}\right)\exp\left(\frac{t_j' - u'}{b}\right) - 2(n\pi - n_a)\left(\frac{\tau_0 - u'}{b^3}\right)\exp\left(\frac{\tau_0 - u'}{b}\right),$$

$$\frac{\partial^2 l}{\partial u^2} = -n_u\sum_{i=1}^{n_u}\exp\left(\frac{t_i - u}{b^3}\right) - \exp\sum_{j=1}^{n_a}\left(\frac{t_j' - u'}{b^3}\right) - \frac{(n\overline{\pi} - n_u)}{b^2}\exp\left(\frac{\tau_0 - u}{b}\right) - \frac{(n\pi - n_a)}{b^2}\exp\left(\frac{\tau_0 - u'}{b}\right),$$

and

$$\begin{aligned}\frac{\partial^2 l}{\partial b^2} = 2\sum_{i=1}^{n_u}\left(\frac{t_i - u}{b^3}\right) &- \sum_{i=1}^{n_u}\exp\left(\frac{t_i - u}{b}\right)\left\{\left(\frac{t_i - u}{b^2}\right)^2 + 2\left(\frac{t_i - u}{b^3}\right)\right\} \\ &+ 2\sum_{j=1}^{n_a}\left(\frac{t_j' - u'}{b^3}\right) - \sum_{j=1}^{n_a}\exp\left(\frac{t_j' - u'}{b}\right)\left\{\left(\frac{t_j' - u'}{b^2}\right)^2 + 2\left(\frac{t_j' - u'}{b^3}\right)\right\} \\ &- (n\overline{\pi} - n_u)\exp\left(\frac{\tau_0 - u}{b}\right)\left\{\left(\frac{\tau_0 - u}{b^2}\right)^2 + 2\left(\frac{\tau_0 - u}{b^3}\right)\right\} \\ &- (n\pi - n_a)\exp\left(\frac{\tau_0 - u'}{b}\right)\left\{\left(\frac{\tau_0 - u'}{b^2}\right)^2 + 2\left(\frac{\tau_0 - u'}{b^3}\right)\right\},\end{aligned}$$

$$\frac{\partial^2 l}{\partial u'^2}=\frac{-1}{b^2}\exp\sum_{j=1}^{n_a}\left(\frac{t_j'-u'}{b}\right)-\frac{\left(n\pi-n_a\right)}{b^2}\exp\left(\frac{\tau_0-u'}{b}\right),$$

$$E\left[\frac{-\partial^2 l}{\partial u\partial u'}\right]=E\left\{\frac{n_a}{b^2}\exp\sum_{j=1}^{n_a}\left(\frac{t_j'-u'}{b}\right)+\frac{\left(n\pi-n_a\right)}{b^2}\exp\left(\frac{\tau-u'}{b}\right)\right\}>0,$$

$$E\left[\frac{-\partial^2 l}{\partial u^2}\right]=E\{\frac{n_u}{b^2}\sum_{i=1}^{n_u}\exp\left(\frac{t_i-u}{b}\right)+\frac{1}{b^2}\exp\sum_{j=1}^{n_a}\left(\frac{t_j'-u'}{b}\right)$$

$$\times\frac{\left(n\bar{\pi}-n_u\right)}{b^2}\exp\left(\frac{\tau_0-u}{b}\right)+\frac{\left(n\pi-n_a\right)}{b^2}\exp\left(\frac{\tau_0-u'}{b}\right)\}>0,$$

$$E\left[\frac{-\partial^2 l}{\partial u'^2}\right]=E\left[\frac{1}{b^2}\exp\sum_{j=1}^{n_a}\left(\frac{t_j'-u'}{b}\right)+\frac{\left(n\pi-n_a\right)}{b^2}\exp\left(\frac{\tau_0-u'}{b}\right)\right]>0$$

$$E\left[\frac{-\partial^2 l}{\partial b^2}\right]=E\{-2\sum_{i=1}^{n_u}\left(\frac{t_i-u}{b^3}\right)+\sum_{i=1}^{n_u}\exp\left(\frac{t_i-u}{b}\right)\left\{\left(\frac{t_i-u}{b^2}\right)^2+2\left(\frac{t_i-u}{b^3}\right)\right\}$$

$$-2\sum_{j=1}^{n_a}\left(\frac{t_j'-u'}{b^3}\right)+\sum_{j=1}^{n_a}\exp\left(\frac{t_j'-u'}{b}\right)\left\{\left(\frac{t_j'-u'}{b^2}\right)^2+2\left(\frac{t_j'-u'}{b^3}\right)\right\}$$

$$+\left(n\bar{\pi}-n_u\right)\exp\left(\frac{\tau_0-u}{b}\right)\left[\left(\frac{\tau_0-u}{b^2}\right)^2+2\left(\frac{\tau_0-u}{b^3}\right)\right]$$

$$+\left(n\pi-n_a\right)\exp\left(\frac{\tau_0-u'}{b}\right)\left[\left(\frac{\tau_0-u'}{b^2}\right)^2+2\left(\frac{\tau_0-u'}{b^3}\right)\right]\},$$

Further simplifying the first and second parts,

$$-2\sum_{i=1}^{n_u}\left(\frac{t_i-u}{b^3}\right)+\sum_{i=1}^{n_u}\exp\left(\frac{t_i-u}{b}\right)\left\{\left(\frac{t_i-u}{b^2}\right)^2+2\left(\frac{t_i-u}{b^3}\right)\right\}$$

$$=-2\sum_{i=1}^{n_u}\left(\frac{t_i-u}{b^3}\right)+\sum_{i=1}^{n_u}\left[1+\left(\frac{t_i-u}{b}\right)+\frac{1}{2!}\left(\frac{t_i-u}{b}\right)^2+\cdots\right]\left[\left(\frac{t_i-u}{b^2}\right)^2+2\left(\frac{t_i-u}{b^3}\right)\right]$$

$$=-2\sum_{i=1}^{n_u}\left(\frac{t_i-u}{b^3}\right)$$

$$+\sum_{i=1}^{n_u}\left[\left[\left(\frac{t_i-u}{b^2}\right)^2+2\left(\frac{t_i-u}{b^3}\right)\right]+\left[\left(\frac{t_i-u}{b^2}\right)^2+2\left(\frac{t_i-u}{b^3}\right)\right]\left(\frac{t_i-u}{b}\right)+\left[\left(\frac{t_i-u}{b^2}\right)^2+2\left(\frac{t_i-u}{b^3}\right)\right]\frac{1}{2!}\left(\frac{t_i-u}{b}\right)^2+\cdots\right]$$

$$=-2\sum_{i=1}^{n_u}\left(\frac{t_i-u}{b^3}\right)+2\sum_{i=1}^{n_u}\left(\frac{t_i-u}{b^3}\right)$$

$$+\sum_{i=1}^{n_u}\left[\left[\left(\frac{t_i-u}{b^2}\right)^2\right]+\left[\left(\frac{t_i-u}{b^2}\right)^2+2\left(\frac{t_i-u}{b^3}\right)\right]\left(\frac{t_i-u}{b}\right)+\left[\left(\frac{t_i-u}{b^2}\right)^2+2\left(\frac{t_i-u}{b^3}\right)\right]\frac{1}{2!}\left(\frac{t_i-u}{b}\right)^2+\cdots\right]$$

$$=\sum_{i=1}^{n_u}\left[\left(\frac{t_i-u}{b^2}\right)^2+\left[\left(\frac{t_i-u}{b^2}\right)^2+2\left(\frac{t_i-u}{b^3}\right)\right]\left(\frac{t_i-u}{b}\right)+\left[\left(\frac{t_i-u}{b^2}\right)^2+2\left(\frac{t_i-u}{b^3}\right)\right]\frac{1}{2!}\left(\frac{t_i-u}{b}\right)^2+\cdots\right]$$

$$=\sum_{i=1}^{n_u}\left[\left(\frac{t_i-u}{b^2}\right)^2+\left[\left(\frac{t_i-u}{b^2}\right)^2+2\left(\frac{t_i-u}{b^3}\right)\right]\left(\frac{t_i-u}{b}\right)+\left[\left(\frac{t_i-u}{b^2}\right)^2+2\left(\frac{t_i-u}{b^3}\right)\right]\frac{1}{2!}\left(\frac{t_i-u}{b}\right)^2+\cdots\right]$$

$$=\sum_{i=1}^{n_u}\left(\frac{t_i-u}{b^2}\right)^2+\sum_{i=1}^{n_u}\left[\left(\frac{t_i-u}{b^2}\right)^2+2\left(\frac{t_i-u}{b^3}\right)\right]\left\{\left(\frac{t_i-u}{b}\right)+\frac{1}{2!}\left(\frac{t_i-u}{b}\right)^2+\frac{1}{3!}\left(\frac{t_i-u}{b}\right)^3+\cdots\right\}.$$

Similar si9mplification of the third and fourth parts yield:

$$E\left[\frac{-\partial^2 l}{\partial b^2}\right]=E\{\sum_{i=1}^{n_u}\left(\frac{t_i-u}{b^2}\right)^2+\left(n\bar{\pi}-n_u\right)\exp\left(\frac{\tau_0-u}{b}\right)\left[\left(\frac{\tau_0-u}{b^2}\right)^2+2\left(\frac{\tau_0-u}{b^3}\right)\right]$$

$$+\sum_{i=1}^{n_u}\left[\left(\frac{t_i-u}{b^2}\right)^2+2\left(\frac{t_i-u}{b^3}\right)\right]\left\{\left(\frac{t_i-u}{b}\right)+\frac{1}{2!}\left(\frac{t_i-u}{b}\right)^2+\frac{1}{3!}\left(\frac{t_i-u}{b}\right)^3+\cdots\right\}$$

$$+\sum_{j=1}^{n_a}\left(\frac{t_j'-u'}{b^2}\right)^2+\left(n\pi-n_a\right)\exp\left(\frac{\tau_0-u'}{b}\right)\left[\left(\frac{\tau_0-u'}{b^2}\right)^2+2\left(\frac{\tau_0-u'}{b^3}\right)\right]$$

$$+\sum_{j=1}^{n_a}\left[\left(\frac{t_j'-u'}{b^2}\right)^2+2\left(\frac{t_j'-u'}{b^3}\right)\right]\left\{\left(\frac{t_j'-u'}{b}\right)+\frac{1}{2!}\left(\frac{t_j'-u'}{b}\right)^2+\frac{1}{3!}\left(\frac{t_j'-u'}{b}\right)^3+\cdots\right\}\}>0,$$

$$E\left[\frac{-\partial^2 l}{\partial u\partial b}\right]=E\{\frac{-n_u-n_a}{b^2}+\frac{1}{b}\sum_{i=1}^{n_u}\left(\frac{t_i-u}{b^2}\right)\exp\left(\frac{t_i-u}{b}\right)+\frac{1}{b^2}\sum_{i=1}^{n_u}\exp\left(\frac{t_i-u}{b}\right)$$

$$+\frac{1}{b}\sum_{j=1}^{n_a}\left(\frac{t_j'-u'}{b^2}\right)\exp\left(\frac{t_j'-u'}{b}\right)+\frac{1}{b^2}\sum_{j=1}^{n_a}\exp\left(\frac{t_j'-u'}{b}\right)$$

$$+\frac{\left(n\bar{\pi}-n_u\right)}{b}\left(\frac{\tau_0-u}{b^2}\right)\exp\left(\frac{\tau_0-u}{b}\right)+\frac{\left(n\bar{\pi}-n_u\right)}{b^2}\exp\left(\frac{\tau_0-u}{b}\right)$$

$$+\frac{\left(n\pi-n_a\right)}{b}\left(\frac{\tau_0-u'}{b}\right)\exp\left(\frac{\tau_0-u'}{b}\right)+\frac{\left(n\pi-n_a\right)}{b^2}\exp\left(\frac{\tau_0-u'}{b}\right)\}.$$

Expanding the second and forth parts using the Taylor series we obtain,

$$\frac{1}{b^2}\sum_{i=1}^{n_u}\exp\left(\frac{t_i-u}{b}\right)=\frac{1}{b^2}\sum_{i=1}^{n_u}\left\{1+\left(\frac{t_i-u}{b}\right)+\frac{1}{2!}\left(\frac{t_i-u}{b}\right)^2+\cdots\right\}$$

$$=\left\{\frac{n_u}{b^2}+\sum_{i=1}^{n_u}\left(\frac{t_i-u}{b}\right)+\frac{1}{2!}\sum_{i=1}^{n_u}\left(\frac{t_i-u}{b}\right)^2+\cdots\right\},$$

$$\frac{1}{b^2}\sum_{j=1}^{n_a}\exp\left(\frac{t'_j-u'}{b}\right)=\frac{1}{b^2}\sum_{j=1}^{n_a}\left\{1+\left(\frac{t'_j-u'}{b}\right)+\frac{1}{2!}\left(\frac{t'_j-u'}{b}\right)^2+\cdots\right\}$$

$$=\left\{\frac{n_a}{b^2}+\sum_{j=1}^{n_a}\left(\frac{t'_j-u'}{b}\right)+\frac{1}{2!}\sum_{j=1}^{n_a}\left(\frac{t'_j-u'}{b}\right)^2+\cdots\right\},$$

and

$$E\left[\frac{-\partial^2 l}{\partial u\partial b}\right]=E\{\frac{1}{b}\sum_{i=1}^{n_u}\left(\frac{t_i-u}{b^2}\right)\exp\left(\frac{t_i-u}{b}\right)-\frac{n_u}{b^2}+\frac{n_u}{b^2}+\sum_{i=1}^{n_u}\left(\frac{t_i-u}{b}\right)+\frac{1}{2!}\sum_{i=1}^{n_u}\left(\frac{t_i-u}{b}\right)^2+\cdots$$

$$+\frac{1}{b}\sum_{j=1}^{n_a}\left(\frac{t'_j-u'}{b^2}\right)\exp\left(\frac{t'_j-u'}{b}\right)-\frac{n_a}{b^2}+\frac{n_a}{b^2}+\sum_{j=1}^{n_a}\left(\frac{t'_j-u'}{b}\right)+\frac{1}{2!}\sum_{j=1}^{n_a}\left(\frac{t'_j-u'}{b}\right)^2+\cdots$$

$$+\frac{\left(n\bar{\pi}-n_u\right)}{b}\left(\frac{\tau_0-u}{b^2}\right)\exp\left(\frac{\tau_0-u}{b}\right)+\frac{\left(n\bar{\pi}-n_u\right)}{b^2}\exp\left(\frac{\tau_0-u}{b}\right)$$

$$+\frac{\left(n\pi-n_a\right)}{b}\left(\frac{\tau_0-u'}{b}\right)\exp\left(\frac{\tau_0-u'}{b}\right)+\frac{\left(n\pi-n_a\right)}{b^2}\exp\left(\frac{\tau_0-u'}{b}\right)\},$$

Thus, $$E\left[\frac{-\partial^2 l}{\partial u\partial b}\right]=E\{\frac{1}{b}\sum_{i=1}^{n_u}\left(\frac{t_i-u}{b^2}\right)\exp\left(\frac{t_i-u}{b}\right)+\sum_{i=1}^{n_u}\left(\frac{t_i-u}{b}\right)+\frac{1}{2!}\sum_{i=1}^{n_u}\left(\frac{t_i-u}{b}\right)^2+\cdots$$

$$+\frac{1}{b}\sum_{j=1}^{n_a}\left(\frac{t'_j-u'}{b^2}\right)\exp\left(\frac{t'_j-u'}{b}\right)+\sum_{j=1}^{n_a}\left(\frac{t'_j-u'}{b}\right)+\frac{1}{2!}\sum_{j=1}^{n_a}\left(\frac{t'_j-u'}{b}\right)^2+\cdots$$

$$+\frac{\left(n\bar{\pi}-n_u\right)}{b}\left(\frac{\tau_0-u}{b^2}\right)\exp\left(\frac{\tau_0-u}{b}\right)+\frac{\left(n\bar{\pi}-n_u\right)}{b^2}\exp\left(\frac{\tau_0-u}{b}\right)$$

$$+\frac{\left(n\pi-n_a\right)}{b}\left(\frac{\tau_0-u'}{b}\right)\exp\left(\frac{\tau_0-u'}{b}\right)+\frac{\left(n\pi-n_a\right)}{b^2}\exp\left(\frac{\tau_0-u'}{b}\right)\}>0.$$

Thus,

$$E\left[\frac{\partial^2 l}{\partial b\partial u'}\right]=E\left[\frac{-n_a}{b^2}+2\sum_{j=1}^{n_a}\left(\frac{t'_j-u'}{b^3}\right)\exp\left(\frac{t'_j-u'}{b}\right)+2\left(n\pi-n_a\right)\left(\frac{\tau_0-u'}{b^3}\right)\exp\left(\frac{\tau_0-u'}{b}\right)\right]$$

because $$2\left(n\pi - n_a\right)\left(\frac{\tau_0 - u'}{b^3}\right)\exp\left(\frac{\tau_0 - u'}{b}\right) = 2\left(n\pi - n_a\right)\left(\frac{\tau_0 - u'}{b^3}\right)$$
$$\times\left\{1 + \left(\frac{\tau_0 - u'}{b}\right) + \frac{1}{2!}\left(\frac{\tau_0 - u'}{b}\right)^2 + \cdots\right\}$$
$$= \frac{2\left(n\pi - n_a\right)}{b^2}\left(\frac{\tau_0 - u'}{b}\right) + \frac{\left(n\pi - n_a\right)}{b^2}\left(\frac{\tau_0 - u'}{b}\right)^2 + \cdots$$

Because $\frac{2\left(n\pi - n_a\right)}{b^2} > \frac{n_a}{b^2}$ and $\frac{\tau_0 - u'}{b} > 0$ we obtain,

$$E\left[\frac{\partial^2 l}{\partial b \partial u'}\right] = E\{\frac{-n_a}{b^2} + 2\sum_{j=1}^{n_a}\left(\frac{t'_j - u'}{b^3}\right)\exp\left(\frac{t'_j - u'}{b}\right)$$
$$+ \frac{2\left(n\pi - n_a\right)}{b^2}\left(\frac{\tau_0 - u'}{b}\right) + \frac{\left(n\pi - n_a\right)}{b^2}\left(\frac{\tau_0 - u'}{b}\right)^2 + \cdots\} > 0.$$

This completes the proof of Result 2.1.

BIBLIOGRAPHY

[1] Abdel-Hamid, A. (2009). "Constant-partially accelerated life tests for Burr type-XII distribution with progressive type-II censoring," Computational Statistics & Data Analysis 53(7), 2511-2523.

[2] Alferink, A. and Samaranayake, V. A. (2011). "Lifetime Predictive Density Estimation in Accelerated Degradation Testing for Lognormal Response Distributions with Arrhenius Rate Relationship". *JSM Proceedings, Quality and Productivity Section*. Alexandria, VA: American Statistical Association, 4373-4385.

[3] Bain, L. J. (1976*). Statistical Analysis of Reliability and Life Testing Model*. Marcel and Dekker Inc., New York.

[4] Engelhardt, M. (1975). "On Simple Estimation of Parameters of the Weibull or Extreme-Value Distribution", *Technometrics*, 17(3), 369-374.

[5] Escobar, L.A. and Meeker, W.Q. (2000). The Asymptotic Equivalence of the Fisher Information matrices for Type I and Type II Censored data from Location-Scale Families. *Statistics Preprints.* Paper 8, http://lib.dr.iastate.edu/stat_las_preprints/8.

[6] Gupta, R. D. and Kundu, D. (1999). Generalized exponential distribution. *Australian and New Zealand Journal of Statistics*, 41(2), 173-188.

[7] Gupta, R. D. and Kundu, D. (2001a). Exponentiated exponential distribution, an alternative to Gamma and Weibull distributions. *Biometrical Journal*, 43(1), 117-130.

[8] Gupta, R. D. and Kundu, D. (2001b). Generalized exponential distributions: different methods of estimation. *Journal of Statistical Computation and Simulation*, 69(4), 315-338.

[9] Mann, N. R., Schafer, R. E., and Singapurwalla, N. D. (1974). *Methods for Statistical Analysis of Reliability and Life Data*, Wiley, New York.

[10] Meeker, W. Q. and Escobar, L. A. (1998). *Statistical Methods for Reliability Data*. Wiley, New York.

[11] Nelson, W. (1980) "Accelerated Life Testing-Step-Stress Models and Data Analyses," *IEEE Transactions on Reliability*, R-29(2), 103-108.

[12] Nelson, W. (1990). *Accelerated Testing: Statistical Models, Test Plans and Data Analyses.* Wiley, New York.

[13] Ismail, A. A. (2013). “Estimating The Generalized Exponential Distribution Parameters and The Acceleration Factor Under Constant-Stress Partially Accelerated Life Testing With Type-II Censoring,” *Strength of Materials*, 45(6), 693-702.

[14] Jayawardhana, A. A .and Samaranayake, V. A. (2003). “Prediction Bounds in Accelerated Life Testing: Weibull Models with Inverse Power Relationship” *Journal of Quality Technology*, 35(1), 89-103.

[15] Jayawardhana, A. A. and Samaranayake, V. A. (2014). “Predictive Density Estimation in Accelerated Life Testing for Lognormal Life Distributions,” in *JSM Proceedings, Quality and Productivity Section*. Alexandria, VA: American Statistical Association. 2325-2338.

[16] Lawless, J.F. (1982). *Statistical models and methods for lifetime data*, Wiley, New York

[17] Ragab, M. Z. and Ahsanullah, M. (2001). “Estimation of the location and parameters of the generalized exponential distribution based on order statistics,” *Journal of Statistical Computation and Simulation*, 69, 109-124.

[18] Tobias, P. A. and Trindade, D. (2011). *Applied Reliability*, Third Edition, CRC Press, New York.

[19] Viertl, R. (1988). *Statistical Methods in Accelerated Life Testing*, Vandenhoeck & Ruprecht, Göttingen, Germany.

[20] M. Kamal, S. Z. and Islam, A. (2013). “Constant Stress Partially Accelerated Life Test Design for Inverted Weibull Distribution with Type-I Censoring” *Algorithms Research* 2(2): 43-49.

[21] Saxena, S. and Zarrin, S. (2013). “Estimation of Partially Accelerated Life Tests for the Extreme Value Type-III Distribution with Type-I Censoring,” *International Journal of Probability and Statistics*, 2(1): 1-8.

[22] Preeti Wanti Srivastava,. Neha Mittal, (2010). “Optimum step-stress partially accelerated life tests for the truncated logistic distribution with censoring” *Applied Mathematical Modelling* 34, 3166–3178

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