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Holomorphic extensions in toric varieties

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HOLOMORPHIC EXTENSIONS IN TORIC VARIETIES

by

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ABSTRACT

The dissertation describes the Hartogs and the Hartogs-Bochner extension phenomena in smooth toric varieties and their connection with the first cohomology group with compact support and sheaf coefficients. The affirmative and negative results are proved for toric surfaces and for line bundles over toric varieties using topological, analytic, and algebraic methods.

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1 INTRODUCTION

This dissertation treats mainly classical complex analysis problems considered for toric varieties. These problems were originally posed and solved on $\mathbb{C}^n$ for $n \geq 2$; and they are very difficult if considered for an arbitrary complex manifold. Manifolds equipped with additional structure, might be a good place to attack these problems, but two fundamental questions remain. What kind of structure would permit solution of the problems? What kind of methods to be applied? Well known examples are manifolds, which allow a vector bundle structure [10], or manifolds with a foliation [32]. This paper works with Hartogs and Hartogs-Bochner phenomena on toric varieties, which actually allow neither structure, but the ideas are taken directly from the types of manifolds mentioned above. Although perhaps not visible in all proofs, the geometric intuition on toric surfaces comes from the existence or nonexistence of some families of curves that play foliation-like role, except that there is a reducible curve among them. This foliation-like geometrical image exhibits the global structure of a surface which is necessary when considering compact sets or the Hartogs-Bochner phenomena on a manifold. The main problem with compact sets, which appear in the Hartogs phenomena, is that a compact set cut into a particular coordinate patch does not have to remain compact. Therefore, a global view of compact sets is absolutely necessary for this problem. Similarly, the manifolds or domains, which do not allow the Hartogs-Bochner phenomena contain a large collection of projective curves, which makes the sheaf of germs of holomorphic functions relatively small.

1.1 OVERVIEW

Section 2 provides a short introduction to toric varieties. The definition is purely combinatorial and is related to a polygonal object in an Euclidean space, called a fan. An affine toric variety X_σ is defined by the convex hull of a finite number of vectors, called a cone:

$$\sigma = v_1\mathbb{R}_{\geq 0} + \ldots + v_k\mathbb{R}_{\geq 0}.$$

It is required that σ is strictly convex, i.e. does not contain a line. A toric variety X_Σ is defined by a finite collection of cones, called a fan:

$$\Sigma = \{\sigma_1, \ldots, \sigma_d\}.$$

The support of the fan in particular is defined as $|\Sigma| = \bigcup_{\sigma\in\Sigma} \sigma$. Fans with strictly convex support (i.e., those that are convex and do not contain a line) are particularly important in this research. It is this definition which makes toric varieties so attractive. Many well known manifolds, like $\mathbb{C}^n$, $\mathbb{P}^n$, or $(\mathbb{C}^*)^n$, as well as their products, are toric varieties. The product of manifolds or even fiber bundles can be expressed in terms of fans. All problems considered in the following chapters have answers expressed in terms of fans. The last subsection in Section 2 contains some information about ends, which connect the topological properties of toric surfaces with the Hartogs phenomena. The following result will be important for the first cohomology group with compact support:

Theorem 1.1.1 ***(Example 2.3.10)*** *A toric surface X_Σ with a strictly convex fan has exactly one end.*

In fact, much material that could be mentioned in connection with ends is omitted. The ends are are closely related to compactification problems, the theory of cobordism, complements of varieties, plurisubharmonic functions, etc. All these problems can be successfully considered for toric varieties, which makes an excellent subject for future research.

For a compact subset K of a complex manifold X we consider an arbitrary function f analytic on the connected set $X \setminus K$. If each such f has a holomorphic extension to all of X, then we say that the Hartogs phenomenon holds in X. The Hartogs phenomenon is considered in sections 3 and 5, starting with the following result.

Theorem 1.1.2 ***(Theorem 3.0.8)*** *Let $f(z,w)$ be a holomorphic function on $\mathbb{C}^2 \setminus V$, where V is defined as follows:*

$$V = \{(z,w) \in \mathbb{C}^2 : \left|z^\beta w^\alpha\right| \leq M, |w| \leq N\},$$

where $\alpha, \beta \in \mathbb{Z}_{\geq 1}$ and $M, N \in \mathbb{R}_{>0}$. Then f has holomorphic extension to $\mathbb{C}^1 \times \mathbb{C}^*$.

The main result from section 3 is:

Theorem 1.1.3 ***(Theorem 3.2.2)*** *If X_Σ is a smooth toric surface with a strictly convex fan, then the Hartogs phenomenon holds in X_Σ.*

On the other hand, examples are provided to show that the Hartogs phenomenon does not hold if the fan of X does not fulfill the required assumption. The proof of Theorem 3.2.2 uses the idea of the ends of a topological space. Ends describe the "holes" at infinity and are in some sense separated from compact sets. Using a comparison with a well known situation, any compact set in $\mathbb{C}^n$ is enclosed in a polydisc with finite radii (or a ball), which means that it does not "meet" infinity.

Section 4 analyzes the properties of the fans associated with fiber bundles. In particular the product of fans is defined:

Definition 1.1.1 ***(Product of fans)*** *Let (Δ_1, N_1) and (Δ_2, N_2) be fans. If*

$$\sigma = \mathbb{R}_{\geq 0} + ... + v_k\mathbb{R}_{\geq 0} \qquad \textit{and} \qquad \tau = w_1\mathbb{R}_{\geq 0} + ... + w_n\mathbb{R}_{\geq 0},$$

then the fan $(\Delta_1 \times \Delta_2, N_1 \times N_2)$, which is their product, is defined by

$$\Delta_1 \times \Delta_2 = \{\sigma \times \tau : \sigma \in \Delta_1, \tau \in \Delta_2\},$$

where $\sigma \times \tau = v_1\mathbb{R}_{\geq 0} + ... + v_k\mathbb{R}_{\geq 0} + w_1\mathbb{R}_{\geq 0} + ... + v_n\mathbb{R}_{\geq 0}$.

We have the following result:

Theorem 1.1.4 ***([14], Exercise, p. 22)*** *Let (Σ, N) be a fan associated with a toric variety X, (Δ, N'') with F, and (Π, N') with B. Then X is a product of the toric varieties F and B if and only if $(\Sigma, N) = (\Delta \times \Pi, N'' \times N')$.*

Similarly, the sum of fans can be defined as follows:

Definition 1.1.2 *(**Sum of fans**) Let (Δ_1, N) and (Δ_2, N) be fans such that $\Delta_1 \cap \Delta_2 = \{0\}$. Then the fan $(\Delta_1 + \Delta_2, N)$, which is their sum, is defined by*

$$\Delta_1 + \Delta_2 = \{\sigma + \tau : \sigma \in \Delta_1, \tau \in \Delta_2\}$$

where $\sigma + \tau = v_1\mathbb{R}_{\geq 0} + ... + v_k\mathbb{R}_{\geq 0} + w_1\mathbb{R}_{\geq 0} + ... + w_n\mathbb{R}_{\geq 0}$ for $\sigma = v_1\mathbb{R}_{\geq 0} + ... + v_k\mathbb{R}_{\geq 0}$ and $\tau = w_1\mathbb{R}_{\geq 0} + ... + w_n\mathbb{R}_{\geq 0}$.

The following theorem, which characterizes toric varieties with fiber bundle structure is proved at the end of Section 4.

Theorem 1.1.5 ***(Theorem 4.2.3)*** *Let (Σ, N) be a fan associated with a toric variety X and (Δ, N'') with a toric variety F, where Δ is a subfan of Σ and N'' is a sublattice of N. Then X is a fiber bundle with fiber F if and only if there exists such a subfan Π' in Σ that $\Sigma = \Delta + \Pi'$ exists and $N'' + \Pi'$ exists.*

Section 5 formulates a simplified version of this theorem for line bundles:

Theorem 1.1.6 ***(Theorem 5.0.4)*** *Let (Σ, N) be a fan associated with a toric variety X. Then X is a line bundle if and only if there exists a subfan Π' of Σ and $v\mathbb{R}_{\geq 0} \in \Sigma$, such that $\Sigma = v\mathbb{R}_{\geq 0} + \Pi'$ and $N'' + \Pi'$ exist, where N'' is a sublattice in N generated by v.*

A detailed description of the fan Σ offered in Section 4 allows the following result in Section 5:

Theorem 1.1.7 ***(Theorem 5.4.2)*** *Let X_Σ be a line bundle with a compact base. If $|\Sigma|$ is strictly convex, then X_Σ allows the Hartogs phenomenon.*

Section 5 contains the following result for holomorphic extension in $\mathbb{C}^n$ for $n \geq 2$:

Theorem 1.1.8 ***(Theorem 5.4.1)*** *Let $f(z_1, \ldots, z_{n-1}, w)$ be a holomorphic function on $\mathbb{C}^n \setminus V$ for V defined as:*

$$V = \{(z_1, \ldots, z_{n-1}, w) \in \mathbb{C}^n : \left|z_1^\beta w\right| \leq M, |w| \leq N\},$$

where $\beta \in \mathbb{Z}_{>0}$ and $M, N \in \mathbb{R}_{>0}$. Then f has a holomorphic continuation to $\mathbb{C}^{n-1} \times \mathbb{C}^*$.

Section 6 contains a short introduction to Cauchy-Riemann (CR) theory. It contains definitions of CR manifolds and CR functions. Roughly speaking, a CR manifold is a real submanifold M in a complex manifold X for which the dimension of the complex tangent space T_pM at each point $p \in M$ is independent of p. A CR function is a differentiable function that fulfills the tangential CR equations. Both CR manifolds and CR functions play a key role in a natural question about holomorphic extensions of CR functions from CR submanifolds. Cauchy-Riemann (sub)manifolds and CR functions turn out to be a reasonable class to consider in this problem. Section 6 also provides a definition of cohomology groups with compact support. This cohomology theory seems to be more natural for complex manifolds than other theories.

Let U be a domain (i.e., an open, connected, relatively compact set with a smooth boundary) in a complex manifold X, and let f be an arbitrary smooth CR function on ∂U. If each such f can be extended holomorphically to U, then we say that the Hartogs-Bochner phenomenon holds for U. If it holds for any U in X, then we say that the Hartogs-Bochner phenomenon holds in X. For a compact manifold X, a real hypersurface M is the boundary of two domains. In this case, we say that the Hartogs-Bochner phenomenon holds in X if for any smooth real hypersurface M any CR function on M can be holomorphically extended to either one side of M. The very first result in Section 6 requires some work with cones and fans in the plane. The subsections which appear before the result, are actually the steps of the proof of the classification theorem for smooth toric surfaces. Unfortunately, neither the classification itself nor the σ-process used in it, helps with the problem. We obtain the following theorem:

Theorem 1.1.9 ***(Theorem 6.4.1)*** *For every smooth, compact toric surface X_Σ, with Σ such that $\Sigma(1)$, the subfan of dimension 1, consists of four or more cones, there exists a compact, connected, hypersurface M and a CR function on M that does not have a holomorphic extension on either side of M.*

This theorem indicates that the Hartogs-Bochner phenomenon cannot be obtained in smooth compact toric surfaces. However, an affirmative answer for some domains in those surfaces can be obtained. For this theorem, the irreducible projective curves $D_1, \ldots, D_k$ are associated with the cones $v_1\mathbb{R}_{\geq 0}, \ldots, v_k\mathbb{R}_{\geq 0} \in \Sigma(1)$, and the subfan $\widetilde{\Sigma}$ of Σ is generated by

$$\widetilde{\Sigma} = \{0, v_0\mathbb{R}_{\geq 0}, \ldots, v_{k+1}\mathbb{R}_{\geq 0}, v_0\mathbb{R}_{\geq 0} + v_1\mathbb{R}_{\geq 0}, \ldots, v_k\mathbb{R}_{\geq 0} + v_{k+1}\mathbb{R}_{\geq 0}\}.$$

The following remark from the Hartogs phenomenon will be crucial for the Hartogs-Bochner phenomenon in compact toric surfaces:

Theorem 1.1.10 ***(Theorem 6.2.2)*** *If X_Σ is a smooth toric surface with a strictly convex fan, then $H^1_c(X_\Sigma, \mathscr{O}) = 0$ and the Hartogs-Bochner phenomenon holds in X_Σ.*

Now, the following theorem can be formulated:

Theorem 1.1.11 ***(Theorem 6.6.1)*** *Let U be a domain that contains a connected, reducible curve $C = D_1 \cup \ldots \cup D_k$, where $D_1, \ldots, D_k$ are projective curves defined by the vectors $v_1\mathbb{R}_{\geq 0}, \ldots, v_k\mathbb{R}_{\geq 0} \in \Sigma(1)$. Then:*

(i) if $\left|\widetilde{\Sigma}\right|$ *covers at least a half plane, then the Hartogs-Bochner phenomenon does not hold in* U*;*

(ii) if $\left|\widetilde{\Sigma}\right|$ *covers less than a half plane and* $\overline{U}$ *does not meet any projective curves associated with any other one-dimensional cones from* Σ*, then the Hartogs-Bochner phenomenon holds in* U*.*

The important assumption in (ii) is that $\overline{U}$ does not meet any other projective curves of this type which implies that $\overline{U} \subset X_{\widetilde{\Sigma}}$. Now we are able to use the first cohomology group with compact support to prove part (ii) of the theorem. Still interesting is part (i), where we use a family of projective curves inside U to prove that there are not enough holomorphic functions in $\mathscr{O}(U)$ to extend any CR function from the boundary of U.

The following remark can be obtained from the Hartogs phenomenon on line bundles with strictly convex fan:

Theorem 1.1.12 ***(Theorem 6.7.2)*** *Let* X_Σ *be a line bundle with a compact base and strictly convex fan. Then* $H^1_c(X_\Sigma, \mathscr{O}) = 0$*, and the Hartogs-Bochner phenomenon holds in* X_Σ*.*

As the most elegant, the first cohomology group with compact support, was chosen to be a bridge between the topology and the analytic properties of toric surfaces. One could just, as well choose the first Chern class [6], the generalized σ-process in the sense of Grauert [16], the Levi form [24], the existence of a convex system of neighborhoods or convex plurisubharmonic functions. This is, of course, a matter of choice and taste but, it is worthwhile to notice that the cohomology theory applied in the reducible case mentioned above appears with the same difficulty as in the irreducible case.

This work may appear not to get involved much in the details of CR manifolds and CR functions. But the following example, computed by hand, is the most important in the paper and encourage the right geometric intuitive response about the projective curves in a domain U.

Example 1.1.1 Let $M \subset \mathbb{P}^1 \times \mathbb{P}^1$ be a real manifold described by the equations $|w_0| = |w_1|$ in the coordinates $(z, w) = (z_0, z_1, w_0, w_1) \in \mathbb{P}^1 \times \mathbb{P}^1$. Then M is a cylinder over the unit circle with fiber $\mathbb{P}^1$. At each point on M, the tangential complex direction is simply along the other copy of $\mathbb{P}^1$. Moreover, M is a hypersurface in $\mathbb{P}^1 \times \mathbb{P}^1$ and thus, it is a CR manifold with $\dim_{\mathbb{C}} H_p M = 1$. Consider the function $f : M \to \mathbb{C}$ defined as $f(z, w) = \frac{w_0}{w_1} + \frac{w_1}{w_0}$. Then the derivative of f with respect to z is 0; therefore, f is CR on M. This function does not have a holomorphic extension to either side of the unit circle in $\mathbb{P}^1$, thus it has no holomorphic extension to either side of M. This example is particularly interesting because it works similarly in Hirzebruch surfaces and in all compact smooth toric surfaces.

1.2 MAIN RESULTS OF THE DISSERTATION

To summarize, the most important results of this work comprise three topic groups:

(i) the Hartogs phenomenon: Theorem 3.2.2 and 5.4.2,

(ii) the first cohomology group with compact support: Theorem 6.2.2 and 6.7.2,

(iii) the Hartogs-Bochner phenomenon: Theorem 6.2.2, 6.6.1, 6.4.1 and 6.7.2.

The results in Theorem 3.0.8 and 5.4.1 are interesting versions of the Hartogs figure in $\mathbb{C}^n$ for $n \geq 2$. These theorems are used to obtain the Hartogs phenomenon for toric varieties, but they could form an interesting topic for independent investigation.

1.3 FURTHER RESEARCH

Further research will continue the flow of extension problems in fiber bundles and smooth toric varieties of an arbitrary dimension. The following conjecture can be formulated for a future project related to the Hartogs phenomenon:

Conjecture 1.3.1 *Let X be a smooth toric variety. If the complement of its fan contains at least one connected component, which is concave, then the Hartogs phenomenon holds in X.*

Similarly, a conjecture related to the first compactly supported cohomology group and the Hartogs-Bochner phenomenon can be formulated as follows:

Conjecture 1.3.2 *Let X be a smooth toric variety. If the complement of its fan has one connected component, which is concave, then $H_c^1(X, \mathcal{O}) = 0$ and the Hartogs-Bochner phenomenon holds in X.*

2 BASICS OF TORIC VARIETIES

The theory of toric varieties was introduced in the early 1970s and since that time has progressed far; today it is still active, providing a basis for fresh ideas. Toric varieties give rise to interesting applications with their rich structure and relatively easy combinatorics. However, toric varieties are normal, rational, and not necessarily projective, which makes them good candidates for examples or counter-examples in a wider class of varieties. Examples of their benefits are:

- Toric varieties are trivial from the minimal model theory point of view; however, they offer an excellent means to explain its main ideas.

- Fano toric varieties are easier to handle, but they are still an interesting subclass of Fano varieties.

- A pair of mirror Calabi-Yau threefolds can be constructed using "reflexive" polytopes.

- Other benefits are related to combinatorial geometry, error-correcting codes, Gromov-Witten invariants, Lagrangian torus fibrations, symplectic geometry, etc.

2.1 AFFINE TORIC VARIETIES

Affine toric varieties play basically the same role for toric varieties as open subsets of $\mathbb{C}^n$ (or $\mathbb{R}^n$) for analytic (real) varieties. An affine toric variety can be associated with a cone.

2.1.1. Lattices and Cones. Consider an r-dimensional lattice N that can be identified with $\mathbb{Z}^r$, and let $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ be its dual lattice. We can define scalar extensions of N and M as: $N_{\mathbb{R}} = N \otimes_{\mathbb{R}} \mathbb{R}$ and $M_{\mathbb{R}} = M \otimes_{\mathbb{R}} \mathbb{R}$. Figure 2.1 shows an example of a cone. Now, the definition of a cone can be provided.

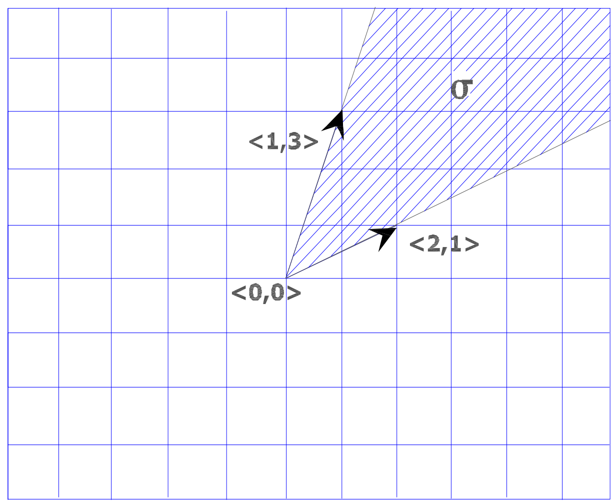

Figure 2.1: The cone $\sigma = (2e_1 + e_2)\mathbb{R}_{\geq 0} + (e_1 + 3e_2)\mathbb{R}_{\geq 0} \subset \mathbb{R}^2$

Definition 2.1.1 *(**Rational polyhedral cones**) A subset $\sigma \subset N_{\mathbb{R}}$ is a rational polyhedral cone with apex at the origin 0 if there are $a_1, ..., a_s \in N$ such that*

$$\sigma = a_1\mathbb{R}_{\geq 0} + ... + a_s\mathbb{R}_{\geq 0} = \{a_1t_1 + ... + a_st_s : \forall_{1\leq j\leq s}\, t_j \in \mathbb{R}_{\geq 0}\},$$

where $\mathbb{R}_{\geq 0}$ is a set of nonnegative real numbers. A cone σ is strictly convex if it is convex as a subset of $N_{\mathbb{R}}$ and does not contain a straight line.

If a point $p \in \sigma$ has a representation $p = a_1 t_1 + \ldots + a_s t_s$ and all $t_j > 0$, for $1 \leq j \leq s$, then p belongs to the relative interior of σ.

In further discussions, except where clarification is needed, this paper will refer to strictly convex rational polyhedral cones simply as cones. It is important to imagine how cones look. The origin $\{0\} \subset N_{\mathbb{R}}$ is a cone. It can be represented as $\sigma = 0\mathbb{R}_{\geq 0}$ and in further discussion the cone $0\mathbb{R}_{\geq 0}$ will be denoted as 0. Since the lattice M is dual to N, there is a dual product denoted as $(,) : N \times M \longrightarrow \mathbb{Z}$.

Definition 2.1.2 *(**Dual cones**) For any cone $\sigma \subset N_{\mathbb{R}}$, we can define its dual cone as $\sigma^{\vee} \subset M_{\mathbb{R}}$: $\sigma^{\vee} = \{u \in M_{\mathbb{R}} : \forall_{v \in \sigma} (v, u) \geq 0\}$.*

Figure 2.2 shows an example of a cone and its dual.

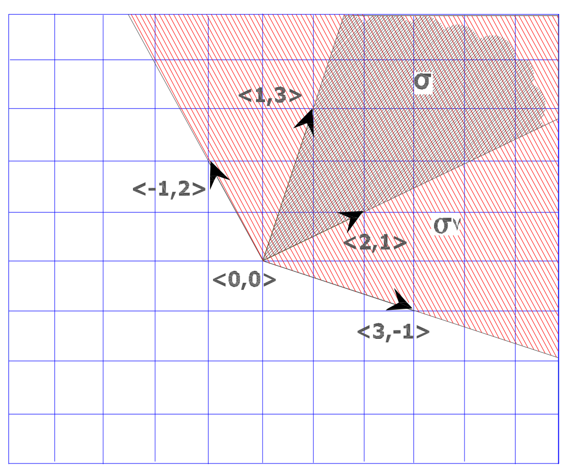

Figure 2.2: A cone and its dual

Example 2.1.1 Consider the cone $\sigma = (2e_1 + e_2)\mathbb{R}_{\geq 0} + (e_1 + 3e_2)\mathbb{R}_{\geq 0} \subset \mathbb{R}^2$. Then its dual is $\sigma^\vee = (3e_1^* - e_2^*)\mathbb{R}_{\geq 0} + (-e_1^* + 2e_2^*)\mathbb{R}_{\geq 0} \subset \mathbb{R}^2$. ▮

Notice that if $\sigma \subset N_\mathbb{R}$ is a strictly convex rational polyhedral cone with apex at the origin then $\sigma^\vee \subset M_\mathbb{R}$ is a rational polyhedral cone with apex at the origin, but it is not necessarily strictly convex. As an example, consider the zero cone $0 \subset N_\mathbb{R}$. Then $(0)^\vee = \{u \in M_\mathbb{R} : \forall_{v \in 0}\, (v, u) \geq 0\} = \{u \in M_\mathbb{R} : (u, 0) \geq 0\} = M_\mathbb{R}$.

2.1.2. Semigroups and Gordan's Lemma. The dual cone $\sigma^\vee$ allows us to define a semigroup $S_\sigma = \sigma^\vee \cap M$ associated with cone σ. The semigroup S_σ is, in fact, finitely generated, which is a key condition in the theory of toric varieties. Consider the following lemma:

Lemma 2.1.1 *(Gordan's lemma)* ***([1], Lec. 1, Prop. 5.4)*** *If σ is a rational polyhedral cone, then S_σ is a finitely generated additive semigroup, i.e., there exists $m_1, \ldots, m_t \in S_\sigma$ so that*

$$S_\sigma = m_1\mathbb{Z}_{\geq 0} + \ldots + m_t\mathbb{Z}_{\geq 0}. \text{ ▮}$$

Figure 2.3 shows the generators of the semigroup.

2.1.3. Semigroup Algebras and Toric Ideals. Any finitely generated semigroup S_σ defines $\mathbb{C}$-algebra $\mathbb{C}[S_\sigma]$ as follows. With an element $u \in S_\sigma$ we associate an element $\chi_u \in \mathbb{C}[S_\sigma]$, which we call a character. If $u = u_1 e_1^* + \ldots + u_n e_n^*$, and if $(t_1, \ldots, t_n)$ are local coordinates, then

$$\chi_u(t_1, \ldots, t_n) = t_1^{u_1} \ldots t_n^{u_n}$$

The algebra $\mathbb{C}[S_\sigma]$ is generated by characters $\{\chi_{u_i}\}_{i \in I}$, where $\{u_i\}_{i \in I}$ are generators of S_σ. Any element of $\mathbb{C}[S_\sigma]$ is a finite linear combination of the form $\sum_{i \in I} n_i \chi_{u_i}$, where $n_i \in \mathbb{C}$.

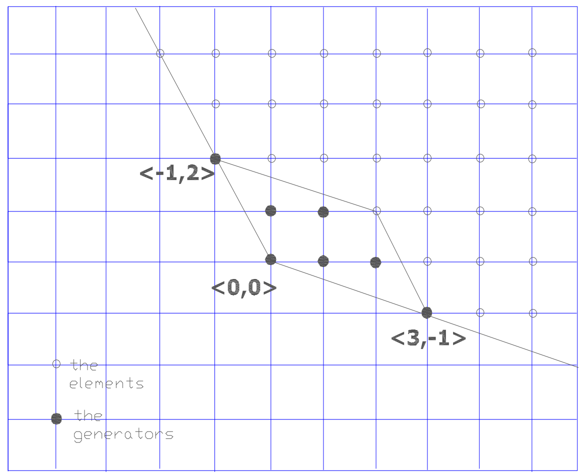

Figure 2.3: The semigroup and its generators

Notice that for any $u_1, u_2 \in S_\sigma$, we have $\chi_{u_1} \cdot \chi_{u_2} = \chi_{u_1+u_2}$. The following examples show some important cones and their algebras.

Example 2.1.2 Consider $0 \in N_{\mathbb{R}}$, where $\dim N_{\mathbb{R}} = n$. Then $0^\vee = \{u \in M_{\mathbb{R}} : (u, 0) \geq 0\} = M_{\mathbb{R}}$, so $S_0 = M$ and $\mathbb{C}[S_0] = \mathbb{C}[M] = \mathbb{C}[\mathbb{Z}^n] = \mathbb{C}[z_1, \ldots, z_n, \frac{1}{z_1 \ldots z_n}]$. Notice that the algebra $\mathbb{C}[\mathbb{Z}^n]$ can be equivalently written as $\mathbb{C}[z_1, \ldots, z_n, \frac{1}{z_1}, \ldots, \frac{1}{z_n}]$, which depends on a choice of generators of $\mathbb{Z}^n$.

Example 2.1.3 For $\sigma = \mathbb{R}_{\geq 0} \subset \mathbb{R}$, we have $\sigma^\vee = \mathbb{R}_{\geq 0}$ and $\mathbb{C}[S_\sigma] = \mathbb{C}[\mathbb{N}] = \mathbb{C}[z]$. For $\sigma = e_1\mathbb{R}_{\geq 0} + \ldots + e_n\mathbb{R}_{\geq 0} \subset \mathbb{R}^n$, we have $\sigma^\vee = e_1\mathbb{R}_{\geq 0} + \ldots + e_n\mathbb{R}_{\geq 0} \subset \mathbb{R}^n$ (where $e_1, \ldots, e_n$ is a standard basis of $\mathbb{R}^n$) and $\mathbb{C}[S_\sigma] = \mathbb{C}[\mathbb{N}^n] = \mathbb{C}[z_1, \ldots, z_n]$.∎

With any algebra $\mathbb{C}[S_\sigma]$ defined by a cone (or with any cone $\sigma \subset N = \mathbb{Z}^n$), we can associate a toric ideal $\mathcal{I}_\sigma$. As noted above, $\mathbb{C}[S_\sigma]$ is generated by characters $\{\chi^{u_i}\}_{i\in I}$, where $\{u_i\}_{i\in I}$ are generators of S_σ. Therefore, the ideal $\mathcal{I}_\sigma$ expresses relations between generators of $\mathbb{C}[S_\sigma]$. Notice that linear relations between elements from S_σ: $\sum a_i u_i = \sum b_j u_j$, where $a_i, b_j \in \mathbb{Z}_{>0}$, turn into multiplicative relations between elements of $\mathbb{C}[S_\sigma]$: $\prod \chi_{u_i}^{a_i} = \prod \chi_{u_j}^{b_j}$. On the other hand, a toric ideal $\mathcal{I}_\sigma$ is a kernel of the homomorphism $\mathbb{C}[\mathbb{N}^k] \to \mathbb{C}[S_\sigma]$, where k is a number of generators of S_σ. The next example shows how to obtain $\mathcal{I}_\sigma$ as a kernel and specifically, how to obtain it from linear relations, which are, in fact, the same thing.

Example 2.1.4 Let $\sigma^\vee = (3e_1 - 1e_2)\mathbb{R}_{\geq 0} + (-1e_1 + 2e_2)\mathbb{R}_{\geq 0} \subset \mathbb{R}^2$. Then $\mathbb{C}[S_\sigma] = \mathbb{C}[x, y, \frac{x^3}{y}, \frac{y^2}{x}]$, and the kernel of the homomorphism $\mathbb{C}[a,b,c,d] \xrightarrow{\phi} \mathbb{C}[x, y, \frac{x^3}{y}, \frac{y^2}{x}]$, which sends $a \mapsto x$, $b \mapsto y$, $c \mapsto \frac{x^3}{y}$, $d \mapsto \frac{y^2}{x}$, is generated by $cb - a^3$ and $da - b^2$. Thus $\mathcal{I}_\sigma = (cb - a^3, da - b^2)$. For linear relations from S_σ, its generators can be chosen as: $e_1^*, e_2^*, 3e_1^* - e_2^*, -e_1^* + 2e_2^*$ with relations: $(3e_1^* - e_2^*) + e_2^* = 3e_1^*$ and $(-e_1^* + 2e_2^*) + e_1^* = 2e_2^*$. Using notation $\chi_{e_1^*} = a$, $\chi_{e_2^*} = b$, $\chi_{3e_1^* - e_2^*} = c$, $\chi_{-e_1^* + 2e_2^*} = d$, we obtain the multiplicative relations $cb = a^3$ and $da = b^2$.

2.1.4. Affine Toric Varieties. From this point an affine toric variety U_σ associated with a cone σ can be defined in many equivalent ways. Most convenient in the present context is to define U_σ as a set of zeros of generators of a toric ideal $\mathcal{I}_\sigma$. This approach treats U_σ is treated as an algebraic set in $\mathbb{C}^{n_\sigma}$, where n_σ is the number of generators of the semigroup S_σ. Equivalently, points of U_σ could be identified with homomorphisms $\mathbb{C}[S_\sigma] \to \mathbb{C}$ or with maximal ideals of algebra $\mathbb{C}[S_\sigma]$. Our final object not only consists of an affine toric variety U_σ, but it is a pair $(U_\sigma, \mathbb{C}[S_\sigma])$ of an affine toric variety U_σ and its algebra of regular functions.

Definition 2.1.3 ***(Algebraic variety associated with a cone)*** *Let $\sigma \subset N_{\mathbb{R}}$ be a cone. The algebraic variety U_σ associated with σ is defined as a set of zeros of polynomials of the form*

$$\left\{\prod \chi_{u_i}^{a_i} - \prod \chi_{u_j}^{b_j}\right\}, \qquad \textit{where} \qquad \left\{\sum a_i u_i = \sum b_j u_j,\ a_i, b_j \in \mathbb{Z}_{>0}\right\}$$

are relations between the generators of the semigroup $S_\sigma = \sigma^\vee \cap M$.

Example 2.1.5 For $0 \subset N_{\mathbb{R}}$, where $\dim N_{\mathbb{R}} = n$, we have $\mathbb{C}[S_0] = \mathbb{C}[z_1, \ldots, z_n, \frac{1}{z_1 \ldots z_n}]$. Thus, $\mathcal{I}_0 = (z_1 ... z_{n+1} - 1)$ and $U_0 = (\mathbb{C}^*)^n$. Because 0 is a special cone and its affine toric variety $U_0 = (\mathbb{C}^*)^n$ plays a crucial role in the theory of toric varieties, we will use notation $U_0 = (\mathbb{C}^*)^n = T^n$ and call T^n an algebraic torus of dimension n.

Example 2.1.6 Consider $\sigma = e_1\mathbb{R}_{\geq 0} + \ldots + e_n\mathbb{R}_{\geq 0} \subset N_{\mathbb{R}}$, where $e_1, ..., e_n$ is a basis of $N_{\mathbb{R}} = \mathbb{R}^n$. Then $\sigma^\vee = \{u \in M_{\mathbb{R}} : \forall_{v\in\sigma}\, (u, v) \geq 0\} = e_1^*\mathbb{R}_{\geq 0} + \ldots + e_n^*\mathbb{R}_{\geq 0} = M_{\mathbb{R}}$, where $e_1^*, ..., e_n^*$ is a dual basis in $M_{\mathbb{R}} = \mathbb{R}^n$. Thus, $S_\sigma = \mathbb{N}^n$ and $\mathbb{C}[S_\sigma] = \mathbb{C}[\mathbb{N}^n] = \mathbb{C}[z_1, ..., z_n]$. The ideal is $\mathcal{I}_\sigma = (0)$, and we finally obtain $U_\sigma = \mathbb{C}^n$.

Example 2.1.7 Let $\sigma = e_1\mathbb{R}_{\geq 0} + ... + e_d\mathbb{R}_{\geq 0} \subset N_{\mathbb{R}}$, where $e_1, ..., e_d$ is a part of a basis of $N_{\mathbb{R}} = \mathbb{R}^n$ and $d < n$. Then $\sigma^\vee = \{u \in M_{\mathbb{R}} : \forall_{v\in\sigma}\, (u, v) \geq 0\} = e_1^*\mathbb{R}_{\geq 0} + ... + e_d^*\mathbb{R}_{\geq 0} + e_{d+1}^*\mathbb{R}_{\geq 0} + (-e_{d+1}^*)\mathbb{R}_{\geq 0} + \ldots + e_n^*\mathbb{R}_{\geq 0} + (-e_n^*)\mathbb{R}_{\geq 0} \subset M_{\mathbb{R}}$; therefore, $S_\sigma = \mathbb{N}^d \times \mathbb{Z}^{n-d}$ and $\mathbb{C}[S_\sigma] = \mathbb{C}[\mathbb{N}^d \times \mathbb{Z}^{n-d}] = \mathbb{C}[z_1, \ldots, z_n, \frac{1}{z_{d+1} \ldots z_n}]$. Then $\mathcal{I}_\sigma = (z_{d+1} \ldots z_{n+1} - 1)$ and $U_\sigma = \mathbb{C}^d \times (\mathbb{C}^*)^{n-d}$. ▮

Notice that in each of these examples, the affine toric variety U_σ is a product of other toric varieties.

2.2 GLUING AFFINE TORIC VARIETIES

This Section explains how to glue affine toric varieties along open and dense subsets which are, in fact, affine toric varieties. These subvarieties are related to faces of a cone.

2.2.1. Faces. Any face of a cone is determined by a hyperplane and a half-space. First, therefore, we recall their definitions. For $0 \neq u \in M_{\mathbb{R}}$, we define a hyperplane $H_u = \{v \in N_{\mathbb{R}} : (u, v) = 0\}$ and the half-space $H_u^+ = \{v \in N_{\mathbb{R}} : (u, v) \geq 0\}$.

Definition 2.2.1 ***(Face of a cone)*** *A subset $\tau \subset \sigma$ is a face of σ if $\tau = H_u \cap \sigma$ for some $0 \neq u \in M_{\mathbb{R}}$ and $\sigma \subset H_u^+$. We will use notation $\tau \prec \sigma$ for faces of σ.*

In the following theorems the cone σ is considered its own face. Hyperplanes and half-spaces define not only faces of a cone, but the whole cone. Obviously, the finite intersection of (closed) half-spaces is a convex polyhedral cone, but there is a much stronger result, which claims that any n-dimensional cone is an intersection of half-spaces determined by its $(n-1)$-dimensional faces:

Theorem 2.2.1 ***([1], Lec. 1, Prop. 3.4)*** *Let σ be a convex n-dimensional cone and let τ_i, $i = 1, ..., k$ be its $(n-1)$-dimensional faces, such that $\tau_i = \sigma \cap H_{u_i}$ for some collection of $u_i \in M_{\mathbb{R}}$. Then $\sigma = \bigcap_{i=1}^{k} H_{u_i}^+$.* ▮

Of course, any face is a cone itself; and σ_0 is a face of any cone. The natural questions are: Which affine toric varieties are associated with faces? And how are they related to the affine variety defined by the cone? First, we define a dual to the face τ.

Definition 2.2.2 *(**Dual faces**) Let $\tau \prec \sigma$; then the dual to τ is: $\tau^* = \{u \in \sigma^\vee : \forall_{v \in \tau}\, (u, v) = 0\}$.*

Proposition 2.2.1 ***([1], Lec. 1, Prop. 3.6)*** *If τ^* is a face of $\sigma^\vee$, then the correspondence $\tau \to \tau^*$ between faces of σ and faces of $\sigma^\vee$ is 1-1.* ▮

2.2.2. Fans and Toric Varieties. This subsection provides the definition of a fan, which is a set of cones. This definition allows us to glue affine toric varieties. Notice that the cone $\{0\}$ belongs to any fan. Figure 2.4 shows an example of a fan.

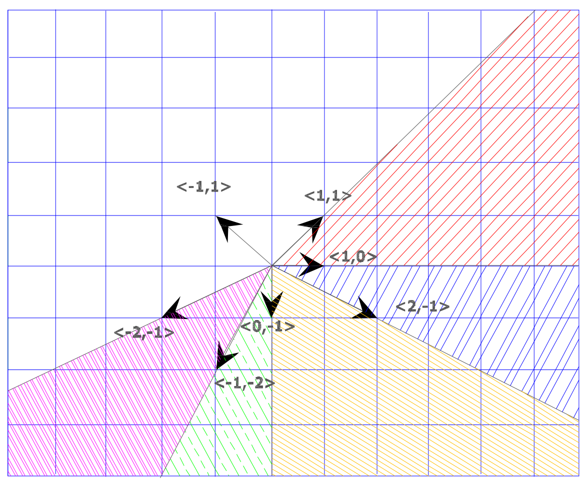

Figure 2.4: Example of a fan in $\mathbb{R}^2$

Definition 2.2.3 ***(Fan)*** *Let N be a lattice. A fan (Σ, N) is a finite, nonempty set of strictly convex rational polyhedral cones in $N_{\mathbb{R}}$ satisfying the following conditions:*

1. *If $\sigma \in \Sigma$ and $\tau \prec \sigma$, then $\tau \in \Sigma$.*

2. *If $\sigma_1, \sigma_2 \in \Sigma$, then $\sigma_1 \cap \sigma_2 \prec \sigma_1$ and $\sigma_1 \cap \sigma_2 \prec \sigma_2$.*

Particularly, we say that Π is a subfan of a fan Σ if Π is a fan and $\Pi \subset \Sigma$.

Definition 2.2.4 ***(Support of a fan)*** *If (Σ, N) is a fan, then we can define its support as:*

$$|\Sigma| = \bigcup_{\sigma \in \Sigma} \sigma.$$

The fan (Σ, N) with $N = \mathbb{Z}^n$ is convex if $|\Sigma|$ is convex as a subset of $N_{\mathbb{R}} = \mathbb{R}^n$. The fan (Σ, N) is strictly convex, if $|\Sigma|$ is convex and does not contain a straight line.

The following two propositions prepare us to glue affine toric varieties along common affine toric subvarieties.

Proposition 2.2.2 ***([1], Lec. 1, Prop. 5.6)*** *If $\tau_1, \tau_2 \prec \sigma$ are faces such that $\tau_1 \cap \tau_2 \prec \sigma$, then $S_{\tau_1 \cap \tau_2} = S_{\tau_1} + S_{\tau_2}$.* ▮

Here, the notation $(U_{\sigma_1})_{\chi_u}$ is used to describe the subset of U_{σ_1}, where the character χ_u does not vanish. Similarly, $(U_{\sigma_2})_{\chi_{-u}}$ describes the subset of U_{σ_2}, where the character χ_{-u} does not vanish.

Proposition 2.2.3 ***([29], Prop. 1.3)*** *If $\tau \prec \sigma_1$ and $\tau \prec \sigma_2$, then both $U_\tau \hookrightarrow U_{\sigma_2}$ and $U_\tau \hookrightarrow U_{\sigma_1}$ are open embeddings, and*

$$\tau = H_u \cap \sigma_1 \quad \text{for} \quad u \in S_{\sigma_1} \quad \text{and} \quad \tau = H_{-u} \cap \sigma_2 \quad \text{for} \quad -u \in S_{\sigma_2};$$

therefore,

$$U_\tau = (U_{\sigma_1})_{\chi_u} \subset U_{\sigma_1} \quad \text{and} \quad U_\tau = (U_{\sigma_2})_{\chi_{-u}} \subset U_{\sigma_2}. ▮$$

Using analytic language, the propositions state that if $\varphi_1 : U_\tau \to U_{\sigma_1}$ and $\varphi_2 : U_\tau \to U_{\sigma_2}$ are open embeddings, then the images of points from U_τ can be identified. Therefore, the map is determined by $\varphi_2 \circ \varphi_1^{-1} : \varphi_1(U_\tau) \to \varphi_2(U_\tau)$, where $\varphi_2 \circ \varphi_1^{-1}$ is an n-tuple of Laurent monomials (i.e. $\varphi_2 \circ \varphi_1^{-1}(z_1, \ldots, z_n) = \left(z_1^{\alpha_{1,1}} \ldots z_n^{\alpha_{1,n}}, \ldots, z_1^{\alpha_{n,1}} \ldots z_n^{\alpha_{n,n}}\right)$ with

$\alpha_{i,j} \in \mathbb{Z}$ for $i, j = 1, \ldots, n$ and $\det(\alpha_{i,j}) = \pm 1$). The following definition ([29], Theorem 1.4) clarifies this idea.

Definition 2.2.5 ***(Toric variety)*** *Let (Σ, N) be a fan. Then the toric variety X_Σ associated with Σ is defined as follows. For any cone $\sigma \in \Sigma$, take an affine toric variety U_σ with its algebra of regular functions $\mathbb{C}[S_\sigma]$. And for such a collection $\{U_\sigma, \mathbb{C}[S_\sigma]\}_{\sigma \in \Sigma}$, notice that conditions described above imply that affine toric varieties can be glued along affine toric varieties associated with their common faces. This construction gives the toric variety associated with the fan Σ.*

Toric varieties are Hausdorff complex analytic spaces as described in [29], Theorem 1.4.

Moreover, nonsingularity conditions of a toric variety can be expressed in terms of the fan. In the next theorem, $\mathbb{Z}$-basis means a basis with coefficients in $\mathbb{Z}$ that is also invertible over $\mathbb{Z}$. To be precise, $\{n_1, \ldots, n_r\}$ is a $\mathbb{Z}$-basis if with the notation $n_i = n_{i,1}e_1 + \ldots + n_{i,r}e_r$ for $i = 1, \ldots, r$, the coefficients fulfill:

(i) $n_{i,j} \in \mathbb{Z}$ for $i, j = 1, \ldots, r$

(ii) and the matrix

$$A = \begin{bmatrix} n_{1,1} & \ldots & n_{1,r} \\ \vdots & & \vdots \\ n_{r,1} & \ldots & n_{r,r} \end{bmatrix}$$

is invertible over $\mathbb{Z}$ (i.e., $\det A = \pm 1$).

Theorem 2.2.2 ***([29], Theorem 1.10)*** *The toric variety X_Σ associated with a fan Σ in N is nonsingular, i.e., a complex manifold, if and only if for each $\sigma \in \Sigma$ there exists a $\mathbb{Z}$-basis $\{n_1, \ldots, n_r\}$ of N and $s \leq r$ such that $\sigma = n_1\mathbb{R}_{\geq 0} + \ldots + n_s\mathbb{R}_{\geq 0}$.* ▮

In other words, a smooth toric variety over complex numbers consists of affine varieties of the type:

$$U_\sigma \simeq \mathbb{C}^r \times (\mathbb{C}^*)^{r-s}$$

glued along the relations defined by the generators of their cones σ.

2.2.3. Torus Action and Orbit Decomposition. As mentioned above, $\{0\} \in \mathbb{R}^n$ is a face of any cone and belongs to any fan. Thus, any toric variety contains an algebraic torus $T^n = U_{\{0\}} = (\mathbb{C}^*)^n$ as an open and dense subset ([12], Part 2, Section VI, Lemma 3.4). The algebraic torus T^n admits a structure of a multiplicative group and acts on itself by transitions. For $t = (t_1, \ldots, t_n) \in T^n$ and $z = (z_1, \ldots, z_n) \in T^n$, the multiplication $t \cdot z$ is defined as:

$$t \cdot z = (t_1 z_1, \ldots, t_n z_n) \in T^n$$

Moreover, the action can be extended naturally and continuously to the whole toric variety X_Σ as described in [12], Part 2, Section VI, Theorem 5.2 and 5.3.

Definition 2.2.6 ***(An orbit)*** *Let G be a group that acts on a set X. An orbit O_p of a point $p \in X$ is defined as follows:*

$$O_p = \{x \in X : x = g \cdot p \quad \textit{for some} \quad g \in G\}$$

where $g \cdot p$ describes an action of $g \in G$ on $p \in X$.

Since the torus T^n itself is an open orbit, other orbits are contained in its closure. (See [12], Part 2, Section VI, Theorem 5.3.) There is a notion of invariant subsets, which are always a sum of orbits. On toric varieties, the orbits are described by the cones and their faces. Let O_τ be an orbit defined by a cone $\tau \in \Sigma$. Then the orbit defined by τ is a torus as well, but of lower dimension:

Lemma 2.2.1 ***([3], Lecture 5, Lemma 1.2)*** *For $\tau \in \Sigma \subset N$ with* $\mathrm{dim}N = n$, $\mathrm{dim}O_\tau + \mathrm{dim}\tau = n$ *and* $O_\tau \simeq \mathbb{C}^{n-\mathrm{dim}\tau}$. ▮

There are no orbits in X_Σ other than those defined by the cones $\tau \in \Sigma$:

Lemma 2.2.2 ***([3], Lecture 5, Lemma 1.3)*** *Every orbit of the torus action on X_Σ is of the form O_τ for some $\tau \in \Sigma$.* ▮

Notice that the closures of orbits $V(\tau) = \overline{O}_\tau$ consist of tori of lower dimension than $\mathrm{dim}O_\tau$ and are invariant subsets of X_Σ. Particularly, the closure of the open orbit T^n is the whole toric variety X_Σ.

Theorem 2.2.3 ***([3], Lecture 5, Theorem 1.9)*** *The orbits O_τ, the orbits closures $V(\tau)$, and the affine open subset U_σ of a toric variety X_Σ are related as follows:*

(i) $U_\sigma = \bigcup_{\tau \prec \sigma} O_\tau$;

(ii) $V(\tau) = \bigcup_{\tau \prec \gamma} O_\gamma$;

(iii) $O_\tau = V(\tau) \setminus \bigcup_{\tau \prec \gamma} V(\gamma)$. ▮

Important for the research on toric surfaces are the orbit closures $V(\tau)$ for $\tau = v\mathbb{R}_{\geq 0} \in \Sigma(1)$. Theorem 2.2.3 part (ii) implies that if the cone $\tau = v\mathbb{R}_{\geq 0}$ is a face of two 2-dimensional cones, then $V(\tau) \simeq \mathbb{P}^1$. In this case, it is convenient to say that v defines the projective curve $D_v = V(\tau)$ or that this curve is associated with v. Consider the following example of a smooth 2-dimensional toric variety E_k with a projective curve defined by the cone $e_2\mathbb{R}_{\geq 0}$. The fan of E_2 is shown in Figure 2.5.

Example 2.2.1 The toric variety E_k for $k \in \mathbb{Z}$ is described by the fan:

$$\Sigma = \{0, e_1\mathbb{R}_{\geq 0}, e_2\mathbb{R}_{\geq 0}, (-1e_1 + ke_2)\mathbb{R}_{\geq 0}, e_1\mathbb{R}_{\geq 0} + e_2\mathbb{R}_{\geq 0}, e_2\mathbb{R}_{\geq 0} + (-1e_1 + ke_2)\mathbb{R}_{\geq 0}\}.$$

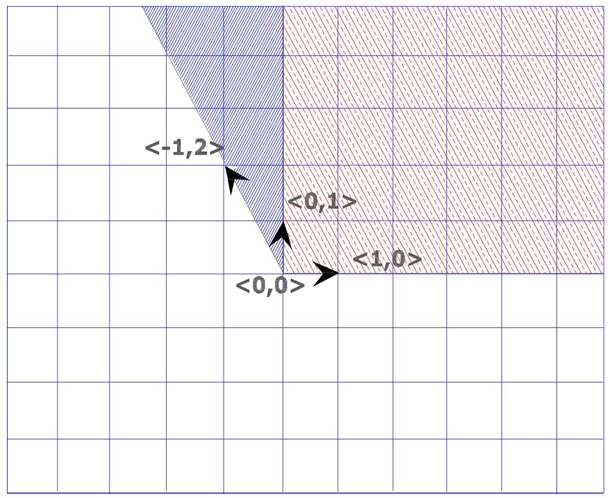

Figure 2.5: The fan of the toric variety E_2

The variety E_k consists of two patches X_0 and X_1, associated respectively with 2-dimensional cones $\sigma_0 = e_1\mathbb{R}_{\geq 0} + e_2\mathbb{R}_{\geq 0}$ and $\sigma_1 = e_2\mathbb{R}_{\geq 0} + (-1e_1 + ke_2)\mathbb{R}_{\geq 0}$. The coordinates $(z, w) \in X_0 \simeq \mathbb{C}^2$ and $(z_1, w_1) \in X_1 \simeq \mathbb{C}^2$ are related on $X_0 \cap X_1 \simeq \mathbb{C}^* \times \mathbb{C}^1$ according to the rule:

$$z_1 = \frac{1}{z} \quad \text{and} \quad w_1 = z^k w$$

E_k contains a projective curve, which is the orbit closure of O_τ with $\tau = e_2\mathbb{R}_{\geq 0}$. Since τ is a face of the cones $\sigma_0 = e_1\mathbb{R}_{\geq 0} + e_2\mathbb{R}_{\geq 0}$ and $\sigma_1 = e_2\mathbb{R}_{\geq 0} + (-1e_1 + ke_2)\mathbb{R}_{\geq 0}$, we obtain $V(\tau) \simeq \mathbb{P}^1$.

2.2.4. Mappings Between Toric Varieties. For a complete view on toric varieties, we must define mappings between them and maps between the associated fans.

Definition 2.2.7 ***(Map of fans)*** *$\varphi : (\Delta_1, N_1) \to (\Delta_2, N_2)$ is a map of fans if it is a $\mathbb{Z}$ linear homomorphism $\varphi : N_1 \to N_2$ that satisfies the property that for any $\sigma \in \Delta_1$ there exists $\tau \in \Delta_2$ such that $\varphi(\sigma) \subset \tau$.*

A map between fans allows us to define a map between toric varieties in a covariant way. The algebraic torus T, if considered in different lattices, needs a subscript. The next theorem uses the notation: $T_{N_i} = N_i \otimes_{\mathbb{Z}} \mathbb{C}^* \simeq (\mathbb{C}^*)^{\dim N_i}$ for $i = 1, 2$.

Theorem 2.2.4 ***([29], Theorem 1.13)*** *A map of fans $\varphi : (\Sigma_1, N_1) \to (\Sigma_2, N_2)$ gives rise to a holomorphic map $\varphi_* : X(\Sigma_1) \to X(\Sigma_2)$ whose restriction to the open subset T_{N_1} coincides with the homomorphism of algebraic tori $\varphi \otimes 1 : T_{N_1} \to T_{N_2}$ arising from φ. Through this homomorphism, φ_* is equivariant with respect to the actions of T_{N_1} and T_{N_2} on the toric varieties. Conversely, suppose $f : T_{N_1} \to T_{N_2}$ is a homomorphism of algebraic tori, and $\varphi_* : X_{\Sigma_1} \to X_{\Sigma_2}$ is a holomorphic map equivariant with respect to f. Then there exists a unique $\mathbb{Z}$ linear homomorphism $\varphi : N_1 \to N_2$, which gives rise to a map of fans $\varphi : (\Sigma_1, N_1) \to (\Sigma_2, N_2)$ such that $f = \varphi_*$.* ▮

It is worth noting at this point that, particularly if Σ_1 is a subfan of Σ_2, then the embedding $\varphi : \Sigma_1 \to \Sigma_2$ induces an embedding of toric varieties $\varphi_* : X_{\Sigma_1} \to X_{\Sigma_2}$.

2.3 THE ENDS

This Section shows how to count the ends of toric varieties. This material does not appear in the literature, so there are no citations except for the definition. The main goal is to show that we can apply Proposition 3.4.1.(b) to noncompact smooth toric surfaces associated with a strictly convex fan. This discussion will show that if X is such a surface, then X has exactly one end. Actually, we can prove a much stronger result for toric varieties using methods similar to those for smooth surfaces.

2.3.1. Definition and Examples.

The following definition of ends was widely used by Freudenthal in [13], Hopf in [21] and others, for example in [15]. An end intuitively describes "a hole at infinity" of a topological space.

Definition 2.3.1 ***(Ends of a topological space)*** *Let X be a connected topological space. Consider the family $\mathscr{F}$ of sequences $\{U_s\}_{s\in\mathbb{N}}$ such that*

(i) U_s is an open, connected subset of X with (nonempty) compact boundary

(ii) $U_{s+1} \subset U_s$ for every $s \in \mathbb{N}$

(iii) $\bigcap_{s\in\mathbb{N}} \overline{U}_s = \emptyset$

In $\mathscr{F}$ we introduce the equivalence relation $\sim$ given by: $\{U_n\} \sim \{V_m\}$ if and only if for every $m \in \mathbb{N}$ there exists $n \in \mathbb{N}$ such that $U_n \subset V_m$. The set of equivalence classes $\mathscr{F}/\sim$ represent the ends of X.

The fact that $\sim$ is an equivalence relation can be found in [13] and implies that, equivalently, we could express it as follows: that for every $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $V_m \subset U_n$.

Before we apply this definition to toric varieties in general, we will see how to describe ends of some well known manifolds.

Example 2.3.1 The sequence of open sets $U_s = \{z \in \mathbb{C}^1 : |z| > s\} \subset \mathbb{C}^1$, $s \in \mathbb{N}$ clearly fulfills (i)-(iii) and defines an end of $\mathbb{C}^1$.

Example 2.3.2 We will show distinct sequences defining ends of $\mathbb{C}^*$. Consider the two sequences of open sets: $U_s = \{z \in \mathbb{C} : |z| > s\}$ and $V_m = \{z \in \mathbb{C} : |z| < \frac{1}{m}\}$. For any s and m, we have $U_s \not\subset V_m$; therefore, the sequences represent different equivalence classes in $\mathscr{F}$.

Example 2.3.3 No compact manifold has ends because there is no sequence $\{U_s\}$ with $U_{s+1} \subset U_s$ such that $\bigcap_{s\in\mathbb{N}} \overline{U}_s = \emptyset$; thus, the family $\mathscr{F}$ is empty. ▮

The following theorem explains how to separate compact sets from ends.

Theorem 2.3.1 *Let K be a compact set in a Hausdorff space X, and let $\{U_s\}_{s\in\mathbb{N}}$ be a sequence of open sets, which defines an end. Then $\exists s \in \mathbb{N}$ such that $K \cap \overline{U}_s = \emptyset$.*

Proof: Notice that since K is a compact subset of a Hausdorff space, K is closed in X from [34], Theorem 17.5.b. Assume that the conclusion is not true. Then for any $s \in \mathbb{N}$, we have $K \cap \overline{U}_s \neq \emptyset$, which together with the condition $U_{s+1} \subset U_s$ implies that $K \cap \bigcap_{s\in\mathbb{N}} \overline{U}_s = \bigcap_{s\in\mathbb{N}} (K \cap \overline{U}_s) \neq \emptyset$ as a decreasing sequence of closed subsets of the compact set K. But then $\bigcap_{s\in\mathbb{N}} \overline{U}_s \neq \emptyset$, which contradicts the definition of an end. Therefore, the conclusion is true and each compact set can be separated from ends. ▮

Particularly interesting for us are manifolds with one end. The following theorem will simplify our work in the next sections:

Theorem 2.3.2 *Let X be a Hausdorff space and let $\{U_s\}_{s\in\mathbb{N}}$ define an end. If for any $s \in \mathbb{N}$ the complement $X \setminus U_s$ is compact, then X has exactly one end.*

Proof: Assuming that X has more ends, we will prove that for some $s \in \mathbb{N}$ the set $X \setminus U_s$ is not compact. Let $\{V_m\}_{m\in\mathbb{N}}$ be a sequence nonequivalent to $\{U_s\}_{s\in\mathbb{N}}$. Then there exists $s \in \mathbb{N}$, so that for each $m \in \mathbb{N}$, $V_m \not\subseteq U_s$, which means that $V_m \setminus U_s \neq \emptyset$. Since $\overline{V}_{m+1} \setminus U_s \subset \overline{V}_m \setminus U_s \subset X \setminus U_s$ and $\bigcap_{m\in\mathbb{N}} \overline{V}_m = \emptyset$, we obtain a contradiction that $X \setminus U_s$ is compact. ▮

More generally, a similar theorem holds for an arbitrary number of ends. Here, D parameterizes the set of ends.

Theorem 2.3.3 *Let X be a Hausdorff space, and for each $d \in D$ let the disjoint sequences $\{U_s^d\}_{s\in\mathbb{N}}$ define distinct ends. If for any $s \in \mathbb{N}$ the complement $X \setminus \bigcup_{d\in D} U_s^d$ is compact, then X has exactly $|D|$ ends.*

Proof: Let $\{W_k\}_{k\in\mathbb{N}}$ be a sequence nonequivalent to any of $\{U_s^d\}_{s\in\mathbb{N}}$ for $d \in D$. Then for any $d \in D$ there exists $s \in \mathbb{N}$ so that for any $k \in \mathbb{N}$ we have $W_k \not\subseteq U_s^d$, which can be written as $W_k \setminus U_s^d \neq \emptyset$. Then we can claim that $W_k \setminus \bigcup_{d\in D} U_s^d \neq \emptyset$. Assume otherwise, i.e., that $W_k \setminus \bigcup_{d\in D} U_s^d = \emptyset$. Then $W_k \subset \bigcup_{d\in D} U_s^d$, and since $U_s^{d_1} \cap U_s^{d_2} = \emptyset$ for $d_1 \neq d_2$, we find that W_k is not connected, which contradicts the definition of an end. Then $\overline{W}_k \setminus \bigcup_{d\in D} U_s^d \neq \emptyset$ is a sequence of closed sets which fulfill $\overline{W}_{k+1} \setminus \bigcup_{d\in D} U_s^d \subset \overline{W}_k \setminus \bigcup_{d\in D} U_s^d \subset X \setminus \bigcup_{d\in D} U_s^d$ and have empty intersection since $\bigcap_{k\in\mathbb{N}} \overline{W}_k = \emptyset$. Then $X \setminus \bigcup_{d\in D} U_s^d$ cannot be compact and the theorem is proven. ▌

Example 2.3.4 Using the results from Examples 2.3.1 and 2.3.2 we can claim that $e(\mathbb{C}^1) = 1$ and $e(\mathbb{C}^*) = 2$.

2.3.2. One-parameter Subgroups and Limit Points. Let $\dim N = n$. The characters $\chi_u \in S_\sigma$ for each $\sigma \in \Sigma$ have already been described in paragraph 2.1.3., but let us recall that each lattice element $u = u_1 e_1^* + \ldots + u_n e_n^* \in S_\sigma \subset N^* = M$ defines a character $\chi_u \in \mathbb{C}[X_\sigma]$, which can be seen particularly as $\chi_u : X_\sigma \to \mathbb{C}^*$, where $\chi_u(t_1, \ldots, t_n) = t_1^{u_1} \cdot \ldots \cdot t_n^{u_n}$. Or, without specification of the element u, we can say that $\chi : (\mathbb{C}^*)^n \to \mathbb{C}^*$, since $(\mathbb{C}^*)^n \subset X_\sigma$ for each $\sigma \in \Sigma$. On the other hand, we can consider a dual object, called a one-parameter subgroup, denoted by $\lambda : \mathbb{C}^* \to (\mathbb{C}^*)^n$ and defined for each $v = m_1 e_1 + \ldots + m_n e_n \in N$ as the group $\lambda_v(t) = (t^{m_1}, \ldots, t^{m_n})$. The duality of characters and one-parameter subgroups can be described by the duality of the lattices

N and M. If $v \in N$ and $u \in M$, then

$$\lambda_v(t)(u) = \chi_u(\lambda_v(t)) = \chi_u(t^{m_1}, \dots, t^{m_n}) = t^{m_1 u_1} \cdot \ldots \cdot t^{m_n u_n} = t^{(v,u)}$$

Each one-parameter subgroup $\lambda_v(t)$ with $t \in \mathbb{C}^*$ might have limits with $t \to 0$ or $t \to \infty$ in X_Σ. Particularly,

$$\lim_{t \to 0} \lambda_v(t) = \lim_{t \to \infty} \lambda_{-v}(t)$$

The one-parameter group $\lambda_v(t)$ has a limit with $t \to 0$ in X_σ only if $v \in \sigma$. Before this result is formulated as a theorem, we need a formal definition of the relative interior of a cone.

Definition 2.3.2 ***(Relative interior of a cone)*** *Let σ be a cone in Σ. Define $\mathrm{Int}_{rel}\sigma$ as the set of all points that do not belong to the faces of σ, i.e., $\mathrm{Int}_{rel}\sigma = \sigma \setminus \bigcup_{\tau \neq \sigma, \tau \prec \sigma} \tau$.*

The limit points on a toric variety X_Σ are defined by relative interiors of the cones $\sigma \in \Sigma$. The point defined by $\mathrm{Int}_{rel}\sigma$ is denoted by x_σ and lies in X_σ. The element x_σ can be seen as the semigroup homomorphism $x_\sigma : \sigma^\vee \cap M \to \mathbb{C}$ defined as:

$$x_\sigma(u) = \begin{cases} 1 & \text{if } -u \in \sigma^\vee \cap M, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Lemma 2.3.1 ***([14] 2.3, Claim 1)*** *Let $v \in Int_{rel}\sigma$ for some $\sigma \in \Sigma$. Then $\lim_{t \to 0} \lambda_v(t) = x_\sigma$.* ▮

Now, the limit $\lim_{t \to 0} \lambda_v(t)$ can be specified to exist in X_Σ if $v \in |\Sigma|$. And the limit $\lim_{t \to \infty} \lambda_v(t)$ exists in X_Σ if $-v \in |\Sigma|$. The following provides an example of a one-parameter subgroup which has both limits. An example with E_{-2} is shown in Figure 2.6.

Example 2.3.5 Let us consider the line bundle E_{-k} with $k = 1, 2, \dots$ described by the fan

$$\Sigma = \{0, e_1\mathbb{R}_{\geq 0}, e_2\mathbb{R}_{\geq 0}, (-1e_1 - ke_2)\mathbb{R}_{\geq 0}, e_1\mathbb{R}_{\geq 0} + e_2\mathbb{R}_{\geq 0}, e_2\mathbb{R}_{\geq 0} + (-1e_1 - ke_2)\mathbb{R}_{\geq 0}\}.$$

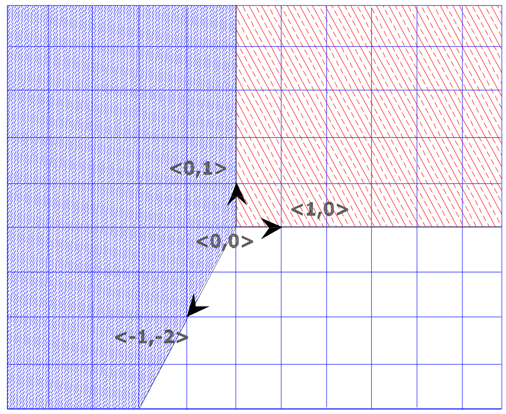

Figure 2.6: The fan of the toric variety E_{-2}

Let the vector $v = 2e_1 + e_2$ define the one-parameter subgroup $\lambda_v(t) = (t^2, t)$. The toric variety E_{-k} consists of two patches, X_0 and X_1, associated respectively with 2-dimensional cones $\sigma_0 = e_1\mathbb{R}_{\geq 0} + e_2\mathbb{R}_{\geq 0}$ and $\sigma_1 = e_2\mathbb{R}_{\geq 0} + (-1e_1 - ke_2)\mathbb{R}_{\geq 0}$. The coordinates $(z, w) \in X_0$ and $(z_1, w_1) \in X_1$ are related on $X_0 \cap X_1$ according to the rule:

$$z_1 = \frac{1}{z} \quad \text{and} \quad w_1 = z^{-k}w.$$

The closure of the subgroup $\lambda_v(t) = (t^2, t)$ in X_0 can be described as the set of solutions of the equation $z = w^2$. For $t = 0$, we get simply the point $(0, 0) \in X_0$. And the closure of the subgroup $\lambda_v(t) = (t^2, t)$ in X_1 can be described as the set of solutions of the equation $z_1^{2k-1} = w_1^2$. Since $k \geq 1$, we find $2k - 1 \geq 1$; therefore, the other limit point of the one-parameter subgroup is simply the point $(0, 0) \in X_1$. Thus, $\overline{\{\lambda_v(t), t \in \mathbb{C}^*\}} \simeq \mathbb{P}^1$ in E_{-k}. ▮

In a discussion of the existence of limit points, the following compactness characterization of toric varieties is worth mentioning:

Theorem 2.3.4 ***([29], Theorem 1.11)*** *The toric variety X_Σ associated with a fan Σ in the lattice N is compact if and only if Σ is complete i.e., if the support $|\Sigma| = N_{\mathbb{R}}$.* ▮

2.3.3. Examples of Ends on Toric Varieties. Now, the "holes" on toric varieties can be described in details. First, we must determine under which conditions the limit points do not exist in a toric variety X_Σ.

Lemma 2.3.2 ***([14] 2.3, Claim 2)*** *If $v \in N$ is not in any cone of Σ, then $\lim_{t\to 0} \lambda_v(t)$ does not exist in X_Σ.* ▮

The set of lattice elements v, which are not in Σ, define a "hole" in X_Σ; therefore, $N_{\mathbb{R}} \setminus \Sigma$ determines the ends. Moreover, as expected, different ends are defined by different connected components of $N_{\mathbb{R}} \setminus \Sigma$. In the following examples, if not clearly indicated, the closures, interiors, and boundaries are taken in the whole toric variety.

Example 2.3.6 The toric variety $\mathbb{C}^*$ is described by the fan $\Sigma = \{0\}$ in the lattice $N = \mathbb{Z}$. The set $N_{\mathbb{R}} \setminus \Sigma$ has two connected components $S_1 = \mathbb{R}_{>0}$ and $S_2 = \mathbb{R}_{<0}$, which define two distinct ends on $\mathbb{C}^*$. The end defined by S_1 is given by the neighborhoods $U_n = \{z \in \mathbb{C}^* : |z| > n\}$, and the end defined by S_2 is given by $U'_n = \{z \in \mathbb{C}^* : |z| < \frac{1}{n}\}$. ▮

More examples of ends on toric varieties must be presented, especially for those toric varieties described by a fan, consisting of several cones. There are interesting examples for line bundles over $\mathbb{P}^1$. The fan of the trivial bundle is presented in Figure 2.7.

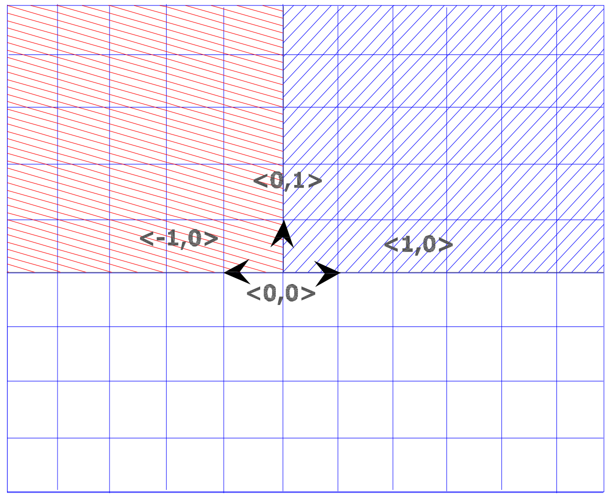

Figure 2.7: The fan of $\mathbb{P}^1 \times \mathbb{C}^1$

Example 2.3.7 The trivial bundle $E_0 = \mathbb{P}^1 \times \mathbb{C}^1$ is described by the fan

$$\Sigma = \{0, e_1\mathbb{R}_{\geq 0}, e_2\mathbb{R}_{\geq 0}, (-1)e_1\mathbb{R}_{\geq 0}, e_1\mathbb{R}_{\geq 0} + e_2\mathbb{R}_{\geq 0}, e_2\mathbb{R}_{\geq 0} + (-1)e_1\mathbb{R}_{\geq 0}\},$$

and the toric variety E_0 consists of two patches, X_0 and X_1, associated respectively with 2-dimensional cones $\sigma_0 = e_1\mathbb{R}_{\geq 0} + e_2\mathbb{R}_{\geq 0}$ and $\sigma_1 = e_2\mathbb{R}_{\geq 0} + (-1)e_1\mathbb{R}_{\geq 0}$. The coordinates $(z, w) \in X_0 \simeq \mathbb{C}^2$ and $(z_1, w_1) \in X_1 \simeq \mathbb{C}^2$ are related on $X_0 \cap X_1 \simeq \mathbb{C}^* \times \mathbb{C}^1$ according to the rule:

$$z_1 = \frac{1}{z} \quad \text{and} \quad w_1 = w.$$

Define the open sets in each patch for $n \geq 1$:

$$U_{0,n} = \{(z, w) \in X_0 : |w| > n\} \subset X_0$$

and

$$U_{1,n} = \{(z_1, w_1) \in X_1 : |w_1| > n\} \subset X_1.$$

Clearly, the condition $|w| = n$ is equivalent to $|w_1| = n$ on $X_0 \cap X_1$, and the set $U_n = U_{0,n} \cup U_{1,n}$ can then be written as

$$U_n = \mathbb{P}^1 \times \left(\mathbb{C}^1 \setminus \overline{\Delta(0,n)}\right)$$

in X_Σ. Thus, U_n is open, connected and $U_{n+1} \subset U_n$. Clearly, $\bigcap_{n\geq 1} \overline{U}_n = \emptyset$, and the boundary of U_n is:

$$\partial U_n = \mathbb{P}^1 \times \{w \in \mathbb{C}^1 : |w| = n\},$$

so it is compact in X_Σ. Figure 2.8 presents another important fact worth noting at this point.

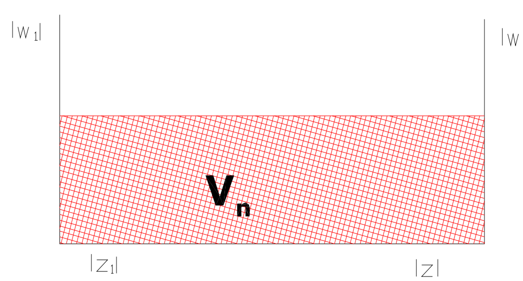

Figure 2.8: A compact set in E_0

The sets $V_n = E_0 \setminus U_n = \mathbb{P}^1 \times \overline{\Delta(0,n)}$ are compact in E_0, and theorem 2.3.2 implies that $\mathbb{P}^1 \times \mathbb{C}^1$ has one end. ▮

The next example explains how to define the end in the line bundle E_{-k} with $k = 1, 2, \ldots$. Figure 2.9 presents the fan of E_{-2}.

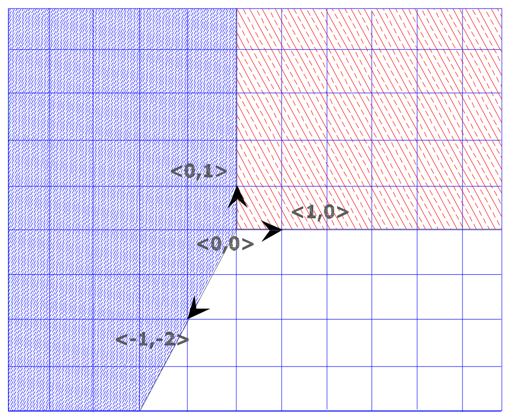

Figure 2.9: The fan of E_{-2}

Example 2.3.8 The line bundle E_{-k} with $k = 1, 2, \ldots$ is described by the fan:

$$\Sigma = \{0, e_1\mathbb{R}_{\geq 0}, e_2\mathbb{R}_{\geq 0}, (-1e_1 - ke_2)\mathbb{R}_{\geq 0}, e_1\mathbb{R}_{\geq 0} + e_2\mathbb{R}_{\geq 0}, e_2\mathbb{R}_{\geq 0} + (-1e_1 - ke_2)\mathbb{R}_{\geq 0}\}.$$

The toric variety E_{-k} consists of two patches X_0 and X_1, associated respectively with 2-dimensional cones $\sigma_0 = e_1\mathbb{R}_{\geq 0} + e_2\mathbb{R}_{\geq 0}$ and $\sigma_1 = e_2\mathbb{R}_{\geq 0} + (-1e_1 - ke_2)\mathbb{R}_{\geq 0}$. The coordinates $(z, w) \in X_0 \simeq \mathbb{C}^2$ and $(z_1, w_1) \in X_1 \simeq \mathbb{C}^2$ are related on $X_0 \cap X_1 \simeq \mathbb{C}^* \times \mathbb{C}^1$ according to the rule:

$$z_1 = \frac{1}{z} \quad \text{and} \quad w_1 = z^{-k}w.$$

Define the sets in each patch for $n \geq 1$:

$$V_{0,n} = \{(z, w) \in X_0 : |w| \leq n\} = \mathbb{C}^1 \times \overline{\Delta(0, n)} \subset X_0$$

and

$$V_{1,n} = \{(z_1, w_1) \in X_1 : |w_1| \leq n\} = \mathbb{C}^1 \times \overline{\Delta(0, n)} \subset X_1.$$

Then $V_n = V_{0,n} \cup V_{1,n}$ and Figure 2.10 shows a sketch of V_n.

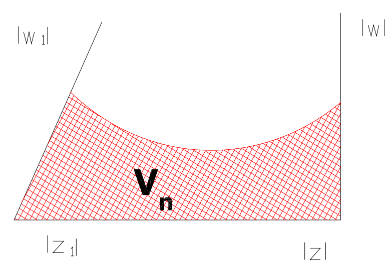

Figure 2.10: A compact set in E_{-k}

The lines $\mathbb{C}^1$ in $V_{0,n} \subset X_0$ have a limit point in X_1; that is, if $w = c$ for some $c \in \mathbb{C}$, then in X_1 we have $z_1^{-k} w_1 = c$. Therefore,

$$w_1 = c z_1^k \qquad \mathrm{i}f \qquad c \in \mathbb{C} \setminus \{0\}$$

and

$$w_1 = 0 \qquad if \qquad c = 0.$$

Then all lines $\mathbb{C}^1$ in $V_{0,n}$ have $(0,0)$ as their limit point in $V_{1,n} \subset X_1$. Similarly, the lines $\mathbb{C}^1$ in $V_{1,n} \subset X_1$ have the limit point $(0,0) \in V_{0,n} \subset X_0$. Then the set $V_n = V_{0,n} \cup V_{1,n}$ is a sum of two compact sets, making it compact and closed in E_{-k}. Now, the open sets can be defined in E_k as $U_n = E_{-k} \setminus V_n$ and then $U_{n+1} \subset U_n$ since $V_n \subset V_{n+1}$. And $\bigcap_{n \geq 1} \overline{U}_n = \emptyset$ since $\bigcup_{n \geq 1} V_n = E_{-k}$. The boundary $\partial U_n = \partial V_n$ is a nonempty compact set. In particular, Theorem 2.3.2 implies that E_{-k} has one end. Figure 2.11 shows projective curves inside V_n. Again, since ∂V_n is a closed subset of a compact set V_n, it must be compact. ▮

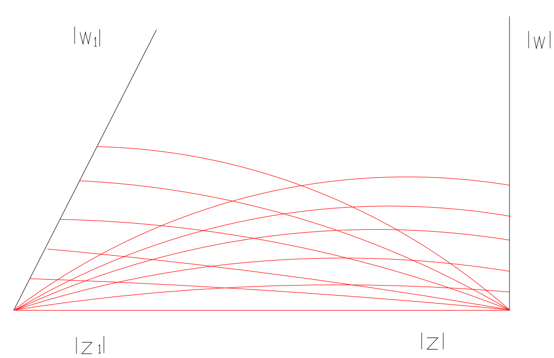

Figure 2.11: Projective curves in the set V_n

Even for line bundles, defining the end is not an easy task. The ends on the bundles E_k with $k = 1, 2, \ldots$ require even more attention. The fan of E_2 is shown in Figure 2.12.

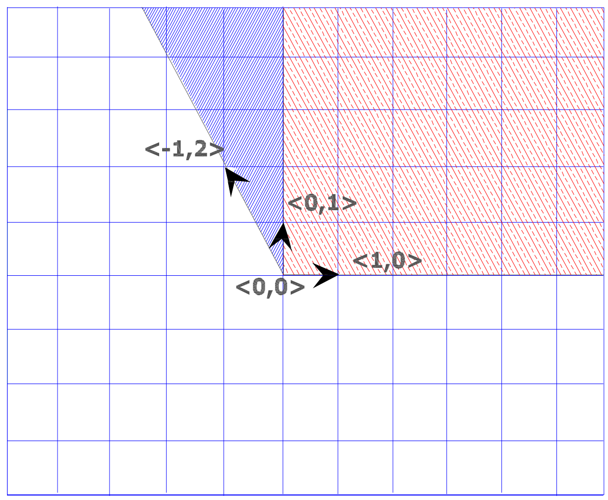

Figure 2.12: The fan of E_2

Example 2.3.9 The line bundle E_k with $k = 1, 2, \ldots$ is described by the fan:

$$\Sigma = \{0, e_1\mathbb{R}_{\geq 0}, e_2\mathbb{R}_{\geq 0}, (-1e_1 + ke_2)\mathbb{R}_{\geq 0}, e_1\mathbb{R}_{\geq 0} + e_2\mathbb{R}_{\geq 0}, e_2\mathbb{R}_{\geq 0} + (-1e_1 + ke_2)\mathbb{R}_{\geq 0}\}.$$

The toric variety E_k consists of two patches X_0 and X_1, associated respectively with 2-dimensional cones $\sigma_0 = e_1\mathbb{R}_{\geq 0} + e_2\mathbb{R}_{\geq 0}$ and $\sigma_1 = e_2\mathbb{R}_{\geq 0} + (-1e_1 + ke_2)\mathbb{R}_{\geq 0}$. The coordinates $(z, w) \in X_0 \simeq \mathbb{C}^2$ and $(z_1, w_1) \in X_1 \simeq \mathbb{C}^2$ are related on $X_0 \cap X_1 \simeq \mathbb{C}^* \times \mathbb{C}^1$ according to the rule:

$$z_1 = \frac{1}{z} \quad \text{and} \quad w_1 = z^k w.$$

Define the open sets in each patch for $n \geq 1$:

$$U_{0,n} = \{(z, w) \in X_0 : |w| > n\} \subset X_0$$

and

$$U_{1,n} = \{(z_1, w_1) \in X_1 : |w_1| > n\} \subset X_1.$$

Then the set $U_n = U_{0,n} \cup U_{1,n}$ is open and connected in E_k, since $U_{0,n}$ and $U_{1,n}$ are open, connected, and have a nonempty intersection. Clearly, $U_{n+1} \subset U_n$, since $U_{0,n+1} \subset U_{0,n}$ and $U_{1,n+1} \subset U_{1,n}$. Now, the fact that $\bigcap_{n \geq 1} \overline{U}_n = \emptyset$, must be justified. Notice that the inequality $|w| \geq n$, which describes $\overline{U}_{0,n}$ in X_0, does not bring any new points if considered in E_k. This observation comes from the fact that the vectors e_1 and $-e_1$ are not in σ_1; thus, the one-parameter subgroup $(t, 1)$ does not have limits in X_1, as claimed in Lemma 2.3.2. Similarly, the closure of $U_{1,n}$ taken in X_1 is equal to its closure in E_k. Then it can be claimed that

$$\bigcap_{n \geq 1} \overline{U}_n = \bigcup_{A \dot{\cup} B = \mathbb{N}} \left(\bigcap_{i \in A} \overline{U}_{0,i} \cap \bigcap_{j \in B} \overline{U}_{1,j} \right), \tag{2}$$

where $A \dot{\cup} B$ denotes the disjoint union of sets A and B. Since $A \dot{\cup} B = \mathbb{N}$, at least one set from each pair is infinite. For infinite sets of indices, the intersections $\bigcap_{i \in A} \overline{U}_{0,i}$ and $\bigcap_{j \in B} \overline{U}_{1,j}$ are empty. Therefore, each factor of the sum in equation (2) is empty, making the sum is empty and $\bigcap_{n \geq 1} \overline{U}_n = \emptyset$. The boundary ∂U_n is a nonempty set, but its compactness is not obvious. First, notice that the set $V_n = E_k \setminus U_n$ is closed and compact in E_k. To prove this, we will represent V_n as a union of compact sets in X_0 and X_1. Notice that the intersection of the real hypersurfaces $|w| = n$ and $|w_1| = n$ lies inside the real hypersurface $|z| = 1$ because we have:

$$n = |w_1| = \left| z^k w \right| = \left| z^k \right| n,$$

which gives

$$|z| = 1.$$

Then

$$V_n = (V_n \cap \{(z, w) \in X_0 : |z| \leq 1\}) \cup (V_n \cap \{(z_1, w_1) \in X_1 : |z_1| \leq 1\})$$

is a sum of two compact sets, making it compact. Now, since ∂V_n is a closed subset of a compact set V_n, it must be compact. Again, theorem 2.3.2 implies that E_k has one end. The set V_n is presented in Figure 2.13. ∎

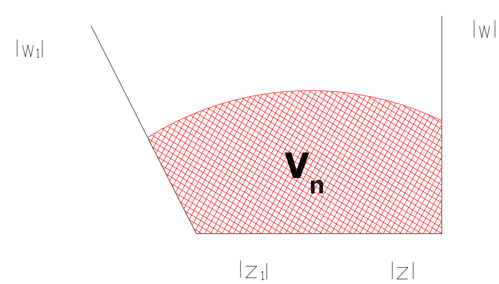

Figure 2.13: A compact set in E_k

It is no coincidence that the methods for defining the ends for E_k and E_{-k} are not comparable. Describing an end on a toric variety with more patches is more complicated and depends on the number of ends and the combinatorial structure of the fan. The end of a nonsingular toric surface with a strictly convex fan and d patches must be described in detail. In the following example, the result for $d = 2$ will appear as the first step of the mathematical induction. Figure 2.14 shows a strictly convex fan, and Figure 2.15 shows

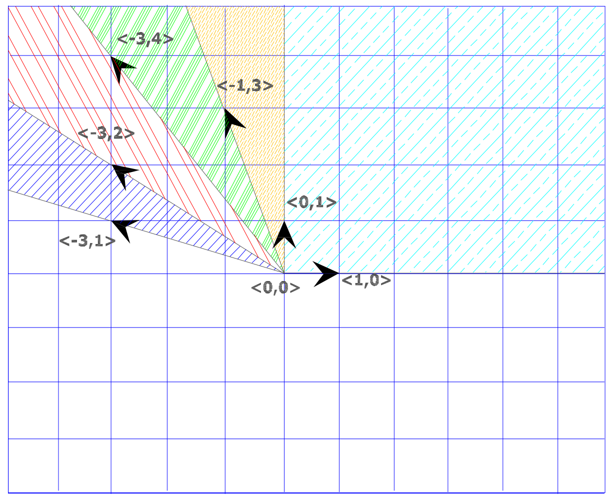

Figure 2.14: An example of a strictly convex fan

a sketch of a compact set V_n in a toric surface with a strictly convex fan.

Example 2.3.10 Let X_Σ be described by a strictly convex fan:

$$\Sigma = \{0, v_0\mathbb{R}_{\geq 0}, v_1\mathbb{R}_{\geq 0}, \ldots, v_d\mathbb{R}_{\geq 0}, v_0\mathbb{R}_{\geq 0} + v_1\mathbb{R}_{\geq 0}, \ldots, v_{d-1}\mathbb{R}_{\geq 0} + v_d\mathbb{R}_{\geq 0}\},$$

where $d \geq 2$ is fixed. We can assume that $v_0 = e_1$, $v_1 = e_2$. Then, for $j = 2, 3, \ldots, d$, we find that $v_j = -\alpha_j e_1 + \beta_j e_2$ for some $\alpha_j \in \mathbb{Z}_{\geq 1}$ and $\beta_j \in \mathbb{Z}_{\geq 1}$, which fulfill $-\alpha_{j-1}\beta_j + \alpha_j\beta_{j-1} = 1$, since the surface is smooth. The surface X_Σ consists of d patches $X_j \simeq \mathbb{C}^2$, with $j = 1, \ldots, d$ related according to the rule:

$$z_1 = \frac{1}{z_j^{\alpha_{j-1}} w_j^{\alpha_j}} \qquad \text{and} \qquad w_1 = z_j^{\beta_{j-1}} w_j^{\beta_j},$$

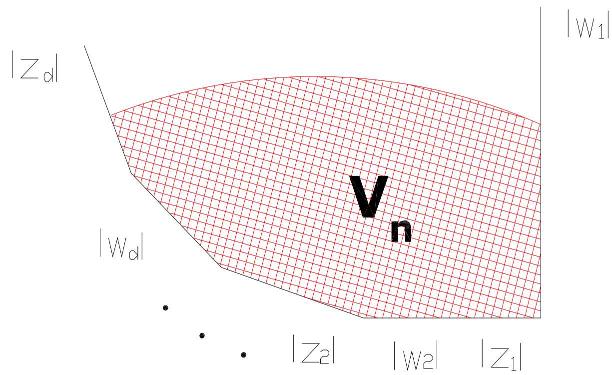

Figure 2.15: A compact set in a toric surface with a strictly convex fan

or conversely,

$$z_j = z_1^{\beta_j} w_1^{\alpha_j} \qquad \text{and} \qquad w_j = \frac{1}{z_1^{\beta_{j-1}} w_1^{\alpha_{j-1}}},$$

where (z_j, w_j) are coordinates in X_j. Direct computations also show:

$$z_d = z_j^{\beta_{j-1}\alpha_d - \alpha_{j-1}\beta_d} w_j^{\beta_j\alpha_d - \alpha_j\beta_d} \qquad \text{and} \qquad w_d = \frac{1}{z_j^{\beta_{j-1}\alpha_{d-1} - \alpha_{j-1}\beta_{d-1}} w_j^{\beta_j\alpha_{d-1} - \alpha_j\beta_{d-1}}}.$$

Define the open sets $U_{1,n}$ and $U_{d,n}$ for $n \geq 1$:

$$U_{1,n} = \{(z_1, w_1) \in X_1 : |w_1| > n\} \subset X_1,$$

and

$$U_{d,n} = \{(z_d, w_d) \in X_d : |z_d| > n^{\beta_d}\} \subset X_d.$$

Then, as before, the set $U_n = U_{0,n} \cup U_{d,n}$ is open and connected in X_Σ and $U_{n+1} \subset U_n$. The property $\bigcap_{n \geq 1} \overline{U}_n = \emptyset$ comes from the following observations: Notice that the inequality

$|w_1| \geq n$ that describes $\overline{U}_{1,n}$ in X_1 does not bring any new points if considered in X_Σ because the vectors e_1 and $-e_1$ are not in σ_j for $j = 2, \ldots, d$. Thus, one-parameter subgroup $(t, 1)$ does not have limits in X_j as claimed in Lemma 2.3.2. Similarly, the closure of $U_{d,n}$ taken in X_d is described by the inequality $|z_d| \geq n^{\beta_d}$ and transformed to coordinates $(z_1, w_1) \in X_1$ gives:

$$\left| z_1^{\beta_d} w_1^{\alpha_d} \right| \geq n^{\beta_d}.$$

We can deduce, that the closure of $U_{d,n}$ taken in X_d is equal to its closure in X_Σ, since one parameter subgroup $(t^{-\alpha_d}, t^{\beta_d})$ does not have limits in X_j for $j = 1, \ldots, d-1$. This is the result of Lemma 2.3.2 and the fact that the vectors $-\alpha_d e_1 + \beta_d e_2$ and $\alpha_d e_1 + (-\beta_d) e_2$ do not lie in σ_j for $j = 1, \ldots, d-1$. We can then claim that

$$\bigcap_{n \geq 1} \overline{U}_n = \bigcup_{A \dot\cup B = \mathbb{N}} \left(\bigcap_{i \in A} \overline{U}_{0,i} \cap \bigcap_{j \in B} \overline{U}_{d,j} \right),$$

where $A \dot\cup B$ denotes the disjoint union of sets A and B. Since $A \dot\cup B = \mathbb{N}$, at least one set from each pair is infinite. For infinite sets of indices, the intersections $\bigcap_{i \in A} \overline{U}_{0,i}$ and $\bigcap_{j \in B} \overline{U}_{d,j}$ are empty, maing each factor of the sum above empty. Thus, the sum is empty and $\bigcap_{n \geq 1} \overline{U}_n = \emptyset$. The boundary ∂U_n is a nonempty set, but its compactness is not obvious. We want to prove that the set $V_n = X_\Sigma \setminus U_n$ is compact for any $n \geq 1$. To do so, we will represent V_n as a union of compact sets. First, let us define the following hypersurfaces, which are the boundaries of the sets $U_{1,n}$ and $U_{d,n}$, respectively:

$$H_{1,n} = \{(z_1, w_1) \in X_1 : |w_1| = n\}$$

and

$$H_{d,n} = \{(z_d, w_d) \in X_d : |z_d| = n^{\beta_d}\}. \tag{3}$$

The equation of $H_{1,n}$ in coordinates (z_d, w_d) can be represented as follows:

$$\left| z_d^{\beta_{d-1}} w_d^{\beta_d} \right| = n.$$

By plugging this expression into equation (3), we obtain

$$\left|z_d^{\beta_{d-1}} w_d^{\beta_d}\right| = n^{\beta_{d-1}\beta_d} \left|w_d^{\beta_d}\right| n^{\beta_d} = n.$$

Therefore, the intersection $H_{1,n} \cap H_{d,n}$ in terms of (z_d, w_d) is as follows:

$$H_{1,n} \cap H_{d,n} = \{(z_d, w_d) \in X_d : |z_d| = n^{\beta_d} \, |w_d| = n^{\frac{1-\beta_{d-1}\beta_d}{\beta_d}}\}.$$

Let us denote by H_n the hypersurface described by the equation $|w_d| = n^{\frac{1-\beta_{d-1}\beta_d}{\beta_d}}$, and let us define H_n^+ as

$$H_n^+ = \{(z_d, w_d) \in X_d : |w_d| < n^{\frac{1-\beta_{d-1}\beta_d}{\beta_d}}\}$$

and $H_n^- = X_\Sigma \setminus \overline{H}_n^+$. Then, $X_\Sigma = \overline{H}_n^+ \cup \overline{H}_n^-$, and in particular

$$V_n = (V_n \cap \overline{H}_n^+) \cup (V_n \cap \overline{H}_n^-).$$

Notice that

$$V_n \cap \overline{H}_n^+ = \{(z_d, w_d) \in X_d : |z_d| \leq n^{\beta_d}, |w_d| \leq n^{\frac{1-\beta_{d-1}\beta_d}{\beta_d}}\} \subset X_d$$

is simply a polydisc in X_d, and thus compact. Now, $V_n \cap \overline{H}_n^-$ is a subset of another smooth toric variety $X_{\Sigma'}$, where

$$\Sigma' = \{0, v_0\mathbb{R}_{\geq 0}, v_1\mathbb{R}_{\geq 0}, v_2\mathbb{R}_{\geq 0}, \ldots, v_{d-1}\mathbb{R}_{\geq 0}, v_0\mathbb{R}_{\geq 0} + v_1\mathbb{R}_{\geq 0}, \ldots, v_{d-2}\mathbb{R}_{\geq 0} + v_{d-1}\mathbb{R}_{\geq 0}\}$$

is a fan which consists of $d-1$ patches. Since the compactness of this type of set for $d = 2$ has already been proved in Example 2.3.9, the mathematical induction justifies that V_n is compact, as a finite sum of compact sets. Since ∂V_n is a closed subset of a compact set V_n, it must be compact. Again, Theorem 2.3.2 implies that X_Σ has one end. ▌

Clearly, describing an end on a toric variety with more patches is complicated and depends on the number of ends and the combinatorial structure of the fan.

3 THE HARTOGS PHENOMENON IN TORIC SURFACES

For a complex manifold we can consider the following problem:

Definition 3.0.3 *The Hartogs phenomenon ($\mathscr{H}$) holds in a complex manifold X if for any compact set K such that $X \setminus K$ is connected, any holomorphic function defined on $X \setminus K$ can be holomorphically extended to X.*

The following theorem is one of the most fundamental in the analysis of several complex variables. Here, a domain is an open, connected set.

Theorem 3.0.5 ***([26], Theorem 1.2.6)*** *Let K be a compact subset of a domain $D \subset \mathbb{C}^n$ for $n \geq 2$. If f is holomorphic on the connected set $D \setminus K$, then there exists a holomorphic extension of f on D.* ▌

In particular, D could be equal to the whole $\mathbb{C}^n$ as in the following version:

Theorem 3.0.6 ***([31], Section 3, Lemma 2)*** *Let K be a compact subset of $\mathbb{C}^n$ for $n \geq 2$. If f is holomorphic on the connected set $\mathbb{C}^n \setminus K$, then there exists a holomorphic extension of f on $\mathbb{C}^n$.* ▌

Let $\Delta(0, R) = \{z \in \mathbb{C} : |z| < R\}$ be an open disc. The notation $C(0, R)$ will be used for the boundary of $\Delta(0, R)$. Let $A(0; \rho, R) = \{z \in \mathbb{C} : \rho < |z| < R\}$ be an annulus in $\mathbb{C}$. We require that $0 \leq \rho < R \leq \infty$. The following Hartogs Continuity Theorem allows a slightly more general extension phenomenon. Here, the set on which the function is not defined does not have to be compact. The accompanying picture shown in Figure 3.1 is often called the Hartogs Figure.

Theorem 3.0.7 ***([25], Section II, Theorem 2.61)*** *Let f be a holomorphic function on a domain $D \subset \mathbb{C}^n$ (with $n \geq 2$) of the form*

$$D = [D' \times A(0; \rho, R)] \cup [D'_0 \times \Delta(0, R)],$$

where D' is a connected domain in $\mathbb{C}^{n-1}$ and D'_0 is a nonempty subdomain of D'. Then f has a holomorphic continuation to $D' \times \Delta(0, R)$. ▮

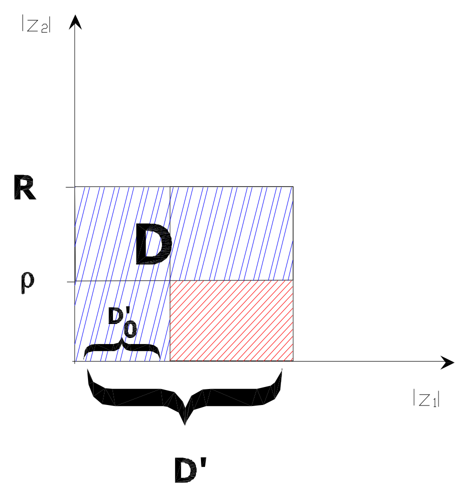

Figure 3.1: The Hartogs figure

This Section determines for which noncompact smooth toric surfaces the Hartogs phenomena holds. This work requires the following version of the Hartogs figure in $\mathbb{C}^2$. Figure 3.2 presents a sketch of the set V.

Theorem 3.0.8 *Let $f(z, w)$ be a holomorphic function on $\mathbb{C}^2 \setminus V$, where V is defined as follows:*

$$V = \{(z, w) \in \mathbb{C}^2 : \left|z^\beta w^\alpha\right| \leq M, |w| \leq N\},$$

where $\alpha, \beta \in \mathbb{Z}_{\geq 1}$ and $M, N \in \mathbb{R}_{>0}$. Then f has holomorphic extension to $\mathbb{C}^1 \times \mathbb{C}^*$.

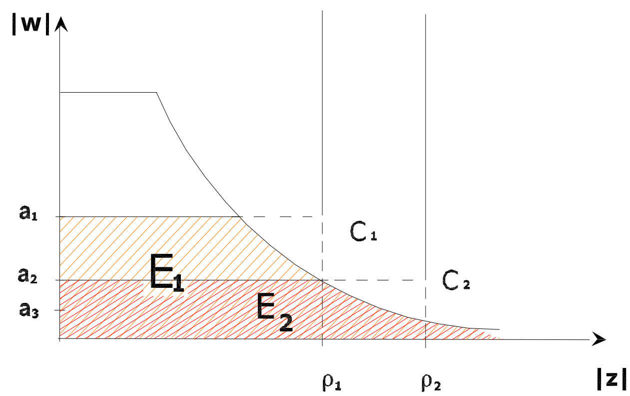

Figure 3.2: Another version of the Hartogs figure

Proof: Let us define sequences of real numbers $a_s = \frac{N}{2^s}$ and $\rho_s = \left(\frac{2^{s+1}}{N}\right)^{\frac{\alpha}{\beta}} M^{\frac{1}{\beta}}$ with $s = 0, 1, \ldots$. For fixed w so that $a_s \leq |w|$, we define C_s by the parametrization $t \mapsto (\rho_s e^{it}, w)$ with $t \in [-\pi, \pi]$. Let $E_s = \{(z, w) \in \mathbb{C}^2 : \left|z^\beta w^\alpha\right| \leq M, |w| \leq a_s\}$. Then the function f_s is defined on $\mathbb{C}^2 \setminus E_s$:

$$f_s(z, w) = \frac{1}{2\pi i} \int_{C_s} \frac{f(\xi, w)}{\xi - z} d\xi$$

The functions f_s and f_{s+1} agree on the set $\mathbb{C}^2 \setminus E_s$:

$$f_s(z,w) - f_{s+1}(z,w) = \frac{1}{2\pi i}\int_{C_s} \frac{f(\xi,w)}{\xi - z} d\xi - \frac{1}{2\pi i}\int_{C_{s+1}} \frac{f(\xi,w)}{\xi - z} d\xi = 0,$$

since the function f is holomorphic for (z,ξ) so that $\rho_s \leq |z|$ and $a_{s+1} \leq |\xi|$.

Since $a_s \to 0$ for $s \to \infty$, the function f can be extended to $\mathbb{C}^1 \times \mathbb{C}^*$.▮

3.1 GLOBAL HOLOMORPHIC FUNCTIONS ON MANIFOLDS

If X is a complex manifold with an atlas $\{X_j, \phi_j\}$, where X_j is an open subset in $\mathbb{C}^n$, then any global function f on X fulfills the natural conditions that the functions $f_j = f\mid_{X_j}$ agree on $X_i \cap X_j$ (i.e., $f_j\mid_{X_i\cap X_j} = f_i\mid_{X_i\cap X_j}$). Clearly, the Hartogs phenomenon makes a nontrivial problem only if the manifold allows nonconstant global functions; that is, if, $\Gamma(X,\mathscr{O}) \neq \mathbb{C}$. Further research will require a description of global functions on smooth toric surfaces.

3.1.1. Global Holomorphic Functions on E_k for $k = 1, 2, \ldots$. Recall that E_k is a smooth toric surface associated with the fan

$$\Sigma = \{0, e_1\mathbb{R}_{\geq 0}, e_2\mathbb{R}_{\geq 0}, (-e_1 + ke_2)\mathbb{R}_{\geq 0}, e_1\mathbb{R}_{\geq 0} + e_2\mathbb{R}_{\geq 0}, e_2\mathbb{R}_{\geq 0} + (-e_1 + ke_2)\mathbb{R}_{\geq 0}\},$$

where the chart X_1 is defined by the cone $\sigma_1 = e_1\mathbb{R}_{\geq 0} + e_2\mathbb{R}_{\geq 0}$ and the chart X_2 by the cone $\sigma_2 = e_2\mathbb{R}_{\geq 0} + (-e_1 + ke_2)\mathbb{R}_{\geq 0}$. Let (z_1, w_1) be coordinates in X_1 and (z_2, w_2) in X_2. Then $z_1 = \frac{1}{z_2}$ and $w_1 = z_2^k w_2$ on $X_1 \cap X_2$.

The holomorphic function f is global on E_k if and only if $f_1 = f\mid_{X_1}$ is holomorphic on X_1 and $f_2 = f\mid_{X_2}$ is holomorphic on X_2, and only if they agree on the intersection $X_1 \cap X_2$. We have $X_1 \simeq \mathbb{C}^2$, $X_2 \simeq \mathbb{C}^2$, so f_1 and f_2 have series representation. Let

$$f_1(z_1, w_1) = \sum_{i,s=0}^{\infty} a_{is} z_1^i w_1^s,$$

and

$$f_2(z_2, w_2) = \sum_{l,m=0}^{\infty} b_{lm} z_2^l w_2^m.$$

Then on $X_1 \cap X_2$ we have:

$$f_1(z_1, w_1) = \sum_{i,s=0}^{\infty} a_{is} z_1^i w_1^s = \sum_{i,s=0}^{\infty} a_{is} \left(\frac{1}{z_2}\right)^i \left(z_2^k w_2\right)^s = \sum_{i,s=0}^{\infty} a_{is} z_2^{(ks-i)} w_2^s$$

$$= \sum_{l,m=0}^{\infty} b_{lm} z_2^l w_2^m = f_2(z_2, w_2)$$

Hence, the necessary and sufficient conditions for f_1 to define a holomorphic function on E_k is that all pairs (i, s) with $a_{is} \neq 0$ fulfill the condition $ks - i = l \geq 0$. Using the matrix notation for the coefficients a_{is}, the function f_1 can be represented as follows:

$$\begin{bmatrix} \vdots & & \vdots & & & \\ a_{o,s} & \dots & \dots & a_{ks,s} & 0 & \dots \\ \vdots & \vdots & & & & \\ a_{0,1} & \dots & a_{k,1} & 0 & \dots & \\ a_{0,0} & 0 & \dots & & \dots & \end{bmatrix} \tag{4}$$

On the other hand, f_2 fulfils a similar condition, since we have on $X_1 \cap X_2$:

$$f_2(z_2, w_2) = \sum_{l,m=0}^{\infty} b_{lm} z_2^l w_2^m = \sum_{l,m=0}^{\infty} b_{lm} \left(\frac{1}{z_1}\right)^l \left(z_1^k w_1\right)^m = \sum_{l,m=0}^{\infty} b_{lm} z_1^{(km-l)} w_1^m$$

$$= \sum_{i,s=0}^{\infty} a_{is} z_1^i w_1^s = f_1(z_1, w_1).$$

This implies that $km - l = i \geq 0$ for those indices (l, m), where $b_{lm} \neq 0$. As before, in matrix notation, the coefficients b_{lm} can be represented as follows:

$$\begin{bmatrix} \vdots & & \vdots & & & \\ b_{0,l} & \dots & \dots & b_{kl,l} & 0 & \dots \\ \vdots & \vdots & & & & \\ b_{0,1} & \dots & b_{k,1} & 0 & \dots & \\ b_{0,0} & 0 & \dots & & \dots & \end{bmatrix} \tag{5}$$

Notice that E_k contains a projective curve $D \simeq \mathbb{P}^1$ associated with the cone $v_1\mathbb{R}_{\geq 0}$ (as explained in Theorem 2.2.3 and mentioned in Example 2.2.1). The projective curve D can be described in the local coordinates as $w_1 = 0$ in X_1 and $w_2 = 0$ in X_2. Any function that is holomorphic on E_k must be constant on D, as stated in [19] Chap. 5, Sec. B, Theorem 6 and Corollary 7.

3.1.2. Global Holomorphic Functions on Toric Surfaces. Let X be a smooth toric surface associated with a strictly convex fan:

$$\Sigma = \{0, v_0\mathbb{R}_{\geq 0}, v_1\mathbb{R}_{\geq 0}, \dots, v_s\mathbb{R}_{\geq 0}, v_0\mathbb{R}_{\geq 0} + v_1\mathbb{R}_{\geq 0}, \dots, v_{d-1}\mathbb{R}_{\geq 0} + v_d\mathbb{R}_{\geq 0}\},$$

where d is fixed. An example is shown in Figure 3.3. We can assume without loss of generality that $v_0 = e_1$ and $v_1 = e_2$. Then for $j = 2, 3, \dots, d$,

$$v_j = -\alpha_j e_1 + \beta_j e_2 \quad \text{for some} \quad \alpha_j \in \mathbb{Z}_{\geq 1} \quad \text{and} \quad \beta_j \in \mathbb{Z}_{\geq 1}.$$

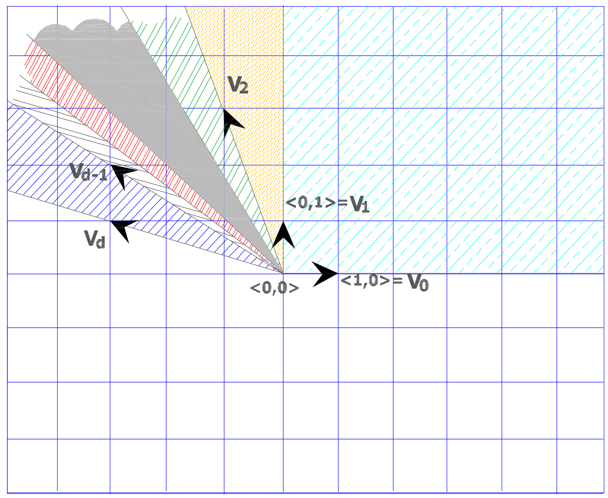

Figure 3.3: A strictly convex fan

Let $\sigma_j = v_{j-1}\mathbb{R}_{\geq 0} + v_j\mathbb{R}_{\geq 0}$ for $j = 1, \ldots, d$ be 2-dimensional cones associated with the charts X_j, which have coordinates (z_j, w_j). Then for σ_j generated by $v_{j-1} = -\alpha_{j-1}e_1 + \beta_{j-1}e_2$ and $v_j = -\alpha_j e_1 + \beta_j e_2$, we find that $\sigma_j^\vee$ is generated by the vectors $\beta_j e_1^* + \alpha_j e_2^*$ and $-\beta_{j-1}e_1^* - \alpha_{j-1}e_2^*$. Then

$$z_j = z_1^{\beta_j} w_1^{\alpha_j} \qquad \text{and} \qquad w_j = \frac{1}{z_1^{\beta_{j-1}} w_1^{\alpha_{j-1}}}.$$

Because X_j is smooth, we know that $-\alpha_{j-1}\beta_j + \alpha_j\beta_{j-1} = 1$, which gives

$$z_1 = \frac{1}{z_j^{\alpha_{j-1}} w_j^{\alpha_j}} \qquad \text{and} \qquad w_1 = z_j^{\beta_{j-1}} w_j^{\beta_j}.$$

Moreover, direct computations prove that

$$z_j = \frac{1}{z_d^{\alpha_j\beta_{d-1} - \beta_j\alpha_{d-1}} w_d^{\alpha_j\beta_d - \beta_j\alpha_d}} \qquad \text{and} \qquad w_j = z_d^{\beta_{j-1}\alpha_{d-1} - \alpha_{j-1}\beta_{d-1}} w_d^{\beta_{j-1}\alpha_d - \alpha_{j-1}\beta_d}.$$

The function f is holomorphic on X if for $j, i = 1, \ldots, d$ we find that $f_j = f\mid_{X_j}$ is holomorphic on X_j and $f_j = f_i$ on $X_j \cap X_i$. Again, $X_j \simeq \mathbb{C}^2$, so $f_j = f\mid_{X_j}$ has a power series expansion. Let

$$f_1(z_1, w_1) = \sum_{i,s=0}^{\infty} a_{is} z_1^i w_1^s.$$

On $X_1 \cap X_j$, we have

$$f_1(z_1, w_1) = \sum_{i,s=0}^{\infty} a_{is} z_1^i w_1^s = \sum_{i,s=0}^{\infty} a_{is} \frac{1}{(z_j^{\alpha_{j-1}} w_j^{\alpha_j})^i} (z_j^{\beta_{j-1}} w_j^{\beta_j})^s$$

$$= \sum_{i,s=0}^{\infty} a_{is} z_j^{s\beta_{j-1} - i\alpha_{j-1}} w_j^{s\beta_j - i\alpha_j} = f_j(z_j, w_j).$$

Since the function f_j has a power series expansion, the necessary and sufficient condition for f to define a holomorphic function on X is that for all powers (i, s) so that $a_{is} \neq 0$, the following conditions are fulfilled for each cone $\sigma_j = v_{j-1}\mathbb{R}_{\geq 0} + v_j \mathbb{R}_{\geq 0}$ for all $j = 1, 2, \ldots, d$.

$$s\beta_{j-1} - i\alpha_{j-1} \geq 0 \quad \text{and} \quad s\beta_j - i\alpha_j \geq 0$$

Since the cones are ordered counterclockwise and

$$\frac{\alpha_{j-1}}{\beta_{j-1}} < \frac{\alpha_j}{\beta_j} \tag{6}$$

for $j = 2, \ldots, d$, the strongest condition is obtained for $j = d$, which is $s\beta_d - i\alpha_d \geq 0$. This inequality can be written as $\left[\frac{s\beta_d}{\alpha_d}\right] \geq i$, where $\left[\frac{s\beta_d}{\alpha_d}\right]$ denotes the integer part of $\frac{s\beta_d}{\alpha_d}$. In matrix notation the coefficients $a_{i,s}$ of f_1 can be represented as follows:

$$\begin{bmatrix} \vdots & & \vdots & & & \\ a_{o,s} & \ldots & \ldots & a_{\left[\frac{s\beta_d}{\alpha_d}\right],s} & 0 & \ldots \\ \vdots & \vdots & & & & \\ a_{0,1} & \ldots & a_{\left[\frac{\beta_d}{\alpha_d}\right],1} & 0 & \ldots & \\ a_{0,0} & 0 & \ldots & & \ldots & \end{bmatrix} \tag{7}$$

Again, a similar condition can be obtained for f_d. Let

$$f_d(z_d, w_d) = \sum_{l,m=0}^{\infty} b_{lm} z_d^l w_d^m.$$

On $X_1 \cap X_d$, we have

$$f_d(z_d, w_d) = \sum_{l,m=0}^{\infty} b_{lm} z_d^l w_d^m = \sum_{l,m=0}^{\infty} b_{lm} (z_1^{\beta_d} w_1^{\alpha_d})^l \frac{1}{(z_1^{\beta_{d-1}} w_1^{\alpha_{d-1}})^m}$$

$$= \sum_{l,m=0}^{\infty} b_{lm} z_1^{(l\beta_d - m\beta_{d-1})} w_1^{(l\alpha_d - m\alpha_{d-1})} = f_1(z_1, w_1).$$

Since f_1 admits a power series expansion, it can be determined that $l\beta_d - m\beta_{d-1} \geq 0$ and $l\alpha_d - m\alpha_{d-1} \geq 0$ for those (l, m) with $b_{l,m} \neq 0$. One of these inequalities is always stronger. Since $\beta_d, \alpha_d > 0$, equation (6) implies that

$$\frac{\beta_{d-1}}{\beta_d} > \frac{\alpha_{d-1}}{\alpha_d}.$$

Thus, $l\beta_d - m\beta_{d-1} \geq 0$ is stronger. The coefficients of f_d in the matrix notation might be viewed as follows:

$$\begin{bmatrix} \vdots & \vdots & & 0 & \\ & 0 & & b_{l,\left[\frac{l\beta_d}{\beta_{d-1}}\right]} & \\ \vdots & b_{1,\left[\frac{\beta_d}{\beta_{d-1}}\right]} & & \vdots & \cdots \\ 0 & \vdots & \cdots & \vdots & \\ b_{0,0} & b_{1,0} & \cdots & b_{l,0} & \cdots \end{bmatrix} \tag{8}$$

Similar computations for $f_j = f\mid_{X_j}$ with $j = 2, \ldots, d-1$ prove that on $X_1 \cap X_j$:

$$f_j(z_j, w_j) = \sum_{i,s=0}^{\infty} c_{is} z_j^i w_j^s = \sum_{i,s=0}^{\infty} c_{is} \left(z_1^{\beta_j} w_1^{\alpha_j}\right)^i \left(\frac{1}{z_1^{\beta_{j-1}} w_1^{\alpha_{j-1}}}\right)^s$$

$$= \sum_{i,s=0}^{\infty} c_{is} z_1^{i\beta_j - s\beta j-1} w_1^{i\alpha_j - s\alpha_{j-1}} = f_(z_1, w_1),$$

which gives the conditions for the powers with indices (i, s) so that $a_{i,s} \neq 0$:

$$i\beta_j - s\beta_{j-1} \geq 0.$$

On $X_i \cap X_d$, we obtain

$$f_j(z_j, w_j) = \sum_{i,s=0}^{\infty} c_{is} z_j^i w_j^s =$$

$$= \sum_{i,s=0}^{\infty} c_{is} \left(\frac{1}{z_d^{\alpha_j\beta_{d-1} - \beta_j\alpha_{d-1}} w_d^{\alpha_j\beta_d - \beta_j\alpha_d}}\right)^i \left(z_d^{\beta_{j-1}\alpha_{d-1} - \alpha_{j-1}\beta_{d-1}} w_d^{\beta_{j-1}\alpha_d - \alpha_{j-1}\beta_d}\right)^s =$$

$$= \sum_{i,s=0}^{\infty} c_{is} z_d^{s(\beta_{j-1}\alpha_{d-1} - \alpha_{j-1}\beta_{d-1}) - i(\alpha_j\beta_{d-1} - \beta_j\alpha_{d-1})} w_d^{s(\beta_{j-1}\alpha_d - \alpha_{j-1}\beta_d) - i(\alpha_j\beta_d - \beta_j\alpha_d)} = f_d(z_d, w_d),$$

which gives:

$$i(-\beta_j\alpha_d + \alpha_j\beta_d) + s(\beta_{j-1}\alpha_d - \alpha_{j-1}\beta_d) \geq 0.$$

Since the vectors $v_j = -\alpha_j e_1 + \beta_j e_2$ in the fan Σ are counted counterclockwise, the last inequality can be viewed as

$$s(\beta_{j-1}\alpha_d - \alpha_{j-1}\beta_d) \geq i(\beta_j\alpha_d - \alpha_j\beta_d),$$

where both expressions $\beta_j\alpha_d - \alpha_j\beta_d$ and $\beta_{j-1}\alpha_d - \alpha_{j-1}\beta_d$ are positive. Then the condition for the indices (i, s) can be written as:

$$\frac{s(\beta_{j-1}\alpha_d - \alpha_{j-1}\beta_d)}{\beta_j\alpha_d - \alpha_j\beta_d} \geq i \geq \frac{s\beta_{j-1}}{\beta_j}$$

Again, notice that the toric variety X contains projective curves $D_1, \dots, D_{d-1}$ associated with the cones $v_1\mathbb{R}_{\geq 0}, \dots, v_{d-1}\mathbb{R}_{\geq 0}$. Each projective curve D_j is defined in X_j by the equation $w_j = 0$ and in X_{j+1} by $z_{j+1} = 0$. Any function f, which is holomorphic on X, must be constant on each D_j for $j = 1, \dots, d-1$. Moreover, because the curves intersect each other, f has the same value on all of them, as stated in [19] Chap. 5, Sec. B, Theorem 6 and Corollary 7.

3.2 THE HARTOGS PHENOMENON IN TORIC SURFACES

Let K be a compact set in a noncompact smooth toric surface X. We must verify under which conditions for X any function f that is holomorphic on $X \setminus K$ can be extended to a holomorphic function on X. It is well known that the Hartogs phenomenon holds for bundles E_k for $k = 1, 2, \dots$. However, we must show another proof of this that can be extended to a more general case. Clearly, the structure of a bundle cannot be used.

The only reasonable approach appears to involve the structure of a manifold. But a set K, compact in a manifold X, might not remain compact if cut into the patches, which are open subsets in $\mathbb{C}^n$. Therefore, the classical Hartogs extension phenomena on $\mathbb{C}^n$ with $n \geq 2$ become useless. The other versions of the Hartogs Figure, like Theorem 3.0.8, do not require compactness, but nor do they give extension to the whole patch. As we will see, the extension to whole patches is not necessary for the whole manifold. Further, some patches may not allow extensions at all (or it is not clear that they would), but the global extension on X exists.

3.2.1. The Hartogs Phenomenon in E_k for $k = 1, 2, \dots$. Before we approach the theorem, recall the description of compact sets in E_k for $k = 1, 2, \dots$ given in

Example 2.3.9 and the description of global holomorphic functions on E_k given in Section 3.1.1. The decomposition of V_n into $V_n \cap X_1$ and $V_n \cap X_2$ is shown in Figure 3.4.

Theorem 3.2.1 *The Hartogs phenomenon holds in E_k for $k = 1, 2, \ldots$*

Proof: We must recall that E_k consists of two patches $X_1 \simeq \mathbb{C}^2$ and $X_2 \simeq \mathbb{C}^2$ with coordinates (z_1, w_1) and (z_2, w_2), respectively, related on $X_1 \cap X_2$ according to the rule $z_1 = \frac{1}{z_2}$ and $w_1 = z_2^k w_2$. Let K be a compact subset in $X = E_k$ for $k = 1, 2, \ldots$ so that $X \setminus K$ is connected. And let f be holomorphic on $X \setminus K$. Notice that according to

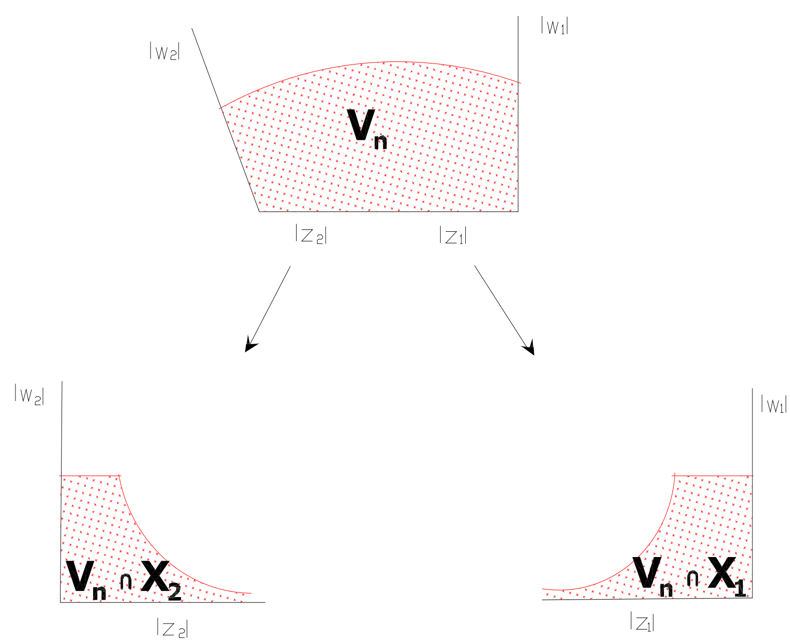

Figure 3.4: The decomposition of V_n

Example 2.3.9 each compact set is enclosed in the set V_n for some $n \in \mathbb{Z}_{\geq 1}$. Therefore, it is sufficient to prove the theorem for sets V_n. The decomposition of V_n into $V_n \cap X_1$ and

$V_n \cap X_2$ can be described as:

$$V_n \cap X_1 = \{(z_1, w_1) : |w_1| \leq n, \left|z_1^k w_1\right| \leq n\}$$

and

$$V_n \cap X_2 = \{(z_2, w_2) : |w_2| \leq n, \left|z_2^k w_2\right| \leq n\}.$$

Notice that after we apply Theorem 3.0.8 to $f_1 = f\mid_{X_1 \setminus V_n}$ and $f_2 = f\mid_{X_2 \setminus V_n}$, both extensions remain equal on $X_1 \cap X_2$ because of the uniqueness of extensions. Now they admit the following Laurent expansions for $(z_1, w_1) \in \mathbb{C}^1 \times \mathbb{C}^* \subset X_1$ and $(z_2, w_2) \in \mathbb{C}^1 \times \mathbb{C}^* \subset X_2$:

$$f_1(z_1, w_1) = \sum_{i=0, s=-\infty}^{\infty} a_{is} z_1^i w_1^s$$

and

$$f_2(z_2, w_2) = \sum_{l=0, m=-\infty}^{\infty} b_{lm} z_2^l w_2^m.$$

Since $f_1 = f_2$ on $X_1 \cap X_2$:

$$f_1(z_1, w_1) = \sum_{i=0, s=-\infty}^{\infty} a_{is} z_1^i w_1^s = \sum_{i=0, s=-\infty}^{\infty} a_{is} (\frac{1}{z_2})^i (z_2^k w_2)^s = \sum_{i=0, s=-\infty}^{\infty} a_{is} z_2^{(ks-i)} w_2^s$$

$$= \sum_{l=0, m=-\infty}^{\infty} b_{lm} z_2^l w_2^m = f_2(z_2, w_2),$$

which proves that $s \geq \frac{i}{k} \geq 0$, meaning that the expansion of f_1 contains no negative powers of w_1. Therefore, f_1 can be extended to a holomorphic function on X_1. On the other hand, starting with f_2 we obtain:

$$f_2(z_2, w_2) = \sum_{l=0, m=-\infty}^{\infty} b_{lm} z_2^l w_2^m = \sum_{l=0, m=-\infty}^{\infty} b_{lm} (\frac{1}{z_1})^l (z_1^k w_1)^m$$

$$= \sum_{l=0, m=-\infty}^{\infty} b_{lm} z_1^{(km-l)} w_1^m = \sum_{i=0, s=-\infty}^{\infty} a_{is} z_1^i w_1^s = f_1(z_1, w_1).$$

Thus, $km - l = i \geq 0$ and $m \geq \frac{l}{k} \geq 0$, which means that the expansion of f_2 contains no negative powers of w_2 and is holomorphic on X_2.

Based on the comparison of expansions $ks - i \geq 0$ and $km - l \geq 0$, the functions f_1 and f_2 define a global function on E_k with $k \in \mathbb{Z}_{\geq 1}$. ▮

3.2.2. The Hartogs Phenomenon in Toric Surfaces with a Strictly Convex Fan. Let X be a toric variety with a strictly convex fan Σ. And let K be a compact set in X. We will prove that any function holomorphic in a connected set $X \setminus K$ can be extended holomorphically to X.

A method similar to that in the previous section will be applied here. Let

$$\Sigma = \{0, v_0\mathbb{R}_{\geq 0}, v_1\mathbb{R}_{\geq 0}, \ldots, v_s\mathbb{R}_{\geq 0}, v_0\mathbb{R}_{\geq 0} + v_1\mathbb{R}_{\geq 0}, \ldots, v_{d-1}\mathbb{R}_{\geq 0} + v_d\mathbb{R}_{\geq 0}\}$$

be the fan Σ associated with X. It can be assumed without loss of generality that $v_0 = e_1$ and $v_1 = e_2$. Then for $j = 2, 3, \ldots, d$, $v_j = -\alpha_j e_1 + \beta_j e_2$ for some $\alpha_j \in \mathbb{Z}_{\geq 1}$ and $\beta_j \in \mathbb{Z}_{\geq 1}$, since $|\Sigma|$ is strictly convex. The chart X_j has coordinates (z_j, w_j) and

$$z_1 = \frac{1}{z_j^{\alpha_{j-1}} w_j^{\alpha_j}} \qquad \text{and} \qquad w_1 = z_j^{\beta_{j-1}} w_j^{\beta_j}.$$

on $X_j \cap X_1$. Because each X_j is smooth, we know that $-\alpha_{j-1}\beta_j + \alpha_j\beta_{j-1} = 1$ which gives

$$z_1 = \frac{1}{z_j^{\alpha_{j-1}} w_j^{\alpha_j}} \qquad \text{and} \qquad w_1 = z_j^{\beta_{j-1}} w_j^{\beta_j}.$$

Remember the description of compact sets from Example 2.3.10 and the global function from Section 3.1.2..

Theorem 3.2.2 *If X_Σ is a smooth toric surface with a strictly convex fan, then the Hartogs phenomenon holds in X_Σ.*

Proof: Let K be a compact set in a smooth toric surface X_Σ with a strictly convex fan. And let f be holomorphic on a connected set $X_\Sigma \setminus K$. Following the proof for E_k

with $k = 1, 2, \ldots$, notice, that each compact set in X_Σ is enclosed in a compact set V_n described in Example 2.3.10. The extension of the function f can be described in terms of extensions of the functions $f_j = f\mid_{X_j \setminus V_n}$ on $X_j \simeq \mathbb{C}^2$ with $j = 1, \ldots, d$. In fact, it is sufficient to use only the extensions for functions f_1 and f_d. (Here, d is the number of 2-dimensional cones and is fixed.)

$$V_n \cap X_1 = \{(z_1, w_1) \in X_1 : \left|z_1^{\beta_d} w_1^{\alpha_d}\right| \leq n^{\alpha_d}, |w_1| \leq n\}$$

and for $j = 2, \ldots, d-1$

$$V_n \cap X_j = \{(z_j, w_j) \in X_j : \left|z_j^{\beta_{j-1}\alpha_d - \alpha_{j-1}\beta_d} w_j^{\beta_j \alpha_d - \alpha_j \beta_d}\right| \leq n^{\alpha_d}, \left|z_j^{\beta_{j-1}} w_j^{\beta_j}\right| \leq n\}.$$

For $j = d$ we have the following:

$$V_n \cap X_d = \{(z_d, w_d) \in X_d : |z_d| \leq n^{\alpha_d}, \left|z_d^{\beta_{d-1}} w_d^{\beta_d}\right| \leq n\}$$

Although the left sides appear not to depend on d, in fact they do, since the definition of V_n depends on d. Sketches of this decomposition are shown in Figure 3.5. From Theorem 3.0.8, the function f_1 extends to $\mathbb{C}^1 \times \mathbb{C}^* \subset X_1$ and the function f_d to $\mathbb{C}^* \times \mathbb{C}^1 \subset X_d$. Since $f_1 = f_j$ on $X_1 \cap X_j$, we have:

$$\begin{aligned} f_1(z_1, w_1) &= \sum_{i=0, s=-\infty}^{\infty} a_{is} z_1^i w_1^s = \sum_{i=0, s=-\infty}^{\infty} a_{is} \left(\frac{1}{z_j^{\alpha_{j-1}} w_j^{\alpha_j}}\right)^i (z_j^{\beta_{j-1}} w_j^{\beta_j})^s \\ &= \sum_{i=0, s=-\infty}^{\infty} a_{is} z_j^{s\beta_{j-1} - i\alpha_{j-1}} w_j^{s\beta_j - i\alpha_j} = f_j(z_j, w_j) \end{aligned} \tag{9}$$

In particular, for $j = d$, we have

$$f_1(z_1, w_1) = \sum_{i=0, s=-\infty}^{\infty} a_{is} z_d^{s\beta_{d-1} - i\alpha_{d-1}} w_d^{s\beta_d - i\alpha_d} = f_d(z_d, w_d),$$

but f_d admits the following Laurent expansion:

$$f_d(z_d, w_d) = \sum_{l=-\infty, m=0}^{\infty} b_{lm} z_d^l w_d^m$$

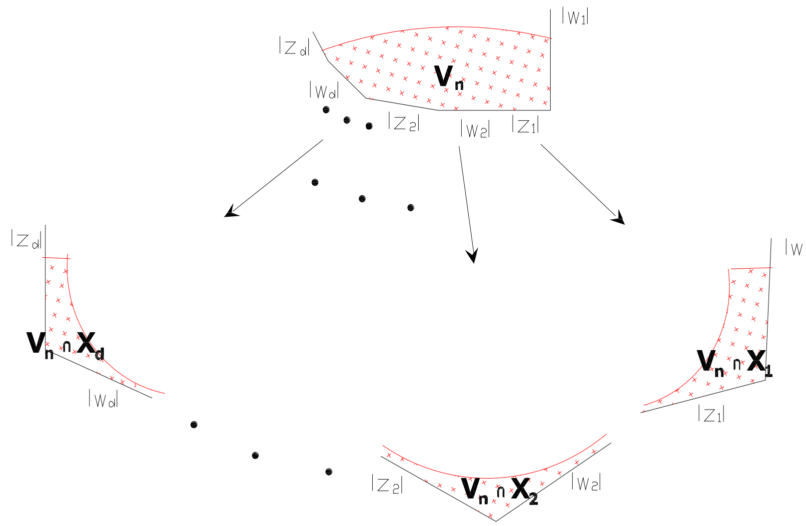

Figure 3.5: The decomposition of V_n in a toric surface with a strictly convex fan

Thus, $s\beta_d - i\alpha_d \geq 0$, which implies that $s \geq \frac{i\alpha_d}{\beta_d} \geq 0$.

Because the expansion of f_1 does not contain any negative powers of w_1, f_1 can be extended to a holomorphic function on X_1. Actually, the condition $s\beta_d - i\alpha_d \geq 0$ is the strongest condition for f_1 to define a holomorphic function on the whole X_Σ.

Following the same procedure, we find that f_d has holomorphic extension to X_d. For f_j with $j = 2, \ldots, d-1$, notice that the conditions $f_j = f_1$ and $f_j = f_d$ on $X_j \cap X_1$ and $X_d \cap X_j$, respectively, imply that f_j are holomorphic on X_j. Moreover, from the uniqueness of the extensions, $f_i = f_j$ on $X_i \cap X_j$. Thus, the Hartogs phenomenon holds for a toric surface with a strictly convex fan. ▮

This proves that the Hartogs phenomenon holds for a toric surface with a strictly convex fan with at least one 2- dimensional cone. This assumption can be relaxed somewhat. If $|\Sigma|$ covers less than a half space, then some of the 2 dimensional cones might be missing. However, missing cones simply remove points from the charts. For example, the toric variety $\mathbb{C}^2 \setminus \{0\}$ described by the fan $\Sigma = \{0, e_1\mathbb{R}_{\geq 0}, e_2\mathbb{R}_{\geq 0}\}$ allows Hartogs phenomenon to occur even if the fan is not convex. Similarly, for $\mathbb{C}^1 \times \mathbb{C}^*$, which is associated with the fan $\Sigma = \{0, e_1\mathbb{R}_{\geq 0}\}$ in $\mathbb{R}^2$, the Hartogs phenomenon holds even if the fan does not contain 2-dimensional cones.

3.2.3. The Hartogs Phenomenon in Other Toric Surfaces. It may be interesting to understand why Hartogs phenomena do not hold in a noncompact smooth toric surface with a fan that contains a line. Below are examples of E_k for $k = 0$ and $k = -1$.

Example 3.2.1 Because $E_0 = \mathbb{P}^1 \times \mathbb{C}^1$, we can consider coordinates $(z, w) \in \mathbb{P}^1 \times \mathbb{C}^1$, where z is the projective coordinate, $z = (z_0, z_1)$, and w is the affine coordinate. Consider the compact set K described in E_0 by $w = 0$. Then the function $f(z, w) = \frac{1}{w}$ is holomorphic on $E_0 \setminus K$ but is not holomorphic on E_0. Therefore, the Hartogs phenomenon does not hold in E_0. ▮

A similar example can be shown for E_{-1}.

Example 3.2.2 The toric surface E_{-1} is associated with the fan

$$\Sigma = \{0, e_1\mathbb{R}_{\geq 0}, e_2\mathbb{R}_{\geq 0}, (-e_1 - e_2)\mathbb{R}_{\geq 0}, e_1\mathbb{R}_{\geq 0} + e_2\mathbb{R}_{\geq 0}, e_2\mathbb{R}_{\geq 0} + (-e_1 - e_2)\mathbb{R}_{\geq 0}\}$$

and consists of two charts X_1 and X_2 with coordinates (z_1, w_1) and (z_2, w_2), respectively. Then:

$$z_1 = \frac{1}{z_2} \qquad \text{and} \qquad w_1 = \frac{w_2}{z_2}.$$

Notice that the set K described by $w_1 = 0$ in X_1 and $w_2 = 0$ in X_2 is compact. This set is actually a projective line. Consider the function f defined on E_{-1} as follows:

$$f = f_1(z_1, w_1) = \frac{1}{w_1} \qquad \text{on} \qquad X_1$$

$$f = f_2(z_2, w_2) = \frac{z_2}{w_2} \qquad \text{on} \qquad X_2.$$

Then f is holomorphic on $E_{-1} \setminus K$, but f is not holomorphic on E_{-1}. We see that the Hartogs phenomena does not hold in E_{-1}.

4 FIBER BUNDLES

This section treats toric varieties with a fiber bundle structure. In terms of fans, we will specify which toric varieties admit this structure and determine how to find the fan of the base and of the fiber.

4.1 DEFINITION AND EXAMPLES

Let X, B, and F be toric varieties, and let π be a mapping between toric varieties X and B with fiber F.

Definition 4.1.1 *(**Fiber bundle**) $\pi : X \to B$ is a fiber bundle with fiber F if for any $x \in B$ there exists an open set U containing x such that $\varphi : \pi^{-1}(U) \to F \times U$ is an isomorphism and $\pi \circ \varphi^{-1}(f, u) = u$, where $u \in U$ and $f \in F$.*

We call X the total space of the bundle, B the base, and F the fiber. Here, π is a projection. We will keep the notation in which the total space X is associated with a fan Σ, the base B with a fan Π, and the fiber F with a fan Δ. The present section is devoted to the description of the geometric properties of the fans of fiber bundles. The main theorem can be found on the end of the section.

Example 4.1.1 These bundles have been mentioned already. Line bundles over $\mathbb{P}^1$ are important examples of smooth noncompact toric surfaces. They are denoted here as E_k with $k \in \mathbb{Z}$. Here,

$$\Sigma = \{0, e_1\mathbb{R}_{\geq 0}, e_2\mathbb{R}_{\geq 0}, ((-1)e_1 + ke_2)\mathbb{R}_{\geq 0}, e_1\mathbb{R}_{\geq 0} + e_2\mathbb{R}_{\geq 0}, e_2\mathbb{R}_{\geq 0} + ((-1)e_1 + ke_2)\mathbb{R}_{\geq 0}\},$$

$\Delta = \{0, e_2\mathbb{R}_{\geq 0}\}$ and $\Pi = \{0, e_1\mathbb{R}_{\geq 0}, -e_1\mathbb{R}_{\geq 0}\}$. An example with $k = 3$ is shown in Figure 4.1. Notice that we consider Σ in the 2-dimensional lattice but Δ and Π in 1-dimensional lattices. Moreover, the fan Π is a subfan of Σ only for $k = 0$. We must remain aware of that while describing the fans associated with bundles.

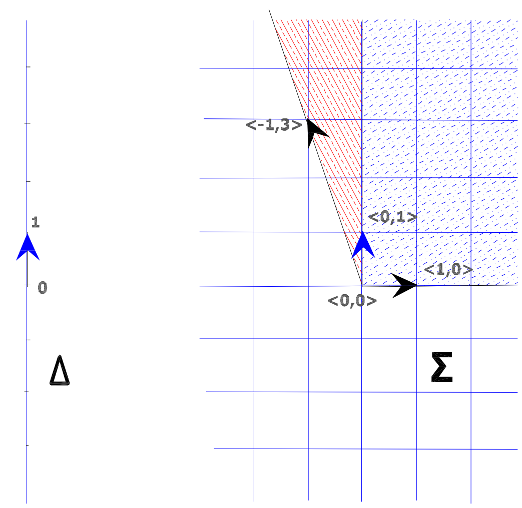

Figure 4.1: The fan of the fiber and the bundle E_k for $k = 3$

Example 4.1.2 ***(Hirzebruch surface)*** Hirzebruch surfaces are compact, smooth toric surfaces that allow a structure of a bundle over $\mathbb{P}^1$ with fiber $\mathbb{P}^1$. Figure 4.2 shows the fan of a Hirzebruch surface with $k = 3$. Notice that Δ and Π are simply fans of $\mathbb{P}^1$. Let

$$\Sigma = \{0, e_1\mathbb{R}_{\geq 0}, e_2\mathbb{R}_{\geq 0}, ((-1)e_1 + ke_2)\mathbb{R}_{\geq 0}, -e_2\mathbb{R}_{\geq 0}, \sigma_1, \sigma_2, \sigma_3, \sigma_4\},$$

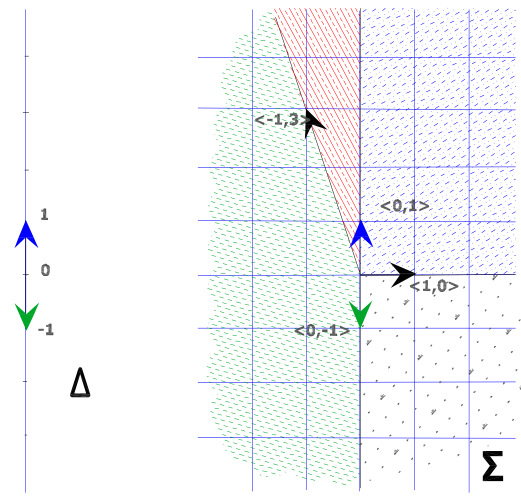

Figure 4.2: The fan of the fiber and the Hirzebruch surface with $k = 3$

where $\sigma_1, \sigma_2, \sigma_3$ and σ_4 are 2-dimensional cones:

$$\sigma_1 = e_1\mathbb{R}_{\geq 0} + e_2\mathbb{R}_{\geq 0},$$

$$\sigma_2 = (-e_1 + ke_2)\mathbb{R}_{\geq 0} + e_2\mathbb{R}_{\geq 0},$$

$$\sigma_3 = (-e_1 + ke_2)\mathbb{R}_{\geq 0} + (-e_2)\mathbb{R}_{\geq 0},$$

$$\sigma_4 = e_1\mathbb{R}_{\geq 0} + (-e_2)\mathbb{R}_{\geq 0}.$$

Clearly, $\Delta = \{0, e_2\mathbb{R}_{\geq 0}, -e_2\mathbb{R}_{\geq 0}\}$ is a subfan of Σ, but

$$\Pi = \{0, e_2\mathbb{R}_{\geq 0}, (-e_2)\mathbb{R}_{\geq 0}\}$$

is not a subfan of Σ unless $k = 0$.

4.2 FIBER BUNDLES THAT ARE TORIC VARIETIES

Clearly, the structure of a fan Σ associated with a toric variety X, which is a fiber bundle, must somehow be related to fans associated with the base B and the fiber F. Before we formulate a theorem that describes fiber bundles in general, we must focus on trivial bundles, i.e., products of toric varieties.

4.2.1. Fans of Products.

The bundle $\pi : X \to B$ with fiber F and base B is called trivial if $X = F \times B$, i.e., if X is a product of F and B.

It is important for further research to know exactly which toric varieties are products of other toric varieties. This will be explained in term of fans. Let us consider the following example, which gives an idea of their appearance.

Example 4.2.1 Let $\Sigma = \{0, e_1\mathbb{R}_{\geq 0}\}$ be a 1-dimensional fan considered in 2-dimensional lattice N. Then

$$(e_1\mathbb{R}_{\geq 0})^\vee = e_1^*\mathbb{R}_{\geq 0} + e_2^*\mathbb{R}_{\geq 0} + (-e_2^*)\mathbb{R}_{\geq 0}$$

and

$$S_{e_1\mathbb{R}_{\geq 0}} = \mathbb{C}\left[z_1, z_2, \frac{1}{z_2}\right],$$

so $X_\Sigma = \mathbb{C}^1 \times \mathbb{C}^*$. ▮

The necessary condition to obtain a product with $\mathbb{C}^*$ is that $\dim\Sigma < \dim N$. Consider the following theorem, which completely characterizes all possible products with $(\mathbb{C}^*)^k$, and notice that the fan, which describes the variety, is embedded in a smaller lattice. The proof of this theorem can be found in Appendix A.

Theorem 4.2.1 ***([14], Exercise, p. 22)*** *Let X be an n-dimensional toric variety with the fan (Σ, N). Then $X = (\mathbb{C}^*)^k \times B$ for some $(n-k)$-dimensional toric variety B if and only if $\Sigma \subset N'_{\mathbb{R}}$, where N' is a $(n-k)$-dimensional sublattice of N.* ▮

Characterization of all possible products requires the following definition:

Definition 4.2.1 *(**Product of fans**) Let (Δ_1, N_1) and (Δ_2, N_2) be fans. If $\sigma = v_1\mathbb{R}_{\geq 0} + ... + v_k\mathbb{R}_{\geq 0}$ and $\tau = w_1\mathbb{R}_{\geq 0} + ... + w_n\mathbb{R}_{\geq 0}$, then the fan $(\Delta_1 \times \Delta_2, N_1 \times N_2)$, which is their product, is defined by*

$$\Delta_1 \times \Delta_2 = \{\sigma \times \tau : \sigma \in \Delta_1, \tau \in \Delta_2\},$$

where $\sigma \times \tau = v_1\mathbb{R}_{\geq 0} + ... + v_k\mathbb{R}_{\geq 0} + w_1\mathbb{R}_{\geq 0} + ... + w_n\mathbb{R}_{\geq 0}$.

As expected, this product of fans defines a product of toric varieties. The proof of this theorem can be found in Appendix A.

Theorem 4.2.2 ***([14], Exercise, p. 22)*** *Let (Σ, N) be a fan associated with a toric variety X, (Δ, N'') with F, and (Π, N') with B. Then X is a product of toric varieties F and B if and only if $(\Sigma, N) = (\Delta \times \Pi, N'' \times N')$.* ▮

4.2.2. Fans of Fiber Bundles. The description of fiber bundles is not as easy as that for products. Clearly, there must be some kind of nonsymmetry because the base and the fiber are usually not reversible as they are for the components of a product. Before examine the relationship between the subfans of a base and a fiber in a fan of a fiber bundle, we must consider the following definition and a few examples.

Definition 4.2.2 *(**Sum of fans**) Let (Δ_1, N) and (Δ_2, N) be fans such that $\Delta_1 \cap \Delta_2 = \{0\}$. Then the fan $(\Delta_1 + \Delta_2, N)$, which is their sum, is defined by*

$$\Delta_1 + \Delta_2 = \{\sigma + \tau : \sigma \in \Delta_1, \tau \in \Delta_2\},$$

where $\sigma + \tau = v_1\mathbb{R}_{\geq 0} + ... + v_k\mathbb{R}_{\geq 0} + w_1\mathbb{R}_{\geq 0} + ... + w_n\mathbb{R}_{\geq 0}$ for $\sigma = v_1\mathbb{R}_{\geq 0} + ... + v_k\mathbb{R}_{\geq 0}$ and $\tau = w_1\mathbb{R}_{\geq 0} + ... + w_n\mathbb{R}_{\geq 0}$.

The difference between the sum and the product of two fans is crucial. For the sum, we consider the cones in the same lattice; for the product, we consider those in the product of lattices. Moreover, although the product always exists, the sum does not necessarily. For example, if $\Delta_1 = \{0, e_1\mathbb{R}_{\geq 0}, e_2\mathbb{R}_{\geq 0}\}$ and $\Delta_2 = \{0, -e_1\mathbb{R}_{\geq 0}, -e_2\mathbb{R}_{\geq 0}\}$, then the cone spanned by e_1 and $-e_1$ is not strictly convex.

Definition 4.2.3 *(**Existence of the sum of fans**) For two fans (Δ_1, N) and (Δ_2, N) we say that the sum $\Delta_1 + \Delta_2$ exists if the collection of cones $\sigma + \tau$, where $\sigma \in \Delta_1$ and $\tau \in \Delta_2$ define a fan in N.*

Let us examine a few examples in which the sum of fans exists.

Example 4.2.2 We can continue the example with line bundles. If $\Delta = \{0, e_2\mathbb{R}_{\geq 0}\}$ and $\Pi' = \{0, e_1\mathbb{R}_{\geq 0}, (-e_1 + ke_2)\mathbb{R}_{\geq 0}\}$, then $\Sigma = \Delta + \Pi'$ and

$$\Sigma = \{0, e_1\mathbb{R}_{\geq 0}, e_2\mathbb{R}_{\geq 0}, (-1e_1 + ke_2)\mathbb{R}_{\geq 0}, e_1\mathbb{R}_{\geq 0} + e_2\mathbb{R}_{\geq 0}, e_2\mathbb{R}_{\geq 0} + (-1e_1 + ke_2)\mathbb{R}_{\geq 0}\}.$$

Clearly, the fan $\Pi = \{0, e_1\mathbb{R}_{\geq 0}, -e_1\mathbb{R}_{\geq 0}\}$ cannot be a subfan of Σ unless $k = 0$. However, there is a projection $P : N \to N$ such that $P(\Pi') = \Pi$ and $P(\Delta) = \{0\}$. The situation is shown in Figure 4.3. ▮

Example 4.2.3 Similarly, for Hirzebruch surfaces, let $\Delta = \{0, e_2\mathbb{R}_{\geq 0}, (-e_2)\mathbb{R}_{\geq 0}\}$ and $\Pi' = \{0, e_1\mathbb{R}_{\geq 0}, (-e_1 + ke_2)\mathbb{R}_{\geq 0}\}$. Then $\Sigma = \Delta + \Pi'$ and Σ is described with details in Example 4.1.2. The fan $\Pi = \{0, e_1\mathbb{R}_{\geq 0}, -e_1\mathbb{R}_{\geq 0}\}$ is not a subfan of Σ, but can be obtained as an image of a projection $P : N \to N$ with $P(\Pi') = \Pi$, and $P(\Delta) = \{0\}$. Figure 4.4 shows the situation for $k = 3$. ▮

The following lemma describes a connection between the components and the sum of two fans. We will use it in the next theorem, which characterizes fiber bundles in terms of

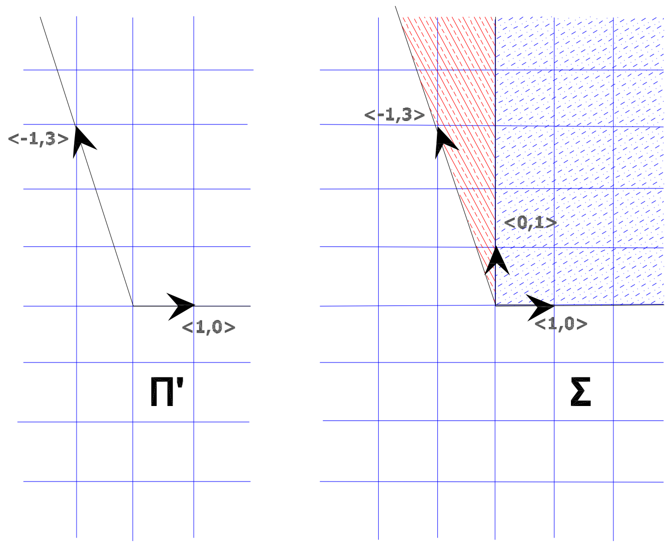

Figure 4.3: The fan of the base in E_k for $k = 3$

fans associated with the base and the fiber. Similar lemma can be found in [12], Section VI, lemma 6.6, but the author assumes that the fiber is compact, which does not give the full characterization of all fiber bundles. In this lemma, we assume that $\dim \Pi' = n - k$, but Π' is a fan considered in an n-dimensional lattice, $\dim \Delta = k$, and Δ is a fan in a k-dimensional lattice. A 3-dimensional example is presented in Figure 4.5.

Lemma 4.2.1 *Let $\Sigma = \Delta + \Pi'$, where Σ is an n-dimensional fan in an n-dimensional lattice N and Δ is a k-dimensional fan in an k-dimensional lattice N'', which is a sublattice of N. Let Π' be an $(n-k)$-dimensional fan in an n-dimensional lattice N. Then there exists an orthogonal projection $P : N_{\mathbb{R}} \to N_{\mathbb{R}}$ with $\ker P = N''_{\mathbb{R}}$, which sends Π' bijectively onto an $(n-k)$-dimensional fan Π in an $(n-k)$-dimensional lattice $N'_{\mathbb{R}} = N_{\mathbb{R}}/N''_{\mathbb{R}}$.*

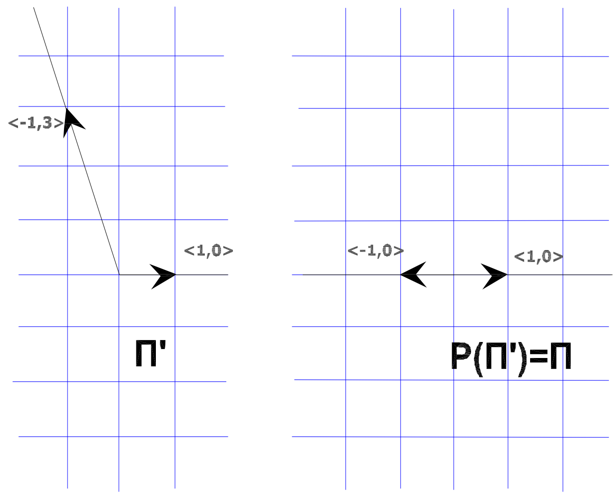

Figure 4.4: The projection of the fan of the base for the Hirzebruch surface

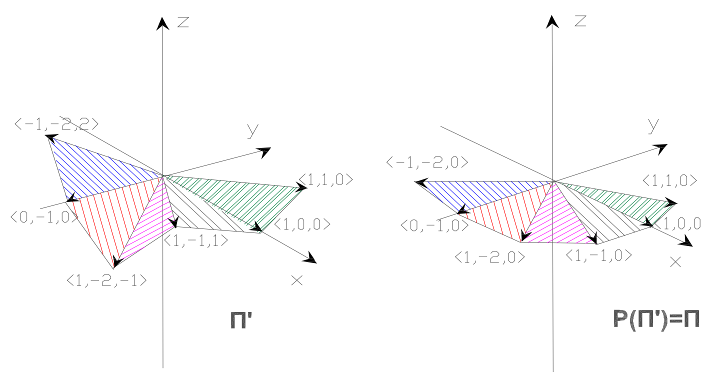

Figure 4.5: An example of a projection of a 2-dimensional fan

Proof: Based on the assumption of the lemma, we know that $\Delta \subset N''_{\mathbb{R}}$, where N'' is a sublattice of N and $N''_{\mathbb{R}}$ is a linear subspace of $N_{\mathbb{R}}$. So, let P be orthogonal projection $N_{\mathbb{R}} \to N_{\mathbb{R}}$ with $\ker P = N''_{\mathbb{R}}$, and let $N'_{\mathbb{R}} = N_{\mathbb{R}}/N''_{\mathbb{R}}$. Here, $N'_{\mathbb{R}}$ can be treated as a linear subspace of $N_{\mathbb{R}}$. First, notice that if $\tau \in \Pi'$ has representation

$$\tau = v_1\mathbb{R}_{\geq 0} + \ldots + v_s\mathbb{R}_{\geq 0},$$

for $1 \leq s \leq n-k$, then

$$P(\tau) = P(v_1)\mathbb{R}_{\geq 0} + \ldots + P(v_s)\mathbb{R}_{\geq 0},$$

and $\dim P(\tau) = \dim \tau = s$. Assume that this is not true, i.e., that the vectors $P(v_1), \ldots, P(v_s)$ are linearly dependent over $\mathbb{R}$. Then there exists $a_1, \ldots, a_s \in \mathbb{R}$ such that

$$a_1P(v_1) + \ldots + a_sP(v_s) = 0.$$

Then the vector $a_1v_1 + \ldots + a_sv_s \in N''_{\mathbb{R}}$. However, from the definition of the sum, $\Pi' \cap N'' = \{0\}$; thus, $a_1v_1 + \ldots + a_sv_s = 0$. Because the vectors $v_1, \ldots, v_s$ are linearly independent in $N_{\mathbb{R}}$ (as we assumed in Section 2, the cones are always simplicial), we conclude that $a_1 = \ldots = a_s = 0$, so $P(v_1), \ldots, P(v_s)$ are linearly independent in N' and generate the simplicial cone $P(\tau)$. The proof actually went further because it is clear now that if a face of σ is generated by a subset of $\{v_1, \ldots, v_s\}$, then its image is generated by images of the generators. Thus, P sends faces of σ to faces of $P(\sigma)$, which proves that $P(\Pi')$ is a fan in N''. We must now show that P sends Π' bijectively onto $P(\Pi') = \Pi$. Let $\sigma_1, \sigma_2 \in \Pi'$ be two cones such that $P(\sigma_1) = P(\sigma_2)$. Since the projection P keeps the number of generators unchanged, we can assume that σ_1 and σ_2 have the same number of generators:

$$\sigma_1 = v_1\mathbb{R}_{\geq 0} + \ldots + v_j\mathbb{R}_{\geq 0}$$

and

$$\sigma_2 = w_1\mathbb{R}_{\geq 0} + \ldots + w_j\mathbb{R}_{\geq 0}$$

for some linearly independent vectors $v_1, \ldots, v_j$ and $w_1, \ldots, w_j$, with $1 \leq j \leq n-k$. Since σ_1 and σ_2 are simplicial cones, after some denumeration in

$$P(\sigma_1) = P(v_1)\mathbb{R}_{\geq 0} + \ldots + P(v_j)\mathbb{R}_{\geq 0}$$

or

$$P(\sigma_2) = P(w_1)\mathbb{R}_{\geq 0} + \ldots + P(w_j)\mathbb{R}_{\geq 0},$$

we can assume that $P(v_i) = P(w_i)$ for $i = 1, \ldots, j$. Then, because P is linear, we find that $v_i - w_i \in \mathrm{Ker}P = N''_{\mathbb{R}}$. Now, we must prove that $v_i - w_i = 0$ for all $i = 1, \ldots, j$. As we have assumed above, $\dim\Delta = k$; therefore, there exists a k-dimensional simplicial cone $\delta \in \Delta(k)$:

$$\delta = u_1\mathbb{R}_{\geq 0} + \ldots + u_k\mathbb{R}_{\geq 0}.$$

Then $\sigma_1 + \delta$ and $\sigma_2 + \delta$ are cones in Σ with generators as follows:

$$\sigma_1 + \delta = v_1\mathbb{R}_{\geq 0} + \ldots + v_j\mathbb{R}_{\geq 0} + u_1\mathbb{R}_{\geq 0} + \ldots + u_k\mathbb{R}_{\geq 0}$$

and

$$\sigma_2 + \delta = w_1\mathbb{R}_{\geq 0} + \ldots + w_j\mathbb{R}_{\geq 0} + u_1\mathbb{R}_{\geq 0} + \ldots + u_k\mathbb{R}_{\geq 0}.$$

Since $\Delta + \Pi'$ exists, we can claim that $\sigma_1 + \delta, \sigma_2 + \delta \in \Sigma(j+k)$. Set $v_i - w_i = r_i \in N''$ for $i = 1, \ldots, j$. Then

$$\sigma_1 + \delta = (w_1 + r_1)\mathbb{R}_{\geq 0} + \ldots + (w_j + r_j)\mathbb{R}_{\geq 0} + u_1\mathbb{R}_{\geq 0} + \ldots + u_k\mathbb{R}_{\geq 0}.$$

If there exists $r_i \neq 0$, then $\sigma_1 \neq \sigma_2$. However, the relative interiors of $\sigma_1 + \delta$ and $\sigma_2 + \delta$ intersect nontrivially since we find $a_i, b_i, c_i, d_i > 0$, so that:

$$\sum_{i=1}^{n} a_i(w_i + r_i) + \sum_{i=1}^{k} b_i(u_i) = \sum_{i=1}^{n} c_i(w_i) + \sum_{i=1}^{k} d_i(u_i).$$

Simply choose $a_i = c_i = 1$ and notice that each r_i belongs to $\mathrm{Span}\{u_1, \ldots, u_k\}$. This contradicts the existence of $\Pi' + \Delta$; thus, all $r_i = 0$ and $\sigma_1 = \sigma_2$. Thus, we prove that P sends bijectively Π' onto $\Pi = P(\Pi')$. ▮

Notice that the assumptions $\dim\Pi' = n - k$ and $\dim\Delta = k$ can be somewhat relaxed. We could say that $\dim\Delta \leq k$, which might allow some extra structure of $\mathbb{C}^*$ bundle on F. The details regarding the fiber $(\mathbb{C}^*)^k$ are not pertinent here, but they are very educational.

Let us work a detailed example:

Example 4.2.4 Let $\Pi' = \{0, e_1\mathbb{R}_{\geq 0}, e_3\mathbb{R}_{\geq 0}, (e_1 + e_3)\mathbb{R}_{\geq 0}\} \subset \mathbb{R}^3$ and $\Delta = \{0\} \subset \mathbb{R}^3$ be two fans considered in the same lattice. Then $\Delta + \Pi'$ describes a trivial bundle with fiber $\mathbb{C}^*$, but the lattice N'', which contains Δ, cannot be chosen freely. It is not true that for any projection $P : \mathbb{Z}^3 \to \mathbb{Z}^3$ the fan $P(\Pi')$ describes the base and that P sends Π' bijectively onto Π. If, for example, the projection P is defined as follows: $P(e_1) = 0$, $P(e_2) = e_2$, $P(e_3) = e_3$, then $P(\Pi') = \{0, e_3\mathbb{R}_{\geq 0}\}$ is a fan. However, P is not a bijection since $P(e_1\mathbb{R}_{\geq 0}) = P((e_1 + e_3)\mathbb{R}_{\geq 0}) = e_3\mathbb{R}_{\geq 0}$. Here, we must define the projection P as $P(e_1) = e_1$, $P(e_2) = 0$, $P(e_3) = e_3$. ▮

The difficulty, then, is not in the dimension of Δ, but rather in the support $|\Delta|$. If $|\Delta| \neq N''_{\mathbb{R}}$, then P might not be unique. Notice that since N'' is a sublattice of N, with $N'_{\mathbb{R}} = N_{\mathbb{R}}/N''_{\mathbb{R}}$ for some sublattice N' in N, we can treat N'' as a complete fan defined by generators of N'' over $\mathbb{N}$. We are ready to formulate the following definition:

Definition 4.2.4 *(**Existence of the sum of a fan and a lattice**) For a sublattice N'' of N and a fan (Π', N) we say that the sum $\Pi' + N''$ exists if for any fan Δ so that $|\Delta| = N''_{\mathbb{R}}$, the sum $\Pi + \Delta$ exists.*

It may be not clear at this point that the existence of $\Pi' + N''$ is well defined since changing the fans (Δ, N'') we might obtain different results. However, the existence of the projection P claimed in Lemma 4.2.1 does not depend on the fan, only on its support. In particular, two complete fans give the same answer in Definition 4.2.4.

Consider the following example, where the sum of fans does not describe a fiber bundle and the sum of a lattice and a fan does not exists.

Example 4.2.5 Let (Π', N) be a 1-dimensional fan in 2-dimensional lattice N defined as:

$$\Pi' = \{0, e_1\mathbb{R}_{\geq 0}, e_2\mathbb{R}_{\geq 0}, (e_1 + e_2)\mathbb{R}_{\geq 0}\}.$$

If $\Delta\{0\}$ is considered in the same lattice N, then $\Pi' + \Delta$ exists. However, there is no projection $P : N \to N$ such that the set $P(\Pi')$ describes a fan because the sum $\Pi' + N''$ does not exist for any choice of a 1-dimensional sublattice N'' in N. Assume otherwise. If $\{0, v\mathbb{R}_{\geq 0}, -v\mathbb{R}_{\geq 0}\}$ is the sublattice, then $\Pi' + N''$ contains the 2-dimensional cones $e_1\mathbb{R}_{\geq 0} + v\mathbb{R}_{\geq 0}$, $e_2\mathbb{R}_{\geq 0} + v\mathbb{R}_{\geq 0}$, and $(e_1 + e_2)\mathbb{R}_{\geq 0} + v\mathbb{R}_{\geq 0}$ and the interiors of at least two of them have nontrivial intersections. Thus, $\Pi' + N''$ cannot exist. ▮

Now we can formulate a more general version of the lemma:

Lemma 4.2.2 *Let $\Sigma = \Delta + \Pi'$, where Σ is an n-dimensional fan in an n-dimensional lattice N, Δ is at most a k-dimensional fan in a k-dimensional lattice N'', which is a sublattice of N, and let Π' be at most an $(n-k)$-dimensional fan in an n-dimensional lattice N. If $N''+\Pi'$ is a fan in N, then there exists an orthogonal projection $P : N_{\mathbb{R}} \to N_{\mathbb{R}}$ with $\ker P = N''_{\mathbb{R}}$, which sends Π' bijectively onto an $(n-k)$-dimensional fan Π in an $(n-k)$-dimensional lattice $N'_{\mathbb{R}} = N_{\mathbb{R}}/N''_{\mathbb{R}}$.*

Proof: Since in the proof of Lemma 4.2.1, the projection P depends on the sublattice N'' rather than on the subfan Δ, we can use the same proof as above. ▮

We can formulate the following theorem, which characterizes toric varieties with a fiber bundle structure:

Theorem 4.2.3 *Let (Σ, N) be a fan associated with toric variety X, and (Δ, N'') a fan associated with toric variety F, where Δ is a subfan of Σ and N'' is a sublattice of N.*

Then X is a fiber bundle with fiber F if and only if there exists such a subfan Π' in Σ that $\Sigma = \Delta + \Pi'$ exists and $N'' + \Pi'$ exists. ▌

Since the lengthy proof of this theorem follows the ideas presented in [12], it is shown in Appendix A. The assumption used in [12] that the fiber is compact was removed here, otherwise we would not be able to apply this theorem to vector bundles.

Example 4.2.6 Notice that n-dimensional bundles with fiber $(\mathbb{C}^*)^k$ are described by a fan (Σ, N), with $\dim\Sigma \leq n - k$, and the sum $\Sigma + N''$ exists for some sublattice N'' with $\dim N'' = k$. Since $\Sigma + N''$ exists, we know that Σ is, in fact, a fan in the lattice

$$N'_{\mathbb{R}} = N_{\mathbb{R}}/N''_{\mathbb{R}}$$

with $\dim N'' = n - k$.

5 HOLOMORPHIC EXTENSIONS IN LINE BUNDLES

The Hartogs phenomenon for line bundles over toric varieties, is more complicated than for vector bundles with higher-dimensional fibers. However, in line bundles we can still solve the $\overline{\partial}$-problem, namely $\overline{\partial}u = \omega$ for a closed $(0,1)$ form, compactly supported along the fibers, and a solution u can be chosen that it vanishes along the fibers. The method shown in this section is independent from this result. Figure 5.1 shows a sketch of a fan of a line bundle.

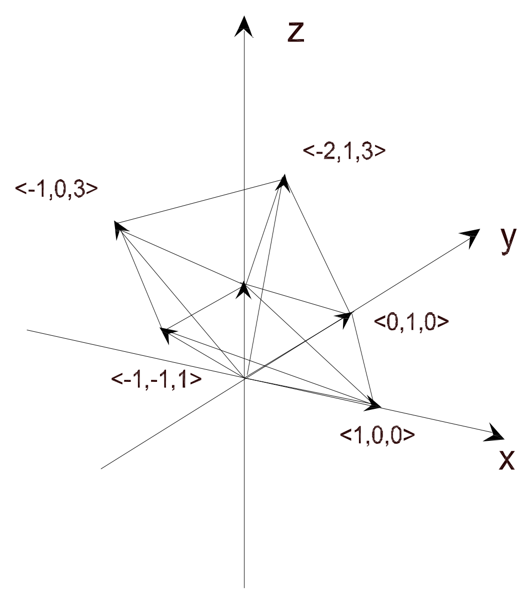

Figure 5.1: A fan of a line bundle

Based on Theorem 4.2.3, we can formulate the following theorem for line bundles:

Theorem 5.0.4 *Let (Σ, N) be a fan associated with a toric variety X. Then X is a line bundle if and only if there exists a subfan Π' of Σ and $v\mathbb{R}_{\geq 0} \in \Sigma$, such that $\Sigma = v\mathbb{R}_{\geq 0} + \Pi'$ and $N'' + \Pi'$ exists, where N'' is a sublattice in N generated by v.* ▮

We must describe the fan Σ of a line bundle in detail with the additional assumptions that the base B is compact and that Σ is strictly convex. The notation, again, is as follows: the fan (Π, N') is associated with the base B, and (Π', N) is the subfan of (Σ, N) so that $P(\Pi') = \Pi$, where P is a projection, as described in Lemma 4.2.2. And $\Pi(k)$ describes the set of cones from Π with dimension k.

5.1 FANS

This subsection presents explicit formulas for the generators of the dual cone $\sigma^\vee$ in terms of the generators of the cone $\sigma \subset \Sigma$. We start with an $(n-1)$-dimensional cone $\tau \in \Pi$ so that $P(\tau') = \tau$ and $\tau' \in \Pi' \subset \Sigma$. Here, P is a projection along N'' as described in the proof of Lemma 4.2.1. Let $e_1, \ldots, e_{n-1}$ be the standard basis for $\mathbb{R}^{n-1} = N'_\mathbb{R} = N_\mathbb{R}/N''_\mathbb{R}$. We can always assume that the vector, which generates N'' is $v = e_n$. Let a cone $\tau \in \Pi(n-1)$ be generated by positively oriented vectors $v_1, \ldots, v_{n-1}$, namely

$$\tau = v_1\mathbb{R}_{\geq 0} + \ldots + v_{n-1}\mathbb{R}_{\geq 0},$$

with

$$v_1 = v_{1,1}e_1 + \ldots + v_{1,n-1}e_{n-1}$$
$$\vdots$$
$$v_{n-1} = v_{n-1,1}e_1 + \ldots + v_{n-1,n-1}e_{n-1}.$$

Because $\tau' \in \Pi'$ fulfils $P(\tau') = \tau$, where P is an orthogonal projection along N'', we can write

$$\tau' = v_1'\mathbb{R}_{\geq 0} + \ldots + v_s'\mathbb{R}_{\geq 0},$$

where

$$\begin{aligned} v_1' &= v_{1,1}e_1 + \ldots + v_{1,n-1}e_{n-1} + v_{1,n}e_n \\ &\vdots \\ v_{n-1}' &= v_{n-1,1}e_1 + \ldots + v_{n-1,n-1}e_{n-1} + v_{n-1,n}e_n. \end{aligned}$$

Then the cone $\sigma = \tau' + e_n\mathbb{R}_{\geq 0} \in \Sigma$ is generated by

$$\sigma = v_1'\mathbb{R}_{\geq 0} + \ldots + v_{n-1}'\mathbb{R}_{\geq 0} + e_n\mathbb{R}_{\geq 0}.$$

And $\sigma^\vee \in M$ is described by the inequalities

$$\begin{gathered} v_{1,1}x_1 + \ldots + v_{1,n-1}x_{n-1} + v_{1,n}x_n \geq 0 \\ \vdots \\ v_{n-1,1}x_1 + \ldots + v_{n-1,n-1}x_{n-1} + v_{n-1,n}x_n \geq 0 \\ x_n \geq 0, \end{gathered}$$

where $(x_1, \ldots, x_n)$ are coordinates in $M_{\mathbb{R}}$.

Since $\dim\tau = n - 1$ and $\sigma = \tau + e_n\mathbb{R}_{\geq 0}$, then $\dim\sigma = n$. The generators of $\sigma^\vee$ lie on the intersections of the hypersurfaces described above. The generators of $\sigma^\vee$ can be described in detail. Let A_σ be the following matrix associated with the cone σ:

$$A_\sigma = \begin{bmatrix} v_{1,1} & \ldots & & v_{1,n} \\ \vdots & & & \vdots \\ v_{n-1,1} & \ldots & & v_{n-1,n} \\ 0 & \ldots & 0 & 1 \end{bmatrix}$$

Denote by $A_{\sigma,j}$ the matrix A_σ with the jth row removed, A^i_σ the matrix with the ith column removed, and $A^i_{\sigma,j}$ the matrix with the jth row and the ith column removed. Let $e^*_1, \ldots, e^*_n$ be the basis in M dual to $e_1, \ldots, e_n$ and let:

$$\beta_{\sigma,j} = \beta^1_{\sigma,j} e^*_1 + \ldots + \beta^n_{\sigma,j} e^*_n$$

be the generator of $\sigma^\vee$ so that $(v'_j, \beta_{\sigma,j}) > 0$, $(v'_k, \beta_{\sigma,j}) = 0$ for $k \neq j$ and $(e_n, \beta_{\sigma,j}) = 0$. Clearly, then, $\beta_{j,n} = 0$ for all $j = 1, \ldots, n-1$. Other coordinates can be expressed as determinants of some matrices:

Lemma 5.1.1 *With the notation used above, $\beta^i_{\sigma,j} = (-1)^{i+j} \det A^i_{\sigma,j}$ for $j = 1, \ldots, n-1$ and $i = 1, \ldots, n-1$. The sign is chosen for $\beta_{\sigma,j}$ so that $(v'_j, \beta_{\sigma,j}) > 0$.*

Proof: We must prove that $(v'_k, \beta_{\sigma,j}) = 0$ for $k \neq j$. Notice that

$$(v'_k, \beta_{\sigma,j}) = v_{k,1}(-1)^{j+1} \det A^1_{\sigma,j} + \ldots + v_{k,n}(-1)^{j+n} \det A^n_{\sigma,j} =$$

$$\det[v'_k, v'_1, \ldots, v'_{j-1}, v'_{j+1}, \ldots, v'_{n-1}, e_n] = 0,$$

since v'_k appears twice. ▮

The other generator lies on the intersection of the following half spaces:

$$v_{1,1}x_1 + \ldots + v_{1,n-1}x_{n-1} + v_{1,n}x_n \geq 0$$

$$\vdots$$

$$v_{n-1,1}x_1 + \ldots + v_{n-1,n-1}x_{n-1} + v_{n-1,n}x_n \geq 0.$$

It can also be described in detail. We will denote it as α_σ to distinguish from the others.

Lemma 5.1.2 *If X_τ is smooth, then $\alpha_\sigma = \alpha^1_\sigma e^*_1 + \ldots + \alpha^{n-1}_\sigma e^*_{n-1} + \alpha^n_\sigma e^*_n$ is the generator* *of* $\sigma^\vee$ *with*

$$\alpha_\sigma^i = (-1)^{n-i}\det A_\sigma^i = (-1)^{n-i}\det \begin{bmatrix} v_{1,1} & \dots & v_{1,i-1} & v_{1,i+1} & \dots & v_{1,n} \\ \vdots & & \vdots & & & \vdots \\ v_{n-1,1} & \dots & v_{n-1,i-1} & v_{n-1,i+1} & \dots & v_{n-1,n} \end{bmatrix}$$

for $i = 1, \dots, n$. *In particular,* $\alpha_n = 1$.

Proof: It is sufficient to prove that $\alpha_\sigma = \alpha_\sigma^1 e_1^* + \dots + \alpha_\sigma^{n-1} e_{n-1}^* + \alpha_\sigma^n e_n^*$ fulfills the following conditions: $(\alpha_\sigma, e_n) > 0$ and $(\alpha_\sigma, v_j') = 0$ for $j = 1, \dots, n-1$. Clearly, $(\alpha_\sigma, e_n) = \alpha_n = \det[v_1, \dots, v_{n-1}] = 1$, since $v_1, \dots, v_n$ are positively oriented. Now, notice that for any j:

$$(\alpha_\sigma, v_j') = \alpha_\sigma^1 v_{j,1} + \dots + \alpha_\sigma^n v_{j,n} = (-1)^n \det[v_j', v_1', \dots, v_{n-1}'] = 0,$$

which comes from the expansion of $\det[v_j', v_1', \dots, v_{n-1}']$ with respect to v_j'. ▮

Thus we have proved that:

Lemma 5.1.3 *If rows of the matrix* A_σ *generate the cone* $\sigma \subset N$, *then the rows of the matrix* $(A_\sigma^{-1})^t$ *generate the dual cone* $\sigma^\vee \subset M$. ▮

5.1.1. Strictly Convex Fans of Line Bundles.

This subsection addresses the question of under what conditions a fan of a line bundle is strictly convex. We address this question for a case in which the base is compact, which is equivalent to the condition that its fan is complete.

Let $\Sigma = \Pi' + e_n\mathbb{R}_{\geq 0}$ be a fan of a line bundle X_Σ over a compact base $B = X_\Pi$. The convexity of the fan Σ can be expressed in terms of the generators of the fan Π'. If $P : N \to N$ is a projection defining the line bundle, then $P(\Pi') = \Pi$. Let us list all generators in both fans:

$$\Pi(1) = \{v_1, \dots, v_d\}$$

and

$$\Pi'(1) = \{v'_1, \ldots, v'_d\}$$

Then, as in the preceding subsection:

$$v_1 = v_{1,1}e_1 + \ldots + v_{1,n-1}e_{n-1}$$

$$\vdots$$

$$v_d = v_{d,1}e_1 + \ldots + v_{d,n-1}e_{n-1}.$$

and

$$v'_1 = v_{1,1}e_1 + \ldots + v_{1,n-1}e_{n-1} + v_{1,n}e_n$$

$$\vdots$$

$$v'_d = v_{d,1}e_1 + \ldots + v_{d,n-1}e_{n-1} + v_{d,n}e_n.$$

Thus, Σ is convex as a positive linear combination of vectors in $\mathbb{R}^n$ if none of the vectors v'_i, belongs to the convex hull created by some of the vectors $v'_1, \ldots, v'_{i-1}, v'_{i+1}, \ldots, v'_d, e_n$. Consider the following lemmas, which convert the convexity of the fan Σ into the properties of the generators of $\Pi(1)$. The first condition describes the situation when vector v_i belongs to the convex hull created by some other vectors from $\Pi(1)$, and its position in the fan Π' is restricted by the positions of those vectors. This condition describes a convex fan. The second condition prevents the fan Σ from containing a line created by vector $v'_i \in \Pi'$ and a convex hull of some other vectors from Π'. Those conditions combined together describe a strictly convex fan of a line bundle over a compact base.

Lemma 5.1.4 *The fan $\Sigma = \Pi' + e_n\mathbb{R}_{\geq 0} \subset \mathbb{R}^n$ is convex if and only if the following condition is satisfied:*

1) If for some $v_i \in \Pi(1)$ we have $v_i = \sum_{j \neq i} b_j v_j$ for $v_j \in \Pi(1)$ and $b_j \geq 0$, then $\sum_{j \neq i} b_j v_{j,n} - v_{i,n} > 0$.

The convex fan $\Sigma = \Pi' + e_n\mathbb{R}_{\geq 0} \subset \mathbb{R}^n$ *is strictly convex if the following condition is satisfied:*

2) If for some $v_i \in \Pi(1)$ *we have* $-v_i = \sum_{j\neq i} b_j v_j$ *for* $v_j \in \Pi(1)$ *and* $b_j \geq 0$*, then* $\sum_{j\neq i} b_j v_{j,n} + v_{i,n} > 0$*.*

Proof: First, we will prove that if 1) is not fulfilled, then the fan Σ is not convex. For 1) let $v_i \in \Pi(1)$ be a nontrivial positive linear combination of some other vectors from $\Pi(1)$:

$$v_i = \sum_{j\neq i} b_j v_j,$$

for $b_j \geq 0$. If the difference

$$\sum_{j\neq i} b_j v_{j,n} - v_{i,n} \geq 0,$$

then the fan Σ is convex. If

$$\sum_{j\neq i} b_j v_{j,n} - v_{i,n} < 0,$$

then the vector v_i' can be represented as a positive sum:

$$v_i' = \sum_{j\neq i} b_j v_j' + \left(v_{i,n} - \sum_{j\neq i} b_j v_{j,n} \right) e_n;$$

therefore, Σ is not convex.

Now, we will prove that if 2) is not fulfilled, then Σ is not strictly convex. Let $v_i \in \Pi(1)$ be such that:

$$-v_i = \sum_{j\neq i} b_j v_j,$$

with $b_j \geq 0$. If the sum

$$\sum_{j\neq i} b_j v_{j,n} + v_{i,n} \leq 0,$$

then the fan Σ is not strictly convex, since

$$-v_i' = \sum_{j\neq i} b_j v_j' + \left(-v_{i,n} - \sum_{j\neq i} b_j v_{j,n} \right) e_n.$$

Next, we will prove that if the fan Σ is not convex, then 1) is not fulfilled. If the fan Σ is not convex, then there exists $v'_i \in \Pi'(1)$ so that:

$$v'_i = \sum_{j \neq i} b_j v'_j$$

for some $b_j \geq 0$. Then

$$v'_i = \sum_{j \neq i,n} b_j v'_j + b_n e_n$$

with $b_n \geq 0$, so

$$v_i = \sum_{j \neq i} b_j v_j$$

and

$$v_{i,n} = \sum_{j \neq i} b_j v_{j,n} + b_n.$$

Since

$$v_{i,n} - \sum_{j \neq i} b_j v_{j,n} = b_n \geq 0,$$

the condition 1) is not fulfilled. We will now prove that if the fan Σ is not strictly convex, then 2) is not fulfilled. Since Σ is a sum of a positive linear combination of its generators (a sum of strictly convex cones), then the property that it contains a line implies that for some vector $v'_i \in \Pi'(1)$, we have $-v'_i \mathbb{R}_{\geq 0} \in |\Sigma|$, i.e.,

$$-v'_i = \sum_{j \neq i} b_j v'_j.$$

with $b_j \geq 0$. Then

$$-v_i = \sum_{j \neq i} b_j v_j,$$

and

$$-v_{i,n} = \sum_{j \neq i} b_j v_{j,n},$$

which proves that

$$v_{i,n} + \sum_{j \neq i} b_j v_{j,n} = 0,$$

and 2) is not fulfilled. ▮

Remark 5.1.1 *Note that, particularly if Σ is strictly convex, then Σ is a subset of a half-space. If we assume that one cone in $\Sigma(n)$ is generated by the standard basis $e_1, \ldots, e_n$, then $\Sigma \subset \{(x_1, \ldots, x_n) \in N_{\mathbb{R}} = \mathbb{R}^n : x_n \geq 0\}$, which implies that $v_{i,n} > 0$ for all $i = 1, \ldots, n-1$ and for all cones $\sigma \in \Sigma$ (except that generated by standard basis vectors).* ▌

5.1.2. Properties of Cones that Share a Face. This paragraph presents technical details necessary for further investigation.

A complex manifold with d coordinate patches admits $d(d-1)$ changes of coordinates. In the case of holomorphic extension problems in line bundles over a compact base, we can narrow our work to only few such changes. For a fixed affine variety X_σ associated with n-dimensional cone $\sigma \in \Sigma$, it is sufficient to consider those coordinate patches $X_{\overline{\sigma}}$ that are defined by those cones $\overline{\sigma} \in \Sigma$ that share an $(n-1)$-dimensional face with σ. In a complete n-dimensional fan, each simplicial cone $\sigma \in \Sigma(n)$ has exactly n faces of dimension $n-1$. In the case of line bundles over compact bases, for $n-1$ faces, there exists a cone that shares this face with σ. We will denote those cones as $\overline{\sigma}_1, \ldots, \overline{\sigma}_{n-1}$. Since we can always assume that one n-dimensional cone in a fan Σ is generated by the standard basis vectors, we will analyze the situation for the cone $\sigma = e_1\mathbb{R}_{\geq 0} + \ldots + e_n\mathbb{R}_{\geq 0}$. The cones that share a $(n-1)$-dimensional face with σ are as follows:

$$
\begin{aligned}
\overline{\sigma}_1 &= u_1\mathbb{R}_{\geq 0} + e_2\mathbb{R}_{\geq 0} + \ldots + e_n\mathbb{R}_{\geq 0} \\
&\vdots \\
\overline{\sigma}_s &= e_1\mathbb{R}_{\geq 0} + \ldots + e_{s-1}\mathbb{R}_{\geq 0} + u_s\mathbb{R}_{\geq 0} + e_{s+1}\mathbb{R}_{\geq 0} + \ldots + e_n\mathbb{R}_{\geq 0} \\
&\vdots \\
\overline{\sigma}_{n-1} &= e_1\mathbb{R}_{\geq 0} + e_2\mathbb{R}_{\geq 0} + \ldots + u_{n-1}\mathbb{R}_{\geq 0} + e_n\mathbb{R}_{\geq 0},
\end{aligned}
$$

where

$$u_1 = u_{1,1}e_1 + \ldots + u_{1,n}e_n$$
$$\vdots$$
$$u_s = u_{s,1}e_1 + \ldots + u_{1,n}e_n$$
$$\vdots$$
$$u_{n-1} = u_{n-1,1}e_1 + \ldots + u_{n-1,n}e_n$$

with $u_{i,j} \in \mathbb{Z}$. Since $\overline{\sigma}_s$ for $s = 1, \ldots, n-1$ are simplicial cones, we can find the coefficients $u_{s,s}$. For $s = 1$ we have:

$$\det \begin{bmatrix} u_{1,1} & u_{1,2} & \ldots & u_{1,n} \\ 0 & 1 & \ldots & 0 \\ \vdots & & & \vdots \\ 0 & & \ldots & 1 \end{bmatrix} = u_{1,1},$$

which proves that $u_{1,1} = -1$, since the sequence of vectors $u_1, e_2, \ldots, e_n$ has a negative orientation. Similar computations for other cones prove that

$$u_1 = (-1)e_1 + u_{1,2}e_2 + \ldots + u_{1,n}e_n$$
$$\vdots$$
$$u_s = u_{s,1}e_1 + \ldots + u_{s,s-1}e_{s-1} + (-1)e_s + u_{s,s+1}e_{s+1} + \ldots + u_{s,n}e_n$$
$$\vdots$$
$$u_{n-1} = u_{n-1,1}e_1 + \ldots + (-1)e_{n-1} + u_{n-1,n}e_n$$

If the fan Σ is strictly convex, then the condition $u_{s,n} \in \mathbb{Z}_{>0}$ is fulfilled for each $s = 1, \ldots, n-1$.

Lemma 5.1.5 *Let σ be a cone in standard position and let $\overline{\sigma}_1, \ldots, \overline{\sigma}_{n-1}$ share a face with σ. With notation as above, the dual cones can be described as follows:*

$$\overline{\sigma}_1^{\vee} = (-1)e_1^*\mathbb{R}_{\geq 0} + (u_{1,2}e_1^* + e_2^*)\,\mathbb{R}_{\geq 0} + \ldots + (u_{1,n}e_1^* + e_n^*)\,\mathbb{R}_{\geq 0}$$

$$\vdots$$

$$\overline{\sigma}_s^{\vee} = (u_{s,1}e_s^* + e_1^*)\,\mathbb{R}_{\geq 0} + \ldots + \left(u_{s,s-1}e_s^* + e_{s-1}^*\right)\mathbb{R}_{\geq 0} + (-1)e_s^*\mathbb{R}_{\geq 0} +$$

$$\left(u_{s,s+1}e_s^* + e_{s+1}^*\right)\mathbb{R}_{\geq 0} + \ldots + (u_{s,n}e_s^* + e_n^*)\,\mathbb{R}_{\geq 0}$$

$$\vdots$$

$$\overline{\sigma}_{n-1}^{\vee} = \left(u_{n-1,1}e_{n-1}^* + e_1^*\right)\mathbb{R}_{\geq 0} + \ldots + (-1)e_{n-1}^*\mathbb{R}_{\geq 0} + \left(u_{n-1,n}e_{n-1}^* + e_n^*\right)\mathbb{R}_{\geq 0}.$$

Proof: The proof is an immediate result of Lemma 5.1.3. ∎

This result will be useful in the next subsection, which describes the mappings between patches X_σ and $X_{\overline{\sigma}_s}$ for $s = 1, \ldots, n-1$. Later, it will help to define the end on X_Σ.

5.1.3. Systems of Coordinates. Now we can specify the mapping between the patch X_σ associated with the cone $\sigma = e_1\mathbb{R}_{\geq 0} + \ldots + e_n\mathbb{R}_{\geq 0}$ and another patch X_γ associated with the cone $\gamma = v_1'\mathbb{R}_{\geq 0} + \ldots + v_{n-1}'\mathbb{R}_{\geq 0} + e_n\mathbb{R}_{\geq 0}$. Assume that the variable in X_γ is $(z_\gamma, w_\gamma) = (z_{\gamma,1}, \ldots, z_{\gamma,n-1}, w_\gamma)$, where $z_\gamma = (z_{\gamma,1}, \ldots, z_{\gamma,n-1})$ is the variable in $\mathbb{C}^{n-1}$ and w_γ is the variable in the fiber $X_{e_n\mathbb{R}_{\geq 0}} \simeq \mathbb{C}^1$. Then:

$$\mathbb{C}[S_\gamma] = \mathbb{C}[z_{\gamma,1}, \ldots, z_{\gamma,n-1}, w_\gamma] = \mathbb{C}[\chi_1(z,w), \ldots, \chi_{n-1}(z,w), \chi_n(z,w)],$$

where $\chi_1(z,w), \ldots, \chi_n(z,w)$ are characters. As described in Lemma 5.1.1 and Lemma 5.1.2, those functions have an important property. All generators of $\gamma^\vee$ but one have the last coordinate 0; therefore, the functions $\chi_1(z,w), \ldots, \chi_{n-1}(z,w)$ depend only on the variables z. Moreover, $\chi_n(z,w) = g(z)w$. Finally; therefore,

$$\mathbb{C}[S_\gamma] = \mathbb{C}[\chi_1(z), \ldots, \chi_{n-1}(z), g(z)w].$$

Using the notation:

$$A_\gamma = \begin{bmatrix} v_{1,1} & \ldots & & v_{1,n} \\ \vdots & & & \vdots \\ v_{n-1,1} & \ldots & & v_{n-1,n} \\ 0 & \ldots & 0 & 1 \end{bmatrix}$$

and

$$(A_\gamma^{-1})^t = \begin{bmatrix} \beta_{1,1} & \ldots & \beta_{1,n-1} & 0 \\ \vdots & & & \vdots \\ \beta_{n-1,1} & \ldots & \beta_{n-1,n-1} & 0 \\ \alpha_1 & \ldots & \alpha_{n-1} & 1 \end{bmatrix}$$

with $\beta_\gamma = (\beta_{i,j})_{i,j=1}^{n-1}$ and $\alpha_\gamma = (\alpha_1, \ldots, \alpha_{n-1})$, we can express $\mathbb{C}[S_\gamma]$ as follows:

$$\mathbb{C}[S_\gamma] = \mathbb{C}[z^{\beta_\gamma}, z^{\alpha_\gamma} w],$$

where $z^{\beta_\gamma} = \left(z_1^{\beta_{1,1}} \ldots z_{n-1}^{\beta_{1,n-1}}, \ldots, z_1^{\beta_{n-1,1}} \ldots z_{n-1}^{\beta_{n-1,n-1}} \right)$ and $z^{\alpha_\gamma} = z_1^{\alpha_1} \ldots z_{n-1}^{\alpha_{n-1}}$. This observation yields the following lemma:

Lemma 5.1.6 *The mapping $\phi_{\sigma,\gamma} : X_\sigma \to X_\gamma$ is defined by*

$$\phi_{\sigma,\gamma}(z, w) = (z^{\beta_\gamma}, z^{\alpha_\gamma} w) = (z_\gamma, w_\gamma). \blacksquare$$

Example 5.1.1 Note that, particularly if $\gamma = \overline{\sigma}_1 = u_1\mathbb{R}_{\geq 0} + e_2\mathbb{R}_{\geq 0} + \ldots + e_n\mathbb{R}_{\geq 0}$ with $u_1 = (-1)e_1 + u_{1,2}e_2 + \ldots + u_{1,n}e_n$ is a cone that shares an $(n-1)$-dimensional face with $\sigma = e_1\mathbb{R}_{\geq 0} + e_2\mathbb{R}_{\geq 0} + \ldots + e_n\mathbb{R}_{\geq 0}$, then we know based on Lemma 5.1.5, that:

$$\phi_{\sigma,\overline{\sigma}_1}(z, w) = (z_1^{-1}, z_1^{u_{1,2}} z_2, \ldots, z_1^{u_{1,n-1}} z_{n-1}, z_1^{u_{1,n}} w) = (z_{\overline{\sigma}_1}, w_{\overline{\sigma}_1}).$$

Since in this case $A_\gamma^t = (A_\gamma^{-1})^t$, the inverse mapping is defined by the same characters (but with variables $(z_{\overline{\sigma}_1}, w_{\overline{\sigma}_1})$).

On the other hand, describing the mapping $\phi_{\gamma,\sigma} : X_\gamma \to X_\sigma$, we need the inverse matrix of $(A_\gamma^{-1})^t$, which is simply A_γ^t. Let us begin with the same notation as above:

$$E_\gamma = \begin{bmatrix} v_{1,1} & \dots & v_{n-1,1} \\ \vdots & & \vdots \\ v_{1,n-1} & \dots & v_{n-1,n-1} \end{bmatrix}$$

and $H_\gamma = (v_{1,n}, \dots, v_{n-1,n})$. The matrix E_γ is simply a $(n-1) \times (n-1)$ block of matrix A_γ, and the vector H_γ consists of the first $(n-1)$ terms of the last column of A_γ. Thus,

Lemma 5.1.7 *The mapping $\phi_{\gamma,\sigma} : X_\gamma \to X_\sigma$ is defined by*

$$\phi_{\gamma,\sigma}(z_\gamma, w_\gamma) = (z_\gamma^{E_\gamma}, z_\gamma^{H_\gamma} w_\gamma) = (z, w). \blacksquare$$

5.2 ENDS OF LINE BUNDLES WITH STRICTLY CONVEX FANS

We must now describe the ends of line bundles especially those with strictly convex fans. Strict convexity is not necessary for one end, but for the particular description of it. The complements of the sets U_N defining the end play the same role in the manifold X as the closed polydiscs in $\mathbb{C}^n$. Moreover, we will prove that each $X \setminus U_N$ is a finite sum of closed polydiscs in coordinate patches.

Theorem 5.2.1 *Let X be a line bundle with a compact base and a strictly convex fan Σ. Then the end of X can be described by the sequence of open sets $\{U_N\}_{N=1}^{\infty}$, where $U_N = \bigcup_{\gamma \in \Sigma} U_{\gamma,N}$ with*

$$U_{\gamma,N} = \{(z_{\gamma,1}, \dots, z_{\gamma,n-1}, w_\gamma) \in X_\gamma : |w_\gamma| > N\}.$$

Proof: For $N = 1, 2, \ldots$ let us define the sequence of open sets $U_{\gamma,N} \subset X_\gamma \simeq \mathbb{C}^n$ as follows:

$$U_{\gamma,N} = \{(z_{\gamma,1}, \ldots, z_{\gamma,n-1}, w_\gamma) \in X_\gamma : |w_\gamma| > N\}.$$

Then $U_N = \bigcup_{\gamma \in \Sigma} U_{\gamma,N}$ is open in X. Moreover, $\{U_N\}_{N=1}^{\infty}$ defines the end on X. Clearly $U_{N+1} \subset U_N$ and $\bigcap_{N=1}^{\infty} \overline{U}_N = \emptyset$, since those conditions are true for each X_γ. To prove that ∂U_N is compact, we must note that $V_N = X \setminus U_N$ is a compact set in X. We will actually prove that V_N is a closed subset of a compact set in X.

First, we must describe the decompositions of V_N into patches X_γ of X. Notice that for the cone $\sigma = e_1\mathbb{R}_{\geq 0} + \ldots + e_n\mathbb{R}_{\geq 0}$ we have:

$$V_N \cap X_\sigma = \{(z_1, \ldots, z_{n-1}, w) \in X_\sigma : |w| \leq N, |z^{\alpha_\gamma} w| \leq N, \gamma \in \Sigma(n) \setminus \sigma\}.$$

On the other hand, the compact base B with the fan Π can be decomposed into a sum of closed polydiscs, $B = \bigcup_{\tau \in \Pi(n-1)} R_\tau$, where

$$R_\tau = \{(z_{\tau,1}, \ldots, z_{\tau,n-1}) \in B_\tau \simeq \mathbb{C}^{n-1} : |z_{\tau,1}| \leq 1, \ldots, |z_{\tau,n-1}| \leq 1\}.$$

Let $D_\gamma \subset X_\gamma$ for $\gamma \in \Sigma(n)$ be defined as follows

$$D_\gamma = \{(z_{\gamma,1}, \ldots, z_{\gamma,n-1}, w_\gamma) \in X_\gamma \simeq \mathbb{C}^n : |z_{\gamma,1}| \leq 1, \ldots, |z_{\gamma,n-1}| \leq 1, |w_\gamma| \leq N\}.$$

Then $V_N \subset \bigcup_{\gamma \in \Sigma(n)} D_\gamma$. Since each D_γ is a closed polydisc in $X_\gamma \simeq \mathbb{C}^n$, then D_γ is compact in X_γ and in X. Thus, V_N is a closed subset of a compact set $\bigcup_{\gamma \in \Sigma(n)} D_\gamma$ in a Hausdorff space, making V_N compact. In addition, Theorem 2.3.2 implies that X has one end. ▮

The sets $V_N \cap X_\gamma$ are described by a large family of inequalities, but working with all of them is not necessary for the Hartogs phenomena. This description can be replaced by a simpler one. If the cone γ shares an $(n-1)$-dimensional face with the cone σ, then using the notation and the coordinates from Example 5.1.1, we find that the change of coordinates along the fiber $w_\gamma = z^{\alpha_\gamma} w$ depends on w and the only variable among $z_1, \ldots, z_{n-1}$. In particular, if $\gamma = \overline{\sigma}_1 = u_1\mathbb{R}_{\geq 0} + e_2\mathbb{R}_{\geq 0} + \ldots + e_n\mathbb{R}_{\geq 0}$ with $u_1 = u_{1,1}e_1 + \ldots + u_{1,n}e_n$, then the character $z^{\alpha_\gamma}w$ takes the form $z_1^{u_{1,n}}w$. Let us formulate the following remark:

Remark 5.2.1 *Let $\gamma = \overline{\sigma}_1 = u_1\mathbb{R}_{\geq 0} + e_2\mathbb{R}_{\geq 0} + \ldots + e_n\mathbb{R}_{\geq 0}$ with $u_1 = u_{1,1}e_1 + \ldots + u_{1,n}e_n$ then*

$$V_N \cap X_\sigma \subset \{(z_1, \ldots, z_{n-1}, w) \in X_\sigma : |w| \leq N, \left|z_1^{u_{1,n}}w\right| \leq N\}. \blacksquare$$

This remark will be useful (in Theorem 5.4.2) for strictly convex fans in which $u_{1,n} > 0$.

5.3 GLOBAL HOLOMORPHIC FUNCTIONS

This subsection identifies the necessary and sufficient conditions for the existence of holomorphic functions on line bundles with compact bases. If $\Sigma = e_n\mathbb{R}_{\geq 0} + \Pi'$ is the fan describing the line bundle X_Σ, then we can assume that one of its cones, say σ, is generated by standard basis vectors, $\sigma = e_1\mathbb{R}_{\geq 0} + \ldots + e_n\mathbb{R}_{\geq 0}$. Let $\gamma \in \Sigma$ be any other cone. Using the notation from the previous sections

$$\gamma = v_1'\mathbb{R}_{\geq 0} + \ldots + v_{n-1}'\mathbb{R}_{\geq 0} + e_n\mathbb{R}_{\geq 0},$$

where

$$v_1' = v_{1,1}e_1 + \ldots + v_{1,n-1}e_{n-1} + v_{1,n}e_n$$

$$\vdots$$

$$v_{n-1}' = v_{n-1,1}e_1 + \ldots + v_{n-1,n-1}e_{n-1} + v_{n-1,n}e_n$$

with

$$E_\gamma = \begin{bmatrix} v_{1,1} & \ldots & v_{n-1,1} \\ \vdots & & \vdots \\ v_{1,n-1} & \ldots & v_{n-1,n-1} \end{bmatrix}$$

and $H_\gamma = (v_{1,n}, \dots, v_{n-1,n})$. Then the mapping $\phi_{\gamma,\sigma} : X_\gamma \to X_\sigma$ is defined as

$$\phi_{\gamma,\sigma}(z_\gamma, w_\gamma) = (z_\gamma^{E_\gamma}, z_\gamma^{H_\gamma} w_\gamma) = (z_1, \dots, z_{n-1}, w) = (z, w),$$

where $(z_\gamma, w_\gamma) = (z_{\gamma,1}, \dots, z_{\gamma,n-1}, w_\gamma) \in X_\gamma$, $z = (z_1, \dots, z_{n-1})$ and $(z, w) = (z_1, \dots, z_{n-1}, w) \in X_\sigma$. Here, $z_\gamma^{E_\gamma} = \left(z_1^{v_{1,1}} \dots z_{n-1}^{v_{n-1,1}}, \dots, z_1^{v_{1,n-1}} \dots z_{n-1}^{v_{n-1,n-1}}\right)$ and $z_\gamma^{H_\gamma} = z_1^{v_{1,n}} \dots z_{n-1}^{v_{n-1,n}}$. For $i = (i_1, \dots, i_{n-1})$ let

$$f_\sigma(z, w) = \sum_{i,s=0}^{\infty} a_{is} z^i w^s,$$

be a holomorphic function on $X_\sigma \simeq \mathbb{C}^n$. And let

$$f_\gamma(z_\gamma, w_\gamma) = \sum_{l,m=0}^{\infty} b_{lm} z_\gamma^l w_\gamma^m$$

be holomorphic on $X_\gamma \simeq \mathbb{C}^n$, where $l = (l_1, \dots, l_{n-1})$ and $z_\gamma = (z_{\gamma,1}, \dots, z_{\gamma,n-1})$. Then on $X_\sigma \cap X_\gamma$

$$f_\sigma(z, w) = \sum_{i,s=0}^{\infty} a_{is} z^i w^s = \sum_{i,s=0}^{\infty} a_{is} z_\gamma^{i \cdot E_\gamma} \left(z_\gamma^{H_\gamma} w_\gamma\right)^s = \sum_{i,s=0}^{\infty} a_{is} z_\gamma^{i \cdot E_\gamma + sH_\gamma} w_\gamma^s$$

$$= \sum_{l,m=0}^{\infty} b_{lm} z_\gamma^l w_\gamma^m = f_\gamma(z_\gamma, w_\gamma)$$

Here, $i \cdot E_\gamma$ is a multiplication of a vector and a matrix, and sH_γ is a multiplication of a scalar and a vector. Notice that the multi-index i applied to the character $z_\gamma^{E_\gamma}$ converts to the multi-index $i \cdot E_\gamma$ applied to the multivariable z_γ, which is a result of the following calculations:

$$z^i = z_1^{i_1} \dots z_{n-1}^{i_{n-1}} = \left(z_{\gamma,1}^{v_{1,1}} \dots z_{\gamma,n-1}^{v_{n-1,1}}\right)^{i_1} \dots \left(z_{\gamma,1}^{v_{n-1,1}} \dots z_{\gamma,n-1}^{v_{n-1,n-1}}\right)^{i_{n-1}}$$

$$= z_{\gamma,1}^{(i_1 v_{1,1} + \dots + i_{n-1} v_{n-1,1})} \dots z_{\gamma,n-1}^{(i_1 v_{n-1,1} + \dots + i_{n-1} v_{n-1,n-1})} = z_\gamma^{i \cdot E_\gamma}.$$

The vector $i \cdot E_\gamma + sH_\gamma$ is an $(n-1)$-dimensional vector, and the expression $i \cdot E_\gamma + sH_\gamma \geq 0$ means that all entries are nonnegative with at least one positive.

Lemma 5.3.1 *Let X_Σ be a smooth toric variety with a line bundle structure over a compact base. Let $X_\Sigma = \bigcup_{\gamma \in \Sigma} X_\gamma$ be its decomposition into coordinate patches. Then the following conditions hold:*

(i) If f_σ is holomorphic in X_σ, then it is holomorphic in X_γ for $\gamma \in \Sigma$ if and only if $i \cdot E_\gamma + sH_\gamma \geq 0$.

(ii) The condition $i \cdot E_\gamma + sH_\gamma \geq 0$ for all $\gamma \in \Sigma$ in all chosen coordinates $(z_\sigma, w_\sigma) \in X_\sigma$ with $\sigma \in \Sigma$ is necessary and sufficient for f to be holomorphic on X_Σ. ▌

5.4 EXTENSION PHENOMENA

This subsection proves that line bundles with compact bases and strictly convex fans allow the Hartogs phenomenon. The key observation lies in the following version of the Hartogs figure in $\mathbb{C}^n$ for $n \geq 2$. Figure 5.2 shows a picture related to this theorem.

Theorem 5.4.1 *Let $f(z_1, \ldots, z_{n-1}, w)$ be a holomorphic function on $\mathbb{C}^n \backslash V$ for V defined as:*

$$V = \{(z_1, \ldots, z_{n-1}, w) \in \mathbb{C}^n : \left|z_1^\beta w\right| \leq M, |w| \leq N\},$$

where $\beta \in \mathbb{Z}_{>0}$ and $M, N \in \mathbb{R}_{>0}$. Then f has holomorphic continuation to $\mathbb{C}^{n-1} \times \mathbb{C}^*$.

Proof: Let us define sequences of radii. Let $a_s = \frac{N}{2^s}$ and $\rho_s = \left(\frac{2^{s+1}M}{N}\right)^{\frac{1}{\beta}}$ for $s = 0, 1, \ldots$. For fixed $z_2, \ldots, z_{n-1}, w$ with $a_s \leq |w|$, define C_s as $t \mapsto (\rho_s e^{it}, z_2, \ldots, z_{n-1}, w)$ for $t \in [-\pi, \pi]$. Let $E_s = \{(z_1, \ldots, z_{n-1}, w) \in \mathbb{C}^n : \left|z_1^\beta w\right| \leq M, |w| \leq a_s\}$. The function f_s is defined on $\mathbb{C}^n \setminus E_s$ by the integral formula:

$$f_s(z_1, \ldots, z_{n-1}, w) = \frac{1}{2\pi i} \int_{C_s} \frac{f(\xi, z_2, \ldots, z_{n-1}, w)}{\xi - z_1} d\xi.$$

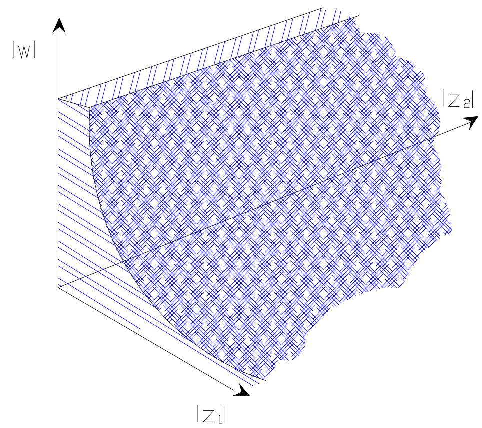

Figure 5.2: Another version of the Hartogs figure

The variables $z_2, \ldots, z_{n-1}$ appear in the integral as parameters. Since the dependence is holomorphic, the function $\lim\limits_{s\to\infty} f_s$ is holomorphic in $\mathbb{C}^{n-1} \times \mathbb{C}^*$ just as in the proof of Theorem 3.0.8.∎

5.4.1. The Hartogs Phenomenon. Here we formulate the main result.

Theorem 5.4.2 *Let X_Σ be a smooth toric variety with a line bundle structure over a compact base. If $|\Sigma|$ is strictly convex, then the Hartogs phenomenon holds in X_Σ.*

Proof: Let K be a compact set in X_Σ and let f be a holomorphic function defined on $X_\Sigma \setminus K$. We will show that f can be extended to X_Σ using ideas similar to those expressed in Theorem 3.2.1. First, notice that if K is compact and if the sequence $\{U_k\}_{k=1}^{\infty}$ defines the end of X_Σ, as in Theorem 5.2.1, then from Theorem 2.3.1 $K \subset V_N$ for some $N \in \mathbb{Z}_{>0}$, where $V_N = X_\Sigma \setminus U_N$. Since f is defined on $X_\Sigma \setminus K$, it is particularly defined on $X_\Sigma \setminus V_N$. Let us consider the decomposition of $X_\Sigma \setminus V_N$ into the patches $X_\gamma \simeq \mathbb{C}^n$, where X_γ is the affine toric variety associated with $\gamma \in \Sigma(n)$. Remark 5.2.1 justifies that

$$V_N \cap X_\gamma \subset \{(z_{\gamma,1}, \ldots, z_{\gamma,n-1}, w_\gamma) \in X_\gamma : |w_\gamma| \leq N, \left|z_{\gamma,1}^{u_{1,n}} w_\gamma\right| \leq N\};$$

therefore, we can apply Theorem 5.4.1 to the functions $f_\gamma = f\mid_{X_\gamma \setminus V_N}$ to obtain extensions to $\mathbb{C}^{n-1} \times \mathbb{C}^*$. The uniqueness of extensions proves that they agree on the intersections of their domains. Since they admit the Laurent expansion with respect to the coordinate w_γ, and series expansion with respect to the coordinates z_γ, we can write them in the following form:

$$f_\gamma(z_\gamma, w_\gamma) = \sum_{j_\gamma=0, s_\gamma=-\infty}^{\infty} a_{j_\gamma s_\gamma} z_\gamma^{j_\gamma} w_\gamma^{s_\gamma}, \tag{10}$$

where $(z_\gamma, w_\gamma) \in X_\gamma \simeq \mathbb{C}^n$ and $j_\gamma = (j_{\gamma,1}, \ldots, j_{\gamma,n-1})$. According to Lemma 5.1.7, the change of coordinates $\phi_{\gamma,\sigma} : X_\gamma \cap X_\sigma \to X_\sigma$ can be expressed as

$$\phi_{\gamma,\sigma}(z_\gamma, w_\gamma) = (z_\gamma^{E\gamma}, z_\gamma^{H_\gamma} w_\gamma) = (z, w),$$

where $E_\gamma \in \mathbb{Z}^{n-1} \times \mathbb{Z}^{n-1}$. Further, according to Theorem 5.1.1, $H_\gamma \geq 0$ (i.e., all entries of H_γ are nonnegative and at least one positive), since Σ is strictly convex. Thus, on $X_\gamma \cap X_\sigma$ we have:

$$\begin{aligned} f_\sigma(z, w) = \sum_{j=0, s=-\infty}^{\infty} a_{j,s} z^j w_i^s &= \sum_{j=0, s=-\infty}^{\infty} a_{j,s} z_\gamma^{j \cdot E_\gamma} \left(z_\gamma^{H_\gamma} w_\gamma\right)^s \\ &= \sum_{j=0, s=-\infty}^{\infty} a_{j,s} z_\gamma^{j \cdot E_\gamma + sH_\gamma} w_\gamma^s = f_\gamma(z_\gamma, w_\gamma) \end{aligned} \tag{11}$$

Since f_γ is holomorphic with respect to z_γ, the conditions

$$j \cdot E_\gamma + sH_\gamma \geq 0$$

are fulfilled for any $\gamma \in \Sigma$ (or $P(\gamma) \in \Pi$). If there exists a vector $v \in \Pi(1)$ with all nonpositive entries, then for some positive scalar u (a positive entry of H_γ) we have:

$$su \geq -j \cdot v,$$

which implies that $s \geq 0$, since $j \geq 0$. Even if there is no such a vector $v \in \Pi(1)$, then some nonnegative linear combination of other vectors from $\Pi(1)$ has only nonpositive entries, i.e., there exists $v \in |\Pi|$ so that

$$v = \sum_{q=1}^{p} c_q v_q,$$

with $v_q \in \Pi(1)$ and $c_q \geq 0$. The existence of v comes from the fact that Π is complete and consists of strictly convex cones. Then, if u is a positive scalar (a positive entry of H_γ), the conditions

$$su \geq -j \cdot v_q$$

for $q = 1, \ldots, p$ multiplied by c_q:

$$c_q su \geq -c_q j \cdot v_q$$

and added together give:

$$\sum_{q=1}^{p} c_q su \geq -\sum_{q=1}^{p} c_q j \cdot v_q.$$

Thus,

$$su \sum_{q=1}^{p} c_q \geq -j \cdot v,$$

and since $c_q \geq 0$, $\sum_{q=1}^{p} c_q, u > 0$ and $-v$ has only nonnegative entries, we find that $s \geq 0$. The function f_σ is then holomorphic with respect to w. To complete the proof it is sufficient to note that the conditions for the exponents of f_σ are equivalent to those obtained in Lemma 5.3.1. ▮

5.4.2. Examples with Strictly Convex Fans. The main idea can be illustrated with an example of a line bundle over projective space. Figure 5.3 shows an example, where the base is the projective plane.

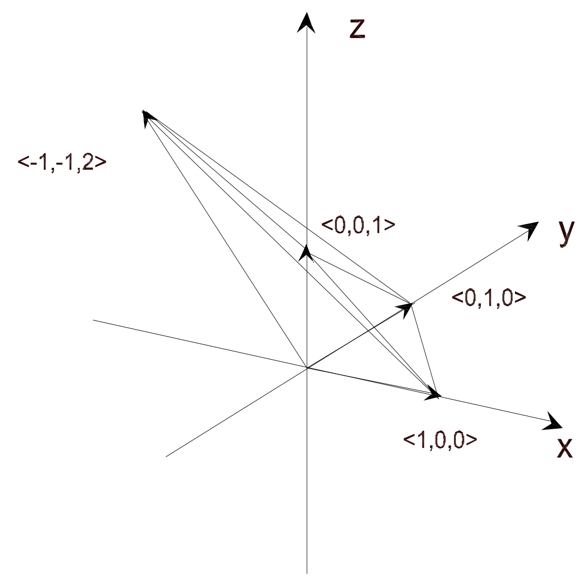

Figure 5.3: A fan of a line bundle over the projective plane

Example 5.4.1 In this example, we show that the Hartogs phenomenon holds for line bundles over $\mathbb{P}^{n-1}$ (for $n \geq 2$) with strictly convex fans. The fan associated with $\mathbb{P}^{n-1}$

contains n cones with dimension $(n-1)$. That is, $\Pi(n-1) = \{\tau_1, \ldots, \tau_n\}$, where

$$\tau_1 = e_1\mathbb{R}_{\geq 0} + \ldots + e_{n-1}\mathbb{R}_{\geq 0}$$

$$\tau_2 = ((-1)e_1 + \ldots + (-1)e_{n-1})\mathbb{R}_{\geq 0} + e_2\mathbb{R}_{\geq 0} + \ldots + e_{n-1}\mathbb{R}_{\geq 0}$$

$$\vdots$$

$$\tau_s = e_1\mathbb{R}_{\geq 0} + \ldots + e_{s-2}\mathbb{R}_{\geq 0} + ((-1)e_1 + \ldots + (-1)e_{n-1})\mathbb{R}_{\geq 0} + e_s\mathbb{R}_{\geq 0} + \ldots + e_{n-1}\mathbb{R}_{\geq 0}$$

$$\vdots$$

$$\tau_n = e_1\mathbb{R}_{\geq 0} + \ldots + e_{n-2}\mathbb{R}_{\geq 0} + ((-1)e_1 + \ldots + (-1)e_{n-1})\mathbb{R}_{\geq 0}.$$

A line bundle can be described in terms of fans by a projection $P : \mathbb{Z}^n \to \mathbb{Z}^n$, which sends the fan Π' onto the fan Π. Here Π' is generated in the lattice N by the following vectors:

$$\Pi'(1) = \{e_1\mathbb{R}_{\geq 0}, \ldots, e_{n-1}\mathbb{R}_{\geq 0}, (-e_1 - \ldots - e_{n-1} + ae_n)\mathbb{R}_{\geq 0}\}.$$

Any line bundle over $\mathbb{P}^{n-1}$ is defined by $\Sigma(n) = \{\sigma_1, \ldots, \sigma_n\}$, where:

$$\sigma_1 = e_1\mathbb{R}_{\geq 0} + \ldots + e_{n-1}\mathbb{R}_{\geq 0} + e_n\mathbb{R}_{\geq 0}$$

$$\sigma_2 = ((-1)e_1 + \ldots + (-1)e_{n-1} + ae_n)\mathbb{R}_{\geq 0} + e_2\mathbb{R}_{\geq 0} + \ldots + e_{n-1}\mathbb{R}_{\geq 0} + e_n\mathbb{R}_{\geq 0}$$

$$\vdots$$

$$\sigma_s = e_1\mathbb{R}_{\geq 0} + \ldots + e_{s-2}\mathbb{R}_{\geq 0} + ((-1)e_1 + \ldots + (-1)e_{n-1} + ae_n)\mathbb{R}_{\geq 0} + e_s\mathbb{R}_{\geq 0} + \ldots$$
$$+ e_{n-1}\mathbb{R}_{\geq 0} + e_n\mathbb{R}_{\geq 0}$$

$$\vdots$$

$$\sigma_n = e_1\mathbb{R}_{\geq 0} + \ldots + e_{n-2}\mathbb{R}_{\geq 0} + ((-1)e_1 + \ldots + (-1)e_{n-1} + ae_n)\mathbb{R}_{\geq 0} + e_n\mathbb{R}_{\geq 0}.$$

with $a \in \mathbb{Z}_{>0}$ (since Σ is strictly convex). The dual cones are given by:

$$\sigma_1^\vee = e_1^*\mathbb{R}_{\geq 0} + \ldots + e_{n-1}^*\mathbb{R}_{\geq 0} + e_n^*\mathbb{R}_{\geq 0}$$

$$\sigma_2^\vee = (-1)e_1^*\mathbb{R}_{\geq 0} + (-e_1^* + e_2^*)\mathbb{R}_{\geq 0} + \ldots + (-e_1^* + e_{n-1}^*)\mathbb{R}_{\geq 0} + (ae_1^* + e_n^*)\mathbb{R}_{\geq 0}$$

$$\vdots$$

$$\sigma_s^\vee = (e_1^* - e_{s-1}^*)\mathbb{R}_{\geq 0} + \ldots + (e_{s-2}^* - e_{s-1}^*)\mathbb{R}_{\geq 0} + (-e_{s-1}^*)\mathbb{R}_{\geq 0} + (e_s^* - e_{s-1}^*)\mathbb{R}_{\geq 0} + \ldots$$

$$+(e_{n-1}^* - e_{s-1}^*)\mathbb{R}_{\geq 0} + (ae_{s-1}^* + e_n^*)\mathbb{R}_{\geq 0}$$

$$\vdots$$

$$\sigma_n^\vee = (e_1^* - e_{n-1}^*)\mathbb{R}_{\geq 0} + \ldots + (e_{n-2}^* - e_{n-1}^*)\mathbb{R}_{\geq 0} + (-1)e_{n-1}^*\mathbb{R}_{\geq 0} + (ae_{n-1}^* + e_{n)}^*\mathbb{R}_{\geq 0}.$$

We will use the notation $(z_{\sigma_1,1}, \ldots, z_{\sigma_1,n-1}, w_{\sigma_1}) = (z_1, \ldots, z_{n-1}, w) \in X_{\sigma_1}$. The systems of coordinates in $(z_{\sigma_s,1}, \ldots, z_{\sigma_s,n-1}, w_{\sigma_s}) \in X_{\sigma_s}$ for $s = 2, \ldots, n$ is given as follows:

$$(z_{\sigma_2,1}, \ldots, z_{\sigma_2,n-1}, w_{\sigma_2}) = \left(\frac{1}{z_1}, \frac{z_2}{z_1} \ldots, \frac{z_{n-1}}{z_1}, z_1^a w\right)$$

$$\vdots$$

$$(z_{\sigma_s,1}, \ldots, z_{\sigma_s,n-1}, w_{\sigma_s}) = \left(\frac{z_1}{z_{s-1}}, \ldots, \frac{z_{s-2}}{z_{s-1}}, \frac{1}{z_{s-1}}, \frac{z_s}{z_{s-1}}, \ldots, \frac{z_{n-1}}{z_{s-1}}, z_{s-1}^a w\right)$$

$$\vdots$$

$$(z_{\sigma_n,1}, \ldots, z_{\sigma_n,n-1}, w_{\sigma_n}) = \left(\frac{z_1}{z_{n-1}}, \ldots, \frac{z_{n-2}}{z_{n-1}}, \frac{1}{z_{n-1}}, z_{n-1}^a w\right).$$

If $a > 0$, then the bundle is positive; if $a < 0$, then it is negative; if $a = 0$, then it is trivial. The Hartogs phenomenon holds only for positive line bundles; therefore, we will retain this assumption for this section. The end was already described, but we must examine it more carefully. Let $U_{\sigma_s,N} \subset X_{\sigma_s}$ be defined for $N \in \mathbb{Z}_{\geq 1}$ as:

$$U_{\sigma_s,N} = \{(z_{\sigma_s,1}, \ldots, z_{\sigma_s,n-1}, w_{\sigma_s}) \in X_{\sigma_s} : |w_{\sigma_s}| > N\}.$$

The set $U_N = \bigcup_{s=1}^{\infty} U_{\sigma_s,N}$ is the open in X_Σ. Then, $X_\Sigma \setminus U_N$ is closed and compact in X_Σ, so ∂U_N is compact and the sequence $\{U_N\}_{N=1}^{\infty}$ defines the end on X_Σ. The computations prove that:

$$V_N \cap X_{\sigma_s} = \{(z_{\sigma_s,1}, \ldots, z_{\sigma_s,n-1}, w_{\sigma_s}) \in X_{\sigma_s} : \left|z^a_{\sigma_s,1} w_{\sigma_s}\right| \leq N, \ldots, \left|z^a_{\sigma_s,n-1} w_{\sigma_s}\right| \leq N, |w_{\sigma_s}| \leq N\}$$

As demonstrated by Theorem 5.4.1, there exists the holomorphic extension from each $X_{\sigma_s} \setminus V_N$ to $\mathbb{C}^{n-1} \times C^* = \{(z_{\sigma_s,1}, \ldots, z_{\sigma_s,n-1}, w_{\sigma_s}) : w_{\sigma_s} \neq 0\} \subset X_{\sigma_s}$. Now, therefore, we can assume that each function $f \mid X_{\sigma_s} = f_s(z_{\sigma_s}, w_s)$ has Laurent expansion with $I = (i_1, \ldots, i_{n-1})$:

$$f_s(z_{\sigma_s}, w_s) = \sum_{I=0,k=-\infty}^{\infty} a_{I,k} z^I_{\sigma_s} w^k_{\sigma_s}$$

$$= \sum_{I=0,k=-\infty}^{\infty} a_{I,k} \left(\frac{z_1}{z_{s-1}}\right)^{i_1} \ldots \left(\frac{z_{s-2}}{z_{s-1}}\right)^{i_{s-2}} \left(\frac{1}{z_{s-1}}\right)^{i_{s-1}} \left(\frac{z_s}{z_{s-1}}\right)^{i_s} \ldots \left(\frac{z_{n-1}}{z_{s-1}}\right)^{i_{n-1}} (z^a_{s-1} w)^k$$

$$= \sum_{I=0,k=-\infty}^{\infty} a_{I,k} z_1^{i_1} \ldots z_{s-2}^{i_{s-2}} z_{s-1}^{-|I|+ka} z_s^{i_s} \ldots z_{n-1}^{i_{n-1}} w^k = f_1(z, w),$$

where $|I| = i_1 + \ldots + i_{n-1}$. Since $f_1(z, w)$ has Laurent expansion

$$f_1(z, w) = \sum_{I=0,k=-\infty}^{\infty} a_{I,k} z^I w^k,$$

the index $-|I| + ka$ fulfills $-|I| + ka \geq 0$, which implies that $ka \geq |I|$. In particular, therefore $k \geq 0$, and f_{σ_s} is holomorphic on the whole $X_{\sigma_s} \simeq \mathbb{C}^n$.

Example 5.4.2 This example explains why the Hartogs phenomenon holds in line bundles over $\mathbb{P}^1 \times \mathbb{P}^1$ with strictly convex fans. Note that this result can be easily generated for line bundles over $(\mathbb{P}^1)^{n-1}$. Recall, too, that $\Pi(2) = \{\tau_1, \tau_2, \tau_3, \tau_4\}$, where

$$\tau_1 = e_1\mathbb{R}_{\geq 0} + e_2\mathbb{R}_{\geq 0}$$

$$\tau_2 = e_1\mathbb{R}_{\geq 0} + (-e_2)\mathbb{R}_{\geq 0}$$

$$\tau_3 = (-e_1)\mathbb{R}_{\geq 0} + (-e_2)\mathbb{R}_{\geq 0}$$

$$\tau_4 = (-e_1)\mathbb{R}_{\geq 0} + e_2\mathbb{R}_{\geq 0}.$$

Any line bundle over $\mathbb{P}^1 \times \mathbb{P}^1$ is determined by a projection $P : \mathbb{Z}^3 \to \mathbb{Z}^3$, and since we can always assume that one of the 3-dimensional cones in Σ is generated by the standard basis vectors, the set $\Sigma(3)$ consists of the following cones:

$$\sigma_1 = e_1\mathbb{R}_{\geq 0} + e_2\mathbb{R}_{\geq 0} + e_3\mathbb{R}_{\geq 0}$$

$$\sigma_2 = e_1\mathbb{R}_{\geq 0} + (-e_2 + ae_3)\mathbb{R}_{\geq 0} + e_3\mathbb{R}_{\geq 0}$$

$$\sigma_3 = (-e_1 + be_3)\mathbb{R}_{\geq 0} + (-e_2 + ae_3)\mathbb{R}_{\geq 0} + e_3\mathbb{R}_{\geq 0}$$

$$\sigma_4 = (-e_1 + be_3)\mathbb{R}_{\geq 0} + e_2\mathbb{R}_{\geq 0} + e_3\mathbb{R}_{\geq 0},$$

where $a, b \in \mathbb{Z}_{>0}$, because the fan is strictly convex. Then the dual cones are described as follows:

$$\sigma_1^\vee = e_1^*\mathbb{R}_{\geq 0} + e_2^*\mathbb{R}_{\geq 0} + e_3^*\mathbb{R}_{\geq 0}$$

$$\sigma_2^\vee = e_1^*\mathbb{R}_{\geq 0} + (-e_2^*)\mathbb{R}_{\geq 0} + (ae_2^* + e_3^*)\mathbb{R}_{\geq 0}$$

$$\sigma_3^\vee = (-e_1^*)\mathbb{R}_{\geq 0} + (-e_2^*)\mathbb{R}_{\geq 0} + (be_1^* + ae_2^* + e_3^*)\mathbb{R}_{\geq 0}$$

$$\sigma_4^\vee = (-e_1^*)\mathbb{R}_{\geq 0} + e_2^*\mathbb{R}_{\geq 0} + (be_1^* + e_3^*)\mathbb{R}_{\geq 0},$$

We will use the notation $(z_{\sigma_1,1}, z_{\sigma_1,2}, w_{\sigma_1}) = (z_1, z_2, w) \in X_{\sigma_1}$. Thus,

$$(z_{\sigma_2,1}, z_{\sigma_2,2}, w_{\sigma_2}) = \left(z_1, \frac{1}{z_2}, z_2^a w\right)$$

$$(z_{\sigma_3,1}, z_{\sigma_3,2}, w_{\sigma_3}) = \left(\frac{1}{z_1}, \frac{1}{z_2}, z_1^b z_2^a w\right)$$

$$(z_{\sigma_4,1}, z_{\sigma_4,2}, w_{\sigma_4}) = \left(\frac{1}{z_1}, z_2, z_1^b w\right)$$

Define open sets $U_{\sigma_s,N} \subset X_{\sigma_s}$ for $N \in \mathbb{Z}_{\geq 1}$ as follows:

$$U_{\sigma_s,N} = \{(z_{\sigma_s,1}, z_{\sigma_s,2}, w_{\sigma_s}) \in X_{\sigma_s} : |w_{\sigma_s}| > N\}.$$

Then $U_N = \bigcup_{\sigma_s \in \Sigma} U_{\sigma_s,N}$ is open for each $N \in \mathbb{Z}_{\geq 1}$. Moreover, $V_N = X_\Sigma \setminus U_N$ is compact and

$$V_N \cap X_{\sigma_1} = \{(z_1, z_2, w) \in X_{\sigma_1} : |w| \leq N, |z_2^a w| \leq N, \left|z_1^b w\right| \leq N, \left|z_1^b z_2^a w\right| \leq N\}$$

As indicated by Theorem 5.4.1, there exists the holomorphic extension from each $X_{\sigma_s} \setminus V_N$ to $\mathbb{C}^2 \times C^* = \{(z_{\sigma_s,1}, z_{\sigma_s,2}, w_{\sigma_s}) : w_{\sigma_s} \neq 0\} \subset X_{\sigma_s}$. Now we can assume, therefore, that each function $f \mid_{X_{\sigma_s}} = f_s(z_{\sigma_s}, w_s)$ has Laurent expansion with $I = (i_1, i_2)$:

$$f_2(z_{\sigma_2}, w_2) = \sum_{I=0,k=-\infty}^{\infty} a_{I,k} z_{\sigma_2}^I w_{\sigma_2}^k = \sum_{I=0,k=-\infty}^{\infty} a_{I,k} (z_1)^{i_1} \left(\frac{1}{z_2}\right)^{i_2} (z_2^a w)^k$$

$$= \sum_{I=0,k=-\infty}^{\infty} a_{I,k} z_1^{i_1} z_2^{ka-i_2} w^k = f_1(z, w),$$

$$f_3(z_{\sigma_3}, w_3) = \sum_{I=0,k=-\infty}^{\infty} a_{I,k} z_{\sigma_3}^I w_{\sigma_3}^k \sum_{I=0,k=-\infty}^{\infty} a_{I,k} \left(\frac{1}{z_1}\right)^{i_1} \left(\frac{1}{z_2}\right)^{i_2} (z_1^b z_2^a w)^k =$$

$$= \sum_{I=0,k=-\infty}^{\infty} a_{I,k} z_1^{bk-i_1} z_2^{ka-i_2} w^k = f_1(z, w),$$

$$f_4(z_{\sigma_4}, w_4) = \sum_{I=0,k=-\infty}^{\infty} a_{I,k} z_{\sigma_4}^I w_{\sigma_4}^k = \sum_{I=0,k=-\infty}^{\infty} a_{I,k} \left(\frac{1}{z_1}\right)^{i_1} (z_2)^{i_2} (z_1^b w)^k$$

$$= \sum_{I=0,k=-\infty}^{\infty} a_{I,k} z_1^{bk-i_1} z_2^{i_2} w^k = f_1(z, w),$$

and f_1 has power expansion, which proves that $ak - i_2 \geq 0$ and $bk - i_1 \geq 0$. Since $a, b > 0$, we find that $k \geq 0$, which means that all functions f_s are holomorphic. The Hartogs phenomenon thus holds in a line bundle over $\mathbb{P}^1 \times \mathbb{P}^1$ with $a, b > 0$.

Example 5.4.3 A more complicated example with a line bundle over a compact surface B associated with a 2-dimensional fan Π shows the nature of the Hartogs problem. Let $\Pi(2) = \{\tau_1, \tau_2, \tau_3, \tau_4, \tau_5\}$, where

$$\tau_1 = e_1\mathbb{R}_{\geq 0} + e_2\mathbb{R}_{\geq 0}$$

$$\tau_2 = (-e_1)\mathbb{R}_{\geq 0} + e_2\mathbb{R}_{\geq 0}$$

$$\tau_3 = (-e_1)\mathbb{R}_{\geq 0} + (-e_1 - e_2)\mathbb{R}_{\geq 0}$$

$$\tau_4 = (-e_1 - e_2)\mathbb{R}_{\geq 0} + (-e_2)\mathbb{R}_{\geq 0}$$

$$\tau_5 = e_1\mathbb{R}_{\geq 0} + (-e_2)\mathbb{R}_{\geq 0}$$

In fact, B is a blow up of $\mathbb{P}^1 \times \mathbb{P}^1$, since its fan Π is a subdivision of the fan that describes $\mathbb{P}^1 \times \mathbb{P}^1$ (Proposition 7.4 from [28]). Then $\Sigma(3) = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5\}$, where

$$\sigma_1 = e_1\mathbb{R}_{\geq 0} + e_2\mathbb{R}_{\geq 0} + e_3\mathbb{R}_{\geq 0}$$

$$\sigma_2 = (-e_1 + ae_3)\mathbb{R}_{\geq 0} + e_2\mathbb{R}_{\geq 0} + e_3\mathbb{R}_{\geq 0}$$

$$\sigma_3 = (-e_1 + ae_3)\mathbb{R}_{\geq 0} + (-e_1 - e_2 + be_3)\mathbb{R}_{\geq 0} + e_3\mathbb{R}_{\geq 0}$$

$$\sigma_4 = (-e_1 - e_2 + be_3)\mathbb{R}_{\geq 0} + (-e_2 + ce_3)\mathbb{R}_{\geq 0} + e_3\mathbb{R}_{\geq 0},$$

$$\sigma_5 = e_1\mathbb{R}_{\geq 0} + (-e_2 + ce_3)\mathbb{R}_{\geq 0} + e_3\mathbb{R}_{\geq 0},$$

for $a, b, c \in \mathbb{Z}_{>0}$. Note at this point that Lemma 5.1.4 proves that Σ is strictly convex if and only if $b \geq a > 0$, $b \geq c > 0$ and $a + c \geq b$. The dual cones are described as follows:

$$\sigma_1^\vee = e_1^*\mathbb{R}_{\geq 0} + e_2^*\mathbb{R}_{\geq 0} + e_3^*\mathbb{R}_{\geq 0}$$

$$\sigma_2^\vee = (-e_1^*)\mathbb{R}_{\geq 0} + e_2^*\mathbb{R}_{\geq 0} + (ae_1^* + e_3^*)\mathbb{R}_{\geq 0}$$

$$\sigma_3^\vee = (-e_1^* + e_2^*)\mathbb{R}_{\geq 0} + (-e_2^*)\mathbb{R}_{\geq 0} + (ae_1^* + (b-a)e_2^* + e_3^*)\mathbb{R}_{\geq 0}$$

$$\sigma_4^\vee = (-e_1^*)\mathbb{R}_{\geq 0} + (-e_1^* + e_2^*)\mathbb{R}_{\geq 0} + ((b-c)e_1^* + ce_2^* + e_3^*)\mathbb{R}_{\geq 0},$$

$$\sigma_5^\vee = e_1^*\mathbb{R}_{\geq 0} + (-e_2^*)\mathbb{R}_{\geq 0} + (ce_2^* + e_3^*)\mathbb{R}_{\geq 0},$$

where $b - c \geq 0$ and $b - a \geq 0$. With the notation $(z_{\sigma_1,1}, z_{\sigma_1,2}, w_{\sigma_1}) = (z_1, z_2, w) \in X_{\sigma_1}$, we have:

$$(z_{\sigma_2,1}, z_{\sigma_2,2}, w_{\sigma_2}) = \left(\frac{1}{z_1}, z_2, z_1^a w\right)$$

$$(z_{\sigma_3,1}, z_{\sigma_3,2}, w_{\sigma_3}) = \left(\frac{z_2}{z_1}, \frac{1}{z_2}, z_1^a z_2^{b-a} w\right)$$

$$(z_{\sigma_4,1}, z_{\sigma_4,2}, w_{\sigma_4}) = \left(\frac{1}{z_1}, \frac{z_1}{z_2}, z_1^{(b-c)} z_2^c w\right)$$

$$(z_{\sigma_5,1}, z_{\sigma_4,2}, w_{\sigma_4}) = \left(z_1, \frac{1}{z_2}, z_2^c w\right).$$

Since $b - c \geq 0$ and $b - a \geq 0$, we can define the end of X_Σ as in the previous examples. Let

$$U_{\sigma_s,N} = \{(z_{\sigma_s,1}, z_{\sigma_s,2}, w_{\sigma_s}) \in X_{\sigma_s} : |w_{\sigma_s}| > N\}.$$

Then $U_N = \displaystyle\bigcup_{\sigma_s \in \Sigma} U_{\sigma_s,N}$ is open for each $N \in \mathbb{Z}_{\geq 1}$. Moreover, $V_N = X_\Sigma \setminus U_N$ is compact, and

$$V_N \cap X_{\sigma_1} = \{(z_1, z_2, w) \in X_{\sigma_1} : |w| \leq N, |z_1^a w| \leq N, |z_2^c w| \leq N, \left|z_1^a z_2^{b-a} w\right| \leq N, \left|z_1^{b-c} z_2^c w\right| \leq N\}$$

Based on Theorem 5.4.1, there exists the holomorphic extension from each $X_\sigma \setminus V_N$ to $\mathbb{C}^2 \times C^* = \{(z_{\sigma_s,1}, z_{\sigma_s,2}, w_{\sigma_s}) : w_{\sigma_s} \neq 0\} \subset X_{\sigma_s}$. We can now assume, therefore, that each function $f\mid_{X_{\sigma_s}} = f_s(z_{\sigma_s}, w_s)$ has Laurent expansion with $I = (i_1, i_2)$:

$$f_2(z_{\sigma_2}, w_2) = \sum_{I=0,k=-\infty}^{\infty} a_{I,k} z_{\sigma_2}^I w_{\sigma_2}^k = \sum_{I=0,k=-\infty}^{\infty} a_{I,k} (z_1)^{i_1} \left(\frac{1}{z_1}\right)^{i_1} z_2^{i_2} (z_1^a w)^k$$

$$= \sum_{I=0,k=-\infty}^{\infty} a_{I,k} z_1^{ak-i_1} z_2^{ak+i_2} w^k = f_1(z, w).$$

Since f_1 is analytic with respect to z_1, the index $ak - i_1$ is nonnegative, and $k \geq \frac{i_1}{a} \geq 0$. Thus, f_2 is analytic on X_2. Moreover,

$$f_3(z_{\sigma_3}, w_3) = \sum_{I=0,k=-\infty}^{\infty} a_{I,k} z_{\sigma_3}^I w_{\sigma_3}^k = \sum_{I=0,k=-\infty}^{\infty} a_{I,k} \left(\frac{z_2}{z_1}\right)^{i_1} \left(\frac{1}{z_2}\right)^{i_2} (z_1^a z_2^{(b-a)} w)^k$$

$$= \sum_{I=0,k=-\infty}^{\infty} a_{I,k} z_1^{ak-i_1} z_2^{(b-a)k+i_1-i_2} w^k = f_1(z, w).$$

Again f_1 is analytic with respect to z_1, and the index $ak - i_1$ is nonnegative. Thus, $k \geq \frac{i_1}{a} \geq 0$ and f_3 is analytic on X_3. Moreover,

$$f_4(z_{\sigma_4}, w_4) = \sum_{I=0,k=-\infty}^{\infty} a_{I,k} z_{\sigma_4}^I w_{\sigma_4}^k = \sum_{I=0,k=-\infty}^{\infty} a_{I,k} \left(\frac{1}{z_1}\right)^{i_1} \left(\frac{z_1}{z_2}\right)^{i_2} (z_1^{(b-c)} z_2^c w)^k$$

$$= \sum_{I=0,k=-\infty}^{\infty} a_{I,k} z_1^{(b-c)k-i_1+i_2} z_2^{ck-i_2} w^k = f_1(z, w),$$

Since f_1 is analytic with respect to z_2, the index $ck - i_2$ is nonnegative, and $k \geq \frac{i_2}{c} \geq 0$.

Thus, f_4 is analytic on X_4. Moreover,

$$f_5(z_{\sigma_5}, w_5) = \sum_{I=0,k=-\infty}^{\infty} a_{I,k} z_{\sigma_5}^I w_{\sigma_4}^k = \sum_{I=0,k=-\infty}^{\infty} a_{I,k} (z_1)^{i_1} \left(\frac{1}{z_2}\right)^{i_2} (z_2^c w)^k$$

$$= \sum_{I=0,k=-\infty}^{\infty} a_{I,k} z_1^{i_1} z_2^{ck-i_2} w^k = f_1(z,w),$$

Since f_1 is analytic with respect to z_2, the index $ck - i_2$ is nonnegative, and $k \geq \frac{i_2}{c} \geq 0$. Thus, f_5 is analytic on X_5. Since the functions f_j for $j = 1, \ldots 5$ fulfill the properties for a global holomorphic function on X_Σ, the Hartogs phenomenon holds.

6 THE HARTOGS-BOCHNER PHENOMENON

This section presents a short overview of CR (Cauchy-Riemann) theory and the Hartogs-Bochner extension phenomenon.

6.1 BASICS OF CR THEORY

Although the definition of CR manifolds and functions appeared in Greenfield's 1968 paper ([17]), the early motivations can be found in a 1907 by H. Poincaré [30]. The idea was later extended by E. Cartan in [4] and [5], by S.S. Chern and J. Moser in [7], and by N. Tanaka in [33].

It is intuitively clear that a real submanifold $M \subset \mathbb{C}^n$, for $n \geq 2$, might carry more than only the real structure. Cauchy-Riemann theory describes the geometry induced on M by the complex structure from $\mathbb{C}^n$ or, in a more general version, a complex manifold. Moreover, some functions $f : M \to \mathbb{C}$ fulfill extra conditions, called tangential CR equations, that are closely related to this structure. We will call these CR functions.

6.1.1. CR Manifolds. The following presents the definition of CR manifolds, which comes from complex vector fields. Remember that any complex vector field L in $\mathbb{C}^n$ of type $(1,0)$ can be written as

$$L = a_1 \frac{\partial}{\partial z_1} + \ldots + a_n \frac{\partial}{\partial z_n}.$$

And its complex conjugate $\overline{L}$, a complex vector field of type $(0,1)$, is written as

$$\overline{L} = \overline{a}_1 \frac{\partial}{\partial \overline{z}_1} + \ldots + \overline{a}_n \frac{\partial}{\partial \overline{z}_n}.$$

If a k-dimensional real, differentiable submanifold M in $\mathbb{C}^n$ is locally defined by the equations:

$$\rho_1(z,\overline{z}) = 0, \ldots, \rho_k(z,\overline{z}) = 0,$$

then we can consider the tangent vector spaces to M:

$$H_p^{1,0}(M) = \{L = a_1\frac{\partial}{\partial z_1} + \ldots + a_n\frac{\partial}{\partial z_n}\ ;\ L\rho_j\big|_p = 0 \ \text{ for } j = 1,\ldots,k\} \tag{12}$$

$$H_p^{0,1}(M) = \{\overline{L} = \overline{a}_1\frac{\partial}{\partial \overline{z}_1} + \ldots + \overline{a}_n\frac{\partial}{\partial \overline{z}_n}\ ;\ \overline{L}\rho_j\big|_p = 0 \ \text{ for } j = 1,\ldots,k\} \tag{13}$$

Note, that we have $H_p^{0,1}(M) = \overline{H_p^{1,0}(M)}$. If we denote

$$H^{1,0}(M) = \bigcup_{p\in M} H_p^{1,0}(M)$$

and

$$H^{0,1}(M) = \bigcup_{p\in M} H_p^{0,1}(M),$$

then

$$H^{1,0}(M) = \overline{H^{1,0}(M)}.$$

Moreover, if M is CR, we have the following properties of these vector bundles ([2], II.7, Lemma 3):

$$\begin{gathered} H^{1,0}(M) \cap H^{0,1}(M) = \{0\}, \\ H^{1,0}(M) \quad \text{and} \quad H^{0,1}(M) \quad \text{are involutive,} \end{gathered} \tag{14}$$

i.e., $[L_1, L_2]$ is a section in $H^{1,0}(M)$ for sections L_1, L_2 of $H^{1,0}(M)$, and similarly for $H^{0,1}(M)$.

Now we are ready to consider the following definitions of CR manifold:

Definition 6.1.1 ***(CR manifolds)*** *A differentiable manifold M is CR if* $\dim_{\mathbb{C}} H_p^{1,0}(M)$ *remains constant for all $p \in M$.*

If M is a smooth real hypersurface in a complex manifold X with $\dim_{\mathbb{C}}(X) = n$, then $\dim_{\mathbb{C}} H_p^{1,0}(M) = n-1$; therefore, all those hypersurfaces are CR manifolds.

Note that the following example can be considered on any smooth compact toric surface, not only on $\mathbb{C}^2$.

Example 6.1.1 Let $M \subset \mathbb{C}^2$ be a real manifold described by the equation $|z| = 1$ in coordinates $(z, w) \in \mathbb{C}^2$. Then M is a cylinder over the unit circle. At each point p the tangential complex direction is simply along w. Moreover, M is a real hypersurface in $\mathbb{C}^2$, so it is a CR manifold with $\dim_{\mathbb{C}} H_p M = 1$.

6.1.2. CR Functions. Before we define CR functions precisely, let us first present an intuitive notion. This notion is local, so it is enough to consider the situation in $\mathbb{C}^n$. Holomorphic functions defined on an open set $U \subset \mathbb{C}^n$ satisfy a system of the CR equations, that can be written as

$$\frac{\partial f}{\partial \overline{z}_1} = 0, \ldots, \frac{\partial f}{\partial \overline{z}_n} = 0,$$

or equivalently as

$$a_1 \frac{\partial f}{\partial \overline{z}_1} + \ldots + a_n \frac{\partial f}{\partial \overline{z}_n} = 0 \quad \text{for any } \ a_1, \ldots, a_n \in \mathbb{C}. \tag{15}$$

If the function f is defined on a real submanifold M, then f can be expected, possibly, to satisfy the CR equations tangent to M. Thus, we should adjust the coefficients $a_1, \ldots, a_n$ from (15) in such a way that the vector on the left side of (15) is tangent to M.

Definition 6.1.2 ***(CR functions)*** *Let $f : M \longrightarrow \mathbb{C}$ be a differentiable function on a CR manifold $M \subset \mathbb{C}^n$. We say that f is CR if $\overline{L} f = 0$ on M for every $\overline{L} \in H^{0,1}(M)$.*

Here, we will continue the example with a cylinder.

Example 6.1.2 Let $M \subset \mathbb{C}^2$ be described by $|z| = 1$ in the coordinates $(z, w) \in \mathbb{C}^2$. Consider the function $f : M \to \mathbb{C}$ defined as $f(z, w) = \frac{1}{z} + z$ (note that this example works the same way with any function that depends on z only). Then $\frac{\partial f}{\partial \overline{w}} = 0$. Because w is the only complex tangential direction in TM, we see that f is CR on M.

6.2 DEFINITION

Let X be a complex manifold, and let the domain U be an open, connected, relatively compact set with a smooth connected boundary. Consider the following definitions:

Definition 6.2.1 *The Hartogs-Bochner phenomenon holds for a domain $U \Subset X$, (or $\mathscr{HB}-U$) if any smooth CR function on ∂U can be holomorphically extended to U and smoothly up to the boundary.*

Definition 6.2.2 *The Hartogs-Bochner phenomenon (or $\mathscr{HB}$) holds in a complex manifold X if $\mathscr{HB}-U$ holds for any domain $U \subset X$.*

The extension phenomena are closely related to the cohomology groups with compact support.

Definition 6.2.3 (The Dolbeault cohomology groups with compact support) *The Dolbeaut cohomology groups with compact support of the domain D are the complex vector spaces:*

$$\mathfrak{H}_c^{p,q}(D) = \frac{\{\overline{\partial}\text{-closed forms with compact support of bidegree } (p,q) \text{ in } D\}}{\{\overline{\partial}\text{-exact forms with compact support of bidegree } (p,q) \text{ in } D\}}.$$

The following theorem shows the relationship between the cohomology groups with compact supports and the Dolbeault cohomology groups with compact support.

Theorem 6.2.1 (Dolbeault's Theorem, [8]) *If D is an open domain in the space of n complex variables, $\mathscr{O}$ is the sheaf of germs of holomorphic functions on D, and $\mathfrak{H}_c^{p,q}(D)$*

is the Dolbeault cohomology group with compact support of bidegree (p, q) *for* D*. Then* $H_c^q(D, \mathscr{O}) = \mathfrak{H}_c^{0,q}(D)$.▮

The following proposition, which can be found in [9], plays a key role here. It ties the first cohomology groups (regular and compact) together with $\overline{\partial}$-problem, the Hartogs phenomenon and the Hartogs-Bochner phenomenon.

Proposition 6.2.1 ***([9], Proposition 2.1)*** *Let* X *be a complex manifold. Then the following apply.*

(a) The group $H_c^1(X, \mathscr{O}) = 0$ *if and only if for any smooth closed* $(0, 1)$ *form* ω *on* X *with compact support there exists a compactly supported solution* u *of* $\overline{\partial} u = \omega$.

(b) The compact cohomology group $H_c^1(X, \mathscr{O})$ *is naturally mapped into the standard cohomology group* $H^1(X, \mathscr{O})$*. If* $\mathscr{H}$ *holds for* X *and* X *has one end, then the mapping is injective.*

(c) Let X *be a noncompact complex manifold. If* $H_c^1(X, \mathscr{O}) = 0$*, then* $\mathscr{HB}$ *holds in* X.

(d) Let X *be a noncompact complex manifold with one end. We suppose that* $\mathscr{H}$ *holds for* X *and that* $\overline{\partial}$*-problem has always a solution. Then* $H_c^1(X, \mathscr{O}) = 0$ *and* $\mathscr{HB}$ *holds in* X.▮

The following are consequences of Theorem 3.2.2 and Proposition 6.2.1:

Theorem 6.2.2 *If* X_Σ *is a smooth toric surface with a strictly convex fan, then* $H_c^1(X_\Sigma, \mathscr{O}) = 0$ *and the Hartogs-Bochner phenomenon holds in* X_Σ.

Proof: Notice that Example 2.3.10 proves that X_Σ has one end and the Hartogs phenomenon holds on X_Σ based on Theorem 3.2.2. Moreover, $H^1(X_\Sigma, \mathscr{O}) = 0$ from [1], Lecture 16-17, Corollary 4.3; therefore, part (b) of Proposition 6.2.1 justifies that $H_c^1(X_\Sigma, \mathscr{O}) =$

0. Part (c) of Proposition 6.2.1 then proves that the Hartogs-Bochner phenomenon holds in X_Σ. ▌

In general, the phenomenon is difficult to establish in compact complex manifolds. Confirming this difficulty, the extension problem of CR functions from real CR hypersurfaces in $\mathbb{P}^2$ requires some work and is still not solved in full generality. We know that $\mathbb{P}^2$ is a toric surface determined by a fan that consists of three vectors.

Definition 6.2.4 *(**Globally minimal manifold**) A CR manifold M is called globally minimal if any two points can be joined by a piecewise smooth curve running in complex tangential directions.*

Theorem 6.2.3 ***([11], Theorem 1.1)*** *Let M be a compact connected $\mathcal{C}^2$-smooth real hypersurface in $\mathbb{P}^2$ that divides the projective space into two open parts U^- and U^+. If M is globally minimal, then*

(1) There exists a side, U^- or U^+, to which every continuous CR function on M extends holomorphically.

(2) All holomorphic functions on the other side of M that are continuous up to M are constant. ▌

The above theorem remains true if, instead of global minimality of M, we have holomorphic functions defined in a neighborhood of M (not just CR functions on M). The theorem is also valid for $\mathbb{P}^n$ for $n \geq 2$.

The question arises whether the assumption of global minimality can be removed. The conjecture is that it can, but this is still not proven. In further work we make no assumption about global minimality.

6.3 FANS OF SMOOTH COMPACT TORIC SURFACES

This subsection discusses some properties of fans associated with smooth compact toric surfaces. Let $\Sigma(1) = \{v_0\mathbb{R}_{\geq 0}, \ldots, v_{d-1}\mathbb{R}_{\geq 0}\}$ be a set of 1-dimensional cones that generate the fan Σ. Obviously, we can denumerate this set and choose any cone to be $v_0\mathbb{R}_{\geq 0}$. With this notation, we assume that $\Sigma(1)$ comes with counterclockwise order, and to simplify some expressions we can use $v_0 = v_d$. Figure 6.1 shows an example of $\Sigma(1)$. For a

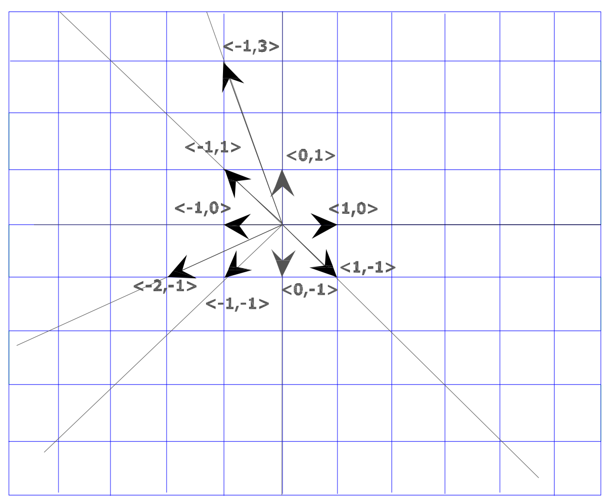

Figure 6.1: An example of $\Sigma(1)$

smooth cone, the determinant of its generators is equal to 1 or -1. Because the angle between the adjacent vectors is always less than 180^o we find that for $v_i = x_i e_1 + y_i e_2$ and

$v_{i+1} = x_{i+1}e_1 + y_{i+1}e_2$

$$\det[v_i, v_{i+1}] = \det \begin{bmatrix} x_i & x_{i+1} \\ y_i & y_{i+1} \end{bmatrix} = x_i y_{i+1} - y_i x_{i+1} = 1,$$

where $i \in \{0, ..., d-1\}$, $x_i, y_{i+1}, y_i, x_{i+1} \in \mathbb{Z}$, and $\{e_1, e_2\}$ is a standard basis of the lattice N.

Notice that any two consecutive vectors, let us say $v_i = x_i e_1 + y_i e_2$ and $v_{i+1} = x_{i+1}e_1 + y_{i+1}e_2$, can be fixed as e_1 and e_2 by applying a linear mapping to N. Clearly, the linear map $\phi : N \to N$ defined by the matrix $\phi = \begin{bmatrix} x_i & x_{i+1} \\ y_i & y_{i+1} \end{bmatrix}$ sends e_1 and e_2 to v_i and v_{i+1}, respectively, thus the map $\phi^{-1} = \begin{bmatrix} y_{i+1} & -x_{i+1} \\ -y_i & x_i \end{bmatrix}$ sends v_i and v_{i+1} to e_1 and e_2:

$\phi^{-1}(v_i) = (y_{i+1}x_i - x_{i+1}y_i)e_1 + (-y_i x_i + x_i y_i)e_2 = e_1$

and

$\phi^{-1}(v_{i+1}) = (y_{i+1}x_{i+1} - x_{i+1}y_{i+1})e_1 + (-y_i x_{i+1} + x_i y_{i+1})e_2 = e_2.$

6.3.1. Opposite Vectors in a Fan. The following observation will be helpful for future work. Assume, as before, that the vectors $v_0\mathbb{R}_{\geq 0}, \ldots, v_{d-1}\mathbb{R}_{\geq 0}$ generate the fan Σ. For the sake of simplicity we can use $v_d = v_0$. Moreover, without loss of generality, we can assume that $v_0 = e_1$ and $v_1 = e_2$, which means that there are no vectors in the interior of the first quadrant.

Lemma 6.3.1 *Let $i \in \{2, \ldots, d-1\}$ and let v_i lie in the interior of the third quadrant. Then:*

(1) the ancestor v_{i-1} does not lie in the interior of the second quadrant, and

(2) the successor v_{i+1} does not lie in the interior of the fourth quadrant.

Proof: Assume that the following situation is possible: Let

$$\begin{aligned} v_{i-1} &= (-\alpha_{i-1})e_1 + \beta_{i-1}e_2 \\ v_i &= (-\alpha_i)e_1 + (-\beta_i)e_2, \qquad \text{for} \quad \alpha_{i-1}, \alpha_i, \alpha_{i+1}, \beta_{i-1}, \beta_i, \beta_{i+1} \in \mathbb{Z}_{\geq 1}. \\ v_{i+1} &= \alpha_{i+1}e_1 + (-\beta_{i+1})e_2 \end{aligned}$$

Thus,

$$\det[v_{i-1}, v_i] = \alpha_{i-1}\beta_i + \beta_{i-1}\alpha_i \geq 2 \quad \text{and} \quad \det[v_i, v_{i+1}] = \alpha_i\beta_{i+1} + \beta_i\alpha_{i+1} \geq 2,$$

which contradicts the required condition that determinants are equal to 1 or -1 and thus proves the statement. ▮

This lemma implies that a vector from the interior of the third quadrant has the adjacents only in the interior of the third quadrant or on the axes. We can reformulate it as a more general statement:

Lemma 6.3.2 *Let v_s and v_{s+1} be two consecutive vectors in a fan Σ that describes a smooth, compact toric surface. If there is a vector v_k that lies in the interior of $(-v_s)\mathbb{R}_{\geq 0} + (-v_{s+1})\mathbb{R}_{\geq 0}$, then the vector v_{k-1} does not lie in the interior of $v_{s+1}\mathbb{R}_{\geq 0} + (-v_s)\mathbb{R}_{\geq 0}$, and the vector v_{k+1} does not lie in the interior of $(-v_{s+1})\mathbb{R}_{\geq 0} + v_s\mathbb{R}_{\geq 0}$.*

Proof: Choose the vectors v_s and v_{s+1} as a basis of the lattice N. Then v_k lies in the interior of the third quadrant and based on Lemma 6.3.1, the vectors v_{k-1} and v_{k+1} cannot lie in the interiors of the second and the fourth quadrants, respectively. In terms of the cones, this property can be expressed as:

$$v_{k-1} \notin \operatorname{Int}\left[(-v_s)\mathbb{R}_{\geq 0} + (-v_{s+1})\mathbb{R}_{\geq 0}\right]$$

$$v_{k+1} \notin \operatorname{Int}\left[v_{s+1}\mathbb{R}_{\geq 0} + (-v_s)\mathbb{R}_{\geq 0}\right],$$

which proves the statement. ▮

We will use this fact to prove the following proposition (which is proved in Appendix B). The result is shown in Figure 6.2.

Proposition 6.3.1 ***([14], Exercise, p.44)*** *If $d \geq 4$, then $v_i = -v_j$ for some $i, j \in \{0, \ldots, d-1\}$.*▮

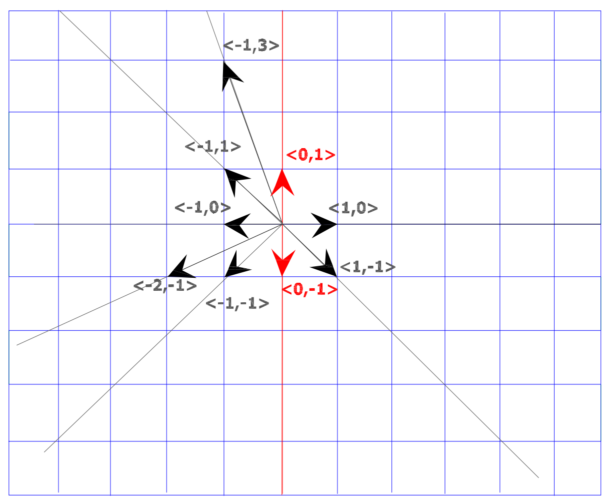

Figure 6.2: Opposite vectors in a fan

6.3.2. Three Consecutive Vectors in a Fan. The proposition and the following lemma (proof of which can be found in Appendix B) clarify the structure of $\Sigma(1)$. Keep in mind that the angle between two consecutive vectors is less than 180^o.

Lemma 6.3.3 ***([14], Section 2.5)*** *For each $v_i \in \Sigma(1)$, $i \in \{0, \ldots, d-1\}$, there exists $a_i \in \mathbb{Z}$ such that $a_i v_i = v_{i-1} + v_{i+1}$.*▮

Even if $a_i v_i = v_{i-1} + v_{i+1}$, the sum $v_{i-1} + v_{i+1}$ does not have to be a multiple of v_i. If $a_i = 0$, then $0 = v_{i-1} + v_{i+1}$, a situation described in Proposition 6.3.1. Such a situation occurs, for example, for a Hirzebruch surface. The following theorem describes a fan for all smooth, compact toric surfaces with $d \geq 5$; its proof is presented in Appendix B.

Theorem 6.3.1 ***([14], Claim, page 43)*** *If $d \geq 5$, then $v_i = v_{i-1} + v_{i+1}$ for some $i \in \{0, \ldots, d-1\}$.* ▮

6.4 THE HARTOGS-BOCHNER PHENOMENON

Before we approach the general problem, let us consider the following example with a sketch shown in Figure 6.3:

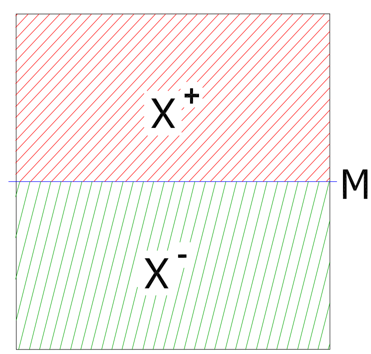

Figure 6.3: $M \subset \mathbb{P}^1 \times \mathbb{P}^1$

Example 6.4.1 In $X = \mathbb{P}^1 \times \mathbb{P}^1$, consider $M = \{(z, w) = (z_0, z_1, w_0, w_1) \in \mathbb{P}^1 \times \mathbb{P}^1 : |w_0| = |w_1|\} = \mathbb{P}^1 \times S^1$. Then $X^+ = \mathbb{P}^1 \times D(0)$ and $X^- = \mathbb{P}^1 \times D(\infty)$, where $D(0)$ ($D(\infty)$) is the unit disk with its center at $w_0 = 0$ ($w_1 = 0$). Notice that the function $f(z, w) = \frac{w_0}{w_1} + \frac{w_1}{w_0}$ obviously fulfills CR equations on M but cannot be extended to either X^+ or X^-. Such an extension would be constant on each fiber $\mathbb{P}^1$, which reduces the problem to a 1-dimensional case. But the function $f(z, w)$ has a pole at $w_0 = 0$ as well as at $w_1 = 0$, so it cannot be extended holomorphically from S^1 to either $D(0)$ or $D(\infty)$. ▌

This example clearly shows that no result can be obtained similar to that for $\mathbb{P}^2$. The following corollary, which is implied by Proposition 6.3.1, shows where the problem occurs. Figure 6.4 presents a sketch of the situation.

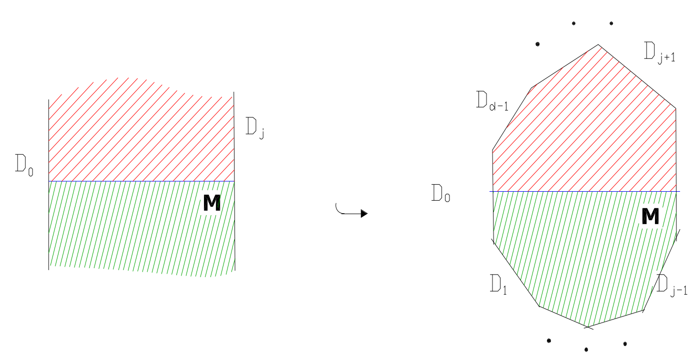

Figure 6.4: The embedding into any smooth compact toric surface with $d \geq 4$

Corollary 6.4.1 *For any smooth, compact toric surface X_Σ with the fan (Σ, N) such that $\Sigma(1)$ consists of 4 or more cones, there exists an embedding $\mathbb{P}^1 \times \mathbb{C}^* \hookrightarrow X_\Sigma$.*

Proof: As indicated by Proposition 6.3.1 there exist antipodal vectors: $v_0 = e_1 = -v_j$ for some $j \in \{2, \ldots, d-1\}$. Then $\Pi = \{0, v_0\mathbb{R}_{\geq 0}, v_j\mathbb{R}_{\geq 0}\}$ is a subfan of Σ, and the identity map of N induces an embedding $\Pi \to \Sigma$. Based on Theorem 2.2.4 there exists an embedding of toric varieties $X_\Pi \hookrightarrow X_\Sigma$. Moreover, $X_\Pi = \mathbb{P}^1 \times \mathbb{C}^*$, since that fan $\Pi = \{0, v_0\mathbb{R}_{\geq 0}, v_j\mathbb{R}_{\geq 0}\}$ with $v_j = -v_0$ is considered in a 2-dimensional lattice $N = \mathbb{Z}^2$. ∎

Notice that the hypersurface $M = \{(z, w) \in \mathbb{P}^1 \times \mathbb{C}^* : |w| = 1\} = \mathbb{P}^1 \times S^1$ divides $X_\Pi = \mathbb{P}^1 \times \mathbb{C}^*$ into two open, disjoint subsets, $X_\Pi^+ = \mathbb{P}^1 \times [D(0) \setminus \{0\}]$ and $X_\Pi^- = \mathbb{P}^1 \times [D(\infty) \setminus \{\infty\}]$. It is now clear that we can consider a similar hypersurface in any smooth, compact toric surface which contains $\mathbb{P}^1 \times \mathbb{C}^*$.

Theorem 6.4.1 *For every smooth, compact toric surface X_Σ with Σ such that $\Sigma(1)$ consists of four or more cones, there exists a compact, connected, C^2-differentiable hypersurface M and a CR function on M that has no holomorphic extension on either side of M.*

Proof: Let $\Sigma(1) = \{v_0\mathbb{R}_{\geq 0}, \ldots, v_{d-1}\mathbb{R}_{\geq 0}\}$, $d \geq 4$, and $v_0 = e_1$, $v_1 = e_2$. From Proposition 6.3.1 we know that $v_0 = -v_j$ for some $j \in \{2, \ldots, d-2\}$. Then the embedding of $\Pi = \{0, v_0\mathbb{R}_{\geq 0}, v_j\mathbb{R}_{\geq 0}\}$ into Σ implies the embedding $X_\Pi = \mathbb{P}^1 \times \mathbb{C}^* \hookrightarrow X_\Sigma$. Consider the following real hypersurface in $X_\Pi = \mathbb{P}^1 \times \mathbb{C}^*$ defined as $M = \{(z, w) \in \mathbb{P}^1 \times \mathbb{C}^* : |w| = 1\} = \mathbb{P}^1 \times S^1$. Let us recall that

$$\left(\mathbb{P}^1 \times \mathbb{C}^*\right)^+ = \{(z, w) \in \mathbb{P}^1 \times \mathbb{C}^* : |w| > 1\}$$

and

$$\left(\mathbb{P}^1 \times \mathbb{C}^*\right)^- = \{(z, w) \in \mathbb{P}^1 \times \mathbb{C}^* : |w| < 1\}.$$

With the notation that D_i is the projective curve defined by $v_i\mathbb{R}_{\geq 0} \in \Sigma(1)$ (in the sense of Theorem 2.2.3), we thus find that

$$X_\Sigma^+ = \left(\mathbb{P}^1 \times \mathbb{C}^*\right)^+ \cup D_{j+1} \cup \ldots \cup D_{d-1}$$

and

$$X_\Sigma^- = \left(\mathbb{P}^1 \times \mathbb{C}^*\right)^- \cup D_1 \cup \ldots \cup D_{j-1}.$$

Thus, we can claim that

$$X_\Sigma = X_\Sigma^+ \cup M \cup X_\Sigma^-,$$

since

$$\overline{X}_\Sigma^+ \cap \overline{X}_\Sigma^- = M.$$

Moreover, X_Σ^+ and X_Σ^- are open in X_Σ, since they are disjoint with their boundary M. The function $f(z,w) = w + \frac{1}{w}$ is clearly continuous and CR on M. We must prove that there is no holomorphic extension of f to either side of M. Let $v_{d-1} = ke_1 + (-1)e_2$, $k \in \mathbb{Z}$, and let the charts $X_1 \simeq \mathbb{C}^2$ and $X_d \simeq \mathbb{C}^2$ be associated with 2-dimensional cones $\sigma_1 = v_0\mathbb{R}_{\geq 0} + v_1\mathbb{R}_{\geq 0} = e_1\mathbb{R}_{\geq 0} + e_2\mathbb{R}_{\geq 0}$ and $\sigma_d = v_{d-1}\mathbb{R}_{\geq 0} + v_0\mathbb{R}_{\geq 0} = v_{d-1}\mathbb{R}_{\geq 0} + e_1\mathbb{R}_{\geq 0}$, respectively. If (z_1, w_1) are coordinates in X_1 and (z_d, w_d) are coordinates in X_d, then $w = w_1$ and $(z_1, w_1) = \left(z_d w_d^k, w_d^{-1}\right)$. The function f then is defined on $\mathbb{C}^1 \times S^1 \subset X_1$ as follows

$$f(z_1, w_1) = w_1 + \frac{1}{w_1}$$

and does not admit a holomorphic extension to $\mathbb{C}^1 \times D(0) \subset X_1$, since it has a pole at $(z_1, w_1) = 0$. On the other hand, the function f is defined on $\mathbb{C}^1 \times S^1 \subset X_d$ as

$$f(z_d, w_d) = \frac{1}{w_d} + w_d$$

and does not admit a holomorphic extension to $\mathbb{C}^1 \times D(0) \subset X_d$, since it has a pole at $(z_d, w_d) = (0,0)$. Because the points $(z_1, w_1) = 0$ and $(z_d, w_d) = 0$ lie on different sides of M, we claim that f does not have a holomorphic extension to either side of M. ▮

Corollary 6.4.2 *The Hartogs-Bochner phenomenon does not hold for a smooth, compact toric surfaces with a fan containing at least four* 1*-dimensional cones.* ▮

6.5 THE HARTOGS-BOCHNER PHENOMENON FOR A DOMAIN

Although the Hartogs-Bochner phenomenon does not hold for smooth compact toric surfaces, except $\mathbb{P}^2$, we can still consider the Hartogs-Bochner phenomenon in certain domains. We must remember that if X_Σ is a smooth compact toric surface, with $\Sigma(1) = \{v_0\mathbb{R}_{\geq 0}, \ldots, v_{d-1}\mathbb{R}_{\geq 0}\}$, then for each $i \in \{0, \ldots, d-1\}$, the cone $v_i\mathbb{R}_{\geq 0} \in \Sigma(1)$ defines the projective line $\mathbb{P}^1 \simeq D_i$ in X_Σ (in the sense of the closure of the orbit as stated in Theorem 2.2.3). On the other hand, each $v_i\mathbb{R}_{\geq 0} \in \Sigma(1)$ determines an integer a_i, as mentioned in Lemma 6.3.3.We express the Hartogs-Bochner phenomenon in a domain U in terms of the projective curves D_i contained in it, particularly the integers a_i. For example, Figure 6.5 shows projective curves inside the set U for $a_i < 0$. In this section, we assume that a

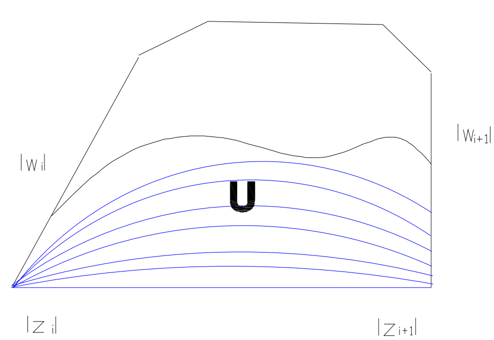

Figure 6.5: Projective curves for $a_i < 0$

fan associated with a smooth, compact toric variety contains at least four 1-dimensional cones.

6.5.1. $a_i < 0$**.** Here, a case when $a_i < 0$ is considered.

Lemma 6.5.1 *Let X_Σ be a smooth compact toric surface with $\Sigma(1) = \{v_0\mathbb{R}_{\geq 0}, \ldots, v_{d-1}\mathbb{R}_{\geq 0}\}$, and let U be a domain in X_Σ. If $\mathbb{P}^1 \simeq D_i \subset U$ for D_i which is associated with 1-dimensional cone $v_i\mathbb{R}_{\geq 0}$ with $a_i < 0$, then there is a family of curves $\{C_\lambda\}_{\lambda\in\Lambda}$, $C_\lambda \simeq \mathbb{P}^1$ such that $C_\lambda \subset U$ and $\bigcap_{\lambda\in\Lambda} C_\lambda \neq \emptyset$.*

Proof: The toric variety X_Σ is smooth; therefore, we can assume for the sake of simplicity that v_{i-1} and v_i are the standard basis vectors, i.e., $v_{i-1} = e_1$ and $v_i = e_2$. Then $v_{i+1} = -e_1 + (-k)e_2$, where $k = -a_i \in \mathbb{Z}_{\geq 1}$. The 2-dimensional cones $\sigma_i = v_{i-1}\mathbb{R}_{\geq 0} + v_i\mathbb{R}_{\geq 0}$ and $\sigma_{i+1} = v_i\mathbb{R}_{\geq 0} + v_{i+1}\mathbb{R}_{\geq 0}$ give the charts X_i and X_{i+1} with coordinates (z_i, w_i) and (z_{i+1}, w_{i+1}), respectively. Notice that on $X_i \cap X_{i+1}$, as indicated in Example 2.3.8, we have $(z_{i+1}, w_{i+1}) = \left(\frac{1}{z_i}, \frac{w_i}{z_i^k}\right)$. We can define the family of affine curves C_λ, with $\lambda \in \mathbb{C}$ in X_{i+1} as $w_{i+1} = \lambda$. Then in X_i the curves are defined by $w_i = \lambda z_i^k$. Since $k \in \mathbb{Z}_{\geq 1}$, each curve C_λ contains the point $(z_{i+1}, w_{i+1}) = (0, \lambda) \in X_{i+1}$ and the point $(z_i, w_i) = (0, 0) \in X_i$. Thus, $C_\lambda \simeq \mathbb{P}^1$ in X_Σ. Let U be an open set, which contains the projective curve D_i, which is actually equal to C_0. Then for λ such that $|\lambda|$ is small enough, the curves $C_\lambda \subset U$; therefore, $\Lambda = \{\lambda \in \mathbb{C} : C_\lambda \subset U\}$. Notice that all curves meet at the point $(z_i, w_i) = (0, 0) \in X_i$, so the intersection is nonempty. ▮

Corollary 6.5.1 *If a domain U contains a family of projective curves C_λ such that $C_{\lambda_1} \cap C_{\lambda_2} \neq \emptyset$ for $\lambda_1 \neq \lambda_2$ then global functions on U are constant, i.e., $\Gamma(U, \mathscr{O}) = \mathbb{C}$.*

Proof: Let $f \in \Gamma(U, \mathscr{O})$. Then f is constant on any projective curve C_λ. Because $C_{\lambda_1} \cap C_{\lambda_2} \neq \emptyset$ for $\lambda_1 \neq \lambda_2$ the value of f on C_{λ_2} is equal to the value of f on C_{λ_2}. Then f is constant on U, so $\Gamma(U, \mathscr{O}) = \mathbb{C}$. ▮

It is now clear that the only functions that could be extended to the whole U are constant. Thus we have the following corollary.

Corollary 6.5.2 *The Hartogs-Bochner phenomenon does not hold in a domain U that contains a projective curve $D_i \simeq \mathbb{P}^1$ defined by $v_i \in \mathbb{R}_{\geq 0} \in \Sigma(1)$ with $a_i < 0$.*

Proof: It is sufficient to show an example of a smooth real hypersurface M in U and a nonconstant CR function defined on M. If X_i and X_{i+1} have coordinates (z_i, w_i) and (z_{i+1}, w_{i+1}), respectively, then on $X_i \cap X_{i+1}$ we have $(z_{i+1}, w_{i+1}) = \left(\frac{1}{z_i}, \frac{w_i}{z_i^k}\right)$, where $k = -a_i > 0$. The hypersurface M is defined in X_i by the equation

$$1 + |z_i|^{2k} = r^2 |w_i|^2$$

and in X_{i+1} by

$$1 + |z_{i+1}|^{2k} = r^2 |w_{i+1}|^2 .$$

Here, $r > 0$ is sufficiently small, so $M \subset U$. In particular, if we define

$$M_i^+ = \{(z_i, w_i) \in X_i : 1 + |z_i|^{2k} > r^2 |w_i|^2\}$$

and

$$M_{i+1}^+ = \{(z_{i+1}, w_{i+1}) \in X_{i+1} : 1 + |z_{i+1}|^{2k} > r^2 |w_{i+1}|^2\},$$

then $M^+ = M_i^+ \cup M_{i+1}^+$ is open in $X_i \cup X_{i+1}$ and $\partial M^+ = M$.

First, we prove that M is smooth. Let us introduce $x_i = \mathrm{Re} z_i$, $y_i = \mathrm{Im} z_i$, $c_i = \mathrm{Re} w_i$, and $d_i = \mathrm{Im} w_i$. Then in real coordinates $(x_i, y_i, c_i, d_i) \in X_i$, the hypersurface M is described as

$$1 + \left(x_i^2 + y_i^2\right)^k - r^2 \left(c_i^2 + d_i^2\right) = 0.$$

Its gradient vector is then as follows:

$$\left[2kx_i \left(x_i^2 + y_i^2\right)^{k-1}, 2ky_i \left(x_i^2 + y_i^2\right)^{k-1}, -2c_i r^2, -2d_i r^2\right].$$

In particular, if $c_i = d_i = 0$, then for points on M we have

$$1 + \left(x_i^2 + y_i^2\right)^k = 0,$$

which is not possible, so $x_i \neq 0$ or $y_i \neq 0$. Thus, M is smooth. Now we must show a nonconstant holomorphic function on M. Consider the function f defined in coordinates $(z_i, w_i) \in X_i$ as

$$f_i(z_i, w_i) = \frac{1}{w_i}$$

and in $(z_{i+1}, w_{i+1}) \in X_{i+1}$ as

$$f_{i+1}(z_{i+1}, w_{i+1}) = \frac{z_{i+1}^k}{w_{i+1}}.$$

Then

$$f = f_i\left(0, \frac{1}{r}\right) = \frac{r}{\sqrt{2}}$$

and

$$f = f_i\left(1, \frac{1}{r}\right) = r;$$

therefore, f is a nonconstant function on M. Specifically, f does not allow holomorphic extension to M^+, since M^+ contains the curve D_i defined by $w_i = 0$ in X_i and $w_{i+1} = 0$ in X_{i+1}. ▮

6.5.2. $a_i = 0$**.** Here, a case when $a_i = 0$ is considered. This case is particularly interesting because locally the toric variety appears like a product of a disc and the projective line. In particular, Figure 6.6 shows projective curves inside U.

Lemma 6.5.2 *Let X_Σ be a smooth compact toric surface with $\Sigma(1) = \{v_0\mathbb{R}_{\geq 0}, ..., v_{d-1}\mathbb{R}_{\geq 0}\}$, and let U be a domain in X_Σ. If there exists $\mathbb{P}^1 \simeq D_i \subset U$ with $v_i\mathbb{R}_{\geq 0} \in \Sigma(1)$ such that $a_i = 0$, then there exists $\{C_\lambda\}_{\lambda \in \Lambda}$, a family of projective curves in U, such that $C_{\lambda_1} \cap C_{\lambda_2} = \emptyset$ for $\lambda_1 \neq \lambda_2$.*

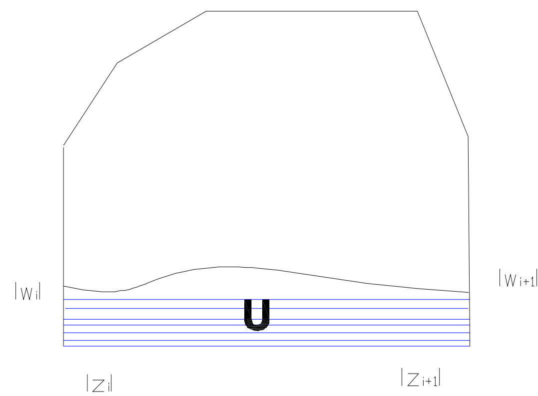

Figure 6.6: The projective curves for $a_i = 0$

Proof: From Lemma 6.3.3, we know that $a_i v_i = v_{i-1} + v_{i+1}$, so $a_i = 0$ gives $0 = v_{i-1} + v_{i+1}$. Again, we can assume that the vectors v_{i-1} and v_i are the standard basis vectors, i.e., $v_{i-1} = e_1$, $v_i = e_2$. Then $v_{i+1} = -e_1$. With two 2-dimensional cones $\sigma_i = e_1\mathbb{R}_{\geq 0} + e_2\mathbb{R}_{\geq 0}$ and $\sigma_{i+1} = e_2\mathbb{R}_{\geq 0} + (-e_1)\mathbb{R}_{\geq 0}$ we can consider two charts X_i and X_{i+1} with coordinates (z_i, w_i) and (z_{i+1}, w_{i+1}), respectively. In $X_i \cap X_{i+1}$ we then have $(z_{i+1}, w_{i+1}) = \left(\frac{1}{z_i}, w_i\right)$, as computed in Example 2.3.7. The family C_λ is defined as $w_i = \lambda$ in X_i and $w_{i+1} = \lambda$ in X_{i+1}. Each C_λ contains the point $(z_i, w_i) = (0, \lambda) \in X_i$ and the point $(z_{i+1}, w_{i+1}) = (0, \lambda) \in X_{i+1}$; thus, $C_\lambda = \mathbb{P}^1$ in $X_i \cup X_{i+1}$. If an open domain U contains the projective curve $D_i \simeq C_0$, then it clearly contains the whole family C_λ for λ such that $|\lambda|$ is small enough. Since C_λ are in fact projective lines in $X_i \cup X_{i+1}$, the open set U, which contains D_i, contains the product of a projective line and a disc. ▮

Corollary 6.5.3 *The Hartogs-Bochner phenomenon does not hold in a domain U that contains a projective curve $D_i \simeq \mathbb{P}^1$ defined by $v_i\mathbb{R}_{\geq 0}$ with $a_i = 0$.*

Proof: If $a_i = 0$, then U contains a product of the projective line and a closure of a small disk:

$$\mathbb{P}^1 \times \overline{D(0,\epsilon)} \subset U$$

Then X_Σ can be divided into two disjoint open sets $X_\Sigma^+ = \mathbb{P}^1 \times D(0,\epsilon)$ and

$$X_\Sigma^- = X_\Sigma \setminus \left[\mathbb{P}^1 \times \overline{D(0,\epsilon)}\right].$$

If f is a holomorphic function on U, then it is constant along the projective lines. The Hartogs-Bochner phenomenon in U can then be reduced to the phenomenon in a disc. Generally, the answer is negative, so we find that the Hartogs-Bochner phenomenon does not hold in U. ▌

6.5.3. $a_i > 0$. The affirmative result obtained in this subsection requires additional theory. Notice that if $v_i\mathbb{R}_{\geq 0} \in \Sigma(1)$, then the cones $v_{i-1}\mathbb{R}_{\geq 0}$, $v_i\mathbb{R}_{\geq 0}$ and $v_{i+1}\mathbb{R}_{\geq 0}$ generate another fan:

$$\widetilde{\Sigma} = \{0, v_{i-1}\mathbb{R}_{\geq 0}, v_i\mathbb{R}_{\geq 0}, v_{i+1}\mathbb{R}_{\geq 0}, v_{i-1}\mathbb{R}_{\geq 0} + v_i\mathbb{R}_{\geq 0}, v_i\mathbb{R}_{\geq 0} + v_{i+1}\mathbb{R}_{\geq 0}\},$$

which is actually a subfan of Σ. Moreover, the toric variety $\widetilde{X}$ defined by $\widetilde{\Sigma}$ is smooth, noncompact and $\widetilde{X} \subset X$. Let $X_i \simeq \mathbb{C}^2$ and $X_{i+1} \simeq \mathbb{C}^2$ be the patches defined by the cones $\sigma_i = v_{i-1}\mathbb{R}_{\geq 0} + v_i\mathbb{R}_{\geq 0}$ and $\sigma_{i+1} = v_i\mathbb{R}_{\geq 0} + v_{i+1}\mathbb{R}_{\geq 0}$, respectively. The assumption that $\overline{U}$ does not meet any other projective curves associated with 1-dimensional cones from Σ is then equivalent to the condition that U does not have limit points outside $X_i \cup X_{i+1}$, i.e., that $\overline{U} \subset X_i \cup X_{i+1}$. Or, equivalently, that $\overline{U} \subset \widetilde{X}$.

Theorem 6.5.1 *If a domain $\overline{U}$ does not meet or contain any projective curves other than D_i defined by $v_i\mathbb{R}_{\geq 0} \in \Sigma(1)$ with $a_i > 0$, then the Hartogs-Bochner phenomenon holds in U.*

Proof: Because $\overline{U}$ does not meet any projective curves associated with cones from $\Sigma(1)$ other than D_i, we find that $\overline{U} \subset \widetilde{X}$. Based on Theorem 6.2.2, we then find that $H^1_c(\widetilde{X}, \mathscr{O}) = 0$; thus Theorem 6.2.1 proves that the Hartogs-Bochner phenomenon holds in U. ▮

This answer need not remain positive if there are more projective curves of this type contained in U. This problem is discussed in the next section.

6.6 REDUCIBLE CASE

Let U be a domain in a compact, smooth toric variety X associated with the fan:

$$\Sigma = \{0, v_0\mathbb{R}_{\geq 0}, \ldots, v_{d-1}\mathbb{R}_{\geq 0}, v_0\mathbb{R}_{\geq 0} + v_1\mathbb{R}_{\geq 0}, \ldots, v_{d-2}\mathbb{R}_{\geq 0} + v_{d-1}\mathbb{R}_{\geq 0}, v_{d-1}\mathbb{R}_{\geq 0} + v_0\mathbb{R}_{\geq 0}\},$$

which contains at least four 1-dimensional cones, i.e., $d \geq 3$. Assume that U contains a connected, reducible curve C that admits the decomposition $C = D_1 \cup \ldots \cup D_k$ into irreducible projective curves defined by $v_1\mathbb{R}_{\geq 0}, \ldots, v_k\mathbb{R}_{\geq 0} \in \Sigma(1)$, as claimed in Theorem 2.2.3. Notice that those cones define the subfan $\widetilde{\Sigma}$ of Σ as follows:

$$\widetilde{\Sigma} = \{0, v_0\mathbb{R}_{\geq 0}, \ldots, v_{k+1}\mathbb{R}_{\geq 0}, v_0\mathbb{R}_{\geq 0} + v_1\mathbb{R}_{\geq 0}, \ldots, v_k\mathbb{R}_{\geq 0} + v_{k+1}\mathbb{R}_{\geq 0}\},$$

where $0 \geq k \geq d-1$. If $\widetilde{X}$ is the toric variety defined by $\widetilde{\Sigma}$, then clearly $\widetilde{X}$ is smooth, noncompact and $\widetilde{X} \subset X$. Moreover, if we assume that $\overline{U}$ does not meet any projective curves associated with other 1-dimensional cones from Σ, then $\overline{U} \subset \widetilde{X}$.

As before, for the sake of simplicity, we assume that $v_0 = e_1$ and $v_1 = e_2$.

Now we are ready to formulate the theorem.

Theorem 6.6.1 *Let U be a domain that contains a connected, reducible curve $C = D_1 \cup \ldots \cup D_k$, where $D_1, \ldots, D_k$ are projective curves defined by the vectors $v_1\mathbb{R}_{\geq 0}, \ldots, v_k\mathbb{R}_{\geq 0} \in \Sigma(1)$. Then*

(i) If $\left|\widetilde{\Sigma}\right|$ covers at least a half plane, then the Hartogs-Bochner phenomenon does not hold in U.

(ii) If $\left|\widetilde{\Sigma}\right|$ covers less than a half plane and $\overline{U}$ does not meet any projective curves associated with other 1-dimensional cones from Σ, then the Hartogs-Bochner phenomenon holds in U.

Proof: For (i) we will prove that there is a family of projective curves in U. Figure 6.7 shows a sketch of those curves. Assume that $v_0 = e_1$ and $v_1 = e_2$. We must consider three possible cases.

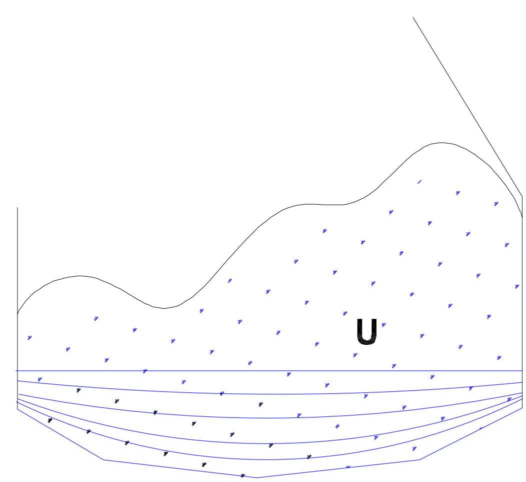

Figure 6.7: Case 1

CASE 1. Assume that there exists $v_j\mathbb{R}_{\geq 0} \in \widetilde{\Sigma}(1)$ such that $v_j = -v_0$. If v_{j-1} proceeds v_j, then smoothness indicates that $v_{j-1} = -\alpha_{j-1}e_1 + e_2$ for some $\alpha_{j-1} \in \mathbb{Z}_{\geq 1}$. Let $\sigma_1 = v_0\mathbb{R}_{\geq 0} + v_1\mathbb{R}_{\geq 0} = e_1\mathbb{R}_{\geq 0} + e_2\mathbb{R}_{\geq 0}$ and consider the chart X_1 with coordinates (z_1, w_1) and the family of curves $w_1 = \lambda$ for $\lambda \in \mathbb{C}$ and $|\lambda|$ small enough. Notice that if the chart X_j, defined by $\sigma_j = v_{j-1}\mathbb{R}_{\geq 0} + v_j\mathbb{R}_{\geq 0}$ has coordinates (z_j, w_j), then the family of curves transforms to $w_j = \lambda$ since the change of coordinates can be express as follows:

$$z_j = \frac{1}{z_1^{\alpha_{j-1}} w_1} \quad \text{and} \quad w_j = w_1.$$

Clearly, in this chart for a small enough value of $|\lambda|$, the curves fit in an arbitrarily small neighborhood of the projective curve D_{j-1}. As demonstrated by Lemma 2.3.2, every curve of the form $w_1 = \lambda$ in X_1 has the point at infinity in the chart X_j; therefore, we can conclude that $w_1 = \lambda$ defines a family of projective curves in $\widetilde{X}$. The vectors $v_0, v_j \in \widetilde{\Sigma}(1)$ which fulfill $v_j = -v_0$ define the embedding $\mathbb{P}^1 \times \mathbb{C}^* \hookrightarrow \widetilde{X}$, and Theorem 6.4.1 shows the existence of a nonconstant CR function on a CR hypersurface.

CASE 2. For all cones $v_j\mathbb{R}_{\geq 0} \in \widetilde{\Sigma}$, $v_j \neq -v_0$, and there are no 1-dimensional cones in the interior of the second quadrant. Then v_2 lies in the interior of the third quadrant, which makes $a_1 < 0$. We considered this problem in Corollary 6.5.2 and proved that the Hartogs-Bochner phenomenon does not hold.

CASE 3. For all cones $v_j\mathbb{R}_{\geq 0} \in \widetilde{\Sigma}$, $v_j \neq -v_0$, and there is a 1-dimensional cone in the interior of the second quadrant. Lemma 6.3.1 then proves that there are no cones in the interior of the third quadrant, and from Lemma 6.3.2, we conclude that there are opposite vectors in this fan. We have already considered this in case 1. For (ii) because $\overline{U}$ does not meet any projective curves associated with other 1-dimensional cones from Σ, we have that $\overline{U} \subset \widetilde{X}$. The fan $\widetilde{\Sigma}$ covers less than a half plane, particularly the curve $C \subset U$ is connected and $\widetilde{\Sigma}$ is strictly convex. Theorem 6.2.2 thus proves that $H^1_c(\widetilde{X}, \mathscr{O}) = 0$. Since $\widetilde{X}$ is noncompact, Theorem 6.2.1, part (c), implies that the Hartogs-Bochner phenomenon holds in $\widetilde{X}$; therefore, it holds for U. Figure 6.8 shows a sketch of a connected reducible curve in U, which is a sum of its irreducible components. ▮

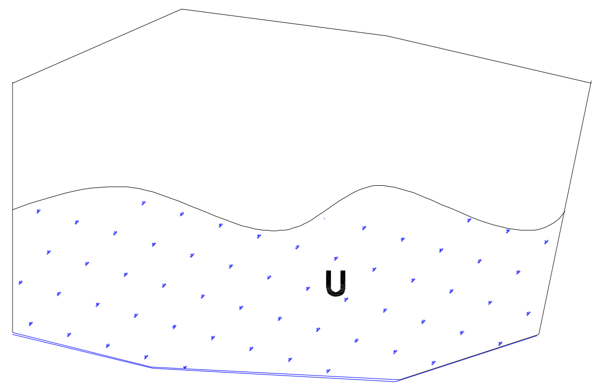

Figure 6.8: Reducible curve inside U in part (ii)

6.7 THE HARTOGS-BOCHNER PHENOMENON FOR LINE BUNDLES

This section considers the Hartogs-Bochner phenomenon in toric varieties X_Σ with a line bundle structure. In particular, if $|\Sigma|$ is strictly convex, then we will prove that the Hartogs-Bochner phenomenon holds in X_Σ.

Theorem 6.7.1 ***([1], Lecture 16-17, Corollary 4.3)*** *If X_Σ is a toric variety with a convex fan then $H^1(X_\Sigma, \mathscr{O}) = 0$.* ▮

Since strictly convex sets are particularly convex, we can use this result in the following theorem:

Theorem 6.7.2 *Let X_Σ be a toric variety with a line bundle structure over a compact base. If Σ is strictly convex then $H^1_c(X_\Sigma, \mathscr{O}) = 0$, and the Hartogs-Bochner phenomenon holds in X_Σ.*

Proof: Note that Theorem 5.2.1 proves that X_Σ has one end, and Theorem 5.4.2 shows that and the Hartogs phenomenon holds in X_Σ. Moreover, $H^1(X_\Sigma, \mathscr{O}) = 0$ from Theorem 6.7.1; therefore, part (b) of Proposition 6.2.1 proves that $H^1_c(X_\Sigma, \mathscr{O}) = 0$. Part (c) of Proposition 6.2.1 then proves that the Hartogs-Bochner phenomenon holds in X_Σ. ▌

6.8 HOLOMORPHIC EXTENSIONS IN VECTOR BUNDLES

Here, we formulate results about holomorphic extensions of CR functions and the $\overline{\partial}$-problem in vector bundles over arbitrary complex manifolds.

Let $\pi : X \longrightarrow B$ be a complex vector bundle, where X and B are complex manifolds and $\pi^{-1}(p) \simeq \mathbb{C}^k$ for $p \in B$.

Theorem 6.8.1 ***([10], Theorem. 6.1)*** *Let X be a complex fiber bundle with fiber dimension k. Let ω be a closed $(0, 1)$ form compactly supported along the fibers. Then there exists a unique smooth function u such that $\overline{\partial} u = \omega$ with the following properties:*

1. *If $k = 1$, then u vanishes at infinity along the fibers.*
2. *If $k > 1$, then u is compactly supported along the fibers.*
3. *If $k > 1$ and the form ω has compact support, then u has compact support.* ▌

Corollary 6.8.1 ***([10], Corollary 7.1)*** *Let X be a complex fiber bundle, and let M be a compact, connected, real hypersurface (without a boundary) that divides X into connected* *open subsets X^+ and X^-. Then any CR function f on M can be represented as $f = f^+ - f^-$, where f^+ and f^- are holomorphic in X^+ and X^-, respectively, and smooth up to M.* ▌

Corollary 6.8.2 ***([10], Corollary 7.2)*** *Let X be a complex fiber bundle with fiber dimension $k > 1$. Let M be a compact, connected, real hypersurface (without a boundary) that divides X into connected open subsets $X^+ \Subset X$ and X^-. Then any CR function f on M can be holomorphically extended to X^+ smoothly up to M.* ▌

Corollary 6.8.3 ***([10], Corollary 6.4)*** *Let X be a complex fiber bundle with a fiber dimension greater or equal to 2. Then the first compactly supported cohomology group $H^1_c(X, \mathscr{O})$ is equal to zero.* ▌

6.9 EXAMPLE AND CONJECTURES

At this point, we can pose the following question: Is strict convexity of a fan Σ a necessary condition for the extension phenomena? The following example of a vector bundle shows that it is not. From Theorem 6.8.1 we know that the $\overline{\partial}$-problem has a solution in all vector bundles with the dimension of the fiber greater or equal to two. Then the first cohomology group with compact support is trivial. Here, we present an example of a vector bundle, whose fan is not convex. Moreover, its support is not even a subset of a half space.

Example 6.9.1 Let X be a toric variety described by the fan $\Sigma \subset N_{\mathbb{R}} = \mathbb{R}^3$, which contains the following 3-dimensional cones σ_0 and σ_1, together with their faces:

$$\sigma_0 = e_1\mathbb{R}_{\geq 0} + e_2\mathbb{R}_{\geq 0} + e_3\mathbb{R}_{\geq 0},$$

and

$$\sigma_1 = e_1\mathbb{R}_{\geq 0} + (-e_1 - e_2 - e_3)\mathbb{R}_{\geq 0} + e_3\mathbb{R}_{\geq 0}.$$

Then $\Sigma(3) = \{\sigma_0, \sigma_1\}$. Notice that the projection $P : \mathbb{R}^3 \to \mathbb{R}^3$ defined as $P(x_1, x_2, x_3) = x_1$ defines the structure of a vector bundle on X over $\mathbb{P}^1$, with a fiber dimension 2. This can also be observed as well by inspection of the change of coordinates. Note, that the dual cones are as follows:

$$\sigma_0^\vee = e_1^*\mathbb{R}_{\geq 0} + e_2^*\mathbb{R}_{\geq 0} + e_3^*\mathbb{R}_{\geq 0},$$

and

$$\sigma_1^\vee = -e_1^*\mathbb{R}_{\geq 0} + (-e_1^* + e_2^*)\mathbb{R}_{\geq 0} + (-e_1^* + e_3^*)\mathbb{R}_{\geq 0}.$$

If $(z, v, w) \in X_0$ and $(z_1, v_1, w_1) \in X_1$, then on $X_0 \cap X_1$ we have:

$$\begin{aligned} z_1 &= \frac{1}{z}, \\ v_1 &= \frac{v}{z}, \\ w_1 &= \frac{w}{z}, \end{aligned}$$

and

$$\begin{aligned} z &= \frac{1}{z_1}, \\ v &= \frac{v_1}{z_1}, \\ w &= \frac{w_1}{z_1}. \end{aligned}$$

Then, clearly, the mappings are linear with respect to the coordinates $(v, w) \in \mathbb{C}^2$. Thus, X is a vector bundle with dimension of the fiber 2, moreover X has one end. From Corollary 6.8.3, the Hartogs-Bochner phenomenon holds in X and the Hartogs phenomenon holds as well, since any function defined outside of a compact set is particularly defined on a CR hypersurface. From Corollary 6.8.2, $H^1_c(X, \mathscr{O}) = 0$, but the fan Σ is not convex.

We can now formulate the following conjecture related to the Hartogs phenomenon:

Conjecture 6.9.1 *Let X be a smooth toric variety. If the complement of its fan contains at least one connected component, which is concave, then the Hartogs phenomenon holds in X.*

Similarly, we can formulate the following conjecture related to the cohomology group with compact support and the Hartogs-Bochner phenomenon:

Conjecture 6.9.2 *Let X be a smooth toric variety. If the complement of its fan has one connected component, which is concave, then $H^1_c(X, \mathscr{O}) = 0$, and the Hartogs-Bochner phenomenon holds in X.*

These appendices contain detailed proofs of some well known facts regarding fans of toric varieties. In most cases, these facts are left as exercises or those proofs offered in the literature require particular assumptions, that cannot be made in the cases addressed here. The numbering used in the body of this work is maintained here..

APPENDIX A

FANS OF FIBER BUNDLES

Appendix A contains proofs from Section 5 related to fans of fiber bundles.

Theorem 4.2.1 ([14], Exercise, p. 22) Let X be an n-dimensional toric variety with the fan (Σ, N). Then $X = (\mathbb{C}^*)^k \times B$ for some $(n-k)$-dimensional toric variety B if and only if $\Sigma \subset N'_{\mathbb{R}}$, where N' is an $(n-k)$-dimensional sublattice of N.

Proof: Let the n-dimensional toric variety X be associated with at most $(n-k)$-dimensional fan (Σ, N), with dim$N = n$ and $\Sigma \subset N'_{\mathbb{R}}$, where N' is a $(n-k)$-dimensional sublattice of N. If $e_{k+1}, \ldots, e_n$ is a basis for $N'_{\mathbb{R}}$ over $\mathbb{Z}$, then there exists a basis of N over $\mathbb{Z}$, that completes it; that is, $e_1, \ldots, e_k, e_{k+1}, \ldots, e_n$ is a basis of $N_{\mathbb{R}}$ over $\mathbb{Z}$. Let $e_1^*, \ldots, e_n^*$ be the dual basis to $e_1, \ldots, e_n$ in $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$. We know that for each $\sigma \in \Sigma$, in fact $\sigma \subset N'_{\mathbb{R}}$; therefore, the dual to σ is of the form:

$$\sigma^\vee = e_1^*\mathbb{R}_{\geq 0} + \ldots + e_k^*\mathbb{R}_{\geq 0} + (-1)e_1^*\mathbb{R}_{\geq 0} + \ldots + (-1)e_k^*\mathbb{R}_{\geq 0} + \tau^\vee,$$

where $\tau = (\sigma, N')$ is the cone σ but considered in the lattice N', and $\tau^\vee$ is the dual to τ in $M' = \mathrm{Hom}_{\mathbb{Z}}(N', \mathbb{Z})$. Then $S[\sigma] = \mathbb{C}[z_1, \ldots, z_k, \frac{1}{z_1}, \ldots, \frac{1}{z_k}, \tau$, which makes $X_\sigma = (\mathbb{C}^*)^k \times X_\tau$. Since for different cones in Σ, the basis $e_1, \ldots, e_n$ of N is chosen uniformly, we have $X = (\mathbb{C}^*)^k \times B$, where B is described by (Σ, N').

Now, let an n-dimensional toric variety X described by a fan (Σ, N) be the product of $(\mathbb{C}^*)^k$ and B, where B is a toric variety with a fan (Π, N'). We have $X = \bigcup_{\sigma \in \Sigma} X_\sigma$ and for each $\sigma \in \Sigma$ the affine toric variety variety X_σ fulfils $X_\sigma = (\mathbb{C}^*)^k \times B_\tau$ with $B = \bigcup_{\tau \in \Pi} B_\tau$. Then $\sigma^\vee = e_1^* + \ldots + e_k^* + (-1)e_1^* + \ldots + (-1)e_k^* + \tau^\vee$, where $\tau \in (\Pi, N')$, and $\tau^\vee$ is taken in $M' = \mathrm{Hom}_{\mathbb{Z}}(N', \mathbb{Z})$. Since the basis $e_1, \ldots, e_k$ is chosen uniformly for all $\sigma \in \Sigma$, the lattice N' is spanned by $e_{k+1} \ldots, e_n$, and each $\sigma \in \Sigma$ is actually in N', which proves that Σ is at most $(n-k)$-dimensional fan and $\Sigma \subset N'$. ▮

Theorem 4.2.2 ([14], Exercise, p. 22) Let (Σ, N) be a fan associated with a toric variety X, (Δ, N'') a fan associated with F, and (Π, N') a fan associated with B. Then X is a product of toric varieties F and B if and only if $(\Sigma, N) = (\Delta \times \Pi, N'' \times N')$.

Proof: If $(\Sigma, N) = (\Delta \times \Pi, N'' \times N')$, then for any $\sigma \in (\Sigma, N)$ we have $\sigma = \gamma \times \tau$, where $\gamma \in (\Delta, N'')$ and $\tau \in (\Pi, N')$. Moreover, $\sigma^\vee = \gamma^\vee \times \tau^\vee$, where $\gamma^\vee$ is the dual cone to γ taken in $M'' = \mathrm{Hom}_{\mathbb{Z}}(N'', \mathbb{Z})$ and $\tau^\vee$ is the dual cone to τ taken in $M' = \mathrm{Hom}_{\mathbb{Z}}(N', \mathbb{Z})$. Then $X_\sigma = X_\gamma \times X_\tau$. Since $M = M'' \times M'$, we can claim that $X = X_\Delta \times X_\Pi = F \times B$. On the other hand, if $X = F \times B$, then for any cone $\sigma \in (\Sigma, N)$ we find that $X_\sigma = F_\gamma \times B_\tau$ for some $\gamma \in (\Delta, N'')$ and $\tau \in (\Pi, N')$. Since $F_\gamma \times B$ and $F \times B_\tau$ are toric varieties, the lattices N'' and N' can be chosen for all cones in Δ and Π, respectively. Then $\sigma = \gamma \times \tau$ and $N = N'' \times N'$, which imply that $(\Sigma, N) = (\Delta \times \Pi, N'' \times N')$. ▮

Theorem 4.2.3 Let (Σ, N) be a fan associated with toric variety X, and (Δ, N'') a fan associated with toric variety F, where Δ is a subfan of Σ and N'' is a sublattice of N. Then X is a fiber bundle with fiber F if and only if there exists such a subfan Π' in Σ that $\Sigma = \Delta + \Pi'$ exists and $N'' + \Pi'$ exists.

Proof: Let us first assume that $\Sigma = \Delta + \Pi'$ is a sum of two fans, so that $N'' + \Pi'$ exists, where $\Delta \subset N''_{\mathbb{R}}$. Consider the orthogonal projection $P : N_{\mathbb{R}} \to N_{\mathbb{R}}$ along $N''_{\mathbb{R}}$ as in Lemma 4.2.1, defining the exact sequence of lattices:

$$0 \to N'' \to N \to N' \to 0,$$

which induces the exact sequence of fans:

$$0 \to (\Delta, N'') \xrightarrow{\alpha} (\Sigma, N) \xrightarrow{\beta} (P(\Pi'), N') \to 0,$$

where α and β are maps of fans, α is an embedding, and β is onto the fan $P(\Pi')$. We will use the notation, where $P : N \to N'$ defines a bijection between the fan Π' and its image, the fan $P(\Pi') = \Pi$. Notice, that the map β defines the mapping between toric varieties $\pi : X \to B$. If $\gamma \in \Pi$, then

$$\beta^{-1}(\gamma) = \Delta + P^{-1}(\gamma),$$

which means that the inverse image of the cone γ, $\beta^{-1}(\gamma) \subset \Sigma$, which is a collection of cones in Σ, can be represented as a sum of the fan Δ and the cone $P^{-1}(\gamma) \in \Pi'$. Then the toric variety associated with the fan $\beta^{-1}(\gamma)$ is a product of F and an affine toric variety $X_{P^{-1}(\gamma)}$. Moreover, we have the following:

$$\pi^{-1}(X_\gamma) = X_{\beta^{-1}(\gamma)} = F \times X_{P^{-1}(\gamma)} \simeq F \times X_\gamma.$$

Specifically, $X_{P^{-1}(\gamma)} \simeq X_\gamma$, since P is a bijection between fans defining the mapping φ as in the definition of a fiber bundle:

$$\varphi : \pi^{-1}(X_\gamma) \to F \times X_\gamma.$$

Since $\beta^{-1}(\gamma) \in \Sigma$,

$$\beta(\beta^{-1}(\gamma)) = \beta(\Delta + P^{-1}(\gamma)) = \beta(\Delta) + \beta(P^{-1}(\gamma)) = \gamma \in \Pi,$$

which implies that the mapping

$$\Delta + \gamma \to \gamma$$

is trivial on Δ. Then $\pi \circ \varphi^{-1}(f, u) = u$ and X has a a fiber bundle structure, which finishes the proof from right to left.

Now let $\pi : X \to B$ be an n-dimensional toric variety with a structure of a fiber bundle with k-dimensional fiber F. Let X be defined by the fan (Σ, N), let F be defined by (Δ, N'') and B by (Π, N'), where $\dim N'' = k$ and $\dim N' = n-k$. The map $\pi : X \to B$ with fiber F defines the following map of fans:

$$(\Sigma, N) \xrightarrow{\beta} (\Pi, N').$$

For any $\gamma \in \Pi$, the affine toric variety X_γ admits the following property: There exists φ so that $\varphi : \pi^{-1}(X_\gamma) \to F \times X_\gamma$, which means that β is actually onto, and for any $\gamma \in \Pi$

its inverse image $\beta^{-1}(\gamma) \subset \Sigma$ is a subfan in Σ and has a structure of a sum of a cone and a fan, i.e.,

$$\beta^{-1}(\gamma) = \Delta + \gamma',$$

where γ' is a cone in Σ. Here, γ' is determined uniquely since the sum $\Delta + \gamma'$ is equal to the entire inverse image $\beta^{-1}(\gamma) \subset \Sigma$. Let us denote the collection of those cones γ' as Π'. We must prove that Π' is a subfan of Σ. Note first, that $0 \in \Pi'$, since $\beta^{-1}(0) = \Delta + 0$. Let now $\gamma_1', \gamma_2' \in \Pi'$ have nonempty intersection, then $\gamma_1 \cap \gamma_2 \neq \emptyset$ and, since Π is a fan, we have that $\gamma_1 \cap \gamma_2 \prec \gamma_1$ and $\gamma_1 \cap \gamma_2 \prec \gamma_2$. Moreover,

$$\beta^{-1}(\gamma_1 \cap \gamma_2) = \Delta + (\gamma_1 \cap \gamma_2)',$$

with $(\gamma_1 \cap \gamma_2)' \prec \gamma_1'$ and $(\gamma_1 \cap \gamma_2)' \prec \gamma_2'$, which proves that Π' is a fan. Notice that $\beta\mid_{\Pi'}$ is a bijection between fans Π' and Π. We will denote $\beta\mid_{\Pi'}$ as P to distinct between $\beta : \Sigma \to \Pi$ and $P : \Pi' \to \Pi$. Since for any $\gamma' \in \Pi'$ we have $\Delta \cap \gamma' = \{0\}$, we conclude that $\Delta \cap \Pi' = \{0\}$.

Since the map $\pi \circ \varphi^{-1} : F \times X_\gamma \to X_\gamma$ fulfills $\pi \circ \varphi^{-1}(f, u) = u$, we see that the maps of fans β and P fulfill:

$$\beta(\beta^{-1}(\gamma)) = \beta(\Delta + \gamma') = \beta(\Delta) + P(\gamma') = \gamma,$$

which implies that $\Sigma = \Delta + \Pi'$, since the inverse images $\beta^{-1}(\gamma)$ for $\gamma \in \Pi$ cover the entire Σ. Moreover, the map $\beta : N_{\mathbb{R}} \to N_{\mathbb{R}}'$ is a map of lattices with $\mathrm{Ker}\beta = N_{\mathbb{R}}''$, which proves that $N'' + \Pi'$ exists. This finishes the proof from left to right. ▮

APPENDIX B

FANS OF TORIC SURFACES

Appendix B contains proofs of facts related to smooth compact toric surfaces discussed in Section 6.

Proposition 6.3.1 ([14], Exercise, p.44) If $d \geq 4$, then $v_i = -v_j$ for some $i, j \in \{0, \ldots, d-1\}$.

Proof: Assuming that the above statement is false, we arrive at a contradiction. Let $v_0 = e_1, v_1 = e_2$. We consider two cases, the first contradicting the assumption that $d \geq 4$. The second contradicts the previous lemma.

CASE 1: Assume that one of the vectors v_i for $i \in \{2, \ldots, d-1\}$ lies in the interior of the third quadrant. Since there are no vectors on the negative axes, we can deduce from the previous lemma that all other vectors also lie in the interior of the third quadrant. Notice that for $v_2 = (-\alpha_2)e_1 + (-\beta_2)e_2$ with $\alpha_2, \beta_2 \in \mathbb{Z}_{\geq 1}$ we have

$$\det[v_1, v_2] = \det \begin{bmatrix} 0 & -\alpha_2 \\ 1 & -\beta_2 \end{bmatrix} = \alpha_2 = 1,$$

which gives $v_2 = (-1)e_1 + (-\beta_2)e_2$. Similarly for $v_{d-1} = (-\alpha_{d-1})e_1 + (-\beta_{d-1})e_2$ with $\alpha_{d-1}, \beta_{d-1} \in \mathbb{N}$ we have

$$\det[v_{d-1}, v_0] = \det \begin{bmatrix} -\alpha_{d-1} & 1 \\ -\beta_{d-1} & 0 \end{bmatrix} = \beta_{d-1} = 1 \text{ gives } v_{d-1} = (-\alpha_{d-1})e_1 + (-1)e_2.$$

Since the positions of the vectors v_2 and v_{d-1} agree with the counterclockwise orientation, we find that $\det[v_2, v_{d-1}] \geq 0$, which expands as:

$$\det[v_2, v_{d-1}] = \det \begin{bmatrix} -1 & -\alpha_{d-1} \\ -\beta_2 & -1 \end{bmatrix} = 1 - \alpha_{d-1}\beta_2 \geq 0.$$

Then $1 \geq \alpha_{d-1}\beta_2$, which implies that $\alpha_{d-1} = \beta_2 = 1$, since α_{d-1} and β_2 are positive integers. Then $v_2 = v_{d-1}$, which implies that $d = 3$ and contradicts the assumption that $d \geq 4$. Specifically, we have proved that the variety associated with this fan is $\mathbb{P}^2$.

CASE 2: Now assume that none of v_i for $i \in \{2, \ldots, d-1\}$ lies in the interior of the third quadrant. Let vectors $v_2, \ldots, v_k$ lie in the interior of the second quadrant and vectors $v_{k+1}, \ldots, v_{d-1}$ lie in the interior of the fourth quadrant. Because the angle between two consecutive vectors in the fan is less than $180°$, we have at least one vector in the interior of the second and in the interior of the fourth quadrant. Moreover, if v_k is the last vector in the interior of the second quadrant, then $-v_k$ lies in the interior of the fourth quadrant. Because according to the assumption $-v_k \neq v_i$ for $i \in \{k+1, \ldots, d\}$ there exist two consecutive vectors, say v_s and v_{s+1}, for some $s \in \{k+1, \ldots, d\}$ such that $-v_k$ lies in the interior of $v_s\mathbb{R}_{\geq 0} + v_{s+1}\mathbb{R}_{\geq 0}$. Thus, v_k lies in the interior of $(-v_s)\mathbb{R}_{\geq 0} + (-v_{s+1})\mathbb{R}_{\geq 0}$. Notice that v_1 lies in the interior of $(-v_{s+1})\mathbb{R}_{\geq 0} + v_s\mathbb{R}_{\geq 0}$, which is impossible based on Lemma 6.3.2. The proposition is thus proved. ▮

Lemma 6.3.3 ([14], Section 2.5) For each $v_i \in \Sigma(1)$, $i \in \{0, \ldots, d-1\}$ there exists $a_i \in \mathbb{Z}$ such that $a_i v_i = v_{i-1} + v_{i+1}$.

Proof: Notice that we can express the vector v_{i+1} using the basis $\{v_{i-1}, v_i\}$ as $v_{i+1} = -\alpha v_{i-1} + \beta v_i$ for some $\alpha \in \mathbb{Z}_{\geq 1}$ and $\beta \in \mathbb{Z}$. The first coordinate is negative because v_{i+1} lies on the side of v_i opposite to v_{i-1}. Similarly, v_{i-1} can be expressed with the basis $\{v_i, v_{i+1}\}$ as $v_{i-1} = \gamma v_i + (-\delta)v_{i+1}$ for some $\delta \in \mathbb{Z}_{\geq 1}$ and $\gamma \in \mathbb{Z}$. Plugging the second equation into the first, we then obtain $v_{i+1} = -\alpha(\gamma v_i + (-\delta)v_{i+1}) + \beta v_i$ and finally, $0 = (-\alpha\gamma + \beta)v_i + (\alpha\delta - 1)v_{i+1}$. Because v_i and v_{i+1} are linearly independent, we find that $\alpha\gamma = \beta$ and $\alpha\delta = 1$, which gives $\alpha = 1$, $\delta = 1$ and $\gamma = \beta$,; therefore, $v_{i+1} = -v_{i-1} + \beta v_i$ for some $\beta \in \mathbb{Z}$. ▮

Theorem 6.3.1 ([14], Claim, page 43) If $d \geq 5$, then $v_i = v_{i-1} + v_{i+1}$ for some $i \in \{0, \ldots, d-1\}$.

Proof: Using Proposition 6.3.1, we can assume that $v_0 = e_1$, $v_j = -e_1$ for some $j \in \{3, \ldots, d-2\}$. Using the previous lemma we then have $a_i v_i = v_{i-1} + v_{i+1}$ for

$i = 1, 2, \ldots, j-1$. The sum of all equations gives

$$\sum_{i=1}^{j-1} a_i v_i = \sum_{i=1}^{j-1} (v_{i-1} + v_{i+1}) = \sum_{i=1}^{j-1} v_{i-1} + \sum_{i=1}^{j-1} v_{i+1} = v_0 + v_1 + v_{j-1} + v_j + 2\sum_{i=2}^{j-2} v_i,$$

which can be rewritten as

$$\sum_{i=1}^{j-1} a_i v_i = v_1 + v_{j-1} + 2\sum_{i=2}^{j-2} v_i,$$

because $v_0 = -v_j$. Finally, we obtain

$$0 = (a_1 - 1)v_1 + (a_{j-1} - 1)v_{j-1} + \sum_{i=2}^{j-2} (a_i - 2)v_i. \qquad (*)$$

Note that because all v_i for $i = 1, 2, \ldots, j-1$ lie in the upper half plane, all a_i in the equations $a_i v_i = v_{i-1} + v_{i+1}$ are positive. We must prove that $a_i = 1$ for some $i = 1, 2, \ldots, j-1$. Let, therefore, assume that each $a_i \geq 2$. Because for all vectors $v_i = x_i e_1 + y_i e_2$, $i = 1, 2, \ldots, j-1$, we have $y_i \in \mathbb{Z}_{\geq 1}$, we find that

$$(a_1 - 1)y_1 + (a_{j-1} - 1)y_{j-1} + \sum_{i=2}^{j-2} (a_i - 2)y_i \geq 1,$$

which contradicts $(*)$. Thus, we have $a_i = 1$ for some $i = 1, 2, ..., j-1$, which proves the theorem. ▌

BIBLIOGRAPHY

[1] G. Barthel, L. Bonavero, M. Brion and D. Cox, *Geometry of Toric Varieties*, Lectures 1 - 20. Summer School (2000), Grenoble, France.

[2] A. Boggess, *CR Manifolds and the Tangential Cauchy-Riemann Complex*, CRC Press (1991).

[3] L. Bonavero and M. Brion (Eds.), *Geometry of toric varieties. Lectures from the Summer School held in Grenoble*, June 19–July 7, 2000. Séminaires et Congres, 6. Société Mathématique de France, Paris (2002).

[4] É. Cartan, *Sur l'quivalence pseudo-conforme des hypersurfaces de l'espace de deux variables complexes I.*, Ann. Mathm. 11 (1932) , 17-90.

[5] É. Cartan, *Sur l'quivalence pseudo-conforme des hypersurfaces de l'espace de deux variable complexes II.*, Ann. Scuola Norm. Sup. Pisa 1 (1932) , 333-354.

[6] S.S. Chern, *Complex Manifolds without Potential Theory*, Springer-Verlag, (1979).

[7] S.S. Chern and J. Moser, *Real hypersurfaces in complex manifolds*, Acta Math. 133 (1974), 219-271.

[8] P. Dolbeault, *Sur la cohomologie des variétés analytiques complexes* C. R. Acad. Sci. Paris, 236 (1953) 175 - 177.

[9] R. Dwilewicz, *Additive Riemann-Hilbert problem in line bundles over* $\mathbb{CP}^1$, Canadian Math. Bull. **49** (2006), 72 - 81.

[10] R. Dwilewicz, *Holomorphic extensions in complex fiber bundles*, J. Math. Analysis and Appl. **322** (2006), 556 - 565.

[11] R. Dwilewicz and J. Merker, *On the Hartogs-Bochner phenomenon for CR functions in* $\mathbb{CP}^2$, Proc. AMS, **130** (2002), 1975 - 1980.

[12] G. Ewald, *Combinatorial Convexity and Algebraic Geometry*, Springer-Verlag (1996).

[13] H. Freudenthal, *Über die Enden topologischer Räume und Gruppen*, Math. Z. 33 (1931), 692-713.

[14] W. Fulton, *Introduction to Toric Varieties*, Princeton Univ. Press, NJ, (1993).

[15] B. Gilligan and A.T. Huckleberry, *Complex homogeneous manifolds with two ends*, Michigan Math. J. (1981).

[16] H. Grauert, *Über Modificationen und exzeptionelle analytishe Mengen*, Math. Annalen 146, (1962), 331-368.

[17] S.J. Greenfield, *Cauchy-Riemann equations in several variables*, Ann. della Sc. Norm. Sup. di Pisa, 22, no 2 (1968), 275-314.

[18] R.C. Gunning, *Introduction to Holomorphic Functions of Several Variables*, 3 vols., Wadsworth and Brooks/Cole, (1990).

[19] R.C. Gunning and H. Rossi, *Analytic Functions of Several Complex Variables*, Prentice-Hall, Inc., (1965).

[20] F. Hartogs, *Zur Theorie der analytischen Functionen mehrener unabhangiger Veränderlichen insbesondere über die Darstellung derselben durch Reihen, welche nach Potenzen einer Veränderlichen fortschreiten*, Math. Ann. 62 (1906), 1-88.

[21] H. Hopf, *Enden offener Räume unendliche diskontinuierliche Gruppen*, Comment. Math. Helv. 16, (1943/4), 81–100.

[22] J. Jurkiewicz, *Torus embeddings, polyhedra, k^*-actions and homology*, Dissertationes Math. (Rozprawy Matematyczne) **236** (1985).

[23] J. Jurkiewicz, *Chow ring of projective nonsingular torus embedding*, Colloq. Math. 43 (1980), no. 2, 261–270.

[24] J. J. Kohn and H. Rossi, *On the extension of holomorphic functions from the boundary of a complex manifold*, Ann. of Math. **81** (1965), 451–472.

[25] J. Korevaar and J. Wiegierinck, *Lecture Notes in Several Complex Variables* available at http://staff.science.uva.nl/ janwieg/edu/scv/scv.pdf

[26] S. G. Krantz, *Function Theory of Several Complex Variables*, AMS Chelsea Publishing, (2001).

[27] C. Laurent-Thiébaut, *Phénomène de Hartogs-Bochner dans les variétés CR*, Topics in Complex Analysis, Banach Center Publications vol. **31** (1995), 233–247.

[28] T. Oda, *Lectures on Torus Embeddings and Applications*, Springer-Verlag (1978).

[29] T. Oda, *Convex Bodies and Algebraic Geometry, An Introduction to the Theory of Toric Varieties*, Springer-Verlag (1988).

[30] H. Poincaré, *Les functions analytiques de deux variables et la reprsentation conforme*, Rend. Circ. Mat. Palermo 23 (1907) , 185-220.

[31] R. M. Range, *Complex Analysis: A Brief Tour into Higher Dimensions*, Lecture Notes, available at http://www.adwan.net/Complex-Analysis/Range.pdf

[32] F. Sarkis, *Hartogs-Bochner type theorem in projective space*, Ark. Mat. **41** (2003), 151 - 163.

[33] N. Tanaka, *On the pseudo-conformal geometry of hypersurfaces of the space of complex variables*, J. Math. Soc. Japan 14 (1962) , 397-429.

[34] S. Willard, *General Topology*, Addison-Wesley Series of Mathematics, Addison-Wesley Publishing Company, (1970).

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