Summer 2017

Local holomorphic extension of Cauchy Riemann functions

Brijitta Antony

LOCAL HOLOMORPHIC EXTENSION OF CAUCHY RIEMANN FUNCTIONS

by

BRIJITTA ANTONY

A DISSERTATION

Presented to the Faculty of the Graduate School of the

MISSOURI UNIVERSITY OF SCIENCE AND TECHNOLOGY

In Partial Fulfillment of the Requirements for the Degree

DOCTOR OF PHILOSOPHY

in

MATHEMATICS

2017

Approved by

Dr. Roman Dwilewicz, Advisor
Dr. Adam Harris
Dr. David Grow
Dr. Vy Le
Dr. John Singler
Dr. Stephen Clark

ABSTRACT

The purpose of this dissertation is to give an analytic disc approach to the CR extension problem. Analytic discs give a very convenient tool for holomorphic extension of CR functions. The type function is introduced and showed how these type functions have direct application to important questions about CR extension. In this dissertation the CR extension theorem is proved for a rigid hypersurface M in $\mathbb{C}^2$ given by $y = (\operatorname{Re} w)^m (\operatorname{Im} w)^n$ where m and n are non-negative integers. If the type function is identically zero at the origin, then there is no CR extension. In such case the hypersurface is foliated by complex curves. If the type function is not identically zero at the origin, then CR function either locally extend to one or both sides of the hypersurface depending on the values of m and n. Finally using this result a more precise description of the extension set is given for CR functions defined on the hypersurface M.

DEDICATION

This dissertation is dedicated to my advisor Dr. Roman Dwilewicz who passed away just short of the goal line of completion of my dissertation. All my work was done under his guidance and mentorship. I could never have done this without his incredible support and encouragement. I thank him for his unrelenting dedication over these years.

ACKNOWLEDGMENTS

After a period of seven years, as I look back, it has been a journey of great learning for me not only in the area of complex analysis but also on a personal level. I would like to reflect on the people who have initiated, supported and helped me so much throughout this period.

Firstly, I would like to express thanks to my advisor Dr. Dwilewicz, for his willingness to guide me throughout my dissertation. His sincerity and dedication towards research is a great inspiration to me. My heartfelt thanks to Dr. Al Boggess for his valuable comments and suggestions. You definitely provided me with the tools I needed to choose the right direction and successfully complete my dissertation. I am also thankful to Dr. Adam Harris for his advice and assistance towards my dissertation. I want to express my sincere thanks to Dr. Stephen Clark, department chair, for his timely help and coordination with all the committee members and staff at Missouri University of Science and Technology. I would like to thank Dr. David Grow, Dr. Vy Le and Dr. John Singler for serving as my committee members. Each of your input and guidance is greatly appreciated, without which this milestone would not have been possible.

I wish to thank my family for all their love and encouragement. For my children, Hephzibah and Daniel who supported me in all my difficult times. And lastly, I am thankful to my beloved husband Obed who initiated and encouraged me to pursue my PhD.

TABLE OF CONTENTS

Page

LIST OF ILLUSTRATIONS

1. INTRODUCTION

Research on the subject of CR manifolds has branched into many areas in the past decades. One interesting area is the holomorphic extension of CR functions. Most of the work dealing with holomorphic extensions of CR functions takes place on a CR submanifold of $\mathbb{C}^n$. The first result is due to Hans Lewy [15]. He proved that under certain convexity assumptions on a real hypersurface in $\mathbb{C}^2$, CR functions locally extend to holomorphic functions. This result was generalized by many authors such as Hill, Taiani, Boggess-Polking, Dwilewicz, Trepreau, and Tumanov for arbitrary dimensions. Over the years, many refinements have been made to CR extension theory. It now includes manifolds of higher codimension with weaker convexity assumptions. The analytic disc approach to CR extension use an approximation theorem by Baouendi and Treves, which states that a CR function on a CR submanifold M can be uniformly approximated on an open set by a sequence of entire functions. Then the problem of obtaining the CR extension from analytic discs was reduced to a simple maximum principle argument. The purpose of my dissertation is to learn about the CR extension on a hypersurface in $\mathbb{C}^2$ given by $y = h(w)$ under certain convexity assumptions.

There are several important motivations to study CR extension. The restriction of holomorphic function to a CR manifold is a CR function. But the converse of this statement is not necessarily true in general. An interesting problem is to find conditions as to when the converse is true. There are also important consequences for the regularity properties of CR functions. If the CR functions extend analytically to both sides of the hypersurface, then all CR functions are guaranteed to be as smooth as the hypersurface, since the analytic functions are infinitely smooth. But one-sided analytic extensions do not guarantee that CR functions are smooth.

We start Section 2 with the definitions of CR manifolds and CR functions. In Section 3, we present the analytic disc approach to CR extension. To find a single analytic disc with boundary on a CR submanifold M, it is necessary to solve a certain system of integral equations called Bishop equation. We also mention about some of the terms involved in Bishop equation namely the Hilbert transform and center of an analytic disc. We present results from the paper of Hill-Taiani [13] about the properties of solutions of Bishop equations. In Section 3.6 we solve a linear Bishop equation which is similar to that of [18]. The description of type function and related theorems taken from [8] given in Section 4 are the basic tools used throughout Section 5.

In Section 5, we consider a rigid hypersurface in $\mathbb{C}^2$ given by $y = h(w)$. To investigate this case, we use some theorems of Globevnik- Rudin [11], [12] and Dwilewicz - Hill [8] about characterization of harmonic functions. Based on these results we prove some theorems and corollaries. This will be essential in proving our main theorem in Section 5.2, which is the main Section of the dissertation. Here we consider a special type of hypersurface M in $\mathbb{C}^2$ given by $y = (\operatorname{Re} w)^m (\operatorname{Im} w)^n$ where m and n are non-negative integers, and discuss about the CR extension on this hypersurface. We divide the theorem into four parts. In part (a), for certain values of m and n, the type function is identically zero at the origin. In this case, there is no holomorphic extension, so the hypersurface is foliated by complex curves. For the remaining parts the type function is not identically zero at the origin. In part (b), if m and n are even integers, then CR functions defined on M extend to one side of M. In part (c), if one of the integers is even and the second one is odd, then CR functions extend to both sides of the hypersurface. In part (d), if both integers are odd, then CR functions extend to both sides of the hypersurface. Finally using these results we give a more precise description of the extension set for CR functions defined on the hypersurface M.

2. DEFINITIONS AND NOTATIONS

In this Section, we present the definitions of CR manifold, CR functions and discuss some of their properties.

2.1. DEFINITION OF CAUCHY-RIEMANN MANIFOLDS

Let X be a complex manifold and let M be a real submanifold of class C^1 embedded in a complex manifold X. Take a point $p \in X$ and let T_pX be the complex tangent space to X at p. We note that T_pX has a natural complex structure generated by the complex structure on X, and we denote by

$$J_p : T_pX \longrightarrow T_pX, \qquad J_p^2 = -\mathrm{id}_{T_pX},$$

$$J_p^2 = J_p \circ J_p = -I \qquad \textit{where} \quad J\left(\frac{\partial}{\partial x_j}\right) = \frac{\partial}{\partial y_j}, \quad J\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial x_j}$$

the complex structure operator. Now take a point $p \in M \subset X$ and let T_pM denote the real tangent space to M at p. The space T_pM can be considered as a real vector subspace of the complex space T_pX, therefore there exists the maximal complex subspace or holomorphic tangent space H_pM, $H_pM \subset T_pM \subset T_pX$, which is equal to $H_pM = T_pM \cap J_p(T_pM)$, $(\det J_p)^2 = (-1)^k$ *where* $k = \dim_{\mathbb{C}} H_pM$ (Fig 2.1)

Definition 2.1.1 (Cauchy-Riemann manifold) *The manifold M is Cauchy-Riemann (CR) if $\dim_{\mathbb{C}} H_pM \equiv \mathrm{const}$ (independent of p). This constant is called the CR dimension of M and is denoted by $\dim_{CR} M$. If $\dim_{CR} M = 0$, then M is called a totally real submanifold of X.*

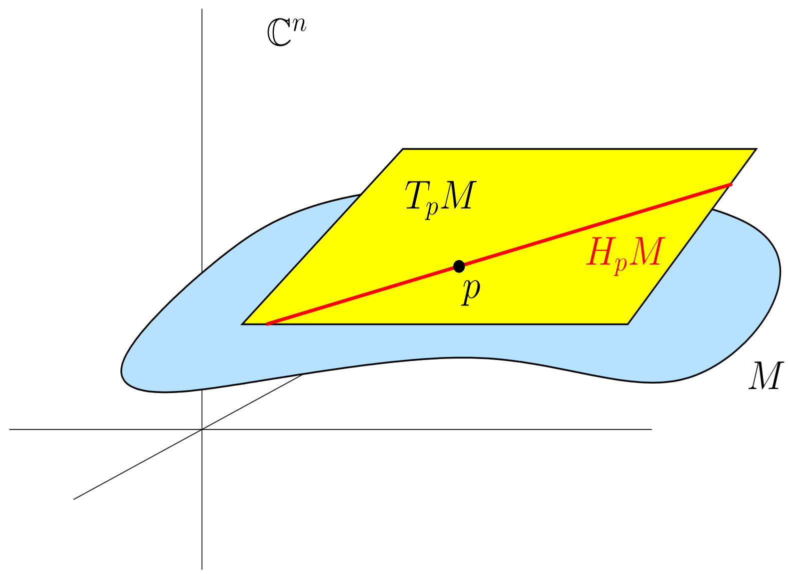

Figure 2.1: Tangent Space

We note that for any real submanifold M of X with $\mathrm{codim}_{\mathbb{R}} M = k$ and $\dim_{\mathbb{C}} X = n$, and M not necessary CR, we have

$$\max(n-k,0) \leq \dim_{\mathbb{C}} H_pM \leq n - \frac{k}{2}$$

To see this, because $H_pM \subset T_pM$, we have

$$\dim_{\mathbb{C}} H_pM \leq \frac{1}{2}\dim_{\mathbb{R}} T_pM = \frac{1}{2}(2n-k) = n - \frac{k}{2}$$

For the second inequality, since

$$T_pX \supset H_pM = T_pM \cap J_p(T_pM),$$

$$\dim_{\mathbb{R}} T_pX \geq \dim T_pM + \dim J(T_pM) - \dim_{\mathbb{C}} H_pM.$$

Since J is an isomorphism, from the above inequality we obtain,

$$2n \geq 4n - 2k - \dim_{\mathbb{R}} H_p M.$$

$$2n - 2k \leq \dim_{\mathbb{R}} H_p M.$$

$$2n - 2k \leq \dim_{\mathbb{R}} H_p M \leq 2n - k.$$

$$n - k \leq \dim_{\mathbb{C}} H_p M \leq n - \frac{k}{2}.$$

In particular, if M is CR, then

$$n - k \leq \max(n - k, 0) \leq \dim_{CR} M \leq n - \frac{k}{2}.$$

If $\dim_{CR} M = n - k$, then M is called *generically embedded* into X or a *generic submanifold of* X. It implies that $0 \leq k \leq n$.

Example 2.1.1 (1) *Any complex submanifold M of X is CR.*

(2) *$M = \mathbb{R}^n \subset \mathbb{C}^n$ is a totally real submanifold of $\mathbb{C}^n$.*

(3) *Any real C^1 hypersurface M of X is CR. No additional assumption is needed. The condition from Definition* **??** *is automatically satisfied.*

(4) *The boundary $M = \partial Y$ of a complex submanifold Y of X is a CR submanifold of X, provided M is of class C^1.*

(5) *Let N be a real C^1 embedded submanifold of $\mathbb{R}^n$, then $N \times i\mathbb{R}^n$ is a CR submanifold of $\mathbb{C}^n$. Such manifolds are called tubular manifolds.(Fig 2.2)*

Less abstractly, since the concept of CR manifold is local, we explain this using local coordinates. Assume that M is given locally by a system of equations:

$$\rho_1(z) = \ldots = \rho_k(z) = 0 \quad \text{where} \quad d\rho_1 \wedge \ldots d\rho_k \neq 0 \quad \text{on} \quad M.$$

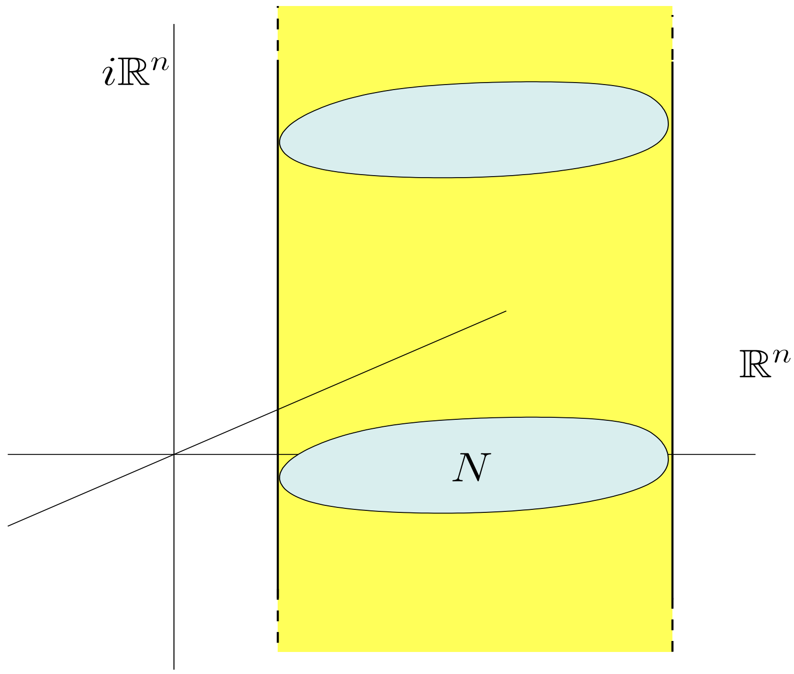

Figure 2.2: Tubular Manifold

The latter condition is equivalent to

$$\operatorname{rank}\begin{bmatrix} \frac{\partial\rho_1}{\partial z_1} & \dots & \frac{\partial\rho_1}{\partial z_n} & \frac{\partial\rho_1}{\partial\overline{z}_1} & \dots & \frac{\partial\rho_1}{\partial\overline{z}_n} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \frac{\partial\rho_k}{\partial z_1} & \dots & \frac{\partial\rho_k}{\partial z_n} & \frac{\partial\rho_k}{\partial\overline{z}_1} & \dots & \frac{\partial\rho_k}{\partial\overline{z}_n} \end{bmatrix} = k.$$

Here $\quad d\rho_k = \partial\rho_k + \overline{\partial}\rho_k, \quad \partial\rho_k = \sum_{i=1}^n \frac{\partial\rho_k}{\partial z_i} dz_i, \quad \overline{\partial}\rho_k = \sum_{i=1}^n \frac{\partial\rho_k}{\partial\overline{z_i}} d\overline{z_i}$

The complex tangent space H_pM can be naturally identified with

$$H_p^{(1,0)}M = \left\{ L = a_1\frac{\partial}{\partial z_1} + \dots + a_n\frac{\partial}{\partial z_n} \in T_p^{(1,0)}\mathbb{C}^n \;;\; L\rho_\alpha(p) = 0,\ \alpha = 1, \dots, k \right\}.$$

Also we denote

$$H_p^{(0,1)}M = \overline{H_p^{(1,0)}M}, \qquad H^{(0,1)}M = \bigcup_p H_p^{(0,1)}M, \qquad H^{(1,0)}M = \bigcup_p H_p^{(1,0)}M.$$

Lemma 2.1.1 (CR manifolds)

If

$$rank\left(\left.\frac{\partial\rho_\alpha}{\partial z_\beta}\right|_p\right)_{\alpha=1,\dots,k;\,\beta=1,\dots,n} = rank\left(\left.\frac{\partial\rho_\alpha}{\partial \overline{z}_\beta}\right|_p\right)_{\alpha=1,\dots,k;\,\beta=1,\dots,n} \equiv const,$$

then M is CR.

This is equivalent to

$$\text{rank}\begin{bmatrix} \frac{\partial\rho_1}{\partial z_1} & \dots & \frac{\partial\rho_1}{\partial z_n} \\ \dots & \dots & \dots \\ \frac{\partial\rho_k}{\partial z_1} & \dots & \frac{\partial\rho_k}{\partial z_n} \end{bmatrix} = \text{rank}\begin{bmatrix} \frac{\partial\rho_1}{\partial \overline{z}_1} & \dots & \frac{\partial\rho_1}{\partial \overline{z}_n} \\ \dots & \dots & \dots \\ \frac{\partial\rho_k}{\partial \overline{z}_1} & \dots & \frac{\partial\rho_k}{\partial \overline{z}_n} \end{bmatrix} = \text{const.}$$

If the above rank is equal to k, then M is called generic at p. Equivalently, if

$$\overline{\partial}\rho_1 \wedge \dots \wedge \overline{\partial}\rho_k \neq 0 \quad \text{on} \quad M,$$

then M is generic. It is an open condition, it means that if M is generic at p, then it is generic in a neighborhood of p. If M is not generic at p, then always locally we can find a generic CR manifold $\tilde{M}$ of a complex vector space such that M and $\tilde{M}$ are locally CR isomorphic.

2.2. DEFINITION OF CAUCHY-RIEMANN FUNCTIONS AND MAPPINGS

A function $f : M \longrightarrow \mathbb{C}$ of class C^1 is called a *CR function* if it is annihilated by any vector $\overline{L} \in H_p^{(0,1)}M$, i.e.,

$$\overline{L}f\big|_p = 0 \quad \text{for} \quad \overline{L} \in H_p^{(0,1)}M, \quad p \in M.$$

If $\tilde{M}$ is a CR manifold, then a C^1 mapping $F : M \longrightarrow \tilde{M}$ is CR if

$$dF|_p(H_pM) \subset H_{F(p)}\tilde{M}.$$

If $\tilde{M}$ is a CR submanifold of $\mathbb{C}^n$, then a C^1 mapping $F : M \longrightarrow \tilde{M}$ is CR if each component of F is a CR function.

Example 2.2.1 *Let U be a neighborhood of a CR manifold M in a complex manifold X. Then the restriction of any holomorphic function F on U to M is CR.(Fig 2.3) The opposite statement is not true in general.*

For example if $M = \{z = (z_1, z_2) \in \mathbb{C}^2 / Imz_1 = 0\}$ is a CR manifold, then the CR function $f(x_1, z_2)$ is holomorphic in z_2 but no condition on x_1.

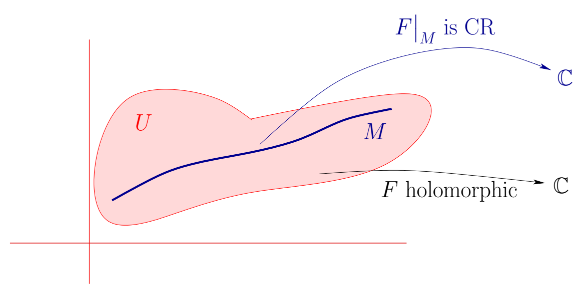

Figure 2.3: Restriction of a holomorphic function to a CR submanifold

Holomorphic extension to a full neighborhood of CR functions from CR manifolds play a very important role in complex analysis. It allows to prove many theorems for such CR manifolds and CR functions defined on them, which exclusively hold for holomorphic functions. As examples of such vast applications the reader is referred to papers of Hill and Nacinovich. Many of their results hold under weaker assumptions, namely when replaced by assumptions of the main theorem of this paper in Section 5.

3. ANALYTIC DISC METHOD IN EXTENSION OF CR FUNCTIONS

Every holomorphic function on $\mathbb{C}^n$ restricts to a CR function on a CR submanifold M of $\mathbb{C}^n$. Some geometric conditions on M guarantee that CR functions on M extend as holomorphic functions on an open set in $\mathbb{C}^n$. We are especially interested in CR extension to an open set that is function independent. Here we present the analytic disc approach to CR extension which is an idea pioneered by Hans Lewy and Bishop. This result was generalized by many others such as Hill, Taiani, Boggess-Polking, Dwilewicz, Trepreau, and Tumanov for arbitrary dimensions. Baouendi and Treves published their general approximation theorem in 1981, which states that CR functions on a CR manifold M can be locally approximated by entire functions. To extend a given CR function to an open set in $\mathbb{C}^n$, we need to show that the sequence of entire functions is uniformly convergent on the openset. This can be accomplished by the use of analytic discs. The idea behind analytic discs is to show that each point in the open set is contained in the image of an analytic disc whose boundary image is contained in M. From the maximum principle for analytic functions, the sequence of entire functions must also converge uniformly on the open set. So our extension theorem is reduced to a theorem about analytic discs for hypersurfaces. In this Section, we discuss about analytic discs, lifted analytic discs, Bishop's equation, Hilbert transform and center of an analytic disc. We will also discuss some of the properties of solutions of Bishop's equation taken from [13].

3.1. MOTIVATION OF USING ANALYTIC DISCS

An analytic disc is a holomorphic mapping

$$g : D = \{z \in \mathbb{C};\ |z| < 1\} \longrightarrow X, \qquad X \textit{ is a complex manifold}$$

with some differentiability class up to the boundary of D. More precise definition will be given later on.(Fig 3.1)

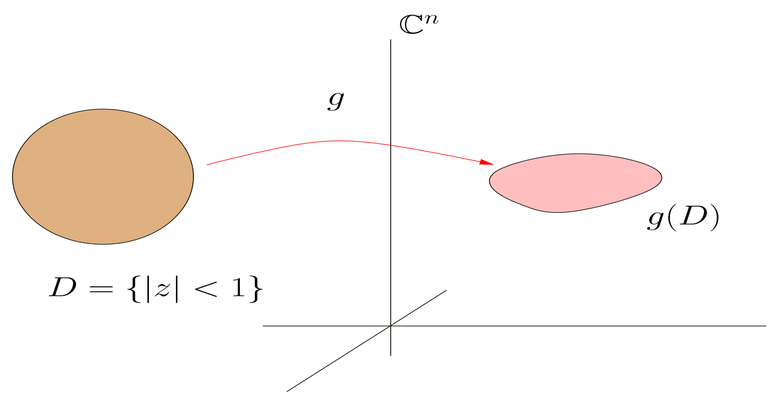

Figure 3.1: Analytic disc

In a famous paper of Hans Lewy [15] in 1956 he considered a strictly pseudo-convex hypersurface M in $\mathbb{C}^2$ and showed that any CR function defined on M can be holomorphically extended to the pseudoconvex side of M.(Fig 3.2) The same sort of extension phenomenon can also occur, as Lewy demonstrated in 1960 in [16], when the hypersurface M is replaced by a real submanifold M in $\mathbb{C}^n$ whose codimension is greater than one. In the papers of Hans Lewy, the region where CR functions extend

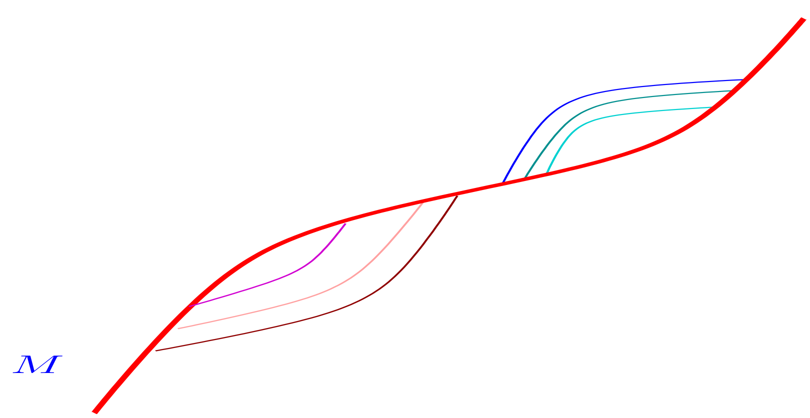

Figure 3.2: Analytic discs with boundaries on M

is the region swept out by the interiors of appropriately chosen family of complex one-dimensional analytic discs in $\mathbb{C}^n$ whose boundaries lie on M. The holomorphic extension can be obtained via the Cauchy integral formula by integrating the CR function around the boundary of each analytic disc. In the hypersurface case, the family of analytic discs can be obtained simply by an elementary slicing technique, using an appropriate system of local holomorphic coordinates.(Fig 3.3) In the codimension greater than one, it does not work so easily. In fact, in order to find even a

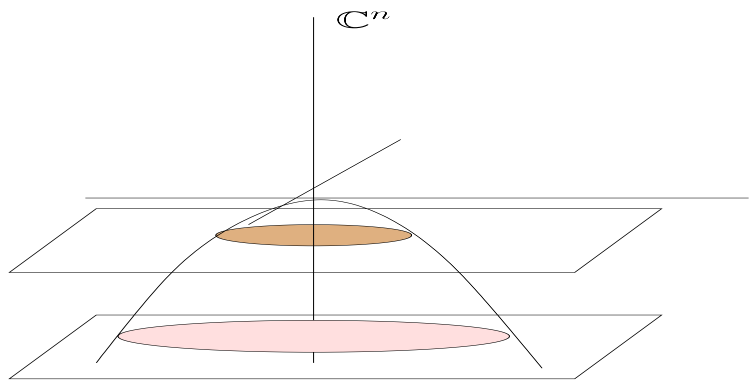

Figure 3.3: Family of analytic discs by slicing technique

single analytic disc with boundary on M, it is necessary to solve a certain system of nonlinear singular integral equations.

Following the work of Lewy, E. Bishop introduced a functional equation (an equation that specifies a function in implicit form) involving the Hilbert transform T on the unit circle, which must be solved in order to produce such a disc. Also he showed how to solve the functional equation by the method of successive approximations, working in the Sobolev space $H_1(S^1)$ and using the boundedness of T on $L^2(S^1)$. He thereby produced a particular family of analytic discs with boundaries on M which depends on certain parameters involved in the construction.

3.2. DEFINITION OF ANALYTIC DISCS

Definition 3.2.1 (Analytic discs) *Let* $D = \{z \in \mathbb{C};\ |z| < 1\}$ *be the open unit disc in* $\mathbb{C}$ *centered at the origin. Consider a map* $g : \overline{D} \longrightarrow \mathbb{C}^n$ *which is holomorphic in* $D \subset \mathbb{C}$ *and belongs to some differentiability class on its closure* $\overline{D}$*. Then* g*, or sometimes the image* $g(D)$*, will be called an analytic disc in* $\mathbb{C}^n$*. The restriction of* g *to* $S^1 = \partial D$*, or sometimes* $g(S^1)$*, will be called the boundary of the disc. The point* $g(0)$ *is called the center of the disc.*

Definition 3.2.2 (Analytic disc attached to a submanifold) *Let* M *be a CR manifold in a complex manifold* X*. We say that an analytic disc* $\mathcal{D}$ *is attached to* M *if its boundary is contained in* M*.(Fig 3.4)*

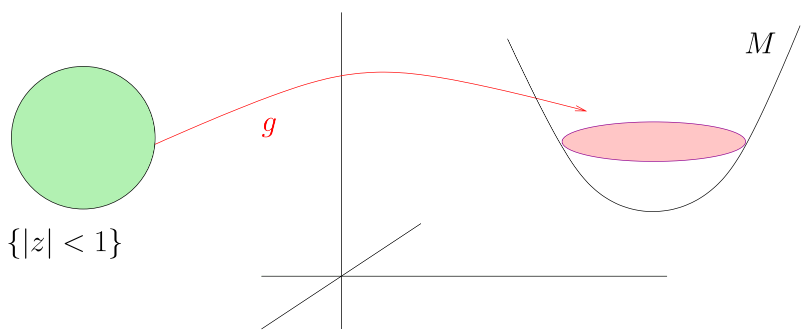

Figure 3.4: Analytic disc attached to a submanifold M

3.2.1. Lifting of Analytic Discs.

The lifting of discs, considered by Bishop is a local problem. Therefore we can assume that M is generically embedded into $\mathbb{C}^{k+n}$, has codimension k, and consider M in a neighborhood of a point p which can be assumed to be the origin, i.e., $p = 0$.(Fig 3.5)

We denote the coordinates on $\mathbb{C}^{k+n}$ by $(z, w) = (z_1, \ldots, z_k, w_1, \ldots, w_n)$, $z_\alpha = x_\alpha + iy_\alpha$, $\alpha = 1, \ldots, k$. Moreover, we can assume that

$$T_0M = \{y_1 = \ldots = y_k = 0\}, \quad H_0M = \{z_1 = \ldots = z_k = 0\}.$$

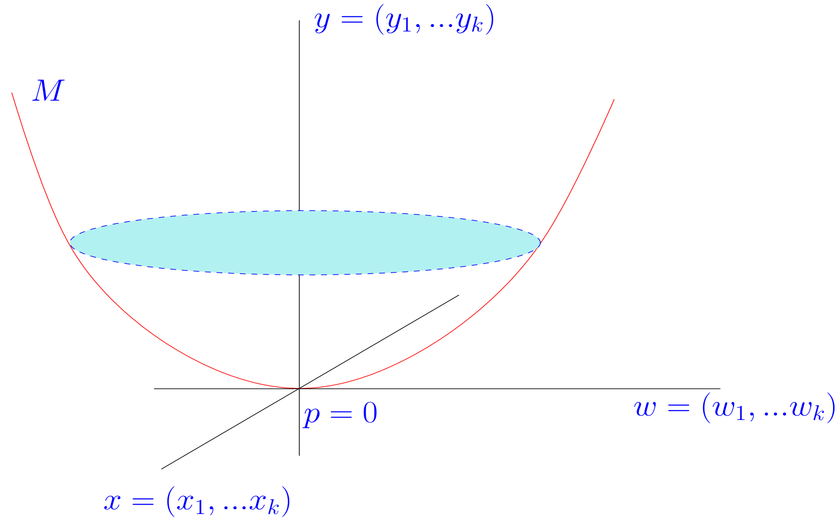

Figure 3.5: Lifting of analytic discs

We can express M locally as a graph over its tangent space, so that

$$M = \{(z, w) = (z_1, z_2, \ldots z_k, w_1, w_2, \ldots w_n)\ ;\ y_\alpha = h_\alpha(x_1, x_2, \ldots x_k, w_1, \ldots w_n)\}$$

or in the short way

$$M = \{(z, w)\ ;\ y = h(x, w)\}, \quad h = (h_1, \ldots, h_k),$$

where h_α, $\alpha = 1, \ldots, k$, are real-valued functions which vanish to the second order at the origin. So $h_\alpha(0) = 0$ and $Dh_\alpha(0) = 0$.

3.2.2. Bishop Equation. The construction of analytic discs requires the solution of a non linear integral equation called Bishop's equation. In order to derive the equation for discs, Bishop first noted that if $h = (h_1, \ldots, h_k)$ is identically zero, then $z_1, \ldots, z_k$ are all real on M and must, therefore, be real and constant on any analytic disc with boundary on M. Thus for $h \equiv 0$, each analytic disc in $\mathbb{C}^{k+n}$ with boundary on M is uniquely determined by an analytic (holomorphic) disc in $\mathbb{R}^k \times \mathbb{C}^n$ in the variables $z_1, \dots z_k, w_1, \dots w_n$ which implies that the first k holomorphic functions $z_1, \dots, z_k$ are real constants.

In general, suppose g is an analytic disc whose boundary lies on M. Then we have

$$g(S^1) = \{(x(e^{i\theta}) + ih(x(e^{i\theta}), w(e^{i\theta})), w(e^{i\theta})) \; ; \quad 0 \leq \theta \leq 2\pi\}.$$

Consider a $\mathbb{R}^k$- valued harmonic function U in D which belongs to an appropriate differentiability class $\overline{D}$. Let V be the unique conjugate harmonic function such that $V(0) = 0$. Let T be the operator (acting componentwise) which takes the boundary values of U to the boundary values of V. We talk about this operator T in the next Section.

Looking at the boundary values of the analytic disc

$$(x(e^{i\theta}) + ih(x(e^{i\theta}), w(e^{i\theta})) \quad \text{or}$$

$$-i(x(e^{i\theta}) + ih(x(e^{i\theta}), w(e^{i\theta}))) = h(x(e^{i\theta}), w(e^{i\theta})) - ix(e^{i\theta})$$

we obtain

$$T[h(x(\cdot), w(\cdot))] = -x(\cdot) + \text{const},$$

or

$$x(\cdot) = c - T[h(x(\cdot), w(\cdot))] \quad \text{or} \quad x = c - Th(x, w) \quad \text{for short.} \tag{3.1}$$

On the other hand, suppose that $c \in \mathbb{R}^k$ is prescribed and $w : \overline{D} \longrightarrow \mathbb{C}^n$ denotes an analytic disc in $\mathbb{C}^n$. If $x = x(\cdot)$ satisfies (3.1), then

$$f(e^{i\theta}) = x(e^{i\theta}) + ih(x(e^{i\theta}), w(e^{i\theta}))$$

is the boundary value of a holomorphic function $f : \overline{D} \longrightarrow \mathbb{C}^k$ such that $\mathrm{Re} f(0) = c$. Consequently, the function

$$g(\zeta) = (f(\zeta), w(\zeta)), \quad \zeta \in \overline{D},$$

defines an analytic disc $g(D)$ in $\mathbb{C}^{k+n}$ whose boundary $g(S^1)$ lies on M. We summarize the above discussion in the following:

Proposition 3.2.1 *An analytic disc $g(D)$ in $\mathbb{C}^{k+n}$ whose boundary $g(S^1)$ lies on M exists if and only if $x(e^{i\theta})$ is a solution of (3.1) corresponding to some choice of the constant $c \in \mathbb{R}^k$ and to some analytic disc $w(D)$ in $\mathbb{C}^n$.*

3.3. HILBERT TRANSFORM ON THE CIRCLE

Let T be the operator from the previous Section. Assume that $x = x(e^{i\theta})$ is a real-valued function and has the Fourier series of the form

$$x(e^{i\theta}) = \frac{a_0}{2} + \sum_{j=1}^{\infty}(a_j \cos j\theta + b_j \sin j\theta),$$

then Tx is the conjugate Fourier series

$$Tx(e^{i\theta}) = \sum_{j=1}^{\infty}(-b_j \cos j\theta + a_j \sin j\theta).$$

Alternately, T can be written as the limit of a convolution operator with the conjugate Poisson kernel

$$Q_r(\theta) = \mathrm{Im}\left(\frac{1 + re^{i\theta}}{1 - re^{i\theta}}\right)$$

$$\begin{aligned} Tx(e^{i\theta}) &= \lim_{r \to 1} \frac{1}{2\pi} \int_{-\pi}^{\pi} x(e^{i\varphi}) Q_r(\theta - \varphi)\, d\varphi \\ &= \frac{1}{2\pi}\, p.v. \int_{-\pi}^{\pi} x(e^{i(\theta-\varphi)}) \mathrm{Im}\left(\frac{1 + re^{i\varphi}}{1 - re^{i\varphi}}\right)\, d\varphi \end{aligned}$$

We shall call T the Hilbert transform on the unit circle. We shall be concerned with the properties of T on the space $C^{l,\alpha}(K)$.

For any $K \subset \mathbb{R}^n$, K compact, and $0 \le \alpha \le 1$, $C^\alpha(K) = C^{0,\alpha}(K)$ is defined by

$$C^\alpha(K) = \left\{ u\; K \longrightarrow \mathbb{R}\ ;\ |u|_\alpha \equiv \sup_{x\in K} |u(x)| + \sup_{x,y\in K, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\alpha} < \infty \right\}.$$

For any non-negative integer l, we define

$$C^{l,\alpha}(K) = \left\{ u : K \longrightarrow \mathbb{R}\ ;\ |u|_{l,\alpha} \equiv \sum_{|\beta|\le l} |D^\beta u|_\alpha < \infty \right\}.$$

We note that $C^{l,\alpha}(K)$ is a Banach algebra under the norm $|\cdot|_{l,\alpha}$.

We have the following proposition which, in the case $l = 0$, is the classical theorem of Privaloff.

Proposition 3.3.1 *Let $g(z) = U(z) + iV(z)$ be holomorphic in in the unit disc $D = \{z \in \mathbb{C};\ |z| < 1\}$ and $0 < \alpha < 1$. If $V(x,y) \in C^{l,\alpha}(S^1) \cap C^l(\overline{D})$ with norm $|V|_{l,\alpha}$ on S^1, then $g \in C^{l,\alpha}(\overline{D})$ and $|g|_{l,\alpha} \le c|V|_{l,\alpha}$, where c depends only on l and α. Thus $T : C^{l,\alpha}(S^1) \longrightarrow C^{l,\alpha}(S^1)$ is a bounded linear operator. Moreover, $\|T\|_{l,\alpha} \le \|T\|_\alpha$.*

3.4. CENTERS OF ANALYTIC DISCS

The centers of analytic discs can give an essential information about CR extension of CR functions from a CR manifold. This in particular gives an additional justification of the notion of type functions introduced by the authors in [7] and [9]. We talk more about type functions in the next Section. For any function $x = x(\cdot) \in C(S^1)$ we denote its mean value by

$$E(x) = \frac{1}{2\pi} \int_0^{2\pi} x(e^{i\theta})\, d\theta. \qquad (3.2)$$

If x is a matrix-valued function, then we apply E component-wise. We have the following simple but useful lemma.

Lemma 3.4.1 *For $x, u \in C^{\alpha}(S^1)$ matrix-valued functions the following relations hold:*

$$E(Tx) = 0, \tag{3.3}$$

$$T(xu - (Tx)(Tu)) = xTu + (Tx)u, \tag{3.4}$$

$$E(xu - (Tx)(Tu)) = 0 \quad \textit{if} \quad E(x) = 0, \tag{3.5}$$

$$E(xTu + (Tx)u) = 0, \tag{3.6}$$

$$\textit{If} \quad u = Tx \quad \textit{then} \quad x = -Tu + E(x). \tag{3.7}$$

Proof. The first relation (3.3) is obvious from the definition of T:

$$E(Tx) = \frac{1}{2\pi}\int_0^{2\pi} Tx(e^{i\theta})\, d\theta \tag{3.8}$$

$$E(Tx) = \frac{1}{2\pi}\int_0^{2\pi} \sum_{j=1}^{\infty}(-b_j \cos j\theta + a_j \sin j\theta)\, d\theta \tag{3.9}$$

$$= \frac{1}{2\pi}\sum_{j=1}^{\infty}(-b_j)\int_0^{2\pi} \cos j\theta\, d\theta + \frac{1}{2\pi}\sum_{j=1}^{\infty}(-a_j)\int_0^{2\pi} \sin j\theta\, d\theta \tag{3.10}$$

$$= 0. \tag{3.11}$$

To see why the other relations hold, let $x + iTx$ and $u + iTu$ be the boundary values of holomorphic functions. We note that the values of the corresponding holomorphic functions at zero are real. Consequently the product of these functions is

$$(x + iTx)(u + iTu) = (xu - (Tx)(Tu)) + i(xTu + (Tx)u)$$

and from here we get immediately (3.4), (3.5), and (3.6). The last relation, (3.7) is obvious since by definition of T we have that $T(Tx) = -x + \text{const}$, and this constant should be $E(x)$.

3.5. PROPERTIES OF SOLUTIONS OF THE BISHOP EQUATION FOR LIFTED ANALYTIC DISCS

Lifted analytic disc is one of the important tools in our main theorem (5.2.1) in Section 5. As mentioned earlier in Section 3.2.2 this can be constructed by solving the Bishop's equation. In this Section we formulate some results from the paper of Hill-Taiani [13] about existence, uniqueness,and the class of solutions of the Bishop equation, especially from the point of view of lifted analytic discs.

Let $h = (h_1, \ldots, h_k) = h(x, w)$, be a vector valued function defined in a neighborhood of the origin in $\mathbb{R}^k \times \mathbb{C}^n$, and let B be a neighborhood of the origin in this space. We define a partial Lipschitz constant by

$$\mathrm{Lip}(h) = \mathrm{Lip}^B(h) = \sup \frac{|h(x_1, w) - h(x_2, w)|}{|x_1 - x_2|},$$

where sup is taken over (x_1, w) and (x_2, w) from B and $x_1 \neq x_2$.

Just recall the Bishop equation

$$x = c - Th(x, w), \quad x = x(\cdot),\ w = w(\cdot).$$

In order the composition $h(x, w)$ is well-defined we shall consider only x and w with $|x|_\infty$, $|w|_\infty$ sufficiently small so that their values lie in B.

Lemma 3.5.1 (uniqueness of the solution of the Bishop equation) *(Proposition 5.1 from [13]). If $h \in C_k^{0,1}(B)$ and $Lip^B(h) < 1$, then the solution $x(\cdot)$ of the Bishop equation is unique.*

3.5.1. Hölder Class Solutions. Next we state our main theorems concerning the existence and dependence upon parameters of solutions to the Bishop equation in the spaces $C^{l,\alpha}$. It will be convenient to denote the space of parameters $p = (c, w)$ by $P = \mathbb{R}^k \times \mathcal{D}_n^{l,\alpha}$, where

$$\mathcal{D}_n^{l,\alpha} = \mathcal{D}^{l,\alpha} \times \ldots \times \mathcal{D}^{l,\alpha}, \qquad \mathcal{D}^{l,\alpha} = \mathcal{O}(D) \cap C^{l,\alpha}(\overline{D}).$$

Here $\mathcal{O}(D)$ is the space of holomorphic functions in the open unit disc D. $\mathcal{D}^{l,\alpha}$ is a Banach space with the norm

$$|p|_{l,\alpha} = |(c,w)|_{l,\alpha} = |c| + |w|_{l,\alpha},$$

where $|c|$ is the Euclidean norm in $\mathbb{R}^k$.

Theorem 3.5.1 (Hölder class solutions) *(Theorem 5.1 from [13]) Let* $k \geq 0$ *be an integer and* $0 < \alpha < 1$.

(a) There exists a positive constant $C = C(k)$ *such that for* $h \in C_k^{l+s+1}(B)$, $s \geq 1$, *and* $Lip^B(h) < [C\,\|T\|_\alpha]^{-1}$, *there is a unique solution* $x \in C_k^{l,\alpha}(S^1)$ *of the Bishop equation. Moreover, there exists a neighborhood* $U = U(l,\alpha)$ *of the origin in* P *such that* x *is given by a map* $x : U \longrightarrow C_k^{l,\alpha}(S^1)$, $x = x(c, w(\cdot))$ *is of class* C^s. *If* $h \in C_k^{l+s+1,1}(B)$, *then the dependence on parameters is of class* $C^{s,1}$.

(b) If $h \in C_k^{l+1,1}(B)$ *and* $dh(0) = 0$ *then there is a local unique solution* $x \in C_k^{l,\alpha}(S^1)$ *of the Bishop equation such that* $|x|_{l,\alpha}$ *is Lipschitz continuous in its dependence on the parameters* $p = (c, w)$ *measured in the norm* $|p|_{l,\alpha}$.

3.5.2. Stability with Respect to the Perturbation of the CR Manifold. It is also of interest to have a stability theorem which exhibits the dependence of the solution x on the defining function h as well.

Let $C_k^{l+1,1}(B,0)$ denote the Banach subspace of functions $h \in C_k^{l+1,1}(B)$ such that $h(0) = 0$. We will also introduce a new space of parameters $(p, h) = (c, w, h) \in \mathcal{P} = \mathbb{R}^k \times \mathcal{D}_n^{l,\alpha} \times C_k^{l+1,1}(B,0)$. In the following theorem, $\mathcal{U}$ will denote a neighborhood of the point $(0, h_0) = (0, 0, h_0)$ in $\mathcal{P}$, and x_1, x_2 will be the unique solutions of the Bishop equation which correspond to the parameters (c_1, w_1, h_1) and (c_2, w_2, h_2) respectively.

Theorem 3.5.2 (stability theorem in the Hölder class) *(Theorem 5.2 from [13]) Let* $l \geq 0$ *be an integer and* $0 < \alpha < 1$*. Assume that* $h_0 \in C_k^{l+1,1}(B,0)$ *and* $dh_0(0) = 0$*. Then there is a neighborhood* $\mathcal{U}$ *of the point* $(0, h_0)$ *and there is a constant* $C = C(l, \alpha, h_0)$ *such that*

$$|x_1 - x_2|_{l,\alpha} \leq C\{|c_1 - c_2| + |w_1 - w_2|_{l,\alpha}^{\overline{D}} + |h_1 - h_2|_{l+1,1}^{B}\}$$

for all $(c_1, w_1, h_1), (c_2, w_2, h_2) \in \mathcal{U}$*. Moreover,* $x = x(\cdot, c, w, h)$ *is strongly differentiable as a function of* (c, w, h) *at the point* $(0, 0, h_0)$ *with respect to the norms indicated above.*

3.5.3. Real Analytic Class Solutions. Let $\mathfrak{U}(B)$, $\mathfrak{U}(S^1)$ be the space of real-valued analytic functions on B or S^1, respectively, and let $\mathfrak{U}_k(B)$, $\mathfrak{U}_k(S^1)$ denote their k-fold Cartesian products.

Theorem 3.5.3 (real analytic class) *(Theorem 6.1 from [13]) Let* $h \in \mathfrak{U}_k(B)$ *and* $0 < \delta < \delta_0 \leq 1$*. There exists a positive constant* $C = C(k, \alpha, \delta)$ *such that if* $Lip^B(h) < [C\|T\|_\alpha]^{-1}$ *then there is a local unique solution* $x \in \mathfrak{U}_k(S^1)$ *of the Bishop equation sthat* x *is real analytic in its dependence upon the parameters* $p = (c, w)$ *measured in the norm* $\|p\|$*, i.e., there exists a neighborhood* $U = U(\alpha, \delta)$ *of the origin in* P *such that* x *is given by the values on* S^1 *of a real analytic map* $x : U \longrightarrow \mathcal{A}_{k,\delta}^{\alpha}$*.*

3.5.4. C^∞ Class of Solutions. Let $M = \{M_l\}_{l=0}^{\infty}$ be any sequence of positive real numbers. We define the normed linear space $B^\alpha\{M\}$ where $0 \le \alpha \le 1$, of all real (or complex) valued C^∞ functions f defined on S^1 for which the norm

$$\|f\|_M \equiv \sup_{l \ge 0} \frac{|D_\theta^l f|_\alpha^{S^1}}{M_l} < \infty.$$

Theorem 3.5.4 **(C^∞ class of solutions)** *(Theorem 7.1 from[13]) Let $h \in C_k^\infty(B)$ and $0 < \alpha < 1$. Then there exists a positive constant $C = C(k)$, and there exists a sequence $M = \{M_l\}$ such that*

(a) If $Lip^B(h) < [C\|T\|_\alpha]^{-1}$ then there is a local unique real-valued C^∞ solution $x \in B_k^\alpha\{M\}$ of the Bishop equation; i.e., there is a neighborhood $U = U(\alpha, M)$ of the origin in $\mathbb{R}^k \times B_n^\alpha\{M\}$ such that x is given by a map $x : U \longrightarrow B_k^\alpha\{M\}$ where $\|x\|_M$ depends in a C^∞ way on the parameters $\|p\|_M = |c| + \|w\|_M$ for $p = (c, w) \in U$.

(b) For any fixed $\delta_0 > 0$ the sequence $M = \{M_k\}$ can be chosen so that x is defined for the parameters in a suitable neighborhood $U_0 = U_0(\alpha, \delta_0, M)$ of the origin in $\mathbb{R}^k \times \mathcal{D}_n^\alpha(D_{\delta_0})$. Moreover $\|u\|_M$ is C^∞ in its dependence on the parameters measured in the norm $\|p\| = |c| + |w|_\alpha^{\overline{D}_{\delta_0}}$.

3.6. SOLUTION TO THE LINEAR BISHOP EQUATION

The explicit solution of Bishop equation will be needed for investigation of the type function. We define type function and formulate the relation between type function and the holomorphic extension of CR functions in the next Section.

In this Section we solve (3.6.1 Proposition) the linear Bishop equation of the form

$$\eta = -T(a\eta + b) \tag{3.12}$$

where a, b are matrix-valued functions, respectively $k \times k$ and $k \times 1$, defined on S^1, with $|a|_\alpha$ sufficiently small. The unknown η is the column vector, i.e., matrix $k \times 1$. The method of solving (3.12) is similar to that of Tumanov [18] (see also [10]). Also we evaluate the expression $E(a\eta + b)$. In the paper of Tumanov [18], another type of "mean" operator was introduced, and some additional vanishing assumptions for a and b were needed to evaluate $E(a\eta + b)$. In our case, no additional assumptions for a and b are required except of those which follow naturally from the context.

Proposition 3.6.1 *Let $a = a(e^{i\theta})$ and $b = b(e^{i\theta})$ be matrix-valued C^α functions on the unit circle with $|a|_\alpha$ sufficiently small. Let G be the $k \times k$ matrix solution to the equation*

$$G = I + T(Ga) \qquad \text{where} \qquad I \text{ is the } k \times k \text{ unit matrix.}$$

Then the solution η to the equation $\eta = -T(a\eta + b)$ is of the form

$$\eta = -[G(I + a^2)]^{-1} [T(Gb) + Gab - E(Ga)E(a\eta + b)]. \tag{3.13}$$

Proof. Let G be as in the assumptions of the proposition. We note that such G exits, for instance taking the Neumann series of the system which will converge to the solution. We have also $E(G) = 1$. Since we are dealing with matrices, we should be careful with the order of multiplication. Put $\varphi = a\eta + b$, then, by using (3.7), we get

$$\eta = -T\varphi, \qquad \varphi = T\eta + E(\varphi). \tag{3.14}$$

From the definition of G we get

$$TG = -Ga + E(Ga). \qquad (\text{Since } TT(Ga) = -Ga) \tag{3.15}$$

Note that $E(Ga)$ is small because $|a|_\infty$ is small. Making use of (3.14) and (3.15) we have

$$\begin{aligned} Gb &= G(\varphi - a\eta) = G\varphi - Ga\eta = G\varphi - (TG - E(Ga))(T\varphi) \\ &= G\varphi - (TG)(T\varphi) + E(Ga)(T\varphi). \end{aligned} \tag{3.16}$$

Now we apply the operator T to the extreme sides of (3.16) and use (3.4) and (3.14)

$$T(Gb) = T(G\varphi - T(G)(T\varphi) + T(E(Ga)T\varphi).$$

$$T(Gb) = GT\varphi + (TG)\varphi + E(Ga)(-\varphi + E(\varphi)).$$

In the above equation we plug back $\varphi = a\eta + b$, $T\varphi$ from (3.14) and TG from (3.15):

$$T(Gb) = -G\eta + (-Ga + E(Ga))(a\eta + b) - E(Ga)(a\eta + b) + E(Ga)E(a\eta + b).$$

$$T(Gb) = -G\eta + -Ga(a\eta + b) + E(Ga)E(a\eta + b).$$

$$T(Gb) = -[G + Ga^2]\eta - Gab + E(Ga)E(a\eta + b).$$

Calculating η from this equation we immediately get (3.13). The proposition is proved.

Theorem 3.6.1 *With the notation and assumptions of 2.6.1 Proposition, we have the formula*

$$E(a\eta + b) = E(Gb). \tag{3.17}$$

If b is of the form $b = cq$, where $c = c(e^{i\theta})$ is a $k \times n$ matrix C^α function and $q = q(e^{i\theta})$ is the C^α boundary value of an arbitrary holomorphic $n \times 1$ matrix function with sufficiently small $|q|_\infty$ and moreover $E(a\eta + cq) = 0$ for any q, then the solution of the equation $\eta = -T(a\eta + cq)$ is

$$\eta = -[iI + a]^{-1}cq.$$

Proof. As in the proof of 2.6.1 Proposition, we write $\varphi = a\eta + b$. Then, from (3.16), we get

$$\varphi - Gb = (1-G)\varphi - (T(1-G))(T\varphi) - E(Ga)(T\varphi).$$

By using (3.5), (3.7), and the fact that $E(1-G) = 0$, we immediately get that $E(\varphi - Gb) = 0$, which proves (3.17).

To prove the second part, we note that if $E(a\eta + cq) = 0$ for any q, then (3.17) implies that Gcq is the boundary value of a holomorphic function and consequently $T(Gcq) = -iGcq$. Plugging all this in (3.13), we get

$$\eta = -[I + a^2]^{-1}G^{-1}[-iGcq + Gacq] \tag{3.18}$$

$$= -[I + a^2]^{-1}[-iI + a]cq \tag{3.19}$$

$$= -[iI + a]^{-1}[-iI + a]^{-1}[-iI + a]cq \tag{3.20}$$

$$= -[iI + a]^{-1}cq. \tag{3.21}$$

which completes the proof.

4. TYPE FUNCTIONS

In the past decades many papers in the Cauchy-Riemann theory were devoted to the notion of type of points on CR manifolds and to applications of this notion to different problems of complex analysis. There are several approaches to the notion of type of points. Our goal here is to introduce an analytic disc approach to the notion of type of points. Although this approach, with some modifications, can be carried out for general CR submanifolds of $\mathbb{C}^n$, we first explain it in the simplest case namely real hypersurfaces in $\mathbb{C}^2$. For general dimension and codimension, there are several type functions, all of which coincide for a hypersurface in $\mathbb{C}^2$. We define the type function $\Phi = \Phi(p, \delta)$, $p \in M$, $0 \leqslant \delta \leqslant \delta_0$, which is continuous with respect to both variables. Here M denotes a real, smooth, embedded hypersurface in $\mathbb{C}^2$. The advantage of our approach to type of points is that the type function Φ classifies not only points of finite or infinite type but also distinguishes between points of the classical type. We start this Section with the definition of type function and formulate some of the properties of type function from the paper [7] which will be used in our main theorem in Section 5 [Theorem 5.2.1]

4.1. DESCRIPTION OF THE TYPE FUNCTION

Let M be a CR manifold of type (n, k) which is locally generically embedded in $\mathbb{C}^{n+k}$. This means that $CRdim_{\mathbb{C}}M = n$ and $codim_{\mathbb{C}}M = k$. Consider a point p on M. The total type function $\Phi(p, \delta)$is designed to measure the maximum distance that the center of an analytic disc can have from p, provided that the boundary of the disc lies on M, and that the disc has radius less than or equal to δ, when measured in a Holder norm α, $0 < \alpha < 1$.

Let us assume that p is the origin, and the local defining equations for M are

$$y = h(x, w)$$

where $h(0,0) = 0$ and $dh(0,0) = 0$, where h is a smooth map: $\mathbb{R}^k \times \mathbb{C}^n \longrightarrow \mathbb{R}^k$ defined near the origin. The total type function of M at the origin is defined by

$$\Phi(0, \delta) = \sup |y(0)[w]|,$$

where

$$y(0)[w] = \frac{1}{2\pi}\int_0^{2\pi} h(x(e^{i\theta})[w], w(e^{i\theta}))d\theta \text{.........................}(4.1)$$

Here the supremum is taken over all $w(.) \in [\mathbb{O}(D) \bigcap C^{\alpha}(\bar{D})]^n$ with $w(0) = 0$ and $|w|_\alpha \leqslant \delta$, where D is the unit disc in $\mathbb{C}$, and $|y(0)[w]|$ denotes the Euclidean norm of the vector $y(0)[w]$ in $\mathbb{R}^k$. Here $|w|_\alpha$ indicates the α norm over S^1, which is equivalent to the α norm $|w|_\alpha^{\bar{D}}$ over $\bar{D}$. Note that $y(0)[w]$ is the center of the *"lifted analytic disc"*, with boundary on M,and $x(e^{i\theta})[w]$ is the unique solution of the Bishop equation $x = c - T[h(x, w)]$ corresponding to the parameters $c = 0$, $w = w(\zeta) = (w_1(\zeta), ..., w_n(\zeta)) \in [\mathbb{O}(D) \bigcap C^{\alpha}(\bar{D})]^n$. In fact each lifted disc is of the form $g(\zeta) = (z(\zeta), w(\zeta))$, and the imaginary part of its first component $z(\zeta) = (z_1(\zeta), ..., z_k(\zeta))$ has mean value given by 4.1.

Next consider a point p on M near the origin. In a small neighborhood of the origin we can fix a smooth mapping $\Omega_p : M_{loc} \longrightarrow AU(\mathbb{C}^{n+k})$ where $AU(\mathbb{C}^{n+k})$ is the space of affine unitary transformation, such that for each $p \in M_{loc}, \omega_p : \mathbb{C}^{n+k} \longrightarrow \mathbb{C}^{n+k}$ arranges that $\Omega_p(p) = (0, 0)$, $\quad \Omega_{p^*}(T_pM) = \{y = 0\}$, $\quad \Omega_{p^*}(H_pM) = \{z = 0\}$, where $(z, w) \in \mathbb{C}^k \times \mathbb{C}^n$ and $z = x + iy$.

Using the standard inner product on $\mathbb{C}^k \times \mathbb{C}^n$, we have a natural choice of normal space at p, with $N_pM = \Omega_{p^*}^{-1}(\{x = 0, \quad w = 0\})$

Then for each $p \in M_{loc}$, we can express $\Omega_p(M) : y = h^p(x, w)$ where $h^p : \mathbb{R}^k \times \mathbb{C}^n \longrightarrow \mathbb{R}^k$ is a smooth map defined near the origin. Then the total type function, at the point $p \in M$, is defined by

$$\Phi(p, \delta) = sup|y^p(0)[w]|,$$

where $y^p(0)[w]$ is given by 4.1, in which h is replaced by h^p, and x is replced by x^p. Here x^p denotes the unique solution to the Bishop equation for h^p, corresponding to the same parameters as before. We first list some properties of $\Phi(p, \delta)$ that follow immediately from the definition:

$$\Phi(p, 0) = 0, \quad \Phi(p, \delta) \geqslant 0, \quad \Phi(p, \delta) \, \text{is monotonically decreasing as} \, \delta \searrow 0.$$

Using the stability results of[[13], Section 5] concerning the continuous dependence of solutions to the Bishop equation with respect to perturbations of the function h, it follows that the total type function $\Phi(p, \delta)$ is jointly continuous with respect to both variables (p, δ).

4.2. RELATION BETWEEN THE TYPE FUNCTION AND THE CLASSICAL TYPE OF POINTS

As was already mentioned in the introduction, there are well known definitions of the type of points for CR manifolds. In the case of hypersurfaces in $\mathbb{C}^2$ all these notions are equivalent and assign to a point $p \in M$ as natural number or positive infinity. The following proposition and theorem demonstrates the relation between our type function and the classical type of points.

Proposition 4.2.1 *(5.1 from [7])*

(a) If a point $p \in M$ is of finite type m in the classical sense, then the limit $\lim_{\delta \to 0} \frac{\Phi(p,\delta)}{\delta^m}$ exists, is finite, and different than zero.

(b) If there exist a natural number n, and a constant $c > 0$ such that $\Phi(p,\delta) \geq c\delta^n$ for small positive δ, then p is of finite type m for some m, $2 \leq m \leq n$, on the classical sense, and the limit above exists.

(c) A point $p \in M$ is of finite type in the classical sense if and only if for any natural number m there exists a constant c_m such that $\Phi(p,\delta) \leq c_m \delta^m$ for $0 \leq \delta \leq \delta_0$, for some $\delta_0 > 0$.

Theorem 4.2.1 *(Theorem 1 from [7])*

A point $p \in M$ is of finite type m in the classical sense (Kohn, Bloom, D'Angelo) if and only if the limit $\lim_{\delta \to 0} \frac{\Phi(p,\delta)}{\delta^m}$ exists, is finite, and different from zero. A point p is of finite type in the classical sense in and only if the function $\delta \in [0, \delta_0] \longrightarrow \Phi(p,\delta)$ vanishes to infinite order at zero.

4.3. HOLOMORPHIC EXTENSION OF CR FUNCTIONS BY USING THE TYPE FUNCTION

The main property of the type function we formulate here is the relation between the type function Φ and the holomorphic extension of CR functions. Now we formulate the results from the paper [10]. First consider the case where $\Phi(p,\delta) \equiv 0$.

Dealing with holomorphic extensions of CR functions, Tumanov[18] introduced the notion of minimality of CR manifolds at a point. In the case of a real hypersurface M in $\mathbb{C}^2$, M is minimal at $p \in M$, if and only if there is no holomorphic curve in M containing p.

Theorem 4.3.1 *([10], Theorem 2) A hypersurface $M \subset \mathbb{C}^2$ is not minimal at $p \in M$ (i.e., M contains a non-constant holomorphic curve through p) if and only if $\Phi(p,\delta) \equiv 0$ for $\delta > 0$ sufficiently small.*

M is said to be extendable at p if and only if every function which is CR in a fixed neighborhood of p in M extends holomorphically to a fixed neighborhood lying on atleast one side of M.

From the definition of the type function Φ, and by using the results of the paper [13], we can get the following Corollary(for details see [10],Corollary 3).

Corollary 4.3.1 *Every CR function in a neighborhood of p in M can be holomorphically extended to at least one side of M if and only if $\Phi(p, \delta) > 0$ for all $\delta > 0$ sufficiently small.*

Let a hypersurface M be given by the equation $y = h(w), h(0) = 0, dh(0) = 0$, where the function h does not depend on the variable x. Then M is said to be a rigid hypersurface.

In the next Section we consider a rigid hypersurface in $\mathbb{C}^2$ given by $y = h(w)$. We will discuss about the holomorphic extension of CR functions on a particular rigid hypersurface M given by $y = (\operatorname{Re} w)^m (\operatorname{Im} w)^n$.

5. HYPERSURFACE CASE

In this Section we consider hypersurfaces of the form

$$y = h(w), \qquad h(0) = 0, \tag{5.1}$$

where $h = h(w)$ is a continuous function defined in a neighborhood of $w = 0$.

5.1. HYPERSURFACES IN $\mathbb{C}^2$ GIVEN BY $Y = H(W)$

To investigate this case, we will use results of Globevnik and Rudin [12], [11],[8] about characterization of harmonic functions. We formulate here these results.

Theorem 5.1.1 *(Theorem from [12]) Suppose that $\varphi : U \to \mathbb{R}$ is continuous and that*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \varphi(f(e^{i\theta}))\, d\theta = \varphi(0) \tag{5.2}$$

holds for every continuous one-to-one function $f : \overline{U} \to U$, with $f(0) = 0$, which is holomorphic in U and for which $f(U)$ is smoothly bounded convex region. Then φ is harmonic in U.

The theorem of [8] uses a much smaller testing class of functions. Namely we have

Theorem 5.1.2 *(Theorem from [8]) Let D be the unit disc in $\mathbb{C}$ and $h : D \to \mathbb{R}$, $h \in C(D)$. Suppose that*

$$\frac{1}{2\pi}\int_{0}^{2\pi} h(ae^{i\theta} + b_k e^{ik\theta})\, d\theta = h(0) \tag{5.3}$$

for any $w = w(\zeta) = a\zeta + b_k\zeta^k$ *which maps* $\overline{D}$ *into* D, $k = 1, 2, 3, \ldots$. *Then* h *is harmonic in* D.

Here we prove the following theorem:

Theorem 5.1.3 *Let* $M \subset \mathbb{C}^2$ *be a hypersurface given by a class* C^1 *function* $y = h(w)$ *with* $h(0) = 0$ *in a neighborhood of the origin. If the center of any analytic disc* $\zeta \to (z(\zeta), w(\zeta))$ *obtained by lifting* $w(\zeta)$, $w(0) = 0$, *with the zero* x*-coordinate at zero is also zero, i.e.,* $y(0) = \mathrm{Im}(w(0)) = 0$, *then the function* $h = h(w)$ *is harmonic in a neighborhood of the origin.*

Proof. When lifting the disc $\zeta \to w(\zeta)$ with parameter $x(0) = 0$, we have the formula

$$y(0) = \frac{1}{2\pi} \int_0^{2\pi} h(w(e^{i\theta}))\, d\theta.$$

By assumptions of the theorem, this integral is zero. Applying theorem [12], we get that the above integral is zero for any function $w(\zeta)$ which satisfy the assumptions of the theorem. We can also apply theorem [8] and take a special form of functions $w(\zeta) = a\zeta + b\zeta^k$, $k = 1, 2, \ldots$, the constants a and b are sufficiently small, and we get the same conclusion, namely the function $h = h(w)$ is harmonic in a neighborhood of the origin. □

It is obvious that if the hypersurface M is given by a harmonic function $y = h(w)$, $h(0) = 0$, in a neighborhood of the origin, then the lift of any analytic disc $w(\zeta)$, $w(0) = 0$, is contained in M. Consequently we get

Corollary 5.1.1 *Let* $M \subset \mathbb{C}^2$ *be a hypersurface given by a class* C^1 *function* $y = h(w)$ *with* $h(0) = 0$ *in a neighborhood of the origin. If the center of any analytic disc* $\zeta \to (z(\zeta), w(\zeta))$ *obtained by lifting* $w(\zeta)$, $w(0) = 0$, *with the zero* x*-coordinate at zero is also zero, i.e.,* $y(0) = \mathrm{Im}(w(0)) = 0$, *if and only if the function* $h = h(w)$ *is harmonic in a neighborhood of the origin.*

Corollary 5.1.2 *Let $M \subset \mathbb{C}^2$ be a hypersurface given by a class C^1 function $y = h(w)$ with $h(0) = 0$ in a neighborhood of the origin. If the center of any analytic disc $\zeta \to (z(\zeta), w(\zeta))$ obtained by lifting $w(\zeta)$, $w(0) = 0$, with the zero x-coordinate at zero is non-zero, i.e., $y(0) = \operatorname{Im}(w(0)) \neq 0$, then the function is not harmonic in any neighborhood of the origin.*

5.2. HYPERSURFACES IN $\mathbb{C}^2$ GIVEN BY $Y = (\mathrm{RE}\, W)^M (\mathrm{IM}\, W)^N$

In this Section we prove the following theorem

Theorem 5.2.1 *Suppose that M is a hypersurface in $\mathbb{C}^2$ given locally near the origin $(0,0)$ by the equation*

$$y = (\operatorname{Re} w)^m (\operatorname{Im} w)^n, \quad \text{where } m, n \text{ are integers such that } m \geq 0,\ n \geq 0, (m,n) \neq (0,0). \tag{5.4}$$

Then

(a) *In each case (1) $m = 1,\ n = 0$; (2) $m = 0,\ n = 1$; (3) $m = 1,\ n = 1$ the type function $\Phi(p, \delta)$ is identically zero at the origin $p = 0$. As a consequence, not all CR functions defined in a neighborhood of the origin in M can be holomorphically extended to at least one side of M.*

(b) *If m and n are even integers, then all CR functions defined on M extend holomorphically to one side of M.*

(c) *If one of the integers is even and the second one is odd with the condition that $m + n \geq 2$, then all CR functions defined on M extend holomorphically to a full neighborhood of the origin.*

(d) *If both integers are odd with the condition $m + n \geq 3$, then all CR functions defined on M extend holomorphically to a full neighborhood of the origin.*

We split the proof of the theorem into few lemmas and corollaries. These are essential in proving the above theorem.

5.2.1. Harmonic Polynomials of the Form $h(w) = (\mathrm{Re}\, w)^m (\mathrm{Im}\, w)^n$. We prove the following

Lemma 5.2.1 *Let $h(w) = (\mathrm{Re}\, w)^m (\mathrm{Im}\, w)^n$, where m and n are non-negative integers. The function h is harmonic on $\mathbb{C}$ if and only if*

$$\text{(i) } m = 0,\ n = 0 \text{ or (ii) } m = 1,\ n = 0 \text{ or (iii) } m = 0,\ n = 1, \text{ or (iv) } m = 1,\ n = 1. \tag{5.5}$$

Proof. We denote $u = \mathrm{Re}\, w,\ v = \mathrm{Im}\, w$. Then the function h can be written as $h(w) = h(u, v) = u^m v^n$. To check harmonicity, we calculate the Laplacian

$$\begin{aligned}
\Delta(h(w)) &= \left(\frac{\partial^2}{\partial u^2} + \frac{\partial^2}{\partial v^2}\right)(u^m v^n) \\
&= m(m-1)u^{m-2}v^n + n(n-1)u^m v^{n-2} \\
&= u^{m-2}v^{n-2}\left[m(m-1)v^2 + n(n-1)u^2\right]
\end{aligned}$$

Since the Laplacian has to be zero everywhere on $\mathbb{C}$, the only possibilities are the three cases mentioned in the lemma. □

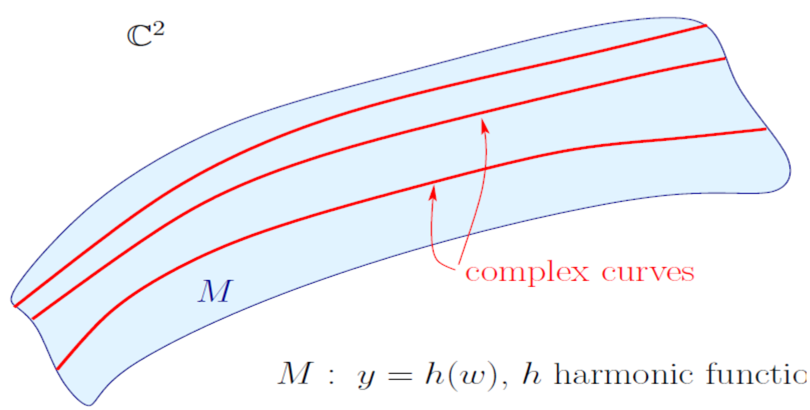

Figure 5.1: Hypersurface M foliated by complex curves

As an immediate consequence, we have the following two corollaries:

Corollary 5.2.1 *Let M be a rigid hypersurface in $\mathbb{C}^2$ given by the equation $y = (\operatorname{Re} w)^m (\operatorname{Im} w)^n$, where m, n are non-negative integers and $(m, n) \neq (0, 0)$. Then the type function of the hypersurface near the origin is*

1. $\Phi(0, \delta) \equiv 0$ *in the cases* (ii), (iii), (iv) *from* (5.5)*;*

2. $\Phi(0, \delta) \not\equiv 0$ *in the remaining cases.*

Corollary 5.2.2 *Let M be a rigid hypersurface in $\mathbb{C}^2$ given by the equation $y = (\operatorname{Re} w)^m (\operatorname{Im} w)^n$, where m, n are as in cases* (ii), (iii), (iv) *from* (5.5)*. Then there exists a CR function defined in a neighborhood of the origin which cannot be holomorphically extended to at least one-side neighborhood of M in $\mathbb{C}^2$.*

The above corollary proves part (a) of theorem 5.2.1.(Fig 5.1)

5.2.2. Analytic Discs Lifted to a Convex Hypersurface. Now we give the proof of part(b) of theorem 5.2.1. If the hypersurface M is given by equation (5.4) and m, n are even integers, then it is clear that the type function $\Phi(0, \delta)$ is positive for $\delta > 0$. It follows from the fact that the hypersurface M is convex and does not

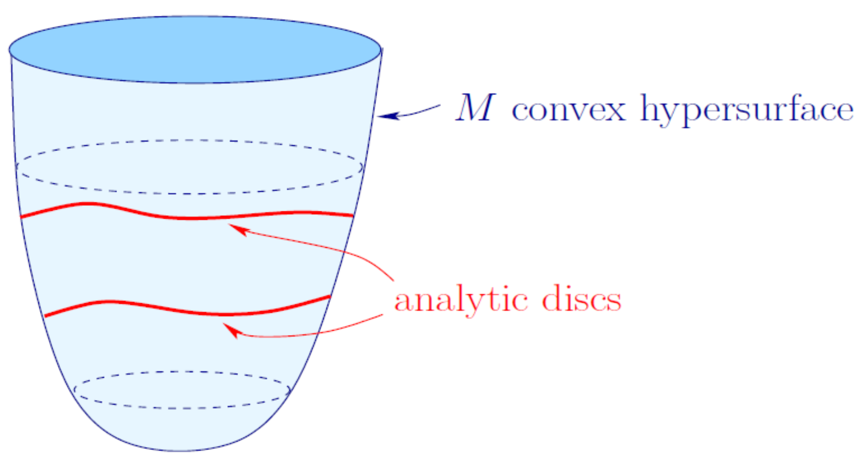

Figure 5.2: Analytic discs lifted to a convex hypersurface

contain a complex curve passing through the origin. So any CR function defined in a neighborhood of the origin holomorphically extends to the convex side.(Fig 5.2)

Moreover, M is strictly pseudoconvex, and consequently, there exists a CR function defined in a neighborhood of the origin in M that cannot be holomorphically extended to the concave side.

So part (b) of theorem 5.2.1 is proved.

5.2.3. Analytic Discs Lifted on Both Sides of a Saddle Hypersurface. Next we give the proof of part (c) of theorem 5.2.1. Let $m = 2\alpha$ be even and $n = 2\beta+1$ be odd. The case is symmetric with respect to m and n and because of that we can fix the above case. Since $m + n \geq 2$, we exclude the case $m = 0$ and $n = 1$. So we can write the equation for the hypersurface as

$$y = u^{2\alpha} v^{2\beta+1} =: h(u, v) = h(w). \tag{5.6}$$

and the family of analytic discs as

$$w(\zeta) = a\zeta + b_k\zeta^k, \quad a, b_k \in \mathbb{C}, k = 1, 2, 3, \ldots\ .$$

Since the function $h(w)$ from (5.6) is not harmonic, for some choice of the coefficients a and b_k the y-coordinate of the center of the lifted analytic disc is not zero: it can be positive or negative. Assume that it is positive, this means, the integral

$$y(0) = \int_0^{2\pi} [\mathrm{Re}(ae^{i\theta} + b_k e^{ik\theta})]^{2\alpha}\ [\mathrm{Im}(ae^{i\theta} + b_k e^{ik\theta})]^{2\beta+1} d\theta > 0\ .$$

Now, keeping the same coefficients a and b_k, we replace

$$w(\zeta) = a\zeta + b_k\zeta^k \qquad by \qquad w_{-}(\zeta) = -a\zeta - b_k\zeta^k$$

in the formula for the y-coordinate of the center

$$\begin{aligned} y_-(0) &= \int_0^{2\pi} [\mathrm{Re}(-ae^{i\theta} - b_k e^{ik\theta})]^{2\alpha} \, [\mathrm{Im}(-ae^{i\theta} - b_k e^{ik\theta})]^{2\beta+1} d\theta \\ &= -\int_0^{2\pi} [\mathrm{Re}(ae^{i\theta} + b_k e^{ik\theta})]^{2\alpha} \, [\mathrm{Im}(ae^{i\theta} + b_k e^{ik\theta})]^{2\beta+1} d\theta \\ &< 0 \end{aligned}$$

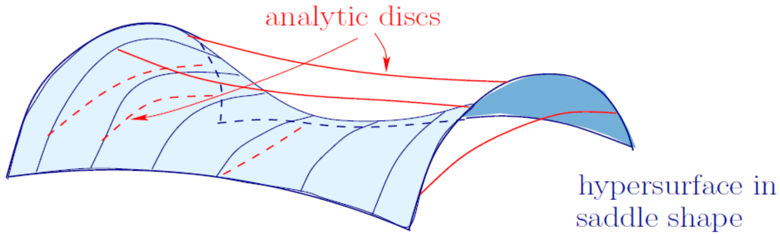

Figure 5.3: Analytic discs lifted on both sides of a saddle hypersurface

Consequently, the type function $\Phi(0, \delta)$ takes both positive and negative values for $\delta >$ small, which implies that any CR function defined on M near the origin can be holomorphically extended to both sides of M, i.e., to a full neighborhood of M.(Fig 5.3) The case (c) is proved.

5.2.4. Behavior of the Type Function Near a Saddle Point. Finally we give the proof of part (d) of theorem 5.2.1.

Let $m = 2\alpha + 1$ and $n = 2\beta + 1$ be both odd. Obviously the case is symmetric with respect to m and n. Also we exclude the possibilities when $m + n \leq 2$, so we assume that $m + n \geq 3$. Without any loss of generality, we can assume that

$$n = 2\beta + 1 \geq 2\alpha + 1 = m \quad \textit{which is equivalent to} \quad \beta \geq \alpha. \tag{5.7}$$

The hypersurface is given by

$$y = u^{2\alpha+1}v^{2\beta+1} =: h(u,v) = h(w). \tag{5.8}$$

Since the function h is not harmonic, there exists a choice of the coefficients a and b_k such that the y-coordinate of the center of the disc is not zero: it can be positive or negative. To simplify calculations, we drop the index k at b_k and denote

$$H(a,b) = \int_0^{2\pi} [\mathrm{Re}(ae^{i\theta} + be^{ik\theta})]^{2\alpha+1}\, [\mathrm{Im}(ae^{i\theta} + be^{ik\theta})]^{2\beta+1} d\theta\ . \tag{5.9}$$

We want to prove that this integral takes both positive and negative values as the coefficients a, $b = b_k$ vary.

We split the proof into few steps.

1° $\alpha = \beta$. In this case we have

$$\int_0^{2\pi} [\mathrm{Re}(ae^{i\theta} + be^{ik\theta})]^{2\alpha+1}\, [\mathrm{Im}(ae^{i\theta} + be^{ik\theta})]^{2\alpha+1} d\theta \tag{5.10}$$

and if instead of the disc $w(\zeta) = a\zeta + b\zeta^k$ we take the disc $iw(\zeta) = ia\zeta + ib\zeta^k$, so we have

$$\mathrm{Re}\big(i(a\zeta + b\zeta^k)\big) = -\mathrm{Im}(a\zeta + b\zeta^k)$$

$$\mathrm{Im}\big(i(a\zeta + b\zeta^k)\big) = \quad \mathrm{Re}(a\zeta + b\zeta^k)$$

and the integral (5.10) becomes

$$-\int_0^{2\pi} [\mathrm{Re}(ae^{i\theta} + be^{ik\theta})]^{2\alpha+1}\, [\mathrm{Im}(ae^{i\theta} + be^{ik\theta})]^{2\alpha+1} d\theta\ .$$

Consequently, the integral (5.10) takes both positive and negative values when $w(\zeta)$ varies. The case 1° is proved.

2° $H(a,0)=0$. In this part we will show that for the analytic disc $w(\zeta)=a\zeta$, the integral is zero, i.e., $H(a,0)=0$.

$$\begin{aligned}
H(a,0) &= \int_0^{2\pi} [\mathrm{Re}(ae^{i\theta})]^{2\alpha+1}[\mathrm{Im}(ae^{i\theta})]^{2\beta+1} d\theta \\
&= \int_0^{2\pi} \left(\frac{ae^{i\theta}+\bar{a}e^{-i\theta}}{2}\right)^{2\alpha+1} \cdot \left(\frac{ae^{i\theta}-\bar{a}e^{-i\theta}}{2i}\right)^{2\beta+1} d\theta \\
&= \frac{1}{2^{2(\alpha+\beta+1)} i^{2\beta+1}} \int_0^{2\pi} (ae^{i\theta}+\bar{a}e^{-i\theta})^{2\alpha+1}(ae^{i\theta}-\bar{a}e^{-i\theta})^{2\beta+1} d\theta \\
&= \frac{1}{2^{2(\alpha+\beta+1)} i^{2\beta+1}} \int_0^{2\pi} \left[\sum_{s=0}^{2\alpha+1} \binom{2\alpha+1}{s} a^s e^{is\theta} \bar{a}^{2\alpha+1-s} e^{-i(2\alpha+1-s)\theta}\right] \times \\
&\quad \times \left[\sum_{p=0}^{2\beta+1} \binom{2\beta+1}{p} a^p e^{ip\theta} (-1)^{2\beta+1-p} \bar{a}^{2\beta+1-p} e^{-i(2\beta+1-p)\theta}\right] d\theta \\
&= \frac{1}{2^{2(\alpha+\beta+1)} i^{2\beta+1}} \int_0^{2\pi} \left[\sum_{s=0}^{2\alpha+1} \binom{2\alpha+1}{s} a^s \bar{a}^{2\alpha+1-s} e^{i(-2\alpha-1+2s)\theta}\right] \times \\
&\quad \times \left[\sum_{p=0}^{2\beta+1} \binom{2\beta+1}{p} a^p \bar{a}^{2\beta+1-p} (-1)^{2\beta+1-p} e^{i(-2\beta-1+2p)\theta}\right] d\theta \\
&= \frac{1}{2^{2(\alpha+\beta+1)} i^{2\beta+1}} \int_0^{2\pi} \left[\sum_{s=0}^{2\alpha+1} \sum_{p=0}^{2\beta+1} \binom{2\alpha+1}{s} \binom{2\beta+1}{p} a^{s+p} \bar{a}^{2(\alpha+\beta+1)-s-p} \times \right. \\
&\quad \left. \times (-1)^{2\beta+1-p} e^{-2i(\alpha+\beta+1-s-p)\theta}\right] d\theta \\
&= \frac{1}{2^{2(\alpha+\beta+1)} i^{2\beta+1}} \sum_{s=0}^{2\alpha+1} \sum_{p=0}^{2\beta+1} \binom{2\alpha+1}{s} \binom{2\beta+1}{p} a^{s+p} \bar{a}^{2(\alpha+\beta+1)-s-p} \times \\
&\quad \times (-1)^{2\beta+1-p} \int_0^{2\pi} e^{-2i(\alpha+\beta+1-s-p)\theta} d\theta
\end{aligned}$$

The above sum simplifies because the integral is zero if the exponent is different than zero, namely

$$\int_0^{2\pi} e^{-2i(\alpha+\beta+1-s-p)\theta} d\theta = \begin{cases} 0 & \text{if} \quad \alpha+\beta+1-s-p \neq 0 \\ 2\pi & \text{if} \quad \alpha+\beta+1-s-p = 0 \end{cases} \tag{5.11}$$

Without any loss of generality, we can assume that $\beta \geq \alpha$. The proof of the other case is exactly the same.

$$\int_0^{2\pi} e^{-2i(\alpha+\beta+1-s-p)\theta} d\theta = 2\pi \quad \text{if} \quad p = \alpha+\beta+1-s, \quad 0 \leq s \leq 2\alpha+1.$$

The last sum in the above long computations becomes

$$\begin{aligned} H(a,0) &= \frac{2\pi}{2^{2(\alpha+\beta+1)} i^{2\beta+1}} \sum_{s=0}^{2\alpha+1} \begin{pmatrix} 2\alpha+1 \\ s \end{pmatrix} \begin{pmatrix} 2\beta+1 \\ \alpha+\beta+1-s \end{pmatrix} \times \\ &\quad \times a^{s+\alpha+\beta+1-s} \, \bar{a}^{2(\alpha+\beta+1)-s-(\alpha+\beta+1-s)} (-1)^{2\beta+1-(\alpha+\beta+1-s)} \\ &= \frac{2\pi \, a^{\alpha+\beta+1} \, \bar{a}^{\alpha+\beta+1}}{2^{2(\alpha+\beta+1)} i^{2\beta+1}} \sum_{s=0}^{2\alpha+1} \begin{pmatrix} 2\alpha+1 \\ s \end{pmatrix} \begin{pmatrix} 2\beta+1 \\ \alpha+\beta+1-s \end{pmatrix} (-1)^{\beta-\alpha+s} \end{aligned}$$

In order to calculate the above sum (without the coefficient in front), we note that

$$\begin{aligned} &\text{term for index } s \text{ is} && \begin{pmatrix} 2\alpha+1 \\ s \end{pmatrix} \begin{pmatrix} 2\beta+1 \\ \alpha+\beta+1-s \end{pmatrix} (-1)^{\beta-\alpha+s} \\ &\text{term for index } 2\alpha+1-s \text{ is} && \begin{pmatrix} 2\alpha+1 \\ 2\alpha+1-s \end{pmatrix} \begin{pmatrix} 2\beta+1 \\ \beta-\alpha+s \end{pmatrix} (-1)^{\beta+\alpha+1-s} \end{aligned}$$

Comparing these two terms we see that

$$\begin{pmatrix} 2\alpha+1 \\ s \end{pmatrix} \begin{pmatrix} 2\beta+1 \\ \alpha+\beta+1-s \end{pmatrix} (-1)^{\beta-\alpha+s} = - \begin{pmatrix} 2\alpha+1 \\ 2\alpha+1-s \end{pmatrix} \begin{pmatrix} 2\beta+1 \\ \beta-\alpha+s \end{pmatrix}$$
$$(-1)^{\beta+\alpha+1-s}$$

so the sum for $H(a,0) = 0$ because the term for index s cancels with the term for the index $2\alpha+1-s$, $s = 0,1,\dots,\alpha$, consequently the terms for indices $s = 0,1,\dots,\alpha$ cancel all terms for indices $s = \alpha+1, \alpha+2, \dots, 2\alpha+1$. The case 2° is proved.

3° Here we prove that the function $H(a,b)$ takes positive and negative values.

We note that the function $H(a,b)$ is a polynomial with respect of two variables a and b. Later on we will see what is the degree of this polynomial, but actually this is not essential for our goal. We can write

$$H(a,b) = \sum_{s,s',p,p'} c_{s,s',p,p'}\, a^s \bar{a}^{s'} b^p \bar{b}^{p'}$$

In part 2° we proved that $H(a,0) = 0$, so in the above notation

$$H(a,0) = \sum_{s,s'} c_{s,s',0,0}\, a^s \bar{a}^{s\prime} = 0.$$

To prove that the function $H(a,b)$ takes positive and negative values, it is enough to show that the first order part with respect to powers of b is not zero, namely

$$H(a,b) = \underbrace{H(a,0)}_{=0} + H_1(a,b) + H_2(a,b) + \dots \tag{5.12}$$

where

$$H_n(a,b) = \sum_{p+p'=n} c_{s,s',p,p'}\, a^s \bar{a}^{s'} b^p \bar{b}^{p'}$$

If we show that $H_1(a,b) \neq 0$, then $H(a,b)$ takes positive and negative values.

To prove that $H_1(a,b)$ is not zero, it is enough to prove this for convenient analytic discs $w(\zeta) = a\zeta + b\zeta^k$. Let us calculate

$$\begin{aligned} H(a,b) &= \int_0^{2\pi} [\mathrm{Re}(ae^{i\theta} + be^{ik\theta})]^{2\alpha+1} [\mathrm{Im}(ae^{i\theta} + be^{ik\theta})]^{2\beta+1} d\theta \\ &= \int_0^{2\pi} \left[\frac{(ae^{i\theta}+\bar{a}e^{-i\theta})+(be^{ik\theta}+\bar{b}e^{-ik\theta})}{2}\right]^{2\alpha+1} \left[\frac{(ae^{i\theta}-\bar{a}e^{-i\theta})+(be^{ik\theta}-\bar{b}e^{-ik\theta})}{2i}\right]^{2\beta+1} d\theta \\ &= \frac{1}{2^{2\alpha+1}(2i)^{2\beta+1}} \int_0^{2\pi} \left[(ae^{i\theta} + \bar{a}e^{-i\theta}) + (be^{ik\theta} + \bar{b}e^{-ik\theta})\right]^{2\alpha+1} \times \\ &\qquad \times \left[(ae^{i\theta} - \bar{a}e^{-i\theta}) + (be^{ik\theta} - \bar{b}e^{-ik\theta})\right]^{2\beta+1} \end{aligned}$$

To simplify calculations, we drop the coefficient in front of the integral, so the last integral becomes

$$\int_0^{2\pi} \left[\sum_{s=0}^{2\alpha+1} \binom{2\alpha+1}{s} (ae^{i\theta} + \bar{a}e^{-i\theta})^{2\alpha+1-s} (be^{ik\theta} + \bar{b}e^{-ik\theta})^s\right] \times$$
$$\times \left[\sum_{p=0}^{2\beta+1} \binom{2\beta+1}{p} (ae^{i\theta} - \bar{a}e^{-i\theta})^{2\beta+1-p} (be^{ik\theta} - \bar{b}e^{-ik\theta})^p\right] d\theta$$

Since we are interested only in the terms with the first power of b or $\bar{b}$, we extract these terms from the above sum

$$\int_0^{2\pi} \left[\binom{2\alpha+1}{0} (ae^{i\theta} + \bar{a}e^{-i\theta})^{2\alpha+1} \binom{2\beta+1}{1} (ae^{i\theta} - \bar{a}e^{-i\theta})^{2\beta} (be^{ik\theta} - \bar{b}e^{-ik\theta}) + \right.$$
$$\left. + \binom{2\alpha+1}{1} (ae^{i\theta} + \bar{a}e^{-i\theta})^{2\alpha} (be^{ik\theta} + \bar{b}e^{-ik\theta}) \binom{2\beta+1}{0} (ae^{i\theta} - \bar{a}e^{-i\theta})^{2\beta+1}\right] d\theta$$

Taking into account the identities

$$\binom{2\alpha+1}{0}=1,\quad \binom{2\beta+1}{1}=2\beta{+}1,\quad \binom{2\alpha+1}{1}=2\alpha{+}1,\quad \binom{2\beta+1}{0}=1$$

the last integral becomes

$$\int_0^{2\pi}\big[(ae^{i\theta}+\bar a e^{-i\theta})^{2\alpha+1}(2\beta+1)(ae^{i\theta}-\bar a e^{-i\theta})^{2\beta}(be^{ik\theta}-\bar b e^{-ik\theta})+$$

$$+\,(2\alpha+1)(ae^{i\theta}+\bar a e^{-i\theta})^{2\alpha}(be^{ik\theta}+\bar b e^{-ik\theta})(ae^{i\theta}-\bar a e^{-i\theta})^{2\beta+1}\big]\,d\theta$$

$$=\int_0^{2\pi}(ae^{i\theta}+\bar a e^{-i\theta})^{2\alpha}(ae^{i\theta}-\bar a e^{-i\theta})^{2\beta}\times$$

$$\times\Big[(2\beta+1)(ae^{i\theta}+\bar a e^{-i\theta})(be^{ik\theta}-\bar b e^{-ik\theta})+(2\alpha+1)(ae^{i\theta}-\bar a e^{-i\theta})(be^{ik\theta}+\bar b e^{-ik\theta})\Big]d\theta$$

$$=\int_0^{2\pi}(ae^{i\theta}+\bar a e^{-i\theta})^{2\alpha}(ae^{i\theta}-\bar a e^{-i\theta})^{2\beta}\times$$

$$\times\Big[(2\beta+1)\left(ab\,e^{i(k+1)\theta}-a\bar b\,e^{-i(k-1)\theta}+\bar a b\,e^{i(k-1)\theta}-\bar a\bar b\,e^{-i(k+1)\theta}\right)+$$

$$+(2\alpha+1)\left(ab\,e^{i(k+1)\theta}+a\bar b\,e^{-i(k-1)\theta}-\bar a b\,e^{i(k-1)\theta}-\bar a\bar b\,e^{-i(k+1)\theta}\right)\Big]d\theta$$

$$=\int_0^{2\pi}(ae^{i\theta}+\bar a e^{-i\theta})^{2\alpha}(ae^{i\theta}-\bar a e^{-i\theta})^{2\beta}\times$$

$$\times\Big[2(\alpha+\beta+1)ab\,e^{i(k+1)\theta}+2(\alpha-\beta)a\bar b\,e^{-i(k-1)\theta}+$$

$$+\,2(\beta-\alpha)\bar a b\,e^{i(k-1)\theta}-2(\alpha+\beta+1)\bar a\bar b\,e^{-i(k+1)\theta}\Big]d\theta$$

Finally we want to prove that the above integral is not zero for some convenient choice of an analytic disc $w(\zeta)=a\zeta+b\zeta^k$. From the formula (5.11) we know that only with the zero exponent in $e^{in\theta}$ the integral is not zero. It is possible to prove that the integral is not zero for any analytic disc $w(\zeta)=a\zeta+b\zeta^k$ where k is running from $k=2$ to some k_0. However, the easiest calculations are when we choose $k-1=2\alpha+2\beta$.

Why is such choice? In the product

$$(ae^{i\theta}+\bar{a}e^{-i\theta})^{2\alpha}(ae^{i\theta}-\bar{a}e^{-i\theta})^{2\beta}$$

after taking powers and after multiplications, the exponential function runs

$$e^{-i(2\alpha+2\beta)\theta},\ldots,e^{i(2\alpha+2\beta)\theta}.$$

Because of that, to minimize the number of non-zero terms in the integral when multiplying by the expression in the square brackets, the best choice is as above, i.e., $k-1=2\alpha+2\beta$. Most of the terms after integration will be zero because the exponent in $e^{in\theta}$ will not be zero. So it is enough to calculate the terms in the integrand when there is chance to get the zero exponent. So we have

$$\begin{aligned}&\left(a^{2\alpha}e^{2i\alpha\theta}+\bar{a}^{2\alpha}e^{-2i\alpha\theta}\right)\left(a^{2\beta}e^{2i\beta\theta}+\bar{a}^{2\beta}e^{-2i\beta\theta}\right)\left[2(\alpha-\beta)a\bar{b}\,e^{-2i(\alpha+\beta)\theta}+2(\beta-\alpha)\bar{a}b\,e^{2i(\alpha+\beta)\theta}\right]\\&=\left(a^{2(\alpha+\beta)}e^{2i(\alpha+\beta)\theta}+a^{2\alpha}\bar{a}^{2\beta}e^{2i(\alpha-\beta)\theta}+\bar{a}^{2\alpha}a^{2\beta}e^{2i(\beta-\alpha)\theta}+\bar{a}^{2(\alpha+\beta)}e^{-2i(\alpha+\beta)\theta}\right)\times\\&\qquad\times\left[2(\alpha-\beta)a\bar{b}\,e^{-2i(\alpha+\beta)\theta}+2(\beta-\alpha)\bar{a}b\,e^{2i(\alpha+\beta)\theta}\right]\end{aligned}\tag{5.13}$$

It is sufficient to multiply in such a way that to get only terms of the exponential function with the zero exponent. Here we have to consider two cases: (1) $\alpha>0$ and (2) $\alpha=0$. In the first case we can ignore terms that contain $\beta-\alpha$ or $\alpha-\beta$ in the exponent, because $|\beta-\alpha|<\alpha+\beta$. So we have

$$\begin{aligned}a^{2(\alpha+\beta)}2(\alpha-\beta)a\bar{b}+\bar{a}^{2(\alpha+\beta)}2(\beta-\alpha)\bar{a}b &= 2(\alpha-\beta)a^{2(\alpha+\beta)+1}\bar{b}+2(\beta-\alpha)\bar{a}^{2(\alpha+\beta)+1}b\\&= 2(\alpha-\beta)(a^{2(\alpha+\beta)+1}\bar{b}-\bar{a}^{2(\alpha+\beta)+1}b)\\&= 4i(\alpha-\beta)\mathrm{Im}\,(a^{2(\alpha+\beta)+1}\bar{b})\end{aligned}$$

By our assumption $\beta > \alpha$, and obviously we can choose a and b in such a way that $\operatorname{Im}(a^{2(\alpha+\beta)+1}\bar{b}) \neq 0$.

If $\alpha = 0$, then multiplying in (5.13) we obtain

$$2\left(a^{2\beta}e^{2i\beta\theta} + \bar{a}^{2\beta}e^{-2i\beta\theta}\right)\left(-2\beta a\bar{b}\,e^{-2i\beta\theta} + 2\beta\bar{a}b\,e^{2i\beta\theta}\right) =$$
$$= 4\beta(-a^{2\beta+1}\bar{b} + \bar{a}^{2\beta+1}b + a^{2\beta}\bar{a}b\,e^{4i\beta\theta} - \bar{a}^{2\beta}a\bar{b}\,e^{-4i\beta\theta})$$

Because of the formulas (5.11), it is enough to take only terms that do not contain the factors $e^{in\theta}$, so we get

$$4\beta(-a^{2\beta+1}\bar{b} + \bar{a}^{2\beta+1}b) = 4\beta(\bar{a}^{2\beta+1}b - a^{2\beta+1}\bar{b}) = 8i\beta\,\operatorname{Im}(\bar{a}^{2\beta+1}b)$$

Obviously we can choose a and b in such a way that $\operatorname{Im}(\bar{a}^{2\beta+1}b) \neq 0$, so the second case is proved too.

This completes the proof of case 3^{o} and also of theorem 5.2.1.

5.2.5. Qualitative Description of the Extension Set. Using Theorem 5.2.1 we can give a more precise description of the extension set for CR functions defined on a real hypersurface in $\mathbb{C}^2$ given by the equation $y = (\operatorname{Re} w)^m(\operatorname{Im} w)^n$. Actually we will use some formulas from the proof of the theorem. The most interesting case is (d), $m = 2\alpha+1$, $n = 2\beta+1$, where α and β are nonnegative integers with the property $\alpha + \beta \geq 2$; for other cases the results and proofs are similar, obviously with suitable adjustments.

In the proof of part (d) we calculated the function

$$H(a,b) = \int_0^{2\pi} [\operatorname{Re}(ae^{i\theta} + be^{ik\theta})]^{2\alpha+1}[\operatorname{Im}(ae^{i\theta} + be^{ik\theta})]^{2\beta+1}d\theta$$

and we obtained that

$$H(a,b) = H(a,0) + H_1(a,b) + H_2(a,b) + \ldots + H_N(a,b), \tag{5.14}$$

for some N, where $H_s(a,b)$ is the homogeneous polynomial of degree s with respect to b and $\bar{b}$. Of course, for the extension property, the most important is $H_1(a,b)$ if this term is not zero, because $H_s(a,b)$ for $s \geq 2$ are higher order terms and if $|b|$ is small, then these terms are small compared to $H_1(a,b)$.

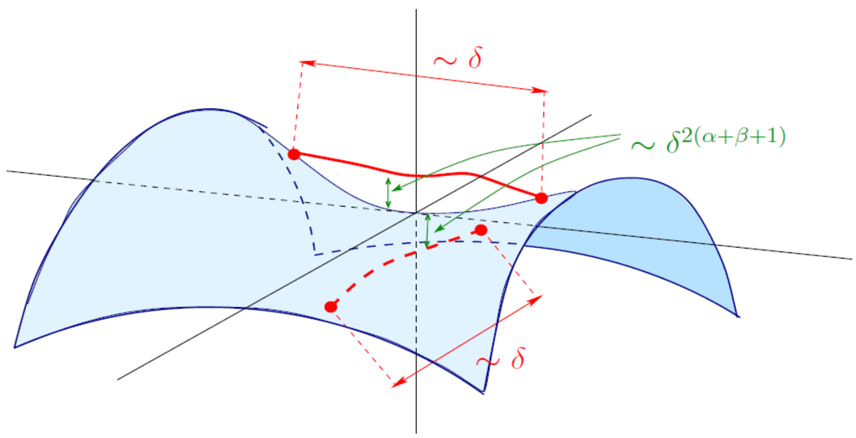

Figure 5.4: Behavior of the type function near a saddle point

We proved that $H(a,0) = 0$ for $a \in \mathbb{C}$. The formula for $H_1(a,b)$ when we take the analytic disc $w(\zeta) = a\zeta + b\zeta^{2(\alpha+\beta)+1}$ is

$$H_1(a,b) = \begin{cases} \dfrac{4i}{2^{2\alpha+1}(2i)^{2\beta+1}}(\alpha-\beta)\mathrm{Im}\,(a^{2(\alpha+\beta)+1}\bar{b}) & \text{if} \quad \alpha = 1, 2, \ldots \\[2ex] \dfrac{8i}{2(2i)^{2\beta+1}}\beta\,\mathrm{Im}\,(\bar{a}^{2\beta+1}b) & \text{if} \quad \alpha = 0 \end{cases}$$

or equivalently,

$$H_1(a,b) = \begin{cases} \dfrac{4i}{2^{2\alpha+1}(2i)^{2\beta+1}}(\beta-\alpha)\mathrm{Im}\,(\bar{a}^{2(\alpha+\beta)+1}b) & \text{if} \quad \alpha = 1,2,\ldots \\ \\ \dfrac{4i}{(2i)^{2\beta+1}}\beta\,\mathrm{Im}\,(\bar{a}^{2\beta+1}b) & \text{if} \quad \alpha = 0 \end{cases} \tag{5.15}$$

So if $m = 2\alpha+1$, $n = 2\beta+1$, we have

$$H(a,b) = H_1(a,b) + H_2(a,b) + \ldots + H_N(a,b)$$

and $H_1(a,b)$ is given in (5.15). If we choose $a = \delta$ to be real and $b = ic\delta(c < 1, real)$, then we see that

$$H_1(\delta, ic\delta) = \ k_1 c\delta^{2(\alpha+\beta+1)}, k_1 \neq 0$$

$$H_j(\delta, ic\delta) = k_j c^j \delta^{2(\alpha+\beta+1)}$$

$$H(a,b) = H_1(\delta, ic\delta) + \sum_{j=2}^{N} H_j(\delta, ic\delta) = \left[k_1 c + \sum_{j=2}^{N} k_j c^j\right]\delta^{2(\alpha+\beta+1)} = (\text{const})\delta^{2(\alpha+\beta+1)}$$

where the constant is not equal to zero provided c is chosen sufficiently small. By taking c both positive and negative with sufficiently small absolute value, it is clear that $H_1(a,b)$ can be made both positive and negative. Translating this in terms of the type function, if the disc $w(\zeta) = a\zeta + b\zeta^k$ has norm comparable to δ,then the centers can go outside the hypersurface M the distance which is proportional to $\delta^{2(\alpha+\beta+1)}$ on both sides.(Fig 5.4)

5.3. HYPERSURFACES IN $\mathbb{C}^2$ GIVEN BY $Y = (\mathrm{RE}\,W)^M(\mathrm{IM}\,W)^N$ + HIGHER ORDER TERMS

If a hypersurface M in $\mathbb{C}^2$ is given by equation

$$y = h(w) = (\mathrm{Re}\,w)^m(\mathrm{Im}\,w)^n + \text{ higher order terms}$$

the situation is practically the same as for hypersurfaces

$$y = h(w) = (\operatorname{Re} w)^m (\operatorname{Im} w)^n$$

because the dominant term is the first one. All considerations from the previous section apply here, including the behavior of the type function.

5.4. HYPERSURFACES IN $\mathbb{C}^2$ GIVEN BY $Y = H(W)$ WHERE $H(W)$ IS A POLYNOMIAL OF W AND $\bar{W}$

$$y = h(w) = (\operatorname{Re} w)^m (\operatorname{Im} w)^n = \frac{1}{2^m (2i)^n} \left[(w + \bar{w})^m (w - \bar{w})^n \right]$$

This contains pure terms namely w^{m+n} and $\bar{w}^{m+n}$. To write the graphing function in the normal form, we need to get rid of the pure terms. This can be done by defining the holomorphic change of coordinates. To find a convenient coordinate description of a CR submanifold, one can refer[8], lemma 1 and theorem 1 of Section 7.2.

6. CONCLUSION

The extension problem for CR functions has appeared to be one of the central problems in Cauchy-Riemann theory. It started with a famous paper H. Poincaré in 1907, then H.Lewy [15] in 1956 and culminated in the results of J.M.Trepreau [17] in 1985 and A.Tumanov [18] in 1988. Professor Roman Dwilewicz has made a significant contribution in this field of CR extensions that we can be proud of. He demonstrated his passion by exploring many results in this area. However still many questions remain open and still many properties are waiting to be proved. Following are some of the results related to our main theorem in Section 5:

1.Generalize results to the case of a hypersurface in $\mathbb{C}^2$ given by $y = h(w)$ where $h(w)$ is a homogeneous polynomial, such as $h(u,v) = \sum_{j+k=m} a_{jk} u^j v^k$

2. Hypersurface in $\mathbb{C}^2$ given by $y = h(w)$ where $h(w)$ is a real analytic function.

3. Non rigid hypersurface in $\mathbb{C}^2$ given by $y = h(x,w)$ where $h(x,w)$ is a C^∞ function in a neighborhood of the origin.

4. Finding an analytic disc proof of the sector property theorem in Baouendi-Treves- Rothschild.

BIBLIOGRAPHY

[1] M.S. Baouendi, L.P. Rothschild, and J.M. Trepreau, *On the geometry of analytic discs attached to real manifolds*, J. Differential Geom. 39 (1994), 379–405.

[2] M.S. Baouendi and F.Treves, *A property of the functions and distributions annihilated by a locally integrable system of complex vector fields*, Ann. Math 113 (1981), 387–421.

[3] E. Bishop, *Differentiable manifolds in complex Euclidean space* Duke Math. J. 32 (1965), 1–22.

[4] T. Bloom, and I. Graham, *On type conditions for generic submanifolds of* $\mathbb{C}^n$, Invent. Math 40 (1977), 217–243.

[5] A. Boggess, *CR manifolds and the tangential Cauchy-Riemann complex*, Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1991.

[6] J. D'Angelo, *Finite–type conditions for real hupersurfaces in* $\mathbb{C}^n$, Lecture Notes in Math 1268 (1987), 83–102.

[7] R.Dwilewicz and C.D. Hill, *An analytic disc approach to the notion of type of points*, Indiana Univ. Math. J. 41 (1992), 713–739.

[8] R.Dwilewicz and C.D. Hill, *A characterization of harmonic functions and points of finite type*, Indag. Mathem. 4 (1993), 39–50.

[9] R.Dwilewicz and C.D. Hill, *Spinning analytic discs and domains of dependence*, Manuscripta Math, 97 (1998), 407–427.

[10] R.Dwilewicz and C.D. Hill, *The conormal type function for CR manifolds*, Publ.Math. Debrecen 60 (2002), 245–282.

[11] J.Globevnikl, *Zero integrals on circles and characterization of harmonic and analytic functions*, Trans. Amer. Math. Soc. 317 (1990), no 1, 313–330.

[12] J.Globevnik and W. Rudin *A characterization of harmonic functions*, Nederl. Akad. Wetenxch. Indag. Math 50 (1988), no 4, 419–426.

[13] C.D. Hill and G.Taiani, *Families of analytic discs in* $\mathbb{C}^n$ *with boundaries on a prescribed CR submanifold*, Ann. Scuola Norm. Sup. Pisa 5 (1978), 327–380.

[14] C.D. Hill and G.Taiani, *Real analytic approximation of locally embeddable CR manifolds*, Composito Math. 44 (1981), 113–131.

[15] H. Lewy, *On the local character of an atypical linear differential equation in three variables and a related theorem for regular functions of two complex variables*, Ann. Math 64 (1956), 514–522.

[16] H. Lewy, *On hulls of holomorphy*, Comm. Pure Appl. Math. 13 (1960), 587–591.

[17] J.M. Trepreau, *Sur le prolongement holomorphe des fonctions C–R definies sur une hypersurface reelle de classe* $\mathbb{C}^2$ *dans* $\mathbb{C}^n$, Invent. Math. 83 (1986), 583–592.

[18] A.E. Tumanov, *Extension of CR functions into a wedge from a manifold of finite type* Math. Sb. Nov. Ser. 136 (1988), 128–139; English transl. Math USSR Sbornik 64 (1989), 129–140.

VITA

Brijitta Antony was born in Tamil Nadu, India. She received her B.S. with Honors in Mathematics from the University of Madras, India in 1982, and her M.S in Mathematics with Honors from Madurai Kamaraj University, India in 1984. Her love for mathematics, and particularly teaching, led her to pursue two more degrees M.S in Philosophy, Mathematics in 1985 and M.S. in Education in 1987 from the same institutions, respectively. After 23 years of serving in the academia field, she took on her life-long dream of pursuing a doctorate degree. In July 2017, she received her PhD in Mathematics from the Missouri University of Science and Technology, Rolla, Missouri USA. Here she conducted research and advanced philosophies of thought in the area of Complex Analysis.

Brijitta Antony has rich teaching experience both in India and United States of America. She served as Assistant Professor in Auxillium College, Vellore, India and later for 17 years (1987 2004) she served as the Head of the Department, Mathematics at Dewan Bahadhur Padmarao Mudhaliar Womens College, Secunderabad, India. While at DBPM, she also served as the Academic Coordinator and Chief Superintendent of Exams. She then moved to ICFAI, Institute of Chatered and Financial Analysts of India as a Faculty Member for 5 years (2004 2009).

Brijitta Antonys commitment and passion for teaching has not gone unnoticed as she received several recognition awards throughout her career. In 1993 she was presented as Best Teacher Award by Lions Club, Secunderabad, India. While at ICFAI, she was awarded the Best Faculty Member Award for two consecutive years, 2007 and 2008. And lastly, in 2016, she won the Merit Honorable Award for teaching at the Missouri University of Science and Technology, Rolla, Missouri.