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T-closed sets, multivalued inverse limits, and hereditarily irreducible maps

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$\mathcal{T}$-CLOSED SETS, MULTIVALUED INVERSE LIMITS, AND HEREDITARILY IRREDUCIBLE MAPS

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ABSTRACT

This dissertation consists of three subjects: $\mathcal{T}$-closed sets, inverse limits with multivalued functions, and hereditarily irreducible maps.

For a subset A of a continuum X define $\mathcal{T}(A) = X \setminus \{x \in X$: there exists a subcontinuum K of X such that $x \in \mathrm{int}_X(K) \subset K \subset X \setminus A\}$. This function was defined by F. Burton Jones and extensively investigated in the book [20] by Sergio Macías. A subset A of a continuum X is called $\mathcal{T}$-closed set if $\mathcal{T}(A) = A$. A characterization of $\mathcal{T}$-closed set is given using generalized continua. We also give a counterexample to a hypothesis by David P. Bellamy, Leobardo Fernández and Sergio Macías about $\mathcal{T}$-closed sets if $\mathcal{T}$ is idempotent.

We construct a monotone multi-valued bound function $f : [0,1]^2 \to 2^{[0,1]^2}$ such that the inverse limit of the inverse sequence using f as the only bounding function is not locally connected. This answers James P. Kelly's question in [18]. We also give a negative answer to W. T. Ingram's question in [14]. Precisely, we give an example of an inverse limit sequence on $[0,1]$ with a single upper semi-continuous set-valued bonding function f such that $G(f^n)$ is an arc for each positive integer n but the inverse limit is not connected.

A mapping $f : X \longrightarrow Y$ from a continuum X onto a continuum Y is called hereditarily irreducible if $f(A) \subsetneq f(B)$ for any subcontinua A and B such that $A \subsetneq B$. We investigate properties of hereditarily irreducible maps between continua. Among other things, we introduce a new notion of an order of a point in a continuum that is a bit different than the notion of of an order in the classical sense. The two notions coincide for graphs, but are different in more general locally connected continua. Moreover we prove some theorems about hereditarily irreducible maps with $[0,1]$ as the domain. Thanks to those theorems we may determine if some continua admit hereditarily irreducible maps from $[0,1]$. We have both necessary conditions and sufficient conditions, so in many cases we may exclude the existence of such maps or prove their existence.

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1. INTRODUCTION

The investigation of inverse limits started in 1950's with an article by C. E. Capel [7]. Inverse limits allow people to construct complicated examples in continuum theory using quite simple spaces. Let us start with the definitions. Given a sequence of topological spaces $X_1, X_2, \ldots$ and continuous functions $f = f_1, f_2, \ldots$ such that $f_i : X_{i+1} \to X_i$ for $i \in \{1, 2, \ldots\}$ we define the inverse limit of $\boldsymbol{f}$ by $\varprojlim \boldsymbol{f} = \{x \in \Pi_{i=1}^{\infty} X_i : x_i \in f_i(x_{i+1})$ for each positive integer $i\}$.

It is known that the inverse limit of compact spaces is compact. Even if the inverse limit of connected spaces does not have to be connected; the inverse limit of continua is continuum.

The inverse limit became a very popular tool when Sibe Mardešíc and Jack Segal [22] proved in 1963 that every compact metric space can be represented as the inverse limit of polyhedra.

In 2004, William S. Mahavier generalized the notion of inverse limit to multivalued functions. In place of a continuous function $f_i : X_{i+1} \to X_i$, he considered an upper semi continuous function $F_i : X_{i+1} \to 2^{X_i}$. The subject became very popular among mathematicians. W. T. Ingram and Mahavier wrote a book [16] published in 2012 , where they considered both cases, single valued and multi-valued, functions.

Many theorems known for single valued functions are not true any more if a single valued functions are replaced by multi-valued ones. For example, for single valued functions the limit of a subsequence is homeomorphic to the limit of the original sequence. For multi-valued functions, changing the first bonding function changes the limit.

Many articles have been written that investigate connectedness of inverse limits. As mentioned before, the inverse limit of continua with single valued bonding mappings is a continuum, but it is not true any more for multi-valued functions. In [15] Ingram and Mahavier proved that the inverse limit of continua is a continuum if the value of every point is connected. Van Nall generalized this result showing that the inverse limit remains connected if the bounding function are unions of (an arbitrary family) of single valued mappings.

An interesting subject is what topological properties are preserved under the inverse limit operation. For single valued functions it is known that, for example, the dimension, hereditary unicoherence, trivial shape, etc., are preserved, while other properties are preserved if we assume addition conditions on the bonding mappings. For example the property of being a Kelley continuum is preserved if the bonding functions are confluent [6]. Similarly, Capel proved that local connectedness is preserved if the bonding functions are monotone. Therefore a natural problem is to generalized those results to multi-valued functions. The first obstacle is to generalize the definition of monotone, confluent, weakly confluent, etc., to multi-valued settings. This was solved by James Kelly [18]. He defined a multi-valued function $F_i : X \to 2^Y$ to be monotone (confluent, weakly confluent, respectively) if the projection from the graph of F in $X \times Y$ to both factor spaces X and Y are such. Then he proved that the inverse limit of $[0, 1]$ with monotone multi-valued bonding functions is a locally connected continuum. Looking at his proof one can generalize his result for dendrites (i.e locally connected continua that contain no simple closed curves) in place of $[0, 1]$. He posed a question in his article if the theorem can be generalized further to arbitrary locally connected continua as in the case of single valued functions. We answered his question in the negative by showing an example of monotone multi-valued functions between squares $[0, 1]^2$ with a non-locally connected inverse limit.

As mentioned before, an important part of research concerning multi-valued inverse limits concerns their connectedness. One of the attempts was to verify if the inverse limit has to be connected if the graphs of functions and their compositions are connected. Recently, Iztok Banič and Judy Kennedy have investigated some conditions under which the generalized inverse limits on [0, 1] with a single surjective upper semicontinuous bonding function whose graph is an arc [1]. In their paper, they showed that if $f : [0, 1] \to 2^{[0,1]}$ is a surjective upper semicontinuous function with graph $G(f)$ being an arc, then $\varprojlim \boldsymbol{f}$ is not totally disconnected and asked whether having an upper semi-continuous function whose graph $G(f)$ is an arc and $G(f^n)$ is connected for each positive number n will produce a connected inverse limit. Their question was answered in the negative by Ingram in [14]. In that paper, Ingram posed a question: if $f : [0, 1] \to 2^{[0,1]}$ is an upper semi-continuous function such that $G(f^n)$ is an arc for each positive number n, is $\varprojlim \boldsymbol{f}$ connected? We have answered his question in the negative.

The set function $\mathcal{T}$ was defined by F. Burton Jones [17] in order to study aposyndetic continua. Since then the properties of this function have been studied by several topologists to investigate properties of continua. Given a continuum X, the set function $\mathcal{T}$ is a set-valued function from the power set $\mathcal{P}(X)$ into itself. If A is a subset of X, then $\mathcal{T}(A) = X \setminus \{x \in X :$ there exists a subcontinuum W of X such that $x \in \mathrm{int}_X(W) \subset W \subset X \setminus A\}$. Thus $\mathcal{T}$ maps a subset of X onto a closed subset of X. It is an operator which can be used to describe continua. For instance, a continuum X is indecomposable continuum if only if $\mathcal{T}(A) = X$ for any subset A of X. It is known that $\mathcal{T}(A)$ is a continuum for any subcontinuum A of X [9] (Corollary 1.1, p. 115).

A subset A of X is a $\mathcal{T}$-closed set provided that $\mathcal{T}(A) = A$. This type of set has been used to study decompositions of continua [12], [29]. Many of properties of $\mathcal{T}$-closed set and its family was presented by David P. Bellamy, Leobardo Fernández and Sergio Macías

[3]. They gave two necessary conditions for a set to be $\mathcal{T}$-closed set and proved that these conditions are sufficient for the class of continua with the property of Kelley. We give a characterization of a $\mathcal{T}$-closed sets using the notion of an exhaustive σ-continuum.

In [3] the authers proved that if X is a continuum and $\mathcal{T}(A) = A$, then every component of $X \setminus A$ is open and continuumwise connected. They also showed that the converse of their theorem is not true and asked if the converse is true under an additional assumption that T is idempotent on X. We gave a negative answer to their question by constructinga respective counterexample.

A map $f : X \to Y$ between continua X and Y is called a hereditarily irreducible map if for any two subcontinua A and B of X with $A \subsetneq B$ we have $f(A) \subsetneq f(B)$. As observed by S. B. Naddler in [24] (1.212.3) f is a hereditarily irreducible map if and only if the induced mapping $C(f) : C(X) \to C(Y)$ is light; i.e. the preimages of points are zero-dimensional. Hereditarily irreducible maps with domain $[0, 1]$ were called arcwise increasing. An important article about arcwise increasing maps was written by B. Espinoza and E. Matsuhashi [11]. In this dissertation we generalize their results to more general domains and we answer some of their problems.

In Section 3 of the article we introduce a new notion of an order of a point in a continuum that is a bit different than the notion of an order in the classical sense. The two notions coincide for graphs, but are different in more general locally connected continua.

In Section 4 we prove some theorems about hereditarily irreducible maps with $[0, 1]$ as the domain. Thanks to those theorems we may determine if some continua admit hereditarily irreducible maps from $[0, 1]$. We have both necessary conditions and sufficient conditions, so in many cases we may exclude the existence of such maps or prove their existence.

Section 6 is devoted to maps onto dendrites admitting hereditarily irreducible maps from graphs.

PAPER

I. $\mathcal{T}$-CLOSED SETS

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ABSTRACT

A subset A of a continuum X is called a $\mathcal{T}$-closed set if $\mathcal{T}(A) = A$, where $\mathcal{T}$ denotes the Jones $\mathcal{T}$- function. We give a characterization of $\mathcal{T}$-closed sets and then we show a counterexample to a hypothesis by David P. Bellamy, Leobardo Fernández and Sergio Macías about $\mathcal{T}$-closed sets if $\mathcal{T}$ is idempotent on X.

Keywords: continuum, continuumwise connected, exhaustive σ-continuum, the set function $\mathcal{T}$, $\mathcal{T}$-closed sets.

1. INTRODUCTION

The set function $\mathcal{T}$ was defined by F. Burton Jones in [4]. Since then it has been used by several topologists to study decompositions of continua, especially homogeneous continua (see [3] and [7]). One can find information about the set function $\mathcal{T}$ in the book [6], Chapters 3 and 5. An interesting work has been done in [1], where the authors study the family of $\mathcal{T}$-closed sets. Here we generalize one of the theorems from that article as well as give a negative answer to one of the questions. The generalization we mentioned uses the notion of an exhaustive σ-continuum, a notion defined by Tomás Fernández-Bayort and Antonio Quintero. We provide several conditions for a generalized continuum that are equivalent to being an exhaustive σ-continuum.

2. DEFINITIONS AND NOTATION

A *continuum* is a non-empty, compact, connected, metric space and a *map* is a continuous function. A *generalized continuum* is a locally compact, connected metric space. If A is a subset of a continuum X, the *interior* of A is denoted by $\text{int}_X(A)$ and the *closure* of A is denoted by $\text{cl}_X(A)$. If A is an arc in X, we denote by $E(A)$ the *set of endpoints* of A, and we say that A is a *free arc* in X if $A \setminus E(A)$ is open in X. We use A_{ab} to denote an arc such that $E(A) = \{a, b\}$. A metric space X is *continuumwise connected* provided that for any x_1, x_2 in X, there is a subcontinuum K of X such that $x_1, x_2 \in K$. It is *arcwise connected* if we can choose K to be an arc. Given a continuum X, we define the set function $\mathcal{T}$ as follows: If $A \subset X$, then $\mathcal{T}(A) = X \setminus \{x \in X :$ there exists a subcontinuum K of X such that $x \in \text{int}_X(K) \subset K \subset X \setminus A\}$. A subset A of X is a $\mathcal{T}$*-closed set* provided that $\mathcal{T}(A) = A$. We say that $\mathcal{T}$ is *idempotent* on X provided that $\mathcal{T}^2(A) = \mathcal{T}(A)$ for any subset A of X.

3. GENERALIZED CONTINUA

It is known that for every generalized continuum X there is an exhaustive sequence of compacta i.e. a sequence $\{E_n\}_{n=1}^{\infty}$ of compacta such that $E_n \subset \text{int}_X(E_{n+1})$ and $X = \bigcup_{n=1}^{\infty} E_n$. A generalized continuum is called *an exhaustive σ-continuum* if there is an exhaustive sequence of continua in X. Notice that if X is the sin(1/x)-continuum with a point on the limit bar deleted, then X is a generalized continuum that is not an exhaustive σ-continuum. The notion of an exhaustive σ-continuum was defined by Tomás Fernández-Bayort and

Antonio Quintero. The second named author has learned about the notion from the doctoral dissertation of Tomás Fernández-Bayort directed by Antonio Quintero. The characterization of $\mathcal{T}$-closed sets that we are going to prove use the notion of an exhaustive σ-continuum, so we want to familiarize the reader with this notion.

The following Theorem was shown by Tomás Fernández-Bayort and Antonio Quintero. Since we have not seen it published, we provide its proof for completeness. The equivalence (5) $\Longleftrightarrow$ (6) is shown in Theorem 3.1 of [2].

Theorem 3.1. *Let X be a generalized continuum. Then the following conditions are equivalent:*

1. *X is an exhaustive σ-continuum;*
2. *For any $p, q \in X$ there is a subcontinuum K of X such that $p, q \in \operatorname{int}(K)$;*
3. *X is continuumwise connected and for any p in X, there is a subcontinuum K of X such that $p \in \operatorname{int}(K)$;*
4. *For any p in X there is a subcontinuum K of X such that $p \in \operatorname{int}(K)$;*
5. *For any compact subset K of X, there is a subcontinuum W of X such that $K \subset \operatorname{int}(W)$;*
6. *For any compact subset K of X, there is a subcontinuum W of X such that $K \subset W$;*
7. *The hyperspace $K(X)$ of all compact subsets of X is arcwise connected;*
8. *The hyperspace $K(X)$ of all compact subsets of X is continuumwise connected.*

Proof. The implications (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4), (5) $\Longrightarrow$ (6), and (6) $\Longrightarrow$ (8) follow from the definitions. Now we will show that (4) $\Longrightarrow$ (3); this implication is used in the next one (4) $\Longrightarrow$ (5). For every point $p \in X$ denote by $C(p)$ the composant of X that contains the point p, i.e. $C(p)$ is the union of all continua that contain the point p. Note that (4) implies that every composant is an open set. Since composants form a decomposition of X, they are closed as well. By connectedness of X we have only one composant, i.e. X is continuumwise connected. To see that (4) $\Longrightarrow$ (5), let K be a compact subset of X. For every point $x \in K$, let W_x be a continuum such that $x \in \mathrm{int}(W_x)$. Then the collection $\{\mathrm{int}(W_x) : x \in K\}$ is an open cover of K, so there are points $x_1, x_2, \ldots x_n$ such that $K \subset \mathrm{int}(W_{x_1}) \cup \cdots \cup \mathrm{int}(W_{x_n})$. Choose a point $a \in X$. Since (4) $\Longrightarrow$ (3), X is continuumwise connected, so there are continua C_n that join the point a and x_n. Then $W_{x_1} \cup C_1 \cup \cdots \cup W_{x_n} \cup C_n$ is the required continuum that contains K in its interior. Now we will show that (5) $\Longrightarrow$ (1). We will proceed by induction. Let $E_0, E_1, \ldots$ be an exhaustive sequence of compacta in X. Define W_0 as a continuum satisfying $E_0 \subset W_0$. Suppose that $W_0, W_1, \ldots W_n$ are continua such that $W_i \subset \mathrm{int}(W_{i+1})$ for $i \in \{0, 1, \ldots, n-1\}$. Define W_{n+1} as a continuum that contains $E_{n+1} \cup W_n$. One can verify that $W_0, W_1, \ldots$ is an exhaustive sequence of continua. Now we will show that (5) $\Longrightarrow$ (6). Let K_1 and K_2 be two compact subsets of X, and let W be a continuum that contains $K_1 \cup K_2$. By [5], Theorem 15.3, p. 120 there are order arcs $\mathcal{A}$ from K_1 to W and $\mathcal{B}$ from K_2 to W. Then $\mathcal{A} \cup \mathcal{B}$ contains an arc that joins K_1 and K_2. Finally, we will show that (8) $\Longrightarrow$ (5). Let K be a compact subset of X and choose a point a in X. Since $K(X)$ is continuumwise connected, there is a continuum $\mathcal{A} \in K(X)$ containing K and $\{p\}$; by [5], 15.9 (2) the union $\bigcup \mathcal{A}$ is a continuum that contains K. □

4. CHARACTERIZATION

In [1] the authors show the following Theorem (Theorem 4.3 of [1]).

Theorem 4.1. *If X is a continuum, A is a subset of X, and $\mathcal{T}(A) = A$, then every component of $X \setminus A$ is open and continuumwise connected.*

Here , in Theorem 4.3 we give a stronger condition that in fact is equivalent to the condition $\mathcal{T}(A) = A$. To prove it we will need the following Theorem. Its proof can be found in [6], page 163. Recall that $\mathcal{T}$ is idempotent on X if $\mathcal{T}^2(A) = \mathcal{T}(A)$ for all subset A of X.

Theorem 4.2. *Let X be a continuum. Then $\mathcal{T}$ is idempotent on X if only if for each subcontinuum K of X and for each point $p \in \mathrm{int}_X(K)$, there exists a subcontinuum W of X such that $x \in \mathrm{int}_X(W) \subset W \subset \mathrm{int}_X(K)$.*

The following Theorem gives a characterization of $\mathcal{T}$-closed sets. It generalizes Theorem 4.1.

Theorem 4.3. *For a continuum X and a closed subset A of X, $\mathcal{T}(A) = A$ if only if every component of $X \setminus A$ is open in X and it is an exhaustive σ-continuum.*

Proof. Let A be a closed subset of X and suppose that $\mathcal{T}(A) = A$. Let C be a component of $X \setminus A$. By Theorem 4.1, C is an open subset of X. We will show that C is an exhaustive σ-continuum. Since $\mathcal{T}(A) = A$ and $A \cap C = \emptyset$, then for any x in C there is a continuum neighborhood Y_x such that $Y_x \cap A = \emptyset$. This shows that condition (4) of Theorem 3.1 is satisfied, so C is an exhaustive σ-continuum. The other implication follows from the definition of the set function $\mathcal{T}$. $\square$

5. COUNTEREXAMPLE

In [1] the authors proved Theorem 4.1 and then they asked if the converse is true if we additionally assume that the set function $\mathcal{T}$ is idempotent on X, (Question 4.6 in [1]). Here we provide a negative answer to that question.

Example 5.1. *Let X be a continuum as shown in Figure 1. The continua X_i shown in the right part of Figure 1, are disjoint copies of $\sin(1/x)$-continuum converging to the arc A_{ab}. The points a_i are endpoints of X_i and they are converging to the point a. The arcs $A_{a_i a_{i+1}}$ are free arcs in X converging to $\{a\}$.*

Let $A = \{p\}$, then $X \setminus A$ is an open subset of X and it is continuumwise connected. We will use Theorem 4.2 to show that $\mathcal{T}$ is idempotent on X. Since $X_i \setminus \{a_i\}$ is an open component of X and X is locally connected at each point of $A_{a_i a_{i+1}}$, then it is enough to

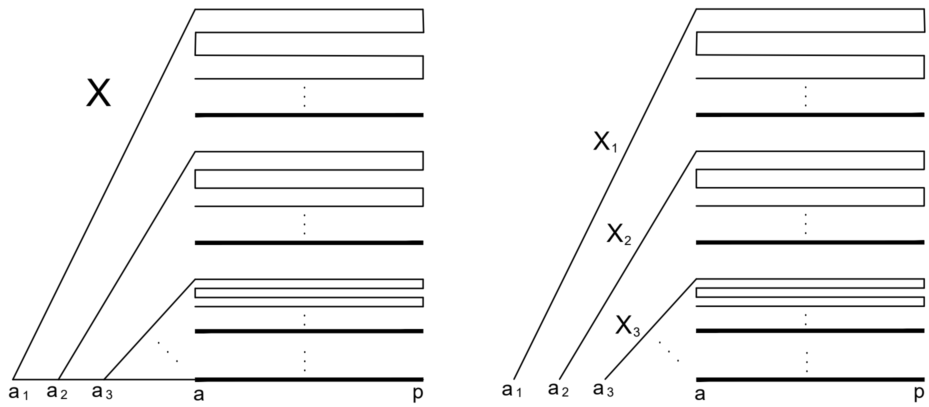

Figure 1. The example.

show that Theorem 4.2 *holds for any point in A_{ap}. Let b be a point in A_{ap}, and let K be a subcontinuum of X such that $b \in \mathrm{int}_X(K)$. We will show that there exists a subcontinuum W of X such that $b \in \mathrm{int}_X(W) \subset W \subset \mathrm{int}_X(K)$. Since $b \in A_{ap}$, $\{X_i\}_{i=1}^{\infty}$ converges to A_{ap}, and K is a subcontinuum of X containing b in its interior, then there is an integers m such that $X_i \subset K$ for each $i \geq m$. Since K is a subcontinuum of X, then $A_{ap} \subset K$ and $A_{a_i a_{i+1}} \subset K$ for each $i \geq m$. Let $W = A_{ap} \cup (\bigcup_{i=m+1}^{\infty} A_{a_i a_{i+1}}) \cup (\bigcup_{i=m+1}^{\infty} X_i)$, then W is a subcontinuum of X and $p \in \mathrm{int}_X(W) \subset W \subset \mathrm{int}_X(K)$, therefore, $\mathcal{T}$ is idempotent on A_{ap}, and hence, it is idempotent on X. Since any subcontinuum of X containing a in its interior must contain p, then $\mathcal{T}(A) \neq A$. So A is not $\mathcal{T}$-closed set.*

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II. LOCALLY CONNECTEDNESS AND INVERSE LIMITS

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ABSTRACT

We show an example of an inverse sequence of squares $[0,1]^2$ with monotone multi-valued bounding functions whose inverse limit is not locally connected. This answers a question posed by James Kelly.

Keywords: continuum, locally connected, monotone, upper semi-continuous function, inverse limits.

1. INTRODUCTION

In [2] James Kelly introduced a notion of monotone upper semi-continuous multivalued function. This was in response to a request by W. J. Charatonik to generalize the notion of a monotone (single valued) map between continua. J. Kelly showed that the inverse limit of arcs with monotone multivalued functions is locally connected (see [2] Theorem 3.12) and he asked if an analogous result would be true if we replace the arcs as the factor spaces by any locally connected continua (see [2], Question 1) . This an important question because it would generalize an old result by C. E. Capel stating that the inverse limit of locally connected continua with monotone bounding (single valued) maps is locally connected, see [1]. In this article we construct an example showing that the answer to Kelly's question is negative. We construct an inverse limit whose factor spaces are $[0,1]^2$ with monotone bounding functions and such that the limit is not locally connected. In our example all but the first function are identities on $[0,1]^2$, so one can consider the inverse limit as the Mahavier product with only two factor spaces.

2. DEFINITIONS AND NOTATION

We will use the terminology from the Ingram and Mahavier book [4]. One can also look at the book [3] for information about inverse limits. A continuum means a non-empty, compact, connected, metric space.

A function $f : X \to 2^Y$ from a continuum X into the hyperspace 2^Y is said to be upper semi-continuous if for every $x_0 \in X$ and every open subset U of Y such that $f(x_0) \subset U$, $\{x \in X : f(x) \subset U\}$ is an open subset of X. The graph of the function $f : X \to 2^Y$ is $G(f) = \{(x, y) : y \in f(x)\}$, and for a subset A of X, we define $f(A) = \{y \in Y : y \in f(x) \text{ for some } x \in A\}$. A continuous function $f : X \to Y$ between continua X and Y is called monotone, if for every $y \in Y$, the preimage $f^{-1}(y)$ is a non-empty continuum. In particular monotone functions are surjective.

If $f : X \to 2^Y$ is an upper semi-continuous from a continuum X into the hyperspace 2^Y, then

a) f is said to be surjective, if $f(X) = Y$.

b) f is said to be monotone, if the projection map $\pi_x : G(f) \to X$ and $\pi_y : G(f) \to Y$ are monotone.

3. EXAMPLES

The proof of the following proposition can be found in [2].

Proposition 3.1. *Suppose X and Y are compact Hausdorff spaces, and $f : X \to 2^Y$ is an upper semi-continuous function. Then f is a monotone if only if for each $x \in X$ and each $y \in Y$, $f(x)$ and $f^{-1}(y)$ are connected.*

The following example plays a key role in the construction of an inverse sequence with non-locally connected inverse limit.

Example 3.2. *There is a monotone upper semi-continuous function $f : [0,1]^2 \to 2^{[0,1]^2}$ with all the values $f(x,y)$ and $f^{-1}(a,b)$ being locally connected continua such that $G(f)$ is not locally connected.*

Define a function $f : [0,1]^2 \to 2^{[0,1]^2}$ by

$$f(x,y) = \begin{cases} \{(x, \sin^2(1/x)\} & if\ x > 0, y < 1; \\ \{0\} \times [0,1] & if\ x = 0, y < 1; \\ [0,1]^2 & if\ y = 1. \end{cases}$$

The function f is surjective thanks to the third condition. To see that f is upper semi-continuous, it is enough to observe that the graph of f is a closed subset of $[0,1]^2 \times [0,1]^2$ *(see [4], Theorem 105). The image of a point is either one-point set, an interval, or a square. So, it is locally connected. To verify that f is monotone, we will use Proposition* 3.1. *The images $f(x,y)$ are connected by the definition of f. To show that $f^{-1}(a,b)$ is locally connected for any $(a,b) \in [0,1]^2$, we need to consider the following cases.*

Case 1. If $a \neq 0$ and $b = \sin^2(1/a)$, then $f^{-1}(a,b) = \{a\} \times [0,1) \cup [0,1] \times \{1\}$.

Case 2. If $a \neq 0$ and $b \neq \sin^2(1/a)$, then $f^{-1}(a,b) = [0,1] \times \{1\}$.

Case 3. If $a = 0$, then $f^{-1}(a,b) = \{0\} \times [0,1) \cup [0,1] \times \{1\}$.

In all cases the preimage $f^{-1}(a,b)$ *is connected and locally connected. Therefore* f *is monotone. We will show that the graph* $G(f)$ *is not locally connected. Suppose the contrary, then there exist a continuum* K *in* $G(f)$ *such that* $((0,0),(0,0)) \in \mathrm{int}(K)$ *and* $K \subset [0,1/2]^4$. *Let* $\pi_Y : G(f) \to [0,1]^2$ *denote the projection of* $G(f)$ *onto the range square. Then,* $\pi_Y(K)$ *is a continuum in* $[0,1/2]^2$ *such that* $(0,0) \in \mathrm{int}\,\pi_Y(K)$ *and* $\pi_Y(K)$ *is contained in* $\{0\} \times [0,1/2] \cup \{(x, \sin^2(1/x)) : x \in (0,1/2]\}$. *Since there is no continuum in* $[0,1/2]^2$ *satisfying the above conditions, we have a contradiction.*

Example 3.3. *There is an inverse sequence of the squares* $[0,1]^2$ *with monotone upper semi-continuous bonding functions such that both images and preimages of points are locally connected continua and such that the inverse limit is not locally connected. It is enough to consider the sequence* $[0,1]^2 \xleftarrow{f} [0,1]^2 \xleftarrow{id} [0,1]^2 \xleftarrow{id} [0,1]^2 \xleftarrow{id} \cdots$, *where* f *is the function from Example* 3.2. *Then the inverse limit* $\varprojlim\{[0,1]^2, f_i\}$ *is homeomorphic to the graph* $G(f)$, *and thus it is not locally connected.*

Example 3.4. *The inverse limit of the inverse sequence* $[0,1]^2 \xleftarrow{f} [0,1]^2 \xleftarrow{f} [0,1]^2 \xleftarrow{f} [0,1]^2 \xleftarrow{f} \cdots$ *is also not locally connected. To see this, let* $X = \varprojlim\{[0,1]^2, f\}$ *and let* $\pi_{1,2} : X \to [0,1]^2 \times [0,1]^2$ *be the projection onto the first two coordinates. Then* $\pi_{1,2}(X) = G(f)$, *so it is not locally connected and consequently* X *is not locally connected.*

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III. NONCONNECTED INVERSE LIMITS

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ABSTRACT

In this paper we give an example of an inverse limit sequence on [0, 1] with a single upper semi-continuous set-valued bonding function f such that $G(f^n)$ is an arc for each positive integer n, but the inverse limit is not connected. This answers a question posed by W. T. Ingram.

Keywords: generalized inverse limits, set valued functions.

1. INTRODUCTION

In [1] Iztok Banič and Judy Kennedy pose a question: if $f : [0,1] \to 2^{[0,1]}$ is an upper semi-continuous function such that $G(f)$ is an arc and $G(f^n)$ is connected for each positive number n, is $\varprojlim \boldsymbol{f}$ connected? In [2] W. T. Ingram answered their question in the negative (see Example 1) and asked whether f produces connected inverse limit in case $G(f^n)$ is an arc for each positive number n. In this paper we give a negative answer to this question.

2. DEFINITIONS AND NOTATION

A *continuum* is a non-empty compact connected metric space. If X is a continuum, $2^X = \{A \subseteq X : A \text{ is non-empty closed in } X\}$ denotes the *hyperspace* of X. If X and Y are continua, a function $f : X \to 2^Y$ is said to be *upper semi-continuous* if for every $x_0 \in X$ and every open subset U of Y such that $f(x_0) \subset U$, we have $\{x \in X : f(x) \subset U\}$ is an open subset of X. The *graph* of the function $f : X \to 2^Y$ is $G(f) = \{(x, y) : y \in f(x)\}$, and for a subset A of X, we define $f(A) = \{y \in Y : y \in f(x) \text{ for some } x \in A\}$. If $f : X \to 2^X$, then we denote the composition $f \circ f$ by f^2 and, for nay $n > 2$, $f^n = f^{n-1} \circ f$. Let $\boldsymbol{X}$ be a sequence $\{X_i\}_{i=1}^{\infty}$ of continua, and let $\boldsymbol{f}$ be a sequence $\{f_i\}_{i=1}^{\infty}$ of functions such the $f_n : X_{i+1} \to 2^{X_i}$, then the subspace $\varprojlim \boldsymbol{f} = \{x \in \Pi_{i=1}^{\infty} X_i : x_i \in f_i(x_{i+1}) \text{ for each postive integer } i\}$ of the product topology $\Pi_{i=1}^{\infty} X_i$ is called the *inverse limit* of $\boldsymbol{f}$. In this paper we will use inverse limits with a single upper semi-continuous set-valued bounding function. More information about inverse limits can be found in [3] and [4].

The following Lemma is known (see [6] or [4], Theorem 116, page 85).

Lemma 2.1. *Suppose X is a Hausdorff continuum, $f : X \to 2^X$ is an upper semi-continuous set-valued function, and, for each n, G_n is the set of all $(x_1, x_2, ..., x_n) \in \Pi_{i=1}^{n} X_i$ such that $x_i \in f(x_{i+1})$ for $i = 1, ..., n_1$. Then $\varprojlim \boldsymbol{f}$ is connected if and only if G_n is connected for each n.*

3. EXAMPLES

The following example by W. T. Ingram answers Iztok Banič and Judy Kennedy question. We recall it here for completeness.

Example 3.1. *Let $f : [0, 1] \to 2^{[0,1]}$ be the function whose graph consists of five straight line intervals, one from $(1/4, 1/4)$ to $(0, 0)$, one from $(0, 0)$ to $(1/2, 0)$, one from $(1/2, 0)$ to $(1, 1/2)$, one from $(1, 1/2)$ to $(1, 1)$, and one from $(1, 1)$ to $(3/4, 3/4)$ (see Figure 1). Then G(f) is an arc and $G(f^n)$ is connected for each positive number n, but $\varprojlim \boldsymbol{f}$ is not connected.*

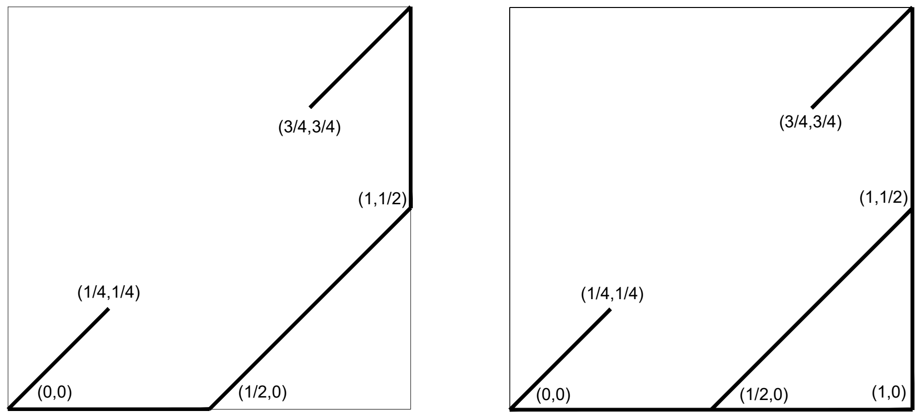

Figure 1. The graph of the bounding function f (left) and f^2 (right).

Example 3.2. *Let $f : [0,1] \to 2^{[0,1]}$ be a function defined by $f(0) = [0,1]$, $f(x) = \{x, 1-x\}$ for $0 < x < 1/4$, $f(x) = \{1/4, 3/4\}$ for $1/4 \leq x \leq 3/4$, and $f(x) = \{x\}$ if $3/4 < x \leq 1$. Then $G(f^n) = G(f)$, so $G(f^n)$ is an arc for any positive integer n (see Figure 2), but $\varprojlim \boldsymbol{f}$ is not connected.*

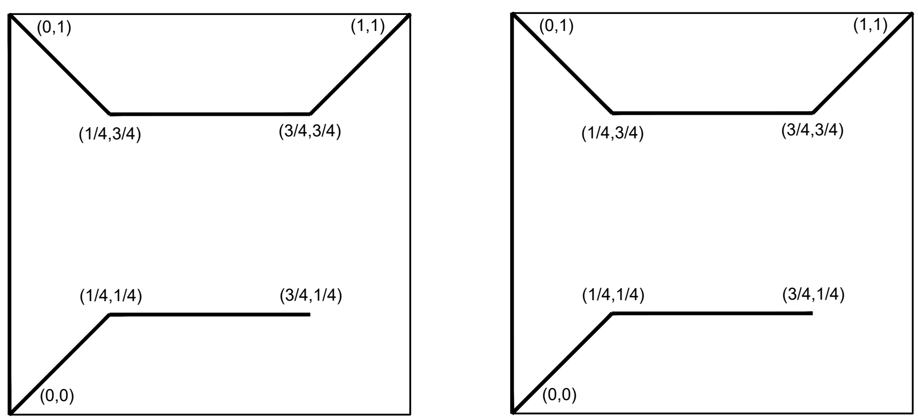

Figure 2. Graphs of the bounding functions f and f^n.

Proof. It is not hard to verify that $G(f^n) = G(f)$, so $G(f^n)$ is an arc for any positive integer n. To show that $\varprojlim \boldsymbol{f}$ is not connected, we will use Lemma 2.1. Let $A = \{1/4\} \times \{3/4\} \times \{1/4\} \times [1/4, 3/4]$, then A is a closed subset of G_4. We will show that A is a clopen subset of G_4. Let $0 < \epsilon < 1/4$ and put $U = (1/4-\epsilon, 1/4+\epsilon) \cup (3/4-\epsilon, 3/4+\epsilon) \cup (1/4-\epsilon, 1/4+$

$\epsilon) \cup (1/4 - \epsilon, 3/4 + \epsilon)$, and let $(x_1, x_2, x_3, x_4) \in G_4 \cap U$.

Case 1: If $1/4 - \epsilon < x_4 < 1/4$, then $x_3 \in f(x_4) = \{x_4, 1 - x_4\}$. Since $x_3 \in (1/4 - \epsilon, 1/4 + \epsilon)$, it follows that $x_3 = x_4$. So, $x_2 \in f(x_3) = f(x_4) = \{x_4, 1 - x_4\}$, but $x_2 \in (3/4 - \epsilon, 3/4 + \epsilon)$; therefore, $x_2 = 1 - x_4$ and $x_1 \in f(x_2) = \{1 - x_4\}$. But this contradicts $x_1 \in (1/4 - \epsilon, 1/4 + \epsilon)$. So, $x_4 \notin (1/4 - \epsilon, 1/4)$.

Case 2: If $3/4 < x_4 < 1/4 + \epsilon$, then $x_3 \in f(x_4) = \{x_4\}$, but this contradicts $x_3 \in (1/4 - \epsilon, 1/4 + \epsilon)$. So, $x_4 \notin (3/4, 3/4 + \epsilon)$. Since for any $x \in [1/4, 3/4]$ we have $f(x) = \{1/4, 3/4\}$, we can conclude that $G_4 \cap U = A$. Therefore, A is a clopen subset of G_3. Thus, G_4 is not connected and by Lemma 2.1, $\varprojlim \boldsymbol{f}$ is not connected. $\square$

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IV. HEREDITARILY IRREDUCIBLE MAPS

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ABSTRACT

A map $f : X \longrightarrow Y$ from a continuum X onto a continuum Y is said to be *hereditarily irreducible* map, if $f(A) \subsetneq f(B)$ for any subcontinua A and B such that $A \subsetneq B$. We show that if X is a locally connected continuum contains no free arcs, then there is a hereditarily irreducible map from a graph G onto X. We investigate properties of hereditarily irreducible maps between continua.

Keywords: continuum, hereditarily irreducible map, order of a point.

1. INTRODUCTION

The notion of hereditarily irreducible map generalizes the notion of an arcwise increasing maps. In fact the two notions coincide if the domain is an arc. Therefore our investigation of hereditarily irreducible map is an extension of the work done by B. Espinoza and E. Matsuhashi [4]. We generalize some of their theorems to more general settings and we answer problems from their article.

2. PRELIMINARIES

In this section we introduce notions that will be used throughout this paper. A space X is called a *continuum* if it is non-empty compact connected metric space. A subset of a continuum X which is itself a continuum is called a *subcontinuum* of X. An *arc* is a homeomorphic image of the closed unit interval $[0,1]$ and a *simple closed curve* is a homeomorphic image of a circle. If X is a continuum and $h : [0,1] \to X$ is a homeomorphism onto its image, we call $h(0) = a$ and $h(1) = b$ the *endpoints* of the arc given by h and denote this arc as A_{ab}. The arc A_{ab} is a *free arc* in X provided that $A_{ab} \setminus \{a,b\}$ is an open subset of X. If h is a homeomorphism on $(0,1)$ and $h(0) = h(1)$, we call the image of h a *loop* in X. A loop is a *free loop* in X if $h([0,1]) \setminus \{h(0)\}$ is an open subset of X. If A is a subset of a continuum X, then the *interior* of A in X is denoted by $\mathrm{int}_X(A)$, the *closure* of A in X is denoted by $\mathrm{cl}_X(A)$, and the *cardinality* of the set A is denoted by $\mathrm{card}(A)$. If X is a continuum with metric d, $A \subseteq X$, $x \in X$, and $\delta > 0$, then $\mathrm{diam}(A)$ denotes the diameter of A, $d(x,A)$ denotes the infimum of the set $\{d(x,a) : a \in A\}$, and $B_\delta(x)$ denotes the set $\{y \in X : d(x,y) < \delta\}$. By a *map* we mean a continuous function. If $f : X \to Y$ is a map, and $A \subseteq X$, then $f|A$ denotes the restriction of f to A. A continuum X is said to be *dendrite* provided that it is locally connected and contains no simple closed curve. A map $f : X \longrightarrow Y$ from a continuum X onto a continuum Y is said to be *hereditarily irreducible* map, if $f(A) \subsetneq f(B)$ for any subcontinua A and B such that $A \subsetneq B$. A map f from a continuum X onto a continuum Y is said to be *open* if it maps every open subset of X onto an open subset of Y.

3. ORDER OF A POINT AND FUNCTIONS BETWEEN GRAPHS

In this section we introduce a new concept of an order of a point. It generalizes the concept of order in the classical sense. The two concepts coincide for graphs, but are different even for locally connected continua.

Let us recall the classical definition of an order.

Definition 3.1. *Let X be a continuum and p be a point in X, and let α be a cardinal number. We say that* $\mathrm{cord}_X(p) \geq \alpha$ *if there are arcs A_γ for $0 \leq \gamma \leq \alpha$ in X such that p is an endpoint of A_γ and $A_\gamma \cap A_\delta = \{p\}$ for $0 \leq \gamma, \delta \leq \alpha$, $\gamma \neq \delta$. Finally, we let* $\mathrm{cord}_X(p) = \alpha$ *if* $\mathrm{cord}_X(p) \geq \alpha$ *and* $\mathrm{cord}_X(p) \geq \beta$ *is not true for any $\beta > \alpha$. The point p is called an endpoint of X if* $\mathrm{cord}_X(p) = 1$ *and a ramification point of X if* $\mathrm{cord}_X(p) \geq 3$*; the set of all endpoints of X is denoted by $E(X)$ and the set of all ramification points of X by $R(X)$.*

Our new definition is very similar to the classical one.

Definition 3.2. *Let X be a continuum and p be a point in X, and let α be a cardinal number. We say that* $\mathrm{ord}_X(p) \geq \alpha$ *if there are arcs A_γ for $0 \leq \gamma \leq \alpha$ in X such that p is an endpoint of A_γ and* $\mathrm{int}(A_\gamma) \cap \mathrm{int}(A_\delta) = \emptyset$ *for $0 \leq \gamma, \delta \leq \alpha$, $\gamma \neq \delta$. Finally, we let* $\mathrm{ord}_X(p) = \alpha$ *if* $\mathrm{ord}_X(p) \geq \alpha$ *and* $\mathrm{ord}_X(p) \geq \beta$ *is not true for any $\beta > \alpha$.*

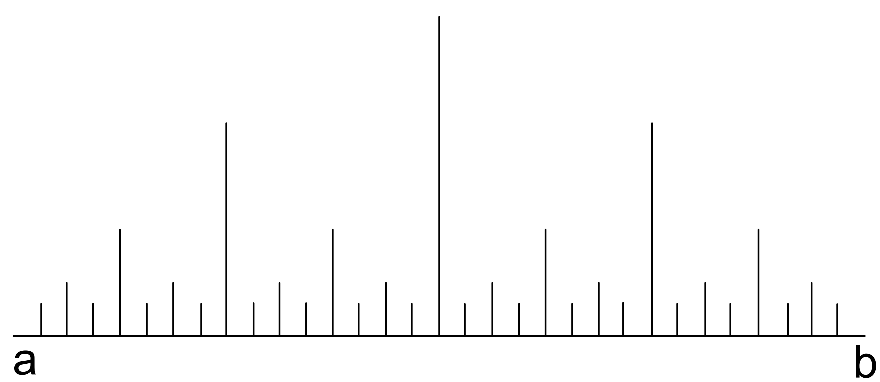

Figure 1. $\mathrm{cord}_D(a) = 1$ and $\mathrm{ord}_D(a) = \mathfrak{c}$.

Let us show an example where the two notions of order do not coincide.

Example 3.3. *Consider an arc A_{ab} and a dense countable subset Q of A_{ab}. At each point of Q erect an arc such that diameters of those arcs converge to* 0 *(see Figure 1). The union of A_{ab} and the erected arcs is a dendrite D. For the point $a \in D$ we have* $\mathrm{cord}_D(a) = 1$ *and* $\mathrm{ord}_D(a) = \mathfrak{c}$.

The following lemma plays a crucial role in proving our next Theorem.

Lemma 3.4. *If $f : X \longrightarrow Y$ is a hereditarily irreducible map from a locally connected X onto a locally connected continuum Y and A, B are arcs satisfying* $\mathrm{int}_X(A) \cap \mathrm{int}_X(B) = \emptyset$, *then* $\mathrm{int}_Y(f(A)) \cap \mathrm{int}_Y(f(B)) = \emptyset$.

Proof. Suppose on the contrary that $\mathrm{int}_X(f(A)) \cap \mathrm{int}_X(f(B)) \neq \emptyset$. Denote by U the set $\mathrm{int}_Y(f(A)) \cap \mathrm{int}_Y(f(B))$. Choose two points p, q in $(A \cap f^{-1}(U)) \setminus B$ such that the arc A_{pq} is contained in $A \cap f^{-1}(U)$. Let R be the shortest arc in X intersecting $\{p, q\}$ and B. Observe that only one point of $\{p, q\}$ is in R. Without loss of generality, assume $q \in R$. Then we have $(R \cup B) \subsetneq (A_{pq} \cup R \cup B)$ and $f(R \cup B) = f(A_{pq} \cup R \cup B)$ contrary to our assumption f was hereditarily irreducible. $\square$

Theorem 3.5. *If $f : X \to Y$ is a hereditarily irreducible map from a locally connected X onto a locally connected continuum Y, then* $\mathrm{ord}_X(x) \leq ord_Y(f(x))$ *for all x in X.*

Proof. Let $x \in X$, and let $\mathrm{ord}_X(x) = \alpha$. By our assumption, there are arcs A_γ for $0 \leq \gamma \leq \alpha$ in X such that $x \in E(A_\gamma)$ for each γ, and $\mathrm{int}(A_\gamma) \cap \mathrm{int}(A_\delta) = \emptyset$ for $0 \leq \gamma, \delta \leq \alpha$, $\gamma \neq \delta$. We have, $f(A_\gamma)$ is a locally connected continuum for all γ, therefore, for each γ there is an arc B_γ in $f(A_\gamma)$ such that $f(x) \in E(B_\gamma)$. By Lemma 3.4, $\mathrm{int}(B_\gamma) \cap \mathrm{int}(B_\delta) = \emptyset$ for $\gamma \neq \delta$. Thus, $\mathrm{ord}_Y(f(x)) \geq \alpha$, and hence, $\mathrm{ord}_X(x) \leq ord_Y(f(x))$. $\square$

Theorem 3.6. *If $f : X \to Y$ is a hereditarily irreducible map from a locally continuum X onto a continuum Y and y is a point in Y, then*

$$\mathrm{ord}_Y(y) \geq \sum_{x \in f^{-1}(y)} \mathrm{ord}_X(x).$$

Proof. Let α be a cardinal number, and let $\sum_{x\in f^{-1}(y)} \operatorname{ord}_X(x) = \alpha$. Then, there are arcs A_γ for $0 \leq \gamma \leq \alpha$ in X such that one of the endpoints of each A_γ is in $f^{-1}(y)$ and $\operatorname{int}_X(A_\gamma) \cap \operatorname{int}_X(A_\delta) = \emptyset$ for $\gamma \neq \delta$. Since $f(A_\gamma)$ is a locally connected continuum, then there is an arc B_γ in $f(A_\gamma)$ such that $y \in E(B_\gamma)$. By Lemma 3.4, we have $\operatorname{int}_Y(f(A_\gamma)) \cap \operatorname{int}_Y(f(A_\delta)) = \emptyset$ for $\gamma \neq \delta$. Since $B_\gamma \subseteq f(A_\gamma)$, then $\operatorname{int}_Y(B_\gamma) \cap \operatorname{int}_Y(B_\delta) = \emptyset$ for $\gamma \neq \delta$. Therefore, $\operatorname{ord}_Y(y) \geq \alpha$, and hence, $\operatorname{ord}_Y(y) \geq \sum_{x\in f^{-1}(y)} \operatorname{ord}_X(x)$. . □

Let us notice that for graphs the two notions of order coincide.

Observation 3.7. *For every graph G and a point $p \in G$ we have* $\operatorname{cord}_G(p) = \operatorname{ord}_G(p)$.

Our next Theorem shows that Theorem 3.6 can be strengthen if the domain and the range of our functions are graphs.

Theorem 3.8. *If $f : X \to Y$ is a hereditarily irreducible map from a graph X onto a graph Y and y is a point in Y, then* $\operatorname{ord}_Y(y) = \sum_{x\in f^{-1}(y)} \operatorname{ord}_X(x)$.

Proof. Suppose there is a hereditarily irreducible map f from a graph X onto a graph Y, and let y be a point in Y. By Theorem 3.6, we have $\operatorname{ord}_Y(y) \geq \sum_{x\in f^{-1}(y)} \operatorname{ord}_X(x)$. We will show that $\operatorname{ord}_Y(y) \leq \sum_{t\in f^{-1}(y)} \operatorname{ord}_X(x)$. If $\operatorname{ord}_Y(y) = n$ for some positive integer n, then there are n arcs $A_1, A_2, ..., A_n$ in Y such that $y \in E(A_i)$ for all i, $i = 1, 2, ..., n$, and $\operatorname{int}_Y(A_i) \cap \operatorname{int}_Y(A_j) = \emptyset$ for $i \neq j$. Since Y is a graph, we may assume that $A_i \cap A_j = \{y\}$, $i \neq j$, and $\operatorname{ord}_Y(z) = 2$ for all $z \in A_i \setminus \{y\}$. For each i, there is a nondegenerate component C_i of $f^{-1}(A_i)$ such that $f(C_i) \subseteq A_i$ and $y \in C_i$. By Theorem 3.5, $\operatorname{ord}_X(x) \leq 2$ for all $x \in C_i \setminus f^{-1}(y)$. Since f is a hereditarily irreducible, then C_i contains no simple closed curve, therefore, C_i is an arc in X. Since $A_i \cap A_j = \{y\}$, then $\operatorname{int}_X(C_i) \cap \operatorname{int}_X(C_j) = \emptyset$ for $i \neq j$. For each i, we have $E(C_i) \cap f^{-1}(y) \neq \emptyset$ since f is a hereditarily irreducible map. It follows that $\sum_{x\in f^{-1}(y)} \operatorname{ord}_X(x) \geq n$. Thus, $\operatorname{ord}_Y(y) \leq \sum_{x\in f^{-1}(y)} \operatorname{ord}_X(x)$, and hence, $\operatorname{ord}_Y(y) = \sum_{x\in f^{-1}(y)} \operatorname{ord}_X(x)$. □

Example 3.9. *The assumption that X and Y are graphs in Theorem* 3.8 *is essential. In general the equality does not hold. In fact there is a hereditarily irreducible map f from* $[0, 1]$ *onto a locally connected continuum X pictured in Figure 2 such that* $f^{-1}(f(1)) = 1$ *and* $\operatorname{ord}_X(f(1)) = 3 > 1 = \operatorname{ord}_{[0,1]}(1)$*. The map is sketched in Figure 3.*

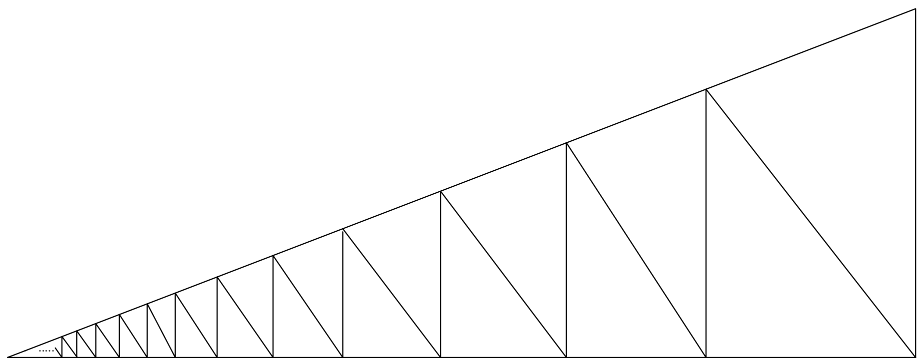

Figure 2. The locally connected continuum X.

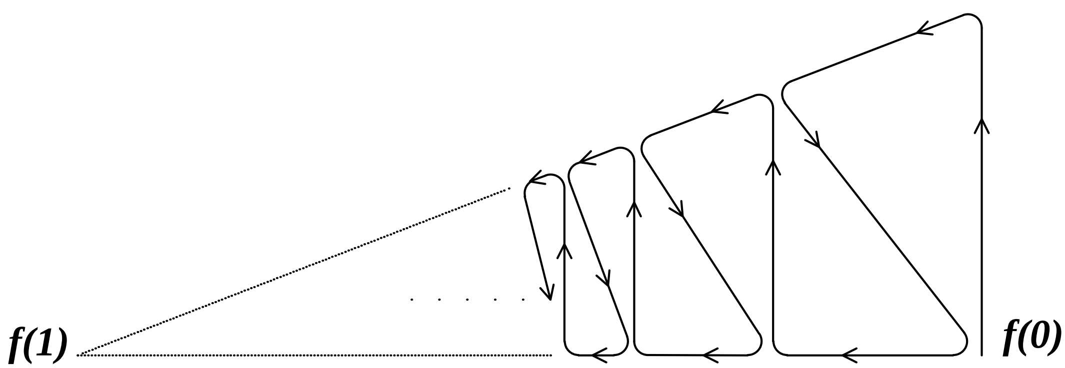

Figure 3. The map f.

Corollary 3.10. *If $f : X \to Y$ is a hereditarily irreducible map between graphs X and Y, then the number of points of odd order of Y is less then or equal to the number of points of odd order of X.*

Proof. By Theorem 3.8 each point of odd order in Y has at least one point in its preimage of odd order. □

Problem 3.11. *Is the conclusion of Corollary 3.10 true for any hereditarily irreducible map between locally connected continua X and Y?*

Our next result is an immediate consequence of Theorem 3.8

Corollary 3.12. *If X is a graph, $f : X \longrightarrow X$ is hereditarily irreducible map, then f is a homeomorphism.*

Proof. Let X be a graph, and let $f : X \longrightarrow X$ be a hereditarily irreducible map. Note that $E(X)$ and $R(x)$ are finite sets and f is onto, therefore, it follows from Theorem 3.8 that f maps endpoints to endpoints and ramification points to ramification points, and f is one-to-one on $E(X)$ and $R(X)$. Since f maps endpoints to endpoints, then, using Theorem 3.6, we can conclude that f maps points of order 2 to points of order 2 and f is one-to-one on $X \setminus (E(X) \cup R(X))$. Thus, f is a homeomorphism. □

The following example shows Corollary 3.12 doesn't hold if we remove the assumption that X is a graph.

Example 3.13. *Let $X = \cup_{i=1}^{\infty}(C_i \cup L_i)$, as shown in Figure 4, where C_i is a circle,* $\operatorname{diam}(C_i) > \operatorname{diam}(C_{i+1})$, $\operatorname{diam}(C_i) \longrightarrow 0$ *as $i \longrightarrow 0$ and L_i is an arc,* $\operatorname{diam}(L_i) > \operatorname{diam}(L_{i+1})$, $\operatorname{diam}(L_i) \longrightarrow 0$ *as $i \longrightarrow 0$. Let f be a map from X onto X define as following. Let $f(L_1) = C_1$, image of the endpoints of L_1 under f is equal to p, and $f(C_i) = C_{i+1}$, $i \geq 1$, $f(L_i) = L_{i+1}$, $i \geq 2$. Then, f is hereditarily irreducible map but f is not one-to-one.*

In some cases the converse to Corollary 3.10 is also true. This is the famous Euler's theorem about Königsberg bridges, see e.g. [4, Theorem 3.6, p. 79].

Theorem 3.14. *If G is a graph, then there is a hereditarily irreducible map from* $[0, 1]$ *onto G if and only if G has at most two points of odd order.*

Similarly, we can characterize hereditarily irreducible images of the simple closed curve.

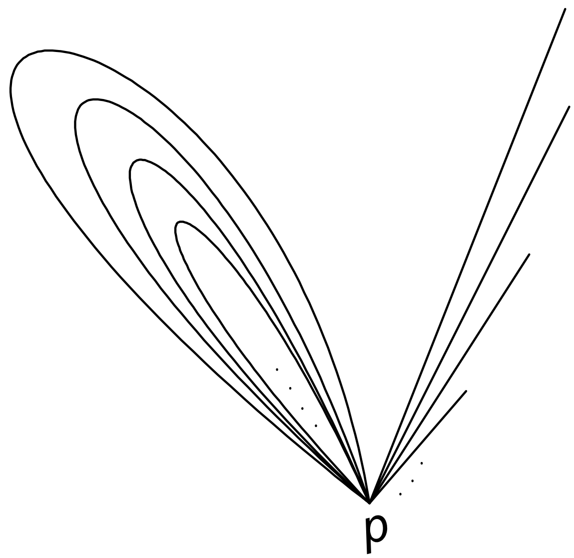

Figure 4. The continuum X.

Theorem 3.15. *A graph G is an image of S^1 under a hereditarily irreducible map if and only if each point of G has even order.*

Proof. The proof is very similar to the proof of Theorem 3.14. □

Generalizations of Theorems 3.14 and 3.15 to arbitrary locally connected continua are not true. This can be seen by the following example.

Example 3.16. *The continuum X pictured in Figure 5 is locally connected, every subcontinuum of X has nonempty interior, X has only two points of odd order, but there is no hereditarily irreducible map from $[0, 1]$ onto X.*

The reason why Example 3.16 works is the existence of points of infinite orders. Therefore the following problems seems interesting.

Problem 3.17. *Suppose X is a locally connected continuum in which the union of free arcs is dense, all points of X are of finite order, and X has at most two points of odd order. Does there exist a hereditarily irreducible map from $[0, 1]$ onto X?*

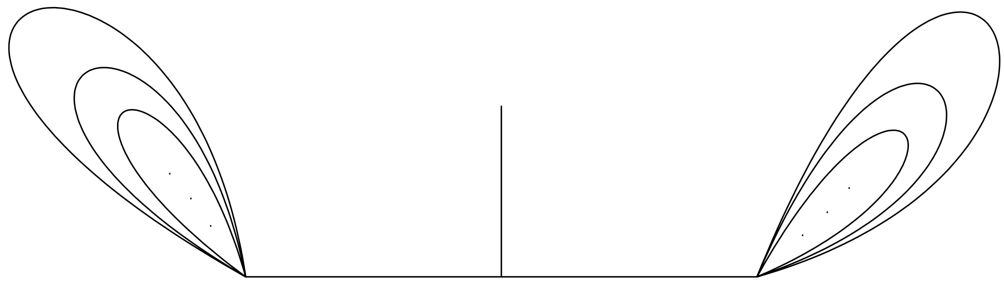

Figure 5. The continuum X has only two points of odd order.

Characterizations of hereditarily irreducible maps of other graphs are much less obvious. Our next example shows that the invariants in Theorem 3.5 and Corollary 3.10 are not enough to characterize hereditarily irreducible images of a simple triod.

Example 3.18. *There is a graph G with the following properties:*

1. *G has one ramification point of odd order;*
2. *G has three endpoints;*
3. *G is not an image of a simple triod under a hereditarily irreducible map.*

Proof. Let G be the graph in Figure 6, and let r be the only point of order three. Suppose there is a hereditarily irreducible map from a simple triod T onto G. Then the three endpoints of T have to go to the three endpoints of G and, consequently, by Theorem 3.8 the image of the vertex of T is r. Let p be a point of the interior of the free arc left to r. Then there are at least three points of order two in the preimage of p. This contradicts Theorem 3.8. $\square$

The next theorem gives another characterization of hereditarily irreducible maps between graphs.

Theorem 3.19. *If $f : X \to Y$ is a surjective map between graphs, then the following conditions are equivalent:*

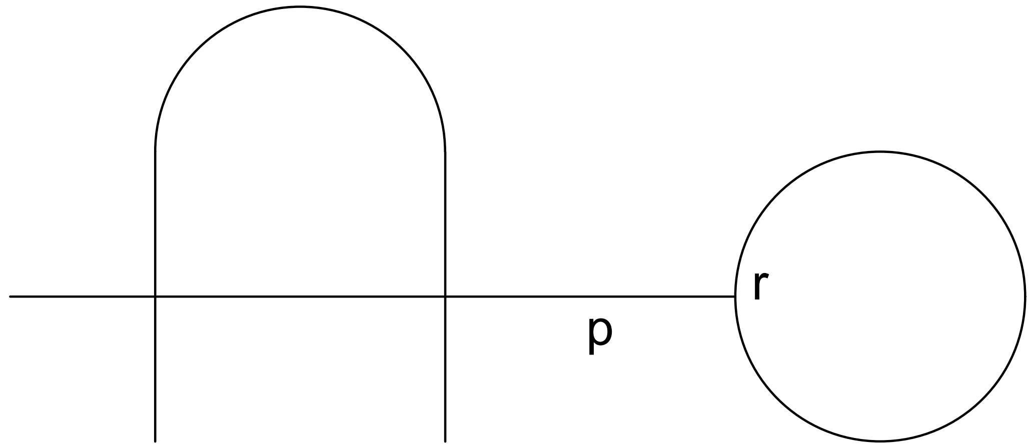

Figure 6. The graph G with one ramifcation point of odd order.

1. *f is a hereditarily irreducible map;*

2. $\mathrm{card}(f^{-1}(y)) < \aleph_0$ *for any $y \in Y$ and* $\mathrm{card}(\{y \in Y : f^{-1}(y)$ *is nondegenerate*$\}) < \aleph_0$.

Proof. To show that (1) $\implies$ (2), let $y \in Y$. Since Y is a graph, then $\mathrm{ord}_Y(y) < \aleph_0$. By Theorem 3.8, we have $\mathrm{ord}_Y(y) = \sum_{t \in f^{-1}(y)} \mathrm{ord}_X(x)$, therefore, $\mathrm{card}(f^{-1}(y)) < \aleph_0$.

Now we will show that $\mathrm{card}(\{y \in Y : f^{-1}(y)$ is nondegenerate$\}) < \aleph_0$. Let y be any point of Y such that $f^{-1}(y)$ is nondegenerate. It follows from Theorem 3.8 that $\mathrm{ord}_Y(y) \geq 2$ and $y \in Y \setminus E(Y)$. Since Y is a graph then $\mathrm{card}(R(Y)) < \aleph_0$. Therefore, $\mathrm{card}(\{y \in R(Y) : f^{-1}(y)$ is nondegenerate$\}) < \aleph_0$. If $y \notin R(Y)$, then it clear that $f^{-1}(y) \subseteq E(X)$. Since X is a graph, then $\mathrm{card}(E(X)) < \aleph_0$, and therefore, $\mathrm{card}(\{y \in Y \setminus (E(Y) \cup R(Y)) : f^{-1}(y)$ is nondegenerate$\}) < \aleph_0$. So $\mathrm{card}(\{y \in Y : f^{-1}(y)$ is nondegenerate$\}) < \aleph_0$.

To show that (2) $\implies$ (1), let A and B be two subcontinua of X such that $A \subsetneq B$. Since $\mathrm{card}(B \setminus A) > \aleph_0$, $\mathrm{card}(f^{-1}(f(a))) < \aleph_0$ for any $a \in A$, and $\mathrm{card}\{a \in A : f^{-1}(f(a))$ is nondegenerate$\}) < \aleph_0$, then $f(A) \subsetneq f(B)$. Thus, f is a hereditarily irreducible map. $\square$

The assumption that Y is a graph in Theorem 3.19 is essential. In general Condition 1 in Theorem 3.19 does not imply 2 (see Example 3.9).

Theorem 3.20. *If f is a hereditarily irreducible map from a locally connected continuum X onto a graph Y, then X is a graph.*

Proof. Let f be a hereditarily irreducible map from a locally continuum X onto a graph Y. By Theorem 3.5, $\mathrm{ord}_X(x) \leq \mathrm{ord}_Y(f(x))$ for each $x \in X$. Since Y is a graph, then $\mathrm{ord}_Y(y) < \aleph_0$ for every $y \in Y$. Therefore, $\mathrm{ord}_X(x) < \aleph_0$ for each $x \in X$. So to show that X is a graph, it is enough to show that $\mathrm{card}(R(X)) < \aleph_0$. Since Y is a graph, then $\mathrm{card}(R(Y)) < \aleph_0$. It follows from Theorem 3.6 that $\mathrm{card}(R(X)) < \aleph_0$. Thus X is a graph. □

4. HEREDITARILY IRREDUCIBLE MAPS FROM AN ARC

In this section we give two characterizations of hereditarily irreducible images of arc. In the case when the image is a graph we have a full characterization by Theorem 3.14. Also if the image is a continuum that does not contain free arc, we always have such mapping by [4, Theorem 4.21, p. 87].

Theorem 4.1. *Suppose X and Y are locally connected continua such that $Y \subseteq X$. If Y contains all free arcs of X and there is a hereditarily irreducible map from* $[0, 1]$ *onto Y, then there is a hereditarily irreducible map from* $[0, 1]$ *onto X.*

Proof. Let X be a locally connected continuum, d a convex metric on X, and let Y be a locally connected subcontinuum of X containing all free arcs of X. Suppose there is a hereditarily irreducible map f from $[0, 1]$ onto Y. Let $\{A_\lambda : \lambda \in \Lambda\}$ be the set of interiors of all maximal free arcs in X; here the set Λ is either finite or countable infinite. Put $U = \bigcup_{\lambda \in \Lambda} f^{-1}(A_\lambda)$, and let T be a component of $[0, 1] \setminus U$. Define $F_T = \{x \in X : d(x, f(T)) = d(x, Y)\}$. We will show that F_T is a connected without free arcs. To see that let $x \in F_T$, and let y be a point in $f(T)$ such that $d(x, y) = d(x, f(T)) = d(x, Y)$. Let A be a convex arc joining x and y. Then for every $z \in A$ we have $d(z, Y) = d(z, f(T)) = d(z, y) \leq d(x, y)$, so $z \in F_T$. This shows F_T is arcwise connected.

To see that F_T contains no free arc, it is enough to notice that $T \subset [0,1] \setminus U$ and $f(U)$ contains all free arcs in X. Observe that if $F_T \cap (X \setminus Y) \neq \emptyset$, then $\mathrm{int}(F_T) \cap (X \setminus Y) \neq \emptyset$. Moreover, if T and T' are two different componenets of $[0,1] \setminus U$, then $\mathrm{int}(F_T) \cap \mathrm{int}(F_{T'}) = \emptyset$. Note also that if F_T is nondegenerate for some component T of $[0,1] \setminus U$, then $\mathrm{int}(F_T) \neq \emptyset$. Therefore there is at most countable set $\mathcal{J}$ such that F_{T_j} is nondegenerate for each $j \in \mathcal{J}$. Let $\alpha : [0,1] \to [0,1]$ be a monotone map such that if F_{T_j} is nondegenerate, then $\alpha^{-1}(T_j)$ is nondegenerate. For each $j \in \mathcal{J}$, let D_j be a dense countable subset of $\alpha^{-1}(T_j)$. By Theorem 4.21 [4], there is a hereditarily irreducible map $g_i : \alpha^{-1}(T_j) \longrightarrow F_{T_j}$ that is one-to-one on D_j. Define $g : [0,1] :\longrightarrow X$ by

$$g(t) = \begin{cases} g_i(t) & if\ t \in \alpha^{-1}(T_i) \text{ for some } i \in \mathcal{J}; \\ f(\alpha(t)) & \text{otherwise.} \end{cases}$$

Observe that g is a hereditarily irreducible map from $[0,1]$ onto X, as required. □

As an illustration of an application of Theorem 4.1 let us have the following example.

Example 4.2. *There is a hereditarily irreducible map from* $[0,1]$ *onto the continuum* X *pictured in left part of Figure 7.*

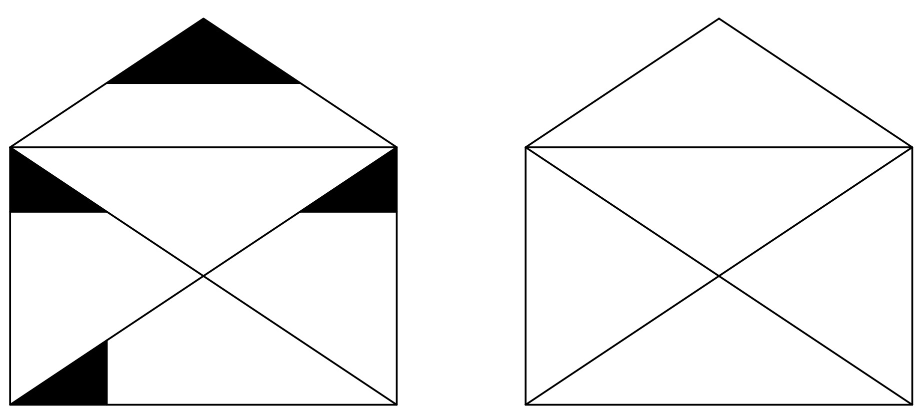

Figure 7. The continuum X (left) and the graph G (right).

Proof. Note that the graph G pictured in the right part of Figure 7 has only two point of odd order, so it is a hereditarily irreducible image of $[0,1]$ by Theorem 3.14. Note also that G contains all free arcs of X, therefore the existence of the required map follows from Theorem 4.1. $\square$

Definition 4.3. *Let X be a locally connected continuum. We define an equivalence relation $\sim$ on X by letting $x \sim y$ if only if there is an arc with empty interior that contains both x and y or $x = y$.*

Theorem 4.4. *Let X be a locally connected continuum. If there is a hereditarily irreducible map from $[0,1]$ onto X, then there is a hereditarily irreducible map from $[0,1]$ onto $X/_\sim$.*

Proof. Suppose there is a hereditarily irreducible map f from $[0,1]$ onto X. We will construct a hereditarily irreducible map from $[0,1]$ onto $X/_\sim$. For any $s,t \in [0,1]$, we define $s \sim_1 t$ if $f(s) \sim f(x) \sim f(t)$ in $X/_\sim$ for any $x \in [s,t]$. Note that $\sim_1$ is a monotone relation on $[0,1]$, so $[0,1]/_{\sim_1}$ is homeomorphic to $[0,1]$. Define $g : [0,1]/_{\sim_1} \longrightarrow X/_\sim$ by $g(t) = [f(t)]_\sim$. To show that g is hereditairly irreducible map, let A be a subcontinuum of $[0,1]/_{\sim_1}$ and B be a proper subcontinuum of A, and let q, q' be the natural quotient maps on X and $[0,1]$ respectively. Since B is proper subcontinuum of A, then $q'^{-1}(B) \subsetneq q'^{-1}(A)$. Since q' is a monotone map, the sets $\sim_1$, $q'^{-1}(A)$ and $q'^{-1}(B)$ are continua. Also, $f(q'^{-1}(B)) \subsetneq f(q'^{-1}(A))$ since f is hereditarily irreducible. Since $B \neq A$, then there is $a \in A$ such that $f(a) \nsim f(b)$ for all $b \in B$. Therefore, $g(B) = q(f(q'^{-1}(B))) \subsetneq g(A) = q(f(q'^{-1}(A)))$, and thus g is a hereditarily irreducible map. $\square$

As an illustration of an application of Theorem 4.4 let us have the following example.

Example 4.5. *There is no hereditarily irreducible map from $[0,1]$ onto the continuum X pictured in left part of Figure 8.*

Proof. The continuum $X/_{\sim}$ pictured in the right part of Figure 8 is just a simple triod, so it contains four points of odd order, therefore, by Theorem 3.14, there is no hereditarily irreducible map from $[0, 1]$ onto $X/_{\sim}$. Consequently, by Theorem 4.4 there is no hereditarily irreducible map from $[0, 1]$ onto X. □

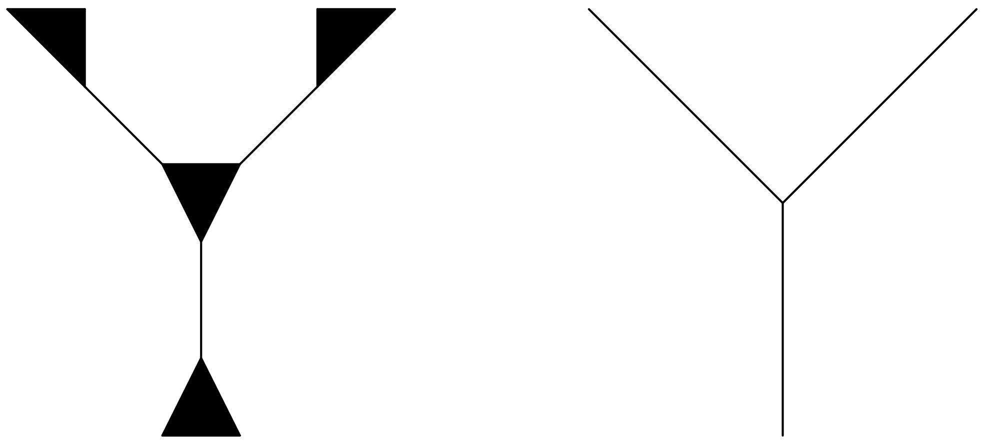

Figure 8. The continuum X (left) and $X/_{\sim}$ (right).

We do not know if the converse Theorem 4.4 is true.

Problem 4.6. *Suppose X is a locally connected continuum and there is a hereditarily irreducible map from* $[0, 1]$ *onto $X/_{\sim}$. Can we prove that there is hereditarily irreducible map from* $[0, 1]$ *onto X?*

To show our next Theorem we need to recall the definition and properties of a dendrite called D_3. This is a dendrite with a dense set of ramification points, each point of classical order 3. It is known, (see e.q. [1, (6), p. 490]), that the above properties characterize D_3, i.e. any two dendrites with a dense set of ramification points, each of classical order 3, are homeomorphic. Note that every point of D_3 has infinite order in the new definition. Here we will use the fact that any dendrite with a dense set of ramification points contains D_3.

Theorem 4.7. *Let $f : [0,1] \to X$ be a hereditarily irreducible map from $[0,1]$ onto a locally connected continuum X. If X is not arc, then X contains either a simple closed curve or a copy of D_3.*

Proof. If X contains a simple closed curve, we are done. Otherwise we will show that X contains a copy of D_3. On contrary, suppose that X contains no copy of D_3 . Since X is not an arc, there are two elements t_1 and t_2 in the unit interval $[0,1]$ such that $t_1 < t_2$ and $f(t_1) = f(t_2)$. Since $f([t_1,t_2])$ is a locally connected continuum and X contains no copy of D_3, then $f([t_1,t_2])$ contains a free arc A. Let a, b be two distinct elements in the interior of A. By Theorem 3.5, f is one-to-one on $f^{-1}(ab)$. Choose t_3, t_4 in $[t_1,t_2]$ such that $f(t_3) = a$ and $f(t_4) = b$. By symmetry, we may assume that $t_3 < t_4$. Since X is locally connected containing no simple closed curve, then it is a dendrite, and therefore, $f([t_1,t_2]) \setminus \mathrm{int}(ab)$ has two components. If $f(t_1)$ and a are in the same component, then there is an element $s \in [t_4,t_2]$ such that $f(s) = a$; since $f(t_1) = f(t_2)$ and for the same reason if $f(t_1)$ and b are in the same component, then there is an element $t \in [t_1,t_3]$ such that $f(t) = b$, but this contradicts the facts that $f^{-1}(a) = \{t_3\}$ and $f^{-1}(b) = \{t_4\}$. Thus, X contains a copy of D_3.

□

As a corollary we get the following result.

Corollary 4.8. *If there is a hereditarily irreducible map from S^1 onto a continuum X, then X contains either a simple closed curve or a copy of D_3.*

5. HEREDITARILY IRREDUCIBLE IMAGES OF GRAPHS

In [4] the authors proved that for every locally connected continuum X without free arcs there is a hereditarily irreducible map from $[0,1]$ onto X. The goal of this section is to generalize the result to have any graph in the domain, not just $[0,1]$.

We have to start with the definitions of necessary symbols. Let X and Y be continua, $\bar{x} = \{x_i\}_{i=1}^n \subset X$, $\bar{y} = \{y_i\}_{i=1}^n \subset Y$ be any subsets with n elements, not necessarily different, and let $C(X,Y)$ denote the set of all maps from X to Y. We define the following sets

1. $\mathcal{S}(X,Y) = \{f \in C(X,Y) : f$ is a surjective map$\}$.

2. $C(X,Y,\bar{x},\bar{y}) = \{f \in C(X,Y) : f(x_i) = y_i$ for all $i \leq n\}$.

3. $\mathcal{S}(X,Y,\bar{x},\bar{y}) = \mathcal{S}(X,Y) \cap C(X,Y,\bar{x},\bar{y})$.

4. $\mathcal{A}_F(X,Y) = \{f \in C(X,Y) : f^{-1}(f(x)) = \{x\}$ for each $x \in F\}$

Let us recall two important results from [4].

Theorem 5.1. *[4, Theorem 4.13, p.85]. Let X be a 1-dimensional continuum, Y a non-degenerate locally connected continuum without free arcs, F a 0-dimensional closed subset of X, $\bar{x} \subset X$, and $\bar{y} \subset Y$. If $F \cap \bar{x} = \emptyset$, then $\mathcal{S}(X,Y,\bar{x},\bar{y}) \cap \mathcal{A}_F(X,Y)$ is a dense G_δ-subset of $\mathcal{S}(X,Y,\bar{x},\bar{y})$.*

Corollary 5.2. *[4] Let X be a 1-dimensional continuum, Y a non-degenerate locally connected continuum without free arcs, T a 0-dimensional F_σ-subset of X. Then $\mathcal{S}(X,Y) \cap \mathcal{A}_T(X,Y)$ is a dense G_δ-subset of $\mathcal{S}(X,Y)$.*

Theorem 5.3. *Let X be a graph, and let Y be a nondegenerate locally connected continuum without free arcs, then there is a hereditarily irreducible map from X onto Y.*

Proof. Let X be a graph, and let Y a non-degenerate locally connected continuum without free arcs. Let T be a countable dense subset of X. By Corollary 5.2, $\mathcal{S}(X,Y) \cap \mathcal{A}_T(X,Y)$ is a dense G_δ- subset of $\mathcal{S}(X,Y)$. Let $f \in \mathcal{S}(X,Y) \cap \mathcal{A}_T(X,Y)$. We will show that f is a hereditarily irreducible map. Let A, B be two subcontinua of X such that $A \subsetneq B$. Since T is dense subset of X, then $T \cap (B \setminus A) \neq \emptyset$. Let $t \in T \cap (B \setminus A)$, then $f^{-1}(f(t)) = \{t\}$. Therefore, $f(A) \subsetneq f(B)$, and hence, f is hereditarily irreducible map. $\square$

6. HEREDITARILY IRREDUCIBLE MAPS ONTO DENDRITES

Theorem 6.1. *Let D be a dendrite containing no copy of D_3. If there is a hereditarily irreducible map f from a locally connected continuum X onto D, then f is a homeomorphism.*

Proof. let x_1, x_2 be two distinct elements of X, and let A be an arc in X irreducible between $\{x_1, x_2\}$. Since D is a dendrite contains no copy of D_3, then $f(A)$ contains no simple closed curve and no copy of D_3. By Proposition 4.7, f is a homeomorphism on A. So, $f(x_1) \neq f(x_2)$, and therefore, f is one-to-one on X. Since f is one-to-one map from a compact space X onto a Hausdorff space D, then f is a homeomorphism.

□

Corollary 6.2. *If $f : X \to Y$ is a hereditarily irreducible map between locally connected continua, and X is not homeomorphic to Y, then Y contains either a simple closed curve or a copy of D_3.*

The following example shows that the assumption of local connectedness in Corollary 6.2 is essential.

Example 6.3. *Let f be a hereditarily irreducible map from a continuum X onto a continuum Y pictured in Figure 9. Y is not homeomorphic to X and contains no simple closed curve nor a copy of D_3.*

Theorem 6.4. *A dendrite D is an image of* $[0, 1]$ *under hereditarily irreducible map if only if there is an arc containing all free arc of D.*

Proof. First, suppose that there is an arc containing all free arcs of D. Then the existence of a hereditarily irreducible map from $[0, 1]$ onto D follows from Theorem 4.1.

Second, suppose $f : [0, 1] \to D$ is a hereditarily irreducible map. We will show that the arc $f(0)f(1)$ contains all free arcs of D. Suppose the contrary, let a, b be two distinct interior points of a free arc in D. Let t_a, t_b be two points in $[0, 1]$ such that $f(t_a) = a$ and

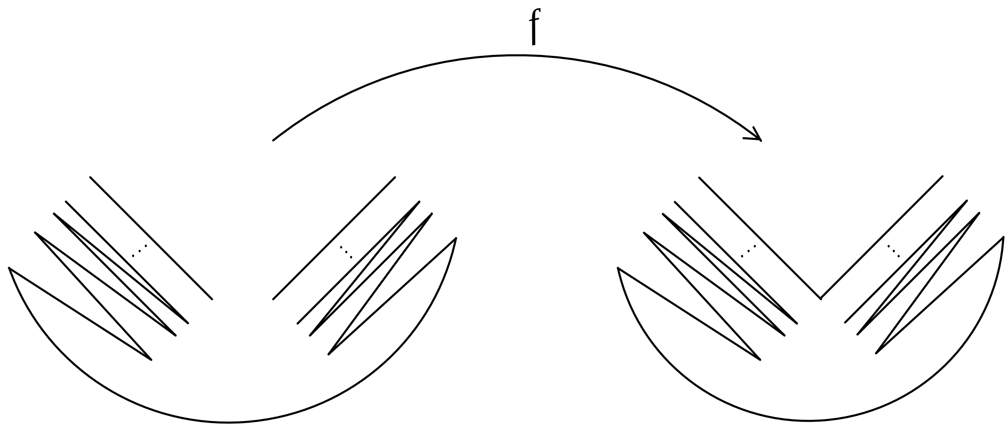

Figure 9. The map f from X onto Y.

$f(t_b) = b$. The difference $D \setminus ab$ has two components, say C_1, C_2. Because a and b are not in the arc $f(0)f(1)$, the arc $f(0)f(1)$ is in one of these components. we may assume $f(0)f(1) \subset C_1$. Let s be a point such that $f(s) \in C_2$; then there is a point $s' \in (s, 1)$ such that $f(s') = a$. Thus $f(t_a) = f(s') = a$, contrary to Theorem 3.6. □

Theorem 6.5. *For a dendrite D the following conditions are equivalent.*

a) *$R(D)$ is dense in D.*

b) *D contains no free arc.*

c) *$E(D)$ is dense in D.*

d) *For any graph G there is a hereditarily irreducible map from G onto D.*

e) *There is a hereditarily irreducible map f from S^1 onto D.*

f) *There is a graph G with no cut points and a hereditarily irreducible map from G onto D.*

Proof. The equivalence of conditions (a), (b), and (c) are shown in [2, Theorem 4.6, p. 10]. The implication $(c) \Rightarrow (d)$ follows from Theorem 5.3. The implications $(d) \Rightarrow (e)$ and $(e) \Rightarrow (f)$ are trivial. So we only need to show $(e) \Rightarrow (b)$. Suppose on the contrary that $f : S^1 \to D$ is hereditarily irreducible map and D contains a free arc. Let c be an interior point of that arc, then $D \setminus \{c\}$ has two comoponents. Let us choose $t_0, t_1 \in S^1$

such that $f(t_0)$ and $f(t_1)$ are in different components of $D \setminus \{c\}$. Let A and B be arcs in S^1 such that $A \cap B = \{t_0, t_1\}$. Since $f(A)$ and $f(B)$ contain the point c, so there are points $s_1 \in A$ and $s_2 \in B$ such that $f(s_1) = f(s_2) = c$. Then $\mathrm{ord}_{S^1}(s_1) = \mathrm{ord}_{S^1}(s_2) = 2$, while $\mathrm{ord}_D(c) = 2 < \mathrm{ord}_{S^1}(s_1) + \mathrm{ord}_{S^1}(s_2)$, contrary to Theorem 3.6. □

7. MAPPINGS

Theorem 7.1. *Let f be a hereditarily irreducible map from a locally connected continuum X onto a locally connected continuum Y, then f is an open map if only if f is a homeomorphism.*

Proof. Suppose, on the contrary, that f is not a homeomorphism. Then by our assumption, there are x_1 and x_2 in X such that $x_1 \neq x_2$ and $f(x_1) = f(x_2) = y$. Let Z be a continuum neighborhood of y in Y such that x_1 and x_2 lying in different components of $f^{-1}(Z)$. Let C_1 and C_2 be components of $f^{-1}(Z)$ that contain x_1 and x_2 respectively. Since f is open map, then f maps C_1 and C_2 onto Z. Let A be an arc in X irreducible between C_1 and C_2, then $C_1 \cup A$ and $C_1 \cup C_2 \cup A$ are subcontinua of X, $C_1 \cup A \subsetneq C_1 \cup C_2 \cup A$, and $f(C_1 \cup A) = f(C_1 \cup C_2 \cup A)$ but this contradicts our assumption that f is a hereditarily irreducible map. Thus, f is a homeomorphism.

□

The assumption that X and Y are locally connected continuum in Theorem 7.1 is essential. The following example shows that without the assumption of connectedness Theorem 7.1 does not hold.

Example 7.2. *Let S^1 be the unit circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$. Define $f, g : S^1 \to S^1$ by $f(z) = z^3$ and $g(z) = z^2$. The inverse limit space $\varprojlim\{S^1, f\}$ with one bonding mapping is called a triadic solenoid Σ_3. The commutative diagram below induces a map $g^* : \Sigma_3 \to \Sigma_3$ (See [3], p. 101).*

$$\begin{tikzcd}
S^1 \arrow[d, "g"'] & S^1 \arrow[l, "f"'] \arrow[d, "g"'] & S^1 \arrow[l, "f"'] \arrow[d, "g"'] & \cdots \arrow[l] \\
S^1 & S^1 \arrow[l, "f"] & S^1 \arrow[l, "f"] & \cdots \arrow[l]
\end{tikzcd}$$

We will show that the map g^* is hereditarily irreducible and open, but not a homeomorphism. To see it is not a homeomorphism, let us observe that $g^*((1, 1, 1, ...)) = g^*((-1, -1, -1, ...)) = (1, 1, 1, ...) \in \Sigma_3$.

To show that g^* is hereditarily irreducible, let A and B be two continua in Σ_3 such that A is a proper subcontinuum of B. Let $\pi_n : \Sigma_3 \longrightarrow S^1$ be the projection onto the n-th factor, i.e $\pi_n((x_1, x_2, ...)) = x_n$. Since $A \neq B$, there is an index n such that $\pi_n(A) \neq \pi_n(B)$. Since $\pi_n(A)$ and $\pi_n(B)$ are continua, thus arcs in S^1, the length of $\pi_n(A)$ is less than 2π. Observe that the length of $\pi_{n+1}(A)$ is three times less than the length of $\pi_n(A)$, so it is less than $2\pi/3$. Thus $\pi_{n+1}(A) \subsetneq \pi_{n+1}(B)$ and the length of $\pi_{n+1}(A)$ is less than $2\pi/3$. Applying the function g, we see that $g(\pi_{n+1}(A))$ has length less than $4\pi/3$ and that $g(\pi_{n+1}(A) \subsetneq g(\pi_{n+1}(B)$. This implies that $g^*(A) \subsetneq g^*(B)$ as required. To see that g^* is open, observe that the map g is open and that the diagram above is exact. Then the conclusion follows from Theorem 4 in [9].

8. ANSWERS

In [4, Theorem 4.21.3, Question 6.4, p. 92] B. Espinoza and E. Matsuhashi proved that for any locally connected continuum X contains no free arcs and for any two points $p, q \in X$, there is a hereditarily irreducible map f from $[0, 1]$ onto X such that $f(0) = p$ and $f(1) = q$ and posed a question: Is the converse of this Theorem true? We answered their question in the positive.

Theorem 8.1. *If X is a locally connected continuum and for any two points $p, q \in X$ there is a hereditarily irreducible map f from $[0, 1]$ onto X such that $f(0) = p$ and $f(1) = q$, then X contains no free arc.*

Proof. Suppose on the contrary that X contains a free arc A. Choose two distinct points p, q in the interior of A and let f be a hereditarily irreducible map from $[0, 1]$ onto X such that $f(0) = p$ and $f(1) = q$. Choose two distinct points a, b in the interior of A such that $p \in \mathrm{int}(ab)$ and $q \notin ab$. Let $t_1, t_2 \in [0, 1]$ such that $f(t_1) = a$ and $f(t_2) = b$. Note that p or q is in the interior of $f([t_1, t_2])$. Without loss of generality, assume $p \, \mathrm{int}\, f([t_1, t_2])$, so there is a point t_3 in the open interval (t_1, t_2) such that $f(t_3) = p$. Then $\mathrm{ord}_X(p) = 2 < \mathrm{ord}_{[0,1]}(t_3) + \mathrm{ord}_{[0,1]}(0)$, but this contradicts Theorem 3.6. Thus X contains no free arc. □

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SECTION

2. CONCLUSIONS

In this dissertation, we studied $\mathcal{T}$-closed sets, inverse limits with multivalued functions, and hereditarily irreducible maps between continua.

We gave a negative answer to a question posted by David P. Bellamy, Leobardo Fernández and Sergio Macías in [3] about $\mathcal{T}$-closed sets if $\mathcal{T}$ and a characterization of $\mathcal{T}$-closed set. We presented necessary and sufficient conditions for a set to be $\mathcal{T}$-closed set using generalized continua.

We answered James P. Kelly's question in [18] in the negative. Specifically, we constructed an example of an inverse limit sequence with a single monotone multi-valued bound function $f : [0, 1]^2 \to 2^{[0,1]^2}$ such that the inverse limit is not locally connected. We also showed that there is an inverse limit sequence on $[0, 1]$ with a single upper semi-continuous set-valued bonding function f such that $G(f^n)$ is an arc for each positive integer n, but the inverse limit is not connected. This answered W. T. Ingram's question in [14] in the negative.

We studied properties of hereditarily irreducible maps between continua. We introduced a new notion of an order of a point in a continuum. Precisely, we generalized the concept of order in the classical sense. We gave necessary conditions for a map to be hereditarily irreducible between continua. Moreover, we presented both necessary conditions and sufficient conditions for a map to be hereditarily irreducible between graphs. We proved the existance of hereditarily irreducible map from a graph onto a locally connected continuum contains no free arcs.

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