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Stochastic dynamic equations

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STOCHASTIC DYNAMIC EQUATIONS

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ABSTRACT

We propose a new area of mathematics, namely stochastic dynamic equations, which unifies and extends the theories of stochastic differential equations and stochastic difference equations. After giving a brief introduction to the theory of dynamic equations on time scales, we construct Brownian motion on isolated time scales and prove some of its properties. Then we define stochastic integrals on isolated time scales. The main contribution of this dissertation is to give explicit solutions of linear stochastic dynamic equations on isolated time scales. We illustrate the theoretical results for dynamic stock prices and Ornstein—Uhlenbeck dynamic equations. Finally we study almost sure asymptotic stability of stochastic dynamic equations and mean-square stability for stochastic dynamic Volterra type equations.

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NOMENCLATURE

Symbol	Description
$\mathbb{T}$	Time Scales
$\mathbb{R}$	Set of Real Numbers
$\mathbb{N}$	Set of Natural Numbers
$\mathbb{N}_0$	Set of Whole Numbers
$\mathbb{N}_0^2$	The Set $\{0, 1, 4, 9, 16, \ldots\}$
$h\mathbb{Z}$	The Set $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$
$\mathbb{C}$	Set of Complex Numbers
$\overline{q^{\mathbb{Z}}}$	The Set $\{\ldots, q^{-2}, q^{-1}, 1, q, q^2 \ldots\}$ for $q > 1$.
σ	Forward Jump Operator
ρ	Backward Jump Operator
μ	Graininess Function
Δ	Delta Derivative Operator
Δ	Forward Difference Operator
ξ_h	Cylinder Transformation
$\mathcal{R}$	Set of Regressive Functions
$\mathcal{R}_W$	Set of Stochastic Regressive Functions
$\mathcal{R}^+$	Set of Positively Regressive Functions
$\mathcal{R}_W^+$	Set of Stochastic Positively Regressive Functions

$\oplus$	Addition in Time Scales
$\oplus_W$	Stochastic Addition in Time Scales
$\ominus$	Subtraction in Time Scales
$\ominus_W$	Stochastic Subtraction in Time Scales
$\odot$	Multiplication in Time Scales
$\odot_W$	Stochastic Multiplication in Time Scales
$e_p(\cdot,\cdot)$	Exponential Function in Time Scales
$E_b(\cdot,\cdot)$	Stochastic Exponential in Time Scales
Ω	Arbitrary Space
$\mathcal{F}$	σ-algebra
$\mathbb{P}$	Probability Measure
$\mathbb{E}$	Expectation
$\mathbb{V}$	Variance
$\mathbb{C}ov$	Covariance
W	Brownian Motion or Wiener Process
$\mathcal{N}$	Gaussian Distribution
$L^2_\Delta(\mathbb{T})$	Space of L^2 Functions on $\mathbb{T}$
$\tilde{b}$	Shift or Delay of b
$b * r$	Convolution of Functions b and r
$t \wedge s$	Minimum of t and s

1. INTRODUCTION

The theory of time scales was introduced by Stefan Hilger [44] in 1998 in order to unify continuous and discrete analysis. This dissertation deals with stochastic dynamic equations on time scales. Many results concerning stochastic differential equations carry over quite easily to corresponding results in stochastic difference equations, while other results seem to be completely different in nature from their continuous counterparts. The study of stochastic dynamic equations reveals such discrepancies, and helps avoid proving results twice, once for stochastic differential equations and once for stochastic difference equations. The general idea is to prove a result for a stochastic dynamic equation, where the domain of the unknown function is a so-called time scale, which is an arbitrary nonempty closed subset of the reals. By choosing the time scale to be the set of real numbers, the general result yields a result concerning a stochastic differential equation. On the other hand, by choosing the time scale to be the set of integers, the same result yields a result in stochastic difference equations. However, since there are many other time scales than just the set of real numbers or the set of integers, one has a much more general result. We may summarize the above and state that **Unification** and **Extension** of stochastic equations are the two main features of this dissertation.

The results concerning Brownian motion given in this dissertation have been investigated from 1827 onward by pioneers like Robert Brown, Louis Bachelier, Langevin, Einstein, Smoluchowski, Fokker, Planck, Wiener, Uhlenbeck and many others [14, 31, 36, 96]. The theory of stochastic dynamic equations that has been developed in this dissertation closely follows the work of Itô [49–52] and others.

In Section 2 the time scale calculus is introduced. A time scale $\mathbb{T}$ is an arbitrary nonempty subset of reals. For functions $f : \mathbb{T} \to \mathbb{R}$ we define the derivative and integrals. Fundamental results, e.g., the product rule and the quotient rule, are also given. Generalized polynomials and exponential functions $e_p(t, s)$ for $\mathbb{T}$ are also defined and examples are given.

In Section 3 we give a brief introduction about stochastic differential equations. We list the problems that we attempt to generalize in subsequent sections.

In Section 4 we define and discuss basic properties of Brownian motion on time scales. We also give the corresponding Haar and Schauder functions for time scales and use them to construct Brownian motion.

In Section 5 we discuss stochastic integrals for time scales. We construct stochastic integrals for random step functions. For technical reason this result is not extended to general time scales. Next we define the quadratic variation of Brownian motion and use it to prove two product rules, one involving an arbitrary function and a random variable function and another involving two random variable functions.

In Section 6 we introduce stochastic dynamic equations which are the hybrid of stochastic differential equations and stochastic difference equations. We define the stochastic exponential function $E_b(\cdot, t_0)$ and give explicit solutions of stochastic dynamic equations (SΔE) in terms of $E_b(t, t_0)$ and $e_p(t, t_0)$, the exponential function on the time scale. We apply the theory of SΔE to stochastic volatility equations and show that the expected stock price is given by $\mathbb{E}[S(t)] = S_0 e_\alpha(t, t_0)$. We also present expectation and variance of the solution of the Ornstein–Uhlenbeck dynamic equation. In our theory we do not use Itô's calculus as is standard and they agree with known results when $\mathbb{T} = \mathbb{R}$. Lastly, an existence and uniqueness theorem is proved.

In Section 7 we give necessary and sufficient conditions for the almost sure asymptotic stability of solutions of some stochastic dynamic equations.

In Section 8 we first introduce the convolution on time scale and prove some basic results. Then we give stochastic dynamic equations of Volterra type and prove a result about the mean-square stability of its solution.

Thus, the setup of this dissertation is as follows. In Section 2 we introduce the notion of a time scale. In Section 3 we give a brief introduction about stochastic differential equations. In Section 4 we construct a one dimensional Wiener process for

isolated time scales. In Section 5, we introduce stochastic Itô integrals and prove some of its properties. In Section 6, stochastic dynamic equations (SΔEs) are introduced and an existence and uniqueness theorem is presented. We also give two examples involving stochastic dynamic equations, namely an equation governing a stock price (stochastic volatility) and the Ornstein–Uhlenbeck equation. In Section 7, we present some results about almost sure stability of SΔEs. In Section 8, we introduce convolution and present some results about mean-square stability of SΔEs of Volterra type.

2. TIME SCALES

In this section we introduce the basic results that we should know before reading the new results obtained in the remaining sections. The theory of measure chains was introduced by Stefan Hilger in his PhD dissertation [44] in 1988 in order to unify continuous and discrete analysis.

2.1. BASIC DEFINITIONS

Definition 2.1. A *time scale* (measure chain) $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$, where we assume $\mathbb{T}$ has the topology that it inherits from the real numbers $\mathbb{R}$ with the standard topology.

Aulbach and Hilger [13] gave a more general definition of a measure chain, but we will only consider the special case given in Definition 2.1. There are other time scales such as $h\mathbb{Z}$ ($h > 0$), the Cantor set, the set of harmonic numbers $\left\{ \sum_{k=1}^{n} \frac{1}{k} : n \in \mathbb{N} \right\}$, and so on. One is usually concerned with step size h, but in some cases one is interested in variable step size. A population of a species where all the adults die out before the babies are born is an example that could lead to a time scale which is the union of disjoint closed intervals. Any dynamic equation on $\mathbb{T} = \overline{q^{\mathbb{Z}}} := \{q^k : k \in \mathbb{Z}\} \cup \{0\}$, for some $q > 1$, is called a q-difference equation. These q-difference equations have been studied by Bézivin [16], Trijtzinsky [92], Zhang [59]. Also Derfel, Romanenko, and Sharkovsky [35] are concerned with the asymptotic behavior of solutions of nonlinear q-difference equations. Bohner and Lutz [27] investigate the asymptotic behavior of dynamic equations on time scales and also consider some q-difference equations.

The sets $\mathbb{T}^{\kappa}$ and $\mathbb{T}_{\kappa}$ are derived from $\mathbb{T}$ as follows: If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^{\kappa} = \mathbb{T} \setminus \{m\}$. Otherwise, $\mathbb{T}^{\kappa} = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum n, then $\mathbb{T}_{\kappa} = \mathbb{T} \setminus \{n\}$. Otherwise $\mathbb{T}_{\kappa} = \mathbb{T}$. Obviously a time scale $\mathbb{T}$ may or may not be connected. Therefore we introduce the concept of forward and backward

jump operators as follows.

Definition 2.2. Let $\mathbb{T}$ be a time scale and define the *forward jump operator* σ on $\mathbb{T}^{\kappa}$ by

$$\sigma(t) := \inf\{s > t : s \in \mathbb{T}\} \tag{2.1}$$

for all $t \in \mathbb{T}^{\kappa}$.

Definition 2.3. The *backward jump operator* ρ on $\mathbb{T}_{\kappa}$ is defined by

$$\rho(t) := \sup\{s < t : s \in \mathbb{T}\} \tag{2.2}$$

for all $t \in \mathbb{T}_{\kappa}$.

If $\sigma(t) > t$, we say t is *right-scattered*, while if $\rho(t) < t$, we say t is *left-scattered*. Points that are right-scattered and left-scattered at the same time are called *isolated*. If $\sigma(t) = t$, we say t is *right-dense*, while if $\rho(t) = t$, we say t is *left-dense*. In this dissertation, we make the blanket assumption that $\mathbb{T}$ refers to an isolated time scale which we define next.

Definition 2.4. We say a time scale $\mathbb{T}$ is *isolated* provided all the points in $\mathbb{T}$ are isolated.

Definition 2.5. The *graininess* function μ is a function $\mu : \mathbb{T}^{\kappa} \to \mathbb{R}$ defined by

$$\mu(t) := \sigma(t) - t \tag{2.3}$$

for all $t \in \mathbb{T}^{\kappa}$.

Table 2.1 gives a classification of points in $\mathbb{T}$ while Table 2.2 gives the forward, backward operators and the graininess function for some well known time scales.

Definition 2.6. The interval $[a, b]$ is the intersection of the real interval $[a, b]$ with the given time scale, that is $[a, b] \cap \mathbb{T}$.

Table 2.1: Classification of Points

t right-scattered	$t < \sigma(t)$
t right-dense	$t = \sigma(t)$
t left-scattered	$\rho(t) < t$
t left-dense	$\rho(t) = t$
t isolated	$\rho(t) < t < \sigma(t)$
t dense	$\rho(t) = t = \sigma(t)$

Table 2.2: Examples of Time Scales

$\mathbb{T}$	$\mu(t)$	$\sigma(t)$	$\rho(t)$
$\mathbb{R}$	0	t	t
$\mathbb{Z}$	1	$t + 1$	$t - 1$
$h\mathbb{Z}$	h	$t + h$	$t - h$
$\overline{q^{\mathbb{Z}}}$	$(q-1)t$	qt	$\frac{t}{q}$
$\overline{2^{\mathbb{Z}}}$	t	$2t$	$\frac{t}{2}$
$\mathbb{N}_0^2$	$2\sqrt{t} + 1$	$(\sqrt{t} + 1)^2$	$(\sqrt{t} - 1)^2,\ t \neq 0$

2.2. DIFFERENTIATION

Definition 2.7 (Hilger [45]). Assume $f : \mathbb{T} \to \mathbb{R}$ and let $t \in \mathbb{T}^{\kappa}$. Then we define $f^{\Delta}(t)$ to be the number (provided it exists) with the property that given any $\varepsilon > 0$, there is a neighborhood U of t such that

$$| [f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s] | \leq \varepsilon \, | \, \sigma(t) - s \, | \tag{2.4}$$

for all $s \in U$. We call $f^{\Delta}(t)$ the *delta derivative* of f at t. We say that $f : \mathbb{T} \to \mathbb{R}$ is (delta) differentiable if it is delta differentiable at any $t \in \mathbb{T}$.

Choosing the time scale to be the set of real numbers corresponds to the continuous case where $^{\Delta}$ is the usual derivative, and choosing the time scale to be isolated corresponds to the case where $^{\Delta}$ is the forward difference operator Δ defined by

$$\Delta f(t) = f(\sigma(t)) - f(t). \tag{2.5}$$

In the next two theorems we give some important properties of the delta derivative.

Theorem 2.8 (Hilger [45], Bohner and Peterson [28]). *Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^{\kappa}$. Then we have the following:*

(i) If f is differentiable at t, then f is continuous at t.

(ii) If f is continuous at t and t is right-scattered, then f is differentiable at t with

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}. \tag{2.6}$$

(iii) If f is differentiable at t and t is right-dense, then

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}. \tag{2.7}$$

(iv) If f is differentiable at t, then

$$f(\sigma(t)) = f(t) + \mu(t)f^{\Delta}(t). \tag{2.8}$$

Theorem 2.9 (Hilger [45], Bohner and Peterson [28]). *Assume $f, g : \mathbb{T} \to \mathbb{R}$ are delta differentiable at $t \in \mathbb{T}^{\kappa}$. Then*

(i) $f + g : \mathbb{T} \to \mathbb{R}$ is differentiable at t with

$$(f + g)^{\Delta}(t) = f^{\Delta}(t) + g^{\Delta}(t). \tag{2.9}$$

(ii) For any constant k, $kf : \mathbb{T} \to \mathbb{R}$ is differentiable at t with

$$(kf)^{\Delta}(t) = kf^{\Delta}(t). \tag{2.10}$$

(iii) $f, g : \mathbb{T} \to \mathbb{R}$ is differentiable at t with

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) = g^{\Delta}(t)f(t) + g(\sigma(t))f^{\Delta}(t). \tag{2.11}$$

(iv) If $f(t)f(\sigma(t)) \neq 0$, then $1/f$ is differentiable at t with

$$\left(\frac{1}{f}\right)^{\Delta}(t) = -\frac{f^{\Delta}(t)}{f(t)f(\sigma(t))}. \tag{2.12}$$

(v) If $g(t)g(\sigma(t)) \neq 0$, then f/g is differentiable at t and

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{g(t)f^{\Delta}(t) - f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}. \tag{2.13}$$

2.3. INTEGRATION

Definition 2.10. We say $f : \mathbb{T} \to \mathbb{R}$ is *right-dense continuous* (rd-continuous) provided f is continuous at each right-dense point $t \in \mathbb{T}$ and whenever $t \in \mathbb{T}$ is left-dense,

$$\lim_{s \to t^-} f(s)$$

exists as a finite number.

For example, the function $\mu : \mathbb{T} \to \mathbb{R}$ in case $\mathbb{T} = [0,1] \cup \mathbb{N}$ is rd-continuous but not continuous at 1. Note that if $\mathbb{T} = \mathbb{R}$, then $f : \mathbb{R} \to \mathbb{R}$ is rd-continuous on $\mathbb{T}$ if and only if f is continuous on $\mathbb{T}$. Also note that if $\mathbb{T} = \mathbb{Z}$, then any function $f : \mathbb{Z} \to \mathbb{R}$ is rd-continuous. We now state some elementary results concerning rd-continuous functions.

Theorem 2.11. *(i) Any continuous function on $\mathbb{T}$ is also rd-continuous on $\mathbb{T}$.*

(ii) If f is rd-continuous on $\mathbb{T}$, then $f \circ \sigma$ is rd-continuous on $\mathbb{T}^\kappa$.

(iii) If f and g are rd-continuous on $\mathbb{T}$, then $f + g$ and fg are rd-continuous on $\mathbb{T}$.

(iv) If f is continuous and g is rd-continuous, then $f \circ g$ is rd-continuous.

Definition 2.12. A function $F : \mathbb{T} \to \mathbb{R}$ is called a *delta antiderivative* of $f : \mathbb{T} \to \mathbb{R}$ provided $F^\Delta(t) = f(t)$ holds for all $t \in \mathbb{T}^\kappa$. In this case we define the integral of f by

$$\int_a^t f(s)\Delta s = F(t) - F(a)$$

for all $t \in \mathbb{T}$.

Hilger [45] proved that every rd-continuous function on $\mathbb{T}$ has a delta antiderivative. Using the different properties of differentiation, one can prove the following properties of the integral.

Theorem 2.13 (Bohner and Peterson [28])**.** *Assume $f, g : \mathbb{T} \to \mathbb{R}$ are rd-continuous. Then the following hold.*

(i) $\int_a^b [f(t) + g(t)]\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t$,

(ii) $\int_a^b kf(t)\Delta t = k \int_a^b f(t)\Delta t$,

(iii) $\int_a^b f(t)\Delta t = - \int_b^a f(t)\Delta t$,

(iv) $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t$,

(v) $\int_a^b f(\sigma(t))g^\Delta(t)\Delta t = [f(t)g(t)]_a^b - \int_a^b f^\Delta(t)g(t)\Delta t$,

(vi) $\int_a^b f(t)g^\Delta(t)\Delta t = [f(t)g(t)]_a^b - \int_a^b f^\Delta(t)g(\sigma(t))\Delta t$,

(vii) $\int_a^a f(t)\Delta t = 0$,

where $a, b, c \in \mathbb{T}$.

In the following theorem we give a well-known formula that we use frequently in later sections.

Theorem 2.14. *Assume $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous and $t \in \mathbb{T}^\kappa$. Then*

$$\int_t^{\sigma(t)} f(\tau)\Delta\tau = \mu(t)f(t). \tag{2.14}$$

Theorem 2.15 (Hilger [45])**.** *Assume $a, b \in \mathbb{T}$ and $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous. Then the integral has the following properties.*

(i) *If $\mathbb{T} = \mathbb{R}$, then $\int_a^b f(t)\Delta t = \int_a^b f(t)dt$, where the integral on the right-hand side is the Riemann integral.*

(ii) *If $\mathbb{T}$ consists of isolated points, then*

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{t\in[a,b)} f(t)\mu(t) & \text{if} \quad a < b \\ 0 & \text{if} \quad a = b \\ -\sum_{t\in[b,a)} f(t)\mu(t) & \text{if} \quad a > b. \end{cases}$$

(iii) If $\mathbb{T} = h\mathbb{Z}$, where $h > 0$, then

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{k=\frac{a}{h}}^{\frac{b}{h}-1} f(kh)h & if \quad a < b \\ 0 & if \quad a = b \\ -\sum_{k=\frac{b}{h}}^{\frac{a}{h}-1} f(kh)h & if \quad a > b. \end{cases}$$

(iv) If $\mathbb{T} = \mathbb{Z}$, then

$$\int_a^b f(t)\Delta t = \begin{cases} \sum_{t=a}^{b-1} f(t) & if \quad a < b \\ 0 & if \quad a = b \\ -\sum_{t=b}^{a-1} f(t) & if \quad a > b. \end{cases}$$

(v) If $\mathbb{T} = q^{\mathbb{N}_0}$, where $q > 1$, then

$$\int_a^b f(t)\Delta t = \begin{cases} (q-1)\sum_{t\in[a,b)} tf(t) & if \quad a < b \\ 0 & if \quad a = b \\ -(q-1)\sum_{t\in[b,a)} tf(t) & if \quad a > b. \end{cases}$$

2.4. GENERALIZED POLYNOMIALS

The generalized polynomials g_k, h_k [1,28] are the functions $g_k, h_k : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$, $k \in \mathbb{N}_0$, defined recursively as follows. The functions g_0 and h_0 are

$$g_0(t,s) = h_0(t,s) \equiv 1 \quad \text{for all} \quad s,t \in \mathbb{T}, \tag{2.15}$$

and given g_k and h_k for $k \in \mathbb{N}_0$, the functions g_{k+1} and h_{k+1} are

$$g_{k+1}(t,s) = \int_s^t g_k(\sigma(\tau),s)\Delta\tau \quad \text{for all} \quad s,t \in \mathbb{T} \tag{2.16}$$

and

$$h_{k+1}(t, s) = \int_s^t h_k(\tau, s) \Delta\tau \quad \text{for all} \quad s, t \in \mathbb{T}. \tag{2.17}$$

If we let $h_k^\Delta(t, s)$ denote for each fixed s the derivative of $h_k(t, s)$ with respect to t, then

$$h_k^\Delta(t, s) = h_{k-1}(t, s) \quad \text{for} \quad k \in \mathbb{N}, \ t \in \mathbb{T}^\kappa. \tag{2.18}$$

Similarly,

$$g_k^\Delta(t, s) = g_{k-1}(\sigma(t), s) \quad \text{for} \quad k \in \mathbb{N}, \ t \in \mathbb{T}^\kappa. \tag{2.19}$$

Here are some examples of polynomials in different time scales.

Example 2.16 (Bohner and Peterson [28]). (i) If $\mathbb{T} = \mathbb{R}$ and $k \in \mathbb{N}_0$, then

$$g_k(t, s) = h_k(t, s) = \frac{(t - s)^k}{k!} \quad \text{for all} \quad s, t \in \mathbb{R}.$$

(ii) If $\mathbb{T} = \mathbb{Z}$ and $k \in \mathbb{N}_0$, then

$$h_k(t, s) = \binom{t - s}{k} \quad \text{for all} \quad s, t \in \mathbb{Z}$$

and

$$g_k(t, s) = \binom{t - s + k - 1}{k} \quad \text{for all} \quad s, t \in \mathbb{Z}.$$

Here $\binom{\alpha}{\beta}$ is the *binomial coefficient* defined by $\binom{\alpha}{\beta} = \frac{\alpha^{(\beta)}}{\Gamma(\beta+1)}$ for all $\alpha, \beta \in \mathbb{C}$ such that the right-hand side of this equation makes sense, where Γ is the *gamma function* and $\alpha^{(\beta)}$ is the *factorial function* defined by $\alpha^{(\beta)} := \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\beta+1)}$ whenever the right-hand side is defined.

(iii) If $\mathbb{T} = \overline{q^{\mathbb{Z}}}$ and $q > 1$, then

$$h_k(t, s) = \prod_{i=0}^{k-1} \frac{t - q^i s}{\sum_{j=0}^i q^j} \quad \text{for all} \quad s, t \in \mathbb{T}$$

and

$$g_k(t, s) = \prod_{i=0}^{k-1} \frac{q^i s - t}{\sum_{j=0}^{i} q^j} \quad \text{for all} \quad s, t \in \mathbb{T}.$$

2.5. EXPONENTIAL FUNCTIONS

We will start with some technical notions given by Hilger [45] to define the exponential function on a general measure chain. He studies the complex exponential function on a measure chain as well. For $h > 0$, let $\mathbb{Z}_h$ be

$$\mathbb{Z}_h := \left\{ z \in \mathbb{C} : -\frac{\pi}{h} < \text{Im}(z) \leq \frac{\pi}{h} \right\},$$

and let $\mathbb{C}_h$ be defined by

$$\mathbb{C}_h := \left\{ z \in \mathbb{C} : z \neq -\frac{1}{h} \right\}.$$

For $h = 0$, let $\mathbb{Z}_0 = \mathbb{C}_0 = \mathbb{C}$, the set of complex numbers.

Definition 2.17. For $h > 0$, the *cylinder transformation* ξ_h is defined by

$$\xi_h(z) = \frac{1}{h} \text{Log}(1 + zh),$$

where Log is the principal logarithm function. For $h = 0$, we define $\xi_0(z) = z$ for all $z \in \mathbb{Z}_0 = \mathbb{C}$.

Definition 2.18. We say that a function $p : \mathbb{T} \to \mathbb{R}$ is *regressive* on $\mathbb{T}$ provided

$$1 + \mu(t)p(t) \neq 0 \quad \text{for all} \quad t \in \mathbb{T}.$$

The set of all regressive functions $\mathcal{R}$ (Bohner and Peterson [29]) on a time scale $\mathbb{T}$ forms an Abelian group under the addition $\oplus$ defined by

$$p \oplus q := p + q + \mu pq.$$

The additive inverse in this group is denoted by

$$\ominus p := -\frac{p}{1 + \mu p}.$$

We then define subtraction $\ominus$ on the set of regressive functions by

$$p \ominus q := p \oplus (\ominus q).$$

It can be shown that

$$p \ominus q = \frac{p - q}{1 + \mu q}.$$

Definition 2.19. We define the set $\mathcal{R}^+$ of all *positively regressive* elements of $\mathcal{R}$ by

$$\mathcal{R}^+ = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0 \text{ for all } t \in \mathbb{T}\}.$$

Definition 2.20. If $p : \mathbb{T} \to \mathbb{R}$ is regressive and rd-continuous, then we define the *exponential function* $e_p(\cdot, \cdot)$ by

$$e_p(t, s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right)$$

for $t \in \mathbb{T}$, $s \in \mathbb{T}^\kappa$, where ξ_h is the cylinder transformation.

Definition 2.21. The first order linear dynamic equation

$$y^\Delta = p(t)y \tag{2.20}$$

is said to be *regressive* provided p is regressive and rd-continuous on $\mathbb{T}$.

Theorem 2.22 (Hilger [45]). *Assume the dynamic equation* (2.20) *is regressive and fix* $t_0 \in \mathbb{T}^\kappa$. *Then* $e_p(\cdot, t_0)$ *is the unique solution of the initial value problem*

$$y^\Delta = p(t)y, \quad y(t_0) = 1 \tag{2.21}$$

on $\mathbb{T}$.

Theorem 2.23. *Let $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}$. The unique solution of the initial value problem*

$$y^\Delta = p(t)y, \quad y(t_0) = y_0 \tag{2.22}$$

is given by

$$y = e_p(\cdot, t_0)y_0. \tag{2.23}$$

We next give the variation of constants formulas for first order linear equations.

Theorem 2.24 (Bohner and Peterson [28]). *Suppose $p \in \mathcal{R}$ and $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous. Let $t_0 \in \mathbb{T}$ and $x_0 \in \mathbb{R}$. The unique solution of the initial value problem*

$$x^\Delta = -p(t)x^\sigma + f(t), \quad x(t_0) = x_0 \tag{2.24}$$

is given by

$$x(t) = e_{\ominus p}(t, t_0)x_0 + \int_{t_0}^t e_{\ominus p}(t, \tau)f(\tau)\Delta\tau. \tag{2.25}$$

Theorem 2.25 (Bohner and Peterson [28]). *Suppose $p \in \mathcal{R}$ and $f : \mathbb{T} \to \mathbb{R}$ is rd-continuous. Let $t_0 \in \mathbb{T}$ and $y_0 \in \mathbb{R}$. The unique solution of the initial value problem*

$$y^\Delta = p(t)y + f(t), \quad y(t_0) = y_0 \tag{2.26}$$

is given by

$$y(t) = e_p(t, t_0)y_0 + \int_{t_0}^t e_p(t, \sigma(\tau))f(\tau)\Delta\tau. \tag{2.27}$$

We next give some important properties of the exponential function.

Theorem 2.26 (Bohner and Peterson [28]). *Assume $p, q : \mathbb{T} \to \mathbb{R}$ are regressive and rd-continuous. Then the following hold.*

   (i) $e_0(t, s) \equiv 1$ and $e_p(t, t) \equiv 1$,

   (ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$,

   (iii) $1/e_p(t, s) = e_p(s, t) = e_{\ominus p}(t, s)$,

(iv) $e_p(t,s)e_p(s,r) = e_p(t,r)$ *(semigroup property),*

(v) $e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s),$

(vi) $e_p(t,s)/e_q(t,s) = e_{p\ominus q}(t,s).$

Here are some examples of exponential functions.

Example 2.27 (Bohner and Peterson [28]).   (i) If $\mathbb{T} = \mathbb{R}$, then

$$e_p(t,s) = \exp\left\{\int_s^t p(\tau)d\tau\right\}$$

for continuous p,

$$e_\alpha(t,s) = e^{\alpha(t-s)}$$

for constant α, and

$$e_1(t,0) = e^t.$$

(ii) If $\mathbb{T} = \mathbb{Z}$, then

$$e_p(t,s) = \prod_{\tau=s}^{t-1}(1 + p(\tau))$$

if p is never -1 (and for $s < t$),

$$e_\alpha(t,s) = (1+\alpha)^{t-s}$$

for constant α, and

$$e_1(t,0) = 2^t.$$

(iii) If $\mathbb{T} = h\mathbb{Z}$ for $h > 0$, then

$$e_p(t,0) = \prod_{j=0}^{\frac{t}{h}-1}[1 + hp(jh)],$$

for regressive p (and for $t > 0$),

$$e_\alpha(t, s) = (1 + h\alpha)^{\frac{t-s}{h}}$$

for constant α, and

$$e_1(t, 0) = (1 + h)^{\frac{t}{h}}.$$

(iv) If $\mathbb{T} = q^{\mathbb{N}_0} = \{q^k : k \in \mathbb{N}_0\}$, where $q > 1$, then it is easy to show that

$$e_p(t, 1) = \sqrt{t} \exp\left(\frac{-\ln^2(t)}{2\ln(q)}\right)$$

if $p(t) := (1 - t)/((q - 1)t^2)$.

(v) If $\mathbb{T} = \mathbb{N}_0^2 = \{k^2 : k \in \mathbb{N}_0\}$, then

$$e_1(t, 0) = 2^{\sqrt{t}}(\sqrt{t})!$$

(vi) If H_n are the harmonic numbers

$$H_0 = 0 \quad \text{and} \quad H_n = \sum_{k=1}^n \frac{1}{n} \quad \text{for} \quad n\mathbb{N}$$

and

$$\mathbb{T} = \{H_n : n \in \mathbb{N}_0\},$$

then

$$e_\alpha(H_n, 0) = \binom{n + \alpha}{n}.$$

3. STOCHASTIC DIFFERENTIAL EQUATION

In this section we give some basic results from stochastic differential equations, which we attempt to extend to time scales in the subsequent sections.

A stochastic process is a phenomenon which evolves with time in a random way. Thus, a stochastic process is a family of random variables $X(t)$, indexed by time (or in a more general framework by a set T). A realization or sample function of a stochastic process $\{X(t)\}_{t \in T}$ is an assignment, to each $t \in T$, of a possible value of $X(t)$. So we obtain a random curve which is referred to as a trajectory or a path of X.

A basic but very important example of a stochastic process is the Brownian motion process, whose name derives from the observation in 1827 by Robert Brown of the motion of the pollen particles in a liquid [31].

3.1. PROBABILITY THEORY

In this subsection we state some concepts from general probability theory. We refer the reader to [43, 62, 97] for more information.

Definition 3.1. If Ω is a given set, then a σ-algebra on Ω is a family $\mathcal{F}$ of subsets of Ω with the following properties:

(i) $\emptyset \in \mathcal{F}$,

(ii) $F \in \mathcal{F}$ implies $F^C \in \mathcal{F}$, where $F^C = \Omega \backslash F$ is the complement of F in Ω,

(iii) $A_1, A_2, \ldots \in \mathcal{F}$ implies $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

The pair $(\Omega, \mathcal{F})$ is called a measurable space.

Definition 3.2. A probability measure $\mathbb{P}$ on a measurable space $(\Omega, \mathcal{F})$ is a function $\mathbb{P} : \mathcal{F} \to [0, 1]$ such that

(i) $\mathbb{P}(\emptyset) = 1$, $\mathbb{P}(\Omega) = 1$,

(ii) if $A_1, A_2, \ldots \in \mathcal{F}$ and $\{A_i\}_{i=1}^{\infty}$ is disjoint (i.e., $A_i \cap A_j = \emptyset$ if $i \neq j$), then

$$\mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

The triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a probability space. It is called a complete probability space if $\mathcal{F}$ contains all subsets G of Ω with $\mathbb{P}$-outer measure zero, i.e., with

$$\mathbb{P}^*(G) := \inf\{\mathbb{P}(G) : F \in \mathcal{F}, G \subset F\} = 0.$$

We note that any probability space can be made complete by adding to $\mathcal{F}$ all sets of outer measure 0 and by extending $\mathbb{P}$ accordingly. The subsets F of Ω which belong to $\mathcal{F}$ are called $\mathcal{F}$-measurable sets.

Definition 3.3. If $(\Omega, \mathcal{F}, \mathbb{P})$ is a given probability space, then a function $X : \Omega \to \mathbb{R}$ is called $\mathcal{F}$-measurable if

$$X^{-1}(U) := \{\omega \in \Omega : X(\omega) \in U\} \in \mathcal{F}$$

for all open sets $U \subset \mathbb{R}$.

In the following we let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a given complete probability space. A random variable X is an $\mathcal{F}$-measurable function $X : \Omega \to \mathbb{R}$. Every random variable induces a probability measure λ_X on $\mathbb{R}$, defined by

$$\lambda_X(B) = \mathbb{P}(X^{-1}(B)).$$

λ_X is called the distribution of X.

Definition 3.4. If $\int_{\Omega} |X(\omega)| d\mathbb{P}(\omega) < \infty$, then the number

$$\mathbb{E}[X] := \int_{\Omega} X(\omega) d\mathbb{P}(\omega) = \int_{R} x d\lambda_X(x)$$

is called the expectation $\mathbb{E}$ of X (w.r.t. $\mathbb{P}$).

Definition 3.5. If $\int_\Omega |X(\omega)|^2 d\mathbb{P}(\omega) < \infty$, then the variance of a random variable X is given by

$$\mathbb{V}[X] = \mathbb{E}\left[(X - \mathbb{E}[X])^2\right] = \mathbb{E}\left[X^2\right] - (\mathbb{E}[X])^2.$$

Definition 3.6. The covariance between two random variables X and Y is given by

$$\mathbb{C}ov\left[X, Y\right] = \mathbb{E}\left[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])\right] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y].$$

Definition 3.7. Two subsets $A, B \in \mathcal{F}$ are called independent if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A) \cdot \mathbb{P}(B).$$

A collection $\mathcal{A} = \{\mathcal{H}_i : i \in I\}$ of families of $\mathcal{H}_i$ of measurable sets is called independent if

$$\mathbb{P}(H_{i_1} \cap H_{i_2} \cap \cdots \cap H_{i_k}) = \mathbb{P}(H_{i_1}) \cdots \mathbb{P}(H_{i_k})$$

for all choices of $H_{i_1} \in \mathcal{H}_{i_1}, \cdots, H_{i_k} \in \mathcal{H}_{i_k}$ with different indices $i_1, \ldots, i_k$. A collection of random variables $\{X_i : i \in I\}$ is called independent if the collection of generated σ-algebras $\mathcal{H}_{X_i}$ is independent.

If two random variables $X, Y : \Omega \to \mathbb{R}$ are independent, then $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$, provided that $\mathbb{E}[|X|] < \infty$ and $\mathbb{E}[|Y|] < \infty$.

Next we discuss conditional expectation.

Definition 3.8. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $X : \Omega \to \mathbb{R}$ be a random variable such that $\mathbb{E}[|X|] < \infty$. If $\mathcal{H} \subset \mathcal{F}$ is a σ-algebra, then the conditional expectation of X given $\mathcal{H}$, is defined as $\mathbb{E}[X|\mathcal{H}] =: Y$, where Y is a random variable satisfying

(i) $\mathbb{E}[|Y|] < \infty$,

(ii) $\mathbb{E}[X|\mathcal{H}]$ is $\mathcal{H}$-measurable,

(iii) $\int_H \mathbb{E}[X|\mathcal{H}] \, d\mathbb{P} = \int_H X d\mathbb{P}$ for all $H \in \mathcal{H}$.

We list some of the basic properties of the conditional expectation.

Theorem 3.9. *Suppose $Y : \Omega \to \mathbb{R}$ is another random variable with $\mathbb{E}[Y] < \infty$ and let $a, b \in \mathbb{R}$. Then*

(i) $\mathbb{E}[aX + bY|\mathcal{H}] = a\mathbb{E}[X|\mathcal{H}] + b\mathbb{E}[Y|\mathcal{H}]$,

(ii) $\mathbb{E}[\mathbb{E}[X|\mathcal{H}]] = \mathbb{E}[X]$,

(iii) $\mathbb{E}[X|\mathcal{H}] = X$ *if X is $\mathcal{H}$-measurable,*

(iv) $\mathbb{E}[X|\mathcal{H}] = \mathbb{E}[X]$ *if X is independent of $\mathcal{H}$,*

(v) $\mathbb{E}[YX|\mathcal{H}] = Y\mathbb{E}[X|\mathcal{H}]$ *if Y is $\mathcal{H}$-measurable.*

Next we define filtration and martingales.

Definition 3.10. A filtration on $(\Omega, \mathcal{F})$ is a family $\mathcal{M} = \{\mathcal{M}(t)\}_{t \in \mathbb{T}}$ of σ-algebras $\mathcal{M}(t) \subset \mathcal{F}$ such that

$$t_0 \leq s < t \quad \text{implies} \quad \mathcal{M}(s) \subset \mathcal{M}(t),$$

i.e., $\{\mathcal{M}(t)\}$ is increasing.

Definition 3.11. A stochastic process $\{M(t)\}_{t \in \mathbb{T}}$ on $(\Omega, \mathcal{F}, \mathbb{P})$ is called a martingale with respect to a filtration $\{\mathcal{M}(t)\}_{t \in \mathbb{T}}$ and with respect to $\mathbb{P}$ if

(i) $M(t)$ is $\mathcal{M}(t)$-measurable for all $t \in \mathbb{T}$

(ii) $\mathbb{E}[|M(t)|] < \infty$ for all $t \in \mathbb{T}$ and

(iii) $\mathbb{E}[M(s)|\mathcal{M}(t)] = M(t)$ for all $s, t \in \mathbb{T}$ with $s \geq t$.

If (iii) above is replaced by

$$\mathbb{E}[M(s)|\mathcal{M}(t)] \leq M(t) \quad \text{for all} \quad s, t \in \mathbb{T} \quad \text{with} \quad s \geq t$$

then $\{M(t)\}_{t \in \mathbb{T}}$ is called a supermartingale and if (iii) above is replaced by

$$\mathbb{E}[M(s)|\mathcal{M}(t)] \geq M(t) \quad \text{for all} \quad s, t \in \mathbb{T} \quad \text{with} \quad s \geq t$$

then $\{M(t)\}_{t \in \mathbb{T}}$ is called a submartingale.

3.2. STOCHASTIC DIFFERENTIAL EQUATIONS

In this subsection we give a brief introduction to stochastic differential equations. Let us fix $x_0 \in \mathbb{R}$ and for $t > 0$ consider the ordinary differential equation

$$\frac{dx}{dt} = a(x(t)), \quad x(0) = x_0, \tag{3.1}$$

where $a : \mathbb{R} \to \mathbb{R}$ is given and the solution is the trajectory $x : [0, \infty) \to \mathbb{R}$.

In many applications, the experimentally measured trajectories of systems modeled by (3.1) do not behave as predicted. Hence, it is reasonable to modify (3.1), somehow to include the possibility of random effects disturbing the system. A formal way to do so is to write

$$\frac{dX}{dt} = a(X(t)) + b(X(t))\zeta(t), \quad X(0) = X_0 \tag{3.2}$$

where $b : \mathbb{R} \to \mathbb{R}$ and ζ is *white noise.*

This approach presents us with these mathematical problems:

- Define what it means for X to solve (3.2).

- Show (3.2) has a solution, discuss asymptotic behavior, dependence upon X_0, a, b, etc.

If we let $X_0 = 0$, $a \equiv 0$, and $b \equiv 1$, then the solution of (3.2) turns out to be the Wiener process or Brownian motion denoted by W. Thus, we may symbolically write $dW/dt = \zeta$, thereby asserting that *white noise is the time derivative of the Wiener process.* Returning to (3.2), we have

$$\frac{dX}{dt} = a(X(t)) + b(X(t))\frac{dW}{dt},$$

which gives us

$$dX = a(X(t))dt + b(X(t))dW, \quad X(0) = X_0. \tag{3.3}$$

This expression is a *stochastic differential equation.* We say that X solves (3.3) provided

$$X(t) = X_0 + \int_0^t a(X(s))\, ds + \int_0^t b(X(s))\, dW \tag{3.4}$$

for all $t > 0$. Now we must

- Construct W.

- Define the stochastic integral.

- Find explicit solutions in special cases.

Next we look at the chain rule in stochastic calculus.

Definition 3.12. We denote by $\mathbb{L}^p(0, T)$, for $p \geq 1$, the space of all real-valued, progressively measurable stochastic processes X such that

$$\mathbb{E}\left[\int_0^T |X|^p(t)\, dt\right] < \infty. \tag{3.5}$$

Theorem 3.13 (Itô's Lemma). *Suppose that X has a stochastic differential*

$$dX = F(t)dt + G(t)dW,$$

for $F \in \mathbb{L}^1(0, T)$, $G \in \mathbb{L}^2(0, T)$. Assume $u : \mathbb{R} \times [0, T] \to \mathbb{R}$ is continuous and that $\partial u/\partial t$, $\partial u/\partial x$, $\partial^2 u/\partial x^2$ exist and are continuous. Set $Y(t) := u(X(t), t)$. Then Y

has the stochastic differential

$$
\begin{aligned}
dY &= \frac{\partial u(X(t), t)}{\partial t} dt + \frac{\partial u(X(t), t)}{\partial x} dX + \frac{1}{2} \frac{\partial^2 u(X(t), t)}{\partial x^2} G^2 dt \\
&= \left(\frac{\partial u(X(t), t)}{\partial t} + \frac{\partial u(X(t), t)}{\partial x} F(t) + \frac{1}{2} \frac{\partial^2 u(X(t), t)}{\partial x^2} G^2(t) \right) dt \\
&\quad + \frac{\partial u(X(t), t)}{\partial x} G(t) \, dW.
\end{aligned}
\tag{3.6}
$$

Example 3.14. Let us suppose that g is a continuous function. Then the unique solution of

$$
dY = g(t)Y \, dW, \quad Y(0) = 1
\tag{3.7}
$$

is

$$
Y(t) = \exp \left(-\frac{1}{2} \int_0^t g^2(s) \, ds + \int_0^t g(s) \, dW \right)
\tag{3.8}
$$

for $0 \le t \le T$. To verify this, note that

$$
X(t) := -\frac{1}{2} \int_0^t g^2(s) \, ds + \int_0^t g(s) \, dW
$$

satisfies

$$
dX = -\frac{1}{2} g^2(t) \, dt + g(t) \, dW.
$$

Thus, Itô's lemma for $u(x) = e^x$ gives

$$
\begin{aligned}
dY &= \frac{\partial u(X(t))}{\partial x} dX + \frac{1}{2} \frac{\partial^2 u(X(t))}{\partial x^2} g^2(t) \, dt \\
&= e^{X(t)} \left(-\frac{1}{2} g^2(t) \, dt + g(t) \, dW + \frac{1}{2} g^2(t) \, dt \right) \\
&= g(t)Y \, dW,
\end{aligned}
$$

as claimed.

Example 3.15. Similarly, the unique solution of

$$
dY = f(t)Y \, dt + g(t)Y \, dW, \quad Y(0) = 1
\tag{3.9}
$$

is

$$Y(t) = \exp\left(\int_0^t \left(f - \frac{1}{2}g^2\right)(s)\, ds + \int_0^t g(s)\, dW\right) \tag{3.10}$$

for $0 \leq t \leq T$.

Example 3.16. Let $S(t)$ denote the price of a stock at time t. We can model the evolution of $S(t)$ in time by supposing that $\frac{dS}{S}$, the relative change of price, evolves according to the SDE

$$\frac{dS}{S} = \alpha dt + \beta dW$$

for certain constants $\alpha > 0$ and β, called the drift and volatility of the stock. Hence,

$$dS = \alpha S dt + \beta S dW, \tag{3.11}$$

and so by Itô's formula

$$\begin{aligned}
d(\log(S)) &= \frac{dS}{S} - \frac{1}{2}\frac{\beta^2 S^2 dt}{S^2} \\
&= \left(\alpha - \frac{\beta^2}{2}\right) dt + \beta dW.
\end{aligned}$$

Consequently

$$S(t) = S_0 \exp\left(\beta W(t) + \left(\alpha - \frac{\beta^2}{2}\right)t\right).$$

The mean of $S(t)$ is given by

$$\mathbb{E}\left[S(t)\right] = S_0 \exp\left(\alpha(t - t_0)\right) \tag{3.12}$$

and its variance is

$$\mathbb{V}\left[S(t)\right] = S_0^2 \exp\left(2\alpha(t - t_0)\right)\left[\exp\left(\beta^2(t - t_0)\right) - 1\right]. \tag{3.13}$$

We refer to [56, 93] for further applications of stochastic differential equations. For a short history of stochastic integration and mathematical finance we refer to [53], and for Stratonovič stochastic integrals we refer to [88–90].

4. CONSTRUCTION OF BROWNIAN MOTION

In this section we construct Brownian motion on an isolated time scale. We also present some of the basic properties of Brownian motion.

4.1. BROWNIAN MOTION

4.1.1. Historical Remarks and Basic Definitions. In 1828, Robert Brown published a brief account of the microscopical observations made in the months of June, July and August, 1827 on the particles contained in the pollen of plants [31]. In 1900, Bachelier [14] postulated that stock prices execute Brownian motion, and he developed a mathematical theory which was similar to the theory which Einstein [36] developed. In 1923, Norbert Wiener proved the existence of Brownian motion and made significant contributions to related mathematical theories, so Brownian motion is often called a Wiener process [96].

This new branch of mathematics blossomed from the pioneering work of Kiyosi Itô [49–52]. Probably his most influential contribution was the development of an equation that describes the evolution of a random variable driven by Brownian motion. Itô's lemma, as mathematicians now call it, is a series expansion of a stochastic function giving the total differential.

The mathematical theory of Brownian motion has been applied in contexts ranging far beyond the movement of particles in fluids. In 1973, Fischer Black, Myron Scholes and Robert Merton [18, 60] used stochastic analysis and an equilibrium argument to compute a theoretical value for an options' price. This is now called the Black and Scholes option price formula or Black and Scholes model.

This brief list, of course, does not do justice to the work of many other people who have written about Brownian motion.

4.1.2. Stochastic Processes. We begin our study by defining a stochastic process on a time scale.

Definition 4.1. A stochastic process is a parameterized collection of random variables

$$\{X(t)\}_{t \in \mathbb{T}}$$

defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and assuming values in $\mathbb{R}$.

The parameter space $\mathbb{T}$ is usually the half line $[0, \infty)$, but it may also be an interval $[a, b]$, the nonnegative integers and even subsets of $\mathbb{R}$. In this dissertation, we focus on those parameter spaces for which $\rho(t) < t < \sigma(t)$ for all $t \in \mathbb{T}$. Such a parameter space is called an isolated time scale (Definition 2.4). We will denote an isolated time scale by $\mathbb{T}$ throughout.

An important class of stochastic processes are those with *independent increments*, that is, for which the random variables $\{\Delta X(t)\}_{t \in \mathbb{T}}$ are independent for any finite combination of time instants in $\mathbb{T}$. A *Brownian motion* or a *standard Wiener process* $W = \{W(t)\}_{t \in \mathbb{T}}$ is an example of a stochastic process with independent increments which we define next.

Definition 4.2. A real-valued stochastic process W is called a Brownian motion or Wiener process on $\mathbb{T}$ if

  (i) $W(t_0) = 0$ a.s.,

  (ii) $W(t) - W(s) \sim \mathcal{N}(0, t - s)$ for all $t_0 \leq s \leq t \in \mathbb{T}$,

  (iii) for all times $t_{i_0} < t_{i_1} < t_{i_2} < \ldots < t_{i_n}$, the random variables

$$W(t_{i_0}), W(t_{i_1}) - W(t_{i_0}), \ldots, W(t_{i_n}) - W(t_{i_{n-1}})$$

    are independent (*independent increments*),

for $t_0, t, s \in \mathbb{T}$ and $\mathcal{N}(0, t - s)$ is the normal distribution with mean 0 and variance $t - s$.

Theorem 4.3. *For an isolated time scale* $\mathbb{T} = \{t_0, t_1, t_2, \ldots\}$, W *is Brownian motion if and only if*

    (i) $W(t_0) = 0$ *a.s.,*

    (ii) $\Delta W(t) \sim \mathcal{N}(0, \mu(t))$ *for all* $t \in \mathbb{T}$,

    (iii) for all $t \in \mathbb{T}$, *the random variables* $\Delta W(t)$ *are independent (*independent incre-ments*)*.

Proof. It is obvious that Definition 4.2 reduces to the assumptions of this theorem if we choose $t_{j_i} = t_j$ for $j \in \mathbb{N}_0$. To see that Definition 4.2 follows from the assumption of this theorem we observe that for $t_0 < t_m < t_n$, $\sum_{i=m}^{n-1} \mathcal{N}(0, \mu(t_i))$ has the same distribution as $\mathcal{N}\left(0, \sum_{i=m}^{n-1} \mu(t_i)\right)$ or $\mathcal{N}(0, t_n - t_m)$. $\qquad\square$

4.1.3. Properties of Brownian Motion. In this part, we prove some of the basic properties of Brownian motion which we use in subsequent sections.

Lemma 4.4. $\mathbb{E}[W(t)] = 0$ *and* $\mathbb{E}[W^2(t)] = t - t_0$ *for each time* $t \geq t_0$.

Proof. We observe that $W(t) - W(t_0) \sim \mathcal{N}(0, t - t_0)$ and that

$$\mathbb{E}[W(t) - W(t_0)] = \mathbb{E}[W(t)] = 0$$

and

$$
\begin{aligned}
\mathbb{E}[W^2(t)] &= \mathbb{E}[W^2(t)] - (\mathbb{E}[W(t)])^2 \\
&= \mathbb{V}[W(t)] \\
&= \mathbb{V}[W(t) - W(t_0)] \\
&= t - t_0.
\end{aligned}
$$

This concludes the proof. $\qquad\square$

Definition 4.5. For $t, s \in \mathbb{T}$, we define $t \wedge s$ as the minimum of t and s.

Lemma 4.6. *Suppose W is a one-dimensional Brownian motion. Then*

$$\mathbb{E}[W(t)W(s)] = (t \wedge s) - t_0 \quad \text{for all} \quad t, s \in \mathbb{T}. \tag{4.1}$$

Proof. Let us assume $t_0 \leq s < t$. Then

$$
\begin{aligned}
\mathbb{C}ov[W(t), W(s)] = \mathbb{E}[W(t)W(s)] &= \mathbb{E}[(W(s) + W(t) - W(s))W(s)] \\
&= \mathbb{E}[W^2(s)] + \mathbb{E}[(W(t) - W(s))W(s)] \\
&= s - t_0 + \mathbb{E}[W(t) - W(s)]\underbrace{\mathbb{E}[W(s)]}_{=0} \\
&= s - t_0 \\
&= (t \wedge s) - t_0,
\end{aligned}
$$

since $W(s) \sim \mathcal{N}(0, s)$ and $W(t) - W(s)$ is independent of $W(s)$. $\qquad\square$

Theorem 4.7. *Brownian motion $\{W(t)\}_{t \in \mathbb{T}}$ is a martingale w.r.t. the σ-algebras $\mathcal{F}(t)$ generated by $\{W(s) : s \leq t\}$.*

Proof. We show that W satisfies the conditions given in Definition 3.11. From Cauchy–Schwarz inequality we have,

$$(\mathbb{E}[W(t)])^2 \leq \mathbb{E}[|W(t)|^2] = t - t_0.$$

Also, for all $t_0 \leq s \leq t < \infty$ and $t_0, s, t \in \mathbb{T}$, we have

$$
\begin{aligned}
\mathbb{E}[W(t)|\mathcal{F}(s)] &= \mathbb{E}[W(s) + W(t) - W(s)|\mathcal{F}(s)] \\
&= \mathbb{E}[W(s)|\mathcal{F}(s)] + \mathbb{E}[(W(t) - W(s))|\mathcal{F}(s)] \\
&= W(s) + 0 = W(s).
\end{aligned}
$$

Here we have used that $\mathbb{E}[W(t) - W(s)|\mathcal{F}(s)] = 0$ since $W(t) - W(s)$ is independent

of $\mathcal{F}(t)$ and we have used that $\mathbb{E}\left[W(s)|\mathcal{F}(s)\right] = W(s)$ since $W(s)$ is $\mathcal{F}(s)$-measurable.

□

Theorem 4.8. $W^2(t) - t$ *is a martingale.*

Proof. For $t > s > t_0$ we have

$$
\begin{aligned}
\mathbb{E}\left[W^2(t) - t|\mathcal{F}(s)\right] &= \mathbb{E}\left[W^2(t)|\mathcal{F}(s)\right] - t \\
&= \mathbb{E}\left[(W(s) + W(t) - W(s))^2|\mathcal{F}(s)\right] - t \\
&= \mathbb{E}\left[W^2(s)|\mathcal{F}(s)\right] - 2\mathbb{E}\left[W(s)(W(t) - W(s))|\mathcal{F}(s)\right] \\
&\quad + \mathbb{E}\left[(W(t) - W(s))^2|\mathcal{F}(s)\right] - t \\
&= W^2(s) + 2W(s)\mathbb{E}\left[W(t) - W(s)|\mathcal{F}(s)\right] \\
&\quad + \mathbb{E}\left[(W(t) - W(s))^2|\mathcal{F}(s)\right] - t \\
&= W^2(s) + 0 + t - s - t \\
&= W^2(s) - s,
\end{aligned}
$$

where on the fourth equality we have used Definition 4.2. □

Theorem 4.9. *Suppose $c > 0$. Let W_c be a Brownian motion on $\mathbb{T}_c := \{c^2t : t \in \mathbb{T}\}$ with $W_c(c^2t_0) = 0$. Then $W(t) := c^{-1}W_c(c^2t)$ is a Brownian motion on $\mathbb{T}$.*

Proof. We have $\mathbb{E}\left[W(t)\right] = c^{-1}\mathbb{E}\left[W_c(c^2t)\right] = 0$ and $\mathbb{V}\left[W(t)\right] = c^{-2}c^2t = t$. Also

$$
\begin{aligned}
\mathbb{C}ov\left[W(t), W(s)\right] &= c^{-1}c^{-1}\mathbb{C}ov\left[W_c(c^2t), W_c(c^2s)\right] \\
&= c^{-2}\left[(c^2t \wedge c^2s) - c^2t_0\right] \\
&= (t \wedge s) - t_0,
\end{aligned}
$$

where in the second equality we have used Lemma 4.6. □

Next we give some possible directions about constructing a Wiener process on isolated time scales.

4.2. BUILDING A ONE-DIMENSIONAL BROWNIAN MOTION

The existence of the Brownian motion process follows from Kolmogorov's existence theorem [17]. Our method will be to develop a formal expansion of ΔW in terms of an orthonormal basis of $L^2_\Delta(\mathbb{T})$ functions on $\mathbb{T}$. We then integrate the resulting expression in time and prove then that we have built a Wiener process.

Theorem 4.10 (Agarwal, Otero-Espinar, Perera, Vivero [2]). *Let $J^o = [t_0, t) \cap \mathbb{T}$, $t_0, t \in \mathbb{T}$, $t_0 < t$, be an arbitrary closed interval of $\mathbb{T}$. Then, the set $L^p_\Delta(J^o)$ is a Banach space together with the norm defined for every $f \in L^p_\Delta(J^o)$ as*

$$||f||_{L^p_\Delta} := \left[\int_{J^o} |f|^p(\tau)\Delta\tau \right]^{1/p} \quad for \quad p \in \mathbb{R}. \tag{4.2}$$

Moreover, $L^2_\Delta(J^o)$ is a Hilbert space together with the inner product given for every $(f, g) \in L^2_\Delta(J^o) \times L^2_\Delta(J^o)$ by

$$(f, g)_{L^2_\Delta} := \int_{J^o} f(\tau)g(\tau)\Delta\tau. \tag{4.3}$$

Definition 4.11. Two functions $f, g : \mathbb{T} \to \mathbb{R}$ are orthonormal over $J^o = [t_0, t) \cap \mathbb{T}$ if

(i) $(f, g)_{L^2_\Delta} = \int_{J^o} f(\tau)g(\tau)\Delta\tau = 0$, and

(ii) $||f||_{L^2_\Delta} = ||g||_{L^2_\Delta} = \left[\int_{J^o} |f|^2(\tau)\Delta\tau \right]^{1/2} = \left[\int_{J^o} |g|^2(\tau)\Delta\tau \right]^{1/2} = 1$.

4.2.1. Haar Functions. The Haar function is the first known wavelet and was proposed in 1909 by Alfréd Haar [41]. We use Haar functions on isolated time scales to construct Brownian motion.

Definition 4.12. The family $\{h_{mn}\}_{m,n \in \mathbb{N}_0}$ of *Haar functions* is defined for $t \in \mathbb{T}$ as follows:

$$h_{00}(t) = \frac{1}{\sqrt{\sum_{t_i \in \mathbb{T}} \mu(t_i)}} \quad for \quad t \in \mathbb{T}.$$

For $n \in \mathbb{N}$, we let $n' = n - 1$. Then

$$
h_{0n}(t) = \begin{cases} \sqrt{\frac{\mu(t_{2n'+1})}{\mu(t_{2n'})[\mu(t_{2n'})+\mu(t_{2n'+1})]}} & \text{if } t = t_{2n'} \\[3ex] -\sqrt{\frac{\mu(t_{2n'})}{\mu(t_{2n'+1})[\mu(t_{2n'})+\mu(t_{2n'+1})]}} & \text{if } t = t_{2n'+1} \\[3ex] 0 & \text{otherwise,} \end{cases}
$$

$$
h_{1n}(t) = \begin{cases} \sqrt{\frac{\mu(t_{4n'+2})+\mu(t_{4n'+3})}{[\mu(t_{4n'})+\mu(t_{4n'+1})][\mu(t_{4n'})+\mu(t_{4n'+1})+\mu(t_{4n'+2})+\mu(t_{4n'+3})]}} & \text{if } t = t_{4n'}, t_{4n'+1} \\[3ex] -\sqrt{\frac{\mu(t_{4n'})+\mu(t_{4n'+1})}{[\mu(t_{4n'+2})+\mu(t_{4n'+3})][\mu(t_{4n'})+\mu(t_{4n'+1})+\mu(t_{4n'+2})+\mu(t_{4n'+3})]}} & \text{if } t = t_{4n'+2}, t_{4n'+3} \\[3ex] 0 & \text{otherwise.} \end{cases}
$$

In general for $m \in \mathbb{N}_0$, $n \in \mathbb{N}$ and $n' = n - 1$, we have

$$
h_{mn}(t) = \begin{cases} \sqrt{\frac{\sum_{i=k}^{2k-1} \mu(t_{i+2n'k})}{\sum_{i=0}^{k-1} \mu(t_{i+2n'k}) \sum_{i=0}^{2k-1} \mu(t_{i+2n'k})}} & \text{if } t_{2n'k} \le t \le t_{2n'k+k-1} \\[3ex] -\sqrt{\frac{\sum_{i=0}^{k-1} \mu(t_{i+2n'k})}{\sum_{i=k}^{2k-1} \mu(t_{i+2n'k}) \sum_{i=0}^{2k-1} \mu(t_{i+2n'k})}} & \text{if } t_{2n'k+k} \le t \le t_{2n'k-2k-1} \\[3ex] 0 & \text{otherwise,} \end{cases}
$$

where $k = 2^m$.

Example 4.13. When $\mathbb{T} = \mathbb{Z}$ we have $\mu(t) = 1$. In this case the Haar functions are

given by

$$
h_{0n}(t) = \begin{cases} \sqrt{\frac{1}{2}} & \text{if } t = t_{2n} \\[2ex] -\sqrt{\frac{1}{2}} & \text{if } t = t_{2n-1} \\[2ex] 0 & \text{otherwise,} \end{cases}
$$

$$
h_{1n}(t) = \begin{cases} \sqrt{\frac{1}{4}} & \text{if } t = t_{4n}, t_{4n+1} \\[2ex] -\sqrt{\frac{1}{4}} & \text{if } t = t_{4n+2}, t_{4n+3} \\[2ex] 0 & \text{otherwise.} \end{cases}
$$

In general, we have

$$
h_{mn}(t) = \begin{cases} \sqrt{\frac{1}{2^{m+1}}} & \text{if } t_{n2^{m+1}} \le t \le t_{n2^{m+1}+2^m-1} \\[2ex] -\sqrt{\frac{1}{2^{m+1}}} & \text{if } t_{n2^{m+1}+2^m} \le t \le t_{n2^{m+1}+2^{m+1}-1} \\[2ex] 0 & \text{otherwise.} \end{cases}
$$

Lemma 4.14. *The functions* $\{h_{mn}\}_{m,n\in\mathbb{N}_0}$ *form an orthonormal basis of* $L_\Delta^2(\mathbb{T})$.

Proof. We have

$$
\begin{aligned}
\int_{\mathbb{T}} h_{mn}^2(t)\Delta t &= \sum_{i=0}^{2^m-1} \mu(t_{i+n2^{m+1}}) \left[\frac{\sum_{i=2^m}^{2^{m+1}-1} \mu(t_{i+n2^{m+1}})}{\sum_{i=0}^{2^m-1} \mu(t_{i+n2^{m+1}}) \sum_{i=0}^{2^{m+1}-1} \mu(t_{i+n2^{m+1}})} \right] \\
&\quad + \sum_{i=2^m}^{2^{m+1}-1} \mu(t_{i+n2^{m+1}}) \left[\frac{\sum_{i=0}^{2^m-1} \mu(t_{i+n2^{m+1}})}{\sum_{i=2^m}^{2^{m+1}-1} \mu(t_{i+n2^{m+1}}) \sum_{i=0}^{2^{m+1}-1} \mu(t_{i+n2^{m+1}})} \right] \\
&= 1.
\end{aligned}
$$

Also for $m' > m$, either $h_{mn}h_{m'n} = 0$ for all t or else h_{mn} is constant on the support of $h_{m'n}$. In this second case,

$$\int_{\mathbb{T}} h_{mn}(t)h_{m'n}(t)\Delta t = h_{mn}\int_{\mathbb{T}} h_{m'n}(t)\Delta t = 0.$$

This completes the proof. $\qquad\square$

Example 4.15. For Haar functions in $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$, we refer to Table 4.1. To make the table compact, we let $p = q - 1$ and $[n] = \sum_{k=0}^{n-1} q^k$.

Table 4.1: Haar Functions for $\mathbb{T} = \{1, q, q^2, q^3, q^4, q^5, q^6, q^7\}$.

	$h_{00}(t)$	$h_{20}(t)$	$h_{10}(t)$	$h_{11}(t)$	$h_{01}(t)$	$h_{02}(t)$	$h_{03}(t)$	$h_{04}(t)$
1	$\frac{1}{\sqrt{[8]p}}$	$\frac{q^2}{\sqrt{[8]p}}$	$\frac{q}{\sqrt{[4]p}}$	0	$\sqrt{\frac{q}{[2]p}}$	0	0	0
q	$\frac{1}{\sqrt{[8]p}}$	$\frac{q^2}{\sqrt{[8]p}}$	$\frac{q}{\sqrt{[4]p}}$	0	$\frac{-1}{\sqrt{[2]pq}}$	0	0	0
q^2	$\frac{1}{\sqrt{[8]p}}$	$\frac{q^2}{\sqrt{[8]p}}$	$\frac{-1}{q\sqrt{[4]p}}$	0	0	$\frac{1}{\sqrt{[2]pq}}$	0	0
q^3	$\frac{1}{\sqrt{[8]p}}$	$\frac{q^2}{\sqrt{[8]p}}$	$\frac{-1}{q\sqrt{[4]p}}$	0	0	$\frac{-1}{\sqrt{[2]pq^3}}$	0	0
q^4	$\frac{1}{\sqrt{[8]p}}$	$\frac{-1}{q^2\sqrt{[8]p}}$	0	$\frac{1}{q\sqrt{[4]p}}$	0	0	$\frac{-1}{\sqrt{[2]pq^3}}$	0
q^5	$\frac{1}{\sqrt{[8]p}}$	$\frac{-1}{q^2\sqrt{[8]p}}$	0	$\frac{1}{q\sqrt{[4]p}}$	0	0	$\frac{-1}{\sqrt{[2]pq^5}}$	0
q^6	$\frac{1}{\sqrt{[8]p}}$	$\frac{-1}{q^2\sqrt{[8]p}}$	0	$\frac{-1}{q^3\sqrt{[4]p}}$	0	0	0	$\frac{1}{\sqrt{[2]pq^5}}$
q^7	$\frac{1}{\sqrt{[8]p}}$	$\frac{-1}{q^2\sqrt{[8]p}}$	0	$\frac{-1}{q^3\sqrt{[4]p}}$	0	0	0	$\frac{-1}{\sqrt{[2]pq^7}}$

4.2.2. Schauder Functions and Wiener Processes.

Definition 4.16. For $m, n \in \mathbb{N}_0$,

$$s_{mn}(t) := \int_{t_0}^{t} h_{mn}(\tau)\Delta\tau \tag{4.4}$$

is called the mn^{th} Schauder function.

Let us assume that $k = 2^m$. Then the graph of s_{mn} is an *open tent* lying above the interval $[t_{2nk}, t_{2nk+2k}]$. The highest point on this tent can be found in the following manner.

$$
\begin{aligned}
\max_{t \in \mathbb{T}} |s_{mn}(t)| &= \int_{t_0}^{t_{2nk+k}} h_{mn}(\tau) \Delta\tau \\
&= \int_{t_{2nk}}^{t_{2nk+k}} h_{mn}(\tau) \Delta\tau \\
&= \sum_{i=0}^{k-1} \mu(t_{i+2nk}) \sqrt{\frac{\sum_{i=k}^{2k-1} \mu(t_{i+2nk})}{\sum_{i=0}^{k-1} \mu(t_{i+2nk}) \sum_{i=0}^{2k-1} \mu(t_{i+2nk})}} \\
&= \sqrt{\frac{\sum_{i=0}^{k-1} \mu(t_{i+2nk}) \sum_{i=k}^{2k-1} \mu(t_{i+2nk})}{\sum_{i=0}^{2k-1} \mu(t_{i+2nk})}}.
\end{aligned}
$$

Next we define

$$
W(t) := \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} Z_{mn}(\omega) s_{mn}(t)
$$

for times $t \in \mathbb{T}$, where the coefficients $\{Z_{mn}\}_{m,n \in \mathbb{N}_0}$ are independent and $\mathcal{N}(0,1)$ random variables defined on some probability space. This series does not converge for all $\mathbb{T}$. For those for which this series does converge, the following holds.

Lemma 4.17. *We have*

$$
\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} s_{mn}(t) s_{mn}(s) = (t \wedge s) - t_0
$$

for each $t, s \in \mathbb{T}$.

Proof. For each $s \in \mathbb{T}$, let us define

$$
\phi_s(\tau) = \begin{cases} 1 & \text{if } t_0 \leq \tau \leq s \\ \\ 0 & \text{otherwise.} \end{cases}
$$

Then using Definition 4.16 and Lemma 4.14, we have

$$
\begin{aligned}
\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} s_{mn}(t)s_{mn}(s) &= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\int_{t_0}^{t} h_{mn}(\tau)\Delta\tau \int_{t_0}^{s} h_{mn}(\tilde{\tau})\Delta\tilde{\tau}\\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\int_{t_0}^{\infty} \phi_t(\tau)h_{mn}(\tau)\Delta\tau \int_{t_0}^{\infty} \phi_s(\tilde{\tau})h_{mn}(\tilde{\tau})\Delta\tilde{\tau}\\
&= \int_{t_0}^{\infty}\int_{t_0}^{\infty} \phi_t(\tau)\phi_s(\tilde{\tau})\left[\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} h_{mn}(\tau)h_{mn}(\tilde{\tau})\right]\Delta\tau\Delta\tilde{\tau}\\
&= \int_{t_0}^{\infty} \phi_t(\tau)\phi_s(\tau)\Delta\tau\\
&= \int_{t_0}^{t\wedge s}\Delta\tau\\
&= (t\wedge s) - t_0,
\end{aligned}
$$

where we observe that for fixed $m,n\in\mathbb{N}$, the above sums and integrals are finite thereby permitting us to interchange the integrations with summations. $\qquad\square$

Theorem 4.18. *Let $\{Z_{mn}\}_{m,n\in\mathbb{N}_0}$ be a sequence of independent and $\mathcal{N}(0,1)$ random variables defined on the same probability space. Then the sum*

$$
W(t,\omega) := \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} Z_{mn}(\omega)s_{mn}(t),
$$

is a Brownian motion for $t\in\mathbb{T}$.

Proof. To prove W is a Brownian motion, we first note that clearly $W(t_0) = 0$ a.s. We assert that $W(t) - W(s) \sim \mathcal{N}(0, t-s)$ for all $s,t\in\mathbb{T}$ such that $s\leq t$. To prove this let us compute

$$
\begin{aligned}
&\mathbb{E}\left[\exp\left(i\lambda(W(t)-W(s))\right)\right]\\
&= \mathbb{E}\left[\exp\left(i\lambda\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} Z_{mn}(s_{mn}(t)-s_{mn}(s))\right)\right]\\
&= \prod_{m=0}^{\infty}\prod_{n=0}^{\infty}\mathbb{E}\left[\exp\left(i\lambda Z_{mn}(s_{mn}(t)-s_{mn}(s))\right)\right]
\end{aligned}
$$

$$= \prod_{m=0}^{\infty} \prod_{n=0}^{\infty} \exp\left(-\frac{\lambda^2}{2}(s_{mn}(t) - s_{mn}(s))^2\right)$$

$$= \exp\left(-\frac{\lambda^2}{2} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty}(s_{mn}^2(t) - 2s_{mn}(t)s_{mn}(s) + s_{mn}^2(s))\right)$$

$$= \exp\left(-\frac{\lambda^2}{2}(t - t_0 - 2(s - t_0) + s - t_0)\right)$$

$$= \exp\left(-\frac{\lambda^2}{2}(t - s)\right),$$

where second equality follows from independence and for the third equality we have used the fact that Z_{mn} is $\mathcal{N}(0, 1)$. By the uniqueness of characteristic functions, the increment $W(t) - W(s)$ is $\mathcal{N}(0, t - s)$ distributed, as asserted. Next we claim for all $p \in \mathbb{N}$ and for all $t_0 < t_1 < t_2 < \ldots < t_p$, that

$$\mathbb{E}\left[\exp\left(i \sum_{j=1}^{p} \lambda_j(W(t_j) - W(t_{j-1}))\right)\right] = \prod_{j=1}^{p} \exp\left(-\frac{\lambda_j^2}{2}(t_j - t_{j-1})\right). \qquad (4.5)$$

Once this is proved, we will know from uniqueness of characteristic functions that

$$F_{W(t_1),\ldots,W(t_p)-W(t_{p-1})}(x_1, \ldots, x_p) = F_{W(t_1)}(x_1) \cdots F_{W(t_p)-W(t_{p-1})}(x_p)$$

for all $x_1, x_2, \ldots, x_p \in \mathbb{R}$. This proves that

$$W(t_1), \ldots, W(t_p) - W(t_{p-1}) \quad \text{are independent.}$$

Thus, (4.5) will establish the theorem. Now in the case $p = 2$, we have

$$\mathbb{E}\left[\exp\left(i[\lambda_1 W(t_1) + \lambda_2(W(t_2) - W(t_1))]\right)\right]$$

$$= \mathbb{E}\left[\exp\left(i[(\lambda_1 - \lambda_2)W(t_1) + \lambda_2 W(t_2)]\right)\right]$$

$$= \mathbb{E}\left[\exp\left(i(\lambda_1 - \lambda_2)\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}Z_{mn}s_{mn}(t_1) + i\lambda_2\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}Z_{mn}s_{mn}(t_2)\right)\right]$$

$$= \prod_{m=0}^{\infty}\prod_{n=0}^{\infty}\mathbb{E}\left[\exp\left(iZ_{mn}((\lambda_1 - \lambda_2)s_{mn}(t_1) + \lambda_2 s_{mn}(t_2))\right)\right]$$

$$= \prod_{m=0}^{\infty} \prod_{n=0}^{\infty} \exp\left(-\frac{1}{2}\left[(\lambda_1 - \lambda_2)s_{mn}(t_1) + \lambda_2 s_{mn}(t_2)\right]^2\right)$$

$$= \exp\left(-\frac{1}{2}\sum_{m=0}^{\infty}\sum_{m=0}^{\infty}\left[(\lambda_1 - \lambda_2)^2 s_{mn}^2(t_1) + 2(\lambda_1 - \lambda_2)\lambda_2 s_{mn}(t_1)s_{mn}(t_2)\right]\right)$$

$$+ \exp\left(-\frac{1}{2}\sum_{m=0}^{\infty}\sum_{m=0}^{\infty}\lambda_2^2 s_{mn}^2(t_2)\right)$$

$$= \exp\left(-\frac{1}{2}\left[(\lambda_1 - \lambda_2)^2(t_1 - t_0) + 2(\lambda_1 - \lambda_2)\lambda_2(t_1 - t_0) + \lambda_2^2(t_2 - t_0)\right]\right)$$

$$= \exp\left(-\frac{1}{2}\left[\lambda_1^2(t_1 - t_0) + \lambda_2^2(t_2 - t_1)\right]\right), \tag{4.6}$$

where on the sixth equality we have used Lemma 4.17. We observe that (4.6) is same as (4.5) for $p = 2$. The general case follows similarly. $\square$

In Figures 4.1, 4.2, 4.3, 4.4 we plot the Haar functions for $\mathbb{T} = \{1, 2, 4, 8\}$ while the corresponding Schauder functions are given in Figures 4.5, 4.6, 4.7, 4.8 and in Figure 4.9 we plot the generated Wiener process. In Figure 4.10 we plot the Haar functions for $\mathbb{T} = \{1, 2, 4, 8, 16, 32, 64, 128\}$.
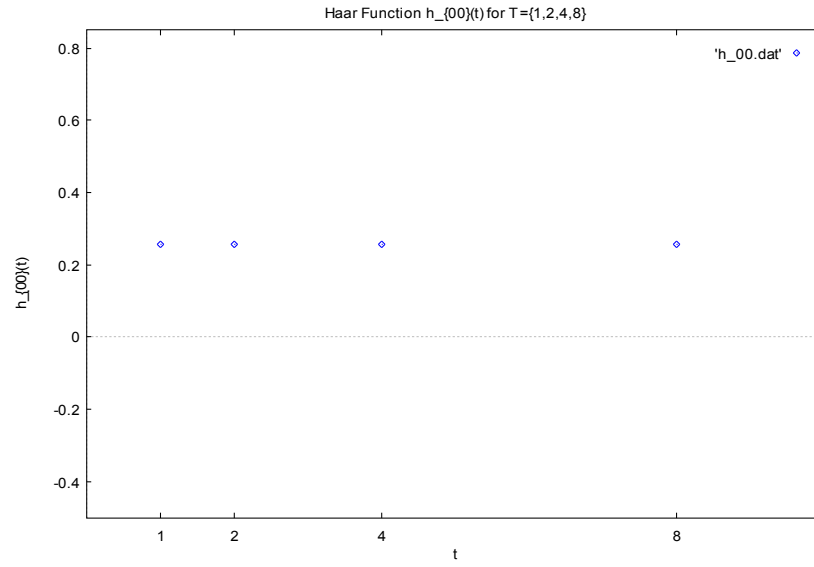
Figure 4.1: Haar Function $h_{00}(t)$ for $\mathbb{T} = \{1, 2, 4, 8\}$.
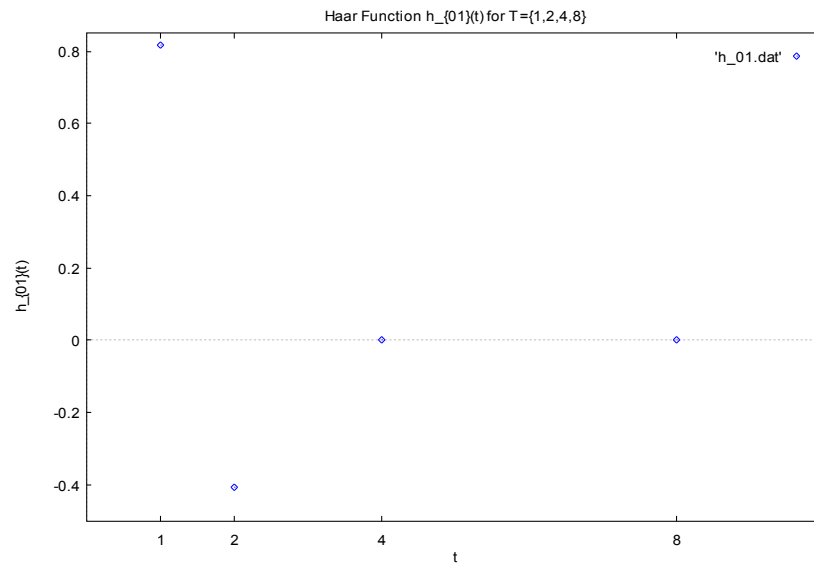
Figure 4.2: Haar Function $h_{01}(t)$ for $\mathbb{T} = \{1, 2, 4, 8\}$.
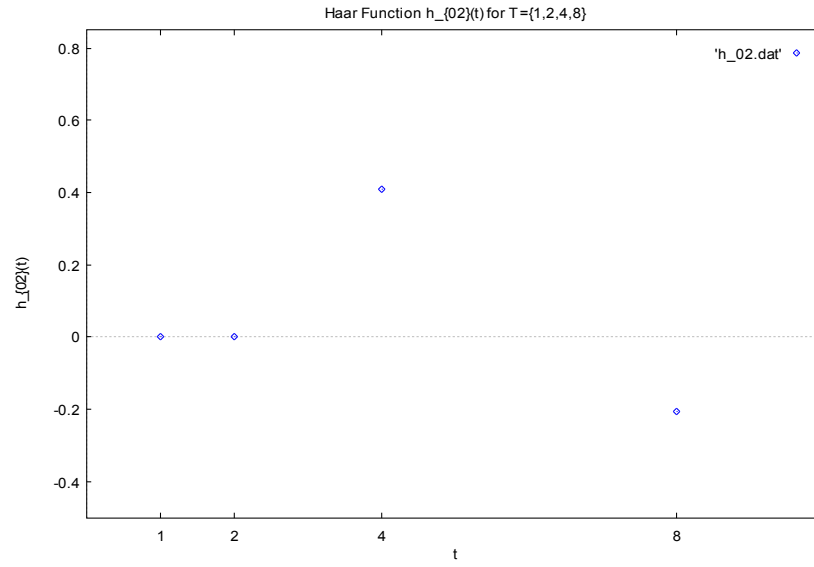
Figure 4.3: Haar Function $h_{02}(t)$ for $\mathbb{T} = \{1, 2, 4, 8\}$.
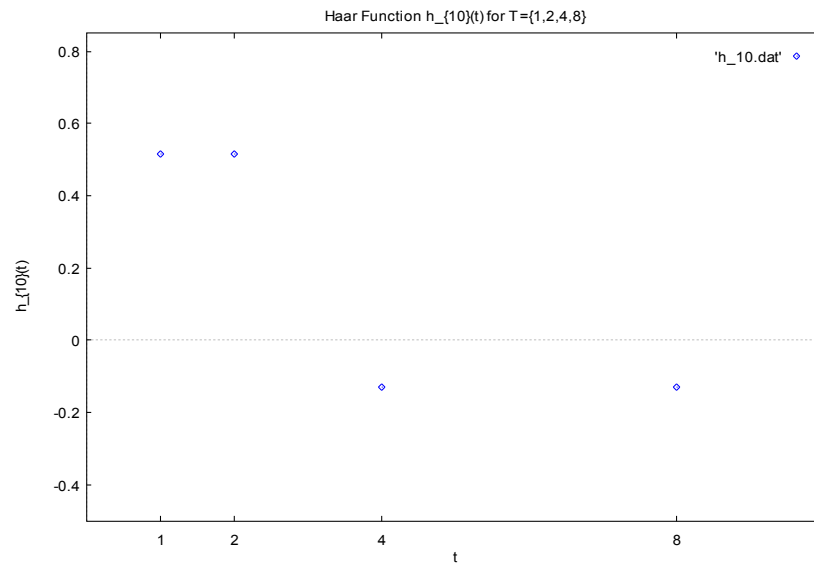
Figure 4.4: Haar Function $h_{10}(t)$ for $\mathbb{T} = \{1, 2, 4, 8\}$.
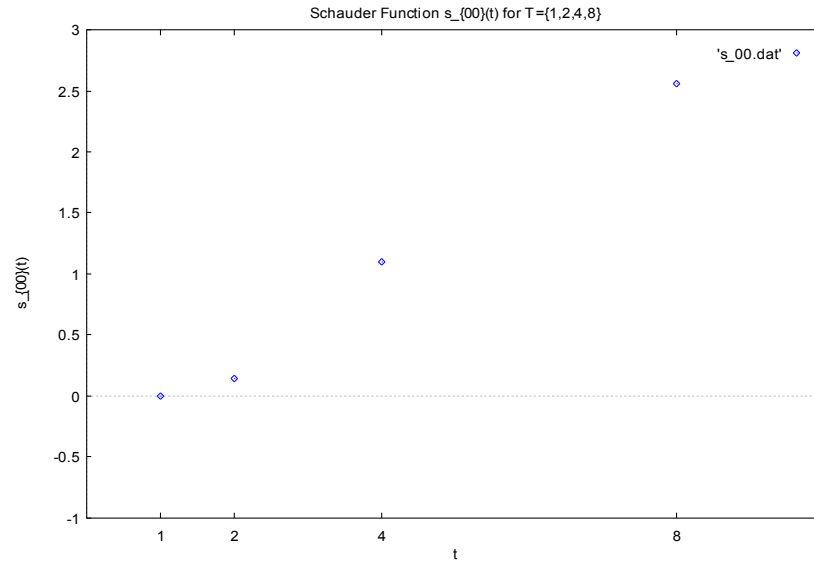
Figure 4.5: Schauder Function $s_{00}(t)$ for $\mathbb{T} = \{1, 2, 4, 8\}$.
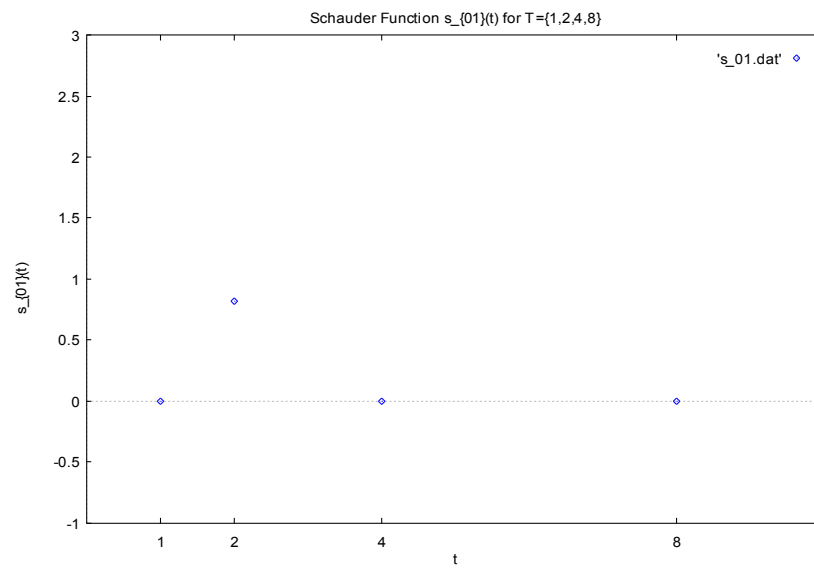
Figure 4.6: Schauder Function $s_{01}(t)$ for $\mathbb{T} = \{1, 2, 4, 8\}$.
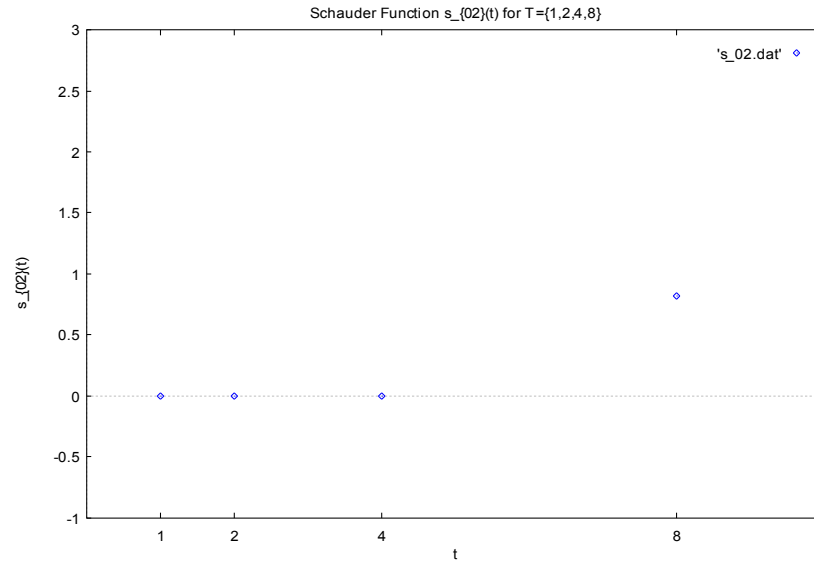
Figure 4.7: Schauder Function $s_{02}(t)$ for $\mathbb{T} = \{1, 2, 4, 8\}$.
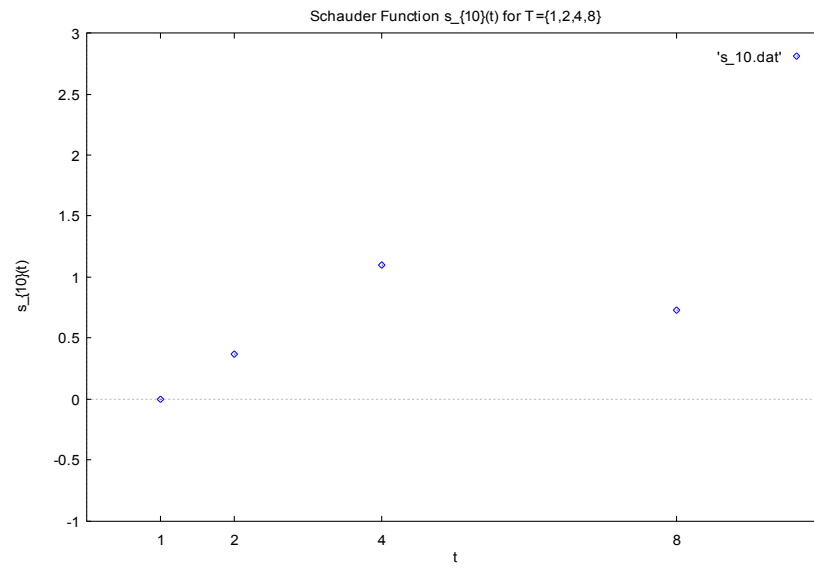
Figure 4.8: Schauder Function $s_{10}(t)$ for $\mathbb{T} = \{1, 2, 4, 8\}$.
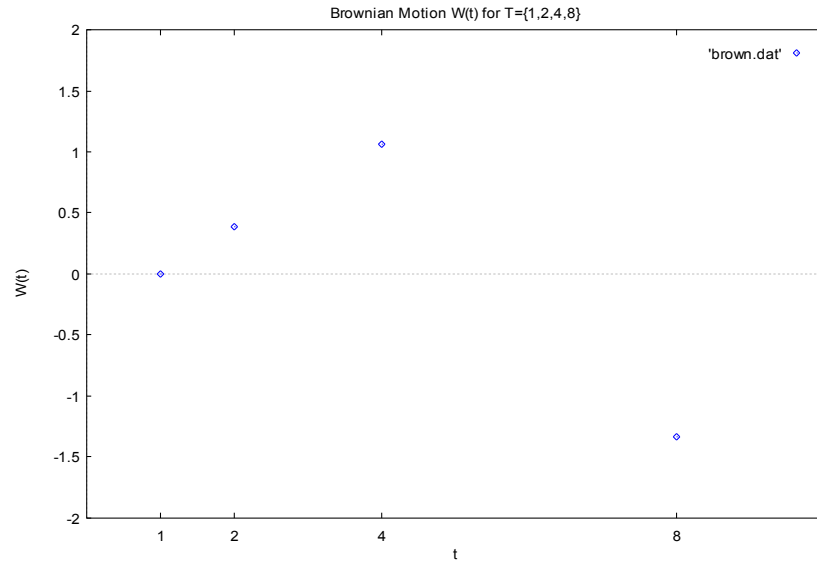
Figure 4.9: Generated Brownian Motion $W(t)$ for $\mathbb{T} = \{1, 2, 4, 8\}$.
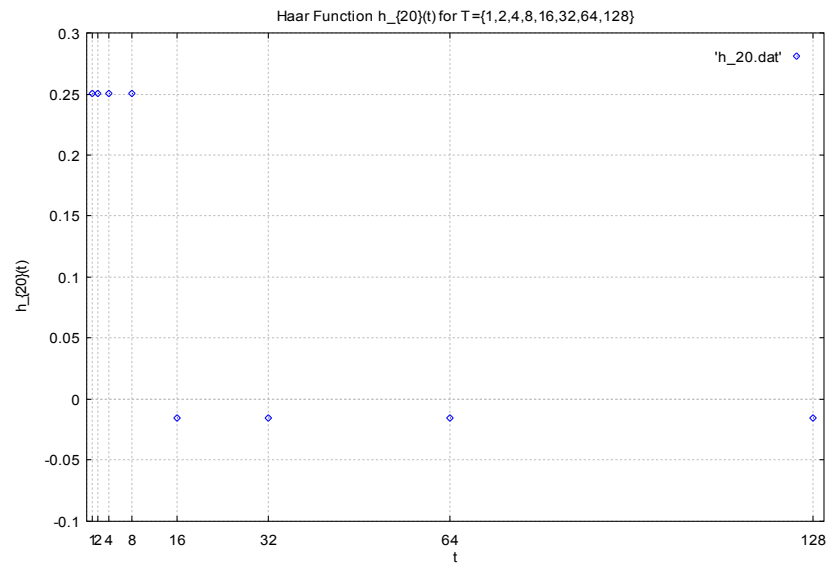
Figure 4.10: Generated Haar Function $h_{20}(t)$ for $\mathbb{T} = \{1, 2, 4, 8, 16, 32, 64, 128\}$.

5. STOCHASTIC INTEGRALS

This section provides an introduction to stochastic calculus, in particular to stochastic integration.

5.1. INTRODUCTION

The stochastic calculus of Itô originated with his investigation of conditions under which the local properties (drift and the diffusion coefficient) of a Markov process could be used to characterize this process. This has been used earlier by Kolmogorov to derive the partial differential equations for the transition probabilities of a diffusion process. Kiyosi Itô's [49–52] approach focussed on the functional form of the processes themselves and resulted in a mathematically meaningful formulation of stochastic differential equations. A similar theory was developed independently at about the same time by Gikhman [38–40].

5.2. CONSTRUCTION OF ITÔ INTEGRAL

An ordinary dynamic equation

$$x^\Delta = a(t, x) \tag{5.1}$$

may be thought of as a degenerate form of a stochastic dynamic equation in the absence of randomness. We could write (5.1) in the symbolic Δ-differential form

$$\Delta x = a(t, x)\Delta t, \tag{5.2}$$

or more accurately a Δ-integral equation

$$x(t) = x_0 + \int_{t_0}^{t} a(\tau, x(\tau))\Delta\tau, \tag{5.3}$$

where x is a solution satisfying the initial condition $x(t_0) = x_0$. Stochastic equations can be written in the form

$$\Delta X(t) = a(t, X(t))\Delta t + b(t, X(t))\xi(t)\Delta t, \qquad (5.4)$$

where the deterministic or average drift term (5.1) is perturbed by a noisy term $b(t, X(t))\xi(t)$, $\xi(t)$ are standard Gaussian random variables for each t, and $b(t, X(t))\xi(t)$ is a space-time dependent intensity factor. Equation (5.4) is then interpreted as

$$X(t) = X(t_0) + \int_{t_0}^{t} a(\tau, X(\tau))\Delta\tau + \int_{t_0}^{t} b(\tau, X(\tau))\xi(\tau)\Delta\tau \qquad (5.5)$$

for each sample path. For the special case of (5.5) with $a \equiv 0$ and $b \equiv 1$, we see that $\xi(t)$ should be the Δ of a Wiener process W, thus suggesting that we could write (5.5) alternatively as

$$X(t) = X(t_0) + \int_{t_0}^{t} a(\tau, X(\tau))\Delta\tau + \int_{t_0}^{t} b(\tau, X(\tau))\Delta W(\tau). \qquad (5.6)$$

For constant $b(t, x) \equiv b$, we would expect the second integral in (5.6) to be $b(W(t) - W(t_0))$. To fix ideas, we shall consider such an integral of a random function X over $\mathbb{T}$, denoting it by $I(X)$, where

$$I(X) = \int_{t_0}^{t} X(\tau)\Delta W(\tau). \qquad (5.7)$$

For a nonrandom step function $X(t) = X_t$ for $t \in \mathbb{T}$, we take

$$I(X) = \sum_{\tau \in [t_0, t)} X_\tau \Delta W(\tau) \quad \text{a.s.} \qquad (5.8)$$

This is a random variable with zero mean since it is the sum of random variables with zero mean. Let $\{\mathcal{F}(t)\}_{t \in \mathbb{T}}$ be an increasing family of σ-algebras such that $W(t)$

is $\mathcal{F}(t)$-measurable for each $t \geq t_0$. We consider a random step function

$$X(t) = X_t$$

for $t \in \mathbb{T}$ such that X_t is $\mathcal{F}(t)$-measurable. We also assume that each X_t is mean-square integrable over Ω. Hence,

$$\mathbb{E}\left[X_t^2\right] < \infty$$

for $t \in \mathbb{T}$. Since

$$\mathbb{E}\left[\Delta W(\tau) | \mathcal{F}(\tau)\right] = 0 \quad \text{a.s.},$$

it follows that the product $X_\tau \Delta W(\tau)$ is $\mathcal{F}(\sigma(\tau))$-measurable, integrable, and

$$\mathbb{E}\left[X_\tau \Delta W(\tau)\right] = \mathbb{E}\left[X_\tau \Delta W(\tau) | \mathcal{F}(\tau)\right] = 0$$

for each $\tau \in \mathbb{T}$. Analogously to (5.8), we define the integral $I(X)$ by

$$I(X) = \sum_{\tau \in [t_0, t)} X_\tau \Delta W(\tau) \quad \text{a.s.} \tag{5.9}$$

Since the X_τ is $\mathcal{F}(\sigma(\tau))$-measurable and hence $\mathcal{F}(t)$-measurable, it follows that $I(X)$ is $\mathcal{F}(t)$-measurable. In addition, $I(X)$ is integrable over Ω, has zero mean. It is also mean-square integrable with

$$
\begin{aligned}
\mathbb{E}\left[(I(X))^2\right] &= \mathbb{E}\left[\left(\sum_{\tau \in [t_0, t)} X_\tau \Delta W(\tau)\right)^2 \Bigg| \mathcal{F}(\tau)\right] \\
&= \sum_{\tau \in [t_0, t)} \mathbb{E}\left[X_\tau^2\right] \mathbb{E}\left[(\Delta W(\tau))^2 \big| \mathcal{F}(\tau)\right] \\
&= \sum_{\tau \in [t_0, t)} \mathbb{E}\left[X_\tau^2\right] (\sigma(\tau) - \tau) \\
&= \sum_{\tau \in [t_0, t)} \mathbb{E}\left[X_\tau^2\right] \mu(\tau) \tag{5.10}
\end{aligned}
$$

on account of the mean-square property of the increments $W(\sigma(\tau)) - W(\tau)$ for $\tau \in \mathbb{T}$. Finally, from (5.9) we have

$$I\left(\alpha X + \beta Y\right) = \alpha I(X) + \beta I(Y) \tag{5.11}$$

a.s. for $\alpha, \beta \in \mathbb{R}$ and any random step functions X, Y satisfying the above properties, that is, the integration operator I is linear in the integrand.

5.3. QUADRATIC VARIATION

Definition 5.1. If $(W(t))_{t \in \mathbb{T}}$ is a Brownian motion defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, then the quadratic variation $\langle W, W \rangle_t$ is defined by

$$\langle W, W \rangle_t := \sum_{\tau \in [t_0, t)} (\Delta W(\tau))^2, \tag{5.12}$$

for $t \in \mathbb{T}$.

Lemma 5.2. *For a Brownian motion W, we have*

$$\langle W, W \rangle_t = W^2(t) - W^2(t_0) - 2 \sum_{\tau \in [t_0, t)} W(\tau) \Delta W(\tau) \tag{5.13}$$

and $\langle W, W \rangle_t = \chi(t)$, where $\chi(t)$ is a random variable with

$$\mathbb{E}\left[\chi(t)\right] = \sum_{\tau \in [t_0, t)} \mu(\tau) = \int_{t_0}^{t} \Delta \tau \tag{5.14}$$

and

$$\mathbb{V}\left[\chi(t)\right] = 2 \sum_{\tau \in [t_0, t)} \mu^2(\tau) = 2 \int_{t_0}^{t} \mu(\tau) \Delta \tau. \tag{5.15}$$

Proof. We use Definition 5.1 to find

$$\langle W, W \rangle_t \;\; = \;\; \sum_{\tau \in [t_0, t)} (\Delta W(\tau))^2$$

$$\begin{aligned}
&= \sum_{\tau \in [t_0,t)} (W^2(\sigma(\tau)) + W^2(\tau)) - \sum_{\tau \in [t_0,t)} 2W(\sigma(\tau))W(\tau) \\
&= \sum_{\tau \in [t_0,t)} (W^2(\sigma(\tau)) - W^2(\tau)) - 2\sum_{\tau \in [t_0,t)} W(\tau)\Delta W(\tau) \\
&= W^2(t) - W^2(t_0) - 2\sum_{\tau \in [t_0,t)} W(\tau)\Delta W(\tau).
\end{aligned}$$

Next we notice that

$$\mathbb{E}\left[(\Delta W(t))^2\right] = \mathbb{V}\left[\Delta W(t)\right] = \mu(t)$$

as the expected value of $\Delta W(t)$ is zero by definition. Therefore, we have,

$$\begin{aligned}
\mathbb{E}[\langle W, W \rangle_t] &= \sum_{\tau \in [t_0,t)} \mathbb{E}\left[(\Delta W(\tau))^2\right] \\
&= \sum_{\tau \in [t_0,t)} \mu(\tau).
\end{aligned}$$

For the variance, we first compute

$$\begin{aligned}
\mathbb{V}[(\Delta W(\tau))^2] &= \mathbb{E}[(\Delta W(\tau))^4] - (\mu(\tau))^2 \\
&= 2(\mu(\tau))^2
\end{aligned}$$

as $\mathbb{E}[(\Delta W(\tau))^4] = 3(\mu(\tau))^2$ since the fourth moment of a normally distributed random variable with zero mean is three times its variance squared (normal kurtosis). With this we get

$$\begin{aligned}
\mathbb{V}[\langle W, W \rangle_t] &= \mathbb{V}\left[\sum_{\tau \in [t_0,t)} (\Delta W(\tau))^2\right] \\
&= \sum_{\tau \in [t_0,t)} \mathbb{V}\left[(\Delta W(\tau))^2\right] \\
&= 2\sum_{\tau \in [t_0,t)} \mu^2(\tau),
\end{aligned}$$

where the second equality follows on the one hand from independence of the increments of the Wiener process. On the other hand, we can use the fact that if two

random variables are independent, then measurable functions of them are again in-
dependent random variables. □

To give a better notation of Lemma 5.2, we first define the following integrals.

Definition 5.3. For the sums in Lemma 5.2, we write

$$\int_{t_0}^{t} X(\tau)\Delta W(\tau) = \sum_{\tau \in [t_0,t)} X(\tau)\Delta W(\tau) \tag{5.16}$$

and

$$\int_{t_0}^{t} X(\tau)\Delta \tau = \sum_{\tau \in [t_0,t)} X(\tau)\mu(\tau). \tag{5.17}$$

With this we get the next corollary.

Corollary 5.4. *For a Wiener process W, we can write*

$$W^2(t) = \langle W, W \rangle_t + W^2(t_0) + 2\int_{t_0}^{t} W(\tau)\Delta W(\tau), \tag{5.18}$$

where $\langle W, W \rangle_t = \chi(t)$ and $\chi(t)$ has the same properties as in Lemma 5.2.

Proof. The results follow directly from Lemma 5.2 and by using the first part of
Definition 5.3. □

For most of the calculations it is easier to use a differential notation than the
integral notation we use in Lemma 5.2. We observe that the differential of $\chi(t)$ has
mean

$$\Delta \sum_{\tau \in [t_0,t)} \mu(\tau) = \mu(t) - \mu(t_0)$$

and variance

$$\Delta \sum_{\tau \in [t_0,t)} \mu^2(\tau) = 2(\mu^2(t) - \mu^2(t_0)).$$

This means that we can write $\Delta \chi(t) = (\Delta W(t))^2$, where $(\Delta W(t))^2$ is a random
variable. With this notation, we get the following corollary of Lemma 5.2.

Corollary 5.5. *In the differential notation we have*

$$\Delta((W(t))^2) = \Delta\chi(t) + 2W(t)\Delta W(t), \tag{5.19}$$

where $\Delta\chi(t) = (\Delta W(t))^2$.

Proof. Use Lemma 5.2 and the results for the random variable χ we just derived. □

Motivated by Definition 5.3, we state the following lemma which we use in subsequent sections.

Lemma 5.6. *If* $\{W(t)\}_{t\in\mathbb{T}}$ *is a Brownian motion defined on some probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *and* $X(t)$ *is* $\mathcal{F}(t)$-*measurable, then*

$$\mathbb{E}\left[\int_{t_0}^t X(\tau)\Delta W(\tau)\right] = 0, \tag{5.20}$$

$$\mathbb{E}\left[\int_{t_0}^t X(\tau)\Delta\tau\right] = \int_{t_0}^t \mathbb{E}\left[X(\tau)\right]\Delta\tau \tag{5.21}$$

and

$$\mathbb{E}\left[\left(\int_{t_0}^t X(\tau)\Delta W(\tau)\right)^2\right] = \int_{t_0}^t \mathbb{E}\left[X^2(\tau)\right]\Delta\tau. \tag{5.22}$$

Proof. Let $\mathcal{W}^+(t)$ be the σ-algebra generated by $W(\tau)$, $\tau > t$. Then to prove (5.20) we observe that

$$
\begin{aligned}
\mathbb{E}\left[\int_{t_0}^t X(\tau)\Delta W(\tau)\right] &= \mathbb{E}\left[\sum_{\tau\in[t_0,t)} X(\tau)\Delta W(\tau)\right] \\
&= \sum_{\tau\in[t_0,t)} \mathbb{E}\left[X(\tau)\Delta W(\tau)\right] \\
&= \sum_{\tau\in[t_0,t)} \mathbb{E}\left[X(\tau)\right]\mathbb{E}\left[\Delta W(\tau)\right] \\
&= 0,
\end{aligned}
$$

since $X(\tau)$ is $\mathcal{F}(\tau)$-measurable and $\mathcal{F}(\tau)$ is independent of $\mathcal{W}^+(\tau)$. On the other hand,

$\Delta W(\tau)$ is $W^+(\tau)$-measurable, and so $X(\tau)$ is independent of $\Delta W(\tau)$. Likewise,

$$
\begin{aligned}
\mathbb{E}\left[\int_{t_0}^t X(\tau)\Delta\tau\right] &= \mathbb{E}\left[\sum_{\tau\in[t_0,t)} X(\tau)\mu(\tau)\right] \\
&= \sum_{\tau\in[t_0,t)} \mathbb{E}\left[X(\tau)\right]\mu(\tau) \\
&= \int_{t_0}^t \mathbb{E}\left[X(\tau)\right]\Delta\tau,
\end{aligned}
$$

which proves (5.21). Next we observe that

$$
\begin{aligned}
\mathbb{E}\left[\left(\int_{t_0}^t X(\tau)\Delta W(\tau)\right)^2\right] &= \mathbb{E}\left[\left(\sum_{\tau\in[t_0,t)} X(\tau)\Delta W(\tau)\right)^2\right] \\
&= \sum_{\tau_1\in[t_0,t)}\sum_{\tau_2\in[t_0,t)} \mathbb{E}\left[X(\tau_1)X(\tau_2)\Delta W(\tau_1)\Delta W(\tau_2)\right].
\end{aligned}
$$

Now if $\tau_1 < \tau_2$, then $\Delta W(\tau_2)$ is independent of $X(\tau_1)X(\tau_2)\Delta W(\tau_1)$. Thus,

$$
\mathbb{E}\left[X(\tau_1)X(\tau_2)\Delta W(\tau_1)\Delta W(\tau_2)\right] = \mathbb{E}\left[X(\tau_1)X(\tau_2)\Delta W(\tau_1)\right]\mathbb{E}\left[\Delta W(\tau_2)\right] = 0.
$$

Consequently

$$
\begin{aligned}
\mathbb{E}\left[\left(\int_{t_0}^t X(\tau)\Delta W(\tau)\right)^2\right] &= \sum_{\tau\in[t_0,t)} \mathbb{E}\left[X^2(\tau)\right]\mathbb{E}\left[(\Delta W(\tau))^2\right] \\
&= \sum_{\tau\in[t_0,t)} \mathbb{E}\left[X^2(\tau)\right]\mu(\tau) \\
&= \int_{t_0}^t \mathbb{E}\left[X^2(\tau)\right]\Delta\tau.
\end{aligned}
$$

This concludes the proof. $\square$

To continue with our study of stochastic Δ-integrals with random integrands,

let us think what might be an appropriate definition for

$$\int_{t_0}^{t} W(\tau) \Delta W(\tau),$$

where W is a one-dimensional Brownian motion. A reasonable procedure will be to construct a Riemann sum. Let $\mathbb{T} = \{t_0, t_1, t_2, \ldots, t_n = t\}$ with $t_0 \geq 0$ and let us set

$$\langle W, W \rangle_t = \sum_{\tau \in [t_0, t)} (\Delta W(\tau))^2. \tag{5.23}$$

Then

$$\langle W, W \rangle_t - (t - t_0) = \sum_{\tau \in [t_0, t)} \left((\Delta W(\tau))^2 - \mu(\tau) \right).$$

Hence,

$$\mathbb{E}\left[(\langle W, W \rangle_t - (t - t_0))^2 \right] = $$
$$\sum_{\tau_1 \in [t_0, t)} \sum_{\tau_2 \in [t_0, t)} \mathbb{E}\left[\left((\Delta W(\tau_1))^2 - \mu(\tau_1) \right) \left((\Delta W(\tau_2))^2 - \mu(\tau_2) \right) \right].$$

For $\tau_1 \neq \tau_2$, the term in the double sum is

$$\mathbb{E}\left[\left((\Delta W(\tau_1))^2 - \mu(\tau_1) \right) \left((\Delta W(\tau_2))^2 - \mu(\tau_2) \right) \right],$$

according to independent increments, and thus equal to 0, as $W(t) - W(s) \sim \mathcal{N}(0, t - s)$ for all $t, s \in \mathbb{T}$ and $t \geq s \geq t_0$. Hence,

$$
\begin{aligned}
\mathbb{E}\left[(\langle W, W \rangle_t - (t - t_0))^2 \right] &= \sum_{\tau \in [t_0, t)} \mathbb{E}\left[(Y^2(\tau) - 1)^2 \mu^2(\tau) \right] \\
&= \sum_{\tau \in [t_0, t)} \mathbb{E}\left[Y^4(\tau) - 2Y^2(\tau) + 1 \right] \mu^2(\tau) \\
&= \sum_{\tau \in [t_0, t)} [3 - 2 + 1] \mu^2(\tau) \\
&= 2 \sum_{\tau \in [t_0, t)} \mu^2(\tau)
\end{aligned}
$$

$$\neq \quad 0,$$

where

$$Y(\tau) := \frac{W(\sigma(\tau)) - W(\tau)}{\sqrt{\mu(\tau)}} \quad \sim \quad \mathcal{N}(0,1).$$

If we assume that $\langle W, W \rangle_t$ is of the form $\alpha(t-t_0)+\beta$, where α and β are deterministic, then we have the following:

$$
\begin{aligned}
\mathbb{E}\left[(\langle W, W \rangle_t - \alpha(t - t_0) - \beta)^2\right] \\
= \quad & \sum_{\tau \in [t_0,t)} \mathbb{E}\left[\left((\Delta W(\tau))^2 - \alpha\mu(\tau) - \beta\right)^2\right] \\
= \quad & \sum_{\tau \in [t_0,t)} \mathbb{E}\left[\left(Y^2(\tau) - \alpha - \frac{\beta}{\mu(\tau)}\right)^2\right] \mu^2(\tau) \\
= \quad & \sum_{\tau \in [t_0,t)} \mathbb{E}\left[Y^4(\tau) + \alpha^2 + \frac{\beta^2}{\mu^2(\tau)} - 2\alpha Y^2(\tau) - \frac{2\beta}{\mu(\tau)}Y^2(\tau) + \frac{2\alpha\beta}{\mu(\tau)}\right] \mu^2(\tau) \\
= \quad & \alpha^2 \sum_{\tau \in [t_0,t)} \mu^2(\tau) + n\beta^2 - 2\alpha \sum_{\tau \in [t_0,t)} \mu^2(\tau) - 2(t - t_0)\beta \\
& + 2(t - t_0)\alpha\beta + 3 \sum_{\tau \in [t_0,t)} \mu^2(\tau).
\end{aligned}
$$

So when (α, β) lies on the curve

$$x^2 \sum_{\tau \in [t_0,t)} \mu^2(\tau) + ny^2 - 2x \sum_{\tau \in [t_0,t)} \mu^2(\tau) - 2(t - t_0)y + 2(t - t_0)xy + 3 \sum_{\tau \in [t_0,t)} \mu^2(\tau) = 0,$$

$$(5.24)$$

we have

$$\mathbb{E}\left[(\langle W, W \rangle_t - \alpha(t - t_0) - \beta)^2\right] = 0,$$

implying that

$$\langle W, W \rangle_t = \sum_{\tau \in [t_0,t)} (\Delta W(\tau))^2 = \alpha(t - t_0) + \beta \quad \text{a.s.} \qquad (5.25)$$

Next we analyze the curve given by (5.24). Let

$$
D = \det \begin{pmatrix} \sum_{\tau \in [t_0,t)} \mu^2(\tau) & t - t_0 & -\sum_{\tau \in [t_0,t)} \mu^2(\tau) \\ \\ t - t_0 & n & -(t - t_0) \\ \\ -\sum_{\tau \in [t_0,t)} \mu^2(\tau) & -(t - t_0) & 3\sum_{\tau \in [t_0,t)} \mu^2(\tau) \end{pmatrix}
$$

$$
= 2 \sum_{\tau \in [t_0,t)} \mu^2(\tau) \left(n \sum_{\tau \in [t_0,t)} \mu^2(\tau) - (t - t_0)^2 \right),
$$

and

$$
J = \det \begin{pmatrix} \sum_{\tau \in [t_0,t)} \mu^2(\tau) & t - t_0 \\ \\ t - t_0 & n \end{pmatrix} = n \sum_{\tau \in [t_0,t)} \mu^2(\tau) - (t - t_0)^2.
$$

Now, if

$$
n \sum_{\tau \in [t_0,t)} \mu^2(\tau) = (t - t_0)^2, \tag{5.26}
$$

then $D = 0 = J$ and

$$
\det \begin{pmatrix} n & -(t - t_0) \\ \\ -(t - t_0) & 3\sum_{\tau \in [t_0,t)} \mu^2(\tau) \end{pmatrix} = 3n \sum_{\tau \in [t_0,t)} \mu^2(\tau) - (t - t_0)^2
$$

$$
= 2n \sum_{\tau \in [t_0,t)} \mu^2(\tau) > 0,
$$

implying that (5.24) represents an imaginary pair of parallel lines [91, Page 145]. If

$$
n \sum_{\tau \in [t_0,t)} \mu^2(\tau) > (t - t_0)^2,
$$

then $D > 0$, $J > 0$ and $D\sum_{\tau\in[t_0,t)}\mu^2(\tau) > 0$, implying that (5.24) again represents an imaginary conic. On the other hand if

$$n\sum_{\tau\in[t_0,t)}\mu^2(\tau) < (t-t_0)^2, \tag{5.27}$$

then $D \neq 0$ and $J < 0$, implying (5.24) represents a hyperbola. But in this case there is no time scale which satisfies (5.27). For if we let t_0 as the first point and consider the case $n = 2$, then we have

$$2\left(\mu^2(t_0) + \mu^2(t_1)\right) < (t_2 - t_1)^2 = (\mu(t_0) + \mu(t_1))^2$$

which reduces to

$$(\mu(t_0) - \mu(t_1))^2 < 0,$$

a contradiction to the fact that the graininess function μ is real and nonnegative.

Theorem 5.7. *There is no $\alpha, \beta \in \mathbb{R}$ such that*

$$\int_{t_0}^t W(\tau)\Delta W(\tau) = \frac{W^2(t)}{2} - \frac{1}{2}[\alpha(t-t_0) + \beta] \tag{5.28}$$

holds.

Proof. It follows from the above discussion and the fact that

$$
\begin{aligned}
\int_{t_0}^t W(\tau)\Delta W(\tau) &= \sum_{\tau\in[t_0,t)} W(\tau)\Delta W(\tau) \\
&= \frac{1}{2}\sum_{\tau\in[t_0,t)}\left(W^2(\sigma(\tau)) - W^2(\tau)\right) - \frac{1}{2}\sum_{\tau\in[t_0,t)}(\Delta W(\tau))^2 \\
&= \frac{1}{2}\left(W^2(t) - W^2(t_0)\right) - \frac{1}{2}\sum_{\tau\in[t_0,t)}(\Delta W(\tau))^2 \\
&= \frac{1}{2}W^2(t) - \frac{1}{2}\langle W, W\rangle_t.
\end{aligned}
$$

This concludes the proof. □

5.4. PRODUCT RULES

In this subsection we prove the following two product rules for stochastic processes.

Theorem 5.8. *For an arbitrary nonrandom function f and a Wiener process W, we have*

$$\Delta(f(t)W(t)) = f(\sigma(t))\Delta W(t) + (\Delta f(t))W(t) \tag{5.29}$$

and

$$\Delta(f(t)W(t)) = f(t)\Delta W(t) + (\Delta f(t))W(\sigma(t)). \tag{5.30}$$

Proof. By using the properties of the Δ-differentials, we get

$$
\begin{aligned}
\Delta(f(t)W(t)) &= f(\sigma(t))W(\sigma(t)) - f(t)W(t) \\
&= f(\sigma(t))W(\sigma(t)) - f(\sigma(t))W(t) + f(\sigma(t))W(t) - f(t)W(t)
\end{aligned}
$$

and therefore

$$\Delta(f(t)W(t)) = f(\sigma(t))\Delta W(t) + (\Delta f(t))W(t). \tag{5.31}$$

For (5.30) we just add and subtract the term $f(t)W(\sigma(t))$ instead of $f(\sigma(t))W(t)$, so that

$$
\begin{aligned}
\Delta(f(t)W(t)) &= f(\sigma(t))W(\sigma(t)) - f(t)W(t) \\
&= f(\sigma(t))W(\sigma(t)) - f(t)W(\sigma(t)) + f(t)W(\sigma(t)) - f(t)W(t)
\end{aligned}
$$

and so again

$$\Delta(f(t)W(t)) = (\Delta f(t))W(\sigma(t)) + f(t)\Delta W(t).$$

Hence, both (5.29) and (5.30) hold. $\qquad\square$

Theorem 5.9. *For two stochastic processes X_1 and X_2 with*

$$X_i(t) = X_i(t_0) + a_i t + b_i W(t) \quad for \quad i = 1, 2$$

and

$$\Delta X_i(t) = a_i \Delta t + b_i \Delta W(t) \quad for \quad i = 1, 2, \tag{5.32}$$

we have

$$\Delta(X_1 X_2) = X_1(\Delta X_2) + X_2(\Delta X_1) + (\Delta X_1)(\Delta X_2). \tag{5.33}$$

Proof. We have

$$
\begin{aligned}
X_1(t)X_2(t) &= [X_1(t_0) + a_1 t + b_1 W(t)] [X_2(t_0) + a_2 t + b_2 W(t)] \\
&= X_1(t_0)X_2(t_0) + [X_1(t_0)a_2 + X_2(t_0)a_1] t + a_1 a_2 t^2 \\
&\quad + [X_1(t_0)b_2 + X_2(t_0)b_1] W(t) + [a_1 b_2 + a_2 b_1] t W(t) \\
&\quad + b_1 b_2 W^2(t).
\end{aligned}
$$

If we now take the differential on both sides and using (5.29) and (5.19), we obtain

$$
\begin{aligned}
\Delta &\left(X_1(t)X_2(t) \right) \\
&= [X_1(t_0)a_2 + X_2(t_0)a_1] \Delta t + [X_1(t_0)b_2 + X_2(t_0)b_1] \Delta W(t) \\
&\quad + a_1 a_2 (t + \sigma(t))\Delta t + [a_1 b_2 + a_2 b_1] [\sigma(t)\Delta W(t) + W(t)\Delta t] \\
&\quad + b_1 b_2 [\Delta \chi(t) + 2W(t)\Delta W(t)] \\
&= [X_1(t_0)a_2 + X_2(t_0)a_1 + (a_1 b_2 + a_2 b_1)W(t) + a_1 a_2(t + \sigma(t))] \Delta t \\
&\quad + [X_1(t_0)b_2 + X_2(t_0)b_1 + (a_1 b_2 + a_2 b_1)\sigma(t) + 2b_1 b_2 W(t)] \Delta W(t) \\
&\quad + b_1 b_2 \Delta \chi(t)
\end{aligned}
$$

and

$$
X_1(t)\Delta X_2(t) = [X_1(t_0) + a_1 t + b_1 W(t)][a_2 \Delta t + b_2 \Delta W(t)]
$$

$$
\begin{aligned}
= \quad & [a_2 X_1(t_0) + a_1 a_2 t + a_2 b_1 W(t)]\Delta t \\
& + [b_2 X_1(t_0) + a_1 b_2 t + b_1 b_2 W(t)]\Delta W(t),
\end{aligned}
$$

and (by switching X_1 and X_2) as above

$$
\begin{aligned}
X_2(t)\Delta X_1(t) \quad = \quad & [a_1 X_2(t_0) + a_1 a_2 t + a_1 b_2 W(t)]\Delta t \\
& + [b_1 X_2(t_0) + a_2 b_1 t + b_1 b_2 W(t)]\Delta W(t)
\end{aligned}
$$

as well as

$$
(\Delta X_1(t))(\Delta X_2(t)) = a_1 a_2(\Delta t)^2 + (a_1 b_2 + a_2 b_1)\Delta W(t)\Delta t + b_1 b_2(\Delta W(t))^2.
$$

Therefore we can express $\Delta(X_1(t)X_2(t))$ as

$$
\begin{aligned}
\Delta(X_1(t)X_2(t)) \quad = \quad & [X_1(t)a_2 + a_1 a_2 t + a_2 b_1 W(t)]\Delta t \\
& + [b_2 X_1(t_0) + a_1 b_2 t + b_1 b_2 W(t)]\Delta W(t) \\
& + [a_1 X_2(t) + a_1 a_2 t + a_1 b_2 W(t)]\Delta t \\
& + [b_1 X_2(t_0) + a_2 b_1 t + b_1 b_2 W(t)]\Delta W(t) \\
& + [b_1 b_2 \chi(t) - a_1 a_2 t \Delta t + a_1 a_2 \sigma(t)\Delta t] \\
& + [-a_2 b_1 t - a_1 b_2 t + (a_1 b_2 + a_2 b_1)\sigma(t)]\Delta W(t) \\
= \quad & X_1(t)\Delta X_2(t) + X_2(t)\Delta X_1(t) + b_1 b_2(\Delta W(t))^2 \\
& + a_1 a_2(\sigma(t) - t)\Delta t + (a_1 b_2 + a_2 b_1)(\sigma(t) - t)\Delta W(t) \\
= \quad & X_1(t)\Delta X_2(t) + X_2(t)\Delta X_1(t) + b_1 b_2(\Delta W(t))^2 \\
& + a_1 a_2(\Delta t)^2 + (a_1 b_2 + a_2 b_1)\Delta t \Delta W(t) \\
= \quad & X_1(t)\Delta X_2(t) + X_2(t)\Delta X_1(t) + (\Delta X_1(t))(\Delta X_2(t)),
\end{aligned}
$$

where we have added and subtracted the terms $a_1 a_2 t \Delta t$ and $(b_1 a_2 + a_2 b_1)\Delta W(t)$ in the first equality and wrote again $(\Delta W(t))^2$ instead of $\Delta \chi(t)$ in the second equality. $\quad\square$

Motivated by Theorem 5.9, we now evaluate $\Delta((W(t))^m)$, where $m \in \mathbb{N}$.

Theorem 5.10. *For a Wiener process W, we have*

$$\Delta W^m = \sum_{k=1}^{m} \binom{m}{k} W^{m-k} (\Delta W)^k. \tag{5.34}$$

Proof. Using the fact that $W(t) + \Delta W(t) = W(\sigma(t))$, we have

$$
\begin{aligned}
\Delta((W(t))^m) &= (W(\sigma(t)))^m - (W(t))^m \\
&= (W(t) + \Delta W(t))^m - (W(t))^m \\
&= \sum_{k=0}^{m} \binom{m}{k} (W(t))^{m-k} (\Delta W(t))^k - (W(t))^m \\
&= \sum_{k=1}^{m} \binom{m}{k} W^{m-k}(t) (\Delta W(t))^k,
\end{aligned}
$$

i.e., (5.34) holds. $\square$

6. STOCHASTIC DYNAMIC EQUATIONS (SΔE)

The theory of stochastic dynamic equations is introduced in this section. The emphasis is on Itô stochastic dynamic equations, for which an existence and uniqueness theorem is proved and properties of their solutions are investigated. Techniques for solving linear stochastic dynamic equations are presented.

6.1. LINEAR STOCHASTIC DYNAMIC EQUATIONS

Stochastic dynamic equations (SΔE) are introduced in this section. Techniques for solving linear stochastic dynamic equations are also presented. The general form of a scalar *linear stochastic dynamic equation* is

$$\Delta X = [a(t)X + c(t)]\,\Delta t + [b(t)X + d(t)]\,\Delta W, \tag{6.1}$$

where the coefficients a, b, c, d are specified functions of $t \in \mathbb{T}$ which may be constants.

6.1.1. Stochastic Exponential.

Definition 6.1. Let W be Brownian motion on $\mathbb{T}$. Then we say a random variable $A : \mathbb{T} \to \mathbb{R}$ defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is *stochastic regressive* (with respect to W) provided

$$1 + A(t)\Delta W(t) \neq 0 \quad \text{a.s.}$$

for all $t \in \mathbb{T}^\kappa$. The set of stochastic regressive functions will be denoted by $\mathcal{R}_W$.

Theorem 6.2. *If we define the "stochastic circle plus" addition $\oplus_W$ on $\mathcal{R}_W$ by*

$$(A \oplus_W B)(t) := A(t) + B(t) + A(t)B(t)\Delta W(t) \quad \text{for all} \quad t \in \mathbb{T}^\kappa, \tag{6.2}$$

then $(\mathcal{R}_W, \oplus_W)$ is an Abelian group.

Proof. To prove that we have closure under the addition $\oplus_W$, we note that, for $A, B \in$

$\mathcal{R}_W$, $A \oplus_W B$ is a function from $\mathbb{T}$ to $\mathbb{R}$. It only remains to show that for all $t \in \mathbb{T}$, $(A \oplus_W B)(t) \neq -1/\Delta W(t)$ a.s., but this follows from

$$
\begin{aligned}
1 + (A \oplus_W B)(t)\Delta W(t) &= 1 + (A(t) + B(t) + A(t)B(t)\Delta W(t))\Delta W(t) \\
&= 1 + A(t)\Delta W(t) + B(t)\Delta W + A(t)B(t)(\Delta W(t))^2 \\
&= (1 + A(t)\Delta W(t))(1 + B(t)\Delta W(t)) \\
&\neq 0 \quad \text{a.s.}
\end{aligned}
$$

Hence, $\mathcal{R}_W$ is closed under the addition $\oplus_W$. Since

$$
(A \oplus_W 0)(t) = (0 \oplus_W A)(t) = A(t),
$$

0 is the additive identity for $\oplus_W$. For $A \in \mathcal{R}_W$, to find the additive inverse of A under $\oplus_W$, we must solve

$$
(A \oplus_W B)(t) = 0 \quad \text{a.s.,}
$$

for B. Hence, we must solve

$$
A(t) + B(t) + A(t)B(t)\Delta W(t) = 0 \quad \text{a.s.,}
$$

for B. Thus,

$$
B(t) = -\frac{A(t)}{1 + A(t)\Delta W(t)} \quad \text{for all} \quad t \in \mathbb{T}
$$

is the additive inverse of A under the addition $\oplus_W$. That the associative law holds follows from the fact that,

$$
\begin{aligned}
((A \oplus_W B) \oplus_W C)(t) &= ((A + B + AB\Delta W) \oplus_W C)(t) \\
&= (A(t) + B(t) + A(t)B(t)\Delta W(t)) + C(t) \\
&\quad + (A(t) + B(t) + A(t)B(t)\Delta W(t))C(t)\Delta W(t) \\
&= A(t) + B(t) + A(t)B(t)\Delta W(t) + C(t) + A(t)C(t)\Delta W(t) \\
&\quad + B(t)C(t)\Delta W(t) + A(t)B(t)C(t)(\Delta W(t))^2
\end{aligned}
$$

$$
\begin{aligned}
&= A(t) + (B(t) + C(t) + B(t)C(t)\Delta W(t)) \\
&\quad + A(t)(B(t) + C(t) + B(t)C(t)\Delta W(t))\Delta W(t) \\
&= (A \oplus_W (B \oplus_W C))(t)
\end{aligned}
$$

for $A, B, C \in \mathcal{R}_W$ and $t \in \mathbb{T}^\kappa$. Hence, $(\mathcal{R}_W, \oplus_W)$ is a group. Since

$$
\begin{aligned}
(A \oplus_W B)(t) &= A(t) + B(t) + A(t)B(t)\Delta W(t) \\
&= B(t) + A(t) + A(t)B(t)\Delta W(t) \\
&= (B \oplus_W A)(t),
\end{aligned}
$$

the commutative law holds, and hence $(\mathcal{R}_W, \oplus_W)$ is an Abelian group. $\quad\square$

Definition 6.3. If $n \in \mathbb{N}$ and $A \in \mathcal{R}_W$, then we define the "stochastic circle dot" multiplication $\odot_W$ by

$$
(n \odot_W A)(t) = (A \oplus_W A \oplus_W \oplus_W \ldots \oplus_W A)(t) \quad \text{for all} \quad t \in \mathbb{T}^\kappa,
$$

where we have n terms on the right-hand side of this last equation.

In the proof of Theorem 6.2, we saw that if $A \in \mathcal{R}_W$, then the additive inverse of A under the operation $\oplus_W$ is

$$
(\ominus_W A)(t) := \frac{-A(t)}{1 + A(t)\Delta W(t)} \quad \text{for all} \quad t \in \mathbb{T}^\kappa. \tag{6.3}
$$

Lemma 6.4. If $A \in \mathcal{R}_W$, then $(\ominus_W(\ominus_W A))(t) = A(t)$ for all $t \in \mathbb{T}^\kappa$.

Proof. Using (6.3), we observe that for all $t \in \mathbb{T}^\kappa$,

$$
\begin{aligned}
(\ominus_W(\ominus_W A))(t) &= \left(\ominus_W \left(\frac{-A}{1 + A\Delta W} \right) \right)(t) \\
&= \frac{\left(\frac{-A(t)}{1 + A(t)\Delta W(t)} \right)}{1 + \left(\frac{-A(t)}{1 + A(t)\Delta W(t)} \right) \Delta W(t)} \\
&= A(t),
\end{aligned}
$$

where on the first and second equality we have used (6.3). □

Definition 6.5. We define the "stochastic circle minus" subtraction $\ominus_W$ on $\mathcal{R}_W$ by

$$(A \ominus_W B)(t) := (A \oplus_W (\ominus_W A))(t) \tag{6.4}$$

for all $t \in \mathbb{T}^\kappa$.

Theorem 6.6. *If $A, B \in \mathcal{R}_W$, then*

$$(A \ominus_W B)(t) = \frac{A(t) - B(t)}{1 + B(t)\Delta W(t)} \tag{6.5}$$

for all $t \in \mathbb{T}^\kappa$.

Proof. From Definition 6.5 and (6.3) we have,

$$
\begin{aligned}
(A \ominus_W B)(t) &= (A \oplus_W (\ominus_W B))(t) \\
&= \left(A \oplus_W \left(\frac{-B}{1 + B\Delta W} \right) \right)(t) \\
&= A(t) + \left(\frac{-B(t)}{1 + B(t)\Delta W(t)} \right) + A(t) \left(\frac{-B(t)}{1 + B(t)\Delta W(t)} \right) \Delta W(t) \\
&= \frac{A(t)(1 + B(t)\Delta W(t)) - B(t) - A(t)B(t)\Delta W(t)}{1 + B(t)\Delta W(t)} \\
&= \frac{A(t) - B(t)}{1 + B(t)\Delta W(t)},
\end{aligned}
$$

as claimed. □

Theorem 6.7. *If $A, B \in \mathcal{R}_W$, then*

(i) $A \ominus_W A = 0$,

(ii) $A \ominus_W B \in \mathcal{R}_W$,

(iii) $\ominus_W (A \ominus_W B) = B \ominus_W A$,

(iv) $\ominus_W (A \oplus_W B) = (\ominus_W A) \oplus_W (\ominus_W B)$.

Proof. Part (i). We observe that

$$(A \ominus_W A)(t) = \frac{A(t) - A(t)}{1 + A(t)\Delta W(t)} = 0.$$

Part (ii). By using (6.5) we have

$$\begin{aligned}
1 + (A \ominus_W B)(t)\Delta W(t) &= 1 + \frac{A(t) - B(t)}{1 + B(t)\Delta W(t)}\Delta W(t) \\
&= \frac{1 + A(t)\Delta W(t)}{1 + B(t)\Delta W(t)} \\
&\neq 0 \quad \text{a.s.,}
\end{aligned}$$

since $A, B \in \mathcal{R}_W$.

Part (iii). We observe that

$$\begin{aligned}
(\ominus_W(A \ominus_W B))(t) &= \left(\ominus_W\left(\frac{A - B}{1 + B\Delta W}\right)\right)(t) \\
&= \frac{-\left(\frac{A(t) - B(t)}{1 + B(t)\Delta W(t)}\right)}{1 + \left(\frac{A(t) - B(t)}{1 + B(t)\Delta W(t)}\right)\Delta W(t)} \\
&= \frac{B(t) - A(t)}{1 + A(t)\Delta W(t)} \\
&= (B \ominus_W A)(t),
\end{aligned}$$

where on the first equality we have used (6.5) and on the second equality we have used (6.3).

Part (iv). We observe that

$$\begin{aligned}
((\ominus_W A) \oplus_W (\ominus_W B))(t) &= \left(\frac{-A}{1 + A\Delta W} \oplus_W \frac{-B}{1 + B\Delta W}\right)(t) \\
&= \frac{-A(t)}{1 + A(t)\Delta W(t)} + \frac{-B(t)}{1 + B(t)\Delta W(t)} + \frac{A(t)B(t)}{(1 + A(t)\Delta W(t))(1 + B(t)\Delta W(t))} \\
&= \frac{-A(t)(1 + B(t)\Delta W(t)) - B(t)(1 + A(t)\Delta W(t)) + A(t)B(t)\Delta W(t)}{(1 + A(t)\Delta W(t))(1 + B(t)\Delta W(t))} \\
&= \frac{-(A(t) + B(t) + A(t)B(t)\Delta W(t))}{1 + (A(t) + B(t) + A(t)B(t)\Delta W(t))\Delta W(t)}
\end{aligned}$$

$$= \frac{-(A \oplus_W B)(t)}{1 + (A \oplus_W B)(t)\Delta W(t)}$$
$$= (\ominus_W (A \oplus_W B))(t),$$

where we have used (6.2), (6.3) and (6.5). $\qquad\qquad\square$

Definition 6.8. If $t_0 \in \mathbb{T}$ and $B \in \mathcal{R}_W$, then the unique solution of

$$\Delta X = B(t)X\Delta W, \quad X(t_0) = 1 \tag{6.6}$$

is denoted by

$$X = E_B(\cdot, t_0). \tag{6.7}$$

We call $E_B(\cdot, t_0)$ the stochastic exponential.

Definition 6.9. If $B \in \mathcal{R}_W$, then the first order linear stochastic dynamic equation

$$\Delta X = B(t)X\Delta W \tag{6.8}$$

is called *stochastic regressive*.

Lemma 6.10. *Let $f, g : \mathbb{T} \to \mathbb{R}$ be functions defined on an isolated time scale. If $f(t) = f(t_0) + \sum_{\tau \in [t_0, t)} f(\tau)g(\tau)$ holds for all $t \geq t_0$, then*

$$S(t): \quad f(t) = f(t_0) \prod_{\tau \in [t_0, t)} [1 + g(\tau)]$$

holds for all $t \geq t_0$.

Proof. We prove the lemma using the induction principle given in [28, Theorem 1.7]. We observe that $S(t_0)$ is trivially satisfied. Now, assuming that $S(t)$ holds, we have

$$\begin{aligned}
f(\sigma(t)) &= f(t_0) + \sum_{\tau \in [t_0, \sigma(t))} f(\tau)g(\tau) \\
&= f(t_0) + f(t)g(t) + \sum_{\tau \in [t_0, t)} f(\tau)g(\tau)
\end{aligned}$$

$$
\begin{aligned}
&= \ f(t)g(t) + f(t) = [1 + g(t)]f(t) \\
&= \ [1 + g(t)]f(t_0) \prod_{\tau \in [t_0, t)} [1 + g(\tau)] \\
&= \ f(t_0) \prod_{t \in [t_0, \sigma(t))} [1 + g(\tau)] \, .
\end{aligned}
$$

Therefore $S(\sigma(t))$ holds. $\qquad\square$

Theorem 6.11. $E_b(\cdot, t_0)$ *defined in Definition 6.8 is given by*

$$
E_B(t, t_0) = \prod_{\tau \in [t_0, t)} [1 + B(\tau)\Delta W(\tau)] \, . \tag{6.9}
$$

Proof. Denoting the right-hand side of (6.9) by $X(t)$, we find that

$$
\begin{aligned}
\Delta X(t) \ &= \ X(\sigma(t)) - X(t) \\
&= \ \prod_{\tau \in [t_0, \sigma(t))} [1 + B(\tau)\Delta W(\tau)] - \prod_{\tau \in [t_0, t)} [1 + B(\tau)\Delta W(\tau)] \\
&= \ [(1 + B(t)\Delta W(\tau)) - 1] \prod_{\tau \in [t_0, t)} [1 + B(\tau)\Delta W(\tau)] \\
&= \ B(t)\Delta W(t) \prod_{\tau \in [t_0, t)} [1 + B(\tau)\Delta W(\tau)] \\
&= \ B(t)X(t)\Delta W(t).
\end{aligned}
$$

Conversely, let X be a solution of (6.6). Then

$$
\begin{aligned}
X(t) \ &= \ X(t_0) + \int_{t_0}^{t} B(\tau)X(\tau)\Delta W(\tau) \\
&= \ 1 + \sum_{\tau \in [t_0, t)} B(\tau)X(\tau)\Delta W(\tau) \\
&= \ \prod_{\tau \in [t_0, t)} [1 + B(\tau)\Delta W(\tau)] \, ,
\end{aligned}
$$

where in the last equality we have used Lemma 6.10 with $f(t_0) = 1$. $\qquad\square$

Example 6.12.   (i) If $\mathbb{T} = \mathbb{Z}$, then

$$E_B(t, t_0) = \prod_{\tau=t_0}^{t-1} \left[1 + B(\tau)(W(\tau+1) - W(\tau))\right].$$

(ii) If $\mathbb{T} = h\mathbb{Z}$ for $h > 0$, then

$$E_B(t, t_0) = \prod_{\tau=\frac{t_0}{h}}^{\frac{t}{h}-1} \left[1 + B(h\tau)(W(h\tau + h) - W(h\tau))\right].$$

(iii) If $\mathbb{T} = q^{\mathbb{N}_0} = \{q^k : k \in \mathbb{N}_0\}$, where $q > 1$, then

$$E_B(t, t_0) = \prod_{\tau=\frac{\ln t_0}{\ln q}}^{\frac{\ln t}{\ln q}-1} \left[1 + B\left(q^\tau\right)\left(W\left(q^{\tau+1}\right) - W\left(q^\tau\right)\right)\right].$$

(iv) If $\mathbb{T} = \mathbb{R}$ and $B(t) = b(t)$ is deterministic, then $E_b(t, t_0)$ is the solution of the stochastic differential problem

$$dX = b(t)X\,dW, \quad X(t_0) = 1,$$

whose solution from Subsection 3.2 is given by

$$X(t) = \exp\left(-\frac{1}{2} \int_{t_0}^{t} b^2(s)ds + \int_{t_0}^{t} b(s)dW\right) \tag{6.10}$$

for $t \in \mathbb{T}$.

Theorem 6.13. *If $A, B \in \mathcal{R}_W$, then*

(i) $E_A(\sigma(t), t_0) = (1 + A(t)\Delta W(t))E_A(t, t_0),$

(ii) $\frac{1}{E_A(t,t_0)} = E_{\ominus_W A}(t, t_0),$

(iii) $E_A(t, t_0)E_B(t, t_0) = E_{A \oplus_W B}(t, t_0),$

(iv) $\frac{E_A(t,t_0)}{E_B(t,t_0)} = E_{A \ominus_W B}(t, t_0),$

(v) $\Delta \left(\frac{1}{E_A(t,t_0)} \right) = -\frac{A(t)\Delta t}{E_A(\sigma(t),t_0)}.$

Proof. Part (i). By Theorem 6.11 we have

$$
\begin{aligned}
E_A(\sigma(t), t_0) &= \prod_{\tau \in [t_0, \sigma(t))} [1 + A(\tau)\Delta W(\tau)] \\
&= (1 + A(t)\Delta W(t)) \prod_{\tau \in [t_0, t)} [1 + A(\tau)\Delta W(\tau)] \\
&= (1 + A(t)\Delta W(t))E_A(t, t_0).
\end{aligned}
$$

Part (ii). By Theorem 6.11 we have

$$
\begin{aligned}
E_{\ominus_W A}(t, t_0) &= \prod_{\tau \in [t_0, t)} [1 + (\ominus_W A)(\tau)\Delta W(\tau)] \\
&= \prod_{\tau \in [t_0, t)} \left[1 - \frac{A(\tau)}{1 + A(\tau)\Delta W(\tau)}\Delta W(\tau) \right] \\
&= \frac{1}{\prod_{\tau \in [t_0, t)} [1 + A(\tau)\Delta W(\tau)]} \\
&= \frac{1}{E_A(t, t_0)}.
\end{aligned}
$$

Part (iii). We observe that

$$
\begin{aligned}
&E_A(t, t_0)E_B(t, t_0) \\
&= \prod_{\tau \in [t_0, t)} [1 + A(\tau)\Delta W(\tau)] \prod_{\tau \in [t_0, t)} [1 + B(\tau)\Delta W(\tau)] \\
&= \prod_{\tau \in [t_0, t)} \left[1 + A(\tau)\Delta W(\tau) + B(\tau)\Delta W(\tau) + A(\tau)B(\tau)(\Delta W(\tau))^2 \right] \\
&= \prod_{\tau \in [t_0, t)} [1 + (A(\tau) + B(\tau) + A(\tau)B(\tau)\Delta W(\tau))\Delta W(\tau)] \\
&= \prod_{\tau \in [t_0, t)} [1 + (A \oplus_W B)(\tau)\Delta W(\tau)] \\
&= E_{A \oplus_W B}(t, t_0).
\end{aligned}
$$

Part (iv). By Theorem 6.6 we have

$$
\begin{aligned}
E_{A\ominus_W B}(t, t_0) &= \prod_{\tau\in[t_0,t)} \left[1 + \left(\frac{A(\tau) - B(\tau)}{1 + B(\tau)\Delta W(\tau)} \right) \Delta W(\tau) \right] \\
&= \frac{\prod_{\tau\in[t_0,t)} \left[1 + A(\tau)\Delta W(\tau) \right]}{\prod_{\tau\in[t_0,t)} \left[1 + B(\tau)\Delta W(\tau) \right]} \\
&= \frac{E_A(t, t_0)}{E_B(t, t_0)}.
\end{aligned}
$$

Part (v). We calculate

$$
\begin{aligned}
\Delta\left(\frac{1}{E_A(t, t_0)} \right) &= \Delta\left(E_{\ominus_W A}(t, t_0) \right) \\
&= (\ominus_W A)(t) E_{\ominus_W A}(t, t_0)\Delta t \\
&= \frac{-A(t)}{1 + A(t)\Delta W(t)} \frac{1}{E_A(t, t_0)} \Delta t \\
&= \frac{-A(t)\Delta t}{E_A(\sigma(t), t_0)},
\end{aligned}
$$

where we have used parts (i) and (ii) of this theorem. $\qquad\square$

Theorem 6.14. *If* $A, B \in \mathcal{R}_W$, *then*

$$
\Delta E_{A\ominus_W B}(t, t_0) = (A(t) - B(t))\frac{E_A(t, t_0)}{E_B(\sigma(t), t_0)}\Delta t.
$$

Proof. We have

$$
\begin{aligned}
\Delta E_{A\ominus_W B}(t, t_0) &= (A \ominus_W B)(t) E_{A\ominus_W B}(t, t_0)\Delta t \\
&= \frac{A(t) - B(t)}{1 + B(t)\Delta W(t)} \frac{E_A(t, t_0)}{E_B(t, t_0)}\Delta t \\
&= \frac{(A(t) - B(t))E_A(t, t_0)}{E_B(\sigma(t), t_0)}\Delta t
\end{aligned}
$$

where we have used Theorem 6.13 (i) and (iv). $\qquad\square$

Definition 6.15. We define the set $\mathcal{R}_W^+$ of all *stochastic positively regressive* elements

of $\mathcal{R}_W$ by

$$\mathcal{R}_W^+ = \{A \in \mathcal{R}_W : 1 + A(t)\Delta W(t) > 0 \text{ a.s., for all } t \in \mathbb{T}\}.$$

Theorem 6.16. *$\mathcal{R}_W^+$ is a subgroup of $\mathcal{R}_W$.*

Proof. Obviously we have $\mathcal{R}_W^+ \subset \mathcal{R}_W$ and that $0 \in \mathcal{R}_W^+$. Now let $A, B \in \mathcal{R}_W^+$. Then

$$1 + A(t)\Delta W(t) > 0 \quad \text{a.s.,} \quad \text{and} \quad 1 + B(t)\Delta W(t) > 0 \quad \text{a.s.}$$

for all $t \in \mathbb{T}$. Therefore

$$1 + (A \oplus_W B)(t)\Delta W(t) = (1 + A(t)\Delta W(t))(1 + B(t)\Delta W(t)) > 0 \quad \text{a.s.}$$

for all $t \in \mathbb{T}$. Hence, we have

$$A \oplus_W B \in \mathcal{R}_W^+.$$

Next, let $A \in \mathcal{R}_W^+$. Then

$$1 + A(t)\Delta W(t) > 0 \quad \text{a.s.}$$

for all $t \in \mathbb{T}$. This implies that

$$1 + (\ominus_W A)(t)\Delta W(t) = 1 - \frac{A(t)\Delta W(t)}{1 + A(t)\Delta W(t)} = \frac{1}{1 + A(t)\Delta W(t)} > 0 \quad \text{a.s.}$$

for all $t \in \mathbb{T}$. Hence,

$$\ominus_W A \in \mathcal{R}_W^+.$$

These calculations establish that $\mathcal{R}_W^+$ is a subgroup of $\mathcal{R}$. $\qquad\square$

Theorem 6.17. *If $B \in \mathcal{R}_W^+$, then $E_B(t, t_0) > 0$ a.s.*

Proof. From Definition 6.15 we have

$$1 + B(t)\Delta W(t) > 0 \quad \text{a.s.,}$$

for all $t \in \mathbb{T}$. Hence,

$$E_B(t, t_0) = \prod_{\tau \in [t_0, t)} [1 + B(\tau) \Delta W(\tau)] > 0 \quad \text{a.s.,} \tag{6.11}$$

for all $t \in \mathbb{T}$. $\qquad \square$

Theorem 6.18. *If $E_B(\cdot, t_0)$ is defined as in Definition (6.8), and $B(t)$ and $\Delta W(t)$ is independent for all $t \in \mathbb{T}$, then*

$$\mathbb{E}\left[E_B(t, t_0)\right] = 1 \tag{6.12}$$

and

$$\mathbb{V}\left[E_B(t, t_0)\right] = e_{\mathbb{E}[B^2]}(t, t_0) - 1. \tag{6.13}$$

Proof. From (6.9) we have

$$
\begin{aligned}
\mathbb{E}\left[E_B(t, t_0)\right] &= \mathbb{E}\left[\prod_{\tau \in [t_0, t)} [1 + B(\tau) \Delta W(\tau)]\right] \\
&= \prod_{\tau \in [t_0, t)} \left(1 + \mathbb{E}\left[B(\tau) \Delta W(\tau)\right]\right) \\
&= \prod_{\tau \in [t_0, t)} \left(1 + \mathbb{E}\left[B(\tau)\right] \mathbb{E}\left[\Delta W(\tau)\right]\right) \\
&= 1, \tag{6.14}
\end{aligned}
$$

where on the third equality we have used the independence of B and ΔW. Likewise,

$$
\begin{aligned}
\mathbb{E}\left[E_B^2(t, t_0)\right] &= \mathbb{E}\left[\prod_{\tau \in [t_0, t)} \left(1 + B(\tau)\left(W(\sigma(\tau)) - W(\tau)\right)\right)^2\right] \\
&= \prod_{\tau \in [t_0, t)} \left(1 + \mathbb{E}\left[2B(\tau)\Delta W(\tau)\right] + \mathbb{E}\left[B^2(\tau)(\Delta W(\tau))^2\right]\right) \\
&= \prod_{\tau \in [t_0, t)} \left(1 + \mu(\tau)\mathbb{E}\left[B^2(\tau)\right]\right) \\
&= e_{\mathbb{E}[B^2]}(t, t_0). \tag{6.15}
\end{aligned}
$$

Now using (6.14) and (6.15) we have

$$\mathbb{V}\left[E_B(t, t_0)\right] = \mathbb{E}\left[E_B^2(t, t_0)\right] - \left(\mathbb{E}\left[E_B(t, t_0)\right]\right)^2 = e_{\mathbb{E}[B^2]}(t, t_0) - 1, \qquad (6.16)$$

as claimed. □

6.1.2. Initial Value Problems   In this subsubsection we study the first order nonhomogeneous linear stochastic dynamic equation

$$\Delta X = c(t)\Delta t + b(t)X\Delta W \qquad (6.17)$$

and the corresponding homogeneous equation

$$\Delta X = b(t)X\Delta W \qquad (6.18)$$

on a time scale $\mathbb{T}$, where $b, c : \mathbb{T} \to \mathbb{R}$ are deterministic functions. The results from Subsubsection 6.1.1 yield the following theorems.

Theorem 6.19. *Suppose (6.18) is regressive. Let $t_0 \in \mathbb{T}$ and $X_0 \in \mathbb{R}$. Then the solution of the initial value problem*

$$\Delta X = b(t)X\Delta W, \quad X(t_0) = X_0 \qquad (6.19)$$

is given by

$$X(t) = X_0 E_b(t, t_0).$$

Proof. Let us assume X is a solution of (6.19) and let us consider the quotient $X/E_b(\cdot, t_0)$. Then we have

$$
\begin{aligned}
\Delta\left(\frac{X(t)}{E_b(t, t_0)}\right) &= \frac{(\Delta X(t))\, E_b(t, t_0) - X(t)\Delta E_b(t, t_0)}{E_b(t, t_0)E_b(\sigma(t), t_0)} \\
&= \frac{b(t)X(t)E_b(t, t_0)\Delta W(t) - X(t)b(t)E_b(t, t_0)\Delta W(t)}{E_b(t, t_0)E_b(\sigma(t), t_0)} \\
&= 0.
\end{aligned}
$$

Hence,

$$\frac{X(t)}{E_b(t, t_0)} \equiv \frac{X(t_0)}{E_b(t_0, t_0)} = X_0$$

and therefore $X(t) = X_0 E_b(t, t_0)$. □

Theorem 6.20. *Suppose $b \in \mathcal{R}_W$. Let $t_0 \in \mathbb{T}$ and $X_0 \in \mathbb{R}$. The unique solution of the initial value problem*

$$\Delta X = -b(t) X^\sigma \Delta W, \quad X(t_0) = X_0 \tag{6.20}$$

is given by

$$X(t) = X_0 E_{\ominus_W b}(t, t_0).$$

Proof. Let us assume X is a solution of (6.20) and let us consider the quotient $X E_b(\cdot, t_0)$. Then we have

$$
\begin{aligned}
\Delta \left[X(t) E_b(t, t_0) \right] &= e_B(t, t_0) \Delta X + b(t) E_b(t, t_0) X(\sigma(t)) \Delta W(t) \\
&= E_b(t, t_0) \left[\Delta X(t) + b(t) X(\sigma(t)) \Delta W(t) \right] \\
&= 0.
\end{aligned}
$$

Hence,

$$X(t) E_b(t, t_0) \equiv X(t_0) E_b(t_0, t_0) = X_0$$

and therefore $X(t) = X_0 E_{\ominus_W b}(t, t_0)$. □

We now turn our attention to the nonhomogeneous problem

$$\Delta X = c(t) \Delta t - b(t) X^\sigma \Delta W, \quad X(t_0) = X_0. \tag{6.21}$$

Let us assume that X is a solution of (6.21). We multiply both sides of the stochastic

dynamic equation in (6.21) by the so-called *integrating factor* $E_b(t, t_0)$ and obtain

$$
\begin{aligned}
\Delta \left[E_b(\cdot, t_0) X \right] &= E_b(t, t_0) \Delta X(t) + b(t) E_b(t, t_0) X(\sigma(t)) \Delta W(t) \\
&= E_b(t, t_0) \left[\Delta X(t) + b(t) X(\sigma(t)) \Delta W(t) \right] \\
&= E_b(t, t_0) c(t) \Delta t,
\end{aligned}
$$

and now we integrate both sides from t_0 to t to conclude

$$
E_b(t, t_0) X(t) - E_b(t_0, t_0) X(t_0) = \int_{t_0}^{t} E_b(\tau, t_0) c(\tau) \Delta \tau. \tag{6.22}
$$

Definition 6.21. The equation (6.17) is called *stochastic regressive* provided (6.18) is regressive and $c : \mathbb{T} \to \mathbb{R}$ is rd-continuous.

Theorem 6.22. *Suppose* (6.17) *is regressive. Let* $t_0 \in \mathbb{T}$ *and* $X_0 \in \mathbb{R}$. *The solution of the initial value problem*

$$
\Delta X = c(t) \Delta t - b(t) X^{\sigma} \Delta W, \quad X(t_0) = X_0 \tag{6.23}
$$

is given by

$$
X(t) = E_{\ominus_W b}(t, t_0) X_0 + \int_{t_0}^{t} E_{\ominus_W b}(t, \tau) c(\tau) \Delta \tau. \tag{6.24}
$$

Proof. To verify that X given by (6.24) solves the initial value problem (6.23), we observe that

$$
\begin{aligned}
X(\sigma(t)) &= E_{\ominus_W b}(\sigma(t), t) X_0 + \int_{t_0}^{\sigma(t)} E_{\ominus_W b}(\sigma(t), \tau) c(\tau) \Delta \tau \\
&= (1 + (\ominus_W b)(t) \Delta W(t)) E_{\ominus_W b}(t, t) X_0 + \int_{t_0}^{t} E_{\ominus_W b}(\sigma(t), \tau) c(\tau) \Delta \tau \\
&\quad + \int_{t}^{\sigma(t)} E_{\ominus_W b}(\sigma(t), \tau) c(\tau) \Delta \tau \\
&= (1 + (\ominus_W b)(t) \Delta W(t)) \left(E_{\ominus_W b}(t, t_0) X_0 + \int_{t_0}^{t} E_{\ominus_W b}(t, \tau) c(\tau) \Delta \tau \right) \\
&\quad + E_{\ominus_W b}(\sigma(t), t) c(t) \Delta t \\
&= (1 + (\ominus_W b)(t) \Delta W(t)) \left(X(t) + c(t) \Delta t \right), \tag{6.25}
\end{aligned}
$$

where on the second equality we have used Theorem 6.13 (i). Now since

$$1 + (\ominus_W b)(t)\Delta W(t) = 1 - \frac{b(t)}{1 + b(t)\Delta W(t)}\Delta W(t) = \frac{1}{1 + b(t)\Delta W(t)},$$

(6.25) reduces to

$$(1 + b(t)\Delta W(t)X(\sigma(t))) = X(t) + c(t)\Delta t$$

or

$$X(\sigma(t)) - X(t) = c(t)\Delta t - b(t)X(\sigma(t))\Delta W(t)$$

which is same as (6.23). Next, if X is a solution of (6.23), then we have seen above that (6.22) holds. Hence, we obtain

$$E_b(t, t_0)X(t) = X_0 + \int_{t_0}^t E_b(\tau, t_0)c(\tau)\Delta\tau.$$

We solve for X and apply Theorem 6.13 to arrive at the final formula given in the theorem. $\qquad\square$

Theorem 6.23. *Suppose* (6.17) *is regressive. Let* $t_0 \in \mathbb{T}$ *and* $X_0 \in \mathbb{R}$. *The solution of the initial value problem*

$$\Delta X = c(t)\Delta t + b(t)X\Delta W, \quad X(t_0) = X_0 \tag{6.26}$$

is given by

$$X(t) = E_b(t, t_0)X_0 + \int_{t_0}^t E_b(t, \sigma(\tau))c(\tau)\Delta\tau. \tag{6.27}$$

Proof. We equivalently rewrite $\Delta X = c(t)\Delta t + b(t)X\Delta W$ as

$$\Delta X = c(t)\Delta t + b(t)[X^\sigma - \Delta X]\Delta W,$$

i.e,

$$(1 + b(t)\Delta W)\Delta X = c(t)\Delta t + b(t)X^\sigma\Delta W,$$

whence using the fact that $b \in \mathcal{R}_W$ we obtain

$$\Delta X = \frac{c(t)\Delta t}{1 + b(t)\Delta W} - (\ominus_W b)(t) X^\sigma \Delta W.$$

Next we apply Theorem 6.22 and the fact that $(\ominus_W(\ominus_W b))(t) = b(t)$ to find the solution of (6.26) as

$$X(t) = X_0 E_b(t, t_0) + \int_{t_0}^t E_b(t, \tau) \frac{c(\tau)}{1 + b(\tau)\Delta W(\tau)} \Delta\tau.$$

For the final calculation

$$\frac{E_b(t, \tau)}{1 + b(\tau)\Delta W(\tau)} = \frac{E_b(t, \tau)}{E_b(\sigma(\tau), \tau)} = E_b(t, \sigma(\tau)),$$

we use Theorem 6.13. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

6.1.3. Gronwall's Inequality. In this subsubsection we present a dynamic form of Gronwall's inequality involving the stochastic exponential. Throughout we let $t_0 \in \mathbb{T}$.

Theorem 6.24. *Let $b \in \mathcal{R}_W^+$. Then*

$$\Delta X(t) \leq c(t)\Delta t + b(t)X(t)\Delta W(t) \quad a.s. \tag{6.28}$$

for all $t \in \mathbb{T}$ implies

$$X(t) \leq X(t_0)E_b(t, t_0) + \int_{t_0}^t E_b(t, \sigma(\tau))c(\tau)\Delta\tau \quad a.s. \tag{6.29}$$

for all $t \in \mathbb{T}$.

Proof. We use Theorem 6.13 to calculate

$$\Delta\left[X(t)E_{\ominus_W b}(t, t_0)\right] = (\Delta X(t))E_{\ominus_W b}(\sigma(t), t_0) + X(t)(\ominus_W b)(t)E_{\ominus_W b}(t, t_0)\Delta W(t)$$

$$\begin{aligned}
&= (\Delta X(t))E_{\ominus_W b}(\sigma(t), t_0)\\
&\quad + X(t)\frac{(\ominus_W b)(t)}{1 + (\ominus_W b)(t)\Delta W(t)}E_{\ominus_W b}(\sigma(t), t_0)\Delta W(t)\\
&= \left[\Delta X(t) - (\ominus_W(\ominus_W b))(t)X(t)\Delta W(t)\right]E_{\ominus_W b}(\sigma(t), t_0)\\
&= \left[\Delta X(t) - b(t)X(t)\Delta W(t)\right]E_{\ominus_W b}(\sigma(t), t_0).
\end{aligned}$$

Since $b \in \mathcal{R}_W^+$, we have $\ominus_W b \in \mathcal{R}_W^+$ by Theorem 6.16. This implies $E_{\ominus_W b} > 0$ a.s., by Theorem 6.17. Now using (6.28) we have

$$\begin{aligned}
X(t)E_{\ominus_W b}(t, t_0) - X(t_0) &\leq \int_{t_0}^t c(\tau)E_{\ominus_W b}(\sigma(\tau), t_0)\Delta\tau \quad \text{a.s.}\\
&= \int_{t_0}^t E_b(t_0, \sigma(\tau))c(\tau)\Delta\tau \quad \text{a.s.,}
\end{aligned}$$

and hence the assertion follows by applying Theorem 6.13. □

Corollary 6.25. *Let $b \in \mathcal{R}_W^+$ with $b \geq 0$. Then*

$$\Delta X(t) \leq b(t)X(t)\Delta W(t) \quad \text{a.s.,} \tag{6.30}$$

for all $t \in \mathbb{T}$ implies

$$X(t) \leq X(t_0)E_b(t, t_0) \quad \text{a.s.,} \tag{6.31}$$

for all $t \in \mathbb{T}$.

Proof. This is Theorem 6.24 with $c(t) \equiv 0$. □

6.1.4. Geometric Brownian Motion. A geometric Brownian motion is a continuous-time stochastic process in which the logarithm of the randomly varying quantity follows a Brownian motion, or a Wiener process. It is applicable to mathematical modeling of some phenomena in financial markets. It is used particularly in the field of option pricing because a quantity that follows a geometric Brownian motion may take any value strictly greater than zero, and only the fractional changes of the random variate are significant. This is a reasonable approximation of stock

price dynamics.

A stochastic process S_t is said to follow a geometric Brownian motion if it satisfies the stochastic differential equation

$$dS_t = \alpha S_t dt + \beta S_t dW_t \tag{6.32}$$

where $\{W_t\}$ is a Wiener process or Brownian motion and α and β are constants.

In this subsubsection we construct and study the properties of geometric Brownian motion in time scales $\mathbb{T}$. We observe that when $c(t) \equiv 0$ and $d(t) \equiv 0$, (6.1) reduces to the homogeneous linear SΔE

$$\Delta X = a(t)X\Delta t + b(t)X\Delta W. \tag{6.33}$$

Obviously, $X(t) \equiv 0$ is a solution of (6.33).

Theorem 6.26. *If* $t_0 \in \mathbb{T}$, $a \in \mathcal{R}$ *and* $\frac{b}{1+\mu a} \in \mathcal{R}_W$, *then the solution of*

$$\Delta X = a(t)X\Delta t + b(t)X\Delta W, \quad X(t_0) = X_0. \tag{6.34}$$

is given by

$$X = X_0 e_a(\cdot, t_0) E_{\frac{b}{1+\mu a}}(\cdot, t_0). \tag{6.35}$$

Proof. Let X be given by (6.35). Then by (5.29),

$$
\begin{aligned}
\Delta X(t) &= X_0 \left(\Delta e_a(t, t_0)\right) E_{\frac{b}{1+\mu a}}(t, t_0) + X_0 e_a(\sigma(t), t_0)\Delta E_{\frac{b}{1+\mu a}}(t, t_0) \\
&= X_0 a(t)e_a(t, t_0) E_{\frac{b}{1+\mu a}}(t, t_0)\Delta t \\
&\quad + X_0(1 + \mu(t)a(t))e_a(t, t_0)\frac{b(t)}{1 + \mu(t)a(t)} E_{\frac{b}{1+\mu a}}(t, t_0)\Delta W(t) \\
&= X_0 a(t)e_a(t, t_0) E_{\frac{b}{1+\mu a}}(t, t_0)\Delta t + X_0 b(t)e_a(t, t_0) E_{\frac{b}{1+\mu a}}(t, t_0)\Delta W(t) \\
&= a(t)X(t)\Delta t + b(t)X(t)\Delta W(t).
\end{aligned}
$$

Conversely, let X be a solution of (6.34). Then

$$
\begin{aligned}
X(t) &= X(t_0) + \int_{t_0}^t a(\tau)X(\tau)\Delta\tau + \int_{t_0}^t b(\tau)X(\tau)\Delta W(\tau) \\
&= X_0 + \sum_{\tau\in[t_0,t)} \mu(\tau)a(\tau)X(\tau) + \sum_{\tau\in[t_0,t)} b(\tau)X(\tau)\Delta W(\tau) \\
&= X_0 + \sum_{\tau\in[t_0,t)} X(\tau)\left[\mu(\tau)a(\tau) + b(\tau)\Delta W(\tau)\right] \\
&= X_0 \prod_{\tau\in[t_0,t)} \left[1 + \mu(\tau)a(\tau) + b(\tau)\Delta W(\tau)\right] \\
&= X_0 \prod_{\tau\in[t_0,t)} \left[1 + \mu(\tau)a(\tau)\right] \prod_{\tau\in[t_0,t)} \left[1 + \frac{b(\tau)}{1 + a(\tau)\mu(\tau)}\Delta W(\tau)\right] \\
&= X_0 e_a(t,t_0) E_{\frac{b}{1+\mu a}}(t,t_0),
\end{aligned}
$$

where on the fourth equality we have used Lemma 6.10. $\qquad\square$

In the proof above we have not used Itô's lemma which is standard while solving such equations.

When $d(t)\equiv 0$ in (6.1), the SΔE has the form

$$\Delta X = (a(t)X + c(t))\Delta t + b(t)\Delta W, \tag{6.36}$$

that is, the noise appears additively. The homogeneous equation obtained from (6.36) is then an ordinary dynamic equation

$$\Delta X = a(t)X\Delta t \tag{6.37}$$

and its fundamental solution is given by $e_a(\cdot,t_0)$. Taking the Δ of $e_{\ominus a}(t,t_0)X(t)$, we obtain

$$
\begin{aligned}
\Delta\left[e_{\ominus a}(t,t_0)X(t)\right] &= \left(\Delta e_{\ominus a}(t,t_0)\right)X(t)\Delta t + e_{\ominus a}(\sigma(t),t_0)\Delta X(t) \\
&= -a(t)e_{\ominus a}(\sigma(t),t_0)X(t)\Delta t \\
&\quad + e_{\ominus a}(\sigma(t),t_0)\left[(a(t)X(t) + c(t))\Delta t + b(t)\Delta W(t)\right]
\end{aligned}
$$

$$= c(t)e_{\ominus a}(\sigma(t), t_0)\Delta t + b(t)e_{\ominus a}(\sigma(t), t_0)\Delta W(t).$$

We can now integrate to get

$$e_{\ominus a}(t, t_0)X(t) = e_{\ominus a}(t_0, t_0)X(t_0) + \int_{t_0}^t c(\tau)e_{\ominus a}(\sigma(\tau), t_0)\Delta\tau$$

$$+ \int_{t_0}^t b(\tau)e_{\ominus a}(\sigma(\tau), t_0)\Delta W(\tau).$$

Since $e_a(t_0, t_0) = 1$, this leads to the solution

$$X(t) = e_a(t, t_0)\left[X(t_0) + \int_{t_0}^t c(\tau)e_{\ominus a}(\sigma(\tau), t_0)\Delta\tau\right]$$

$$+ e_a(t, t_0)\int_{t_0}^t b(\tau)e_{\ominus a}(\sigma(\tau), t_0)\Delta W(\tau) \tag{6.38}$$

of the SΔE (6.36).

Theorem 6.27. *If X is a solution of (6.36), then X is given by (6.38) and*

$$\mathbb{E}[X(t)] = e_a(t, t_0)\left[\mathbb{E}[X(t_0)] + \int_{t_0}^t c(\tau)e_{\ominus a}(\sigma(\tau), t_0)\Delta\tau\right]. \tag{6.39}$$

Proof. That X is given by (6.38) is a solution of (6.36) follows from the discussion above. For (6.39), we observe that

$$\mathbb{E}[X(t)] = \mathbb{E}\left[e_a(t, t_0)\left(X(t_0) + \int_{t_0}^t c(\tau)e_{\ominus a}(\sigma(\tau), t_0)\Delta\tau\right)\right]$$

$$+ \mathbb{E}\left[e_a(t, t_0)\int_{t_0}^t b(\tau)e_{\ominus a}(\sigma(\tau), t_0)\Delta W(\tau)\right]$$

$$= e_a(t, t_0)\left[\mathbb{E}[X(t_0)] + \int_{t_0}^t c(\tau)e_{\ominus a}(\sigma(\tau), t_0)\Delta\tau\right]$$

$$+ e_a(t, t_0)\mathbb{E}\left[\int_{t_0}^t b(\tau)e_{\ominus a}(\sigma(\tau), t_0)\Delta W(\tau)\right]$$

$$= e_a(t, t_0)\left[\mathbb{E}[X(t_0)] + \int_{t_0}^t c(\tau)e_{\ominus a}(\sigma(\tau), t_0)\Delta\tau\right],$$

where in the third equality we have used Lemma 5.6. $\qquad\square$

Example 6.28. Let us consider the SΔE

$$\Delta X = a(t)(X - 1)\Delta t + b(t)\Delta W, \tag{6.40}$$

where $a, b \in \mathcal{R}$. First, we observe that (6.40) is of the form (6.36) with $c(t) = -a(t)$. Therefore, from (6.39) we have

$$
\begin{aligned}
\mathbb{E}[X(t)] &= e_a(t, t_0) \left[\mathbb{E}[X(t_0)] - \int_{t_0}^{t} a(\tau) e_{\ominus a}(\sigma(\tau), t_0) \Delta \tau \right] \\
&= e_a(t, t_0) \left[\mathbb{E}[X(t_0)] + \int_{t_0}^{t} \left(\frac{1}{e_a(\cdot, t_0)} \right)^{\Delta} (\tau) \Delta \tau \right] \\
&= e_a(t, t_0) \left[\mathbb{E}[X(t_0)] + \frac{1}{e_a(t, t_0)} - \frac{1}{e_a(t_0, t_0)} \right] \\
&= 1 + e_a(t, t_0) \left(\mathbb{E}[X(t_0)] - 1 \right),
\end{aligned}
$$

where on the second equality we have used Theorem 2.26 and on the third equality we have used Definition 2.12. An important conclusion from above is $\mathbb{E}[X(t)] \equiv 1$ for all $t \in \mathbb{T}$ if $\mathbb{E}[X(t_0)] = 1$.

Example 6.29. When $\mathbb{T} = \mathbb{R}$, (6.6) is given by

$$dX = b(t)X \, dW, \quad X(t_0) = 1 \tag{6.41}$$

whose solution from (3.8) is given by

$$X(t) = \exp \left(-\frac{1}{2} \int_{t_0}^{t} b^2(s) ds + \int_{t_0}^{t} b(s) dW \right) \tag{6.42}$$

for $t \in \mathbb{T}$. We observe that (6.42) gives us $E_b(t, t_0)$ when $\mathbb{T} = \mathbb{R}$. Likewise we observe that when $\mathbb{T} = \mathbb{R}$, (6.34) becomes

$$dX = a(t)X \, dt + b(t)X \, dW, \quad X(t_0) = 1 \tag{6.43}$$

whose solution is given by

$$X(t) = \exp\left(\int_{t_0}^t \left(a(s) - \frac{1}{2}b^2(s)\right)ds + \int_{t_0}^t b(s)dW\right). \tag{6.44}$$

From the above discussion we conclude that (6.35) with $X(t_0) = 1$ is also true when $\mathbb{T} = \mathbb{R}$. To observe this we note that $\mu(t) \equiv 0$ in this case and (6.35) becomes

$$
\begin{aligned}
X(t) &= e_a(t, t_0)E_b(t, t_0) \\
&= \exp\left(\int_{t_0}^t a(s)ds\right)\exp\left(-\frac{1}{2}\int_{t_0}^t b^2(s)ds + \int_{t_0}^t b(s)dW\right) \\
&= \exp\left(\int_{t_0}^t \left(a(s) - \frac{1}{2}b^2(s)\right)ds + \int_{t_0}^t b(s)dW\right),
\end{aligned}
$$

which is the same as (6.44).

6.2. STOCK PRICE

Let $S(t)$ denote the price of stock at time t and $S(t_0) = S_0$ the current price of the stock. Then the evolution of $S(t)$ in time is modeled by supposing that $\Delta S/S$, the relative change in price, evolves according to the SΔE

$$\frac{\Delta S}{S} = \alpha(t)\Delta t + \beta(t)\Delta W, \quad S(t_0) = S_0 > 0$$

for certain $\alpha \in \mathcal{R}$ and $\beta : \mathbb{T} \to \mathbb{R}$, called the *drift* and the *volatility* of the stock. Then

$$\Delta S = \alpha(t)S\Delta t + \beta(t)S\Delta W, \tag{6.45}$$

and so by (6.35) we have

$$S(t) = S_0 e_\alpha(t, t_0)E_{\frac{\beta}{1+\mu\alpha}}(t, t_0). \tag{6.46}$$

Thus,

$$
\begin{aligned}
\mathbb{E}[S(t)] &= \mathbb{E}\left[S_0 e_\alpha(t, t_0) E_{\frac{\beta}{1+\mu\alpha}}(t, t_0)\right] \\
&= S_0 e_\alpha(t, t_0) \mathbb{E}\left[E_{\frac{\beta}{1+\mu\alpha}}(t, t_0)\right] \\
&= S_0 e_\alpha(t, t_0),
\end{aligned}
\tag{6.47}
$$

where on the second equality we have used Theorem 6.18. We can also arrive at (6.47) by observing that

$$
S(t) = S(t_0) + \int_{t_0}^t \alpha(\tau) S(\tau) \Delta\tau + \int_{t_0}^t \beta(\tau) S(\tau) \Delta W(\tau)
$$

and therefore,

$$
\begin{aligned}
\mathbb{E}[S(t)] &= \mathbb{E}[S(t_0)] + \mathbb{E}\left[\int_{t_0}^t \alpha(\tau) S(\tau) \Delta\tau\right] + \mathbb{E}\left[\int_{t_0}^t \beta(\tau) S(\tau) \Delta W(\tau)\right] \\
&= S_0 + \int_{t_0}^t \alpha(\tau) \mathbb{E}[S(\tau)] \Delta\tau,
\end{aligned}
$$

where we have used Lemma 5.6. If we take $y(t) = \mathbb{E}[S(t)]$, then this is a first-order homogeneous linear dynamic equation of the form $y^\Delta = \alpha(t)y$, $y(t_0) = y_0$, whose solution from Theorem 2.23 is $y(t) = e_\alpha(t, t_0)y_0$. Using this fact we conclude that

$$
\mathbb{E}[S(t)] = S_0 e_\alpha(t, t_0).
\tag{6.48}
$$

For the variance of stock price, we observe that

$$
\begin{aligned}
\mathbb{V}[S(t)] &= \mathbb{E}[S^2(t)] - (\mathbb{E}[S(t)])^2 \\
&= S_0^2 e_\alpha^2(t, t_0) \mathbb{E}\left[E_{\frac{\beta}{1+\mu\alpha}}^2(t, t_0)\right] - (S_0 e_\alpha(t, t_0))^2 \\
&= S_0^2 e_\alpha^2(t, t_0) e_{\frac{\beta^2}{(1+\mu\alpha)^2}}(t, t_0) - (S_0 e_\alpha(t, t_0))^2 \\
&= S_0^2 e_\alpha^2(t, t_0) \left(e_{\frac{\beta^2}{(1+\mu\alpha)^2}}(t, t_0) - 1\right),
\end{aligned}
\tag{6.49}
$$

where on the third equality we have used Theorem 6.18. We note that when $\mathbb{T} = \mathbb{R}$, (6.49) reduces to

$$\begin{aligned} \mathbb{V}[S(t)] &= S_0^2 e_\alpha^2(t, t_0)\left(e_{\beta^2}(t, t_0) - 1\right) \\ &= S_0^2 \exp\left(2\alpha(t - t_0)\right)\left[\exp\left(\beta^2(t - t_0)\right) - 1\right], \end{aligned}$$

which matches with the standard result regarding the variance of stock price [93, Page 231].

Example 6.30. From (6.48), the expected value of the stock price at time t for different time scales are the following.

(i) If $\mathbb{T} = \mathbb{Z}$, then

$$\mathbb{E}[S(t)] = S_0 \prod_{\tau=t_0}^{t-1} (1 + \alpha(\tau))$$

if α is never -1, and

$$\mathbb{E}[S(t)] = S_0(1 + \alpha)^{t - t_0}$$

for constant $\alpha \neq -1$.

(ii) If $\mathbb{T} = h\mathbb{Z}$ for $h > 0$, then

$$\mathbb{E}[S(t)] = S_0 \prod_{\tau=\frac{t_0}{h}}^{\frac{t}{h}-1} (1 + h\alpha(h\tau))$$

for α regressive, and

$$\mathbb{E}[S(t)] = S_0(1 + h\alpha)^{\frac{t - t_0}{h}}$$

for constant $\alpha \neq -1/h$.

(iii) If $\mathbb{T} = q^{\mathbb{N}_0}$ where $q > 1$, then

$$\mathbb{E}[S(t)] = S_0 \prod_{\tau=\frac{\ln t_0}{\ln q}}^{\frac{\ln t}{\ln q}-1} (1 + (q - 1)q^\tau \alpha\left(q^\tau\right))$$

for regressive α.

(iv) If $\mathbb{T} = \mathbb{R}$, then

$$\mathbb{E}[S(t)] = S_0 \exp\left(\int_{t_0}^{t} \alpha(\tau)d\tau\right)$$

for continuous α, and

$$\mathbb{E}[S(t)] = S_0 e^{\alpha(t-t_0)}$$

for constant α.

6.3. ORNSTEIN–UHLENBECK DYNAMIC EQUATION

In 1930, Langevin initiated a train of thought that culminated in a new theory of Brownian motion by Leonard S. Ornstein and George Eugene Uhlenbeck [94]. For ordinary Brownian motion the predictions of the Ornstein–Uhlenbeck theory are numerically indistinguishable from those of the Einstein–Smoluchowski theory. However, the Ornstein–Uhlenbeck theory is a truly dynamical theory and represents great progress in the understanding of Brownian motion [61,95]. In this subsection we consider Ornstein–Uhlenbeck type dynamic equation

$$\begin{cases} \Delta\left(Y^\Delta(t)\right) = -\alpha\Delta Y(t) + \beta\Delta W(t) \\ \\ Y(t_0) = Y_0, \quad Y^\Delta(t_0) = Y_1, \end{cases} \tag{6.50}$$

where $Y(t)$ is the *position* of a Brownian particle at time t, Y_0 and Y_1 are given random variables, while $\alpha > 0$ is the friction coefficient and β is the diffusion coefficient. If we substitute

$$X(t) = Y^\Delta(t), \tag{6.51}$$

then X is the *velocity* of the Brownian particle at time t and (6.50) reduces to

$$\begin{cases} \Delta X(t) = -\alpha X(t)\Delta t + \beta \Delta W(t) \\ \\ X(t_0) = Y_1. \end{cases} \tag{6.52}$$

Theorem 6.31. *Let $\alpha \in \mathbb{R}^+$, $\beta \in \mathbb{R}$ and let W be the Wiener process on $\mathbb{T}$. The solution of* (6.52) *for $t > t_0$ is*

$$X(t) = e_{-\alpha}(t, t_0)\left(Y_1 + \beta \int_{t_0}^t e_{\ominus(-\alpha)}(\sigma(\tau), t_0)\Delta W(\tau)\right). \tag{6.53}$$

The random variables $X(t)$ has mean

$$\mathbb{E}[X(t)] = \mathbb{E}[Y_1]e_{-\alpha}(t, t_0), \tag{6.54}$$

variance

$$\mathbb{V}[X(t)] = e^2_{-\alpha}(t, t_0)\left(\mathbb{V}[Y_1] + \beta^2 \int_{t_0}^t e^2_{\ominus(-\alpha)}(\sigma(\tau), t_0)\Delta \tau\right), \tag{6.55}$$

and covariance

$$\mathbb{C}ov[X(t), X(s)] = e_{-\alpha}(t, t_0)e_{-\alpha}(s, t_0)\left(\mathbb{V}[Y_1] + \beta^2 \int_{t_0}^{t \wedge s} e^2_{\ominus(-\alpha)}(\sigma(\tau), t_0)\Delta \tau\right). \tag{6.56}$$

Proof. If we take $a(t) = -\alpha$, $b(t) = \beta$ and $c(t) \equiv 0$ in (6.36), then from (6.38) we have

$$X(t) = e_{-\alpha}(t, t_0)\left(Y_1 + \beta \int_{t_0}^t e_{\ominus(-\alpha)}(\sigma(\tau), t_0)\Delta W(\tau)\right) \tag{6.57}$$

as the solution of (6.52). Now taking expectation on both sides of (6.57), we have

$$\begin{aligned} \mathbb{E}[X(t)] &= e_{-\alpha}(t, t_0)\left(\mathbb{E}[Y_1] + \mathbb{E}\left[\beta \int_{t_0}^t e_{\ominus(-\alpha)}(\sigma(\tau), t_0)\Delta W(\tau)\right]\right) \\ &= \mathbb{E}[Y_1]e_{-\alpha}(t, t_0), \end{aligned} \tag{6.58}$$

where on the second equality we have used Lemma 5.6. Also from (6.57),

$$
\begin{aligned}
\mathbb{E}\left[X(t)X(s)\right] &= e_{-\alpha}(t,t_0)e_{-\alpha}(s,t_0)\mathbb{E}\left[Y_1^2 + \beta Y_1 \int_{t_0}^{t} e_{\ominus(-\alpha)}(\sigma(\tau_1),t_0)\Delta W(\tau_1)\right.\\
&\quad + \beta Y_1 \int_{t_0}^{s} e_{\ominus(-\alpha)}(\sigma(\tau_2),t_0)\Delta W(\tau_2)\\
&\quad \left.+ \beta^2 \int_{t_0}^{t}\int_{t_0}^{s} e_{\ominus(-\alpha)}(\sigma(\tau_1),t_0)e_{\ominus(-\alpha)}(\sigma(\tau_2),t_0)\Delta W(\tau_1)\Delta W(\tau_2)\right]\\
&= e_{-\alpha}(t,t_0)e_{-\alpha}(s,t_0)\left[\mathbb{E}\left[Y_1^2\right] + \beta^2 \int_{t_0}^{t\wedge s} e_{\ominus(-\alpha)}^2(\sigma(\tau),t_0)\Delta\tau\right]. \ (6.59)
\end{aligned}
$$

For $t = s$, this is

$$
\mathbb{E}\left[X^2(t)\right] = e_{-\alpha}^2(t,t_0)\left(\mathbb{E}\left[Y_1^2\right] + \beta^2 \int_{t_0}^{t} e_{\ominus(-\alpha)}^2(\sigma(\tau),t_0)\Delta\tau\right). \qquad (6.60)
$$

Thus, from (6.58) and (6.60) we have

$$
\begin{aligned}
\mathbb{V}\left[X(t)\right] &= e_{-\alpha}^2(t,t_0)\left(\mathbb{E}\left[Y_1^2\right] + \beta^2 \int_{t_0}^{t} e_{\ominus(-\alpha)}^2(\sigma(\tau),t_0)\Delta\tau\right)\\
&\quad - \left(\mathbb{E}\left[Y_1\right]\right)^2 e_{-\alpha}^2(t,t_0)\\
&= e_{-\alpha}^2(t,t_0)\left(\mathbb{V}\left[Y_1\right] + \beta^2 \int_{t_0}^{t} e_{\ominus(-\alpha)}^2(\sigma(\tau),t_0)\Delta\tau\right).
\end{aligned}
$$

The covariance of X is given by

$$
\begin{aligned}
\mathbb{C}ov\left[X(t),X(s)\right] &= \mathbb{E}\left[X(t)X(s)\right] - \mathbb{E}\left[X(t)\right]\mathbb{E}\left[X(s)\right]\\
&= e_{-\alpha}(t,t_0)e_{-\alpha}(s,t_0)\left(\mathbb{E}\left[Y_1^2\right] + \beta^2 \int_{t_0}^{t\wedge s} e_{\ominus(-\alpha)}^2(\sigma(\tau),t_0)\Delta\tau\right)\\
&\quad - \left(\mathbb{E}\left[Y_1\right]\right)^2 e_{-\alpha}(t,t_0)e_{-\alpha}(s,t_0)\\
&= e_{-\alpha}(t,t_0)e_{-\alpha}(s,t_0)\left(\mathbb{V}\left[Y_1\right] + \beta^2 \int_{t_0}^{t\wedge s} e_{\ominus(-\alpha)}^2(\sigma(\tau),t_0)\Delta\tau\right),
\end{aligned}
$$

where on the second equality we have used (6.58) and (6.59). $\qquad \square$

Example 6.32. For $\mathbb{T} = \mathbb{R}$, $t_0 = 0$ and nonrandom Y_1, (6.54) reduces to

$$
\mathbb{E}\left[X(t)\right] = Y_1 e^{-\alpha t},
$$

while (6.55) reduces to

$$\mathbb{V}\left[X(t)\right] = \beta^2 e^{-2\alpha t} \int_0^t e^{2\alpha\tau} d\tau = \frac{\beta^2}{2\alpha}\left(1 - e^{-2\alpha t}\right)$$

and (6.56) reduces to

$$
\begin{aligned}
\mathbb{C}ov\left[X(t)X(s)\right] &= \beta^2 e^{-\alpha t} e^{-\alpha s} \int_0^{t\wedge s} e^{2\alpha\tau} d\tau \\
&= \frac{\beta^2}{2\alpha} e^{-\alpha(t+s)} \left(e^{2\alpha(t\wedge s)} - 1\right) \\
&= \frac{\beta^2}{2\alpha} e^{-\alpha(t+s)} \left(e^{\alpha(t+s)} e^{-\alpha|t-s|} - 1\right) \\
&= \frac{\beta^2}{2\alpha} \left(e^{-\alpha|t-s|} - e^{-\alpha(t+s)}\right),
\end{aligned}
$$

which matches with known result given in $[61, 94]$.

Example 6.33. If $\mathbb{T} = h\mathbb{Z}$ for $h > 0$ and Y_1 is deterministic, then $\mu(t) \equiv h$ for all $t \in \mathbb{T}$, and (6.54) reduces to

$$\mathbb{E}\left[X(t)\right] = Y_1(1 - h\alpha)^{\frac{t-t_0}{h}}.$$

Likewise (6.55) reduces to

$$\mathbb{V}\left[X(t)\right] = \beta^2 e_p(t, t_0) \int_{t_0}^t e_q(\sigma(\tau), t_0)\Delta\tau,$$

where

$$p = (-\alpha) \oplus (-\alpha) = \alpha(h\alpha - 2)$$

and

$$q = (\ominus(-\alpha)) \oplus (\ominus(-\alpha)) = \frac{\alpha(2 - h\alpha)}{(1 - h\alpha)^2}.$$

Thus,

$$\mathbb{V}\left[X(t)\right] = \beta^2 e_p(t, t_0) \int_{t_0}^t (1 + hq)e_q(\tau, t_0)\Delta\tau$$

$$= \beta^2 e_p(t, t_0) \frac{(1 + hq)}{q} \int_{t_0}^t q e_q(\tau, t_0) \Delta\tau$$

$$= \frac{(1 + hq)\beta^2}{q} e_p(t, t_0) \left(e_q(t, t_0) - 1 \right)$$

$$= \frac{\beta^2}{\alpha(2 - h\alpha)} \left(1 - e_p(t, t_0) \right)$$

$$= \frac{\beta^2}{\alpha(2 - h\alpha)} \left(1 - (1 + h\alpha(h\alpha - 2))^{\frac{t - t_0}{h}} \right)$$

$$= \frac{\beta^2}{\alpha(2 - h\alpha)} \left(1 - (1 - h\alpha)^{\frac{2(t - t_0)}{h}} \right),$$

where on the fourth equality we have used the fact that $1 + hq = 1/(1 - h\alpha)^2$ and $p \oplus q = 0$. Next we observe that $p \in \mathcal{R}^+$ and thus $p = \alpha(h\alpha - 2) < 0$ would imply that

$$\lim_{t \to \infty} \mathbb{V}[X(t)] = \frac{\beta^2}{\alpha(2 - h\alpha)}$$

as in this case $e_p(t, t_0) \to 0$ as $t \to \infty$. Likewise, if $\mathbb{T} = \mathbb{N}_0$, $t_0 = 0$ and Y_1 is nonrandom, we have

$$\mathbb{E}[X(t)] = Y_1(1 - \alpha)^t$$

and

$$\mathbb{V}[X(t)] = \frac{\beta^2}{\alpha(2 - \alpha)} \left(1 - (1 - \alpha)^{2t} \right).$$

Theorem 6.34. *Let $X(t)$ be as in Theorem 6.31, and let*

$$Y(t) = Y(t_0) + \int_{t_0}^t X(\tau) \Delta\tau. \tag{6.61}$$

Then $Y(t)$ has mean

$$\mathbb{E}[Y(t)] = \mathbb{E}[Y_0] + \left(\frac{1 - e_{-\alpha}(t, t_0)}{\alpha} \right) \mathbb{E}[Y_1] \tag{6.62}$$

and variance

$$\mathbb{V}[Y(t)] = \mathbb{V}[Y_0] + \left(\frac{1 - e_{-\alpha}(t, t_0)}{\alpha} \right)^2 \mathbb{V}[Y_1] + \frac{\beta^2}{\alpha^2}(t - t_0)$$

$$+ \frac{2\beta^2}{\alpha^3}\left(e_{-\alpha}(t, t_0) - 1\right) + \frac{\beta^2}{\alpha^2}\int_{t_0}^{t} e_{-\alpha}^2(t, \sigma(\tau))\Delta\tau. \qquad (6.63)$$

Proof. If we take expectation on both sides of (6.61), then

$$
\begin{aligned}
\mathbb{E}[Y(t)] &= \mathbb{E}[Y_0] + \int_{t_0}^{t} \mathbb{E}[X(\tau)]\Delta\tau \\
&= \mathbb{E}[Y_0] + \int_{t_0}^{t} e_{-\alpha}(\tau, t_0)\mathbb{E}[Y_1]\Delta\tau \\
&= \mathbb{E}[Y_0] - \frac{\mathbb{E}[Y_1]}{\alpha}\left(e_{-\alpha}(t, t_0) - 1\right) \\
&= \mathbb{E}[Y_0] + \left(\frac{1 - e_{-\alpha}(t, t_0)}{\alpha}\right)\mathbb{E}[Y_1], \qquad (6.64)
\end{aligned}
$$

where on the second equality we have used (6.58). Thus,

$$\mathbb{E}\left[Y(t) - Y_0\right] = \frac{\mathbb{E}\left[Y_1\right]}{\alpha}\left(1 - e_{-\alpha}(t, t_0)\right). \qquad (6.65)$$

This can be interpreted as the distance traveled by the Brownian particle in the time $t - t_0$ with the mean velocity $\mathbb{E}[Y_1]e_{-\alpha}(t, t_0)$. Likewise,

$$\mathbb{E}\left[(Y(t) - Y(t_0))^2\right] = \mathbb{E}\left[(Y(t) - Y_0)^2\right] = \mathbb{E}\left[\left(\int_{t_0}^{t} X(\tau)\Delta\tau\right)^2\right]. \qquad (6.66)$$

Now using (6.65) and (6.66), we have

$$
\begin{aligned}
\mathbb{V}\left[Y(t) - Y_0\right] &= \mathbb{E}\left[(Y(t) - Y_0)^2\right] - \left(\mathbb{E}\left[Y(t) - Y_0\right]\right)^2 \\
&= \mathbb{E}\left[\left(\int_{t_0}^{t} X(\tau)\Delta\tau\right)^2\right] - \left(\frac{\mathbb{E}\left[Y_1\right]}{\alpha}\right)^2\left(1 - e_{-\alpha}(t, t_0)\right)^2. \qquad (6.67)
\end{aligned}
$$

We can further simplify the expression involving X in (6.67) by observing that

$$
\begin{aligned}
\mathbb{E}\left[\left(\int_{t_0}^{t} X(\tau)\Delta\tau\right)^2\right] &= \int_{t_0}^{t}\int_{t_0}^{t} \mathbb{E}\left[X(\tau_1)X(\tau_2)\right]\Delta\tau_1\Delta\tau_2 \\
&= \int_{t_0}^{t}\left(\int_{t_0}^{\tau_2} \mathbb{E}\left[X(\tau_1)X(\tau_2)\right]\Delta\tau_1 + \int_{\tau_2}^{t} \mathbb{E}\left[X(\tau_1)X(\tau_2)\right]\Delta\tau_1\right)\Delta\tau_2
\end{aligned}
$$

$$
\begin{aligned}
= {} & \int_{t_0}^{t} \left\{ \int_{t_0}^{\tau_2} e_{-\alpha}(\tau_1, t_0) e_{-\alpha}(\tau_2, t_0) \left(\mathbb{E}\left[Y_1^2 \right] + \beta^2 \int_{t_0}^{\tau_1} e_{\ominus(-\alpha)}^2(\sigma(\tau), t_0) \Delta\tau \right) \Delta\tau_1 \right. \\
& \left. + \int_{\tau_2}^{t} e_{-\alpha}(\tau_2, t_0) e_{-\alpha}(\tau_1, t_0) \left(\mathbb{E}\left[Y_1^2 \right] + \beta^2 \int_{t_0}^{\tau_2} e_{\ominus(-\alpha)}^2(\sigma(\tau), t_0) \Delta\tau \right) \Delta\tau_1 \right\} \Delta\tau_2 \\
= {} & \mathbb{E}\left[Y_1^2 \right] \int_{t_0}^{t} \int_{t_0}^{t} e_{-\alpha}(\tau_1, t_0) e_{-\alpha}(\tau_2, t_0) \Delta\tau_1 \Delta\tau_2 \\
& + \beta^2 \int_{t_0}^{t} \left\{ \int_{t_0}^{\tau_2} \int_{t_0}^{\tau_1} e_{-\alpha}(\tau_1, t_0) e_{-\alpha}(\tau_2, t_0) e_{\ominus(-\alpha)}^2(\sigma(\tau), t_0) \Delta\tau \Delta\tau_1 \right. \\
& \left. + \int_{\tau_2}^{t} \int_{t_0}^{\tau_2} e_{-\alpha}(\tau_1, t_0) e_{-\alpha}(\tau_2, t_0) e_{\ominus(-\alpha)}^2(\sigma(\tau), t_0) \Delta\tau \Delta\tau_1 \right\} \Delta\tau_2 \\
= {} & \left(\frac{e_{-\alpha}(t, t_0) - 1}{\alpha} \right)^2 \mathbb{E}\left[Y_1^2 \right] \\
& + \beta^2 \int_{t_0}^{t} \left\{ \int_{t_0}^{\tau_2} \int_{t_0}^{\tau_1} e_{-\alpha}(\tau_2, t_0) e_{-\alpha}(\tau_1, t_0) e_{\ominus(-\alpha)}^2(\sigma(\tau), t_0) \Delta\tau \Delta\tau_1 \right\} \Delta\tau_2 \\
& + \beta^2 \int_{t_0}^{t} e_{-\alpha}(\tau_2, t_0) \left\{ \int_{\tau_2}^{t} \int_{t_0}^{\tau_2} e_{-\alpha}(\tau_1, t_0) e_{\ominus(-\alpha)}^2(\sigma(\tau), t_0) \Delta\tau \Delta\tau_1 \right\} \Delta\tau_2 \\
= {} & \left(\frac{e_{-\alpha}(t, t_0) - 1}{\alpha} \right)^2 \mathbb{E}\left[Y_1^2 \right] \\
& + \beta^2 \int_{t_0}^{t} \left\{ \int_{t_0}^{\tau_2} \int_{t_0}^{\tau_1} e_{-\alpha}(\tau_2, \sigma(\tau)) e_{-\alpha}(\tau_1, \sigma(\tau)) \Delta\tau \Delta\tau_1 \right\} \Delta\tau_2 \\
& + \beta^2 \int_{t_0}^{t} e_{-\alpha}(\tau_2, t_0) \left\{ \int_{\tau_2}^{t} \int_{t_0}^{\tau_2} e_{-\alpha}(\tau_1, t_0) e_{-\alpha}^2(t_0, \sigma(\tau)) \Delta\tau \Delta\tau_1 \right\} \Delta\tau_2 \\
= {} & \left(\frac{e_{-\alpha}(t, t_0) - 1}{\alpha} \right)^2 \mathbb{E}\left[Y_1^2 \right] \\
& + \beta^2 \int_{t_0}^{t} \left\{ \int_{t_0}^{\tau_2} \int_{\sigma(\tau)}^{\tau_2} e_{-\alpha}(\tau_2, \sigma(\tau)) e_{-\alpha}(\tau_1, \sigma(\tau)) \Delta\tau_1 \Delta\tau \right\} \Delta\tau_2 \\
& + \beta^2 \int_{t_0}^{t} e_{-\alpha}(\tau_2, t_0) \left\{ \int_{\tau_2}^{t} e_{-\alpha}(\tau_1, t_0) \Delta\tau_1 \int_{t_0}^{\tau_2} e_{-\alpha}^2(t_0, \sigma(\tau)) \Delta\tau \right\} \Delta\tau_2 \\
= {} & \left(\frac{e_{-\alpha}(t, t_0) - 1}{\alpha} \right)^2 \mathbb{E}\left[Y_1^2 \right] \\
& + \beta^2 \int_{t_0}^{t} \left\{ \int_{t_0}^{t} e_{-\alpha}(\tau_2, \sigma(\tau)) \left(\int_{\sigma(\tau)}^{\tau_2} e_{-\alpha}(\tau_1, \sigma(\tau)) \Delta\tau_1 \right) \Delta\tau \right\} \Delta\tau_2 \\
& + \frac{\beta^2}{\alpha} \int_{t_0}^{t} \int_{t_0}^{\tau_2} e_{-\alpha}^2(t_0, \sigma(\tau)) \left(e_{-\alpha}(\tau_2, t_0) - e_{-\alpha}(t, t_0) \right) e_{-\alpha}(\tau_2, t_0) \Delta\tau \Delta\tau_2 \\
= {} & \left(\frac{e_{-\alpha}(t, t_0) - 1}{\alpha} \right)^2 \mathbb{E}\left[Y_1^2 \right]
\end{aligned}
$$

$$+ \frac{\beta^2}{\alpha} \int_{t_0}^{t} \int_{t_0}^{\tau_2} e_{-\alpha}(\tau_2, \sigma(\tau)) \left(1 - e_{-\alpha}(\tau_2, \sigma(\tau))\right) \Delta\tau \Delta\tau_2$$

$$+ \frac{\beta^2}{\alpha} \int_{t_0}^{t} \int_{\sigma(\tau)}^{t} e_{-\alpha}^2(t_0, \sigma(\tau)) \left(e_{-\alpha}(\tau_2, t_0) - e_{-\alpha}(t, t_0)\right) e_{-\alpha}(\tau_2, t_0) \Delta\tau_2 \Delta\tau$$

$$= \left(\frac{e_{-\alpha}(t, t_0) - 1}{\alpha}\right)^2 \mathbb{E}\left[Y_1^2\right] + \frac{\beta^2}{\alpha} \int_{t_0}^{t} \int_{\sigma(\tau)}^{t} e_{-\alpha}(\tau_2, \sigma(\tau)) \Delta\tau_2 \Delta\tau$$

$$- \frac{\beta^2}{\alpha} \int_{t_0}^{t} \int_{\sigma(\tau)}^{t} e_{-\alpha}^2(\tau_2, \sigma(\tau)) \Delta\tau_2 \Delta\tau + \frac{\beta^2}{\alpha} \int_{t_0}^{t} \int_{\sigma(\tau)}^{t} e_{-\alpha}^2(\tau_2, \sigma(\tau)) \Delta\tau_2 \Delta\tau$$

$$- \frac{\beta^2}{\alpha} \int_{t_0}^{t} \int_{\sigma(\tau)}^{t} e_{-\alpha}(t, \sigma(\tau)) e_{-\alpha}(\tau_2, \sigma(\tau)) \Delta\tau_2 \Delta\tau$$

$$= \left(\frac{e_{-\alpha}(t, t_0) - 1}{\alpha}\right)^2 \mathbb{E}\left[Y_1^2\right] + \frac{\beta^2}{\alpha^2} \int_{t_0}^{t} \left(1 - e_{-\alpha}(t, \sigma(\tau))\right) \Delta\tau$$

$$- \frac{\beta^2}{\alpha} \int_{t_0}^{t} e_{-\alpha}(t, \sigma(\tau)) \left(\int_{\sigma(\tau)}^{t} e_{-\alpha}(\tau_2, \sigma(\tau)) \Delta\tau_2\right) \Delta\tau$$

$$= \left(\frac{e_{-\alpha}(t, t_0) - 1}{\alpha}\right)^2 \mathbb{E}\left[Y_1^2\right] + \frac{\beta^2}{\alpha^2}(t - t_0) - \frac{\beta^2}{\alpha^3}\left(1 - e_{-\alpha}(t, \sigma(\tau))\right)$$

$$+ \frac{\beta^2}{\alpha^2} \int_{t_0}^{t} e_{-\alpha}^2 t, \sigma(\tau)) \Delta\tau - \frac{\beta^2}{\alpha^2} \int_{t_0}^{t} e_{-\alpha}(t, \sigma(\tau)) \Delta\tau$$

$$= \left(\frac{e_{-\alpha}(t, t_0) - 1}{\alpha}\right)^2 \mathbb{E}\left[Y_1^2\right] + \frac{\beta^2}{\alpha^2}(t - t_0) + \frac{2\beta^2}{\alpha^3}\left(e_{-\alpha}(t, t_0) - 1\right)$$

$$+ \frac{\beta^2}{\alpha^2} \int_{t_0}^{t} e_{-\alpha}^2(t, \sigma(\tau)) \Delta\tau. \tag{6.68}$$

Now combining (6.67) and (6.68) we have

$$\mathbb{V}\left[Y(t) - Y_0\right] = \left(\frac{1 - e_{-\alpha}(t, t_0)}{\alpha}\right)^2 \mathbb{V}\left[Y_1\right] + \frac{\beta^2}{\alpha^2}(t - t_0) + \frac{2\beta^2}{\alpha^3}\left(e_{-\alpha}(t, t_0) - 1\right)$$

$$+ \frac{\beta^2}{\alpha^2} \int_{t_0}^{t} e_{-\alpha}^2(t, \sigma(\tau)) \Delta\tau, \tag{6.69}$$

which concludes the proof. $\qquad\square$

Example 6.35. For $\mathbb{T} = \mathbb{R}$, $t_0 = 0$ and nonrandom Y_0 and Y_1, (6.64) reduces to

$$\mathbb{E}\left[Y(t)\right] = Y_0 + \frac{Y_1}{\alpha}\left(1 - e^{-\alpha t}\right)$$

while (6.69) reduces to

$$
\begin{aligned}
\mathbb{V}\left[Y(t)\right] &= \frac{\beta^2}{\alpha^2}t + \frac{2\beta^2}{\alpha^3}\left(e^{-\alpha t} - 1\right) + \frac{\beta^2}{\alpha^2}\int_0^t e^{-2\alpha(t-\tau)}d\tau \\
&= \frac{\beta^2}{\alpha^2}t + \frac{\beta^2}{2\alpha^3}\left(-3 + 4e^{-\alpha t} - e^{-2\alpha t}\right),
\end{aligned}
$$

which matches with known result given in [61, 94].

Example 6.36. If $\mathbb{T} = h\mathbb{Z}$ for $h > 0$, Y_0 and Y_1 is nonrandom then (6.64) reduces to

$$
\mathbb{E}\left[Y(t)\right] = Y_0 + \frac{Y_1}{\alpha}\left(1 - (1 - h\alpha)^{\frac{t-t_0}{h}}\right)
$$

and (6.69) reduces to

$$
\begin{aligned}
\mathbb{V}\left[Y(t)\right] &= \frac{\beta^2}{\alpha^2}(t - t_0) + \frac{2\beta^2}{\alpha^3}\left(e_{-\alpha}(t, t_0) - 1\right) + \frac{\beta^2}{\alpha^2}e_p(t, t_0)\int_{t_0}^t e_q(\sigma(\tau), t_0)\Delta\tau \\
&= \frac{\beta^2}{\alpha^2}(t - t_0) + \frac{2\beta^2}{\alpha^3}\left(e_{-\alpha}(t, t_0) - 1\right) + \frac{\beta^2}{\alpha^2}e_p(t, t_0)\frac{(1 + hq)}{q}\left(e_q(t, t_0) - 1\right) \\
&= \frac{\beta^2}{\alpha^2}(t - t_0) + \frac{2\beta^2}{\alpha^3}\left((1 - h\alpha)^{\frac{t-t_0}{h}} - 1\right) + \frac{\beta^2}{\alpha^3(2 - h\alpha)}\left(e_q(t, t_0) - 1\right) \\
&= \frac{\beta^2}{\alpha^2}(t - t_0) + \frac{2\beta^2}{\alpha^3}\left((1 - h\alpha)^{\frac{t-t_0}{h}} - 1\right) \\
&\quad + \frac{\beta^2}{\alpha^3(2 - h\alpha)}\left(\left(1 + \frac{h\alpha(2 - h\alpha)}{(1 - h\alpha)^2}\right)^{\frac{t-t_0}{h}} - 1\right) \\
&= \frac{\beta^2}{\alpha^2}(t - t_0) + \frac{2\beta^2}{\alpha^3}\left((1 - h\alpha)^{\frac{t-t_0}{h}} - 1\right) \\
&\quad + \frac{\beta^2}{\alpha^3(2 - h\alpha)}\left((1 - h\alpha)^{\frac{-2(t-t_0)}{h}} - 1\right).
\end{aligned}
$$

Likewise, if $\mathbb{T} = \mathbb{N}_0$, $t_0 = 0$ and Y_0 and Y_1 is nonrandom, we have

$$
\mathbb{E}\left[Y(t)\right] = Y_0 + \frac{Y_1}{\alpha}\left(1 - (1 - \alpha)^t\right)
$$

and

$$
\mathbb{V}\left[Y(t)\right] = \frac{\beta^2}{\alpha^2}t + \frac{2\beta^2}{\alpha^3}\left((1 - \alpha)^t - 1\right) + \frac{\beta^2}{\alpha^3(2 - \alpha)}\left((1 - \alpha)^{-2t} - 1\right).
$$

6.4. AN EXISTENCE AND UNIQUENESS THEOREM

We now turn to the existence and uniqueness question. For that we need Gronwall's lemma which we state next.

Lemma 6.37 (Bohner and Peterson [28])**.** *Let $\phi \in C_{rd}$, $f \in \mathcal{R}^+$, $f \geq 0$, and let $C_0 \in \mathbb{R}$. Then*

$$\phi(t) \leq C_0 + \int_{t_0}^t f(s)\phi(s)\Delta s \quad \text{for all} \quad t_0 \leq t \leq T$$

implies

$$\phi(t) \leq C_0 e_f(t, t_0) \quad \text{for all} \quad t_0 \leq t \leq T.$$

Theorem 6.38. *Let us consider the time scale $\mathbb{T} = \{t_0, t_1, \ldots, t_n = T\}$ and suppose $b, B : \mathbb{R} \times \mathbb{T} \to \mathbb{R}$ satisfy the conditions*

$$|b(x_1, t) - b(x_2, t)| \leq L|x_1 - x_2|, \tag{6.70}$$

$$|B(x_1, t) - B(x_2, t)| \leq L|x_1 - x_2|, \tag{6.71}$$

and

$$|b(x, t)| \leq L(1 + |x|), \tag{6.72}$$

$$|B(x, t)| \leq L(1 + |x|) \tag{6.73}$$

for all $t_0 \leq t \leq T$ and $x, x_1, x_2 \in \mathbb{R}$ for some constant L. Let X_0 be any real-valued random variable such that $\mathbb{E}[|X_0|^2] < \infty$ and X_0 is independent of $W(t)$ for $t > t_0$, where W is a given one-dimensional Brownian motion. Then for $t_0 \leq t \leq T$, there exists a unique solution X of the stochastic dynamic equation

$$\Delta X = b(X, t)\Delta t + B(X, t)\Delta W, \quad X(t_0) = X_0 \tag{6.74}$$

such that

$$\mathbb{E}\left[\int_{t_0}^t X^2(\tau)\Delta\tau\right] < \infty. \tag{6.75}$$

Proof. 1. *Uniqueness.* Suppose X and $\hat{X}$ are solutions of (6.74). Then for all $t_0 \leq t \leq T$, $t, t_0, T \in \mathbb{T}$,

$$\begin{aligned}
X(t) - \hat{X}(t) &= \int_{t_0}^t \left(b(X(s), s) - b(\hat{X}(s), s)\right)\Delta s \\
&\quad + \int_{t_0}^t \left(B(X(s), s) - B(\hat{X}(s), s)\right)\Delta W(s). \tag{6.76}
\end{aligned}$$

Since $(a + b)^2 \leq 2a^2 + 2b^2$, we can estimate

$$\begin{aligned}
\mathbb{E}\left[\left|X(t) - \hat{X}(t)\right|^2\right] &= 2\mathbb{E}\left[\left|\int_{t_0}^t (b(X(s), s) - b(\hat{X}(s), s))\Delta s\right|^2\right] \\
&\quad + 2\mathbb{E}\left[\left|\int_{t_0}^t (B(X(s), s) - B(\hat{X}(s), s))\Delta W(s)\right|^2\right].
\end{aligned}$$

The Cauchy–Schwarz inequality [28, Page 260] implies that

$$\left|\int_{t_0}^t f(s)\Delta s\right|^2 \leq (t - t_0)\int_{t_0}^t |f(s)|^2 \Delta s$$

for any $t \geq t_0$ and $f : \mathbb{T} \to \mathbb{R}$. We use this to estimate

$$\begin{aligned}
2\mathbb{E}&\left[\left|\int_{t_0}^t (b(X(s), s) - b(\hat{X}(s), s))\Delta s\right|^2\right] \\
&\leq 2(T - t_0)\mathbb{E}\left[\int_{t_0}^t \left|b(X(s), s) - b(\hat{X}(s), s)\right|^2 \Delta s\right] \\
&\leq 2L^2(T - t_0)\int_{t_0}^t \mathbb{E}\left[\left|X(s) - \hat{X}(s)\right|^2\right]\Delta s.
\end{aligned}$$

Furthermore,

$$2\mathbb{E}\left[\left|\int_{t_0}^t (B(X(s), s) - B(\hat{X}(s), s))\Delta W(s)\right|^2\right]$$

$$= 2\mathbb{E}\left[\int_{t_0}^t \left|B(X(s),s) - B(\hat{X}(s),s)\right|^2 \Delta s\right]$$

$$\leq 2L^2(T - t_0)\int_{t_0}^t \mathbb{E}\left[\left|X(s) - \hat{X}(s)\right|^2\right]\Delta s,$$

where on the first equality we have used (5.22). Therefore, for $C = 4L^2(T - t_0)$, we have

$$\mathbb{E}\left[\left|X(t) - \hat{X}(t)\right|^2\right] \leq C\int_{t_0}^t \mathbb{E}\left[\left|X(s) - \hat{X}(s)\right|^2\right]\Delta s$$

provided $t_0 \leq t \leq T$. If we now set

$$\phi(t) := \mathbb{E}\left[\left|X(t) - \hat{X}(t)\right|^2\right],$$

then the foregoing reads

$$\phi(t) \leq C\int_{t_0}^t \phi(s)\Delta s \quad \text{for all} \quad t_0 \leq t \leq T.$$

Therefore Gronwall's lemma (Lemma 6.37), with $C_0 = 0$, implies $\phi \equiv 0$. Thus,

$$X(t) = \hat{X}(t) \quad \text{a.s. for all} \quad t_0 \leq t \leq T.$$

2. *Existence.* We will utilize the iterative scheme. Let us define

$$\begin{cases} X^0(t) \quad := X_0 \\ \\ X^{n+1}(t) \quad := X_0 + \int_{t_0}^t b(X^n(s),s)\Delta s + \int_{t_0}^t B(X^n(s),s)\Delta W(s) \end{cases}$$

for $n \in \mathbb{N}_0$ and $t_0 \leq t \leq T$. Let us also define

$$\delta^n(t) := \mathbb{E}\left[\left|X^{n+1}(t) - X^n(t)\right|\right].$$

We claim that for some constant M, depending on L, T and X_0,

$$\delta^n(t) \leq M^{n+1} h_{n+1}(t, t_0) \quad \text{for all} \quad n \in \mathbb{N}_0, \ t_0 \leq t \leq T,$$

where h_n are the generalized polynomials defined in Subsection 2.4. Indeed for $n = 0$, we have

$$
\begin{aligned}
\delta^0(t) &= \mathbb{E}\left[\left|X^1(t) - X^0(t)\right|^2\right] \\
&= \mathbb{E}\left[\left|\int_{t_0}^t b(X_0, s)\Delta s + \int_{t_0}^t B(X_0, s)\Delta W(s)\right|^2\right] \\
&\leq 2\mathbb{E}\left[\left|\int_{t_0}^t L(1 + |X_0|)\Delta s\right|^2\right] + 2\mathbb{E}\left[\int_{t_0}^t L^2(1 + |X_0|)^2 \Delta s\right] \\
&\leq (t - t_0)M \\
&= M h_1(t, t_0)
\end{aligned}
$$

for $M = 4L^2(1 + |X_0|)^2$. This confirms the claim for $n = 0$. Next we assume the claim is valid for some $n - 1$. Then

$$
\begin{aligned}
\delta^n(t) &= \mathbb{E}\left[\left|X^{n+1}(t) - X^n(t)\right|^2\right] \\
&= \mathbb{E}\left[\left|\int_{t_0}^t (b(X^n(s), s) - b(X^{n-1}(s), s))\Delta s \right.\right. \\
&\qquad\qquad \left.\left. + \int_{t_0}^t (B(X^n(s), s) - B(X^{n-1}(s), s))\Delta W(s)\right|^2\right] \\
&\leq 2\mathbb{E}\left[\left|\int_{t_0}^t (b(X^n(s), s) - b(X^{n-1}(s), s))\Delta s\right|^2\right] \\
&\qquad + 2\mathbb{E}\left[\left|\int_{t_0}^t (B(X^n(s), s) - B(X^{n-1}(s), s))\Delta W(s)\right|^2\right] \\
&\leq 2\mathbb{E}\left[\int_{t_0}^t \left|(b(X^n(s), s) - b(X^{n-1}(s), s))\right|^2 \Delta s\right] \\
&\qquad + 2\mathbb{E}\left[\int_{t_0}^t \left|(B(X^n(s), s) - B(X^{n-1}(s), s))\right|^2 \Delta s\right] \\
&\leq 2(T - t_0)L^2 \mathbb{E}\left[\int_{t_0}^t \left|X^n(s) - X^{n-1}(s)\right|^2 \Delta s\right]
\end{aligned}
$$

$$+ 2L^2 \mathbb{E} \left[\int_{t_0}^t \left| X^n(s) - X^{n-1}(s) \right|^2 \Delta s \right]$$

$$\leq 2L^2(T - t_0 + 1) \mathbb{E} \left[\int_{t_0}^t \left| X^n(s) - X^{n-1}(s) \right| \Delta s \right]$$

$$= 2L^2(T - t_0 + 1) \int_{t_0}^t \delta^{n-1}(\tau) \Delta \tau$$

$$\leq 2L^2(T - t_0 + 1) \int_{t_0}^t M^n h_n(s, t_0) \Delta s$$

$$\leq M^{n+1} h_{n+1}(t, t_0),$$

provided we choose $M \geq 2L^2(T - t_0 + 1)$. This proves the claim.

Next using (6.76) and (6.70) we have

$$\sup_{t \in [t_0, T]} |X^{n+1}(t) - X^n(t)|^2 \leq 2(T - t_0)L^2 \int_{t_0}^T |X^n(s) - X^{n-1}(s)|^2 \Delta s$$

$$+ 2 \sup_{t \in [t_0, T]} \left| \int_{t_0}^T \left(B(X^n(s), s) - B(X^{n-1}(s), s) \right) \Delta W(s) \right|^2.$$

Consequently the martingale inequality [62] implies

$$\mathbb{E} \left[\sup_{t \in [t_0, T]} |X^{n+1}(t) - X^n(t)|^2 \right] \leq 2(T - t_0)L^2 \int_{t_0}^T \mathbb{E} \left[|X^n(s) - X^{n-1}(s)|^2 \right] \Delta s$$

$$+ 8L^2 \int_{t_0}^T \mathbb{E} \left[|X^n(s) - X^{n-1}(s)|^2 \right] \Delta s$$

$$\leq C M^n h_n(T, t_0),$$

by the claim above, where $C = 2L^2(T - t_0 + 4)$. The Borel–Cantelli lemma [62] thus applies, since

$$\mathbb{P} \left[\sup_{t \in [t_0, T]} |X^{n+1}(t) - X^n(t)| > \frac{1}{2^n} \right] \leq 4^n \mathbb{E} \left[\sup_{t \in [t_0, T]} |X^{n+1}(t) - X^n(t)|^2 \right]$$

$$\leq 4^n C M^n h_n(T, t_0)$$

and

$$\sum_{n=1}^{\infty} 4^n C M^n h_n(T, t_0) < \infty.$$

Thus,

$$\mathbb{P}\left[\sup_{t \in [t_0, T]} |X^{n+1}(t) - X^n(t)| > \frac{1}{2^n} \text{ i.o.}\right] = 0.$$

In light of this, for almost every ω

$$X^n = X^0 + \sum_{j=0}^{n-1} (X^{j+1} - X^j)$$

converges on $[t_0, T]$ to the process $X(\cdot)$. Thus, if we let $n \to \infty$ in the definition of $X^{n+1}(\cdot)$, then we have

$$X(t) = X_0 + \int_{t_0}^t b(X, s)\Delta s + \int_{t_0}^t B(X, s)\Delta W(s).$$

That is, (6.74) holds for all times $t_0 \leq t \leq T$. Next we show that (6.75) holds. We have

$$\begin{aligned}
\mathbb{E}\left[|X^{n+1}(t)|^2\right] &\leq C\mathbb{E}\left[|X_0|^2\right] + C\mathbb{E}\left[\left|\int_{t_0}^t b(X^n(s), s)\Delta s\right|^2\right] \\
&\quad + C\mathbb{E}\left[\left|\int_{t_0}^t B(X^n(s), s)\Delta W(s)\right|^2\right] \\
&\leq C(1 + \mathbb{E}[|X_0|^2]) + C\int_{t_0}^t \mathbb{E}\left[|X^n|^2\right]\Delta s,
\end{aligned}$$

where, as usual, C will denote various constants. By induction, therefore,

$$\mathbb{E}\left[|X^{n+1}(t)|^2\right] \leq \left[C + C^2 h_1(t, t_0) + \ldots + C^{n+2} h_{n+1}(t, t_0)\right]\left(1 + \mathbb{E}\left[|X_0|^2\right]\right)$$

Consequently,

$$\mathbb{E}\left[|X^{n+1}(t)|^2\right] \leq C(1 + \mathbb{E}\left[|X_0|^2\right]) e_C(t, t_0).$$

Let $n \to \infty$. Then

$$\mathbb{E}\left[|X(t)|^2\right] \leq C\left(1 + \mathbb{E}\left[|X_0|^2\right]\right) e_C(t, t_0) \quad \text{for all} \quad t_0 \leq t \leq T$$

and hence

$$\begin{aligned}
\mathbb{E}\left[\int_{t_0}^t |X(\tau)|^2 \Delta\tau\right] &= \int_{t_0}^t \mathbb{E}\left[|X(\tau)|^2\right]\Delta\tau \\
&\leq \left(1 + \mathbb{E}\left[|X_0|^2\right]\right)\int_{t_0}^t Ce_C(\tau, t_0)\Delta\tau \\
&= \left(1 + \mathbb{E}\left[|X_0|^2\right]\right)\left(e_C(t, t_0) - 1\right) \\
&< \infty,
\end{aligned}$$

which proves (6.75). $\qquad\qquad\square$

7. STABILITY

Conditions which guarantee almost sure asymptotic stability of solutions of stochastic equations are crucial in diverse applications. Among such applications we can mention asset price evolution in discrete markets and population dynamics in mathematical biology.

Solutions of stochastic equations have been subjected to detailed study. Stochastic functional-integral equations have been discussed in [12, 65–67, 69]. Boundedness and stability of stochastic equations have been discussed in [7, 47, 57, 58, 70, 74–76, 78, 79, 82, 83, 86]. Convergence and asymptotic properties have been studied in [6, 9–11, 15, 42, 71–73, 81, 85]. For dynamic equations, stability and asymptotic properties have been studied in [19, 27, 30, 46, 55, 63, 64].

7.1. ASYMPTOTIC BEHAVIOUR

In this subsection we consider a linear stochastic dynamic equation without drift

$$\Delta X = \alpha X \Delta \xi, \quad X(t_0) = X_0 \neq 0, \tag{7.1}$$

where $\Delta \xi(t)$ are random variables such that $\mathbb{E}\left[\Delta \xi(t)\right] = 0$,

$$\lim_{t \to \infty} \ln |1 + \alpha \Delta \xi(t)| \neq 0, \tag{7.2}$$

$$\mathbb{V}\left[\ln |1 + \alpha \Delta \xi(t)|\right] < K < \infty \quad \text{for all} \quad t \in \mathbb{T}, \tag{7.3}$$

$\alpha \in \mathcal{R}_\xi$, $t \in \mathbb{T}$ with $\sup \mathbb{T} = \infty$ and obtain necessary and sufficient conditions for the fulfillment of the following:

(i) $\lim_{t \to \infty} X(t) = 0$ holds a.s.

(ii) $\lim_{t \to \infty} X(t) = \infty$ holds a.s.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a filtered probability space and $\{\Delta\xi(t)\}_{t\in\mathbb{T}}$ be independent and identically distributed (i.i.d.) random variables. We also suppose that the filtration $\{\mathcal{F}(t)\}_{t\in\mathbb{T}}$ is naturally generated, i.e., $\mathcal{F}(t)$ is the σ-algebra generated by $\{\xi(t)\}_{t\in\mathbb{T}}$. We use the standard abbreviation a.s. for the wordings *almost surely* with respect to the fixed probability measure $\mathbb{P}$.

We start by observing that the modulus of the solution X of (7.1) is given by

$$
\begin{aligned}
|X(t)| &= |X_0| \prod_{\tau\in[t_0,t)} |1 + \alpha\Delta\xi(\tau)| \\
&= |X_0| \exp\left(\sum_{\tau\in[t_0,t)} \ln|1 + \alpha\Delta\xi(\tau)|\right).
\end{aligned}
\tag{7.4}
$$

From the above representation we obtain that

$$
\lim_{t\to\infty} X(t) = 0 \quad \text{if and only if} \quad \sum_{\tau\in[t_0,\infty)} \ln|1 + \alpha\Delta\xi(\tau)| = -\infty
\tag{7.5}
$$

and

$$
\lim_{t\to\infty} X(t) = \infty \quad \text{if and only if} \quad \sum_{\tau\in[t_0,\infty)} \ln|1 + \alpha\Delta\xi(\tau)| = \infty.
\tag{7.6}
$$

Since $\lim_{t\to\infty} \ln|1 + \alpha\Delta\xi(t)| \neq 0$ we observe from (7.4) that $X(t)$ can be either 0 or ∞ as $t \to \infty$. Now we derive the conditions which insure fulfillment of one of the following:

$$
\sum_{\tau\in[t_0,\infty)} \ln|1 + \alpha\Delta\xi(\tau)| = -\infty
$$

or

$$
\sum_{\tau\in[t_0,\infty)} \ln|1 + \alpha\Delta\xi(\tau)| = \infty.
$$

Let us define

$$
\begin{aligned}
\kappa(\tau) &:= \ln|1 + \alpha\Delta\xi(\tau)|, \\
S(t) &:= \sum_{\tau\in[t_0,t)} \kappa(\tau),
\end{aligned}
$$

$$a \ := \ \mathbb{E}\left[\kappa(\tau)\right],$$

$$\theta \ := \ \mathbb{V}\left[\kappa(\tau)\right].$$

Let $n_t = |[t_0, t)|$ be the number of points in the interval $[t_0, t)$ and $\mathbb{T}$ be such that

$$\sum_{\tau \in \mathbb{T}} \frac{1}{n_\tau^2} < \infty. \tag{7.7}$$

Then the random variables $\{\kappa(\tau)\}_{\tau \in \mathbb{T}}$ are identically distributed and

$$\sum_{\tau \in \mathbb{T}} \frac{\mathbb{V}\left[\kappa(\tau)\right]}{n_\tau^2} \leq K \sum_{\tau \in \mathbb{T}} \frac{1}{n_\tau^2} < \infty,$$

where on the first inequality we have used (7.3). So from Kolmogorov's strong law of large numbers [87, Page 389], we have

$$\frac{S(t) - \mathbb{E}[S(t)]}{n_t} = \frac{\sum_{\tau \in [t_0, t)} \kappa(\tau) - n_t a}{n_t} = \frac{\sum_{t \in [t_0, t)} \kappa(\tau)}{n_t} - a \to 0. \tag{7.8}$$

Theorem 7.1. *Assume that $a \neq 0$ and $\mathbb{T}$ is such that (7.7) is saisfied. Then*

(i) $\lim_{t \to \infty} X(t) = 0$ *holds a.s. for the solution $\{X(t)\}_{t \in \mathbb{T}}$ to equation (7.1) if and only if*

$$a = \mathbb{E}\left[\ln|1 + \alpha \Delta \xi(\tau)|\right] < 0 \quad \text{for all} \quad \tau \in \mathbb{T}. \tag{7.9}$$

(ii) $\lim_{t \to \infty} X(t) = \infty$ *holds a.s. for the solution $\{X(t)\}_{t \in \mathbb{T}}$ to equation (7.1) if and only if*

$$a = \mathbb{E}\left[\ln|1 + \alpha \Delta \xi(\tau)|\right] > 0 \quad \text{for all} \quad \tau \in \mathbb{T}. \tag{7.10}$$

Proof. Case (i), sufficiency. If $a < 0$, then from (7.8) we can find $N_1 = N(\omega, a)$ such that for $t > N_1$ we have

$$\frac{\sum_{\tau \in [t_0, t)} \kappa(\tau) - n_t a}{n_t} \leq -\frac{a}{2}$$

and therefore,

$$\sum_{\tau \in [t_0, t)} \kappa(\tau) \leq \frac{a}{2} n_t \to -\infty$$

when $t \to \infty$. Now the result is immediately obtained from (7.5).

Necessity. Suppose that $\lim_{t \to \infty} X(t) = 0$ which, according to (7.5), is equivalent to $\sum_{\tau \in [t_0, t)} \kappa(\tau) \to -\infty$. Let us assume the contrary, i.e., that $a > 0$. Then there is $N_2 = N_2(\omega, a)$ such that for $t > N_2$

$$\frac{\sum_{\tau \in [t_0, t)} \kappa(\tau) - n_t a}{n_t} \geq -\frac{a}{2}.$$

Then

$$\infty \leftarrow n_t \frac{a}{2} \leq \sum_{\tau \in [t_0, t)} \kappa(\tau) \to -\infty \quad \text{as} \quad t \to \infty,$$

which is a contradiction to our assumption.

Case (ii), sufficiency. Let us suppose that $a \not> 0$. Then from (i) of this theorem we have $\lim_{t \to \infty} X(t) = 0$ implying that $\lim_{t \to \infty} X(t) \neq \infty$. Hence, we conclude that $a > 0$ implies $\lim_{t \to \infty} X(t) = \infty$.

Necessity. Let us suppose that $\lim_{t \to \infty} X(t) \neq \infty$. Then from the fact that $\lim_{t \to \infty} X(t)$ can either be 0 or ∞, we have $\lim_{t \to \infty} X(t) = 0$ and hence from (i) of this theorem we have $a < 0$ or $a \not> 0$. But this means that $\lim_{t \to \infty} X(t) = \infty$ implies $a > 0$. $\qquad \square$

Remark 7.2. Suppose there exists some $k \in (0, 1)$ such that for any t

$$|\alpha \Delta \xi(t)| < k. \tag{7.11}$$

Then $\mathbb{E}\left[\ln |1 + \alpha \Delta \xi(t)|\right] < 0$. Indeed, from (7.11) we have

$$0 < 1 - k < 1 + \alpha \Delta \xi(t) < 1 + k,$$

so that

$$\ln|1 + \alpha\Delta\xi(t)| = \ln(1 + \alpha\Delta\xi(t)).$$

Expanding $\ln(1 + u)$ in a Taylor series, we get

$$\ln(1 + \alpha\Delta\xi(t)) = \alpha\Delta\xi(t) - \frac{\alpha^2(\Delta\xi(t))^2}{2(1 + \gamma)^2},$$

where $|\gamma| = |\gamma(t)| \in (0, |\alpha\Delta\xi(t)|)$. Using the estimates

$$1 + \gamma < 1 + k \quad\text{and}\quad -\frac{1}{(1 + \gamma)^2} < -\frac{1}{(1 + k)^2},$$

we arrive at

$$
\begin{aligned}
\mathbb{E}\left[\ln|1 + \alpha\Delta\xi(t)|\right] &= \mathbb{E}\left[\alpha\Delta\xi(t)\right] - \mathbb{E}\left[\frac{\alpha^2(\Delta\xi(t))^2}{2(1 + \gamma)^2}\right] \\
&\leq \mathbb{E}\left[\alpha\Delta\xi(t)\right] - \mathbb{E}\left[\frac{\alpha^2(\Delta\xi(t))^2}{2(1 + k)^2}\right] \\
&= 0 - \frac{\alpha^2}{2(1 + k)^2}\mathbb{E}\left[(\Delta\xi(t))^2\right] \\
&< 0
\end{aligned}
$$

(note that if $\mathbb{E}\left[(\Delta\xi(t))^2\right] = 0$, then together with $\mathbb{E}\left[\Delta\xi(t)\right] = 0$, makes $\Delta\xi$ deterministic with zero mean and variance). This shows that when $|\alpha(\Delta\xi(t))| < 1$ for all $t \in \mathbb{T}$, condition (7.9) is automatically fulfilled.

7.2. ALMOST SURE ASYMPTOTIC STABILITY

In this subsection we prove a theorem on the almost sure asymptotic stability of the solutions of the stochastic equation

$$X(\sigma(t)) = X(t)\left[1 + a(t)f(X(t)) + b(t)g(X(t))\xi(\sigma(t))\right], \quad t \in \mathbb{T}, \tag{7.12}$$

where $\xi(\sigma(t))$ are independent random variables, $\mathbb{E}[\xi(t)] = 0$, $\mathbb{E}[\xi^2(t)] = 1$, $a, b :$ $\mathbb{T} \to \mathbb{R}$, and $f, g : \mathbb{R} \to \mathbb{R}$ are continuous. We also assume that $X(t_0) = X_0 > 0$. Throughout this subsection we assume that $\sup \mathbb{T} = \infty$.

Definition 7.3. A stochastic process $\{X(t)\}_{t\in\mathbb{T}}$ is said to be an $\mathcal{F}(t)$-martingale-difference, if $\mathbb{E}[X(t)] < \infty$ and $\mathbb{E}[X(\sigma(t))|\mathcal{F}(t)] = 0$ a.s. for all $t \in \mathbb{T}$.

Definition 7.4. A stochastic process $\{X(t)\}_{t\in\mathbb{T}}$ is said to be increasing if

$$\Delta X(t) = X(\sigma(t)) - X(t) > 0 \quad \text{a.s.}$$

for all $t \in \mathbb{T}$.

Lemma 7.5. *If $\{X(t)\}_{t\in\mathbb{T}}$ is increasing with $\mathbb{E}[X(t)] < \infty$ for all $t \in \mathbb{T}$, then $\{X(t)\}_{t\in\mathbb{T}}$ is a submartingale as defined in Definition 3.11.*

Proof. If $\{X(t)\}_{t\in\mathbb{T}}$ is increasing, then from Definition 7.4 we have

$$\mathbb{E}[X(\sigma(t)) - X(t)|\mathcal{F}(t)] \geq 0.$$

Then, $\{X(t)\}_{t\in\mathbb{T}}$ is a submartingale by the fact that

$$\mathbb{E}[X(\sigma(t))|\mathcal{F}(t)] \geq X(t)$$

for all $t \in \mathbb{T}$. $\qquad\square$

The following is a variant of the Doob decomposition theorem (cf., e.g., [87]).

Theorem 7.6. *Suppose that $\{X(t)\}_{t\in\mathbb{T}}$ is an $\mathcal{F}(t)$-submartingale. Then there exists an $\mathcal{F}(t)$-martingale $\{M(t)\}_{t\in\mathbb{T}}$ and an increasing $\mathcal{F}(\rho(t))$-measurable stochastic process $\{A(t)\}_{t\in\mathbb{T}}$ such that for all $t \in \mathbb{T}$*

$$X(t) = X(t_0) + M(t) + A(t), \quad a.s. \tag{7.13}$$

Proof. If $X(t)$ is a submartingale, then

$$X(\sigma(t)) = X(t_0) + \sum_{\tau \in [t_0, t]} (X(\sigma(\tau)) - X(\tau)).$$

By adding and subtracting $\mathbb{E}\left[X(\sigma(\tau))|\mathcal{F}(\tau)\right]$, we obtain the Doob decomposition

$$\begin{aligned}
X(\sigma(t)) &= X(t_0) + \sum_{\tau \in [t_0, t]} (X(\sigma(\tau)) - \mathbb{E}\left[X(\sigma(\tau))|\mathcal{F}(\tau)\right]) \\
&+ \sum_{\tau \in [t_0, t]} (\mathbb{E}\left[X(\sigma(\tau))|\mathcal{F}(\tau)\right] - X(\tau)),
\end{aligned}$$

where the martingale and the increasing process are given by

$$M(\sigma(t)) = \sum_{\tau \in [t_0, t]} (X(\sigma(\tau)) - \mathbb{E}\left[X(\sigma(\tau))|\mathcal{F}(\tau)\right])$$

and

$$A(\sigma(t)) = \sum_{\tau \in [t_0, t]} (\mathbb{E}\left[X(\sigma(\tau))|\mathcal{F}(\tau)\right] - X(\tau))$$

respectively. Here $A(t)$ is an increasing process due to the submartingale property, $\mathbb{E}\left[X(\sigma(\tau))|\mathcal{F}(\tau)\right] - X(\tau) \geq 0$ for all $\tau \in \mathbb{T}$ and Definition 7.4. $\qquad \square$

Lemma 7.7. *Let* $\{X(t)\}_{t \in \mathbb{T}}$ *be a nonnegative* $\mathcal{F}(t)$*-measurable process,* $\mathbb{E}[X(t)] < \infty$ *for all* $t \in \mathbb{T}$ *and*

$$X(\sigma(t)) \leq X(t) + u(t) - v(t) + p(\sigma(t)), \qquad (7.14)$$

where $\{p(t)\}_{t \in \mathbb{T}}$ *is an* $\mathcal{F}(t)$*-martingale-difference,* $\{u(t)\}_{t \in \mathbb{T}}$, $\{p(t)\}_{t \in \mathbb{T}}$ *are nonnegative* $\mathcal{F}(t)$*-measurable processes,* $\mathbb{E}[u(t)]$, $\mathbb{E}[v(t)] < \infty$ *for all* $t \in \mathbb{T}$. *Then*

$$\left\{\omega : \sum_{t \in \mathbb{T}} u(t) < \infty\right\} \subseteq \left\{\omega : \sum_{t \in \mathbb{T}} v(t) < \infty\right\} \cap \{X(t) \to\}. \qquad (7.15)$$

Here by $\{X(t) \to\}$ *we denote the set of all* $\omega \in \Omega$ *for which* $\lim_{t \to \infty} X(t)$ *exists and is finite.*

Proof. We have

$$\begin{aligned}
X(\sigma(t)) &= X(t) + u(t) - v(t) + p(\sigma(t)) \\
&\quad - (X(t) - X(\sigma(t)) + u(t) - v(t) + p(\sigma(t))) \\
&= X(t) + u(t) - v(t) + p(\sigma(t)) - w(\sigma(t)), \tag{7.16}
\end{aligned}$$

where by (7.14)

$$w(\sigma(t)) = X(t) - X(\sigma(t)) + u(t) - v(t) + p(\sigma(t)) \geq 0$$

and $w(t)$ is an $\mathcal{F}(t)$-measurable process. Since $\overline{w}(t) := \sum_{\tau \in [\sigma(t_0), t]} w(\tau)$ is increasing and is $\mathcal{F}(t)$-measurable with

$$\mathbb{E}[\overline{w}(t)] = \mathbb{E}\left[\sum_{\tau \in [\sigma(t_0), t]} w(\tau) \right] = \sum_{\tau \in [\sigma(t_0), t]} \mathbb{E}[w(\tau)] < \infty,$$

we conclude from Lemma 7.5 that $\overline{w}(t)$ is an $\mathcal{F}(t)$-submartingale. Therefore, from Theorem 7.6, we have the representation

$$\overline{w}(\sigma(t)) = \sum_{\tau \in [\sigma(t_0), \sigma(t)]} w(\tau) = \overline{w}(\sigma(t_0)) + M^{\dagger}(\sigma(t)) + C(t), \tag{7.17}$$

where $\{M^{\dagger}(t)\}_{t \in \mathbb{T}}$ is an $\mathcal{F}(t)$-martingale and $\{C(t)\}_{t \in \mathbb{T}}$ is an $\mathcal{F}(t)$-measurable and increasing process. From these observations and summing (7.16), we obtain

$$\begin{aligned}
\sum_{\tau \in [t_0, t]} X(\sigma(\tau)) &= \sum_{\tau \in [t_0, t]} X(\tau) + \sum_{\tau \in [t_0, t]} u(\tau) - \sum_{\tau \in [t_0, t]} v(\tau) \\
&\quad + \sum_{\tau \in [t_0, t]} p(\sigma(\tau)) - \sum_{\tau \in [t_0, t]} w(\sigma(\tau))
\end{aligned}$$

which reduces to

$$X(\sigma(t)) = X(t_0) + U(t) - V(t) + M(\sigma(t)) - \left(\overline{w}(\sigma(t_0)) + M^{\dagger}(\sigma(t)) + C(t) \right)$$

$$\begin{aligned} = \quad & X(t_0) - \overline{w}(t_0) + U(t) - (V(t) + C(t)) \\ & + \left(M(\sigma(t)) - M^\dagger(\sigma(t)) \right), \end{aligned} \qquad (7.18)$$

where on the first equality we have used (7.17) and

$$U(t) = \sum_{\tau \in [t_0, t]} u(\tau),$$

$$V(t) = \sum_{\tau \in [t_0, t]} v(\tau),$$

$$M(t) = \sum_{\tau \in [\sigma(t_0), t]} p(\tau).$$

We define

$$\overline{M}(t) = M(t) - M^\dagger(t)$$

and

$$\overline{U}(t) = X(t_0) - \overline{w}(\sigma(t_0)) + U(t).$$

Then from (7.18) we see that for all $t \in \mathbb{T}$

$$X(\sigma(t)) + (V(t) + C(t)) = \overline{U}(t) + \overline{M}(\sigma(t)) =: Y(\sigma(t)). \qquad (7.19)$$

The process $\{Y(\sigma(t))\}_{t \in \mathbb{T}}$ is a nonnegative $\mathcal{F}(\sigma(t))$-submartingale, and it can be decomposed uniquely into the sum of the $\mathcal{F}(\sigma(t))$-martingale $\{\overline{M}(\sigma(t))\}_{t \in \mathbb{T}}$ and $\mathcal{F}(t)$-measurable and increasing sequence $\{\overline{U}(t)\}_{t \in \mathbb{T}}$, namely,

$$Y(\sigma(t)) = \overline{U}(t) + \overline{M}(\sigma(t)).$$

Now we let $\lim_{t \to \infty} \overline{U}(t) = \overline{U}_\infty$. Then, from martingale convergence theorem [87, Page 551], we conclude that

$$\Omega_1 = \{\overline{U}_\infty < \infty\} \subseteq \{Y(t) \to\} \quad \text{a.s.} \qquad (7.20)$$

This means that $\lim_{t\to\infty} Y(t)$ exists a.s. on Ω_1 and therefore $Y(\sigma(t))$ is a.s. bounded from above on Ω_1. From the left-hand side of (7.19) we have another representation for $Y(\sigma(t))$, namely

$$Y(\sigma(t)) = X(\sigma(t)) + (V(t) + C(t)). \tag{7.21}$$

Since $Y(\sigma(t))$ is a.s. bounded from above on Ω_1 and the process $X(\sigma(t))$ is nonnegative, the process $V(t) + C(t)$ is also a.s. bounded from above on Ω_1. Since $V(t)$ and $C(t)$ are increasing, both have a.s. finite limits $\lim_{t\to\infty} V(t)$ and $\lim_{t\to\infty} C(t)$ on Ω_1. Therefore the $\lim_{t\to\infty} X(t)$ also exists on Ω_1. $\qquad\square$

Theorem 7.8. *Suppose that there exist some $L, L_0 \in (0,\infty)$ such that for all $t \in \mathbb{T}$, $u \in \mathbb{R}$*

$$-1 < a(t)f(u) + b(t)g(u)\xi(\sigma(t)) \le L \quad a.s., \tag{7.22}$$

$$g(u) \ne 0 \quad when \quad u \ne 0, \tag{7.23}$$

$$a(t) \le L_0 b^2(t)g^2(u), \tag{7.24}$$

$$2L_0(1 + L)^2 < 1, \tag{7.25}$$

$$\sum_{\tau \in \mathbb{T}} b^2(\tau) = \infty \tag{7.26}$$

are fulfilled. Let X be a solution of (7.12). Then

$$\lim_{t\to\infty} X(t) = 0 \quad a.s. \tag{7.27}$$

Proof. We observe that the solution X of (7.12) can be represented in the form

$$X(\sigma(t)) = X(t_0) \prod_{\tau \in [t_0, t]} \left[1 + a(\tau)f(X(\tau)) + b(\tau)g(X(\tau))\xi(\sigma(\tau))\right]. \tag{7.28}$$

By the assumption that $X(t_0) = X_0 > 0$ and (7.22), we see from (7.28) that $X(t) > 0$

for all $t \in \mathbb{T}$. Also from (7.22) and (7.28), we have

$$
\begin{aligned}
\mathbb{E}[|X(\sigma(t))|^p] &= \mathbb{E}\left[X_0^p \prod_{\tau \in [t_0,t]} (1 + a(\tau)f(X(\tau)) + b(\tau)g(X(\tau))\xi(\sigma(\tau)))^p\right] \\
&\leq \mathbb{E}\left[X_0^p \prod_{\tau \in [t_0,t]} (1+L)^p\right] < \infty
\end{aligned}
$$

for all $t \in \mathbb{T}$ and all $p > 0$. Let $\alpha \in (0,1)$. Applying the Taylor expansion of the function $y = (1+u)^\alpha$ up to the third term gives

$$
(1+u)^\alpha = 1 + \alpha u + \frac{\alpha(\alpha-1)}{2}(1+\theta)^{\alpha-2}u^2, \tag{7.29}
$$

where θ lies between 0 and u. Taking into account (7.22), we can estimate the expression $\frac{\alpha(\alpha-1)}{2}(1+\theta)^{\alpha-2}$ when $u = a(t)f(X(t)) + b(t)g(X(t))\xi(\sigma(t))$, according to

$$
1 + \theta \leq 1 + |u| \leq 1 + L, \quad \frac{\alpha(\alpha-1)}{2(1+\theta)^{2-\alpha}} \leq \frac{\alpha(\alpha-1)}{2(1+L)^{2-\alpha}}. \tag{7.30}
$$

Applying (7.22), (7.29) and (7.30) we get

$$
\begin{aligned}
X^\alpha(\sigma(t)) &= X^\alpha(t)\left[1 + a(t)f(X(t)) + b(t)g(X(t))\xi(\sigma(t))\right]^\alpha \\
&= X^\alpha(t)\left[1 + \alpha\left(a(t)f(X(t)) + b(t)g(X(t))\xi(\sigma(t))\right)\right] \\
&\quad + X^\alpha(t)\left[\frac{\alpha(\alpha-1)}{2(1+\theta)^{2-\alpha}}\left(a(t)f(X(t)) + b(t)g(X(t))\xi(\sigma(t))\right)^2\right] \\
&\leq X^\alpha(t)\left[1 + \alpha\left(a(t)f(X(t)) + b(t)g(X(t))\xi(\sigma(t))\right)\right] \\
&\quad + X^\alpha(t)\left[\frac{\alpha(\alpha-1)}{2(1+L)^{2-\alpha}}\left(a(t)f(X(t)) + b(t)g(X(t))\xi(\sigma(t))\right)^2\right] \\
&= X^\alpha(t)\left[1 + \alpha a(t)f(X(t))\right] + P(\sigma(t)) \\
&\quad + X^\alpha(t)\left[\frac{\alpha(\alpha-1)}{2(1+L)^{2-\alpha}}\left(a^2(t)f^2(X(t)) + b^2(t)g^2(X(t))\right)\right], \tag{7.31}
\end{aligned}
$$

where

$$
P(\sigma(t)) = \alpha b(t)X^\alpha(t)g(X(t))\xi(\sigma(t)) + \frac{\alpha(\alpha-1)}{2(1+L)^{2-\alpha}}a^2(t)X^\alpha(t)g^2(X(t))Q(\sigma(t))
$$

$$+ \frac{\alpha(\alpha - 1)}{(1 + L)^{2-\alpha}} a(t)b(t)X^{\alpha}(t)f(X(t))g(X(t))\xi(\sigma(t)) \tag{7.32}$$

and $Q(t) = \xi^2(t) - 1$. From (7.31), we get the estimate

$$X^{\alpha}(\sigma(t)) - X^{\alpha}(t) \le \alpha X^{\alpha}(t) \left[a(t)f(X(t)) - \frac{1-\alpha}{2(1+L)^{2-\alpha}} b^2(t)g^2(X(t)) \right] + P(\sigma(t)). \tag{7.33}$$

We substitute condition (7.24) into (7.33) and get

$$\begin{aligned}
X^{\alpha}(\sigma(t)) &\le X^{\alpha}(t) \left[1 + \alpha L_0 b^2(t)g^2(X(t)) - \frac{\alpha(1-\alpha)}{2(1+L)^{2-\alpha}} b^2(t)g^2(X(t)) \right] \\
&\quad + P(\sigma(t)) \\
&\le X^{\alpha}(t) - \alpha X^{\alpha}(t)b^2(t)g^2(X(t)) \left[\frac{1-\alpha}{2(1+L)^{2-\alpha}} - L_0 \right] \\
&\quad + P(\sigma(t)). \tag{7.34}
\end{aligned}$$

By (7.25), we have

$$0 < \frac{1}{2} - L_0(1 + L)^2 < 1.$$

Let us define

$$\alpha = \frac{1}{2} - L_0(1 + L)^2.$$

Then we have

$$\frac{1-\alpha}{2(1+L)^{2-\alpha}} - L_0 \ge \frac{1-\alpha}{2(1+L)^2} - L_0 = \frac{\alpha}{2(1+L)^2}. \tag{7.35}$$

Substituting (7.35) in (7.34) we arrive at

$$X^{\alpha}(\sigma(t)) \le X^{\alpha}(t) - \frac{\alpha^2}{2(1+L)^2}X^{\alpha}(t)b^2(t)g^2(X(t)) + P(\sigma(t)). \tag{7.36}$$

We can now apply Lemma 7.7 by making the identification

$$X(t) = X^{\alpha}(t),$$
$$u(t) \equiv 0,$$

$$v(t) = \frac{\alpha^2}{2(1+L)^2} \sum_{\tau \in [t_0, t]} b^2(\tau) X^\alpha(\tau) g^2(X(\tau)),$$

$$p(t) = P(t).$$

to conclude that

$$\lim_{t \to \infty} X^\alpha(t) \in [0, \infty) \quad \text{exists a.s.}$$

and also that

$$\sum_{\tau \in \mathbb{T}} b^2(\tau) X^\alpha(\tau) g^2(X(\tau)) < \infty \quad \text{a.s.} \tag{7.37}$$

We put

$$\Omega_1 = \{\omega : \lim_{t \to \infty} X(t, \omega) = 0\} \quad \text{and} \quad \Omega_2 = \{\omega : \lim_{t \to \infty} X(t, \omega) > 0\}.$$

We note that $\mathbb{P}(\Omega_1 \cup \Omega_2) = 1$ since $X(t) > 0$ for all $t \in \mathbb{T}$. Using (7.37), we get for almost every $\omega \in \Omega_2$

$$\sum_{\tau \in \mathbb{T}} b^2(\tau) \le c \sum_{\tau \in \mathbb{T}} b^2(\tau) X^\alpha(\tau) g^2(X(\tau)) < \infty,$$

where $c = c(\omega) > 0$ is some a.s. finite random variable. This contradicts the assumption (7.26) if $\mathbb{P}(\Omega_2) > 0$. In other words, we must have $\mathbb{P}(\Omega_2) = 0$ whence $\mathbb{P}(\Omega_1) = 1$ as desired. $\qquad\square$

8. STOCHASTIC EQUATION OF VOLTERRA TYPE

In this section we consider the mean square stability of linear stochastic dynamic equations of the form

$$\Delta X = (a * X)(t)\Delta t + (b * X)(t)\Delta V, \quad X(t_0) = X_0, \tag{8.1}$$

where $a, b : \mathbb{T} \to \mathbb{R}$, $a * X$ is the convolution of a and X defined in Definition 8.3 and V is the solution of

$$\Delta V = \sqrt{\mu(t)}\Delta W. \tag{8.2}$$

In (8.2), W is one-dimensional Brownian motion. Since $V^\Delta(t) = \Delta V(t)/\Delta t = \Delta W(t)/\sqrt{\mu(t)}$, we observe that $\{V^\Delta(t)\}_{t\in\mathbb{T}}$ are i.i.d. random variables which generate the natural filtration $\{\mathcal{F}(t)\}_{t\in\mathbb{T}}$ on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{E}[V^\Delta(t)] = 0$ and $\mathbb{E}\left[\left(V^\Delta(t)\right)^2\right] = 1$. We also assume that $X(\tau)$ is independent of $V^\Delta(t)$ for $\tau \in [t_0, t)$.

For basic concepts of integral equations of Volterra type we refer to [32]. Stability and convergence of solutions of Volterra equations, likewise, has been discussed in [3–5, 8, 33, 34, 37, 48, 54, 68, 77, 80, 84]. For improper integrals and multiple integration on time scales we refer to Bohner and Guseinov [20, 21, 23, 24, 26], and for partial differentiation on time scales we refer to [22].

8.1. CONVOLUTION

Convolution on time scales was introduced by Bohner and Guseinov in [25]. Let $\sup \mathbb{T} = \infty$ and fix $t_0 \in \mathbb{T}$.

Definition 8.1. For $b : \mathbb{T} \to \mathbb{R}$, the *shift* (or delay) $\tilde{b}$ of b is the function $\tilde{b} : \mathbb{T}\times\mathbb{T} \to \mathbb{R}$ given by

$$\tilde{b}^{\Delta_t}(t, \sigma(s)) = -\tilde{b}^{\Delta_s}(t, s), \quad t, s \in \mathbb{T}, \ t \geq s \geq t_0, \tag{8.3}$$

$$\tilde{b}(t, t_0) = b(t), \quad t \in \mathbb{T}, \, t \geq t_0,$$

where $\frac{\Delta}{t}$ is the partial Δ-derivative with respect to t.

For the forward difference operator, (8.3) reduces to

$$\mu(s)\Delta_t\tilde{b}(t, \sigma(s)) = -\mu(t)\Delta_s\tilde{b}(t, s), \quad t, s \in \mathbb{T}, \, t \geq s \geq t_0, \tag{8.4}$$
$$\tilde{b}(t, t_0) = b(t), \quad t \in \mathbb{T}, \, t \geq t_0,$$

In the case $\mathbb{T} = \mathbb{R}$, the problem (8.3) takes the form

$$\frac{\partial \tilde{b}(t, s)}{\partial t} = -\frac{\partial \tilde{b}(t, s)}{\partial s}, \quad \tilde{b}(t, t_0) = b(t), \tag{8.5}$$

and its unique solution is $\tilde{b}(t, s) = b(t - s + t_0)$. In the case $\mathbb{T} = \mathbb{Z}$, (8.3) becomes

$$\tilde{b}(t + 1, s + 1) - \tilde{b}(t, s + 1) = -\tilde{b}(t, s + 1) + \tilde{b}(t, s), \quad \tilde{b}(t, t_0) = f(t), \tag{8.6}$$

and its unique solution is again $\tilde{b}(t, s) = b(t - s + t_0)$.

Lemma 8.2. *If $\tilde{b}$ is the shift of b, then $\tilde{b}(t, t) = b(t_0)$ for all $t \in \mathbb{T}$.*

Proof. By putting $B(t) = \tilde{b}(t, t)$, we find $B(t_0) = \tilde{b}(t_0, t_0) = b(t_0)$ due to the initial condition in (8.3) and $B^\Delta(t) = \tilde{b}^{\Delta_t}(t, \sigma(t)) + \tilde{b}^{\Delta_s}(t, t) = 0$ due to the dynamic equation in (8.3), where we have used [22, Theorem 7.2]. $\square$

Definition 8.3. The convolution of two functions $b, r : \mathbb{T} \to \mathbb{R}$, $b * r$ is defined as

$$(b * r)(t) = \int_{t_0}^t \tilde{b}(t, \sigma(s))r(s)\Delta s, \quad t \in \mathbb{T}, \tag{8.7}$$

where $\tilde{b}$ is given by (8.3).

Theorem 8.4. *The shift of a convolution is given by the formula*

$$\widetilde{(b * r)}(t, s) = \int_s^t \tilde{b}(t, \sigma(l))\tilde{r}(l, s)\Delta l. \tag{8.8}$$

Proof. We fix $t_0 \in \mathbb{T}$. Let us consider $F(t,s) = \int_s^t \tilde{b}(t,\sigma(l))\tilde{r}(l,s)\Delta l$. Then

$$
\begin{aligned}
F(t,t_0) &= \int_{t_0}^t \tilde{b}(t,\sigma(l))\tilde{r}(l,t_0)\Delta l \\
&= \int_{t_0}^t \tilde{b}(t,\sigma(l))r(l)\Delta l \\
&= (b * r)(t).
\end{aligned}
$$

Next, we calculate

$$
\begin{aligned}
&F^{\Delta_t}(t,\sigma(s)) + F^{\Delta_s}(t,s) \\
&= \int_{\sigma(s)}^t \tilde{b}^{\Delta_t}(t,\sigma(l))\tilde{r}(l,\sigma(s))\Delta l + \tilde{b}(\sigma(t),\sigma(t))\tilde{r}(t,\sigma(s)) \\
&\quad + \int_s^t \tilde{b}(t,\sigma(l))\tilde{r}^{\Delta_s}(l,s)\Delta l - \tilde{b}(t,\sigma(s))\tilde{r}(s,\sigma(s)) \\
&= -\int_{\sigma(s)}^t \tilde{b}^{\Delta_s}(t,l)\tilde{r}(l,\sigma(s))\Delta l + b(t_0)\tilde{r}(t,\sigma(s)) \\
&\quad + \int_s^t \tilde{b}(t,\sigma(l))\tilde{r}^{\Delta_s}(l,s)\Delta l - \tilde{b}(t,\sigma(s))\tilde{r}(s,\sigma(s)) \\
&= -\tilde{b}(t,l)\tilde{r}(l,\sigma(s))\Big|_{l=\sigma(s)}^{l=t} + \int_{\sigma(s)}^t \tilde{b}(t,\sigma(l))\tilde{r}^{\Delta_t}(l,\sigma(s))\Delta l + b(t_0)\tilde{r}(t,\sigma(s)) \\
&\quad + \int_s^t \tilde{b}(t,\sigma(l))\tilde{r}^{\Delta_s}(l,s)\Delta l - \tilde{b}(t,\sigma(s))\tilde{r}(s,\sigma(s)) \\
&= -\tilde{b}(t,t)\tilde{r}(t,\sigma(s)) + \tilde{b}(t,\sigma(s))\tilde{r}(\sigma(s),\sigma(s)) + \int_{\sigma(s)}^t \tilde{b}(t,\sigma(l))\tilde{r}^{\Delta_t}(l,\sigma(s))\Delta l \\
&\quad + b(t_0)\tilde{r}(t,\sigma(s)) + \int_s^t \tilde{b}(t,\sigma(l))\tilde{r}^{\Delta_s}(l,s)\Delta l - \tilde{b}(t,\sigma(s))\tilde{r}(s,\sigma(s)) \\
&= \int_{\sigma(s)}^t \tilde{b}(t,\sigma(l))\tilde{r}^{\Delta_t}(l,\sigma(s))\Delta l + \tilde{b}(t,\sigma(s))r(t_0) \\
&\quad + \int_s^t \tilde{b}(t,\sigma(l))\tilde{r}^{\Delta_s}(l,s)\Delta l - \tilde{b}(t,\sigma(s))\tilde{r}(s,\sigma(s)) \\
&= \tilde{b}(t,\sigma(s))r(t_0) - \int_{\sigma(s)}^t \tilde{b}(t,\sigma(l))\tilde{r}^{\Delta_s}(l,s)\Delta l \\
&\quad + \int_s^t \tilde{b}(t,\sigma(l))\tilde{r}^{\Delta_s}(l,s)\Delta l - \tilde{b}(t,\sigma(s))\tilde{r}(s,\sigma(s)) \\
&= \tilde{b}(t,\sigma(s))r(t_0) + \int_{\sigma(s)}^s \tilde{b}(t,\sigma(l))\tilde{r}^{\Delta_s}(l,s)\Delta l - \tilde{b}(t,\sigma(s))\tilde{r}(s,\sigma(s))
\end{aligned}
$$

$$= \tilde{b}(t, \sigma(s))r(t_0) + \mu(s)\tilde{b}(t, \sigma(s))\tilde{r}^{\Delta_s}(s, s) - \tilde{b}(t, \sigma(s))\tilde{r}(s, \sigma(s))$$

$$= \tilde{b}(t, \sigma(s))r(t_0) + \tilde{b}(t, \sigma(s))\left[\tilde{r}(s, \sigma(s)) - \tilde{r}(s, s)\right] - \tilde{b}(t, \sigma(s))\tilde{r}(s, \sigma(s))$$

$$= 0,$$

where on the eighth equality we have used Theorem 2.14. $\square$

Theorem 8.5. *The convolution is associative, that is,*

$$(a * f) * r = a * (f * r). \tag{8.9}$$

Proof. We use Theorem 8.4. Then

$$
\begin{aligned}
((a * f) * r)(t) &= \int_{t_0}^t \widetilde{(a * f)}(t, \sigma(s))r(s)\Delta s \tag{8.10}\\
&= \int_{t_0}^t \int_{\sigma(s)}^t \tilde{a}(t, \sigma(u))\tilde{f}(u, \sigma(s))r(s)\Delta u \Delta s\\
&= \int_{t_0}^t \int_{t_0}^u \tilde{a}(t, \sigma(u))\tilde{f}(u, \sigma(s))r(s)\Delta s \Delta u\\
&= \int_{t_0}^t \tilde{a}(t, \sigma(u))(f * r)(u)\Delta u \tag{8.11}\\
&= (a * (f * r))(t),
\end{aligned}
$$

where on the second equality we have used (8.8). Hence, the associative property holds. $\square$

Theorem 8.6. *If r is delta differentiable, then*

$$(r * f)^\Delta = r^\Delta * f + r(t_0)f \tag{8.12}$$

and if f is delta differentiable, then

$$(r * f)^\Delta = r * f^\Delta = rf(t_0). \tag{8.13}$$

Proof. First note that

$$(r * f)^{\Delta}(t) = \int_{t_0}^{t} r^{\Delta_t}(t, \sigma(t)) f(s) \Delta s + \tilde{r}(\sigma(t), \sigma(t)) f(t). \tag{8.14}$$

From here, since $\tilde{r}(\sigma(t), \sigma(t)) = r(t_0)$ by Lemma 8.2, and since

$$\widetilde{r^{\Delta}}(t, s) = \tilde{r}^{\Delta_t}(t, s), \tag{8.15}$$

the first equal sign of the statement follows. For the second equal sign, we use the definition of $\tilde{r}$ and integration by parts:

$$
\begin{aligned}
(r * f)^{\Delta}(t) &= -\int_{t_0}^{t} \tilde{r}^{\Delta_s}(t, s) f(s) \Delta s + r(t_0) f(t) \\
&= -\int_{t_0}^{t} \left((\tilde{r}(t, \cdot) f)^{\Delta} - \tilde{r}(t, \sigma(s)) f^{\Delta}(s) \right) \Delta s + r(t_0) f(t) \\
&= -\tilde{r}(t, t) f(t) + \tilde{r}(t, t_0) f(t_0) + \int_{t_0}^{t} \tilde{r}(t, \sigma(s)) f^{\Delta}(s) \Delta s + r(t_0) f(t) \\
&= (r * f^{\Delta})(t) + r(t) f(t_0).
\end{aligned}
\tag{8.16}
$$

This completes the proof. $\qquad\square$

8.2. MEAN-SQUARE STABILITY

Theorem 8.7. *If $X(t)$ is represented as*

$$X(t) = r(t) X_0 + (r * f)(t), \tag{8.17}$$

where

$$r^{\Delta}(t) = (a * r)(t), \quad r(t_0) = 1 \tag{8.18}$$

and

$$f(t) = (b * X)(t) V^{\Delta}(t). \tag{8.19}$$

then X is a solution of the scalar Volterra dynamic problem

$$\Delta X = (a * X)(t)\Delta t + (b * X)(t)\Delta V, \quad X(t_0) = X_0, \tag{8.20}$$

Proof. From (8.17) we have

$$
\begin{aligned}
\Delta X(t) &= r^\Delta(t)X_0\Delta t + (r * f)^\Delta(t)\Delta t \\
&= (a * r)(t)X_0\Delta t + (r^\Delta * f)(t)\Delta t + f(t)\Delta t \\
&= (a * (rX_0))(t)\Delta t + (r^\Delta * f)(t)\Delta t + f(t)\Delta t \\
&= (a * (X - r * f))(t)\Delta t + (r^\Delta * f)(t)\Delta t + f(t)\Delta t \\
&= (a * X)(t)\Delta t - (a * (r * f))(t)\Delta t + ((a * r) * f)(t)\Delta t + f(t)\Delta t \\
&= (a * X)(t)\Delta t + f(t)\Delta t \\
&= (a * X)(t)\Delta t + (b * X)(t)\Delta V(t),
\end{aligned}
$$

where on the second equality we have used (8.12) and on the sixth equality we have used Theorem 8.5. □

Lemma 8.8. *If f is given by (8.19), then $\mathbb{E}[f(t)] = 0$ and*

$$
\mathbb{E}[f(t)f(s)] =
\begin{cases}
(b * \mathbb{E}[X])^2(t) =: \phi(t) & \text{if } s = t \\
\\
0 & \text{if } s \neq t.
\end{cases}
$$

Proof. We first note that

$$
\begin{aligned}
\mathbb{E}[f(t)] &= \mathbb{E}\left[\int_{t_0}^t \tilde{b}(t, \sigma(\tau))X(\tau)V^\Delta(t)\Delta\tau\right] \\
&= \int_{t_0}^t \tilde{b}(t, \sigma(\tau))\mathbb{E}[X(\tau)V^\Delta(t)]\Delta\tau \\
&= \int_{t_0}^t \tilde{b}(t, \sigma(\tau))\mathbb{E}\left[X(\tau)\right]\mathbb{E}[V^\Delta(t)]\Delta\tau \\
&= 0,
\end{aligned}
$$

by the assumption that $X(\tau)$ is independent of $V^{\Delta}(t)$ for $\tau \in [t_0, t)$ and $\mathbb{E}\left[V^{\Delta}(t)\right] = 0$.

Next, we consider

$$\mathbb{E}[f(t)f(s)]$$

$$= \mathbb{E}\left[\int_{t_0}^{t} \tilde{b}(t, \sigma(t_1))X(t_1)V^{\Delta}(t)\Delta t_1 \int_{t_0}^{s} \tilde{b}(s, \sigma(t_2))X(t_2)V^{\Delta}(s)\Delta t_2\right]$$

$$= \mathbb{E}\left[\int_{t_0}^{t} \int_{t_0}^{s} \tilde{b}(t, \sigma(t_1))\tilde{b}(s, \sigma(t_2))X(t_1)X(t_2)V^{\Delta}(t)V^{\Delta}(s)\Delta t_1 \Delta t_2\right]$$

$$= \int_{t_0}^{t} \int_{t_0}^{s} \tilde{b}(t, \sigma(t_1))\tilde{b}(s, \sigma(t_2))\mathbb{E}\left[X(t_1)X(t_2)V^{\Delta}(t)V^{\Delta}(s)\right]\Delta t_1 \Delta t_2$$

$$= \int_{t_0}^{t} \int_{t_0}^{s} \tilde{b}(t, \sigma(t_1))\tilde{b}(s, \sigma(t_2))\mathbb{E}\left[X(t_1)X(t_2)\right]\mathbb{E}\left[V^{\Delta}(t)V^{\Delta}(s)\right]\Delta t_1 \Delta t_2$$

$$= \begin{cases} \int_{t_0}^{t} \int_{t_0}^{t} \tilde{b}(t, \sigma(t_1))\tilde{b}(t, \sigma(t_2))\mathbb{E}\left[X(t_1)\right]\mathbb{E}\left[X(t_2)\right]\Delta t_1 \Delta t_2 & \text{if } s = t \\ \\ 0 & \text{if } s \neq t \end{cases}$$

$$= \begin{cases} (b * \mathbb{E}[X])^2 (t) & \text{if } s = t \\ \\ 0 & \text{if } s \neq t, \end{cases}$$

where on the third equation we have used the assumption that $X(\tau)$ is independent of $V^{\Delta}(t)$ for $\tau \in [t_0, t)$ and on fourth equation we have used $\mathbb{E}[V^{\Delta}(t)] = 0$ and $\mathbb{E}[(V^{\Delta}(t))^2] = 1 > 0$. $\qquad \square$

Lemma 8.9. *If $X(t) = r(t)X_0 + (r * f)(t)$, then*

$$\mathbb{E}[X(l)X(m)] = r(l)r(m)X_0^2 + \int_{t_0}^{l \wedge m} \tilde{r}(l, \sigma(s))\tilde{r}(m, \sigma(s))\phi(s)\Delta s,$$

where ϕ is as in Lemma 8.8 and $l \wedge m$ as in Definition 4.5.

Proof. From (8.17) we have,

$$
\begin{aligned}
\mathbb{E}[X(l)X(m)] &= \mathbb{E}\left[\{r(l)X_0 + (r * f)(l)\}\{r(m)X_0 + (r * f)(m)\}\right] \\
&= r(l)r(m)X_0^2 \\
&\quad + \int_{t_0}^{l} \int_{t_0}^{m} \tilde{r}(l, \sigma(s_1))\tilde{r}(m, \sigma(s_2))\mathbb{E}\left[f(s_1)f(s_2)\right] \Delta s_1 \Delta s_2 \\
&= r(l)r(m)X_0^2 + \int_{t_0}^{l \wedge m} \tilde{r}(l, \sigma(s))\tilde{r}(m, \sigma(s))\mathbb{E}\left[f^2(s)\right] \Delta s \\
&= r(l)r(m)X_0^2 + \int_{t_0}^{l \wedge m} \tilde{r}(l, \sigma(s))\tilde{r}(m, \sigma(s))\phi(s)\Delta s,
\end{aligned}
$$

where on the second equality we have used the fact that $\mathbb{E}\left[f(t)\right] = 0$ and on the third equality we have used Lemma 8.8. $\qquad\square$

Lemma 8.10. ϕ *defined in Lemma 8.8 is given by*

$$
\begin{aligned}
\phi(t) &= (b * r)^2(t)X_0^2 + \int_{t_0}^{t} \left(\int_{\sigma(s)}^{t} \tilde{b}(t, \sigma(l))\tilde{r}(l, \sigma(s))\Delta l\right)^2 \phi(s)\Delta s \\
&= (b * r)^2(t)X_0^2 + \int_{t_0}^{t} \widetilde{(b * r)}^2(t, \sigma(s))\phi(s)\Delta s.
\end{aligned}
$$

Proof. Using Lemma 8.8, Lemma 8.9 and (8.7), we have

$$
\begin{aligned}
\phi(t) &= (b * \mathbb{E}[X])^2(t) \\
&= \int_{t_0}^{t} \int_{t_0}^{t} \tilde{b}(t, \sigma(l))\tilde{b}(t, \sigma(m))\mathbb{E}\left[X(l)X(m)\right] \Delta l \Delta m \\
&= \int_{t_0}^{t} \int_{t_0}^{t} \tilde{b}(t, \sigma(l))\tilde{b}(t, \sigma(m))r(l)r(m)X_0^2 \Delta l \Delta m \\
&\quad + \int_{t_0}^{t} \int_{t_0}^{t} \tilde{b}(t, \sigma(l))\tilde{b}(t, \sigma(m)) \int_{t_0}^{l \wedge m} \tilde{r}(l, \sigma(s))\tilde{r}(m, \sigma(s))\phi(s)\Delta s \Delta l \Delta m \\
&= \int_{t_0}^{t} \int_{t_0}^{t} \tilde{b}(t, \sigma(l))\tilde{b}(t, \sigma(m))r(l)r(m)X_0^2 \Delta l \Delta m \\
&\quad + \int_{t_0}^{t} \int_{t_0}^{t} \int_{t_0}^{l \wedge m} \tilde{b}(t, \sigma(l))\tilde{b}(t, \sigma(m))\tilde{r}(l, \sigma(s))\tilde{r}(m, \sigma(s))\phi(s)\Delta s \Delta l \Delta m \\
&= \left(\int_{t_0}^{t} \tilde{b}(t, \sigma(l))r(l)\Delta l\right)^2 X_0^2
\end{aligned}
$$

$$+ \int_{t_0}^{t} \int_{\sigma(s)}^{t} \int_{\sigma(s)}^{t} \tilde{b}(t, \sigma(l))\tilde{b}(t, \sigma(m))\tilde{r}(l, \sigma(s))\tilde{r}(m, \sigma(s))\phi(s)\Delta m \Delta l \Delta s$$

$$= (b*r)^2(t)X_0^2 + \int_{t_0}^{t} \left(\int_{\sigma(s)}^{t} \tilde{b}(t, \sigma(l))\tilde{r}(l, \sigma(s))\Delta l \right)^2 \phi(s)\Delta s$$

$$= (b*r)^2(t)X_0^2 + \int_{t_0}^{t} \widetilde{(b*r)}^2(t, \sigma(s))\phi(s)\Delta s,$$

where on the last equality we have used Theorem 8.4. □

Theorem 8.11. *If X is a solution of* (8.20)*, then*

$$\mathbb{E}\left[X^2(t)\right] = r^2(t)X_0^2 + \int_{t_0}^{t} \tilde{r}^2(t, \sigma(s))\phi(s)\Delta s.$$

Proof. Squaring both sides of (8.17), we have

$$\begin{aligned}
X^2(t) &= r^2(t)X_0^2 + 2r(t)X_0(r*f)(t) \\
&\quad + \int_{t_0}^{t} \tilde{r}(t, \sigma(s_1))f(s_1)\Delta s_1 \int_{t_0}^{t} \tilde{r}(t, \sigma(s_2))f(s_2)\Delta s_2 \\
&= r^2(t)X_0^2 + 2r(t)X_0\,(r*f)(t) \\
&\quad + \int_{t_0}^{t} \int_{t_0}^{t} \tilde{r}(t, \sigma(s_1))\tilde{r}(t, \sigma(s_2))f(s_1)f(s_2)\Delta s_1 \Delta s_2.
\end{aligned}$$

Now taking the expectation on both sides of the above expression, we have

$$\begin{aligned}
\mathbb{E}\left[X^2(t)\right] &= r^2(t)X_0^2 + 2r(t)X_0 \int_{t_0}^{t} \tilde{r}(t, \sigma(s))\mathbb{E}[f(s)]\Delta s \\
&\quad + \int_{t_0}^{t} \int_{t_0}^{t} \tilde{r}(t, \sigma(s_1))\tilde{r}(t, \sigma(s_2))\mathbb{E}[f(s_1)f(s_2)]\Delta s_1 \Delta s_2 \\
&= r^2(t)X_0^2 + \int_{t_0}^{t} \tilde{r}^2(t, \sigma(s))\phi(s)\Delta s,
\end{aligned}$$

where on the second equality we have used Lemma 8.8. □

Theorem 8.12. *Suppose that X is the solution of* (8.20) *and r is the solution of* (8.18)*. Then*

$$r, \; \tilde{r}(\cdot, s), \; b*r \in L^2_\Delta(\mathbb{T})$$

and

$$\int_{\sigma(s)}^{\infty} (\widetilde{b * r})^2(t, \sigma(s))\Delta t \leq k < 1$$

for all $s \in \mathbb{T}$, implies that

$$\int_{\mathbb{T}} \mathbb{E}\left[X^2(t)\right] \Delta t < \infty.$$

Proof. From Lemma 8.10, we have

$$
\begin{aligned}
\int_{t_0}^{\infty} \phi(t)\Delta t &= X_0^2 \int_{t_0}^{\infty} (b * r)^2(t)\Delta t + \int_{t_0}^{\infty} \int_{t_0}^{t} (\widetilde{b * r})^2(t, \sigma(s))\phi(s)\Delta s \Delta t \\
&= X_0^2 \int_{t_0}^{\infty} (b * r)^2(t)\Delta t + \int_{t_0}^{\infty} \int_{\sigma(s)}^{\infty} (\widetilde{b * r})^2(t, \sigma(s))\phi(s)\Delta t \Delta s \\
&\leq X_0^2 \int_{t_0}^{\infty} (b * r)^2(t)\Delta t + k \int_{t_0}^{\infty} \phi(s)\Delta s.
\end{aligned}
$$

Simplifying and using the fact that $b * r \in \mathbb{L}_\Delta^2(\mathbb{T})$, we have

$$\int_{t_0}^{\infty} \phi(t)\Delta t \leq \frac{X_0^2}{1-k} \int_{t_0}^{\infty} (b * r)^2(t)\Delta t < \infty, \tag{8.21}$$

which implies that $\phi \in L_\Delta^1(\mathbb{T})$. Then from Theorem 8.11, we have

$$
\begin{aligned}
\int_{t_0}^{\infty} \mathbb{E}\left[X^2(t)\right]\Delta t &= X_0^2 \int_{t_0}^{\infty} r^2(t)\Delta t + \int_{t_0}^{\infty} \int_{t_0}^{t} \tilde{r}^2(t, \sigma(s))\phi(s)\Delta s \Delta t \\
&\leq \alpha + \int_{t_0}^{\infty} \int_{\sigma(s)}^{\infty} \tilde{r}^2(t, \sigma(s))\phi(s)\Delta t \Delta s \\
&\leq \alpha + \beta \int_{t_0}^{\infty} \phi(s)\Delta s \\
&< \infty,
\end{aligned}
$$

where $\alpha, \beta \in \mathbb{R}$ such that $X_0^2 \int_{t_0}^{\infty} r^2(t)\Delta t < \alpha$ and $\int_{\sigma(s)}^{\infty} \tilde{r}^2(t, \sigma(s))\Delta t < \beta$. $\qquad\square$

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