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The Kalman filter on time scales

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THE KALMAN FILTER ON TIME SCALES

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DEDICATION

This dissertation is dedicated in loving memory to

Debra Ann Wintz

September 7, 1955 – November 7, 2008

and

Ryan David Hall

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ABSTRACT

In this work, we study concepts in optimal control for dynamic equations on time scales, which unifies the discrete and continuous cases. After a brief introduction of dynamic equations on time scales, we will examine controllability and observability for linear systems. Then we construct and solve the linear quadratic regulator for arbitrary time scales. Here, we seek to find an optimal control that minimizes a given cost function associated with a linear system. We will find such an input under two different settings; when the final state is fixed and when it is free. Later, we extend these results to deal with linear quadratic tracking on time scales. The main contribution of this dissertation is the construction of the Kalman filter on time scales. In this setting, we seek to find an optimal estimate of a linear stochastic system whose state is corrupted by noisy measurements. Finally, we will make an argument that the linear quadratic regulator and the Kalman filter are mathematically dual to each other.

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1. INTRODUCTION

1.1. A BRIEF HISTORY OF FILTERING THEORY

The modern theory of filtering is commonly considered to have been developed by the independent work of Kolmogorov [37–39] and Wiener [28, 54, 55] in the 1940s. Both began by considering stationary processes with scalar signals and noise. Their efforts, particularly those of Wiener, were not purely academic. The immediate need for this theory was determining where to aim anti-aircraft guns at evading airplanes during World War II. As a forecasting problem, the desire was to determine where to aim the gun so that the shell travels near an aircraft with the smallest error. In continuous time, this required solving the well-known Wiener–Hopf integral equation. In order to solve this equation, Wiener had to transform the equation into the frequency domain, decomposing the resulting power densities using spectral factorization, and combining certain factors to find a time-invariant filter over an infinite observation time in the form of a frequency response. As a result of the war effort, much of Wiener's work remained classified until after 1945.

While Wiener's method was elegant, it was equally restrictive. Using Kolmogorov's theory, Levinson [41, 42] showed in the discrete case that the entire theory could be reduced to least squares. A few years later, Bode and Shannon [9] would introduce a more simplified method of solving the Wiener–Hopf equation by introducing the concept of a shaping filter. In 1952, Booten generalized the Wiener–Hopf equation for nonstationary processes and time-varying filters (see [13]). Follin and Carlton [25] independently of Hanson [26] soon afterward began investigating the filter problem for a finite observation time. Their work showed that the optimal parameters for corresponding time-varying filters of stationary processes had to satisfy certain differential equations. In 1959, Bucy [18] proved that this method could be applied for nonstationary processes as well. Much of these results were later considered in discrete time, particulary by Swerling [52].

Then in 1960, Kalman extended Wiener's theory for nonstationary processes using the concept of state-space techniques he developed earlier [32]. Kalman invented a recursive algorithm involving a system of difference equations for the filter and its gain matrices. In the discrete case, the Kalman filter is essentially a predictor-corrector type estimator. The first step in this algorithm is to predict an estimate of the state based from a previous measurement. This prediction is then associated with an error covariance. This step is sometimes referred to as the "time update" portion of the algorithm. Then the next step is to calculate a correction of the state estimate based on the prediction and the new measurement along with its associated error covariance. This new corrected estimate depends on a residual whose weighting coefficient represents the Kalman gain. This last step is sometimes called the "measurement update" portion. The algorithm then repeats itself. A year later Kalman and Bucy created a corresponding filter in the continuous case [35]. This case was also formulated in state-space representation to derive a time-varying matrix form of the Wiener–Hopf equation. Using this equation, Kalman and Bucy derived a similar filter involving a system of differential equations. While Kalman is credited with developing the Kalman filter (Swerling invented a similar algorithm earlier), it was actually Schmidt who first found an application of the filter. Following Kalman's visit to the NASA Ames Research Center, Schmidt applied Kalman's work to the problem of navigating to the moon with the Apollo program. Sometimes the Kalman filter is called the Linear Quadratic Estimator (LQE).

Since then, the Kalman filter has been implemented for numerous applications in one form or another. One such example is the extended Kalman filter which is used for nonlinear systems. Applications for this form of the filter include ballistics, neural networks, and GPS navigation [8, 27, 58]. However, this form is not particularly reliable as an optimal estimator. An improvement for this design is the unscented filter, which uses deterministic sampling techniques to find the true mean and variances. There has also been interest in comparing discrete with continuous measurements when the filter design is given as a continuous process [22, 49, 57]. Applications here include biomechanical models, particularly for cardiac kinetics estimation. Such filters are sometimes called "hybrid" filters, although this term is generic. Despite its various incarnations, each filter design strikingly resembles Kalman's original filter.

1.2. OUTLINE OF DISSERTATION

In 1988 Stefan Hilger [29], under the direction of Bernd Aulbach, introduced the theory of time scales in order to unify discrete and continuous analysis. As a result, one can generalize a process to account for both cases, or any combination of the two provided we restrict ourselves to closed, nonempty subsets of the reals (a time scale). However, this generalized process can handle many other time scales than just the set of real numbers and the set of integers, and we are left with a more general result. This thesis deals with the theory of optimal control on time scales. The theory of optimal control leads itself extremely well to the theory of time scales, as many discrete processes often look strikingly similar to their continuous counterparts. Our aim is to unify and extend such processes to dynamic equations on time scales.

In Section 2, we introduce some basic concepts of time scale calculus. A more formal definition of time scales is given along with examples. We will also consider the derivative and integral of functions on an arbitrary time scale, along with some basic properties of each. The so-called "simple useful formula," which we will use to derive many of our results in Sections 4 and 5 is discussed. Also, since we are mainly concerned with linear systems, we will examine the matrix exponential and its properties.

For Section 3, we will unify and extend the concepts of controllability and observability for the adjoint equation in the time invariant setting. These concepts are attributed to Kalman in the real and discrete cases (see [31, 33, 34, 36]). We will compare these results with the controllability and observability results of a similar linear system. We also consider controllability and observability for linear time-varying (LTV) systems as well.

In Section 4, we will introduce and unify the concept of the linear quadratic regulator (LQR) on time scales. This concept can also be attributed to Kalman in the discrete and continuous cases. In this section, we seek to find an optimal control that minimizes a quadratic cost function associated with a linear system. Depending on the final state, this optimal control can take on two completely different forms. If the final state is fixed we have an open-loop control, meaning that the input is not in terms of the current state. On the other hand, if the final state is not fixed, we have a closed-loop control. This means that the optimal input is in terms of the current state (i.e., state feedback). In the second situation, we will use the simple useful formula to find a Riccati equation whose solution gives us the optimal control.

Section 5 is essentially an extension of the LQR. In this section, we introduce and unify the concept of the linear quadratic tracker (LQT) on time scales. Using the concept of the tracker, we find an optimal control when the final state is fixed using the same mechanics as we did when the final state was free.

In Section 6, we introduce the Kalman filter or linear quadratic estimator (LQE) for time scales. Yet again this concept can be attributed to Kalman in the discrete case (see [32]) and the continuous case (with Bucy, see [35]). However this one is up for debate! One can see [51] for more details. In this section, we consider the Kalman filter in both its discrete and continuous forms. Next we introduce the concept of an observer for linear systems on time scales. An observer is a linear system that estimates a system when the system is not entirely available for measurement. Then when the linear system is also stochastic, we introduce the Kalman filter as an observer that estimates the system when the state is corrupted by noisy measurements. Finally, we make an argument that LQR and LQE are dual problems of each other, since the Riccati equations and gains that describe each optimal problem look mathematically similar to each other.

In Section 7, we consider some open problems dealing with the LQR and LQE on time scales.

2. INTRODUCTION TO TIME SCALES

In 1988, Stefan Hilger under the direction of Bernd Aulbach introduced calculus on time scales in his PhD thesis [4,29]. The study of dynamic equations on time scales unifies both continuous and discrete mathematical analysis. As a result, one can generalize a process to account for both cases, or any combination of the two. Since its inception, this area of mathematics has gained a great deal of international attention. Researchers have since found applications of time scales to include heat transfer, population dynamics, and economics. In further sections, we will extend our results toward applications found in electrical engineering. For a more in-depth study of time scales, see Bohner and Peterson's books [11,12].

2.1. BASIC DEFINITIONS

In this subsection, we will introduce the basic results on time scales that we will use in later sections.

Definition 2.1. A *time scale* $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers.

Next, we consider some common examples of time scales.

Example 2.2. Some common time scales include

a. $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$;

b. $\mathbb{T} = h\mathbb{Z} := \{hz : z \in \mathbb{Z}\}$ for $h > 0$;

c. $\mathbb{T} = q^{\mathbb{Z}} := \{q^k : k \in \mathbb{Z}\}$ for $q > 1$;

d. $\mathbb{T} = 2^{\mathbb{Z}} := \{2^k : k \in \mathbb{Z}\}$;

e. The so-called harmonic numbers, $\left\{H_n = \sum_{k=1}^{n} \frac{1}{k} : \ n \in \mathbb{N}_0\right\}$;

f. $\mathbb{T} = \mathbb{N}_0^2 := \{n^2 : n \in \mathbb{N}_0\}$;

g. The Cantor set.

Any time scale that is a combination of any of the above sets is called a hybrid time scale. On the other hand, sets such as (a, b) and $\mathbb{C}$ are not time scales.

Next, we define the forward and backward jump operators.

Definition 2.3. For $t \in \mathbb{T}$ we define the following:

a. The *forward jump operator* $\sigma : \mathbb{T} \to \mathbb{T}$ is given by

$$\sigma(t) := \inf \{s \in \mathbb{T} : \ s > t\}. \tag{2.1}$$

b. The *backward jump operator* $\rho : \mathbb{T} \to \mathbb{T}$ is defined by

$$\rho(t) := \sup \{s \in \mathbb{T} : \ s < t\}. \tag{2.2}$$

Definition 2.4. For any function $f : \mathbb{T} \to \mathbb{R}$, we will define the function $f^\sigma : \mathbb{T} \to \mathbb{R}$ to be

$$f^\sigma(t) = f(\sigma(t)) \quad \text{for all } t \in \mathbb{T}, \tag{2.3}$$

i.e., $f^\sigma = f \circ \sigma$.

Remark 2.5. We classify points as follows. If $\sigma(t) > t$, then t is said to be right-scattered. Similarly if $\rho(t) < t$, t is said to be left-scattered. If a point is both left and right-scattered, it is said to be isolated. On the other hand, if $\sigma(t) = t$, then t is said to be right-dense. Similarly if $\rho(t) = t$, t is said to be left-dense. If a point is both left and right-dense, it is said to be dense. Table 2.1 gives a classification of points.

Definition 2.6. If $\mathbb{T}$ is a time scale with a left-scattered maximum m, then the set $\mathbb{T}^\kappa = \mathbb{T} \setminus \{m\}$. Otherwise $\mathbb{T}^\kappa = \mathbb{T}$.

Definition 2.7. The *graininess function* $\mu : \mathbb{T} \to [0, \infty)$ is defined by

$$\mu(t) := \sigma(t) - t. \tag{2.4}$$

$t < \sigma(t)$	t is right-scattered
$\rho(t) < t$	t is left-scattered
$\rho(t) < t < \sigma(t)$	t is isolated
$\sigma(t) = t$	t is right-dense
$\rho(t) = t$	t is left-dense
$\rho(t) = t = \sigma(t)$	t is dense

Table 2.1. Classification of Points

In the Table 2.2, we define the forward and backward jump operators and the graininess function for some common time scales.

$\mathbb{T}$	$\mu(t)$	$\sigma(t)$	$\rho(t)$
$\mathbb{R}$	0	t	t
$\mathbb{Z}$	1	$t+1$	$t-1$
$h\mathbb{Z}$	h	$t+h$	$t-h$
$\overline{q^{\mathbb{Z}}}$	$(q-1)t$	qt	$\frac{t}{q}$
$\overline{2^{\mathbb{Z}}}$	t	$2t$	$\frac{t}{2}$
$\mathbb{N}_0^2$	$1+2\sqrt{t}$	$(\sqrt{t}+1)^2$	$(\sqrt{t}-1)^2$
H_n	$\frac{1}{n+1}$	H_{n+1}	H_{n-1}

Table 2.2. Examples of Times Scales

2.2. DIFFERENTIATION

Next we define the delta (or Hilger) derivative.

Definition 2.8. Let $f : \mathbb{T} \to \mathbb{R}$. The *delta derivative* $f^{\Delta}(t)$ is the number (when it exists) such that given any $\varepsilon > 0$, there is a neighborhood U of t such that

$$\big|[f(\sigma(t)) - f(s)] - f^{\Delta}(t)[\sigma(t) - s]\big| \leq \varepsilon|\sigma(t) - s| \quad \text{for all } s \in U.$$

In the next two theorems, we will consider some properties of the delta derivative.

Theorem 2.9. *[11, Theorem 1.16] Suppose $f : \mathbb{T} \to \mathbb{R}$ is a function and let $t \in \mathbb{T}^{\kappa}$. Then we have the following:*

a. If f is differentiable at a point t, then f is continuous at t.

b. If f is continuous at t, where t is right-scattered, then f is differentiable at t and

$$f^{\Delta}(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}. \tag{2.5}$$

c. If f is differentiable at t, where t is right-dense, then

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s}. \tag{2.6}$$

d. If f is differentiable at t, then

$$f(\sigma(t)) = f(t) + \mu(t)f^{\Delta}(t). \tag{2.7}$$

The last equation is sometimes called the "simple useful formula." We will need this equation to develop the "sweep method" used extensively in later sections.

Remark 2.10. Note the following examples.

a. When $\mathbb{T} = \mathbb{R}$, then (if the limit exists)

$$f^{\Delta}(t) = \lim_{s \to t} \frac{f(t) - f(s)}{t - s} = f'(t).$$

b. When $\mathbb{T} = \mathbb{Z}$, then

$$f^{\Delta}(t) = f(t+1) - f(t) = \Delta f(t).$$

c. When $\mathbb{T} = q^{\mathbb{Z}}$ for $q > 1$, then

$$f^{\Delta}(t) = \frac{f(qt) - f(t)}{(q-1)t}.$$

Next we consider the linearity property as well as the product and quotient rules.

Theorem 2.11. *[11, Theorem 1.20] Let $f, g : \mathbb{T} \to \mathbb{R}$ be differentiable at $t \in \mathbb{T}^{\kappa}$. Then we have the following:*

a. For any constants α and β, the sum $(\alpha f + \beta g) : \mathbb{T} \to \mathbb{R}$ is differentiable at t with

$$(\alpha f + \beta g)^{\Delta}(t) = \alpha f^{\Delta}(t) + \beta g^{\Delta}(t). \tag{2.8}$$

b. The product $fg : \mathbb{T} \to \mathbb{R}$ is differentiable at t with

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f^{\sigma}(t)g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g^{\sigma}(t). \tag{2.9}$$

c. If $g(t)g(\sigma(t)) \neq 0$, then the quotient $f/g : \mathbb{T} \to \mathbb{R}$ is differentiable at t with

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}. \tag{2.10}$$

2.3. INTEGRATION

We will now consider when functions are integrable on an arbitrary time scale. However, we must first introduce the following two concepts.

Definition 2.12. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be *regulated* if its left and right-sided limits exist at all left and right-dense points in $\mathbb{T}$, respectively.

Definition 2.13. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be *rd-continuous* if it is continuous at right-dense points in $\mathbb{T}$ and its left-sided limits exist at left-dense points in $\mathbb{T}$. The class of rd-continuous functions $f : \mathbb{T} \to \mathbb{R}$ is given by

$$C_{\mathrm{rd}} = C_{\mathrm{rd}}(\mathbb{T}) = C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}). \tag{2.11}$$

From the previous two definitions, we have the following theorem.

Theorem 2.14. *[11, Theorem 1.60] Let $f : \mathbb{T} \to \mathbb{R}$.*

a. If f is continuous, then it is also rd-continuous.

b. If f is rd-continuous, then it is also regulated.

c. The jump operator σ is rd-continuous.

d. If f is regulated or rd-continuous, then so is f^σ.

e. Assume f is continuous. If $g : \mathbb{T} \to \mathbb{R}$ is regulated or rd-continuous, so is $f \circ g$.

Definition 2.15. A continuous function $f : \mathbb{T} \to \mathbb{R}$ is said to be *pre-differentiable* with (region of differentiation) D, provided $D \subset \mathbb{T}^\kappa$, $\mathbb{T}^\kappa \setminus D$ is countable and contains no right-scattered elements of $\mathbb{T}$, and f is differentiable at each point $t \in D$.

Next, we consider when the existence of pre-antiderivatives are guaranteed.

Theorem 2.16. *[11, Theorem 1.70] Let $f : \mathbb{T} \to \mathbb{R}$ be a regulated function. Then there exists a function F which is pre-differentiable with region of differentiation D such that*

$$F^\Delta(t) = f(t) \quad \textit{for all} \quad t \in D.$$

Any such function F is called a pre-antiderivative *of f.*

Definition 2.17. Let $f : \mathbb{T} \to \mathbb{R}$ be a regulated function and let F be a pre-antiderivative of f. Then the Cauchy integral integral of f is given by

$$\int_a^b f(t)\,\Delta t = F(b) - F(a) \qquad \text{for all } a, b \in \mathbb{T}.$$

Example 2.18. Let $a, b \in \mathbb{T}$ and f be rd-continuous. Note the following examples.

a. When $\mathbb{T} = \mathbb{R}$, then

$$\int_a^b f(t)\,\Delta t = \int_a^b f(t)\,dt.$$

b. When $[a, b]$ contains only isolated points, then

$$\int_a^b f(t)\,\Delta t = \begin{cases} \sum_{t\in[a,b)} \mu(t)f(t) & \text{if} \quad a < b \\ 0 & \text{if} \quad a = b \\ -\sum_{t\in[a,b)} \mu(t)f(t) & \text{if} \quad a > b. \end{cases}$$

c. When $\mathbb{T} = \mathbb{Z}$, then

$$\int_a^b f(t)\,\Delta t = \begin{cases} \sum_{t=a}^{b-1} f(t) & \text{if} \quad a < b \\ 0 & \text{if} \quad a = b \\ -\sum_{t=b}^{a-1} f(t) & \text{if} \quad a > b. \end{cases}$$

c. When $\mathbb{T} = h\mathbb{Z}$, then

$$\int_a^b f(t)\,\Delta t = \begin{cases} \sum_{k=a/h}^{b/h-1} hf(hk) & \text{if} \quad a < b \\ 0 & \text{if} \quad a = b \\ -\sum_{k=b/h}^{a/h-1} hf(hk) & \text{if} \quad a > b. \end{cases}$$

In the next theorem, we consider basic properties of integration on time scales.

Theorem 2.19. *[11, Theorem 1.77] If $a, b, c \in \mathbb{T}$, $\alpha \in \mathbb{R}$, and $f, g \in C_{\mathrm{rd}}$, then*

a. $\int_a^b [f(t) + g(t)]\Delta t = \int_a^b f(t)\Delta t + \int_a^b g(t)\Delta t$;

b. $\int_a^b (\alpha f)(t)\Delta t = \alpha \int_a^b f(t)\Delta t$;

c. $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t$;

d. $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t$;

e. $\int_a^b [f^\sigma(t)g^\Delta(t)]\Delta t = (fg)(b) - (fg)(a) + \int_a^b f^\Delta(t)g(t)\Delta t$;

f. $\int_a^b [f(t)g^\Delta(t)]\Delta t = (fg)(b) - (fg)(a) + \int_a^b f^\Delta(t)g^\sigma(t)\Delta t$;

g. $\int_a^a f(t)\Delta t = 0$.

Finally, we consider a generalized form of the Leibniz rule.

Theorem 2.20. *[11, Theorem 1.117] Let $a \in \mathbb{T}^\kappa$, $b \in \mathbb{T}$, and assume $f : \mathbb{T} \times \mathbb{T}^\kappa \to \mathbb{R}$ is continuous at (t, t), where $t \in \mathbb{T}^\kappa$ with $t > a$. Also assume that $f^\Delta(t, \cdot)$ is rd-continuous on $[a, \sigma(t)]$. Suppose that for each $\epsilon > 0$ there exists a neighborhood U of t independent of $\tau \in [a, \sigma(t)]$, such that*

$$|f(\sigma(t), \tau) - f(s, \tau) - f^\Delta(t, \tau)(\sigma(t) - s)| \leq \epsilon|\sigma(t) - s| \quad \textit{for all} \quad s \in U,$$

where f^Δ denotes the derivative of f with respect to the first variable. Then

a. $g(t) := \int_a^t f(t, \tau)\Delta\tau$ *implies* $g^\Delta(t) = \int_a^t f^\Delta(t, \tau)\Delta\tau + f(\sigma(t), t)$;

b. $h(t) := \int_t^b f(t, \tau)\Delta\tau$ *implies* $h^\Delta(t) = \int_t^b f^\Delta(t, \tau)\Delta\tau - f(\sigma(t), t)$.

2.4. EXPONENTIAL FUNCTIONS

In this section, we introduce exponential functions on time scales. First we offer regressive functions on time scales.

Definition 2.21. We say that the function $p : \mathbb{T} \to \mathbb{R}$ is *regressive* provided that

$$1 + \mu(t)p(t) \neq 0 \quad \text{for all} \quad t \in \mathbb{T}^\kappa.$$

The set of all regressive and rd-continuous functions is given by

$$\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R}).$$

Now we will introduce some special operations for regressive functions on time scales. We will use the next three definitions to introduce some properties of the exponential function on time scales.

Definition 2.22. Let $p, q \in \mathcal{R}$. Then we define "circle plus" addition $\oplus$ by

$$(p \oplus q)(t) = p(t) + (1 + \mu(t)p(t))q(t) \quad \text{for all} \quad t \in \mathbb{T}^\kappa. \tag{2.12}$$

Definition 2.23. Let $p, q \in \mathcal{R}$. Then we define "circle minus" subtraction $\ominus$ by

$$(p \ominus q)(t) = \frac{p(t) - q(t)}{1 + \mu(t)q(t)} \quad \text{for all} \quad t \in \mathbb{T}^\kappa. \tag{2.13}$$

Definition 2.24. Let $n \in \mathbb{N}$ and $p \in \mathcal{R}$. Then we define the "circle dot" multiplication $\odot$, denoted by

$$n \odot p = p \oplus p \oplus p \oplus \dots \oplus p,$$

where there are n terms on the right-hand side of the equation.

Next, we introduce the notion of the Hilger complex plane.

Definition 2.25. For $h > 0$, we define the *Hilger complex numbers* by

$$\mathbb{C}_h \;:=\; \left\{ z \in \mathbb{C} : z \neq -\frac{1}{h} \right\}.$$

When $h = 0$, let $\mathbb{C}_0 = \mathbb{C}$.

We will now express the exponential function in terms of what is known as the cylinder transformation whose range is the set Z_h defined as follows.

Definition 2.26. For $h > 0$, we define the strip

$$\mathbb{Z}_h \;:=\; \left\{ z \in \mathbb{C} : -\frac{\pi}{h} < \operatorname{Im}(z) \leq \frac{\pi}{h} \right\}.$$

When $h = 0$, let $\mathbb{Z}_0 := \mathbb{C}$.

Definition 2.27. For $h > 0$, we define the *cylinder transformation* $\xi_h : \mathbb{C}_h \to \mathbb{Z}_h$ by

$$\xi_h(z) = \frac{1}{h}\operatorname{Log}(zh + 1),$$

where Log represents the principal logarithm function. For $h = 0$, we define $\xi_0(z) = z$ for all $z \in \mathbb{C}$.

Next, the generalized exponential function is given as follows.

Definition 2.28. If $p \in \mathcal{R}$, then we define the exponential function by

$$e_p(t,s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right) \quad \text{for} \quad s,t \in \mathbb{T}. \tag{2.14}$$

where the cylinder transformation $\xi_h(z)$ is the same as in Definition 2.27.

Definition 2.29. If $p \in \mathcal{R}$, then the linear dynamic equation

$$y^{\Delta}(t) = p(t)y(t) \tag{2.15}$$

is regressive.

Theorem 2.30. *[11, Theorem 2.33] Suppose that* (2.15) *is regressive and fix* $t_0 \in \mathbb{T}$. *Then the solution to the initial value problem*

$$y^{\Delta}(t) = p(t)y(t), \quad y(t_0) = 1 \tag{2.16}$$

is given by $e_p(\cdot, t_0)$.

Our next theorem addresses the uniqueness of the solution for (2.16).

Theorem 2.31. *[11, Theorem 2.35] If* (2.15) *is regressive, then the only solution of* (2.16) *is given by* $e_p(\cdot, t_0)$.

Now we will state some properties of the exponential function.

Theorem 2.32. *[11, Theorem 2.36] and [12, Theorem 2.44] If* $p, q \in \mathcal{R}$, *then*

a. $e_0(t,s) \equiv 1$ *and* $e_p(t,t) \equiv 1$;

b. $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t,s)$;

c. $e_{\ominus p}(t,s) = \frac{1}{e_p(t,s)}$;

d. $e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$;

e. $e_p(t,s)e_p(s,r) = e_p(t,r)$;

f. $e_p(t,s)e_q(t,s) = e_{p\oplus q}(t,s)$;

g. $\dfrac{e_p(t,s)}{e_q(t,s)} = e_{p\ominus q}(t,s)$;

h. $\left(\dfrac{1}{e_p(\cdot,s)}\right)^{\Delta} = -\dfrac{p(t)}{e_p^{\sigma}(\cdot,s)}$;

i. $e_p^n(t,s) = e_{n\odot p}(t,s)$.

We will use the next result in Section 4.

Theorem 2.33. *[11, Theorem 2.39] If $p \in \mathcal{R}$ and $a, b, c \in \mathbb{T}$, then*

$$[e_p(c,\cdot)]^{\Delta} = -p[e_p(c,\cdot)]^{\sigma}$$

and

$$\int_a^b p(\tau)e_p(c,\sigma(\tau))\Delta\tau = e_p(c,a) - e_p(c,b).$$

2.5. MATRIX EXPONENTIAL

Before we introduce the matrix exponential, we consider the notion of regressive matrices.

Definition 2.34. Let A be an $m \times n$ matrix-valued function defined on $\mathbb{T}$. If every entry of A is rd-continuous on $\mathbb{T}$, then A is said to be rd-continuous on $\mathbb{T}$.

It should be noted that the class of rd-continuous matrix-valued functions is abbreviated by

$$C_{\mathrm{rd}} = C_{\mathrm{rd}}(\mathbb{T}) = C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^{m\times n}). \tag{2.17}$$

Remark 2.35. Consider the linear system of dynamic equations

$$x^{\Delta}(t) = A(t)x(t), \tag{2.18}$$

where A is an $n \times n$ matrix defined on $\mathbb{T}$. We say that the vector-valued function $v : \mathbb{T} \to \mathbb{R}$ is a solution to (2.18) provided that $v^{\Delta}(t) = A(t)v(t)$ holds for all $t \in \mathbb{T}^{\kappa}$. Now in order to discuss this system subject to some initial condition, we need the following definition.

Definition 2.36. Let A be an $n \times n$ matrix-valued function defined on $\mathbb{T}$. A is said to be *regressive* if $I + \mu(t)A(t)$ is invertible for all $t \in \mathbb{T}^{\kappa}$, where I is the identity matrix.

The class of all rd-continuous and regressive matrix-valued functions is given by

$$\mathcal{R} = \mathcal{R}(\mathbb{T}) = \mathcal{R}(\mathbb{T}, \mathbb{R}^{m \times n}). \tag{2.19}$$

The system (2.18) is said to be regressive provided that $A \in \mathcal{R}$. Before we consider the solution to an initial value problem for (2.18), we offer the existence and uniqueness theorem as follows.

Theorem 2.37. *[11, Theorem 5.8] Let $A \in \mathcal{R}$ be an $n \times n$ matrix-valued function defined on $\mathbb{T}$. Suppose that $f : \mathbb{T} \to \mathbb{R}^n$, $t_0 \in \mathbb{T}$, and $x_0 \in \mathbb{R}$. Then the initial value problem*

$$x^{\Delta}(t) = A(t)x(t) + f(t), \quad x(t_0) = x_0 \tag{2.20}$$

has a unique solution $z : \mathbb{T} \to \mathbb{R}^n$.

Next, we introduce two special operations we will use with the matrix exponential.

Definition 2.38. Let A and B be regressive $n \times n$ matrix-valued functions defined on $\mathbb{T}$. Then we define "circle plus" addition $\oplus$ by

$$(A \oplus B)(t) = A(t) + (I + \mu(t)A(t))B(t) \quad \text{for all} \quad t \in \mathbb{T}^{\kappa}. \tag{2.21}$$

We define the additive inverse $\ominus$ (read "circle minus") by

$$(\ominus)A(t) = -[I + \mu(t)A(t)]^{-1}A(t) \quad \text{for all} \quad t \in \mathbb{T}^\kappa \tag{2.22}$$

$$= -A(t)[I + \mu(t)A(t)]^{-1}. \tag{2.23}$$

Next, we will consider the matrix exponential on the time scale $\mathbb{T}$ and some of its properties.

Definition 2.39. Suppose that A is regressive and rd-continuous. Then the unique $n \times n$ matrix-valued solution to the IVP

$$X^\Delta(t) = A(t)X(t), \quad X(t_0) = I$$

is called the matrix exponential function and denoted is by $e_A(\cdot, t_0)$.

Example 2.40. Assume that A is an $n \times n$ matrix.

a. If $\mathbb{T} = \mathbb{Z}$, then

$$e_A(t, t_0) = \begin{cases} \displaystyle\prod_{\tau=t_0}^{t-1} [I + A(\tau)] & \text{if} \quad A(t) \text{ is never } -I \\ (I + A)^{t-t_0} & \text{if} \quad I + A \text{ is a constant and invertible.} \end{cases}$$

b. If $\mathbb{T} = \mathbb{R}$, then

$$e_A(t, t_0) = \begin{cases} \exp\left\{\displaystyle\int_{t_0}^{t} A(\tau)d\tau\right\} & \text{if } A \text{ is continuous and} \\ & A(s)A(t) = A(t)A(s) \text{ for all } s, t \in \mathbb{T} \\ e^{A(t-t_0)} & \text{if } A(t) \text{ is constant.} \end{cases}$$

c. If $\mathbb{T} = h\mathbb{Z}$, then

$$e_A(t, t_0) = \begin{cases} \displaystyle\prod_{\tau=t_0}^{t/h-1} [I + hA(h\tau)] & \text{if} \quad A(t) \text{ is regressive} \\ (I + hA)^{\frac{t-t_0}{h}} & \text{if} \quad I + hA \text{ is a constant and invertible.} \end{cases}$$

d. If $\mathbb{T} = q^{\mathbb{N}_0}$ for $q > 1$, then

$$e_A(t,1) = \prod_{\tau \in \mathbb{T} \cap (0,t)} [I + (q-1)\tau A(\tau)].$$

Theorem 2.41. *[11, Theorem 5.21] Let $e_A(\cdot, t_0)$ be as in Definition 2.39. Then for $r, s, t \in \mathbb{T}$, we have the following:*

a. $e_A(t,t) = e_0(t,s) \equiv I$.

b. $e_A(\sigma(t), s) = (I + \mu(t)A(t))e_A(t,s)$.

c. $e_A^{-1}(t,s) = e_A(s,t) = e^T_{\ominus A^T}(t,s)$.

d. $e_A(t,s)e_A(s,r) = e_A(t,r)$.

Next we will find the solution (state response) to our linear systems using variation of parameters.

Theorem 2.42. *[11, Theorem 5.24] Let $A \in \mathcal{R}$ be an $n \times n$ matrix-valued function on $\mathbb{T}$ and suppose that $f : \mathbb{T} \to \mathbb{R}^n$ is rd-continuous. Let $t_0 \in \mathbb{T}$ and $x_0 \in \mathbb{R}^n$. Then the solution of the initial value problem*

$$x^\Delta(t) = A(t)x(t) + f(t), \quad x(t_0) = x_0$$

is given by

$$x(t) = e_A(t,t_0)x_0 + \int_{t_0}^{t} e_A(t, \sigma(\tau))f(\tau)\Delta\tau.$$

Theorem 2.43. *[11, Theorem 5.27] Let $A \in \mathcal{R}$ be an $n \times n$ matrix-valued function on $\mathbb{T}$ and suppose that $f : \mathbb{T} \to \mathbb{R}^n$ is rd-continuous. Then for $t_0 \in \mathbb{T}$, $x_0 \in \mathbb{R}^n$, the solution of the initial value problem*

$$x^\Delta(t) = -A(t)x^\sigma(t) + f(t), \quad x(t_0) = x_0$$

is given by

$$\begin{aligned} x(t) &= e_{\ominus A}(t,t_0)x_0 + \int_{t_0}^{t} e_{\ominus A}(t,\tau)f(\tau)\Delta\tau \\ &= e_{A^T}^T(t_0,t)\left[x_0 + \int_{t_0}^{t} e_{A^T}^T(\tau,t_0)f(\tau)\Delta\tau\right]. \end{aligned}$$

2.6. LYAPUNOV AND RICCATI EQUATIONS

In much of the later sections, we will use the notion of Lyapunov and Riccati equations on time scales to obtain our results.

Definition 2.44. A square matrix-valued function A is said to *symmetric* if it is equal to its transpose, i.e. $A = A^T$.

Definition 2.45. A symmetric matrix-valued function A is said to *positive definite* (denoted $A > 0$) if $x^T Ax > 0$ for any nonzero vector x. A symmetric matrix-valued function A is said to *positive semi-definite* (denoted $A \geq 0$) if $x^T Ax \geq 0$ for any nonzero vector x.

In the next lemma, we consider a Lyapunov function on time scales associated with the autonomous dynamic equation

$$x^\Delta(t) = Ax(t). \tag{2.24}$$

Definition 2.46. Let $S \in C^1_{\mathrm{rd}}(\mathbb{T}, R^{n\times n})$ be symmetric. A generalized Lyapunov function is given by

$$x^T(t)S(t)x(t). \tag{2.25}$$

Lemma 2.47. *The derivative of the generalized Lyapunov function is given by*

$$\begin{aligned} (x^T Sx)^\Delta(t) &= x^T(t)[A^T S(t) + (I+\mu(t)A^T)S(t)A \\ &\quad +(I+\mu(t)A^T)S^\Delta(t)(I+\mu(t)A)]x(t). \end{aligned} \tag{2.26}$$

Proof. Using the product rule, we have

$$\begin{aligned}(x^TSx)^\Delta(t) &= (x^TS)^\Delta(t)x^\sigma(t) + (x^TS)(t)x^\Delta(t)\\ &= [(x^T)^\Delta(t)S(t) + (x^T)^\sigma(t)S^\Delta(t)]x^\sigma(t) + (x^TS)(t)Ax(t).\end{aligned}$$

Now using the simple useful formula, we have

$$\begin{aligned}(x^TSx)^\Delta(t) &= [(x^T(t)A^TS(t) + (x+\mu x^\Delta)^T(t)S^\Delta(t)](x+\mu x^\Delta)(t) + (x^TS)(t)Ax(t)\\ &= [(x^T(t)A^TS(t) + x^T(t)(I+\mu(t)A)^TS^\Delta(t)](I+\mu(t)A)x(t)\\ &\quad +(x^TS)(t)Ax(t)\\ &= x^T[A^TS(t)(I+\mu(t)A) + (I+\mu(t)A)^TS^\Delta(t)(I+\mu(t)A)]x(t)\\ &\quad +(x^TS)(t)Ax(t)\\ &= x^T(t)[A^TS(t) + (I+\mu(t)A^T)S(t)A\\ &\quad +(I+\mu(t)A^T)S^\Delta(t)(I+\mu(t)A)]x(t).\end{aligned}$$

This gives the result as desired. □

While Lyapunov functions are often used to establish a stability criterion (see DaCunha's dissertation [19]), we will use them to find an optimal cost in Sections 4 and 5. Next, we consider two forms of the matrix Riccati equation on time scales.

Definition 2.48. Let X be positive definite and let A, B, C be constant matrices. Then a Riccati equation of the first form is given by

$$\begin{aligned}X^\Delta(t) &= C + AX(t) + (I+\mu(t)A)X(t)A^T\\ &\quad -(I+\mu(t)A)X(t)B^T(C+\mu(t)BX(t)B^T)^{-1}BX(t)(I+\mu(t)A^T)\end{aligned} \tag{2.27}$$

and a Riccati equation of the second form is given by

$$-X^\Delta(t) = C + AX^\sigma(t) + (I+\mu(t)A)X^\sigma(t)A^T$$

$$-(I+\mu(t)A)X^{\sigma}(t)B^{T}(C+\mu(t)BX^{\sigma}(t)B^{T})^{-1}BX^{\sigma}(t)(I+\mu(t)A^{T}). \tag{2.28}$$

It should be noted that we will derive both forms using widely different methods in Sections 4 and 6. However, we will use the solutions of both forms in the same manner. The solution to the first form will be used to determine the gain for the Kalman filter. Similarly, the solution of the second form will be used to determine the gain for the linear quadratic regulator.

3. CONTROLLABILITY AND OBSERVABILITY

3.1. INTRODUCTION

In the early 1960s, R.E. Kalman introduced two concepts that have since become the backbone of modern control theory (see [31, 33, 34, 36]). With "controllability" and "observability," one can classify a control system without first finding the solution in closed form. A linear system is said to be controllable if there exists at least one input that drives the state vector to the origin. On the other hand, a linear system is said to be observable if there exists at least one output such that the initial state can be determined. These properties have been studied in depth in both the continuous and discrete cases, where one can see striking similar, if not identical, results. Yet until recently there did not exist a method to relate these results in one case with the results in the other.

The purpose of this section is to lay down the foundation of linear control systems on time scales. Here we examine controllability and observability in both time-invariant and time-dependent cases. It should be noted that there have been other excellent attempts to do so, e.g., in [6, 7, 24]. Both examine the linear system

$$\begin{aligned} x^{\Delta}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t), \end{aligned} \tag{3.1}$$

in an effort to generalize controllability and observability for dynamic equations. At first, this seems to be a very natural extension from the continuous and discrete cases. However, as one examines the rank condition for controllability in the classical sense, one must assume that the graininess function is differentiable, an assumption that is not satisfied in general for all time scales (see [11, Example 1.56]). To side step this issue, we have altered the linear system so that it appears as

$$\begin{aligned} x^{\Delta}(t) &= -A(t)x^{\sigma}(t) + B(t)u(t) \\ y(t) &= C(t)x^{\sigma}(t). \end{aligned}$$

The motivation for modifying the system follows from the generalized variation of parameters formula. However, just as with the other system, one must assume that the graininess function is differentiable when one examines the rank condition for observability in the classical means. Instead, we will examine the controllability of the above and the observability portion of (3.1) and then draw a parallel between both systems. As a result, we see that this connection of controllability and observability for linear systems on time scales is more compelling than previously realized. Also, the proofs of this thesis follow a similar pattern as the proofs found for the continuous and discrete cases.

It should be noted, however, that in later sections we will restrict ourselves to the system (3.1). Now we will examine controllability and observability of linear systems on time scales.

3.2. THE TIME-INVARIANT CASE FOR THE ADJOINT EQUATION

Now let us consider the general state space representation of a linear dynamic system to be

$$\begin{aligned} x^{\Delta}(t) &= -Ax^{\sigma}(t) + Bu(t) \\ y(t) &= Cx^{\sigma}(t), \end{aligned} \tag{3.2}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the input (control), and $y \in \mathbb{R}^r$ is the output. Here u is assumed to be rd-continuous. Note that A, B, and C are real-valued matrices of dimensions $n \times n$, $n \times m$, and $r \times n$ respectively. A is assumed to be regressive. C is assumed to be of rank n. In Subsection 3.4, we will consider $A(t)$, $B(t)$, and $C(t)$ for (3.2) instead of constant matrices.

Note that for the controllability portion of this work, we will look solely at the matrices A and B while we will examine the matrices A and C for the observability portion.

3.2.1. Controllability and Reachability. Throughout our discussion, we will make the assumptions that $t_0, t_f \in \mathbb{T}$ and $t_f > \sigma^n(t_0)$. As controllability does not rely on the output equation, we will only consider the state equation

$$x^{\Delta}(t) = -Ax^{\sigma}(t) + Bu(t) \tag{3.3}$$

in this section. When we refer to a linear system being "controllable," we mean there exist inputs such that the state vector "can be driven" to the origin for any given initial condition. If all of the states of the linear system are controllable, then we have the following definition for complete controllability (see [15]).

Definition 3.1. The state equation (3.3) is said to be *completely controllable* on $[t_0, t_f]$ if for all $x_0 \in \mathbb{R}^n$, there exists u such that the solution x of (3.3) with $x(t_0) = x_0$ satisfies $x(t_f) = 0$.

Next we give the generalized controllability criterion as follows.

Theorem 3.2. *The state equation* (3.3) *is completely controllable if and only if the controllability Gramian* $W_C[t_0, t_f]$ *is invertible where*

$$W_C[t_0, t_f] := \int_{t_0}^{t_f} e_{A^T}^T(\tau, t_0) B B^T e_{A^T}(\tau, t_0) \Delta\tau.$$

Proof. First assume that (3.3) is completely controllable and let $\alpha \in \operatorname{Ker} W_C[t_0, t_f]$. Then

$$\begin{aligned} 0 &= \alpha^T W_C[t_0, t_f] \alpha \\ &= \int_{t_0}^{t_f} \alpha^T e_{A^T}^T(\tau, t_0) B B^T e_{A^T}(\tau, t_0) \alpha \Delta\tau \\ &= \int_{t_0}^{t_f} \left\| B^T e_{A^T}(\tau, t_0) \alpha \right\|^2 \Delta\tau, \end{aligned}$$

which implies $B^T e_{A^T}(\tau, t_0)\alpha = 0$ for all $\tau \in [t_0, t_f) \cap \mathbb{T}$. Then there exists a u such that the solution x of (3.3) with $x(t_0) = \alpha$ satisfies $x(t_f) = 0$. It follows from Theorem 2.43 that

$$\alpha = -\int_{t_0}^{t_f} e_{A^T}^T(\tau, t_0) B u(\tau) \Delta\tau.$$

Then

$$\|\alpha\|^2 = \alpha^T\alpha = -\int_{t_0}^{t_f} u^T(\tau)B^T e_{A^T}(\tau, t_0)\alpha\Delta\tau = 0,$$

which implies $\alpha = 0$. Hence $\operatorname{Ker} W_C[t_0, t_f] = \{0\}$. Therefore $W_C[t_0, t_f]$ is invertible.

Now assume that $W_C[t_0, t_f]$ is invertible and let $x_0 \in \mathbb{R}^n$. Define u by

$$u(t) = -B^T e_{A^T}(t, t_0)W_C^{-1}[t_0, t_f]x_0.$$

Then by Theorem 2.43, the solution x of (3.3) with $x(t_0) = x_0$ satisfies

$$\begin{aligned} x(t_f) &= e_{A^T}^T(t_f, t_0)\left[x_0 + \int_{t_0}^{t_f} e_{A^T}^T(\tau, t_0)Bu(\tau)\Delta\tau\right] \\ &= e_{A^T}^T(t_0, t_f)\left[x_0 - \int_{t_0}^{t_f} e_{A^T}^T(\tau, t_0)BB^T e_{A^T}(\tau, t_0)W_C^{-1}[t_0, t_f]x_0\Delta\tau\right] \\ &= e_{A^T}^T(t_0, t_f)\left[x_0 - W_C[t_0, t_f]W_C^{-1}[t_0, t_f]x_0\right] \\ &= 0, \end{aligned}$$

which tells us that the linear system is completely controllable. □

Now we will look at the generalized Kalman rank condition for controllability of linear systems on time scales.

Theorem 3.3. *The state equation* (3.3) *is completely controllable if and only if the* $n \times (nm)$ *controllability matrix* $\Gamma_C[A, B]$ *has full rank* n*, where*

$$\Gamma_C[A, B] := [\ B \quad AB \quad A^2B \quad \cdots \quad A^{n-1}B\ ].$$

Proof. First assume that (3.3) is completely controllable. Let $x_0 \in \mathbb{R}^n$. Then there exists u such that the solution of x of (3.3) with $x(t_0) = x_0$ satisfies $x(t_f) = 0$. It follows from Theorem 2.43 that

$$x_0 = -\int_{t_0}^{t_f} e_{A^T}^T(\tau, t_0)Bu(\tau)\Delta\tau. \tag{3.4}$$

Now by DaCunha [20] or Zafer [56] (using the Cayley–Hamilton theorem), there exist r_j such that

$$e_{A^T}(t,t_0) = \sum_{j=0}^{n-1} r_j(t,t_0)(A^T)^j. \tag{3.5}$$

Next, we define H_j by

$$H_j = -\int_{t_0}^{t_f} r_j(\tau,t_0)u(\tau)\Delta\tau.$$

Now substituting H_j and (3.5) into (3.4), we have

$$\begin{aligned}
x_0 &= -\int_{t_0}^{t_f} e_{A^T}^T(\tau,t_0)Bu(\tau)\Delta\tau \\
&= -\int_{t_0}^{t_f} \left[\sum_{j=0}^{n-1} r_j(\tau,t_0)A^j\right] Bu(\tau)\Delta\tau \\
&= \sum_{j=0}^{n-1} A^j B H_j \\
&= \Gamma_C[A,B]\begin{bmatrix} H_0 \\ H_1 \\ \vdots \\ H_{n-1}\end{bmatrix} \in \operatorname{Im}\Gamma_C[A,B].
\end{aligned}$$

Then $\mathbb{R}^n \subset \operatorname{Im}\Gamma_C[A,B] \subset \mathbb{R}^n$ and thus $\operatorname{rank}\Gamma_C[A,B]=n$.

Now assume that $\operatorname{rank}\Gamma_C[A,B]=n$ and let $\alpha \in \operatorname{Ker} W_C[t_0,t_f]$. Then

$$\begin{aligned}
0 &= \alpha^T W_C[t_0,t_f]\alpha \\
&= \int_{t_0}^{t_f} \alpha^T e_{A^T}^T(\tau,t_0)BB^T e_{A^T}(\tau,t_0)\alpha\Delta\tau \\
&= \int_{t_0}^{t_f} \left\|B^T e_{A^T}(\tau,t_0)\alpha\right\|^2 \Delta\tau,
\end{aligned}$$

which implies

$$B^T e_{A^T}(\tau,t_0)\alpha = 0 \quad \text{for all} \quad \tau \in [t_0,t_f)\cap\mathbb{T}. \tag{3.6}$$

Now differentiating (3.6) m times, where $0 \leq m \leq n-1$, we have

$$B^T (A^m)^T e_{A^T}(\tau, t_0)\alpha = 0 \quad \text{for all} \quad \tau \in [t_0, t_f)^{\kappa^m},$$

hence

$$(A^m B)^T e_{A^T}(\tau, t_0)\alpha = 0 \quad \text{for all} \quad \tau \in [t_0, t_f)^{\kappa^m}.$$

Then picking $\tau = t_0 \in [t_0, t_f)^{\kappa^m}$ for all $0 \leq m \leq n-1$ (since $t_f > \sigma^n(t_0)$) and using Theorem 2.41 part (a), we have

$$(A^m B)^T \alpha = 0 \quad \text{for all} \quad 0 \leq m \leq n-1,$$

which implies $\Gamma_C^T[A, B]\alpha = 0$. Then $\alpha \in \operatorname{Ker} \Gamma_C^T[A, B] \neq \{0\}$. Hence $\operatorname{Ker} W_C[t_0, t_f] = \{0\}$ and thus $W_C[t_0, t_f]$ is invertible. Then by Theorem 3.2, (3.3) is completely controllable. $\square$

Next, we introduce a similar concept to controllability: reachability.

Definition 3.4. The state equation (3.3) is said to be *completely reachable* on $[t_0, t_f]$ if for all $x_f \in \mathbb{R}^n$, there exists u such that the solution x of (3.3) with $x(t_0) = 0$ satisfies $x(t_f) = x_f$.

Now we will give the relationship between controllability and reachability in the following theorem.

Theorem 3.5. *The state equation* (3.3) *is controllable if and only if it is reachable.*

Proof. First assume that (3.3) is controllable. Let $x_f \in \mathbb{R}^n$. Then there exists u such that the solution x of (3.3) with $x(t_0) = -e_{A^T}^T(t_f, t_0)x_f$ satisfies $x(t_f) = 0$. Then by Theorem 2.43, we have

$$\begin{aligned} 0 &= x(t_f) \\ &= e_{A^T}^T(t_0, t_f)\left[-e_{A^T}^T(t_f, t_0)x_f + \int_{t_0}^{t_f} e_{A^T}^T(\tau, t_0)Bu(\tau)\Delta\tau\right] \end{aligned}$$

$$= \quad -x_f + \int_{t_0}^{t_f} e_{A^T}^T(\tau, t_f) B u(\tau) \Delta\tau,$$

which implies that the solution $\tilde{x}$ of (3.3) with $\tilde{x}(t_0) = 0$ satisfies

$$\tilde{x}(t_f) = \int_{t_0}^{t_f} e_{A^T}^T(\tau, t_f) B u(\tau) \Delta\tau = x_f.$$

Therefore (3.3) is reachable.

Now assume that (3.3) is reachable. Let $x_0 \in \mathbb{R}^n$. Then there exists an input u such that the solution x of (3.3) with $x(t_0) = 0$ satisfies $x(t_f) = -e_{A^T}^T(t_f, t_0)x_0$. Then by Theorem 2.43, we have

$$-e_{A^T}^T(t_f, t_0)x_0 = x(t_f) = \int_{t_0}^{t_f} e_{A^T}^T(\tau, t_f) B u(\tau) \Delta\tau,$$

which implies that the solution $\tilde{x}$ of (3.3) with $\tilde{x}(t_f) = 0$ satisfies

$$\begin{aligned}
0 &= \tilde{x}(t_0) \\
&= e_{A^T}^T(t_0, t_0) \left[\tilde{x}(t_0) + \int_{t_0}^{t_f} e_{A^T}^T(\tau, t_0) B u(\tau) \Delta\tau \right] \\
&= -\int_{t_0}^{t_f} e_{A^T}^T(\tau, t_0) B u(\tau) \Delta\tau \\
&= x_0,
\end{aligned}$$

which implies (3.3) is controllable. $\square$

Remark 3.6. In the continuous case, controllability and reachability are always equivalent. This is not necessarily true in the discrete case. In fact, reachability is considered stronger condition than controllability. However, Theorem 3.5 holds since A is always assumed to be regressive. As a result, we will use these terms interchangeably.

Theorem 3.7. *The state equation* (3.3) *is completely reachable if and only if the reachability Gramian $W_R[t_0, t_f]$ is invertible, where*

$$W_R[t_0, t_f] := \int_{t_0}^{t_f} e_{A^T}^T(\tau, t_f) B B^T e_{A^T}(\tau, t_f) \Delta\tau.$$

Proof. Note that

$$W_R[t_0,t_f] = e^T_{A^T}(t_0,t_f)W_c[t_0,t_f]e_{A^T}(t_0,t_f),$$

which implies that $W_R[t_0,t_f]$ and $W_C[t_0,t_f]$ are congruent. Then by Sylvester's law of inertia, $W_R[t_0,t_f]$ is invertible if and only if $W_C[t_0,t_f]$ is invertible. Then the statement follows from Theorem 3.2. □

Next we revisit the generalized Kalman rank condition on time scales for reachability. The proof is omitted as it again relies on Sylvester's law of inertia as found in Theorem 3.7. Then using Theorem 3.3, we have the following result.

Theorem 3.8. *The state equation* (3.3) *is completely reachable if and only if the* $n\times(nm)$ *controllability matrix* $\Gamma_C[A,B]$ *has full rank* n*.*

3.2.2. Observability. We refer to a linear system being "observable" if given the output y and input u, we can find our initial condition x_0. If this is true regardless of the initial time and initial state, we have the following definition for complete observability.

Definition 3.9. The linear system (3.2) is said to be *completely observable* on $[t_0,t_f]$ where, if for any $x(t_0)$ and a known u, $x(t_0)$ can be uniquely determined by $y(t)$.

On the other hand, the linear system (3.2) is said to be unobservable if given $x(t_0) = x_0$ and the input $u(t) = 0$, there exists a finite time t_f such that $y(t) = 0$ for all $t \in [t_0,t_f)$.

Theorem 3.10. *The linear system* (3.2) *is completely observable if and only if the observability Gramian* $W_O[t_0,t_f]$ *is invertible, where*

$$W_O[t_0,t_f] := \int_{t_0}^{t_f} e^T_{\ominus A}(\sigma(\tau),t_0)C^TCe_{\ominus A}(\sigma(\tau),t_0)\Delta\tau. \tag{3.7}$$

Proof. Assume (3.1) is completely observable and let $\alpha \in \operatorname{Ker} W_O[t_0,t_f]$. Then

$$\begin{aligned} 0 &= \alpha^T W_O[t_0,t_f]\alpha \\ &= \int_{t_0}^{t_f} \alpha^T e^T_{\ominus A}(\sigma(\tau),t_0)C^TCe_{\ominus A}(\sigma(\tau),t_0)\alpha\Delta\tau \end{aligned}$$

$$= \int_{t_0}^{t_f} \|Ce_{\ominus A}(\sigma(\tau), t_0)\alpha\|^2 \Delta\tau,$$

which implies $Ce_{\ominus A}(\sigma(\tau), t_0)\alpha = 0$ for all $t \in [t_0, t_f) \cap \mathbb{T}$. Now picking $x_0 = \alpha$ and $u = 0$, we have $y = 0$ for all $t \in [t_0, t_f) \cap \mathbb{T}$. This implies that (3.1) is not completely observable, a contradiction.

Now suppose that $W_O^{-1}[t_0, t_f]$ exists. Then for any output y, we have

$$\begin{aligned} W_O[t_0, t_f]x_0 &= \int_{t_0}^{t_f} e_{\ominus A}^T(\sigma(\tau), t_0)C^T C e_{\ominus A}(\sigma(\tau), t_0)x_0 \Delta\tau \\ &= \int_{t_0}^{t_f} e_{\ominus A}^T(\sigma(\tau), t_0)C^T y(\tau)\Delta\tau, \end{aligned}$$

which implies $x_0 = W_O^{-1}[t_0, t_f] \int_{t_0}^{t_f} e_{\ominus A}^T(\sigma(\tau), t_0)C^T y(\tau)\Delta\tau$. Therefore the system is completely observable. $\square$

Next, we will construct the generalized Kalman rank condition for observability of linear systems on time scales. However the Gramian is not particularly useful in establishing this result. Instead, we will appeal to the simple useful formula.

Theorem 3.11. *The linear system* (3.2) *is completely observable if and only if the* $(nr) \times n$ *observability matrix* $\Gamma_O[A, C]$ *has full rank* n*, where*

$$\Gamma_O[A, C] := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}.$$

Proof. First assume $B = 0$ since the input does not affect observability and that $y(t) = 0$ for all $t \geq 0$ while $\operatorname{rank} \Gamma_O[A, C] = n$. We will further assume that $\mu(t) \neq 0$ for all $t \geq 0$.

Then using the simple useful formula $n-1$ times, the output equation can rewritten as

$$\begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & -\mu(t) & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & (-\mu(t))^{n-1} \end{bmatrix} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} x(t) = 0, \tag{3.8}$$

where $x(t) = e_{\ominus A}(t, t_0)x_0$. Note that equation (3.8) then implies $x_0 = 0$. Thus (3.2) is completely observable.

Next, suppose that $\operatorname{rank}\Gamma_O[A, C] < n$. Then there exists a vector $\alpha \in \mathbb{R}^n \setminus \{0\}$ such that $\Gamma_O[A, C]\alpha = 0$. This is equivalent to the system equation

$$\begin{bmatrix} CP_0 \\ CP_1 \\ \vdots \\ CP_{n-1} \end{bmatrix} \alpha = 0,$$

or $CP_0\alpha = 0, \dots, CP_{n-1}\alpha = 0$ for $t \geq t_0$. This implies that $y(t) = Ce_{\ominus A}(t, t_0)\alpha = 0$. So for $x_0 = \alpha$, the linear system (3.2) is not observable. □

3.3. THE TIME-INVARIANT CASE FOR THE CLASSIC EQUATION

3.3.1. Controllability. For the sake of completeness, we will include the study of controllability of the state equation

$$x^{\Delta}(t) = Ax(t) + Bu(t). \tag{3.9}$$

First, we give a similar controllability criterion as the one for the adjoint equation.

Theorem 3.12. *The state equation* (3.3) *is completely controllable if and only if the controllability Gramian* $W_C[t_0, t_f]$ *is invertible where*

$$W_C[t_0, t_f] := \int_{t_0}^{t_f} e_A(t_0, \sigma(\tau))BB^T e_A^T(t_0, \sigma(\tau))\Delta\tau. \tag{3.10}$$

While we will appeal to the Gramian in later sections, it is not particularly useful in deriving the rank condition in this case. This is due to the way the matrix exponential here is defined. We will use the argument first presented by Davis, Gravagne, Jackson, Marks, and Ramos in [21, 30]. This argument is made possible using the Laplace transform and convolution on time scales. In the next lemma, we define the input as a delay function.

Lemma 3.13. *[30, Lemma 3.1] Let $u = u_{x_0}(t_f, \sigma(\tau))$ be any rd-continuous function, then*

$$\operatorname{span}\left\{\int_{t_0}^{t_f} e_A(\tau, t_0) B u_{x_0}(t_f, \sigma(\tau))\Delta\tau\right\} = \operatorname{span}\{B, AB, A^2B, \cdots, A^{n-1}B\}$$

Proof. The proof can found in Jackson's dissertation [30]. □

Theorem 3.14. *[30, Theorem 3.2] The state equation* (3.9) *is completely controllable if and only if the $n \times (nm)$ controllability matrix $\Gamma_C[A, B]$ has full rank n, where*

$$\Gamma_C[A, B] := \begin{bmatrix} B & AB & A^2B & \cdots & A^{n-1}B \end{bmatrix}.$$

Proof. The proof can found in Jackson's dissertation [30]. □

Remark 3.15. Note that (3.3) was also completely controllable if and only if the same controllability matrix had full rank! Therefore it can be argued that (3.3) is completely controllable if and only if (3.9) is completely controllable.

3.3.2. Observability. It should be noted that the following theorems in observability follow much the same way as the controllability portion for the adjoint equation, only with a few changes in notation. First we will give the generalized observability criterion as follows.

Theorem 3.16. *The linear system* (3.1) *is completely observable if and only if the observability Gramian $W_O[t_0, t_f]$ is invertible, where*

$$W_O[t_0, t_f] := \int_{t_0}^{t_f} e_A^T(\tau, t_0) C^T C e_A(\tau, t_0)\Delta\tau.$$

Proof. Assume (3.1) is completely observable and let $\alpha \in \operatorname{Ker} W_O[t_0, t_f]$. Then

$$\begin{aligned}
0 &= \alpha^T W_O[t_0, t_f]\alpha \\
&= \int_{t_0}^{t_f} \alpha^T e_A^T(\tau, t_0) C^T C e_A(\tau, t_0)\alpha \Delta\tau \\
&= \int_{t_0}^{t_f} \|C e_A(\tau, t_0)\alpha\|^2 \Delta\tau,
\end{aligned}$$

which implies $Ce_A(\tau, t_0)\alpha = 0$ for all $t \in [t_0, t_f) \cap \mathbb{T}$. Now picking $x_0 = \alpha$ and $u = 0$, we have $y = 0$ for all $t \in [t_0, t_f) \cap \mathbb{T}$. This implies that (3.1) is not completely observable, a contradiction.

Now suppose that $W_O^{-1}[t_0, t_f]$ exists. Then for any output y, we have

$$\begin{aligned}
W_O[t_0, t_f]x_0 &= \int_{t_0}^{t_f} e_A^T(\tau, t_0) C^T C e_A(\tau, t_0) x_0 \Delta\tau \\
&= \int_{t_0}^{t_f} e_A^T(\tau, t_0) C^T y(\tau) \Delta\tau,
\end{aligned}$$

which implies $x_0 = W_O^{-1}[t_0, t_f] \int_{t_0}^{t_f} e_A^T(\tau, t_0) C^T y(\tau)\Delta\tau$. Therefore the system is completely observable. $\square$

Next, we will give the generalized Kalman rank condition for observability of linear systems on time scales.

Theorem 3.17. *The linear system* (3.1) *is completely observable if and only if the* $(nr) \times n$ *observability matrix* $\Gamma_O[A, C]$ *has full rank* n*, where*

$$\Gamma_O[A, C] := \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}.$$

Proof. First assume (3.1) is completely observable. Then for any $x_0 \in \mathbb{R}^n$, we have

$$W_O[t_0, t_f]x_0 = \int_{t_0}^{t_f} e_A^T(\tau, t_0) C^T y(\tau)\Delta\tau.$$

Next define F_j such that

$$F_j = \int_{t_0}^{t_f} r_j(\tau, t_f) y(\tau) \Delta\tau.$$

Now we have

$$\begin{aligned} W_O[t_0, t_f]x_0 &= \int_{t_0}^{t_f} \left[\sum_{j=0}^{n-1} r_j(\tau, t_f)(A^j)^T \right] C^T y(\tau)\Delta\tau \\ &= \sum_{j=0}^{n-1} (CA^j)^T F_j \\ &= \Gamma_O^T[A, C] \begin{bmatrix} F_0 \\ F_1 \\ \vdots \\ F_{n-1} \end{bmatrix}. \end{aligned}$$

Then $W_O[t_0, t_f]x_0 \in \operatorname{Im}\Gamma_O^T[A, C] \subset \mathbb{R}^n$, which then implies that $\operatorname{rank}\Gamma_O^T[A, C] = n$. Finally, $\operatorname{rank}\Gamma_O^T[A, C] = \operatorname{rank}\Gamma_O[A, C]$. Now assume that $\operatorname{rank}\Gamma_O[A, C] = n$, while the system is not completely observable. Using the previous theorem, there exists $\alpha \in \mathbb{R}^n \backslash \{0\}$ such that $W_O[t_0, t_f]\alpha = 0$. Then

$$\begin{aligned} 0 &= \alpha^T W_O[t_0, t_f]\alpha \\ &= \int_{t_0}^{t_f} \alpha^T e_A^T(\tau, t_0) C^T C e_A(\tau, t_0)\alpha\Delta\tau \\ &= \int_{t_0}^{t_f} \|C e_A(\tau, t_0)\alpha\|^2 \Delta\tau, \end{aligned}$$

which implies that $Ce_A(\tau, t_0)\alpha = 0$ for all $\tau \in [t_0, t_f] \cap \mathbb{T}$. Now taking m derivatives, where $0 \leq m \leq n-1$ we have

$$CA^m e_A(\tau, t_0)\alpha = 0 \quad \text{for all} \quad \tau \in [t_0, t_f)^{\kappa^m}.$$

Then picking $\tau = t_0$, we obtain

$$CA^m\alpha = 0 \quad \text{for all} \quad \tau \in [t_0, t_f)^{\kappa^m},$$

which can be rewritten as $\Gamma_O[A, C]\alpha = 0$, and implies $\alpha = 0$. □

Remark 3.18. When examining linear systems in the continuous and discrete case, there can occur what is referred to as a duality between controllability and observability. Next, we wish to see if this relationship is preserved in the generalized case. We will do this by comparing (3.1) with its adjoint. The theorem below can be referred to as the duality principle theorem.

Theorem 3.19. *The linear system* (3.1) *is completely controllable (observable) over* $[t_0, t_f]$ *if and only if the linear system*

$$\begin{aligned} x^{\Delta}(t) &= A^T x(t) + C^T u(t) \\ y(t) &= B^T x(t) \end{aligned} \tag{3.11}$$

is completely observable (controllable).

Proof. Assume the linear system (3.1) is completely controllable. Note that from Theorem 3.14, this is true if and only if

$$\Gamma_C[A, B] = \begin{bmatrix} B & AB & A^2B & \dots & A^{n-1}B \end{bmatrix}$$

has full rank n. Note that this is true if and only if

$$\Gamma_C^T[A, B] = \begin{bmatrix} B^T \\ B^T A^T \\ \vdots \\ B^T (A^{n-1})^T \end{bmatrix}$$

has full rank n, where $\Gamma_C^T[A, B] = \Gamma_O[A^T, B^T]$ is the observability matrix associated with (3.11). □

We will use Theorem 3.19 later in Subsection 3.5. We will also appeal to this result when we compare the linear quadratic regulator with the Kalman filter on time scales in Section 6.

3.4. THE TIME-VARYING CASE

3.4.1. The Adjoint Equation. Now let us consider the general state space representation of a linear dynamic system to be

$$\begin{aligned} x^{\Delta}(t) &= -A(t)x^{\sigma}(t) + B(t)u(t) \\ y(t) &= C(t)x(t), \end{aligned} \tag{3.12}$$

where the assumptions on $A(t)$, $B(t)$, and $C(t)$ are the same as their counterparts in Section 3.2. We will only consider the controllability of this system.

Theorem 3.20. *The linear system* (3.12) *is completely controllable if and only if the controllability Gramian* $W_C[t_0, t_f]$ *is invertible over* $[t_0, t_f]$*, where*

$$W_C[t_0, t_f] = \int_{t_0}^{t_f} e_{A^T}^T(\tau, t_0)B(\tau)B^T(\tau)e_{A^T}(\tau, t_0)\Delta\tau.$$

Proof. The proof is the same as found in [24], only with e_{A^T} in place of Φ. □

Theorem 3.21. *Let* $B(t)$ *be* m *times* Δ*-differentiable and* $A(t)$ *be* $m-1$ *times* Δ*-differentiable. Then the linear system* (3.12) *is completely controllable for* $t_f > \sigma(t_0)$ *if and only if there exist* $K_j(t)$ *that satisfies*

$$\begin{aligned} K_0(t) &= B^T(t) \\ K_j(t) &= K_{j-1}^{\Delta}(t) + K_{j-1}^{\sigma}(t)A^T(t) \end{aligned}$$

such that $\operatorname{rank} Q_c(t) = n$*, where*

$$Q_c(t) = \begin{bmatrix} K_0^T(t) & K_1^T(t) & K_2^T(t) & \dots & K_{n-1}^T(t) \end{bmatrix}.$$

Proof. By definition of $K_j(t)$, $B^T(t)e_{A^T}(t,t_0) = K_0(t)e_{A^T}(t,t_0)$. Then

$$\begin{aligned}
[B^T(t)e_{A^T}(t,t_0)]^\Delta &= (B^T)^\Delta(t)e^\sigma_{A^T}(t,t_0) + B^T(t)e^\Delta_{A^T}(t,t_0) \\
&= [K_0^\Delta(t)(I+\mu(t)A^T(t)) + K_0(t)A^T(t)]e_{A^T}(t,t_0) \\
&= [K_0^\Delta(t) + K_0^\sigma(t)A^T(t)]e_{A^T}(t,t_0) \\
&= K_1(t)e_{A^T}(t,t_0).
\end{aligned}$$

Then by induction, we have $[K_i(t)e_{A^T}(t,t_0)]^\Delta = K_{i+1}(t)e_{A^T}(t,t_0)$. Now assume that $\operatorname{rank} Q_c(t) = n$, while the linear system (3.12) is not completely controllable. Then by the previous theorem, there exists a nonzero vector $\alpha \in \mathbb{R}^n$ such that

$$\begin{aligned}
0 &= \alpha^T W_C[t_0,t_f]\alpha \\
&= \int_{t_0}^{t_f} \alpha^T e^T_{A^T}(\tau,t_0)B(\tau)B^T(\tau)e_{A^T}(\tau,t_0)\alpha\Delta\tau \\
&= \int_{t_0}^{t_f} \left\|B^T(\tau)e_{A^T}(\tau,t_0)\alpha\right\|^2 \Delta\tau,
\end{aligned}$$

which implies $B^T(t)e_{A^T}(t,t_0)\alpha = 0$ for all $t \in [t_0,t_f] \cap \mathbb{T}$. Now differentiating m times where $0 \le m \le n-1$ we have

$$\begin{aligned}
0 &= [B^T(t)e_{A^T}(t,t_0)\alpha]^{\Delta^m} \quad \text{for all} \quad t \in [t_0,t_f)^{\kappa^m} \\
&= K_m(t)e_{A^T}(t,t_0)\alpha,
\end{aligned}$$

hence

$$[K_0^T(\tau) \quad K_1^T(t) \quad K_2^T(t) \quad \ldots \quad K_{n-1}^T(t)]^T e_{A^T}(t,t_0)\alpha = 0.$$

Then picking $t = t_0$ we have

$$Q_c^T(t)\alpha = 0 \quad \text{for all} \quad t \in [t_0,t_f)^{\kappa^m},$$

which implies $\alpha = 0$. Therefore the linear system is completely controllable. Now suppose that the linear system is completely controllable. Define $\Theta(t) = B^T(t)e_{A^T}(t, t_0)$. From [50, 53] we see that by Theorem 3.20 the linear system is completely controllable if and only if the rows of $\Theta(t)$ are linearly independent over $[t_0, t_f]$. Now define the Wronskian, $W(t)$ to be such that

$$\begin{aligned} W(t) &= [\Theta(t) \quad \Theta^{(1)}(t) \quad \Theta^{(2)}(t) \quad \dots \quad \Theta^{(n-1)}(t)] \\ &= [K_0^T(t) \quad K_1^T(t) \quad K_2^T(t) \quad \dots \quad K_{n-1}^T(t)]^T e_{A^T}(t, t_0) \\ &= Q_c^T(t)e_{A^T}(t, t_0). \end{aligned}$$

Since $e_{A^T}(t, t_0)$ is nonsingular, it follows that $\operatorname{rank} W(t) = \operatorname{rank} Q_c(t)$. Since the rows of $W(t)$ are linearly independent, it follows that the $\operatorname{rank} W(t) = n$. Then as a consequence, $Q_c(t)$ will also have full rank. □

3.4.2. The Classic Equation. We will now consider the general time-varying linear system

$$\begin{aligned} x^\Delta(t) &= A(t)x(t) + B(t)u(t) \\ y(t) &= C(t)x(t). \end{aligned} \tag{3.13}$$

Theorem 3.22. *[30, Theorem 3.1] The linear system* (3.13) *is completely controllable if and only if the controllability Gramian $W_C[t_0, t_f]$ is invertible over $[t_0, t_f]$, where*

$$W_C[t_0, t_f] = \int_{t_0}^{t_f} e_A(t_0, \sigma(\tau))B(\tau)B^T(\tau)e_A^T(t_0, \sigma(\tau))\Delta\tau.$$

Proof. The proof can be found in Jackson's dissertation [30]. □

Similar to the adjoint equation, there is a rank condition theorem. However we must assume that μ is sufficiently differentiable.

Theorem 3.23. *[30, Theorem 3.2] Suppose that m is a positive integer such that, for $t \in [t_0, t_f]$, both $A(t)$ and $\mu(t)$ are $(m-1)$-times Δ-differentiable and $B(t)$ is m-times Δ-differentiable. Then* (3.13) *is completely controllable on $[t_0, t_f]$ if for some $t_c \in [t_0, t_f)$,* *we have*

$$\operatorname{rank}\begin{bmatrix} K_0(t_c) & K_1(t_c) & K_2(t_c) & \dots & K_m(t_c) \end{bmatrix} = n$$

where

$$K_j(t) = \frac{\partial}{\Delta s^j}\left[e_A(\sigma(t),\sigma(s))B(s)\right]\Big|_{s=t}, \quad j = 0,1,2...m.$$

Proof. The proof can be found in [30]. □

Next, we introduce a stronger notion of controllability for linear time-varying systems when restrictions are imposed on the Gramian.

Definition 3.24. The linear system (3.13) is said to be *uniformly completely controllable* if there exist positive integers α_0, α_1 such that for any t_0,

$$\alpha_0 I \leq W_C(t_0, t_f) \leq \alpha_1 I. \tag{3.14}$$

Note that the above definition implies that the control that drives the state is roughly independent of the initial time. In the time-invariant case uniformly completely controllable and completely controllable are equivalent. Next, we consider the observability conditions for (3.13).

Theorem 3.25. *[30, Theorem 3.7] The linear system* (3.13) *is completely observable if and only if the observability Gramian $W_O[t_0, t_f]$ is invertible over $[t_0, t_f]$, where*

$$W_O[t_0, t_f] = \int_{t_0}^{t_f} e_A^T(\tau, t_0)C^T(\tau)C(\tau)e_A(\tau, t_0)\Delta\tau.$$

Proof. The proof can be found in Jackson's dissertation [30]. □

Now we consider the observability rank condition for linear time-varying systems.

Theorem 3.26. *[30, Theorem 3.8] Suppose that m is a positive integer such that, for $t \in [t_0, t_f]$, both $A(t)$ and $\mu(t)$ are $(m-1)$-times Δ-differentiable and $C(t)$ is m-times*

Δ-differentiable. Then (3.13) *is completely observable on* $[t_0, t_f]$ *if for some* $t_c \in [t_0, t_f)$, *we have*

$$\text{rank}\begin{bmatrix} L_0(t_c) \\ L_1(t_c) \\ \vdots \\ L_{n-1}(t_c) \end{bmatrix} = n,$$

where

$$Lj(t) = \frac{\partial}{\Delta s^j}\big[C(t)e_A(t,s)\big]\Big|_{s=t}, \quad j = 0, 1, 2...m.$$

Proof. The proof can be found in [30]. □

As before, we now impose restriction on the Gramian.

Definition 3.27. The linear system (3.13) is said to be *uniformly completely observable* if there exist positive integers α_0, α_1 such that for any t_f,

$$\alpha_0 I \leq W_O(t_0, t_f) \leq \alpha_1 I. \tag{3.15}$$

Uniform observability guarantees that the state can be found within roughly the same time. For time-invariant systems the uniform observability and complete observability are the same.

3.5. KALMAN DECOMPOSITIONS

In this subsection, we will examine our canonical forms for linear time invariant systems as seen in Brogan and Kwakernaak and Sivan [15, 40]. We will use these forms in a later section on stability. In the previous sections, we considered the conditions necessary for our system to be completely controllable. Now suppose that system is in fact not completely controllable. Then becomes of great interest to find what part of state can be controlled.

Theorem 3.28. *Consider the state equation $x^{\Delta}(t) = Ax(t) + Bu(t)$, where the controllability matrix $\Gamma_C[A, B]$ has rank $q < n$. Then there exists a completely controllable subsystem*

$$\overline{x}_1^{\Delta}(t) = \overline{A}_{11}\overline{x}_1 + \overline{B}_1 u. \tag{3.16}$$

Proof. Let $T = (T_1, T_2)$ be a nonsingular matrix transformation, where the columns of T_1 represent a basis for the q-dimensional controllable subspace. Next we define the new transformed state by

$$\overline{x}(t) = T^{-1}x(t). \tag{3.17}$$

Then the new state equation can be written as $\overline{x}^{\Delta}(t) = \overline{A}\overline{x}(t) + \overline{B}u(t)$, where

$$\overline{A} = T^{-1}AT = \begin{bmatrix} \overline{A}_{11} & \overline{A}_{12} \\ 0 & \overline{A}_{22} \end{bmatrix} \quad \text{and} \quad \overline{B} = T^{-1}B = \begin{bmatrix} \overline{B}_1 \\ 0 \end{bmatrix}.$$

Note that $\overline{A}_{11}$ is $n \times q$ and $\overline{A}_{22}$ is $(n-q) \times (n-q)$. Now $(\overline{A}_{11}, \overline{B}_1)$ is completely controllable since any state of the form $[x_0, 0]^T$ is in the controllable subspace of the original state equation. □

Remark 3.29. Now if we rewrite $\overline{x}$ as

$$\overline{x} = \begin{bmatrix} \overline{x}_1 \\ \overline{x}_2 \end{bmatrix}, \quad \text{where} \quad \overline{x}_1 \in \mathbb{R}^q \quad \text{and} \quad \overline{x}_2 \in \mathbb{R}^{n-q},$$

then the transformed system can be written as

$$\begin{bmatrix} \overline{x}_1 \\ \overline{x}_2 \end{bmatrix}^{\Delta} = \begin{bmatrix} \overline{A}_{11} & \overline{A}_{12} \\ 0 & \overline{A}_{22} \end{bmatrix} \begin{bmatrix} \overline{x}_1 \\ \overline{x}_2 \end{bmatrix} + \begin{bmatrix} \overline{B}_1 \\ 0 \end{bmatrix} u. \tag{3.18}$$

Note that (3.18) is our original state equation transformed into controllability canonical form.

Next, we will show that controllability is invariant under a nonsingular matrix transformation.

Corollary 3.30. *The linear system* (3.1) *is completely controllable if and only if* (3.18) *is completely controllable.*

Proof. (3.1) is completely controllable if and only if $\Gamma_C[A,B]$ has rank n. Now let T be a nonsingular matrix as described in Theorem 3.28. Then $\Gamma_C[A,B]$ has rank n if and only if

$$[\ T^{-1}B \quad T^{-1}ATT^{-1}B \quad T^{-1}A^2TT^{-1}B \quad \cdots \quad T^{-1}A^{n-1}TT^{-1}B\ ]$$

has rank n. $\square$

Theorem 3.31. *Consider the linear system* (3.1) *where the observability matrix* $\Gamma_O[A,C]$ *has rank* $l<n$*. Then there exists a completely observable subsystem*

$$\begin{aligned} \overline{x}_1^{\Delta}(t) &= \overline{A}_{11}\overline{x}_1(t)+\overline{B}_1u(t) \\ \overline{y}(t) &= \overline{C}_1\overline{x}_1(t). \end{aligned} \tag{3.19}$$

Proof. Let $U=(U_1,U_2)$ be a nonsingular matrix transformation where the rows of U_1 represent a basis for the l-dimensional observable subspace. Now we define the transformed state to be

$$\overline{x}(t)=Ux(t).$$

The transformed system can be rewritten as

$$\begin{aligned} \overline{x}^{\Delta}(t) &= \overline{A}\overline{x}(t)+\overline{B}u(t) \\ \overline{y}(t) &= \overline{C}\overline{x}(t), \end{aligned} \tag{3.20}$$

where

$$\overline{A}=UAU^{-1}=\begin{bmatrix}\overline{A}_{11} & 0\\ \overline{A}_{21} & \overline{A}_{22}\end{bmatrix}, \overline{B}=UB=\begin{bmatrix}\overline{B}_1\\ \overline{B}_2\end{bmatrix}, \text{and } \overline{C}=CU^{-1}=\begin{bmatrix}\overline{C}_1 & 0\end{bmatrix}.$$

Note that $\overline{A}_{11}$ is $l \times l$ and the pair $(\overline{A}_{11}, \overline{C}_1)$ is completely observable. This follows from the fact that if $\overline{x}(t_0)$ produces a zero input response identical to zero, it must be of the form $\overline{x}(t_0) = [0, \overline{x}_2(t_0)]^T$. $\square$

Remark 3.32. Now if we rewrite $\overline{x}$ as

$$\overline{x} = \begin{bmatrix} \overline{x}_1 \\ \overline{x}_2 \end{bmatrix}, \quad \text{where} \quad \overline{x}_1 \in \mathbb{R}^l \quad \text{and} \quad \overline{x}_2 \in \mathbb{R}^{n-l},$$

then the transformed system can be written as

$$\begin{aligned} \begin{bmatrix} \overline{x}_1 \\ \overline{x}_2 \end{bmatrix}^{\Delta} &= \begin{bmatrix} \overline{A}_{11} & 0 \\ \overline{A}_{21} & \overline{A}_{22} \end{bmatrix} \begin{bmatrix} \overline{x}_1 \\ \overline{x}_2 \end{bmatrix} + \begin{bmatrix} \overline{B}_1 \\ \overline{B}_2 \end{bmatrix} u \\ \overline{y} &= \begin{bmatrix} \overline{C}_1 & 0 \end{bmatrix} \overline{x}. \end{aligned} \tag{3.21}$$

Note that (3.21) is the original system (3.1) transformed into observability canonical form.

Finally, our last corollary states that observability is invariant under a nonsingular matrix transformation.

Corollary 3.33. *The linear system* (3.1) *is completely observable if and only if* (3.21) *is completely observable.*

Proof. This is true by Theorem 3.19. $\square$

4. OPTIMIZATION OF LINEAR SYSTEMS ON TIME SCALES

In this section, we will extend and unify the optimal control problem for systems on time scales. In Subsection 4.2, we will derive the optimal control for general systems. Next, we turn our attention to the special case of linear systems associated with a quadratic cost function in Subsection 4.3. Then in Subsection 4.4 we will consider the optimal regulator problem in three parts. First, we will consider the form of our cost function in the absence of an input. Second, we find an optimal control when the final state is fixed, resulting in an open-loop control. Finally, we will consider a free final state, resulting in a closed-loop control.

To understand the difference between these control schemes, consider the following example. Suppose that we want to heat a room through a heating vent to a desired temperature. The room and vent describe a control system (plant) we wish to control. This desired temperature can be thought of as a reference value. If the input heats the room without regard to the current temperature (state) of the room, it is called an open-loop control. This control system will run uninterrupted unless some disturbance is introduced. While such control scheme is simple to set up, it is not self-correcting. Suppose we now include a sensor in the room that reads the current temperature of the room. If the control that now heats the room is based on the reading of the current temperature, we say we have a closed-loop system. In this case, the reading of the current temperature describes what is called state feedback. In this set up, the heating vent can be turned on and off as necessary.

In order to derive an optimal control law needed to minimize the quadratic performance index, we must first describe the notion of calculus of variations on time scales.

4.1. CALCULUS OF VARIATIONS ON TIME SCALES

In this subsection, we will consider a generalization of calculus. Variational theory is concerned with the existence and determination for which a given function has a stationary value, namely a minimum or maximum. Bohner introduced the following variational problem in [10]

$$\mathcal{L}(y) = \int_a^b L(t, y^\sigma(t), y^\Delta(t))\Delta t \to \min, \quad y(a) = \alpha, \quad y(b) = \beta, \tag{4.1}$$

where $a, b \in \mathbb{T}$ with $a < b$; $\alpha, \beta \in \mathbb{R}^n$ and $L : \mathbb{T} \times \mathbb{R}^{2n} \to \mathbb{R}$.

Definition 4.1. For $f \in C^1_{\mathrm{rd}}$, we define the norm

$$||f|| = \max_{t\in[a,b]^\kappa} |f^\sigma(t)| + \max_{t\in[a,b]^\kappa} |f^\Delta(t)|. \tag{4.2}$$

A function $\hat{y} \in C^1_{\mathrm{rd}}$ is said to be a (*weak*) *local minimum* of (4.1) provided that there exists $\delta > 0$ such that $||y - \hat{y}|| < \delta$ and $\mathcal{L}(\hat{y}) \leq \mathcal{L}(y)$ for all $y \in C^1_{rd}$ satisfying the given boundary conditions. Next $\eta \in C^1_{\mathrm{rd}}$ is said to be an *admissible variation* if $\eta(a) = \eta(b) = 0$.

Next, we consider a generalized definition of the first and second derivatives.

Definition 4.2. Let η be an admissible function. We define the function $\Phi : \mathbb{R} \to \mathbb{R}$ such that

$$\Phi(\epsilon) = \Phi(\epsilon; y, \eta) = \mathcal{L}(y + \epsilon\eta), \quad \epsilon \in \mathbb{R}. \tag{4.3}$$

Then the first variation of the variational problem (4.1) is defined by

$$\mathcal{L}_1(y, \eta) = \dot{\Phi}(0; y, \eta). \tag{4.4}$$

Similarly, the second variation of the variational problem (4.1) is defined by

$$\mathcal{L}_2(y, \eta) = \ddot{\Phi}(0; y, \eta). \tag{4.5}$$

Finally, we consider a generalized form of Dubois–Reymond's theorem on time scales.

Theorem 4.3. *[10, Lemma 4.1] Let $g \in C_{\mathrm{rd}}$, where $g : [a, b] \to \mathbb{R}^n$. Then*

$$\int_a^b g^T(\tau)\eta^\Delta(\tau)\Delta\tau = 0 \textit{ for all } \eta \in C^1_{\mathrm{rd}} \textit{ with } \eta(a) = \eta(b) = 0$$

holds if and only if

$$g(t) \equiv c \text{ on } [a,b]^\kappa \text{ for some } c \in \mathbb{R}^n.$$

4.2. SOLUTION TO THE GENERAL OPTIMIZATION PROBLEM

In this subsection, we will derive the control law for a state equation (plant) associated with a general cost function. Assume that the plant can be described on the interval $[t_0, t_f]$ by the time-varying equation (see [44])

$$x^\Delta(t) = f(t, x(t), u(t)), \tag{4.6}$$

where x describes the state and u represents the input. For the given plant, we shall consider the cost function

$$J(t_0) = \phi(x(t_f), t_f) + \int_{t_0}^{t_f} L(t, x(t), u(t))\Delta t. \tag{4.7}$$

Note that $L(t, x, u)$ represents the weighting function which depends on the state and input for intercessory times in $[t_0, t_f]$ while the final weighting function $\phi(x(t_f), t_f)$ depends only on the final state and time. Now the goal is to find the input u^* that not only drives our plant along a trajectory x^* such that (4.7) will be minimized, but also ensures that

$$\Psi(x(t_f), t_f) = 0 \tag{4.8}$$

for some given function $\Psi \in \mathbb{R}^p$. This can be thought of as some final time constraint. To solve the above optimal control problem, we will introduce the Lagrange multipliers $\lambda \in \mathbb{R}^n$ and $\nu \in \mathbb{R}^p$, associated with (4.6) and (4.8) respectively. Now adding (4.6) and (4.8) to (4.7), we have the augmented cost function

$$\begin{aligned} J^+(x, u, \lambda, t_0) &= \nu^T \Psi(x(t_f), t_f) + \phi(x(t_f), t_f) \\ &\quad + \int_{t_0}^{t_f} [L(\cdot, x, u) - (\lambda^\sigma)^T f(\cdot, x, u) - x^\Delta)](\tau)\Delta\tau \end{aligned}$$

$$= \quad \nu^T \Psi(x(t_f), t_f) + \phi(x(t_f), t_f) + \int_{t_0}^{t_f} [H(\cdot, x, u, \lambda^\sigma) - (\lambda^\sigma)^T x^\Delta](\tau)\Delta\tau, \tag{4.9}$$

where

$$H(t, x, u, \lambda) = L(t, x, u) + \lambda^T f(t, x, u) \tag{4.10}$$

is called the Hamiltonian function. We will refer to λ as the costate since it is related to the state equation through the Hamiltonian.

While the augmented cost (4.9) is essentially the same equation as (4.7), it will be easier to calculate the first variation in this form. Using [10], we introduce $\eta_1, \eta_2, \eta_3 \in C^1_{\mathrm{rd}}$ where η_i is an admissible variation. Next, we define a function $\Phi : \mathbb{R} \to \mathbb{R}$ such that

$$\Phi(\epsilon) = \Phi(\epsilon; x, u, \lambda, \eta_1, \eta_2, \eta_3) = J^+(x + \epsilon\eta_1, u + \epsilon\eta_2, \lambda^\sigma + \epsilon\eta_3^\sigma, t_0), \quad \epsilon \in \mathbb{R}.$$

Calculating $\dot{\Phi}(\epsilon)$, we have

$$\begin{aligned} \dot{\Phi}(\epsilon) \quad = \quad & [\Psi_x^T(x(t_f) + \epsilon\eta_1(t_f), t_f)\nu + \phi_x(x(t_f) + \epsilon\eta_1(t_f), t_f)]^T \eta_1(t_f) \\ & + \int_{t_0}^{t_f} H_x(\cdot, x + \epsilon\eta_1, u + \epsilon\eta_2, \lambda^\sigma + \epsilon\eta_3^\sigma)^T \eta_1(\tau)\Delta\tau \\ & + \int_{t_0}^{t_f} H_u(\cdot, x + \epsilon\eta_1, u + \epsilon\eta_2, \lambda^\sigma + \epsilon\eta_3^\sigma)^T \eta_2(\tau)\Delta\tau \\ & + \int_{t_0}^{t_f} [H_\lambda(\cdot, x + \epsilon\eta_1, u + \epsilon\eta_2, \lambda^\sigma + \epsilon\eta_3^\sigma)^T \eta_3^\sigma \Delta\tau \\ & - \int_{t_0}^{t_f} [(\eta_3^\sigma)^T (x + \epsilon\eta_1)^\Delta - (\lambda^\sigma + \epsilon\eta_3^\sigma)^T \eta_1^\Delta](\tau)\Delta\tau, \end{aligned}$$

provided that the Hamiltonian is differentiable with respect to x, u, and λ and that the derivative can be brought inside the integral. Then the first variation is given by

$$\begin{aligned} \dot{\Phi}(0) \quad = \quad & [\Psi_x^T(x(t_f), t_f)\nu + (\phi_x(x(t_f), t_f)]^T \eta_1(t_f) \\ & + \int_{t_0}^{t_f} [H_x^T(\cdot, x, u, \lambda^\sigma)\eta_1 + H_u^T(\cdot, x, u, \lambda^\sigma)\eta_2 + H_\lambda^T(\cdot, x, u, \lambda^\sigma)\eta_3^\sigma](\tau)\Delta\tau \\ & - \int_{t_0}^{t_f} [(\eta_3^\sigma)^T x^\Delta + (\lambda^\sigma)^T \eta_1^\Delta](\tau)\Delta\tau. \end{aligned}$$

Recall that

$$\int_{t_0}^{t_f} (\lambda^\sigma)^T(\tau)\eta_1^\Delta(\tau)\Delta\tau = \lambda^T(t_f)\eta_1(t_f) - \lambda^T(t_0)\eta_1(t_0) - \int_{t_0}^{t_f} (\lambda^\Delta)^T(\tau)\eta_1(\tau)\Delta\tau. \quad (4.11)$$

Using (4.11), the first variation can be rewritten as

$$\begin{aligned} \dot{\Phi}(0) &= \left[\Psi_x^T(x(t_f),t_f)\nu + \phi_x(x(t_f),t_f) - \lambda(t_f)\right]^T \eta_1(t_f) + \lambda^T(t_0)\eta_1(t_0) \\ &\quad + \int_{t_0}^{t_f} \left[\left(H_x(\cdot,x,u,\lambda^\sigma) + \lambda^\Delta\right)^T \eta_1 + H_u^T(\cdot,x,u,\lambda^\sigma)\eta_2\right](\tau)\Delta\tau \\ &\quad + \int_{t_0}^{t_f} \left[\left(H_\lambda(\cdot,x,u,\lambda^\sigma) - x^\Delta\right)^T \eta_3^\sigma - (\lambda^\sigma)^T\eta_1^\Delta\right](\tau)\Delta\tau. \end{aligned}$$

Now in order for $\dot{\Phi}(0) = 0$, we will set each coefficient of independent increments $\eta_1, \eta_2, \eta_3^\sigma$ equal to zero. This yields the necessary conditions for a minimum of (4.7). First, the state equation (plant) can be found to be

$$x^\Delta(t) = H_\lambda(t,x(t),u(t),\lambda^\sigma(t)) = f(t,x(t),u(t)). \quad (4.12)$$

Similarly, the costate equation can be found using

$$-\lambda^\Delta(t) = H_x(t,x(t),u(t),\lambda^\sigma(t)) = f_x^T(t,x(t),u(t))\lambda^\sigma(t) + L_x(t,x(t),u(t)). \quad (4.13)$$

Finally, the stationary condition is given by

$$0 = H_u(t,x(t),u(t),\lambda^\sigma(t)) = L_u(t,x(t),u(t)) + f_u^T(t,x(t),u(t))\lambda^\sigma(t). \quad (4.14)$$

This final condition yields our control law. Note that our optimal control can be found simply by the knowledge of the costate. For our purposes, we will assume that the initial state, $x(t_0)$ is fixed to some given number x_0. Therefore, our boundary condition will be determined by

$$[\Psi_x^T(x(t_f),t_f)\nu + \phi_x(x(t_f),t_f) - \lambda(t_f)]^T\eta_1(t_f) = 0. \quad (4.15)$$

We will use this second boundary condition later to determine the "sweep method."

4.3. EXAMPLES

Example 4.4. Let $\theta(t)$ represent the temperature of some object. Assume that θ_m represents the temperature of the surrounding medium held constant. Let $u(t)$ be rate of heat supply to the medium. Then the rate of change in the object's temperature may be modeled by the dynamic equation

$$\theta^{\Delta}(t) = a(t)(\theta(t) - \theta_m) + b(t)u(t). \tag{4.16}$$

Suppose that we want to find a minimum input needed to heat the object over the interval $[t_0, t_f]$. First, define the state to be the difference between the object's temperature and its surrounding environment, that is

$$x(t) = \theta(t) - \theta_m. \tag{4.17}$$

Then the state equation is given by

$$x^{\Delta}(t) = a(t)x(t) + b(t)u(t). \tag{4.18}$$

Next, define the associated cost function

$$J(t_0) = \frac{1}{2}\int_{t_0}^{t_f} u^2(\tau)\Delta\tau. \tag{4.19}$$

Then from the general form of the Hamiltonian, we have

$$H(t, x(t), u(t), \lambda(t)) = \frac{Ru^2(t)}{2} + \lambda(t)(a(t)x(t) + b(t)u(t)), \tag{4.20}$$

where R is a constant. Finally, the state and costate equations and stationary condition are given by

$$x^{\Delta}(t) = H_{\lambda}(t, x(t), u(t), \lambda^{\sigma}(t)) = a(t)x(t) + b(t)u(t), \tag{4.21}$$

$$-\lambda^{\Delta}(t) = H_{x}(t, x(t), u(t), \lambda^{\sigma}(t)) = a(t)\lambda^{\sigma}(t), \tag{4.22}$$

and

$$0 = H_{u}(t, x(t), u(t), \lambda^{\sigma}(t)) = Ru(t) + b(t)\lambda^{\sigma}(t). \tag{4.23}$$

Rewriting (4.23), we find the optimal control to be

$$u(t) = -\frac{b(t)\lambda^{\sigma}(t)}{R}. \tag{4.24}$$

Now plugging the optimal control into the state-costate equations, we have

$$x^{\Delta}(t) = a(t)x(t) - \frac{b^2(t)\lambda^{\sigma}(t)}{R}, \tag{4.25}$$

$$\lambda^{\Delta}(t) = -a(t)\lambda^{\sigma}(t). \tag{4.26}$$

Solving for $\lambda(t)$, we have

$$\lambda(t) = e_{\ominus a}(t, t_f)\lambda(t_f), \tag{4.27}$$

where $\lambda(t_f)$ will be determined later. Before solving for x note that

$$\lambda^{\sigma}(t) = \lambda(t) + \mu(t)\lambda^{\Delta}(t) = \lambda(t) - a(t)\mu(t)\lambda^{\sigma}(t),$$

which implies that

$$\lambda^{\sigma}(t) = \frac{\lambda(t)}{1 + \mu(t)a(t)} \tag{4.28}$$

provided that $1 + \mu(t)a(t) \neq 0$. Then the state equation is given by

$$x^{\Delta}(t) = a(t)x(t) - \frac{b^2(t)}{R(1+\mu(t)a(t))} e_{\ominus a}(t, t_f)\lambda(t_f). \tag{4.29}$$

Now the exponential can be rewritten as

$$\begin{aligned} e_{\ominus a}(t, t_f) &= e_a(t_f, \sigma(t))e_a(\sigma(t), t) = e_a(t_f, \sigma(t))(1+\mu(t)a(t))e_a(t, t) \\ &= e_a(t_f, \sigma(t))(1+\mu(t)a(t)). \end{aligned} \tag{4.30}$$

Solving (4.25) and using Theorem 2.32 yields

$$\begin{aligned} x(t) = e_a(t,t_0)x(t_0) &- \frac{b^2(t)}{R}\int_{t_0}^{t} \frac{e_a(t,\sigma(\tau))e_{\ominus a}(\tau, t_f)\lambda(t_f)}{(1+\mu(\tau)a(\tau))}\Delta\tau \\ = \quad e_a(t,t_0)x(t_0) &- \frac{b^2(t)}{R}\int_{t_0}^{t} \frac{(1+\mu(\tau)a(\tau))e_a(t,\sigma(\tau))e_a(t_f,\sigma(\tau))}{(1+\mu(\tau)a(\tau))}\Delta\tau\lambda(t_f) \\ = \quad e_a(t,t_0)x(t_0) &- \frac{b^2(t)}{R}\int_{t_0}^{t} e_a(t,\sigma(\tau))e_a(t_f,\sigma(\tau))\Delta\tau\lambda(t_f) \\ = \quad e_a(t,t_0)x(t_0) &- \frac{b^2(t)}{R}e_a(t,t_f)\int_{t_0}^{t} e_a^2(t_f,\sigma(\tau))\Delta\tau\lambda(t_f) \\ = \quad e_a(t,t_0)x(t_0) &- \frac{b^2(t)}{R}e_a(t,t_f)\int_{t_0}^{t} e_{2\odot a}(t_f,\sigma(\tau))\Delta\tau\lambda(t_f). \end{aligned} \tag{4.31}$$

In order to find our state and costate equations, we must find $\lambda(t_f)$. Now we consider two different control schemes that can be used to determine $\lambda(t_f)$.

Case 1 (Fixed Final State): Assume that the temperature of the object is originally the same as the temperature of its surrounding medium, $\theta_m = 70^0$ F (i.e., $x(t_0) = 0$). Now we are looking for an optimal control such that the final temperature of the object is $\theta(t_f) = 100^0$ F. This means that the value that the final state must take on is given by $x(t_f) = 30^0$ F. Since both the initial and final states are fixed, the boundary conditions are given by $\eta_1(t_0) = \eta_1(t_f) = 0$. Note that by (4.31), the final state can also be written as

$$x(t_f) = 30 = -\frac{b^2(t_f)}{R}\int_{t_0}^{t_f} e_{2\odot a}(t_f, \sigma(\tau))\Delta\tau\lambda(t_f). \tag{4.32}$$

Solving for the final costate, we have

$$\lambda(t_f) = \frac{-30R}{b^2(t_f)\displaystyle\int_{t_0}^{t_f} e_{2\odot a}(t_f, \sigma(\tau))\Delta\tau}. \tag{4.33}$$

Now by (4.27), the final costate is given by

$$\lambda^*(t) = \frac{-30R}{b^2(t_f)\displaystyle\int_{t_0}^{t_f} e_{2\odot a}(t_f, \sigma(\tau))\Delta\tau} e_{\ominus a}(t, t_f). \tag{4.34}$$

Using (4.24) and (4.28), the optimal control is given by

$$\begin{aligned} u^*(t) &= -\frac{b(t)}{R}\left[\frac{-30R}{b^2(t_f)(1+\mu(t)a(t))\displaystyle\int_{t_0}^{t_f} e_{2\odot a}(t_f, \sigma(\tau))\Delta\tau}\right] e_{\ominus a}(t, t_f) \\ &= \frac{30b(t)}{b^2(t_f)\displaystyle\int_{t_0}^{t_f} e_{2\odot a}(t_f, \sigma(\tau))\Delta\tau} e_{\ominus a}(\sigma(t), t_f). \end{aligned} \tag{4.35}$$

Finally, the optimal trajectory is given by

$$x^*(t) = \frac{30b^2(t)e_a(t, t_f)\displaystyle\int_{t_0}^{t} e_{2\odot a}(t_f, \sigma(\tau))\Delta\tau}{b^2(t_f)\displaystyle\int_{t_0}^{t_f} e_{2\odot a}(t_f, \sigma(\tau))\Delta\tau}. \tag{4.36}$$

Case 2 (Free Final State): Now suppose that the final state is not quite 30^0 F. However, we would like the final state to be as close to 30^0 F as possible. Thus to make the difference $x(t_f) - 30$ small, we will include it in the cost function

$$J(t_0) = \frac{1}{2}s(x(t_f) - 30)^2 + \frac{1}{2}\int_{t_0}^{t_f} u^2(\tau)\,\Delta\tau, \tag{4.37}$$

where s is some constant. Here, we would like to find a minimum input that minimizes both (4.37) and $|x(t_f) - 30|$, later which can be thought of as an error term. From (4.7), this has required us to add the term

$$\phi(x,t) = \frac{1}{2}s(x-30)^2. \tag{4.38}$$

Note that since the integrand remains unchanged, the Hamiltonian is still given by (4.20). Thus the equations for the state, costate, and input are preserved. Next, we need to determine our boundary conditions. As before, $x(t_0)$ is given, so $\eta_1(t_0) = 0$. On the other hand, $x(t_f)$ is not fixed, which means that $\eta_1(t_f) \neq 0$. Therefore we require that

$$\lambda(t_f) = \left.\frac{\partial \phi}{\partial x}\right|_{t=t_f} = s(x(t_f) - 30). \tag{4.39}$$

Now by (4.32), the final costate can be rewritten as

$$\lambda(t_f) = \frac{-30Rs}{R + sb^2(t_f)\displaystyle\int_{t_0}^{t_f} e_{2\odot a}(t_f, \sigma(\tau))\Delta\tau}. \tag{4.40}$$

Using (4.27), we have the optimal costate

$$\lambda^*(t) = \frac{-30Rs}{R + sb^2(t_f)\displaystyle\int_{t_0}^{t_f} e_{2\odot a}(t_f, \sigma(\tau))\Delta\tau} e_{\ominus a}(t, t_f). \tag{4.41}$$

From the optimal costate, we have the optimal control

$$u^*(t) = \frac{30sb(t)}{R + sb^2(t_f)\displaystyle\int_{t_0}^{t_f} e_{2\odot a}(t_f, \sigma(\tau))\Delta\tau} e_{\ominus a}(\sigma(t), t_f). \tag{4.42}$$

Finally, the optimal state is given by

$$x^*(t) = \frac{30sb^2(t)e_a(t,t_f)\displaystyle\int_{t_0}^{t} e_{2\odot a}(t_f, \sigma(\tau))\Delta\tau}{R + sb^2(t_f)\displaystyle\int_{t_0}^{t_f} e_{2\odot a}(t_f, \sigma(\tau))\Delta\tau}. \tag{4.43}$$

Remark 4.5. In the previous example, we were unable to evaluate the integral over a general time scale due to the way the exponential is defined. In the next example, we will consider a specific state coefficient.

Example 4.6. Let $\mathbb{T}$ be a general time scale and pick a to be such that $2 \odot a = c$ for some constant c. Then the state and costate equations and stationary condition are the same as found in Example 4.4. Now solving for the state, we have

$$\begin{aligned} x(t) &= -\frac{b^2(t)}{R} e_a(t,t_f) \int_{t_0}^{t} e_{2\odot a}(t_f, \sigma(\tau))\Delta\tau \lambda(t_f) \\ &= -\frac{b^2(t)}{R(2\odot a)} e_a(t,t_f) \int_{t_0}^{t} (2\odot a) e_{2\odot a}(t_f, \sigma(\tau))\Delta\tau \lambda(t_f) \\ &= -\frac{b^2(t)}{R(2\odot a)} e_a(t,t_f) \int_{t_0}^{t} [e_{2\odot a}(t_f, \cdot)]^{\Delta} \Delta\tau \lambda(t_f) \\ &= -\frac{b^2(t)}{R(2\odot a)} e_a(t,t_f) [e_{2\odot a}(t_f, t_0) - 1]. \end{aligned}$$

<u>Case 1</u> (Fixed Final State): The optimal control is given by

$$u^*(t) = \frac{30(2\odot a)}{[e_{2\odot a}(t_f,t_0) - 1]} e_{\ominus a}(\sigma(t), t_f). \tag{4.44}$$

The optimal state is given by

$$x^*(t) = \frac{30 e_a(t,t_f)[e_{2\odot a}(t_f,t_0) - e_{2\odot a}(t_f,t)]}{[e_{2\odot a}(t_f,t_0) - 1]}. \tag{4.45}$$

<u>Case 2</u> (Free Final State): Here we have the optimal control

$$u^*(t) = \frac{30bs(2\odot a)}{R(2\odot a) + b^2 s[e_{2\odot a}(t_f,t_0) - 1]} e_{\ominus a}(\sigma(t), t_f). \tag{4.46}$$

Finally, the optimal state is given by

$$x^*(t) = \frac{30b^2 s e_a(t,t_f)[e_{2\odot a}(t_f,t_0) - 1]}{R(2\odot a) + b^2 s[e_{2\odot a}(t_f,t_0) - 1]}. \tag{4.47}$$

Example 4.7. Let $\mathbb{T} = \mathbb{R}$ with $2 \odot a \equiv c$. Then the state and costate equations and stationary condition are given by

$$\dot{x}(t) = a(t)x(t) + b(t)u(t), \tag{4.48}$$

$$-\dot{\lambda}(t) = a(t)\lambda(t), \tag{4.49}$$

and

$$0 = Ru(t) + b(t)\lambda(t). \tag{4.50}$$

Case 1 (Fixed Final State): In this case, the optimal control is given by

$$u^*(t) = \frac{30c}{[e^{c(t_f - t_0)} - 1]e^{a(t-t_f)}},$$

while the optimal state is given by

$$x^*(t) = \frac{30e^{a(t-t_f)}[e^{c(t_f-t_0)} - e^{c(t_f-t)}]}{[e^{c(t_f-t_0)} - 1]}. \tag{4.51}$$

Case 2 (Free Final State): In this setting we have the optimal control

$$u^*(t) = \frac{30bsc}{Rc + b^2 s[e^{c(t_f-t_0)} - 1]e^{a(t-t_f)}} \tag{4.52}$$

with the optimal state given by

$$x^*(t) = \frac{30b^2 s e^{a(t-t_f)}[e^{c(t_f-t_0)} - 1]}{Rc + b^2 s[e^{c(t_f-t_0)} - 1]}. \tag{4.53}$$

Example 4.8. Let $\mathbb{T} = q^{\mathbb{N}_0}$ with $q > 1$ and $2 \odot a \equiv c$. Then the state and costate equations and stationary condition are given by

$$x^{\Delta}(t) = a(t)x(t) + b(t)u(t), \tag{4.54}$$

$$-\lambda^{\Delta}(t) = a(t)\lambda(qt), \tag{4.55}$$

and

$$0 = Ru(t) + b(t)\lambda(qt). \tag{4.56}$$

<u>Case 1</u> (Fixed Final State): The optimal control is given by

$$\begin{aligned} u^*(t) &= \frac{30c}{[e_c(t_f, t_0) - 1]} e_{\ominus a}(\sigma(t), t_f) \\ &= \frac{30c}{[e_c(t_f, t_0) - 1](1 + (q-1)ta(t))e_a(t, t_f)}, \end{aligned}$$

where

$$e_c(t_f, t_0) = \prod_{\tau \in \mathbb{T} \cap [t_0, t_f)} (1 + (q-1)c\tau).$$

Similarly, the optimal state is given by

$$x^*(t) = \frac{30e_a(t, t_f)[e_c(t_f, t_0) - e_c(t_f, t)]}{[e_c(t_f, t_0) - 1]}. \tag{4.57}$$

<u>Case 2</u> (Free Final State): In this setting we have the optimal control

$$u^*(t) = \frac{30bsc}{Rc + b^2 s[e_c(t_f, t_0) - 1](1 + (q-1)ta(t))e_a(t, t_f)}. \tag{4.58}$$

Finally, the optimal state is given by

$$x^*(t) = \frac{30b^2 s e_a(t, t_f)[e_c(t_f, t_0) - 1]}{Rc + b^2 s[e_c(t_f, t_0) - 1]}. \tag{4.59}$$

4.4. THE LINEAR QUADRATIC REGULATOR ON TIME SCALES

While the previous subsection provides us with a control law for general time varying systems, this may be difficult to calculate for nonlinear systems. In this subsection, we will examine the special case where we are given a linear system associated with a quadratic performance index. For convenience, we shall only consider the time-invariant case. Consider the state equation

$$x^{\Delta}(t) = Ax(t) + Bu(t) \tag{4.60}$$

with the associated quadratic performance index

$$J(t_0) = \frac{1}{2}x^T(t_f)S(t_f)x(t_f) + \frac{1}{2}\int_{t_0}^{t_f}(x^TQx + u^TRu)(\tau)\Delta\tau, \tag{4.61}$$

where $S(t_f)$, $Q \geq 0$ and $R > 0$. In Subsubsections 4.4.2 and 4.4.3, we will determine an optimal control u^* that minimizes (4.61) over the interval $[t_0, t_f]$. We will assume that $x(t_0)$ is given and that the final time $t_f \in \mathbb{T}$ is a fixed known number. Additionally, we assume that there is no function Ψ that must be augmented to (4.61). Next, define the Hamiltonian to be

$$H(x, u, \lambda) = \frac{1}{2}(x^TQx + u^TRu) + \lambda^T(Ax + Bu),$$

where $\lambda \in \mathbb{R}^n$ is a multiplier to be determined in later subsections. Using the above Hamiltonian, we have the following state and costate equations and stationary condition

$$x^{\Delta} = H_{\lambda}(x, u, \lambda^{\sigma}) = Ax + Bu, \tag{4.62}$$

$$-\lambda^{\Delta} = H_x(x, u, \lambda^{\sigma}) = Qx + A^T\lambda^{\sigma}, \tag{4.63}$$

and

$$0 = H_u(x, u, \lambda^{\sigma}) = Ru + B^T\lambda^{\sigma}. \tag{4.64}$$

Solving the stationary equation for u yields

$$u = -R^{-1}B^T\lambda^{\sigma}. \tag{4.65}$$

Eliminating the input from the state equation, the Hamiltonian system is given by

$$\begin{bmatrix} x \\ \lambda \end{bmatrix}^{\Delta}(t) = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x \\ \lambda^{\sigma} \end{bmatrix}(t).$$

A block diagram of this control scheme is found in Figure 4.1.

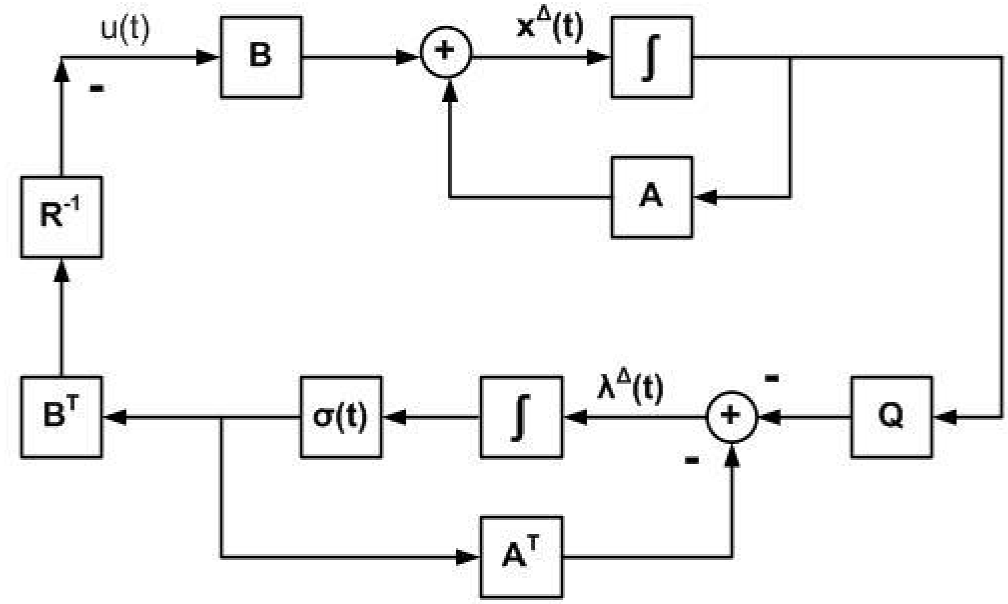

Figure 4.1. State-costate formulation of dynamic linear quadratic optimal controller

Example 4.9. When $\mathbb{T} = \mathbb{Z}$, the Hamiltonian can be rewritten as

$$\begin{bmatrix} x_k \\ \lambda_k \end{bmatrix} = \begin{bmatrix} (I+A)^{-1} & (I+A)^{-1}BR^{-1}B^T \\ Q(I+A)^{-1} & Q(I+A)^{-1}BR^{-1}B^T - (I+A^T) \end{bmatrix} \begin{bmatrix} x_{k+1} \\ \lambda_{k+1} \end{bmatrix}.$$

Note that from the above system, the equations for the state and costate can be written so that they operate "backward" in time. From Example 4.4, we saw that if we know $x(t_f)$ and $\lambda(t_f)$, we can find x and λ. Thus once the state and costate are known, we can determine the optimal control. However, we are only given $x(t_0)$, not $\lambda(t_f)$.

Recall in Subsection 4.2, we found the necessary conditions to find an optimal control. Now we want to show the sufficient conditions such that (4.65) is an optimal control that minimizes (4.61). This requires finding the second variation. We will restrict ourselves to the linear quadratic case. First note that

$$\begin{aligned}
\Phi(\epsilon) &= \frac{1}{2}(x+\epsilon\eta_1)^T(t_f)S(t_f)(x+\epsilon\eta_1)(t_f) \\
&\quad +\frac{1}{2}\int_{t_0}^{t_f}[(x+\epsilon\eta_1)^T Q(x+\epsilon\eta_1)+(u+\epsilon\eta_2)^T R(u+\epsilon\eta_2)](\tau)\Delta\tau \\
&\quad +\int_{t_0}^{t_f}[(\lambda^\sigma+\epsilon\eta_3^\sigma)^T A(x+\epsilon\eta_1)+(\lambda^\sigma+\epsilon\eta_3^\sigma)^T B(u+\epsilon\eta_2)](\tau)\Delta\tau \\
&\quad -\int_{t_0}^{t_f}(\lambda^\sigma+\epsilon\eta_3^\sigma)^T(x+\epsilon\eta_1)^\Delta(\tau)\Delta\tau.
\end{aligned}$$

Then

$$\begin{aligned}
\dot{\Phi}(\epsilon) &= \frac{1}{2}\eta_1^T(t_f)S(t_f)(x+\epsilon\eta_1)(t_f)+\frac{1}{2}(x+\epsilon\eta_1)^T(t_f)S(t_f)\eta_1(t_f) \\
&+\frac{1}{2}\int_{t_0}^{t_f}[\eta_1^T Q(x+\epsilon\eta_1)+(x+\epsilon\eta_1)^T Q\eta_1](\tau)\Delta\tau \\
&+\frac{1}{2}\int_{t_0}^{t_f}[\eta_2^T R(u+\epsilon\eta_2)+(u+\epsilon\eta_2)^T R\eta_2](\tau)\Delta\tau \\
&+\int_{t_0}^{t_f}[(\eta_3^\sigma)^T A(x+\epsilon\eta_1)+(\lambda^\sigma+\epsilon\eta_3^\sigma)^T A\eta_1+(\eta_3^\sigma)^T B(u+\epsilon\eta_2)](\tau)\Delta\tau \\
&+\int_{t_0}^{t_f}[(\lambda^\sigma+\epsilon\eta_3^\sigma)^T B\eta_2](\tau)\Delta\tau-\int_{t_0}^{t_f}[(\eta_3^\sigma)^T(x+\epsilon\eta_1)^\Delta+(\lambda^\sigma+\epsilon\eta_3^\sigma)^T\eta_1^\Delta](\tau)\Delta\tau.
\end{aligned}$$

Thus the first variation can be written as

$$\begin{aligned}
\dot{\Phi}(0) &= [S(t_f)x(t_f)-\lambda(t_f)]^T\eta_1(t_f)+\lambda^T(t_0)\eta_1(t_0) \\
&\quad +\int_{t_0}^{t_f}[(A\lambda^\sigma+Qx+\lambda^\Delta)^T\eta_1+(Ru+B^T\lambda^\sigma)^T\eta_2](\tau)\Delta\tau \\
&\quad +\int_{t_0}^{t_f}[(Ax+Bu-x^\Delta)^T\eta_3^\sigma](\tau)\Delta\tau.
\end{aligned}$$

Next,

$$\ddot{\Phi}(\epsilon) = \frac{1}{2}\eta_1^T(t_f)S(t_f)\eta_1(t_f)+\frac{1}{2}\eta_1^T(t_f)S(t_f)\eta_1(t_f)$$

$$
\begin{aligned}
& +\frac{1}{2}\int_{t_0}^{t_f}[\eta_1^T Q\eta_1 + \eta_1^T Q\eta_1 + \eta_2^T R\eta_2 + \eta_2^T R\eta_2](\tau)\Delta\tau \\
& +\int_{t_0}^{t_f}[(\eta_3^\sigma)^T A\eta_1 + (\eta_3^\sigma)^T A\eta_1 + (\eta_3^\sigma)^T B\eta_2](\tau)\Delta\tau \\
& +\int_{t_0}^{t_f}[(\eta_3^\sigma)^T B\eta_2](\tau)\Delta\tau - \int_{t_0}^{t_f}[(\eta_3^\sigma)^T\eta_1^\Delta + (\eta_3^\sigma)^T\eta_1^\Delta](\tau)\Delta\tau \\
= \; & \eta_1^T(t_f)S(t_f)\eta_1(t_f) + \int_{t_0}^{t_f}[\eta_1^T Q\eta_1 + \eta_2^T R\eta_2](\tau)\Delta\tau \\
& +\int_{t_0}^{t_f}[(A\eta_1 + B\eta_2 - \eta_1^\Delta)^T\eta_3^\sigma](\tau)\Delta\tau.
\end{aligned}
$$

If we assume that η_1 and η_2 satisfy the constraint

$$
\eta_1^\Delta = A\eta_1 + B\eta_2, \tag{4.66}
$$

the second variation is given by

$$
\ddot{\Phi}(0) = \eta_1^T(t_f)S(t_f)\eta_1(t_f) + \int_{t_0}^{t_f}[\eta_1^T Q\eta_1 + \eta_2^T R\eta_2](\tau)\Delta\tau. \tag{4.67}
$$

Note that (5.57) closely resembles (4.61), where x and u have been replaced by their variations. Now if $\eta_3 \neq 0$, then (5.57) is guaranteed to be positive. As such, (4.65) now represents a control that locally minimizes (4.61).

4.4.1. Zero Input and the Observability Lyapunov Equation. In this part, we will find the form of our cost function (4.61) in the absence of a given input. In this setting, we will use a generalized form of the Lyapunov equation to rewrite (4.61). Here we will make no assumptions as to whether or not the final state is fixed, though this a large role later when we seek an optimal input that minimizes (4.61). Now consider the state equation when $u(t) = 0$. Let S be an $n \times n$ matrix solution to

$$
-S^\Delta(t) = \tilde{A}^T(t)[A^T S(t) + (I + \mu(t)A^T)S(t)A + Q]\tilde{A}(t), \tag{4.68}
$$

where

$$
\tilde{A}(t) = (I + \mu(t)A)^{-1} \tag{4.69}
$$

and $S(t_f)$ is as in (4.61). As (4.68) describes the interaction between the plant and cost in the absence of a driving force, it can be called an "observability" Lyapunov equation.

Lemma 4.10. *A solution to* (4.68) *is given by*

$$S(t) = e_A^T(t_f,t)S(t_f)e_A(t_f,t) + \int_t^{t_f} e_A^T(\tau,t)Qe_A(\tau,t)\Delta\tau.$$

Proof. Before we differentiate S, we need to know $e_A^\Delta(t_f,t)$ as well as $e_A(t_f,\sigma(t))$. First note that $e_A(t,t_f)e_A(t_f,t) = I$. Then

$$\begin{aligned} 0 &= [e_A(\cdot,t_f)e_A(t_f,\cdot)]^\Delta(t) \\ &= e_A^\Delta(t,t_f)e_A(t_f,t) + e_A(\sigma(t),t_f)e_A^\Delta(t_f,t) \\ &= A + (I+\mu(t)A)e_A(t,t_f)e_A^\Delta(t_f,t). \end{aligned}$$

Solving for $e_A^\Delta(t_f,t)$ yields

$$\begin{aligned} e_A^\Delta(t_f,t) &= -[(I+\mu(t)A)e_A(t,t_f)]^{-1}A = -e_A(t_f,t)(I+\mu(t)A)^{-1}A \\ &= -e_A(t_f,t)A(I+\mu(t)A)^{-1}. \end{aligned} \tag{4.70}$$

Applying the simple useful formula, we have

$$\begin{aligned} e_A(t_f,\sigma(t)) &= e_A(t_f,t) + \mu(t)e_A^\Delta(t_f,t) \\ &= e_A(t_f,t)[I-\mu(t)(I+\mu(t)A)^{-1}A] \\ &= e_A(t_f,t)(I+\mu(t)A)^{-1}. \end{aligned} \tag{4.71}$$

Now using Theorem 2.20 part (b), we differentiate S to get

$$\begin{aligned} S^\Delta(t) &= (e_A^T)^\Delta(t_f,t)S(t_f)e_A(t_f,\sigma(t)) + e_A^T(t_f,t)S(t_f)e_A^\Delta(t_f,t) \\ &\quad + \int_t^{t_f} \left[e_A^T(\tau,\cdot)Qe_A(\tau,\cdot)\right]^\Delta(t)\Delta\tau - e_A^T(t,\sigma(t))Qe_A(t_f,\sigma(t)). \end{aligned}$$

Note that because of (4.70) and (4.71), the first term can be rewritten as

$$\begin{aligned}
&\left[-e_A(t_f,t)(I+\mu(t)A)^{-1}A\right]^T S(t_f)e_A(t_f,t)(I+\mu(t)A)^{-1} \\
&\quad = \quad -A^T(I+\mu(t)A^T)^{-1}e_A^T(t_f,t)S(t_f)e_A(t_f,t)(I+\mu(t)A)^{-1} \\
&\quad = \quad -(I+\mu(t)A^T)^{-1}A^Te_A^T(t_f,t)S(t_f)e_A(t_f,t)(I+\mu(t)A)^{-1}.
\end{aligned}$$

Similarly the second term can be rewritten as

$$-e_A^T(t_f,t)S(t_f)e_A(t_f,t)(I+\mu(t)A)^{-1}A \quad = \quad -e_A^T(t_f,t)S(t_f)e_A(t_f,t)A(I+\mu(t)A)^{-1}.$$

Finally, the last two terms can restated as

$$\begin{aligned}
& -\int_t^{t_f} [(I+\mu(\cdot)A^T)^{-1}A^Te_A^T(\tau,\cdot)Qe_A(\tau,\cdot)(I+\mu(\cdot)A)^{-1}](t)\Delta\tau \\
&\qquad -\int_t^{t_f} [e_A^T(\tau,\cdot)Qe_A(\tau,\cdot)A(I+\mu(\cdot)A)^{-1}](t)\Delta\tau + (I+\mu(t)A^T)^{-1}Q(I+\mu(t)A)^{-1} \\
& = \quad -(I+\mu(t)A^T)^{-1}A^T\int_t^{t_f} [e_A^T(\tau,\cdot)Qe_A(\tau,\cdot)](t)\Delta\tau(I+\mu(t)A)^{-1} \\
&\qquad -\int_t^{t_f} [e_A^T(\tau,\cdot)Qe_A(\tau,\cdot)](t)\Delta\tau A(I+\mu(t)A)^{-1} + (I+\mu(t)A^T)^{-1}Q(I+\mu(t)A)^{-1}.
\end{aligned}$$

Now combining like terms, we have

$$\begin{aligned}
-S^\Delta(t) \quad &= \quad -(I+\mu(t)A^T)^{-1}A^TS(t)(I+\mu(t)A)^{-1} - S(t)A(I+\mu(t)A)^{-1} \\
&\quad\quad -(I+\mu(t)A^T)^{-1}Q(I+\mu(t)A)^{-1} \\
&= \quad -(I+\mu(t)A^T)^{-1}[A^TS(t) + (I+\mu(t)A^T)S(t)A + Q](I+\mu(t)A)^{-1}.
\end{aligned}$$

This concludes the proof. □

It should be noted that DaCunha examined the limiting case of a similar equation in his thesis (see [19]). Now we rewrite the quadratic performance index (4.61) as follows.

Theorem 4.11. *Consider the state equation $x^\Delta = Ax$ associated the cost function* (4.61)*. Then the cost function on $[t, t_f]$ can be rewritten as*

$$J(t) = \frac{1}{2}x^T(t)S(t)x(t).$$

Proof. First note that

$$\frac{1}{2}\int_{t_0}^{t_f}(x^TSx)^\Delta(\tau)\Delta\tau = \frac{1}{2}x^T(t_f)S(t_f)x(t_f) - \frac{1}{2}x^T(t_0)S(t_0)x(t_0). \tag{4.72}$$

Then using (2.26), we have

$$\begin{aligned}
J(t_0) &= \frac{1}{2}x^T(t_f)S(t_f)x(t_f) + \frac{1}{2}\int_{t_0}^{t_f}(x^TQx)(t)\Delta t \\
&= \frac{1}{2}x^T(t_f)S(t_f)x(t_f) + \frac{1}{2}\int_{t_0}^{t_f}(x^TQx)(t)\Delta t + \frac{1}{2}\int_{t_0}^{t_f}(x^TSx)^\Delta(t)\Delta t \\
&\quad -\frac{1}{2}x^T(t_f)S(t_f)x(t_f) + \frac{1}{2}x^T(t_0)S(t_0)x(t_0) \\
&= \frac{1}{2}x^T(t_0)S(t_0)x(t_0) + \frac{1}{2}\int_{t_0}^{t_f}(x^TQx + \{x^TSx\}^\Delta)(t)\Delta t \\
&= \frac{1}{2}x^T(t_0)S(t_0)x(t_0) \\
&\quad +\frac{1}{2}\int_{t_0}^{t_f}x^T[Q + (I+\mu A^T)S^\Delta(I+\mu A) + A^TS(I+\mu A) + SA]x(t)\Delta t \\
&= \frac{1}{2}x^T(t_0)S(t_0)x(t_0).
\end{aligned}$$

This gives the result as stated. □

S can be referred here as the cost kernel function. Since S does not depend on x, it can be "pre-computed" and "stored off–line," i.e., in order to find the cost function, we need only to compute the current state.

4.4.2. Fixed-Final-State and Open-Loop Control. Now we want to determine an optimal control u^* on $[t_0, t_f]$ that minimizes (4.61). Recall the state and costate equations are given by the system

$$\begin{bmatrix} x \\ \lambda \end{bmatrix}^{\Delta}(t) = \begin{bmatrix} A & -BR^{-1}B^T \\ -Q & -A^T \end{bmatrix} \begin{bmatrix} x \\ \lambda^\sigma \end{bmatrix}(t). \tag{4.73}$$

We need to solve (4.73) with the following boundary conditions. As before we will assume that $x(t_0)$ is some known value x_0. Further, we will assume $\eta_1(t_f) = 0$. Then the control objective is to drive the state exactly to a given fixed reference value $r(t_f)$ at the final time t_f, i.e., $x(t_f) = r(t_f)$. Note that since $x(t_f)$ is fixed at $r(t_f)$, it is redundant to have a final-state weighting in the performance index, so we pick $S(t_f) = 0$. To find a useful solution, we pick $Q = 0$ so that (4.61) becomes

$$J(t_0) = \frac{1}{2}\int_{t_0}^{t_f} (u^T R u)(\tau)\Delta\tau. \tag{4.74}$$

In other words, we are looking for a "minimum control" that drives $x(t_0)$ to $x(t_f) = r(t_f)$.

Theorem 4.12. *Consider the linear system* (4.73) *subject to the boundary conditions*

$$\begin{cases} x(t_0) = x_0 \\ x(t_f) = r(t_f). \end{cases}$$

Assume that

$$G_C(t_0, t_f) := \int_{t_0}^{t_f} e_A(t_f, \sigma(\tau))BR^{-1}B^T e_A^T(t_f, \sigma(\tau))\Delta\tau \tag{4.75}$$

is invertible. Then the control that minimizes (4.74) *is given*

$$u^*(t) = R^{-1}B^T(I + \mu(t)A^T)^{-1}e_A^T(t_f, t)G_C^{-1}(t_0, t_f)[r(t_f) - e_A(t_f, t_0)x(t_0)]. \tag{4.76}$$

Proof. Using the above conditions, (4.73) becomes

$$\begin{aligned} x^\Delta(t) &= Ax(t) - BR^{-1}B^T\lambda^\sigma(t) \\ \lambda^\Delta(t) &= -A^T\lambda^\sigma(t). \end{aligned} \tag{4.77}$$

Solving the costate equation for $\lambda(t)$, we get

$$\lambda(t) = e_{\ominus A^T}(t, t_f)\lambda(t_f), \tag{4.78}$$

where $\lambda(t_f)$ is still unknown. Before solving for $x(t)$, we will eliminate $\lambda^\sigma(t)$ from the equation. Note that

$$\lambda^\sigma(t) = \lambda(t) + \mu(t)\lambda^\Delta(t) = \lambda(t) - \mu(t)A^T\lambda^\sigma(t), \tag{4.79}$$

which gives

$$\lambda^\sigma(t) = (I + \mu(t)A^T)^{-1}\lambda(t) = (I + \mu(t)A^T)^{-1}e_{\ominus A^T}(t, t_f)\lambda(t_f). \tag{4.80}$$

Using (4.80), the state equation now becomes

$$x^\Delta(t) = Ax(t) - BR^{-1}B^T(I + \mu(t)A^T)^{-1}e_{\ominus A^T}(t, t_f)\lambda(t_f). \tag{4.81}$$

Now solving (4.81) at time $t = t_f$, we get

$$\begin{aligned}
x(t_f) &= e_A(t_f, t_0)x(t_0) - \int_{t_0}^{t_f} e_A(t_f, \sigma(\tau))BR^{-1}B^T(I + \mu(t)A^T)^{-1}e_{\ominus A^T}(\tau, t_f)\lambda(t_f)\Delta\tau \\
&= e_A(t_f, t_0)x(t_0) \\
&\quad - \int_{t_0}^{t_f} e_A(t_f, \sigma(\tau))BR^{-1}B^T(I + \mu(t)A^T)^{-1}(I + \mu(t)A^T)e_A^T(t_f, \sigma(\tau))\Delta\tau\lambda(t_f) \\
&= e_A(t_f, t_0)x(t_0) - G_C(t_0, t_f)\lambda(t_f),
\end{aligned}$$

where (4.75) represents a weighted controllability Gramian. Now solving for $\lambda(t_f)$ and using the given boundary conditions, we have

$$\lambda(t_f) = -G_C^{-1}(t_0, t_f)[x(t_f) - e_A(t_f, t_0)x(t_0)] = -G_C^{-1}(t_0, t_f)[r(t_f) - e_A(t_f, t_0)x(t_0)].$$

Then the optimal control is given by

$$\begin{aligned} u^*(t) &= -R^{-1}B^T\lambda^\sigma(t) \\ &= -R^{-1}B^T(I+\mu(t)A^T)^{-1}e_{\ominus A^T}(t,t_f)\lambda(t_f) \\ &= R^{-1}B^T(I+\mu(t)A^T)^{-1}e_A^T(t_f,t)G_C^{-1}(t_0,t_f)[r(t_f)-e_A(t_f,t_0)x(t_0)]. \end{aligned} \tag{4.82}$$

This gives our optimal control as desired. □

Here u^* represents the minimum-energy control that drives the given initial state $x(t_0)$ to the desired final reference value of $x(t_f) = r(t_f)$. Note that when $u^*(t) = 0$, the final state is given by $x(t_f) = e_A(t_f, t_0)x(t_0)$. This implies that the optimal control is proportional to the difference between the zero input solution and the desired final state.

Since $G_C^{-1}(t_0, t_f)$ is needed in order to find u^*, the optimal control exists for arbitrary $x(t_0)$ and $r(t_f)$ if and only if $G_C(t_0, t_f)$ is invertible. However, note that this is just a restatement of the controllability of the plant. Also, the optimal control (4.82) is called an *open–loop* control since while u^* depends on both the initial and final states, it does not rely on the current state. As a result, this control can be pre–computed and applied for all $t \in [t_0, t_f]$.

Next, we want to determine the optimal cost $J^*(t_0)$.

Theorem 4.13. *Let* (4.76) *be the optimal control that minimizes the cost function* (4.74)*. Assume that* (4.75) *is invertible. Then the optimal cost is given by*

$$J^*(t_0) = \frac{1}{2}d^T(t_0, t_f)G_C^{-1}(t_0, t_f)d(t_0, t_f), \tag{4.83}$$

where $d(t_0, t_f) := r(t_f) - e_A(t_f, t_0)x(t_0)$.

Proof. First we will define the final state difference to be

$$d(t_0, t_f) = r(t_f) - e_A(t_f, t_0)x(t_0). \tag{4.84}$$

Again, the second term represents the final state when $u^*(t) = 0$. Then $J^*(t_0)$ is given by

$$
\begin{aligned}
J^*(t_0) &= \frac{1}{2}\int_{t_0}^{t_f} (u^{*T} R u^*)(\tau)\Delta\tau \\
&= \frac{1}{2}\int_{t_0}^{t_f} d^T(t_0,t_f)G_C^{-1}(t_0,t_f)\big[e_A(t_f,\cdot)(I+\mu(\cdot)A)^{-1}BR^{-1}RR^{-1}B^T \\
&\quad (I+\mu(\cdot)A^T)^{-1}e_A^T(t_f,\cdot)\big](\tau)G_C^{-1}(t_0,t_f)d(t_0,t_f)\Delta\tau.
\end{aligned}
$$

Now since neither $d(t_0,t_f)$ nor $G_C^{-1}(t_0,t_f)$ depend on t, $J^*(t_0)$ can be rewritten as

$$
\begin{aligned}
J^*(t_0) &= \frac{1}{2}d^T(t_0,t_f)G_C^{-1}(t_0,t_f)\int_{t_0}^{t_f} e_A(t_f,\tau)(I+\mu(\tau)A)^{-1}BR^{-1}B^T\cdot \\
&\quad (I+\mu(\tau)A^T)^{-1}e_A^T(t_f,\tau)\Delta\tau G_C^{-1}(t_0,t_f)d(t_0,t_f) \\
&= \frac{1}{2}d^T(t_0,t_f)G_C^{-1}(t_0,t_f)\int_{t_0}^{t_f} e_A(t_f,\sigma(\tau))BR^{-1}B^T e_A^T(t_f,\sigma(\tau))\Delta\tau G_C^{-1}(t_0,t_f)d(t_0,t_f) \\
&= \frac{1}{2}d^T(t_0,t_f)G_C^{-1}(t_0,t_f)G_C(t_0,t_f)G_C^{-1}(t_0,t_f)d(t_0,t_f) \\
&= \frac{1}{2}d^T(t_0,t_f)G_C^{-1}(t_0,t_f)d(t_0,t_f).
\end{aligned}
$$

This concludes the proof. $\square$

While an open-loop control is not usually in terms of the current state, we can show that with some additional assumptions that it can. This will require a knowledge of the mechanics in the next subsection. We will revisit this case in Section 5.

4.4.3. Free-Final-State and Closed-Loop Control. In the previous subsubsection, we found an optimal control when the final time was fixed. In this case, we shall develop an optimal control law in the form of state feedback. Note that the state and costate equations are the same as found in (4.73). In considering the boundary conditions, note that $x(t_0)$ is known (meaning $\eta_1(t_0) = 0$) while $x(t_f)$ is free (meaning $\eta_1(t_f) \neq 0$). Thus the coefficient on $\eta_1(t_f)$ must be zero. Let $\phi_x(x(t_f),t) - \lambda(t_f)$ be the coefficient of $\eta_1(t_f)$, where

$$
\phi(x,t) = \frac{1}{2}x^T S(t)x. \tag{4.85}
$$

Then the terminal condition on the costate is given by

$$\lambda(t_f) = \left.\frac{\partial \phi}{\partial x}\right|_{t=t_f} = S(t_f)x(t_f). \tag{4.86}$$

Remark 4.14. Now we will make the assumption that x and λ satisfy a linear relationship similar to (4.86) for all $t \in [t_0, t_f]$, i.e.,

$$\lambda(t) = S(t)x(t). \tag{4.87}$$

This condition (4.87) is called a "sweep condition," a term used by Bryson and Ho in [17]. Since $S(t_f) \geq 0$, it is natural to assume that $S \geq 0$ as well.

Next, we use this sweep condition to derive our Riccati equation.

Theorem 4.15. *Assume that* $\left(I + \mu(t)BR^{-1}B^T S^\sigma(t)\right)$ *is invertible. Further assume that the condition* (4.87) *holds. Then* S *satisfies a Riccati equation (of the second form)*

$$-S^\Delta = Q + A^T S^\sigma + (I + \mu(t)A^T)S^\sigma \left(I + \mu(t)BR^{-1}B^T S^\sigma\right)^{-1} \left(A - BR^{-1}B^T S^\sigma\right), \tag{4.88}$$

for all $t \leq t_f$ *with boundary condition* $S(t_f)$.

Proof. First, we will rewrite the state equation solely in terms of one variable.

$$\begin{aligned}
x^\Delta(t) &= Ax(t) - BR^{-1}B^T S^\sigma(t)x^\sigma(t) \\
&= Ax(t) - BR^{-1}B^T S^\sigma(t)\left(x(t) + \mu(t)x^\Delta(t)\right) \\
&= Ax(t) - BR^{-1}B^T S^\sigma(t)x(t) - \mu(t)BR^{-1}B^T S^\sigma(t)x^\Delta(t) \\
&= \left(I + \mu(t)BR^{-1}B^T S^\sigma(t)\right)^{-1}\left(A - BR^{-1}B^T S^\sigma(t)\right)x(t)
\end{aligned}$$

if $\left(I + \mu(t)BR^{-1}B^T S^\sigma(t)\right)$ is invertible. Next, we rewrite the costate equation in terms of x:

$$\begin{aligned}
\lambda^\Delta &= -Qx - A^T S^\sigma x^\sigma \\
&= -Qx - A^T S^\sigma\left(x + \mu x^\Delta\right)
\end{aligned}$$

$$\begin{aligned} &= -\left(Q + A^T S^\sigma\right) x - \mu A^T S^\sigma x^\Delta \\ &= -[Q + A^T S^\sigma + \mu A^T S^\sigma \left(I + \mu B R^{-1} B^T S^\sigma\right)^{-1} \left(A - B R^{-1} B^T S^\sigma\right)]x. \end{aligned} \quad (4.89)$$

But using the product rule instead, we get

$$\begin{aligned} \lambda^\Delta &= S^\Delta x + S^\sigma x^\Delta \\ &= S^\Delta x + S^\sigma \left(I + \mu B R^{-1} B^T S^\sigma\right)^{-1} \left(A - B R^{-1} B^T S^\sigma\right) x. \end{aligned} \quad (4.90)$$

Comparing equations (4.89) and (4.90) and combining like terms, we have

$$\left[S^\Delta + Q + A^T S^\sigma + (I + \mu A^T) S^\sigma \left(I + \mu B R^{-1} B^T S^\sigma\right)^{-1} \left(A - B R^{-1} B^T S^\sigma\right)\right] x = 0.$$

Then the above equation must be true for all $t \leq t_f$ and for any choice of $x(t_0)$. Note that S does not depend on x for any time t. It follows that the above equation must be true for all values of x. As a result, this means that S must satisfy (4.88). This concludes the proof. □

At first glance (4.88) does not resemble a Riccati equation of the second form as we defined it by (2.28). We will later show that they are in fact equivalent. But first, we must define a state feedback gain.

Definition 4.16. A matrix-valued function

$$K(t) = (R + \mu(t) B^T S^\sigma(t) B)^{-1} B^T S^\sigma(t) (I + \mu(t) A) \quad (4.91)$$

is called a state feedback or Kalman gain, provided that $R + \mu(t) B^T S^\sigma(t) B$ is invertible.

Corollary 4.17. *The Riccati equation* (4.88) *is a Riccati equation of the second form provided that* $BR^{-1}B^T S^\sigma(t)$ *is regressive.*

Proof. First note that (4.91) can also be written as

$$K(t) = R^{-1} B^T S^\sigma(t) (I + \mu(t) B R^{-1} B^T S^\sigma(t))^{-1} (I + \mu(t) A). \quad (4.92)$$

Then (4.88) can be written as

$$
\begin{aligned}
-S^{\Delta} = Q + A^T S &+ (I+\mu A^T)S^\sigma (I+\mu BR^{-1}B^T S^\sigma)^{-1} A \\
&-(I+\mu A^T)S^\sigma BR^{-1}B^T S^\sigma (I+\mu BR^{-1}B^T S^\sigma)^{-1}.
\end{aligned}
$$

Next, we multiply the last term by I in the form $I-\mu A+\mu A$ on the right. Then plugging in (4.92) yields

$$
\begin{aligned}
-S^{\Delta} &= Q + A^T S + (I+\mu A^T)S^\sigma (I+\mu BR^{-1}B^T S^\sigma)^{-1} A \\
&\quad -(I+\mu A^T)S^\sigma BK + \mu(I+\mu A^T)S^\sigma BR^{-1}B^T S^\sigma (I+\mu BR^{-1}B^T S^\sigma)^{-1} A \\
&= Q + A^T S - (I+\mu A^T)S^\sigma BK \\
&\quad +(I+\mu A^T)S^\sigma (I+\mu BR^{-1}B^T S^\sigma)(I+\mu BR^{-1}B^T S^\sigma)^{-1} A.
\end{aligned}
$$

Finally, plugging in (4.91), we have

$$
-S^{\Delta} = Q + A^T S + (I+\mu A^T)S^\sigma A - (I+\mu A^T)S^\sigma B(R+\mu B^T S^\sigma B)^{-1} B^T S^\sigma (I+\mu A). \tag{4.93}
$$

Note that the above equation is a Riccati equation of the second form. □

When an optimal control is in terms of the current state, it is said to be a closed-loop control. We now consider the form of an optimal control that minimizes (4.61).

Theorem 4.18. *Let u^* represent an optimal control that minimizes* (4.61)*. Then u^* can be written in the form*

$$
u^*(t) = -K(t)x(t), \tag{4.94}
$$

where the matrix $K(t)$ is given by (4.91) *provided that $R+\mu(t)B^T S^\sigma(t)B$ is invertible.*

Proof. Using (4.82) and the state equation (4.60), we have

$$
\begin{aligned}
u^*(t) &= -R^{-1}B^T S^\sigma(t)\left(x(t)+\mu(t)x^{\Delta}(t)\right) \\
&= -R^{-1}B^T S^\sigma(t)(I+\mu(t)A)x(t) - \mu(t)R^{-1}B^T S^\sigma(t)Bu^*(t).
\end{aligned}
$$

Now combining like terms and pre-multiplying by R we have

$$(R + \mu(t)B^T S^\sigma(t)B)u^*(t) = -B^T S^\sigma(t)(I + \mu(t)A)x(t),$$

which implies

$$u^*(t) = -(R + \mu(t)B^T S^\sigma(t)B)^{-1}B^T S^\sigma(t)(I + \mu(t)A)x(t).$$

This gives our optimal control as desired. $\square$

The input (4.94) is sometimes referred to as the control law that minimizes the cost function (4.61). Note in order to determine the closed-loop control that gives u^*, we need only K and the solution to the Riccati equation, S. But recall that we obtained S from the sweep condition, not from the Riccati equation. At first, it would appear as though the derivation of (4.88) was pointless. However as we will see, the Riccati equation will be used to find an optimal cost. As neither K nor S depend on x, they can be computed and stored off–line. It should be noted that even for a time invariant plant the gain here will be time varying, thus (4.94) is also called a time-varying state feedback.

Now the closed-loop plant is given by

$$x^\Delta(t) = (A - BK(t))x(t), \tag{4.95}$$

which can be used to find an optimal trajectory x^* for any given $x(t_0)$. A block diagram describing this control scheme appears in Figure 4.2.

Next, we will consider alternative forms of our Riccati equation (of the second form). As we will see in this section and Section 5, these other forms will be much more useful in finding an optimal cost than (4.93). First we want to rewrite our Riccati equation in terms of the Kalman gain.

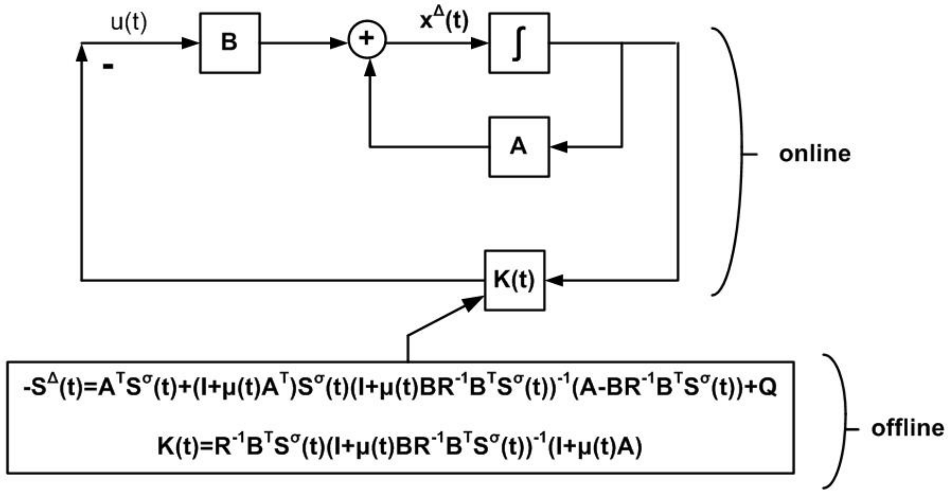

Figure 4.2. The free-final-state LQ regulator

Corollary 4.19. *The Riccati equation* (4.93) *can be written in terms of the Kalman gain as*

$$-S^{\Delta}(t) = A^T S^{\sigma}(t) + (I + \mu(t)A^T)S^{\sigma}(t)(A - BK(t)) + Q. \tag{4.96}$$

Proof. This proof is a direct result from Corollary 4.17. Using (4.91), we have

$$\begin{aligned}
-S^{\Delta} &= Q + A^T S + (I + \mu A^T)S^{\sigma} A \\
&\quad -(I + \mu A^T)S^{\sigma} B(R + \mu B^T S^{\sigma} B)^{-1} B^T S^{\sigma}(I + \mu A) \\
&= Q + A^T S + (I + \mu A^T)S^{\sigma} A - (I + \mu A^T)S^{\sigma} BK \\
&= Q + A^T S + (I + \mu A^T)S^{\sigma}(A - BK).
\end{aligned}$$

This gives the desired result. □

Next, we will introduce another form of the Riccati equation in terms of the Kalman gain. We will use this form to find an optimal cost.

Corollary 4.20. *The Riccati equation* (4.93) *can be written in terms of the Kalman gain as*

$$-S^{\Delta}(t) = Q + A^T S^{\sigma}(t) + (I + \mu(t)A^T)S^{\sigma}(t)A - K^T(t)(R + \mu(t)B^T S^{\sigma}(t)B)K(t). \quad (4.97)$$

Proof. Using (4.91), we have

$$\begin{aligned}
-S^{\Delta} &= A^T S^{\sigma} + (I + \mu A^T)S^{\sigma}A - (I + \mu A^T)S^{\sigma}BK + Q \\
&= A^T S^{\sigma} + (I + \mu A^T)S^{\sigma}A + Q \\
&\quad -(I + \mu A^T)S^{\sigma}B(R + \mu B^T S^{\sigma}B)^{-1}(R + \mu B^T S^{\sigma}B)K \\
&= Q + A^T S^{\sigma} + (I + \mu A^T)S^{\sigma}A - K^T(R + \mu B^T S^{\sigma}B)K.
\end{aligned}$$

This concludes the proof. □

Finally, we rewrite the Riccati equation in terms of the closed-loop plant matrix. This form of the Riccati equation is called the (generalized) Joseph stabilized form.

Corollary 4.21. *The Riccati equation* (4.93) *can be written in terms of the closed-loop plant as*

$$\begin{aligned}
-S^{\Delta}(t) &= Q + (A - BK(t))^T S^{\sigma}(t) + (I + \mu(t)(A - BK(t))^T)S^{\sigma}(t)(A - BK(t)) \\
&\quad +K^T(t)RK(t). \qquad (4.98)
\end{aligned}$$

Proof. Rewriting (4.97), we have

$$\begin{aligned}
-S^{\Delta} &= Q + (A - BK + BK)^T S^{\sigma} + [I + \mu(A - BK + BK)^T]S^{\sigma}A \\
&\quad -K^T(R + \mu B^T S^{\sigma}B)K \\
&= Q + (A - BK)^T S^{\sigma} + (I + \mu(A - BK)^T)S^{\sigma}(A - BK) + K^T B^T S^{\sigma}(I + \mu A) \\
&\quad +(I + \mu(A - BK)^T)S^{\sigma}BK - K^T(R + \mu B^T S^{\sigma}B)K.
\end{aligned}$$

Now using (4.91) yields

$$\begin{aligned}
-S^{\Delta} &= Q + (A - BK)^T S^{\sigma} + (I + \mu(A - BK)^T)S^{\sigma}(A - BK) + K^T(R + \mu B^T S^{\sigma} B)K \\
&\quad + (I + \mu A^T)S^{\sigma} BK - \mu K^T B^T S^{\sigma} BK - K^T(R + \mu B^T S^{\sigma} B)K \\
&= Q + (A - BK)^T S^{\sigma} + (I + \mu(A - BK)^T)S^{\sigma}(A - BK) + K^T(R + \mu B^T S^{\sigma} B)K \\
&\quad - \mu K^T B^T S^{\sigma} BK \\
&= Q + (A - BK)^T S^{\sigma} + (I + \mu(A - BK)^T)S^{\sigma}(A - BK) + K^T RK.
\end{aligned}$$

This gives the result as desired. $\square$

We will use this form in Section 5.

Example 4.22. Here we rewrite the Riccati equation (4.98) for various time scales.

a. If $\mathbb{T} = \mathbb{R}$, then (4.98) becomes

$$-\dot{S}(t) = Q + (A - BK(t))^T S(t) + S(t)(A - BK(t)) + K^T(t)RK(t),$$

where $K(t) = R^{-1}B^T S(t)$.

b. If $\mathbb{T} = \mathbb{Z}$, then (4.98) is given by

$$\begin{aligned}
S(t) - S(t+1) &= Q + (A - BK(t))^T S(t+1) + K^T(t)RK(t) \\
&\quad + (I + (A + BK(t))^T)S(t+1)(A - BK(t)),
\end{aligned}$$

where $K(t) = (R + B^T S(t+1)B)^{-1}B^T S(t+1)(I + A)$.

c. If $\mathbb{T} = h\mathbb{Z}$, then (4.98) turns into

$$\begin{aligned}
\frac{S(t) - S(t+h)}{h} &= Q + (A - BK(t))^T S(t+h) + K^T(t)RK(t) \\
&\quad + (I + h(A + BK(t))^T)S(t+h)(A - BK(t)),
\end{aligned}$$

where $K(t) = (R + hB^T S(t+h)B)^{-1}B^T S(t+h)(I + hA)$.

d. If $\mathbb{T} = q^{\mathbb{N}_0}$ for $q > 1$, then (4.98) is given by

$$\begin{aligned}\frac{S(t) - S(qt)}{(q-1)t} &= Q + (A - BK(t))^T S(qt) + K^T(t) R K(t) \\ &\quad + (I + (q-1)t(A + BK(t))^T) S(qt)(A - BK(t)),\end{aligned}$$

where $K(t) = (R + (q-1)tB^T S(qt)B)^{-1} B^T S(qt)(I + (q-1)tA)$.

Finally, now that we have an optimal control, let us determine the optimal cost.

Theorem 4.23. *Let* (4.94) *be an optimal control that minimizes* (4.61)*. Then the optimal cost is given by*

$$J^*(t) = \frac{1}{2} x^T(t) S(t) x(t), \quad t \leq t_f,$$

provided that $R + \mu(t) B^T S^\sigma(t) B$ is invertible.

Proof. First note that

$$\begin{aligned}(x^T S x)^\Delta &= (x^T S)^\Delta x + (x^T S)^\sigma x^\Delta \\ &= (x^\Delta)^T S^\sigma x + x^T S^\Delta x + (x^T S)^\sigma x^\Delta \\ &= (x^T A^T + u^T B^T) S^\sigma x + x^T S^\Delta x + (x + \mu x^\Delta)^T S^\sigma x^\Delta \\ &= (x^T A^T + u^T B^T) S^\sigma x + x^T S^\Delta x + \left[x^T(I + \mu A^T) + \mu u^T B^T\right] S^\sigma (Ax + Bu).\end{aligned}$$

Combining like terms gives us

$$\begin{aligned}(x^T S x)^\Delta &= x^T \left[A^T S^\sigma + S^\Delta + (I + \mu A^T) S^\sigma A\right] x + u^T B^T S^\sigma (I + \mu A) x \\ &\quad + x^T (I + \mu A^T) S^\sigma B u + \mu u^T B^T S^\sigma B u.\end{aligned}$$

Now the equation for the cost over $[t, t_f]$ is given by

$$\begin{aligned}J(t) &= \frac{1}{2} x^T(t) S(t) x(t) + \frac{1}{2} \int_t^{t_f} [(x^T S x)^\Delta + x^T Q x + u^T R u](\tau) \Delta\tau \\ &= \frac{1}{2} x^T(t) S(t) x(t) + \frac{1}{2} \int_t^{t_f} [x^T (A^T S^\sigma + S^\Delta + (I + \mu A^T) S^\sigma A + Q) x](\tau) \Delta\tau\end{aligned}$$

$$+\frac{1}{2}\int_t^{t_f}[u^TB^TS^\sigma(I+\mu A)x+x^T(I+\mu A^T)S^\sigma Bu](\tau)\Delta\tau$$
$$+\frac{1}{2}\int_t^{t_f}[u^T(R+\mu B^TS^\sigma B)u](\tau)\Delta\tau.$$

Since S satisfies (4.97), the first integrand can be rewritten as

$$x^T(t)K^T(t)(R+\mu(t)B^TS^\sigma(t)B)K(t)x(t). \tag{4.99}$$

Then the cost function can be written as

$$\begin{aligned}
J(t) = {} & \frac{1}{2}x^T(t)S(t)x(t)+\frac{1}{2}\int_t^{t_f}[x^TK^T(R+\mu B^TS^\sigma B)Kx](\tau)\Delta\tau \\
& +\frac{1}{2}\int_t^{t_f}[u^TB^TS^\sigma(I+\mu A)x+x^T(I+\mu A^T)S^\sigma Bu](\tau)\Delta\tau \\
& +\frac{1}{2}\int_t^{t_f} u^T(R+\mu B^TS^\sigma B)u(\tau)\Delta\tau \\
= {} & \frac{1}{2}x^T(t)S(t)x(t)+\frac{1}{2}\int_t^{t_f}\left[x^TK^T(R+\mu B^TS^\sigma B)Kx\right](\tau)\Delta\tau \\
& +\frac{1}{2}\int_t^{t_f}[u^T(R+\mu B^TS^\sigma B)Kx+x^TK^T(R+\mu B^TS^\sigma B)u](\tau)\Delta\tau \\
& +\frac{1}{2}\int_t^{t_f}\left[u^T(R+\mu B^TS^\sigma B)u\right](\tau)\Delta\tau \\
= {} & \frac{1}{2}x^T(t)S(t)x(t)+\frac{1}{2}\int_t^{t_f}||Kx+u||^2_{(R+\mu B^TS^\sigma B)}(\tau)\Delta\tau,
\end{aligned}$$

where the integrand is the perfect square of a norm of $Kx+u$ with respect to $R+\mu B^TS^\sigma B$. Letting u satisfy (4.94), the cost function over $[t,t_f]$ is

$$J^*(t)=\frac{1}{2}x^T(t)S(t)x(t).$$

This conclude the proof. $\square$

From Theorem 4.23, if the current state and S are known, we can determine the optimal cost over $[t,t_f]$ before we apply the optimal control or even calculate it!

5. THE TRACKING PROBLEM

In this section, we will consider optimal problems that are natural extensions to those we encountered in the last section. For Subsection 5.1, we will find an optimal control where the cost function is in terms of the output. As a result, in order to consider the state, we will make the assumption that the linear system associated the cost function is completely observable. Again, we will assume that the initial state is known. Furthermore, we will make the assumption that the final state is free, resulting in closed-loop control. Next, in Subsections 5.2 and 5.3 we will consider a more general problem: tracking. In Subsection 5.3, we will introduce an optimal control that contains a feedforward term.

The difference between feedback and feedforward control is given by the following example. In the previous section, we considered heating a room to a desired temperature. Assume that the state represents the temperature of the room and the control represents the heating vent. Furthermore, suppose that it is snowing outside. If the control that heats the room is based on the current temperature of the room, we have an example of state feedback. As we have seen earlier, feedback control is simple to design. Unfortunately, this control scheme can only account and correct errors after they have occurred. Now suppose the heater reads the temperature outside instead. As a result, the control that heats the room will be activated when it cools outside, rather than wait for the room to cool. This is an example of feedforward control. By comparison, this control scheme not only anticipates deviations in our model, but corrects them before they can occur. However, the physics of the model must be well known in advance in order to use feedforward control. The optimal control we find in Subsection 5.3 will contain both a feedback and feedforward term.

Finally in Subsection 5.4, we will revisit the fixed final state case from Subsubsection 4.4.2. In this portion we will use the mechanics of the tracker to find an optimal control in terms of the current state as well as some desired final reference signal. In this setting, we will consider a more general cost function, thus giving us a stronger result than the one we have in Subsubsection 4.4.2.

5.1. OUTPUT QUADRATIC REGULATOR

In the previous section, we examined the LQR in terms of the state. For this section, we will use the concept of observability to minimize a cost function in terms of the output (see [2, 44]). First, let us consider the completely observable system

$$\begin{aligned} x^{\Delta}(t) &= Ax(t) + Bu(t), \quad x(t_0) = x_0 \\ y(t) &= Cx(t). \end{aligned} \tag{5.1}$$

Now assume the system is associated with the cost functional

$$J(t_0) = \frac{1}{2} y^T(t_f) F y(t_f) + \frac{1}{2} \int_{t_0}^{t_f} (y^T Q y + u^T R u)(\tau) \Delta\tau, \tag{5.2}$$

where $R > 0$ and F, $Q \geq 0$. Now before we find an optimal control that minimizes (5.2), note that (5.2) can be rewritten in terms of the state as

$$J(t_0) = \frac{1}{2} x^T(t_f) C^T F C x(t_f) + \frac{1}{2} \int_{t_0}^{t_f} (x^T C^T Q C x + u^T R u)(\tau) \Delta\tau. \tag{5.3}$$

Comparing (5.3) with (4.61), note that $S(t_f)$ and Q are replaced by $C^T F C$ and $C^T Q C$. In the next theorem, we will show the correspondence between (4.61) and (5.2).

Theorem 5.1. *Let F, $Q \geq 0$ and let* (5.1) *be completely observable. Then the matrices $C^T F C$ and $C^T Q C$ are positive semidefinite.*

Proof. Since F and Q are symmetric, $C^T F C$ and $C^T Q C$ are symmetric as well. Also, since (5.1) is assumed to be completely observable, C^T cannot be zero. Now if $Q \geq 0$, then $y^T Q y \geq 0$ for any choice of y, which means that $x^T C^T Q C x \geq 0$. But observability implies that since each y is given by a unique x, then $x^T C^T Q C x \geq 0$ for all x. Therefore $C^T Q C \geq 0$. □

Next, we define the Hamiltonian to be

$$H(x,u,\lambda) = \frac{1}{2}(x^T C^T Q C x + u^T R u) + \lambda^T (Ax + Bu).$$

Using the above Hamiltonian, the state and costate equations and stationary condition are given by

$$x^\Delta = H_\lambda(x,u,\lambda^\sigma) = Ax + Bu,$$

$$-\lambda^\Delta = H_x(x,u,\lambda^\sigma) = C^T Q C x + A^T \lambda^\sigma,$$

and

$$0 = H_u(x,u,\lambda^\sigma) = Ru + B^T \lambda^\sigma,$$

which yields

$$u = -R^{-1} B^T \lambda^\sigma \tag{5.4}$$

If we assume that η_1 and η_2 satisfy the constraint

$$\eta_1^\Delta = A\eta_1 + B\eta_2, \tag{5.5}$$

the second variation is given by

$$\ddot{\Phi}(0) = \eta_1^T(t_f) C^T F C \eta_1(t_f) + \int_{t_0}^{t_f} [\eta_1^T C^T Q C \eta_1 + \eta_2^T R \eta_2](\tau)\Delta\tau. \tag{5.6}$$

Note that if $\eta_3 \neq 0$, then (5.4) is an optimal control that locally minimizes (5.3). Now since the final state is free,

$$\phi(x(t),t) = \frac{1}{2} x^T(t) C^T F(t) C x(t).$$

Then the terminal condition on the costate is given by

$$\lambda(t_f) = \left.\frac{\partial \phi}{\partial x}\right|_{t=t_f} = C^T F C x(t_f). \tag{5.7}$$

Next, we will use the sweep condition

$$\lambda(t) = S(t)x(t). \tag{5.8}$$

Now we will use our sweep condition to derive a Riccati equation.

Theorem 5.2. *Assume that* $\left(I + \mu(t) B R^{-1} B^T S^\sigma(t)\right)$ *is invertible and that the condition* (5.8) *holds. Then* S *satisfies a Riccati equation (of the second form)*

$$\begin{aligned} -S^\Delta &= A^T S^\sigma + (I + \mu(t) A^T) S^\sigma \left(I + \mu(t) B R^{-1} B^T S^\sigma\right)^{-1} \left(A - B R^{-1} B^T S^\sigma\right) \\ &\quad + C^T Q C, \\ S(t_f) &= C^T F C. \end{aligned}$$

Proof. The proof here is same as for Theorem 4.15, just replace Q with $C^T Q C$. □

Next, we will establish a control law similar to the state regulator.

Theorem 5.3. *Let* u^* *represent an optimal control that minimizes* (5.3)*. Then* u^* *can be written in the form*

$$u^*(t) = -K(t)x(t), \tag{5.9}$$

where the matrix $K(t)$ *is given by* (4.91) *provided that* $R + \mu(t) B^T S^\sigma(t) B$ *is invertible.*

Proof. The proof is identical to the proof of Theorem 4.18. □

At first, it may be surprising that the optimal control here is written in terms of the state rather than the output. However since the system is observable, each x can be computed based on the knowledge of y. If the system were not observable, we could not construct an optimal control since the state could not be determined from the output. The fact that x is generally of a larger dimension than y only complicates matters. However, it should not be surprising that the feedback gain is same as before, since the proof for Theorem 5.3 is identical to that of Theorem 4.18. Then under the influence of this control law, the closed-loop system becomes

$$x^{\Delta}(t) = (A - BK(t))x(t).$$

Now that we have an optimal input, we will find an optimal cost.

Theorem 5.4. *Let* (5.9) *be an optimal control that minimizes* (5.3)*. Then the optimal cost is given by*

$$J^*(t) = \frac{1}{2}x^T(t)S(t)x(t), \quad t \leq t_f,$$

provided that $R + \mu(t)B^T S^{\sigma}(t)B$ *is invertible.*

Proof. The proof is the same as for Theorem 4.23. □

Now that we have reconciled any conceptual issues with the input, we turn to a more pressing question. While we needed the LQR in terms of the state to determine our control law, what purpose does the output regulator serve? It turns out that the output regulator is a special case of a more general concept: tracking. A block diagram of the output regulator is given in Figure 5.1.

5.2. THE GENERAL TIME-VARYING CASE

In this subsection, we will study a natural extension to the LQR. We would like to find an optimal control law that drives the plant to track a desired reference signal $r(t)$ over $[t_0, t_f]$. Consider the system

$$\begin{aligned} x^{\Delta}(t) &= f(x(t), u(t)) \\ y(t) &= Cx(t). \end{aligned}$$

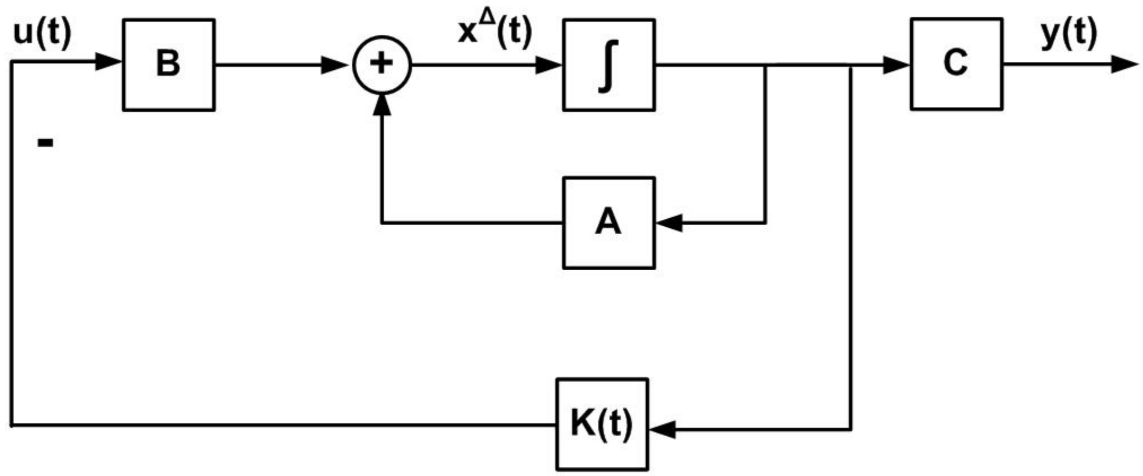

Figure 5.1. The optimal output regulator

Then the quadratic cost function close to this reference signal is

$$J(t_0) = \frac{1}{2}(Cx-r)^T(t_f)P(Cx-r)(t_f)+\frac{1}{2}\int_{t_0}^{t_f} [(Cx-r)^T Q(Cx-r)+u^T Ru](\tau)\Delta\tau, \quad (5.10)$$

where P, $Q \geq 0$ and $R > 0$. Note that

$$e(t) = y(t) - r(t) \quad (5.11)$$

represents the error term. Now we define the Hamiltonian to be

$$H(t, x, u, \lambda) = L(t, x, u) + \lambda^T f(t, x, u). \quad (5.12)$$

Then the state, costate, and stationary equations are given by

$$x^{\Delta}(t) = H_{\lambda}(t, x(t), u(t), \lambda^{\sigma}(t)) = f(x(t), u(t)),$$

$$-\lambda^{\Delta}(t) = H_x(t, x(t), u(t), \lambda^{\sigma}(t)) = f_x^T(t, x(t), u(t))\lambda^{\sigma}(t) + C^T QCx(t) - C^T Qr(t),$$

and

$$0 = H_u(t, x(t), u(t), \lambda^\sigma(t)) = Ru(t) + f_u^T(t, x(t), u(t))\lambda^\sigma(t).$$

Similarly, the boundary conditions are given by

$$\begin{cases} x(t_0) = x_0 \\ \lambda(t_f) = C^T P(Cx(t_f) - r(t_f)). \end{cases} \tag{5.13}$$

Note that the terminal condition for the costate is called an *affine* condition since it is terms of the final state plus some additional term. Also, the input in this case depends on x in a nonlinear matter. As a result, this control is in general a nonlinear feedback of the state and costate.

5.3. THE LINEAR QUADRATIC CASE

In this subsection, we will make the blanketed assumption that the system is completely observable (see [2,44]). Now if we assume the plant is linear, then the Hamiltonian becomes

$$H(x, u, \lambda) = \frac{1}{2}[(Cx - r)Q(Cx - r) + u^T Ru] + \lambda^T(Ax + Bu). \tag{5.14}$$

Thus, the state, costate, and stationary equations are given by

$$\begin{aligned} x^\Delta(t) &= Ax(t) + Bu(t) = Ax(t) - BR^{-1}B^T\lambda^\sigma(t) && (5.15) \\ -\lambda^\Delta(t) &= A^T\lambda^\sigma(t) + C^TQ(Cx(t) - r(t)) && (5.16) \\ u(t) &= -R^{-1}B^T\lambda^\sigma(t). && (5.17) \end{aligned}$$

If we assume that η_1 and η_2 satisfy the constraint

$$\eta_1^\Delta = A\eta_1 + B\eta_2, \tag{5.18}$$

the second variation is given by

$$\ddot{\Phi}(0) = \eta_1^T(t_f)C^T P C \eta_1(t_f) + \int_{t_0}^{t_f} [\eta_1^T C^T Q C \eta_1 + \eta_2^T R \eta_2](\tau)\Delta\tau. \tag{5.19}$$

Note that if $\eta_3 \neq 0$, then (5.17) is an optimal control that locally minimizes (5.10). Now removing the input from the above equations, the forced Hamiltonian system driven by $-C^T Q r(t)$ becomes

$$\begin{bmatrix} x \\ \lambda \end{bmatrix}^{\Delta}(t) = \begin{bmatrix} A & -BR^{-1}B^T \\ -C^T Q C & -A^T \end{bmatrix} \begin{bmatrix} x(t) \\ \lambda^\sigma(t) \end{bmatrix} + \begin{bmatrix} 0 \\ C^T Q \end{bmatrix} r(t). \tag{5.20}$$

This additional term was not present when we were seeking a control in Section 4.

Remark 5.5. Since the final state is free, it is natural to assume that we can use the same sweep condition as found in (5.8). Unfortunately, this condition cannot account for the final reference signal. Instead, we will use an affine sweep condition

$$\lambda(t) = S(t)x(t) - v(t), \tag{5.21}$$

where output $v(t)$ of the adjoint of the closed-loop plant driven by $r(t)$. This second term anticipates the final reference signal.

Next we will use (5.21) to find equations for S and v.

Theorem 5.6. *Assume that $I + \mu(t)BR^{-1}B^T S^\sigma(t)$ is invertible. Furthermore, assume that the condition* (5.21) *holds. Then S satisfies a Riccati equation (of the second form)*

$$\begin{aligned} -S^\Delta &= C^T Q C + A^T S^\sigma \\ &\quad + (I + \mu A^T) S^\sigma \left(I + \mu B R^{-1} B^T S^\sigma\right)^{-1} \left(A - BR^{-1}B^T S^\sigma\right), \qquad (5.22) \\ S(t_f) &= C^T P C \qquad (5.23) \end{aligned}$$

and v satisfies

$$-v^\Delta = [A^T - (I + \mu A) S^\sigma (I + \mu B R^{-1} B^T S^\sigma)^{-1} B R^{-1} B^T] v^\sigma + C^T Q r, \tag{5.24}$$

$$v(t_f) \;=\; C^T Pr(t_f). \tag{5.25}$$

Proof. First, we will rewrite the state equation solely in terms of x and v.

$$\begin{aligned}
x^\Delta &= Ax - BR^{-1}B^T(Sx-v)^\sigma \\
&= Ax - BR^{-1}B^TS^\sigma\left(x+\mu x^\Delta\right) + BR^{-1}B^Tv^\sigma \\
&= (A - BR^{-1}B^TS^\sigma)x - \mu BR^{-1}B^TS^\sigma x^\Delta + BR^{-1}B^Tv^\sigma \\
&= \left(I+\mu BR^{-1}B^TS^\sigma\right)^{-1}\left(A-BR^{-1}B^TS^\sigma\right)x \\
&\quad + \left(I+\mu BR^{-1}B^TS^\sigma\right)^{-1}BR^{-1}B^Tv^\sigma,
\end{aligned} \tag{5.26}$$

provided that $I+\mu(t)BR^{-1}B^TS^\sigma(t)$ is invertible. Now rewriting (5.16), we have

$$\begin{aligned}
-\lambda^\Delta &= A^T\lambda^\sigma + C^TQCx - C^TQr \\
&= A^T(Sx-v)^\sigma + C^TQCx - C^TQr \\
&= A^TS^\sigma(x+\mu(t)x^\Delta) - A^Tv^\sigma + C^TQCx - C^TQr.
\end{aligned}$$

Using (5.26), we have

$$\begin{aligned}
-\lambda^\Delta &= A^TS^\sigma(I+\mu\left(I+\mu BR^{-1}B^TS^\sigma\right)^{-1}\left(A-BR^{-1}B^TS^\sigma\right)]x \\
&\quad +\mu A^TS^\sigma\left(I+\mu BR^{-1}B^TS^\sigma\right)^{-1}BR^{-1}B^Tv^\sigma - A^Tv^\sigma + C^TQCx - C^TQr \\
&= [A^TS^\sigma + \mu A^TS^\sigma\left(I+\mu BR^{-1}B^TS^\sigma\right)^{-1}\left(A-BR^{-1}B^TS^\sigma\right) + C^TQC]x \\
&\quad +[-A^T+\mu A^TS^\sigma\left(I+\mu BR^{-1}B^TS^\sigma\right)^{-1}BR^{-1}B^T]v^\sigma - C^TQr.
\end{aligned} \tag{5.27}$$

However, note that by (5.26)

$$\begin{aligned}
-\lambda^\Delta &= -(Sx-v)^\Delta \\
&= -S^\Delta x - S^\sigma x^\Delta + v^\Delta \\
&= [-S^\Delta - S^\sigma\left(I+\mu BR^{-1}B^TS^\sigma\right)^{-1}\left(A-BR^{-1}B^TS^\sigma\right)]x \\
&\quad -S^\sigma\left(I+\mu BR^{-1}B^TS^\sigma\right)^{-1}BR^{-1}B^Tv^\sigma + v^\Delta.
\end{aligned} \tag{5.28}$$

When comparing (5.27) and (5.28), and combining like terms we have

$$\begin{aligned}
&-[S^{\Delta}+C^TQC+A^TS^{\sigma}+(I+\mu A^T)S^{\sigma}\left(I+\mu BR^{-1}B^TS^{\sigma}\right)^{-1}\left(A-BR^{-1}B^TS^{\sigma}\right)]x \\
&= \quad v^{\Delta}+[A^T-(I+\mu A)S^{\sigma}(I+\mu BR^{-1}B^TS^{\sigma})^{-1}BR^{-1}B^T]v^{\sigma}+C^TQr.
\end{aligned}$$

Then the above equation must be true for all $t \leq t_f$ and for any choice of $x(t_0)$. Note that S does not depend on x for any time t. It follows that the above equation must be true for all values of x. As a result, this means that S must satisfy (5.22) while v satisfies (5.24). This yields the result as desired. □

Before we find an optimal control that minimizes (5.10), we will first introduce a matrix that represents the feedforward gain.

Definition 5.7. A matrix-valued function

$$K_v(t)=(R+\mu(t)B^TS^{\sigma}(t)B)^{-1}B^T. \tag{5.29}$$

is called a feedforward gain, provided that $R+\mu(t)B^TS^{\sigma}(t)B$ is invertible.

Theorem 5.8. *Let u^* represent an optimal control that minimizes* (5.10)*. Assume that $(R+\mu(t)B^TS^{\sigma}(t)B)$ is invertible. Then u^* can be written in the form*

$$u^*(t)=-K(t)x(t)+K_v(t)v^{\sigma}(t), \tag{5.30}$$

where $K(t)$ represents is given by (4.91) *and $K_v(t)$ is given by* (5.29)*.*

Proof. Recall from (5.17) that we have

$$\begin{aligned}
u^*(t) &= -R^{-1}B^T(S(t)x(t)-v(t))^{\sigma} \\
&= -R^{-1}B^TS^{\sigma}(t)[x(t)+\mu(t)x^{\Delta}(t)]+R^{-1}B^Tv^{\sigma}(t) \\
&= -R^{-1}B^TS^{\sigma}(t)[(I+\mu(t)A)x(t)+\mu(t)Bu(t)]+R^{-1}B^Tv^{\sigma}(t).
\end{aligned}$$

Combining like terms, we have

$$(I+\mu(t)R^{-1}B^TS^\sigma(t)B)u(t) = -R^{-1}B^TS^\sigma(t)(I+\mu(t)A)x(t) + R^{-1}B^Tv^\sigma(t).$$

Now premultiplying by R, we have

$$(R+\mu(t)B^TS^\sigma(t)B)u(t) = -B^TS^\sigma(t)(I+\mu(t)A)x(t) + B^Tv^\sigma(t).$$

Assuming that $(R+\mu(t)B^TS^\sigma(t)B)$ is invertible, we get

$$u = -(R+\mu(t)B^TS^\sigma B)^{-1}B^TS^\sigma(I+\mu(t)A)x + (R+\mu(t)B^TS^\sigma B)^{-1}B^Tv^\sigma.$$

Then the feedback gain is given by

$$K(t) = (R+\mu(t)B^TS^\sigma(t)B)^{-1}B^TS^\sigma(t)(I+\mu(t)A) \tag{5.31}$$

while the feedforward gain is

$$K_v(t) = (R+\mu(t)B^TS^\sigma(t)B)^{-1}B^T. \tag{5.32}$$

This concludes the proof. □

Such an optimal input is called an *affine state feedback*. It should be noted that the gain of this additional term is independent of the state. This control (5.30) can also be called a control-tracker law that minimizes (5.10). Now under the control-tracker law, the closed-loop plant can be written as

$$\begin{aligned} x^\Delta(t) &= Ax(t) + B(-K(t)x(t) + K_v(t)v^\sigma(t)) \\ &= (A-BK(t))x(t) + B(R+\mu(t)B^TS^\sigma(t)B)^{-1}B^Tv^\sigma(t). \end{aligned} \tag{5.33}$$

Now we want to rewrite our Riccati equation (of the second form) and output equation in terms of the Kalman gain.

Corollary 5.9. *The Riccati equation* (5.22) *can be written in terms of the Kalman gains as*

$$-S^{\Delta}(t) = A^T S^{\sigma}(t) + (I + \mu(t)A^T)S^{\sigma}(t)(A - BK(t)) + C^T QC \tag{5.34}$$

while the output equation (5.24) *is equivalent to*

$$-v^{\Delta}(t) = (A - BK(t))^T v^{\sigma}(t) + C^T Qr(t). \tag{5.35}$$

Proof. This proof is a direct result from Corollary 4.17. Using (4.91), we have

$$\begin{aligned} -S^{\Delta} &= C^T QC + A^T S + (I + \mu A^T)S^{\sigma} A \\ &\quad -(I + \mu A^T)S^{\sigma} B(R + \mu B^T S^{\sigma} B)^{-1} B^T S^{\sigma}(I + \mu A) \\ &= C^T QC + A^T S + (I + \mu A^T)S^{\sigma} A - (I + \mu A^T)S^{\sigma} BK \\ &= C^T QC + A^T S + (I + \mu A^T)S^{\sigma}(A - BK). \end{aligned}$$

Similarly, using (4.91) we have

$$\begin{aligned} -v^{\Delta} &= (A^T - K^T B^T)v^{\sigma} + C^T Qr \\ &= (A - BK(t))^T v^{\sigma}(t) + C^T Qr. \end{aligned}$$

This yields our equivalent forms. $\square$

A block diagram of the affine control scheme is found in Figure 5.2. In the next theorem, we will find an optimal cost function.

Theorem 5.10. *Let* (5.30) *be an optimal control that minimizes* (5.10)*. Then the optimal cost is given by*

$$J^*(t) = \frac{1}{2}x^T(t)S(t)x(t) - x^T(t)v(t) + w(t), \quad t \le t_f, \tag{5.36}$$

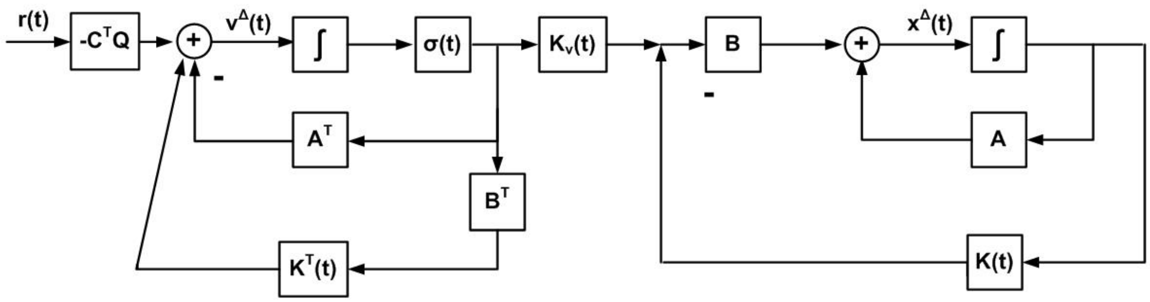

Figure 5.2. LQT as affine state feedback

where the auxiliary function w satisfies

$$\begin{aligned} -w^{\Delta}(t) &= \frac{1}{2}r^T(t)Qr(t) - \frac{1}{2}v^{\sigma T}(t)BK_v(t)v^{\sigma}(t) \\ w(t_f) &= \frac{1}{2}r^T(t_f)Pr(t_f). \end{aligned}$$

Proof. First note that

$$\begin{aligned} &(x^TSx)^{\Delta} = (x^TS)^{\Delta}x + (x^TS)^{\sigma}x^{\Delta} \\ &= (x^{\Delta T}S^{\sigma} + x^TS^{\Delta})x + (x+\mu x^{\Delta})^TS^{\sigma}x^{\Delta} \\ &= [(A-BK)x + BK_vv^{\sigma}]^TS^{\sigma}x + x^TS^{\Delta}x \\ &\quad +[x+\mu(A-BK)x + \mu BK_vv^{\sigma}]^TS^{\sigma}x^{\Delta} \\ &= x^T(A-BK)^TS^{\sigma}x + v^{\sigma T}K_v^TB^TS^{\sigma}x + x^TS^{\Delta}x \\ &\quad +x^T[I+\mu(A-BK)^T]S^{\sigma}[(A-BK)x + BK_vv^{\sigma}] \\ &\quad +\mu v^{\sigma T}K_v^TB^TS^{\sigma}[(A-BK)x+BK_vv^{\sigma}] \\ &= x^T[(A-BK)^TS^{\sigma} + S^{\Delta} + (I+\mu(A-BK)^T)S^{\sigma}(A-BK)]x \\ &\quad +x^T[I+\mu(A-BK)^T]S^{\sigma}BK_vv^{\sigma} + v^{\sigma T}K_v^TB^TS^{\sigma}[I+\mu(A-BK)^T]x \\ &\quad +\mu v^{\sigma T}K_v^TB^TS^{\sigma}BK_vv^{\sigma}. \end{aligned}$$

Now using (4.72), the quadratic cost becomes

$$
\begin{aligned}
J(t) &= \frac{1}{2}[Cx(t_f)-r(t_f)]^T P[Cx(t_f)-r(t_f)] + \frac{1}{2}\int_t^{t_f} [(Cx-r)^T Q(Cx-r)](\tau)\Delta\tau \\
&+\frac{1}{2}\int_t^{t_f} [u^T Ru + (x^T Sx)^{\Delta}](\tau)\Delta\tau - \frac{1}{2}x^T(t_f)S(t_f)x(t_f) + \frac{1}{2}x^T(t)S(t)x(t).
\end{aligned}
$$

From our boundary conditions (5.23) and (5.25), we have

$$
\begin{aligned}
J(t) &= \frac{1}{2}x^T(t)S(t)x(t) - \frac{1}{2}x^T(t_f)C^T Pr(t_f) - \frac{1}{2}r^T(t_f)PCx(t_f) + \frac{1}{2}r^T(t_f)Pr(t_f) \\
&+\frac{1}{2}\int_t^{t_f} [(Cx-r)^T Q(Cx-r) + (x^T Sx)^{\Delta} + u^T Ru](\tau)\Delta\tau.
\end{aligned}
$$

Now using the boundary conditions (5.23) and (5.25) and (5.30), we expand the integrand to get

$$
\begin{aligned}
J(t) &= \frac{1}{2}x^T(t)S(t)x(t) - \frac{1}{2}x^T(t_f)v(t_f) - \frac{1}{2}v^T(t_f)x(t_f) + \frac{1}{2}r^T(t_f)Pr(t_f) \\
&+\frac{1}{2}\int_t^{t_f} \left[x^T\left[(A-BK)^T S^{\sigma} + [I+\mu(A-BK)^T]S^{\sigma}(A-BK)\right]x\right](\tau)\Delta\tau \\
&+\frac{1}{2}\int_t^{t_f} \left[x^T[S^{\Delta} + C^T Q^T C + K^T RK]x\right](\tau)\Delta\tau \\
&+\frac{1}{2}\int_t^{t_f} \left[x^T[(I+\mu(A-BK))^T S^{\sigma}B - K^T R]K_v v^{\sigma}\right](\tau)\Delta\tau \\
&+\frac{1}{2}\int_t^{t_f} \left[v^{\sigma T}K_v^T[B^T S^{\sigma}(I+\mu(A-BK)) - RK]x\right](\tau)\Delta\tau \\
&+\frac{1}{2}\int_t^{t_f} \left[v^{\sigma T}K_v^T[R+\mu B^T S^{\sigma}B]K_v v^{\sigma}\right](\tau)\Delta\tau \\
&+\frac{1}{2}\int_t^{t_f} [r^T Qr - x^T C^T Qr - r^T QCx](\tau)\Delta\tau.
\end{aligned}
$$

Repeating the calculation shown to get (4.98), we can rewrite (5.34) as

$$
\begin{aligned}
-S^{\Delta}(t) &= (A-BK(t))^T S^{\sigma}(t) + [I+\mu(t)(A-BK(t))^T]S^{\sigma}(A-BK(t)) \\
&+C^T Q^T C + K^T(t)RK(t).
\end{aligned}
\tag{5.37}
$$

Therefore the first two integrands sum up to zero. Then J can be rewritten as

$$\begin{aligned} J(t) &= \frac{1}{2}x^T(t)S(t)x(t) - x^T(t_f)v(t_f) + \frac{1}{2}r^T(t_f)Pr(t_f) \\ &+ \int_t^{t_f} \left[x^T[(I+\mu(\tau)(A-BK))^T S^\sigma B - K^T R]K_v v^\sigma - x^T C^T Q r\right](\tau)\Delta\tau \\ &+ \frac{1}{2}\int_t^{t_f} \left[v^{\sigma T} K_v^T [R + \mu(\tau)B^T S^\sigma B]K_v v^\sigma\right](\tau)\Delta\tau + \frac{1}{2}\int_t^{t_f} (r^T Q r)(\tau)\Delta\tau. \end{aligned}$$

Next, note that

$$\begin{aligned} (x^T v)^\Delta &= x^{\Delta T} v^\sigma + x^T v^\Delta \\ &= [(A - BK(t))x + BK_v(t)v^\sigma]^T v^\sigma - x^T[(A - BK(t))^T v^\sigma + C^T Q r] \\ &= v^{\sigma T} K_v^T(t) B^T v^\sigma - x^T C^T Q r \end{aligned} \tag{5.38}$$

and

$$\int_t^{t_f} (x^T v)^\Delta(\tau)\Delta\tau = x^T(t_f)v(t_f) - x^T(t)v(t). \tag{5.39}$$

Then adding and subtracting (5.39) to J yields

$$\begin{aligned} J(t) = &\frac{1}{2}x^T(t)S(t)x(t) - x^T(t)v(t) + \frac{1}{2}r^T(t_f)Pr(t_f) \\ &+ \int_t^{t_f} \left[x^T[(I+\mu(A-BK))^T)S^\sigma B - K^T R]K_v v^\sigma - (x^T v)^\Delta - x^T C^T Q r\right](\tau)\Delta\tau \\ &+ \frac{1}{2}\int_t^{t_f} \left[v^{\sigma T} K_v^T (R + \mu B^T S^\sigma B)K_v v^\sigma\right](\tau)\Delta\tau + \frac{1}{2}\int_t^{t_f} (r^T Q r)(\tau)\Delta\tau \\ = &\frac{1}{2}x^T(t)S(t)x(t) - x^T(t)v(t) + \frac{1}{2}r^T(t_f)Pr(t_f) \\ &\int_t^{t_f} \left[x^T[(I+\mu(A-BK))^T S^\sigma B - K^T R]K_v v^\sigma\right](\tau)\Delta\tau + \frac{1}{2}\int_t^{t_f} (r^T Q r)(\tau)\Delta\tau \\ &+ \frac{1}{2}\int_t^{t_f} \left[v^{\sigma T}[K_v^T(R + \mu B^T S^\sigma B)K_v - K_v^T B^T - BK_v(t)]v^\sigma\right](\tau)\Delta\tau. \end{aligned}$$

Now using the feedback gain (4.91), note that

$$(I + \mu(t)(A - BK(t))^T)S^\sigma(t)B = (I + \mu(t)A^T)S^\sigma(t)B - \mu(t)K^T(t)B^T S^\sigma(t)B$$

$$
\begin{aligned}
&= (I+\mu(t)A^T)S^\sigma(t)B(R+\mu(t)B^TS^\sigma(t)B)^{-1}(R+\mu(t)B^TS^\sigma(t)B) \\
&\quad -\mu(t)K^T(t)B^TS^\sigma(t)B \\
&= \mu(t)K^T(t)B^TS^\sigma(t)B+K^T(t)R-\mu(t)K^T(t)B^TS^\sigma(t)B \\
&= K^T(t)R.
\end{aligned}
$$

The cost can now be written as

$$
\begin{aligned}
J(t) &= \frac{1}{2}x^T(t)S(t)x(t)-x^T(t)v(t)+\frac{1}{2}r^T(t_f)Pr(t_f)+\frac{1}{2}\int_t^{t_f}(r^TQr)(\tau)\Delta\tau \\
&\quad +\frac{1}{2}\int_t^{t_f}\left[v^{\sigma T}[K_v^T(R+\mu B^TS^\sigma B)K_v-K_v^TB^T-BK_v]v^\sigma\right](\tau)\Delta\tau.
\end{aligned}
$$

Using (5.29), the first integrand can be rewritten as

$$
\begin{aligned}
v^{\sigma T}[K_v^T(R+\mu B^TS^\sigma B)K_v-K_v^TB^T-BK_v]v^\sigma &= v^{\sigma T}[K_v^TB^T-K_v^TB^T-BK_v]v^\sigma \\
&= -v^{\sigma T}BK_vv^\sigma.
\end{aligned}
$$

Finally, we define a function w such that w satisfies

$$
\begin{aligned}
-w^\Delta(t) &= \frac{1}{2}r^T(t)Qr(t)-\frac{1}{2}v^{\sigma T}(t)BK_v(t)v^\sigma(t) \\
w(t_f) &= \frac{1}{2}r^T(t_f)Pr(t_f).
\end{aligned}
$$

This concludes the proof. □

Example 5.11. Let $\mathbb{T}=q^{\mathbb{N}_0}$ with $q>1$. Now consider (5.1) associated with the cost function (5.10). Then the state, costate, and stationary equations are given by

$$
\begin{aligned}
x^\Delta(t) &= Ax(t)+Bu(t)=Ax(t)-BR^{-1}B^T\lambda(qt) \\
-\lambda^\Delta(t) &= A^T\lambda(qt)+C^TQ(Cx(t)-r(t)) \\
0 &= B^T\lambda(qt)+Ru(t),
\end{aligned}
$$

subject to the boundary conditions

$$\begin{cases} x(t_0) = x_0 \\ \lambda(t_f) = C^T P(Cx(t_f) - r(t_f)). \end{cases}$$

Next, we find that the feedback and feedforward gains are given to be

$$K(t) = (R + (q-1)tB^T S(qt)B)^{-1} B^T S(qt)(I + (q-1)tA)$$

and

$$K_v(t) = (R + (q-1)tB^T S(qt)B)^{-1} B^T,$$

respectively, so that the optimal control can be written in the form (5.30). Now under the influence of the control-tracker law, the closed-loop plant can be written as

$$x^{\Delta}(t) = (A - BK(t))x(t) - BK_v(t)v(qt).$$

Then the closed-loop Riccati and output equations can be written as

$$\frac{S(t) - S(qt)}{(q-1)t} = A^T S(qt) + (I + (q-1)tA^T)S(qt)(A - BK(t)) + C^T QC$$

and

$$\frac{v(t) - v(qt)}{(q-1)t} = (A - BK(t))^T v(qt) + C^T Qr(t),$$

respectively. Now using (5.30), the optimal cost becomes

$$J^*(t) \;=\; \frac{1}{2}x^T(t)S(t)x(t) - x^T(t)v(t) + w(t),$$

where w satisfies

$$\begin{aligned}\frac{w(t)-w(qt)}{(q-1)t} &= \frac{1}{2}r^T(t)Qr(t)-\frac{1}{2}v^T(qt)BK_v(t)v(qt)\\ w(t_f) &= \frac{1}{2}r^T(t_f)Pr(t_f).\end{aligned}$$

Example 5.12. (The Scalar LQT) Note that the scalar control system can be written as

$$\begin{aligned}x^\Delta(t) &= ax(t)+bu(t)\\ y(t) &= cx(t).\end{aligned}$$

Next, define the cost function to be

$$J(t_0)=\frac{1}{2}p(cx(t_f)-r(t_f))^2+\frac{1}{2}\int_{t_0}^{t_f}[q(Cx-r)^2+Ru^2](\tau)\Delta\tau.$$

where r represents the desired reference signal and R is the weighting constant on u. Now the Hamiltonian is given by the scalar equation

$$H(x,u,\lambda)=\frac{1}{2}(q(cx-r)^2+Ru^2)+\lambda(ax+bu).$$

Note that the state and stationary equations are the same as in Example 4.4. However the costate equation is now given by

$$-\lambda^\Delta(t)=a\lambda^\sigma(t)+c^2qx(t)-cqr(t),$$

where the terminal condition is given by $\lambda(t_f)=c^2px(t_f)-cpr(t_f)$. Now assuming that the costate is linear in x and v, we define

$$\lambda(t)=s(t)x(t)-v(t).$$

Next, we use the sweep method to remove the costate from the state equation.

$$\begin{aligned} x^{\Delta} &= ax - \frac{b^2}{R}(sx - v)^{\sigma} \\ &= ax - \frac{b^2 s^{\sigma}}{R}(x + \mu x^{\Delta}) + \frac{b^2}{R} v^{\sigma} \\ &= \left(a - \frac{b^2}{R} s^{\sigma}\right) x - \frac{\mu b^2 s^{\sigma}}{R} x^{\Delta} + \frac{b^2}{R} v^{\sigma}. \end{aligned}$$

Pre-multiplying by R and combining like terms yields

$$x^{\Delta} = \left(\frac{aR - b^2 s^{\sigma}}{R + \mu b^2 s^{\sigma}}\right) x + \left(\frac{b^2}{R + \mu b^2 s^{\sigma}}\right) v^{\sigma},$$

provided that $R + \mu b^2 s^{\sigma} \neq 0$. Now, note that

$$\begin{aligned} -\lambda^{\Delta} &= a(sx - v)^{\sigma} + c^2 qx - cqr \\ &= as^{\sigma} x + \mu a s^{\sigma} \left[\left(\frac{aR - b^2 s^{\sigma}}{R + \mu b^2 s^{\sigma}}\right) x + \left(\frac{b^2}{R + \mu b^2 s^{\sigma}}\right) v^{\sigma}\right] \\ &\quad -av^{\sigma} + c^2 qx - cqr \\ &= \left[as^{\sigma} + \mu a s^{\sigma}\left(\frac{aR - b^2 s^{\sigma}}{R + \mu b^2 s^{\sigma}}\right)\right] x + \left[-a + \mu a s^{\sigma}\left(\frac{b^2}{R + \mu b^2 s^{\sigma}}\right)\right] v^{\sigma} \\ &\quad -cqr. \end{aligned} \tag{5.40}$$

But

$$\begin{aligned} -\lambda^{\Delta} &= -(sx - v)^{\Delta} \\ &= -s^{\Delta} x - s^{\sigma} x^{\Delta} + v^{\Delta} \\ &= \left[-s^{\Delta} - s^{\sigma}\left(\frac{aR - b^2 s^{\sigma}}{R + \mu b^2 s^{\sigma}}\right)\right] x - \left(\frac{b^2 s^{\sigma}}{R + \mu b^2 s^{\sigma}}\right) v^{\sigma} + v^{\Delta}. \end{aligned} \tag{5.41}$$

Comparing (5.40) and (5.41) in terms of x, we have a Riccati equation

$$-s^{\Delta} = as^{\sigma} + (1 + \mu a) s^{\sigma}\left(\frac{aR - b^2 s^{\sigma}}{R + \mu b^2 s^{\sigma}}\right) + c^2 q, \quad s(t_f) = c^2 p.$$

Similarly, (5.40) and (5.41) in terms of v and r yields the output equation

$$-v^{\Delta} = \left[a - (1+\mu a)\left(\frac{b^2 s^{\sigma}}{R+\mu b^2 s^{\sigma}}\right)\right] v^{\sigma} + cqr, \quad v(t_f) = cpr(t_f).$$

Now the optimal control can by rewritten

$$\begin{aligned} u &= -\frac{b}{R}(sx - v)^{\sigma} \\ &= -\frac{b}{R}[(1+\mu a)x + \mu(t)bu] + \frac{b}{R}v^{\sigma} \\ &= -\left(\frac{(1+\mu a)bs^{\sigma}(t)}{R+\mu b^2 s^{\sigma}(t)}\right)x + \frac{b}{R+\mu b^2 s^{\sigma}}v^{\sigma}. \end{aligned}$$

This implies that the optimal control is of the form

$$u = -kx + k_v v^{\sigma},$$

where the feedback gain is given by

$$k = \frac{(1+\mu a)bs^{\sigma}}{R+\mu b^2 s^{\sigma}} \tag{5.42}$$

and the feedforward gain is given by

$$k_v = \frac{b}{R+\mu b^2 s^{\sigma}}. \tag{5.43}$$

Under the above control law, the closed-loop system becomes

$$x^{\Delta} = (a - bk)x + bk_v v^{\sigma}. \tag{5.44}$$

Then the Riccati equation in terms of the closed-loop system is given by

$$-s^{\Delta} = as^{\sigma}(t) + (1+\mu a)s^{\sigma}(a - bk) + c^2 q.$$

Similarly, the output equation in terms of the reference signal is

$$-v^{\Delta} = (a - bk)v^{\sigma} + cqr.$$

Note that just as in the vector case, the minimum cost is given by

$$J^*(t) = \frac{1}{2}s(t)x^2(t) - x(t)v(t) + w(t), \tag{5.45}$$

where w is a new auxiliary function that satisfies the equations

$$\begin{aligned} -w^{\Delta}(t) &= \frac{1}{2}qr^2(t) - \frac{1}{2}bk_v(t)v^{2\sigma}(t) && (5.46)\\ w(t_f) &= \frac{1}{2}pr^2(t_f). && (5.47) \end{aligned}$$

Next, we consider the case in which we have a known disturbance (see [23, 44]).

Example 5.13. In this example, we will consider the state equation

$$n^{\Delta} = An + Bu,$$

where n is the given state. Suppose that we want a more desirable state, r. Assuming that r is known for $t \leq t_f$, when we plug the substitution $x = n - r$ into the previous state equation, we have

$$\begin{aligned} x^{\Delta} &= n^{\Delta} - r^{\Delta} \\ &= An + Bu - r^{\Delta} \\ &= A(x + r) + Bu - r^{\Delta} \\ &= Ax + Bu + d, \end{aligned}$$

where $d = Ar - r^{\Delta}$ is a known disturbance. Let the cost function over $[t_0, t_f]$ be given by

$$J(t_0) = \frac{1}{2}x^T(t_f)S(t_f)x(t_f) + \frac{1}{2}\int_{t_0}^{t_f} (x^TQx + u^TRu)(\tau)\Delta\tau.$$

The Hamiltonian here is given by

$$H(x,u,\lambda) = \frac{1}{2}(x^TQx + u^TRu) + \lambda^T(Ax + Bu + d).$$

Then the state, costate, and stationary equations are given by

$$\begin{aligned} x^\Delta &= H_\lambda(x,u,\lambda^\sigma) = Ax + Bu + d \\ -\lambda^\Delta &= H_x(x,u,\lambda^\sigma) = A^T\lambda^\sigma + Qx \\ 0 &= H_u(x,u,\lambda^\sigma) = B^T\lambda^\sigma + Ru. \end{aligned}$$

Similarly the boundary conditions are given by

$$\begin{cases} x(t_0) = x_0 \\ \lambda(t_f) = S(t_f)x(t_f). \end{cases}$$

Now we will assume our sweep condition is

$$\lambda = Sx - v. \tag{5.48}$$

Next we want to find equations for S and v so that (5.48) is valid. Rewriting x^Δ as before, we have

$$\begin{aligned} x^\Delta &= \left(I + \mu BR^{-1}B^TS^\sigma\right)^{-1}\left(A - BR^{-1}B^TS^\sigma\right)x \\ &\quad + \left(I + \mu BR^{-1}B^TS^\sigma\right)^{-1}BR^{-1}B^Tv^\sigma + (I + \mu BR^{-1}B^TS^\sigma)^{-1}d. \end{aligned}$$

Similarly, the costate equation can be rewritten as

$$\begin{aligned} -\lambda^\Delta &= [A^TS^\sigma + \mu A^TS^\sigma(I + \mu BR^{-1}B^TS^\sigma)^{-1}(A - BR^{-1}B^TS^\sigma) + Q]x \\ &\quad + [-A^T + \mu A^TS^\sigma(I + \mu BR^{-1}B^TS^\sigma)^{-1}BR^{-1}B^T]v^\sigma \\ &\quad + \mu A^TS^\sigma(I + \mu BR^{-1}B^TS^\sigma)^{-1}d. \end{aligned} \tag{5.49}$$

But differentiating (5.48), we have

$$\begin{aligned}
-\lambda^{\Delta} &= -(Sx - v)^{\Delta} \\
&= [-S^{\Delta} - S^{\sigma}(I + \mu BR^{-1}B^{T}S^{\sigma})^{-1}(A - BR^{-1}B^{T}S^{\sigma})]x \\
&\quad +[-S^{\sigma}(I + \mu BR^{-1}B^{T}S^{\sigma})^{-1}BR^{-1}B^{T}]v^{\sigma} \\
&\quad -S^{\sigma}(I + \mu(t)BR^{-1}B^{T}S^{\sigma})^{-1}d + v^{\Delta}. \qquad (5.50)
\end{aligned}$$

Now comparing (5.49) and (5.50) we have

$$\begin{aligned}
&-[S^{\Delta} + A^{T}S^{\sigma} + (I + \mu A^{T})S^{\sigma}\left(I + \mu BR^{-1}B^{T}S^{\sigma}\right)^{-1}\left(A - BR^{-1}B^{T}S^{\sigma}\right) + Q]x \\
&= v^{\Delta} + [-A^{T} + (I + \mu A^{T})S^{\sigma}(I + \mu BR^{-1}B^{T}S^{\sigma})^{-1}BR^{-1}B^{T}]v^{\sigma} \\
&\quad +(I + \mu A^{T})S^{\sigma}(I + \mu BR^{-1}B^{T}S^{\sigma})^{-1}d.
\end{aligned}$$

Then the above equation must be true for all $t \leq t_f$ and for any $x(t_0)$. Also, S does not depend on x for any time t. It follows that the above equation must be true for all values of x. As a result, this means that S must satisfy

$$-S^{\Delta} = A^{T}S^{\sigma} + (I + \mu A^{T})S^{\sigma}\left(I + \mu BR^{-1}B^{T}S^{\sigma}\right)^{-1}\left(A - BR^{-1}B^{T}S^{\sigma}\right) + Q.$$

Similarly, v must satisfy

$$\begin{aligned}
v^{\Delta} &= [-A^{T} + (I + \mu A^{T})S^{\sigma}(I + \mu BR^{-1}B^{T}S^{\sigma})^{-1}BR^{-1}B^{T}]v^{\sigma} \\
&\quad +(I + \mu A^{T})S^{\sigma}(I + \mu BR^{-1}B^{T}S^{\sigma})^{-1}d. \qquad (5.51)
\end{aligned}$$

Finally, we find an optimal control to be

$$\begin{aligned}
u^{*} &= -R^{-1}B^{T}\lambda^{\sigma} \\
&= -R^{-1}B^{T}S^{\sigma}[(I + \mu A)x + \mu Bu + \mu d] + R^{-1}B^{T}v^{\sigma} \\
&= -(R + \mu B^{T}S^{\sigma}B)^{-1}B^{T}S^{\sigma}(I + \mu A)x \\
&\quad +(R + \mu B^{T}S^{\sigma}B)^{-1}B^{T}(v^{\sigma} - \mu S^{\sigma}d)
\end{aligned}$$

$$= \quad -Kx + K_v(v^\sigma - \mu S^\sigma d). \tag{5.52}$$

The previous example can be referred to as a disturbance-rejection problem. In Section 6, we will consider our state equation when we have an unknown, stochastic disturbance. This leads us to the development of the Kalman filter on time scales.

5.4. REGULATOR WITH FUNCTION OF FINAL STATE FIXED

Recall in Subsubsection 4.4.2 we sought to find an optimal control that would drive the final state to some fixed reference signal. In order to find this open-loop control, we needed to know the value of the final costate. This in turn required the existence of the inverse of some weighted controllability Gramian. However, note that in the development of linear quadratic tracking thus far we have assumed that the final state has been free. As a result, the optimal control with a function of the final state fixed will closely resemble the control-tracker law (5.30). However we will need to consider a boundary value problem similar to (5.40), which previously was only necessary to find the optimal cost. Now consider the plant

$$x^\Delta(t) = Ax(t) + Bu(t) \tag{5.53}$$

associated with the cost functional

$$J(t_0) = \frac{1}{2}x^T(t_f)S(t_f)x(t_f) + \frac{1}{2}\int_{t_0}^{t_f}(x^TQx + u^TRu)(\tau)\Delta\tau, \tag{5.54}$$

where $R > 0$ and $S(t_f)$, $Q \geq 0$. Now we seek an optimal control u^* that not only minimizes $J(t_0)$, but also guarantees

$$\Psi(x(t_f), t_f) = Cx(t_f) - r(t_f) = 0, \tag{5.55}$$

where $r(t_f) \in \mathbb{R}^p$ and C are known. It should be noted that the state, costate, stationary equations are the same as with the LQR. If we assume that η_1 and η_2 satisfy the constraint

$$\eta_1^\Delta = A\eta_1 + B\eta_2, \tag{5.56}$$

the second variation is given by

$$\ddot{\Phi}(0) = \eta_1^T(t_f)S(t_f)\eta_1(t_f) + \int_{t_0}^{t_f} [\eta_1^T Q\eta_1 + \eta_2^T R\eta_2](\tau)\Delta\tau. \tag{5.57}$$

Note that if $\eta_3 \neq 0$, (5.17) is an optimal control that locally minimizes (5.54). Next we assume the boundary conditions are given by

$$\begin{cases} x(t_0) = x_0 \\ \lambda(t_f) = S(t_f)x(t_f) + C^T\alpha, \end{cases} \tag{5.58}$$

where α is an unknown constant multiplier. Furthermore, we assume that $r(t_f) = Cx(t_f)$ is a linear combination of x and α. In other words, r is given by

$$r(t_f) = U(t)x(t) + P(t)\alpha, \tag{5.59}$$

where U and P are still unknown. Next, we assume that we have the affine sweep condition

$$\lambda(t) = S(t)x(t) + V(t)\alpha, \tag{5.60}$$

where V is not necessarily a square matrix. Before we determine an optimal control, we use the condition (5.60) to determine equations for S and V.

Theorem 5.14. *Assume that* $\left(I + \mu(t)BR^{-1}B^TS^\sigma(t)\right)$ *is invertible and that the condition* (5.60) *holds. Then* S *satisfies a Riccati equation (of the second form)*

$$-S^\Delta = Q + A^TS^\sigma + (I + \mu A^T)S^\sigma\left(I + \mu BR^{-1}B^TS^\sigma\right)^{-1}\left(A - BR^{-1}B^TS^\sigma\right), \tag{5.61}$$

for all $t \leq t_f$ with boundary condition $S(t_f)$ and V satisfies

$$\begin{aligned} -V^{\Delta} &= [A^T - (I+\mu A^T)S^{\sigma}(I+\mu BR^{-1}B^TS^{\sigma})^{-1}BR^{-1}B^T]V^{\sigma}, \qquad (5.62)\\ V(t_f) &= C^T. \qquad (5.63) \end{aligned}$$

Proof. First assume that $\left(I+\mu(t)BR^{-1}B^TS^{\sigma}(t)\right)$ is invertible. Then the state equation can be rewritten as

$$\begin{aligned} x^{\Delta} &= Ax - BR^{-1}B^T(Sx+V\alpha)^{\sigma}\\ &= (A - BR^{-1}B^TS^{\sigma})x - \mu BR^{-1}B^TS^{\sigma}x^{\Delta} - BR^{-1}B^TV^{\sigma}\alpha\\ &= \left(I+\mu BR^{-1}B^TS^{\sigma}\right)^{-1}\left(A - BR^{-1}B^TS^{\sigma}\right)x\\ &\quad - \left(I+\mu BR^{-1}B^TS^{\sigma}\right)^{-1}BR^{-1}B^TV^{\sigma}\alpha. \end{aligned}$$

Next, we rewrite the costate equation as follows

$$\begin{aligned} -\lambda^{\Delta} &= A^T(Sx+V\alpha)^{\sigma} + Qx\\ &= [A^TS^{\sigma} + \mu(t)A^TS^{\sigma}(I+\mu(t)BR^{-1}B^TS^{\sigma})^{-1}(A - BR^{-1}B^TS^{\sigma}) + Q]x\\ &\quad + [A^T - \mu(t)A^TS^{\sigma}(I+\mu(t)BR^{-1}B^TS^{\sigma})^{-1}BR^{-1}B^T]V^{\sigma}\alpha. \qquad (5.64) \end{aligned}$$

Now differentiating (5.60), we have

$$\begin{aligned} -\lambda^{\Delta} &= -(Sx+V\alpha)^{\Delta}\\ &= [-S^{\Delta} - S^{\sigma}(I+\mu BR^{-1}B^TS^{\sigma})^{-1}(A - BR^{-1}B^TS^{\sigma})]x\\ &\quad + S^{\sigma}(I+\mu BR^{-1}B^TS^{\sigma})^{-1}BR^{-1}B^T]V^{\sigma}\alpha - V^{\Delta}\alpha. \qquad (5.65) \end{aligned}$$

Comparing (5.64) and (5.65) and combining like terms, we have

$$\begin{aligned} &[S^{\Delta} + A^TS^{\sigma} + (I+\mu A^T)S^{\sigma}\left(I+\mu BR^{-1}B^TS^{\sigma}\right)^{-1}\left(A - BR^{-1}B^TS^{\sigma}\right) + Q]x\\ &= -[V^{\Delta} + [A^T - (I+\mu A^T)S^{\sigma}(I+\mu BR^{-1}B^TS^{\sigma})^{-1}BR^{-1}B^T]V^{\sigma}]\alpha. \end{aligned}$$

Then the above equation must be true for all $t \leq t_f$ and for any choice of $x(t_0)$ and α. Note that S does not depend on x for any time t. Similarly, V does not depend on α. It follows that the above equation must be true for all values of x. As a result, this means that S must satisfy (5.61) while V satisfies (5.62). This concludes the proof. □

Next, we will find an optimal control that minimizes our cost function.

Theorem 5.15. *Let u^* represent an optimal control that minimizes* (5.54)*. Then u^* can be written in the form*

$$u^*(t) = -K(t)x(t) + K_v(t)V^\sigma(t)\alpha, \tag{5.66}$$

where $K(t)$ represents is given by (4.91) *and $K_v(t)$ is given by* (5.29)*.*

Proof. Using (5.17) and (5.60), we have

$$\begin{aligned} u^*(t) &= -R^{-1}B^T(S(t)x(t) + V(t)\alpha)^\sigma(t) \\ &= -R^{-1}B^TS^\sigma(t)[(I + \mu(t)A)x(t) + \mu(t)Bu(t)] + R^{-1}B^TV^\sigma(t)\alpha \\ &= -(R + \mu(t)B^TS^\sigma(t)B)^{-1}B^TS^\sigma(t)(I + \mu(t)A)x(t) \\ &\quad -(R + \mu(t)B^TS^\sigma(t)B)^{-1}B^TV^\sigma(t)\alpha \\ &= -K(t)x(t) + K_v(t)V^\sigma(t)\alpha. \end{aligned} \tag{5.67}$$

This gives the optimal control as desired. □

Note that (5.66) is in terms of the current state whereas the optimal control (4.76) in Subsubsection 4.4.2 was only in terms of a final state difference. Now under this control law, the closed plant can be written as

$$\begin{aligned} x^\Delta(t) &= Ax(t) + B(-K(t)x(t) - K_v(t)V^\sigma(t)\alpha) \\ &= (A - BK(t))x(t) + B(R + \mu(t)B^TS^\sigma(t)B)^{-1}B^TV^\sigma(t)\alpha. \end{aligned} \tag{5.68}$$

Next we want to rewrite our Riccati equation (of the second form) and output equation in terms of the Kalman gain.

Corollary 5.16. *The Riccati equation* (5.61) *is equivalent to*

$$-S^{\Delta}(t) = A^T S^{\sigma}(t) + (I + \mu(t)A^T)S^{\sigma}(t)(A - BK(t)) + Q \tag{5.69}$$

while the output equation (5.62) *is equivalent to*

$$-V^{\Delta}(t) = (A - BK(t))^T V^{\sigma}(t). \tag{5.70}$$

Proof. This proof is a direct result from Corollary 4.17. Using (4.91), we have

$$\begin{aligned}
-S^{\Delta} &= Q + A^T S + (I + \mu A^T)S^{\sigma} A \\
&\quad -(I + \mu A^T)S^{\sigma} B(R + \mu B^T S^{\sigma} B)^{-1} B^T S^{\sigma}(I + \mu A) \\
&= Q + A^T S + (I + \mu A^T)S^{\sigma} A - (I + \mu A^T)S^{\sigma} BK \\
&= Q + A^T S + (I + \mu A^T)S^{\sigma}(A - BK).
\end{aligned}$$

Similarly, using (4.91) we have

$$-V^{\Delta} = (A^T - K^T B^T)V^{\sigma} = (A - BK(t))^T V^{\sigma}(t).$$

This yields our equivalent forms. □

Now looking back at (5.66), note that the feedforward term represents the term that anticipates the final state being equal to some final reference signal. As a result, we want to rewrite the Lagrange multiplier α in terms of this final reference signal. This gives us the following form of our optimal control.

Theorem 5.17. *Let u^* represent an optimal control that minimizes* (5.54)*. Furthermore, assume that* (5.59) *holds. Now suppose that*

$$P(t) := -\int_t^{t_f} V^{\sigma T}(\tau)B(R + \mu(\tau)B^T S^{\sigma}(\tau)B)^{-1} B^T V^{\sigma}(\tau)\Delta\tau.$$

is invertible. Then u^* can be written in the form

$$u^*(t) = -(K(t) - K_v(t)V^\sigma(t)P^{-1}(t)V^T(t))x(t) - K_v(t)V^\sigma(t)P^{-1}(t)r(t_f). \tag{5.71}$$

Proof. Recall earlier that we assumed that $r(t_f) = Cx(t_f)$ could be written as a linear combination of x and α as in (5.59). Note that $P(t_f) = 0$ and $U(t_f) = C$. Now differentiating (5.59) and plugging in (5.68), we have

$$\begin{aligned} 0 &= [r(t_f)]^\Delta \\ &= U^\Delta(t)x(t) + U^\sigma(t)x^\Delta(t) + P^\Delta(t)\alpha \\ &= [U^\Delta(t) + U^\sigma(t)(A - BK(t))]x(t) + [P^\Delta(t) - U^\sigma(t)BK_v(t)V^\sigma(t)]\alpha. \end{aligned} \tag{5.72}$$

Then the terminal value problem for U is given by

$$U^\Delta(t) = -U^\sigma(t)(A - BK(t)), \tag{5.73}$$

subject to $U(t_f) = C$. It follows from the boundary conditions that $U(t) = V^T(t)$ for $t \leq t_f$. Similarly, we can find an equation for P by

$$\begin{aligned} P^\Delta(t) &= U^\sigma(t)BK_v(t)V^\sigma(t) \\ &= V^{\sigma T}(t)B(R + \mu(t)B^T S^\sigma(t)B)^{-1}B^T V^\sigma(t). \end{aligned} \tag{5.74}$$

Then

$$P(t) = -\int_t^{t_f} V^{\sigma T}(\tau)B(R + \mu(\tau)B^T S^\sigma(\tau)B)^{-1}B^T V^\sigma(\tau)\Delta\tau.$$

Now since P is a sort of weighted controllability Gramian, it is a natural to assume that $\det P(t) \neq 0$ for $t \geq t_0$. Then r can be written as

$$r(t_f) = V^T(t)x(t) + P(t)\alpha,$$

which implies that

$$\alpha = P^{-1}(t)[r(t_f) - V^T(t)x(t)].$$

Finally, the optimal control in terms of r is given by

$$\begin{aligned} u^*(t) &= -K(t)x(t) - K_v(t)V^\sigma(t)P^{-1}(t)[r(t_f) - V^T(t)x(t)] \\ &= -(K(t) - K_v(t)V^\sigma(t)P^{-1}(t)V^T(t))x(t) - K_v(t)V^\sigma(t)P^{-1}(t)r(t_f). \end{aligned}$$

This gives the optimal control as desired. □

Note that just as in Subsubsection 4.4.2, the optimal control depends on the inverse of a weighted controllability Gramian. If $\det P(t) = 0$ for all $t \in [t_0, t_f]$, then the problem is said to be *abnormal* and there is no solution. If we pick $C = 0$, the problem reduces to the free final state case in Subsubsection 4.4.3. On the other hand, if we pick $C = I$, the problem reduces to the fixed final state case similar to Subsubsection 4.4.2. However, in Subsubsection 4.4.2, we found an optimal control where $S(t_f) = Q = 0$. As a result, we have found a stronger result than the one we had in Subsubsection 4.4.2.

6. THE KALMAN FILTER ON TIME SCALES

It is often desirable to estimate system states that can only be observed indirectly or inaccurately by the given system. For instance, it is possible that noise from an electromagnetic signal may corrupt some radio communication signal. In such a case, it is necessary to determine a filter that not only removes the noise but retains relevant information. What makes the Kalman filter attractive compared to other filters is that it is designed to minimize the variance of the estimation error. While the Kalman filter is most famous for its role in putting a man on the moon, it has numerous other useful applications in engineering as well. These include analysis of economic systems (see [1,5]) and airborne target tracking systems (see [3,47,48]).

In determining the best estimate for the state, there are two concerns that need to be addressed. First, we know how the system behaves according to the state equation. However the state is not immediately available to us. Therefore we want the expected value of our state estimate to be equal to the true state's expected value. As a result, this will give us an estimate that is not biased in one way or another. Second, we want our state estimate to vary as little as possible away from the true state. Mathematically, we want to find an estimator that ensures that we have the smallest error covariance possible. These two concepts together describe the theory of mean square estimation. As it turns out, the Kalman filter is such an estimator.

In this section, we will consider the discrete and continuous cases of the Kalman filter. We will restrict ourselves to studying processes that can be described by linear systems. We will also assume that our process and measurement noises are independent of each other. First, let us consider the discrete case as introduced by Kalman. In this case, the filter becomes a recursive estimator. There are two phases for this filter design. In the predict phase, the state is estimated from the previous time step to produce a new state estimate at the current time step. Associated with this new estimate is a predicted error covariance matrix. In the update phase, measurement information at the current time step is used to refine our prediction to determine a new state estimate. This new estimate is now the sum of the previous state and the measurement (innovation) at the current time step. For a more in depth introduction of this filter design, one should see Maybeck's book [46, Chapter 1] or Sorenson [51], which includes a detailed historical perspective of least square estimation from Gauss to Kalman. A summary of this filter can be found in Table 6.1.

System: $x(t+1) = Ax(t) + Bu(t) + Gw(t), \quad x(t_0) = x_0$ Measurement: $y(t) = Cx(t) + v(t)$
Assumptions: $x_0 \sim (\overline{x}_0, P_0)$, $w \sim (0, Q\delta(t-s))$, $v \sim (0, R\delta(t-s))$, which are mutually uncorrelated, $R > 0$
Initialization Initial estimate: $\hat{x}(t_0) = \overline{x}_0$ Error covariance: $P(t_0) = \mathbb{E}[(x_0 - \hat{x}_0)(x_0 - \hat{x}_0)^T] = P_0$
Time update (the effect of system dynamics) Estimate update: $\hat{x}(t+1\|t) = A\hat{x}(t\|t) + Bu(t)$ Error covariance: $P(t+1\|t) = AP(t\|t)A^T + GQG^T$
Measurement update (the effect from measurement y) Estimate update: $\hat{x}(t+1\|t+1) = A\hat{x}(t+1\|t) + Bu(t) + K(t+1)[y(t+t) - C\hat{x}(t+1\|t)]$ Error covariance update: $P(t+1) = AP(t)A^T - AP(t)C^T(R + CP(t)C^T)^{-1}CP(t)A^T + GQG^T$
Kalman gain: $K(t+1) = P(t+1\|t)C^T(R + CP(t+1\|t)C^T)^{-1}$

Table 6.1. The Kalman Filter for $\mathbb{T} = \mathbb{Z}$

While the discrete Kalman filter is usually written with separate time and measurement updates, this is not necessarily always the case. In Table 6.2 we consider an alternative form (the predictive case). Just as before the propagation of the error covariance can be calculated recursively. However note that the equation for the error covariance is not in terms of either the measurement or the input (see [14, 43]). As a result, it is possible that both the error covariance and the Kalman gain can be computed *a priori*, meaning that terms can be determined before the filter is implemented. We write this alternative form in delta notation,

System: $\Delta x(t) = Ax(t) + Bu(t) + Gw(t), \quad x(t_0) = x_0$ Measurement: $y(t) = Cx(t) + v(t)$
Assumptions: $x_0 \sim (\overline{x}_0, P_0)$, $w \sim (0, Q\delta(t-s))$, $v \sim (0, R\delta(t-s))$, which are mutually uncorrelated, $R > 0$
Initialization Initial estimate: $\hat{x}(t_0) = \overline{x}_0$ Error covariance: $P(t_0) = \mathbb{E}[(x_0 - \hat{x}_0)(x_0 - \hat{x}_0)^T] = P_0$
Estimate update: $\Delta\hat{x}(t\|t) = A\hat{x}(t\|t-1) + Bu(t) + K(t)[y(t) - C\hat{x}(t\|t-1)]$
Kalman gain: $K(t) = AP(t)C^T(R + CP(t)C^T)^{-1}$
Error covariance update: $\Delta P(t) = AP(t)A^T - AP(t)C^T(R + CP(t)C^T)^{-1}CP(t)A^T + GQG^T$

Table 6.2. The (Predictive) Kalman Filter for $\mathbb{T} = \mathbb{Z}$

It should be noted that the continuous (Kalman–Bucy) filter can be derived from the discrete filter. The filter for the continuous case is found in Table 6.3 below.

However, mathematically speaking, the filtering in continuous time is a more advanced problem than filtering in discrete time. While the discrete filter could be formulated and solved with elementary probability theory and algebraic methods, this is not the case for the continuous filter. Vector processes in discrete time can simply be interpreted as a multidimensional random variable. As a result, the calculation of the stochastic difference equations and propagation of the error covariance is not numerically cumbersome. White noise in discrete time does not pose any problems as well. Vector processes in continuous time become much more tricky since the number of points considered becomes a continuum (uncountable infinity). White noise also presents a problem as it forces the filter to be formulated and solved using Ito differentials and Brownian motion if certain

System: $\dot{x}(t) = Ax(t) + Bu(t) + Gw(t), \quad x(t_0) = x_0$ Measurement: $y(t) = Cx(t) + v(t)$
Assumptions: $x_0 \sim (\overline{x}_0, P_0)$, $w \sim (0, Q\delta(t-s))$, $v \sim (0, R\delta(t-s))$, which are mutually uncorrelated, $R > 0$
Initialization Initial estimate: $\hat{x}(t_0) = \overline{x}_0$ Error covariance: $P(t_0) = \mathbb{E}[(x_0 - \hat{x}_0)(x_0 - \hat{x}_0)^T] = P_0$
Estimate update: $\dot{\hat{x}}(t) = A\hat{x}(t) + Bu(t) + K(t)[y(t) - C\hat{x}(t)]$
Kalman gain: $K(t) = P(t)C^T R^{-1}$
Error covariance update: $\dot{P}(t) = AP(t) + P(t)A^T - P(t)C^T R^{-1} CP(t) + GQG^T$

Table 6.3. The Kalman Filter for $\mathbb{T} = \mathbb{R}$

precautions are not taken. Fortunately, since we will only consider the linear case, we can avoid this issue.

While the error covariances in the discrete represent a sampling of the error covariance in the continuous time, the same cannot be said regarding the gains. Also unlike the discrete case, the continuous Kalman filter cannot be decomposed into separate time and measurement updates. This is due to the fact that all of the error covariances in the discrete case tend to the same error covariance in limit. As a result, we must also assume that our measurement is given in "real-time." Otherwise if the error covariance and gain are known beforehand, they can be determined a priori. Despite their differences, both filters look strikingly similar. Now we would like to extend the Kalman filter onto time scales to unify both cases. Table 6.4 summarizes our results.

In this section, we will assume that there exists a $\delta(\cdot, \cdot)$ such that

$$\int_{t_0}^{t_f} \int_{t_0}^{t_f} x^T(\tau_1) Q \delta(\tau_1, \tau_2) x(\tau_2) \Delta\tau_1 \Delta\tau_2 = \int_{t_0}^{t_f} x^T(\tau) Q x(\tau) \Delta\tau. \tag{6.1}$$

This assumption is valid for isolated time scales as well as the reals with δ being the usual Dirac delta. Since the continuous Kalman filter cannot be decomposed into separate time

System: $x^{\Delta}(t) = Ax(t) + Bu(t) + Gw(t), \quad x(t_0) = x_0$ Measurement: $y(t) = Cx(t) + v(t)$
Assumptions: $x_0 \sim (\overline{x}_0, P_0)$, $w \sim (0, Q\delta(t,s))$, $v \sim (0, R\delta(t,s))$, which are mutually uncorrelated, $R > 0$
Initialization Initial estimate: $\hat{x}(t_0) = \overline{x}_0$ Error covariance: $P(t_0) = \mathbb{E}[(x_0 - \hat{x}_0)(x_0 - \hat{x}_0)^T] = P_0$
Estimate update: $\hat{x}^{\Delta}(t) = A\hat{x}(t) + Bu(t) + K(t)[y(t) - C\hat{x}(t)]$
Kalman gain: $K(t) = (I + \mu(t)A)P(t)C^T(R + \mu(t)CP(t)C^T)^{-1}$
Error covariance update: $P^{\Delta}(t) = AP(t) + (I + \mu(t)A)P(t)A^T$ $-(I + \mu(t)A)P(t)C^T(R + \mu(t)CP(t)C^T)^{-1}CP(t)(I + \mu(t)A^T) + GQG^T$

Table 6.4. The Kalman Filter for $\mathbb{T}$

and measurement updates, the error covariance in our hybrid filter will be though the integrator just as it is for the Kalman–Bucy filter. For this to be, it should be understood that we will assume that the measurement is being updated in "real-time." This means that there is a measurement at the next available point in the time scale. To side step this issue with the error covariances, we will first introduce the observer.

6.1. OBSERVERS

In this subsection, we use the terminology first introduced in the discrete and continuous cases by Luenberger [45]. Consider the linear system

$$\begin{aligned} x^{\Delta}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t). \end{aligned} \tag{6.2}$$

In Sections 4 and 5 concerning the LQR and LQT, it was assumed that the state can be accurately measured. However, this is not always realistic. It is possible that the measurement is incomplete or corrupted by noisy measurements. In this subsection, we will introduce a system that estimates the state, which we will call an observer.

Definition 6.1. The n-dimensional system

$$\hat{x}^{\Delta}(t) = F(t)\hat{x}(t) + G(t)y(t) + I(t)u(t) \tag{6.3}$$

is a (full order) *observer* for (6.2) if $\hat{x}(t_0) = x(t_0)$ implies that the solutions of (6.2) and (6.3) are equal for any u for all $t \geq t_0$.

Theorem 6.2. *The system* (6.3) *is an observer for* (6.2) *if and only if*

$$\begin{cases} F(t) &= A - K(t)C \\ G(t) &= K(t) \\ I(t) &= B. \end{cases} \tag{6.4}$$

As a result, the observer is given by

$$\begin{aligned} \hat{x}^{\Delta}(t) &= (A - K(t)C)\hat{x}(t) + K(t)y(t) + Bu(t) \\ &= A\hat{x}(t) + Bu(t) + K(t)[y(t) - C\hat{x}(t)]. \end{aligned} \tag{6.5}$$

Proof. First, assume that (6.3) is an observer for (6.2). Now let x be a solution to (6.2) and let $\hat{x}$ be a solution to (6.3). Now

$$\begin{aligned} x^{\Delta}(t) - \hat{x}^{\Delta}(t) &= (Ax(t) + Bu(t)) - (F(t)\hat{x}(t) + G(t)y(t) + I(t)u(t)) \\ &= (A - G(t)C)x(t) - F(t)\hat{x}(t) + (B - I(t))u(t) \\ &= 0. \end{aligned} \tag{6.6}$$

Then $\hat{x}(t) = x(t)$ for all u for $t \geq t_0$. Thus (6.4) is true.

Conversely, assume that (6.4) holds. Then

$$\begin{aligned} x^{\Delta}(t) - \hat{x}^{\Delta}(t) &= (Ax(t) + Bu(t)) - ((A - K(t)C)x(t) + B(t)u(t) + K(t)Cx(t)) \\ &= (A - K(t)C)(x(t) - \hat{x}(t)). \end{aligned} \tag{6.7}$$

Now if $\hat{x}(t_0) = x(t_0)$, then $\hat{x}(t) = x(t)$ for all u, $t \geq t_0$. $\square$

The structure of (6.6) follows from plugging (6.4) into (6.3). Thus the full order observer consists of a model of a systems with an extra driving force proportional to the difference $y(t) - \hat{y}(t)$ where $\hat{y} = C\hat{x}(t)$. A block diagram for the observer scheme can be found in Figure 6.1. Note, thus far the choice of the observer gain $K(t)$ is still arbitrary.

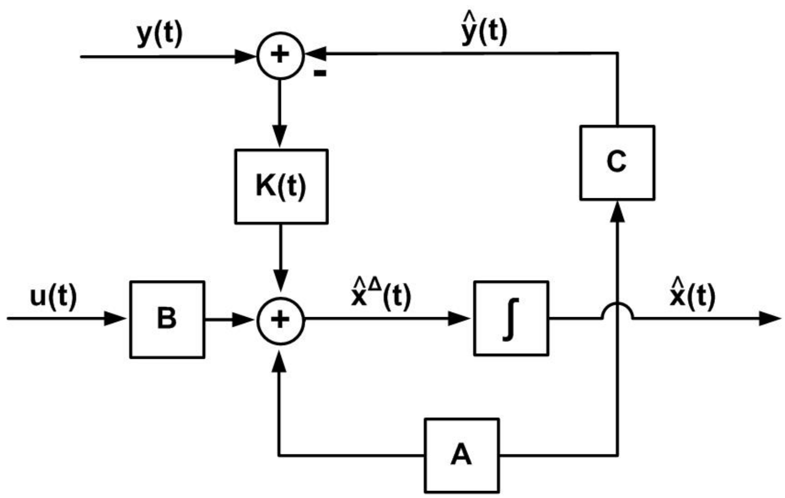

Figure 6.1. The observer design

Next we consider a concept similar to state feedback: output feedback. In Section 4, we considered the linear system (6.2) under the influence of the control law

$$u(t) = -L(t)x(t), \tag{6.8}$$

where x here could be accurately measured and L represents the state feedback gain (4.91) found in Sections 4 and 5. As previously stated, this is not always the case. Therefore, if the state is not available for measurement, it is a natural assumption that the observer (6.5) can be joined by the estimated control law

$$u(t) = -L(t)\hat{x}(t), \tag{6.9}$$

where $L(t)$ is the same matrix as above. The block diagram in Figure 6.2 describes the interconnection of the plant, the observer, and the estimated control law.

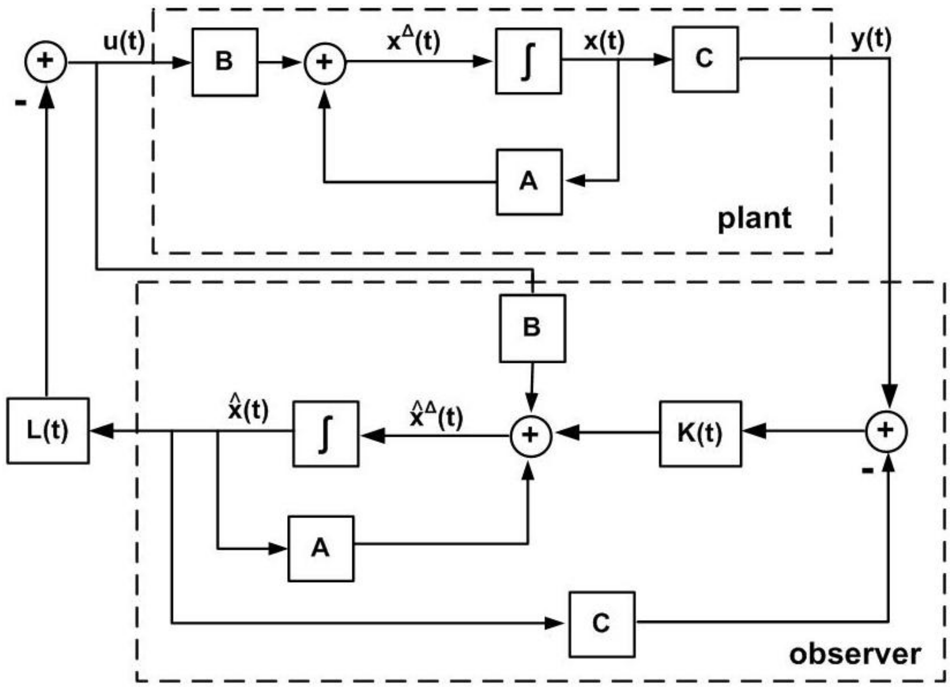

Figure 6.2. The output feedback design

Now substituting this control law into the observer, we get the controller equations

$$\begin{aligned} \hat{x}^{\Delta}(t) &= (A - BL(t) - K(t)C)\hat{x}(t) + K(t)y(t) \\ u(t) &= -L(t)\hat{x}(t). \end{aligned} \tag{6.10}$$

Next, interconnecting the plant with the controller yields the closed-loop system

$$\begin{bmatrix} x^{\Delta}(t) \\ \hat{x}^{\Delta}(t) \end{bmatrix} = \begin{bmatrix} A & -BL(t) \\ K(t)C & A - K(t)C - BL(t) \end{bmatrix} \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix}. \tag{6.11}$$

Note that the dimension of the above system is $2n$, as the dimension of the state is n. Next, we denote the error as

$$e(t) = x(t) - \hat{x}(t). \tag{6.12}$$

Now by using (6.2), (6.5), and (6.12), we get the error system

$$e^{\Delta}(t) = (A - K(t)C)e(t). \tag{6.13}$$

6.2. LINEAR STOCHASTIC SYSTEMS

Before we consider the Kalman filter on time scales, we will consider mean square estimation as well as the propagation of means and variances. Some of the notation used here can be found in Appendix A. In particular, we will denote the expected value of X by $\mathbb{E}(X)$ as well as by $\overline{X}$.

6.2.1. Mean-Square Estimation. In this portion, we will consider the argument on mean-square estimation by Lewis [43]. We will consider a stochastic vector $X \in \mathbb{R}^n$ that cannot be directly determined through measurement. While X is unknown, we will assume we can find the initial statistics of X prior to our measurement. Through measurement we can find another stochastic vector $Z \in \mathbb{R}^p$ that is in some way related to X. Now by knowing Z, we can find some information on X through the joint statistics of X, Z.

Given Z, we would like to determine an estimate $\hat{X}(Z)$ of the value of X such that the estimation error

$$e = X - \hat{X} \tag{6.14}$$

can be made small.

Definition 6.3. Given an estimate $\hat{X}(Z)$, the *mean-square error* is the expected value of the Euclidean squared norm of e,

$$J = \mathbb{E}(e^T e) = \mathbb{E}[(X - \hat{X})^T (X - \hat{X})]. \tag{6.15}$$

The purpose of mean-square estimation can be stated as follows. First note that $\hat{X}$ does not depend on X. Using all available information, we seek to find such an estimate $\hat{X}$ that minimizes

$$J = \mathbb{E}(X^T X) - \hat{X}^T \overline{X} - \overline{X}^T \hat{X} + \hat{X}^T \hat{X} = \mathbb{E}(X^T X) - 2\hat{X}^T \overline{X} + \hat{X}^T \hat{X}. \tag{6.16}$$

Now suppose that before we determine a measurement Z, we know only the *a priori* statistics of X. Differentiating with respect to $\hat{X}$ and setting equal to 0 yields

$$\frac{\partial J}{\partial \hat{X}} = -2\overline{X} + 2\hat{X} = 0, \tag{6.17}$$

implying $\hat{X} = \overline{X}$. Now note that

$$\mathbb{E}(X - \overline{X}) = \overline{X} - \overline{X} = 0. \tag{6.18}$$

This leads us to the following definition.

Definition 6.4. An estimate $\hat{X}$ is said to be *unbiased* if

$$\mathbb{E}(X - \hat{X}) = 0. \tag{6.19}$$

This is equivalent to

$$\mathbb{E}(\hat{X}) = \overline{X}. \tag{6.20}$$

Intuitively this tells us if only the statistics of X are known, the best estimate $\hat{X}$ is the mean of X. Then we can write the a priori error covariance as

$$P_e = \mathbb{E}[ee^T] = \mathbb{E}[(X-\hat{X})(X-\hat{X})^T] = \mathbb{E}[(X-\overline{X})(X-\overline{X})^T], \tag{6.21}$$

which is just the covariance of X. If this error covariance is small, then the associated estimate is a good one.

Now suppose that in addition to knowing the statistics of X, we know a random variable Z through measurement. We can call the estimate $\hat{X}(Z)$ after taking the measurement the *a posteriori* estimate of X. Note that as Z changes, so does $\hat{X}$. Now the problem requires that we find an estimate $\hat{X}$ that minimizes the *conditional mean-square error covariance*

$$\begin{aligned} J' &= \mathbb{E}[(X-\hat{X}(Z))^T(X-\hat{X}(Z))|Z] \\ &= \mathbb{E}(X^TX|Z) - 2\hat{X}(Z)\mathbb{E}(X|Z) + \hat{X}(Z)^T\hat{X}(Z). \end{aligned} \tag{6.22}$$

Again, differentiating with respect to $\hat{X}(Z)$ and setting equal to 0, we have

$$\frac{\partial J'}{\partial \hat{X}} = -2\overline{X|Z} + 2\hat{X}(Z) = 0, \tag{6.23}$$

implying that $\hat{X}(Z) = \overline{X|Z}$. Equation (6.23) is sometimes called the *conditional mean estimate* of X. Note that this estimate is unbiased since

$$\mathbb{E}(\overline{(X-\hat{X})|Z}) = \mathbb{E}[(X-\overline{X|Z})|Z] = \mathbb{E}[\overline{X|Z} - \mathbb{E}(\overline{X|Z}|Z)] = \mathbb{E}[\overline{X|Z} - \overline{X|Z}] = 0. \tag{6.24}$$

Now the error covariance associated with the conditional mean estimate is given by

$$P_e = \mathbb{E}[(X-\hat{X})(X-\hat{X})^T] = \mathbb{E}[(X-\overline{X|Z})(X-\overline{X|Z})^T|Z] = \mathbb{E}[P_{X|Z}]. \tag{6.25}$$

6.2.2. Propagation of Means and Variances. For the rest of this dissertation, we will use lower case letters to denote vector-valued random variables. We also consider time scales such that the assumption in (6.1) is valid. Before we consider the means and variances of the state and measurements, we categorize the vectors that comprise them by the following definitions.

Definition 6.5. A random vector is said to be *stationary* if all of its statistical properties do not vary with time. Processes whose statistical properties do change are referred to as *nonstationary*.

The type of stationary processes we consider is white noise, which we define as follows.

Definition 6.6. A random vector v is said to be a white random vector if and only if

a. $\mathbb{E}(v(t)) = 0$;

b. $\mathbb{E}(v(t)v^T(s)) = R\delta(t,s)$,

where $\delta(\cdot,\cdot)$ is as in (6.1).

Next, we consider when two vectors are orthogonal to each other.

Definition 6.7. Two vector-valued functions $w, v : \mathbb{T} \to \mathbb{R}^m$ are said to be mutually uncorrelated if

$$\mathbb{E}[w(t)v^T(s)] = 0$$

for any $s, t \in \mathbb{T}$.

In this portion, we will assume that both the state and measurement are Gaussian and examine how their statistics will vary with time. Now consider the linear stochastic system

$$\begin{aligned} x^{\Delta}(t) &= Ax(t) + Bu(t) + Gw(t), \quad x(t_0) = x_0, \\ y(t) &= Cx(t) + v(t), \end{aligned} \tag{6.26}$$

where

a. the state $x \in \mathbb{R}^n$ is a nonstationary random variable with mean $\overline{x}$ and covariance $P_x = \mathbb{E}[(x - \overline{x})(x - \overline{x})^T]$.

b. the input $u \in \mathbb{R}^m$ is deterministic.

c. the output $y \in \mathbb{R}^p$ is a nonstationary random variable with mean $\overline{y}$ and covariance $P_y = \mathbb{E}[(y - \overline{y})(y - \overline{y})^T]$.

d. the process noise $w \in \mathbb{R}^l$ is stationary white noise with mean 0 and covariance $\mathbb{E}[ww^T] = Q\delta(t,s)$.

e. the measurement noise $v \in \mathbb{R}^p$ is stationary white noise with mean 0 and covariance $\mathbb{E}[vv^T] = R\delta(t,s)$.

f. x_0, w, and v are assumed to be mutually uncorrelated.

g. P_{x_0}, Q, and R are all positive definite.

h. we can interchange the expectation and integration operations, i.e.,

$$\mathbb{E}\left[\int S\Delta\tau\right] = \int \mathbb{E}[S]\Delta\tau.$$

Theorem 6.8. *[11, Theorem 5.24] The solution of the initial value problem*

$$x^{\Delta}(t) = Ax(t) + Bu(t) + Gw(t), \quad x(t_0) = x_0 \tag{6.27}$$

is given by

$$x(t) = e_A(t,t_0)x_0 + \int_{t_0}^{t} e_A(t,\sigma(\tau))Bu(\tau)\Delta\tau + \int_{t_0}^{t} e_A(t,\sigma(\tau))Gw(\tau)\Delta\tau.$$

Corollary 6.9. *The mean of the solution for* (6.27) *is given by*

$$\overline{x}(t) = e_A(t,t_0)\overline{x}_0 + \int_{t_0}^{t} e_A(t,\sigma(\tau))Bu(\tau)\Delta\tau. \tag{6.28}$$

and the difference between the solution of (6.27) *and its mean is given by*

$$x(t) - \overline{x}(t) = e_A(t, t_0)(x_0 - \overline{x}_0) + \int_{t_0}^{t} e_A(t, \sigma(\tau))Gw(\tau)\Delta\tau.$$

Proof. First, using the assumptions (a), (b), (d), and (h), we have

$$\begin{aligned}
\mathbb{E}(x(t)) &= \mathbb{E}\left[e_A(t, t_0)x_0 + \int_{t_0}^{t} e_A(t, \sigma(\tau))Bu(\tau)\Delta\tau + \int_{t_0}^{t} e_A(t, \sigma(\tau))Gw(\tau)\Delta\tau\right] \\
&= e_A(t, t_0)\overline{x}_0 + \int_{t_0}^{t} \mathbb{E}[e_A(t, \sigma(\tau))Bu(\tau)]\Delta\tau + \int_{t_0}^{t} \mathbb{E}[e_A(t, \sigma(\tau))Gw(\tau)]\Delta\tau \\
&= e_A(t, t_0)\overline{x}_0 + \int_{t_0}^{t} e_A(t, \sigma(\tau))Bu(\tau)\Delta\tau.
\end{aligned}$$

Then the difference between the solution of (6.27) and its mean is as given above. □

Theorem 6.10. *The covariance for the state is given by*

$$P(t) = e_A(t, t_0)\left[P_0 + \int_{t_0}^{t} e_A(t_0, \sigma(\tau))GQG^T e_A^T(t_0, \sigma(\tau))\Delta\tau\right] e_A^T(t, t_0). \tag{6.29}$$

Proof. Using the definition of covariance, Corollary 6.9 and the assumptions (a), (b), (d), and (h), we have

$$\begin{aligned}
P(t) &= \mathbb{E}[(x(t) - \overline{x}(t))(x(t) - \overline{x}(t))^T] \\
&= \mathbb{E}[e_A(t, t_0)(x_0 - \overline{x}_0)(x_0 - \overline{x}_0)^T e_A^T(t, t_0)] \\
&\quad + \mathbb{E}\left[e_A(t, t_0)(x_0 - \overline{x}_0)\int_{t_0}^{t} w^T(\tau)G^T e_A^T(t, \sigma(\tau))\Delta\tau\right] \\
&\quad + \mathbb{E}\left[\int_{t_0}^{t} e_A(t, \sigma(\tau))Gw(\tau)(x_0 - \overline{x}_0)^T e_A^T(t, t_0)\Delta\tau\right] \\
&\quad + \mathbb{E}\left[\int_{t_0}^{t}\int_{t_0}^{t} e_A(t, \sigma(\tau_1))Gw(\tau_1)w^T(\tau_2)G^T e_A^T(t, \sigma(\tau_2))\Delta\tau_1\Delta\tau_2\right] \\
&= e_A(t, t_0)P_0 e_A^T(t, t_0) + \int_{t_0}^{t}\int_{t_0}^{t} e_A(t, \sigma(\tau_1))GQ\delta(\tau_2, \tau_1)G^T e_A^T(t, \sigma(\tau_2))\Delta\tau_1\Delta\tau_2 \\
&= e_A(t, t_0)\left[P_0 + \int_{t_0}^{t} e_A(t_0, \sigma(\tau))GQG^T e_A^T(t_0, \sigma(\tau))\Delta\tau\right] e_A^T(t, t_0).
\end{aligned}$$

This shows (6.29) as desired. □

Note that the mean and covariance of the state should actually be considered as conditional. Next we find the propagation of the state covariance matrix.

Corollary 6.11. *The propagation of the state is given by*

$$P^{\Delta}(t) = AP(t)(I+\mu(t)A)^T + P(t)A^T + GQG^T. \tag{6.30}$$

Proof. Using Theorem 2.20 part (a), we differentiate (6.29) to get

$$\begin{aligned}
P^{\Delta}(t) = & \left[e_A(t,t_0)\left[P_0 + \int_{t_0}^{t} e_A(t_0,\sigma(\tau))GQG^T e_A^T(t_0,\sigma(\tau))\Delta\tau\right]\right]^{\Delta} e_A^T(\sigma(t),t_0) \\
& + e_A(t,t_0)\left[P_0 + \int_{t_0}^{t} e_A(t_0,\sigma(\tau))GQG^T e_A^T(t_0,\sigma(\tau))\Delta\tau\right](e_A^{\Delta}(t,t_0))^T \\
= \; & e_A(\sigma(t),t_0)e_A(t_0,\sigma(t))GQG^T e_A^T(t_0,\sigma(t))e_A^T(\sigma(t),t_0) \\
& + Ae_A(t,t_0)\left[P_0 + \int_{t_0}^{t} e_A(t_0,\sigma(\tau))GQG^T e_A^T(t_0,\sigma(\tau))\Delta\tau\right] e_A^T(t,t_0)(I+\mu(t)A)^T \\
& + e_A(t,t_0)\left[P_0 + \int_{t_0}^{t} e_A(t_0,\sigma(\tau))GQG^T e_A^T(t_0,\sigma(\tau))\Delta\tau\right] e_A^T(t,t_0)A^T \\
= \; & AP(t)(I+\mu(t)A)^T + P(t)A^T + GQG^T.
\end{aligned}$$

This concludes the proof. $\square$

Note that the mean of the measurement is given by

$$\overline{y}(t) = C\overline{x}(t). \tag{6.31}$$

Then the cross-covariance between the state and measurement is given by

$$\begin{aligned}
P_{xy}(t) &= \mathbb{E}[(x-\overline{x})(y-\overline{y})^T] \\
&= \mathbb{E}[(x-\overline{x})[C(x-\overline{x})+v]^T] \\
&= P_x(t)C^T,
\end{aligned}$$

due to assumption (g) that x, v are uncorrelated. Finally, the covariance of the measurement is

$$P_y(t) = \mathbb{E}[(y - \overline{y})(y - \overline{y})^T] = CP_x(t)C^T + R\delta(t, s). \tag{6.32}$$

6.3. THE LINEAR QUADRATIC ESTIMATOR

While it is convenient in theory to assume that the system is exactly known and no modeling disturbances are present, nature is not always so cooperative. The Kalman filter can be thought of as an observer in which the state is reconstructed from noisy measurements. Although the initial state is not known, we will assume that the statistics are known. We can symbolize this by

$$x_0 \sim (\overline{x}_0, P_0). \tag{6.33}$$

Now since the initial covariance is given by $P_0 = \mathbb{E}[(x(t_0) - \hat{x}(t_0))(x(t_0) - \hat{x}(t_0))^T]$, it is natural to assume that $\hat{x}(t_0) = \overline{x}(t_0)$ so that the observer is unbiased. This is sometimes referred to as the initialization of the filter.

Definition 6.12. Let the linear stochastic system be given by (6.26). Then an estimator to (6.26) is given by

$$\hat{x}^{\Delta}(t) = A\hat{x}(t) + Bu(t) + K(t)[y(t) - C\hat{x}(t)], \quad \hat{x}(t_0) = \overline{x}(t_0), \tag{6.34}$$

where $K(t) = (I + \mu(t)A)P(t)C^T(R + \mu(t)CP(t)C^T)^{-1}$ is a Kalman gain.

We will derive the form of this gain later on in this section. Now since $\hat{x}$ is the state estimate, we can call $\hat{y} = C\hat{x}$ the output estimate. Just as we desire $\hat{x}$ to be close to x, our observer is working properly if $y - \hat{y}$ is made to be small. This difference is sometimes called the residual or output estimation error. The observer (6.34) can be thought as the measurement update of our state equation. As mentioned before, we will assume that this measurement is being updated in real-time. Thus, the Kalman gain can be thought of as a "blending" factor that fuses the residual with the state estimate.

Lemma 6.13. *Let* (6.34) *be the observer to the stochastic system* (6.26)*. Then the error system is given by*

$$e^{\Delta}(t) = M(t)e(t) + Gw(t) - K(t)v(t), \tag{6.35}$$

where $M(t) = A - K(t)C$*.*

Proof. Taking the difference of (6.26) and (6.34), we have

$$\begin{aligned} e^{\Delta}(t) &= x^{\Delta}(t) - \hat{x}^{\Delta}(t) \\ &= (A - K(t)C)e(t) + Gw(t) - K(t)v(t) \\ &= M(t)e(t) + Gw(t) - K(t)v(t). \end{aligned}$$

This gives the result as desired. □

Remark 6.14. Note that the solution to (6.35) is of the form

$$e(t) = x(t) - \hat{x}(t) = e_M(t, t_0)e_0 + \int_{t_0}^{t} e_A(t_0, \sigma(\tau))[Gw(\tau) - K(\tau)v(\tau)]\Delta\tau. \tag{6.36}$$

Theorem 6.15. *Let the solution to the error system* (6.35) *be* (6.36)*. Then the error covariance is given by*

$$\begin{aligned} P(t) = e_M(t, t_0)\Bigg[P_0 &+ \int_{t_0}^{t} e_M(t_0, \sigma(\tau))GQG^T e_M^T(t_0, \sigma(\tau))\Delta\tau \\ &+ \int_{t_0}^{t} e_M(t_0, \sigma(\tau))K(\tau)RK^T(\tau)e_M^T(t_0, \sigma(\tau))\Delta\tau\Bigg]e_M^T(t, t_0). \end{aligned} \tag{6.37}$$

Proof. Expanding the definition of the error covariance and using assumptions (a) and (d)–(h), we have

$$P(t) = \mathbb{E}[(x(t) - \hat{x}(t))(x(t) - \hat{x}(t))^T]$$

$$
\begin{aligned}
&= \mathbb{E}[e_M(t,t_0)e_0e_0^Te_M^T(t,t_0)] \\
&\quad +\mathbb{E}\left[e_M(t,t_0)e_0\int_{t_0}^t [w^T(\tau)G^T - v^T(\tau)K^T(\tau)]e_M^T(t,\sigma(\tau))\Delta\tau\right] \\
&\quad +\mathbb{E}\left[\int_{t_0}^t e_M(t,\sigma(\tau))[Gw(\tau)-K(\tau)v(\tau)]e_0)^Te_M^T(t,t_0)\Delta\tau\right] \\
&\quad +\mathbb{E}\left[\int_{t_0}^t\int_{t_0}^t e_M(t,\sigma(\tau_1))[Gw(\tau_1)-K(\tau_1)v(\tau_1)][w^T(\tau_2)G^T - v^T(\tau_2)K^T(\tau_2)]\times\right. \\
&\quad \left. e_M^T(t,\sigma(\tau_2))\Delta\tau_1\Delta\tau_2\right] \\
&= e_M(t,t_0)P_0e_M^T(t,t_0) + \int_{t_0}^t e_M(t,\sigma(\tau))GQG^Te_M^T(t,\sigma(\tau))\Delta\tau \\
&\quad + \int_{t_0}^t e_M(t,\sigma(\tau))KRK^Te_M^T(t,\sigma(\tau))\Delta\tau \\
&= e_M(t,t_0)\left[P_0 + \int_{t_0}^t e_M(t_0,\sigma(\tau))GQG^Te_M^T(t_0,\sigma(\tau))\Delta\tau\right. \\
&\quad \left. + \int_{t_0}^t e_M(t_0,\sigma(\tau))K(\tau)RK^T(\tau)e_M^T(t_0,\sigma(\tau))\Delta\tau\right]e_M^T(t,t_0).
\end{aligned}
$$

This concludes the proof. □

Corollary 6.16. *Let the error covariance be as found in* (6.37). *Then the propagation of the error covariance is given by*

$$
\begin{aligned}
P^\Delta &= AP + (I+\mu A)PA^T + K[R+\mu CPC^T]K^T - K[CP+\mu CPA^T] \\
&\quad -[\mu APC^T + PC^T]K^T + GQG^T. \qquad (6.38)
\end{aligned}
$$

Proof. Using Theorem 2.20 part (a), we differentiate the error covariance (6.37) to get

$$
\begin{aligned}
P^\Delta(t) &= e_M(t,t_0)\left[P_0 + \int_{t_0}^t e_M(t_0,\sigma(\tau))GQG^Te_M^T(t_0,\sigma(\tau))\Delta\tau\right. \\
&\quad \left. + \int_{t_0}^t e_M(t_0,\sigma(\tau))K(\tau)RK^T(\tau)e_M^T(t_0,\sigma(\tau))\Delta\tau\right]^\Delta e_M^T(\sigma(t),t_0) \\
&\quad +e_M(t,t_0)\left[P_0 + \int_{t_0}^t e_M(t_0,\sigma(\tau))GQG^Te_M^T(t_0,\sigma(\tau))\Delta\tau\right. \\
&\quad \left. + \int_{t_0}^t e_M(t_0,\sigma(\tau))K(\tau)RK^T(\tau)e_M^T(t_0,\sigma(\tau))\Delta\tau\right](e_M^\Delta(t,t_0))^T \\
&= e_M(\sigma(t),t_0)e_M(t_0,\sigma(t))[GQG^T + K(\tau)RK^T(\tau)]e_M^T(t_0,\sigma(t))e_M^T(\sigma(t),t_0) \\
&\quad +M(t)e_M(t,t_0)\left[P_0 + \int_{t_0}^t e_M(t_0,\sigma(\tau))GQG^Te_M^T(t_0,\sigma(\tau))\Delta\tau\right.
\end{aligned}
$$

$$
\begin{aligned}
&+\int_{t_0}^{t} e_M(t_0,\sigma(\tau))K(\tau)RK^T(\tau)e_M^T(t_0,\sigma(\tau))\Delta\tau\Big]e_M^T(t,t_0)(I+\mu(t)M(t))^T \\
&+e_M(t,t_0)\Big[P_0+\int_{t_0}^{t} e_M(t_0,\sigma(\tau))GQG^T e_M^T(t_0,\sigma(\tau))\Delta\tau \\
&+\int_{t_0}^{t} e_M(t_0,\sigma(\tau))K(\tau)RK^T(\tau)e_M^T(t_0,\sigma(\tau))\Delta\tau\Big]e_M^T(t,t_0)M^T(t) \\
= \;& M(t)P(t)(I+\mu(t)M(t))^T+P(t)M^T(t)+K(t)RK^T(t)+GQG^T \\
= \;& (A-K(t)C)P(t)(I+\mu(t)(A-K(t)C))^T+P(t)(A-K(t)C)^T \\
&+K(t)RK^T(t)+GQG^T \\
= \;& AP(t)+(I+\mu(t)A)P(t)A^T+K(t)[R+\mu(t)CP(t)C^T]K^T(t) \\
&-K(t)[CP(t)+\mu(t)CP(t)A^T]-[\mu(t)AP(t)C^T+P(t)C^T]K^T(t)+GQG^T.
\end{aligned}
$$

This gives us the error propagation as desired. □

Next, we find the gain using a technique found in Sorenson and Brown [16, 51] for the discrete case.

Theorem 6.17. *Suppose that the error propagation is as given in* (6.38). *Then a Kalman gain can found to be in the form*

$$K(t)=(I+\mu(t)A)P(t)C^T(R+\mu(t)CP(t)C^T)^{-1} \tag{6.39}$$

provided that $R+\mu(t)CP(t)C^T>0$.

Proof. First note that the term

$$K(t)[CP(t)+\mu(t)CP(t)A^T]+[\mu(t)AP(t)C^T+P(t)C^T]K^T(t)$$

is linear in $K(t)$ while $K(t)[R+\mu(t)CP(t)C^T]K^T(t)$ is quadratic in $K(t)$. Now assume that $R+\mu(t)CP(t)C^T$ is positive definite. Next we define $D(t)$ such that

$$D(t)D^T(t)=R+\mu(t)CP(t)C^T. \tag{6.40}$$

Then (6.38) can be rewritten as

$$P^{\Delta}(t) = AP(t) + (I + \mu(t)A)P(t)A^T + K(t)D(t)D^T(t)K^T(t)$$
$$-K(t)[CP(t) + \mu(t)CP(t)A^T] - [\mu(t)AP(t)C^T + P(t)C^T]K^T(t) + GQG^T. \quad (6.41)$$

Next, completing the square we get

$$P^{\Delta}(t) = AP(t) + (I + \mu(t)A)P(t)A^T + K(t)D(t)D^T(t)K^T(t)$$
$$+(K(t)D(t) - N(t))(K(t)D(t) - N(t))^T - N(t)N^T(t) + GQG^T. \quad (6.42)$$

Now comparing (6.41) and (6.42), we have

$$K(t)[CP + \mu(t)CPA^T] + [\mu(t)APC^T + PC^T]K^T(t) = K(t)DN^T + ND^T K^T(t),$$

which implies

$$N(t) = (I + \mu(t)A)P(t)C^T(D^T(t))^{-1}.$$

Then to minimize the diagonal terms of $P^{\Delta}(t)$, we want the middle terms of (6.42) to be zero. Setting

$$K(t)D(t) = N(t),$$

we get

$$\begin{aligned} K(t) &= N(t)D^{-1}(t) \\ &= (I + \mu(t)A)P(t)C^T(D^T(t))^{-1}D^{-1}(t) \\ &= (I + \mu(t)A)P(t)C^T(R + \mu(t)CP(t)C^T)^{-1}. \end{aligned}$$

This concludes the proof. $\square$

Now given a Kalman gain in the form (6.39), it is possible to simplify our error propagation. This form will gives us a Riccati equation (of the first form) as found in (2.27).

Corollary 6.18. *Let the Kalman gain be of the form* (6.39)*. Then the error propagation can be rewritten as a Riccati equation (of the first form)*

$$\begin{aligned} P^{\Delta}(t) &= AP(t) + (I+\mu(t)A)P(t)A^T \\ &\quad -(I+\mu(t)A)P(t)C^T(R+\mu(t)CP(t)C^T)^{-1}CP(t)(I+\mu(t)A^T) + GQG^T. \end{aligned} \tag{6.43}$$

Proof. Plugging (6.39) into (6.38), we get

$$\begin{aligned} P^{\Delta}(t) &= AP(t) + (I+\mu(t)A)P(t)A^T \\ &\quad +(I+\mu(t)A)P(t)C^T(R+\mu(t)CP(t)C^T)^{-1}(R+\mu(t)CP(t)C^T)K^T(t) \\ &\quad -(I+\mu(t)A)P(t)C^T(R+\mu(t)CP(t)C^T)^{-1}CP(t)(I+\mu(t)A^T) \\ &\quad -(I+\mu(t)A)P(t)C^TK^T(t) + GQG^T \\ &= AP(t) + (I+\mu(t)A)P(t)A^T \\ &\quad -(I+\mu(t)A)P(t)C^T(R+\mu(t)CP(t)C^T)^{-1}CP(t)(I+\mu(t)A^T) + GQG^T. \end{aligned}$$

This gives our Riccati equation as desired. □

Note that the term $-(I+\mu(t)A)P(t)C^T(R+\mu(t)CP(t)C^T)^{-1}CP(t)(I+\mu(t)A^T)$ represents the decrease in $P(t)$ due to measurement. We can see this in the following example.

Example 6.19. Assume that $C = 0$ such that there are no measurements. Then the propagation of the error covariance of the linear stochastic system

$$x^{\Delta} = Ax + Bu + Gw$$

is given by

$$P^{\Delta} = AP(I+\mu A)^T + PA^T + GQG^T.$$

Since there are no measurements, the observer equation becomes

$$\hat{x}^{\Delta} = A\hat{x} + Bu.$$

Therefore the estimator propagates according to the deterministic version of the system. Next define $W(t) = \mathbb{E}(x(t)x^T(t))$. Now assume that $u = 0$. Then the optimal estimate is given as $\hat{x} = \overline{x}$. Now it follows that

$$P = \mathbb{E}[(x - \overline{x})(x - \overline{x})^T] = W - \overline{xx^T},$$

such that

$$\begin{aligned} P^{\Delta} &= W^{\Delta} - \overline{x^{\Delta}x^T} - \overline{x^{\sigma}x^{\Delta T}} \\ &= W^{\Delta} - A\overline{xx^T} - \overline{(I + \mu A)x(Ax)^T} \\ &= W^{\Delta} - AW - (I + \mu A)WA^T. \end{aligned}$$

Now comparing equations, we have

$$W^{\Delta} = AW(I + \mu A)^T + WA^T + GQG^T. \tag{6.44}$$

Thus, with no measurement and under a deterministic input, W, P must satisfy the same Lyapunov equation.

Now note the form of the Riccati equation

$$\begin{aligned} -S^{\Delta} &= Q + A^T S^{\sigma} \\ &\quad + (I + \mu A^T)S^{\sigma}\left(I + \mu BR^{-1}B^T S^{\sigma}\right)^{-1}\left(A - BR^{-1}B^T S^{\sigma}\right) \\ &= Q + A^T S^{\sigma} + (I + \mu A^T)S^{\sigma}A \\ &\quad - (I + \mu A^T)S^{\sigma}B(R + \mu B^T S^{\sigma}B)^{-1}B^T S^{\sigma}(I + \mu A), \end{aligned}$$

along with the feedback gain

$$K(t) = (R + \mu(t)B^T S^\sigma(t)B)^{-1} B^T S^\sigma(t)(I + \mu(t)A)$$

associated with the linear quadratic regulator (LQR). These equations appear to be the duals to the Riccati equation (6.43) and gain (6.39) associated with the Kalman filter, suggesting that there is a dual relationship between the LQR and the Kalman filter. Intuitively we can see that both mirror two concepts we have shown are dual in Section 3. The LQR mirrors the concept of controllability in that we are seeking an optimal control such that some cost function is minimized. Similarly the LQE mirrors the concepts of observability in that we are seeking an optimal estimate based on previously observed measurements. A comparison of the Riccati equations and gains is given in Table 6.5.

LQR	LQE
$-S^\Delta(t)$	$P^\Delta(t)$
$S^\sigma(t)$	$P(t)$
A^T	A
B^T	C
$R > 0$	$R > 0$
Q	GQG^T

Table 6.5. A comparison of the LQR and LQE

Finally, we show that when the final time t_f is fixed, the optimal estimator problem associated with

$$\begin{aligned} x^\Delta(t) &= Ax(t) + Gw(t), \quad x(t_0) = x_0, \\ y(t) &= Cx(t) + v(t) \end{aligned} \tag{6.45}$$

can be rewritten as an optimal regulator problem. Now in estimating $x(t_f)$, we want to find a number β from the measurement such that

$$\mathbb{E}[(\alpha^T x(t_f) - \beta)(\alpha^T x(t_f) - \beta)^T] \tag{6.46}$$

is a minimum where α is some constant vector. We will refer to β as the minimum variance estimate of $\alpha^T x(t_f)$. Note that since all random variables of (6.45) are assumed to be Gaussian, we can derive β from linear operations of y. In other words, we assume that there exists some function $s(t; \alpha, t_f)$ for $[t_0, t_f]$ such that

$$\beta = \int_{t_0}^{t_f} s^T(\tau; \alpha, t_f) y(\tau) \Delta\tau. \tag{6.47}$$

Now plugging in (6.47), we can show that

$$J = \mathbb{E}\left[\left(\alpha^T x(t_f) - \int_{t_0}^{t_f} s^T(\tau; \alpha, t_f) y(\tau) \Delta\tau\right)^2\right] \tag{6.48}$$

can be rewritten as quadratic performance index. This brings us to our next theorem.

Theorem 6.20. *Let r be a deterministic solution of*

$$r^{\Delta}(t) = -A^T r^{\sigma}(t) + C^T s(t), \quad r(t_f) = \alpha. \tag{6.49}$$

Then (6.49) *is associated with the cost function*

$$J = r^T(t_0) P_0 r(t_0) + \int_{t_0}^{t_f} (r^{\sigma T} G Q G^T r^{\sigma} + s^T R s)(\tau) \Delta\tau. \tag{6.50}$$

Proof. First assume that r is of the same dimension as x. Then using the equations (6.45) and (6.49), we have

$$\begin{aligned} (r^T x)^{\Delta} &= r^{\Delta T} x + r^{\sigma T} x^{\Delta} \\ &= -r^{\sigma T} A x + s^T C x + r^{\sigma T} A x + r^{\sigma T} G w \\ &= s^T y - s^T v + r^{\sigma T} G w. \end{aligned}$$

Integrating both sides from t_0 to t_f, we have

$$r^T(t_f)x(t_f) - r^T(t_0)x(t_0) = \int_{t_0}^{t_f} s^T(\tau)y(\tau)\Delta\tau - \int_{t_0}^{t_f} s^T(\tau)v(\tau)\Delta\tau + \int_{t_0}^{t_f} r^{\sigma T}(\tau)Gw(\tau)\Delta\tau.$$

Now, rearranging terms we have

$$\alpha^T x(t_f) - \int_{t_0}^{t_f} s^T(\tau)y(\tau)\Delta\tau = r^T(t_0)x(t_0) - \int_{t_0}^{t_f} s^T(\tau)v(\tau)\Delta\tau + \int_{t_0}^{t_f} r^{\sigma T}(\tau)Gw(\tau)\Delta\tau.$$

Recalling that x_0, w, v are independent of each other by assumption (f), we can write

$$\begin{aligned} \mathbb{E}\left[\left(\alpha^T x(t_f) - \int_{t_0}^{t_f} s^T(\tau;\alpha,t_f)y(\tau)\Delta\tau\right)^2\right] &= \mathbb{E}[(r^T(t_0)x(t_0))^2] \\ +\mathbb{E}\left[\left(\int_{t_0}^{t_f} s^T(\tau)v(\tau)\Delta\tau\right)^2\right] &+ \mathbb{E}\left[\left(\int_{t_0}^{t_f} r^{\sigma T}(\tau)Gw(\tau)\Delta\tau\right)^2\right]. \end{aligned}$$

Next, we will calculate the expectations of each term on the right-hand side separately. Thus,

$$\begin{aligned} \mathbb{E}[(r^T(t_0)x(t_0))^2] &= \mathbb{E}[r^T(t_0)x(t_0)x^T(t_0)r(t_0)] \\ &= r^T(t_0)\mathbb{E}[x(t_0)x^T(t_0)]r(t_0) \\ &= r^T(t_0)P_0 r(t_0). \end{aligned}$$

Also,

$$\begin{aligned} \mathbb{E}\left[\left(\int_{t_0}^{t_f} s^T(\tau)v(\tau)\Delta\tau\right)^2\right] &= \mathbb{E}\left[\int_{t_0}^{t_f}\int_{t_0}^{t_f} s^T(\tau_1)v(\tau_1)v^T(\tau_2)s(\tau_2)\Delta\tau_1\Delta\tau_2\right] \\ &= \int_{t_0}^{t_f}\int_{t_0}^{t_f} s^T(\tau_1)\mathbb{E}[v(\tau_1)v^T(\tau_2)]s(\tau_2)\Delta\tau_1\Delta\tau_2 \\ &= \int_{t_0}^{t_f}\int_{t_0}^{t_f} s^T(\tau_1)R\delta(\tau_1,\tau_2)s(\tau_2)\Delta\tau_1\Delta\tau_2 \\ &= \int_{t_0}^{t_f} s^T(\tau)Rs(\tau)\Delta\tau. \end{aligned}$$

Similarly,

$$\begin{aligned}
\mathbb{E}\left[\left(\int_{t_0}^{t_f} r^{\sigma T}(\tau)Gw(\tau)\Delta\tau\right)^2\right] &= \mathbb{E}\left[\int_{t_0}^{t_f}\int_{t_0}^{t_f} r^{\sigma T}(\tau_1)Gw(\tau_1)w^T(\tau_2)G^T r^{\sigma}(\tau_2)\Delta\tau_1\Delta\tau_2\right] \\
&= \int_{t_0}^{t_f}\int_{t_0}^{t_f} r^{\sigma T}(\tau_1)G\mathbb{E}[w(\tau_1)w^T(\tau_2)]G^T r^{\sigma}(\tau_2)\Delta\tau_1\Delta\tau_2 \\
&= \int_{t_0}^{t_f}\int_{t_0}^{t_f} r^{\sigma T}(\tau_1)GQ\delta(\tau_1,\tau_2)G^T r^{\sigma}(\tau_2)\Delta\tau_1\Delta\tau_2 \\
&= \int_{t_0}^{t_f} r^{\sigma T}(\tau)GQG^T r^{\sigma}(\tau)\Delta\tau.
\end{aligned}$$

Therefore,

$$J = r^T(t_0)P_0 r(t_0) + \int_{t_0}^{t_f} (r^{\sigma T}GQG^T r^{\sigma} + s^T Rs)(\tau)\Delta\tau.$$

This gives the result as desired. □

Now all of the terms on the right of (6.50) are deterministic. Then the goal of this regulator problem is to find a function s such that (6.50) is minimized. Note that this performance index has an initial weighting function rather than a terminal weighting function. This is due to the fact that the equations that describe the optimal estimator operate "forward in time," whereas the equations that describe the optimal regulator operate "backward in time."

7. CONCLUSIONS

In this portion, we will briefly discuss some open problems concerning the optimal regulator and the optimal estimator on time scales. From Section 6, we determined an optimal estimate for the linear stochastic system

$$\begin{aligned} x^{\Delta}(t) &= Ax(t) + Bu(t) + Gw(t), \quad x(t_0) = x_0, \\ y(t) &= Cx(t) + v(t). \end{aligned} \tag{7.1}$$

As it stands, we have completely ignored the issue of stability. An optimal estimate, in general does not imply anything about the stability of the error system

$$e^{\Delta}(t) = (A - K(t)C)e(t). \tag{7.2}$$

The same can be said of the optimal control we have found to minimize the performance index with regard to the closed-loop system

$$x^{\Delta}(t) = (A - BL(t))x(t), \tag{7.3}$$

where $x_0 \sim (\overline{x}_0, P_0)$, $w \sim (0, W\delta(t,s))$, $v \sim (0, V\delta(t,s))$, which are mutually uncorrelated. It is worth mentioning that more conditions must be in place to consider each problem in the steady-state sense. First, we can restrict ourselves to unbounded time scales with bounded graininess (i.e., $\mu_{\max} < \infty$). Second, we must show that uniform controllability and observability of the systems describing both the optimal regulator and optimal estimator problem are sufficient conditions for stability. However, such conditions for stability are overkill. These conditions must be further refined for two concepts slightly weaker than controllability and observability; stabilizability and detectability, respectively.

Along with steady-state results, it is also of great interest to obtain spectral factorization results in the frequency domain. It is also of interest to derive the Kalman gain and Riccati equation directly from a generalized, nonstationary form of the Wiener–Hopf equation. Preliminary results, while not perfected, have been promising.

Similarly, we can extend our knowledge of the Kalman filter by considering the following. In Section 6, we considered only the case where the process and measurement noise were both white and uncorrelated. However this is not always the case. Also, in this thesis we have restricted ourselves to linear systems. In future, we will explore extending these results for nonlinear systems, namely the extended Kalman filter (EKF). Likewise, we can derive other filtering problems similar in structure to the Kalman filter. For instance, the inverse of the covariance matrix $P(t)$ is called the *information* matrix. Performing similar matrix calculations, we can generalize the information filter for time scales.

Now that the LQR and LQE problems have been unified and extended to dynamic equations on time scales, we can introduce another fundamental problem in optimal control. The linear quadratic Gaussian (LQG) problem concerns stochastic linear systems disturbed by white noise, corrupted measurements of the state, and associated with a quadratic cost function. This problem is essentially a combination of the LQR and LQE. Consider the linear system (7.1) associated with the cost function

$$J(t_0) = \mathbb{E}\left[\frac{1}{2}x^T(t_f)Fx(t_f) + \frac{1}{2}\int_{t_0}^{t_f}(x^TQx + u^TRu)(\tau)\Delta\tau\right], \tag{7.4}$$

where F, $Q \geq 0$ and $R > 0$. The control objective then is to find an optimal control that minimizes (7.4). Next we introduce the controller equations

$$\begin{aligned} \hat{x}^{\Delta}(t) &= A\hat{x}(t) + Bu(t)K(t)[y(t) - C\hat{x}(t)], \quad \hat{x}(t_0) = \overline{x}_0 \\ u(t) &= -L(t)\hat{x}(t). \end{aligned} \tag{7.5}$$

Next, define the error to be

$$e(t) = x(t) - \hat{x}(t). \tag{7.6}$$

Now using the state-estimate feedback in (7.5), the closed-loop can be written as

$$
\begin{aligned}
x^{\Delta}(t) &= Ax(t) - BL(t)\hat{x}(t) + Gw(t) \\
&= (A - BL(t))x(t) + BL(t)e(t) + Gw(t).
\end{aligned}
$$

Similarly, the error system can be written as

$$
\begin{aligned}
e^{\Delta}(t) &= x^{\Delta}(t) - \hat{x}^{\Delta}(t) \\
&= A(x(t) - \hat{x}(t)) + Gw(t) + K(t)[Cx(t) - C\hat{x}(t)] \\
&= (A - K(t)C)e(t) + Gw(t).
\end{aligned}
$$

Now the closed-loop dynamics can be described by the system

$$
\begin{bmatrix} x \\ e \end{bmatrix}^{\Delta}(t) = \begin{bmatrix} A - BL(t) & BL(t) \\ 0 & A - K(t)C \end{bmatrix} \begin{bmatrix} x \\ e \end{bmatrix}(t). \tag{7.7}
$$

The system (7.7) describes the *separability principle* of the LQR and LQE. Intuitively, this means that both can be designed and computed independent of each other. Again, this implies that the LQR and LQE are dual problems of each other. Then the Kalman filter estimates the state based previous measurements and is associated with the Riccati equation

$$
\begin{aligned}
P^{\Delta} &= AP + (I + \mu A)PA^T - (I + \mu A)PC^T(V + \mu CPC^T)^{-1}CP(I + \mu A^T) \\
&\quad + GWG^T, \\
P(t_0) &= \mathbb{E}(x_0 x_0^T).
\end{aligned}
$$

Using the solution P, the Kalman gain becomes

$$
K = (I + \mu A)PC^T(V + \mu CPC^T)^{-1}. \tag{7.8}
$$

Next, the Riccati equation that solves the LQR problem is given by

$$\begin{aligned} -S^{\Delta} &= A^T S^{\sigma} + (I + \mu A^T) S^{\sigma} \left(I + \mu B R^{-1} B^T S^{\sigma} \right)^{-1} \left(A - B R^{-1} B^T S^{\sigma} \right) + Q, \\ S(t_f) &= F. \end{aligned}$$

Using the solution S, the feedback gain becomes

$$L = (R + \mu B^T S^{\sigma} B)^{-1} B^T S^{\sigma} (I + \mu A). \tag{7.9}$$

Finally, the equations (7.4) through (7.9) describe the LQG on time scales. An advantage to the LQG is that the structure of the estimator is given by the process, therefore it does not need to be known beforehand.

APPENDIX A

ELEMENTS OF PROBABILITY THEORY

In this section, we will consider some basic concepts from probability theory.

Definition A.1. Let the set Ω represent the sample space. Then a family $\mathcal{F}$ of subsets of Ω is said to be a σ-algebra on Ω provided that

(i.) $\emptyset \in \mathcal{F}$;

(ii.) if $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$;

(iii.) if $A_n \in \mathcal{F}$ for all $n \in \mathbb{N}$, then $\bigcup_{n\in\mathbb{N}} A_n \in \mathcal{F}$.

Definition A.2. Let Ω be a set and let $\mathcal{F}$ be a σ-algebra of subsets of Ω. Then the function $\mathbb{P} : \mathcal{F} \to [0, 1]$ is said to be a *probability measure* provided

(i.) $0 < \mathbb{P}(\cdot) \leq 1$;

(ii.) $\mathbb{P}(\Omega) = 1$;

(iii.) if $A_n \in \mathcal{F}$ for all $n \in \mathbb{N}$ are disjoint, then $\mathbb{P}\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \mathbb{P}(A_n)$.

Furthermore, the measure is said to be complete if

(iii.) $A \in \mathcal{F}$, $B \subset A$ with $\mathbb{P}(A) = 0$, then $B \in \mathcal{F}$ and $\mathbb{P}(B) = 0$.

Definition A.3. The Borel algebra is the smallest σ-algebra $\mathcal{B}$ of subsets of $\mathbb{R}^n$ containing all open sets.

The triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a probability space (or Kolmogorov triple) where Ω is any set, $\mathcal{F}$ is σ-algebra of subsets of Ω, and $\mathbb{P}$ is probability measure defined on $\mathcal{F}$.

Definition A.4. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, then a function $X : \Omega \to \mathbb{R}^n$ is called $\mathcal{F}$-measurable if

$$X^{-1}(B) = \{\omega \in \Omega : X(\omega) \in B\} \in \mathcal{F} \quad \text{for all} \quad B \in \mathcal{B}.$$

X is also called a random variable.

Definition A.5. Let $X : \Omega \to \mathbb{R}^n$ be a random variable. Then the σ-algebra generated by X is given by

$$\sigma(X) = \{X^{-1}(B) : B \in \mathcal{B}\}.$$

This is the smallest σ-algebra of Ω with respect to which X is measurable.

Definition A.6. Let $X : \Omega \to \mathbb{R}^n$ be a random variable. If $\displaystyle\int_\Omega |X(\omega)| d\mathbb{P}(\omega) < \infty$, then

$$\mathbb{E}(X) = \int_\Omega X(\omega)\, d\mathbb{P}(\omega) = \overline{X}$$

is said to be the *expected value* of X.

We use both overbars and $\mathbb{E}(\cdot)$ in this thesis to denote expectation. Next, we consider the linearity property of expectation.

Theorem A.7. *Let $X, Y : \Omega \to \mathbb{R}^n$ be two random variables and let α be a constant. Then*

$$\mathbb{E}(\alpha X + Y) = \alpha \mathbb{E}(X) + \mathbb{E}(Y).$$

Definition A.8. Let $X : \Omega \to \mathbb{R}^n$ be a random variable. If $\displaystyle\int_\Omega |X(\omega)|^2 d\mathbb{P}(\omega) < \infty$, then

$$\mathbb{V}\mathrm{ar}(X) = \mathbb{E}(|X - \mathbb{E}(X)|^2) = \mathbb{E}(X^2) - (\mathbb{E}(X))^2$$

is said to be the *variance* of X.

Definition A.9. Let $X, Y : \Omega \to \mathbb{R}^n$ be two random variables. Then the *covariance* between X and Y is given by

$$\mathbb{C}\mathrm{ov}(X, Y) = \mathbb{E}(X - \mathbb{E}(X))\mathbb{E}(Y - \mathbb{E}(Y)) = \mathbb{E}(XY) - \mathbb{E}(X)\mathbb{E}(Y).$$

Theorem A.10. *Let $X, Y : \Omega \to \mathbb{R}^n$ be two random variables and let α be a constant. Then* *(i.)* $\mathbb{V}\mathrm{ar}(\alpha X) = \alpha^2\, \mathbb{V}\mathrm{ar}(X)$;

(ii.) $\mathbb{V}\mathrm{ar}(X+Y) = \mathbb{V}\mathrm{ar}(X) + \mathbb{V}\mathrm{ar}(Y) + 2\,\mathbb{C}\mathrm{ov}(X,Y)$.

Definition A.11. Let $\mathcal{F}$ be a σ-algebra and let A, B be two subsets (events) in $\mathcal{F}$. Then A, B are called *independent* if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B).$$

Moreover, if $\mathbb{P}(B) > 0$, then

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

Definition A.12. Let $\mathcal{F}$ be a σ-algebra and let $\mathcal{G}, \mathcal{H} \subset \mathcal{F}$ be sub-σ-algebras of $\mathcal{F}$. Then $\mathcal{G}, \mathcal{H}$ are said to be *independent* if A, B are independent for any $A \in \mathcal{G}$ and $B \in \mathcal{H}$.

Next, we consider when two random variables are independent.

Definition A.13. Let $X, Y : \Omega \to \mathbb{R}^n$ be two random variables. Then X, Y are independent if $\sigma(X)$ and $\sigma(Y)$ are independent.

Theorem A.14. *If X, Y are independent, then $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$ provided that $\mathbb{E}(|X|)$ and $\mathbb{E}(|Y|)$ are both finite.*

Now we will consider conditional expectation.

Definition A.15. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra of $\mathcal{F}$. Let $X : \Omega \to \mathbb{R}^n$ be a random variable where $\mathbb{E}(|X|) < \infty$. Then the *conditional expectation* of X given $\mathcal{G}$ is defined to be $\mathbb{E}(X|\mathcal{G}) \equiv Y$, where Y is a random variable satisfying

(i.) $\mathbb{E}(|Y|) < \infty$;

(ii.) Y is $\mathcal{G}$-measurable;

(iii.) $\displaystyle\int_A Y\, d\mathbb{P} = \int_A X\, d\mathbb{P}$ for any $A \in \mathcal{G}$.

Next, we consider some basic properties of conditional expectation.

Theorem A.16. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra of $\mathcal{F}$. Let $X, Z : \Omega \to \mathbb{R}^n$ be two random variables, where $\mathbb{E}(|X|)$ and $\mathbb{E}(|Z|)$ are both finite, and let α be a constant. Then*

(i.) $\mathbb{E}(X|\mathcal{G}) = \overline{X|\mathcal{G}} \geq 0$ *a.s. if* $X \geq 0$;

(ii.) $\mathbb{E}(\alpha X + Z|\mathcal{G}) = \alpha\mathbb{E}(X|\mathcal{G}) + \mathbb{E}(Z|\mathcal{G})$;

(iii.) $\mathbb{E}(\mathbb{E}(X|\mathcal{G})) = \mathbb{E}(X)$;

(iv.) $\mathbb{E}(X|\mathcal{G}) = X$ *if* X *is* $\mathcal{G}$*-measurable;*

(v.) $\mathbb{E}(X|\mathcal{G}) = \mathbb{E}(X)$ *if* X *is independent of* $\mathcal{G}$;

(vi.) $\mathbb{E}(ZX|\mathcal{G}) = Z\mathbb{E}(X|\mathcal{G})$ *if* Z *is* $\mathcal{G}$*-measurable.*

Remark A.17. Let $X, Z : \Omega \to \mathbb{R}^n$ be two random variables. Then we can write the conditional expectation of X given Z as

$$\mathbb{E}(X|Z) = \mathbb{E}(X|\sigma(Z)),$$

where $\sigma(Z)$ is a σ-algebra generated by Z.

Definition A.18. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra of $\mathcal{F}$. Let $X : \Omega \to \mathbb{R}^n$ be a random variable where $\mathbb{E}(|X|^2) < \infty$. Then the *conditional variance* of X given $\mathcal{G}$ is defined to be a random variable $\mathbb{E}[(X - \mathbb{E}(X|\mathcal{G}))^2|\mathcal{G}]$.

Now we consider some basic properties of conditional variance.

Theorem A.19. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra of $\mathcal{F}$. Let $X : \Omega \to \mathbb{R}^n$ be a random variables where $\mathbb{E}(|X|^2)$ is finite. Then*

(i.) $\mathbb{V}\mathrm{ar}(X|\mathcal{G}) \geq 0$ *a.s. if* $X \geq 0$;

(ii.) $\mathbb{V}\mathrm{ar}(X|\mathcal{G}) = \mathbb{E}(X^2|\mathcal{G}) - (\mathbb{E}(X|\mathcal{G}))^2$;

(iii.) $\mathbb{V}\mathrm{ar}(\mathbb{E}(X|\mathcal{G})) = \mathbb{E}[\mathbb{E}(X|\mathcal{G})^2] - (\mathbb{E}[\mathbb{E}(X|\mathcal{G})])^2$;

(iv.) $\mathbb{E}[\mathbb{V}\mathrm{ar}(X|\mathcal{G})] = \mathbb{E}[\mathbb{E}(X^2|\mathcal{G})] - \mathbb{E}[(\mathbb{E}(X|\mathcal{G}))^2]$.

Remark A.20. Let $X, Z : \Omega \to \mathbb{R}^n$ be two random variables. Then we can write the conditional variance of X given Z as

$$\mathbb{V}\mathrm{ar}(X|Z) = \mathbb{V}\mathrm{ar}(X|\sigma(Z)),$$

where $\sigma(Z)$ is a σ-algebra generated by Z.

APPENDIX B

NOMENCLATURE

Symbol	Description
$\mathbb{T}$	A Time Scale
$\mathbb{R}$	Set of Real Numbers
$\mathbb{N}$	Set of Natural Numbers
$\mathbb{N}_0$	$\mathbb{N} \cup \{0\}$
$\mathbb{N}_0^2$	The Set $\{0, 1, 4, 9, 16, \ldots\}$
$h\mathbb{Z}$	The Set $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$
$\mathbb{C}$	Set of Complex Numbers
$\overline{q^{\mathbb{Z}}}$	The Set $\{\ldots, q^{-2}, q^{-1}, 1, q, q^2 \ldots\}$ for $q > 1$.
σ	Forward Jump Operator
ρ	Backward Jump Operator
μ	Graininess Function
$^{\Delta}$	Delta Derivative Operator
Δ	Forward Difference Operator
$\mathbb{E}$	Expectation
$\mathbb{V}\mathrm{ar}$	Variance
$\mathcal{R}$	Set of Regressive Functions
$A > 0$	A Positive Definite Matrix
$A \geq 0$	A Positive Semi-definite Matrix
$\oplus$	Addition on Time Scales
$\ominus$	Subtraction on Time Scales
$\odot$	Multiplication on Time Scales
$e_A(\cdot, \cdot)$	Matrix Exponential on Time Scales
$W_C[t_0, t_f]$	Controllability Gramian
$W_O[t_0, t_f]$	Observability Gramian
$\Gamma_C[A, B]$	Controllability Matrix
$\Gamma_O[A, C]$	Observability Matrix

Symbol	Description
$J(\cdot)$	Cost Function
$K(\cdot)$	Kalman Gain
$K_v(\cdot)$	Feedforward Gain
x	State Vector
u	Input Vector
y	Output/Measurement Vector
λ	Costate
r	Reference Vector
e	Error Vector

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VITA

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