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Existence and classification of nonoscillatory solutions of two dimensional time scale systems

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EXISTENCE AND CLASSIFICATION OF NONOSCILLATORY SOLUTIONS
OF TWO DIMENSIONAL TIME SCALE SYSTEMS

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ABSTRACT

During the past years, there has been an increasing interest in studying oscillation and nonoscillation criteria for dynamic equations and systems on time scales that harmonize the oscillation and nonoscillation theory for the continuous and discrete cases in order to combine them in one comprehensive theory and eliminate obscurity from both.

We not only classify nonoscillatory solutions of dynamic equations and systems on time scales but also guarantee the (non)existence of such solutions by using the Knaster fixed point theorem, Schauder - Tychonoff fixed point theorem, and Schauder fixed point theorem. The approach is based on the sign of nonoscillatory solutions. A short introduction to the time scale calculus is given as well.

Examples are significant in order to see if nonoscillatory solutions exist or not. Therefore, we give several examples in order to highlight our main results for the set of real numbers $\mathbb{R}$, the set of integers $\mathbb{Z}$, and $q^{\mathbb{N}_0} = \{1, q, q^2, q^3, ...\}$, $q > 1$, which are the most well-known time scales.

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TABLE OF CONTENTS

Page

LIST OF TABLES

LIST OF SYMBOLS

$\mathbb{T}$	A Time Scale
$\mathbb{R}$	A Set of Real Numbers
$\mathbb{N}$	A Set of Natural Numbers
$\mathbb{Z}$	A Set of Integers
$\mathbb{C}$	A Set of Complex Numbers
$q^{\mathbb{N}_0}$	The Set $\{1, q, q^2, q^3, ...\}$ for $q > 1$.
$\emptyset$	Empty Set
σ	Forward Jump Operator
ρ	Backward Jump Operator
μ	Graininess Function
$^{\Delta}$	Delta Derivative Operator
Δ	Forward Difference Operator
Δ_q	q - Difference Operator

1. INTRODUCTION

1.1. PRELIMINARIES

The theory of time scales was first introduced by Stefan Hilger in his Ph.D. thesis in 1988. The main idea is to unify and extend the continuous and discrete theories that are used in mathematical models in population dynamics, economics, and engineering. For example, if the time scale is chosen as the set of real numbers, the general results give the results in differential equations, while if the time scale is chosen as the set of integers, the results hold for difference equations. In this section, basic definitions and the theory of time scales are introduced based on the books by Bohner and Peterson, see [14].

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$ that has the following properties:

(i) $\mathbb{T}$ and $\emptyset$ are closed subsets of $\mathbb{T}$,

(ii) Any intersection of arbitrarily many closed subsets of $\mathbb{T}$ is a also closed subset of $\mathbb{T}$,

(iii) Any union of finitely many closed subsets of $\mathbb{T}$ is a closed subset of $\mathbb{T}$.

The most well known examples for time scales are $\mathbb{R}$, $\mathbb{Z}$, and $q^{\mathbb{N}_0}$. However, $\mathbb{Q}$, $\mathbb{R}\backslash\mathbb{Q}$, $\mathbb{C}$, and the interval $(0,1)$ are not time scales.

Definition 1.1. *[14, Definition 1.1] Let $\mathbb{T}$ be a time scale. For $t \in \mathbb{T}$, we have the following definitions:*

(i) The forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ by

$$\sigma(t) := \inf\{s \in \mathbb{T} : \quad s > t\} \quad \textit{for all} \quad t \in \mathbb{T}.$$

(ii) The backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ *by*

$$\rho(t) := \inf\{s \in \mathbb{T} : \quad s < t\} \quad \textit{for all} \quad t \in \mathbb{T}.$$

(iii) The graininess function $\mu : \mathbb{T} \to [0, \infty)$ *by*

$$\mu(t) := \sigma(t) - t \quad \textit{for all} \quad t \in \mathbb{T}.$$

We define $\inf \emptyset = sup\mathbb{T}$. If $\sigma(t) > t$, then t is called right - scattered, while if $\rho(t) < t$, t is called left - scattered. If t is right and left - scattered at the same time, then we say that t is isolated. If $t < \sup \mathbb{T}$ and $\sigma(t) = t$, then t is called right - dense, while if $t > \inf \mathbb{T}$ and $\rho(t) = t$, we say t is left - dense. Also, if t is right and left - dense at the same time, then we say t is dense.

Tables 1.1 and 1.2 show some examples of the forward and backward jump operators and the graininess function for most known time scales and the classifications for a time scale point.

Table 1.1 Examples of Most Known Time Scales

$\mathbb{T}$	$\sigma(t)$	$\rho(t)$	$\mu(t)$
$\mathbb{R}$	t	t	0
$\mathbb{Z}$	$t+1$	$t-1$	1
$q^{\mathbb{N}_0}$	tq	$\frac{t}{q}$	$(q-1)t$

Table 1.2 Classification of Points

$t < \sigma(t)$	t is right-scattered
$t > \rho(t)$	t is left-scattered
$\rho(t) < t < \sigma(t)$	t is isolated
$t = \sigma(t)$	t is right-dense
$t = \rho(t)$	t is left dense
$\rho(t) = t = \sigma(t)$	t is dense

If $\sup \mathbb{T} < \infty$, then $\mathbb{T}^\kappa = \mathbb{T} \backslash (\rho(\sup \mathbb{T}), \sup \mathbb{T}]$, and $\mathbb{T}^\kappa = \mathbb{T}$ if otherwise. Suppose that $f : \mathbb{T} \to \mathbb{R}$ is a function. Then $f^\sigma : \mathbb{T} \to \mathbb{R}$ is defined by $f^\sigma(t) = f(\sigma(t)) \quad \text{for all} \quad t \in \mathbb{T}$.

Definition 1.2. *[14, Definition 1.10] For any ϵ, if there exists a $\delta > 0$ such that*

$$|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)| \leq \epsilon|\sigma(t) - s| \quad \textit{for all} \quad s \in (t - \delta, t + \delta) \cap \mathbb{T},$$

then f is called delta (or Hilger) differentiable on $\mathbb{T}^\kappa$ and f^Δ is called delta derivative of f.

Theorem 1.3. *[14, Theorem 1.16] Let $f : \mathbb{T} \to \mathbb{R}$ be a function with $t \in \mathbb{T}^\kappa$. Then*

(i) If f is differentiable at t, f is continuous at t.

(ii) If f is continuous at t and t is right-scattered, then f is differentiable at t and

$$f^\Delta(t) = \frac{f(\sigma(t)) - f(t)}{\mu(t)}.$$

(iii) If f is differentiable at t, then $f(\sigma(t)) = f(t) + \mu(t)f^{\Delta}(t)$.

The following theorem presents the product and quotient rules on time scales.

Theorem 1.4. *[14, Theorem 1.20] Let $f, g : \mathbb{T} \to \mathbb{R}$ be differentiable at $t \in \mathbb{T}^{\kappa}$. Then*

(i) If $fg : \mathbb{T} \to \mathbb{R}$ is differentiable at t, then

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t) = f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t)).$$

(ii) If $g(t)g(\sigma(t)) \neq 0$, then $\frac{f}{g}$ is differentaible at t with

$$\left(\frac{f}{g}\right)^{\Delta}(t) = \frac{f^{\Delta}(t)g(t) - f(t)g^{\Delta}(t)}{g(t)g(\sigma(t))}.$$

The following concepts must be introduced in order to define Δ - integrable functions.

Definition 1.5. *[14, Definition 1.58] $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous, denoted by $\mathrm{C}_{\mathrm{rd}}, \mathrm{C}_{\mathrm{rd}}(\mathbb{T}),$ or $\mathrm{C}_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$ if it is continuous at right dense points in $\mathbb{T}$ and its left sided limits exist as a finite number at left dense points in $\mathbb{T}$. Also the set of functions $f : \mathbb{T} \to \mathbb{R}$ which are differentiable and whose derivative is rd-continuous is denoted by $\mathrm{C}^1_{\mathrm{rd}}, \mathrm{C}^1_{\mathrm{rd}}(\mathbb{T}),$ or $\mathrm{C}^1_{\mathrm{rd}}(\mathbb{T}, \mathbb{R})$.*

Theorem 1.6. *[14, Theorem 1.60] Let $f : \mathbb{T} \to \mathbb{R}$.*

(i) If f is continuous, then f is rd-continuous.

(ii) The jump operator σ is rd-continuous.

Also, Cauchy integral is defined by

$$\int_a^b f(t)\Delta t = F(b) - F(a) \quad \text{for all} \quad a, b \in \mathbb{T}.$$

The following theorem presents the existence of antiderivatives.

Theorem 1.7. *[14, Theorem 1.74] Every rd-continuous function has an antiderivative. Moreover, F given by*

$$F(t) = \int_{t_0}^{t} f(s)\Delta s \quad \textit{for} \quad t \in \mathbb{T}$$

is called an antiderivative of f.

Remark 1.8. *[14, Theorem 1.76] If $f^{\Delta} \geq 0$, then f is nondecreasing.*

Theorem 1.9. *[14, Theorem 1.77] Let $a, b, c \in \mathbb{T}, \alpha \in \mathbb{R},$ and $f, g \in \mathrm{C_{rd}}$. Then the following holds:*

(i) $\int_a^b [(\alpha f(t)) + (\alpha g(t))] = \alpha \int_a^b f(t)\Delta t + \int_a^b \alpha g(t)\Delta t.$

(ii) $\int_a^b f(t)\Delta t = -\int_b^a f(t)\Delta t.$

(iii) $\int_a^b f(t)\Delta t = \int_a^c f(t)\Delta t + \int_c^b f(t)\Delta t.$

(iv) $\int_a^a f(t)\Delta t = 0.$

Table 1.3 shows the derivative and integral definitions for the most known time scales for $a, b \in \mathbb{T}$. Finally, we finish the subsection by the following fixed point theorems.

Table 1.3 Derivative and Integrals for Most Common Time Scales

$\mathbb{T}$	$f^{\Delta}(t)$	$\int_a^b f(t)\Delta t$
$\mathbb{R}$	$f'(t)$	$\int_a^b f(t)dt$
$\mathbb{Z}$	$\Delta f(t)$	$\sum_{t=a}^{b-1} f(t)$
$q^{\mathbb{N}_0}$	$\Delta_q f(t)$	$\sum_{t\in[a,b)_{q^{\mathbb{N}_0}}} f(t)\mu(t)$

Theorem 1.10 (Schauder's Fixed Point Theorem)**.** *[50, Theorem 2.A] Let M be a nonempty, closed, bounded, convex subset of a Banach space X, and suppose that $T : M \to M$ is a compact operator. Then, T has a fixed point.*

The following theorem is the alternate version of the Schauder's fixed point theorem, see [50].

Corollary 1.11. *Let M be a nonempty, compact, convex subset of a Banach space X, and suppose that $T : M \to M$ is a continuous operator. Then, T has a fixed point.*

The Schauder fixed point theorem was proved by Juliusz Schauder in 1930. In 1934, Tychonoff proved the same theorem for the case when M is a compact convex subset of a locally convex space X. In the literature, this version is known as the Schauder - Tychonoff fixed point theorem, see [45].

Theorem 1.12 (Schauder - Tychonoff Fixed Point Theorem)**.** *Let M be a compact convex subset of a locally convex (linear topological) space X and T a continuous map of M into itself. Then, T has a fixed point.*

Finally, we provide the Knaster fixed point theorem, see [38].

Theorem 1.13 (Knaster Fixed Point Theorem)**.** *If* $(M, \leq)$ *is a complete lattice and* $T : M \to M$ *is order-preserving (also called monotone or isotone), then* T *has a fixed point. In fact, the set of fixed points of* T *is a complete lattice.*

1.2. INTRODUCTION TO DYNAMIC EQUATIONS AND SYSTEMS

Asymptotic properties of systems of first order dynamic equations on time scales have recently gotten a lot of attention that combines continuous and discrete analyses, which are related but in distinct areas. One special case of systems of dynamic equations is the Emden-Fowler type equation. The equation has several interesting applications such as in astrophysics, gas dynamics and fluid mechanics, relativistic mechanics, nuclear pyhsics, and chemically reacting systems. For example, the fundamental problem in studying the stellar structure for gaseous dynamics in astrophysics was to look into the equilibrium formation of the mass of spherical clouds of gas for the continuous case, proposed by Kelvin and Lane, see [47] and [36]. They considered the equation

$$\frac{1}{t^2}\frac{d}{dt}\left(t^2\frac{du}{dt}\right) + u^n = 0 \tag{1}$$

for $n = 1.5$ and $n = 2.5$. This equation is referred to as the Lane - Emden equation, see [16] - [17]. At that time, astrophysicists were interested in equation (1) for initial conditions $u(0) = 1$ and $u'(0) = 0$. The mathematical foundation for the study of such an equation was made by Fowler in a series of four papers during 1914 - 1931, see [29] - [32]. The other types of dynamic equations on time scales are quasilinear, half - linear and self - adjoint equations. Classification for nonoscillatory solutions of the quasilinear dynamic equation

$$\left[a(t)\Phi_p(x^\Delta)\right]^\Delta = b(t)f(x^\sigma) \tag{2}$$

is considered in [4] and [5], where $a, b \in \mathrm{C}_{\mathrm{rd}}(\mathbb{T}, R^+)$ and $f : \mathbb{R} \to \mathbb{R}$ is continuous with $uf(u) > 0$ for $u \neq 0$ and $\Phi_p(u) = |u|^{p-2}u$ with $p > 1$. When $\mathbb{T} = \mathbb{R}$, equation (2) is reduced to a quasilinear differential equation, see [21]

$$[a(t)\Phi_p(x')]' = b(t)f(x),$$

while if $\mathbb{T} = \mathbb{Z}$, it is reduced to a quasilinear difference equation, see [22]

$$\Delta\left[a_k \Phi_p(\Delta x_k)\right] = b_k f(x_{k+1}).$$

The following half - linear dynamic equation

$$\left[a(t)\Phi_p(x^\Delta)\right]^\Delta = b(t)\Phi_p(x^\sigma) \tag{3}$$

is considered by P. Řehak in [48], where $f = \Phi_p$ in equation (2). The continuous and discrete cases of equation (3)

$$[a(t)\Phi_p(x')]' = b(t)\Phi_p(x') \tag{4}$$

and

$$\Delta\left[a_k \Phi_p(\Delta x_k)\right] = b_k \Phi_p(x_{k+1}). \tag{5}$$

are considered by Došly in [27] and by Řehak in [49], respectively.

In case $p = 2$ in equations (3), (4) and (5), we obtain Sturm-Liouville dynamic, differential, and difference equations

$$\left(a(t)x^\Delta\right)^\Delta = b(t)x^\sigma,$$

$$(a(t)x')' = b(t)x$$

and

$$\Delta\left(a_k \Delta x_k\right) = b_k x_{k+1},$$

respectively, see [14], [34], and [2]. Finally, the case $a(t) = 1$, $p = 2$ and $f = \Phi_q$, $q > 1$ in equation (2) is considered by E. Akın and J. Hoffacker in [8] and [9].

In the first paper, we consider

$$\left[a(t)|x^{\Delta}(t)|^{\alpha} \operatorname{sgn} x^{\Delta}\right]^{\Delta} = b(t)|x^{\sigma}(t)|^{\beta} \operatorname{sgn} x^{\sigma}(t),$$

where $f = \Phi_q(x^{\sigma})$ in equation (2) for $q = \beta + 1$ and $p = \alpha + 1$, and deal with the (non)existence of nonoscillatory solutions by using fixed point theorems and the convergence/divergence of some improper integrals of coefficient functions a and b.

Systems of dynamic equations are more fit for physical applications. Therefore, classification of nonoscillatory solutions is important in order to have enough information about the behavior of solutions in a long term. For example, the study of discrete systems has been motivated by their applications in modeling for population, extinction, and neuron dynamics because their computational costs are very low.

Motivated by [20], we study the classification and existence of nonoscillatory solutions of the Emden - Fowler system of first order dynamic equations

$$\begin{cases} x^{\Delta}(t) = \left(\frac{1}{a(t)}\right)^{\frac{1}{\alpha}} |y(t)|^{\frac{1}{\alpha}}\, sgn y(t) \\ y^{\Delta}(t) = -b(t)\, |x^{\sigma}(t)|^{\beta}\, sgn x^{\sigma}(t), \end{cases} \tag{6}$$

where $\alpha, \beta > 0$ and $a, b \in C_{rd}\left([t_0, \infty)_{\mathbb{T}}, \mathbb{R}^+\right)$.

Systems of delay dynamic equations take a lot of attention in all areas such as population dynamics and epidemiology in biological sciences. For instance, when the birth rate of preys is affected by the previous values rather than current values, a system of delay dynamic equations is used, because the delta derivative at any time depends on solutions at prior times. Therefore, we consider a system of first order delay dynamic equations

$$\begin{cases} x^{\Delta}(t) = a(t)f(y(t)) \\ y^{\Delta}(t) = -b(t)g(x(\tau(t))) \end{cases} \tag{7}$$

where $a, b \in C_{rd}\left([t_0,\infty)_{\mathbb{T}}, \mathbb{R}^+\right)$, $\tau \in C_{rd}\left([t_0,\infty)_{\mathbb{T}}, [t_0,\infty)_{\mathbb{T}}\right)$, $\tau(t) \leq t$, and $\tau(t) \to \infty$ as $t \to \infty$, f and g are nondecreasing functions such that $uf(u) > 0$ and $ug(u) > 0$ for $u \neq 0$ in order to make observations for the (non)existence of nonoscillatory solutions.

We also consider the special cases $\tau(t) = t$ in (7) in order to show the asymptotic behaviors and the (non)existence of nonoscillatory solutions in M^+ and M^- based on the sign of such solutions. Classification of nonoscillatory solutions when $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$ is given in [40] and [39] as

$$\begin{cases} x' = a(t)f(y(t)) \\ y' = -b(t)g(x(t)), \end{cases} \qquad \begin{cases} \Delta x_n = a_n f(y_n) \\ \Delta y_n = -b_n g(x_n), \end{cases}$$

respectively.

I. CLASSIFICATION OF NONOSCILLATORY SOLUTIONS OF NONLINEAR DYNAMIC EQUATIONS ON TIME SCALES

ABSTRACT

We study the asymptotic behavior of nonoscillatory solutions of nonlinear dynamic equations on time scales. More precisely, all eventually monotone solutions of nonlinear dynamic equations can be divided into several disjoint subsets by means of necessary and sufficient integral conditions. Examples are given to illustrate some of our main results.

1. INTRODUCTION

This paper deals with the asymptotic behavior of solutions of the nonlinear dynamic equation

$$\left[a(t)|x^{\Delta}(t)|^{\alpha}\operatorname{sgn}x^{\Delta}\right]^{\Delta} = b(t)|x^{\sigma}(t)|^{\beta}\operatorname{sgn}x^{\sigma}(t), \tag{I.1}$$

where $a, b \in C_{rd}\left([t_0,\infty)_{\mathbb{T}}, \mathbb{R}^+\right)$ and $\alpha, \beta > 0$. A time scale, denoted by $\mathbb{T}$, is a closed subset of real numbers. Throughout this paper, we assume that $\mathbb{T}$ is unbounded above. By a solution we mean a delta differentiable function x satisfying equation (I.1) such that $\left[a(t)|x^{\Delta}(t)|^{\alpha}\operatorname{sgn}x^{\Delta}\right] \in \mathrm{C}^1_{\mathrm{rd}}$, where the set of rd-continuous functions and the set of functions that are differentiable and whose derivative is rd-continuous will be denoted by C_{rd} and $\mathrm{C}^1_{\mathrm{rd}}$, respectively. We also assume that $x(t)$ is a proper solution on $[t_0,T)_{\mathbb{T}}$, i.e., $x(t)$ exists and $x(t) \neq 0$ on $[t_0,T)_{\mathbb{T}}$. Whenever we write $t \geq t_1$, we mean that $t \in [t_1,\infty)_{\mathbb{T}} := [t_1,\infty) \cap \mathbb{T}$.

Equation (I.1) reduces to the nonlinear differential equation, see Cecchi, Došlá, Marini and Vrkoč [8], and Tanigawa [15],

$$[a(t)|x'(t)|^{\alpha}\operatorname{sgn}x']' = b(t)|x(t)|^{\beta}\operatorname{sgn}x \tag{I.2}$$

when $\mathbb{T} = \mathbb{R}$, and the nonlinear difference equation, see Cecchi, Došlá, Marini [9],

$$\Delta(a_n|\Delta x_n|^{\alpha}\operatorname{sgn}\Delta x_n) = b_n|x_{n+1}|^{\beta}\operatorname{sgn}x_{n+1} \tag{I.3}$$

when $\mathbb{T} = \mathbb{Z}$.

Such dynamic equations are studied by Akın-Bohner in [1, 2, 3], by Erbe, Baoguo and Peterson in [12] and Akın-Bohner, Bohner, and Saker in [4]. Such studies are motivated by the dynamics of positive radial solutions of reaction-diffusion (flow through porous media, nonlinear elasticity) problems, see Diaz [11] and Grossinho and Omari [13]. Our results and methods extend those stated and used in the continuous case in [1] and [8], and in the discrete case in [9, 10], see also references therein.

Our goal is to investigate the asymptotic behavior of nonoscillatory solutions of (I.1) by certain types of integrals depending on a, b, α and β. In Section 2, we classify eventually monotone solutions in two types, introduce the sub-classes that are obtained by using equation (I.1) and show the existence and non-existence of nonoscillatory solutions of (I.1). In Section 3, we investigate the convergence and divergence of more general integrals and use those results in Section 4 to show the co-existence of solutions of (I.1) in these sub-classes when $\alpha > \beta$, $\alpha < \beta$ and $\alpha = \beta$. Finally, we construct examples to highlight some of our results in the last section.

An excellent introduction of time scales calculus can be found in [6] and [7] by Bohner and Peterson. Therefore, we only give the preliminary results that we use in our proofs.

Theorem 1.1. *[6, Theorem 1.75]. If $f \in \mathrm{C_{rd}}$ and $t \in \mathbb{T}^{\kappa}$, then*

$$\int_t^{\sigma(t)} f(\tau)\Delta\tau = \mu(t)f(t).$$

Theorem 1.2. *[6, Theorem 1.77] If $a, b \in \mathbb{T}$ and $f, g \in C_{rd}$,then*

$$\int_a^b f(\sigma(t))g^{\Delta}(t) = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(t)\Delta t;$$

or

$$\int_a^b f(t)g^{\Delta}(t) = (fg)(b) - (fg)(a) - \int_a^b f^{\Delta}(t)g(\sigma(t))\Delta t.$$

Theorem 1.3. *[6, Theorem 1.90] Let $f : \mathbb{R} \mapsto \mathbb{R}$ be continuously differentiable and suppose $g : \mathbb{T} \mapsto \mathbb{R}$ is delta differentiable. Then $f \circ g : \mathbb{T} \mapsto \mathbb{R}$ is delta differentiable* *and the formula*

$$(f \circ g)^{\Delta}(t) = \left\{ \int_0^1 f'\left(g(t) + h\mu(t) g^{\Delta}(t)\right) dh \right\} g^{\Delta}(t)$$

holds.

Theorem 1.4. *[6, Theorem 1.98] Assume $\nu : \mathbb{T} \longrightarrow \mathbb{R}$ is strictly increasing and $\tilde{\mathbb{T}} = \nu(\mathbb{T})$ is a time scale. If $f : \mathbb{T} \longrightarrow \mathbb{R}$ is an rd-continuous function and ν is differentiable with rd-continuous derivative, then for $a, b \in \mathbb{T}$*

$$\int_a^b f(t)\nu^{\Delta}(t)\Delta t = \int_{\nu(a)}^{\nu(b)} (f \circ \nu^{-1})(s)\tilde{\Delta} s.$$

Theorem 1.5. *(Integral Minkowski Inequality) [5, Theorem 2.1] Let $(X, \mathscr{M}, \mu_\Delta)$ and $(Y, \mathscr{L}, \nu_\Delta)$ be time scale measure spaces and let u, v and f be nonnegative functions on X, Y, and $X \times Y$, respectively. If $p \geq 1$, then*

$$\left[\int_X \left(\int_Y f(x,y)v(y)d\nu_\Delta(y))\right)^p u(x)d\mu_\Delta(x)\right]^{\frac{1}{p}}$$

$$\leq \int_Y \left(\int_X f^p(x,y)u(x)d\mu_\Delta(x)\right)^{\frac{1}{p}} v(y)d\nu_\Delta(y) \tag{I.4}$$

holds provided all integrals in (I.4) exist. If $0 < p < 1$ and

$$\int_X \left(\int_Y fvd\nu_\Delta\right)^p ud\mu_\Delta > 0, \quad \int_Y fvd\nu_\Delta > 0 \tag{I.5}$$

then (I.4) is reversed. If $f < 0$ and (I.5) and

$$\int_X f^p ud\mu_\Delta > 0$$

hold, then (I.4) is reversed, as well.

Theorem 1.6. *(Hölder's Inequality) [5, Theorem 1.3] For $p \neq 1$, define $q = p/(p-1)$. Let $(E, \mathscr{F}, \mu_\Delta)$ be a time scale measure space. Assume w, f, g are nonnegative functions such that $wf^p, wg^p, w(f+g)^p$ are $\Delta -$integrable on E. If $p > 1$, then*

$$\int_E w(t)f(t)g(t)d\mu_\Delta(t) \leq \left(\int_E w(t)f^p(t)d\mu_\Delta(t)\right)^{\frac{1}{p}} \left(\int_E w(t)g^q(t)d\mu_\Delta(t)\right)^{\frac{1}{q}}. \tag{I.6}$$

If $0 < p < 1$ and $\int_E wg^q d\mu_\Delta > 0$, or if $p < 0$ and $\int_E wf^p d\mu_\Delta > 0$, then (I.6) is reversed.

We also use the algebraic inequality

$$(a+b)^p \leq 2^p(a^p + b^p) \tag{I.7}$$

for $a \geq 0,\ b \geq 0$ and $p > 0$, see [14].

It is shown by Akın-Bohner in [1] that any nontrivial solutions of equation (1) on $[t_0, \infty)_{\mathbb{T}}$ is eventually monotone and belongs to one of the following classes:

$$M^+ := \{x \text{ is a solution of (1)} : \exists\ t_1 \geq t_0 \ \text{ such that } \ x(t)x^\Delta(t) > 0 \text{ for } t \geq t_1\},$$

$$M^- := \{x \text{ is a solution of (1)} : x(t)x^\Delta(t) < 0 \text{ for } t \geq t_0\}.$$

For equation (I.1), M^+ can be empty when $\mathbb{T} = \mathbb{R}$, see [1]. However, it is not true when $\mathbb{T} = \mathbb{Z}$, see [9]. In addition, M^- can be empty when $\mathbb{T} = \mathbb{R}$, see [1], while this is an open problem in the case $\mathbb{T} = \mathbb{Z}$. In this paper, we study the solutions of (I.1) in M^+ and M^- described by the following integrals:

$$J_1 = \lim_{T \to \infty} \int_{t_0}^T \left(\frac{1}{a(t)}\right)^{\frac{1}{\alpha}} \left(\int_{t_0}^t b(s)\Delta s\right)^{\frac{1}{\alpha}} \Delta t, \tag{I.8}$$

$$K_1 = \lim_{T \to \infty} \int_{t_0}^T b(t) \left(\int_{t_0}^t \left(\frac{1}{a(s)}\right)^{\frac{1}{\alpha}} \Delta s\right)^{\beta} \Delta t, \tag{I.9}$$

$$J_2 = \lim_{T\to\infty} \int_{t_0}^{T} \left(\frac{1}{a(t)}\right)^{\frac{1}{\alpha}} \left(\int_{\sigma(t)}^{T} b(s)\Delta s\right)^{\frac{1}{\alpha}} \Delta t, \tag{I.10}$$

$$K_2 = \lim_{T\to\infty} \int_{t_0}^{T} b(t) \left(\int_{\sigma(t)}^{T} \left(\frac{1}{a(s)}\right)^{\frac{1}{\alpha}} \Delta s\right)^{\beta} \Delta t, \tag{I.11}$$

$$J_3 = \lim_{T\to\infty} \int_{t_0}^{T} \left(\frac{1}{a(t)}\right)^{\frac{1}{\alpha}} \Delta t,$$

$$K_3 = \lim_{T\to\infty} \int_{t_0}^{T} b(t)\Delta t.$$

We now present the convergence and divergence relationships between above integrals. One can prove the followings similar to [2, Lemma 2.1].

Lemma 1.7. *For the integrals J_1, K_1, J_2, K_2, J_3 and K_3, we have the following relationships:*

(a) *If $J_1 < \infty$, then $J_3 < \infty$.*

(b) *If $K_1 < \infty$, then $K_3 < \infty$.*

(c) *If $J_1 = \infty$, then $J_3 = \infty$ or $K_3 = \infty$.*

(d) *If $K_1 = \infty$, then $J_3 = \infty$ or $K_3 = \infty$.*

(e) *$J_1 < \infty$ and $K_1 < \infty$ if and only if $J_3 < \infty$ and $K_3 < \infty$,*

(f) *If $J_2 < \infty$, then $K_3 < \infty$.*

(g) *If $K_2 < \infty$, then $J_3 < \infty$.*

(h) *If $J_2 = \infty$, then $J_3 = \infty$ or $K_3 = \infty$.*

(i) *If $K_2 = \infty$, then $J_3 = \infty$ or $K_3 = \infty$.*

(j) *$J_2 < \infty$ and $K_2 < \infty$ if and only if $J_3 < \infty$ and $K_3 < \infty$.*

2. CLASSIFICATION OF NONOSCILLATORY SOLUTIONS OF (I.1) IN M^+ AND M^-

In this section, we obtain the existence and non-existence of solutions of (I.1) in M^+ and M^- depending on J_1, K_1, and J_2, K_2, respectively.

For the convenience, we denote

$$x^{[1]} = a(t)|x^\Delta|^\alpha \operatorname{sgn} x^\Delta, \tag{I.12}$$

so-called the quasi-derivative of x. Let $x(t)$ be a proper solution of (1) in M^+ on $[t_0, \infty)_\mathbb{T}$, and without loss of generality assume that $x(t) > 0$ for $[t_0, \infty)_\mathbb{T}$. By equation (I.1) we have that $x^{[1]}(t)$ is increasing for $t \geq t_0$. Then either there exists $t_1 \geq t_0$ such that $x^{[1]}(t) > 0,\ t \geq t_1$ or $x^{[1]}(t) < 0,\ t \geq t_0$. If $x^{[1]}(t) > 0,\ t \geq t_1$, then $x^\Delta(t) > 0$ for $t \geq t_1$ and $x^{[1]}(t)$ tends to a positive constant or infinity as $t \to \infty$. Clearly, x has a positive limit or infinite limit. Similarly, if $x^{[1]}(t) < 0,\ t \geq t_0$, then $x^\Delta(t) < 0$ for $t \geq t_0$ and so $x^{[1]}(t)$ tends to a non-positive constant as $t \to \infty$ while $x(t)$ goes to a non-negative constant $t \to \infty$.

So in the light of this information, we can have the following lemmas:

Lemma 2.1. *For positive real numbers c and d, M^+ can be a divided into the following sub-classes according to the asymptotic behavior of solution x of (I.1) and $x^{[1]}$:*

$$\begin{aligned}
M^+_{B,B} &= \left\{x \in M^+ : \lim_{t\to\infty} |x(t)| = c, \quad \lim_{t\to\infty} |x^{[1]}(t)| = d\right\}, \\
M^+_{\infty,B} &= \left\{x \in M^+ : \lim_{t\to\infty} |x(t)| = \infty, \quad \lim_{t\to\infty} |x^{[1]}(t)| = d\right\}, \\
M^+_{B,\infty} &= \left\{x \in M^+ : \lim_{t\to\infty} |x(t)| = c, \quad \lim_{t\to\infty} |x^{[1]}(t)| = \infty\right\}, \\
M^+_{\infty,\infty} &= \left\{x \in M^+ : \lim_{t\to\infty} |x(t)| = \infty, \quad \lim_{t\to\infty} |x^{[1]}(t)| = \infty\right\}.
\end{aligned}$$

Lemma 2.2. *For positive real numbers c and d, M^- can be divided into the following sub-classes according to the asymptotic behavior of solution x of (I.1) and $x^{[1]}$:*

$$
\begin{aligned}
M^-_{B,B} &= \left\{x \in M^- : \lim_{t\to\infty} |x(t)| = c, \quad \lim_{t\to\infty} |x^{[1]}(t)| = d\right\}, \\
M^-_{B,0} &= \left\{x \in M^- : \lim_{t\to\infty} |x(t)| = c, \quad \lim_{t\to\infty} |x^{[1]}(t)| = 0\right\}, \\
M^-_{0,B} &= \left\{x \in M^- : \lim_{t\to\infty} |x(t)| = 0, \ \lim_{t\to\infty} |x^{[1]}(t)| = d\right\}, \\
M^-_{0,0} &= \left\{x \in M^+ : \lim_{t\to\infty} |x(t)| = 0, \ \lim_{t\to\infty} |x^{[1]}(t)| = 0\right\}.
\end{aligned}
$$

In the literature, any eventually nontrivial solution $x \in M^+$ is called *regularly (weakly) increasing* if at least one of $\lim_{t\to\infty} |x(t)|$, $\lim_{t\to\infty} |x^{[1]}(t)|$ exists finitely. Otherwise, it is called a *strongly increasing* solution. Similarly, a solution in $M^-_{0,B}$ is called *regularly (weakly) decaying* while a solution in $M^-_{0,0}$ is called *strongly decaying*.

The following theorem gives us the existence of proper solutions of (I.1) in sub-classes of M^+ based on the integrals J_1 and K_1.

Theorem 2.3. *For solutions of (I.1) in M^+, we have the followings:*

(a) $J_1 < \infty$ *and* $K_1 < \infty$ *if and only if* $M^+_{B,B} \neq \emptyset$.

(b) $J_1 < \infty$ *and* $K_1 = \infty$ *if and only if* $M^+_{B,\infty} \neq \emptyset$.

(c) *If* $M^+_{\infty,B} \neq \emptyset$, *then* $J_1 = \infty$ *and* $K_1 < \infty$.

(d) *If* $J_1 = K_1 = \infty$, *then every solution in* M^+ *belongs to* $M^+_{\infty,\infty}$.

Proof. (a) Suppose that there exists a solution of (I.1) in $M^+_{B,B}$. Without loss of generality we assume that $x(t) > 0$ for $t \geq t_1$. Then $x^{[1]}(t)$ is increasing for $t \geq t_1$. By [2, Theorem 3.1], if x has a finite limit, then $J_1 < \infty$. So it is enough to prove that $K_1 < \infty$. Since $x^{[1]}(t)$ is increasing for $t \geq t_1$, $x^{[1]}(t) \geq M$, where $x^{[1]}(t_1) = M \in \mathbb{R}^+$. This implies that

$$x^\Delta(t) \geq M^{\frac{1}{\alpha}} \left(\frac{1}{a(t)}\right)^{\frac{1}{\alpha}}, \ t \geq t_1.$$

Integrating the last inequality from t_1 to t yields

$$x(t) > M^{\frac{1}{\alpha}} \int_{t_1}^{t} \left(\frac{1}{a(s)}\right)^{\frac{1}{\alpha}} \Delta s,\ t \geq t_1$$

or

$$x^{\sigma}(t) > M^{\frac{1}{\alpha}} \int_{t_1}^{t} \left(\frac{1}{a(s)}\right)^{\frac{1}{\alpha}} \Delta s,\ t \geq t_1 \tag{I.13}$$

by the monotonicity of x. Taking the β^{th} power of both sides of (I.13) and multiplying the resulting by b yield

$$(x^{\sigma}(t))^{\beta}\, b(t) > M^{\frac{\beta}{\alpha}} b(t) \left[\int_{t_1}^{t} \left(\frac{1}{a(s)}\right)^{\frac{1}{\alpha}} \Delta s\right]^{\beta},\ t \geq t_1.$$

From (I.1) we get

$$\left[x^{[1]}(t)\right]^{\Delta} > M^{\frac{\beta}{\alpha}} b(t) \left[\int_{t_1}^{t} \left(\frac{1}{a(s)}\right)^{\frac{1}{\alpha}} \Delta s\right]^{\beta},\ t \geq t_1.$$

Finally, integrating the last inequality from t_1 to t yields

$$x^{[1]}(t) > M^{\frac{\beta}{\alpha}} \int_{t_1}^{t} b(s) \left[\int_{t_1}^{s} \left(\frac{1}{a(\tau)}\right)^{\frac{1}{\alpha}} \Delta \tau\right]^{\beta} \Delta s,\ t \geq t_1. \tag{I.14}$$

Since $x^{[1]}$ has a finite limit, $K_1 < \infty$ from the above inequality.

Conversely, suppose that $J_1 < \infty$ and $K_1 < \infty$. Without loss of generality assume that $x(t) > 0$ for $t \geq t_1$. By [2, Theorem 3.1], there exists a solution x of (I.1) such that $\lim_{t\to\infty} x(t) = c$, where $0 < c < \infty$. So it is enough to show that $x^{[1]}(t)$ converges to a finite number as $t \to \infty$. Since $x(t)$ has a finite limit, there exists

$t_2 \geq t_1$ such that $x^\sigma(t) < c$ for $t \geq t_2$. Integrating equation (I.1) from t_2 to t gives

$$x^{[1]}(t) = x^{[1]}(t_2) + \int_{t_2}^{t} b(s) \left(x^\sigma(s)\right)^\beta \Delta s < x^{[1]}(t_2) + c^\beta \int_{t_2}^{t} b(s)\Delta s. \tag{I.15}$$

By Lemma 1.7(b), $K_3 < \infty$. Therefore, taking the limit of both sides of (I.15) as $t \to \infty$ proves the assertion.

(b) Suppose that there exists a solution x of (I.1) in $M^+_{B,\infty}$. It is enough to show that $K_1 = \infty$ since we show in Theorem 2.3(a) that if there exists a bounded solution of (I.1), then $J_1 < \infty$. By Lemma 1.1(b), it is enough to show that $K_3 = \infty$. Without loss of generality, we assume that $x(t) > 0$ for $t \geq t_1$. Integrating equation (1) from t_1 to t yields

$$x^{[1]}(t) = x^{[1]}(t_1) + \int_{t_1}^{t} b(s) \left(x^\sigma(s)\right)^\beta \Delta s \leq x^{[1]}(t_1) + \left(x^\sigma(t)\right)^\beta \int_{t_1}^{t} b(s)\Delta s, \ t \geq t_1.$$

Taking the limit of both sides of the inequality above as $t \to \infty$ gives us that $K_3 = \infty$.

Conversely, suppose that $J_1 < \infty$ and $K_1 = \infty$. By Theorem 2.3(a), we have the existence of a bounded solution x of (I.1) in M^+. By the estimate (I.14) and the divergence of K_1, we obtain that $x^{[1]}$ has an infinite limit. So this completes the proof.

(c) Suppose that there exists a solution of (I.1) in $M^+_{\infty,B}$. By [2, Corollary 3.1], $J_1 = \infty$. So it suffices to prove that $K_1 < \infty$. The proof is very similar to the proof of Theorem 2.3(a). So from estimate (I.14) and since $x^{[1]}$ has a finite limit, we obtain that $K_1 < \infty$.

(d) It follows from Theorem 2.3 (a). $\square$

In the following corollary, we obtain the necessary conditions for the non-existence of solutions in sub-classes of M^+ based on the integrals J_1 and K_1 and the proof follows from Theorem 2.3.

Corollary 2.4. *For solutions of (I.1) in M^+, we have the followings:*

(a) *If $J_1 = \infty$ or $K_1 = \infty$, then $M^+_{B,B} = \emptyset$.*

(b) *If $J_1 = \infty$ or $K_1 < \infty$, then $M^+_{B,\infty} = \emptyset$.*

(c) *If $J_1 < \infty$ or $K_1 = \infty$, then $M^+_{\infty,B} = \emptyset$.*

We finish this section by showing the existence and non-existence of solutions of equation (I.1) in sub-classes of M^-. In order to do that we define the following integral

$$I = \lim_{T\to\infty} \int_{t_0}^{T} \left(\frac{1}{a(t)}\right)^{\frac{1}{\alpha}} \left(\int_t^T b(s)\Delta s\right)^{\frac{1}{\alpha}} \Delta t.$$

The proofs of (b) and (d) below can be found in [3, Theorem 2.1, Theorem 2.3] and [3, Theorem 2.4], respectively. So we only prove parts (a) and (c). We use Schauder-Tychonoff fixed point theorem in order to show some of the existence of solutions in M^-.

Theorem 2.5. *For solutions of (I.1) in M^-, we have the followings:*

(a) *$M^-_{B,B} \neq \emptyset$ if and only if $I < \infty$ and $K_2 < \infty$.*

(b) *$M^-_{0,B} \neq \emptyset$ if and only if $K_2 < \infty$.*

(c) *If $I < \infty$ and $K_2 = \infty$, then $M^-_{B,0} \neq \emptyset$*

(d) *If $J_2 = K_2 = \infty$, then every solution in M^- belongs to $M^-_{0,0}$.*

Proof. (a) Suppose that $M^-_{B,B} \neq \emptyset$. Then for $c > 0$ and $d > 0$, there exists a solution $x \in M^-$ of (I.1) such that $|x(t)| \to c$ and $|x^{[1]}(t)| \to d$ as $t \to \infty$. By [1, Theorem 4.1], we have that $I < \infty$. So it is enough to show that $K_2 < \infty$. Without loss of generality, assume that $x(t) > 0$ for $t \geq t_0$. Then integrating (I.1) from $\sigma(t)$ to ∞ gives us

$$x^\sigma(t) > \int_{\sigma(t)}^{\infty} \left(\frac{1}{a(s)}\right)^{\frac{1}{\alpha}} \left[-x^{[1]}(s)\right]^{\frac{1}{\alpha}} \Delta s > d^{\frac{1}{\alpha}} \int_{\sigma(t)}^{\infty} \left(\frac{1}{a(s)}\right)^{\frac{1}{\alpha}} \Delta s. \tag{I.16}$$

Taking the β^{th} power and multiplying both sides of (I.16) by b yield us

$$\left[-x^{[1]}(t)\right]^{\Delta} > d^{\frac{\beta}{\alpha}}\, b(t) \left[\int_{\sigma(t)}^{\infty} \left(\frac{1}{a(s)}\right)^{\frac{1}{\alpha}} \Delta s\right]^{\beta}. \tag{I.17}$$

Integrating (I.17) from t_0 to t gives us

$$0 < -x^{[1]}(t_0) + (d)^{\frac{\beta}{\alpha}} \int_{t_0}^{t} b(s) \left[\int_{\sigma(s)}^{\infty} \left(\frac{1}{a(\tau)}\right)^{\frac{1}{\alpha}} \Delta\tau\right]^{\beta} \Delta s < -x^{[1]}(t).$$

As $t \to \infty$ the assertion follows.

Conversely, assume that $I < \infty$ and $K_2 < \infty$. Since $J_3 < \infty$ by Lemma 1.7(g), for arbitrarily given $c > 0$ and $d > 0$, take $t_1 \geq t_0$ so large that

$$\int_{t_1}^{\infty} \left(\frac{1}{a(t)}\right)^{\frac{1}{\alpha}} \left[d + (2c)^{\beta} \int_{t}^{\infty} b(s)\Delta s\right]^{\frac{1}{\alpha}} \Delta t \leq c.$$

Define X to be the Frečhet space of all continuous functions on $[t_1, \infty)_{\mathbb{T}}$ endowed with the topology of uniform convergence on compact sub-intervals of $[t_1, \infty)_{\mathbb{T}}$. Let Ω be the nonempty subset of X given by

$$\Omega := \{x \in X : \ c \leq x(t) \leq 2c, \ \ t \geq t_1\}.$$

Define

$$(\mathscr{F}x)(t) = c + \int_{t}^{\infty} \left(\frac{1}{a(s)}\right)^{\frac{1}{\alpha}} \left[d + \int_{s}^{\infty} b(\tau)(x^{\sigma}(\tau))^{\beta}\Delta t\right]^{\frac{1}{\alpha}} \Delta s.$$

Clearly Ω is closed, convex and bounded. One can also show that $\mathscr{F} : \Omega \to \Omega$ is a continuous mapping and relatively compact. Then by the Schauder-Tychonoff fixed point theorem, $\mathscr{F}$ has a fixed element $x \in \Omega$ such that $x = \mathscr{F}(x)$, i.e.,

$$x(t) = (\mathscr{F}x)(t) = c + \int_{t}^{\infty} \left(\frac{1}{a(s)}\right)^{\frac{1}{\alpha}} \left[d + \int_{s}^{\infty} b(\tau)(x^{\sigma}(\tau))^{\beta}\Delta t\right]^{\frac{1}{\alpha}} \Delta s. \tag{I.18}$$

So by (I.18), we have $x^{\Delta}(t) < 0$ for $[t_1, \infty)_{\mathbb{T}}$, i.e., $x(t)x^{\Delta}(t) < 0$ on $[t_1, \infty)_{\mathbb{T}}$. Taking the limit as $t \to \infty$ proves the assertion.

(c) Suppose that $I < \infty$ and $K_2 = \infty$. By [1, Theorem 4.1], we have that there exists a solution x of (I.1) such that $|x(t)| \to c$ as $t \to \infty$. So we only show that $x^{[1]}$ has a zero limit. Since $K_2 = \infty$, by Lemma 1.7(i), $J_3 = \infty$ or $K_3 = \infty$. But since $I < \infty$ implies that $J_2 < \infty$, we have that $K_3 < \infty$ by Lemma 1.7(f). Hence $J_3 = \infty$. Therefore by [3, Lemma 1.3], the proof is complete. □

The following corollary gives us the non-existence of solutions of (1) in subclasses of M^-.

Corollary 2.6. *For solutions of (I.1) in M^-, we have the following results:*

(a) $M^-_{B,B} = \emptyset$ *if and only if* $I = \infty$ *or* $K_2 = \infty$.

(b) $M^-_{0,B} = \emptyset$ *if and only if* $K_2 = \infty$.

(c) *Let* $\beta \geq \alpha$. $M^-_{0,0} = \emptyset$ *if* $I < \infty$ *or* $K_2 < \infty$.

(d) *Let* $\beta \geq \alpha$. *If* $J_2 = \infty$ *or* $K_2 < \infty$, *then* $M^-_{B,0} = \emptyset$.

Proof. (a) and (b) immediately follow from Theorem 2.5(a) and (b), respectively. The part (c) was proved in [3, Theorem 2.2]. For part (d), non-existence of such a solution of (I.1) can be found in [1, Theorem 4.1] and limit behavior of $x^{[1]}$ can be shown with the similar idea as in [3, Theorem 2.2(ii)]. □

3. INTEGRAL RELATIONS

In this section, we introduce more general integrals than J_i and K_i, $i = 1, 2$. The goal is to obtain not only integral relations between these integrals but also some preliminary results in order to investigate the co-existence of solutions in M^+ and M^-.

Let $r, q \in C_{rd}\left([t_0, \infty)_{\mathbb{T}}, \mathbb{R}^+\right)$ and $\lambda, \gamma > 0$.

Define

$$L_\lambda(r,q) = \lim_{T\to\infty} \int_{t_0}^{T} q(t) \left(\int_{t_0}^{t} r(s)\Delta s\right)^{\lambda} \Delta t \tag{I.19}$$

and

$$M_\gamma(r,q) = \lim_{T\to\infty} \int_{t_0}^{T} r(t) \left(\int_{\sigma(t)}^{T} q(s)\Delta s\right)^{\frac{1}{\gamma}} \Delta t. \tag{I.20}$$

We can rewrite the integrals J_1, J_2, K_1 and K_2 by using (I.19) and (I.20) as follows:

$$J_1 = L_{\frac{1}{\alpha}}(b, A),\ J_2 = M_\alpha(A, b),\ K_1 = L_\beta(A, b),\ K_2 = M_{\frac{1}{\beta}}(b, A),$$

where $A = \left(\frac{1}{a}\right)^{\frac{1}{\alpha}}$. It is clear that if

$$\lim_{T\to\infty} \int_{t_0}^{T} q(t)\Delta t = \infty, \tag{I.21}$$

then

$$L_\lambda(r,q) = M_\gamma(r,q) = \infty.$$

The following follows from Theorem 1.2.

Lemma 3.1. *If $\lambda = \gamma = 1$, then $L_1(r,q) = M_1(r,q)$.*

The following lemmas show the convergence and divergence of (I.19) and (I.20) by using λ and γ.

Lemma 3.2. *Let $\lambda = \gamma \leq 1$. If $M_\lambda(r,q) = \infty$, then $L_\lambda(r,q) = \infty$.*

Proof. Let $p = \frac{1}{\lambda}$. So $L_\lambda(r,q)$ and $M_\lambda(r,q)$ can be rewritten as

$$L_{\frac{1}{p}}(r,q) = \lim_{T\longrightarrow\infty} \int_{t_0}^{T} q(t) \left(\int_{t_0}^{t} r(s)\Delta s\right)^{\frac{1}{p}} \Delta t,$$
$$M_{\frac{1}{p}}(r,q) = \lim_{T\longrightarrow\infty} \int_{t_0}^{T} r(t) \left(\int_{\sigma(t)}^{T} q(s)\Delta s\right)^{p} \Delta t.$$

Set

$$r(t,s) = \begin{cases} 0; & s \le \sigma(t) \\ r(t); & s > \sigma(t). \end{cases}$$

Then we have

$$\begin{aligned} \left[\int_{t_0}^{T} r(t) \left(\int_{\sigma(t)}^{T} q(s)\Delta s\right)^{p} \Delta t\right]^{\frac{1}{p}} &= \left[\int_{t_0}^{T} \left(\int_{\sigma(t)}^{T} (r(t))^{\frac{1}{p}} q(s)\Delta s\right)^{p} \Delta t\right]^{\frac{1}{p}} \\ = \left[\int_{t_0}^{T} \left(\int_{t_0}^{T} (r(t,s))^{\frac{1}{p}} q(s)\Delta s\right)^{p} \Delta t\right]^{\frac{1}{p}} &\le \int_{t_0}^{T} q(s) \left(\int_{t_0}^{T} r(t,s)\Delta t\right)^{\frac{1}{p}} \Delta s \\ &= \int_{t_0}^{T} q(s) \left(\int_{t_0}^{s} r(t)\Delta t\right)^{\frac{1}{p}} \Delta s, \end{aligned}$$

where $u = 1$, $f = r^{\frac{1}{p}}$ and $v = q$ in Theorem 1.5. Taking limit as $T \to \infty$ completes the proof. $\square$

Lemma 3.3. *Let $\lambda = \gamma \ge 1$. If $L_\lambda(r,q) = \infty$, then $M_\lambda(r,q) = \infty$.*

Proof. Suppose that $L_\lambda(r,q) = \infty$ and $\lambda \ge 1$. Let

$$q(t,s) = \begin{cases} 0; & s \ge t \\ q(t); & s < t. \end{cases}$$

Then we have

$$\left[\int_{t_0}^{T} q(t)\left(\int_{t_0}^{t} r(s)\Delta s\right)^{\lambda}\Delta t\right]^{\frac{1}{\lambda}} = \left[\int_{t_0}^{T}\left(\int_{t_0}^{t}(q(t))^{\frac{1}{\lambda}} r(s)\Delta s\right)^{\lambda}\Delta t\right]^{\frac{1}{\lambda}}$$

$$= \left[\int_{t_0}^{T}\left(\int_{t_0}^{T}(q(t,s))^{\frac{1}{\lambda}} r(s)\Delta s\right)^{\lambda}\Delta t\right]^{\frac{1}{\lambda}} \leq \int_{t_0}^{T} r(s)\left(\int_{t_0}^{T} q(t,s)\Delta t\right)^{\frac{1}{\lambda}}\Delta s$$

$$= \int_{t_0}^{T} r(s)\left(\int_{\sigma(s)}^{T} q(t)\Delta t\right)^{\frac{1}{\lambda}}\Delta s,$$

where $f = q^{\frac{1}{\lambda}}$, $v = r$ and $u = 1$ in Theorem 1.5. As $T \to \infty$, the assertion follows. □

Now we will obtain similar results for $\lambda \neq \gamma$. But in order to do that we need the following two lemmas.

Lemma 3.4. *Let*

$$Q_T(t) = \int_{t}^{T} q(s)\Delta s. \tag{I.22}$$

If $\eta < 1$ *and*

$$\lim_{T\to\infty}\int_{t_0}^{T} q(s)\Delta s < \infty,$$

then

$$\lim_{T\longrightarrow\infty}\int_{t_0}^{T}\frac{-Q_T^{\Delta}(t)}{[Q_T(\sigma(t)]^{\eta}}\Delta t < \infty.$$

Proof. Set $\nu(t) = -Q_T(t)$ and $f(t) = \frac{1}{[Q_T(\sigma(t)]^{\eta}}$. Since $-Q_T(t)$ is increasing on $[t_0, T)_{\mathbb{T}}$ and $f \in C_{rd}(\mathbb{T}, \mathbb{R})$ on $[t_0, T)_{\mathbb{T}}$, by Theorem 1.4, we have

$$\int_{t_0}^{T}\frac{-Q_T^{\Delta}(t)}{[Q_T(\sigma(t))]^{\eta}}\Delta t = \int_{-\int_{t_0}^{T} q(s)\Delta s}^{0}\frac{dt}{[Q_T((-Q_T)^{-1}(t))]^{\eta}} \qquad \text{for} \quad t \in Range(Q_T).$$

So

$$\int_{t_0}^{T} \frac{-Q_T^{\Delta}(t)}{[Q_T(\sigma(t))]^{\eta}}\Delta t = \int_{-\int_{t_0}^{T} q(s)\Delta s}^{0} \frac{dt}{(-t)^{\eta}} = \lim_{b\to 0^-} \frac{1}{1-\eta}\left[(-b)^{-\eta-1} - \left(\int_{t_0}^{T} q(s)ds\right)\right]^{-\eta+1}$$

$$= -\frac{1}{1-\eta}\left[\int_{t_0}^{T} q(s)ds\right]^{1-\eta}.$$

As $T \to \infty$, the assertion follows, in which $\nu(t) = -Q_T(t)$ and $f(t) = \frac{1}{[Q_T(\sigma(t))]^{\eta}}$ in Theorem 1.4. □

Lemma 3.5. *Let*

$$R_1(t) = 1 + \int_{t_0}^{t} r(s)\Delta s.$$

If $\eta > 1$*, then*

$$\int_{t_0}^{\infty} \frac{R_1^{\Delta}(t)}{R_1^{\eta}(t)}\Delta t < \infty.$$

Proof. Set $\nu(t) = R_1(t)$ and $f(t) = \frac{1}{R_1^{\eta}(t)}$ in Theorem 1.4. Since $R_1(t)$ is strictly increasing on $[t_0, T)_{\mathbb{T}}$ and $f \in C_{rd}([t_0, T)_{\mathbb{T}}, \mathbb{R})$ by Theorem 1.4, we have

$$\int_{t_0}^{T} \frac{R_1^{\Delta}(t)}{R_1^{\eta}(t)}\Delta t = \int_{1}^{1+\int_{t_0}^{T} r(s)\Delta s} \frac{dt}{\left[R_1(R_1^{-1}(t))\right]^{\eta}} \quad \text{for } t \in Range(R_1(t)).$$

So we have

$$\int_{t_0}^{T} \frac{R_1^{\Delta}(t)}{R_1^{\eta}(t)}\Delta t = \int_{1}^{1+\int_{t_0}^{T} r(s)\Delta s} \frac{dt}{t^{\eta}} = \frac{1}{1-\eta}\left[1 - \left(1 + \int_{t_0}^{T} r(s)\Delta s\right)^{-\eta+1}\right].$$

As $T \to \infty$, the assertion follows. □

Lemma 3.6. *Let* $\gamma > \lambda$*. If* $L_{\lambda}(r, q) = \infty$*, then* $M_{\gamma}(r, q) = \infty$*.*

Proof. Suppose that $\gamma > \lambda$. If (I.21) holds, the assertion follows. Since $L_{\lambda}(r, q) = \infty$, we can assume

$$\lim_{T\to\infty} \int_{t_0}^{T} r(t)\Delta t = \infty \quad \text{and} \quad \lim_{T\to\infty} \int_{t_0}^{T} q(t)\Delta t < \infty. \tag{I.23}$$

Denote

$$R_1(t) = 1 + R(t),$$

where

$$R(t) = \int_{t_0}^{t} r(s)\Delta s. \tag{I.24}$$

Consider two cases:

(i) $\gamma \geq 1$ and (ii) $0 < \gamma < 1$

Case(i): Let $t_1 \geq t_0$ be such that $R(t) > 1$ for $t \geq t_1$. Since $Ł_\lambda(r, q) = \infty$, we have

$$\int_{t_1}^{T} q(t) \left(\int_{t_0}^{t} r(s)\Delta s \right)^{\gamma} \Delta t \geq \int_{t_1}^{T} q(t) \left(\int_{t_0}^{t} r(s)\Delta s \right)^{\lambda} \Delta t.$$

As $T \to \infty$, the right hand side goes to infinity, so does the left hand side. Then by Lemma 3.3, we have $M_\gamma(r, q) = \infty$. This completes *Case*(i).

Case(ii)*:* By Theorem 1.2, we have

$$\int_{t_0}^{T} q(t) R_1^{\lambda}(t)\Delta t = Q(t_0) + \int_{t_0}^{T} \left(R_1^{\lambda}(t) \right)^{\Delta} Q_T(\sigma(t))\Delta t.$$

By Theorems 1.2, 1.3, and 1.6, we have

$$\begin{aligned}
\int_{t_0}^{T} q(t) R_1^{\lambda}(t)\Delta t &= Q_T(t_0) + \int_{t_0}^{T} \left\{ \int_0^1 \lambda \left[R_1(t) + h\mu(t) R_1^{\Delta}(t) \right]^{\lambda - 1} dh \right\} R_1^{\Delta}(t) Q_T(\sigma(t))\Delta t \\
&\leq Q_T(t_0) + \int_{t_0}^{T} \lambda \left[R_1(t) \right]^{\lambda - 1} R_1^{\Delta}(t) Q_T(\sigma(t))\Delta t \\
&\leq Q_T(t_0) + \lambda \left[\int_{t_0}^{T} R_1^{\Delta}(t) Q_T^{\frac{1}{\gamma}}(\sigma(t))\Delta t \right]^{\gamma} \left[\int_{t_0}^{T} R_1^{\Delta}(t) \left(R_1(t) \right)^{\frac{\lambda - 1}{1 - \gamma}} \Delta t \right]^{1 - \gamma} \\
&= Q_T(t_0) + \lambda \left[\int_{t_0}^{T} R_1^{\Delta}(t) Q_T^{\frac{1}{\gamma}}(\sigma(t))\Delta t \right]^{\gamma} \left[\int_{t_0}^{T} \frac{R_1^{\Delta}(t)}{\left[R_1(t) \right]^{\frac{1 - \lambda}{1 - \gamma}}} \Delta t \right]^{1 - \gamma}.
\end{aligned}$$

Hence we have

$$\int_{t_0}^{T} q(t)R_1^{\lambda}(t)\Delta t \leq Q_T(t_0) + \lambda \left[\int_{t_0}^{T} R_1^{\Delta}(t)Q_T^{\frac{1}{\gamma}}(\sigma(t))\Delta t\right]^{\gamma} \left[\int_{t_0}^{T} \frac{R_1^{\Delta}(t)}{[R_1(t)]^{\frac{1-\lambda}{1-\gamma}}}\Delta t\right]^{1-\gamma}.$$

Since

$$\int_{t_0}^{\infty} \frac{R_1^{\Delta}(t)}{[R_1(t)]^{\frac{1-\lambda}{1-\gamma}}}\Delta t < \infty$$

for $\frac{1-\lambda}{1-\gamma} > 1$, by Lemma 3.5 the assertion follows as $T \to \infty$. □

Lemma 3.7. *Let $\gamma < \lambda$. If $M_\gamma(r,q) = \infty$, then $L_\lambda(r,q) = \infty$*

Proof. It is clear that if (I.21) holds, there is nothing to show. So since $M_\gamma(r,q) = \infty$, as in the proof in Lemma 3.6, we can assume (I.23) holds.

We will consider two cases:

(i) $\gamma \leq 1$ and (ii) $\gamma > 1$.

Case(i)*:* For $t_1 \geq t_0$, we may suppose $R(t) > 1$ for $t \geq t_1$. Since $M_\gamma(r,q) = \infty$, we have $L_\gamma(r,q) = \infty$ by Lemma 3.2. Hence, similar to the *Case*(i) in proof of Lemma 3.6, the assertion follows.

Case(ii)*:* By (I.22), (I.24) and Theorem 1.2, we have

$$\int_{t_0}^{T} r(t)\left(Q_T(\sigma(t))\right)^{\frac{1}{\gamma}}\Delta t = -\int_{t_0}^{T} \left[\left(Q_T(t)\right)^{\frac{1}{\gamma}}\right]^{\Delta} R(t)\Delta t.$$

Finally, Theorems 1.3 and 1.6 yield

$$\int_{t_0}^{T} r(t)\left(Q_T(\sigma(t))\right)^{\frac{1}{\gamma}}\Delta t = \frac{1}{\gamma}\int_{t_0}^{T}\left\{\int_0^1 \left(Q_T(t) + h\mu(t)Q_T^{\Delta}(t)\right)^{\frac{1-\gamma}{\gamma}} dh\right\} q(t)R(t)\Delta t$$

$$\leq \frac{1}{\gamma}\int_{t_0}^{T} \left(Q_T(\sigma(t))\right)^{\frac{1-\gamma}{\gamma}} q(t)R(t)\Delta t \leq \frac{1}{\gamma}\left[\int_{t_0}^{T} q(t)R^{\lambda}(t)\Delta t\right]^{\frac{1}{\lambda}} \left[\int_{t_0}^{T} -\frac{Q_T^{\Delta}(t)}{\left(Q_T(\sigma(t))\right)^{\xi}}\Delta t\right]^{\frac{\lambda-1}{\lambda}},$$

where $\xi = \frac{(\gamma-1)\lambda}{\gamma(\lambda-1)} < 1, w = q, f = R$ and $g = (Q^\sigma)^{\frac{1-\gamma}{\gamma}}$ in Theorem 1.6. Taking the limit as $T \to \infty$ and using Lemma 3.4 complete the proof. $\square$

4. EXAMPLES

In this section, we give two examples to highlight Theorem 2.5(b).

Example 4.1. *Let* $\mathbb{T} = \mathbb{R}$, $\alpha = 1$, $\beta = \frac{1}{4}$, $a(t) = \frac{1+e^{-4t}}{2e^{-2t}}$ *and* $b(t) = 4e^{\frac{-7t}{2}}$ *in equation (I.1). Then we have*

$$\lim_{T\to\infty}\left(\int_{\sigma(t)}^{T} A(s)ds\right)^{\beta} = \lim_{T\to\infty}\left(\int_{t}^{T} \frac{2e^{-2s}}{1+e^{-4s}}ds\right)^{\frac{1}{4}} < \left(\frac{\pi}{2}\right)^{\frac{1}{4}}$$

and so we obtain

$$\int_{t_0}^{T} b(t)\left(\int_{\sigma(t)}^{T} A(s)ds\right)^{\frac{1}{4}} dt = \int_{t_0}^{T} 4e^{\frac{-7t}{2}}\left(\int_{t}^{T} \frac{2e^{-2s}}{1+e^{-4s}}ds\right)^{\frac{1}{4}} dt < \left(\frac{\pi}{2}\right)^{\frac{1}{4}} \frac{8}{7}e^{\frac{-7t_0}{2}}.$$

As $T \to \infty$, *we have* $K_2 < \infty$. *One can also easily show that* $x(t) = e^{-2t}$ *is a solution of*

$$\left[\frac{1+e^{-4t}}{2e^{-2t}}|x'|\operatorname{sgn} x'\right]' = 4e^{\frac{-7t}{2}}|x|^{\frac{1}{4}}\operatorname{sgn} x$$

such that $\lim_{t\to\infty} x(t) = 0$ *and* $\lim_{t\to\infty} x^{[1]}(t) = -1$, *i.e.,* $M_{0,B}^{-} \neq \emptyset$.

Example 4.2. *Let* $\mathbb{T} = \mathbb{Z}$, $\alpha = 1$, $\beta < 1$, $t_0 \geq 1$, $a_n = \frac{3}{2}(3^n + 1)$ *and* $b_n = 2(3^{n+1})^{\beta-1}$ *in equation (I.1). Letting* $t = n$ *and* $s = m$ *gives us*

$$\begin{aligned}\int_{t_0}^{T} b(t)\left(\int_{\sigma(t)}^{T} A(s)\Delta s\right)^{\beta}\Delta t &= \sum_{n=1}^{T-1} 2(3^{n+1})^{\beta-1}\left(\sum_{m=n+1}^{T-1}\frac{2}{3(3^m+1)}\right)^{\beta}\\ &\leq \frac{2}{3}\sum_{n=1}^{T-1}\left(\frac{1}{3^{1-\beta}}\right)^{n}.\end{aligned}$$

Hence, we have $K_2 < \infty$ *as* $T \to \infty$. *One can show that* $x_n = 3^{-n}$ *is a solution of*

$$\Delta\left[\frac{3}{2}(3^n+1)|\Delta x_n|\operatorname{sgn}\Delta x_n\right] = 2(3^{n+1})^{\beta-1}|x_{n+1}|^{\beta}\operatorname{sgn} x_{n+1}$$

such that $\lim_{n\to\infty} x_n = 0$ *and* $\lim_{n\to\infty} x_n^{[1]} = -1$, *i.e.,* $M_{0,B}^{-} \neq \emptyset$.

5. CONCLUSIONS

In this section, one can obtain the co-existence and non-coexistence of solutions of (I.1) in sub-classes of M^- and M^+ in each of the cases $\alpha = \beta$, $\alpha > \beta$ and $\alpha < \beta$.

The following integral relationships among J_1, K_1, J_2 and K_2 follow directly from Lemmas 3.1-3.3 and 3.6-3.7.

Lemma 5.1. *We have the followings:*

(a) *If* $\alpha = \beta = 1$, *then* $J_1 = K_2$ *and* $J_2 = K_1$.

(b) *If* $\alpha = \beta \leq 1$, *then* $J_2 = \infty \Longrightarrow K_1 = \infty$ *and* $J_1 = \infty \Longrightarrow K_2 = \infty$.

(c) *If* $\alpha = \beta \geq 1$, *then* $K_1 = \infty \Longrightarrow J_2 = \infty$ *and* $K_2 = \infty \Longrightarrow J_1 = \infty$.

(d) *If* $\alpha > \beta$, *then* $K_1 = \infty \Longrightarrow J_2 = \infty$ *and* $J_1 = \infty \Longrightarrow K_2 = \infty$.

(e) *If* $\alpha < \beta$, *then* $J_2 = \infty \Longrightarrow K_1 = \infty$ *and* $K_2 = \infty \Longrightarrow J_1 = \infty$.

In the light of Lemma 5.1, there exist eight cases:

$$(C_1) : J_1 = J_2 = K_1 = K_2 = \infty,$$

$$(C_2) : J_1 = K_2 = \infty,\ J_2 < \infty,\ K_1 < \infty,$$

$$(C_3) : J_1 < \infty,\ K_2 < \infty,\ J_2 = K_1 = \infty,$$

$$(C_4) : J_1 < \infty,\ K_1 < \infty,\ J_2 < \infty,\ K_2 < \infty,$$

$$(C_5) : J_1 = J_2 = K_2 = \infty,\ K_1 < \infty,$$

$$(C_6) : J_1 = J_2 = K_1 = \infty,\ K_2 < \infty,$$

$$(C_7) : J_1 = K_1 = K_2 = \infty,\ J_2 < \infty,$$

$$(C_8) : K_1 = K_2 = J_2 = \infty,\ J_1 < \infty.$$

Note that Cases (C_i), $i=$ (1)-(4) occur for any $\alpha>0$ and $\beta>0$ while (C_5) occurs only for $\alpha=\beta>1$ or $\alpha>\beta$, (C_6) occurs only for $\alpha=\beta>1$ or $\alpha<\beta$, (C_7) occurs only for $\alpha<\beta$ or $\alpha=\beta<1$ and (C_8) occurs only for $\alpha>\beta$ or $\alpha=\beta<1$.

We now investigate the co-existence and co-nonexistence of solutions of (I.1) by using the cases $(C_i), i=$ (1)-(8) and Theorems (2.3), (2.4), (2.5) and (2.6) in the following theorems.

Theorem 5.2. *Let $\alpha=\beta$. For solutions of equation (I.1) in M^+ and M^-, we have the followings:*

(a) *If (C_1) holds, then $M^+=M^+_{\infty,\infty}$ and $M^-=M^-_{0,0}$.*

(b) *If (C_2) holds, then $M^+_{B,B}=M^+_{B,\infty}=\emptyset$ and $M^-_{B,B}=M^-_{0,B}=\emptyset$.*

(c) *If (C_3) holds, then $M^+_{B,\infty}\neq\emptyset, M^+_{B,B}=M^+_{\infty,B}=\emptyset$ and $M^-_{B,B}=M^-_{0,0}=M^-_{B,0}=\emptyset$. Therefore $M^-=M^-_{0,B}$.*

(d) *If (C_4) holds, then $M^+_{B,B}\neq\emptyset,\ M^+_{B,\infty}=M^+_{\infty,B}=\emptyset$ and $M^-_{0,B}\neq\emptyset,\ M^-_{0,0}=M^-_{B,0}=\emptyset$.*

(e) *If (C_5) holds, then $M^+_{B,B}=M^+_{B,\infty}=\emptyset$ and $M^-=M^-_{0,0}$.*

(f) *If (C_6) holds, then $M^+=M^+_{\infty,\infty}$ and $M^-_{B,B}=M^-_{0,0}=M^-_{B,0}=\emptyset$. Therefore, $M^-=M^-_{0,B}$.*

(g) *If (C_7) holds, then $M^+=M^+_{\infty,\infty}$ and $M^-_{B,B}=M^-_{0,B}=\emptyset$.*

(h) *If (C_8) holds, then $M^+_{B,\infty}\neq\emptyset,\ M^+_{B,B}=M^+_{\infty,B}=\emptyset$ and $M^-=M^-_{0,0}$.*

Theorem 5.3. *Let $\alpha>\beta$. For solutions of equation (I.1) in M^+ and M^-, we have the followings:*

(a) *If (C_1) holds, then $M^+=M^+_{\infty,\infty}$ and $M^-=M^-_{0,0}$.*

(b) *If (C_2) holds, then $M^+_{B,B}=M^+_{B,\infty}=\emptyset$ and $M^-_{B,B}=M^-_{0,B}=\emptyset$.*

(c) *If (C_3) holds, then $M^+_{B,\infty}\neq\emptyset,\ M^+_{B,B}=M^+_{\infty,B}=\emptyset$ and $M^-_{0,B}\neq\emptyset,\ \ M^-_{B,B}=\emptyset$.*

(d) *If (C_4) holds, then $M^+_{B,B}\neq\emptyset,\ M^+_{B,\infty}=M^+_{\infty,B}=\emptyset,\quad$ and $M^-_{0,B}\neq\emptyset$.*

(e) *If (C_5) holds, then $M^+_{B,B}=M^+_{B,\infty}=\emptyset$ and $M^-=M^-_{0,0}$.*

(f) *If (C_8) holds, then $M^+_{\infty,\infty}\neq\emptyset,\ \ M^+_{B,B}=\emptyset,\ M^+_{\infty,B}=\emptyset$ and $M^-=M^-_{0,0}$.*

Theorem 5.4. *Let $\alpha < \beta$. For solutions of equation (I.1) in M^+ and M^-, we have the followings:*

(a) *If (C_1) holds, then $M^+ = M^+_{\infty,\infty}$ and $M^- = M^-_{0,0}$.*

(b) *If (C_2) holds, then $M^+_{B,B} = M^+_{B,\infty} = \emptyset$ and $M^-_{B,B} = M^-_{0,B} = \emptyset$.*

(c) *If (C_3) holds, then $M^+_{B,\infty} \neq \emptyset$, $M^+_{B,B} = M^+_{\infty,B} = \emptyset$ and $M^-_{B,B} = M^-_{0,0} = M^-_{B,0} = \emptyset$. Therefore $M^- = M^-_{0,B}$.*

(d) *If (C_4) holds, then $M^+_{B,B} \neq \emptyset$, $M^+_{B,\infty} = M^+_{\infty,B} = \emptyset$ and $M^-_{0,B} \neq \emptyset$, $M^-_{0,0} = M^-_{B,0} = \emptyset$.*

(e) *If (C_6) holds, then $M^+ = M^+_{\infty,\infty}$ and $M^-_{B,B} = M^-_{0,0} = M^-_{B,0} = \emptyset$. Therefore, $M^- = M^-_{0,B}$.*

(f) *If (C_7) holds, then $M^+ = M^+_{\infty,\infty}$ and $M^-_{B,B} = M^-_{0,B} = \emptyset$.*

Our goal for the entire paper has been to classify nonoscillatory solutions of (I.1) depending on J_1, K_1, J_2 and K_2. However, we would like to indicate the following remarks.

Remark 5.5. *When $J_1 = \infty$ and $K_1 < \infty$, we have to assume that*

$$\mu(t) \text{ is differentiable such that } \mu^{\Delta}(t) \geq 0 \text{ and } a^{\sigma}(t) \geq a(t) \text{ for } t \geq t_1 \tag{I.25}$$

to be able to obtain $M^+_{\infty,B} \neq \emptyset$, which follows from [2, Theorem 3.1] and [2, Corollary 5.1]. On the other hand, in case (C_2) or (C_5) holds with $\alpha \geq \beta$, or (C_2) holds with $\alpha < \beta$, we obtain $M^+_{\infty,B} \neq \emptyset$ as well. If $\mathbb{T} = \mathbb{R}$, then (I.25) holds automatically. So our result corresponds with the continuous case. Of course, one can obtain that $M^+_{\infty,B} \neq \emptyset$ by assuming both conditions

$$J_1 = \infty, \qquad \text{and} \qquad \lim_{T\to\infty} \int_{t_0}^{T} b(t) \left(\int_{t_0}^{t} A^{\sigma}(s)\Delta s \right)^{\beta} \Delta t < \infty$$

without (I.25) as in the discrete case, see [9].

Remark 5.6. *When $J_1 < \infty$ or $K_1 < \infty$, we have to assume that*

$$\int_{t_1}^{\infty} b(t)\mu^{\beta}(t)\left(\frac{1}{a(t)}\right)^{\frac{\beta}{\alpha}} \Delta t < \infty, \tag{I.26}$$

where $\alpha > \beta$ to be able to obtain $M^{+}_{\infty,\infty} = \emptyset$ by using [2, Theorem 3.2], Theorem 1.1, inequality (I.7), and Lemma 1.7(b). On the other hand, if we have one of the cases (C_2), (C_3), (C_4), (C_5) and (C_8) with $\alpha > \beta$, then $M^{+}_{\infty,\infty} = \emptyset$ as well. If $\mathbb{T} = \mathbb{R}$, then (I.26) holds automatically. So our result matches with the continuous case. Of course, one can show that $M^{+}_{\infty,\infty} = \emptyset$ by assuming

$$\lim_{T\to\infty} \int_{t_0}^{T} b(t)\left(\int_{t_0}^{t} A^{\sigma}(s)\Delta s\right)^{\beta} \Delta t < \infty, \quad \alpha > \beta$$

without (I.26) as in the discrete case, see [9].

Another reasonable nonlinear dynamic equation is to consider

$$\left[a(t)|x^{\Delta}(t)|^{\alpha} \operatorname{sgn} x^{\Delta}\right]^{\Delta} = -b(t)|x^{\sigma}(t)|^{\beta} \operatorname{sgn} x^{\sigma}(t) \tag{I.27}$$

as our new project because several questions arise. For example, what integral conditions might we have in order to obtain the existence of nonoscillatory solutions of (I.27)? And what sub-classes might occur for nonoscillatory solutions of (I.27) depending on the convergence/divergence of J_3 and K_3? Also what oscillation criteria do we need for (I.27)?

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II. ON NONOSCILLATORY SOLUTIONS OF EMDEN-FOWLER DYNAMIC SYSTEMS ON TIME SCALES

ABSTRACT

We study the existence and asymptotic behavior of nonoscillatory solutions of Emden-Fowler dynamic sytems on time scales. In order to show the existence, we use Schauder, Knaster and Tychonoff Fixed Point Theorems. Some examples are illustrated as well.

1. INTRODUCTION

In this paper, we deal with the classification of nonoscillatory solutions of the Emden-Fowler system of first order dynamic equations

$$\begin{cases} x^{\Delta}(t) = \left(\frac{1}{a(t)}\right)^{\frac{1}{\alpha}} |y(t)|^{\frac{1}{\alpha}} \operatorname{sgn} y(t) \\ y^{\Delta}(t) = -b(t)\, |x^{\sigma}(t)|^{\beta} \operatorname{sgn} x^{\sigma}(t), \end{cases} \tag{II.1}$$

where $\alpha, \beta > 0$ and $a, b \in C_{rd}\left([t_0, \infty)_{\mathbb{T}}, \mathbb{R}^{+}\right)$. Whenever we write $t \geq t_1$, we mean that $t \in [t_1, \infty)_{\mathbb{T}} := [t_1, \infty) \cap \mathbb{T}$. A time scale $\mathbb{T}$, a nonempty closed subset of real numbers, is introduced by Bohner and Peterson in [6] and [7]. Throughout this paper, we assume that $\mathbb{T}$ is unbounded above. We call (x, y) a *proper solution* if it is defined on $[t_0, \infty)_{\mathbb{T}}$ and $sup\{|x(s)|, |y(s)| : s \in [t, \infty)_{\mathbb{T}}\} > 0$ for $t \geq t_0$. A solution (x, y) of (II.1) is said to be nonoscillatory if the component functions x and y are both nonoscillatory, i.e., either eventually positive or eventually negative. Otherwise it is said to be oscillatory. Throughout this paper without loss of generality we assume that x is eventually positive in our proofs. Our results can be obtained similarly for the case that x is eventually negative.

System (II.1) can be easily derived from the Emden Fowler dynamic equation

$$\left(a(t)|x^{\Delta}(t)|^{\alpha} sgn x^{\Delta}(t)\right)^{\Delta} + b(t)|x(t)|^{\beta} sgn x^{\sigma}(t) = 0 \tag{II.2}$$

by letting $x = x$ and $y = |x^{\Delta}|^{\alpha} sgn x^{\Delta}$ in (II.2). If $\alpha = \beta$ in (II.2), then it is called a half-linear dynamic equation.

If $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$, equation (II.2) reduces to the Emden Fowler differential equation

$$(a(t)|x'(t)|^{\alpha} sgn x'(t))' + b(t)|x(t)|^{\beta} sgn x(t) = 0,$$

see [11], and the Emden-Fowler difference equation

$$\Delta\left(a_n|\Delta x_n|^{\alpha} sgn \Delta x_n\right) + b_n|x_{n+1}|^{\beta} sgn x_{n+1} = 0,$$

see [8], respectively.

This paper is motivated by the papers [8], [14] and [10]. The related oscillation and nonoscillation results for two and three dimensional dynamic systems are given in [5], [3], [4], and [2], respectively. The setup of this paper is as follows: In Section 1, we give preliminary lemmas playing an important role in the further sections. In Sections 2 and 3, we show the existence and asymptotic properties of nonoscillatory solutions of system (II.1) by using certain improper integrals and fixed point theorems. In Section 4, we obtain some conclusions. And finally, the paper concludes with some examples.

Let M be the set of all nonoscillatory solutions of system (II.1). One can easily show that any nonoscillatory solution (x, y) of system (II.1) belongs to one of the following classes:

$$M^+ := \{(x, y) \in M : \ x(t)y(t) > 0 \ \text{ eventually}\}$$
$$M^- := \{(x, y) \in M : \ x(t)y(t) < 0 \ \text{ eventually}\}.$$

Lemma 1.1. *[5, Lemma 2.1] Let (x, y) be a solution of system (II.1). Then the component functions x and y are themselves nonoscillatory if (x, y) is a nonoscillatory solution of system (II.1).*

Remark 1.2. *Let (x, y) be a nonoscillatory solution of system (II.1). If $x(t)$ is nonoscillatory for $t \geq t_0$, then the other component function $y(t)$ is also nonoscillatory for sufficiently large t.*

For convenience, let us set

$$Y_a = \int_{t_0}^{\infty} A(t)\Delta t \quad \text{and} \quad Z_b = \int_{t_0}^{\infty} b(t)\Delta t, \tag{II.3}$$

where $A = \left(\frac{1}{a}\right)^{\frac{1}{\alpha}}$.

The following lemma gives some sufficient conditions for oscillation and nonoscillation of system (II.1).

Lemma 1.3. (a)[5, Lemma 2.3] If $Y_a < \infty$ and $Z_b < \infty$, then system (II.1) is nonoscillatory.
(b)[5, Lemma 2.2] If $Y_a = \infty$ and $Z_b = \infty$, then system (II.1) is oscillatory.

In the next two lemmas we show that M^+ and M^- can be empty.

Lemma 1.4. *If $Y_a = \infty$ and $Z_b < \infty$, then any nonoscillatory solution (x, y) of system (II.1) belongs to M^+, i.e $M^- = \emptyset$.*

Proof. Suppose that $Y_a = \infty$ and $Z_b < \infty$. The proof is by contradiction. So assume that there exists a solution (x, y) of system (II.1) such that $(x, y) \in M^-$. Without loss of generality assume that $x(t) > 0$ for $t \geq t_1$. Then by integrating the first equation of system (II.1) from t_1 to t and the monotonicity of y, we have

$$x(t) = x(t_1) - \int_{t_1}^{t} A(s)\left(-y(s)\right)^{\frac{1}{\alpha}} \Delta s \leq x(t_1) - \left(-y(t_1)\right)^{\frac{1}{\alpha}} \int_{t_1}^{t} A(s)\Delta s.$$

As $t \to \infty$, $x \to -\infty$. But this contradicts the positivity of x. Note that the proof can be done without the condition $Z_b < \infty$. However in order for nonoscillatory solutions to exist, we need the assumption $Z_b < \infty$ by Lemma 1.3 (b). $\square$

Lemma 1.5. *If $Y_a < \infty$ and $Z_b = \infty$, then any nonoscillatory solution (x, y) of system (II.1) belongs to M^-, i.e., $M^+ = \emptyset$.*

Proof. Suppose that $Y_a < \infty$ and $Z_b = \infty$. The proof is by contradiction. So assume that there exists a nonoscillatory solution (x, y) of system (II.1) such that $xy > 0$ eventually. Without loss of generality, assume that $x(t) > 0$ for $t \geq t_1$. So by integrating the second equation of system (II.1) from t_1 to t and the monotonicity of x give us

$$y(t) \leq y(t_1) - (x^\sigma(t_1))^\beta \int_{t_1}^{t} b(s)\Delta s.$$

As $t \to \infty$, it follows that $y(t) \to -\infty$. But this contradicts that y is eventually positive. □

The discrete version of the following lemmas can be found in [14].

Lemma 1.6. *Let (x, y) be a nonoscillatory solution of system (II.1).*

(a) If $Y_a < \infty$, then the component function x has a finite limit.

(b) If $Y_a = \infty$ or $Z_b < \infty$, then the component function y has a finite limit.

Proof. (a) Suppose that $Y_a < \infty$ and (x, y) is a nonoscillatory solution of system (II.1). Then by Lemma 1.1, x and y are themselves nonoscillatory. Without loss of generality, assume that there exists $t_1 \geq t_0$ such that $x(t) > 0$ for $t \geq t_1$. If $(x, y) \in M^-$, then by the first equation of system (II.1), $x^\Delta(t) < 0$ for $t \geq t_1$. Therefore, limit of x exists. So let us show that the assertion follows if $(x, y) \in M^+$. From the first equation of system (II.1), we have $x^\Delta(t) > 0$ for $t \geq t_1$. Hence two things might happen: The limit of the component function x exists or blows up. Now let us show that $\lim_{t\to\infty} x(t) = \infty$ cannot happen. Assume $x(t) \to \infty$ as $t \to \infty$. By integrating the first equation of system (II.1) from t_1 to t and using the monotonicity of y, we get

$$x(t) \leq x(t_1) + y^{\frac{1}{\alpha}}(t_1) \int_{t_1}^{t} A(s)\Delta s.$$

Taking the limit as $t \to \infty$, it follows that $Y_a = \infty$, which is a contradiction. This completes the proof.

(b) Suppose that $Y_a = \infty$ or $Z_b < \infty$ and (x, y) is a nonoscillatory solution of system (II.1). The case $Z_b < \infty$ can be proved similar to part (a). For $Y_a = \infty$, assume that x is eventually positive. Then proceeding as in the proof of Lemma 1.4, it can be shown that y is eventually positive. Then by the second equation of system (II.1), it follows that y has a finite limit. $\square$

In the following lemmas, we find upper and lower bounds for the component function x of a nonoscillatory solution (x, y) of system (II.1).

Lemma 1.7. *Let $Y_a < \infty$. If (x, y) is a nonoscillatory solution of system (II.1), then there exist $c, d > 0$ and $t_1 \geq t_0$ such that*

$$c \int_t^\infty A(s)\Delta s \leq x(t) \leq d$$

or

$$-d \leq x(t) \leq -c \int_t^\infty A(s)\Delta s$$

for $t \geq t_1$.

Proof. Suppose that $Y_a < \infty$ and (x, y) is a nonoscillatory solution of system (II.1). Without loss of generality, let us assume that x is eventually positive. Then by Lemma 1.6 (a), we have $x(t) \leq d$ for $t \geq t_1$. If $y(t) > 0$ for $t \geq t_1$, then x is eventually increasing by the first equation of system (II.1). So for large t, the assertion follows. If $y(t) < 0$ for $t \geq t_1$, then integrating the first equation of system (II.1) from t to ∞ and the monotonicity of y give

$$\begin{aligned} x(t) &= x(\infty) + \int_t^\infty A(s)(-y(s))^{\frac{1}{\alpha}}\Delta s \geq \int_t^\infty A(s)(-y(s))^{\frac{1}{\alpha}}\Delta s \\ &\geq (-y(t_1))^{\frac{1}{\alpha}} \int_t^\infty A(s)\Delta s. \end{aligned}$$

Setting $c = (-y(t_1))^{\frac{1}{\alpha}}$ in the last inequality proves the assertion. Assuming x is eventually negative gives the second part of the proof. $\square$

Lemma 1.8. *Let $Y_a = \infty$ and $Z_b < \infty$. If (x, y) is a nonoscillatory solution of system (II.1), then there exist $k_1, k_2 > 0$ and $t_1 \geq t_0$ such that*

$$k_1 \leq x(t) \leq k_2 \int_{t_1}^{t} A(s)\Delta s$$

or

$$-k_2 \int_{t_1}^{t} A(s)\Delta s \leq x(t) \leq k_1$$

for $t \geq t_1$.

Proof. Suppose that $Y_a = \infty$ and $Z_b < \infty$, and (x, y) is a nonoscillatory solution of system (II.1). Then by Lemma 1.1, x and y are themselves nonoscillatory. Without loss of generality let us assume that $x(t) > 0$ for $t \geq t_1$. Then by Lemma 1.4, (x, y) must be in M^+. Hence, there is a constant $k_1 > 0$ such that $x(t) \geq k_1$ for $t \geq t_1$. Integrating the first equation of system (II.1) and the monotonicity of y give

$$\begin{aligned} x(t) &= x(t_1) + \int_{t_1}^{t} A(s) y^{\frac{1}{\alpha}}(s)\Delta s \leq x(t_1) + y^{\frac{1}{\alpha}}(t_1) \int_{t_1}^{t} A(s)\Delta s \\ &= \left(\frac{x(t_1)}{\int_{t_1}^{t} A(s)\Delta s} + y^{\frac{1}{\alpha}}(t_1) \right) \int_{t_1}^{t} A(s)\Delta s. \end{aligned}$$

Since $Y_a = \infty$, we can choose $t_2 \geq t_1$ such that

$$\int_{t_2}^{t} A(t)\Delta t \geq 1 \quad \text{for} \quad t \geq t_2.$$

So this implies that

$$x(t) \leq \left(x(t_1) + y^{\frac{1}{\alpha}}(t_1)\right) \int_{t_1}^{t} A(s)\Delta s$$

and the assertion follows by letting $k_2 = x(t_1)+y^{\frac{1}{\alpha}}(t_1)$. Assuming that x is eventually negative proves the second part of the proof. $\square$

2. THE CASE $Y_a = \infty$ AND $Z_b < \infty$

In this section, we show that M^+ can be divided into some sub-classes under the case $Y_a = \infty$. By Lemma 1.3(b), in order to obtain the existence of nonoscillatory solutions, we also have to assume $Z_b < \infty$. So throughout this section, we suppose that $Y_a = \infty$ and $Z_b < \infty$ hold. Then by Lemma 1.4, $(x, y) \in M^+$. Without loss of generality we suppose that $x > 0$ eventually. Then by the second equation of system (II.1), y is positive and decreasing eventually. In addition to that, by using the first equation of system (II.1) and taking Lemma 1.6(b) into consideration we have that $x(t) \to c$ or ∞, and $y(t) \to d$ or 0 as $t \to \infty$ for $0 < c < \infty$ and $0 < d < \infty$.

Lemma 2.1. *If $x(t) \to c$, then $y(t) \to 0$ as $t \to 0$ for $c < 0 < \infty$.*

Proof. Suppose that $x(t) \to c$ as $t \to \infty$. Assume the contrary. So $y(t) \to d$ for $0 < d < \infty$ as $t \to \infty$. Then since $y(t) > 0$ and decreasing eventually, there exists $t_1 \geq t_0$ such that $y(t) \geq d$ for $t \geq t_1$. By the first equation of system (II.1), we have

$$x^{\Delta}(t) = A(t) y^{\frac{1}{\alpha}}(t) \geq A(t) d^{\frac{1}{\alpha}} \quad \text{for} \quad t \geq t_1. \tag{II.4}$$

Integrating (II.4) from t_1 to t yields

$$x(t) \geq x(t_1) + d^{\frac{1}{\alpha}} \int_{t_1}^{t} A(s) \Delta s.$$

As $t \to \infty$, this gives us a contradiction to the fact $x(t) \to c$. So the assertion follows. $\square$

In light of Lemma 2.1 and the explanation above, we have the following lemma.

Lemma 2.2. *For $0 < c < \infty$ and $0 < d < \infty$, any nonoscillatory solution in M^+ must belong to one of the following sub-classes:*

$$M^+_{B,0} = \left\{ x \in M^+ : \lim_{t\to\infty} |x(t)| = c, \quad \lim_{t\to\infty} |y(t)| = 0 \right\},$$
$$M^+_{\infty,B} = \left\{ x \in M^+ : \lim_{t\to\infty} |x(t)| = \infty, \ \lim_{t\to\infty} |y(t)| = d \right\},$$
$$M^+_{\infty,0} = \left\{ x \in M^+ : \lim_{t\to\infty} |x(t)| = \infty, \ \lim_{t\to\infty} |y(t)| = 0 \right\}.$$

In the literature, solutions in $M^+_{B,0}$, $M^+_{\infty,B}$ and $M^+_{\infty,0}$ are called *subdominant solutions*, *dominant solutions* and *intermediate solutions*, respectively.

The following theorems show the existence of nonoscillatory solutions in sub-classes mentioned above by using the improper integrals:

$$J_\alpha = \int_{t_0}^{\infty} A(t) \left(\int_t^\infty b(s)\Delta s \right)^{\frac{1}{\alpha}} \Delta t \tag{II.5}$$

$$K_\beta = \int_{t_0}^{\infty} b(t) \left(\int_{t_0}^{\sigma(t)} A(s)\Delta s \right)^{\beta} \Delta t. \tag{II.6}$$

Theorem 2.3. *$M^+_{B,0} \neq \emptyset$ if and only if $J_\alpha < \infty$.*

Proof. Suppose that $M^+_{B,0} \neq \emptyset$. Then there exists $(x, y) \in M^+$ such that $|x(t)| \to c > 0$ and $|y(t)| \to 0$ as $t \to \infty$. Without loss of generality let us assume that $x(t) > 0$ for $t \geq t_1$. Integrating the second equation of system (II.1) from t to ∞ gives us

$$y(t) = \int_t^\infty b(s) \left(x^\sigma(s)\right)^\beta \Delta s. \tag{II.7}$$

Solving the first equation of system (II.1) for y, substituting the resulting equation into (II.7) and by the monotonicity of y, we obtain

$$x^\Delta(t) \geq A(t) x^{\frac{\beta}{\alpha}}(t) \left(\int_t^\infty b(s)\Delta s \right)^{\frac{1}{\alpha}}. \tag{II.8}$$

Integrating (II.8) from t_1 to t gives

$$\begin{aligned} x(t) &\geq x(t_1) + \int_{t_1}^{t} A(s) x^{\frac{\beta}{\alpha}}(s) \left(\int_s^\infty b(\tau)\Delta\tau \right)^{\frac{1}{\alpha}} \Delta s \\ &\geq x^{\frac{\beta}{\alpha}}(t_1) \int_{t_0}^{t} A(s) \left(\int_s^\infty b(\tau)\Delta\tau \right)^{\frac{1}{\alpha}} \Delta s. \end{aligned}$$

As $t \to \infty$, the assertion follows.

Conversely, suppose that $J_\alpha < \infty$. Choose $t_1 \geq t_0$ so large that

$$\int_{t_1}^\infty A(t) \left(\int_t^\infty b(s)\Delta s \right)^{\frac{1}{\alpha}} \Delta t < \left(\frac{c}{2} \right) \frac{1}{c^{\frac{\beta}{\alpha}}} \tag{II.9}$$

for arbitrarily given $c > 0$. Let X be the set of all bounded, continuous, real valued functions with the norm $\|x\| = \sup_{t \in [t_1, \infty)_{\mathbb{T}}} \{|x(t)|\}$. It is clear that X is a Banach Space, see [9]. Let us define a subset Ω of X such that

$$\Omega := \{x \in X : \ \frac{c}{2} \leq x(t) \leq c, \quad t \geq t_1\}.$$

It is clear that Ω is closed, bounded and convex. Define an operator $F : \Omega \to X$ by

$$(Fx)(t) = c - \int_t^\infty A(s) \left(\int_s^\infty b(\tau) \left(x^\sigma(\tau) \right)^\beta \Delta\tau \right)^{\frac{1}{\alpha}} \Delta s \quad \text{for} \quad t \geq t_1. \tag{II.10}$$

By inequality (II.9), we have

$$\begin{aligned} c \geq (Fx)(t) &= c - \int_t^\infty A(s) \left(\int_s^\infty b(\tau) \left(x^\sigma(\tau) \right)^\beta \Delta\tau \right)^{\frac{1}{\alpha}} \Delta s \\ &\geq c - c^{\frac{\beta}{\alpha}} \int_t^\infty A(s) \left(\int_s^\infty b(\tau)\Delta\tau \right)^{\frac{1}{\alpha}} \Delta s \geq \frac{c}{2}, \end{aligned}$$

and so $F : \Omega \to \Omega$. Since

$$\begin{aligned}&||(Fx_n)(t) - (Fx)(t)|| \\ &\qquad \le \int_{t_1}^{\infty} A(s) \left| \left(\int_s^{\infty} b(\tau) \left(x_n^{\sigma}(\tau)\right)^{\beta} \Delta\tau \right)^{\frac{1}{\alpha}} - \left(\int_s^{\infty} b(\tau) \left(x^{\sigma}(\tau)\right)^{\beta} \Delta\tau \right)^{\frac{1}{\alpha}} \right| \Delta s,\end{aligned}$$

where x_n is a sequence of functions converging to x. Hence, the Lebesque Dominated Convergence Theorem yields

$$||(Fx_n)(t) - (Fx)(t)|| \to 0,$$

which implies the continuity of F on Ω. Also

$$0 \le -\left[F(x)(t)\right]^{\Delta} = A(t) \left(\int_t^{\infty} b(\tau) \left(x^{\sigma}(\tau)\right)^{\beta} \Delta\tau \right)^{\frac{1}{\alpha}} \le c^{\frac{\beta}{\alpha}} A(t) \left(\int_t^{\infty} b(\tau) \Delta\tau \right)^{\frac{1}{\alpha}} < \infty$$

implies that F is equibounded and equicontinuous. Therefore by Schauder's Fixed Point Theorem, there exists $\bar{x} \in \Omega$ such that $\bar{x} = F\bar{x}$. Then

$$\bar{x}(t) = c - \int_t^{\infty} A(s) \left(\int_s^{\infty} b(\tau) \left(\bar{x}^{\sigma}(\tau)\right)^{\beta} \Delta\tau \right)^{\frac{1}{\alpha}} \Delta s. \qquad \text{(II.11)}$$

So as $t \to \infty$, $\bar{x}(t) \to c$. Note that $\bar{x}^{\Delta}(t) > 0$ for $t \ge t_1$. So it is eventually monotone, i.e., $\bar{x}$ is nonoscillatory. Therefore, taking the derivative of (II.11) and using the first equation of system (II.1) give us

$$\bar{y}(t) = \int_t^{\infty} b(\tau) \left(\bar{x}^{\sigma}(\tau)\right)^{\beta} \Delta\tau.$$

It follows that $\bar{y}(t) > 0$ for $t \ge t_1$, i.e., $(\bar{x}, \bar{y})$ is nonoscillatory and then by Remark 1.2 and Lemma 1.4, $(\bar{x}, \bar{y}) \in M^{+}$. Taking the limit as $t \to \infty$ yields $\bar{y}(t) \to 0$. Hence $M_{B,0}^{+} \neq \emptyset$. $\square$

Theorem 2.4. *$M^+_{\infty,B} \neq \emptyset$ if and only if $K_\beta < \infty$.*

Proof. Suppose that $M^+_{\infty,B} \neq \emptyset$. Then there exists $(x, y) \in M^+$ such that $|x(t)| \to \infty$ and $|y(t)| \to d$, for $0 < d < \infty$. Without loss of generality assume that $x(t) > 0$ for $t \geq t_1$ Integrating the first equation from t_1 to $\sigma(t)$ and the second equation from t_1 to t of system (II.1) give us

$$x^\sigma(t) = x^\sigma(t_1) + \int_{t_1}^{\sigma(t)} A(s) y^{\frac{1}{\alpha}}(s)\Delta s > d^{\frac{1}{\alpha}} \int_{t_1}^{\sigma(t)} A(s)\Delta s. \tag{II.12}$$

and

$$y(t_1) - y(t) = \int_{t_1}^{t} b(s) \left(x^\sigma(s)\right)^\beta \Delta s, \tag{II.13}$$

respectively. Then by (II.12) and (II.13), we have

$$\begin{aligned}\int_{t_1}^{t} b(s) \left(\int_{t_1}^{\sigma(s)} A(\tau)\Delta\tau\right)^\beta \Delta s &< d^{\frac{-\beta}{\alpha}} \int_{t_1}^{t} b(s) \left(x^\sigma(s)\right)^\beta \Delta s \\ &< d^{\frac{-\beta}{\alpha}} \left(y(t_1) - y(t)\right)\end{aligned}$$

So as t goes to ∞, it follows that $K_\beta < \infty$.

Conversely, suppose that $K_\beta < \infty$. Choose $t_1 \geq t_0$ so large that

$$\int_{t_1}^{\infty} b(s) \left(\int_{t_1}^{\sigma(s)} A(\tau)\Delta\tau\right)^\beta \Delta s < \frac{d}{(2d)^\beta} \tag{II.14}$$

for arbitrarily given $d > 0$. Let X be the partially ordered Banach Space of all real-valued continuous functions with the norm $\|x\| = \sup\limits_{t>t_1} \frac{|x(t)|}{\int_{t_1}^{t} A(s)\Delta s}$ and the usual pointwise ordering $\leq$. Define a subset Ω of X as follows:

$$\Omega : \{x \in X : \quad d^{\frac{1}{\alpha}} \int_{t_1}^{t} A(s)\Delta s \leq x(t) \leq (2d)^{\frac{1}{\alpha}} \int_{t_1}^{t} A(s)\Delta s \ \text{ for } \ t > t_1\}.$$

First since every subset of Ω has a supremum and infimum in Ω, $(\Omega, \leq)$ is a complete lattice. Define an operator $F : \Omega \to X$ as

$$(Fx)(t) = \int_{t_1}^{t} A(s) \left(d + \int_{s}^{\infty} b(\tau) \left(x^{\sigma}(\tau) \right)^{\beta} \Delta\tau \right)^{\frac{1}{\alpha}} \Delta s. \tag{II.15}$$

It can be shown that $F : \Omega \to \Omega$ is an increasing mapping for $t \geq t_1$.

So by the Knaster Fixed Point Theorem, we have that there exists $\bar{x} \in \Omega$ such that

$$\bar{x}(t) = \int_{t_1}^{t} A(s) \left(d + \int_{s}^{\infty} b(\tau) \left(\bar{x}^{\sigma}(\tau) \right)^{\beta} \Delta\tau \right)^{\frac{1}{\alpha}} \Delta s \ \text{ for } \ t > t_1. \tag{II.16}$$

Hence $\bar{x}$ is eventually positive, and hence nonoscillatory. Then by taking the derivative of (II.16) and using the first equation of system (II.1) give us

$$\bar{y}(t) = \left(\bar{x}^{\Delta}(t) \right)^{\alpha} a(t) = d + \int_{t}^{\infty} b(\tau) \left(\bar{x}^{\sigma}(\tau) \right)^{\beta} \Delta\tau. \tag{II.17}$$

Then it follows that $\bar{y}$ is eventually positive, i.e., nonoscillatory. Hence, $(\bar{x}, \bar{y})$ is a nonoscillatory solution of system (II.1) and by Lemma 1.4 we have $(\bar{x}, \bar{y}) \in M^{+}$. For $\bar{x} \in \Omega$, we also have

$$\bar{x}(t) \geq \int_{t_1}^{t} A(s) \left[d + \int_{s}^{\infty} b(\tau) \left(d^{\frac{1}{\alpha}} \int_{t_1}^{\sigma(\tau)} A(\lambda) \Delta\lambda \right)^{\beta} \Delta\tau \right]^{\frac{1}{\alpha}} \Delta s$$

As $t \to \infty$, the right hand side of the last inequality goes to ∞ since $Y_a = \infty$. Therefore $\bar{x}(t) \to \infty$ as $t \to \infty$. Taking the limit as $t \to \infty$ of (II.17) gives that y has a finite limit. Therefore $M^{+}_{\infty,B} \neq \emptyset$. $\square$

Theorem 2.5. *If $J_{\alpha} = \infty$ and $K_{\beta} < \infty$, then $M^{+}_{\infty,0} \neq \emptyset$.*

Proof. Suppose that $J_\alpha = \infty$ and $K_\beta < \infty$. Since $Y_a = \infty$, we can choose $t_1, t_2 \geq t_0$ so large that

$$\int_{t_2}^{\infty} b(t) \left(\int_{t_0}^{\sigma(t)} A(s)\Delta s \right)^{\beta} \Delta t \leq 1 \tag{II.18}$$

and

$$\int_{t_1}^{t_2} A(s)\Delta s \geq 1. \tag{II.19}$$

Let X be the Fréchet Space of all continuous functions on $[t_1, \infty)_{\mathbb{T}}$ endowed with the topology of uniform convergence on compact subintervals of $[t_1, \infty)_{\mathbb{T}}$. Set

$$\Omega := \{x \in X : \quad 1 \leq x(t) \leq \int_{t_1}^{t} A(s)\Delta s \;\text{ for }\; t \geq t_1\}$$

and define an operator $T : \Omega \to X$ by

$$(Tx)(t) = 1 + \int_{t_2}^{t} A(s) \left(\int_{s}^{\infty} b(\tau) \left(x^{\sigma}(\tau) \right)^{\beta} \Delta\tau \right)^{\frac{1}{\alpha}}. \tag{II.20}$$

We can show that $T : \Omega \to \Omega$ is continuous on $\Omega \subset X$ by the Lebesque Dominated Convergence Theorem. Since

$$\begin{aligned} 0 \leq [(Tx)(t)]^{\Delta} &= A(t) \left(\int_{t}^{\infty} b(\tau) \left(x^{\sigma}(\tau) \right)^{\beta} \Delta\tau \right)^{\frac{1}{\alpha}} \\ &\leq A(t) \left(\int_{t}^{\infty} b(\tau) \left(\int_{t_1}^{\sigma(\tau)} A(\lambda)\Delta\lambda \right)^{\beta} \Delta\tau \right)^{\frac{1}{\alpha}} < \infty, \end{aligned}$$

it follows that T is equibounded and equicontinuous. Then by Tychonoff Fixed Point Theorem, there exists $\bar{x} \in \Omega$ such that

$$\bar{x}(t) = (T\bar{x})(t) = 1 + \int_{t_2}^{t} A(s) \left(\int_{s}^{\infty} b(\tau) \left(\bar{x}^{\sigma}(\tau) \right)^{\beta} \Delta\tau \right)^{\frac{1}{\alpha}} \;\text{ for }\; t \geq t_2. \tag{II.21}$$

Therefore, it follows that $\bar{x}$ is eventually positive, i.e nonoscillatory. Then integrating (II.21) and by the first equation of system (II.1), we have

$$\bar{y}(t) = a(t)\left(x^{\Delta}(t)\right)^{\alpha} = \int_t^{\infty} b(\tau)\left(\bar{x}^{\sigma}(\tau)\right)^{\beta} \Delta\tau. \tag{II.22}$$

It follows that $\bar{y}$ is eventually positive, and hence (x, y) is a nonoscillatory solution of system (II.1). So by Lemma 1.4 it follows that $(\bar{x}, \bar{y}) \in M^{+}$. Also by monotonicity of $\bar{x}$, we have

$$\bar{x}(t) = 1 + \int_{t_2}^{t} A(s)\left(\int_s^{\infty} b(\tau)\left(\bar{x}^{\sigma}(\tau)\right)^{\beta} \Delta\tau\right)^{\frac{1}{\alpha}} \geq \left(\bar{x}(t_2)\right)^{\beta} \int_{t_2}^{t} A(s)\left(\int_s^{\infty} b(\tau)\Delta\tau\right)^{\frac{1}{\alpha}}.$$

Hence as $t \to \infty$, it follows that $\bar{x}(t) \to \infty$. And by (II.22), we have $\bar{y}(t) \to 0$ as $t \to \infty$. Therefore $M^{+}_{\infty,0} \neq \emptyset$. □

Next we give the integral relationships between J_α, K_β, Y_a, and Z_b and obtain a conclusion for the existence and non-existence of solution (x, y) of system (II.1) based on α and β. The proof of the following lemma is similar to the proofs of Lemma 1.1, Lemma 3.2, Lemma 3.3, Lemma 3.6 and Lemma 3.7 in [13].

Lemma 2.6. (a) If $J_\alpha < \infty$ or $K_\beta < \infty$ then $Z_b < \infty$.

(b) If $K_\beta = \infty$, then $Y_a = \infty$ or $Z_b = \infty$.

(c) If $J_\alpha = \infty$, then $Y_a = \infty$ or $Z_b = \infty$.

(d) Let $\alpha \geq 1$. If $J_\alpha < \infty$, then $K_\alpha < \infty$.

(e) Let $\beta \leq 1$. If $K_\beta < \infty$, then $J_\beta < \infty$.

(f) Let $\alpha < \beta$. If $K_\beta < \infty$, then $J_\alpha < \infty$ and $K_\alpha < \infty$.

(g) Let $\alpha > \beta$. If $J_\alpha < \infty$, then $K_\beta < \infty$ and $J_\beta < \infty$.

The following corollaries give the existence and nonexistence of nonoscillatory solutions (x, y) of system (II.1) in our subclasses by Lemma 2.6 and our main theorems presented in this section.

Corollary 2.7. *Suppose that $Y_a = \infty$ and $Z_b < \infty$. Then*

(a) $M^+_{B,0} \neq \emptyset$ *if any of the followings hold:*

(i) $J_\alpha < \infty$,

(ii) $\alpha < \beta$ *and* $K_\beta < \infty$,

(iii) $\alpha < \beta$, $\beta \geq 1$ *and* $J_\beta < \infty$,

(iv) $\alpha \leq 1$ *and* $K_\alpha < \infty$.

(b) $M^+_{\infty,B} \neq \emptyset$ *if any of the followings hold:*

(i) $K_\beta < \infty$,

(ii) $\alpha > \beta$ *and* $J_\alpha < \infty$,

(iii) $\alpha \geq 1$ *and* $J_\beta < \infty$.

(c) $M^+_{B,0} = \emptyset$ *if any of the followings hold:*

(i) $J_\alpha = \infty$,

(ii) $\alpha > \beta$ *and either* $J_\beta = \infty$ *or* $K_\beta = \infty$,

(iii) $\alpha \geq 1$ *and* $K_\alpha = \infty$.

(d) $M^+_{\infty,B} = \emptyset$ *if any of the followings hold:*

(i) $K_\beta = \infty$,

(ii) $\alpha < \beta$ *and either* $J_\alpha = \infty$ *or* $K_\alpha = \infty$,

(iii) $\beta \leq 1$ *and* $J_\beta = \infty$.

3. THE CASE $Y_a < \infty$ AND $Z_b < \infty$

In this section, we show the existence of a solution (x, y) of system (II.1) by assuming $Y_a < \infty$. Since we investigate a solution (x, y) in M^+, we also have to assume that $Z_b < \infty$ because of Lemma 1.5. Recall that M^+ is the set of nonoscillatory solutions (x, y) such that x and y have the same sign. Without loss of generality let us assume that $x > 0$ eventually. Then by the first equation of system (II.1), x is eventually increasing and by Lemma 1.6 the limit of x approaches a positive constant and the limit of y exists. Also by the second equation of system (II.1) y is eventually decreasing and approaches a nonnegative constant.

In light of this information, one can easily prove the following lemma.

Lemma 3.1. *For $0 < c < \infty$ and $0 < d < \infty$, any nonoscillatory solution in M^+ belongs to the following subclasses:*

$$M^+_{B,B} = \left\{(x, y) \in M^+ : \lim_{t\to\infty} |x(t)| = c, \quad \lim_{t\to\infty} |y(t)| = d\right\}$$
$$M^+_{B,0} = \left\{(x, y) \in M^+ : \lim_{t\to\infty} |x(t)| = c, \quad \lim_{t\to\infty} |y(t)| = 0\right\}.$$

The following theorems show the existence of nonoscillatory solutions (x, y) in these subclasses of M^+.

Theorem 3.2. (a) $M^+_{B,B} \neq \emptyset$ if $Y_a < \infty$ and $Z_b < \infty$.
(b) If $M^+_{B,B} \neq \emptyset$, then $J_\alpha < \infty$.

Proof. (a) Suppose that $Y_a < \infty$ and $Z_b < \infty$. Then $J_\alpha < \infty$ by Lemma 2.6 (c). Since $Y_a < \infty$, for arbitrarily given $c, d > 0$ there exists $t_1 \geq t_0$ such that

$$\int_{t_1}^{t} A(s) \left(d + \int_s^\infty c^\beta b(s)\Delta s\right)^{\frac{1}{\alpha}} \leq \frac{c}{2} \text{ for } t \geq t_1. \tag{II.23}$$

Let X be the Banach space of all real-valued continuous functions endowed with the norm $\|x\| = \sup_{t\in[t_1,\infty)_{\mathbb{T}}} |x(t)|$ and with the usual pointwise ordering $\leq$. Define a subset Ω of X such that

$$\Omega := \left\{ x \in X : \ \frac{c}{2} \leq x(t) \leq c \ \text{ for } \ t \geq t_1 \right\}.$$

For any subset $\tilde{\Omega} \in \Omega$, it is obvious that $\inf \tilde{\Omega} \in \Omega$ and $\sup \tilde{\Omega} \in \Omega$. Define an operator $F : \Omega \to X$ as

$$(Fx)(t) = \frac{c}{2} + \int_{t_1}^{t} A(s) \left(d + \int_{s}^{\infty} b(\tau) \left(x^{\sigma}(\tau)\right)^{\beta} \Delta\tau \right)^{\frac{1}{\alpha}} \Delta s.$$

One can show that $F : \Omega \to \Omega$ and F is an increasing mapping. So by the Knaster Fixed point theorem [12], there exists $\bar{x} \in \Omega$ such that

$$\bar{x}(t) = (F\bar{x})(t) = \frac{c}{2} + \int_{t_1}^{t} A(s) \left(d + \int_{s}^{\infty} b(\tau) \left(\bar{x}^{\sigma}(\tau)\right)^{\beta} \Delta\tau \right)^{\frac{1}{\alpha}} \Delta s. \tag{II.24}$$

Therefore, it follows that $\bar{x}(t) > 0$ for $t \geq t_1$. So by the first equation of system (II.1), we have $\bar{y}(t) > 0$ for $t \geq t_1$, i.e., $(\bar{x}, \bar{y}) \in M^{+}$. From (II.24), we have

$$\bar{x} \leq \frac{c}{2} + \int_{t_1}^{t} A(s) \left(d + c^{\beta} \int_{s}^{\infty} b(\tau)\Delta\tau \right)^{\frac{1}{\alpha}} \Delta s.$$

So as $t \to \infty$, it follows that the limit of $\bar{x}$ is finite. By taking the derivative of (II.24) and the first equation of system (II.1), we have

$$\bar{y}(t) = \left(\bar{x}^{\Delta}(t)\right)^{\alpha} a(t) = d + \int_{t}^{\infty} b(\tau) \left(\bar{x}^{\sigma}(\tau)\right)^{\beta} \Delta\tau. \tag{II.25}$$

Taking the limit of (II.25) as $t \to \infty$ yields that $\bar{y}(t) \to d$. Therefore, we conclude that $(\bar{x}, \bar{y}) \in M^{+}_{B,B} \neq \emptyset$.

(b) Suppose that $M^{+}_{B,B} \neq \emptyset$. Without loss of generality assume that x is eventually positive. Then there exists $t_1 \geq t_0$ and $(x, y) \in M^+$ such that $x \to c$ and $y \to d$ as $t \to \infty$ for $0 < c < \infty$ and $0 < d < \infty$. Integrating the second equation of system (II.1) from t to ∞ and using the monotonicity of x give us

$$y(t) > (x(t))^{\beta} \int_t^{\infty} b(s)\Delta s \ \text{ for } \ t \geq t_1$$

or

$$y^{\frac{1}{\alpha}}(t) > (x(t))^{\frac{\beta}{\alpha}} \left(\int_t^{\infty} b(s)\Delta s \right)^{\frac{1}{\alpha}} \quad \text{for } \ t \geq t_1. \tag{II.26}$$

Substituting (II.26) into the first equation of system (II.1) yields

$$x^{\Delta}(t) > A(t) x^{\frac{\beta}{\alpha}} \left(\int_t^{\infty} b(s)\Delta s \right)^{\frac{1}{\alpha}}. \tag{II.27}$$

Integrating (II.27) from t_1 to t and by the monotonicity of x give us

$$x(t) > x^{\frac{\beta}{\alpha}}(t_1) \int_{t_1}^{t} A(s) \left(\int_s^{\infty} b(\tau)\Delta\tau \right)^{\frac{1}{\alpha}} \Delta s \tag{II.28}$$

As $t \to \infty$, the assertion follows. □

The following theorem can be proved similar to Theorem 2.3.

Theorem 3.3. (a) $M^+_{B,0} \neq \emptyset$ if $Y_a < \infty$ and $Z_b < \infty$.
(b) If $M^+_{B,0} \neq \emptyset$, then $J_\alpha < \infty$.

By Lemma 2.1 and from our main results in Sections 2 and 3, one can have the following corollaries.

Corollary 3.4. *If $Y_a < \infty$ and $Z_b < \infty$, then any nonoscillatory solution in M^+ of system (II.1) belongs to $M^+_{B,B}$ or $M^+_{B,0}$, i.e., $M^+_{\infty,B} = M^+_{\infty,0} = \emptyset$.*

Corollary 3.5. *If $Y_a = \infty$ and $Z_b < \infty$, then $M^+_{B,B} = \emptyset$.*

4. EXAMPLES

In this section, we give three examples to illustrate Theorem 2.4 and Theorem 2.5.

Example 4.1. *Let* $\mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$, $\alpha = 1$, $A(t) = \frac{t}{1+2t}$, $b(t) = \frac{1}{q^{1+\beta}t^{\beta+2}}$, $s = q^m$ *and* $t = q^n$, *where* $m, n \in \mathbb{N}_0$, *in system (II.1). It is easy to show that* $Y_a = \infty$ *and* $Z_b < \infty$. *Let us show that* $K_\beta < \infty$.

$$\int_{t_0}^{T} b(t)\left(\int_{t_0}^{\sigma(t)} A(s)\Delta s\right)^{\beta}\Delta t = \sum_{t=1}^{\rho(T)} \frac{1}{q^{1+\beta}t^{\beta+2}}\left(\sum_{s=1}^{t}\frac{s^2(q-1)}{1+2s}\right)^{\beta}(q-1)t$$
$$< \frac{(q-1)^{\beta+1}}{q^{1+\beta}}\sum_{t=1}^{\rho(T)}\frac{1}{t^{1+\beta}}\left(\sum_{s=1}^{t} s\right)^{\beta} < \frac{q-1}{q}\sum_{t=1}^{\rho(T)}\frac{1}{t}.$$

We also have

$$\lim_{T\to\infty}\sum_{t=1}^{\rho(T)}\frac{1}{t} = \sum_{n=0}^{\infty}\frac{1}{q^n} < \infty$$

by the geometric series test. So we have that $K_\beta < \infty$. *It can be verified that* $(t, \frac{1}{t}+2)$ *is a nonoscillatory solution of*

$$\begin{cases} x^{\Delta} = \frac{t}{1+2t}\,|y|\; sgny \\ y^{\Delta} = -\frac{1}{q^{1+\beta}t^{\beta+2}}\,|x^{\sigma}|^{\beta}\; sgnx \end{cases}$$

in M^+ *such that* $\lim_{t\to\infty} t = \infty$ *and* $\lim_{t\to\infty}\frac{1}{t}+2 = 2$, *i.e.,* $M^+_{\infty,B} \neq \emptyset$.

Example 4.2. *Let* $\mathbb{T} = \mathbb{R}, \alpha > \beta$ *with* $\beta < 1$, $A(t) = e^{2t}$ *and* $b(t) = \alpha e^{-t(\alpha+\beta)}$ *in system (II.1). Clearly,* $Y_a = \infty$ *and* $Z_b < \infty$. *One can show that*

$$J_\alpha = \int_{t_0}^{\infty} e^{2t}\left(\int_{t}^{\infty}\alpha e^{-s(\alpha+\beta)}ds\right)^{\frac{1}{\alpha}}dt = \infty$$

and

$$K_\beta = \int_{t_0}^{\infty} \alpha e^{-t(\alpha+\beta)} \left(\int_{t_0}^{t} e^{2s} ds \right)^{\beta} dt < \infty.$$

It is easy to verify that $(e^t, e^{-\alpha t})$ *is a nonoscillatory solution of*

$$\begin{cases} x' = e^{2t} |y|^{\frac{1}{\alpha}} sgny \\ y' = -\alpha e^{-t(\alpha+\beta)} |x|^{\beta} sgnx \end{cases}$$

in M^+ *such that* $\lim_{t\to\infty} e^t = \infty$ *and* $\lim_{t\to\infty} e^{-\alpha t} = 0$*, i.e.,* $M^+_{\infty,0} \neq \emptyset$*.*

Example 4.3. *Let* $\mathbb{T} = q^{\mathbb{N}_0}$*,* $q > 1$*,* $\alpha = 1$*,* $\beta < 1$*,* $A(t) = 1 + t$*,* $b(t) = \frac{1}{(1+t)(1+tq)^{\beta+1}}$ *in system (II.1). It is easy to verify that* $Y_a = \infty$ *and* $Z_b < \infty$*. Letting* $s = q^m$ *and* $t = q^n$*, where* $m, n \in \mathbb{N}_0$*, gives*

$$\int_{t_0}^{T} A(t) \left(\int_{t}^{T} b(s)\Delta s \right)^{\frac{1}{\alpha}} \Delta t = \sum_{t=1}^{\rho(T)} (1+t) \left(\sum_{s=t}^{\rho(T)} \frac{(q-1)s}{(1+s)(1+sq)^{\beta+1}} \right) (q-1)t$$
$$\geq (q-1)^2 \sum_{t=1}^{\rho(T)} (1+t) \left(\frac{t}{(1+t)(1+tq)^{\beta+1}} \right) t = (q-1)^2 \sum_{t=1}^{\rho(T)} \frac{t^2}{(1+tq)^{\beta+1}}.$$

So we have

$$\lim_{T\to\infty} \sum_{t=1}^{\rho(T)} \frac{t^2}{(1+tq)^{\beta+1}} = \sum_{n=0}^{\infty} \frac{q^{2n}}{(1+q^{n+1})^{\beta+1}} = \infty$$

by the Test for Divergence and $\beta < 1$*. Now let us show that* $K_\beta < \infty$*. One can show that*

$$\int_{t_0}^{\sigma(t)} A(s)\Delta s = \sum_{s=1}^{t} (1+s)(q-1)s \leq tq(1+tq)$$

and so we have

$$\int_{t_0}^{T} b(t)\left(\int_{t_0}^{\sigma(t)} A(s)\Delta s\right)^{\beta}\Delta t \le \sum_{t=1}^{\rho(T)} \frac{1}{(1+t)(1+tq)^{\beta+1}}\left(tq(1+tq)\right)^{\beta} t(q-1)q^{\beta}(q-1)\sum_{t=1}^{\rho(T)}\frac{t^{\beta}}{1+t}.$$

Therefore,

$$\lim_{T\to\infty} q^{\beta}(q-1)\sum_{t=1}^{T}\frac{t^{\beta}}{1+t} = q^{\beta}(q-1)\sum_{n=0}^{\infty}\frac{(q^n)^{\beta}}{(1+q^n)} < \infty$$

by the Ratio Test and $\beta < 1$. *It can also be verified that* $\left(1+t, \frac{1}{t+1}\right)$ *is a nonoscillatory solution of*

$$\begin{cases} x^{\Delta} = (1+t)\,|y|^{\frac{1}{\alpha}}\, sgny \\ y^{\Delta} = -\frac{1}{(1+t)(1+tq)^{\beta+1}}\,|x^{\sigma}|^{\beta}\, sgnx \end{cases}$$

in M^{+} *such that* $\lim_{t\to\infty}(1+t) = \infty$ *and* $\lim_{t\to\infty}\frac{1}{t+1} = 0$, *i.e.,* $M^{+}_{\infty,0} \neq \emptyset$.

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III. NONOSCILLATION CRITERIA FOR TWO-DIMENSIONAL TIME-SCALE SYSTEMS

ABSTRACT

We study the existence and nonexistence of nonoscillatory solutions of a two-dimensional system of first-order dynamic equations on time scales. Our approach is based on the Knaster and Schauder fixed point theorems and some certain integral conditions. Examples are given to illustrate some of our main results.

1. INTRODUCTION

In this paper, we study on the asymptotic behavior of solutions of the nonlinear system of the first-order dynamic equations

$$\begin{cases} x^{\Delta}(t) = a(t)f(y(t)) \\ y^{\Delta}(t) = -b(t)g(x(t)), \end{cases} \tag{III.1}$$

where $f, g \in C(\mathbb{R}, \mathbb{R})$ are nondecreasing such that $uf(u) > 0$, $ug(u) > 0$ for $u \neq 0$ and $a, b \in \mathrm{C_{rd}}\left([t_0, \infty)_{\mathbb{T}}, \mathbb{R}^+\right)$. Whenever we write $t \geq t_1$, we mean that $t \in [t_1, \infty)_{\mathbb{T}} := [t_1, \infty) \cap \mathbb{T}$. A time scale, denoted by $\mathbb{T}$, is a closed subset of real numbers. An excellent introduction of time scales calculus can be found in [2, 3] by Bohner and Peterson. Throughout this paper, we assume that $\mathbb{T}$ is unbounded above. We call (x, y) a *proper solution* if it is defined on $[t_0, \infty)_{\mathbb{T}}$ and $\sup\{|x(s)|, |y(s)| : s \in [t, \infty)_{\mathbb{T}}\} > 0$ for $t \geq t_0$. A solution (x, y) of (III.1) is said to be nonoscillatory if the component functions x and y are both nonoscillatory, i.e., either eventually positive or eventually negative. Otherwise, it is said to be oscillatory. Throughout this paper, without loss of generality, we assume that x is eventually positive. Our results can be shown for that x is eventually negative similarly.

If $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$, equation (III.1) turns out to be system of first-order differential equations and difference equations

$$\begin{cases} x' = a(t)f(y(t)) \\ y' = -b(t)g(x(t)) \end{cases}$$

see [7],

$$\begin{cases} \Delta x_n = a_n f(y_n) \\ \Delta y_n = -b_n g(x_n) \end{cases}$$

see [8], respectively. Oscillation and nonoscillation criteria for two-dimensional time scale systems have been studied by [1], [5], [10], [11, 12].

One can easily show that any nonoscillatory solution (x, y) of system (III.1) belongs to one of the following classes:

$$M^+ := \{(x, y) \in M : \ x(t)y(t) > 0 \ \text{ eventually}\}$$
$$M^- := \{(x, y) \in M : \ x(t)y(t) < 0 \ \text{ eventually}\},$$

where M is the set of all nonoscillatory solutions of system (III.1). In this paper, we only focus on the existence and nonexistence of solutions of system (III.1) in M^-.

The set up of this paper is as follows. In Section 1, we give preliminary lemmas that are used in the proofs of our main theorems. In Section 2, we introduce the subclasses that are obtained by using system (III.1) and show the existence of nonoscillatory solutions of system (III.1) by using the Knaster and Schauder fixed point theorems and certain improper integrals. In Section 3, we show the nonexistence of such solutions by relaxing the monotonicity condition on the functions f and g. We finalize the paper by giving some examples and a conclusion.

The following lemma is shown in [1].

Lemma 1.1. *If (x, y) is a nonoscillatory solution of system (III.1), then the component functions x and y are themselves nonoscillatory.*

For convenience, let us set

$$Y(t) = \int_t^\infty a(t)\Delta t \quad \text{ and } \quad Z(t) = \int_t^\infty b(t)\Delta t. \tag{III.2}$$

The following lemma shows the existence and nonexistence of nonoscillatory solutions of system (III.1) by using convergence/divergence of $Y(t)$ and $Z(t)$.

Lemma 1.2. *Let $t_0 \in \mathbb{T}$. Then we have the following:*

(a) *[1, Lemma 2.3] If $Y(t_0) < \infty$ and $Z(t_0) < \infty$, then system (III.1) is nonoscillatory.*

(b) *[1, Lemma 2.2] If $Y(t_0) = \infty$ and $Z(t_0) = \infty$, then system (III.1) is oscillatory.*

(c) *If $Y(t_0) < \infty$ and $Z(t_0) = \infty$, then any nonoscillatory solution (x, y) of system (III.1) belongs to M^-, i.e., $M^+ = \emptyset$.*

(d) *If $Y(t_0) = \infty$ and $Z(t_0) < \infty$, then any nonoscillatory solution (x, y) of system (III.1) belongs to M^+, i.e., $M^- = \emptyset$.*

Proof. Here we only prove part (c) because (d) can be shown similarly. Suppose that $Y(t_0) < \infty$ and $Z(t_0) = \infty$. So assume that there exists a nonoscillatory solution (x, y) of system (III.1) in M^+ such that $xy > 0$ eventually. Without loss of generality, assume that $x(t) > 0$ for $t \geq t_1$. Then by monotonicity of x and g, there exists a number $k > 0$ such that $g(x(t)) \geq k$ for $t \geq t_1$. Integrating the second equation of system (III.1) from t_1 to t gives us

$$y(t) \leq y(t_1) - k \int_{t_1}^{t} b(s)\Delta s.$$

As $t \to \infty$, it follows that $y(t) \to -\infty$. But this contradicts that y is eventually positive. Proof is by contradiction. $\square$

The following two lemmas are related with the first component function of any nonoscillatory solutions of (III.1) when $Y(t_0) < \infty$.

Lemma 1.3. *Let (x, y) be a nonoscillatory solution of system (III.1) and $Y(t_0) < \infty$. Then the component function x has a finite limit.*

Proof. Suppose that $Y(t_0) < \infty$ and (x, y) is a nonoscillatory solution of system (III.1). Then by Lemma 1.1, x and y are themselves nonoscillatory. Without loss of generality, assume that there exists $t_1 \geq t_0$ such that $x(t) > 0$ for $t \geq t_1$. If $(x, y) \in$

M^-, then by the first equation of system (III.1), $x^\Delta(t) < 0$ for $t \geq t_1$. Therefore, the limit of x exists. So let us show that the assertion follows if $(x, y) \in M^+$. Suppose $(x, y) \in M^+$. Then from the first equation of system (III.1), we have $x^\Delta(t) > 0$ for $t \geq t_1$. Hence two possibilities might happen: The limit of the component function x exists or blows up. Now let us show that $\lim_{t\to\infty} x(t) = \infty$ cannot happen. Integrating the first equation of system (III.1) from t_1 to t and using the monotonicity of y and f yield

$$x(t) \leq x(t_1) + f(y(t_1)) \int_{t_1}^{t} a(s)\Delta s.$$

Taking the limit as $t \to \infty$, it follows that x has a finite limit. This completes the proof. $\square$

Lemma 1.4. *Let $Y(t_0) < \infty$. If (x, y) is a nonoscillatory solution of system (III.1), then there exist $c, d > 0$ and $t_1 \geq t_0$ such that*

$$c \int_{t}^{\infty} a(s)\Delta s \leq x(t) \leq d$$

or

$$-d \leq x(t) \leq -c \int_{t}^{\infty} a(s)\Delta s$$

for $t \geq t_1$.

Proof. Suppose that $Y(t_0) < \infty$ and (x, y) is a nonoscillatory solution of system (III.1). Without loss of generality, let us assume that x is eventually positive. Then by Lemma 1.3, we have $x(t) \leq d$ for $t \geq t_1$ and for some $d > 0$. If $y(t) > 0$ for $t \geq t_1$, then x is eventually increasing by the first equation of system (III.1). So for large t, the assertion follows. If $y(t) < 0$ for $t \geq t_1$, then integrating the first equation of system (III.1) from t to ∞ and the monotonicity of f and y give

$$\begin{aligned} x(t) &= x(\infty) - \int_t^\infty a(s) f(y(s)) \Delta s \\ &\geq -f(y(t_1)) \int_t^\infty a(s) \Delta s. \end{aligned}$$

Setting $c = -f(y(t_1)) > 0$ on the last inequality proves the assertion. $\square$

According to Lemma 1.2 (c), we assume $Y(t_0) < \infty$ and $Z(t_0) = \infty$ from now on. Let (x, y) be a nonoscillatory solution of system (III.1) such that the component function x of the solution (x, y) is eventually positive. Then by the second equation of system (III.1), we have $y < 0$ and eventually decreasing. Then for $d < 0$, we have $y \to d$ or $y \to -\infty$. In view of Lemma 1.3, x has a finite limit. So in light of this information, we obtain the following lemma.

Lemma 1.5. *Any nonoscillatory solution of system (III.1) in M^- belongs to one of the following subclasses:*

$$\begin{aligned} M^-_{0,B} &= \left\{ (x, y) \in M^- : \lim_{t\to\infty} |x(t)| = 0, \quad \lim_{t\to\infty} |y(t)| = d \right\}, \\ M^-_{B,B} &= \left\{ (x, y) \in M^- : \lim_{t\to\infty} |x(t)| = c, \quad \lim_{t\to\infty} |y(t)| = d \right\}, \\ M^-_{0,\infty} &= \left\{ (x, y) \in M^- : \lim_{t\to\infty} |x(t)| = 0, \ \lim_{t\to\infty} |y(t)| = \infty \right\}, \\ M^-_{B,\infty} &= \left\{ (x, y) \in M^- : \lim_{t\to\infty} |x(t)| = c, \ \lim_{t\to\infty} |y(t)| = \infty \right\}, \end{aligned}$$

where $0 < c < \infty$ and $0 < d < \infty$.

2. EXISTENCE OF NONOSCILLATORY SOLUTIONS OF (III.1) IN M^-

The following theorems show the existence of nonoscillatory solutions in subclasses of M^- given in Lemma 1.5.

Theorem 2.1. *$M^-_{0,B} \neq \emptyset$ if and only if*

$$\int_{t_0}^{\infty} b(t) g\left(c_1 \int_t^{\infty} a(s)\Delta s\right)\Delta t < \infty \tag{III.3}$$

for some $c_1 \neq 0$.

Proof. Suppose that there exists a solution $(x, y) \in M^-_{0,B}$ such that $x(t) > 0$ for $t \geq t_0$, $x(t) \to 0$ and $y(t) \to -d$ as $t \to \infty$, where $d > 0$. By Lemma 1.4, there exists $c > 0$ such that

$$x(t) \geq c \int_t^{\infty} a(s)\Delta(s), \quad t \geq t_0. \tag{III.4}$$

By integrating the second equation from t_0 to t, using inequality (III.4) with $c = c_1$ and the monotonicity of g, we have

$$y(t) = y(t_0) - \int_{t_0}^{t} b(s) g(x(s))\Delta s \leq - \int_{t_0}^{t} b(s) g\left(c_1 \int_s^{\infty} a(\tau)\Delta\tau\right)\Delta s.$$

So as $t \to \infty$, the assertion follows since y has a finite limit. (For the case $x < 0$ eventually, the proof can be shown similarly with $c_1 < 0$.)

Conversely, suppose that (III.3) holds for some $c_1 > 0$. (For the case $c_1 < 0$ can be shown similarly.) Then there exist $t_1 \geq t_0$ and $d > 0$ such that

$$\int_{t_1}^{\infty} b(t) g\left(c_1 \int_t^{\infty} a(s)\Delta s\right)\Delta t < d, \quad t \geq t_1, \tag{III.5}$$

where $c_1 = -f(-3d)$. Let X be the space of all continuous and bounded functions on $[t_1,\infty)_{\mathbb{T}}$ with the norm $\|y\| = \sup_{t\in[t_1,\infty)_{\mathbb{T}}} |y(t)|$. Let Ω be the subset of X such that

$$\Omega := \{y \in X : \quad -3d \le y(t) \le -2d, \quad t \ge t_1\}$$

and define an operator $T : \Omega \to X$ such that

$$(Ty)(t) = -3d + \int_t^\infty b(s) g\left(-\int_s^\infty a(\tau) f(y(\tau))\Delta\tau\right)\Delta s.$$

It is easy to see that T maps into itself. Indeed, we have

$$-3d \le (Ty)(t) \le -3d + \int_t^\infty b(s) g\left(-\int_s^\infty a(\tau) f(-3d)\Delta\tau\right)\Delta s \le -2d$$

by (III.5). Let us show that T is continuous on Ω. Let y_n be a sequence in Ω such that $y_n \to y \in \Omega = \bar{\Omega}$. Then

$$\begin{aligned} &|(Ty_n)(t) - (Ty)(t)| \\ &\le \int_{t_1}^\infty b(s) \left|\left[g\left(-\int_s^\infty a(\tau) f(y_n(\tau))\Delta\tau\right) - g\left(-\int_s^\infty a(\tau) f(y(\tau))\Delta\tau\right)\right]\right|\Delta s. \end{aligned}$$

Then the Lebesque dominated convergence theorem and the continuity of g give $\|(Ty_n) - (Ty)\| \to 0$ as $n \to \infty$, i.e., T is continuous. Also since

$$0 < -(Ty)^\Delta(t) = b(t) g\left(-\int_t^\infty a(\tau) f(y(\tau))\Delta\tau\right) < \infty,$$

it follows that $T(\Omega)$ is relatively compact. Then by the Schauder Fixed point theorem, there exists $\bar{y} \in \Omega$ such that $\bar{y} = T\bar{y}$. So as $t \to \infty$, we have $\bar{y}(t) \to -3d < 0$. Setting

$$\bar{x}(t) = -\int_t^\infty a(\tau) f(\bar{y}(\tau))\Delta\tau > 0$$

gives that $\bar{x}(t) \to 0$ ans $t \to \infty$, i.e., $M_{0,B}^{-} \neq \emptyset$. $\square$

Theorem 2.2. *$M_{B,B}^{-} \neq \emptyset$ if and only if*

$$\int_{t_0}^{\infty} b(t) g\left(d_1 - f(c_1) \int_{t}^{\infty} a(s)\Delta s\right) \Delta t < \infty \tag{III.6}$$

for some $c_1 < 0$ and $d_1 > 0$. (Or $c_1 > 0$ and $d_1 < 0$.)

Proof. Suppose that there exists a nonoscillatory solution $(x, y) \in M_{B,B}^{-}$ such that $x > 0$ eventually, $\lim_{t\to\infty} x(t_1) = c_2 > 0$ and $\lim_{t\to\infty} y(t) = d_2 < 0$. Since x and y have finite limits, there exist $t_1 \geq t_0$, $c_3 > 0$ and $d_3 < 0$ such that $c_2 \leq x(t) \leq c_3$ and $d_2 \leq y(t) \leq d_3$ for $t \geq t_1$. Integrating the first equation from t to ∞ gives

$$x(t) = c_2 - \int_{t}^{\infty} a(s) f(y(s))\Delta s \geq c_2 - f(d_3) \int_{t}^{\infty} a(s)\Delta s. \tag{III.7}$$

By integrating the second equation from t_1 to t and using (III.7,) we get

$$y(t) \leq -\int_{t_1}^{t} b(s) g(x(s))\Delta s \leq -\int_{t_1}^{t} b(s) g\left(c_2 - f(d_3) \int_{s}^{\infty} a(\tau)\Delta\tau\right) \Delta s.$$

By setting $c_2 = d_1 > 0$ and $d_3 = c_1 < 0$ and taking the limit of the last inequality as $t \to \infty$, the assertion follows. (The case $x < 0$ eventually can be done similarly with $c_1 > 0$ and $d_1 < 0$.)

Conversely, choose $t_1 \geq t_0$ so large that

$$\int_{t_1}^{\infty} b(t) g\left(d_1 - f(c_1) \int_{t}^{\infty} a(s)\Delta s\right) \Delta t < \frac{-c_1}{2}, \tag{III.8}$$

where $c_1 < 0$ and $d_1 > 0$. (The case $c_1 > 0$ and $d_1 < 0$ can be done similarly.) Let X be the set of all all bounded and continuous functions endowed with the norm $\|y\| = \sup_{t\in[t_1,\infty)_{\mathbb{T}}} |y(t)|$. Clearly $(X, \|\cdot\|)$ is a Banach space, see [4]. Define a subset Ω

of X such that

$$\Omega =: \left\{ y \in X : \quad c_1 \leq y(t) \leq \frac{c_1}{2}, \quad t \geq t_1 \right\}.$$

Define an operator $F : \Omega \to X$ such that

$$(Fy)(t) = c_1 + \int_t^\infty b(s) g\left(d_1 - \int_s^\infty a(\tau) f(y(\tau)) \Delta\tau \right) \Delta s.$$

First, we show that $F : \Omega \to \Omega$.

$$c_1 \leq (Fy)(t) \leq c_1 + \int_t^\infty b(s) g\left(d_1 - \int_s^\infty a(\tau) f(c_1) \Delta\tau \right) \Delta s \leq \frac{c_1}{2}.$$

Second, we show that F is continuous on Ω. Let y_n be a sequence in Ω such that $y_n \to y \in \Omega = \bar{\Omega}$. Then

$$\|Fy_n - Fy\| \leq \int_{t_1}^\infty b(s) \left(\left| g\left(d - \int_s^\infty a(\tau) f(y_n(\tau)) \Delta\tau \right) \right| - \left| g\left(d - \int_s^\infty a(\tau) f(y(\tau)) \Delta\tau \right) \right| \right) \Delta s.$$

By the Lebesque dominated convergence theorem and the continuity of f and g, it follows that F is continuous.

Third, we show that $F(\Omega)$ is relatively compact. Since $Y(t_0) < \infty$, we have

$$0 < -(Fy)^\Delta(t) = b(t) g\left(d_1 - \int_t^\infty a(\tau) f(y(\tau)) \Delta\tau \right) < \infty$$

and therefore F is equibounded and equicontinuous, i.e., relatively compact. So by the Schauder fixed point theorem, there exists $\bar{y} \in X$ such that

$$\bar{y}(t) = F\bar{y}(t) = c_1 + \int_t^\infty b(s) g\left(d_1 - \int_s^\infty a(\tau) f(\bar{y}(\tau)) \Delta\tau \right) \Delta s.$$

Setting $\bar{x}(t) = d_1 - \int_t^\infty a(\tau)f(y(\tau))\Delta\tau$ and taking limit as $t \to \infty$, we have that there exists a nonoscillatory solution in M^- such that $\bar{x}(t) \to d_1 > 0$ and $\bar{y}(t) \to c_1 < 0$, i.e., $M_{B,B}^- \neq \emptyset$.

□

Theorem 2.3. *$M_{B,\infty}^- \neq \emptyset$ if and only if*

$$\int_{t_0}^{\infty} a(s)f\left(g(c_1)\int_{t_0}^{s} b(\tau)\Delta\tau\right)\Delta s < \infty \tag{III.9}$$

for some $c_1 \neq 0$, where f is an odd function.

Proof. Suppose that there exists a nonoscillatory solution $(x, y) \in M_{B,\infty}^-$ such that $x > 0$ eventually, $x(t) \to c_2$ and $y(t) \to -\infty$ as $t \to \infty$, where $0 < c_2 < \infty$. Because of the monotonicity of x and the fact that x has a finite limit, there exist $t_1 \geq t_0$ and $c_3 > 0$ such that

$$c_2 \leq x(t) \leq c_3 \quad \text{for} \quad t \geq t_1. \tag{III.10}$$

Integrating the first equation from t_1 to t gives us

$$c_2 \leq x(t) = x(t_1) + \int_{t_1}^{t} a(s)f(y(s))\Delta s \leq c_3, \quad t \geq t_1.$$

So by taking the limit as $t \to \infty$, we have

$$\int_{t_1}^{\infty} a(s)|f(y(s))|\Delta s < \infty. \tag{III.11}$$

The monotonicity of g, (III.10) and integrating the second equation from t_1 to t yield

$$y(t) \leq y(t_1) - g(c_2)\int_{t_1}^{t} b(s)\Delta s \leq -g(c_2)\int_{t_1}^{t} b(s)\Delta s.$$

Since $f(-u) = -f(u)$ for $u \neq 0$ and by the monotonicity of f, we have

$$|f(y(t))| \geq f\left(g(c_2)\int_{t_1}^{t} b(s)\Delta s\right), \quad t \geq t_1. \tag{III.12}$$

By (III.11) and (III.12), we have

$$\int_{t_1}^{t} a(s)|f(y(s))|\Delta s \geq \int_{t_1}^{t} a(s) f\left(g(c_2)\int_{t_1}^{s} b(\tau)\Delta\tau\right)\Delta s.$$

As $t \to \infty$, the proof is finished. (The case $x < 0$ eventually can be proved similarly with $c_1 < 0$.) □

Conversely, suppose that $\int_{t_0}^{\infty} a(s) f\left(g(c_1)\int_{t_0}^{s} b(\tau)\Delta\tau\right)\Delta s < \infty$ for some $c_1 \neq 0$. Without loss of generality, assume that $c_1 > 0$. (The case $c_1 < 0$ can be done similarly.) Then we can choose $t_1 \geq t_0$ and $d > 0$ such that

$$\int_{t_1}^{\infty} a(s) f\left(g(c_1)\int_{t_1}^{s} b(\tau)\Delta\tau\right)\Delta s < d, \quad t \geq t_1, \tag{III.13}$$

where $c_1 = 2d > 0$. Let X be the partially ordered Banach space of all real-valued continuous functions endowed with supremum norm $\|x\| = \sup_{t\in[t_1,\infty)_{\mathbb{T}}} |x(t)|$ and with the usual pointwise ordering $\leq$. Define a subset Ω of X such that

$$\Omega =: \{x \in X : \quad d \leq x(t) \leq 2d, \quad t \geq t_1\}. \tag{III.14}$$

For any subset B of Ω, $\inf B \in \Omega$ and $\sup B \in \Omega$, i.e., $(\Omega, \leq)$ is complete. Define an operator $F : \Omega \to X$ as

$$(Fx)(t) = d + \int_{t}^{\infty} a(s) f\left(\int_{t_1}^{s} b(\tau) g(x(\tau))\Delta\tau\right)\Delta s, \quad t \geq t_1. \tag{III.15}$$

First, we need to show that $F : \Omega \to \Omega$ is an increasing mapping into itself. It is obvious that it is an increasing mapping and since

$$d \leq (Fx)(t) = d + \int_t^\infty a(s) f\left(\int_{t_1}^s b(\tau) g(x(\tau)) \Delta\tau\right) \Delta s \leq 2d$$

by (III.13), it follows that $F : \Omega \to \Omega$. Then by the Knaster fixed point theorem, there exists $\bar{x} \in \Omega$ such that

$$\bar{x}(t) = (F\bar{x})(t) = d + \int_t^\infty a(s) f\left(\int_{t_1}^s b(\tau) g(\bar{x}(\tau)) \Delta\tau\right) \Delta s, \quad t \geq t_1. \tag{III.16}$$

By taking the derivative of (III.16) and the fact that f is an odd function, we have

$$\bar{x}^\Delta(t) = a(t) f\left(-\int_{t_1}^t b(\tau) g(\bar{x}(\tau)) \Delta\tau\right), \quad t \geq t_1.$$

Setting $\bar{y} = -\displaystyle\int_{t_1}^t b(\tau) g(\bar{x}(\tau)) \Delta\tau$ and using the monotonicity of g give

$$\bar{y}(t) \leq -g(d) \int_{t_1}^t b(\tau) \Delta\tau, \quad t \geq t_1.$$

So we have that $\bar{x}(t) > 0$ and $\bar{y}(t) < 0$ for $t \geq t_1$, and $\bar{x}(t) \to d$ and $\bar{y}(t) \to -\infty$ as $t \to \infty$. This completes the proof.

Theorem 2.4. *If*

$$\int_{t_0}^\infty a(t) f\left(\int_t^\infty b(s) g(c_1) \Delta s\right) \Delta t < \infty$$

and

$$\int_{t_0}^\infty b(t) g\left(d_1 \int_t^\infty a(s) \Delta s\right) \Delta t = \infty \quad (-\infty)$$

for some $c_1 > 0$ and any $d_1 > 0$ ($c_1 < 0$ and $d_1 < 0$), where f is an odd function, then $M_{0,\infty}^- \neq \emptyset$.

Proof. Choose $t_1 \geq t_0$ and $c_1 > 0$ such that

$$\int_{t_1}^{\infty} a(t) f\left(g(c_1)\int_t^{\infty} b(s)\Delta s\right)\Delta t < \frac{c_1}{2}, \quad t \geq t_1. \tag{III.17}$$

Let X be the partially ordered Banach space of all real-valued continuous functions endowed with the norm $\|x\| = \sup_{t\in[t_1,\infty)_{\mathbb{T}}} |x(t)|$ and with the usual pointwise ordering $\leq$. Define a subset Ω of X such that

$$\Omega =: \{x \in X : \quad f(1)\int_t^{\infty} a(s)\Delta s \leq x(t) \leq \frac{c_1}{2}, \quad t \geq t_1\}.$$

It is clear that $(\Omega, \leq)$ is complete. Define an operator $F : \Omega \to X$ such that

$$(Fx)(t) = \int_t^{\infty} a(s) f\left(\int_{t_1}^{s} b(\tau) g(x(\tau))\Delta\tau\right)\Delta s.$$

It is clear that F is an increasing mapping. We also need to show that $F : \Omega \to \Omega$. By (III.17), the monotonicity of g and the fact that $x \in \Omega$, we have

$$(Fx)(t) \leq \int_t^{\infty} a(s) f\left(g(c_1)\int_{t_1}^{s} b(\tau)\Delta\tau\right)\Delta s \leq \frac{c_1}{2}.$$

Also since

$$\int_{t_0}^{\infty} b(t) g\left(d_1 \int_t^{\infty} a(s)\Delta s\right)\Delta t = \infty,$$

we can choose $t_2 \geq t_1$ such that

$$\int_{t_2}^{t} b(s) g\left(d_1 \int_s^{\infty} a(\tau)\Delta\tau\right)\Delta s > 1$$

for $t \geq t_2$ and any $d_1 > 0$. So by setting $f(1) = d_1$, we have

$$(Fx)(t) \geq \int_t^\infty a(s) f\left(\int_{t_1}^s b(\tau) g\left(f(1)\int_\tau^\infty a(\lambda)\Delta\lambda\right)\Delta\tau\right)\Delta s \geq f(1)\int_t^\infty a(s)\Delta s.$$

Then by the Knaster fixed point theorem, there exists $\bar{x} \in \Omega$ such that $\bar{x} = F\bar{x}$. Setting

$$\bar{y}(t) = -\int_{t_1}^t b(\tau) g(\bar{x}(\tau))\Delta\tau,$$

using the fact that $\bar{x} \in \Omega$ and taking the limit of $\bar{x}$ and $\bar{y}$ as $t \to \infty$, the proof is complete. (The case $c_1 < 0$ and $d_1 < 0$ can be shown similarly.)

$\square$

3. NONEXISTENCE OF NONOSCILLATORY SOLUTIONS OF (III.1) IN M^-

In the previous section, we used the monotonicity of the functions f and g in order to show the existence of nonoscillatory solutions of system (III.1). Nonexistence of such solutions in $M^-_{0,B}$, $M^-_{B,B}$, and $M^-_{B,\infty}$ directly follows from Theorems 2.1 - 2.3. In this section, we relax this condition by assuming that there exist positive constants F and G such that

$$\frac{f(u)}{u} \geq F \quad \text{ and } \frac{g(u)}{u} \geq G \quad \text{for} \quad u \neq 0 \tag{III.18}$$

in order to get the emptiness of those subclasses. The following theorems show the nonexistence of such solutions in the subclasses of M^- given in Lemma 1.5.

Theorem 3.1. *Suppose that (III.18) holds. If*

$$\int_{t_1}^{\infty} a(s)\left(\int_{t_1}^{s} b(\tau)\left(\int_{\tau}^{\infty} a(\lambda)\Delta\lambda\right)\Delta\tau\right)\Delta s = \infty, \tag{III.19}$$

then $M^-_{0,\infty} = \emptyset$.

Proof. Assume that there exists a solution $(x, y) \in M^-$ such that $x > 0$ eventually, $x \to 0$ and $y \to -\infty$ as $t \to \infty$. By Lemma 1.4, there exist $c_1 > 0$ and $t_1 \geq t_0$ such that

$$c_1 \int_{t}^{\infty} a(s)\Delta s \leq x(t), \quad t \geq t_1. \tag{III.20}$$

By integrating the second equation from t_1 to t, and using (III.18) and (III.20), there exist $t_2 \geq t_1$ and $G > 0$ such that

$$y(t) \leq -c_1 G \int_{t_1}^{t} b(s)\left(\int_{s}^{\infty} a(\tau)\Delta\tau\right)\Delta s, \quad t \geq t_2. \tag{III.21}$$

By integrating the first equation from t_2 to t, and using (III.21) and (III.18), there exist $t_3 \geq t_2$ and $F > 0$ such that

$$x(t_2) \geq c_1 FG \int_{t_2}^{t} a(s) \left(\int_{t_1}^{s} b(\tau) \left(\int_{\tau}^{\infty} a(\lambda)\Delta\lambda \right) \Delta\tau \right) \Delta s, \quad t \geq t_3. \tag{III.22}$$

As $t \to \infty$, it contradicts to (III.19). So the assertion follows. Proof is by contradiction. □

Theorem 3.2. *Suppose that (III.18) holds. If*

$$\int_{t_0}^{\infty} b(t) \left(\int_{t}^{\infty} a(s)\Delta s \right) \Delta t = \infty, \tag{III.23}$$

then $M_{0,B}^{-} = \emptyset$ *and* $M_{B,B}^{-} = \emptyset$.

Proof. We only show the emptiness of $M_{0,B}^{-}$ since $M_{B,B}^{-} = \emptyset$ can be shown similarly. So assume that there exists a nonoscillatory solution (x, y) in $M_{0,B}^{-}$ such that $x > 0$ eventually, $\lim_{t\to\infty} x(t) = 0$ and $\lim_{t\to\infty} y(t) = d_1 < 0$. By Lemma 1.4, we have that there exist $c_2 > 0$ and $t_1 \geq t_0$ such that

$$c_2 \int_{t}^{\infty} a(s)\Delta s \leq x(t), \quad t \geq t_1. \tag{III.24}$$

Integrating the first equation from t to ∞ gives

$$x(t) = -\int_{t}^{\infty} a(s) f(y(s))\Delta s, \quad t \geq t_1. \tag{III.25}$$

By integrating the second equation from t_1 to t, and by using (III.18) and (III.25), we have that there exists $G > 0$ such that

$$y(t) \leq -Gc_2 \int_{t_1}^{t} b(s) \left(\int_{s}^{\infty} a(\tau)\Delta\tau \right) \Delta s.$$

So as $t \to \infty$, it contradicts to (III.23). Proof is by contradiction. $\square$

Theorem 3.3. *Suppose that (III.18) holds and f is an odd function. If*

$$\int_{t_0}^{\infty} a(s) \left(\int_{t_0}^{s} b(\tau)\Delta\tau \right) \Delta s = \infty, \tag{III.26}$$

then $M_{B,\infty}^{-} = \emptyset$.

Proof. Suppose (III.26) holds and that there exists a nonoscillatory (x, y) solution of (III.1) in $M_{B,\infty}^{-}$ such that $x > 0$ eventually, $x(t) \to c_1 > 0$ and $y(t) \to -\infty$ as $t \to \infty$. Since x has a finite limit, there exist $t_1 \geq t_0$ such that $c_1 \leq x(t)$ for $t \geq t_1$. Integrating the first equation from t_1 to t gives

$$x(t) = x(t_1) + \int_{t_1}^{t} a(s) f(y(s))\Delta s. \tag{III.27}$$

By taking the limit of (III.27) as $t \to \infty$, we have

$$\int_{t_1}^{\infty} a(s)|f(y(s))|\Delta s < \infty. \tag{III.28}$$

By integrating the second equation from t_1 to t, using (III.18) and the fact that $x(t) \geq c_1$ for $t \geq t_1$, we have that there exist $t_2 \geq t_1$ and $G > 0$ such that

$$y(t) = y(t_1) - \int_{t_1}^{t} b(s)g(x(s))\Delta s \leq -Gc_1 \int_{t_1}^{t} b(s)\Delta s, \quad t \geq t_2. \tag{III.29}$$

By (III.29) and the fact that f is an odd function, there exist $t_3 \geq t_2$ and $F > 0$ such that

$$|f(y(t))| \geq f\left(Gc_1 \int_{t_1}^{t} b(s)\Delta s \right) \geq FGc_1 \int_{t_1}^{t} b(s)\Delta s, \quad t \geq t_3. \tag{III.30}$$

Multiplying (III.30) by $a(t)$ and integrating the resulting inequality from t_3 to t give us

$$\int_{t_3}^{t} a(s)|f(y(s))|\Delta s \geq FGc_1 \int_{t_3}^{t} a(s) \left(\int_{t_3}^{s} b(\tau)\Delta\tau \right) \Delta s.$$

By taking the limit of the last inequality as $t \to \infty$ and by (III.28), we obtain a contradiction. So the assertion follows. $\square$

4. EXAMPLES

In this section, we give some examples in order to highlight our main results.

Example 4.1. *Let* $\mathbb{T} = q^{\mathbb{N}_0}$, $t_0 = 1$, $q > 1$, $a(t) = \frac{t^{\frac{1}{3}}}{(t+1)(tq+1)(2t-1)^{\frac{1}{3}}}$, $b(t) = \frac{(t+1)^{\frac{5}{3}}}{qt^2}$, $f(u) = u^{\frac{1}{3}}, c_1 = 1$, $g(u) = u^{\frac{5}{3}}$, $t = q^n$ *and* $s = tq^m$, *where* $n, m \in \mathbb{N}_0$ *in system (III.1). First we need to show* $Y(1) < \infty$ *and* $Z(1) = \infty$.

One can easily show that

$$\int_1^T a(s)\Delta s = (q-1) \sum_{s \in [1,T)_{q^{\mathbb{N}_0}}} \frac{s^{\frac{4}{3}}}{(s+1)(sq+1)(2s-1)^{\frac{1}{3}}} \leq (q-1) \sum_{s \in [1,T)_{q^{\mathbb{N}_0}}} \frac{1}{s^{\frac{2}{3}}}. \tag{III.31}$$

So as $T \to \infty$, *we have that*

$$Y(1) \leq (q-1) \sum_{n=0}^{\infty} \left(\frac{1}{q^{\frac{2}{3}}}\right)^n < \infty.$$

One can also show

$$\int_1^T b(s)\Delta s = \sum_{s \in [1,T)_{q^{\mathbb{N}_0}}} \frac{(s+1)^{\frac{5}{3}}}{qs^2}(q-1)s \geq \frac{q-1}{q} \sum_{s \in [1,T)_{q^{\mathbb{N}_0}}} s^{\frac{2}{3}}.$$

So as $T \to \infty$, *we have*

$$Z(1) = \int_1^{\infty} b(s)\Delta s \geq \frac{q-1}{q} \sum_{m=0}^{\infty} (q^{\frac{2}{3}})^m = \infty.$$

Now let us show that (III.3) holds. First we have

$$\int_t^T a(s)\Delta s \leq (q-1) \sum_{s \in [t,T)_{q^{\mathbb{N}_0}}} \frac{1}{s^{\frac{2}{3}}}$$

by (III.31). So taking the limit as $T \to \infty$, we have

$$\int_t^\infty a(s)\Delta s \leq (q-1) \sum_{s\in[t,\infty)_{q^{\mathbb{N}_0}}} \frac{1}{s^{\frac{2}{3}}} = \frac{q^{\frac{2}{3}}(q-1)}{(q^{\frac{2}{3}}-1)t^{\frac{2}{3}}}.$$

Therefore,

$$\int_1^T b(t)g\left(c_1 \int_t^\infty a(s)\Delta s\right)\Delta t \leq \alpha \sum_{t\in[1,T)_{q^{\mathbb{N}_0}}} \frac{(t+1)^{\frac{5}{3}}}{t^{\frac{19}{10}}},$$

where $\alpha = \dfrac{(q-1)^2 q^{\frac{1}{9}}}{(q^{\frac{2}{3}}-1)^{\frac{5}{3}}}$. *So as $T \to \infty$, we have that (III.3) holds by using the ratio test. One can also show that* $\left(\dfrac{1}{t+1}, -2+\dfrac{1}{t}\right)$ *is a solution of*

$$\begin{cases} \Delta_q x(t) = \frac{t^{\frac{1}{3}}}{(t+1)(tq+1)(2t-1)^{\frac{1}{3}}} y^{\frac{1}{3}}(t) \\ \Delta_q y(t) = -\frac{(t+1)^{\frac{5}{3}}}{qt^2} x^{\frac{5}{3}}(t) \end{cases}$$

such that $x(t) \to 0$ and $y(t) \to -2$, i.e., $M^-_{0,B} \neq \emptyset$ by Theorem 2.1.

Example 4.2. *Let $\mathbb{T} = \mathbb{Z}, t_0 = 0$, $a_n = 2^{\frac{-6n}{5}-1}$, $b_n = \frac{4^n}{1+2^n}$, $c_1 = 1$, $f(u) = u^{\frac{1}{5}}$ and $g(u) = u$. It is clear that $Y(0) < \infty$ and $Z(0) = \infty$. Also note that*

$$\int_0^T a(s)f\left(\int_0^s b(\tau)g(c_1)\Delta\tau\right)\Delta s = \sum_{s=0}^{T-1} 2^{\frac{-6s}{5}-1}\left(\sum_{\tau=0}^{s-1} \frac{4^\tau}{1+2^\tau}\right)^{\frac{1}{5}} \leq \frac{1}{2}\sum_{s=0}^{T-1}\left(\frac{1}{2}\right)^s.$$

So as $T \to \infty$, it follows that

$$\int_0^\infty a(s)f\left(\int_0^s b(\tau)g(c_1)\Delta\tau\right)\Delta s < \infty$$

by the geometric series. It can also be shown that $(x_n, y_n) = (1 + 2^{-n}, -2^n)$ *is a nonosicllatory solution of*

$$\begin{cases} \Delta x_n = 2^{\frac{-6n}{5}-1}(y_n)^{\frac{1}{5}} \\ \Delta y_n = -\frac{4^n}{1+2^n}(x_n) \end{cases}$$

such that $x_n \to 1$ *and* $y_n \to -\infty$ *as* $n \to \infty$, *i.e.,* $M^-_{B,\infty} \neq \emptyset$ *by Theorem 2.3 (or Theorem 10 in [8]).*

5. CONCLUSIONS

In this paper, we consider the case $Y(t_0) < \infty$ and $Z(t_0) = \infty$ in order to show the existence and nonexistence of nonoscillatory solutions in M^-. When we have the case

$$Y(t_0) = \infty \quad \text{and} \quad Z(t_0) < \infty, \tag{III.32}$$

we know from Lemma 1.2(d) that all nonoscillatory solutions belong to M^+. So as a future work, we will consider the case (III.32) in order to show the existence and nonexistence of nonoscillatory solutions in M^+.

Another open problem is to extend our main results to the delay equation

$$\begin{cases} x^{\Delta}(t) = a(t)f(y(t)) \\ y^{\Delta}(t) = -b(t)g(x(\tau(t))), \end{cases} \tag{III.33}$$

where $\tau : \mathbb{T} \to \mathbb{T}$ is an increasing function such that $\tau(t) < t$ and $\tau(t) \to \infty$ as $t \to \infty$. Even though the system

$$\begin{cases} x^{\Delta}(t) = a(t)f(y(t)) \\ y^{\Delta}(t) = -b(t)g(x(t-\tau)), \end{cases} \tag{III.34}$$

where $\tau > 0$, is considered in [11], it is not valid for all time scales, such as $\mathbb{T} = q_0^{\mathbb{N}}$, where $q > 1$.

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IV. ON NONOSCILLATORY SOLUTIONS OF TWO - DIMENSIONAL NONLINEAR TIME - SCALE SYSTEMS WITH DELAY

ABSTRACT

The classification schemes for nonoscillatory solutions of a class of nonlinear two - dimensional systems of first order delay dynamic equations on time scales are studied. Necessary and sufficient conditions are also given in order to show the existence and nonexistence of such solutions, and some of the results are new for the discrete case. Examples are given to illustrate some of the results.

1. INTRODUCTION

A number of oscillation and nonoscillation criteria have already been given for special cases of the system

$$\begin{cases} x^{\Delta}(t) = a(t)f(y(t)) \\ y^{\Delta}(t) = -b(t)g(x(\tau(t))) \end{cases} \tag{IV.1}$$

where $a, b \in C_{rd}([t_0, \infty)_{\mathbb{T}}, \mathbb{R}^+)$, $\tau \in C_{rd}([t_0, \infty)_{\mathbb{T}}, [t_0, \infty)_{\mathbb{T}})$, $\tau(t) \leq t$, and $\tau(t) \to \infty$ as $t \to \infty$, f and g are nondecreasing functions such that $uf(u) > 0$ and $ug(u) > 0$ for $u \neq 0$, see [1], [10], [11]. Motivated by [12] in which $\tau(t) = t - \eta$, $\eta > 0$, the purpose of this study is to obtain the existence and nonexistence of nonoscillatory solutions of (IV.1). According to the current knowledge, not only are the results obtained in [12] improved but some of the results are also new for the discrete case. The theory of time scales, which is a nonempty closed subset of real numbers denoted by $\mathbb{T}$, was introduced by Stefan Hilger in his Ph.D. thesis in 1988 in order to unify continuous and discrete analyses and to extend the results to any time scale (see [2] and [3]). Throughout this paper, it is assumed that $\mathbb{T}$ is unbounded above. We mean by $t \geq t_1$ that $t \in [t_1, \infty)_{\mathbb{T}} := [t_1, \infty) \cap \mathbb{T}$. We call (x, y) a *proper solution* if it is defined on $[t_0, \infty)_{\mathbb{T}}$ and $\sup\{|x(s)|, |y(s)| : s \in [t, \infty)_{\mathbb{T}}\} > 0$ for $t \geq t_0$. A solution (x, y) of (IV.1) is said to be nonoscillatory if the component functions x and y are both nonoscillatory (i.e., either eventually positive or eventually negative). Otherwise, it is said to be oscillatory.

One can easily show that any nonoscillatory solution (x, y) of system (IV.1) belongs to one of the following two classes:

$$M^+ := \{(x, y) \in M : \ xy > 0 \ \text{ eventually}\}$$

$$M^- := \{(x, y) \in M : \ xy < 0 \ \text{ eventually}\},$$

where M is the set of all nonoscillatory solutions of system (IV.1).

For convenience, set

$$A(t) = \int_t^\infty a(s)\Delta s \quad \text{and} \quad B(t) = \int_t^\infty b(s)\Delta s. \tag{IV.2}$$

The set up of this paper is as follows: in Section 1, essential lemmas that are used in proofs of the main results are given. In Section 2, the existence of nonoscillatory solutions of system (IV.1) is shown in some sub-classes of M^+ and M^- by using convergence/divergence of $A(t_0)$ and $B(t_0)$ for $t_0 \in \mathbb{T}$ and some other improper integrals. We also give examples in order to highlight our main results. In Section 3, we show the nonexistence of nonoscillatory solutions of system (IV.1) in M^+ and M^-. Finally, we end up the paper by a conclusion.

As shown in [1], the component functions x and y are themselves nonoscillatory if (x, y) is a nonoscillatory solution of the system (IV.1). The following lemmas show the oscillation and nonoscillation criteria of the system (IV.1). Because system (IV.1) has been considered without a delay term in [11], we refer the reader to [11] for some of the proofs we skip here.

Lemma 1.1. (a) If $A(t_0) < \infty$ and $B(t_0) < \infty$, then system (IV.1) is nonoscillatory. (b) If $A(t_0) = \infty$ and $B(t_0) = \infty$, then system (IV.1) is oscillatory.

Proof. (a) Suppose that $A(t_0) < \infty$ and $B(t_0) < \infty$. Choose $t_1 \in [t_0, \infty)_{\mathbb{T}}$ such that

$$\int_{t_1}^\infty a(t) f(1 + g(2) \int_t^\infty b(s)\Delta s)\Delta t < 1.$$

Let X be the space of all rd-continuous functions on $\mathbb{T}$ with the norm $\|x\| = \sup_{t \in [t_1, \infty)_{\mathbb{T}}} |x(t)|$ and with the usual pointwise ordering $\leq$. Define a subset Ω of X as

$$\Omega := \{x \in X : \quad 1 \leq x(\tau(t)) \leq 2, \quad \tau(t) \geq t_1\}.$$

For any subset S of Ω, $\inf S \in \Omega$ and $\sup S \in \Omega$. Define an operator $F : \Omega \to X$ such that

$$(Fx)(t) = 1 + \int_{t_1}^{t} a(s) f\left(1 + \int_s^\infty b(u) g(x(\tau(u)))\Delta u\right)\Delta s, \quad \tau(t) \geq t_1.$$

By using the monotonicity of f and g and the fact that $x \in \Omega$, we have

$$1 \leq (Fx)(t) \leq 1 + \int_{t_1}^{t} a(s) f(1 + g(2)\int_s^\infty b(u)\Delta u)\Delta s \leq 2, \quad \tau(t) \geq t_1.$$

It is also easy to show that F is an increasing mapping. Therefore, by the Knaster fixed point theorem, there exists $\bar{x} \in \Omega$ such that $F\bar{x} = \bar{x}$. Then

$$\bar{x}^\Delta(t) = a(t) f\left(1 + \int_t^\infty b(u) g(\bar{x}(\tau(u)))\Delta u\right).$$

Setting

$$\bar{y}(t) = 1 + \int_t^\infty b(u) g(\bar{x}(\tau(u)))\Delta u$$

gives

$$\bar{y}^\Delta(t) = -b(t) g(\bar{x}(\tau(t))),$$

i.e., $(\bar{x}, \bar{y})$ is a nonoscillatory solution of (IV.1). $\square$

Lemma 1.2. (a) If $A(t_0) < \infty$ and $B(t_0) = \infty$, then any nonoscillatory solution (x, y) of system (IV.1) belongs to M^-, i.e., $M^+ = \emptyset$.

(b) If $A(t_0) = \infty$ and $B(t_0) < \infty$, then any nonoscillatory solution (x, y) of system (IV.1) belongs to M^+, i.e., $M^- = \emptyset$.

The following lemma shows the limit behaviors of the component functions x and y of solution (x, y) of system (IV.1).

Lemma 1.3. *Let (x, y) be a nonoscillatory solution of system (IV.1).*

(a) If $A(t_0) < \infty$, then the component function x of (x, y) has a finite limit.

(b) If $A(t_0) = \infty$ or $B(t_0) < \infty$, then the component function y of (x, y) has a finite limit.

2. EXISTENCE OF NONOSCILLATORY SOLUTIONS OF (IV.1) IN M^+ AND M^-

This section shows the existence of nonoscillatory solutions of system (IV.1) by considering the convergence/divergence of $A(t_0)$ and $B(t_0)$. Because the system (IV.1) is oscillatory for the case $A(t_0) = \infty$ and $B(t_0) = \infty$, only the other three cases are considered.

2.1. THE CASE $A(t_0) = \infty$ AND $B(t_0) < \infty$

Let (x, y) be a nonoscillatory solution of system (IV.1) such that the component function x of the solution (x, y) is eventually positive. Then by the same discussion in [11], any nonoscillatory solution of system (IV.1) in M^+ belongs to one of the following sub-classes:

$$
\begin{aligned}
M^+_{B,0} &= \left\{(x, y) \in M^+ : \lim_{t\to\infty} |x(t)| = c, \quad \lim_{t\to\infty} |y(t)| = 0\right\},\\
M^+_{\infty,B} &= \left\{(x, y) \in M^+ : \lim_{t\to\infty} |x(t)| = \infty, \ \lim_{t\to\infty} |y(t)| = d\right\},\\
M^+_{\infty,0} &= \left\{(x, y) \in M^+ : \lim_{t\to\infty} |x(t)| = \infty, \ \lim_{t\to\infty} |y(t)| = 0\right\},
\end{aligned}
$$

where $0 < c < \infty$ and $0 < d < \infty$.

Theorem 2.1. $M^+_{B,0} \neq \emptyset$ *if and only if*

$$
\int_{t_0}^{\infty} a(t) f\left(k \int_t^{\infty} b(s)\Delta s\right) \Delta t < \infty \tag{IV.3}
$$

for some nonzero k.

Proof. Suppose that there exists a solution $(x, y) \in M^+_{B,0}$ such that $x(t) > 0$, $x(\tau(t)) > 0$ for $t \geq t_0$, $x(t) \to c_1$ and $y(t) \to 0$ as $t \to \infty$. Because x is eventually increasing, there exist $t_1 \geq t_0$ and $c_2 > 0$ such that $c_2 \leq g(x(\tau(t)))$ for $t \geq t_1$. Integrating the second equation from t to ∞ gives

$$y(t) = \int_t^\infty b(s)g(x(\tau(s)))\Delta s, \quad t \geq t_1. \tag{IV.4}$$

Also, integrating the first equation from t_1 to t, using the monotonicty of g and (IV.4) result in

$$x(t) \geq \int_{t_1}^t a(s)f\left(\int_s^\infty b(u)g(x(\tau(u)))\Delta u\right)\Delta s \geq \int_{t_1}^t a(s)f\left(c_2\int_s^\infty b(u)\Delta u\right)\Delta s$$

Setting $c_2 = k$ and taking the limit as $t \to \infty$ prove the assertion. (For the case $x < 0$ eventually, the proof can be shown similarly with $k < 0$.)

Conversely, suppose that (IV.3) holds for some $k > 0$. (For the case $k < 0$ can be shown similarly.) Then choose $t_1 \geq t_0$ so large that

$$\int_{t_1}^\infty a(t)f\left(k\int_t^\infty b(s)\Delta s\right)\Delta t < \frac{c_1}{2}, \quad t \geq t_1, \tag{IV.5}$$

where $k = g(c_1)$. Let X be the space of all continuous and bounded functions on $[t_1,\infty)_{\mathbb{T}}$ with the norm $\|y\| = \sup_{t\in[t_1,\infty)_{\mathbb{T}}} |y(t)|$. Then, X is a Banach space, (see [4]). Let Ω be the subset of X such that

$$\Omega := \{x \in X : \quad \frac{c_1}{2} \leq x(\tau(t)) \leq c_1, \quad \tau(t) \geq t_1\},$$

and define an operator $F : \Omega \to X$ such that

$$(Fx)(t) = c_1 - \int_t^\infty a(s)f\left(\int_s^\infty b(u)g(x(\tau(u)))\Delta u\right)\Delta s, \quad \tau(t) \geq t_1.$$

It is easy to see that Ω is bounded, convex and a closed subset of X. Now, F has the following properties. In addition, F maps into itself. Indeed, we have

$$c_1 \geq (Fx)(t) \geq c_1 - \int_t^\infty a(s) f\left(g(c_1)\int_s^\infty b(u)\Delta u\right)\Delta s \geq \frac{c_1}{2}, \quad \tau(t) \geq t_1$$

by (IV.5). In order to show that F is continuous on Ω, let x_n be a sequence in Ω such that $x_n \to x \in \Omega = \bar{\Omega}$. Then, for $\tau(t) \geq t_1$

$$\begin{aligned}
&|(Fx_n)(t) - (Fx)(t)| \\
&\leq \int_{t_1}^\infty a(s)\left|\left[f\left(-\int_s^\infty b(u)g(x_n(\tau(u)))\Delta u\right) - f\left(-\int_s^\infty b(u)g(x(\tau(u)))\Delta u\right)\right]\right|\Delta s.
\end{aligned}$$

Then, the Lebesgue Dominated Convergence theorem and the continuity of g give $\|(Fx_n) - (Fx)\| \to 0$ as $n \to \infty$, i.e., F is continuous on Ω. Finally, $F\Omega$ is shown to be precompact. Let $x \in \Omega$ and $s, t \geq t_1$. Without loss of generality, assume $s > t$. Then, we obtain

$$|(Fx)(s) - (Fx)(t)| \leq \int_s^t a(u) f\left(g(c_1)\int_u^\infty b(\lambda)\Delta\lambda\right)\Delta u < \epsilon, \quad \tau(t) \geq t_1,$$

by assumption, which implies that $F\Omega$ is relatively compact. Then, by the Schauder fixed point theorem, there exists $\bar{x} \in \Omega$ such that $\bar{x} = F\bar{x}$. As $t \to \infty$, we get $\bar{x}(t) \to c_1 > 0$. Setting

$$\bar{y}(t) = \int_t^\infty b(u)g(\bar{x}(\tau(u)))\Delta u > 0, \quad \tau(t) \geq t_1$$

shows that $\bar{y}(t) \to 0$ as $t \to \infty$, i.e., $M_{B,0}^+ \neq \emptyset$. $\square$

Example 2.2. *Let* $\mathbb{T} = 2^{\mathbb{N}_0}$, $\tau(t) = \frac{t}{4}$, $t = 2^n$, $s = 2^m$, $m, n \geq 2$, $a(t) = \dfrac{1}{2t^{\frac{4}{5}}}$, $b(t) = \dfrac{3}{4t^2(8t-4)}$, $f(u) = u^{\frac{3}{5}}$, $k = 1$ *and* $g(u) = u$. *First, it must be shown that* $A(t_0) = \infty$

and $B(t_0) < \infty$. *Indeed,*

$$\int_{t_0}^{t} a(s)\Delta s = \frac{1}{2} \sum_{s\in[4,t)_{2^{\mathbb{N}_0}}} s^{\frac{1}{5}}.$$

Therefore,

$$A(t_0) = \frac{1}{2} \lim_{n\to\infty} \sum_{m=2}^{n-1} (2^m)^{\frac{1}{5}} = \infty.$$

Because

$$\int_{t_0}^{t} b(s)\Delta s \leq \frac{3}{16} \sum_{s\in[4,t)_{2^{\mathbb{N}_0}}} \frac{1}{s},$$

we have

$$B(t_0) \leq \frac{3}{16} \lim_{n\to\infty} \sum_{m=2}^{n-1} \frac{1}{2^m} < \infty$$

by the geometric series. Note that

$$\int_{t}^{T} b(s)\Delta s \leq \frac{3}{16} \sum_{s\in[t,T)_{2^{\mathbb{N}_0}}} \frac{1}{s}.$$

This implies that

$$B(t) \leq \frac{3}{16} \lim_{n\to\infty} \sum_{m=2}^{n-1} \frac{1}{2^m} = \frac{3}{8} \lim_{n\to\infty} \left(\frac{1}{t} - \frac{1}{t2^n}\right) = \frac{3}{8t}.$$

Letting $k = 1$ *and using the last inequality gives*

$$\int_{t_0}^{T} a(t) f\left(k \int_{t}^{\infty} b(s)\Delta s\right) \Delta t \leq \int_{t_0}^{T} \frac{1}{2t^{\frac{4}{5}}} \left(\frac{3}{8t}\right)^{\frac{3}{5}} \Delta t = \left(\frac{3}{8}\right)^{\frac{3}{5}} \frac{1}{2} \sum_{t\in[1,T)_{2^{\mathbb{N}_0}}} \frac{1}{t^{\frac{2}{5}}}.$$

Therefore, we have

$$\int_{t_0}^{\infty} a(t) f\left(k \int_t^{\infty} b(s)\Delta s\right) \Delta t \leq \left(\frac{3}{8}\right)^{\frac{3}{5}} \frac{1}{2} \sum_{n=0}^{\infty} \frac{1}{2^{\frac{2n}{5}}} < \infty$$

by the geometric series. One can also show that $(x, y) = \left(8 - \frac{1}{t}, \frac{1}{t^2}\right)$ *is a nonoscillatory solution of*

$$\begin{cases} \Delta_2 x(t) = \dfrac{1}{2t^{\frac{4}{5}}} \left(y(t)\right)^{\frac{3}{5}} \\ \Delta_2 y(t) = -\dfrac{3}{4t^2(8t-4)} x(\dfrac{t}{4}), \end{cases} \tag{IV.6}$$

where $\Delta_2 x$ *is the delta-derivative of* x *in* $2^{\mathbb{N}_0}$*, i.e.,* $\Delta_2 h(t) = \dfrac{h(2t) - h(t)}{t}$ *such that* $x(t) \to 8$ *and* $y(t) \to 0$ *as* $t \to \infty$*, i.e.,* $M_{B,0}^{+} \neq \emptyset$ *by Theorem 2.1.*

When the case $A(t_0) = \infty$ and $B(t_0) < \infty$ holds, it can be shown that $M_{B,\infty}^{+} \neq \emptyset$ with $\tau(t) = t - \eta$ for $\eta \geq 0$, see [12].

2.2. THE CASE $A(t_0) < \infty$ AND $B(t_0) < \infty$

Because the component fuctions x and y have finite limits by Lemma 1.3, only two subclasses in M^{+} can exist by the same discussion in [11]

$$M_{B,0}^{+} = \left\{(x, y) \in M^{+} : \lim_{t\to\infty} |x(t)| = c, \quad \lim_{t\to\infty} |y(t)| = 0\right\},$$
$$M_{B,B}^{+} = \left\{(x, y) \in M^{+} : \lim_{t\to\infty} |x(t)| = c, \ \lim_{t\to\infty} |y(t)| = d\right\},$$

where $0 < c < \infty$ and $0 < d < \infty$. Because the existence of nonoscillatory solutions in $M_{B,0}^{+}$ is shown in the previous subsection, it is only proven for $M_{B,B}^{+}$.

Theorem 2.3. $M_{B,B}^{+} \neq \emptyset$ *if and only if*

$$\int_{t_0}^{\infty} a(s) f\left(d_1 + k \int_s^{\infty} b(u)\Delta u\right) \Delta s < \infty \tag{IV.7}$$

for some $k \neq 0$ *and* $d_1 \neq 0$.

Proof. Suppose that a nonoscillatory solution $(x,y) \in M^+_{B,B}$ exists such that $x > 0$ eventually, $x(t) \to c_1$, and $y(t) \to d_1$ as $t \to \infty$. (For the case $x < 0$ eventually, the proof can be shown similarly.) Because x is eventually positive and increasing, there exist a large $t_1 \geq t_0$ and $c_2 > 0$ such that $c_2 \leq x(\tau(t)) \leq c_1$ for $t \geq t_1$. Integrating the second equation from t to ∞ and the monotonicity of g give

$$y(t) \geq d_1 + g(c_2) \int_t^\infty b(s)\Delta s, \quad t \geq t_1. \tag{IV.8}$$

Integrating the first equation from t_1 to t and using the monotonicity of f yield

$$x(t) \geq \int_{t_1}^t a(s) f\left(d_1 + g(c_2) \int_s^\infty b(\tau)\Delta\tau\right) \Delta s.$$

So, as $t \to \infty$, the assertion follows for $k = g(c_2)$.

Conversely, suppose (IV.7) holds. Choose $t_1 \geq t_0$, $k > 0$ and $d_1 > 0$ such that

$$\int_{t_1}^\infty a(s) f\left(d_1 + k \int_s^\infty b(u)\Delta u\right) \Delta s < d_1, \tag{IV.9}$$

where $k = g(2d_1)$. (The case $k, d_1 < 0$ can be done similarly.) Let X be the Banach space of all continuous real valued functions endowed with the norm $\|x\| = \sup_{t\in[t_1,\infty)_{\mathbb{T}}} |x(t)|$ and with usual pointwise ordering $\leq$. Define a subset Ω of X as

$$\Omega := \{x \in X : \quad d_1 \leq x(\tau(t)) \leq 2d_1, \quad \tau(t) \geq t_1\}.$$

For any subset B of Ω, it is clear that $\inf B \in \Omega$ and $\sup B \in \Omega$. An operator $F : \Omega \to X$ is defined as

$$(Fx)(t) = d_1 + \int_{t_1}^t a(s) f\left(d_1 + \int_s^\infty b(u) g(x(\tau(u)))\Delta u\right) \Delta s, \quad \tau(t) \geq t_1.$$

It is obvious that F is an increasing mapping into itself. Therefore,

$$d_1 \leq (Fx)(t) \leq d_1 + \int_{t_1}^{t} a(s) f\left(d_1 + g(2d_1)\int_{s}^{\infty} b(u)\Delta u\right)\Delta s \leq 2d_1, \quad \tau(t) \geq t_1.$$

Then, by the Knaster fixed point theorem, there exists $\bar{x} \in \Omega$ such that $\bar{x} = F\bar{x}$. By setting

$$\bar{y}(t) = d_1 + \int_{t}^{\infty} b(u) g(\bar{x}(\tau(u))), \quad \tau(t) \geq t_1,$$

we get that

$$\bar{y}^{\Delta}(t) = -b(t) g(\bar{x}(\tau(t))).$$

Therefore, $\bar{x}(t) \to \alpha$ and $\bar{y}(t) \to d_1$ as $t \to \infty$, where $0 < \alpha < \infty$, i.e., $M_{B,B}^{+} \neq \emptyset$. Note that a similar proof can be done for the case $k < 0$ and $d_1 < 0$ with $x < 0$. $\square$

Example 2.4. *Let* $\mathbb{T} = 2^{\mathbb{N}_0}$, $\tau(t) = \frac{t}{4}$, $t = 2^n$, $s = 2^m$, $n \geq 2$, $a(t) = \dfrac{1}{2t^{\frac{5}{3}}(3t+1)^{\frac{1}{3}}}$, $b(t) = \frac{1}{2t(6t-4)}$, $f(u) = u^{\frac{1}{3}}$ *and* $g(u) = u$. *We first show* $A(t_0) < \infty$ *and* $B(t_0) < \infty$.

$$\int_{t_0}^{t} a(s)\Delta s = \frac{1}{2}\sum_{s \in [4,t)_{2^{\mathbb{N}_0}}} \frac{1}{s^{\frac{2}{3}}(3s+1)^{\frac{1}{3}}}.$$

So we have

$$A(t_0) = \frac{1}{2}\lim_{n\to\infty}\sum_{m=2}^{n-1} \frac{1}{(2^m)^{\frac{2}{3}}(3 \cdot 2^m + 1)^{\frac{1}{3}}} < \infty$$

by the ratio test. Similarly,

$$\int_{t_0}^{t} b(s)\Delta s = \frac{1}{2}\sum_{s \in [4,t)_{2^{\mathbb{N}_0}}} \frac{1}{6s-4}.$$

Hence, as $t \to \infty$, *we obtain*

$$B(t_0) = \frac{1}{2}\lim_{n\to\infty}\sum_{m=2}^{n-1} \frac{1}{6.2^m - 4} < \infty.$$

Because $A(t_0) < \infty$ and $B(t_0) < \infty$, it is easy to show that (IV.7) holds. One can also show that $\left(6 - \frac{1}{t}, 3 + \frac{1}{t}\right)$ is a nonoscillatory solution of

$$\begin{cases} \Delta_2 x(t) = \dfrac{1}{2t^{\frac{5}{3}}(3t+1)^{\frac{1}{3}}} y^{\frac{1}{3}}(t) \\ \Delta_2 y(t) = -\dfrac{1}{2t(6t-4)} x\left(\dfrac{t}{4}\right) \end{cases} \tag{IV.10}$$

such that $x(t) \to 6$ and $y(t) \to 3$ as $t \to \infty$, i.e., $M^+_{B,B} \neq \emptyset$ by Theorem 2.3.

2.3. THE CASE $A(t_0) < \infty$ AND $B(t_0) = \infty$

By the similar argument in [11], any nonoscillatory solution of system (IV.1) in M^- belongs to one of the following sub-classes:

$$\begin{aligned} M^-_{0,B} &= \left\{(x,y) \in M^- : \lim_{t\to\infty} |x(t)| = 0, \quad \lim_{t\to\infty} |y(t)| = d\right\}, \\ M^-_{B,B} &= \left\{(x,y) \in M^- : \lim_{t\to\infty} |x(t)| = c, \quad \lim_{t\to\infty} |y(t)| = d\right\}, \\ M^-_{0,\infty} &= \left\{(x,y) \in M^- : \lim_{t\to\infty} |x(t)| = 0, \ \lim_{t\to\infty} |y(t)| = \infty\right\}, \\ M^-_{B,\infty} &= \left\{(x,y) \in M^- : \lim_{t\to\infty} |x(t)| = c, \ \lim_{t\to\infty} |y(t)| = \infty\right\}, \end{aligned}$$

where $0 < c < \infty$ and $0 < d < \infty$.

Theorem 2.5. *$M^-_{B,\infty} \neq \emptyset$ if and only if*

$$\int_{t_0}^{\infty} a(s) f\left(k \int_{t_0}^{s} b(u)\Delta u\right) \Delta s < \infty \tag{IV.11}$$

for some $k \neq 0$, where f is an odd function.

Proof. Suppose that there exists a nonoscillatory solution $(x,y) \in M^-_{B,\infty}$ such that $x(t) > 0$, $x(\tau(t)) > 0$, $t \geq t_1$, $x(t) \to c_2$ and $y(t) \to -\infty$ as $t \to \infty$, where $0 < c_2 < \infty$. Because x is monotonic and has a finite limit, there exist $t_2 \geq t_1$ and $c_3 > 0$ such that

$$c_2 \leq x(\tau(t)) \leq c_3 \quad \text{for} \quad t \geq t_2. \tag{IV.12}$$

Integrating the first equation from t_2 to t gives

$$c_2 \leq x(t) = x(t_1) + \int_{t_1}^{t} a(s)f(y(s))\Delta s \leq c_3, \quad t \geq t_2.$$

By taking the limit as $t \to \infty$, we obtain

$$\int_{t_2}^{\infty} a(s)|f(y(s))|\Delta s < \infty. \tag{IV.13}$$

Using the monotonicity of g, (IV.12) and integrating the second equation from t_2 to t yield

$$y(t) \leq y(t_2) - g(c_2)\int_{t_2}^{t} b(s)\Delta s \leq -g(c_2)\int_{t_2}^{t} b(s)\Delta s.$$

Because $f(-u) = -f(u)$ for $u \neq 0$ and by the monotonicity of f, we have

$$|f(y(t))| \geq f\left(g(c_2)\int_{t_2}^{t} b(s)\Delta s\right), \quad t \geq t_2. \tag{IV.14}$$

By (IV.13) and (III.12), we have

$$\int_{t_2}^{t} a(s)|f(y(s))|\Delta s \geq \int_{t_2}^{t} a(s)f\left(g(c_2)\int_{t_2}^{s} b(u)\Delta u\right)\Delta s.$$

As $t \to \infty$, the assertion follows by setting $g(c_2) = k$. (The case $x < 0$ eventually can be proved similarly with $k < 0$.)

Conversely, without loss of generality, suppose that (IV.11) holds for some $k > 0$. (The case $k < 0$ can be done similarly.) Then one can choose $t_1 \geq t_0$ and

$d > 0$ such that

$$\int_{t_1}^{\infty} a(s) f\left(k \int_{t_1}^{s} b(u)\Delta u\right) \Delta s < d, \quad \tau(t) \geq t_1, \tag{IV.15}$$

where $k = g(2d)$. Let X be the partially ordered Banach space of all real-valued continuous functions endowed with supremum norm $\|x\| = \sup_{t\in[t_1,\infty)_{\mathbb{T}}} |x(t)|$ and with the usual pointwise ordering $\leq$. Define a subset Ω of X such that

$$\Omega := \{x \in X : \quad d \leq x(\tau(t)) \leq 2d, \quad \tau(t) \geq t_1\}. \tag{IV.16}$$

For any subset B of Ω, $\inf B \in \Omega$ and $\sup B \in \Omega$, i.e., $(\Omega, \leq)$ is complete. Define an operator $F : \Omega \to X$ as

$$(Fx)(t) = d + \int_{t}^{\infty} a(s) f\left(\int_{t_1}^{s} b(u) g(x(\tau(u)))\Delta u\right) \Delta s, \quad \tau(t) \geq t_1. \tag{IV.17}$$

It must be shown that $F : \Omega \to \Omega$ is an increasing mapping into itself. It is obvious that it is an increasing mapping and because

$$d \leq (Fx)(t) = d + \int_{t}^{\infty} a(s) f\left(\int_{t_1}^{s} b(u) g(x(\tau(u)))\Delta u\right) \Delta s \leq 2d$$

by (IV.15), it follows that $F : \Omega \to \Omega$. Then, by the Knaster fixed point theorem, there exists $\bar{x} \in \Omega$ such that

$$\bar{x}(t) = (F\bar{x})(t) = d + \int_{t}^{\infty} a(s) f\left(\int_{t_1}^{s} b(u) g(\bar{x}(\tau(u)))\Delta u\right) \Delta s, \quad \tau(t) \geq t_1. \tag{IV.18}$$

Taking the derivative of (IV.18) and the fact that f is an odd function show that

$$\bar{x}^{\Delta}(t) = a(t) f\left(-\int_{t_1}^{t} b(u) g(\bar{x}(\tau(u)))\Delta u\right), \quad \tau(t) \geq t_1.$$

Setting $\bar{y} = -\int_{t_1}^{t} b(u)g(\bar{x}(\tau(u)))\Delta u$ and using the monotonicity of g give

$$\bar{y}(t) \leq -g(d)\int_{t_1}^{t} b(u)\Delta u, \quad \tau(t) \geq t_1.$$

Therefore, $\bar{x}(t) > 0$ and $\bar{y}(t) < 0$ for $t \geq t_1$, and $\bar{x}(t) \to d$ and $\bar{y}(t) \to -\infty$ as $t \to \infty$. This completes the proof. $\square$

Example 2.6. *Let $\mathbb{T} = 2^{\mathbb{N}_0}$, $\tau(t) = \frac{t}{4}$, $t = 2^n$, $s = 2^m$, $m, n \geq 2$, $k = 1$, $a(t) = \dfrac{1}{2t^{\frac{7}{5}}(t^2+1)^{\frac{3}{5}}}$, $b(t) = \dfrac{2t^2-1}{2t^{\frac{9}{5}}(3t+4)^{\frac{1}{5}}}$, $f(u) = u^{\frac{3}{5}}$ and $g(u) = u^{\frac{1}{5}}$. One can easily show $A(t_0) < \infty$ and $B(t_0) = \infty$. To show (IV.11) holds, first we have*

$$\int_{t_0}^{s} b(u)\Delta u = \frac{1}{2}\sum_{u\in[4,s)_{2^{\mathbb{N}_0}}} \frac{2u^2-1}{u^{\frac{4}{5}}(3u+4)^{\frac{1}{5}}} \leq \sum_{u\in[1,s)_{2^{\mathbb{N}_0}}} u = s-1$$

Hence

$$\int_{t_0}^{\infty} a(s)f\left(k\int_{t_0}^{s} b(u)\Delta u\right)\Delta s \leq \int_{t_0}^{T} \frac{1}{2s^{\frac{7}{5}}(s^2+1)^{\frac{3}{5}}}(s-1)^{\frac{3}{5}}\Delta s$$
$$= \frac{1}{2}\sum_{s\in[4,T)_{2^{\mathbb{N}_0}}} \frac{(s-1)^{\frac{3}{5}}}{s^{\frac{2}{5}}(s^2+1))^{\frac{3}{5}}} \leq \sum_{s\in[4,T)_{2^{\mathbb{N}_0}}} \frac{1}{s}.$$

Because

$$\lim_{T\to\infty} \sum_{s\in[4,T)_{2^{\mathbb{N}_0}}} \frac{1}{s} = \sum_{m=2}^{\infty} \frac{1}{2^m} < \infty,$$

it can be shown that (IV.11) holds as $T \to \infty$. It can also be shown that $(3+\frac{1}{t}, -t-\frac{1}{t})$ is a nonoscillatory solution of

$$\begin{cases} \Delta_2 x(t) = \dfrac{1}{2t^{\frac{7}{5}}(t^2+1)^{\frac{3}{5}}}(y(t))^{\frac{3}{5}} \\ \Delta_2 y(t) = -\dfrac{2t^2-1}{2t^{\frac{9}{5}}(3t+4)^{\frac{1}{5}}}\left(x(\dfrac{t}{4})\right)^{\frac{1}{5}} \end{cases} \tag{IV.19}$$

such that $x(t) \to 3$ and $y(t) \to -\infty$ as $t \to \infty$, i.e., $M^{-}_{B,\infty} \neq \emptyset$ by Theorem 2.5.

3. NONEXISTENCE OF NONOSCILLATORY SOLUTIONS OF (IV.1) IN M^+ AND M^-

The nonexistence of nonoscillatory solutions of system (IV.1) in $M^+_{B,0}$, $M^+_{B,B}$ and $M^-_{B,\infty}$ directly follows from Theorems 2.1, 2.3 and 2.5, respectively. Hence, the focus is only on $M^+_{\infty,B}$, $M^+_{\infty,0}$, $M^-_{0,B}$, $M^-_{B,B}$ and $M^-_{0,\infty}$.

3.1. THE CASE $A(t_0) = \infty$ AND $B(t_0) < \infty$

Theorem 3.1. *If*

$$\int_{t_0}^{\infty} b(s) g\left(c_1 \int_{t_0}^{\tau(s)} a(u)\Delta u \right) \Delta s = \infty \tag{IV.20}$$

for some nonzero c_1*, then* $M^+_{\infty,B} = \emptyset$.

Proof. Assume that there exists a solution $(x, y) \in M^+_{\infty,B}$ of (IV.1) such that $x(t) > 0$, $x(\tau(t)) > 0$, $y(t) > 0$ for $t \geq t_0$, $x(t) \to \infty$ and $y(t) \to d_1$ as $t \to \infty$, where $0 < d_1 < \infty$. Because $y(t) > 0$ and decreasing for $t \geq t_0$, there exists $t_1 \geq t_0$ and $d_2 > 0$ such that $d_1 \leq y(t) \leq d_2$ for $t \geq t_1$. Integrating the first equation from t_1 to $\tau(t)$ gives

$$x(\tau(t)) \geq f(d_1) \int_{t_1}^{\tau(t)} a(s)\Delta s. \tag{IV.21}$$

By integrating the second equation form t_1 to t and using (IV.21) yield us

$$y(t_1) \geq \int_{t_1}^{t} b(s) g(x(\tau(s)))\Delta s \geq \int_{t_1}^{t} b(s) g\left(c_1 \int_{t_1}^{\tau(s)} a(u)\Delta u \right) \Delta s, \quad t \geq t_1,$$

where $c_1 = f(d_1)$. As $t \to \infty$, we have a contradiction to (IV.20). The proof can be shown similarly when $x < 0$ eventually with $c_1 < 0$. $\square$

Theorem 3.2. *If*

$$\int_{t_0}^{\infty} a(t) f\left(\int_t^{\infty} b(s) g\left(c_1 \int_{t_0}^{s} a(u)\Delta u\right)\Delta s\right)\Delta t < \infty \tag{IV.22}$$

for some $c_1 \neq 0$, *then* $M^+_{\infty,0} = \emptyset$.

Proof. The proof is by contradiction, so assume that there exists a nonoscillatory solution in $M^+_{\infty,0}$ such that $x(t) > 0$, $x(\tau(t)) > 0$, $y(t) > 0$ for $t \geq t_0$, $x(t) \to \infty$ and $y(t) \to 0$ as $t \to \infty$. Integrating the second equation from t to ∞ gives

$$y(t) = \int_t^{\infty} b(s) g(x(\tau(s)))\Delta s. \tag{IV.23}$$

Because y is eventually decreasing, there exist $t_1 \geq t_0$ and $d_1 > 0$ such that $f(y(t)) \leq d_1$ for $t \geq t_1$. Then by integrating the first equation from t_1 to t and the monotonicity of x and f, we have that

$$x(\tau(t)) \leq x(t) \leq x(t_1) + d_1 \int_{t_1}^{t} a(s)\Delta s \leq c_1 \int_{t_1}^{t} a(s)\Delta s, \quad t \geq t_1, \tag{IV.24}$$

where $c_1 = 1 + max\{x(t_1), d_1\}$. Integrating the first equation from t_1 to t, monotonicty of f and g, (IV.23) and (IV.24) give

$$x(t) \leq x(t_1) + \int_{t_1}^{t} a(s) f\left(\int_s^{\infty} b(u) g\left(c_1 \int_{t_1}^{u} a(\lambda)\Delta\lambda\right)\Delta u\right)\Delta s.$$

As $t \to \infty$, we have a contradiction to $x(t) \to \infty$. The proof can be done similarly when $x < 0$ eventually with $c_1 < 0$. □

3.2. THE CASE $A(t_0) < \infty$ AND $B(t_0) = \infty$

Theorem 3.3. *If*

$$\int_{t_0}^{\infty} b(t) g\left(c_1 \int_t^{\infty} a(s)\Delta s\right)\Delta t = \infty \tag{IV.25}$$

for some $c_1 \neq 0$, *then* $M^-_{0,B} = \emptyset$.

Proof. The proof is by contradiction. Assume that there exists a solution $(x, y) \in M^-_{0,B}$ such that $x(t) > 0$, $x(\tau(t)) > 0$, $y(t) < 0$ for $t \geq t_0$, $x(t) \to 0$ and $y(t) \to -d$ as $t \to \infty$, where $d > 0$. By integrating the first equation of system (IV.1) and using the monotonicity of x, y and f, there exist $c_1 > 0$ and $t_1 \geq t_0$ such that

$$x(\tau(t)) \geq x(t) \geq c_1 \int_t^\infty a(s)\Delta(s), \quad t \geq t_1. \tag{IV.26}$$

By integrating the second equation from t_1 to t, using inequality (IV.26) and the monotonicity of g, we have

$$y(t) = y(t_0) - \int_{t_0}^t b(s)g(x(\tau(s)))\Delta s \leq -\int_{t_0}^t b(s)g\left(c_1 \int_s^\infty a(\tau)\Delta\tau\right)\Delta s.$$

As $t \to \infty$, we have a contradiction to (IV.25). For the case $x < 0$ eventually, the proof can be shown similarly with $c_1 < 0$. □

Theorem 3.4. *If*

$$\int_{t_0}^\infty b(t)g\left(c_1 - d_1 \int_t^\infty a(s)\Delta s\right) = \infty \tag{IV.27}$$

for some $c_1 > 0$ *and* $d_1 < 0$, *(or* $c_1 < 0$ *and* $d_1 > 0$*) then* $M^-_{B,B} = \emptyset$.

Proof. The proof is by contradiction. Hence, assume that there exists a nonoscillatory solution $(x, y) \in M^-_{B,B}$ such that $x(t) > 0$, $x(\tau(t)) > 0$, $y(t) < 0$ for $t \geq t_0$, $\lim_{t\to\infty} x(t) = c_1 > 0$ and $\lim_{t\to\infty} y(t) = d_1 < 0$. Since y is decreasing, there exists $d_2 < 0$ and $t_1 \geq t_0$ such that $f(y(t)) \leq d_2$ for $t \geq t_1$. Integrating the first equation from t to ∞ and the monotonicty of x yield

$$x(\tau(t)) \geq x(t) = c_1 - \int_t^\infty a(s)f(y(s))\Delta s \geq c_1 - d_2 \int_t^\infty a(s)\Delta s, \quad t \geq t_1. \tag{IV.28}$$

By integrating the second equation from t_1 to t and using (IV.28), we have

$$y(t) \leq -\int_{t_1}^{t} b(s)g(x(\tau(s)))\Delta s \leq -\int_{t_1}^{t} b(s)g\left(c_1 - d_2\int_{s}^{\infty} a(u)\Delta u\right)\Delta s,$$

where $d_2 = d_1 < 0$ and taking the limit of the last inequality as $t \to \infty$, we have a contradiction to (IV.27). This completes the proof. Note that the case $x < 0$ eventually can be done similarly with $c_1 < 0$ and $d_1 > 0$. □

Theorem 3.5. *Suppose that f is an odd function. If*

$$\int_{t_0}^{\infty} a(s)f\left(\int_{t_1}^{s} b(u)g\left(c_1\int_{u}^{\infty} a(\lambda)\Delta\lambda\right)\Delta u\right)\Delta s = \infty \tag{IV.29}$$

for some $c_1 \neq 0$, then $M^-_{0,\infty} = \emptyset$.

Proof. The proof is by contradiction. Assume that there exists a nonoscillatory solution $(x, y) \in M^-_{0,\infty}$ such that $x(t) > 0$, $x(\tau(t)) > 0$, $y(t) < 0$ for $t \geq t_0$, $x(t) \to 0$ and $y(t) \to -\infty$ as $t \to \infty$. Inequality (IV.26) and the monotonicity of g yield us that there exists $c_1 > 0$ and $t_1 \geq t_0$ such that

$$g(x(\tau(t))) \geq g(x(t)) \geq g\left(c_1\int_{t}^{\infty} a(s)\Delta s\right), \quad t \geq t_1. \tag{IV.30}$$

Integrating the second equation of system (IV.1) from t_1 to t and using (IV.30) yield

$$y(t) \leq -\int_{t_1}^{t} b(s)g\left(c_1\int_{s}^{\infty} a(u)\Delta u\right)\Delta s, \quad t \geq t_1. \tag{IV.31}$$

By integrating the first equation of system (IV.1) from t_1 to t, (IV.31) and the fact that f is an odd function, we have

$$x(t_1) \geq x(t_1) - x(t) \geq \int_{t_1}^{t} a(s)\left(\int_{t_1}^{s} b(u)g\left(c_1\int_{u}^{\infty} a(\lambda)\Delta\lambda\right)\Delta u\right)\Delta s, \quad t \geq t_1.$$

Taking the limit of the last inequality as $t \to \infty$, we have a contradiction to (IV.29). For the case $x < 0$, the proof can be shown similary with $c_1 < 0$. $\square$

4. CONCLUSION

In this section, we reconsider (IV.1), where $\tau(t) = t$, namely,

$$\begin{cases} x^{\Delta}(t) = a(t)f(y(t)) \\ y^{\Delta}(t) = -b(t)g(x(t)) \end{cases} \tag{IV.32}$$

and investigate the asymptotic properties of nonoscillatory solutions for (IV.32). Because the existence and nonexistence of nonoscillatory solutions of (IV.32) in M^- are considered in [11], we only focus on M^+. Notice that the results are obtained for system (IV.1) in Sections 2 and 3 also hold for system (IV.32). Therefore, we only show the existence of nonoscillatory solutions for (IV.32) in $M^+_{\infty,B}$ and $M^+_{\infty,0}$, which are not acquired for (IV.1). In order to do that, we assume $A(t_0) = \infty$ and $B(t_0) < \infty$ throughout this section.

Theorem 4.1. *$M^+_{\infty,B} \neq \emptyset$ if and only if*

$$\int_{t_0}^{\infty} b(s)g\left(c_1 \int_{t_0}^{s} a(u)\Delta u\right)\Delta s < \infty \tag{IV.33}$$

for some $c_1 \neq 0$.

Proof. The necessity directly follows from Theorem 3.1. For suffiency, suppose that (IV.33) holds. Choose $t_1 \geq t_0$, $c_1 > 0$ and $d_1 > 0$ such that

$$\int_{t_1}^{\infty} b(s)g\left(c_1 \int_{t_1}^{s} b(u)\Delta u\right)\Delta s < d_1, \quad t \geq t_1, \tag{IV.34}$$

where $c_1 = f(2d_1) > 0$. (The case $c_1 < 0$ can be done similarly.) Let X be the partially ordered Banach space of all real-valued continuous functions endowed with supremum norm $\|x\| = \sup\limits_{t\in[t_1,\infty)_{\mathbb{T}}} \frac{|x(t)|}{\int_{t_1}^{t} a(s)\Delta s}$ and with the usual pointwise ordering $\leq$.

Define a subset Ω of X such that

$$\Omega =: \{x \in X : \quad f(d_1) \int_{t_1}^{t} a(s)\Delta s \leq x(t) \leq f(2d_1) \int_{t_1}^{t} a(s)\Delta s, \quad t \geq t_1\}. \tag{IV.35}$$

For any subset B of Ω, $\inf B \in \Omega$ and $\sup B \in \Omega$, i.e., $(\Omega, \leq)$ is complete. Define an operator $F : \Omega \to X$ as

$$(Fx)(t) = \int_{t_1}^{t} a(s) f\left(d_1 + \int_{t}^{\infty} b(u)g(x(u))\Delta u\right)\Delta s, \quad t \geq t_1. \tag{IV.36}$$

First we need to show that $F : \Omega \to \Omega$ is an increasing mapping into itself. It is obvious that it is an increasing mapping, so let us show $F := \Omega \to \Omega$.

$$\begin{aligned} f(d_1) \int_{t_1}^{t} a(s)\Delta s &\leq (Fx)(t) \\ &\leq \int_{t_1}^{t} a(s) f\left(d_1 + \int_{s}^{\infty} b(u) g\left(f(2d_1)\int_{t_1}^{u} a(\lambda)\Delta\lambda\right)\Delta u\right)\Delta s \\ &\leq f(2d_1)\int_{t_1}^{t} a(s)\Delta s \end{aligned}$$

by (IV.34). Then, by the Knaster fixed point theorem, there exists $\bar{x} \in \Omega$ such that

$$\bar{x}(t) = (F\bar{x})(t) = \int_{t_1}^{t} a(s) f\left(d_1 + \int_{s}^{\infty} b(u)g(\bar{x}(u))\Delta u\right)\Delta s, \quad t \geq t_1. \tag{IV.37}$$

By taking the derivative of (IV.37)

$$\bar{x}^{\Delta}(t) = a(t) f\left(d_1 + \int_{t}^{\infty} b(u)g(\bar{x}(u))\Delta u\right), \quad t \geq t_1.$$

Setting $\bar{y}(t) = d_1 + \int_{t}^{\infty} b(u)g(\bar{x}(u))\Delta u$ and taking the limit as $t \to \infty$ show that $\bar{x}(t) > 0$ and $\bar{y}(t) > 0$ for $t \geq t_1$, and $\bar{x}(t) \to \infty$ and $\bar{y}(t) \to d_1 > 0$ as $t \to \infty$, i.e., $M^{+}_{\infty,B} \neq \emptyset$. $\square$

Theorem 4.2. *If*

$$\int_{t_0}^{\infty} a(t) f\left(k \int_{t}^{\infty} b(s)\Delta s\right) \Delta t = \infty \quad (-\infty)$$

and

$$\int_{t_0}^{\infty} b(t) g\left(l \int_{t_0}^{\infty} a(s)\Delta s\right) \Delta t < \infty$$

for any $k > 0$ and some $l > 0$ ($k < 0$ and $l < 0$), then $M^{+}_{\infty,0} \neq \emptyset$.

Proof. Choose $t_1 \geq t_0$ and $c_1 > 0$ such that

$$\int_{t_1}^{\infty} b(t) g\left(l \int_{t_0}^{t} a(s)\Delta s\right) \Delta t < \frac{c_1}{2}, \quad t \geq t_1, \tag{IV.38}$$

where $l = f(c_1)$. Let X be the partially ordered Banach space of all real-valued continous functions endowed with the norm $\|y\| = \sup_{t \in [t_1,\infty)_{\mathbb{T}}} |y(t)|$ and with the usual pointwise ordering $\leq$. Define a subset Ω of X such that

$$\Omega =: \{y \in X : \quad g(1) \int_{t}^{\infty} b(s)\Delta s \leq y(t) \leq \frac{c_1}{2}, \quad t \geq t_1\}.$$

It is clear that $(\Omega, \leq)$ is complete. Define an operator $F : \Omega \to X$ such that

$$(Fy)(t) = \int_{t}^{\infty} b(s) g\left(\int_{t_1}^{s} a(u) f(y(u))\Delta u\right) \Delta s.$$

It is clear that F is an increasing mapping. We also need to show that $F : \Omega \to \Omega$. By (IV.38) and the monotonicity of g, we have

$$(Fy)(t) \leq \int_{t}^{\infty} b(s) g\left(l \int_{t_1}^{s} a(u)\Delta u\right) \Delta s \leq \frac{c_1}{2}$$

for $y \in \Omega$. Since

$$\int_{t_0}^{\infty} a(t) f\left(k \int_t^{\infty} b(s)\Delta s\right) \Delta t = \infty,$$

there exists $t_2 \geq t_1$ such that

$$\int_{t_2}^{t} a(s) f\left(k \int_s^{\infty} b(u)\Delta u\right) \Delta s > 1$$

for $t \geq t_2$ and any $k > 0$, so by setting $k = g(1)$, we have

$$(Fy)(t) \geq \int_t^{\infty} b(s) g\left(\int_{t_1}^{s} a(u) f\left(g(1) \int_u^{\infty} b(\lambda)\Delta\lambda\right) \Delta u\right) \Delta s \geq g(1) \int_t^{\infty} a(s)\Delta s,$$

for $t \geq t_2$. Then, by the Knaster fixed point theorem, there exists $\bar{y} \in \Omega$ such that $\bar{y} = F\bar{y}$. Then we have

$$\bar{y}^{\Delta}(t) = -b(t) g\left(\int_{t_1}^{t} a(u) f(\bar{y}(u))\Delta u\right).$$

Setting

$$\bar{x}(t) = \int_{t_1}^{t} a(u) f(\bar{x}(u))\Delta u$$

and taking the limit as $t \to \infty$ give us that $\bar{x} \to \infty$ and $\bar{y} \to 0$, i.e., $M_{\infty,0}^{+} \neq \emptyset$. The case $k < 0$ and $l < 0$ with $x < 0$ can be shown similarly. $\square$

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SECTION

2. CONCLUSIONS

In this thesis, we investigated the existence and nonexistence of nonoscillatory solutions of equation (I.1) and systems (II.1), (III.1), and (IV.1) in M^+ and M^-. Investigation of classification of nonoscillatory solutions to dynamic equations and systems on time scales is related with the signs of their solutions.

In the first paper, we consider equation (I.1) which can be rewritten as a system of first order dynamic equations

$$\begin{cases} x^{\Delta}(t) = \left(\frac{1}{a(t)}\right)^{\frac{1}{\alpha}} |y(t)|^{\frac{1}{\alpha}} \, sgn y(t) \\ y^{\Delta}(t) = b(t) \left|x^{\sigma}(t)\right|^{\beta} sgn x^{\sigma}(t), \end{cases} \tag{2.1}$$

where $y = x^{[1]}$ defined in (I.12).

The following tables indicate the criteria for nonoscillatory solutions of (I.1) (or system (2.1)) in M^+ and M^- based on the integrals J_1, K_1, J_2, and K_2 defined as (I.8) - (I.11), respectively.

Table 2.1 Classification for (I.1) in M^+

$M^+_{B,B}$	$\neq \emptyset (= \emptyset)$	$J_1 < \infty$ and $K_1 < \infty$ ($J_1 = \infty$ or $K_1 = \infty$)
$M^+_{B,\infty}$	$\neq \emptyset (= \emptyset)$	$J_1 < \infty$ and $K_1 = \infty$ ($J_1 = \infty$ or $K_1 < \infty$)
$M^+_{\infty,B}$	$= \emptyset$	$J_1 < \infty$ or $K_1 = \infty$

Table 2.2 Classification for (I.1) in M^-

$M^-_{B,B}$	$\neq \emptyset \, (= \emptyset)$	$I < \infty$ and $K_2 < \infty$ ($I = \infty$ or $K_2 = \infty$)
$M^-_{0,B}$	$\neq \emptyset \, (= \emptyset)$	$K_2 < \infty \, (K_2 = \infty)$
$M^-_{B,0}$	$\neq \emptyset \, (= \emptyset)$	$I < \infty$ and $K_2 = \infty$ ($J_2 = \infty$ or $K_2 < \infty$ with $\beta \geq \alpha$)
$M^-_{0,0}$	$= \emptyset$	$I < \infty$ or $K_2 < \infty$ with $\beta \geq \alpha$

The second paper is concerned with system (II.1). Table 2.3 presents the classification of nonoscillatory solutions of (II.1) in M^+ by using the integrals Y_a, Z_b, J_α, and K_β defined by (II.3), (II.5), and (II.6), respectively. Here, we assume $Y_a = \infty$ and $Z_b < \infty$.

Table 2.3 Classification for (II.1) in M^+

$M^+_{B,0}$	$\neq \emptyset$	$J_\alpha < \infty$	$\alpha < \beta$ and $K_\beta < \infty$	$\alpha < \beta$, $\beta \geq 1$ and $J_\beta < \infty$	$\alpha \leq 1$ and $K_\alpha < \infty$
$M^+_{\infty,B}$	$\neq \emptyset$	$K_\beta < \infty$	$\alpha > \beta$ and $J_\alpha < \infty$	$\alpha \geq 1$ and $J_\beta < \infty$	
$M^+_{\infty,0}$	$\neq \emptyset$	$J_\alpha = \infty$ and $K_\beta < \infty$			
$M^+_{B,0}$	$= \emptyset$	$J_\alpha = \infty$	$\alpha > \beta$ and either $K_\beta = \infty$ or $J_\beta = \infty$	$\alpha \geq 1$ and $K_\alpha = \infty$	
$M^+_{\infty,B}$	$= \emptyset$	$K_\beta = \infty$	$\alpha < \beta$ and either $J_\alpha = \infty$ or $K_\alpha = \infty$	$\beta \leq 1$ and $J_\beta = \infty$	

We also prove the (non)existence of nonoscillatory solutions of (II.1) without any need of J_α and K_β. Therefore, we have the following table:

Table 2.4 Classification for (II.1) in M^+

$M^+_{B,B}$	$\neq \emptyset$	$Y_a < \infty$ and $Z_b < \infty$
$M^+_{B,0}$	$\neq \emptyset$	$Y_a < \infty$ and $Z_b < \infty$

In order to classify of nonoscillatory solutions of (III.1) and (IV.1), for simplicity we let

$$
\begin{aligned}
I_1 &= \int_{t_0}^{\infty} b(t) g\left(c_1 \int_t^{\infty} a(s)\Delta s\right)\Delta t, \quad I_2 = \int_{t_0}^{\infty} b(t) g\left(k - l\int_t^{\infty} a(s)\Delta s\right)\Delta t,\\
I_3 &= \int_{t_0}^{\infty} a(s) f\left(k \int_{t_0}^{s} b(\tau)\Delta \tau\right)\Delta s, \quad I_4 = \int_{t_0}^{\infty} a(t) f\left(k\int_t^{\infty} b(s)\Delta s\right)\Delta t,\\
I_5 &= \int_{t_0}^{\infty} a(s) f\left(d_1 + k\int_s^{\infty} b(u)\Delta u\right)\Delta s, \quad I_6 = \int_{t_0}^{\infty} b(s) g\left(c_1 \int_{t_0}^{\tau(s)} a(u)\Delta u\right)\Delta s,\\
I_7 &= \int_{t_0}^{\infty} a(t) f\left(\int_t^{\infty} b(s) g\left(c_1 \int_{t_0}^{s} a(u)\Delta u\right)\Delta s\right)\Delta t,\\
I_8 &= \int_{t_0}^{\infty} a(s) f\left(\int_{t_1}^{s} b(u) g\left(c_1 \int_u^{\infty} a(\lambda)\Delta \lambda\right)\Delta u\right)\Delta s,\\
I_9 &= \int_{t_0}^{\infty} b(s) g\left(c_1 \int_{t_0}^{s} a(u)\Delta u\right)\Delta s, \quad I_{10} = \int_{t_0}^{\infty} b(t) g\left(l \int_{t_0}^{\infty} a(s)\Delta s\right)\Delta t.
\end{aligned}
$$

In the third paper, we assume that $Y(t_0) = \infty$ and $Z(t_0) < \infty$ for the following tables, where $Y(t_0)$ and $Z(t_0)$ are defined in (III.2).

Table 2.5 Classification for (III.1) in M^-

$M^-_{0,B}$	$\neq \emptyset\,(=\emptyset)$	$I_1 < \infty (I_1 = \infty)$
$M^-_{B,B}$	$\neq \emptyset\,(=\emptyset)$	$I_2 < \infty\,(I_2 = \infty)$
$M^-_{B,\infty}$	$\neq \emptyset\,(=\emptyset)$	$I_3 < \infty\,(I_3 = \infty)$
$M^-_{0,\infty}$	$\neq \emptyset$	$I_4 < \infty$ and $I_1 = \infty$

In the fourth paper, Tables 2.6 - 2.8 present how we classify nonoscillatory solutions of the delay system (IV.1) in M^+ and M^- and of system (IV.32) in M^+. For Tables 2.6 and 2.8, it is assumed that $A(t_0) = \infty$ and $B(t_0) < \infty$, while for Table 2.7, we assume that $A(t_0) < \infty$ and $B(t_0) = \infty$, where $A(t_0)$ and $B(t_0)$ are defined in (IV.2). Finally, in the case $A(t_0) < \infty$, $B(t_0) < \infty$, and $I_5 < \infty$, we obtaine the existence of nonoscillatory solutions of (IV.1) in $M^+_{B,B}$.

Table 2.6 Classification for (IV.1) in M^+

$M^+_{B,0}$	$\neq \emptyset\,(=\emptyset)$	$I_4 < \infty (I_4 = \infty)$
$M^+_{\infty,B}$	$= \emptyset$	$I_6 = \infty$
$M^+_{\infty,0}$	$= \emptyset$	$I_7 = \infty$

Table 2.7 Classification for (IV.1) in M^-

$M^-_{B,\infty}$	$\neq \emptyset\,(= \emptyset)$	$I_3 < \infty (I_3 = \infty)$
$M^-_{0,B}$	$= \emptyset$	$I_1 = \infty$
$M^-_{B,B}$	$= \emptyset$	$I_2 = \infty$
$M^-_{0,\infty}$	$= \emptyset$	$I_8 = \infty$

Table 2.8 Classification for (IV.32) in M^+

$M^+_{\infty,B}$	$\neq \emptyset\,(= \emptyset)$	$I_9 < \infty\,(I_9 = \infty)$
$M^+_{\infty,0}$	$\neq \emptyset$	$I_4 = \infty$ and $I_{10} < \infty$

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Özkan had a lot of great experiences in pursuing his master's, and hence, decided to join Missouri University of Science and Technology for his Ph.D. with Dr. Elvan Akin in June 2011. After one year, Özkan got a graduate teaching assistant position and taught several undergraduate courses, such as Calculus for Engineers, Trigonometry, and Ordinary Differential Equations. During his Ph.D. he served for the Turkish Student Association as the vice - president in 2012 - 2014 and as the president in 2014 - 2015. He organized several cultural organizations and was put on the Honor's List by the Student Life organization in the fall of 2014. During his teaching, he earned a teaching excellence award in 2012 - 2013 and a most honorable mention award in 2014. Özkan attended several international and national conferences during his Ph.D. in order to widen his knowledge in his field of expertise. In May 2016, he received the degree of Doctor of Philisophy in Mathematics from Missouri University of Science and Technology.