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ESSAYS ON UNIT ROOT TESTING IN TIME SERIES

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ABSTRACT

Unit root tests are frequently employed by applied time series analysts to determine if the underlying model that generates an empirical process has a component that can be well-described by a random walk. More specifically, when the time series can be modeled using an autoregressive moving average ($ARMA$) process, such tests aim to determine if the autoregressive (AR) polynomial has one or more unit roots. The effect of economic shocks do not diminish with time when there is one or more unit roots in the AR polynomial, whereas the contribution of shocks decay geometrically when all the roots are outside the unit circle. This is one major reason for economists' interest in unit root tests. Unit roots processes are also useful in modeling seasonal time series, where the autoregressive polynomial has a factor of the form $(1-z^s)$, and s is the period of the season. Such roots are called seasonal unit roots. Techniques for testing the unit roots have been developed by many researchers since late 1970s. Most such tests assume that the errors (shocks) are independent or weakly dependent. Only a few tests allow conditionally heteroskedastic error structures, such as Generalized Autoregressive Conditionally Heteroskedastic ($GARCH$) error. And only a single test is available for testing multiple unit roots. In this dissertation, three papers are presented. Paper I deals with developing bootstrap-based tests for multiple unit roots; Paper II extends a bootstrap-based unit root test to higher order autoregressive process with conditionally heteroscedastic error; and Paper III extends a currently available seasonal unit root test to a bootstrap-based one while at the same time relaxing the assumption of weakly dependent shocks to include conditional heteroscedasticity in the error structure.

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1. INTRODUCTION

Time series analysis is an important and challenging field of research in statistical science. It has broad applications in many areas, such as economics, finance, engineering, and bio-medical sciences. Extensive work has been done on aspects of estimation and forecasting of time series. The accuracy of the scientific inferences based on such estimation and forecasting is affected significantly by how well the nature of the underlying process that governs an empirical time series is identified. In particular, determining whether a process is stationary or not plays an important role in time series analysis. Stationarity in its weakest sense implies that the first and second moments of a time series remains constant over time. In such a situation, the future will behave very similar to the past and reliable forecasts based on past data can be easily obtained. Instances where an empirical time series shows behavior patterns that suggest non-stationarity, however, is not that uncommon. For example, certain stock prices show “random walk” type behavior. Rather than make regular crossings of its mean value, these empirical processes make extended sojourns above and below the mean. Such behavior, which exhibit one very common type of non-stationarity, can be modeled by what is known as an integrated autoregressive moving average (*ARIMA*) process and are commonly known as “unit root processes” because the autoregressive polynomial associated with the process contains roots that are on the unit circle. Testing for the presence of one or more unit roots, therefore, plays a central role in empirical time series analysis, especially in areas such as economics and finance. Unit roots processes are also useful in modeling seasonal time series, where the autoregressive polynomial has a factor of the form $(1-z^s)$ where s is the period of the season. Such roots are called seasonal unit roots. Testing for seasonal unit roots is also a widely used practice when modeling economic data. The work presented in this dissertation concerns unit root testing using the bootstrap resampling technique under three different scenarios, namely testing for multiple unit roots, testing for a single unit root, and testing for a seasonal unit root, the latter two under the assumption of a conditionally heteroskedastic error structure.

To facilitate a clear understanding of unit root testing, and to illustrate the contribution of the work presented herein, fundamental concepts and definitions concerning stationary time series, nonstationary time series, and unit roots, are provided in Section 1.1. Most unit root testing procedures are developed for empirical time series with independent or weakly dependent errors, while two of the methods developed in this thesis extend this to errors with conditional heteroskedastic volatilities. The autoregressive conditional heteroscedastic ($ARCH$) and generalized autoregressive conditional heteroscedastic ($GARCH$) models are defined in Section 1.2. The historical background of the development of unit root testing procedures, including the utilization of the bootstrap resampling approach for unit root testing, is reviewed in Section 1.3. Although a large amount of work on unit root testing procedures are available in the literature, improvements are still possible and this area presents many unresolved issues one can work on. A few of these topics are tackled in the following work. Section 1.4 briefly describes the outline and organization of the remaining portion of the dissertation.

1.1. UNIT ROOT PROCESSES

The concepts of stochastic processes, time series, stationary time series, nonstationary time series, autoregressive moving average ($\boldsymbol{ARMA}(\boldsymbol{p},\boldsymbol{q})$) processes, and unit roots, are introduced in this section.

Definition 1.1.1. *Stochastic Process: A stochastic process is a family of random variables* $\{X_t, t \in T\}$ *defined on a probability space* $(\Omega, \mathcal{F}, \mathcal{P})$*, where* T *denotes an index set, which is usually a set of real numbers. If* T *denotes a set of points in time, then* $\{X_t, t \in T\}$ *is called a time series. In particular, if* $\{T \subseteq \mathbb{Z}\}$*, then* $\{X_t, t \in T\}$ *is called a discrete time series.*

Note that $\{X_t\}_{t \in T}$ is sometimes used in place of $\{X_t, t \in T\}$ to denote a time series.

Definition 1.1.2. *Stationary Time Series: The time series* $\{X_t, t \in \mathbb{Z}\}$*, is said to be stationary if for all* $t, r, s \in \mathbb{Z}$ *,*

(i) $E[|X_t|^2] < \infty$*;*

(ii) $E[X_t] = m$*;*

(iii) $Cov(X_r\ ,\ X_s) = \gamma_X(r, s) = \gamma_X(r + t, s + t)$.

Such stationarity is sometimes referred as weak stationarity, covariance stationarity, stationarity in the wide sense, or second-order stationarity. Otherwise, the time series $\{X_t, t \in \mathbb{Z}\}$ is nonstationary.

Definition 1.1.3. *Autoregressive Moving Average* $(ARMA(p,q))$ *Process: A real-valued time series* $\{X_t\}_{t\in\mathbb{Z}}$ *is said to be an autoregressive moving average (*$ARMA(p,q)$*) process with mean* μ *if it is stationary and satisfies*

$$\Phi(B)(X_t - \mu) = \theta(B)\varepsilon_t, t \in \mathbb{Z},$$

where $$\Phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \ldots - \phi_p z^p$$

and $$\theta(z) = 1 + \theta_1 z + \theta_2 z^2 \ldots + \theta_q z^q$$

are autoregressive and moving-average polynomials of orders p *and* q*, respectively, with no common roots;* $\{\varepsilon_t\}_{t\in Z}$ *is a white noise error (innovations) process with zero-mean and constant variance* σ^2*;* $\mu = E(X_t)$ *for all* t*;* B *is the back-shift operator defined such that* $B^k X_t = X_{t-k}$ *for all* $k \in \mathbb{N}$*, and* $B^0 X_t = X_t$.

If $p = 0$*,* $\{X_t\}_{t\in\mathbb{Z}}$ *is called a pure moving average process of order* q *(*$MA(q)$*), and if* $q = 0$*, the time series is termed a pure autoregressive process of order* p $(AR(p))$.

Definition 1.1.4. *Unit Root Processes: Given a discrete time series,* $\{X_t\}_{t\in Z}$ *, that can be represented by the ARMA (p, q) model*

$$\Phi(B)(X_t - \mu) = \theta(B)\varepsilon_t, t \in \mathbb{Z},$$

suppose that $m = 1$ *is one of the roots of the characteristic equation based on the autoregressive polynomial given by*

$$\Phi(m) = 1 - \phi_1 m - \phi_2 m^2 - \ldots - \phi_p m^p = 0.$$

Then the time series is said to have a unit root. Moreover, if m=1 is a root of multiplicity r*, then there are* r *unit roots associated with the time series. A time series with one or more unit roots is sometimes called a unit root process.*

A time series is nonstationary as long as one or more unit roots exist. If there is only one unit root, and all the other roots of the characteristic equation lie outside the unit circle (i.e. $|m| > 1$), then the first difference, $\left\{X_t^{(1)} = X_t - X_{t-1}: t \in \mathbb{Z}\right\}$, of the process will be stationary. Similarly, in the presence of two unit roots, the second difference, $\left\{X_t^{(2)} = X_t^{(1)} - X_{t-1}^{(1)}: t \in \mathbb{Z}\right\}$, would be stationary.

Time series analysts routinely employ differencing to achieve stationarity, after which they can utilize the numerous estimation and forecasting techniques developed for stationary time series. If no unit roots are present, however, then differencing the stationary *ARMA* (*p*, *q*) process $\Phi(B)(X_t - \mu) = \theta(B)\varepsilon_t, t \in \mathbb{Z}$, would result in a new process $\{Y_t: t \in \mathbb{Z}\}$ that satisfies

$$\Phi(B)(Y_t) = (1 - B)\theta(B)\varepsilon_t, t \in \mathbb{Z},$$

which has a unit root in the moving average polynomial $(1 - m)\theta(m)$. Not only does this result in a more complicated time series model, the unit root in the *MA* polynomial makes the time series non-invertible. Inevitability allows the representation of the time series by an infinite autoregressive process:

$$X_t = \sum_{j=1}^{\infty} \phi_j X_{t-j} + \varepsilon_t , t \in \mathbb{Z} ,$$

which in turn allows the approximation of the time series by a finite autoregressive process $X_t = \sum_{j=1}^{p} \phi_j X_{t-j} + \varepsilon_t , t \in \mathbb{Z}$, where p is chosen appropriately. Therefore, the testing of unit roots is crucial for determining if the time series needs to be differenced and if so, the number of times such differences should be taken.

1.2. $ARCH$ AND $GARCH$ MODELS

Many unit root testing procedures have been developed for empirical time series with independent errors or weakly dependent errors. Note that the term weakly dependent is used to describe the dependence structure of discrete time series $\{\varepsilon_t : t \in \mathbb{Z}\}$ that are covariance stationary and have the property that $\mathrm{Cov}(\varepsilon_t, \varepsilon_{t+h}) \to 0$ as $h \to \infty$, for all $t \in \mathbb{Z}$. Another assumption is that the conditional variance of ε_t given the past values $\{\varepsilon_j, \ j < t\}$ is constant for all t. Not all error processes associated with time series, however, possess this homoscedastic property. In particular, some time series have errors whose conditional variance given the past depends on the variance of the errors in the recent past. In financial literature, these changes in variances are associated with changing market volatility. Thus this phenomenon is referred to as conditionally heteroskedastic volatility. The autoregressive conditional heteroskedastic (ARCH) and generalized autoregressive conditional heteroskedastic (GARCH) models were developed to describe the structure of such errors.

The $ARCH$ models were first proposed by Engle (1982), and the $GARCH$ models were developed by Bollerslev (1986). These models have numerous applications in econometrics and financial fields. In particular, they are employed to model the empirical time series whose error terms at any time point may have a variance that depends on past volatility. Specifically, $ARCH$ models assume the variance of the current error term or innovation to be a function of the squares of the previous error terms; $GARCH$ models assume the variance of the current error term to be a linear combination of the squares of the previous error terms and the variance of the previous error terms. The definitions of $ARCH(q)$ and $GARCH(p,q)$ are given below.

Definition 1.2.1. *Autoregressive Conditional Heteroscedastic Process of Order* q: *A time series* $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ *is said to be an autoregressive conditional heteroskedastic process of order* q, $ARCH(q)$, *if*

$(i)\ var(\varepsilon_t) = h_t\ , t \in \mathbb{Z};$

$(ii)\ \varepsilon_t = \sqrt{h_t}\,\eta_t\ ,$ *where* $\eta_t \sim iid\ (0,1)$. *In addition, it is usually assumed that* $E[\eta_t^3] = 0, E[\eta_t^4] < \kappa < \infty, t \in \mathbb{Z};$

$(iii)\ h_t\ =\ \omega + \alpha_1\varepsilon_{t-1}^2 + \alpha_2\varepsilon_{t-2}^2 + \cdots + \alpha_q\varepsilon_{t-q}^2 = \omega + \sum_{i=1}^{q}\alpha_i\varepsilon_{t-i}^2\ ,$

where $\omega > 0,\ \ \alpha_i \geq 0,\ \ t \in \mathbb{Z}.$

Note that in $ARCH(q)$ model, the error variance h_t is actually a moving average (MA) process of order q. The coefficients of the $ARCH(q)$ model can be estimated using ordinary least squares (LSE). And the order or the lag length q of the $ARCH$ errors can be tested by a methodology proposed by Engle (1982), which is based on the score test or the Lagrange multiplier test.

Definition 1.2.2. *Generalized Autoregressive Conditional Heteroscedastic Process of Orders* p *And* q: *A time series* $\{\varepsilon_t\}_{t\in\mathbb{Z}}$ *is said to be a generalized autoregressive conditional heteroscedastic process of orders p and* q, $GARCH(p,q)$, *if*

(i) $var(\varepsilon_t) = h_t$, $t \in \mathbb{Z}$;

(ii) $\varepsilon_t = \sqrt{h_t}\,\eta_t$, *where* $\eta_t \sim iid\ (0,1)$. *Usually, the assumptions that* $E[\eta_t^3] = 0, E[\eta_t^4] < \kappa < \infty, t \in \mathbb{Z}$ *are also made.*

$$(iii)\ h_t = \omega + \alpha_1\varepsilon_{t-1}^2 + \cdots + \alpha_q\varepsilon_{t-q}^2 + \beta_1 h_{t-1} + \cdots + \beta_p h_{t-p}$$

$$= \omega + \sum_{i=1}^{q}\alpha_i\varepsilon_{t-i}^2 + \sum_{j=1}^{p}\beta_j\, h_{t-j}\ ,$$

where $\omega > 0,\ \alpha_i \geq 0,\ \beta_j \geq 0,\ t \in \mathbb{Z}$.

It's obvious that in the $GARCH(p,q)$ model, the error variance h_t is an autoregressive moving average process of orders p and q, $ARMA(p,q)$. More precisely, p is the order of the $GARCH$ terms h_t and q is the order of the $ARCH$ terms ε_t^2.

In general, a good test for testing the heteroscedasticity in econometrics is the White test. Additional tests dealing with $ARCH$ and $GARCH$ errors have also been developed. Since many financial time series are known to have heteroskedastic volatilities, a specific $ARCH$ or $GARCH$ model can be applied to those time series during the unit root testing procedure.

1.3. TESTING FOR UNIT ROOTS

A unit root test is not only used to determine whether or not a time series $\{X_t\}_{t\in Z}$ need to be differenced to obtain stationarity, but also has important applications in certain economic hypotheses. For example, Altonji and Ashenfelter (1980) used unit root tests to test an equilibrium hypothesis for wage movements; Nelson and Plosser (1982) applied unit root tests to describe the effect of monetary disturbances on macroeconomic series; Meese and Singleton (1982) explained the importance of unit root testing in the theory of linearized expectations by applying unit root tests to exchange rates. In addition, unit root tests can indicate if the shocks (ε_t) to an economic system have a permanent impact on the future econometric pattern. Specifically, if at least one unit root exists, then each shock does have a permanent effect on the future forecasts; otherwise, the effect is transitory and could be ignored in the long run. For more details, see J. Franke et al. (2010, p. 244). As a result, during the passing decades many researchers have worked in this field and developed a wide variety of unit root tests.

Among all the tests, the most commonly used unit root test for time series was introduced by Dickey and Fuller (1979) and are referred to as the Dickey-Fuller (DF) test. This test was developed for the first order autoregressive processes. Said and Dickey (1984) generalized the Dickey-Fuller test and applied it to $ARMA$ models of unknown orders. Their test is called Augmented Dickey-Fuller (ADF) test. Phillips (1987) and Phillips and Perron (1988) also developed their own tests based on DF and ADF tests. These tests assumed the errors are either independent and identically distributed ($i.i.d.$) or weakly dependent. All the above tests are actually not applicable to many empirical processes that arise in financial and economic fields. More specifically, such tests neglect any underlying volatility structure of the errors. As a remedy to this situation, Ling and Li (1998, 2003), Ling, Li, and McAleer (2003) developed unit root tests under a Generalized Autoregressive Conditional Heteroskedastic ($GARCH$) error structure.

While these asymptotic distribution-based tests were an improvement to existing tests as far as taking care of the underlying volatility structure is concerned, they share the serious size distortion and low power weaknesses that were present in DF, ADF, and related tests. In order to mitigate the size distortion and low power issue, common to most asymptotic tests, the bootstrap resampling procedure was introduced into $AR(1)$ unit root testing by Basawa et al. (1991). Ferretti and Romo (1996) and Datta (1996) also made their contributions to such tests. Moreover, if the bootstrap is applied to residuals obtained using a sieve which is an approximation of an infinite dimensional model by a sequence of finite dimensional models, it's called a sieve bootstrap procedure. The sieve bootstrap was first introduced by Bühlmann (1997). Chang and Park (2003) considered a sieve bootstrap for a unit root test in models driven by general linear processes. Their sieve bootstrap-based ADF unit root tests are shown to be consistent under very general conditions and the asymptotic validity of such tests was established theoretically. Significant improvements on finite sample performance of these tests as compared to asymptotic tests were demonstrated through Monte Carlo simulations. Until recently, however, the bootstrap-based unit root tests were available only for processes with conditionally homoscedastic error.

Gospodinov and Tao (2011) were the first to develop a bootstrap approach to unit root tests for autoregressive (AR) time series with $GARCH(1,1)$ errors and described how this procedure can be carried out for $AR(1)$ processes. Zhong and Samaranayake (2014) adapted Gospodinov-Tao method to unit root tests for general $AR(p)$ processes. Their simulation results show that the proposed method has good size and power properties for higher order autoregressive processes even when the second largest root is close to unity. A more detailed exposition of this work is presented in this thesis.

Although the above tests focus on the regular unit root testing in non-seasonal time series, a large portion of financial and economic time series possess substantial seasonality. Therefore, Box and Jenkins (1970) introduced their famous seasonal time series models based on an autoregressive moving average ($ARMA$) formulation. They and many other time series researchers influenced by their work used a seasonal differencing filter to obtain stationarity. Their formulation assumed that seasonal unit roots may exist in seasonal time series through a factor of the form $(1-Z^s)$ present in the autoregressive polynomial, where s denotes the period of the season. Consequently, unit root tests for seasonal time series were developed. The Dickey-Hasza-Fuller (DHF) test and $HEGY$ (1990) test are two of them. The DHF test was proposed by Dickey, Hasza and Fuller (1984) and the $HEGY$ test was proposed by Hylleberg, Engle, Granger, and Yoo, (1990). The above seasonal unit root tests assume $i.i.d.$ errors, and they all have serious size distortion and low power problems. To solve these problems, Psaradakis (2000) implemented a bootstrap-based unit root test for pure seasonal time series with independent errors and gained higher powers than the DHF test. Psaradakis (2001) was the first to introduce the sieve bootstrap-based unit root test to non-seasonal time series with weakly dependent errors. Chang and Park (2003) also proposed their sieve bootstrap versions of the ADF tests for non-seasonal unit roots. Psaradakis (2001) method is called difference based because it calculates the residuals by fitting an $AR(p)$ model to the differenced non-seasonal time series, whereas the method proposed by Palm, Smeekes, and Urbain (2008) is acknowledged as residual based test because it computes the residuals by fitting the DF regression model to the differenced series. However, like most of the tests developed for testing non-seasonal unit roots, these tests didn't consider any underlying volatility structure of the innovations in seasonal time series.

All the tests reviewed above use the existence of a single unit root as the null hypothesis. And they all assume that the series have at most one unit root. If there are more than one unit root, the suggestion is to apply a sequence of Dickey-Fuller tests to the raw series and the differenced series repeatedly. Dickey and Pantula (1987) and Sen (1985) showed that if there are actually two unit roots, then the method of applying Dickey-Fuller tests to the raw and the differenced series repeatedly is not valid. As matter of fact, the Dickey-Fuller test is based on the assumption of at most one unit root, therefore, at least the first few tests in this sequence cannot be theoretically justified. In order to solve these problems and perform tests on a sound theoretical foundation, Dickey and Pantula (1987) proposed a strategy of carrying out the sequence tests in a different order. This strategy is recognized for its high power.

1.4. OUTLINE AND ORGANIZATION

As explained in Section 1.3, many procedures are available in the field of unit root testing. However, there are still new topics and difficult issues to work on, for example, the size distortion and low power problems of time series with conditional heteroskedastic errors, and the unit root testing of time series that exhibit both seasonal behavior and conditional heteroskedastic errors. Even the only multiple unit root test proposed by Dickey and Pantula has some weaknesses, for example, the tables used to obtain the critical values in Dickey-Pantula tests are not extensive; the Dickey-Pantula method requires first testing H_p where p is the order of the autoregressive process, even when it is reasonable to assume that the number of unit roots is less than p.

This dissertation focuses on some of these topics and issues. The remaining portion of the dissertation is organized in the form of a series of three papers. Paper I is about developing a bootstrap version of Dickey-Pantula test for multiple unit roots, with special attention paid to the case of two unit roots. In Paper II, a bootstrap-based unit root test for higher order autoregressive process where the error process $\{\varepsilon_t : t \in \mathbb{Z}\}$ shows conditional heteroscedasticity is presented. Paper III accommodates $GARCH(1,1)$ errors in seasonal time series and proposes a bootstrap-based seasonal unit root test by extending the DHF test and using the residual-based method. Conclusions are presented after the three papers followed by the bibliography.

PAPER

I. A BOOTSTRAP-BASED TEST FOR MULTIPLE UNIT ROOTS

ABSTRACT

A bootstrap-based test for determining if an autoregressive process has two unit roots is introduced. This contrasts with the standard procedure of determining the number of unit roots by first conducting a unit root test, then differencing the series if the null hypothesis of a unit root process is not rejected and repeating the unit root test on the differenced series. Specifically, we develop a bootstrap test based on a test proposed by Dickey and Pantula in 1987. A Monte Carlo simulation study is carried out to investigate the finite sample properties of the proposed test. Results show that the bootstrap-based Dickey-Pantula test has reasonable properties for moderate samples.

Keywords: Integrated Processes; Unit root Tests; Multiple Unit Roots; Bootstrap; $ARIMA$

1. INTRODUCTION

When modeling empirical time series, it is sometimes necessary to perform unit root tests. One reason for carrying out such tests is to determine if the time series needs differencing to obtain stationarity. More importantly, unit root tests have been applied in the investigation of certain economic hypotheses. For example, Altonji and Ashenfelter (1980) used unit root tests to test an equilibrium hypothesis for wage movements; Nelson and Plosser (1982) applied unit root tests to describe the effect of monetary disturbances on macroeconomic series; Meese and Singleton (1982) explained the importance of unit root testing in the theory of linearized expectations by applying unit root tests to exchange rates. Also, over the last three decades, the unit root tests have drawn more and more attention in many research fields related to economics. In particular, such tests can imply whether or not the shocks to an economic system have a permanent effect on the future econometric pattern. Specifically, if at least one unit root exists, then each shock does have a permanent impact on the future forecasts; otherwise, the impact could be negligible in the long run. For more details, see J. Franke et al. (2010, p. 244).

Given a discrete time series, $\{X_t\}_{t\in Z}$, that can be represented by the *ARMA* (*p, q*) model

$$\Phi(B)(X_{\mathrm{t}} - \mu) = \theta(B)\varepsilon_{\mathrm{t}}, t \in \mathbb{Z} \ ,$$

suppose that $m = 1$ is one of the roots of the characteristic equation of the $AR(p)$ polynomial given by

$$\Phi(m) = 1 - \phi_1 m - \phi_2 m^2 - \ldots - \phi_p m^p = 0.$$

Then the time series is said to have a unit root. Moreover, if $m=1$ is a root of multiplicity r, then there are r unit roots associated with the time series. A time series is nonstationary as long as one or more unit roots exist.

Practically, the existence of unit roots is often suspected by visual inspection of the autocorrelation function (ACF) and data plots. As long as the ACF decays slowly, the time series should be considered having at least one unit root and the operation of differencing the time series may be performed repeatedly to obtain a stationary time series. Many statistical tests for unit roots, including what is proposed herein, are based on autoregression tests of linear dependence. Such tests simply mitigate the subjectivity of visual inspection of autoregression plots; compared to visual inspection, these tests are more helpful in deciding close-call situations.

The most commonly used unit root tests were developed by Dickey and Fuller (1979) and sometimes referred to as Dickey-Fuller (*DF*) tests. Dickey-Fuller tests are based on first-order auto-regressions, that is, an autoregressive model of order 1 ($AR(1)$) is assumed. In addition, the errors of the model are assumed to be independent and identically distributed ($i.i.d.$). However, in general, a time series can be a higher order $AR(p)$ with $p > 1$ and usually unknown. Phillips (1987) and Phillips and Perron (1988) modified the Dickey-Fuller tests to be applicable to the case where the errors are weakly dependent rather than $i.i.d.$. Such a situation arises when the underlying process is $AR(p)$ or $ARMA(p,q)$ but only an $AR(1)$ model is fitted. Said and Dickey (1984) generalized the DF test to accommodate $ARMA$ processes by using autoregressions with lagged differences. They showed that these tests are valid for all finite $ARMA$ procedures with unknown orders if we increase the number of lagged differences appropriately as the sample size grows. These unit root tests are more useful than the tests that assume $p = 1$ and is an alternative to the Phillips and Perron test. However, some researchers such as Leybourne and Newbold (1999) found that these unit root tests have serious size distortion and low power issues in finite samples, especially when the model has a moving average component. Subsequently, bootstrap and sieve bootstrap methods were introduced to improve the finite sample performance of some of the above tests.

Basawa et al. (1991) applied a bootstrap process to $AR(1)$ unit root tests and showed that the unit root must be imposed on the generation of bootstrap samples to achieve consistency of the bootstrap unit root tests. Ferretti and Romo (1996) and Datta (1996) also made their contributions to such tests. If the bootstrap procedure is based on a sieve which is an approximation of an infinite dimensional model by a sequence of finite dimensional models, we get the sieve bootstrap procedure introduced by B$\ddot{u}$hlmann (1997). Specifically, we can approximate any linear process such as AR, MA or $ARMA$ by a finite $AR(\hat{p})$ where $\hat{p}$ increases with the sample size; and resample from the residuals of the approximated auto-regressions. Chang and Park (2003) considered a sieve bootstrap for the test of a unit root in models driven by general linear processes. Their bootstrapped versions of ADF unit root tests are shown to be consistent under very general conditions and the asymptotic validity of such tests were established. Significant improvements on finite sample performance of the tests are also established by Monte Carlo simulations.

On the other hand, all of the above tests assumed that the series have at most one unit root. If there are more than one unit root, a sequence of Dickey-Fuller or Augmented Dickey-Fuller type tests may be applied to the raw series and the differenced series repeatedly. Intuitively, we expect that if there are more than one unit root, the test for one unit root will strongly indicate that the process needs to be differenced. Hence we expect that the null hypothesis of one unit root will be rejected (and the hypothesis of no unit root will be favored) less than 5% of the time when there are more than one unit root present. However, a simulation study done by Dickey and Pantula (1987) doesn't support that intuition. Moreover, Sen (1985) showed that if there are actually two unit roots, then the method of applying Dickey-Fuller tests on the raw and the differenced series repeatedly is not valid. As matter of fact, since the Dickey-Fuller test is based on the assumption of at most one unit root, at least the first few tests in this sequence cannot be theoretically justified. In order to mitigate these problems and perform tests based on a sound theoretical foundation, Dickey and Pantula (1987) proposed a strategy of performing the sequence tests in a different order. In their paper, they propose a method for sequential testing of unit roots. These tests compare a null hypothesis of d unit roots with an alternative of $d-1$ unit roots. Specifically, one starts with the largest d to test and work down if the null hypothesis of having d unit roots is rejected. The sequential testing procedure stops when a null hypothesis cannot be rejected. This test is recognized for its simplicity (it uses existing τ tables given in Fuller (1976)) and high power. However, the τ tables used to obtain the critical values in Dickey-Pantula tests are not complete. For example, these tables include column *df* (n), H_0, H_1, *probabilities*. But only certain degrees of freedoms are given there. In addition, Dickey-Pantula method requires first testing H_p where p is the order of the autoregressive process, even when it is reasonable to assume that the number of unit roots is less than p.

Hence, we adjust Dickey-Pantula tests for multiple unit roots by initiating the sequential testing process with a value of $d \leq p$. In addition, a bootstrap procedure is used to calculate the critical values for Dickey-Pantula multiple unit root tests. That is, our bootstrap-version Dickey-Pantula tests for multiple unit roots are not only a better alternative to the standard procedure of determining the number of unit roots by first performing a unit root test, then differencing the raw time series if the null hypothesis of a unit root process is not rejected and repeating the same unit root test on the differenced series, but also more practical compared to the original Dickey-Pantula tests because it does not depend on a limited number of tabulated critical values.

The rest of the paper is organized as follows. Section 2 introduces Dickey and Pantulas' tests and presents their asymptotic theories. The bootstrap version of their tests is described in Section 3. In Section 4, the Monte Carlo studies for this method are presented. The conclusion is given in Section 5.

2. DICKEY-PANTULA'S TESTS AND THEIR LIMITING DISTRIBUTIONS

Assume the time series $\{X_t\}$ satisfy

$$X_t = \sum_{j=1}^{p} \phi_j X_{t-j} + e_{t,} \;\; t \in \mathbb{Z}, \tag{2.1}$$

where $\{e_t\}$ is a sequence of $i.i.d.$ random variables with mean 0 and variance 1. To make the presentation simple, p is restricted to 3. Extensions for cases where $p > 3$ follow naturally from what is presented here. Let m_1, m_2 and m_3 represent the roots of the characteristic equation

$$m^3 - \phi_1 m^2 - \phi_1 m - \phi_1 = 0. \tag{2.2}$$

Assume that $|m_1| \geq |m_2| \geq |m_3|$. Consider the following four hypotheses: H_0: $|m_1| < 1$; H_1: $m_1 = 1$, $|m_2| < 1$; H_2: $m_1 = 1$, $m_2 = 1$, $|m_3| < 1$; H_3: $m_1 = m_2 = m_3 = 1$. That is, under H_d, $d = 0, 1, 2, 3$, there are d unit roots. After a re-parameterization of model (2.1), we can write

$$W_t = \theta_1 X_{t-1} + \theta_2 Y_{t-1} + \theta_3 Z_{t-1} + e_t, \tag{2.3}$$

where $Y_t = X_t - X_{t-1}$, $Z_t = Y_t - Y_{t-1}$, $W_t = Z_t - Z_{t-1}$, and the $H_d's$ are transformed into: H_3: $\theta_1 = \theta_2 = \theta_3 = 0$; H_2: $\theta_1 = \theta_2 = 0$, $\theta_3 < 0$; H_1: $\theta_1 = 0$, $\theta_2 < 0$, $\theta_3 < 0$; H_0:

$\theta_1 < 0,\ \theta_2 < 0,\ \theta_3 < 0$. The reparameterization is useful because now we can use the usual regression tests for the thetas in (2.3).

The procedure proceeds as follows: perform a regression of W_t over X_{t-1}, Y_{t-1} and Z_{t-1} to get the least squares estimates $\hat{\theta}_i$ and the corresponding t-statistics $t_{i,n}(3)$, $i = 1, 2, 3,$ where $t_{i,n}(p) = t_{i,n} = \frac{\hat{\theta}_i}{s(\hat{\theta}_i)}$, n denotes the sample size, and $s(\hat{\theta}_i)$ is the standard error of $\hat{\theta}_i$ obtained from the regression.

Now, a sequential testing procedure is considered. We test the null hypothesis H_3 against the alternative hypothesis H_2 first by considering the t-statistic $t^*_{3,n}(3)$ obtained by regression of $W_t$on Z_{t-1}. Then, we can test the null hypothesis H_2 against the alternative hypothesis H_1 by considering the t-statistic $t^*_{2,n}(3)$ obtained by regression of W_t on Y_{t-1} and Z_{t-1}. Moreover, let $t^*_{1,n}(3) = t_{1,n}(3)$.

Pantula (1986) proved that the asymptotic distributions of the $t^*_{d,n}$ statistics under H_d for $d = 1, 2, 3$ can be characterized as the distribution of certain functional of a standard Brownian motion. In summary, Dickey and Pantula proposed the following sequential procedure for testing the hypotheses:

1. Reject H_3 of three unit roots and go to Step 2 if $t^*_{3,n}(3) \leq \hat{\tau}_{n,\alpha}$, where $\hat{\tau}_{n,\alpha}$ is given in Fuller (1976).
2. Reject H_2 of two unit roots and go to Step 3 if $t^*_{2,n}(3) \leq \hat{\tau}_{n,\alpha}$, where $\hat{\tau}_{n,\alpha}$ is given in Fuller (1976).
3. Reject H_1 of one unit root in favor of H_0 of no unit roots if $t^*_{1,n}(3) \leq \hat{\tau}_{n,\alpha}$, where $\hat{\tau}_{n,\alpha}$ is given in Fuller (1976).

Note that these critical points are not available for all significance levels and sample sizes. Therefore, the bootstrap-based critical points may be an alternative.

3. THE BOOTSTRAP DICKEY-PANTULA TESTS

In this section, we modify the Dickey-Pantula test in two ways. First we obtain the critical points using the bootstrap. Second, we observe that the Dickey-Pantula method requires first testing H_p where p is the order of the autoregressive process, even when it is reasonable to assume that the number of unit roots is less than p. Therefore we modify their method to accommodate such cases by starting the sequential testing at a value of $d \leq p$.

Let's assume p = 3 for the simplicity of explanation, and the maximum number of unit roots, d, is assumed to be 2. Extension to other values of p and d can be done quite easily.

Define $\{W_t\}$ as the third difference of $\{X_t\}$, $\{Z_t\}$ as the second difference of $\{X_t\}$, $\{Y_t\}$ as the first difference of $\{X_t\}$, where $t = 1, 2, \ldots, n$. Then the transformed model is the same as Equation (2.3): $W_t = \theta_1 X_{t-1} + \theta_2 Y_{t-1} + \theta_3 Z_{t-1} + e_t$. Since we assume d = 2 in this section, the hypotheses to test are: H_2: $\theta_1 = \theta_2 = 0,\ \theta_3 < 0$; H_1: $\theta_1 = 0,\ \theta_2 < 0,\ \theta_3 < 0$; H_0: $\theta_1 < 0,\ \theta_2 < 0,\ \theta_3 < 0$, where $\theta_i, i = 1, 2, 3$, are the coefficients used in the model (2.3). More precisely, $H_i,\ i = 1, 2, 3,$ represents the case where i unit roots exist in the time series under consideration. The three roots of the characteristic polynomial associated with the time series are denoted by $m_{i,} i = 1, 2, 3$. Now, to test $H_2\ vs.\ H_1\ or\ H_0$, we proceed as follows.

1) To get $t^{*}_{2,n}(3)$, fit the regression model: $W_t = \theta_2 Y_{t-1} + \theta_3 Z_{t-1} + e_t$. Then let $t_2 = t^{*}_{2,n}(3) = \frac{\hat{\theta}_2}{s(\hat{\theta}_2)}$.

2) Now, fit the model (under the null hypothesis): $W_t = \theta_3 Z_{t-1} + e_t$. Obtain the centered residuals: $Res(i) = \hat{e}_i - \bar{\hat{e}}$, where $\hat{e}_i = W_i - \widehat{W}_i$, $i = 1, 2, \ldots, n$; and $\bar{\hat{e}} = \frac{1}{n}\sum_{i=1}^{n} \hat{e}_i$.

3) Sample with replacement from all the centered residuals to obtain a bootstrap sample of errors, $\{e_t^{bt}\}_{t=1}^{n+50}$.

4) Then we can compute the bootstrap samples: $\{Z_t^{bt}\}_{t=1}^{n+50}, \{W_t^{bt}\}_{t=1}^{n+50}, \{Y_t^{bt}\}_{t=1}^{n+50}, \{X_t^{bt}\}_{t=1}^{n+50}$ by using the recursive equations: $Z_0^{bt} = 0$; $Z_t^{bt} = \left(1 + \hat{\theta}_3\right) Z_{t-1}^{bt} + e_t^{bt}, t = 1, 2, \ldots, n + 50$; $W_t^{bt} = Z_t^{bt} - Z_{t-1}^{bt}$; $Y_t^{bt} = \sum_{j=1}^{t} Z_j^{bt}$; $X_t^{bt} = \sum_{j=1}^{t} Y_j^{bt}$.

5) Carry out the regression defined in Step 1) with the bootstrap samples obtained in Step 4) and calculate the bootstrap t_2-statistic, t_2^{bt}, as in Step 1).

6) Repeat Step 2) ~ Step 5) B times (e.g., 2,000 times) and determine the critical value τ_2^{bt} which is the 5th percentile of the B t_2^{bt} values.

7) If the t_2 from Step 1) is less than τ_2^{bt}, then reject the hypothesis of two unit roots and let rej_2 equal 1; otherwise, don't reject and let rej_2 equal 0.

8) Repeat Step 1) ~ Step 7), M times (e.g., 2000 times) and calculate the significance level (empirical size) or the power of the test as

$$significance\ level\ (\text{or}\ power) = \frac{\sum rej_2}{M}.$$

The above procedure can be modified to test $H_1\ vs.\ H_0$ as well.

i) To get $t^*_{1,n}(3)$, fit the regression model: $W_t = \theta_1 X_{t-1} + \theta_2 Y_{t-1} + \theta_3 Z_{t-1} + e_t$. Then let $t_1 = t^*_{1,n}(3) = \frac{\hat{\theta}_1}{s(\hat{\theta}_1)}$.

ii) Now, fit the model (under the null hypothesis): $W_t = \theta_2 Y_{t-1} + \theta_3 Z_{t-1} + e_t$. Obtain all the centered residuals: $Res(i) = \hat{e}_i - \bar{\hat{e}}$, where $\hat{e}_i = W_i - \widehat{W}_i$, $i = 1, 2, \ldots, n$; and $\bar{\hat{e}} = \sum_{i=1}^{n} \hat{e}_i / n$.

iii) Sample with replacement from all the centered residuals to obtain a bootstrap sample of errors, $\{e_t^{bt}\}_{t=1}^{n+50}$.

iv) Then we can compute the bootstrap samples: $\{Z_t^{bt}\}_{t=1}^{n+50}$, $\{W_t^{bt}\}_{t=1}^{n+50}$, $\{Y_t^{bt}\}_{t=1}^{n+50}$, $\{X_t^{bt}\}_{t=1}^{n+50}$ easily by using the recursive equations: $Y_0^{bt} = Y_1^{bt} = 0$; $Y_t^{bt} = \left(2 + \hat{\theta}_2 + \hat{\theta}_3\right) Y_{t-1}^{bt} - (1 + \hat{\theta}_3) Y_{t-2}^{bt} + e_t^{bt}, t = 2, 3, \ldots, n+50$; $Z_t^{bt} = Y_t^{bt} - Y_{t-1}^{bt}$; $W_t^{bt} = Z_t^{bt} - Z_{t-1}^{bt}$; $X_t^{bt} = \sum_{j=1}^{t} Y_j^{bt}$.

v) Do the regression defined in Step i) with the bootstrap samples obtained in Step iv) and calculate the bootstrap t_1-statistic, t_1^{bt}, as in Step i).

vi) Repeat Step ii) ~ Step v) B times (e.g., 2,000 times) and determine the critical value τ_1^{bt} which is the 5th percentile of the 2,000 t_1^{bt} values.

vii) If the t_1 from Step i) is less than τ_1^{bt}, then reject the hypothesis of one unit root and let $rej_1 = 1$; otherwise, don't reject and let $rej_1 = 0$.

viii) Repeat Step i) ~ Step vii), M times (e.g., 2000 times) and calculate the significance level (empirical size) or the power of the test as

$$significance\ level\ (\text{or}\ power) = \frac{\sum rej_1}{M}.$$

The simulation results for testing $H_2\ vs.\ H_1\ or\ H_0$ are given in Section 4. Testing $H_1\ vs.\ H_0$ is similar to bootstrap-based tests for one unit root and hence the results are not presented.

4. SIMULATION RESULTS

In order to determine the finite sample properties of these tests we carried out the following setting of experiments: n = 50 and 100; p = 3; d = 2; the number of Monte Carlo simulations M = 2,000; the number of bootstrap samples B = 2,000. The exact hypotheses we are testing here is: $H_2\ vs.\ H_1\ or\ H_0$. That is, 2 unit roots $vs.$ 1 or 0 unit root. We may use another notation, such as, $H_1^*: \theta_1 = \theta_2 = 0,\ \theta_3 < 0$; $H_0^*: \theta_1 \leq 0,\ \theta_2 < 0,\ \theta_3 < 0$. Results of the Monte Carlo study are given in Table 1. Note that m_1, m_2, and m_3 denote the roots of the autoregressive polynomial for the case $p = 3$, with the ordering $m_3 \leq m_2 \leq m_1$.

Table 4.1. The Results of Bootstrap Dickey-Pantula Tests

n	m_1	m_2	m_3	significance level	power
50	1.0	1.0	0.8	0.0610	
50	1.0	1.0	0.2	0.0630	
50	1.0	0.8	0.2		0.2860
50	1.0	0.5	0.2		0.8200
50	1.0	0.2	0.2		0.9760
50	1.0	0.9	0.5		0.1155
50	1.0	0.8	0.5		0.2385
50	0.9	0.9	0.9		0.1205
50	0.9	0.9	0.5		0.2110
50	0.9	0.9	0.2		0.2380
50	0.9	0.5	0.2		0.9200
50	0.9	0.1	0.2		0.9950
100	1.0	1.0	0.8	0.0595	
100	1.0	1.0	0.2	0.0530	
100	1.0	0.8	0.2		0.7705
100	1.0	0.5	0.2		1.0000
100	1.0	0.2	0.2		1.0000
100	1.0	0.9	0.5		0.2780
100	1.0	0.8	0.5		0.6630
100	0.9	0.9	0.9		0.3100
100	0.9	0.9	0.5		0.6605
100	0.9	0.9	0.2		0.7405
100	0.9	0.5	0.2		1.0000
100	0.9	0.1	0.2		1.0000

As seen from the results listed in Table 4.1, the bootstrap version of the Dickey-Pantula tests is good at maintaining the size, even when the sample size is as small as 50.

When the sample size increases from 50 to 100, the empirical size gets slightly closer to the nominal significance level of 0.05. It also shows reasonably good power, especially if the second root is not close to unity. More precisely, the further the second root is away from the unity, the higher the power can achieve. For example, consider $n = 50$. If $m_1 = 1.0, m_2 = 0.5, m_3 = 0.2,$ the power is 0.82; if $m_1 = 1.0, m_2 = 0.2, m_3 = 0.2,$ the power is 0.976. In both cases, the second root (m_2) is not close to the unity and the powers are high; especially, the power increases from 0.82 to 0.976 when m_2 decreases from 0.5 to 0.2. However, if m_2 is close to the unity, the power is low. For example, consider $n = 50$ again. If $m_1 = 1.0, m_2 = 0.8, m_3 = 0.2,$ the power is 0.286; if $m_1 = 1.0, m_2 = 0.9, m_3 = 0.5,$ the power is 0.1155; if $m_1 = 1.0, m_2 = 0.8, m_3 = 0.5,$ the power is 0.2385; if $m_1 = 0.9, m_2 = 0.9, m_3 = 0.2,$ the power is 0.238; if $m_1 = 0.9, m_2 = 0.5, m_3 = 0.2,$ the power is 0.92; if $m_1 = 0.9, m_2 = 0.9, m_3 = 0.9,$ the power is 0.1205; if $m_1 = 0.9, m_2 = 0.9, m_3 = 0.5,$ the power is 0.211. Whereas, if $m_1 = 0.9, m_2 = 0.1, m_3 = 0.2,$ the power is 0.995. The same pattern can be observed for the cases of $n = 100$. Besides that, the power increases significantly as the sample size increases from 50 to 100, up to 1.

We also made a brief comparison between our results with the results presented by Dickey and Pantula (1987). Part of the transformed Dickey-Pantula non-bootstrap test results are given in Table 4.2.

Table 4.2. Part of Transformed Dickey-Pantula
Non-Bootstrap Test Results

n	m_1	m_2	m_3	**sig level**	**power**
50	1.0	1.0	1.0	0.0023	
50	1.0	1.0	0.9	0.0156	
50	1.0	1.0	0.7	0.0463	
50	1.0	1.0	0.0	0.0525	
50	1.0	0.9	0.7		0.2287
50	1.0	0.9	0.0		0.3034
50	1.0	0.7	0.0		0.9082
50	1.0	0.5	0.0		0.9950
50	0.9	0.9	0.9		0.3490
50	0.9	0.9	0.5		0.6902

It's obvious that even for $n = 50$, Dickey-Pantula's test has relatively higher power than our bootstrap-based test. However, there is not much difference in significance level between these two methods.

5. CONCLUSION

In summary, testing for two unit roots in a time series has not received as much attention as the case of testing for one unit root. The only procedure that tests for two unit roots using a single test was proposed by Dickey and Pantula in 1987. This test requires taking p differences of the time series where p is the order of the autoregressive process. We modify this test so that the percentile points are directly derived using the bootstrap. Preliminary results show that the bootstrap version of the Dickey-Pantula test has reasonably good small sample properties including both size and power. In the future, we may assume the value of p is unknown and develop a sieve bootstrap-version of Dickey-Pantula tests for multiple unit roots.

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II. BOOTSTRAP-BASED UNIT ROOT TESTS FOR HIGHER ORDER AUTOREGRESSIVE MODELS WITH $GARCH(1,1)$ ERRORS

ABSTRACT

Bootstrap-based unit root tests are a viable alternative to asymptotic distribution-based procedures and, in some cases, are preferable because of the serious size distortions associated with the latter tests under certain situations. While several bootstrap-based unit root tests exist for *ARMA* processes with homoscedastic errors, only one such test is available when the innovations are conditionally heteroskedastic. The details for the exact implementation of this procedure are currently available only for the first order autoregressive processes. Monte Carlo results are also published only for this limited case. In this paper we demonstrate how this procedure can be extended to higher order autoregressive processes through a transformed series used in augmented Dickey-Fuller unit root tests. We also investigate the finite sample properties for higher order processes through a Monte Carlo study. Results show that the proposed tests have reasonable power and size properties.

Keywords Non-stationary, Conditional volatility, Residual bootstrap, Time series, Random walk

1. INTRODUCTION

The most commonly used unit root tests for time series were introduced by Dickey and Fuller [1] and are referred to as Dickey-Fuller (DF) tests. They were, however, developed for the first order autoregressive processes. Said and Dickey [2] generalized the Dickey-Fuller tests to be applicable to $ARMA$ models of unknown orders. These tests are referred to as Augmented Dickey-Fuller (ADF) tests. Alternatively, Phillips and Perron [3] provided a correction to the Dickey-Fuller tests to account for the presence of higher order terms. Specifically, Phillips and Perron tests accommodate innovations that are weakly dependent as well as heterogeneously distributed.

The above tests, however, ignore any underlying volatility structure of the innovations. More recently, authors such as Ling and Li,[4,5] Ling, Li, and McAleer [6] have proposed unit root tests under Generalized Autoregressive Conditional Heteroskedastic ($GARCH$) innovations. Especially, Ling *et al.* [6] showed that under certain conditions, the unit root tests that take into account the $GARCH$ structure of the innovations produce tests with higher power. One drawback of these newer tests is that, as in the case with the standard DF and ADF tests, they show serious size distortions. Bootstrap-based tests have been proposed as an alternative to asymptotic distribution-based tests in order to overcome this flaw. Gospodinov and Tao [7] were the first to adopt this bootstrap approach to obtain unit root tests for autoregressive (AR) time series with $GARCH$ innovations and showed how this procedure can be implemented for first order processes. They also proved the asymptotic validity of the test for the $AR(1)$ case but indicated that these results can be extended to the general $AR(p)$ case. Their Monte Carlo results, reported for the $AR(1)$ case with $GARCH(1,1)$ errors, show that the bootstrap-based tests do not exhibit the size distortions present in the asymptotic-based procedures. In this paper, we detail how the Gospodinov-Tao method can be adapted to conduct unit root tests for general $AR(p)$ processes with correctly specified order p and present results of a Monte Carlo study. The motivation is to show the applied practitioner a step-by-step procedure for implementing this important methodology to the general autoregressive model. In addition, the Monte Carlo study is employed not only to explore the size and power of the test when the order of the process is greater than one, but also to see if these properties are affected by other roots in the autoregressive polynomial. The results show that the proposed method has good size and power properties for higher order processes even when the second largest root is close to unity.

2. BRIEF HISTORICAL REVIEW

Ling and Li [4] considered the general nonstationary autoregressive moving average time series with general-order $GARCH$ errors, and demonstrated that the maximum likelihood estimators (MLEs) of the relevant autoregressive coefficients are more efficient than the least-squares estimators (LSEs). They also developed the limiting distribution of the relevant local MLEs. Their results require that the fourth-order moments of the errors exist. Assuming that the eighth-order moments of the errors exist, Seo [8] independently derived the limiting distribution of the local MLEs in the nonstationary $AR(p)$ model. Additionally, Ling et al. [6] considered the LSE and the two-step local quasi-maximum likelihood estimator (quasi-MLE) for the unit root $AR(1)$ processes with $GARCH(1,1)$ errors. The relevant asymptotic distributions of the LSE and the two-step quasi-MLE were also derived. Correspondingly, Ling and Li [5] developed the one-step local quasi-MLE and its asymptotic distribution for the unit root $AR(1)$ processes with $GARCH(1,1)$ errors. The distributions obtained by Ling et al. [6] and Ling and Li [5] are the same as that reported in Ling and Li.[4] However, Ling *et al.* [6] and Ling and Li [5] assumed that the scaled conditional errors (the ratio of the error to its conditional standard deviation) follow a symmetric distribution. They also assumed that the second-order moments of the errors exist, which translates to $\alpha + \beta < 1$ in the case of $GARCH(1,1)$ errors with parameters α and β. These assumptions are recognized as the least restrictive ones in the research field of unit root tests with $GARCH$ errors.

While non-bootstrap-based unit root tests with both homoscedastic and heteroskedastic errors have been explored widely, their weaknesses have been also identified. Leybourne and Newbold [9] found that the Phillips-Perron unit root tests have serious size distortion and low power issues in finite samples, especially when the model has a moving average component. In addition, Gospodinov and Tao [7] commented that the applications of the unit root tests developed for processes with $GARCH$ errors have been restricted in financial time series because of several issues, such as the complicated calculating procedure needed for the MLEs of the main parameters and nuisance parameters, the substantial size distortions of the asymptotic distribution-based tests, and so on.

Adaptation of bootstrap-based unit root tests for time series with $GARCH$ errors seems a logical alternative to the existing asymptotic distribution-based tests because they do not have the size distortions exhibited by the latter. Cavaliere and Taylor [10] developed a bootstrap-based unit root test for time series with non-stationary volatility that satisfies the assumption that the time dependent volatility term σ_t follows the rule $\sigma_{[st]} = w(s)$ for $s \in [0,1]$, where $w(.)$ is non-stochastic and strictly positive. They assumed that the time series $\{X_t\}$ is such that $X_t = d_t + Y_t$, with $Y_t = \alpha Y_{t-1} + u_t$, $u_t = \sum_{j=0}^{\infty} c_j \varepsilon_{t-j}$, $\varepsilon_t = \sigma_t e_t$, where $e_t \sim iid(0,1)$ and d_t is a trend component. While their formulation can be generalized to include the case where $\{e_t\}$ follows a $GARCH$ process, the proposed bootstrap procedure does not model the underlying $GARCH$ structure as was done by Gospodinov [11] who derived bootstrap results when testing for nonlinearity in models with a unit root and $GARCH$ errors. Subsequently, Gospodinov and Tao [7] proposed a bootstrap-based unit root test for $AR(1)$ processes with $GARCH(1,1)$ errors. Specifically, they extended the results of Basawa et al.,[12,13] Ferretti and Romo,[14] Heimann and Kreiss,[15] and Park,[16] to unit root models, with conditional heteroscedasticity estimated by maximum likelihood methods. They also followed Ling and Li [5] and derived the consistency of the bootstrap distribution given the finite second-order moments of the errors and the symmetry of the standardized errors. One advantage of this method is that it does not require explicit estimation of the nuisance parameters involved in the distribution of the test statistic. Their simulation results show excellent size and power properties compared to Dickey-Fuller tests. Also, Gospodinov and Tao [7] suggest that the results can be easily extended to processes of higher order. They do not, however, describe how such an extension may be carried out. For example, one may employ the type of model used in Augmented Dickey-Fuller (ADF) unit root test [2] or the version proposed by Phillips and Perron.[3] A detailed step-by-step procedure describing how Gospodinov-Tao test can be extended to the general $AR(p)$ case will be of help to the practitioner and that is the intent of this paper. Moreover, the simulation results and the derivation of the consistency of the bootstrap distribution reported by Gospodinov and Tao [7] are limited to the first order autoregressive case. Our study aims to explore the performance of the test when applied to higher order models. Of special interest is how the size and power of the test is affected by other roots of the AR polynomial.

In Sections 3 we provide the model formulation that will be employed to develop the test procedure for the general $AR(p)$ case and also provide reasons why a test based on the limiting distribution of the test statistic is unsuitable. In Section 4 we show in detail how the test proposed by Gospodinov and Tao [7] can be extended to the general

$AR(p)$ case. The simulation results are given in Section 5. Section 6 provides concluding remarks and discusses future work.

3. MODEL FORMULATION AND THE ASYMPTOTIC DISTRIBUTION OF THE TEST STATISTIC

The proposed procedure adopts the test presented by Gospodinov and Tao [7] to the general $AR(p)$ case with correctly specified p, using the model formulation employed by Said and Dickey.[2] Two equivalent formulations of autoregressive models with order p are considered. Equation (1) is the classical format, and Equation (3) follows the Augmented Dickey-Fuller model. The complete model formulation is:

$$y_t = \Phi_1 y_{t-1} + \Phi_2 y_{t-2} + \dots + \Phi_p y_{t-p} + \varepsilon_t , \quad t = 1, 2, \dots, T, \tag{1}$$

$$m^p - \Phi_1 m^{p-1} - \Phi_2 m^{p-2} - \dots - \Phi_{p-1} m - \Phi_p = 0, \tag{2}$$

$$\nabla y_t = r y_{t-1} + \delta_1 \nabla y_{t-1} + \delta_2 \nabla y_{t-2} + \dots + \delta_{p-1} \nabla y_{t-p+1} + \varepsilon_t , \quad t = 1, 2, \dots, T, \tag{3}$$

$$\varepsilon_t = \sqrt{h_t}\, \eta_t , \quad \eta_t \sim iid\,(0,1) , \quad E[\eta_t^3] = 0, E[\eta_t^4] < \kappa < \infty, \quad t = 1, 2, \dots, T, \tag{4}$$

and $h_t = \omega + \alpha \varepsilon_{t-1}^2 + \beta h_{t-1}$

with $0 < \omega_l < \omega < \omega_u, 0 < \alpha_l < \alpha < \alpha_u, 0 < \beta_l < \beta < \beta_u, \alpha + \beta < 1,$

for $t = 1, 2, \dots, T.$ (5)

Note the assumption in (5) that the parameters $\omega, \alpha,$ and β are bounded below and above by constants, which is stronger than what is prescribed in the standard $GARCH$ formulation. This is used by Gospodinov and Tao [7] to prove their asymptotic results. They also required that $y_0 = 0$ and that h_0 is initialized from its invariant measure. Expression (2) gives the characteristic equation of the autoregressive model described in Equation (1), which is equivalent to $\prod_{i=1}^{p}(m - r_i) = 0$, where $r_i, \ i = 1, 2, \dots, p,$ are the roots of $AR(p)$ polynomial. We assume $|r_1| \leq 1$, and $|r_i| < 1$, for $i \geq 2$. Note that the coefficient r associated with the term y_{t-1} in Equation (3) is zero when $r_1 = 1$. We also let $\rho = \binom{r}{\delta_0}$, with $\delta_0 = (\delta_1, \ \delta_2, \ ..., \ \delta_{p-1})'$, and let $\delta = (\omega, \alpha, \beta)'$. The hypothesis we test is $H_0: r \ = 0 \ vs. \ H_1: r \ < 0$. The test statistic we use is

$$t(\hat{r}) = (\hat{r} - 0) \left(- \sum_{t=1}^{T} \frac{\partial^2 l_t(\rho, \delta)}{\partial r^2} \right)^{\frac{1}{2}}_{\rho = \hat{\rho} \ , \ \delta = \hat{\delta}}, \tag{6}$$

where $\quad l_t(\rho, \ \delta) = l_t(r, \delta_0, \omega, \alpha, \beta) = -\frac{1}{2} \ ln \ h_t - \frac{1}{2} \ \frac{\varepsilon_t^2}{h_t} \ , \ \ t = 1, 2, \dots, T,$

with $\hat{\rho}$ and $\hat{\delta}$ representing the maximum likelihood estimates (MLE) of ρ and δ respectively. Note that this is the "studentized" version of the ADF test under the assumption of correctly specified p. Note that the test is a lower-tail test where the rejection region lies below the critical point.

The limiting distribution of the maximum likelihood estimators of parameters of a process with roots on the unit circle that is more general than the model formulation given in (1), (4), and (5) was derived by Ling and Li [4] under the assumption that $E\left[\varepsilon_t^4\right] < \infty$ and that $\eta_t \sim iid \ (0,1)$. This limiting distribution is expressed in terms of several complicated stochastic and deterministic components and therefore is not reported here for brevity. It suffices to note that the marginal limiting distribution of the MLEs of the autoregressive parameters can be characterized as that of a functional of stochastics integrals of Weiner processes and that it is dependent on the parameters ω, α, and β. There are several drawbacks to constructing a unit root test based on this limiting distribution. These drawbacks can easily be seen by examining the limiting distributions of $(\widehat{\Phi}_1 - 1)$, obtained by Ling and Li [4, 5], and of $t(\hat{r})$ given in (6), obtained by Gospodinov and Tao [7], for the simpler $AR(1)$ case. Note that $\Phi_1 - 1$ in the formulation given in [4, 5] is the same as *r* given in Equation (3). Moreover, in [5] Ling and Li showed that one can assume the less stringent condition $E\left[\varepsilon_t^2\right] < \infty$ if the η_t are restricted to having a normal distribution. Ling and Li showed that under H_0,

$$T\,(\widehat{\Phi}_1 - 1) \xrightarrow{L} \left[\int_0^1 w_1(\tau)dw_2(\tau)\right]\left[F\int_0^1 w_1^2(\tau)d\tau\right]^{-1}$$

where $(w_1(\tau), w_2(\tau))$ is a bivariate zero-mean Weiner process with covariance matrix

$$\tau\Sigma = \tau\begin{bmatrix} E(h_t) & 1 \\ 1 & \left\{E(1/h_t) + k\alpha^2\sum_{k=1}^{\infty}\beta^{2(k-1)}E(\varepsilon_{t-k}^2/h_t^2)\right\}\end{bmatrix}, \tag{7}$$

where $\tau \in [0,1]$, $F = E(1/h_t) + 2\alpha^2\sum_{k=1}^{\infty}\beta^{2(k-1)}E(\varepsilon_{t-k}^2/h_t^2)$ and $\kappa = E(\eta_t^4) - 1$. When η_t are normal, $\kappa = 2$. Applying a suitable transformation to the above result, the authors also showed that the above limiting distribution can be expressed as a functional of two independent Weiner processes $B_1(\tau)$ and $B_2(\tau)$, and after further simplification showed that

$$nc(\widehat{\Phi}_1 - 1) \xrightarrow{L} \left[\rho\int_0^1 B_1(\tau)dB_1(\tau)\right]\left[\int_0^1 B_1^2(\tau)d\tau\right]^{-1} + \sqrt{1-\rho^2}\left[\int_0^1 B_1^2(\tau)d\tau\right]^{-1/2} \xi \;, \quad (8)$$

where $c = \sigma F / \sqrt{K}$, K is the $(2,2)^{th}$ element of Σ in (7), $\sigma^2 = E(h_t)$, $\rho^2 = \left(1/\sigma^2 K\right)$,

and ζ is a standard normal random variable independent of $\left[\int_0^1 B_1^2(\tau)d\tau\right]$. Note that

$\rho^2 \in [0,1]$ and plays the role of a mixing parameter.

Gospodinov and Tao [7] used the above result to obtain the limiting distribution of the *t*-statistic $t(\hat{r})$ under the null hypothesis of $r = 0$ for the $AR(1)$ case. Their results show that under the assumptions given in (4) with $r = 0$,

$$t(\hat{r}) \xrightarrow{L} \sqrt{\frac{K}{F}}\left[\frac{\rho\int_0^1 B_1(\tau)dB_1(\tau)}{\left[\int_0^1 B_1^2(\tau)d\tau\right]^{-1/2}} + \sqrt{1-\rho^2\zeta}\right]. \quad (9)$$

Examination of the above results show that the limiting distributions of $(\widehat{\Phi}_1 - 1)$ and of $t(\hat{r})$ in particular, are (a) dependent on nuisance parameters ω, α, and β and thus non-pivotal, (b) these parameters appear in highly non-linear form, and in addition (c) they are present as part of infinite sums. Replacing these unknown parameters by their estimates can introduce bias and lead to severe size distortions [7] because of (b). Gospodinov and Tao [7] used a Monte Carlo study to illustrate these phenomena, and showed that size distortions increase as $\alpha + \beta$ approaches the unit boundary. In addition, the infinite sums the nuisance parameters appear in has to be truncated in order to obtain a computationally tractable form. Moreover, the critical points have to be obtained using an iterative numerical algorithm. Thus, the bootstrap approach to testing for unit roots is appealing even though the proposed method is somewhat computationally demanding.

An important observation is that when the above test-statistic is obtained using straightforward least squares estimation of the autoregressive parameter Φ_1 ignoring the *GARCH* structure of the error process $\{\varepsilon_t\}$, the limiting distribution reduces to that of

$$\sqrt{\frac{K}{F}}\left[\frac{\int_0^1 B_1(\tau)dB_1(\tau)}{\left[\int_0^1 B_1^2(\tau)d\tau\right]^{-1/2}}\right], \tag{10}$$

which is the same as that of the Dickey-fuller test statistic. [1] Furthermore, ρ is a monotone decreasing function of *K*, which increases with α and β. Thus, as the degree of the conditional heteroscedasticity in the error process increases, the limiting distribution tends more towards the standard normal as stated in [7], since the latter distribution has lower critical values. The net result is an increase in the power of the test when the underlying *GARCH* structure is accounted for when estimating the autoregressive parameters. Gospodinov and Tao [7] has shown that the bootstrap test statistic for the $AR(1)$ case has the same limiting distribution as that of the functional given in (9) and hence this advantage carries over to their bootstrap-based unit root test.

4. PROPOSED BOOTSTRAP METHOD

The main steps for performing a bootstrap-based unit root test on $AR(p)$ models with $GARCH(1,1)$ errors are listed below:

i) Use the least-squares estimates of $\rho = \begin{pmatrix} r \\ \delta_1 \\ . \\ . \\ . \\ \delta_{p-1} \end{pmatrix}$. as initial values for maximum likelihood estimation. Initial values for δ can be obtained by fitting an $ARMA\ (1,1)$ model to the squared residuals obtained by the least squares fit because these residuals are estimates of the ε_t and $\{\varepsilon_t^2\}$ obeys *an* $ARMA\ (1,1)$ process with AR and MA parameters $(\alpha+\beta)$ and β , respectively, with intercept ω. Use these initial values to obtain the maximum likelihood estimates (*MLE*) of both ρ and δ, and record them as $\hat{\rho}$, $\hat{\delta}$, where $\hat{\rho} = \begin{pmatrix} \hat{r} \\ \hat{\delta}_1 \\ . \\ . \\ . \\ \hat{\delta}_{p-1} \end{pmatrix}$, $\hat{\delta} = (\widehat{\omega},\ \hat{\alpha},\ \hat{\beta})'$.

ii) Compute the test statistic,

$$t(\hat{r}) = (\hat{r} - 0)\left(-\sum_{t=1}^{T} \frac{\partial^2 l_t(\rho,\delta)}{\partial r^2}\right)^{\frac{1}{2}}_{\rho=\hat{\rho}\ ,\ \delta=\hat{\delta}}, \quad \text{where}$$

$$l_t(\rho,\ \delta) = l_t(r, \delta_1, \delta_2, \dots, \delta_{p-1}, \omega, \alpha, \beta) = -\frac{1}{2}\ ln\, h_t - \frac{1}{2}\ \frac{\varepsilon_t^2}{h_t}\ ,\ \ t = 1, 2, \dots, T.$$

iii) Compute $\hat{\varepsilon}_t = \nabla y_t - \hat{r} y_{t-1} - \sum_{j=1}^{p-1} \hat{\delta}_j \nabla y_{t-j}$, for $t = p+1,\ p+2,\ \dots,\ T.$

iv) Compute $\hat{h}_{p-1} = \hat{\omega} + \hat{\alpha}\hat{\varepsilon}_{p-2}^2 + \hat{\beta}\,\hat{h}_{p-2}$, where $\hat{\varepsilon}_{p-2}^2 = \hat{h}_{p-2} = \frac{1}{T}\sum_{i=p+1}^{T} \hat{\varepsilon}_i^2$;

$$\hat{h}_t = \hat{\omega} + \hat{\alpha}\hat{\varepsilon}_{t-1}^2 + \hat{\beta}\,\hat{h}_{t-1},\ \ t = p+1,\ p+2,\ \dots,\ T.$$

v) Let $\hat{\eta}_t = \frac{\hat{\varepsilon}_t}{\sqrt{\hat{h}_t}}$, and let $\widetilde{\eta}_t$ be centered $\hat{\eta}_t$, for $t = p+1,\ p+2,\ \dots,\ T.$

vi) Resample η_t^*, $t = 1,\ 2, \dots, 2T$, from $\{\pm\widetilde{\eta}_t\}_{\mathrm{t=p+1}}^{T}$. Note that $\{\pm\widetilde{\eta}_t\}_{\mathrm{t=p+1}}^{T}$ contain both the $\widetilde{\eta}_t$ and the values $\widetilde{\eta}_t$ multiplied by -1. This ensures the symmetry of the underlying distribution that will be resampled.

vii) Compute $h_t^* = \hat{\omega} + (\hat{\alpha}\eta_{t-1}^{*2} + \hat{\beta})h_{t-1}^*$, and let $h_1^* = \hat{\varepsilon}_1^2$ or $h_1^* = \hat{h}_1$, for $t = 2,\ 3,\ \dots,\ 2T.$

viii) Compute $\nabla y_t^* = \sum_{j=1}^{p-1} \hat{\delta}_j \nabla y_{t-j}^* + \sqrt{h_t^*}\eta_t^*$, $t = 2,\ 3,\ \dots,\ 2T,$ with $\nabla y_{t-j}^* = 0$ if $t \le j.$ That is, under $H_0\colon r = 0$, we have

$$y_t^* - y_{t-1}^* = \sum_{j=1}^{p-1} \hat{\delta}_j (y_{t-j}^* - y_{t-j-1}^*) + \sqrt{h_t^*}\eta_t^* ,$$

$$y_t^* = y_{t-1}^* + \sum_{j=1}^{p-1} \hat{\delta}_j (y_{t-j}^* - y_{t-j-1}^*) + \sqrt{h_t^*}\eta_t^* , \quad \text{for}$$

$$t = p+1,\ p+2,\ \dots, 2T,\ \text{and}\ y_1^* = \dots = y_p^* = 0 .$$

ix) To reduce the effect of the initial conditions, drop the first $T - p$ values of y_t^* . Also re-label *t* so the new values read from 1 to *T*. Fit ∇y_t^* against y_{t-1}^* and ∇y_{t-j}^*, $j = 1,\ 2,\ \dots,\ p-1.$ And estimate r^* and δ_j^*, $j = 1,\ 2, \dots,\ p-1,$ using least squares.

x) Use the least-squares estimates as initial values and obtain MLEs of

$$\rho^* = \begin{pmatrix} r^* \\ \delta_1^* \\ . \\ . \\ . \\ \delta_{p-1}^* \end{pmatrix}, \quad \delta^* = (\omega^*, \alpha^*, \beta^*)'$$, and denote these estimates as $\hat{\rho}^*$ and $\hat{\delta}^*$.

xi) Compute the bootstrap test statistic,

$$t^*(\hat{r}^*) = (\hat{r}^* - 0) \left(-\sum_{t=1}^{T} \frac{\partial^2 l_t^*(\rho^*, \delta^*)}{\partial r^{*2}}\right)^{\frac{1}{2}}_{\rho^*=\hat{\rho}^*,\, \delta^*=\hat{\delta}^*},$$ where

$$l_t^*(\rho^*,\, \delta^*) = l_t^*(r^*, \delta_1^*, \delta_2^*, \ldots, \delta_{p-1}^*, \omega^*, \alpha^*, \beta^*) = -\frac{1}{2}\, ln\, h_t^* - \frac{1}{2}\frac{\varepsilon_t^{*2}}{h_t^*}, \quad t = 1,\ 2,\ \ldots,\ T.$$

xii) Repeat Step vi) ~ xi) B times, say $B = 1{,}000$, and calculate the lower 5^{th} percentile of $t^*(\hat{r}^*)$, $t^*_{0.05}$, then compare $t^*_{0.05}$ with $t(\hat{r})$. If $t(\hat{r}) < t^*_{0.05}$, reject H_0 and let rej equal 1; otherwise, do not reject and let rej equal 0.

xiii) Repeat Step i) ~ xii) M times, say $M = 1{,}000$, and calculate the significance level (empirical size) or the power of the test as: $significance\ level$ (or $power$) $= \frac{\sum rej}{M}$.

5. SIMULATION RESULTS

For brevity, we assume $p = 2$ here. To carry out the simulations, we used Expression (1) together with (4) and (5) to generate the raw time series $\{y_t\}_{t=1}^{2T}$, and then threw away the first T values of the series. We also re-labeled t to go from 1 to T. Then fit model (3) to the remaining series of length T and calculated the least-squares estimates of r and other coefficients. The same goes for Step ix) under Proposed Bootstrap Method section.

MATLAB was used to perform Monte Carlo simulations and bootstrap procedures. We considered two types of distributions for the centered and standardized error terms, one is standard normal and the other is t-distribution with 7 degrees of freedom. The simulation results for $T = 200$ and $T = 400$ are given in Tables 5.1- 5.12. We did 1,000 simulations for both $T = 200$ and $T = 400$ cases. The sample sizes chosen are the same as those employed by Gospodinov and Tao.[7] Considering that there are approximately 250 trading days per year, say at the New York Stock Exchange, the sample size of 200 reflects stock return data from less than one year. As such, the sample sizes chosen are not unreasonably large.

For the simulation we considered $AR(2)$ models with roots $r_1 \in \{0.5, 0.9, 1.0\}$, $r_2 \in \{0.2, 0.5, 0.9\}$. The (α, β) combinations considered are (0,0), (0.5,0.4), (0.25,0.7), (0.399,0.6), (0.199,0.8), (0.7,0.25), (0.6,0.399), (0.8,0.199), (0.2,0.4) and (0.4,0.2). These combinations are very similar to those employed by Gospodinov and Tao.[7] As was done by the above authors, many of the (α, β) combinations were intentionally selected so that $\alpha + \beta$ is close to 1 in order to demonstrate that the procedure works well even when these two parameters take values close to the $\alpha + \beta < 1$ threshold needed for stationarity of the $GARCH$ process. Unreported results for cases where $\alpha + \beta \ll 1$ show good power and size properties. To save space, results for all combinations are not reported but are available upon request from the first author.

Table 5.1. Estimated Coverage Probabilities for the Model with
$\alpha = 0, \beta = 0$, and Normal Errors

AR Roots		**Sample Size**			
		200		**400**	
r_1	**r_2**	**Size**	**Power**	**Size**	**Power**
1	0.2	0.04		0.058	
1	0.5	0.04		0.057	
1	0.9	0.044		0.056	
0.9	0.2		0.993		1
0.9	0.5		0.988		1
0.9	0.9		0.757		0.991
0.5	0.5		1		1
0.5	0.2		1		1

Table 5.2. Estimated Coverage Probabilities for the Model with
$\alpha = 0.5, \beta = 0.4$, and Normal Errors

AR Roots		**Sample Size**			
		200		**400**	
r_1	**r_2**	**Size**	**Power**	**Size**	**Power**
1	0.2	0.041		0.054	
1	0.5	0.04		0.052	
1	0.9	0.053		0.047	
0.9	0.2		0.996		1
0.9	0.5		0.992		1
0.9	0.9		0.915		0.996
0.5	0.5		1		1
0.5	0.2		1		1

Table 5.3. Estimated Coverage Probabilities for the Model with
$\alpha = 0.25, \beta = 0.7$, and Normal Errors

***AR* Roots**		**Sample Size**			
		200		**400**	
r_1	r_2	**Size**	**Power**	**Size**	**Power**
1	0.2	0.05		0.048	
1	0.5	0.054		0.047	
1	0.9	0.058		0.047	
0.9	0.2		0.99		1
0.9	0.5		0.987		1
0.9	0.9		0.853		0.998
0.5	0.5		1		1
0.5	0.2		1		1

Table 5.4. Estimated Coverage Probabilities for the Model with
$\alpha = 0.399, \beta = 0.6$, and Normal Errors

***AR* Roots**		**Sample Size**			
		200		**400**	
r_1	r_2	**Size**	**Power**	**Size**	**Power**
1	0.2	0.045		0.047	
1	0.5	0.047		0.045	
1	0.9	0.055		0.049	
0.9	0.2		0.995		1
0.9	0.5		0.992		1
0.9	0.9		0.921		0.998
0.5	0.5		1		1
0.5	0.2		1		1

Table 5.5. Estimated Coverage Probabilities for the Model with
$\alpha = 0.199, \beta = 0.8$, and Normal Errors

***AR* Roots**		**Sample Size**			
		200		**400**	
r_1	r_2	**Size**	**Power**	**Size**	**Power**
1	0.2	0.045		0.051	
1	0.5	0.048		0.054	
1	0.9	0.055		0.05	
0.9	0.2		0.989		1
0.9	0.5		0.983		1
0.9	0.9		0.827		0.999
0.5	0.5		1		1
0.5	0.2		1		1

Table 5.6. Estimated Coverage Probabilities for the Model with
$\alpha = 0.7, \beta = 0.25$, and Normal Errors

***AR* Roots**		**Sample Size**			
		200		**400**	
r_1	r_2	**Size**	**Power**	**Size**	**Power**
1	0.2	0.051		0.056	
1	0.5	0.044		0.054	
1	0.9	0.056		0.05	
0.9	0.2		0.997		1
0.9	0.5		0.997		1
0.9	0.9		0.955		0.999
0.5	0.5		1		1
0.5	0.2		1		1

Table 5.7. Estimated Coverage Probabilities for the Model with
$\alpha = 0.6, \beta = 0.399$, and Normal Errors

***AR* Roots**		**Sample Size**			
		200		**400**	
r_1	r_2	**Size**	**Power**	**Size**	**Power**
1	0.2	0.054		0.056	
1	0.5	0.045		0.054	
1	0.9	0.05		0.053	
0.9	0.2		0.999		1
0.9	0.5		0.999		1
0.9	0.9		0.955		1
0.5	0.5		1		1
0.5	0.2		0.999		1

Table 5.8. Estimated Coverage Probabilities for the Model with
$\alpha = 0.8, \beta = 0.199$, and Normal Errors

***AR* Roots**		**Sample Size**			
		200		**400**	
r_1	r_2	**Size**	**Power**	**Size**	**Power**
1	0.2	0.045		0.048	
1	0.5	0.043		0.048	
1	0.9	0.05		0.048	
0.9	0.2		0.998		1
0.9	0.5		0.999		1
0.9	0.9		0.971		0.997
0.5	0.5		1		1
0.5	0.2		1		1

Table 5.9. Estimated Coverage Probabilities for the Model with
$\alpha = 0.2, \beta = 0.4$, and Normal Errors

AR Roots		**Sample Size**			
		200		**400**	
r_1	r_2	**Size**	**Power**	**Size**	**Power**
1	0.2	0.044		0.056	
1	0.5	0.04		0.055	
1	0.9	0.04		0.06	
0.9	0.2		0.993		1
0.9	0.5		0.989		1
0.9	0.9		0.802		0.993
0.5	0.5		1		1
0.5	0.2		1		1

Table 5.10. Estimated Coverage Probabilities for the Model with
$\alpha = 0.4, \beta = 0.2$, and Normal Errors

AR Roots		**Sample Size**			
		200		**400**	
r_1	r_2	**Size**	**Power**	**Size**	**Power**
1	0.2	0.042		0.046	
1	0.5	0.043		0.046	
1	0.9	0.041		0.047	
0.9	0.2		0.994		1
0.9	0.5		0.991		1
0.9	0.9		0.86		0.993
0.5	0.5		1		1
0.5	0.2		1		1

Table 5.11. Estimated Coverage Probabilities for the Model with
$\alpha = 0.5, \beta = 0.4$, and t_7 Errors

AR Roots		**Sample Size**			
		200		**400**	
r_1	r_2	**Size**	**Power**	**Size**	**Power**
1	0.2	0.047		0.051	
1	0.5	0.043		0.048	
1	0.9	0.049		0.051	
0.9	0.2		0.994		1
0.9	0.5		0.992		1
0.9	0.9		0.947		0.997
0.5	0.5		1		1
0.5	0.2		1		1

Table 5.12. Estimated Coverage Probabilities for the Model with $\alpha = 0.4, \beta = 0.2$, and t_7 Errors

AR Roots		**Sample Size**			
		200		**400**	
r_1	r_2	**Size**	**Power**	**Size**	**Power**
1	0.2	0.046		0.056	
1	0.5	0.049		0.057	
1	0.9	0.044		0.057	
0.9	0.2		0.989		1
0.9	0.5		0.981		0.999
0.9	0.9		0.808		0.989
0.5	0.5		1		1
0.5	0.2		1		1

Specifically, simulation results show that the size of the tests ranges from 0.04 to 0.058. These values are similar to what Gospodinov and Tao [7] obtained in the $AR(1)$ case for tests conducted under the nominal significance level of 5%. There is no discernible pattern with respect to the size of the test and the parameters of the model. Under normal errors with sample size 200, a size above 0.05 is obtained whenever $\alpha + \beta$ is close to 1 and $r_2 = 0.9$, except in two cases (Tables 5.7 & 5.8) when the size equals 0.05. Size greater than 0.05, however, is obtained in other situations as well for sample size 400. If the true significance level is 0.05, we expect approximate 95% confidence limits for the estimates based on 1,000 simulation runs to be approximately $0.05 \pm 1.96\{[0.05 \times 0.95]/1{,}000\}^{1/2} = 0.05 \pm 0.0135$, and hence the slight deviations from 0.05 we observe can very well be due to estimation error. What is more important is to note that severe size distortions are not present. Simulation results for the t-distribution case show similar behavior as far as size is concerned. Note that the theory was developed assuming the standardized errors to be symmetric.

The power of the test increases with decreases in r_1 and r_2. For example, in Table 5.2, one sees that the power is 0.915 when $r_1 = r_2 = 0.9$ but increases to 0.996 when $r_1 = 0.9$ but $r_2 = 0.2$. The power is practically unity when $r_1 = r_2 = 0.5$ or lower. This pattern holds irrespective of the underlying distribution considered in the simulation study. Power also increases with sample size as seen in all of the tables. Power is very close to one even in cases where $r_1 = r_2 = 0.9$ when the sample size is 400. A more interesting result can be observed by comparing the power when $r_1 = r_2 = 0.9$ under the non-heteroskedastic case (Table 5.1) to the power under $\alpha = 0.5,\ \beta = 0.4$. Under homoscedastic errors the power is 0.757, which climbs to 0.915 under heteroskedasticity. A similar phenomenon is also observed when comparing results in Table 5.4 to those in Table 5.9. Table 5.9 looks at the case where $\alpha = 0.2,\ \beta = 0.4$ (so $\alpha + \beta = 0.6$) in contrast to Table 5.4 where $\alpha = 0.399,\ \beta = 0.6$ (so $\alpha + \beta = 0.999$). Increasing $\alpha + \beta$ seems to increase the power, especially for the case with $r_1 = r_2 = 0.9$. The power for this case given in Table 5.9 is 0.802 whereas the power reported for this case in Table 5.4 is 0.921. This pattern is also evident under the t-distribution as seen in Tables 5.11 and 5.12. This is the same phenomenon observed by Ling et al.[6] Overall, the proposed method seems to work well for all cases, maintaining a reasonably near nominal size and producing good power.

6. CONCLUSION AND FUTURE WORK

A bootstrap-based procedure for conducting unit root tests in higher order autoregressive models with $GARCH$ errors was introduced. This procedure is based on the seminal work of two authors who detailed the implementation of the method for first order autoregressive processes. It was shown how this method can be extended to general autoregressive processes using a transformed series. Simulation results indicate that the proposed method mitigates the size distortion issue present in the asymptotic-based tests and achieves high powers at different combinations of the autoregressive roots and $GARCH$ coefficients. An obvious future extension is to develop a bootstrap-based unit root test for the case where the underlying process is $ARIMA$ with unknown orders. Relaxation of the $GARCH$ structure to include asymmetric effects of shocks can also be another potential extension.

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III. BOOTSTRAP-BASED UNIT ROOT TESTING FOR SEASONAL TIME SERIES UNDER $GARCH(1,1)$ ERRORS

ABSTRACT

We introduce a bootstrap-based test for a seasonal unit root of a time series with $GARCH(1,1)$ errors. Seasonal time series based on autoregressive moving average formulation ($ARMA$) was first described by Box and Jenkins in 1970. In 1984, Dickey, Hasza and Fuller proposed their test (DHF test) to determine if a seasonal unit root exists in a time series with independent and identically distributed ($i.i.d.$) errors. In 2000, Psaradakis carried out a bootstrap-based unit root test for pure seasonal time series with independent errors and gained higher powers than the DHF test. His method is recognized as difference based because it calculates the residuals by fitting an $AR(p)$ model to the differenced time series, whereas the method proposed by Palm, Smeekes, and Urbain in 2008 is called residual based because it computes the residuals by fitting the Dickey-Fuller (DF) regression model to the differenced series. In 2014, Rupasinghe and Samaranayake developed their own bootstrap-based seasonal unit root tests using both difference based and residual based methods. Their test focused on the seasonal time series with weakly dependent errors and without considering any underlying conditional heteroskedastic error structure. In this paper, we consider extending the DHF test and developing bootstrap-based unit root test for seasonal time series with $GARCH(1,1)$ errors using the residual-based method. A Monte Carlo simulation study is carried out to investigate the properties of the test. Results show that our bootstrap-based seasonal unit root test has reasonable small sample properties with respect to both size and power.

Keywords: Seasonal time series; $GARCH$; Nonstationary; Seasonal unit root; Bootstrap

1. INTRODUCTION

Many financial and economic time series exhibit substantial seasonality. Therefore, Box and Jenkins (1970) introduced seasonal time series models based on autoregressive moving average ($ARMA$) formulation. They and many other time series analysts influenced by their work used a seasonal differencing filter that implies the presence of seasonal unit roots in the time series. The seasonal unit roots are different from the regular unit roots (latter known as units roots at zero), but their testing procedures share some common features.

Some unit root tests have been applied in the investigation of certain economic hypotheses; for example see Altonji and Ashenfelter (1980), Nelson and Plosser (1982), Meese and Singleton (1982). The Dickey-Fuller (DF) test proposed by Dickey and Fuller (1979) is the most commonly used testing procedure for the existence of at most one non-seasonal unit root in the first order autoregressive, $AR(1)$, processes. Said and Dickey (1984) generalized the DF tests to $ARMA$ models and assumed the orders of the process are unknown. Such tests are referred to as Augmented Dickey-Fuller (ADF) tests. Phillips (1987) and Phillips and Perron (1988) also developed their own tests based on DF test which is an alternative to the ADF test. All these tests assume that the innovations are either $i.i.d.$ or weakly dependent, and ignore conditional heteroscedasticity in the errors that many financial and economic time series commonly exhibit.

To address this shortcoming, Ling and Li (1998, 2003), Ling, Li, and McAleer (2003) developed unit root tests under Generalized Autoregressive Conditional

Heteroskedastic ($GARCH$) innovations. However, as is the case with many other distribution-based unit root tests, both these newer tests suffer from serious size distortion issues. It's recognized that the bootstrap approach is one way to mitigate this situation. As the only researchers to adopt such an approach for processes with conditionally heteroskedastic error, Gospodinov and Tao (2011) developed a bootstrap-based unit root test for autoregressive (AR) time series with $GARCH(1,1)$ innovations and presented how this procedure can be carried out for $AR(1)$. Their Monte Carlo results show that the bootstrap-based tests maintain their size reasonably well. Zhong and Samaranayake (2014) adapted Gospodinov-Tao method to unit root tests for general $AR(p)$ processes. The simulation results suggest that the proposed method has good size and power properties for higher order processes even when the second largest root is close to unity.

Besides these tests, researchers have also developed unit root tests for seasonal time series. The standard seasonal unit root testing procedures include DHF test and the $HEGY$ test. The DHF test was proposed by Dickey, Hasza and Fuller (1984) and is based on the following model

$$Y_t = \alpha Y_{t-s} + u_t \quad , \tag{1.1}$$

where the u_t are either a stationary process with zero mean and constant variance or a martingale difference sequence following the regularity conditions stated in Phillips (1987) and Chan and Wei (1988). If $\alpha = 1$, the seasonal time series $\{Y_t\}$ has s roots on the unit circle, one at frequency zero, one at frequency π, with the others being complex roots. Dickey, Hasza and Fuller (1984) proposed several regression-based tests for the null hypothesis $\alpha = 1$ in model (1.1).

$HEGY$ (1990) developed another seasonal unit root test by introducing a factorization of the seasonal differencing polynomial $\nabla_s \equiv (1 - B^s)$. For example, if $s = 4$, the $HEGY$ test consists in estimating the following regression via OLS:

$$\nabla_4 Y_t = \pi_1 W_{1,t-1} + \pi_2 W_{2,t-1} + \pi_3 W_{3,t-2} + \pi_4 W_{3,t-1} + \varepsilon_t$$

where

$$W_{1t} = (1 + B + B^2 + B^3)\, Y_t \,,$$

$$W_{2t} = -(1 - B + B^2 - B^3)\, Y_t \,,$$

$$W_{3t} = -(1 - B^2)\, Y_t \,. \tag{1.2}$$

Notice that if $\alpha = 1$ in (1.1), W_{1t} , W_{2t} , and W_{3t} have unit roots only at frequency zero, π and $\pi/2$ respectively, which implies that $\pi_i = 0, i = 1, 2, 3$ correspondingly. $HEGY$ also proposed several test statistics for the null hypothesis of $\pi_i = 0, i = 1, 2, 3$ separately or jointly.

These seasonal unit root tests only deal with $i.i.d.$ errors. Although the $HEGY$ tests are more flexible than the DHF tests, they all have the weaknesses such as serious size distortion and low power that are associated with asymptotic distribution based seasonal unit root tests. To solve these problems, Psaradakis (2000) implemented a bootstrap-based unit root test for pure seasonal time series (that is a time series that satisfy (1.1) with no additional lag Y_{t-j} terms) with independent errors, and gained higher powers than the DHF test. Psaradakis (2001) was also the first to introduce the sieve bootstrap-based (residuals are obtained by fitting autoregressive approximations to the time series) unit root tests to non-seasonal time series with weakly dependent errors. Chang and Park (2003) also proposed their sieve bootstrap versions of the ADF tests for non-seasonal unit roots. Psaradakis (2001) method is called as difference based because it calculates the residuals by fitting an $AR(p)$ model to the differenced non-seasonal time series, whereas the method proposed by Palm, Smeekes, and Urbain (2008) is acknowledged as residual based because it computes the residuals by fitting the DF regression model to the differenced series. In addition, Rupasinghe and Samaranayake (2014) developed their bootstrap-based unit root tests for seasonal time series under weakly dependent error. Like most of the tests developed for testing non-seasonal unit roots, these tests didn't consider any underlying volatility structure of the innovations. In this paper, we accommodate $GARCH(1,1)$ errors to seasonal time series and propose a bootstrap-based seasonal unit root test by extending the DHF test and using the residual-based method. A Monte Carlo study is carried out to explore the finite sample properties of the test including both size and power.

In Section 2, the seasonal time series under $GARCH(1,1)$ errors is described. Section 3 introduced our bootstrap-based seasonal unit root test with $GARCH(1,1)$ errors. The Monte Carlo simulation study and results are presented in Section 4. Section 5 concludes with a summary of the simulation results and future work.

2. SEASONAL TIME SERIES UNDER $GARCH(1,1)$ ERRORS

There are plenty of formulations developed for seasonal time series. We use the seasonal time series defined by the following model

$$\Phi(B)Y_t = \varepsilon_t \text{ , where}$$

$$\Phi(B) = (1-\rho B^s)\,\psi(B) \text{ , and } t = 1, 2, \dots, T. \tag{2.1}$$

Assume that the seasonality parameter $s > 1$ and the autoregressive polynomial $\psi(B)$ has all roots outside the unit circle. Let the order of $\psi(B)$ be p_0 , then the order of $\Phi(B)$ is $p_0 + s$. Here B is used to define the "backshift operator" given by $B^k Y_t = Y_{t-k}$ for $k \in \mathbb{N}_0$.

Now $$\Phi(B)Y_t = \varepsilon_t$$

=> $$(1-\rho B^s)\,\psi(B)Y_t = \varepsilon_t$$

=> $$\psi(B)\,(1-\rho B^s)\,Y_t = \varepsilon_t$$

=> $$\psi(B)Y_t - \rho\psi(B)B^s\,Y_t = \varepsilon_t$$

=> $$\psi(B)Y_t - \rho\psi(B)Y_{t-s} = \varepsilon_t$$

=> $$\left(1-\psi_1 B - \cdots - \psi_{p_0}B^{p_0}\right)Y_t - \rho\left(1-\psi_1 B - \cdots - \psi_{p_0}B^{p_0}\right)Y_{t-s} = \varepsilon_t$$

=> $$(Y_t - \psi_1 BY_t - \cdots - \psi_{p_0}B^{p_0}Y_t) - \rho(Y_{t-s} - \psi_1 BY_{t-s} - \cdots - \psi_{p_0}B^{p_0}Y_{t-s}) = \varepsilon_t$$

=> $$(Y_t - \psi_1 Y_{t-1} - \cdots - \psi_{p_0}Y_{t-p_0}) - \rho(Y_{t-s} - \psi_1 Y_{t-s-1} - \cdots - \psi_{p_0}Y_{t-s-p_0}) = \varepsilon_t$$

$$=> \ Y_t = (\psi_1 Y_{t-1} + \cdots + \psi_{p_0} Y_{t-p_0}) + \rho(Y_{t-s} - \psi_1 Y_{t-s-1} - \cdots - \psi_{p_0} Y_{t-s-p_0}) + \varepsilon_t$$

$$=> \ Y_t = \rho Y_{t-s} + (\psi_1 Y_{t-1} + \cdots + \psi_{p_0} Y_{t-p_0}) + (-\rho\psi_1 Y_{t-s-1} - \rho\psi_2 Y_{t-s-2} - \cdots$$

$$- \rho\psi_{p_0} Y_{t-s-p_0}) + \varepsilon_t$$

That is,

$Y_t =$

$$\rho Y_{t-s} + (\psi_1 Y_{t-1} + \cdots + \psi_{p_0} Y_{t-p_0}) + (\xi_1 Y_{t-s-1} + \xi_2 Y_{t-s-2} + \cdots + \xi_{p_0} Y_{t-s-p_0}) + \varepsilon_t$$

where

$$\xi_i = -\rho\psi_i, i = 1, 2, \ldots, p_0, \text{ and } t = s + p_0 + 1, s + p_0 + 2, \ldots, T. \tag{2.2}$$

If $\rho = 1$, that is, if there exists a seasonal unit root, then (2.1) can be transformed to

$$Y_t = Y_{t-s} + \psi_1 \nabla_s Y_{t-1} + \psi_2 \nabla_s Y_{t-2} + \cdots + \psi_{p_0} \nabla_s Y_{t-p_0} + \varepsilon_t$$

or

$$Y_t = Y_{t-s} + \sum_{i=1}^{p_0} \psi_i \nabla_s Y_{t-i} + \varepsilon_t \tag{2.3}$$

For example, if $p_0 = 1$, the model (2.1) and (2.2) become

$$Y_t = \rho Y_{t-s} + \psi_1 Y_{t-1} + \xi_1 Y_{t-1-s} + \varepsilon_t \text{ , where } \xi_1 = -\rho\psi_1 . \tag{2.4}$$

$$Y_t = Y_{t-s} + \psi_1 \nabla_s Y_{t-1} + \varepsilon_t . \tag{2.5}$$

The seasonality of the seasonal time series is determined by s, and $s \geq 2$. For example, $s = 2$, $s = 4$, $s = 7$, and $s = 12$ indicate that the underlying process follows semi-annual, quarterly, weekly and monthly seasonal behaviors respectively. In particular,

$s = 5$ can be used to model the seasonal behavior of stock prices given that there are five trading days each week.

Note that $\xi_i = -\rho\psi_i, i = 1, 2, \dots, p_0$. The model (2.2) is used in Section 3 and Section 4. Also, we focus on the case where ε_t has $GARCH(1,1)$ structure. That is,

$$\varepsilon_t = \sqrt{h_t}\,\eta_t, \quad \eta_t \sim iid\,(0,1), \ E[\eta_t^3] = 0, \ E[\eta_t^4] < \kappa < \infty, \quad t = 1, 2, \dots, T;$$

$$h_t = \omega + \alpha\varepsilon_{t-1}^2 + \beta h_{t-1}, \qquad \text{with}$$

$$0 < \omega_l < \omega < \omega_u, 0 < \alpha_l < \alpha < \alpha_u, 0 < \beta_l < \beta < \beta_u, \alpha + \beta < 1, \ t = 2, 3, \dots, T. \tag{2.6}$$

The term $(1 - \rho B^s)$ in (2.1) signifies a seasonal component of period s. If $\rho = 1$, one obtains a seasonal unit root process, which means the effect of a particular seasonal value in a previous season on the corresponding seasonal value in the current season does not decay with time. Therefore, the null hypothesis of the seasonal unit root testing procedure derived from (2.1) ~ (2.6) is $\rho = 1$, whereas the alternative hypothesis is $|\rho| < 1$. The seasonal stationarity of the underlying process can be determined via such tests.

3. BOOTSTRAP-BASED SEASONAL UNIT ROOT TEST WITH $GARCH(1,1)$ ERRORS

In order to determine the existence of a seasonal unit root in a seasonal time series under $GARCH(1,1)$ errors, we extended the DHF test and developed a bootstrap-based testing procedure in which the residual-based method mentioned in Section 1 is applied. Assume that a realization $\{y_t\}_{t=1}^{T}$ is obtained from the model given in equation (2.2). The main steps for performing a residual-based seasonal unit root test based on the bootstrap method on $AR(p_0 + s)$ models with $GARCH(1,1)$ errors are listed below.

1) Use the least-squares estimates of ρ, ψ_i, and ξ_i, $i = 1, 2, \ldots, p_0$, as initial values and then employ maximum likelihood estimation (MLE) to obtain the estimates of ρ, ψ_i, ξ_i, $i = 1, 2, \ldots, p_0$, and $\delta = (\omega, \alpha, \beta)'$. Record these estimates as $\hat{\rho}$, $\hat{\psi}_i$, $\hat{\xi}_i$, $i = 1, 2, \ldots, p_0$, and $\hat{\delta} = (\hat{\omega}, \hat{\alpha}, \hat{\beta})'$.

   Note: The initial estimates of $\delta = (\omega, \alpha, \beta)'$ can be obtained by least squares fitting of the ARMA representation of the square of the residuals from the $AR(p_0 + s)$ regression. Or use any value of δ that meets the assumptions as the initial estimates. The results are the same.

2) Compute the test statistic,

$t(\hat{\rho}\,) = \ (\hat{\rho} - 1)*$

$$\left(-\sum_{t=1}^{T} \frac{\partial^2 l_t\big(\rho,\ \{\psi_i\}_{i=1}^{p_0},\ \{\xi_i\}_{i=1}^{p_0},\ \delta\big)}{\partial \rho^2}\right)^{\frac{1}{2}}_{\rho=\hat{\rho},\psi_1=\hat{\psi}_1,\psi_2=\hat{\psi}_2,\ldots,\psi_{p_0}=\hat{\psi}_{p_0},\xi_1=\hat{\xi}_1,\xi_2=\hat{\xi}_2,\ldots,\xi_{p_0}=\hat{\xi}_{p_0},\delta=\hat{\delta}}$$

where

$$l_t\big(\rho,\{\psi_i\}_{i=1}^{p_0},\{\xi_i\}_{i=1}^{p_0},\delta\big) = l_t\big(\rho,\psi_1,\psi_2,\ldots,\psi_{p_0},\xi_1,\xi_2,\ldots,\xi_{p_0},\delta\big)$$

$$= -\frac{1}{2}\ ln\, h_t - \frac{1}{2}\ \frac{\varepsilon_t^2}{h_t}\ \ ,$$

for $t = 1, 2, \ldots, T$.

Since $\xi_i = \ -\rho\psi_i\ , i = 1, 2, \ldots,\ p_0$ in equation (2.2), we actually compute the test statistic $t(\hat{\rho}\,)$ based on $\rho = \hat{\rho},\ \psi_i = \hat{\psi}_i,\ \xi_i = -\hat{\rho}\hat{\psi}_i,\ i = 1, 2, \ldots, p_0$, and $\delta = \hat{\delta}$.

3) Compute $\hat{\varepsilon}_t = y_t - \ \hat{\rho} y_{t-s} - \sum_{i=1}^{p_0} \hat{\psi}_i y_{t-i} + \sum_{i=1}^{p_0} \hat{\rho}\hat{\psi}_i y_{t-s-i}$, for $t = s + p_0 + 1,$ $s + p_0 + 2, \ldots\ ,\ T$. Let $\hat{\varepsilon}_t = 0$ for $t = 1, 2, \ldots, s + p_0$.

4) Compute $$\hat{h}_{s+p_0} = \widehat{\omega} + \hat{\alpha}\hat{\varepsilon}_{s+p_0-1}^2 + \ \hat{\beta}\ \hat{h}_{s+p_0-1}\ ,$$ where

$$\hat{\varepsilon}_{s+p_0-1}^2\ =\ \hat{h}_{s+p_0-1} = \frac{1}{T} \sum_{i=s+p_0+1}^{T} \hat{\varepsilon}_i^2\ ;$$

$$\hat{h}_t\ = \widehat{\omega} + \hat{\alpha}\hat{\varepsilon}_{t-1}^2 + \ \hat{\beta}\ \hat{h}_{t-1}\ ,\ \ t =\ s + p_0 + 1,\ s + p_0 + 2,\ \ldots\ ,\ T.$$

5) Let $\hat{\eta}_t = \frac{\hat{\varepsilon}_t}{\sqrt{\hat{h}_t}}$, and let $\widetilde{\eta_t}$ be the centered $\hat{\eta}_t$, for $t = s + p_0 + 1,\ s + p_0 + 2,\ \dots$, T.

6) Resample η_t^* , $t = 1,\ 2, \dots, 2T$, from $\{\pm\widetilde{\eta_t}\}_{t=s+p_0+1}^{T}$. Note that $\{\pm\widetilde{\eta_t}\}_{t=s+p_0+1}^{T}$ contain both the $\widetilde{\eta_t}$ and the values $\widetilde{\eta_t}$ multiplied by -1. This ensures the symmetry of the underlying distribution that will be resampled.

7) Compute $h_t^* = \widehat{\omega} + (\hat{\alpha}\eta_{t-1}^{*2} + \hat{\beta})h_{t-1}^*$, for $t = 2,\ 3,\ \dots,\ 2T$. And let $h_1^* = \hat{\varepsilon}_1^2$, or $h_1^* = \hat{h}_1$.

8) Compute $y_t^* = y_{t-s}^* + \sum_{i=1}^{p_0} \hat{\psi}_i y_{t-i}^* - \sum_{i=1}^{p_0} \hat{\rho}\hat{\psi}_i\, y_{t-s-i}^* + \sqrt{h_t^*}\eta_t^*$,

$t = s + p_0 + 1,\ s + p_0 + 2,\ \dots,\ 2T,$ using

$y_t^* = 0$ for $t = 1, 2, \dots, s + p_0$.

9) To reduce the effect of the initial conditions, drop the first $T - s - p_0$ values of y_t^* . Also re-label *t* so that the new values read from 1 to *T*. Fit y_t^* against y_{t-s}^* , y_{t-i}^* , and y_{t-s-i}^*, $i = 1,\ 2,\ \dots,\ p_0$. And estimate ρ^* , ψ_i^* , and ξ_i^*, $i = 1,\ 2, \dots,\ p_0$, using least squares.

10) Use the least-squares estimates as initial values and obtain MLEs of $\rho^*, \{\psi_i^*\}_{i=1}^{p_0}$,

$\{\xi_i^*\}_{i=1}^{p_0}$, and $\delta^* = (\omega^*, \alpha^*, \beta^*)'$, and denote these estimates as $\hat{\rho}^*, \{\hat{\psi}_i^*\}_{i=1}^{p_0}$, $\{\hat{\xi}_i^*\}_{i=1}^{p_0}$,

and $\hat{\delta}^* = (\hat{\omega}^*, \hat{\alpha}^*, \hat{\beta}^*)'$.

Note: The initial estimates of $\delta^* = (\omega^*, \alpha^*, \beta^*)'$ *can use* $\hat{\delta} = (\hat{\omega}, \hat{\alpha}, \hat{\beta})'$ *obtained in Step 1). Or use any value of* δ^* *that meets the assumptions. The results are the same.*

11) Compute the bootstrap test statistic,

$$t^*(\hat{\rho}^*) = (\hat{\rho}^* - 1)\left(-\sum_{t=1}^{T} \frac{\partial^2 l_t^*\left(\rho^*, \{\psi_i^*\}_{i=1}^{p_0}, \{\xi_i^*\}_{i=1}^{p_0}, \delta^*\right)}{\partial \rho^{*2}}\right)^{\frac{1}{2}}_{\substack{\rho^*=\hat{\rho}^*, \psi_1^*=\hat{\psi}_1^*, \psi_2^*=\hat{\psi}_2^*, \ldots, \psi_{p_0}^*=\hat{\psi}_{p_0}^*, \\ \xi_1^*=\hat{\xi}_1^*, \xi_2^*=\hat{\xi}_2^*, \ldots, \xi_{p_0}^*=\hat{\xi}_{p_0}^*, \delta^*=\hat{\delta}^*}}$$

where

$$l_t^*\left(\rho^*, \{\psi_i^*\}_{i=1}^{p_0}, \{\xi_i^*\}_{i=1}^{p_0}, \delta^*\right) = l_t^*\left(\rho^*, \psi_1^*, \psi_2^*, \ldots, \psi_{p_0}^*, \xi_1^*, \xi_2^*, \ldots, \xi_{p_0}^*, \delta^*\right)$$

$$= -\frac{1}{2} \ln h_t^* - \frac{1}{2} \frac{\varepsilon_t^{*2}}{h_t^*} \ ,$$

and $t = 1, 2, \ldots, T$.

Again, since $\xi_i = -\rho\psi_i$, $i = 1, 2, \ldots, p_0$ in equation (2.2), we compute the test statistic $t^*(\hat{\rho}^*)$ based on $\rho^* = \hat{\rho}^*$, $\psi_i^* = \hat{\psi}_i^*$, $\xi_i^* = \hat{\rho}^*\hat{\psi}_i^*$, $i = 1, 2, \ldots, p_0$, and $\delta^* = \hat{\delta}^*$.

12) Repeat Step 6) ~ 11) B times, say $B = 1{,}000$, and calculate the lower 5^{th} percentile of $t^*(\hat{\rho}^*)$, $t^*_{0.05}$, then compare $t^*_{0.05}$ with $t(\hat{\rho})$. If $t(\hat{\rho}) < t^*_{0.05}$, reject H_0 and let rej equal 1; otherwise, do not reject and let rej equal 0.

13) Repeat Step 1) ~ 12) M times, say $M = 1{,}000$, and calculate the significance level (empirical size) or the power of the test as: $level$ (or $power$) $= \frac{\sum rej}{M}$.

4. MONTE CARLO SIMULATION STUDY

In this section, we assume $p_0 = 1$ and $s \in \{5,12\}$. The extension to other values of p_0 and s is straightforward. Equation (2.2) is employed together with Equation (2.6) to generate the raw time series $\{y_t\}_{t=1}^{2T}$. That is, we performed Monte Carlo simulation study in the following procedure. We used Equation (2.2) and Equation (2.6) to generate $\{y_t\}_{t=1}^{2T}$, and then threw away the first T values of the series and re-labeled *t* to go from 1 to *T*. Then fit the Model (2.2) to the re-labeled time series of length *T* and calculated the least-squares estimates of ρ and other coefficients. The same goes for Step 9) under Section 3.

The Monte Carlo simulations and bootstrap procedures were carried out using MATLAB. Two types of distributions were assumed for the centered error terms, one is standard normal and the other is t-distribution with 7 degrees of freedom. The simulation results for $s = 5$, $T = 100$, $T = 200$, and $T = 400$ are given in Tables 4.1 ~ 4.4; Tables 4.5 ~ 4.8 include the results for $s = 12$, $T = 100$, $T = 200$, and $T = 400$. We did 1,000 simulations and 1,000 bootstraps for each test.

The coefficients used are $\rho \in \{0.5, 0.9, 1.0\}$, $\psi_1 \in \{0.2, 0.5, 0.9\}$, where ρ actually represents the seasonal root. For both $s = 5$ and $s = 12$, (α, β) combinations considered are (0.5,0.4), and (0.4,0.2). The first (α, β) combinations were intentionally selected so that $\alpha + \beta$ is close to 1 in order to demonstrate that the procedure works well even when these two parameters take values close to the $\alpha + \beta < 1$ threshold needed for stationarity of the $GARCH$ process. Unreported results for cases where $\alpha + \beta \ll 1$ show good power and size properties. To save space, results for all combinations are not reported but are available upon request from the first author.

Table 4.1. Estimated Coverage Probabilities for the Model with $s = 5, \alpha = 0.5, \beta = 0.4$, and Normal Errors

Coefficients		**Sample Size**					
		100		**200**		**400**	
$\boldsymbol{\rho}$	$\boldsymbol{\psi_1}$	**Size**	**Power**	**Size**	**Power**	**Size**	**Power**
1	0.2	0.044		0.048		0.062	
1	0.5	0.05		0.049		0.059	
1	0.9	0.052		0.054		0.058	
0.9	0.2		0.938		0.999		1
0.9	0.5		0.94		0.999		1
0.9	0.9		0.925		0.999		1
0.5	0.5		1		1		1
0.5	0.2		1		1		1

Table 4.2. Estimated Coverage Probabilities for the Model with $s = 5, \alpha = 0.4, \beta = 0.2$, and Normal Errors

Coefficients		**Sample Size**					
		100		**200**		**400**	
$\boldsymbol{\rho}$	$\boldsymbol{\psi_1}$	**Size**	**Power**	**Size**	**Power**	**Size**	**Power**
1	0.2	0.049		0.045		0.055	
1	0.5	0.048		0.046		0.055	
1	0.9	0.048		0.053		0.051	
0.9	0.2		0.885		0.997		1
0.9	0.5		0.883		0.997		1
0.9	0.9		0.85		0.997		1
0.5	0.5		1		1		1
0.5	0.2		1		1		1

Table 4.3. Estimated Coverage Probabilities for the Model with $s = 5, \alpha = 0.5, \beta = 0.4$, and t_7 Errors

Coefficients		**Sample Size**					
		100		**200**		**400**	
$\boldsymbol{\rho}$	$\boldsymbol{\psi_1}$	**Size**	**Power**	**Size**	**Power**	**Size**	**Power**
1	0.2	0.046		0.037		0.05	
1	0.5	0.044		0.038		0.049	
1	0.9	0.052		0.041		0.049	
0.9	0.2		0.959		1		1
0.9	0.5		0.959		1		1
0.9	0.9		0.948		1		1
0.5	0.5		1		1		1
0.5	0.2		1		1		1

Table 4.4. Estimated Coverage Probabilities for the Model with $s = 5, \alpha = 0.4, \beta = 0.2$, and t_7 Errors

Coefficients		**Sample Size**					
		100		**200**		**400**	
$\boldsymbol{\rho}$	$\boldsymbol{\psi_1}$	**Size**	**Power**	**Size**	**Power**	**Size**	**Power**
1	0.2	0.057		0.053		0.057	
1	0.5	0.061		0.049		0.056	
1	0.9	0.057		0.054		0.052	
0.9	0.2		0.909		0.997		1
0.9	0.5		0.908		0.997		1
0.9	0.9		0.876		0.997		1
0.5	0.5		1		1		1
0.5	0.2		1		1		1

Table 4.5. Estimated Coverage Probabilities for the Model with $s = 12, \alpha = 0.5, \beta = 0.4$, and Normal Errors

Coefficients		**Sample Size**					
		100		**200**		**400**	
$\boldsymbol{\rho}$	$\boldsymbol{\psi_1}$	**Size**	**Power**	**Size**	**Power**	**Size**	**Power**
1	0.2	0.053		0.049		0.042	
1	0.5	0.053		0.05		0.043	
1	0.9	0.052		0.049		0.041	
0.9	0.2		0.899		0.999		1
0.9	0.5		0.904		0.999		1
0.9	0.9		0.898		0.999		1
0.5	0.5		1		1		1
0.5	0.2		1		1		1

Table 4.6. Estimated Coverage Probabilities for the Model with
$s = 12, \alpha = 0.4, \beta = 0.2$, and Normal Errors

Coefficients		**Sample Size**					
		100		**200**		**400**	
$\boldsymbol{\rho}$	$\boldsymbol{\psi_1}$	**Size**	**Power**	**Size**	**Power**	**Size**	**Power**
1	0.2	0.045		0.053		0.033	
1	0.5	0.047		0.054		0.033	
1	0.9	0.054		0.053		0.035	
0.9	0.2		0.821		0.997		1
0.9	0.5		0.814		0.997		1
0.9	0.9		0.815		0.996		1
0.5	0.5		1		1		1
0.5	0.2		1		1		1

Table 4.7. Estimated Coverage Probabilities for the Model with
$s = 12, \alpha = 0.5, \beta = 0.4$, and t_7 Errors

Coefficients		**Sample Size**					
		100		**200**		**400**	
$\boldsymbol{\rho}$	$\boldsymbol{\psi_1}$	**Size**	**Power**	**Size**	**Power**	**Size**	**Power**
1	0.2	0.04		0.048		0.048	
1	0.5	0.041		0.053		0.05	
1	0.9	0.042		0.052		0.05	
0.9	0.2		0.942		1		1
0.9	0.5		0.948		1		1
0.9	0.9		0.942		1		1
0.5	0.5		1		1		1
0.5	0.2		1		1		1

Table 4.8. Estimated Coverage Probabilities for the Model with
$s = 12, \alpha = 0.4, \beta = 0.2$, and t_7 Errors

Coefficients		**Sample Size**					
		100		**200**		**400**	
$\boldsymbol{\rho}$	$\boldsymbol{\psi_1}$	**Size**	**Power**	**Size**	**Power**	**Size**	**Power**
1	0.2	0.044		0.059		0.044	
1	0.5	0.044		0.061		0.047	
1	0.9	0.048		0.061		0.049	
0.9	0.2		0.864		0.999		1
0.9	0.5		0.868		0.998		1
0.9	0.9		0.857		0.999		1
0.5	0.5		1		1		1
0.5	0.2		1		1		1

It can be observed from the above results that the finite sample properties of the seasonal unit root test are pretty good and reasonable. In general, the size approaches 0.05, the true significance level as the sample size increases from 100 to 400. The approximate 95% confidence limits for the estimated significance level based on 1,000 simulation runs can be calculated as $0.05 \pm 1.96\{(0.05 \times 0.95)/1000\}^{1/2} = 0.05 \pm 0.0135$. Most sizes we obtained are within the approximate 95% confidence limits except for three tests given under normal errors and $s = 12,\ \alpha = 0.4,\ \beta = 0.2,\ n = 400$. In addition, the seasonality, s, has some effect on the size. As s increases, say, from 5 to 12, the size of the test gets a little bit more deviated from 0.05, especially if the sample size is small, say, $n = 100$. For example, compare the results in Table 4.1 vs. Table 4.5, Table 4.2 vs. Table 4.6, and Table 4.3 vs. Table 4.7. The power of the test increases with an increase in $|1-\rho|,\ |1-\psi_1|,$ or the sample size n. It's not obvious that the power of the test is affected by $s,\ \alpha,\ \beta,$ or $\alpha+\beta$ though. The lowest power is 0.814 obtained when testing the seasonal unit root on the time series with $s = 12, \alpha = 0.4, \beta = 0.2, \rho = 0.9, \psi_1 = 0.5,$ and $n = 100$. The highest power is approximately 1.

The pattern described above is similar under the normal and the t-distribution. Overall, the simulation results show that the proposed methods work reasonably well for all combinations of the parameters and coefficients considered, maintaining a near nominal size and achieving high power.

5. CONCLUSION AND FUTURE WORK

A bootstrap-based seasonal unit root test for seasonal time series with $GARCH(1,1)$ innovations is explored by extending the DHF test and employing the residual-based method. The Monte Carlo simulation results show that our bootstrap-based seasonal unit root test achieves reasonable and good small sample properties regarding both size and power at different combinations of the seasonal roots, the regular autoregressive coefficients, and the $GARCH(1,1)$ coefficients. Extensions that are planned are to develop a comprehensive bootstrap-based test to detect both seasonal and non-seasonal unit roots, and to develop a procedure where the knowledge of the order of the autoregression model is not needed.

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SECTION

2. CONCLUSION

Unit root testing is an important research field in time series analysis. It's not only used to detect any possible non-stationarity existing in the time series, but also employed to investigate certain economic and financial hypotheses. The unit root testing procedures developed in the dissertation are based on a very efficient resampling approach, bootstrap. And different types of innovations of the time series are considered, including independent innovations, weakly dependent innovations, and innovations with conditionally heteroskedastic volatility. Monte Carlo simulations are carefully conducted for each model. The simulation results show that each testing procedure had good small sample properties with respect to size and power.

In Paper I, testing for two unit roots in a time series with independent innovations is discussed. The bootstrap version of a sequential testing procedure proposed by Dickey and Pantula is presented with details. The Dickey-Pantula test requires taking p differences of the time series where p is the order of the autoregressive process, and the table of critical values used in the raw test is not complete. The bootstrap version of the raw test overcomes these restrictions. Preliminary results show that the bootstrap version of the Dickey-Pantula test is superior. In the future, we will assume the value of p is unknown and develop a sieve bootstrap-version of Dikey-Pantula tests for multiple unit roots.

In Paper II, a general bootstrap-based procedure for conducting unit root tests in higher order autoregressive models with $GARCH$ errors was introduced. This procedure is based on the seminal work of two authors who detailed the implementation of the method for first order autoregressive processes. Their method is extended to general autoregressive processes using a transformed series in the paper. Simulation results indicate that the proposed method mitigates the size distortion issue present in the asymptotic-based tests and achieves high powers at different combinations of the autoregressive roots and $GARCH$ coefficients. In the future, developing a bootstrap-based unit root test for the case where the underlying process is $ARIMA$ with unknown orders may be considered.

In Paper III, seasonal unit root testing is emphasized. By extending the DHF test and employing the residual-based method, a bootstrap-based seasonal unit root test for seasonal time series with $GARCH(1,1)$ innovations is proposed. The Monte Carlo simulation results show that our bootstrap-based seasonal unit root test achieves reasonable and good small sample properties regarding both size and power at different combinations of the seasonal roots, the regular autoregressive coefficients, and the $GARCH(1,1)$ coefficients. This work can be regarded as one part of detecting both seasonal and non-seasonal multiple unit roots in a more complicated bootstrap-based test, which will be pursued after my graduation.

APPENDIX A

MATLAB ALGORITHM FOR SECOND PAPER

```
%H0: r=0; H1: r<0

B = 1000;
M = 1000;
d = 400;
T = 400;
aalpha = 0.05;

rho1 = 0.5;
rho2 = 0.2;

r = rho1-1;
delta1 = rho2;
%delta1 can be any real number
rr = r;
ddelta1 = delta1;

phi1 = rho1+rho2;
phi2 = -1*rho1*rho2;

alpha = 0.399;
beta = 0.6;
w = 1-alpha-beta;
```

```
theta0 = [rr;ddelta1;w;alpha;beta];

Av = [0 0 0 1 1];

bv = 0.999999999;

lbb = [-999999999;-999999999;0.000000001;0;0];

ubb = [999999999;999999999;1;0.999999999;0.999999999];


opts = optimset('Display', 'off', 'Algorithm', 'sqp');


y = zeros(d+T,1);

h = zeros(d+T,1);

eps = zeros(d+T,1);


z = zeros(T,1);

z_star = zeros(T,1);

z1 = zeros(T,1);

z1_star = zeros(T,1);

z2 = zeros(T,1);

z2_star = zeros(T,1);

zx = zeros(T,2);

zx_star = zeros(T,2);


eps_ml = zeros(T,1);

h_ml = zeros(T,1);
```

```
eps_star_ml = zeros(T,1);
h_star_ml = zeros(T,1);

yita_ml = zeros(T,1);
cyita_ml = zeros(T,1);
ccyita = zeros(2*T,1);
yita_star = zeros(d+T,1);

h_star = zeros(d+T,1);
y_star = zeros(d+T,1);

start1 = zeros(B,1);
sst1 = zeros(B,1);
rej1 = zeros(M,1);
bpt1 = zeros(M,1);
rawt1 = zeros(M,1);
srt1 = zeros(M,1);

allstart1 = zeros(B*M,1);

mcstart1 = zeros(B,M);
mcsst1 = zeros(B,M);
```

```
for MC = 1:M
    rng('default');
    rng(MC);
    yita = randn(d+T,1);

    h(1) = 1;
    for t=2:d+T
        h(t) = w+(beta+alpha*(yita(t-1)^2))*h(t-1);
    end

    eps = sqrt(h).*yita;

    y(1) = 0;
    y(2) = 0;
    for t=3:d+T
        y(t) = phi1*y(t-1)+phi2*y(t-2)+eps(t);
    end

    z(1:T) = diff(y(d:d+T),1);
    z1(1:T) = y(d:d+T-1);
    z2(1:T) = diff(y(d-1:d+T-1),1);
    zx = [z1 z2];
    rawlmf = LinearModel.fit(zx,z,'Intercept',false);
```

```
    r_ls = rawlmf.Coefficients.Estimate(1);

    delta1_ls = rawlmf.Coefficients.Estimate(2);

    rr = r_ls;

    ddelta1 = delta1_ls;


    [theta] =
fmincon(@(theta)AR2Lgarch(theta,z,z1,z2),theta0,Av,bv,[],[]
,lbb,ubb,[],opts);


    r_ml = theta(1);

    rr_star = r_ml;

    delta1_ml = theta(2);

    ddelta1_star = delta1_ml;

    w_ml = theta(3);

    alpha_ml = theta(4);

    beta_ml = theta(5);


    for t=1:T

        eps_ml(t) = z(t)-r_ml*z1(t)-delta1_ml*z2(t);

    end


    h_ml(1) = sum(eps_ml(1:T).^2)/T; %better results
```

```
    for t=2:T
        h_ml(t) = w_ml+alpha_ml*eps_ml(t-
1)^2+beta_ml*h_ml(t-1);
    end;

    [srt1(MC),rawt1(MC)] =
AR2tValue(z1,h_ml,alpha_ml,beta_ml,r_ml,eps_ml);

    yita_ml = eps_ml./sqrt(h_ml);
    cyita_ml = yita_ml-sum(yita_ml)/T;

    theta0_star =
[rr_star;ddelta1_star;w_ml;alpha_ml;beta_ml];
    for i=1:T
        ccyita(2*i) = cyita_ml(i);
        ccyita(2*i-1) = -1*cyita_ml(i);
    end

    for BC = 1:B
        rng('default');
        rng(BC+M);
        randomIndex = randi([1,2*T],d+T,1);
        yita_star = ccyita(randomIndex);
```

```
        h_star(1) = 1;
        for t=2:d+T
            h_star(t) = w_ml+(beta_ml+alpha_ml*yita_star(t-
1)^2)*h_star(t-1);
        end

        eps_star = sqrt(h_star).*yita_star;

        y_star(1) = 0;
        y_star(2) = 0;
        for t = 3:d+T
            y_star(t) = (1+delta1_ml)*y_star(t-1)-
delta1_ml*y_star(t-2)+eps_star(t);
        end

        z_star(1:T) = diff(y_star(d:d+T),1);
        z1_star(1:T) = y_star(d:d+T-1);
        z2_star(1:T) = diff(y_star(d-1:d+T-1),1);
        zx_star = [z1_star z2_star];
        starlmf =
LinearModel.fit(zx_star,z_star,'Intercept',false);
        r_star_ls = starlmf.Coefficients.Estimate(1);
        delta1_star_ls = starlmf.Coefficients.Estimate(2);
```

```
        rr_star = r_star_ls;
        ddelta1_star = delta1_star_ls;

        [theta_star] =
fmincon(@(theta_star)AR2Lgarch(theta_star,z_star,z1_star,z2
_star),theta0_star,Av,bv,[],[],lbb,ubb,[],opts);
        r_star_ml = theta_star(1);
        delta1_star_ml = theta_star(2);
        w_star_ml = theta_star(3);
        alpha_star_ml = theta_star(4);
        beta_star_ml = theta_star(5);

        for t=1:T
           eps_star_ml(t) = z_star(t)-r_star_ml*z1_star(t)-
delta1_star_ml*z2_star(t);
        end

        h_star_ml(1) = sum(eps_star_ml(1:T).^2)/T; %better
results

        for t=2:T
```

```
            h_star_ml(t) =
w_star_ml+alpha_star_ml*eps_star_ml(t-
1)^2+beta_star_ml*h_star_ml(t-1);
        end

        [sst1(BC),start1(BC)] =
AR2tValue(z1_star,h_star_ml,alpha_star_ml,beta_star_ml,r_st
ar_ml,eps_star_ml);

        mcstart1(BC,MC) = start1(BC);
        mcsst1(BC,MC) = sst1(BC);

    end
    bpt1(MC) = prctile(start1,aalpha*100);

    if (rawt1(MC) < bpt1(MC))
        rej1(MC) = 1;
    end
    if (rawt1(MC) >= bpt1(MC))
        rej1(MC) = 0;
    end

end
```

```
sig1 = sum(rej1)/M;


for MC = 1:M
    for BC = 1:B
        allstart1(BC+B*(MC-1)) = mcstart1(BC,MC);
    end;
end


sig1
%rawt1
%allstart1
```

```
function L2 = AR2Lgarch(theta,z,z1,z2)
%qMLE's -1*likelihood: theta = [rho1;rho2;w;alpha;beta];
%H0:rho1=0; H1:rho1<0


rho1 = theta(1);
rho2 = theta(2);
w = theta(3);
alpha = theta(4);
beta = theta(5);
```

```
n = length(z);%the length of time series considered is T,
%not T+2, T=n

eps = zeros(n,1);
for i = 1:n
    eps(i) = z(i)-rho1*z1(i)-rho2*z2(i);
end
eps2 = eps.^2;

h = zeros(n,1);
h(1) = sum(eps2)/n;
for i = 2:n
    h(i) = w+alpha*eps2(i-1)+beta*h(i-1);
end

sqrth = sqrt(h);
x = eps./sqrth; %x is yita

l = -0.5*log(h)-0.5*x.^2;
L2 = sum(-1*l)/n;

end
```

```
function [st,tstat] = AR2tValue(z1,h,alpha,beta,r,eps)
%h=h_hat(1:T);alpha=alpha_hat;beta=beta_hat;phi=phi_hat;
%eps=eps_hat(1:T);
%zt=r*z1t+delta1*z2t+epst;
%H0:r=0; H1:r<0

n = length(z1);
st = -1*z1(1)^2/h(1);
%t=1 and 2 are considered only above

for t=2:n
    st1 = 0;
    st2 = 0;
    for i=1:t-1
        st1 = st1+beta^(i-1)*eps(t-i)*z1(t-i);
        st2 = st2+beta^(i-1)*z1(t-i)^2;
    end
    st = st-z1(t)^2/h(t)+(2*alpha^2/h(t)^2-
4*alpha^2*eps(t)^2/h(t)^3)*st1^2+4*alpha*eps(t)*z1(t)/h(t)^
2*st1+alpha/h(t)*(eps(t)^2/h(t)-1)*st2;
end
tstat = sqrt(-1*st)*r;
end
```

APPENDIX B

MATLAB ALGORITHM FOR THIRD PAPER

```
S = 5;

B = 1000;
M = 1000;

d = 100;
T = 100;

aalpha = 0.05;

rho = 1;
sai1 = 0.9;
ksai1 = -1*rho*sai1;

rrho = rho;
ssai1 = sai1;
kksai1 = ksai1;

alpha = 0.5;
beta = 0.4;
w = 1-alpha-beta;

theta0 = [rrho;ssai1;kksai1;w;alpha;beta];
```

```
rawAv = [0 0 0 0 1 1];
rawbv = 0.999999999;
rawlbb = [-1000;-1000;-1000000;0.000000001;0;0];
rawubb = [1000;1000;1000000;1;0.999999999;0.999999999];

opts = optimset('Display','off','Algorithm','sqp');

y = zeros(d+T,1);
h = zeros(d+T,1);
eps = zeros(d+T,1);

z = zeros(T,1);
z_star = zeros(T,1);

zs1_star = zeros(T,1);

z1 = zeros(T,1);
z1_star = zeros(T,1);
z4 = zeros(T,1);
z4_star = zeros(T,1);
z5 = zeros(T,1);
z5_star = zeros(T,1);
zx = zeros(T,2);
```

```
zx_star = zeros(T,2);

eps_ml = zeros(T,1);
h_ml = zeros(T,1);
eps_star_ml = zeros(T,1);
h_star_ml = zeros(T,1);

yita_ml = zeros(T,1);
cyita_ml = zeros(T,1);
ccyita = zeros(2*T,1);
yita_star = zeros(d+T,1);

h_star = zeros(d+T,1);
y_star = zeros(d+T,1);

start1 = zeros(B,1);
sst1 = zeros(B,1);
rawt1 = zeros(M,1);
srt1 = zeros(M,1);
bpt1 = zeros(M,1);
rej1 = zeros(M,1);

allstart1 = zeros(B*M,1);
```

```
mcstart1 = zeros(B,M);
mcsst1 = zeros(B,M);

for MC = 1:M
    rng('default');
    rng(MC);
    yita = randn(d+T,1);

    h(1) = 1;
    for t=2:d+T
        h(t) = w+(beta+alpha*(yita(t-1)^2))*h(t-1);
    end

    eps = sqrt(h).*yita;

    for t=1:S+1
        y(t) = 0;
    end

    for t = S+2:d+T
        y(t) = rho*y(t-S)+sai1*y(t-1)+ksai1*y(t-S-
1)+eps(t);
    end
```

```
    z(1:T) = y(d+1:d+T);
    z1(1:T) = y(d:d+T-1);
    z4(1:T) = y(d-S+1:d+T-S);
    z5(1:T) = y(d-S:d+T-S-1);

    zx = [z4 z1 z5];
    rawlmf = LinearModel.fit(zx,z,'Intercept',false);
    rho_ls = rawlmf.Coefficients.Estimate(1);
    sai1_ls = rawlmf.Coefficients.Estimate(2);
    ksai1_ls = rawlmf.Coefficients.Estimate(3);
    rrho = rho_ls;
    ssai1 = sai1_ls;
    kksai1 = ksai1_ls;

    [theta] =
fmincon(@(theta)rawSARLgarch(theta,z,z4,z1,z5),theta0,rawAv
,rawbv,[],[],rawlbb,rawubb,[],opts);

    rho_ml = theta(1);
    rrho_star = rho_ml;
    sai1_ml = theta(2);
    ssai1_star = sai1_ml;
    ksai1_ml = theta(3);
```

```
    kksai1_star = ksai1_ml;
    w_ml = theta(4);
    alpha_ml = theta(5);
    beta_ml = theta(6);

    for t = 1:T;
        eps_ml(t) = z(t)-rho_ml*z4(t)-
sai1_ml*z1(t)+rho_ml*sai1_ml*z5(t);
    end
    h_ml(1) = sum(eps_ml(1:T).^2)/T;

    for t=2:T
        h_ml(t) = w_ml+alpha_ml*eps_ml(t-
1)^2+beta_ml*h_ml(t-1);
    end

    [srt1(MC),rawt1(MC)] =
rawSARtValue(z4,z5,h_ml,alpha_ml,beta_ml,rho_ml,sai1_ml,eps
_ml);

    yita_ml = eps_ml./sqrt(h_ml);
    cyita_ml = yita_ml-sum(yita_ml)/T;
```

```
    theta0_star =
[rrho_star;ssai1_star;kksai1_star;w_ml;alpha_ml;beta_ml];

    for i = 1:T
        ccyita(2*i) = cyita_ml(i);
        ccyita(2*i-1) = -1*cyita_ml(i);
    end

    for BC = 1:B
        rng('default');
        rng(BC+M);
        randomIndex = randi([1,2*T],d+T,1);
        yita_star = ccyita(randomIndex);
        h_star(1) = 1;
        for t=2:d+T
            h_star(t) = w_ml+(beta_ml+alpha_ml*yita_star(t-
1)^2)*h_star(t-1);
        end

        eps_star = sqrt(h_star).*yita_star;

        for t=1:S+1
            y_star(t) = 0;
```

```
        end

        for t=S+2:d+T
            y_star(t) = y_star(t-S)+sai1_ml*(y_star(t-1)-
y_star(t-S-1))+eps_star(t);
        end

        z_star(1:T) = y_star(d+1:d+T);
        z1_star(1:T) = y_star(d:d+T-1);
        z4_star(1:T) = y_star(d-S+1:d+T-S);
        z5_star(1:T) = y_star(d-S:d+T-S-1);
        zx_star = [z4_star z1_star z5_star];

        starlmf =
LinearModel.fit(zx_star,z_star,'Intercept',false);
        rho_star_ls = starlmf.Coefficients.Estimate(1);
        sai1_star_ls = starlmf.Coefficients.Estimate(2);
        ksai1_star_ls = starlmf.Coefficients.Estimate(3);
        rrho_star = rho_star_ls;
        ssai1_star = sai1_star_ls;
        kksai1_star = ksai1_star_ls;
```

```
        [theta_star] =
fmincon(@(theta_star)rawSARLgarch(theta_star,z_star,z4_star
,z1_star,z5_star),theta0_star,rawAv,rawbv,[],[],rawlbb,rawu
bb,[],opts);
        rho_star_ml = theta_star(1);
        sai1_star_ml = theta_star(2);
        ksai1_star_ml = theta_star(3);
        w_star_ml = theta_star(4);
        alpha_star_ml = theta_star(5);
        beta_star_ml = theta_star(6);

        for t=1:T
           eps_star_ml(t) = z_star(t)-
rho_star_ml*z4_star(t)-
sai1_star_ml*z1_star(t)+rho_star_ml*sai1_star_ml*z5_star(t)
;
        end

        h_star_ml(1) = sum(eps_star_ml(1:T).^2)/T;

        for t=2:T
```

```
            h_star_ml(t) =
w_star_ml+alpha_star_ml*eps_star_ml(t-
1)^2+beta_star_ml*h_star_ml(t-1);
        end

        [sst1(BC),start1(BC)] =
rawSARtValue(z4_star,z5_star,h_star_ml,alpha_star_ml,beta_s
tar_ml,rho_star_ml,sai1_star_ml,eps_star_ml);
        mcstart1(BC,MC) = start1(BC);
        mcsst1(BC,MC) = sst1(BC);

    end

    bpt1(MC) = prctile(start1,aalpha*100);

    if (rawt1(MC) < bpt1(MC))
        rej1(MC) = 1;
    end
    if (rawt1(MC) >= bpt1(MC))
        rej1(MC) = 0;
    end

end
```

```
sig1 = sum(rej1)/M;

for MC = 1:M
    for BC = 1:B
        allstart1(BC+B*(MC-1)) = mcstart1(BC,MC);
    end;
end

sig1
```

```
function L2 = rawSARLgarch(theta,z,z4,z1,z5)
%qMLE's -1*likelihood: theta =
%[rho;sai1;ksai1;w;alpha;beta];
%H0:rho=1; H1:|rho|<1
%L2 or -1*L2 has to be real numbers

rho = theta(1);
sai1 = theta(2);
w = theta(4);
alpha = theta(5);
beta = theta(6);
n = length(z);
```

```
eps = zeros(n,1);
for i = 1:n
    eps(i) = z(i)-rho*z4(i)-sai1*z1(i)+rho*sai1*z5(i);
end
eps2 = eps.^2;

h = zeros(n,1);
h(1) = sum(eps2)/n;
for i = 2:n
    h(i) = w+alpha*eps2(i-1)+beta*h(i-1);
end

sqrth = sqrt(h);
x = eps./sqrth;

l = -0.5*log(h)-0.5*x.^2;
L2 = sum(-1*l)/n;
%cal the negative log likelihood with L2 - fmincon is used
%to get Minimum Likelihood Estimates and we are looking for
%Maximum Likelihood Estimates

end
```

```
function [st,tstat] =
rawSARtValue(z4,z5,h,alpha,beta,rho,sai1,eps)
%h=h_hat(1:T);alpha=alpha_hat;beta=beta_hat;eps=eps_hat(1:T
%);
%z=rho*z4+sai1*z1+ksai1*z5+epst; or
%z=rho*z4+sai1*(z1-rho*z5)+epst, where ksai1=-1*rho*sai1.
%If rho=1 or H0 holds, z=rho*z4+sai1*zs1+epst, where
%zs1=z1-z5.
%H0:rho=1; H1:|rho|<1
%z4=y(t-s), z5=y(t-s-1)

n = length(z4);
st = -1*(sai1*z5(1)-z4(1))^2/h(1);
%t=1 (and 2) are considered only above

for t=2:n
    st1 = 0;
    st2 = 0;
    for i=1:t-1
        st1 = st1+beta^(i-1)*eps(t-i)*(sai1*z5(t-i)-z4(t-
i));
        st2 = st2+beta^(i-1)*(sai1*z5(t-i)-z4(t-i))^2;
    end
```

```
    st = st-(sai1*z5(t)-z4(t))^2/h(t)+(2*alpha^2/h(t)^2-
4*alpha^2*eps(t)^2/h(t)^3)*st1^2+4*alpha*eps(t)*(sai1*z5(t)
-z4(t))/h(t)^2*st1+alpha/h(t)*(eps(t)^2/h(t)-1)*st2;
end

tstat = sqrt(-1*st)*(rho-1);


end
```

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VITA

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