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# Zero-dimensional spaces and their inverse limits

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ZERO-DIMENSIONAL SPACES AND THEIR INVERSE LIMITS

by

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# ABSTRACT

In this dissertation we investigate zero-dimensional compact metric spaces and their inverse limits. We construct an uncountable family of zero-dimensional compact metric spaces homeomorphic to their Cartesian squares. It is known that the inverse limit on $[0, 1]$ with an upper semi-continuous function with a connected graph has either one or infinitely many points. We show that this result cannot be generalized to the inverse limits on simple triods or simple closed curves. In addition to that, we introduce a class of zero-dimensional spaces that can be obtained as the inverse limits of arcs. We complete by answering a problem by Kelly and Meddaugh about the limits of inverse limits.

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## 1. INTRODUCTION

The origins of the continuum theory date back to the 1880s. A *continuum* usually means a metric compact connected space. The original definition given by Cantor in 1883 was that a subset of a Euclidean space is a continuum, provided it is perfect and connected, see [Cantor, 1883]. Then in 1968 Kuratowski proved the equivalence of the two definitions. A great breakthrough in the history of continuum theory was made in 1890 when Peano proved that $[0, 1]^2$ is a continuous image of $[0, 1]$, see [Peano, 1890]. Generalizing Peano's idea, Hahn [1914] and Mazurkiewicz [1920] characterized continuous images of $[0, 1]$ as locally connected continua. Peano's result also led Urysohn to invent the notion of the topological dimension theory, another branch of topology. Urysohn [1922] and Menger [1923], working independently, defined the dimension inductively. But actually, in 1922–before the dimension was defined–Sierpiński described zero-dimensional spaces when he introduced the totally disconnected topological spaces, see [Sierpiński, 1921].

The theory of hyperspaces started with Hausdorff and Vietoris in the 1900s and the fundamentals of the subject were discovered during the 1920s and 1930s. The paper published in 1931, where Borsuk and Mazurkiewicz [1931] proved that the hyperspaces $2^X$ and $C(X)$ are arcwise connected showed the importance of the subject since it was the first paper about the arc structure of hyperspaces. Another paper, published by Kelley [1942], was an important milestone in the history of hyperspaces, since it was the first paper to study the hyperspace of hereditarily indecomposable continua. This was not, however, the only reason to make this paper very important. In his study, Kelley used Whitney maps for the first time in the investigation of hyperspaces, and Whitney maps became one of the standard tools in the study of hyperspaces. Moreover, he defined what is now known as the property of Kelley, or Kelley continua. In addition to the aforementioned, after Kelley's paper, the interest in hyperspace theory significantly increased among mathematicians, including those in the United States. In 1978, Sam B. Nadler, Jr. published a book, *Hyperspace of Sets*, see [Nadler, 1978]. This book is significant for continuum theorists since it was the first book published about hyperspaces and continuum theory and it influenced so many mathematicians to study continuum theory. Marjanović [1972] proved that there are exactly nine different zero-dimensional, compact, metric spaces $X$ including the point, the Cantor set, and the Pełczyński space, that are homeomorphic to their hyperspaces $2^X$ by generalizing a result published in 1965 by Pełczyński, see [Pełczyński, 1965].

During the 1920s and 1930s, another useful tool in topology and more generally in mathematics, the theory of inverse limits, was established. But the field became popular among mathematicians, when Capel showed in 1954 that the inverse limit of arcs with monotone bonding functions is an arc, see [Capel, 1954]. In 1959 Anderson and Choquet showed how a plane continuum with no two non-degenerate homeomorphic subcontinua can be constructed by using an inverse limit with continuous bonding functions, see [Anderson and Chouquet, 1959]. This was a breakthrough because the idea of inverse limits was being used for the first time to construct such a complicated continuum. Since then inverse limits have become an important tool in continuum theory.

In 2004, William Mahavier introduced the inverse limits with set-valued functions as inverse limits with closed subsets of the unit square, see [Mahavier, 2004] and then a book by Ingram and Mahavier [2010] had been published and the subject became even more popular among mathematicians, especially among continuum theorists. This new form of inverse limits made the subject popular also among researchers of economics and dynamical systems. With so many open problems about set-valued inverse limits the popularity of the subject increases every day.

This dissertation consists of three articles in which we focus on hyperspaces and the inverse limits of zero-dimensional compact metric spaces and present some results about the inverse limits of set-valued functions. In the first paper, motivated by the studies of Marjanović, we show that there exists uncountably many zero-dimensional compact metric spaces $X$ that are homeomorphic to their Cartesian products as well as their hyperspaces $F_n(X)$. In the second paper we show that the result by Roškarič and Tratnik [2015] , and by Banič and Kennedy [2015] cannot be generalized to simple triods and simple closed curves. In the same paper we also introduce a new method to construct a zero-dimensional compact metric space as the inverse limit of a single set-valued function on $[0, 1]$. Finally, in the last paper we provide an answer to a question posted by Kelly and Meddaugh [2015].

# 2. LITERATURE REVIEW

## 2.1. PRELIMINARY DEFINITIONS AND THEOREMS

We begin this section with some preliminary definitions and theorems from continuum theory. For a better understanding of the subject, one can refer to the important book *Continuum Theory* by Nadler [1992].

A topological space $X$ is a *continuum* if it is a nonempty, compact, connected metric space. A subset of a space $X$ that is a continuum is called a *subcontinuum* of $X$. A continuum $X$ is called *decomposable* if it is the union of two proper subcontinua. A non-degenerate continuum that is not decomposable is called *indecomposable*. An indecomposable continuum in which every proper subcontinuum is also indecomposable is called a *hereditarily indecomposable continuum*.

An *arc* is a continuum that is homeomorphic to a closed interval. A *simple closed curve* is a continuum that is homeomorphic to a circle. A *simple triod* is a finite graph that is the union of three arcs emanating from a single point, $v$, and otherwise disjoint from one another. The point $v$ is called the *vertex* of the triod.

Given continua $X$ and $Y$, a continuous function $f : X \to Y$ is called an $\epsilon-$map if for each $y \in Y$, diam $f^{-1}(y) < \epsilon$. A continuum $X$ is said to be *arc-like* if for every $\epsilon > 0$, there exists an $\epsilon-$map such that $f : X \to [0, 1]$.

## 2.2. HYPERSPACES OF ZERO-DIMENSIONAL SPACES

The fundamentals of both hyperspaces and zero-dimensional spaces date back to the 1920s and both subjects have been broadly used in the continuum theory since then.

A *hyperspace* of a topological space $X$ is the set of all closed subsets of $X$ but in this dissertation we consider metric spaces only. For a continuum $X$ define the following hyperspaces,

$$2^X = \{A \subset X | A \text{ is nonempty and closed } \}$$
$$C(X) = \{A \in 2^X | A \text{ is connected } \}$$
$$F_n(X) = \{A \subset X | \operatorname{card}(A) \le n\}$$

Let $X$ be a continuum. If $\epsilon > 0$ and $A \in 2^X$ then we define the *$\epsilon$-neighboorhood of $A$* by

$$N_d(\epsilon, A) = \{x \in X | \ d(x,a) < \epsilon \text{ for some } a \in A\}.$$

If $A, B \in 2^X$, then define the *the Hausdorff distance $H_d$* by the formula,

$$H_d(A,B) = \inf\{\epsilon > 0 | \ A \subset N_d(\epsilon, B) \text{ and } B \subset N_d(\epsilon, A)\}.$$

The inductive definition of dimension by Urysohn and Menger is as follows (see [Engelking, 1978]): To every regular space $X$ one assigns the *small inductive dimension of $X$*, denoted by $\operatorname{ind} X$, which is an integer larger than or equal to $-1$ or the infinite number $\infty$: the definition of the dimension function ind consists in the following conditions:

1. $\operatorname{ind} X = -1$ if and only if $X = \emptyset$;

2. $\operatorname{ind} X \le n$, where $n = 0, 1, \ldots,$ if for every point $x \in X$ and each neighborhood $V \subset X$ of the point $x$ there exists an open set $U \subset X$ such that

$$x \in U \subset V \text{ and } \operatorname{ind} \operatorname{bd}(U) \le n - 1;$$

3. $\operatorname{ind} X = n$ if $\operatorname{ind} X \le n$ and $\operatorname{ind} X > n - 1$, i.e., the inequality $\operatorname{ind} X \le n$ does not hold;

4. $\operatorname{ind} X = \infty$ if $\operatorname{ind} X > n$ for $n = -1, 0, 1, \ldots$

For a regular space $X$ if ind $X = 0$, then $X$ is called as *zero-dimensional*.

In 1922, before the dimension was defined, Sierpiński introduced the totally disconnected topological space as follows (see [Sierpiński, 1921]):

Let $X$ be a topological space. If for every pair $x$, $y$ of distinct points of $X$ there exists a clopen subset $U$ of $X$ such that $x \in U$ and $y \in X \setminus U$ then $X$ is called *totally disconnected*. It should be clear that every zero-dimensional space is totaly disconnected.

**Example 1** The best known examples of zero-dimensional compact metric spaces are finite sets, the Cantor set (Figure 2.1), and the Pełczyński compactum (Figure 2.2), that is obtained by adding a point to the middle of every deleted interval of the Cantor set.

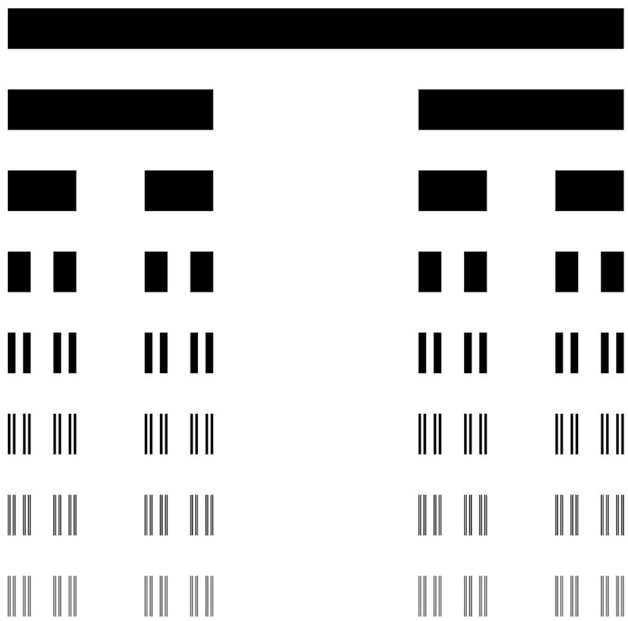

Figure 2.1. Cantor Set.

In 1972, Marjanović published an article about the zero-dimensional compact metric spaces $X$ that are homeomorphic to their hyperspace $2^X$ generalizing a result by Pełczyński [1965]. In his paper Marjanović showed that there exist exactly nine such spaces and the Cantor set, and the Pełczyński space are among those nine spaces. As a continuation to his studies, Marjanović published another paper in 1974, and showed that there exist infinitely many pairs of non-homeomorphic zero-dimensional compact metric spaces having their

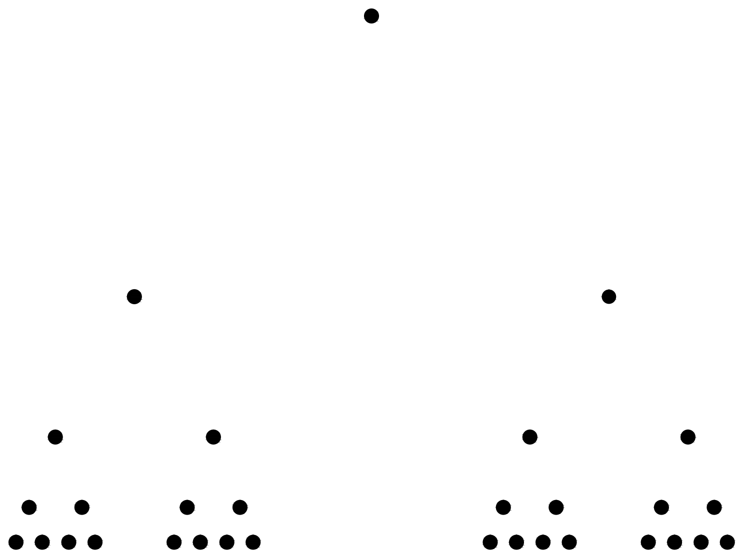

Figure 2.2. Pełczyński compactum.

squares homeomorphic i.e. spaces $X$ and $Y$ are not homeomorphic, while $X^2$ and $Y^2$ are homeomorphic, see [Marjanović, 1974]. Also, at the end of this paper Marjanović introduces a simpler construction of the sequence of zero-dimensional compact metric spaces homeomorphic to their hyperspace $2^X$. This construction is as follows: Let $C_0$ be the set of a point, and $C_1$ be the Cantor set. The space $C_2$, which is the Pełczyński space, is obtained by interpolation of a copy of $C_0$ in each deleted interval of $C_1$. The space $C_n$ is obtained from $C_{n-1}$, by interpolation a copy of $C_{n-2}$ in each deleted intervals of $C_{n-1}$.

The following definition plays an important role in the investigation of zero-dimensional compact spaces.

**Definition 1** The Cantor-Bendixson derivative of order $\alpha$, or $\alpha$-derivative of a compact space $X$ is defined inductively as follows:

1. $X^{(0)} = X$
2. $X^{(\alpha+1)} = \{x \in X^{(\alpha)} : x \text{ is a limit point in } X^{(\alpha)}\}$
3. For limit ordinals $\gamma$: $X^{(\gamma)} = \bigcap_{\beta<\gamma} X^{(\beta)}$

Then, the Cantor-Bendixson rank of a space $X$, denoted by $\operatorname{rank}(X)$ is defined as the least ordinal $\alpha$ such that $X^{(\alpha+1)} = \emptyset$.

The Cantor-Bendixson rank of a point $x$ in a set $X$, denoted by $CB(x)$ is defined, if $x$ has a countable neighborhood, by

$$CB(x) = \min\{ \operatorname{rank}(U) : U \text{ is a countable compact neighborhood of } x \text{ in } X\}.$$

Using this definition as a tool, Cantor and Bendixson constructed the full classification of the compact countable spaces as follows:

**Theorem 1** Let $\alpha$ be an ordinal, and let $X$ and $Y$ be compact countable spaces such that $\operatorname{rank}(X) = \operatorname{rank}(Y) = \alpha$ and $\operatorname{card} X^{\alpha} = \operatorname{card} Y^{\alpha}$, then $X$ and $Y$ are homeomorphic.

But such classification for compact uncountable spaces still remains as an open problem and in this dissertation we provide a partial answer to this.

## 2.3. INVERSE LIMITS

The roots of inverse limits go back to the 1920s and since then the subject have been widely used in the studies of continuum theory. Here by a *mapping*, we mean a continuous function. The definitions for the inverse limits of single-valued functions are as follows: Given a sequence, $\mathbf{X} = (X_i)_{i=1}^{\infty}$, of topological spaces and a sequence, $\mathbf{f} = (f_i)_{i=1}^{\infty}$, of continuous functions such that for each $i \in \mathbb{N}$, $f_i : X_{i+1} \to X_i$, the pair $\{\mathbf{X}, \mathbf{f}\}$ is called an *inverse sequence*. The *inverse limit* of this inverse sequence is defined to be the set

$$\varprojlim \{\mathbf{X}, \mathbf{f}\} = \{\mathbf{x} \in \prod_{i=1}^{\infty} X_i : x_i = f_i(x_{i+1}) \text{ for all } i \in \mathbb{N}\}.$$

Here, the spaces $X_i$ are called *factor spaces*, and the functions $f_i$ are called *bonding maps*.

For each $j$, let $\pi_j : \varprojlim \{\mathbf{X}, \mathbf{f}\} \to X_j$ be defined by $\pi_j\big(\langle x_1, x_2, \ldots\rangle\big) = x_j$ that is, $\pi_j$ is the projection map of $\varprojlim \{\mathbf{X}, \mathbf{f}\}$ to the $j$-th factor space.

Inverse limit constructions became a very important tool in continuum theory since Mardesič and Segal proved in 1967 that every continuum can be represented as the inverse limit of polyhedra, see [Mardešić and Segal, 1967].

In 2004, Mahavier investigated the continua that can be represented as inverse limits of closed subsets of the unit square $I^2 = [0, 1] \times [0, 1]$. This investigation led Mahavier to the generalized notion of an inverse limit. In [Mahavier, 2004], Mahavier showed that some of usual properties for inverse limits with single-valued functions were still valid with this new notion. Later,Ingram and Mahavier [2006] built upon this definition by generalizing it to inverse sequences $(X_1, f_1), (X_2, f_2), \ldots$ where each $X_i$ is a compact Hausdorff space and each $f_i$ is an upper semi-continuous set-valued function from $X_{i+1}$ to $2^{X_i}$. They showed that even if some of the properties still held, many other properties failed to hold for inverse limits with set-valued functions.

The book by Ingram and Mahavier [2010] contains further generalization of the inverse limit to inverse systems when the underlying index set is an arbitrary directed set in place of nonnegative integers. Additionally, Charatonik and Roe generalized the notion further in [Charatonik and Roe, 2014]. They defined Mahavier systems when the underlying index set is an arbitrary preorder, i.e. a transitive and reflexive relation. However, in this dissertation we will restrict our attention to the inverse sequences rather than the inverse systems or the Mahavier systems.

The *graph* $G(f)$ of a function $f : X \to 2^Y$ is the set of all points $\langle x, y\rangle \in X \times Y$ such that $y \in f(x)$. For a positive integer $n$, the *partial graph* $G_n$ defined to be $\{\mathbf{x} \in \prod_{i=1}^{n} X_i | x_i \in f_i(x_{i+1})$ for $1 \leq i < n\}$.

Given compact metric spaces $X$ and $Y$, a function $f : X \to 2^Y$ is *upper semi-continuous* if for each open set $V \subset Y$ the set $\{x \in X | f(x) \subset V\}$ is a an open set in $X$, and it is known that a function between compact Hausdorff spaces is upper semi-continuous if and only if its graph is closed. A set-valued function $f : X \to 2^Y$ is *surjective* if for each $y \in Y$ there exist an $x \in X$ such that $y \in f(x)$. The set-valued function $f^{-1} : Y \to 2^X$ defined as the set of all $x \in X$ such that $y \in f(x)$ is called the *inverse* of $f$ and it is also upper semi-continuous when $f$ is upper semi-continuous.

If $\{X_i : i \in \{1, 2, \dots\}\}$ is a countable collection of compact metric spaces each with a metric $d_i$ bounded by 1, then $\prod_{i=1}^{\infty} X_i$ is the countable product of the collection $\{X_i : i \in \{1, 2, \dots\}\}$ with the metric given by $d(\langle x_1, x_2, \dots\rangle, \langle y_1, y_2, \dots\rangle) = \sum_{i=1}^{\infty} \frac{d_i(x_i, y_i)}{2^i}$.

For each $i$ let $f_i \colon X_{i+1} \longrightarrow 2^{X_i}$ be a set-valued function where $2^{X_i}$ denotes the hyperspace of all nonempty closed subsets of $X_i$. The inverse limit of the sequence of pairs $\{(X_i, f_i)\}$, denoted by $\varprojlim \{X_i, f_i\}$, is defined to be the set of all points $\langle x_1, x_2, \dots\rangle$ in $\prod_{i=1}^{\infty} X_i$ such that $x_i \in f_i(x_{i+1})$. For a finite sequence $\mathbf{x} = \langle x_1, x_2, \dots, x_n\rangle$ and finite or infinite sequence $\mathbf{y} = \langle y_1, y_2, \dots\rangle$, let $\mathbf{x} \oplus \mathbf{y} = \langle x_1, x_2, \dots, x_n, y_1, y_2, \dots\rangle$

Many well-known results about the connectedness of the inverse limit of single-valued functions do not hold when the functions are generalized to the set-valued functions. Mahavier [2004] provided the following graph, Figure2.3, of an upper semi-continuous set-valued function whose inverse limit is not connected.

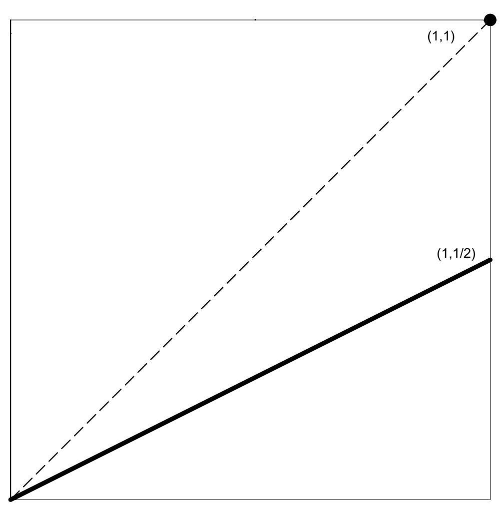


Figure 2.3. A function with disconnected inverse limit.

One can easily see that the graph for this function is not connected thus may think that this is a reason for disconnectedness of the inverse limit. But then Ingram and Mahavier [2006] showed that connectedness of the graph was not a sufficient condition to guarantee connectedness of the inverse limit. Ingram and Mahavier constructed an example whose graph is given by Figure 2.4 and they showed that the inverse limit is not connected although the graph itself is connected, see [Ingram and Mahavier, 2006].

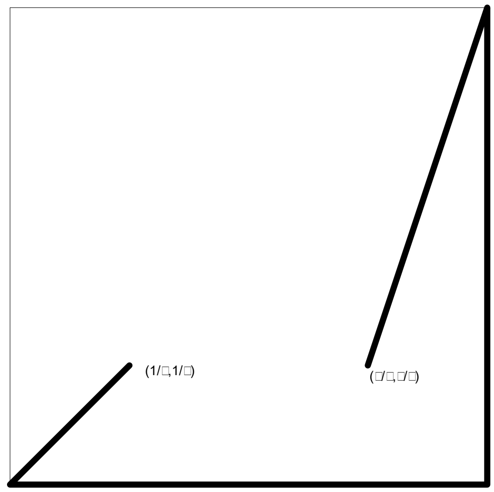

Figure 2.4. A function with connected graph whose inverse limit is not connected.

In the same paper they provided the following important results about the connectedness of inverse limits.

**Theorem 2** [Ingram and Mahavier, 2006, Theorem 4.7] Suppose that for each $i$, $X_i$ is Hausdorff continuum, $f_i : X_i \to 2^{X_i}$ is an upper semi-continuous function, and for each $x$ in $X_{i+1}$, $f_i(x)$ is connected. Then $\varprojlim \mathbf{f}$ is connected.

**Theorem 3** [Ingram and Mahavier, 2006, Theorem 4.8] Suppose that for each $i$, $X_i$ is Hausdorff continuum, $f_i : X_i \to 2^{X_i}$ is an upper semi-continuous function, and for each $x$ in $X_i$ $\{y \in X_{i+1} | x \in f_i(y)\}$ is a non-empty, connected set. Then $\varprojlim \mathbf{f}$ is connected.

In addition to these results, Nall [2012] provided some results about connectivity for a single surjective upper semi-continuous function on $[0, 1]$. These results are as follows:

**Theorem 4** [Nall, 2012, Theorem 3.1] Suppose $X$ is a compact metric space, and $\{F_\alpha\}_{\alpha\in\Lambda}$ is a collection of closed subsets of $X \times X$ such that for each $x \in X$ and each $\alpha \in \Lambda$, the set $\{y \in X|(x, y) \in F_\alpha\}$ is nonempty and connected, and such that $F = \bigcup_{\alpha\in\Lambda} F_\alpha$ is closed connected subset of $X\times X$ such that for each $y \in X$, the set $\{x \in X|(x, y) \in F\}$ is nonempty. Then $\varprojlim F$ is connected.

**Theorem 5** [Nall, 2012, Lemma 3.2] Suppose $X$ is a Hausdorff continuum, $f : X \to 2^X$ is an upper semi-continuous set-valued function. Then $\varprojlim \{X, f\}$ is connected if and only if $G_n$ is connected for each $n$.

**Theorem 6** [Nall, 2012, Theorem 3.3] Suppose $X$ is a Hausdorff continuum and $f : X \to 2^X$ is a surjective upper semi-continuous set-valued function. Then $\varprojlim \{X, f\}$ is connected if and only if $\varprojlim \{X, f^{-1}\}$ is connected.

Greenwood and Kennedy [2012] constructed a function whose graph is given by Figure 2.5. With this example they showed that even though $G(f^n) = G(f \circ f \circ \ldots \circ f)$ is connected for all $n$, the inverse limit still can be disconnected.

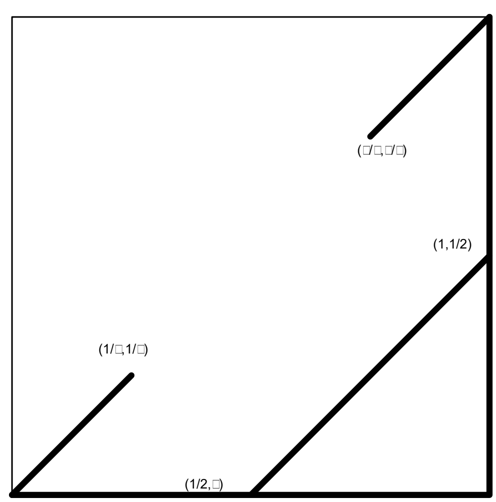


Figure 2.5. A function with connected compositions and disconnected inverse limit.

Banič and Kennedy in 2015 proved that inverse limit of a surjective upper semi-continuous set-valued functions whose graph is an arc on $[0, 1]$ is never totally disconnected, see [Banič and Kennedy, 2015]. But they did not only focus on the connectedness of such a function, but proved a very important result about the cardinality of the inverse limit of such functions without the surjectivity condition. Precisely, they proved the following theorem.

**Theorem 7** [Banič and Kennedy, 2015, Theorem 3.9] Suppose $f : [0, 1] \to 2^{[0,1]}$ is an upper semi-continuous function with connected graph. If $\varprojlim \{[0, 1], f\}$ consists of more than one point, then $\varprojlim \{[0, 1], f\}$ is infinite.

In 2015 Roškarič and Tratnik showed similar results by using the ideas of Banič and Kennedy, independently, see [Roškarič and Tratnik, 2015], and [Banič and Kennedy, 2015]. In this dissertation we have shown that the connectedness of the graph is essential. We have also shown that the set $[0, 1]$ cannot be replaced by a simple triod nor by a simple closed curve.

Banič et al. [2010] and Banič et al. [2011] showed that if a sequence of graphs of upper semi-continuous set-valued functions $f_n : X \to 2^X$ converges to the graph of a continuous single-valued function $f : X \to X$, then the sequence of the inverse limits $\varprojlim \{X, f_n\}$ converges to the inverse limit $\varprojlim \{X, f\}$ by proving the next theorem.

**Theorem 8** [Banič and Kennedy, 2015, Theorem 3.3] Let $X$ be a compact metric space and for each $n \in \mathbb{N}$, let $f_n : X \to 2^X$ be an upper semi-continuous set-valued function, and let $f : X \to X$ be continuous single-valued function, such that $\lim_{n\to\infty} G(f_n) = G(f)$ in $2^{X\times X}$. Then the following are equivalent:

1. $\lim_{n\to\infty} K_n = K$ in $2^X$,
2. $\pi_1(K) \subseteq \liminf_{n\to\infty} \pi_1(K_n)$,
3. $\pi_1(K) = \lim_{n\to\infty} \pi_1 K_n$ in $2^X$.

Then Kelly and Meddaugh proved the following two theorems that relaxes the condition about $f$ being continuous, see [Kelly and Meddaugh, 2015].

**Theorem 9** [Kelly and Meddaugh, 2015, Theorem 1.2] Let $X$ be a compact metric space and $f : X \to 2^X$ be upper semi-continuous. For each $n \in \mathbb{N}$, let $f_n : X \to 2^X$ be an upper semi-continuous function such that $\lim_{n\to\infty} G(f_n) = G(f)$ in $2^{X\times X}$. If $\pi_1(K) \subseteq \liminf_{n\to\infty} \pi_1(K_n)$ and $K$ has the weak compact full projection property, then $\lim_{n\to\infty} K_n = K$ in $2^X$.

**Theorem 10** [Kelly and Meddaugh, 2015, Theorem 1.3] Let $X$ be a compact metric space and $f : X \to 2^X$ be continuous. For each $n \in \mathbb{N}$, let $f_n : X \to 2^X$ be upper semi-continuous. If $\pi_1(K) \subseteq \liminf_{n\to\infty} \pi_1(K_n)$, and there exists a set $A \subseteq X$ such that

1. $A$ is dense in $\pi_1(K)$,
2. for each $a \in A$, $A \cap f(a)$ is dense in $f(a)$,
3. $A \subseteq f(A)$, and
4. for each $a \in A$, $(f_n)_{n=1}^{\infty}$ converges uniformly to $f$ on a neighborhood of $a$

then $\lim_{n\to\infty} K_n = K$ in $2^X$

In the same paper Kelly and Meddaugh posted the following question and in this dissertation we answer this question by providing an example.

**Question 1** [Kelly and Meddaugh, 2015, Question 6.4] Let $f : X \to 2^X$ be a function, and for each $n \in \mathbb{N}$, let $f_n : X \to 2^X$ be upper semi-continuous such that $\lim_{n\to\infty} G(f_n) = G(f)$, and $\pi_1(K) \subseteq \liminf_{n\to\infty} \pi_1(K_n)$. If $f$ is continuous, does it follow that $\lim_{n\to\infty} K_n = K$ in $2^X$?

## PAPER

# I. ZERO-DIMENSIONAL SPACES HOMEOMORPHIC TO THEIR CARTESIAN SQUARES

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**ABSTRACT**

We show that there exists uncountably many zero-dimensional compact metric spaces homeomorphic to their Cartesian squares as well as their $n$-fold symmetric products.

**Keywords:** Cantor-Bendixson rank; Cartesian squares; zero-dimensional spaces

## 1. INTRODUCTION AND PRELIMINARIES

Marjanović [1972] showed that there are exactly nine different zero-dimensional compact metric spaces $X$ which are homeomorphic to $2^X$. In this paper, we look at this subject from a different perspective and show that there exist uncountably many zero-dimensional compact metric spaces homeomorphic to their Cartesian products and furthermore to their $n$-fold symmetric products. As a tool, we use the Cantor-Bendixson rank of a zero-dimensional compact metric space.

For a compact metric space $X$, we define *hyperspaces* $2^X$ and $F_n(X)$ as follows:

$$2^X = \{A \subset X | A \text{ is nonempty and closed}\}$$

$$F_n(X) = \{A \subset X | \operatorname{card}(A) \leq n\}$$

The *derivative of a set* $X$ represents the set of all limit points of $X$ and is denoted by $X'$.

A compact space $X$ is called *zero-dimensional* if every component of $X$ is degenerate.

## 2. CANTOR-BENDIXSON RANK IN CARTESIAN PRODUCT

**Definition 2** The Cantor-Bendixson derivative of order $\alpha$, or $\alpha$-derivative, of a compact space $X$ is defined inductively as follows:

1. $X^{(0)} = X$

2. $X^{(\alpha+1)} = \{x \in X^{(\alpha)} : x \text{ is a limit point in } X^{(\alpha)}\}$

3. For limit ordinals $\gamma$: $X^{(\gamma)} = \bigcap_{\beta<\gamma} X^{(\beta)}$

Then, the Cantor-Bendixson rank of a space $X$, denoted by $\operatorname{rank}(X)$, is defined as the least ordinal $\alpha$ such that $X^{(\alpha+1)} = \emptyset$.

Finally, the Cantor-Bendixson rank of a point $x$ in a set $X$, denoted by $CB(x)$, is defined, if $x$ has a countable neighborhood, as

$$CB(x) = \min\{ \operatorname{rank}(U) : U \text{ is a countable compact neighborhood of } x \text{ in } X \text{ and } x \in U\}.$$

**Observation 1** For compact spaces $X$ and $Y$, we have

$$(X \times Y)' = X' \times Y \cup X \times Y'.$$

Any ordinal number $\alpha$ can be uniquely written as $\omega^{\alpha_1} n_1 + \omega^{\alpha_2} n_2 + \cdots + \omega^{\alpha_k} n_k$, where $\alpha_1, \alpha_2, \ldots, \alpha_k$ are ordinals in a decreasing order and $n_1, n_2, \ldots, n_k$ are integers. This is called *Cantor form*. Let $\alpha = \omega^{\alpha_1} n_1 + \omega^{\alpha_2} n_2 + \cdots + \omega^{\alpha_k} n_k$ and $\beta = \omega^{\alpha_1} m_1 + \omega^{\alpha_2} m_2 + \cdots + \omega^{\alpha_k} m_k$ be two ordinals, where the *natural sum* $\alpha \oplus \beta$ is defined by

$$\alpha \oplus \beta = \omega^{\alpha_1}(n_1 + m_1) + \omega^{\alpha_2}(n_2 + m_2) + \cdots + \omega^{\alpha_k}(n_k + m_k)$$

(see e.g. [Kuratowski and Mostowski, 1976] pg.253).

The following lemmas were proved by Charatonik and Charatonik [2001].

**Lemma 1** Let $\beta$ be a limit ordinal, and for each $\alpha < \beta$, assign two ordinals $\alpha_1, \alpha_2$ such that $\alpha = \alpha_1 \oplus \alpha_2$. Then,

$$\sup\{\alpha_1 | \alpha < \beta\} \oplus \sup\{\alpha_2 | \alpha < \beta\} \geq \beta.$$

**Lemma 2** Let $\alpha$ and $\beta$ be ordinals such that $\alpha < \beta$ and $\beta = \beta_1 \oplus \beta_2$. Then, there exists ordinals $\alpha_1$ and $\alpha_2$ such that

$$\alpha_1 \leq \beta_1, \alpha_2 \leq \beta_2, \text{and } \alpha = \alpha_1 \oplus \alpha_2.$$

Corollary 1 was proposed by J. R. Prajs, but the proof to this Corollary has never been published. In order to present a proof to this corollary we first prove the following theorem.

**Theorem 11** Let $X$ and $Y$ be two compact spaces and let $Z = X \times Y$. Then,

$$Z^{(\alpha)} = \bigcup\{X^{(\alpha_X)} \times Y^{(\alpha_Y)} | \alpha = \alpha_X \oplus \alpha_Y\}.$$

**Proof:**

- If we let $\alpha = 0$, then it is clear that $Z^{(0)} = X^{(0)} \times Y^{(0)}$, so the first condition is satisfied.

- $Z^{(\alpha+1)} = (Z^{(\alpha)})' = \bigcup\{X^{(\alpha_X)} \times Y^{(\alpha_Y)} | \alpha = \alpha_X \oplus \alpha_Y\}' = \bigcup\{(X^{(\alpha_X)})' \times Y^{(\alpha_Y)} \cup X^{(\alpha_X)} \times (Y^{(\alpha_Y)})' | \alpha = \alpha_X \oplus \alpha_Y\} = \bigcup\{X^{(\alpha_X+1)} \times Y^{(\alpha_Y)} \cup X^{(\alpha_X)} \times Y^{(\alpha_Y+1)} | \alpha = \alpha_X \oplus \alpha_Y\} = \bigcup\{X^{(\alpha_X)} \times Y^{(\alpha_Y)} | \alpha + 1 = \alpha_X \oplus \alpha_Y\}$, as required. Here, the third equation is a consequence of Observation 1.

- Now to show that $Z^{(\gamma)} = \bigcup\{X^{(\gamma_X)} \times Y^{(\gamma_Y)} | \gamma = \gamma_X \oplus \gamma_Y\}$ for each limit ordinals $\gamma$, first take $z = \langle x, y\rangle \in Z^{(\gamma)}$. Then, by Definition 2, $\langle x, y\rangle \in Z^{(\gamma)} = \bigcap_{\beta<\gamma} Z^{(\beta)}$. Thus, by the inductive hypothesis, we have $\langle x, y\rangle \in \bigcup\{X^{(\beta_X)} \times Y^{(\beta_Y)} | \beta = \beta_X \oplus \beta_Y\}$. Now, let $\gamma_X = \sup\{\beta_X | \beta < \gamma\}$ and $\gamma_Y = \sup\{\beta_Y | \beta < \gamma\}$. By Lemma 1, $\gamma_X \oplus \gamma_Y \geq \gamma$ and by Lemma 2, there exists ordinals $\gamma'_X$ and $\gamma'_Y$ such that $\gamma'_X < \gamma_X, \gamma'_Y < \gamma_Y$ and $\gamma'_X \oplus \gamma'_Y = \gamma$. Now since $X^{(\gamma_X)} = \bigcap_{\beta<\gamma} X^{(\beta_X)}$ and $Y^{(\gamma_Y)} = \bigcap_{\beta<\gamma} Y^{(\beta_Y)}$ for all $\beta < \gamma$, we conclude that $\langle x, y\rangle \in X^{(\gamma_X)} \times Y^{(\gamma_Y)}$. Therefore, $\langle x, y\rangle \in X^{(\gamma_X)} \times Y^{(\gamma_Y)} \subset X^{(\gamma'_X)} \times Y^{(\gamma'_Y)} \subset \bigcup\{X^{(\gamma'_X)} \times Y^{(\gamma'_Y)} | \gamma = \gamma'_X \oplus \gamma'_Y\}$.

In order to show the other inclusion, let $\langle x, y\rangle \in \bigcup\{X^{(\gamma_X)} \times Y^{(\gamma_Y)} | \gamma = \gamma_X \oplus \gamma_Y\}$. Then by Lemma 1 for any ordinal $\beta$ such that $\beta < \gamma$, we can assign two ordinals $\beta_X$ and $\beta_Y$ such that $\beta_X < \gamma_X, \beta_Y < \gamma_Y$ and $\beta = \beta_X \oplus \beta_Y$. Therefore, $\langle x, y\rangle \in X^{(\gamma_X)} \times Y^{(\gamma_Y)} \subset X^{(\beta_X)} \times Y^{(\beta_Y)} \subset Z^{(\beta)}$. Now by Definition 2, $\langle x, y\rangle \in Z^{(\gamma)}$. So finally the equality holds.

□

**Corollary 1** Let $X$ and $Y$ be two compact spaces. Then,

$$\mathrm{rank}(X \times Y) = \mathrm{rank}(X) \oplus \mathrm{rank}(Y).$$

## 3. CHARACTERIZATION

In this section, we define an uncountable family $\{Z(\alpha) : \alpha < \omega_1\}$ of zero-dimensional compact metric spaces and provide a topological characterization of the spaces $Z(\alpha)$. Also, we prove that uncountably many of them have the property that their Cartesian squares are homeomorphic to their factor spaces.

**Theorem 12** The following two conditions are equivalent for an ordinal $\alpha$:

(3.12.1) For every $\beta, \gamma < \alpha$, we have $\beta \oplus \gamma < \alpha$.

(3.12.2) $\alpha = 0$ or there is an ordinal $\delta$ such that $\alpha = \omega^\delta$.

**Proof:**

(3.12.1) ⇒ (3.12.2) Let $\alpha \neq 0$ and assume there is no $\delta$ such that $\alpha = \omega^\delta$. We may express $\alpha = \omega^{\alpha_1} k_1 + \omega^{\alpha_2} k_2 + \cdots + \omega^{\alpha_n} k_n$, where $\alpha_1 > \alpha_2 > \cdots > \alpha_n$ and $k_1, k_2, \ldots k_n$ are nonzero natural numbers. If $k_1 = 1$, then $n \geq 2$. Now, we take $\beta = \gamma = \omega^{\alpha_1}$, and then we have $\beta \oplus \gamma = \omega^{\alpha_1} \cdot 2 > \alpha$. If $k_1 \neq 1$, we take $\beta = \omega^{\alpha_1}(k_1 - 1) + \omega^{\alpha_2} k_2 + \cdots + \omega^{\alpha_n} k_n$ and

$\gamma = \omega^{\alpha_1}(k_1 - 1)$, and then $\beta \oplus \gamma = \omega^{\alpha_1}(2k_1 - 2) + \omega^{\alpha_2}k_2 + \cdots + \omega^{\alpha_n}k_n \geq \alpha$ which is a contradiction.

(3.12.2) $\Rightarrow$ (3.12.1) If $\alpha = 0$ then the conclusion is true vacuously. Let $\alpha = \omega^{\delta}$ for some ordinal $\delta$. Then let $\beta$ and $\gamma$ be less than $\alpha$. We may express $\beta = \omega^{\beta_1}i_1 + \omega^{\beta_2}i_2 + \cdots + \omega^{\beta_n}i_n$, where $\beta_1 > \beta_2 > \cdots > \beta_n$ and $i_1, i_2, \ldots i_n < \omega$, $\gamma = \omega^{\gamma_1}j_1 + \omega^{\gamma_2}j_2 + \cdots + \omega^{\gamma_m}j_m$ where $\gamma_1 > \gamma_2 > \ldots \gamma_m$ and $j_1, j_2, \ldots, j_m < \omega$. Since $\beta, \gamma < \alpha$, note that $\beta_1, \gamma_1 < \delta$. Then $\beta \oplus \gamma < \omega^{max\{\beta_i, \gamma_j\}+1} < \alpha$. $\square$

**Corollary 2** There exist uncountably many ordinals $\alpha < \omega_1$ satisfying condition 3.12.1 (or equivalently 3.12.2) of Theorem 12.

To prove the next theorem we have used a large part of the proof of Proposition 8.8 in Illanes and Nadler [1996] and added necessary conditions on the ranks of points. Here, $S(X)$ represents the set of all points of $X$ with no countable neighborhood.

**Theorem 13** Let $\alpha$ be a countable ordinal and let $X_1$ and $X_2$ be two compact metric spaces satisfying the following conditions for $i \in \{1, 2\}$:

(3.13.1) If $x$ has a countable neighborhood in $X_i$, then $CB(x) < \alpha$, for every $x \in X_i$.

(3.13.2) For every ordinal $\beta$ such that $\beta < \alpha$ and for every uncountable open subset $U_i$ of $X_i$, there exists an $x \in U_i$ such that $CB(x) = \beta$.

(3.13.3) $S(X_1)$ is homeomorphic to $S(X_2)$.

Then the spaces $X_1$ and $X_2$ are homeomorphic.

**Proof:** Let $I(X_i)$ denote the set of all elements of $X_i$ with a countable neighborhood. Thus, $X_i = S(X_i) \cup I(X_i)$ for $i \in \{1, 2\}$ where $S(X_i)$ is closed and $I(X_i)$ is open. Since $S(X_1)$ is homeomorphic to $S(X_2)$, there exists a homeomorphism $h : S(X_1) \longrightarrow S(X_2)$.

Let $d_1$ and $d_2$ denote the metrics for $X_1$ and $X_2$, respectively.

We will define by induction two sequences $R_k$ and $S_k$ of finite sets such that $I(X_1) = \bigcup\{R_k : k \in \{0, 1, \ldots\}\}$ and $I(X_2) = \bigcup\{S_k : k \in \{0, 1, \ldots\}\}$ and a homeomorphism $h_k : R_k \longrightarrow S_k$ such that $CB(x) = CB(h_k(x))$ for all $x \in R_k$, then the required homeomorphism

$h^* : X_1 \to X_2$ will be defined as follows:

$$h^*(x) = \begin{cases} h(x) & \text{if } x \in S(X_1) \\ h_k(x) & \text{if } x \in R_k \end{cases}$$

To start, define

$$R_0 = S_0 = \emptyset \text{ and } h_0 = \emptyset$$

and assume that the sets $R_l \subset I(X_1)$ and $S_l \subset I(X_2)$ have been defined inductively for all $l$ such that $0 \leq l \leq k-1$. Now, for any $n \in \mathbb{N}$, let $Y_1$ be any infinite subset of $I(X_1)$, and $X_1(n, Y_1)$ be the set of $n$ distinct points of $Y_1$ with the following property:

$$(3.13.4) d_1(x_1, S(X_1)) \leq d_1(x, S(X_1)) \text{ for any } x_1 \in Y_1 \setminus X_1(n, Y_1) \text{ and} x \in X_1(n, Y_1)$$

. Define $X_2(n, Y_2)$, where $n \in \mathbb{N}$ and $Y_2$ is any infinite subset of $I(X_2)$, analogously. Since $S(X_1)$ and $S(X_2)$ are compact there exist nonempty, finite subsets $A_k$ and $B_k$ of $S(X_1)$ and $S(X_2)$, resp., defined as follows:

$$A_k = \{a_{k,1}, a_{k,2}, \ldots, a_{k,n(k)}\}$$
$$B_k = \{b_{k,1}, b_{k,2}, \ldots, b_{k,m(k)}\}$$

such that each point of $S(X_1)$ and $S(X_2)$ is within $1/k$ of a point of $A_k$ and $B_k$ respectively. Observing that $I(X_1) \setminus \cup_{l=0}^{k-1} R_l$ is an infinite set, let

$$P_k = X_1(n(k), I(X_1) \setminus \cup_{l=0}^{k-1} R_l).$$

For each point $p \in P_k$, let

$$\alpha(p) = \min\{j : d_1(p, a_{k,j}) = d_1(p, A_k)\}.$$

We index the points of $P_k$ as $p_{k,1}, p_{k,2}, \ldots, p_{k,n(k)}$ in such a way that if $s < t$, then either (3.13.5) or (3.13.6) holds:

(3.13.5) $\alpha(p_{k,s}) < \alpha(p_{k,t})$, (3.13.6) $\alpha(p_{k,s}) = \alpha(p_{k,t})$ and $d_1(p_{k,s}, a_{k,\alpha(p_{k,s})}) \geq d_1(p_{k,t}, a_{k,\alpha(p_{k,t})})$.

Since $I(X_2) \setminus \cup_{l=0}^{k-1} S_l$ contains points of all $CB$-rank less than $\alpha$, we can define the sets, $P'_k$ and $Q'_k$, as follows: let $p'_{k,1}, p'_{k,2}, \ldots, p'_{k,n(k)}$ be $n(k)$ distinct points of $I(X_2) \setminus \cup_{l=0}^{k-1} S_l$ such that for each $i$,

$$d_2(p'_{k,i}, h(a_{k,\alpha(p_{k,t})})) \leq d_1(p_{k,i}, a_{k,\alpha(p_{k,t})});$$
$$CB(p_{k,i}) = CB(p'_{k,i});$$

then let

$$P'_k = \{p'_{k,1}, p'_{k,2}, \ldots, p'_{k,n(k)}\}$$

and let

$$Q'_k = X_2(m(k), I(X_2) \setminus [P'_k \cup (\cup_{l=0}^{k-1} S_l)]).$$

For each point $q' \in Q'_k$, let

$$\beta(q') = \min\{j : d_2(q', b_{k,j}) = d_2(q', B_k)\}.$$

We index the points of $Q'_k$ as $q'_{k,1}, q'_{k,2}, \ldots, q'_{k,m(k)}$ in such a way that if $s < t$, then, as is analogous to (3.13.5) and (3.13.6) above, either (3.13.7) or (3.13.8) holds:

(3.13.7) $\beta(q'_{k,s}) < \beta(q'_{k,t})$,

(3.13.8) $\beta(q'_{k,s}) = \beta(q'_{k,t})$ and $d_2(q'_{k,s}, b_{k,\beta(q'_{k,t})}) \geq d_2(q'_{k,t}, b_{k,\beta(q'_{k,t})})$.

Now, we define $Q_k$ in terms of $Q'_k$ and $B_k$ with a similar way that we have defined $P'_k$ in terms of $P_k$ and $A_k$ as follows: noting that $I(X_1) \setminus [P_k \cup (\cup_{l=0}^{k-1} R_l)]$ contains points of all $CB$-rank less than $\alpha$, let $q_{k,1}, q_{k,2}, \ldots, q_{k,m(k)}$ be $m(k)$ distinct points of $I(X_1) \setminus [P_k \cup (\cup_{l=0}^{k-1} R_l)$ such that for each $i$,

$$d_1(q_{k,i}, h^{-1}(b_{k,\beta(q'_{k,t})})) \leq d_2(q'_{k,i}, b_{k,\beta(q'_{k,t})}),$$

and

$$CB(q_{k,i}) = CB(q'_{k,i}),$$

and let

$$Q_k = \{q_{k,1}, q_{k,2}, \ldots, q_{k,m(k)}\}.$$

Finally, let

$$R_k = P_k \cup Q_k \text{ and } S_k = P'_k \cup Q'_k,$$

and define $h_k : R_k \longrightarrow S_k$ as follows:

$$h_k(p_{k,i}) = p'_{k,i} \text{ for all } p_{k,i} \in P_k,\ h_k(q_{k,i}) = q'_{k,i} \text{ for all } q_{k,i} \in Q_k.$$

Therefore, $R_k$, $S_k$ and the one-to-one and onto function $h_k : R_k \longrightarrow S_k$ for each $k = 0, 1, 2, \ldots$ have been defined by induction.

To be able to complete the definiton of the homeomorphism $h^*$ of $X_1$ onto $X_2$ we need to prove the following two facts:

(3.13.9) $\cup_{k=0}^{\infty} R_k = I(X_1)$;

(3.13.10) $\cup_{k=0}^{\infty} S_k = I(X_2)$. First suppose that (3.13.9) is false. Then there is a point $x_0 \in X_1 \setminus \cup_{k=0}^{\infty} R_k$. Hence, by the definition of $R_k$ and $P_k$,

$$x_0 \notin \cup_{k=1}^{\infty} P_k = \cup_{k=1}^{\infty} X_1(n(k), I(X_1, \cup_{l=0}^{k-1} R_l)).$$

Thus, since $x_0 \in I(X_1) \setminus \cup_{l=0}^{k-1} R_l$ for each $k$, from (3.13.4) we have the following:

$$(3.13.11) d_1(x_0, S(X_1)) \leq d_1(p, S(X_1)) \text{ for all } p \in \cup_{k=1}^{\infty} P_k.$$

From the definitions of $P_k$ and $R_k$ , we see that the sets $P_1, P_2, \ldots$ are mutually disjoint and nonempty; hence, $\cup_{k=1}^{\infty} P_k$ is an infinite set. Thus, since $X_1$ is compact,

$$\inf\{d_1(p, S(X_1)) : p \in \cup_{k=1}^{\infty} P_k\} = 0.$$

Hence, by (3.13.11), $d_1(x_0, S(X_1)) = 0$; however, since $x_0 \in I(X_1)$, this is impossible and this completes the proof of (3.13.9). The proof of (3.13.10) is similar using the sets $Q'_k$.

We complete the definition of $h^* : X_1 \longrightarrow X_2$. First of all, note that $h^*$ is well defined since the sets $S(X_1), R_1, R_2, \ldots$ are mutually disjoint. Also, by (3.13.9) $h^*$ is defined on all $X_1$ and by (3.13.10) $h^*$ maps onto $X_2$ since $h_k(R_k) = S_k$ for each $k$ and $h[S(X_1)] = S(X_2)$. Due to the fact that the sets $S(X_2), S_1, S_2, \ldots$ are mutually disjoint, $h$ and each $h_k$ is one-to-one $h^*$ is one-to-one.

Finally, the continuity of $h^*$ follows from the uniform continuity of $h$, the properties of the sets $A_k$ and $B_k$ and (3.13.5)-(3.13.8). Therefore, since $X_1$ is compact and $X_2$ is Hausdorff, $h^*$ is a homeomorphism of $X_1$ onto $X_2$. □

**Theorem 14** Let $\alpha$ be a countable ordinal and let $X_1$ and $X_2$ be two zero-dimensional compact metric spaces satisfying the following conditions for $i \in \{1, 2\}$:

1. For every $x \in X_i$, if $x$ has a countable neighborhood in $X_i$, then $CB(x) < \alpha$.

2. For every ordinal $\beta$ such that $\beta < \alpha$ and for every uncountable, open subset $U_i$ of $X_i$, there exist an $x \in U_i$ such that $CB(x) = \beta$.

Then the spaces $X_1$ and $X_2$ are homeomorphic.

**Proof:** In this case $S(X_1)$ and $S(X_2)$ are homeomorphic to the Cantor set, so the condition (3.13.3) of Theorem 13 is satisfied. □

Denote by $Z(\alpha)$ the (topologically unique) metric spaces satisfying the assumptions of Theorem 14. In particular $Z(0)$ is the Cantor set and $Z(1)$ is the Pełczyński space described in p.70 in [Illanes and Nadler, 1996].

**Theorem 15** If $\alpha$ is an ordinal satisfying the condition (3.12.1) (or equivalently (3.12.2)) of Theorem 12, then $Z(\alpha) \times Z(\alpha)$ is homeomorphic to $Z(\alpha)$ and for every natural number $n$, the hyperspace $F_n(Z(\alpha))$ is homeomorphic to $Z(\alpha)$. In particular there are uncountably many compact metric spaces $X$ homeomorphic to their Cartesian products $X^n$ and to their hyperspaces $F_n(X)$.

**Proof:** Note that if $\alpha$ satisfies condition (3.12.1) of Theorem 12 then $Z(\alpha) \times Z(\alpha)$ satisfies the conditions of Theorem 14, so it is homeomorphic to $Z(\alpha)$. □

Note that the spaces $Z(\alpha)$ are not the only ones that are homeomorphic to their Cartesian squares. For example, the disjoint union $Z(0) \cup Z(1)$ also has this property.

**Problem 1** Characterize all zero-dimensional compact metric spaces homeomorphic to their Cartesian squares.

**Problems 1** Is there a zero-dimensional compact metric space $X$ such that $X$ is homeomorphic to $X \times X$, but not homeomorphic to $F_2(X)$ nor to $F_n(X)$, for some $n$? Similarly, is there a zero-dimensional compact metric space $X$ such that $X$ is homeomorphic to $F_n(X)$, for some $n$, but not to $X \times X$? Is there a zero-dimensional compact metric space and two natural numbers $n, m$ such that $X$ is homemomorphic to $F_n(X)$, but not to $F_m(X)$?

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# II. INVERSE LIMITS WITH BONDING FUNCTIONS WHOSE GRAPHS ARE ARCS

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## ABSTRACT

First, answering a question by Roškarič and Tratnik, we present inverse sequences of simple triods or simple closed curves with set-valued bonding functions whose graphs are arcs and the limits are $n$-point sets. Second, we present a wide class of zero-dimensional spaces that can be obtained as the inverse limits of arcs with one set-valued function whose graph is an arc.

**Keywords:** zero-dimensional, inverse limit, set-valued functions

## 1. INTRODUCTION

The inverse limits with upper semicontinuous bonding functions were introduced by Mahavier [2004]. Since then, they became a very popular subject of investigation, especially in the case when all the factor spaces are arcs. Then a book by Ingram and Mahavier [2010] (one of a very few in continuum theory) containing a lot of information about the subject was published and the subject became even more popular.

It is known that the inverse limit of compact nonempty spaces with upper semicontinuous functions is nonempty [Charatonik and Roe, 2012], but in some cases it can be degenerate even if the factor spaces are not. Banič and Kennedy [2015] ,and Roškarič and Tratnik [2015] independently showed that if $f$ is an upper semicontinuous function whose graph is connected, then $\varprojlim \{[0,1], f\}$ is either degenerate or infinite. Here we show by counterexamples that this theorem is no longer true if we replace $[0,1]$ by a circle or by a triod. Moreover, we present a wide class of zero-dimensional spaces that can be obtained as the inverse limits of arcs with one set-valued function. At the end of the two sections some open problems are asked.

## 2. PRELIMINARIES

In this article we consider metric spaces only. A *continuum* is a nonempty, compact and connected metric space.

If $X$ is a continuum, then $2^X$ denotes the family of all nonempty closed subsets of $X$.

The *graph* $G(f)$ of a function $f : X \to 2^Y$ is the set of all points $\langle x, y\rangle \in X \times Y$ such that $y \in f(x)$.

Given compact metric spaces $X$ and $Y$, a function $f : X \to 2^Y$ is *upper semicontinuous* if for each open set $V \subset Y$ the set $\{x \in X | f(x) \subset V\}$ is a an open set in $X$. It is known that a function between compact spaces is upper semicontinuous if and only if its graph is closed.

If $\{X_i : i \in \{1, 2, \ldots\}\}$ is a countable collection of compact metric spaces each with a metric $d_i$ bounded by 1, then $\Pi_{i=1}^{\infty} X_i$ is the countable product of the collection $\{X_i : i \in \{1, 2, \ldots\}\}$ with the metric given by $d(\langle x_1, x_2, \ldots\rangle, \langle y_1, y_2, \ldots\rangle) = \sum_{i=1}^{\infty} \frac{d_i(x_i,y_i)}{2^i}$. For each $j$, let $\pi_j \colon \Pi_{i=1}^{\infty} X_i \to X_j$ be defined by $\pi_j\big(\langle x_1, x_2, \ldots\rangle\big) = x_j$ that is, $\pi_j$ is the projection map onto the $j$-th factor space. For each $i$ let $f_i \colon X_{i+1} \to 2^{X_i}$ be a set valued function where $2^{X_i}$ denotes the hyperspace of all nonempty closed subsets of $X_i$. The inverse limit of the sequence of pairs $\{(X_i, f_i)\}$, denoted by $\varprojlim \{X_i, f_i\}$, is defined to be the set of all points $\langle x_1, x_2, \ldots \rangle$ in $\Pi_{i=1}^{\infty} X_i$ such that $x_i \in f_i(x_{i+1})$. The functions $f_i$ are called *bonding functions*. For a finite sequence $\mathbf{x} = \langle x_1, x_2, \ldots, x_n \rangle$ and finite or infinite sequence $\mathbf{y} = \langle y_1, y_2, \ldots \rangle$, let $\mathbf{x} \oplus \mathbf{y} = \langle x_1, x_2, \ldots, x_n, y_1, y_2, \ldots \rangle$. More information on inverse limits with upper semicontinuous bonding functions can be found for example in the book by Ingram and Mahavier [2010].

A continuum $X$ is called a *dendrite* if it is locally connected and it contains no simple closed curves.

A continuum $X$ is a *hereditarily unicoherent* if for any two subcontinua $A$ and $B$ of $X$ the intersection $A \cap B$ is connected. Consequently, by induction, the intersection of any finite family of subcontinua of $X$ is connected, and since $X$ is compact the intersection of any family of subcontinua of $X$ is connected. As a consequence, for any subset $S$ of $X$ there is a unique continuum $C$ such that $S \subset C$ and $C$ is contained in any continuum that contains $S$. Here $C$ is the intersection of all continua that contain $S$. The continuum $C$ is called the *irreducible continuum containing* $S$.

## 3. COUNTEREXAMPLES

Theorem 16 below has been proved by Banič and Kennedy [2015] and also by Roškarič and Tratnik [2015] independently. In this section, we show that, this theorem cannot be generalized further by replacing $[0, 1]$ by a circle nor by a simple triod. We provide examples of inverse sequences of circles or of simple triods with set-valued bonding functions whose graphs are arcs and the limits are $n$-point sets .

**Theorem 16** Suppose that $f : [0, 1] \to 2^{[0,1]}$ is an upper semicontinuous function whose graph $G(f)$ is connected. Then $\varprojlim \{[0, 1], f\}$ consists of either one or infinitely many points.

**Definition 3** Let $X$ be a compact metric space. For an upper semicontinuous (not necessarily surjective) function $f$ on $X$, and a positive integer $n$, define $P_n(f) = \{x \in X :$ there is $x_n \in X$ such that $\langle x_n, x\rangle \in G(f^n)\}$, and let $P(f) = \bigcap_{n=1}^{\infty} P_n(f)$.

Note that $f|_{P(f)} : P(f) \to 2^{P(f)}$ is surjective and that if $f$ is surjective, then $P(f) = X$.

The following theorem is a generalization of Theorem 3.4 by Banič and Kennedy [2015].

**Theorem 17** Suppose $X$ is a compact metric space and $f : X \to 2^X$ is upper semicontionuous. Then $\varprojlim \{X, f\} = \varprojlim \{P(f), f|_{P(f)}\}$.

**Proof:** It is clear that $\varprojlim \{P(f), f|_{P(f)}\} \subset \varprojlim \{X, f\}$. Let $\langle y_1, y_2, \ldots\rangle \in \varprojlim \{X, f\}$ be any point and let $i$ be a positive integer. For each positive integer $n$ and $x_i = y_{i+n}$, $x_i \in f^n(y_i)$. Thus, $\langle x_i, y_i\rangle \in G(f^n)$. So, this shows that $y_i \in P(f)$ for each nonnegative integer $i$. Obviously $y_i \in f(y_{i+1}) = f|_{P(f)}(y_{i+1})$ for each nonnegative integer $i$. Thus $\langle y_1, y_2, \ldots\rangle \in \varprojlim \{P(f), f|_{P(f)}\}$. □

In the next two examples we show that the assumptions that $X = [0, 1]$ and that $G(f)$ is connected are necessary conditions in Theorem 16.

**Example 2** For every natural number $n$ there is an upper semicontinuous function $f : S^1 \to 2^{S^1}$ whose graph is an arc such that $\varprojlim \{S^1, f\}$ has exactly $n$ points.

Represent $S^1$ as $[0, 1]$ with 0 and 1 identified. For a fixed natural number $n$, define $a_i = \frac{i}{n+1}$ and define $f : S^1 \to 2^{S^1}$ as follows:

$$f(x) = \begin{cases} \{x + a_1\} & : 0 \leq x \leq a_{n-1} \\ \{a_n\} & : a_{n-1} \leq x \leq \frac{a_{n-1}+a_n}{2} \\ \{a_1\} & : \frac{a_{n-1}+a_n}{2} \leq x \leq 1 \end{cases}$$

and note that the only point with a double value is $\frac{a_{n-1}+a_n}{2}$.

To describe the inverse limit $\varprojlim \{S^1, f\}$ we will use Theorem 17. Note that $P_1(f) = [a_1, a_n]$, $P_2(f) = [a_2, a_n] \cup \{a_1\}$, $\ldots$, and $P_k(f) = [a_k, a_n] \cup \{a_1, a_2, \ldots, a_{k-1}\}$. Consequently, $P(f) = P_n(f) = \{a_1, a_2, \ldots, a_n\}$. Observe that, $f|_{P(f)}$ is the permutation defined by $f(a_i) = a_{(i+1)} \bmod n$. So by Theorem 17, the only points of the inverse limit $\varprojlim \{S^1, f\}$ are $\langle a_i, a_{i-1}, \ldots, a_1, a_n, a_{n-1}, \ldots \rangle$ for $i \in \{1, 2, \ldots, n\}$ as required.

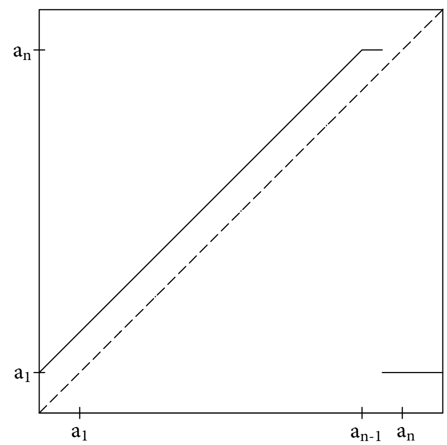


Figure 1. Inverse limit with n points.

**Example 3** Connectedness of the graph $G(f)$ is essential in Theorem 16 . Really, if $X = [0, 1]$ and $f$ is the function pictured in Figure 1, then, arguing as in Example 2 we can see that $\varprojlim \{[0, 1], f\}$ consists of $n$ points.

Next example shows that $X = [0, 1]$ in Theorem 16 cannot be replaced by a simple triod.

**Example 4** Let $X = A \cup B$ where $A$ and $B$ are as shown in Figure 2.

Define $f : A \cup B \to 2^A \subset 2^{A \cup B}$ such that the graph of $f$ as in the Figure 3.

It is clear that the graph $G(f|_A)$ of $f|_A : A \to 2^A$ is as in Figure 1 and the graph of $f|_B : B \to 2^A$ as in Figure 4.

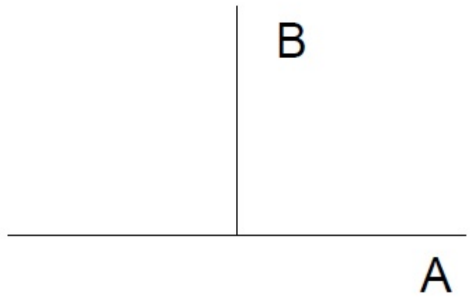


Figure 2. Triod.

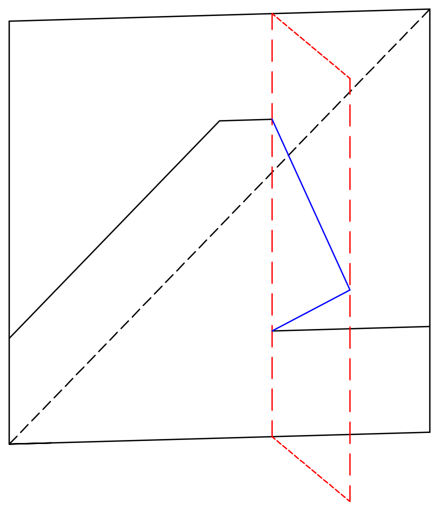

Figure 3. Function of the triod.

With a similar argument as in Example 2 we see that,

$P_1(f) = [a_1, a_n] \subset A$

$P_2(f) = [a_2, a_n] \cup \{a_1\} \subset A, \ldots$

Finally, $P(f) = P_n(f) = \{a_1, a_2, \ldots, a_n\}$ and $\varprojlim \{A \cup B, f\}$ has exactly $n$ points.

**Problem 2** Can Theorem 16 be generalized to an arc-like continuum in place of $[0, 1]$?

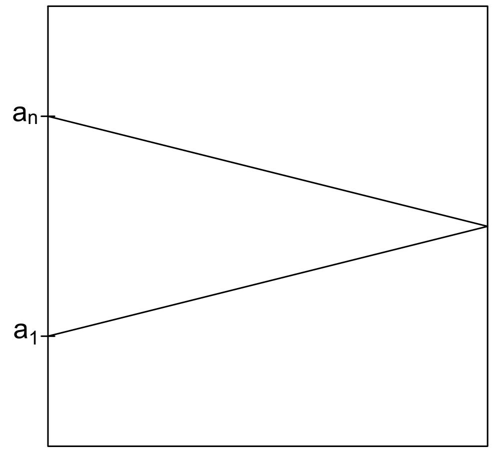


Figure 4. Partial graph.

## 4. ZERO-DIMENSIONAL INVERSE LIMITS

Now we present a class of compact metric zero-dimensional spaces that can be obtained as the inverse limit of intervals with bonding functions whose graphs are arcs.

**Theorem 18** Let $X$ be a nonempty compact subset of $[0, 1]$ and let $a$ and $b$ be the least and the greatest elements of $X$ (in the order of $[0, 1]$), respectively. Define $Y$ to be the one point compactification of the countable union $\bigcup_{i=1}^{\infty} X_i$, where each $X_i$ is a homeomorphic copy of $X$ and $b$ in $X_i$ is identified with $a$ in $X_{i+1}$. Then there is a function $f : [0, 1] \to 2^{[0,1]}$ whose graph is an arc such that $Y$ is homeomorphic to the inverse limit $\varprojlim \{[0, 1], f\}$.

**Proof:** Without loss of generality we may assume $X \subset [a, b] \subsetneq (0, 1)$. For each $c, d \in X$ such that $c < d$ and $[c, d] \bigcap X = \{c, d\}$, define $m(c, d) = \langle a - \frac{1}{2}(d - c), \frac{1}{2}(c + d)\rangle$ and let $L_{c,d}$ be the union of two line segments: from $\langle a, c\rangle$ to $m(c, d)$ and from $m(c, d)$ to $\langle a, d\rangle$. Let $f$ be the set-valued function whose graph (see Figure 5) is the union of the line segment from $\langle 0, a\rangle$ to $\langle a, a\rangle$, the set $\{a\} \times X$, the union of all arcs $L_{c,d}$ for all $c, d$ for which $L_{c,d}$ is defined, and the line segment from $\langle a, b\rangle$ to $\langle 1, b\rangle$.

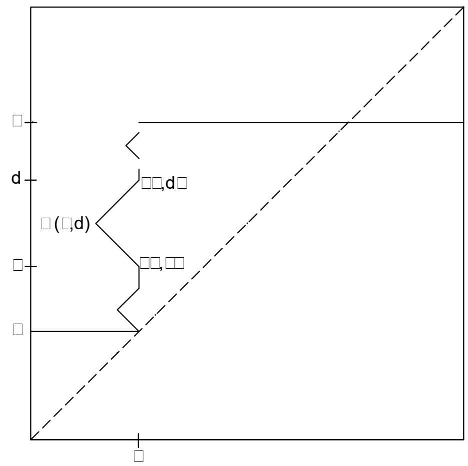

Figure 5. Graph describing Theorem18.

We will show that the inverse limit $\varprojlim \{[0,1], f\}$ is homeomorphic to $Y$. Let us examine all possible threads $\langle x_1, x_2, \ldots\rangle$ in the inverse limit. First observe that all coordinates of any thread are in $[a, b]$. Define $X_n$ to be the set of all threads $\langle x_1, x_2, \ldots\rangle$ such that $x_i = b$ for $i < n$, $x_n \in X$, and $x_i = a$ for $i > n$. Notice that each $X_n$ is homeomorphic to $X$, and that $X_n \cap X_{n+1}$ is the one point set $\{\langle b, b, \ldots, b, a, a, \ldots\rangle\}$, where first $n$ coordinates are equal to $b$. Then the inverse limit $\varprojlim \{[0,1], f\} = \bigcup_{n=1}^{\infty} X_n \cup \{\langle b, b, \ldots\rangle\}$ as required.

□

**Example 5** Marjanović [1972] characterized compact metric zero-dimensional spaces $X$ that are homeomorphic to their hyperspaces $2^X$ and showed there are exactly 9 such spaces. All these 9 spaces can be obtained by using the method given in Theorem 18.

**Theorem 19** Any compact metric zero-dimensional space can be represented as the inverse limit of $[0,1]$ using two bonding functions whose graphs are arcs.

**Proof:** Let first bonding function be the one that we defined in the proof of Theorem 18 and the remaining bonding maps be the constant map at $a$. Then $\langle x, a, a, \ldots\rangle$ is the only kind of thread in the inverse limit. Thus, $\{\langle x, a, a, \ldots\rangle : x \in X\}$ is homeomorphic to $X$. □

**Problem 3** What kind of compact metric zero-dimensional spaces can be obtained as the inverse limits $\varprojlim \{[0,1], f\}$ with one set-valued function $f$? In particular, is the converse of the Theorem 18 true, i.e. if $Y = \varprojlim \{[0,1], f\}$ is a compact metric zero-dimensional space, then does there exist a compact zero-dimensional set $X$ such that $Y$ is the union of infinitely many copies of $X$ with the identifications as in Theorem 18?

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# III. LIMITS OF INVERSE LIMITS - A COUNTEREXAMPLE

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**ABSTRACT**

An answer is provided to the following question that was asked by Kelly and Meddaugh in 2015. Let $X$ be a continuum and $G(f)$ be the graph of an upper semi-continuous functions $f : X \to 2^X$. Denote by $K_n = \varprojlim \{X, f_n\}$ for all $n \in \mathbb{N}$ and $K = \varprojlim \{X, f\}$. Assuming that $\lim_{n\to\infty} G(f_n) = G(f)$, and $\pi_1(K) \subset \liminf_{n\to\infty} \pi_1(K_n)$ where $\pi_1$ denotes the projection from the infinite product $\Pi X$ onto its first coordinate, and $f$ is continuous, does it follow that $\lim_{n\to\infty} K_n = K$ in $2^{\Pi X}$?

**Keywords:** inverse limit, set-valued functions, upper semi-continuous

## 1. INTRODUCTION

By [Banič et al., 2010] and [Banič et al., 2011], it has been discussed under what conditions the sequence of inverse limits obtained from $f_n$, a sequence of upper semi-continuous functions, converges to the inverse limit obtained from a function $f$, when the sequence of the graphs of $f_n$ converges to the graph of $f$. Kelly and Meddaugh [2015] has given two nonequivalent generalizations to the previous answers by [Banič et al., 2010] and [Banič et al., 2011], and they posted the following question. Let $f : X \to 2^X$ be a function, and for each $n \in \mathbb{N}$, let $f_n : X \to 2^X$ be upper semi-continuous such that $\lim_{n\to\infty} G(f_n) = G(f)$, and $\pi_1(K) \subset \liminf_{n\to\infty} \pi_1(K_n)$. If $f$ is continuous, does it follow that $\lim_{n\to\infty}(K_n) = K$ in $2^{\prod X}$? In this paper we provide a negative answer to this question by giving a counterexample.

First, we need to explain the denotations we use. For a compact metric space $X$, let $f : X \to 2^X$ be an upper semi-continuous function, and $G(f) = \{(x, y) \in X \times X | \, y \in f(x)\}$. For each $n \in \mathbb{N}$, let $f_n : X \to 2^X$ be a sequence of upper semi-continuous functions and let $K = \varprojlim \{X, f\}$, $K_n = \varprojlim \{X, f_n\}$ for all $n \in \mathbb{N}$. Denote by $\Pi X$ the infinite product $\prod_{i=1}^{\infty} X$ and by $\pi_1 : \Pi X \to X$ the projection onto the first coordinate.

## 2. EXAMPLE

We construct an example where $X = [0, 1]$, $f : [0, 1] \to 2^{[0,1]}$ is a continuous function, $f_n : [0, 1] \to 2^{[0,1]}$ is an upper semi-continuous function for all $n \in \mathbb{N}$, $\lim_{n\to\infty} G(f_n) = G(f)$ and $\pi_1(K) \subset \liminf_{n\to\infty} \pi_1(K_n)$ but $\lim_{n\to\infty} K_n \neq K$ in $2^{\Pi X}$.

Let $\{\langle a_1, b_1\rangle, \langle a_2, b_2\rangle, \dots\}$ be dense subsets of $[0, 1]^2$ such that all points of $\{a_1, a_2, \dots, b_1, b_2, \dots\}$ are different.

For each $n \in \mathbb{N}$, define $f_n : [0, 1] \to 2^{[0,1]}$ by

$$f_n(x) = \begin{cases} \{x\} & \text{if } x \notin \{a_1, a_2, \dots, a_n\} \\ \{a_i, b_i\} & \text{if } x = a_i \text{ and } i \leq n \end{cases}$$

Thus, the graph of $f_n$ is the diagonal with $n$ points added to it. Let $f : [0, 1] \to 2^{[0,1]}$ be the function defined by $f(x) = [0, 1]$ for all $x \in [0, 1]$.

First observe that $K = \varprojlim \{[0, 1], f\} = [0, 1]^\infty$ and $\pi_1(K) = \pi_1(K_n) = [0, 1]$. To show that $\lim_{n\to\infty} K_n \neq K$ in $2^{\Pi X}$, define $F$ as the set of points of $2^{\Pi X}$ in the form: $\langle a, a, \dots, a, b, b \dots\rangle$ where $a, b \in [0, 1]$. Then $F$ is a closed proper subset of $K$. All points of the inverse limit $K_n$ are the form $\langle a, a, \ldots \rangle$ for $a \in [0, 1]$ or $\langle b_i, b_i, \ldots, b_i, a_i, a_i, \ldots \rangle$ for $i \leq n$. Then, clearly $K_n \subset F$. Since $F$ is closed and $K_n \subset F$, we have that $\lim_{n\to\infty} K_n \subset F$, so $\lim_{n\to\infty} K_n \neq K$ in $2^{\Pi X}$.

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# SECTION

## 3. SUMMARY AND CONCLUSIONS

This dissertation focuses on zero-dimensional compact metric spaces and inverse limits of upper semi-continuous set-valued functions; it consists of three main parts. Firstly, an uncountable family of zero-dimensional compact metric spaces, which are homeomorphic to the their Cartesian squares, is constructed. The full characterization of such spaces is still unknown; and one might possibly find a characterization of such spaces using Marjanovic's technique. Secondly, a wide class of zero-dimensional compact metric spaces, which can be obtained as the inverse limit of upper semi-continuous functions, is found. Lastly, the following results about the inverse limit of upper semi-continuous functions are obtained:

- Suppose that $f : [0,1] \to 2^{[0,1]}$ is an upper semicontinuous function, whose graph $G(f)$ is connected. Then, $\varprojlim \{[0,1], f\}$ consists of either one or infinitely many points. This result cannot be generalized to the cases when (i) the coordinate space $[0,1]$ is replaced by a simple triod or a simple closed curve and (ii) the function's graph $G(f)$ is not connected.

- Let $X$ be a continuum and $G(f)$ be the graph of an upper semi-continuous functions $f : X \to 2^X$. Denote by $K_n = \varprojlim \{X, f_n\}$ for all $n \in \mathbb{N}$ and $K = \varprojlim \{X, f\}$. Assuming that $\lim_{n\to\infty} G(f_n) = G(f)$; $\pi_1(K) \subset \liminf_{n\to\infty} \pi_1(K_n)$, where $\pi_1$ denotes the projection from the infinite product $\Pi X$ onto its first coordinate; and $f$ is continuous. Then, $\lim_{n\to\infty} K_n$ is not equal to $K$ in $2^{\Pi X}$.

Inverse limit of upper semi-continuous functions is a new subject and there are new results, answers, questions coming up all the time. This subject seems interesting and promising in terms of working in the area, collaborating with others, answering problems, and posing questions.

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## VITA

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