Spring 2008

Inverse limits of permutation maps

Robbie A. Beane

Department: Mathematics and Statistics

INVERSE LIMITS OF PERMUTATION MAPS

by

ROBBIE ALLEN BEANE

A DISSERTATION

Presented to the Faculty of the Graduate School of the

MISSOURI UNIVERSITY OF SCIENCE AND TECHNOLOGY

in Partial Fulfillment of the Requirements for the Degree

DOCTOR OF PHILOSOPHY

in

MATHEMATICS

2008

Włodzimierz J. Charatonik, Advisor

Robert P. Roe

Matt Insall

Ilene H. Morgan

W. T. Ingram

ABSTRACT

In this paper we study the topological properties of continua which arise as inverse limits on $[0,1]$ with bonding maps chosen from the permutation family of Markov maps. For such inverse limits, we examine the occurrence of indecomposability, the number of end points in the continuum, and the types of subcontinua present in the continuum. We provide a process for determining the topological structure of the inverse limit generated by a single permutation map, or by the composition of several such maps. Additionally, we show that all such inverse limits are Kelley continua. We will apply these results to study inverse limits on $[0,1]$ with a single bonding map chosen from the one parameter family of logistic mappings. It is known that there is an open and dense subset of the parameter space for which the associated logistic maps have attracting periodic orbits. We show that any continuum generated by such a logistic map is homeomorphic to the inverse limit on $[0,1]$ with some permutation bonding map. We close by providing a sufficient condition for the inverse limit on an interval with a single bonding map to fail to be a Kelley continuum, and applying this information to the logistic family.

ACKNOWLEDGMENT

I wish to thank Professor Włodzimierz J. Charatonik for serving as my advisor, and for introducing me to the beauty of topology. The incredible amount of support and encouragement he has shown me has helped me to find confidence in myself as a mathematician. I would also like to thank him for being more than an advisor. Włodek has also been a good friend, and someone with whom I have been able to share more than just mathematics.

I would also like to thank the other members of my committee - Dr. Robert Roe, Dr. Ilene Morgan, Dr. Matt Insall, and Dr. W. T. Ingram - for agreeing to serve on my committee, for proofreading this dissertation, and for so much more. The time that Dr. Roe has made to discuss my research, and the insights that he has given me, have proven invaluable in the preparation of this dissertation. I am also very grateful to him for the encouragement he has shown me during the many years in which I have known him. I would like to think Dr. Morgan and Dr. Insall for providing me with outlets to explore my interests in algebra. I am indebted to Dr. Ingram for providing the foundations on which this dissertation rests, and for encouraging me to continue to pursue the problems which led to the results presented here.

For listening to my talks and providing me with insights and inspiration, I would like to thank the participants of the topology seminar: Dr. Charatonik, Dr. Roe, Dr. Insall, Dr. Prajs, Chris Jacobsen, and Evan Wright.

I would like to extend my sincere appreciation to Col. Tom Akers for helping me to find confidence in myself as an instructor, and for helping me to improve myself as a teacher. I was very fortunate to have his guidance and his example to look to during my first year as a mathematics instructor.

It would not be possible to list each of the friends and family members to whom I am indebted for their friendship and support, but I would like to mention Bobby, Joyce, and Robert Marcak, John and Pam Nelson, and Jon Swagman. In particular, I want to thank Jason Black, for being my best friend for more than a decade. The last ten years of my life would have been a lot less interesting without him there.

Finally, I want to thank Katie Scanlon for her constant love and support, and for her patience during the last few months. My time with her has been truly wonderful.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

1. INTRODUCTION AND OUTLINE

In 2002 W. T. Ingram introduced a family of Markov maps called permutation maps [22]. Each such map is defined by permuting the elements of its partition in some way, and then extending the map linearly to the rest of the interval. In that paper, Ingram began a study of inverse limits on $[0, 1]$ with bonding maps chosen from this family. The goal of this dissertation is to continue this study. Here we provide an outline of the material contained in this paper.

In Section 2, we provide a brief historical account of related topics. Specifically, we focus on the historical development in the theory of continua, inverse limits, and Kelley continua.

Section 3 serves to provide preliminary information specifically related to our study. In this section, one can find definitions and fundamental results relating to dynamical systems, continuum theory, inverse limits, Markov maps, Kelley continua, and decomposition spaces. The collection of results and definitions supplied in this section is intended to be the smallest such collection which adequately prepares the reader to understand the material presented in later sections.

Section 4 contains the main results of this dissertation. We will begin this section by providing the definition of a permutation map and developing some notation and terminology that will be useful in our study. In Subsection 4.1 we will be concerned with end points of inverse limits with Markov bonding maps. In this subsection we provide a theorem which determines the number of end points in the classical sense which are present in such an inverse limit. Subsection 4.2 deals with indecomposability, and features a theorem which provides conditions under which the inverse limit on an interval with a Markov bonding maps is an indecomposable continua whose only subcontinua are arcs. These indecomposable arc continua will be of special importance in our study, as they (along with the arc) provide the building blocks from which each inverse limit with Markov bonding maps is constructed. Subsection 4.3 provides the primary machinery we will use to determine the overall structure of these particular inverse limits. In Subsection 4.4 we show that inverse limits with permutation maps produce Kelley continua. Finally, in Subsection 4.5, we provide an example of the process we have formulated for determining the topological structure of an inverse limit with permutation bonding maps.

In Section 5 of this paper, we turn our attention to the logistic family of mappings. The logistic family is a one parameter family of mappings which has been intensely studied by dynamicists. We show that logistic maps determined by parameter values in an open and dense subset of the parameter space produce inverse limits which are homeomorphic to inverse limits with permutation bonding maps. We close by providing a sufficient condition for the inverse limit on an interval with a single bonding map to fail to be a Kelley continuum, and applying this information to the logistic family.

2. A BRIEF HISTORY OF RELATED TOPICS

Throughout this paper, the term *continuum* will mean a compact connected metric space. Some authors relax the condition that a continuum be metric, and instead consider compact connected Hausdorff spaces. Such a space is typically referred to as a *Hausdorff continuum*. The term *continuum* was coined by G. Cantor in 1883. Cantor originally defined a continuum to be a perfect subset X of a Euclidean space such that to each $a, b \in X$ and each $\varepsilon > 0$, there corresponds a finite system $a = p_0 < p_1 < ... < p_n = b$ of points in X satisfying $|p_i - p_{i-1}| < \varepsilon$ for $i = 1, 2, ..., n$ [6, p. 576]. For compact metric spaces, the existence of such a finite system of points is equivalent to the notion of connectedness, as is shown, for example, in Kuratowski's Monograph [29, Theorem 0, p. 167]. The core notions involved in the modern definition of a continuum - connectedness, compactness, and metric spaces - were not identified when Cantor first originated the term "continuum". These terms were introduced in the late 19th and early 20th centuries, and their meanings have since evolved to their modern definitions. For a history of the evolution of these terms, and for a detailed history of continuum theory in general, the reader is referred to the article *History of Continuum Theory* [11] by J. J. Charatonik.

2.1. INVERSE LIMITS

An *inverse sequence* is a pair $\{X_i, f_i\}_{i=1}^{\infty}$ where each X_i is a topological space, and each f_i is a mapping (i.e., continuous function) with $f_i : X_{i+1} \to X_i$. The spaces X_i are called *factor spaces*, and the mappings f_i are called *bonding maps*. Given an inverse sequence $\{X_i, f_i\}_{i=1}^{\infty}$, the *inverse limit* of the inverse sequence, denoted by $\varprojlim\{X_i, f_i\}$, is the subset of the product space $\prod_{i=1}^{\infty} X_i$ defined by: $\varprojlim\{X_i, f_i\} = \left\{(x_i)_{i=1}^{\infty} \in \prod_{i=1}^{\infty} X_i : f_i(x_{i+1}) = x_i \text{ for all } i \in \mathbb{Z}^+\right\}$. It is well known that the inverse limit of continua is a continuum, and that the inverse limit of nonempty compact metric spaces is a nonempty compact metric space.

Inverse limits have proven to be a valuable tool in continuum theory, as complicated continua may be constructed as the inverse limit of simple spaces. For instance, it is known that every arc-like continuum, or equivalently, every chainable continuum, is homeomorphic to an inverse limit on intervals. Hence the pseudo-arc, which is an example of a hereditarily indecomposable, hereditarily equivalent, homogeneous arc-like continuum, can be obtained as an inverse limit on intervals. In fact, in 1964 G. W. Henderson showed that the pseudo-arc can be represented using an inverse limit on $[0, 1]$ with a single bonding map [16], though this is not true for arc-like continua in general. In 1967 W. S. Mahavier showed that not every arc-like continuum can be represented by an inverse limit on intervals with only one bonding map, though in the same paper Mahavier established that every arc-like continuum can be embedded in such an inverse limit [30]. In 1969 H. Cook and W. T. Ingram constructed two mappings on $[0, 1]$ such that every arc-like continuum is homeomorphic to the inverse limit on $[0, 1]$ using some sequence of these two maps as the bonding maps [9].

We now provide a brief list of some other well-known results in continuum theory that were established using inverse limit techniques. In 1959 R. D. Anderson and G. Choquet [1] constructed a non-separating plane continuum with the property that no two of its nondegenerate subcontinua are homeomorphic. Then in 1961 J. J. Andrews modified the Anderson-Choquet example to show that there exists an arc-like continuum with the same property [2]. In 1967 H. Cook used a similar method to construct a continuum whose only non-constant self-map is the identity [8]. In 1965 R. M. Schori constructed a universal arc-like continuum, i.e. an arc-like continuum which contains a homeomorphic copy of every arc-like continuum [36]. In 1972 W. T. Ingram constructed an atriodic, tree-like continuum which is not arc-like as the inverse limit on simple triods with a single bonding map [19]. In 1980 D. P. Bellamy constructed a tree-like continuum which admits a fixed-point-free homeomorphism onto itself [4]. The question of whether or not there exists a non-separating plane continuum which admits a fixed-point-free map into itself remains open, and is one of the most famous problems in continuum theory.

2.2. KELLEY CONTINUA

Given a continuum X and a point $p \in X$, X is said to be *Kelley at* p (or alternately, to have the *property of Kelley at* p), provided that for each subcontinuum K of X containing p and for each sequence $\{p_n\}_{n=1}^{\infty}$ converging to p, there is a sequence of subcontinua $\{K_n\}_{n=1}^{\infty}$ converging to K such that $p_n \in K_n$ for each n. A continuum is said to be a *Kelley continuum* (or alternately, to have the *property of Kelley*), if it is Kelley at each of its points.

The property of Kelley was introduced by J. L. Kelley as Property 3.2 in [27] to investigate contractibility of hyperspaces. Kelley showed that the hyperspaces 2^X and $C(X)$ are contractible when X is a Kelley continuum [27, Theorem 3.3]. There are many important classes of continua whose members are Kelley continua. It is known that all locally connected continua are Kelley [18, Example 20.4]. R. W. Wardle showed that homogeneous continua and hereditarily indecomposable continua are all Kelley [38, Theorems 2.3 and 3.1]. J. J. Charatonik generalized Wardle's result concerning homogeneous continua by proving that any continuum homogeneous with respect to open mappings is Kelley [10], and H. Kato later showed that the result can not be further extended to continua homogeneous with respect to confluent mappings. In the same paper as the previously mentioned results, Wardle established that the property of Kelley is preserved under confluent mappings [38, Theorem 4.3], and that any continuum has the property of Kelley at each point of a dense G_δ set [38, Theorem 2.3]. In [12], W. J. Charatonik showed that the inverse limit of Kelley continua with confluent bonding maps is itself a Kelley continua.

3. PRELIMINARY DEFINITIONS AND RESULTS

The material in this section establishes fundamental definitions and results that will be used throughout this dissertation. We will assume some amount of familiarity with basic topological notions, though nothing above what would be covered in a first year graduate level course in general topology.

The material in this section is arranged in the following way: Subsection 3.1 will provide necessary terminology from the study of dynamical systems. In Subsection 3.2 we will introduce some basic notions and results related to the theory of continua. In Subsection 3.3, the definition of an inverse limit is given, and fundamental results dealing with inverse limits are provided. Subsection 3.4 deals with Markov maps, which are of fundamental importance throughout this paper. Subsection 3.5 contains the definition of a Kelley continuum, and a theorem of W. J. Charatonik which provides a sufficient condition for an inverse limit to be a Kelley continuum. Finally, Subsection 3.6 discusses upper semi-continuous decompositions of continua.

The only results in this preliminary section which are due to the author are found in Subsection 3.3. References to the original source of a result is provided when this information is known to the author. For the sake of completion, many proofs are provided for the results in this section, though they may be omitted by the reader.

3.1. DYNAMICS

The theory of discrete dynamical systems is concerned with the analysis of the general behavior, and particularly the long-term behavior, of points in a space under iteration of a continuous function from the space into itself. Results from dynamical systems often provide powerful tools for analyzing inverse limits, which will be introduced in Subsection 3.3, and are the central topic of this dissertation. In this subsection, we will provide the terminology and results from this field which are used in this dissertation. All spaces in this section are assumed to be metric. We will begin with the definition of a mapping.

Definition 3.1. A *map* or a *mapping* is a continuous function.

Notation. The n-fold composition of a mapping $f : X \to X$ with itself is denoted by f^n, with the convention that f^0 denotes the identity on X.

As mentioned in the opening paragraph of this section, in dynamics one is often interested in the behavior of a point in a space under iteration of a mapping on that space. We refer to the set of all iterates of a point as the orbit of that point.

Definition 3.2. Given a space X, a point $p \in X$, and a mapping $f : X \to X$, the *orbit of p under f*, denoted $\mathrm{Orbit}(p, f)$, is the set $\mathrm{Orbit}(p, f) = \{f^n(p) : n$ is a non-negative integer$\}$. Given a subset $A \subseteq X$, we say that the *orbit of A under f*, denoted $\mathrm{Orbit}(A, f)$, is the family $\mathrm{Orbit}(A, f) = \{f^n[A] : n$ is a non-negative integer$\}$.

There are many ways in which a point can behave under iteration. Perhaps the most important type of behavior is that displayed by fixed points, or more generally, periodic points, which we now define.

Definition 3.3. Given a space X, a mapping $f : X \to X$, and a point $p \in X$, we say that *p is periodic under f* if there exists a positive integer n such that $f^n(p) = p$. The least such n is called the *period of p under f*. If p is periodic with period 1, or in other words, $f(p) = p$, then we say that *f is fixed under f*.

It is worth noting that if a point p is periodic under f with period k, then p is a fixed point of the mapping f^k. It is clear that periodic points have a finite orbit. There can be, however, non-periodic points which have finite orbits. These points are not themselves periodic, but under iteration of the mapping, eventually land on a periodic point.

Definition 3.4. Given a space X, a mapping $f : X \to X$, and a point $p \in X$, we say that *p is eventually periodic under f* if p itself is not periodic, but there exists a positive integer n such that $f^n(p)$ is periodic.

In studying the dynamics of a mapping f, one is often interested in the long-term behavior of points near fixed points.

Definition 3.5. Given a space X with metric d, a mapping $f : X \to X$, a point $q \in X$, and a fixed point $p \in X$, we say that p *attracts* q *under* f if $f^n(q) \to p$ as n increases without bound. We say that p is an *attracting fixed point of* f if there is a neighborhood U of p such that p attracts each point in U. The point p is a *repelling periodic point* if there is a neighborhood U of p such that if $x \in U$ and $x \neq p$, then $d(f(x), p) > d(x, p)$.

The notion of an attracting or repelling fixed point can be generalized to include periodic points as well.

Definition 3.6. Consider a space X, a mapping $f : X \to X$, and a periodic point $p \in X$ with period k. The point p is called an *attracting periodic point* if it is an attracting fixed point of f^k. Similarly, p is said to be a *repelling periodic point* if it is a repelling fixed point of f^k.

The definition of an attracting periodic point p tells us that all points sufficiently near p are attracted to p. We now introduce terminology to describe the collection of all points which are attracted to such a periodic point.

Definition 3.7. Consider a space X, a mapping $f : X \to X$, and an attracting periodic point $p \in X$ with period k. The *basin of attraction of* p is the collection of all points in X which are attracted to p under f^k. The *immediate basin of attraction of* p is the connected component of the basin of attraction of p which contains p.

We now extend the definition of a periodic point to allow for the notion of a periodic set.

Definition 3.8. Given a space X, a mapping $f : X \to X$, and a set $A \in X$, we say that A *is periodic under* f if there exists a positive integer n such that $f^n[A] = A$. The least such n is called the *period of* A *under* f. If A is periodic with period 1, or in other words, $f[A] = A$, then we say that f *is fixed, or invariant, under* f.

An important class of invariant sets is provided by ω-limit sets.

Definition 3.9. Let X be a space, and $f : X \to X$ be a mapping. Given a point $p \in X$, the *ω-limit set* of p, denoted by $\omega(p, f)$, is the set of all points $y \in X$ such that some subsequence of $\{f^n(p)\}_{n=1}^{\infty}$ converges to y. Given a set $A \subseteq X$, the *ω-limit set* of p, denoted by $\omega(p, f)$, is the set of all points $y \in X$ such that if U is a neighborhood of y and m is a positive integer, then there exists an integer $n > m$ such that $f^n[A] \cap U \neq \emptyset$.

In this paper, we will only be interested in the ω-limit sets of finite collections of eventually periodic points, and primarily use this notion as a notational convenience. It is clear that the ω-limit set of such a collection A is the finite set containing exactly those periodic points x which lies in the orbit of some point $y \in A$.

We close this subsection with a discussion about the Schwarzian derivative.

Definition 3.10. *The* Schwarzian derivative *of a function* $f : \mathbb{R} \to \mathbb{R}$ *at* x *is given by*

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2$$

The Schwarzian derivative is a useful tool for studying one-dimensional dynamical systems. For results relating to the Schwarzian derivative, see, for example [13, Section 1.11]. If a mapping f has a negative Schwarzian derivative at every point in its domain, we say it has negative Schwarzian derivative, and denote this situation by $Sf < 0$. Our interest in the Schwarzian derivative stems from the following theorem, which we will make use of in Section 5.

Theorem 3.11. (Singer [37, Theorem 2.7]) *If $f : [a, b] \to [a, b]$ is a mapping with negative Schwarzian derivative, then each attracting periodic orbit contains a point p such that the immediate basin of attraction for p contains either a critical point of f or an end point of $[a, b]$. Hence, each attracting periodic orbit of f attracts at least one critical point or end point.*

3.2. CONTINUA

In this subsection we will provide basic notions and results from continuum theory. We will begin with the definition of a continuum.

Definition 3.12. A *continuum* is a nonempty compact connected metric space. A *(proper) subcontinuum* of a continuum X is a (proper) subset of X which is also a continuum.

The next definition introduces two important classes of continuous functions: monotone mappings, and confluent mappings.

Definition 3.13. A mapping $f : X \to Y$ is said to be *monotone* if $f^{-1}(y)$ is connected for every $y \in Y$. A mapping $f : X \to Y$ is said to be *confluent* if for each subcontinuum K of Y, each component of $f^{-1}(K)$ maps onto K under f.

It is well known each monotone map is also confluent. This fact follows directly from the definitions of such mappings. As an immediate consequence, one also sees that the composition of monotone maps is monotone. Next we provide the definition of an indecomposable continuum.

Definition 3.14. A continuum is said to be *decomposable* if it the union of two proper subcontinua. Otherwise it is *indecomposable.*

We now introduce two different notions of an end point of a continuum. End points in the classical sense will be of particular importance to us later.

Definition 3.15. Given a continuum X, a point $p \in X$ is said to be an *end point of* X if given any two subcontinua A and B of X which contain p, either $A \subseteq B$ or $B \subseteq A$. A point $p \in X$ is referred to as an *end point in the classical sense* if it is an end point of every arc which contains it.

Next we provide the definition of a terminal subcontinuum. It should be noted that there are many, sometimes conflicting, definitions attributed to the term “terminal subcontinuum". The definition provided here is in common usage, but should not be considered the standard definition.

Definition 3.16. Given a continuum X, a subcontinuum K of X is said to be *terminal in X* if given any subcontinuum L of X such that $K \cap L \neq \emptyset$, either $K \subseteq L$ or $L \subseteq K$.

In Definition 3.17 we introduce the hyperspace 2^X as well as the Hausdorff metric. The theory of Hyperspaces is a very active and interesting field of research in continuum theory, although we do not delve into this area in this dissertation. We will, however, make occasional use of the Hausdorff metric.

Definition 3.17. Let (X, d) be a bounded metric space. The *hyperspace of closed subsets of X*, denoted 2^X, is the collection of all non-empty closed subsets of X. Let $A \in 2^X$ and $r > 0$. The *generalized open ball of radius r about A* is the set $N_d(r, A) = \{x \in X : d(x, a) < r$ for some $a \in A\}$. The *Hausdorff metric for 2^X induced by d*, denoted by $\mathcal{H}_d$, is defined by $\mathcal{H}(A, B) = \inf\{r > 0 : A \subseteq N_d(r, B)$ and $B \subseteq N_d(r, A)\}$ for all $A, B \in 2^X$.

As indicated by the name, the Hausdorff metric is indeed a metric for 2^X. For a proof of this fact, the reader is referred to [18, Theorem 2.2]. When the metric from which $\mathcal{H}_d$ is induced is obvious, we will adopt the convention of denoting the Hausdorff metric by $\mathcal{H}$.

Definition 3.18. Given a sequence $\{A_i\}_{i=1}^{\infty}$ of subsets of a space X, we define the *limit inferior* (or *lower limit*) of the sequence, denoted $\operatorname{Li} A_i$, by $\operatorname{Li} A_i = \{p \in X :$ if U is an open set containing p then $U \cap A_i \neq \emptyset$ for all but finitely many $i\}$. The *limit superior* (or *upper limit*) of the sequence, denoted $\operatorname{Ls} A_i$, is defined by $LsA_i = \{p \in X :$ if U is an open set containing p then $U \cap A_i \neq \emptyset$ for infinitely many $i\}$. We say that the sequence $\{A_i\}_{i=1}^{\infty}$ is *L-convergent to A in X* (denoted $\operatorname{Lim} A_i = A$) provided that $\operatorname{Li} A_i = \operatorname{Ls} A_i = A$.

When X is a compact Hausdorff space, the concept of L-convergence in X coincides with the notion of convergence in 2^X with respect to the Hausdorff metric [18, Theorem 4.7]. Since all of the spaces considered in this dissertation are compact and metric, we may simply say that a particular sequence of closed subsets *converges*, and there is no danger of ambiguity about the type of convergence to which we refer.

The first theorem we include states that the intersection of a nested family of continua is itself a continuum. This is an extremely useful tool for constructing continua. This theorem is used in the proof of Theorem 3.23, which states that the inverse limit of continua is a continuum.

Theorem 3.19. *Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of continua such that $X_{i+1} \subseteq X_i$ for each $i = 1, 2, ...$, and let $X = \bigcap_{i=1}^{\infty} X_i$. Then X is a continuum.*

Proof. We begin by showing that X is nonempty. To that end, let $p_i \in X_i$ for each positive integer i. Every point of the sequence $\{p_i\}_{i=1}^{\infty}$ is in X_1, and so we may assume that the sequence converges to some point $p \in X_1$, by taking a subsequence if necessary. Further, since for each k we have that $p_i \in X_k$ for all $i \geq k$, we can see that $p \in X_i$ for all i, and hence that that $p \in X$. This completes that proof that X is nonempty. It is clear that X is a closed subset of a metric space, and hence is itself compact and metric.

We have left to show that X is connected. Suppose to the contrary that X is not connected. Then there exist nonempty, closed (and hence compact), disjoint subsets A and B of X such that $A \cup B = X$. By normality of X_1 we may find disjoint open subsets V and W of X_1 containing A and B respectively. Let $U = V \cup W$. We claim that there exists a positive integer N such that $X_i \subseteq U$ for all $i \geq N$. Assuming otherwise, we have that $X_i \nsubseteq U$ for all i, and hence we can find a sequence $\{q_i\}_{i=1}^{\infty}$ in $X_1 - U$ such that $q_i \in X_i - U$ for each i. Since $X_1 - U$ is compact, We may assume that $\{q_i\}_{i=1}^{\infty}$ converges to a point $q \in X_1 - U$. Since $q_i \in X_i$ for each $i = 1, 2, ...$, the point q must be in X, but this contradicts that $q \notin U$. This justifies our claim that there exists a positive integer N such that $X_i \subseteq U$ for all $i \geq N$. Notice that $A, B \subseteq X \subseteq X_N$, and so we see that $X_N \cap V \neq \emptyset$ and $X_N \cap W \neq \emptyset$. It follows that X_N is not connected, which contradicts our assumption that X_N is a continuum. Therefore, we may conclude that X is connected, and hence a continuum. ■

3.3. INVERSE LIMITS

Here we will define the notions of an inverse sequence and the inverse limit of an inverse sequence. Inverse limits of continua represent a powerful tool for constructing continua, and are the central topic of this dissertation. It is worth noting that an inverse sequence is a special case of the more general notion of an inverse system, for which inverse limits are also defined. In this dissertation we work exclusively in the more specific setting of inverse limits of inverse sequences, so we will omit the definition of an inverse system. The curious reader is referred to the paper, *Inverse Limits* [21], by Ingram for the definition of an inverse system, as well as some basic theory relating to inverse limits of inverse systems.

Definition 3.20. An *inverse sequence* is a pair of sequences $\{X_i, f_i\}_{i=1}^{\infty}$ where each X_i is a topological space, and each f_i is a mapping with $f_i : X_{i+1} \to X_i$. The spaces X_i are called *factor spaces*, and the mappings f_i are called *bonding maps*. Given positive integers i,j such that $i < j$, we define the mapping $f_{i,j} : X_j \to X_i$ by composing the appropriate bonding functions; that is, $f_{i,j} = f_i \circ f_{i+i} \circ ... \circ f_{j-1}$. Given an inverse sequence $\{X_i, f_i\}_{i=1}^{\infty}$, the *inverse limit* of the inverse sequence, denoted by $\varprojlim\{X_i, f_i\}$, is the subset of the product space $\prod_{i=1}^{\infty} X_i$ defined by: $\varprojlim\{X_i, f_i\} = \left\{(x_i)_{i=1}^{\infty} \in \prod_{i=1}^{\infty} X_i : f_i(x_{i+1}) = x_i \text{ for all } i \in \mathbb{Z}^+\right\}$. For each positive integer n, we define a function $\pi_n : \varprojlim\{X_i, f_i\} \to X_n$ by $\pi_n((x_i)_{i=1}^{\infty}) = x_n$ for all $(x_i)_{i=1}^{\infty} \in \varprojlim\{X_i, f_i\}$. For a given n, π_n is referred to as the *n^{th}-projection.* Notice that these functions are simply the n^{th}-projection maps defined on the product space $\prod_{i=1}^{\infty} X_i$ restricted to the inverse limit, and as such, they are continuous.

We will always assume that each X_i is equipped with a metric d_i which is bounded by 1, and that the product space $\prod_{i=1}^{\infty} X_i$, and hence the inverse limit, is given the "product metric", given by $d(x, y) = \sum_{i=1}^{\infty} \frac{d_i(\pi_i(x), \pi_i(y))}{2^i}$. Throughout this paper, we will use $\mathcal{H}$ to denote the Hausdorff metric on the inverse limit, and $\mathcal{H}_i$ to denote the Hausdorff metric on the factor space X_i. When each factor space in an inverse sequence is the same space X and each bonding map is the same mapping $f : X \to X$, we will denote the inverse limit of the inverse sequence by $\varprojlim\{X, f\}$. In such a case, f induces a self-homeomorphism of the inverse limit referred to as the *shift homeomorphism*. The following theorem states this well-known fact, and provides the definition of the shift homeomorphism.

Theorem 3.21. *Let $M = \varprojlim\{X, f\}$ where X is a topological space, and $f : X \to X$ is a mapping. The function $\hat{f} : M \to M$ given by $\hat{f}((x_i)_{i=1}^{\infty}) = (f(x_i))_{i=1}^{\infty}$ is a homeomorphism.*

The following theorem, which is often referred to as the Subsequence Theorem, states that given any inverse sequence, we many "throw out" any collection of factor spaces whose complement is not finite without affecting the inverse limit. We will appeal to this theorem often throughout this paper. A proof of this theorem can be found in [21, Corollary 1.7.1].

Theorem 3.22. *Let $\{X_i, f_i\}_{i=1}^{\infty}$ be an inverse sequence, and let $\{i_j\}_{j=1}^{\infty}$ be a strictly increasing sequence of positive integers. Then $\varprojlim\{X_i, f_i\}$ is homeomorphic to $\varprojlim\{X_{i_j}, f_{i_j,i_{j+1}}\}$. In particular, given a single factor space X, and a mapping $f : X \to X$, then $\varprojlim\{X, f\}$ is homeomorphic to $\varprojlim\{X, f^n\}$ for each positive integer n.*

Next we will show that an inverse limit of continua can be realized as the intersection of a nested sequence of subcontinua of the product space, and as such, is itself a continuum.

Theorem 3.23. *Let $\{X_i, f_i\}_{i=1}^{\infty}$ be an inverse sequence for which each factor space X_i is a continuum. Then the inverse limit $X = \varprojlim\{X_i, f_i\}$ is a continuum.*

Proof. For each positive integer n, let $G_n = \Big\{(x_i)_{i=1}^{\infty} \in \prod_{i=1}^{\infty} X_i : f_i(x_{i+1}) = x_i$ for all $i \leq n\Big\}$. For each n, we define a function $h_n : G_n \to \prod_{i=n+1}^{\infty} X_i$ by $h_n((x_i)_{i=1}^{\infty}) = (x_i)_{i=n+1}^{\infty}$ for all $(x_i)_{i=1}^{\infty} \in G_n$. It is easy to verify that h_n is a homeomorphism for each n. Since the space $\prod_{i=n+1}^{\infty} X_i$ is a cartesian product of continua, and hence a continuum itself, G_n is a continuum for each positive integer i. It is clear that $G_{n+1} \subseteq G_n$ for each n, and that $X = \bigcap_{n=1}^{\infty} G_n$. Therefore, X is the intersection of a nested sequence of continua and, by Theorem 3.19, is a continuum. ■

The following lemma is used in the proof of Lemma 3.25, but will not find any other further usage in this paper.

Lemma 3.24. (Ingram [21, Lemma 1.15]) *Let* $X = \varprojlim\{X_i, f_i\}$ *where* X_i *is a compact metric space for each positive integer* i*. Given* $\varepsilon > 0$*, there exists an arbitrarily large positive integer* N *and a positive real number* ε_N *such that if* C *is a subset of* X_N *satisfying* $\operatorname{diam} C < \varepsilon_N$*, then* $\operatorname{diam} \pi_N^{-1}(C) < \varepsilon$.

Proof. The metric d on X is given by $d(x,y) = \sum_{i=1}^{\infty} \frac{d_i(\pi_i(x),\pi_i(y))}{2^i}$. Let $\varepsilon > 0$. There exists a positive integer M such that $\sum_{i=n}^{\infty} 2^{-i} < \frac{\varepsilon}{2}$ for all $n \geq M$. Let N be any integer such that $N \geq M$. For each $i < N$, the mapping $f_{i,N}$ is uniformly continuous, so we may find a positive real number $\varepsilon_N < \frac{\varepsilon}{2}$ such that if $x_N, y_N \in X_N$, then $d_i(f_{i,N}(x_N), f_{i,N}(y_N)) < \frac{\varepsilon}{2}$. Suppose C is a subset of X_N satisfying $\operatorname{diam} C < \varepsilon_N$, and that $x, y \in X$ such that $\pi_N(x), \pi_N(y) \in C$. Then $d_N(\pi_N(x), \pi_N(y)) < \varepsilon_N$, and so $d_i(\pi_i(x), \pi_i(y)) < \frac{\varepsilon}{2}$ for all $i = 1, 2, ..., N$. If follows that $d(x,y) < \varepsilon$. Since x and y were arbitrary points in $\operatorname{diam} \pi_N^{-1}(C)$, we may conclude that $\operatorname{diam} \pi_N^{-1}(C) < \varepsilon$. ■

The following lemma is a special case of a result by Ingram and can be found in [23, Lemma 1.2] in its more general form. We will use Lemma 3.25 in the proofs of theorems 3.45 and 4.20.

Lemma 3.25. (Ingram [23, Lemma 1.2]) *Let* $X = \varprojlim\{X_i, f_i\}$ *where* X_i *is a continuum for each positive integer* i*. Given* $\varepsilon > 0$*, there exists an arbitrarily large positive integer* N *and a positive real number* ε_N *such that if* A *and* B *are subcontinua of* X *satisfying* $\mathcal{H}_N(\pi_N[A], \pi_N[B]) < \varepsilon_N$*, then* $\mathcal{H}(A, B) < \varepsilon$.

Proof. Let $\varepsilon > 0$. By Lemma 3.24, there exists an arbitrarily large positive integer N and a positive real number ε_N such that if C is a subset of X_n satisfying $\operatorname{diam} C < \varepsilon_N$, then $\operatorname{diam} \pi_N^{-1}(C) < \frac{\varepsilon}{2}$. Let A and B be subcontinua of X satisfying $\mathcal{H}_N(\pi_N[A], \pi_N[B]) < \varepsilon_N$ and let $p \in A$. For each positive integer i, let $p_i = \pi_i(p)$,

$A_i = \pi_i[A]$, and $B_i = \pi_i[B]$. Since $p_N \in A_N$ and $\mathcal{H}_N(A_N, B_N) < \varepsilon_N$, there exists a point $q_N \in B_N$ such that $d_N(p_N, q_N) < \varepsilon_N$. Let q be any point in B_N such that $\pi_N(q) = q_N$. Notice that $\operatorname{diam}(\{p_N, q_N\}) = d_N(p_N, q_N) < \varepsilon_N$, so we have that $\operatorname{diam} \pi_N^{-1}(\{p_N, q_N\}) < \frac{\varepsilon}{2}$. Clearly, $p, q \in \operatorname{diam} \pi_N^{-1}(\{p_N, q_N\})$, so $d(p,q) < \frac{\varepsilon}{2}$. Hence, we have that for every $p \in A$, there exists $q \in B$ such that $d(p,q) < \frac{\varepsilon}{2}$, and so $A \subseteq N_d(\frac{\varepsilon}{2}, B)$. A similar argument shows that $B \subseteq N_d(\frac{\varepsilon}{2}, A)$. Therefore we may conclude that $\mathcal{H}(A,B) \leq \frac{\varepsilon}{2} < \varepsilon$. ■

Theorem 3.27, which is a corollary of [32, Theorem 2.7], provides a well known condition which is sufficient for a particular interval mapping to generate an indecomposable inverse limit. Before stating this theorem, we need to define a two pass map.

Definition 3.26. A map $f : [a,b] \to [a,b]$ is referred to as a *two-pass map* if there exists a point $c \in [a,b]$ such that $f[a,c] = f[c,b] = [a,b]$.

Theorem 3.27. *If $X = \varprojlim\{[a,b], f\}$ where f^n is a two-pass map for some n, then X is indecomposable.*

The next theorem, which was proven by C. E. Capel, tells us any inverse limit on intervals with monotone bonding maps produces an arc.

Theorem 3.28. (Capel [7]) *If $X = \varprojlim\{X_i, f_i\}$ where each X_i is a closed interval, and each f_i is monotone and surjective, then X is an arc.*

Next, we state Theorem 3.29, which provides an extremely useful tool for studying inverse limits on intervals. This theorem is originally due to Bennett and appears in its original form in [5]. A proof of the theorem in the form in which it appears here can be found in [20].

Theorem 3.29. (Bennett) *Suppose $f : [a,b] \to [a,b]$ is a surjective mapping and d is a point between a and b such that:*

1. $f[d,b] \subseteq [d,b]$,

2. *f is monotone on $[a, d]$, and*

3. *there is a positive integer n such that $f^n[a, d] = [a, b]$.*

Then $\varprojlim\{[a, b], f\}$ is the union of a topological ray R and a continuum $K = \varprojlim\{[d, b], f|_{[d,b]}\}$ such that $\mathrm{cl}(R) - R = K$.

We close this section by noting that theorems 3.28 and 3.29 can be used to show that the inverse limit of either of the two mappings whose graphs are pictured later in 3.3 is a ray limiting to an arc, similar to the $\sin(\frac{1}{x})$-curve. It is in fact known that either inverse limit is precisely the $\sin(\frac{1}{x})$-curve, though we do not prove that here.

3.4. MARKOV MAPS

In this subsection, we will provide definitions and results relating to an important class of functions referred to as Markov maps.

Definition 3.30. A mapping $f : [a, b] \to [a, b]$ is said to be *Markov with respect to $\mathcal{A}$* for a given partition $\mathcal{A} = \{a = a_1 < a_2 < ... < a_m = b\}$ of $[a, b]$ if $f[\mathcal{A}] \subseteq \mathcal{A}$ and f restricted to $[a_i, a_{i+1}]$ is monotone for each $i = 1, ..., m - 1$. The set $\mathcal{A}$ is called the *Markov partition associated with f*. We say that a mapping $f : [a, b] \to [a, b]$ is *Markov* if there exists a Markov partition $\mathcal{A}$ such that f is Markov with respect to $\mathcal{A}$.

The map whose graph is pictured in Figure 3.1 is an example of a Markov map. The inverse limit of this map is the well-known Brouwer-Janiszewski-Knaster Continuum, also known as the BJK Continuum, or the Buckethandle Continuum. This continuum is an indecomposable continuum, and each of its proper subcontinua is an arc.

The following observation about Markov maps will see much use in Section 4.

Observation 3.31. *If $f : [a, b] \to [a, b]$ is Markov with respect to $\mathcal{A}$, then $\omega(\mathcal{A}, f) \subseteq \mathcal{A}$, and f maps $\mathcal{A}$ bijectively onto itself.*

Proof. It can be seen that $\omega(\mathcal{A}, f) = \bigcap_{i=1}^{\infty} f^i[\mathcal{A}]$, and so $\omega(\mathcal{A}, f) \subseteq \mathcal{A}$. The set $\mathcal{A}$ is finite and invariant under f, and so if follows that f is one-to-one on $\mathcal{A}$. ■

Before further discussing the properties of Markov maps, we introduce the notion of a turning point of an interval map. It can be seen from the following definition that a turning point is a specific type of local extremum.

Definition 3.32. Given a mapping $f : [a, b] \to [c, d]$, we say that $p \in [a, b]$ is a turning point of f if there is a subinterval $J \subseteq [a, b]$ containing p in its interior such that the following conditions are satisfied:

1. $f[J]$ is a nondegenerate interval having $f(p)$ as an end point, and
2. if C is any component of $J - \{p\}$, then $f[C]$ is nondegenerate.

For an example illustrating the turning points of a mapping, refer to Figure 3.2. In Figure 3.2, the turning points of f are p_1, p_4, p_5, and the points in the interval $[p_2, p_3]$.

In this paper, we will primarily be concerned with mappings which have a finite number of turning points. We make the following observations concerning such mappings.

Observation 3.33. Let $f : [a, b] \to [c, d]$ be a non-constant mapping with a finite number of turning points. Note the following:

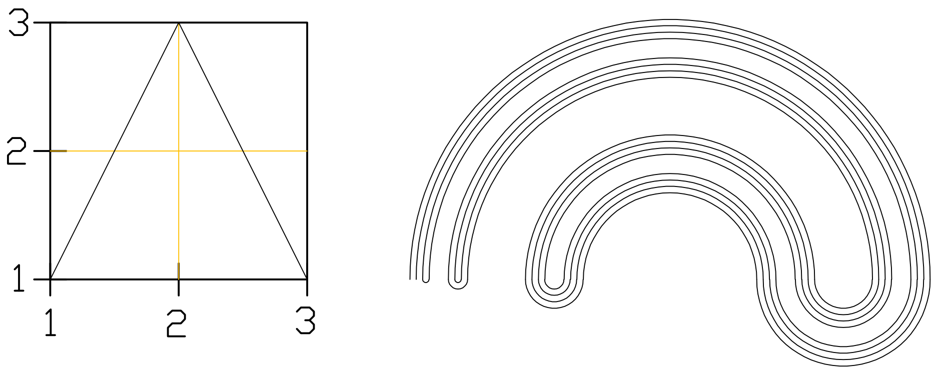

Figure 3.1. The BJK Continuum.

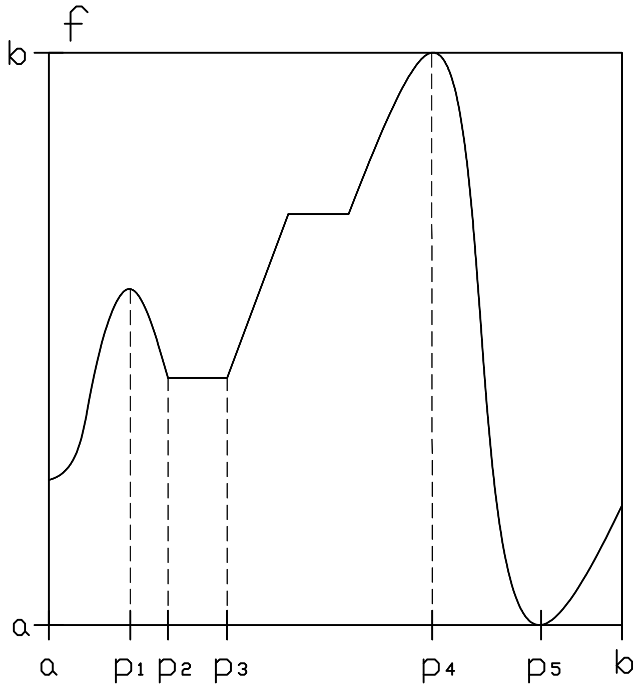

Figure 3.2. Turning points

1. The points a and b are both turning points of f.
2. The mapping f fails to be monotone on some interval $I \subseteq [a,b]$, if and only if I contains a turning point in its interior.
3. If p is a turning point of f, then J may be chosen in such a way that if C_1 and C_2 are components of $J - \{p\}$, then $f[C_1] = f[C_2]$ and f is monotone on C_1 and C_2.
4. If $f : [a,b] \to [a,b]$ is Markov with respect to $\mathcal{A} \subseteq [a,b]$ and p is a turning point of f, then $f(p) \in \mathcal{A}$.

Item 4 of Observation 3.33 addresses what is perhaps the most important characteristic of Markov maps. If f is Markov with respect to $\mathcal{A}$, then every turning point of f maps into $\mathcal{A}$ under f. Since $\mathcal{A}$ is finite and invariant, it follows that each turning point eventually maps onto a periodic point.

Notice that if f is Markov with respect to $\mathcal{A}$, and p is an element of $\mathcal{A}$ whose orbit does not contain any critical points, then the entire orbit of p may be removed from $\mathcal{A}$, and the remaining points will still provide a Markov partition for f. This observation motivates the following definition.

Definition 3.34. Let $f : [a, b] \to [a, b]$ be Markov with respect to $\mathcal{A}$. A point $p \in \mathcal{A}$ is *essential with respect to f* if $\mathrm{Orbit}(p, f)$ contains a turning point of f. Otherwise, we say that p is *inessential.* We say that $\mathcal{A}$ is an *essential Markov partition for f* if each point in $\mathcal{A}$ is essential.

Observation 3.35. *Let $f : [a, b] \to [a, b]$ is a Markov map, and denote by $\mathcal{T}_f$ the collection of all turning points of f. Then $\mathcal{A} = \mathcal{T}_f \cup \left(\bigcup_{n=1}^{\infty} f^n[T_f]\right)$ is the only essential Markov partition for f.*

To simplify the statement of the statement of Theorem 3.37, and the language used in situations where this theorem applies, we introduce the following definition.

Definition 3.36. Let $f : [a, b] \to [a, b]$ and $g : [c, d] \to [c, d]$ be Markov maps. We say that f and g *follow the same pattern* if there exist Markov partitions $\mathcal{A} = \{a = a_1 < a_2 < ... < a_n = b\}$ and $\mathcal{B} = \{c = b_1 < b_2 < ... < b_n = d\}$ for f and g respectively such that $f(a_i) = a_j$ if and only if $g(b_i) = b_j$.

The following theorem, due to Raines, states that any two Markov maps which follow the same pattern will generate homeomorphic inverse limits. In particular, given a Markov map f, we may change the spacing of the partition points and change the slope of f on the components of the complement of the Markov partition in any way that preserves monotonicity, without changing the inverse limit.

Theorem 3.37. (Raines [34, Corollary 3.2.1]) *Let each of $f : [a, b] \to [a, b]$ and $g : [c, d] \to [c, d]$ be Markov maps which follow the same pattern. Then $\varprojlim\{[a, b], f\}$ is homeomorphic to $\varprojlim\{[c, d], g\}$.*

Raines' theorem provides an an extremely useful tool for simplifying the study of inverse limits with Markov bonding maps as it allows us to represent any such inverse

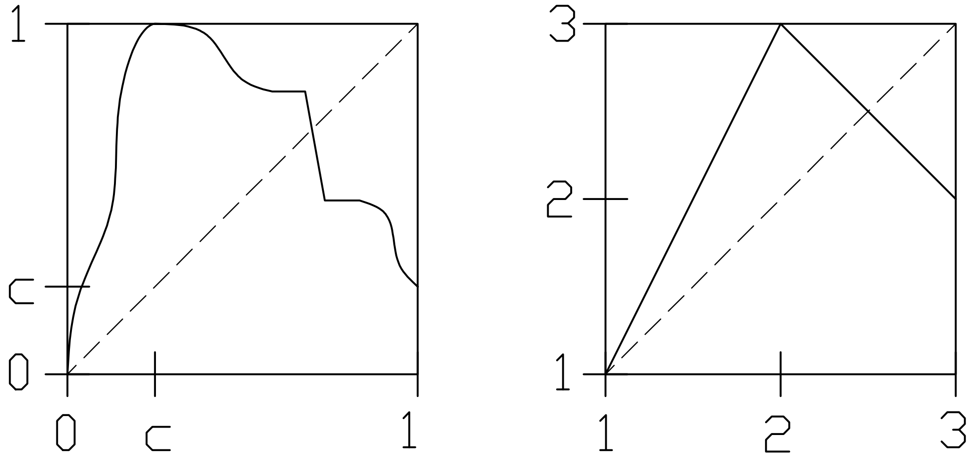

Figure 3.3. Raines's Theorem

limit as an inverse limit with a piecewise linear Markov bonding map (See Figure 3.3). We will use this theorem extensively throughout this paper.

The following technical lemma is used to prove Theorem 3.39, which is a modification of Raines's theorem. Theorem 3.39 will be used later in the proof of Corollary 4.15.

Lemma 3.38. *Let* $f_1, f_2 : [a,b] \to [a,b]$ *and* $g_1, g_2 : [c,d] \to [c,d]$ *be mappings, and let* $\mathcal{A} = \{a = a_1 < a_2 < ... < a_n = b\}$, $\mathcal{B} = \{c = b_1 < b_2 < ... < b_n = d\}$, $\mathcal{C} = \{a = c_1 < c_2 < ... < c_m = b\}$, *and* $\mathcal{D} = \{c = d_1 < d_2 < ... < d_m = d\}$ *be partitions of* $[a,b]$ *and* $[c,d]$. *Assume further that the following conditions are satisfied:*

1. f_1 *and* g_1 *are Markov with respect to* $\mathcal{A}$ *and* $\mathcal{B}$ *respectively,*
2. $f_1(a_i) = a_j$ *if and only if* $g_1(b_i) = b_j$,
3. $f_2[\mathcal{C}] \subseteq \mathcal{A}$ *and* $g_2[\mathcal{D}] \subseteq \mathcal{B}$ *(and thus* $\mathcal{A} \subseteq \mathcal{C}$ *and* $\mathcal{B} \subseteq \mathcal{D}$*),*
4. f_2 *and* g_2 *are Markov with respect to* $\mathcal{C}$ *and* $\mathcal{D}$ *respectively,*
5. *for any* $i = 1, ..., n-1$, *the number of elements of* $\mathcal{C}$ *in* $[a_i, a_{i+1}]$ *is equal to the number of elements of* $\mathcal{D}$ *in* $[b_i, b_{i+1}]$, *and*

6. $f_2(c_i) = a_j$ *if and only if* $g_2(d_i) = b_j$.

Then there are partitions $\mathcal{P} = \{a = p_1 < p_2 < ... < p_k = b\}$ *and* $\mathcal{Q} = \{c = q_1 < q_2 < ... < q_k = d\}$ *such that if* $f = f_2 \circ f_1$ *and* $g = g_2 \circ g_1$*, then:*

i. $f[\mathcal{P}] \subseteq \mathcal{A}$ *and* $g[\mathcal{Q}] \subseteq \mathcal{B}$ *(and thus* $\mathcal{A} \subseteq \mathcal{P}$ *and* $\mathcal{B} \subseteq \mathcal{Q}$*),*

ii. f *and* g *are Markov with respect to* $\mathcal{P}$ *and* $\mathcal{Q}$ *respectively,*

iii. for any $i = 1, ..., n-1$*, the number of elements of* $\mathcal{P}$ *in* $[a_i, a_{i+1}]$ *is equal to the number of elements of* $\mathcal{Q}$ *in* $[b_i, b_{i+1}]$*, and*

iv. $f(p_i) = a_j$ *if and only if* $g(q_i) = b_j$.

Proof. Define $\mathcal{P}$ to be a partition of $[a, b]$ containing every point in $\mathcal{A}$ and exactly one point from each component of $g_1^{-1}(\mathcal{C})$ which does not intersect $\mathcal{A}$. In a similar fashion, define $\mathcal{Q}$ to be a partition of $[c, d]$ containing every point in $\mathcal{B}$ and exactly one point from each component of $f_1^{-1}(\mathcal{D})$ which does not intersect $\mathcal{B}$. Denote the elements of $\mathcal{P}$ and $\mathcal{Q}$ by $\mathcal{P} = \{a = p_1 < p_2 < ... < p_k = b\}$ and $\mathcal{Q} = \{c = q_1 < q_2 < ... < q_k = d\}$. We shall establish claims *i*-*iv* one at a time.

(i) It is clear from the definition of $\mathcal{P}$ and $\mathcal{Q}$ that $f_1[\mathcal{P}] = \mathcal{C}$ and $g_1[\mathcal{Q}] = \mathcal{D}$, and so $f[\mathcal{P}] = f_2[f_1[\mathcal{P}] = f_2[\mathcal{C}] \subseteq \mathcal{A}$ and $g[\mathcal{Q}] = g_2[g_1[\mathcal{Q}] = g_2[\mathcal{D}] \subseteq \mathcal{B}$.

(ii) Notice that $\mathcal{A} \subseteq \mathcal{P}$, and so invariance of $\mathcal{P}$ under f follows from (i). Let $J_1 = [p_i, p_{i+1}]$ for some $i = 1, ..., k-1$, and let $J_2 = f_1[J_1]$.

The only points in the interval J_1 which could be in $\mathcal{A}$ are the end points of J_1, from which it follows that f is monotone on J_1. Also, it is clear from the definition of $\mathcal{P}$ that p_i and p_{j+1} map onto consecutive members of $\mathcal{C}$, so we see that $J_2 = [c_j, c_{j+1}]$ for some $j = 1, 2, ..., m-1$, and hence that f_2 is monotone on J_2. The composition of monotone maps is monotone and $f|_{J_1} = f_2|_{J_2} \circ f_1|_{J_1}$, so we may conclude that f is monotone on J_1. Since our choice of i was arbitrary, we see that f is Markov with respect to $\mathcal{P}$. A similar argument shows that g is Markov with respect to $\mathcal{Q}$.

(iii) Let $I = [a_l, a_{l+1}]$ and $J = [b_l, b_{l+1}]$ for some $l = 1, .., n-1$. Condition (1) of our hypothesis tells us that the end points of $f_1[I]$ and $g_1[J]$ are points in $\mathcal{A}$ and $\mathcal{B}$, respectively. Further, Condition (2) allows us to conclude that if $f_1[I] = [a_N, a_M]$

for some $1 \leq N < M \leq n$, then $g_1[J] = [b_N, b_M]$. Since f_1 is monotone on I, the pre-image of any point in $f_1[I]$ has exactly one component intersecting I, and so it follows from the definition of $\mathcal{P}$ that $|I \cap \mathcal{P}| = |f_1[I] \cap \mathcal{C}| = |[a_N, a_M] \cap \mathcal{C}|$. Similarly, we see that $|J \cap \mathcal{Q}| = |g_1[J] \cap \mathcal{D}| = |[b_N, b_M] \cap \mathcal{D}|$. We may then use Condition (5) to conclude that $|I \cap \mathcal{P}| = |J \cap \mathcal{Q}|$.

(*iv*) Let $I = [a_l, a_{l+1}]$ and $J = [b_l, b_{l+1}]$ for some $l = 1, .., n-1$. Since $\mathcal{A} \subseteq \mathcal{P}$ and $\mathcal{B} \subseteq \mathcal{Q}$, we see that the end points of I and J are members of $\mathcal{P}$ and $\mathcal{Q}$ respectively. Applying (*iii*) we can see that if N and M are integers such that $I = [p_N, p_M]$, then $J = [q_N, q_M]$. Let K and L be positive integers such that $f_1(p_N) = c_K$ and $f_1(p_M) = c_L$. It follows from conditions (2) and (5) that $g_1(q_N) = d_K$ and $g_1(q_M) = d_L$. The mapping f_1 is monotone on I and g_1 is monotone on J, so we may conclude that for $N \leq i \leq M$, we have $f_1(p_i) = c_j$ if and only if $g_1(q_i) = d_j$. In fact, since our initial choice of l was arbitrary, we in fact have that $f_1(p_i) = c_j$ if and only if $g_1(q_i) = d_j$ for any i and j. Applying Condition (6), we see that $f(p_i) = a_j$ if and only if $g(q_i) = b_j$. ∎

Theorem 3.39. *Let $f = f_m \circ f_{m-1} \circ ... \circ f_1$ and $g = g_m \circ g_{m-1} \circ ... \circ g_1$ where for each $i = 1, 2, ..., m$, f_i is a mapping from $[a, b]$ to itself and g_i is a mapping from $[c, d]$ to itself. Let $\mathcal{A} = \{a = a_1 < a_2 < ... < a_n = b\}$ and $\mathcal{B} = \{c = b_1 < b_2 < ... < b_n = d\}$ be partitions of $[a, b]$ and $[c, d]$. Assume that for each i, f_i is Markov with respect to $\mathcal{A}$ and g_i is Markov with respect to $\mathcal{B}$, and for any $j = 1, 2, ..., n$, we have $f_i(a_j) = a_k$ if and only if $g_i(b_j) = b_k$. Then $\varprojlim\{[a, b], f\}$ is homeomorphic to $\varprojlim\{[c, d], g\}$.*

Proof. By repeatedly applying Lemma 3.38, we obtain partitions $\mathcal{P} = \{a = p_1 < p_2 < ... < p_k = b\}$ and $\mathcal{Q} = \{c = q_1 < q_2 < ... < q_k = d\}$ such that f is Markov with respect to $\mathcal{P}$ and g is Markov with respect to $\mathcal{Q}$, and for each i, $f(p_i) = p_j$ if and only if $g(q_i) = q_j$. Applying Theorem 3.37 produces the desired result. ∎

The following theorem provides a necessary and sufficient condition for a point to be a turning point of a mapping which is defined as the composition of mappings, and will be applied later in the proof of Theorem 4.5.

Lemma 3.40. *Let $f = f_k \circ ... \circ f_1$, where $f_1, ..., f_k$ are mappings from $[a, b]$ onto itself such that for each $i = 1, ..., k$, f_i has only finitely many turning points and is nowhere locally constant. Let $p_1 \in [a, b]$, and for each $i = 1, ..., k$, set $p_{i+1} = f_i(p_i)$. Note that $p_{k+1} = f(p_1)$. The point p_1 is a turning point for f if and only if there is an $n = 1, ..., k$ such that p_n is a turning point for f_n.*

Proof. We begin by assuming that there does not exist an integer $n \in \{1, ..., k\}$ such that p_n is a turning point for f_n. We will show that this assumption leads to the conclusion that p_1 is not a turning point of f.

For each $i = 1, 2, ..., k$, f_i has only finitely many turning points, so we may find intervals $J_1, J_2, ..., J_k$ such that for each i, J_i contains p_i in its interior, and contains no turning points. Observation 3.33.2 tells us that f_i is monotone of J_i for each i. Since each f_i is nowhere locally constant, we see that $f_i[J_i]$ is a nondegenerate interval containing $f(p_i) = p_{i+1}$ for each $i = 1, 2, ..., k - 1$. For each such i, the fact that p_i is not a turning point of f_i leads us to deduce that p_{i+1} is not an end point of $f_i[J_i]$, and hence lies in the interior of $f_i[J_i]$. Continuity of the mappings f_i, and the fact that each f_i is monotone on J_i, allows us to define our intervals J_i in such a way that $f_i[J_i] = J_{i+1}$ for each $i = 1, 2, ..., k - 1$. It then follows that $f|_{J_1} = f_k|_{J_k} \circ f_{k-1}|_{J_k - 1} \circ ... f_1|_{J_1}$. Since $f|_{J_1}$ is the composition of monotone mappings, f itself is monotone on J_1. Since p_1 is in the interior of J_1, if follows from Observation 3.33.2 that p_1 is not a turning point for f. Thus, we have proven that if p_1 is a turning point for f, then p_n must be a turning point for f_n for some $n = 1, ..., k$.

Now we assume that there is some $n = 1, ..., k$ such that p_n is a turning point for f_n. Since p_n is a turning point of f_n, we may find an interval L_n containing p_n in its interior such that $f_n[L_n]$ is a nondegenerate interval having $f(p_n)$ as an end point, and if C is any component of $J - \{p\}$, then $f_n[C]$ is nondegenerate and f_n is monotone on C. For each $i \in \{n, n + 1, ..., k\}$, let $L_{i+1} = f_i[L_i]$. For each such i, the interval L_i contains the point p_i, which may be a turning point for f_i. But since each f_i has only finitely many turning points, we may, without loss of generality, choose L_n to be small enough to ensure that for each $i = n, ..., k$, L_i contains at most one turning point for the map f_i, which would have to be p_i, assuming p_i is a turning point of f_i. By the continuity of the functions f_i, we may find an interval J_1 containing p_1 in its interior such that $(f_{n-1} \circ ... \circ f_1)[J_1] \subseteq L_n$. For each $i = 1, ..., k$, let $J_{i+1} = f_i[J_i]$. Again, we may assume that J_1 is small enough that each J_i contains at most one turning point of f_i, which, if it exists, would have to be p_i. One can see that for each $i = n, ..., k$, $J_i \subseteq L_i$, and that p_{n+1} is an end point of the interval J_{n+1}. Since each f_i is nowhere constant, $f[J_1] = J_{k+1}$ is a nondegenerate interval which contains $f(p_1) = p_{k+1}$. It follows from the facts that p_{n+1} is an end point of J_{n+1} and that for any $i = 1, ..., k$, the only possible turning point for f_i in J_i is p_i, that $f(p_1)$ is an end point of $f[J_1]$, and that f is monotone on any component of $J_1 - \{p_1\}$. Hence, we conclude that p_1 is a turning point for f. ■

Theorem 3.41, which is due to Ryden, will provide two separate conditions, each of whose satisfaction by a particular Markov map is both necessary and sufficient for that map to produce an indecomposable inverse limit.

Theorem 3.41. (Ryden [35, Theorem 3.4]) *Let* $X = \varprojlim\{[a, b], f\}$ *where* $f : [a, b] \to [a, b]$ *is a Markov map. The following are equivalent:*

1. X *is indecomposable.*
2. f^n *is a two pass map for some* n.
3. f *has at least three maximal periodic subcontinua.*

The following theorem of Holte does not explicitly relate to inverse limits of Markov maps, or even inverse limits on intervals, though it is a useful tool for proving that certain maps generate inverse limit spaces which are homeomorphic to inverse limits with Markov bonding maps.

Theorem 3.42. (Holte [17, Lemma 1.1]) *Suppose that* f *and* g *are mappings from a metric space* X *into itself, and* $A_1, ..., A_m$ *are closed disjoint subsets of* X *such that:*

1. $f(x) = g(x)$ *for all* $x \in X - \bigcup_{i=1}^m A_i$,
2. $\mathrm{diam}(f^k[A_i]) \to 0$ *and* $\mathrm{diam}(g^k[A_i]) \to 0$ *as* $k \to \infty$ *for* $i = 1, ..., m$,

3. for each $i = 1, ..., m$, *there exists* j *such that* $f(A_i) \cup g(A_i) \subseteq A_j$.

Then the shift homeomorphisms on $\varprojlim\{X, f\}$ and $\varprojlim\{X, g\}$ are topologically conjugate, and hence the inverse limits are homeomorphic.

3.5. KELLEY CONTINUA

Definition 3.43. Given a continuum X and a point $p \in X$, X is said to be *Kelley at* p (or alternately, to have the *property of Kelley at* p), provided that for each subcontinuum K of X containing p and for each sequence $\{p_n\}_{n=1}^{\infty}$ converging to p, there is a sequence of subcontinua $\{K_n\}_{n=1}^{\infty}$ converging to K such that $p_n \in K_n$ for each n. A continuum is said to be a *Kelley continuum* (or alternately, to have the *property of Kelley*), if it is Kelley at each of its points.

Since we assume a continuum to be metric, the following observation follows directly from the previous definition.

Observation 3.44. *A continuum* X *is Kelley at* $p \in X$ *if and only if for each subcontinuum* K *of* X *containing* p *and for each* $\varepsilon > 0$, *there exists* $\delta > 0$ *such that if* $q \in X$ *satisfying* $d(p, q) < \delta$, *then there exists a subcontinuum* L *of* X *containing* q *such that* $\mathcal{H}(K, L) < \varepsilon$, *where* $\mathcal{H}$ *represents the Hausdorff metric on* $C(X)$.

The following theorem states that being a Kelley continuum is preserved under the inverse limit operation when the bonding mappings are confluent. It is due to W. J. Charatonik and originally appeared in the paper *Inverse limits of smooth continua* [12], although the proof given here is different from that given by Charatonik. The theorem also appears in [24], with the proof provided here.

Theorem 3.45. *Let* $\{X_i, f_i\}$ *be an inverse sequence where for each* $i = 1, 2, ...,$ *the factor space* X_i *is a Kelley continuum and the bonding map* f_i *is confluent. Then the inverse limit* $X = \varprojlim\{X_i, f_i\}$ *is a Kelley continuum.*

Proof. Let $p \in X$, let K be a subcontinuum of X containing p, and let ε be a positive real number. For each $i = 1, 2, ...$, we set $p_i = \pi_i(p)$ and $K_i = \pi_i[K]$. Let N be a positive integer and ε_N be a positive real number such that if A and B are subcontinua of X satisfying $\mathcal{H}_N(\pi_N[A], \pi_N[B]) < \varepsilon_N$, then $\mathcal{H}(A, B) < \varepsilon$, as guaranteed by Lemma 3.25. Since X_N is a Kelley continuum we may find $\eta > 0$ such that if $y \in X_N$ satisfying $d_N(p_N, y) < \eta$, then there exists a subcontinuum Y of X_N such that $\mathcal{H}_N(K_N, Y) < \varepsilon_N$. Let $\delta > 0$ such that if $a, b \in X$ satisfying $d(a, b) < \delta$, then $d_N(\pi_N(a), \pi_N(b)) < \eta$. Choose $q \in X$ such that $d(q, p) < \delta$, and for each $i = 1, 2, ...$ set $q_i = \pi_i(q)$. Since $d(q, p) < \delta$, we have that $d_N(p_N, q_N) < \eta$, and hence that there exists a subcontinuum L_N of X_N containing q_N such that $\mathcal{H}_N(K_N, L_N) < \varepsilon_N$. For each $i = 1, 2, ..., N-1$, let $L_i = f_{i,N}[L_N]$ and for each $i = N+1, N+2, ...$, let L_i be the component of $f_{i,N}^{-1}(L_N)$ which contains q_i. Now, define a subcontinuum L of X by $L = \varprojlim\{L_i, f_i|_{L_{i+1}}\}$. Notice that $q \in L$. From the definition of the sets L_i, and the fact that each bonding map f_i is confluent, it follows that $f_i|_{L_{i+1}}$ is surjective for each i, and hence that $\pi_i[L] = L_i$ for each positive integer i. Since $\mathcal{H}_N(K_N, L_N) < \varepsilon_N$, we have that $\mathcal{H}(K, L) < \varepsilon$, and thus may conclude that X is a Kelley continuum. ■

3.6. DECOMPOSITIONS

In this subsection, we will discuss upper semi-continuous decompositions, which provide a useful technique for constructing continua. We begin by defining the notion of a decomposition space.

Definition 3.46. Let X be a topological space. A *partition* of X is a collection $\mathcal{D}$ of nonempty, mutually disjoint subsets of X whose union is X. The collection $T(\mathcal{D}) = \{\mathcal{U} \subseteq \mathcal{D} : \bigcup \mathcal{U} \text{ is open in } X\}$ provides a topology for $\mathcal{D}$. Such a topology is called the *decomposition topology*, and when $\mathcal{D}$ is equipped with this topology, we refer to it as a *decomposition space*, or simply a *decomposition* of X. We define the *natural map* $P : X \to \mathcal{D}$ by setting $P(x)$ equal to the unique element of $\mathcal{D}$ which contains

x. We say that $\mathcal{D}$ is an *upper semi-continuous decomposition* if the natural map P is closed.

The following theorem explains why upper semi-continuous decompositions are of particular importance in the theory of continua. A proof of the theorem can be found in [32].

Theorem 3.47. *Any upper semi-continuous decomposition of a continuum is itself a continuum.*

Let X be a continuum, K be any subcontinuum of X, and $\mathcal{D}_K = \{K\} \cup \{\{p\} : p \in X - K\}$. It is well known that this decomposition space is an upper semi-continuous decomposition. This decomposition space is typically denoted by X/K. Intuitively, one may imagine X/K being obtained from X by shrinking K to a point.

In this paper, we will use a slight generalization of this notion. Let $\mathcal{K}$ be a finite collection of pairwise disjoint subcontinua of a continuum X, and let K denote the union of the members of $\mathcal{K}$. Let $\mathcal{D}_{\mathcal{K}}$ be the partition of X given by $\mathcal{D}_{\mathcal{K}} = \mathcal{K} \cup \{\{x\} : x \in X - K\}$. We denote the decomposition space $(\mathcal{D}_{\mathcal{K}}, T(\mathcal{D}_{\mathcal{K}}))$ by $X/\mathcal{K}$. It follows from the previous paragraph that $X/\mathcal{K}$ is an upper semi-continuous decomposition, and can be obtained from X by shrinking each subcontinuum in $\mathcal{K}$ to a separate point.

Now, assume that X is a continuum, and f is a mapping from K onto itself. Assume further that K is periodic with respect to f and let $\mathcal{K} = \text{Orbit}(K)$. In this case, we define a mapping $f/\mathcal{K}$ from $X/\mathcal{K}$ onto itself in a natural way: If $A, B \in X/\mathcal{K}$, then $(f/\mathcal{K})(A) = B$ if and only if $f[A] \subseteq B$.

4. PERMUTATION MAPS

In [22] Ingram introduced a family of Markov maps whose members are based on permutations. Our goal in this section, and the primary goal if this paper, is to conduct a study of the topological properties of the inverse limits generated by such Markov maps. In particular, we study indecomposability, end points, subcontinua, and the property of Kelley for such inverse limits. We begin by providing the definition of a permutation map. As is standard, for a given positive integer n, we denote the set of all permutations of degree n by S_n.

Definition 4.1. Let $n \geq 2$ be an integer. For $1 \leq i \leq n$, let $a_i = \frac{i-1}{n-1}$ and let $\mathcal{A}_n = \{a_1, ..., a_n\}$. Given $\sigma \in S_n$, we define the *permutation map associated with* σ, $f_\sigma : [0,1] \to [0,1]$, by first setting $f_\sigma(a_i) = a_{\sigma(i)}$ for each $a_i \in \mathcal{A}_n$, and then extending f_σ linearly to the rest of the interval. Let $\mathcal{S}_n$ denote the family of all permutation maps generated by permutations in S_n, and let $\mathcal{S}_n^c$ denote the family of all finite compositions of members of $\mathcal{S}_n$.

Is is clear from the definition that any permutation map $f \in \mathcal{S}_n$ is Markov with respect to $\mathcal{A}_n$, and that $\mathcal{A}_n$ is not only invariant under f, but strongly invariant (that is, $f[\mathcal{A}_n] = \mathcal{A}_n$). It is not the case that every Markov map which maps its Markov partition onto itself is a permutation map. The partition points may not be evenly spaced, or the the function might not be piecewise linear. However, in light of Theorem 3.37, if we are studying an inverse limit using such a Markov map, then we may instead consider the inverse limit using the permutation map which “follows the same pattern” on its partition.

In [22], Ingram began a study of inverse limits on $[0,1]$ with a single bonding map chosen from the permutation family. Though he gave some general results concerning the topology of such continua, Ingram focused his study on inverse limits with maps chosen from $\mathcal{S}_3$, $\mathcal{S}_4$, and $\mathcal{S}_5$. For each of these maps, he gave detailed information about the continuum it generated, including whether or not it is indecomposable, the number of its end points, and a description of its subcontinua. A table containing brief descriptions of each of the continua in $\mathcal{S}_n$ for $2 \leq n \leq 5$ appears in the appendix of this dissertation. In a later paper [24, Section 5] Ingram established that each of these continua are Kelley.

In [33], Ingram posed the problem of classifying the continua which can be represented as an inverse limit on $[0,1]$ using a single permutation map f from $\mathcal{S}_n$ for arbitrary n, and in [23],[24],[25],and [33, page 296], has asked if all permutation maps produce inverse limits which are Kelley continua (though he credits the latter question to W. J. Charatonik). In this section, we address each of these questions. Though we do not provide a complete classification of such inverse limit spaces, we shall provide methods for determining when such an inverse limit space is indecomposable, how many end points it has (in terms of he standard definition, and in the classical sense), and what sort of subcontinua it contains. We also show that each such inverse limit is a Kelley continuum, providing an affirmative answer to Charatonik's question.

Most of the results we develop will apply not only to inverse limits generated by single permutation maps, but also to inverse limits generated by the composition of such maps. We will show that this widening of scope is in fact necessary to be able to fully understand the topological properties of the subcontinua of inverse limits generated by only a single permutation map.

To facilitate our discussion, we introduce the following notation.

Notation. For each $n \geq 2$, we define families of continua $\mathcal{M}_n$ and M_n^c as follows:

$\mathcal{M}_n = \{X : X \text{ is homeomorphic to } \varprojlim\{[0,1], f_\sigma\} \text{ for some } f_\sigma \in \mathcal{S}_n\}$

$\mathcal{M}_n^c = \{X : X \text{ is homeomorphic to } \varprojlim\{[0,1], f\} \text{ for some } f \in \mathcal{S}_n^c\}$

Further, let $\mathcal{M} = \bigcup_{n=1}^{\infty} \mathcal{M}_n$ and $\mathcal{M}^c = \bigcup_{n=1}^{\infty} \mathcal{M}_n^c$.

We begin our investigation of the families $\mathcal{M}$ and $\mathcal{M}^c$ with the following observation.

Observation 4.2. *Assume that* $X = \varprojlim\{[0,1], f\}$ *where* $f \in \mathcal{S}_n^c$ *for some* $n \geq 2$*. If* $p \in \mathcal{A}_n$ *has period* k *and is inessential with respect to* f*, then* $X \in \mathcal{M}_{n-k}^c$*. Additionally, if* $f \in \mathcal{S}_n$*, then* $X \in \mathcal{M}_{n-k}$*.*

Proof. Since p is inessential, we may remove it and each point in its orbit from $\mathcal{A}_n$ and have a Markov partition for f containing $n-k$ points. On the new partition, the mapping f will follow the same pattern as a mapping g in either $\mathcal{S}_{n-k}$ or $\mathcal{S}_{n-k}^c$ (depending on whether we started with a permutation map, or a composition of permutation maps). Then we may apply Theorem 3.37 (Raines' Theorem) to obtain to desired result. ∎

Knowing the smallest n for which a continuum X is in the family $\mathcal{M}_n^c$ will provide us with some useful topological information about X. In particular, if n is the smallest integer such that $X \in \mathcal{M}_n^c$, then X has exactly n end points in the classical sense. This will be established later in Corollary 4.10.

As we shall see, the family $\mathcal{M}$ is not closed with respect to subcontinua. We will formally demonstrate this in observations 4.3 and 4.4 by showing that there is a continuum in $\mathcal{M}^c$ which is not in $\mathcal{M}$, and then showing that every continuum in $\mathcal{M}^c$ appears as a subcontinuum of some member of $\mathcal{M}$. As we show later in Theorem 4.19, the family $\mathcal{M}^c$ is in fact closed with respect to subcontinua. For these reasons, we will focus much of our discussion on this larger family.

Observation 4.3. *There is a continuum* $X \in \mathcal{M}^c$ *such that* $X \notin \mathcal{M}$*.*

Proof. Let f_σ and f_γ be permutation maps from $\mathcal{S}_4$ associated with the permutations $\sigma = (234)$ and $\gamma = (24)$, and let $f \in \mathcal{S}_4^c$ be given by $f = f_\sigma \circ f_\gamma$. The graph of this mapping is pictured in Figure 4.1. Let $X = \varprojlim\{[0,1], f\} \in \mathcal{M}^c$. Applying Bennett's Theorem 3.29, one can see that X is a topological ray limiting to a $\sin(\frac{1}{x})$-curve. We actually do not yet have the machinery in place to show that $X \notin \mathcal{M}$. We will eventually see, by way of Theorem 4.10, that since X has 4 end points in the classical sense, if $X \in \mathcal{M}$, then X would have to be in $\mathcal{M}_4$. In [22] Ingram specified each continuum in $\mathcal{M}_2$, $\mathcal{M}_3$, $\mathcal{M}_4$, and $\mathcal{M}_5$. A table detailing the continua in these families has been provided in the appendix. It can be seen from this table that X is not homeomorphic to any continuum in $\mathcal{M}_4$. ■

Observation 4.4. *If $K \in \mathcal{M}_n^c$ for some integer $n \geq 2$, then there exists an integer $m \geq n$ and an $X \in \mathcal{M}_m$ such that K is homeomorphic to a subcontinuum of X.*

Proof. Let $n \geq 2$ be an integer, and let $K \in \mathcal{M}_n^c$. Then K is homeomorphic to $\varprojlim\{[0,1], f\}$ where $f = f_k \circ f_{k-1} \circ ... \circ f_1$ for some $f_1, ..., f_k \in \mathcal{S}_n$. Let $m = kn$. We will construct $g \in \mathcal{S}_m$ such that $X = \varprojlim\{[0,1], g\}$ contains a homeomorphic copy of K.

Denote the members of $\mathcal{A}_m$ by $\mathcal{A}_m = \{0 = a_1 < a_2 < ... < a_m = 1\}$, and for each integer i such that $1 \leq i \leq k$, let $J_i = [a_{kn-k+1}, a_{kn}]$. The J_i's form a collection of disjoint subintervals of $[0,1]$ whose union contains every element of $\mathcal{A}_m$. For each

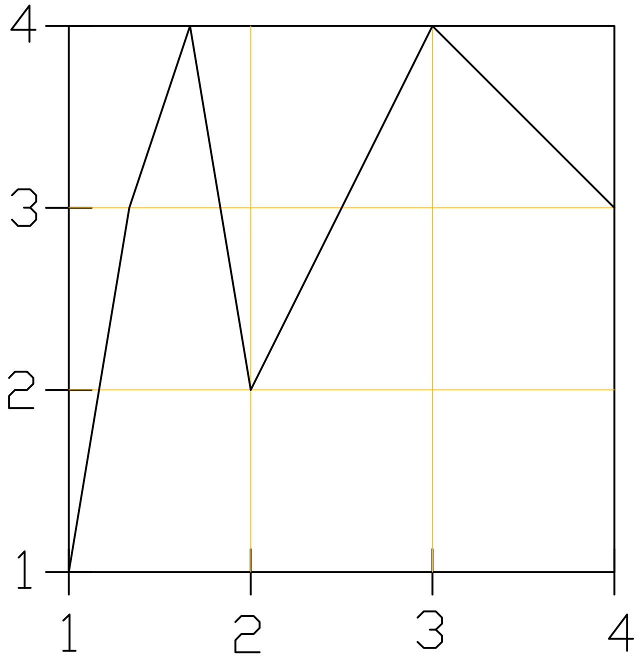

Figure 4.1. Composition of f_σ and f_γ

$1 \le i < k$, set $\alpha(i) = i+1$ if $i < k$, and set $\alpha(k) = 1$. For each such i, let $\phi_i : J_i \to [0,1]$ and $\psi_i : [0,1] \to J_{\alpha(i)}$ be linear homeomorphisms, and define a function $f_i^* : J_i \to J_{\alpha(i)}$ by $f_i^* = \psi_i \circ f_i \circ \phi_i$. We now define $g : [0,1] \to [0,1]$ by setting $g(x) = f_i^*(x)$ if $x \in J_i$ for some i, and then extending g linearly to the rest of the interval. Loosely speaking, the effect of this construction is to place a "copy" of the graph of each f_i in the square $J_i \times J_{\alpha(i)}$. See Figure 4.2.

Let $X = \varprojlim\{[0,1], g\}$. It can be seen that g is a permutation map with Markov partition $\mathcal{A}_m$, and hence that $X \in \mathcal{M}_m$. Further, it is not difficult to see that for each $1 \le i \le k$, the subinterval J_i is mapped back onto itself under g^k, and that $g^k|_{J_1}$ is conjugate to f. It follows that $K = \varprojlim\{[0,1], f\}$ is homeomorphic to $\varprojlim\{J_1, g^k|_{J_1}\}$, and hence to a subcontinuum of X. In fact, one may show that X contains k disjoint, homeomorphic copies of K. ■

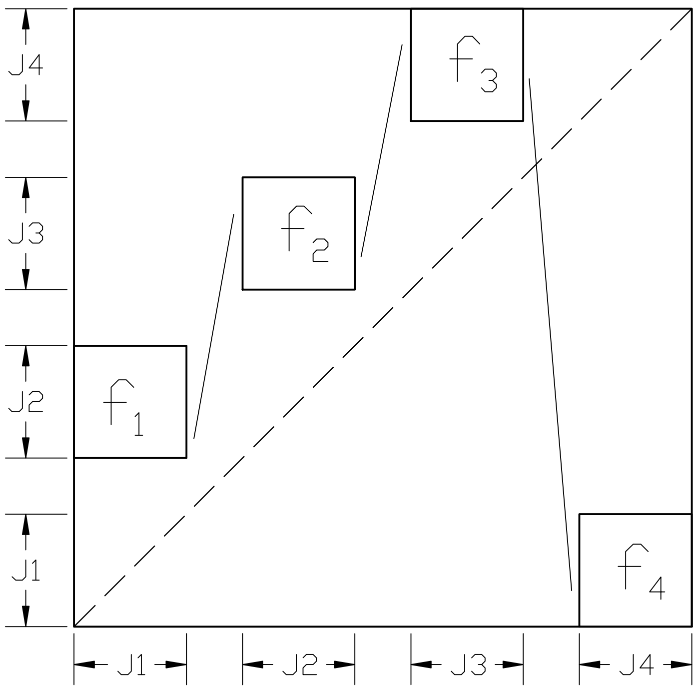

Figure 4.2. Sketch of the graph of g from Observation 4.4 with k=4

It should be made clear that a mapping $f \in \mathcal{S}_n^c$ is not, in general, a permutation map. However, as we note in the following Observation, if $f \in \mathcal{S}_n^c$, then f is Markov with respect to some partition $\mathcal{P}$ such that $\mathcal{A}_n \subseteq \mathcal{P}$, and $\omega(\mathcal{P}, f) = \mathcal{A}_n$.

Observation 4.5. *If $f \in \mathcal{S}_n^c$ for some integer $n \geq 2$, then f is Markov with respect to some partition $\mathcal{P}$ such that $\mathcal{A}_n \subseteq \mathcal{P}$, and $\omega(\mathcal{P}, f) = \mathcal{A}_n$.*

Proof. Fix $n \geq 2$, and let $f \in \mathcal{S}_n^c$. Then $f = f_k \circ f_{k-1} \circ ... \circ f_1$ for some permutation maps $f_1, f_2, ..., f_k \in \mathcal{S}_n$. Since $f_i[A_n] = A_n$ for each $i \in \{1, 2, ..., k\}$, we see that $f[A_n = \mathcal{A}_n$. It follows immediately from Lemma 3.40 that if p is a turning point for f, then $f(p) \in \mathcal{A}_n$. If f_σ is any permutation map and $p \in [0, 1]$, then the pre-image of p under f_σ is finite, from which it follows that $\mathcal{T}_f$, the set of turning points of f, is finite. Let $\mathcal{P} = \mathcal{T}_f \cup \mathcal{A}_n$. Then $f[\mathcal{P}] = f[\mathcal{T}_f] \cup f[\mathcal{A}_n] = \mathcal{A}_n \subseteq \mathcal{P}$. Since $\mathcal{P}$ contains each turning point of f, and maps into itself under f, we see that f is Markov with respect to $\mathcal{P}$. Also, since $f[\mathcal{P}] = \mathcal{A}_n$, we have that $f^m[\mathcal{P}] = \mathcal{A}_n$ for each positive integer m, and hence $\omega(\mathcal{P}, f) = \mathcal{A}_n$. ∎

In the rest of this section, we will provide several results relating to inverse limits with Markov bonding maps, and then apply these results specifically to the cases in which the bonding maps are permutation maps, or compositions of such. The following observation, provides an important tool for working with inverse limits of inverse sequences with Markov bonding maps.

Observation 4.6. *Let $X = \varprojlim\{[a, b], f\}$, where $f : [a, b] \to [a, b]$ is Markov with respect to some partition $\mathcal{P}$, and let $\mathcal{A} = \omega(\mathcal{P}, f)$. If K is a subcontinuum of X, then $|\pi_i[K] \cap \mathcal{A}| \geq |\pi_{i+1}[K] \cap \mathcal{A}|$ for each positive integer i. As a result, $|\pi_i[K] \cap \mathcal{A}|$ is eventually constant.*

Proof. This observation follows immediately from the fact that f is one-to-one on $\mathcal{A}$, which was noted in Observation 3.31. ∎

The results of Observation 4.6 will be used frequently in subsequent proofs. To simplify the language in these proofs, we will introduce some special notation for the limiting value of $|\pi_i[K] \cap \mathcal{A}|$, the number of points of $\mathcal{A}$ which are contained in the projections of K.

Notation. Let $X = \varprojlim\{[a,b], f\}$, where $f : [a,b] \to [a,b]$ is Markov with respect to some partition $\mathcal{P}$, and let $\mathcal{A} = \omega(\mathcal{P}, f)$. Let $C(X)$ denote the collection of all subcontinua of X. We define the function $\Phi : C(X) \to \mathbb{N}$ by $\Phi(K) = \lim |\pi_i[K] \cap \mathcal{A}| = \min\{|\pi_i[K] \cap \mathcal{A}| : i \text{ is a positive integer}\}$.

Before moving on to establish results concerning the inverse limits of Markov maps, we introduce one last piece of notation that we will use throughout this section.

Notation. Let $\mathcal{A} = \{a = a_1 < a_2 < ... < a_n = b\}$ be a partition of the interval $[a,b]$. We denote by $\mathcal{I}(\mathcal{A})$ the collection of all nondegenerate subintervals of $[a,b]$ whose end points are elements of $\mathcal{A}$.

4.1. END POINTS

In this subsection, we will discuss the end points of inverse limits with Markov bonding maps. The main result of this section appears as Corollary 4.10, which provides a method of determining exactly how many end points in the classical sense that such an inverse limit contains.

Theorem 4.7. *Let $X = \varprojlim\{[a,b], f\}$, where $f : [a,b] \to [a,b]$ is a nowhere locally constant Markov map. Denote by $\mathcal{P}$ some Markov partition of f, and set $\mathcal{A} = \omega(\mathcal{P}, f)$. Let K be a subcontinuum of X. If $\Phi(K) = 0$, then K is an arc. If $\Phi(K) = 1$, and $|\pi_N[K] \cap \mathcal{P}| = 1$ for some positive integer N, then K is an arc. Furthermore, if $p = (p_1, p_2, ...) \in K$ such that $p_i \in \mathcal{A}$ for each i, and p_k is a turning point of f for some positive integer k, then p is an end point of K.*

Proof. For each positive integer i, set $K_i = \pi_i[K]$. By using the Subsequence Theorem 3.22, we may assume without loss of generality that $|K_i \cap \mathcal{A}| = \Phi(K)$ for each positive integer i. We first consider the case where $\Phi(K) = 0$, and hence $\mathcal{K}_i \cap \mathcal{A} = \emptyset$ for each i. There exists a positive integer M such that $f^M[\mathcal{P}] = \mathcal{A}$, so $K_i \cap \mathcal{P} = \emptyset$ for each $i > M$.

By again appealing to the Subsequence Theorem, we may assume this to be true for all i. Since each turning point of f must reside in $\mathcal{P}$, we see that for each $i =\in \mathbb{Z}^+$, the interval K_i contains no turning points for f, and hence $f|_{K_i+1}$ is monotone for each positive integer i. It follows from Theorem 3.28 that $K = \varprojlim\{K_i, f|_{K_{i+1}}\}$ is an arc.

Now assume that $\Phi(K) = 1$ and that there exists a positive integer N such that $|K_N \cap \mathcal{P}| = 1$. Applying the Subsequence Theorem, we may as well assume that $N = 1$. For each positive integer i, let p_i denote the unique point in $K_i \cap \mathcal{A}$. Notice that since $f[\mathcal{A}] = \mathcal{A}$, it must be the case that $p_i = f(p_{i+1})$ for each i. Since $|K_1 \cap \mathcal{P}| = |K_1 \cap \mathcal{A}| = 1$, it is clear that $K_1 \cap \mathcal{P} = K_1 \cap \mathcal{A} = \{p_1\}$. Our next step is to use an inductive argument to show that $K_i \cap \mathcal{P} = K_i \cap \mathcal{A} = \{p_i\}$ for each positive integer i. Assume this to be true for some positive integer M. By way of contradiction, assume that K_{M+1} contains points of the partition $\mathcal{P}$ other than p_{i+1}. Since $\mathcal{P}$ is finite, we may choose a point $q \in \mathcal{K}_{M+1} \cap \mathcal{P}$ such that there are no points of $\mathcal{P}$ between q and p_{i+1}. Since $q \in \mathcal{K}_{M+1} \cap \mathcal{P}$, we see that $f(q) \in \mathcal{K}_M \cap \mathcal{P} = \{p_M\}$, and so $f(q) = f(p_{i+1})$. Since f is nowhere locally constant, the fact that q and p_{i+1} map to the same point under f indicates that there must be a turning point between q and p_{i+1}, which is contrary to our selection of q. Hence, we see that if $K_i \cap \mathcal{P} = K_i \cap \mathcal{A} = \{p_i\}$ $i = M$, then the same is true for $i = M + 1$. Since the statement is true for $i = 1$, we see that $K_i \cap \mathcal{P} = K_i \cap \mathcal{A} = \{p_i\}$ for each positive integer i. Thus, for each i, the only point in K_i which could possibly be a turning point of f is p_i.

It is clear that if p_i is not a turning point for any positive integer i, then $f|_{K_{i+1}}$ is monotone for each i, and Theorem 3.28 tells us that K is an arc. So, we assume that there is a positive integer n such that p_n is a turning point of f. Since f acts as a permutation on the members of $\mathcal{A}$, we see that $p_i = p_n$ for infinitely many positive integers i, and so we may choose n to be arbitrarily large. Since p_n is a turning point and there are no other turning points in K_n, we see that p_{n-1} must be an end point of the arc K_{n-1}. Furthermore, as p_i is the only possible turning point in K_i for any given i, it then follows that p_i is an end point of K_i for all $i < n$. Since n may be chosen to be arbitrarily large, we in fact have that p_i is an end point of K_i for every positive integer i. Hence, $f|_{K_{i+1}}$ is monotone for each such i, and K is an arc.

Let $p = (p_1, p_2, ...) \in K$. We have left to show that in the last case, in which we assume that p_n is a turning point for some n, the point p is an end point of the arc K. This follows directly from the fact that p_i is an end point of K_i for each i. If A and B are subcontinua of K, then for each positive integer i, $A_i = \pi_i[A]$ and $B_i = \pi_i[B]$ are subintervals of K_i containing p_i, and so either $A_i \subseteq B_i$ for all i, or $B_i \subseteq A_i$ for all i. ■

Theorem 4.7 has three immediate corollaries. Corollary 4.8 states that each sufficiently small subcontinuum of an inverse limit with Markov bonding maps is an arc. Corollary 4.9 characterizes the number of end points in the classical sense that such an inverse limit contains, and Corollary 4.10 is an application of Corollary 4.9 to inverse limits of permutation maps.

Corollary 4.8. *Let X be as described in Theorem 4.7. There exists an $\varepsilon > 0$ such that if K is a subcontinuum of X satisfying $\operatorname{diam} K < \varepsilon$, then K is an arc.*

Proof. This follows immediately from Theorem 4.7, since we may choose ε in such a way as to ensure that if $\operatorname{diam} K < \varepsilon$, then $\operatorname{diam} \pi_1[K]$ is small enough that $|\pi_1[K] \cap \mathcal{P}| \leq 1$. ■

Corollary 4.9. *Let X be as described in Theorem 4.7. Denote by $\mathcal{P}$ the essential Markov partition of f, and set $\mathcal{A} = \omega(\mathcal{P}, f)$. Let $p = (p_1, p_2, ...) \in X$. The following are equivalent:*

1. *The point p is an end point of X in the classical sense.*
2. *For each positive integer i, $p_i \in \mathcal{A}$.*

Proof. Assume Condition 2 to be true. It follows from Theorem 4.7 that p is an end point of every sufficiently small arc containing it, and so it follows that p is an end point of every arc that contains it. Hence, p is an end point of X in the classical sense.

We now show that that Condition 1 implies Condition 2. Assume that p is an end point of X, and assume that there is a positive integer N such that $p_N \notin \mathcal{A}$. Since

f maps $\mathcal{A}$ onto itself, we see that $p_i \notin \mathcal{A}$ for each $i \geq N$. There exists a positive integer M such that $f^M[\mathcal{P}] = \mathcal{A}$, so we see that $p_i \notin \mathcal{P}$ for each $i > N + M$. By applying the Subsequence Theorem 3.22 if necessary, we may assume that $p_i \notin \mathcal{P}$ for each positive integer i. For each $i = 1, 2, ...$, let J_i be the smallest member of $\mathcal{I}(\mathcal{P})$ containing p_i. Notice that p_i lies in the interior of J_i for each such i.

Let k be an arbitrary positive integer. It is clear that f must map J_{k+1} onto a member of $\mathcal{I}(\mathcal{P})$ which contains $f(p_{k+1}) = p_k$, and hence $f[J_{k+1}]$ must contain J_k. Since J_{k+1} contains no turning points in its interior, f is monotone, and hence confluent, on J_{k+1}. In particular, if I is any subset of J_k which contains p_k, and $C = J_{k+1} \cap f^{-1}(I)$, then $f[C] = I$.

With the observations of the previous paragraph in mind, let K_1 be a subinterval of $[a, b]$ containing p_1 in its interior such that $K_1 \cap \mathcal{P} = \emptyset$. For each positive integer i, we inductively define $K_{i+1} = J_{i+1} \cap f^{-1}(K_i)$. We can see that $\mathcal{K}_i \cap \mathcal{P} = \emptyset$ for each i, and the discussion in the previous paragraph gives us that $f[K_{i+1}] = K_i$ for each i, and so $K = \varprojlim\{K_i, f|_{K_{i+1}}\}$ is a subcontinuum of X containing p such that $\pi_i[K] = K_i$ for each positive integer i. Since $\Phi(K) = 0$, Theorem 4.7 gives us that K is an arc. For each $i \in \mathbb{Z}^+$, the point p_i fails to be an end point of K_i, and so it follows that p is not an end point of the arc K, and thus that p is not an end point of X in the classical sense. We have therefore shown that if p is an end point of X in the classical sense, then $p_i \in \mathcal{A}$ for each positive integer i. ■

Corollary 4.10. *If $f \in \mathcal{S}_n^c$ and each point of $\mathcal{A}_n$ is essential with respect to f, then $X = \varprojlim\{[a, b], f\}$ has exactly n end points in the classical sense.*

Corollary 4.10 provides useful information about the topology of the inverse limit of any permutation maps. Given a mapping $f \in \mathcal{S}_n^c$, we may remove all inessential points of the Markov partition, and then apply Raines' Theorem to represent the inverse limit with f as a single bonding map as an inverse limit using a bonding map $g \in \mathcal{S}_m^C$ such that each point of $\mathcal{A}_m$ is essential with respect to g. This tells us immediately how many end points in the classical sense that we may expect to find in the inverse limit.

4.2. INDECOMPOSABILITY

In this subsection, we discuss indecomposability of inverse limits with permutation bonding maps. Recall that Theorem 3.41 by Ryden has provided a very nice characterization of indecomposability of inverse limits with Markov bonding maps. The goal of this section is to provide sufficient conditions for such an inverse limit to produce an indecomposable arc continuum.

Theorem 4.11. *Let $X = \varprojlim\{[a,b], f\}$ where $f : [a,b] \to [a,b]$ is a Markov map. Denote by $\mathcal{P}$ the essential Markov partition for f, and set $\mathcal{A} = \omega(\mathcal{P}, f)$. Assume that $|\mathcal{P}| \geq 3$, and denote $|\mathcal{A}|$ by n. If there is no proper subinterval J of $[a,b]$ such that $J \in \mathcal{I}(\mathcal{P})$ and $f^m[J] \subseteq J$ for some positive integer m, then X is an indecomposable arc continuum with exactly n end points.*

Proof. We first establish indecomposability of X. Since $|\mathcal{P}| \geq 3$, we may find intervals J_1 and J_2 in $\mathcal{I}(\mathcal{P})$ whose intersection contains at most one point. Since f is Markov, it is the case that $f^i[J_1] \in \mathcal{I}(\mathcal{P})$ and $f^i[J_2] \in \mathcal{I}(\mathcal{P})$ for each positive integer i. Furthermore, by our hypotheses, for $k = 1, 2$ and each positive integer i, $f^{i+1}[J_k]$ properly contains $f^i[J_k]$, unless $f^i[J_k] = [a,b]$. So we see that there exists an N such that $f^N[J_1] = f^N[J_2] = [a,b]$. Applying Theorem 3.41 establishes indecomposability of X.

Next we show that X is an arc continuum. Let K be a proper subcontinuum of X, and set $K_i = \pi_i[K]$ for each positive integer i. Assume that $|K_i \cap \mathcal{P}| \geq 2$ for each such i. Then K_i contains some $J_i \in \mathcal{I}(\mathcal{P})$ for each i. As shown in the previous paragraph, each member of $\mathcal{I}(\mathcal{P})$ maps onto the entire interval $[a,b]$ in finitely many iterations of f. It follows from this fact that $K_i = [a,b]$ for all i. This is a contradiction to the assumption that K is a proper subcontinuum of X. Therefore, there exists a positive integer j such that $|K_i \cap \mathcal{P}| < 2$. It follows from Theorem 4.7 that K is an arc.

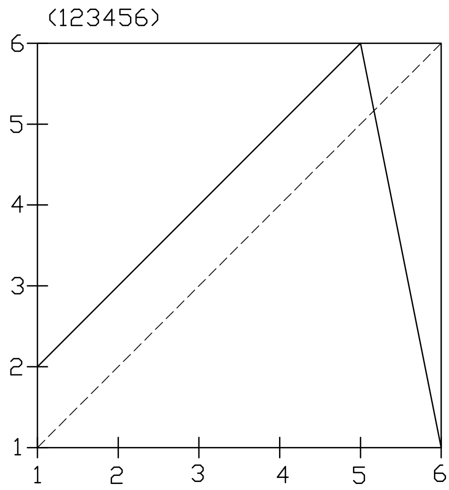

Figure 4.3. Map generating a 6 end point indecomposable arc continuum.

We finish the proof by showing that X has exactly n end points. Recall that f maps $\mathcal{A}$ onto itself in a one-to-one manner. Since $\mathcal{A}$ contains n points which are permuted by f, there are exactly n points $p_1, p_2, ..., p_n \in X$ satisfying the property that $\pi_i(p_j) \in \mathcal{A}$ for each $j = 1, 2, ..., n$ and for each positive integer i. Theorem 4.9 tells us that these are precisely the points of X which are end points in the classical sense. Since each proper subcontinuum of X is an arc, we see that $p_1, p_2, ..., p_n$ are in fact actual end points of X. Since any other end point would also have to be an end point in the classical sense, we conclude that X has exactly n end points. ■

We now apply Theorem 4.11 to inverse limits with permutation bonding maps to obtain Corollary 4.12.

Corollary 4.12. *Let $f \in \mathcal{S}_n^c$ such that $\mathcal{A}_n$ is the essential Markov partition of f, and let $X = \varprojlim\{[0,1], f\}$. If $[a,b]$ is the only member of $\mathcal{I}(\mathcal{A}_n)$ which is periodic under f, then X is an indecomposable arc continuum with exactly n end points.*

As we shall soon see, indecomposable arc continua will play a crucial role in our study of inverse limits with Markov bonding maps. The arc along with indecomposable

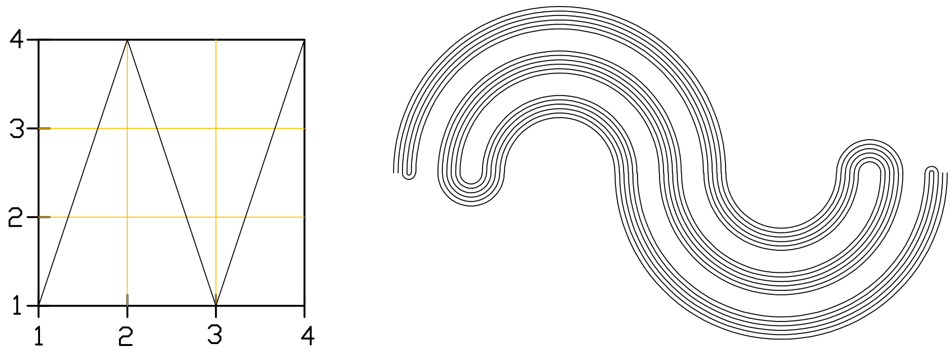

Figure 4.4. Indecomposable arc continuum with exactly two end points.

arc continua will, in some sense provide the building blocks from which all continua generated as inverse limits with Markov bonding maps are constructed.

It is worth noting that there exist n end point, indecomposable arc continua in $\mathcal{M}_n$ for every $n > 2$. For such an n, it is simple to see that the n-cycle $(123...n)$ generates a permutation mapping satisfying the conditions in Corollary 4.12. See Figure 4.3 for an example with $n = 6$. One may not, however, find indecomposable arc continua with only 1 or 2 end points in the family $\mathcal{M}$ of continua generated by permutation maps, or even in $\mathcal{M}^c$, those continua generated by compositions of such maps. Such continua do appear as the inverse limits of Markov maps, however. The familiar BJK Continuum, pictured in Figure 3.1, is an indecomposable arc continuum with only a single end point. A variation of the BJK Continuum with two end points is pictured in Figure 4.4 along with the Markov map which generates it.

4.3. SUBCONTINUA

In Theorem 4.11, we showed that if $f : [a, b] \to [a, b]$ is a Markov map with Markov partition $\mathcal{P}$ such that no member of $\mathcal{I}(\mathcal{P})$ other than $[a, b]$ is periodic under f, then $\varprojlim\{[0, 1], f\}$ is an indecomposable continuum each of whose proper subcontinua is an arc. In this section, we consider what sort of subcontinua appear in $\varprojlim\{[0, 1], f\}$

when such periodic subintervals exist.

We begin with Theorem 4.13, which provides conditions under which we may represent an upper semi-continuous decomposition of an inverse limit space as the inverse limit of a decomposition of the factor spaces. Theorem 4.15 applies this theorem to inverse limits of permutation maps to obtain a central result of this paper. The notation relating to decomposition spaces which is used in these two theorems is explained in subsection 3.6.

Theorem 4.13. *Let* $M = \varprojlim\{X, f\}$ *where* X *is a continuum, and let* K *be a subcontinuum of* X *which is periodic under* f *with period* k*, and such that the elements of* $\mathrm{Orbit}(K, f)$ *are pairwise disjoint. Denote* $\mathrm{Orbit}(K, f)$ *by* $\mathcal{K}$*, and let* $\mathcal{L}$ *be the family of subcontinua of* M *such that* $L \in \mathcal{L}$ *if and only if* $\pi_i[L] \in \mathcal{K}$ *for each* $i \in \mathbb{Z}^+$ *(note that* $\mathcal{K}$ *and* $\mathcal{L}$ *each have* k *elements). The upper semi-continuous decomposition* $M/\mathcal{L}$ *is homeomorphic to* $\varprojlim\{X/\mathcal{K}, f/\mathcal{K}\}$.

Proof. Let $\psi : X \to X/\mathcal{K}$ be the natural projection (or quotient map) from X to $X/\mathcal{K}$. That is, for a given $x \in X$, $\psi(x) = A \in \mathcal{K}$ if and only if $x \in A$. Notice that for any $x \in X$ we have that $(\psi \circ f)(x) = \psi(f(x)) = (f/\mathcal{K})(\psi(x)) = (f/\mathcal{K} \circ \psi)(x)$, and so $\psi \circ f = f/\mathcal{K} \circ \psi$. Hence, the mapping ψ from X to the decomposition space $X/\mathcal{K}$ induces a limit mapping $g : M \to \varprojlim\{X/\mathcal{K}, f/\mathcal{K}\}\}$, which is given by $g((x_i)_{i=1}^{\infty}) = (\psi(x_i))_{i=1}^{\infty}$. Let $\mathcal{D}$ be the decomposition of M given by $\mathcal{D} = \{g^{-1}(p) : p \in \varprojlim\{X/\mathcal{K}, f/\mathcal{K}\}$. In light of [32, Theorem 3.21], our proof will be complete if we can show that $D = M/\mathcal{L}$. In other words, we wish to show that two points $a, b \in M$ are in the same block of the decomposition $\mathcal{D}$ if and only if they are in the same block of the decomposition $M/\mathcal{L}$.

Let $a = (a_i)_{i=1}^{\infty}$ and $b = (b_i)_{i=1}^{\infty}$ be points in M. We wish that show that $g(a) = g(b)$ if and only if $a = b$ or $a, b \in L$ for some $L \in \mathcal{L}$. It is obvious that if $a = b$, then $g(a) = g(b)$. Assume that $a, b \in L$ for some $L \in \mathcal{L}$. Then for each positive integer i, $a_i, b_i \in \pi_i[L] \in \mathcal{K}$, and so $\psi(a_i) = \psi(b_i)$. It follows that $g(a) = g(b)$. We establish the other direction of the implication by assuming that $a \neq b$, and that $g(a) = g(b)$, and showing that this leads to the conclusion that $a, b \in L$ for some $L \in \mathcal{L}$. If $g(a) = g(b)$, then $(\psi(a_i))_{i=1}^{\infty} = (\psi(b_i))_{i=1}^{\infty}$. The points a and b are distinct, so there exists a positive integer N such that $a_i \neq b_i$ for each $i \geq N$. For each such i, $a_i \neq b_i$, but $\psi(a_i) = \psi(b_i)$, so there must be a $K_i \in \mathcal{K}$ such that $a_i, b_i \in K_i$. Since the members of $\mathcal{K}$ are permuted by f, it follows that for each $i \in \mathbb{Z}^+$, there is a $K_i \in \mathcal{K}$ such that $a_i, b_i \in K_i$. Recalling that the members of $\mathcal{K}$ are pairwise disjoint, and that $f(a_{i+1}) = a_i$ for all $i \in \mathbb{Z}^+$, we see that $f[K_{i+1}] = K_i$ for each such i. So, $L = \varprojlim\{K_i, f|K_{i+1}\}$ is a subcontinuum of M containing both a and b. We note that each projection of L onto a factor space is a member of $\mathcal{K}$, and hence $L \in \mathcal{L}$. This completes our justification that $D = M/\mathcal{L}$, and hence the proof of the theorem. ■

We can of course apply Theorem 4.13 to inverse limits on intervals with Markov bonding maps. When we do so, our "decomposed" bonding map will itself be a Markov map on the shrunken interval.

Theorem 4.14. *Let $f : [a, b] \to [a, b]$ be a Markov map with Markov partition $\mathcal{P}$. Assume that $K \in \mathcal{I}(\mathcal{P})$ is periodic and that the members of $\mathcal{K} = \mathrm{Orbit}(K, f)$ are pairwise disjoint. Then the mapping $f/\mathcal{K}$ from $[a, b]/\mathcal{K}$ to itself is a Markov map with Markov partition $\psi[\mathcal{P}]$, where $\psi : [a, b] \to [a, b]/\mathcal{K}$ is the natural projection.*

Proof. Let $\mathcal{B} = \psi[\mathcal{P}]$. We begin by showing that $(f/\mathcal{K})[\mathcal{B}] \subseteq \mathcal{B}$. If $B \in \mathcal{B}$, then either $B = \{p\}$ for some $p \in \mathcal{P}$, or B is a member of $\mathcal{K}$ which contains some $p \in \mathcal{P}$. In either case, $(f/\mathcal{K})(B) = \psi(f(p)) \in \mathcal{B}$. Notice that if C is a component of $[a, b] - \mathcal{P}$, then either C gets shrunk to a point (if it is contains in a member of $\mathcal{K}$), or ϕ is one-to-one on C, and maps each point of C to its associated singleton. It can also be seen that $\mathcal{D} = \phi[C]$ is a component of $[a, b]/\mathcal{K} - \mathcal{B}$, and in fact each component of $[a, b]/\mathcal{K} - \mathcal{B}$ is obtained in such a way from a component of $[a, b] - \mathcal{P}$. The mapping ϕ is monotone and f is monotone on C. Since $f/\mathcal{K} = \phi \circ f$, we thus may conclude that $f/\mathcal{K}$ is monotone on $\mathcal{D}$. ■

Theorem 4.14 shows that $f/\mathcal{K}$ is Markov when f is Markov. The next theorem states that if f is the composition of permutation maps, then $f/\mathcal{K}$ follows the same pattern on its partition as some mapping g which is also the composition of permutation maps.

Theorem 4.15. *Let $f \in \mathcal{S}_n^c$ and $X = \varprojlim\{[0,1], f\}$ and let $J \in \mathcal{I}(\mathcal{A}_n)$ such that J is periodic under f with period k, and such that the elements of $\mathcal{K} = \mathrm{Orbit}(J, f)$ are pairwise disjoint. Let $\mathcal{L}$ be the family of subcontinua of X such that $L \in \mathcal{L}$ if and only if $\pi_i[L] \in \mathcal{K}$ for each positive integer i. Let m denote the number of points in $J \cap \mathcal{A}_n$. Then $X/\mathcal{L}$ is homeomorphic to $\varprojlim\{[0,1], g\}$ for some $g \in S_N^c$, where $N = n - km + k$. Furthermore, if $f \in \mathcal{S}_n$, then g can be chosen from S_N.*

Proof. Since $f \in \mathcal{S}_n^c$, there exist mappings $f_1, f_2, ..., f_l \in \mathcal{S}_n$ such that $f = f_l \circ f_{l-1} \circ ... \circ f_1$. It can be seen then that $f/\mathcal{K} = f_l/\mathcal{K} \circ f_{l-1}/\mathcal{K} \circ ... \circ f_1/\mathcal{K}$. Let $\psi : [0,1] \to [0,1]/\mathcal{K}$ denote the natural projection from X to $X/\mathcal{K}$, and set $\mathcal{B} = \psi[\mathcal{A}_n]$. Notice that km of the n points in $\mathcal{A}_n$ reside in members of $\mathcal{K}$ and ψ maps these points onto the k elements of $\mathcal{K}$. The remaining $n - km$ points in $\mathcal{A}_n$ are each mapped to their respective singletons, and hence ϕ is one-to-one on these points. It follows that $\mathcal{B}$ is a partition of $[0,1]/\mathcal{K}$ containing $N = n - km + k$ points. For each $i \in \{1, 2, ..., l\}$, the mapping $f_i/\mathcal{K}$ is Markov with respect to $\mathcal{B}$, and in fact permutes the elements of $\mathcal{B}$. The mappings $f_i/\mathcal{K}$ might not be permutation maps, as they may contain flat spots, but each such map is a Markov map which "follows the same pattern" as some permutation map $g_i \in \mathcal{S}_N$. It follows from Theorem 3.39 that $\varprojlim\{[0,1]/\mathcal{K}, f/\mathcal{K}\}$ is homeomorphic to $\varprojlim\{[0,1], g\}$ where $g \in \mathcal{S}_N^c$ is given by $g = g_l \circ g_{l-1} \circ ... \circ g_1$. Applying Theorem 4.13, we conclude that $X/\mathcal{L}$ is homeomorphic to $\varprojlim\{[0,1], g\}$. ∎

The combination of Corollary 4.12 and Theorem 4.11 will provide a useful set of tools for determining the topology of any inverse limit with a single Markov bonding map. If there exist periodic elements of $\mathcal{I}(\mathcal{P})$, then we can apply Theorem 4.15 to shrink the associated subcontinua of the inverse limit to points. The continuum resulting from this decomposition will be representable as the inverse limit of some other Markov map g. It f was originally a permutation map, or composition of permutation maps, then g will be as well. We continue to shrink down continua until we no longer have periodic subintervals. At this stage in the process, we remove all of the inessential points in the partition, apply Raines' Theorem 3.37, and then use Corollary 4.12 to determine that our final decomposition space is either an arc, or an indecomposable arc continuum with some number of end points. We then move backwards through the process, growing continua back from the points to which they were shrunk. Of course, there are many ways that this "growing of continua" can occur. To get a handle on this, we have the following two theorems, which address the issue of terminality of subcontinua of inverse limits on intervals.

Theorem 4.16. *Let $X = \varprojlim\{[a,b], f_i\}$ where, for each positive integer i, f_i is a mapping from $[a,b]$ onto itself. Let K be a subcontinuum of X, and for each $i \in \mathbb{Z}^+$, set $K_i = \pi_i[K]$. Assume that there exists a positive integer m such that for any $i > m$, if C is the component of $f^{-1}(K_{i-1})$ which contains K_i, and D is any component of $C - \mathrm{int}(K_i)$, then $f_{i-m,i}[D] = K_{i-m}$. Then K is terminal in X.*

Proof. Let L be a subcontinuum of X such that $L \cap K \neq \emptyset$ and $L \nsubseteq K$. We wish to show then that $K \subseteq L$. For each $i \in \mathbb{Z}^+$, let $L_i = \pi_i[L]$. Since $L \nsubseteq K$, there must exist a positive integer n such that $L_i \nsubseteq K_i$ for each $i \geq n$. We will assume that $n \geq m$. For each $i \in \mathbb{Z}^+$, let C_i be the component of $f^{-1}(K_{i-1})$ which contains K_i. If $L_i \subseteq C_i$ for any $i > 2$, then certainly $L_{i-1} \subseteq K_{i-1}$. Therefore, for each $i > n$, it must be the case that $L_i \nsubseteq C_i$. But $L \cap K \neq \emptyset$, so $L_i \cap K_i \neq \emptyset$ for each i. The facts that for $i > n$ we have $L_i \nsubseteq C_i$ and $L_i \cap K_i \neq \emptyset$ lead us to observe that L_i must completely contain some component D_i of $C_i - \mathrm{int}(K_i)$ for each $i > n$. But since $f_{i-m,i}[D_i] = K_{i-m}$ for all $i > n$, we see that $K_{i-m} \subseteq f_{i-m,i}[L_i] = L_{i-m}$ for each $i > n$. Thus, we have that $K_i \subseteq L_i$ for each positive integer i, and so $K \subseteq L$. ■

In the next theorem we will supply conditions which characterize when certain subcontinua of a continuum $X \in \mathcal{M}^c$ are terminal in X.

Theorem 4.17. *Let $X = \varprojlim\{[0,1], f\}$ where $f \in \mathcal{S}_n^c$. Let K be a subcontinuum of X and for each $i \in \mathbb{Z}^+$, set $K_i = \pi_i[K]$. Assume that for each i, K_i is a periodic element of $\mathcal{I}(\mathcal{A}_n)$. Let m be the order of the permutation $f|_{\mathcal{A}_n}$. The following are equivalent:*

1. *K is terminal in X.*

2. *There is a positive integer i and an interval J_i such that $K_i \subseteq \operatorname{int}(J_i)$ and if D is a component of $J_i - \operatorname{int}(K_i)$, then $f^m[D] = K_i$.*

3. *For each positive integer i, there is an interval J_i such that $K_i \subseteq \operatorname{int}(J_i)$ and if D is a component of $J_i - \operatorname{int}(K_i)$, then $f^m[D] = K_i$.*

Proof. It is clear that Condition 3 implies Condition 2, which in turn implies Condition 1 by Theorem 4.16. We will now show that Condition 1 implies Condition 3. To that end, assume that K is terminal in X. Hoping to achieve a contradiction, we assume that there is a positive integer N such that given any interval J_N which contains K_N in its interior, there is a component D of $J_N - \operatorname{int}(K_N)$ such that $f^m[D] \neq K_N$. By using the Subsequence Theorem 3.22, we see that we lose no generality in assuming that $N = 1$. Notice that since m is the order of $f|_{\mathcal{A}_n}$, each point of $\mathcal{A}_n$ is fixed under f^m. It follows that any periodic element of $\mathcal{I}(\mathcal{A}_n)$ is fixed under f^m. In particular, we have that $f^m[K_1] = K_1$. By again appealing to the Subsequence Theorem 3.22, we see that X is homeomorphic to $X^* = \varprojlim\{[0,1], f^m\}$ and K is homeomorphic to $K^* = \varprojlim\{K_1, f^m|_{K_1}\}$. Furthermore, there is a homeomorphism of X to X^* with take K to K^*. Therefore, it will suffice to show that K^* is terminal in X^*.

Let $c, d \in \mathcal{A}_n$ be such that $K_1 = [c, d]$. We shall first assume that c is not a turning point for f^m.

Since $f^m[c, d] = [c, d]$, and c is not a turning point, f^m must be strictly increasing at c. Let $\mathcal{Q}$ be the Markov partition for f^m, and let u and v be the elements of $\mathcal{Q}$ immediately to the left and right of c, respectively. Since $[u, v] \cap \mathcal{Q} = \{u, c, v\}$, and c is not a turning point for f^m, the mapping f^m is monotone, and hence confluent on $[u, v]$. Furthermore, since $f^m[\{u, v\}] \subseteq f^m[\mathcal{Q}] \subseteq \mathcal{Q}$ and f^m is strictly increasing at c, we see that $f^m(u) \leq u$ and $f^m(v) \geq v$, and hence $[u, v] \subseteq f^m[u, v]$. These observations lead us to conclude that, given any subinterval A of $[u, v]$, if $C = (f^m)^{-1}(A) \cap [u, v]$, then $f^m[C] = A$. With this in mind, let L_1 be any subinterval of (u, v) containing c in its interior, and for each $i > 2$, inductively define intervals L_i by $L_i = (f^m)^{-1}(L_{i-1}) \cap [u, v]$. We see that $f^m[L_{i+1}] = L_i$ and $c \in L_i$ for each $i \in \mathbb{Z}^+$. So the inverse limit $L =$

$\varprojlim\{L_i, f^m|_{L_{i_1}}\}$ is a subcontinuum of X containing the point $(c, c, c...)$, and satisfying $\pi_i[L] = L_i$ for each positive integer i. The mapping f^m is monotone on each L_i, so we have that L is an arc. The subcontinua L and K^* both contain the point $(c, c, c, ...)$, and so they intersect. We chose L_1 in such a way that c is not an end point of L_1, and so that $d \notin L_1$. This allows us to claim that $L_1 \not\subseteq K_1$ and $K_1 \not\subseteq L_1$, and so $L \not\subseteq K^*$ and $K^* \not\subseteq L$. This contradicts the terminality of K^*. Assuming that d is not a turning point for f^m produces a similar contradiction.

We now assume that both c and d are turning points of f^m. If c is a turning point, then for all x sufficiently close to c, it is either the case that $f^m(x) \geq f^m(c) = c$, or $f^m(x) \leq f^m(c) = c$. Since $f^m[c, d] = [c, d]$, we see that we are operating under the former of the two cases. Similar considerations show that $f^m(x) \leq f^m(d) = d$ for all x sufficiently close to d. Let $r = \min\{x : f^m[x, c] \subseteq K_1\}$ and let $s = \max\{x : f^m[d, x] \subseteq K_1\}$. It is clear from these definitions that $f^m[r, c] \subseteq K_1$ and $f^m[d, s] \subseteq K_1$. These observations, combined with the fact that points in $\mathcal{A}_n$ are fixed under f^m, tells us that the intervals $[r, c]$ and $[d, s]$ intersect $\mathcal{A}$ only at $\{c\}$ and $\{d\}$ respectively. We have assumed that if J_1 is an interval containing K_1 in its interior, then there is a component D of $J_1 - \text{int}(K_1)$ such that $f^m[D] \neq K_1$. Taking J_1 to be $[r, s]$, we see that either $f^m[r, c]$ or $f^m[d, s]$ is a proper subset of K_1. We will only consider the case where $f^m[r, c]$ is a proper subset of K_1, as the other case is completely analogous. We can see from the definition of r that $f^m(r) \in \{c, d\}$, and the fact that $f^m[r, c]$ is a proper subset of K_1 gives us that $f^m(r) = c$. The definition of r also tells us that r is not a turning point of f^m, and in fact f^m is strictly increasing at r. Let t be the first turning point of f^m to the left of r. Since f^m is increasing at r, we have that $f(t) < f(r) = c$. Let z be the largest point of $\mathcal{A}_n$ such that $z < c$. Then $f^m(t) \leq z$. Assume that $t > z$. Since $z < t < r < c$ and $f^m(t) \leq z < c = f^m(r)$, we see that there is a fixed point $p \in (t, r)$. Now assume that $t \leq z$. In this case, we will let $p = z$. In either case, we have that $f^m(p) = p$, f^m is monotone on $[p, r]$ and $f^m[p, r] = [p, c]$. Now, notice that since $f^m(r) = f^m(c) = c \in \mathcal{A}_n$, and each turning point of f^m maps into $\mathcal{A}_n$, the image of $[r, c]$ under f^m must be in $\mathcal{I}(\mathcal{A}_n)$. Also, recall that $f^m[r, c]$ is a proper subinterval of $[c, d]$ which contains c. We will denote $f^m[r, c]$ by $[c, q]$. Since every member of $\mathcal{I}(\mathcal{A}_n)$ is fixed by f^m, we have that $f^m[c, q] = [c, q]$. Hence, we have that $f^m[p, q] = f^m[p, r] \cup [r, c] \cup [c, q] = f^m[p, r] \cup f^m[r, c] \cup f^m[c, q] = [p, c] \cup [c, q] \cup [c, q] = [p, q]$. So, the interval $[p, q]$ is fixed under f^m. Let $L = \varprojlim\{[p, q], f^m|_{[p,q]}\}$. The continuum L intersects K^* at the point $(c, c, c, ...)$, but since $[p, q] \not\subseteq [c, d]$ and $[c, d] \not\subseteq [p, q]$ we see that $L \not\subseteq K^*$ and $K^* \not\subseteq L$. Thus we have shown that K^* is not terminal in X^*, and hence that K is not terminal in X, producing the desired contradiction. This completes the argument that Condition 1 implies Condition 3, and thus completes the proof. ■

We shall close this subsection with Theorem 4.19, which was alluded to earlier in this section, and states that the family $\mathcal{M}^c$ of all continua arising as the inverse limit of a mapping $f \in \mathcal{S}_n^c$ is closed with respect to subcontinua. Before proving Theorem 4.19, however, we must first establish the following lemma.

Lemma 4.18. *Let $f \in \mathcal{S}_n$ for some $i \in \mathbb{Z}^+$, and let K_1 and K_2 be subintervals of $[0, 1]$ such that $f[K_2] = K_1$. For each $i \in \{1, 2\}$, set I_i to be the smallest member of $\mathcal{I}(\mathcal{A}_n)$ containing K_i. Let $W = \{x \in \mathcal{I}_2 : f(x) \in I_1\}$. Then W is a closed interval, and $f[W] = I_1$.*

Proof. We begin by proving that W is an interval. It follows from the continuity of f that W is closed. Assume that that W is not connected. Then there exist $a, b \in W$ such that $(a, b) \cap W = \emptyset$. Since f is continuous, it must be the case that $f(a) = f(b)$. Since f is nowhere locally constant, it follows that there must be a turning point $t \in (a, b)$. Each turning point of f in the interior of I_2 lies in K_2. In particular, $t \in K_2$, and so $f(t) \in K_1 \subset I_1$, which indicates that $t \in W$, which is a contradiction. Thus, we conclude that W is connected, and hence a closed interval.

Next we show that $f[W] = I_1$. Let $c, d \in I_1$ such that $f[W] = [c, d]$, and let $a, b \in I_2$ such that $f(a) = c$ and $f(b) = d$. If a is an end point of I_1, then $a \in \mathcal{A}_n$ and hence $c = f(a) \in \mathcal{A}_n$. If a is not an end point, and $f(a) = c$ is not an end point of I_1, then it follows that a is a turning point of f. If a is a turning point, then $a \in \mathcal{A}_n$

and hence $c = f(a) \in \mathcal{A}_n$. So, we see that in any case, $c \in \mathcal{A}_n$. Similar considerations show that $d \in \mathcal{A}_n$, and thus we have that $f[W] = [c, d] \in \mathcal{I}(\mathcal{A}_n)$. It is clear that $K_1 \subseteq f[W] \subseteq I_1$, and so it follows from the definition of I_1 that $f[W] = I_1$. ■

Theorem 4.19. *Let $X = \varprojlim\{[0,1], f\}$ where $f \in \mathcal{S}_n^c$ for some $n \geq 2$. If K is a subcontinuum of X, then K is homeomorphic to $\varprojlim\{[0,1], g\}$ where $g \in \mathcal{S}_N^c$ for some $N \in \{2, 3, ..., n\}$.*

Proof. For ease of reading, we introduce the following notation: Given positive integers i and k, denote by $\alpha(i, k)$ the positive integer n such that $1 \leq n \leq k$ and $(i - n) = 0$ mod k. So, the calculation $\alpha(i, k)$ is quite similar to finding the modulus of i with respect to k, except that when k divides i, $\alpha(i, k)$ is equal to k instead of 0.

We begin the proof by denoting the elements of $\mathcal{A}_n$ by $\mathcal{A}_n = \{a = a_1 < a_2 < ... < a_n = b\}$. Let K be a proper subcontinuum of X. We may assume without loss of generality that $|\pi_i[K] \cap \mathcal{A}_n| = \Phi(K)$ for each positive integer i. Notice that if $\Phi(K) \leq 1$, then by Theorem 4.7, K is an arc, which is homeomorphic to the inverse limit of any map in $\mathcal{S}_2$. So, we will assume henceforth that $\Phi(K) \geq 2$. In this case, for each $i \in \mathbb{Z}^+$, there exists an interval $Z_i \in \mathcal{I}(A_n)$ such that $Z_i \subseteq \pi_i[K]$ and $|Z_i \cap \mathcal{A}_n| = \Phi(K)$ (Z_i is the smallest arc which contains every point in $\pi_i[K] \cap \mathcal{A}_n$). Notice that the following statements are true for each $i \in \mathbb{Z}^+$: $Z_i \in \mathcal{A}_n$, and so $f[Z_i] \in \mathcal{A}_n$; f is one-to-one on $\mathcal{A}_n$, so $|f[Z_i] \cap \mathcal{A}_n| \geq |Z_i \cap \mathcal{A}_n|$; and $Z_{i+1} \subseteq K_{i+1}$, so $f[Z_{i+1}] \subseteq K_i$. It follows from these facts that $f[Z_{i+1}] = Z_i$ for each $i \in \mathbb{Z}^+$. Since the end points of each Z_i lie in $\mathcal{A}_n$, and are hence periodic under f, we see that Z_i is periodic under f for each $i \in \mathbb{Z}^+$. Denote the period of Z_1 by m. Then $f^m[Z_i] = Z_i$ and $Z_i = Z_{\alpha(i,m)}$ for each i. Let $J = \varprojlim\{Z_i, f|_{J^{i+1}}\}$. Since $f[Z_{i+1}] = Z_i$ for each $i \in \mathbb{Z}^+$, we see that $\pi_i[J] = Z_i$ for each such i. Notice that for a given i, it is not necessarily the case that $f|_{Z_i}$ is a Markov map, since its domain and range are different. However, it is not hard to see that one can linearly map each Z_i onto the interval $[0, 1]$, and redefine the bonding maps accordingly to represent J as an inverse limit on $[0, 1]$ with Markov bonding maps. In fact, it can be seen that these new bonding maps will be in $\mathcal{S}^c_{\Phi(K)}$, and so

$J \in \mathcal{M}^c_{\Phi(K)}$. Our goal is to establish a similar result for K, though the proof for K will be more involved since for a given i, the end points of $\pi_i[K]$ are not necessarily in $\mathcal{A}_n$. If they were, then it would be the case that $\pi_i[K] = Z_i$ for each i, and hence K would equal J.

As we continue, it will be useful for us to represent X as the inverse limit of a sequence of permutation maps rather than as the inverse limit with a single bonding map chosen from $\mathcal{S}^c_n$. Since f is a mapping in S^c_n, it can be written as $f = f_1 \circ f_2 \circ ... \circ f_k$, where $f_1, f_2, ..., f_k \in \mathcal{S}_n$. For each $i > k$, set $f_i = f_{\alpha(i,k)}$. In other words $\{f_i\}_{i=1}^{\infty} = \{f_1, f_2, ..., f_k, f_1, f_2, ..., f_k, f_1, ...\}$. By the Subsequence Theorem 3.22, X is homeomorphic to $X^* = \varprojlim\{[0,1], f_i\}$. Further, it can be seen that the mapping $\varphi : X^* \to X$ given by $\varphi((x_1, x_2, ...)) = (x_1, x_{k+1}, x_{2k+1}, x_{3k+1}, ...)$ is a homeomorphism between these two spaces. Define subcontinua K^* and J^* of X^* by $K^* = \varphi^{-1}(K)$ and $J^* = \varphi^{-1}(J)$. From this point on, we will work exclusively with X^* and its subcontinua. Our goal will be to show that K^*, and thus K, is representable as an inverse limit on $[0,1]$ with bonding map chosen from $\mathcal{S}^c_N$ for some N. For each $i \in \mathbb{Z}^+$, let $K_i = \pi_i[K^*]$ and $J_i = \pi_i[J^*]$. Notice that the following statements regarding K_i and J_i are true for each $i \in \mathbb{Z}^+$: (1) $|J_i \cap \mathcal{A}_n| = |K_i \cap \mathcal{A}_n| = |Phi(K)$, (2) $J_i \in \mathcal{I}(\mathcal{A}_n)$, and (3) $J_{(i-i)k+1} = Z_i = Z_{\alpha i,m}$. It follows from this last observation that $Z_1 = J_1 = J_{mk+1} = J_{2mk+1} = ...$, and in fact $J_i = J_{\alpha(i,mk)}$ for each positive integer i.

Next, we claim that for each such i, the number of end points that J_i and K_i have in common is greater than or equal to the number of end points that J_{i+1} and K_{i+1} share. It is clear that if J_{i+1} and K_{i+1} have two end points in common, then $J_{i+1} = K_{i+1}$ and so J_i and K_i are equal, and obviously share two end points. If J_{i+1} and K_{i+1} share exactly one end point, then $K_{i+1} - J_{i+1}$ has one component, on which f_i is monotone, and thus contains no turning points. It follows that the image of the end point shared between J_{i+1} and K_{i+1} is a common end point of J_i and K_i. The claim is obvious when J_{i+1} and K_{i+1} share no end points. Having established our claim, we may then conclude that there must exist a positive integer N such that the number of end points J_i and K_i have in common is constant for all $i > N$. We then lose no generality in assuming that the number of shared end points is constant for all i. We will consider three cases:

A. For each $i \in \mathbb{Z}^+$, J_i and K_i share no end points.

B. For each $i \in \mathbb{Z}^+$, J_i and K_i share exactly one end point.

C. For each $i \in \mathbb{Z}^+$, J_i and K_i share two end points.

Case C is the simplest to deal with. In this case, $J_i = K_i$ for each i, from which it follows that $K^* = J^* \subseteq \mathcal{S}^c_{\Phi(K)}$. The proofs for Case A and Case B are very similar. Since either of these proofs is long on its own, we will consider both simultaneously. As we proceed, we will clearly state any differences in the proofs of the two cases.

For each $i \in \mathbb{Z}^+$, let $I_i = [c_i, d_i]$ be the smallest member of $\mathcal{I}(\mathcal{A}_n)$ containing K_i. For later use, note that since $J_i = J_{\alpha(i,mk)}$ for each i, we also have that $I_i = I_{\alpha(i,mk)}$. Now, recall that $|J_i \cap \mathcal{A}_n| = |K_i \cap \mathcal{A}_n|$ for each i, and then notice the following:

A. If J_i and K_i share no end points, then $|I_i \cap \mathcal{A}_n| = |J_i \cap \mathcal{A}_n| + 2 = \Phi(K) + 2$.

B. If J_i and K_i share exactly one end point, then $|I_i \cap \mathcal{A}_n| = |J_i \cap \mathcal{A}_n| + 1 = \Phi(K) + 1$.

Since $K_i \subseteq I_i$ for each positive integer i, we see that $K^* \subseteq \left(\prod_{i=1}^{\infty} I_i\right) \cap X$. However, it is not necessarily the case for any given i that $f_i[I_{i+1}] \subseteq I_i$, and so $\varprojlim\{I_i, f_i|_{I_{i+1}}\}$ may not be defined. Our goal is to define for each i a surjective mapping $\psi_i : I_{i+1} \to I_i$ which will be based on f_i. We will then argue based on the definitions of our new mappings, that K^* is homeomorphic to $\varprojlim\{I_i, \psi_i\}$, and that $\varprojlim\{I_i, \psi_i\}$ is in turn homeomorphic to $\varprojlim\{[0,1], g\}$, where $g \in \mathcal{S}^c_N$ for some positive integer N.

We now begin working toward the definition of the aforementioned functions ψ_i. For each $i > 2$, let $W_i = \{x \in \mathcal{I}_i : f_{i-1}(x) \in I_{i-1}\}$. So that we have W_i defined for each positive integer, we will arbitrarily set $W_1 = [r_1, s_1] = I_1$. We see from Lemma 4.18 that for each $i \in \mathbb{Z}^+$, W_i is an interval and $f_i[W_{i+1}] = I_i$. For each $i \in \mathbb{Z}^+$, we define $\psi_i : I_{i+1} \to I_i$ as follows: First, set $\psi_i(x) = f_i(x)$ for all $x \in W_{i+1}$. Then, extend ψ to the rest of I_{i+1} by setting it constant on each component of $I_{i+1} - W_{i+1}$ in such a way that ψ is continuous. Let $L = \varprojlim\{I_i, \psi_i\}$. Since $K_i \subseteq W_i$ for each i, it follows that $K^* \subseteq \left(\prod_{i=1}^{\infty} W_i\right) \cap X^*$. But since f_i and ψ_i agree on W_{i+1} for each i, we see that $\left(\prod_{i=1}^{\infty} W_i\right) \cap X^* = \left(\prod_{i=1}^{\infty} W_i\right) \cap L$, and hence that $K^* \subseteq L$. Our next step is to show that K^* is homeomorphic to L.

Notice that since $I_i = I_{\alpha(i,mk)}$ for each i, we may define a mapping $\psi : I_1 \to I_1$ by $\psi = \psi_1 \circ \psi_2 \circ ... \psi_{mk}$. Also recall that for each i, $f_i = f_{\alpha(i,k)}$ from which it immediately follows that $f_i = f_{\alpha(i,mk)}$. Combining the facts that $I_i = I_{\alpha(i,mk)}$ and $f_i = f_{\alpha(i,mk)}$, we can see that $\psi_i = \psi_{\alpha(i,mk)}$. Therefore L is homeomorphic to $\tilde{L} = \varprojlim\{I_1, \psi\}$, by the Subsequence Theorem 3.22. Notice that J_1 is fixed under ψ, and that, again by the Subsequence Theorem, J^* is homeomorphic to $\tilde{J} = \varprojlim\{J_1, \psi|_{J_1}\}$. We wish to show that the decomposition space $\tilde{L}/\tilde{J}$ is an arc. We will do so by showing that the mapping $\psi/J_1 : I_1/J_1 \to I_1/J_1$ is monotone, and then applying Theorem 4.13. Notice that in either Case A or Case B, for any given i, if S is a subinterval of I_i containing J_i, then $\psi_i^{-1}(S)$ is a subinterval of I_{i+1} containing J_{i+1}. From this we may gather that $\psi^{-1}(J_1)$ is a subinterval of I_1, and thus is connected. Additionally, if T is a (possibly degenerate) subinterval of I_{i+1} which does not intersect J_{i+1}, then $\psi_i^{-1}(T)$ is a subinterval of I_{i+1} which does not intersect J_{i+1}. By repeatedly applying this result, we see that if $x \in I_1 - J_1$, then $\psi^{-1}(x)$ is connected. Since $\psi^{-1}(J_1)$ is connected, as is $\psi^{-1}(x)$ for each $x \in I_1 - J_1$, we see that ψ/J_1 is monotone, and so $\varprojlim\{I_1/J_1, \psi/J_1\}$ is an arc. It follows then from Theorem 4.13 that $\tilde{L}/\tilde{J}$ is an arc as well, and thus, so to is L/J^*. Notice that K^* is a subcontinuum of L properly containing J^*, and so K^*/J^* is itself a nondegenerate arc. Let $P : L \to L/J^*$ be the natural map which assigns each point $x \in L$ to the unique member of the decomposition L/J^* which contains x. We now consider cases A and B separately.

A. Assume that for each $i \in \mathbb{Z}^+$, the intervals J_i and K_i share no end points. It follows that J_i and I_i share no endpoints. For each $i \in \mathbb{Z}^+$, it can be seen from the definition of the mapping ψ_i that the end points of the interval I_{i+1} are mapped onto the endpoints of the interval I_i. There are then points a and b in L such that each $i \in \mathbb{Z}^+$, the end points of I_i are the points $\pi_i(a)$ and $\pi_i(b)$. It can be seen that the singletons $\{a\}$ and $\{b\}$ are the end points of the arc L/J^*, and so J^* is a cut point of the arc L/J^*. Let C_1 and C_2 be the components of $L/J^* - \{J^*\}$. Let $R_1 = P^{-1}(C_1)$ and $R_2 = P^{-1}(C_2)$. It is clear that $L = J^* \cup R_1 \cup R_2$, and that J^*, R_1, and R_2 are pairwise disjoint. When restricted to either R_1 or R_2, the natural map P is a homeomorphism, and so R_1 and R_2 are homeomorphic to C_1 and C_2, and each of these spaces is a topological ray. Each of the rays R_1 and R_2 spirals down to some portion of J^*, or in other words, $\mathrm{cl}(R_1) - R_1 \subseteq J^*$ and $\mathrm{cl}(R_2) - R_2 \subseteq J^*$. Using a similar argument, one can see that the point J^* of the decomposition space K^*/J^* is a cut point of the arc K^*/J^*. It follows that the subcontinuum K^* of L contains J^* and intersects each of the rays R_1 and R_2. It can be seen that any subcontinuum of L which contains J^* and a portion of each of the rays R_1 and R_2 must be homeomorphic to L, and in particular that K^* and L are homeomorphic.

B. In this case, J_i and I_i have an end point in common for each positive integer i, and so we can see that J^* is an end point of the arc L/J^*. Using an argument similar to that used in Case A, we can see that $L = R \cup J^*$, where R is a topological ray spiralling down to some subcontinuum of J^*. We can also see that since K^* is a subcontinuum of L properly containing J^*, K^* is homeomorphic to L.

We have established our claim that K^* is homeomorphic to L. We have left to show that $L \in \mathcal{M}_N^c$ for some positive integer N.

Recall that the number of points of $\mathcal{A}_n$ which lie in I_i does not vary with i. Let $N = |I_1 \cap \mathcal{A}_n|$. Notice that in Case A, we have that $N = \Phi(K) + 2$, whereas in Case B, we have $N = \Phi(K) + 1$. It is our goal to show that L is homeomorphic to $\varprojlim\{[0,1], g\}$ for some $g \in \mathcal{S}_N^c$. For each i, let h_i be a linear homeomorphism from I_i onto $[0,1]$. Of course it is the case that the end points of I_i are mapped onto $\{0,1\}$ for each i, but it is also the case that the N points of $I_i \cap \mathcal{A}_n$ are mapped onto $\mathcal{A}_N$. For each positive integer i, let $\xi_i : [0,1] \to [0,1]$ be given by $\xi_i = h_i \circ \psi_i \circ h_{i+1}^{-1}$. The homeomorphisms $h_1, h_2, ...$ induce a limit homeomorphism h between L and $\varprojlim\{[0,1], \xi_i\}$. It can readily be seen from the definition of ξ_i and h_i that for each each $i = 1, 2...$, the mapping ξ_i permutes the members of $\mathcal{A}_N$, and is monotone on each component of $[0,1] - \mathcal{A}_N$. Thus, for each i, ξ_i is a Markov map which follows the same pattern as some permutation map $g_i \in \mathcal{S}_N$. Since $I_i = I_{\alpha(i,mk)}$ for each i, we see that $h_i = h_{\alpha(i,mk)}$ for each i. Furthermore, since $\psi_i = \psi_{\alpha(i,mk)}$ for each i, we may conclude that $\xi_i = \xi_{\alpha(i,mk)}$ and $g_i = g_{\alpha(i,mk)}$ for each i. Let $\xi = \xi_1 \circ \xi_2 \circ ... \circ \xi_{mk}$ and $g = g_1 \circ g_2 \circ ... \circ g_{mk}$. Then $g \in \mathcal{S}_N^c$. It follows from Theorem 3.39 that $\varprojlim\{[0,1], \xi\}$ is homeomorphic to $\varprojlim\{[0,1], g\}$. Using the Subsequence Theorem 3.22, we see that $\varprojlim\{[0,1], \xi\}$ is homeomorphic to $\varprojlim\{[0,1], \xi_i\}$ which is in turn homeomorphic to L. We previously established that L was homeomorphic to K^*, and hence to K. Therefore, we have that K is homeomorphic to $\varprojlim\{[0,1], g\}$ where $g \in \mathcal{S}_N^c$, which completes the proof.

■

4.4. KELLEY CONTINUA

In this subsection, we address the occurrence of Kelley continua as inverse limits on intervals using permutation maps. In Theorem 4.20, we provide an affirmative answer to Charatonik's question as to whether each permutation map generates a Kelley continuum in the inverse limit. In fact, we show that for a fixed $n \geq 2$, the inverse limit with any sequence of maps from $\mathcal{S}_n$ will be a Kelley continuum.

Theorem 4.20. *Let $n \geq 3$ be an integer, and let $X = \varprojlim\{[0,1], f_i\}$ where $f_i \in \mathcal{S}_n$ for each positive integer i. Then X is a Kelley continuum.*

Proof. Let $p \in X$, K be a subcontinuum of X containing p, and $\varepsilon > 0$. We wish to find a $\delta > 0$ such that if $q \in X$ satisfying $d(p,q) < \delta$, then there exists a subcontinuum L of X containing q such that $\mathcal{H}(K,L) < \varepsilon$. For each positive integer i, we will set $K_i = \pi_i[K]$ and $p_i = \pi_i(p)$. Let $\mathcal{A}_n$ be as defined in Definition 4.1. By applying the Subsequence Theorem 3.22, we shall assume that $|K_i \cap \mathcal{A}_n| = \Phi(K)$ for each positive integer i. Let N and ε_N be as guaranteed by Lemma 3.25. Let $0 < \eta < \varepsilon_N$ be such that if J is a subinterval of $[0,1]$ satisfying $K_N \subseteq J$ and $\mathcal{H}_N(J, K_N) < \eta$, then $|J \cap \mathcal{A}_n| = \Phi(K)$. Now let δ be a positive real number such that if $x, y \in X$ with

$d(x,y) < \delta$, then $d_N(\pi_N(x), \pi_N(y)) < \eta$, and let $q \in X$ such that $d(p,q) < \delta$. For each positive integer i, let $q_i = \pi_i(q)$. We will eventually define a subcontinuum L of X, containing q, by $L = \varprojlim\{L_i, f|_{L_{i+1}}\}$ where each L_i will be a subinterval of $[0,1]$ containing q_i. We wish to define the L_i's in such a way that $f|_{L_{i+1}}$ is surjective for each i.

Let L_N be the interval irreducible about $K_N \cup \{q_N\}$ and let $L_i = f_{i,N}[L_N]$ for all $1 \leq i < N$. We will define L_i for $i > N$ inductively. For each such i, we will consider one of two cases in defining L_{i+1}. To begin the process, notice that since $\mathcal{H}_N(K_N, L_N) < \eta$, we have that $|L_N \cap \mathcal{A}_n| = \Phi(K)$. If $\Phi(K) = 0$, proceed to Case 1 with $i = N$, otherwise proceed to Case 2 with $i = N$.

(Case 1) It follows from the definition of a permutation map that if $L_i \cap \mathcal{A}_n = \emptyset$, then every component of $f_i^{-1}(L_i)$ maps onto L_i under f_i. Let L_{i+1} be the component of $f_i^{-1}(L_i)$ which contains q_{i+1}. Notice that $L_{i+1} \cap \mathcal{A}_n = \emptyset$, and repeat Case 1 with i incremented by 1.

(Case 2) In this case, L_i is irreducible about $K_i \cup \{q_i\}$ and $|L_i \cap \mathcal{A}_n| = \Phi(K)$. Since f_i is one-to-one on $\mathcal{A}_n$, we know that $|f_i^{-1}(L_i) \cap \mathcal{A}_n| = |L_i \cap \mathcal{A}_n| = \Phi(K)$. Since $K_i \subseteq L_i$, we can see that $K_{i+1} \subseteq f_i^{-1}(L_i)$. The continuum K_{i+1} must be contained in a single component of $f_i^{-1}(L_i)$. Since $|K_{i+1} \cap \mathcal{A}| = \Phi(K) = |f_i^{-1}(L_i) \cap \mathcal{A}|$, it is the case that if C is a component of $f_i^{-1}(L_i)$, then either $C \cap \mathcal{A}_n = \emptyset$ or $K_{i+1} \subseteq C$. In particular, let C be the component of $f_i^{-1}(L_i)$ which contains q_{i+1}. If $C \cap \mathcal{A}_n = \emptyset$ then it follows from the definition of a permutation map that $f_i[C] = L_i$. In this situation, let $L_{i+1} = C$ and proceed to Case 1 with i incremented by 1. If $C \cap \mathcal{A}_n \neq \emptyset$, then $K_{i+1} \subseteq C$. In this case, let L_{i+1} be the interval irreducible about $K_{i+1} \cup \{q_{i+1}\}$. From the fact that $L_{i+1} \subseteq C \subseteq f_i^{-1}(L_i)$, and L_i is irreducible about $K_i \cup \{q_i\}$, it follows that $f_i[L_{i+1}] = L_i$. It also follows that $|L_{i+1} \cap \mathcal{A}_n| = \Phi(K)$. Next, repeat Case 2 with i incremented by 1.

Continuing this process, we define L_i for every positive integer, and then we set $L = \varprojlim\{L_i, f_i|_{L_{i+1}}\}$. It is the case that $f_i|_{L_{i+1}} : L_{i+1} \to L_i$ is surjective for each i, and so it follows that $\pi_i[L] = L_i$ for all i. Notice that $q \in L$, and since

$\mathcal{H}_N(K_N, L_N) < \eta < \varepsilon_N$, Lemma 3.25 tells us that $\mathcal{H}(K, L) < \varepsilon$. Hence, we conclude that X is a Kelley continuum. ■

4.5. AN EXAMPLE

In this section, we will provide an example of the process we have described for determining the inverse limit generated by a map $f_\sigma \in \mathcal{S}_n^c$. Although our specific example will be generated by a map $f \in \mathcal{S}_{14}$, the same technique can be applied when the bonding map is given as the composition of permutation maps rather than a single such map.

Let f_σ be the permutation map determined by a permutation $\sigma \in S_{14}$, which is represented in cycle notation as $\sigma = (1\ 7)(2\ 9)(3\ 8)(4\ 10)(5\ 14\ 11\ 6\ 13\ 12)$. Notice that in light of Theorem 3.37 (Raines's Theorem), we may, for the sake of convenience, think

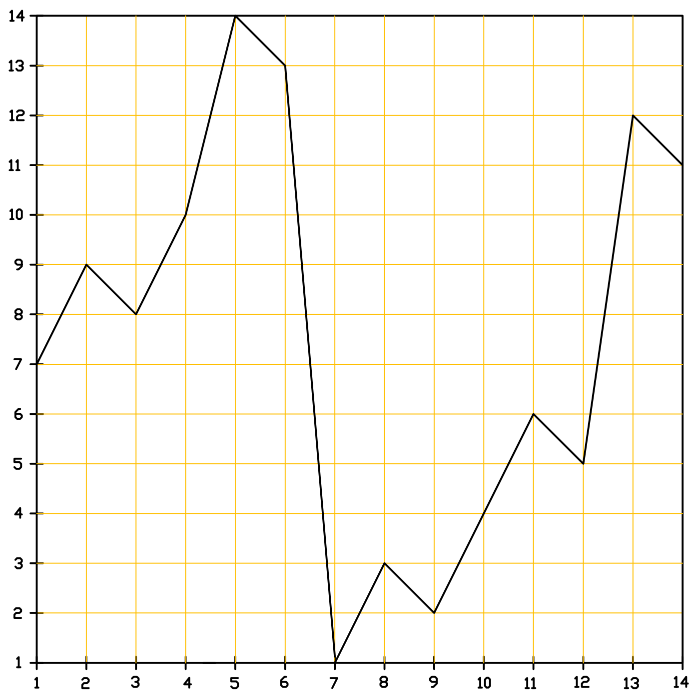

Figure 4.5. f_σ for $\sigma = (1\ 7)(2\ 9)(3\ 8)(4\ 10)(5\ 14\ 11\ 6\ 13\ 12)$

of f_σ as a mapping from $[1, 14]$ onto itself instead of $[0, 1]$. In this case, the Markov partition for f_σ will be given by $\{1, 2, ..., 14\}$. The graph of f_σ is pictured in Figure 4.5. Our goal is to determine the space X which is generated by f_σ.

Observe that the points on the orbit (4 10) are both inessential, so we may remove these points from the Markov partition. Removing these points and then applying Raines' Theorem, we see that we can represent X as the inverse limit with bonding map $f_\rho \in S_{12}$ where $\rho = (1\ 6)(2\ 8)(3\ 7)(4\ 12\ 9\ 5\ 11\ 10)$. The graph of f_ρ (or at least a mapping with is equivalent for our purposes) is pictured in Figure 4.6. Despite the fact that they are homeomorphic, X and $\varprojlim\{[1, 12], f_\rho\}$ are not the "same" space. Nonetheless, to facilitate our discussion, we will now use X to refer specifically to $\varprojlim\{[1, 12], f_\rho\}$.

We can see that each of the 12 points in $\mathcal{A}_n$ is essential with respect to f_ρ, and so

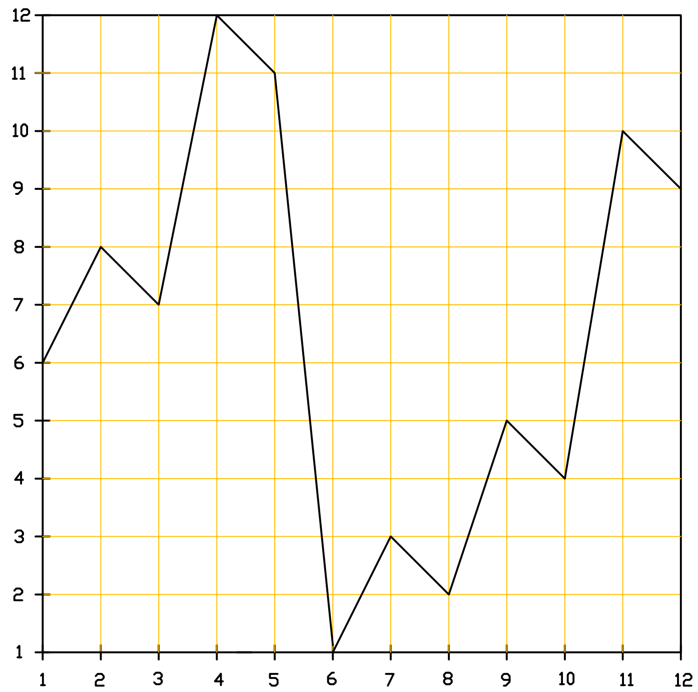

Figure 4.6. f_ρ for $\rho = (1\ 6)(2\ 8)(3\ 7)(4\ 12\ 9\ 5\ 11\ 10)$

we may conclude from Corollary 4.10 that X has exactly 12 end points in the classical sense. The next step in the process of determining X is to identify members of $\mathcal{I}(A_n)$ which are periodic under f_ρ. Notice that under iteration of f_ρ the interval $[1,3]$ is mapped onto $[6,8]$, which is mapped back onto $[1,3]$. Also, we have that $[4,5]$ is taken to $[11,12]$, which goes to $[9,10]$, which is mapped back onto $[4,5]$. We can see then that there are two subcontinua K_1 and K_2 of X such that for each $i \in \mathbb{Z}^+$ and $j = 1, 2$, the projection $\pi_i[K_j]$ is either $[1,3]$ or $[6,8]$. Similarly, there are three subcontinua L_1, L_2, and L_3 of X such that for each $i \in \mathbb{Z}^+$ and $k \in \{1,2,3\}$, the projection $\pi_i[L_k]$ is $[4,5]$, $[9,10]$, or $[11,12]$. Considering the mappings f_ρ^2 and f_ρ^3 we can see that K_1 and K_2 are homeomorphic copies of the familiar $\sin(\frac{1}{x})$ curve, and L_1, L_2, and L_3 are arcs. Also, notice that the shift homeomorphism on X swaps K_1 and K_2 and cycles the arcs L_1, L_2, and L_3.

We do not yet know where these specific subcontinua lie in X. In fact, we don't have a clear picture of the structure of X outside of these subcontinua. However, we can see by way of Theorem 4.17 that each of these subcontinua is terminal in X.

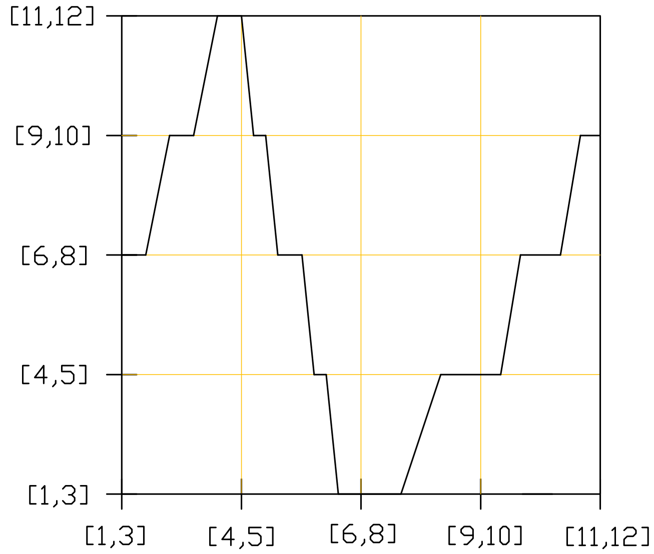

Figure 4.7. The mapping $f_\rho/\mathcal{G}$

Notice also that if p is an end point of any of these terminal subcontinua, then p is an end point of X in the classical sense. It is clear that each of K_1 and K_2 has 3 end points, while each of L_1, L_2, and L_3 has two. This accounts for each of the 12 end points in the classical sense which are present in X.

Let $X/\mathcal{D}$ be the upper semi-continuous decomposition space obtained from X by shrinking each of K_1, K_2, L_1, L_2, and L_3 to separate points. Additionally, let $[1,12]/\mathcal{G}$ be the interval obtained by shrinking each of the 5 periodic members of $\mathcal{I}(\mathcal{A}_n)$ which were observed in the previous paragraph to separate points. Let $f_\rho/\mathcal{G}$ be the mapping from $[1,12]/\mathcal{G}$ to itself determined by f_ρ in the manner described in Section 3.5. The graph of $f_\rho/\mathcal{G}$ is shown in Figure 4.7.

Theorem 4.13 tells us that $X/\mathcal{D}$ is homeomorphic to $\varprojlim\{[1,12]/\mathcal{G}, f_\rho/\mathcal{G}\}$. We use Raines' Theorem to straighten portions of the graph of $f/\mathcal{G}$ without affecting the inverse limit. Doing so, we see that $\varprojlim\{[1,12]/\mathcal{G}, f_\rho/\mathcal{G}\}$ is homeomorphic to $\varprojlim\{[1,5], f_\gamma\}$ where $\gamma \in S_5$ is given by $\gamma = (1\ 3)(2\ 5\ 4)$. The graph of f_γ is pictured in Figure 4.8. Notice that applying Theorem 4.13 and Raines' Theorem in this manner to show that $X/\mathcal{D}$ is homeomorphic to an inverse limit with a permutation bonding map amounts to a applying Theorem 4.15.

It is clear from Corollary 4.12 that $\varprojlim\{[1,5], f_\gamma\}$, and hence $X/\mathcal{D}$, is homeomorphic to an indecomposable arc continuum having exactly 5 end points. The 5 end points of $X/\mathcal{D}$ are precisely the points to which the continua K_1, K_2, L_1, L_2, and L_3 were shrunk. We may now grow these points back to their original continua to arrive back at X. Thus we see that X is constructed from a 5 end point indecomposable arc continuum by replacing two of the end points with terminal $\sin(\frac{1}{x})$ curves, and replacing the other three end points with terminal arcs.

It is worth noting that the shift homeomorphism on $\varprojlim\{[1,5], f_\gamma\}$ swaps the points $(1,3,1,3,...)$ and $(3,1,3,1,...)$ and acts as a 3-cycle on the points $(2,4,5,2,...)$, $(4,5,2,4,...)$, and $(5,2,4,5,...)$. If we imagine growing our continua from the end points of $\varprojlim\{[1,5], f_\gamma\}$ instead of from $X/\mathcal{D}$ (which are of course homeomorphic), the points with period two under the shift homeomorphism would be grown to the $\sin(\frac{1}{2})$

curves, while the other three points would be grown into the arcs.

We should also note that our approach does not provide a method for completely classifying the continua generated by permutation maps, as it does not distinguish between different indecomposable arc continua which have the same number of end points.

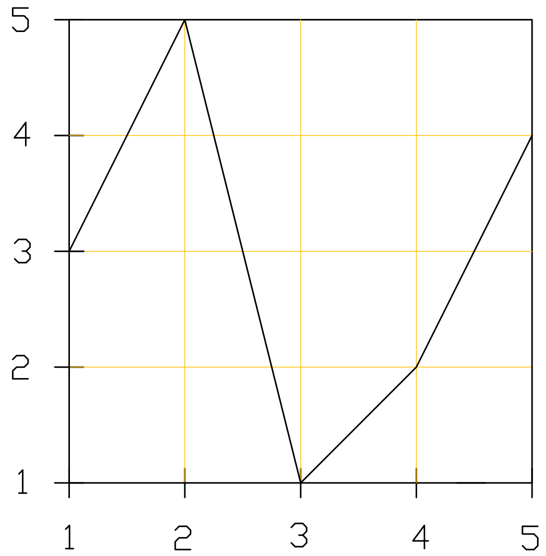

Figure 4.8. f_γ for $\gamma = (1\ 3)(2\ 5\ 4)$

5. LOGISTIC FAMILY

In this section we turn our attention to the logistic family of mappings, given by $f_\lambda = 4\lambda x(1-x)$ where $0 \leq \lambda \leq 1$. This family has been extensively studied by dynamicists, as it is an example of a family of simple functions, some of which exhibit complicated, chaotic dynamics. As the parameter λ increases from 0, the mapping f_λ undergoes a period doubling bifurcation. The parameter values at which the bifurcations occur limit to $\lambda_c \approx .89249...$, which is called the Feigenbaum limit. Beyond this value, there exist uncountably many parameter values λ for which f_λ behaves chaotically. On the other hand, the parameter values for which f_λ has an attracting periodic orbit is an open and dense subset of $[0,1]$ [15]. For information about the dynamics of maps in this family, see, for example, [13, Chapter 1].

Much work has been done in studying inverse limits on $[0,1]$ with a single bonding map taken from the logistic family. See, for example, [3], [14], [24], and [31]. In [3], Barge and Ingram identify the continua that arise as such inverse limits for all values of $\lambda \leq \lambda_c$, and in [24] Ingram shows that each of these continua is a Kelley continuum. Ingram also shows that there are values of $\lambda > \lambda_c$ for which $\varprojlim\{[0,1], f_\lambda\}$ is not Kelley. Our main goal in this section is to establish Corollary 5.3, which states that if f_λ has an attracting periodic orbit, then $\varprojlim\{[0,1], f_\lambda\}$ is homeomorphic to $\varprojlim\{[0,1], f_\sigma\}$ for some permutation map f_σ, and is thus a Kelley continuum. We will do this by first establishing the more general Theorem 5.1. To show that Corollary 5.3 follows from Theorem 5.1, we will need to use the fact that mappings in the logistic family have negative *Schwarzian derivative*, which can be established by direct calculation.

The following theorem provides the main theorem of this section.

Theorem 5.1. *Let $f : [a,b] \to [a,b]$ be a surjective, mapping with N attracting periodic orbits, where N is a positive integer. If each turning point of f in (a,b) lies in the immediate basin of attraction for some point on an attracting orbit, then $\varprojlim\{[a,b], f\}$ is homeomorphic to $\varprojlim\{[0,1], f_\sigma\}$ for some permutation map f_σ.*

Proof. Let $\mathcal{P} = \{p_1, ..., p_m\}$ be the collection of all points in $[a, b]$ which lie on periodic orbits. Note that we are not assuming that $f(p_i) = p_{i+1}$ in general. For each $i = 1, ..., m$, let B_i denote the immediate basin of attraction for p_i. The B_i's are open, pairwise disjoint subintervals of $[a, b]$. For convenience, let α and β be positive integers such that p_α and p_β are respectively the smallest and the largest elements of $\mathcal{P}$.

Since f is surjective, there exist $r, s \in [a, b]$ such that $f(r) = a$ and $f(s) = b$. The points r and s are local extrema, and so each must be either a critical point or an end point. An examination of the possible cases shows that if $d \in \{a, b\}$ then one of the following three conditions must be satisfied: d is a fixed point, d is periodic with period 2, and its orbit is $\{a, b\}$, or d lies on the orbit of some critical point, and hence lies in B_i for some i.

Fix $i = 1, ..., m$. Notice that B_i cannot be the entire interval. If it were, then p_i would be fixed and would attract the entire interval. Since f is surjective, this situation is impossible. The basin B_i can thus contain either a or b, or neither, but not both. Let $a_i = \inf(B_i)$ and $b_i = \sup(B_i)$. Observe the following two facts:

(a) If $a \notin B_i$, then $a_i \in \mathrm{bd}(B_i)$, and hence is not attracted to p_i. In this case, $f(a_i) \neq f(p_i)$, and we may find a point c_i, sufficiently close to a_i, such that c_i is strictly less than every critical point in B_i, and if $x \in [a_i, c_i]$, then $f(x) \neq f(p_i)$.

(b) If $b \notin B_i$, then $b_i \in \mathrm{bd}(B_i)$, and hence is not attracted to p_i. In this case, $f(b_i) \neq f(p_i)$, and we may find a point d_i, sufficiently close to b_i, such that d_i is strictly greater than every critical point in B_i, and if $x \in [d_i, b_i]$, then $f(x) \neq f(p_i)$.

We now define D_i to be a subinterval of B_i as follows:

1. If $a \in B_i$, then $b \notin B_i$. Let $D_i = [a, d_i]$ where d_i is as in (b) above.
2. If $b \in B_i$, then $a \notin B_i$. Let $D_i = [c_i, b]$ where c_i is as in (a) above.
3. If $a, b \notin B_i$, let $D_i = [c_i, d_i]$ where c_i and d_i are as in (a) and (b).

In the same manner, define D_i for all $i = 1, ..., m$. For the purpose of applying Theorem 3.40, we wish to have intervals satisfying the conditions of D_i, but also such that $f[D_i] \subset D_j$ if $f(p_i) = p_j$. We currently have no such guarantee for our intervals. To correct this, we will define intervals A_i such that $D_i \subset A_i \subset B_i$ and $f[A_i] \subset A_j$ if $f(p_i) = p_j$. To that end, let $\mathcal{Q} \subset \mathcal{P}$ be an attracting orbit of f with period n. Without loss of generality, we will assume that $\mathcal{Q} = \{p_1, ..., p_n\}$ and that $f(p_i) = p_{i+1}$ if $i < n$ and $f(p_n) = p_1$. For each $i, j = 1, ..., n$, let $k_{i,j}$ denote the smallest non-negative integer such that $f^{k_{i,j}}(p_i) = p_j$. For each $i = 1, ..., n$, let $A_i = \bigcup_{j=1}^{n} f^{k_{i,j}}[D_i]$. For any $i = 1, ..., n$, the interval D_i is in the basin of attraction B_i, and so $f^n[D_i] \subset D_i$. From this fact, it follows that $f[A_i] \subset A_{i+1}$ if $i < n$, and $f[A_n] \subset A_1$. In the same manner, define A_i for all $i = 1, ..., m$.

We will now define a mapping $g : [a, b] \to [a, b]$ as follows: If $x \notin \bigcup_{i=1}^{m} A_i$ or if $x \in \mathcal{P}$, then set $g(x) = f(x)$. If $g(a)$ was not defined in the first step, then $a \in A_\alpha$. If this is the case, let $g(a) = f(p_\alpha)$. Similarly, if $g(b)$ remains undefined, then $b \in A_\beta$. If this is the case, let $g(b) = f(p_\beta)$. We conclude the definition of g by extending it linearly to the entire interval $[a, b]$. Note that if $x \in [a, b]$ such that $f(x) \neq g(x)$, then it must be the case that $x \in B_i$ for some i, and that x is not periodic under f or g. Notice also that the definition of g guarantees that $\min\{g(x) : x \in [a, b]\}$ is either a or p_α and similarly $\max\{g(x) : x \in [a, b]\}$ is either b or p_β

The mappings f and g, and the intervals $A_1, ..., A_m$ satisfy the conditions of Theorem 3.40, and so $\varprojlim\{[a, b], f\}$ is homeomorphic to $\varprojlim\{[a, b], g\}$. The mapping g permutes the points in $\mathcal{P}$ and is monotone on $[a, b] - \mathcal{P}$. If the end points a and b are periodic under g, then g is a Markov map, and by applying Theorem 3.37 we can conclude that $\varprojlim\{[a, b], g\}$, and thus $\varprojlim\{[a, b], f\}$, is homeomorphic to $\varprojlim\{[0, 1], f_\sigma\}$ for some permutation map f_σ. Thus, assume that one of the end points is not periodic under g. This leaves us with three cases to consider: (Case 1) a is not periodic under g, but b is, (Case 2) b is not periodic under g, but a is, and (Case 3) neither a nor b is periodic under g.

(Case 1) Assume that a is not periodic under g, but b is. Then $g(a) \neq a$ and $g(b) \neq a$. So, if there is a point $c \in [a, b]$ such that $g(c) = a$, then $a < c < b$. Certainly, such a point c would be a local minimum. We can see from the definition of g that this situation can only occur if $c \in \mathcal{P}$, but this would result in both c and a being periodic under f and g, and hence is a contradiction. Thus, there is no $c \in [a, b]$ such that $f(c) = a$. This means that $\min\{g(x) : x \in [a, b]\} = p_\alpha$. Let J denote the interval $[p_\alpha, b]$. We have shown that $g[[a, b]] = J$, and thus that $\varprojlim\{[a, b], g\}$ is homeomorphic to $\varprojlim\{J, g|_J\}$.

(Case 2) Assume that b is not periodic under g, but a is. In this case, we may proceed in a manner similar to Case 1 to show that $\varprojlim\{[a, b], g\}$ is homeomorphic to $\varprojlim\{J, g|_J\}$, with $J = [a, p_\beta]$ in this case.

(Case 3) Assume that neither a nor b is periodic under g. As in Case 1, if $x \in g^{-1}(a) \cup g^{-1}(b)$, then $x \in \{a, b\} \cup \mathcal{P}$. But, if $x \in \mathcal{P}$, then $f(x) \in \{a, b\}$ would be periodic. So, we can see that if $x \in g^{-1}(a) \cup g^{-1}(b)$, then $x \in \{a, b\}$. If $g^{-1}(a)$ and $g^{-1}(b)$ are both nonempty, then $g^{-1}(a) \cup g^{-1}(b) = \{a, b\}$, from which we can conclude that a and b are both periodic, which is a contradiction. So, $g^{-1}(a) \cup g^{-1}(b)$ contains at most one element, either a or b. First assume that $g^{-1}(a) \cup g^{-1}(b) = \emptyset$. Then $f[[a, b]] = J$, and $\varprojlim\{[a, b], g\}$ is homeomorphic to $\varprojlim\{J, g|_J\}$, where $J = [p_\alpha, p_\beta]$. Now assume that $g^{-1}(a) \cup g^{-1}(b) = \{a\}$. Then $f(a) = b$. In this case, $g^{-1}(a) = \emptyset$, so $f[[a, b]] = [p_\alpha, b]$, and then $f^2[[a, b]] = [p_\alpha, p_\beta]$. Similarly, if $g^{-1}(a) \cup g^{-1}(b) = \{b\}$, then $f^2[[a, b]] = [p_\alpha, p_\beta]$. In either case, $\varprojlim\{[a, b], g\}$ is homeomorphic to $\varprojlim\{J, g|_J\}$, with $J = [p_\alpha, p_\beta]$.

In each of the cases above, $g|_J$ is a Markov map which permutes the elements of its Markov partition, and so we may use Theorem 3.37 to conclude that $\varprojlim\{J, g|_J\}$ is homeomorphic to $\varprojlim\{[0, 1], f_\sigma\}$ for some permutation map f_σ. We have already shown that $\varprojlim\{[a, b], f\}$ is homeomorphic to $\varprojlim\{[a, b], g\}$, which is in turn homeomorphic to $\varprojlim\{J, g|_J\}$. Thus $\varprojlim\{[a, b], f\}$ is homeomorphic to $\varprojlim\{[0, 1], f_\sigma\}$. ■

Corollary 5.2. *Let $f : [a, b] \to [a, b]$ be a surjective mapping with negative Schwarzian derivative, N attracting periodic orbits, and N critical points, where N is a positive integer. Assume also that a and b are not attracted to any of the attracting orbits. Then $\varprojlim\{[a, b], f\}$ is homeomorphic to $\varprojlim\{[0, 1], f_\sigma\}$ for some permutation map f_σ.*

Proof. Theorem 3.11 guarantees that every critical point lies in the immediate basin of attraction for a point on one of the attracting periodic orbits, and so we may apply Theorem 5.1. ■

It is clear from the definition of the logistic family that when f_λ has an attracting orbit, it satisfies the conditions of Corollary 5.2, giving us the following corollary, which is the main result of this section.

Corollary 5.3. *If $f_\lambda : [0,1] \to [0,1]$ is a member of the logistic family which has an attracting periodic orbit, then $\varprojlim\{[a,b], f_\lambda\}$ is homeomorphic to $\varprojlim\{[0,1], f_\sigma\}$ for some permutation map f_σ, and is hence a Kelley continuum.*

As mentioned in the first paragraph of this section, $f_\lambda = 4\lambda x(1-x)$ has an attracting periodic orbit, and hence $\varprojlim\{[a,b], f_\lambda\}$ is Kelley, for an open and dense set of parameter values in $[0,1]$. So, in this sense, parameter values λ for which $\varprojlim\{[a,b], f_\lambda\}$ is not Kelley are rare, though as Ingram has shown in [24], they do exist. It is worth noting that Corollary 5.3 does not characterize those values of λ for which $\varprojlim\{[a,b], f_\lambda\}$ is Kelley. The mapping f_{λ_c} does not have an attracting periodic orbit, but does generate a Kelley continuum as its inverse limit [24].

In the set of parameter values for which f_λ has no attracting periodic orbit, there is a countable dense subset of parameter values λ for which the critical point of f_λ eventually maps on to a repelling periodic orbit. We will apply the following theorem to show that for these values of λ, the continuum generated by f_λ is not a Kelley continuum.

Theorem 5.4. *Let $f : [a,b] \to [a,b]$ be a surjective mapping satisfying the following conditions:*

1. *there is a point $t \in (a,b)$ and a point $c \in [a,b]$ such that $f(c) = f(t) = t$,*
2. *there is a subinterval I of $[a,b]$ containing t in its interior such that $I \subset f[I]$ and if $x \in I$, then $|t-x| \le |t - f(x)|$,*
3. *there is a subinterval J of $[a,b]$ containing c in its interior such that $f[J] \subset I$, $t \in \mathrm{bd}(f[J])$, and f is not constant on any component of $J - \{c\}$.*

Then $X = \varprojlim\{[a,b], f\}$ *is not a Kelley continuum.*

Proof. Let $I_1 = [a,t] \cap I$ and $I_2 = [t,b] \cap I$. Notice that condition 2 implies that $I_1 \subset f[I_1]$ and $I_2 \subset f[I_2]$ or $I_1 \subset f[I_2]$ and $I_2 \subset f[I_1]$. Assume first that $I_1 \subset f[I_2]$ and $I_2 \subset f[I_1]$. We may find intervals $I^* \subset I$ and $J^* \subset J$ satisfying condition 2 and 3 above for f^2, and it is certainly true that $f^2(c) = f^2(t) = t$. Notice also that in this case we have that $I_1 \subset f^2[I_1]$ and $I_2 \subset f^2[I_2]$. Therefore, since $\varprojlim\{[a,b], f\}$ is homeomorphic to $\varprojlim\{[a,b], f^2\}$, it is enough to establish the result for the case when $I_1 \subset f[I_1]$ and $I_2 \subset f[I_2]$. Further, notice that condition 3 implies that $f[J] \subset I_i$ for some $i \in \{1, 2\}$. We may assume without loss of generality that $f[J] \subset I_1$.

Let $\beta \in I_1$ such that $\beta \neq t$ and if J^* is a component of $J - \{c\}$, then $\beta \in F[J^*]$ (see Figure 5.1). We will establish that if L is a subinterval of $[a,b]$ containing c such that $f^N[L] \cap I_2 \neq \emptyset$ for some positive integer N, then there exists a positive integer

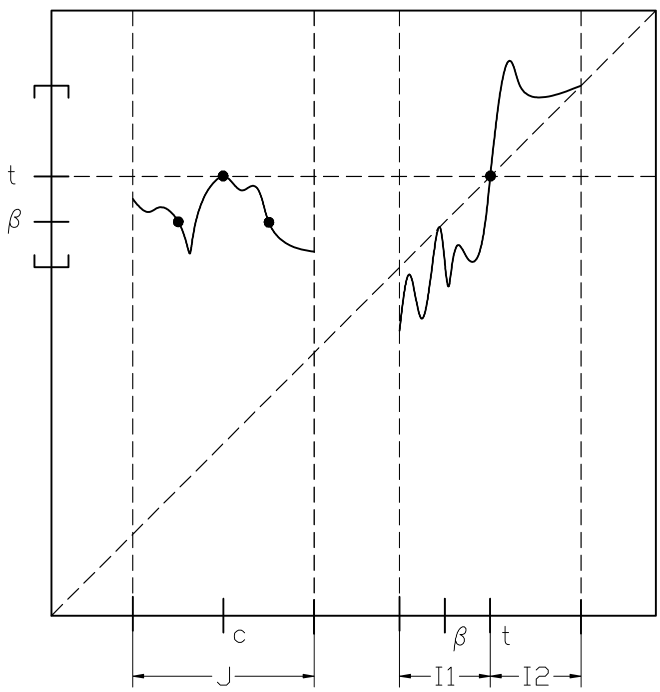

Figure 5.1. Conditions of Theorem 5.4

$M \leq N$ such that $\beta \in f^i[L]$ for all $i \geq M$. We will use this fact later in the proof of the theorem. Assume that L is as described. Observe that $t \in f^i[L]$ for all $i = 1, 2, ...$, and that condition 2 guarantees $[t, \beta] \subset f[[t, \beta]]$. These facts tell us that if $\beta \in f^M[L]$ for some $M = 1, 2, ...$, then $\beta \in f^i[L]$ for all $i \geq M$. If $\beta \notin f[L]$, then it is clear that $L \subset J$, and that $t \in f[L] \subset [t, \beta) \subset I_1$. It readily follows from condition 2 that if $\beta \notin f^i[L]$ for each positive integer i, then $f^i[L] \subset [t, \beta)$ for all such i. This contradicts our assumption that $f^N[L] \cap I_2 \neq \emptyset$ for some positive integer N. Hence, there exists a positive integer $M \leq N$ such that $\beta \in f^i[L]$ for all $i \geq M$.

Let $p = (t, t, t, ...) \in X$ and K be a subcontinuum of X such that $p \in K$ and $\pi_1[K]$ is a nondegenerate subinterval of I_2. Notice that condition 2 of our hypotheses guarantees that such a K does indeed exist. For each $i = 1, 2, ...$, set $K_i = \pi_i[K]$. For each positive integer n, let $p^n \in \pi_n^{-1}(c)$. It is clear that for each $i < n$, $\pi_i(p^n) = t$, and hence it follows that the sequence $\{p^n\}_{n=1}^{\infty}$ converges to p. Let $\{K^n\}_{n=1}^{\infty}$ be a sequence of subcontinua of X such that $p^n \in K^n$ for each n. We will show that $\{K^n\}_{n=1}^{\infty}$ does not converge to K, and hence that X is not Kelley.

Assume to the contrary that $\{K^n\}_{n=1}^{\infty}$ does converge to K. For each pair of positive integers n and i, set $K_i^n = \pi_i[K^n]$. The sequence $\{K_1^n\}_{n=1}^{\infty}$ must converge to K_1, and thus K_1^n must intersect $K_1 \subset I_2$ for all but finitely many n. Without loss of generality, we may assume that $K_1^n \cap I_2 \neq \emptyset$ for each positive integer n. For a given n, we have that $p^n \in K^n$, and hence that $c \in K_n^n$. It then follows from the discussion in the second paragraph that $\beta \in K_1^n$ for all n, and therefore that $\{K_1^n\}_{n=1}^{\infty}$ does not converge to K_1. This provides the desired contradiction to the assumption that $\{K^n\}_{n=1}^{\infty}$ converges to K, and thus completes the proof that X is not a Kelley continuum. ∎

Theorem 5.5. *If $\lambda \in [0, 1]$ is such that $f_\lambda : [0, 1] \to [0, 1]$ has no attracting periodic orbit, and $\frac{1}{2}$ is not periodic under f_λ, but is eventually periodic, then $\varprojlim\{[a, b], f_\lambda\}$ is not a Kelley continuum.*

APPENDIX A

Appendix

The table provided here contains a brief description of the continua in the family $\mathcal{M}_5$. For $2 \leq n \leq 4$, the continua in $\mathcal{M}_n$ are precisely those continua in $\mathcal{S}_5$ which have n or less end points in the classical sense. The results summarized in this table are due to Ingram [22].

Continua in $\mathcal{M}_5$

Continuum	e.p.c.s.*
arc	2
$\sin(\frac{1}{x})$-curve	3
3 endpoint indecomposable arc continuum	3
double $\sin(\frac{1}{x})$-curve	4
pair of $\sin(\frac{1}{x})$-curves	4
ray limiting to a 3 endpoint indecomposable arc continuum	4
4 endpoint indecomposable arc continuum	4
two rays limiting to $\sin(\frac{1}{x})$-curve	5
ray limiting to double $\sin(\frac{1}{x})$-curve	5
ray limiting to pair of $\sin(\frac{1}{x})$-curves	5
union of $\sin(\frac{1}{x})$-curve and double $\sin(\frac{1}{x})$-curve intersecting at one limit bar	5
double ray limiting to an arc on one end, and a 3 endpoint indecomposable arc continuum on other end	5
two rays each limiting to a 3 endpoint indecomposable arc continuum	5
ray limiting to a 4 endpoint indecomposable arc continuum	5
3 endpoint indecomposable arc continuum with a non-endpoint grown into an arc	5
4 endpoint indecomposable arc continuum with an endpoint grown into an arc	5
5 endpoint indecomposable arc continuum	5

* End points in the classical sense

BIBLIOGRAPHY

[1] R. D. Anderson and G. Choquet, *A plane continuum no two of whose nondegenerate subcontinua are homeomorphic: an application of inverse limits*, Proc. Amer. Math. Soc. **10** (1959), 347–353.

[2] J. J. Andrews, *A chainable continuum no two of whose nondegenerate subcontinua are homeomorphic*, Proc. Amer. Math. Soc. **12** (1961), 333–334.

[3] M. Barge and W. T. Ingram, *Inverse limits on* $[0, 1]$ *using logistic bonding maps*, Topology Appl. **72** (1996), 159–172.

[4] D. P. Bellamy, *A tree-like continuum without the fixed point property*, Houston J. Math. **6** (1980), 1–13.

[5] R. Bennett, *On Inverse Limit Sequences*, Master's Thesis, University of Tennessee, 1962

[6] G. Cantor, *Über unendliche lineare punktmannigfaltigkeiten, 5 Fortsetzung*, Math. Ann. **21** (1883), 545–591.

[7] C. E. Capel, *Inverse limit spaces* , Duke Math. J. **21** (1954), 233–245.

[8] H. Cook *Continua which admit only the identity mapping onto nondegenerate subcontinua*, Fund. Math. **60** (1967), 241–249.

[9] H. Cook and W. T. Ingram *Obtaining AR-like continua as inverse limits with only two bonding maps*, Glasnik Mat. **4** (1969), 309–312.

[10] J. J. Charatonik, *The property of Kelley and confluent mappings*, Bull. Polish Acad. Sci. Math. **31** (1983), 375–380.

[11] J. J. Charatonik, *History of Continuum Theory*, The Handbook of the History of General Topology Vol. 2 (C. E. Aull and R. Lowen, eds), Kluwer Academic Publishers, 1998, 703–786

[12] W. J. Charatonik, *Inverse limits of smooth continua*, Comment. Math. Univ. Carolin. **23** (1982), 183–191.

[13] R. Devaney, *An Introduction to Chaotic Dynamical Systems*, Addison-Wesley, 1987

[14] J. Fassett, *Inverse limits and full families*, Topology Appl. **139** (2004), 237–252.

[15] J. Graczyk and G. Światek, *General hyperbolicity in the logistic family*, Annals of Mathematics **146** (1997), 1–52.

[16] G. W. Henderson, *The pseudo-arc as an inverse limit with one binding map*, Duke Math. J. **31** (1964), 421–425.

[17] S. Holte, *Embedding inverse limits of nearly Markov interval maps as attracting sets of planar diffeomorphisms*, Colloquium Mathematicum **68** (1995), 291–296.

[18] A. Illanes and S. B. Nadler, Jr., *Hyperspaces: Fundamentals and Recent Advances*. Monographs and Textbooks in Pure and Applied Mathematics, 216. New York: Marcel Dekker, Inc., 1999.

[19] W. T. Ingram, *An atriodic tree-like continuum with positive span*, Fund. Math. **77** (1872), 99–107.

[20] W. T. Ingram, *Periodicity and Indecomposability*, Proc. Amer. Math. Soc. **123** (1995), 1907–1916.

[21] W. T. Ingram, *Inverse limits*, Sociedad Matemtica Mexicana, Mxico, 2000

[22] W. T. Ingram, *Invariant sets and inverse limits*, Topology Appl. **126** (2002), no. 3, 393–408.

[23] W. T. Ingram, *Inverse limits and a property of J. L. Kelley, I*, Bol. Soc. Mat. Mexicana (3) **8** (2002), no. 1, 83–91.

[24] W. T. Ingram, *Inverse limits and a property of J. L. Kelley, II*, Bol. Soc. Mat. Mexicana (3) **9** (2003), no. 1, 135–150.

[25] W. T. Ingram, *Families of inverse limits on* $[0, 1]$, Topology Proc. **27** (2003), no. 1, 189–201.

[26] H. Kato, *Generalized homogeneity of continua and a question of J. J. Charatonik*, Houston J. Math. **13** (1987), 51–63.

[27] J. L. Kelley, *Hyperspaces of Continua*, Trans. Amer. Math Soc. **52** (1942), 22–36.

[28] K. Kuratowski, *Topology, vol. 1*, Academic Press and PWN, 1966

[29] K. Kuratowski, *Topology, vol. 2*, Academic Press and PWN, 1968

[30] W. S. Mahavier, *A chainable continuum not homeomorphic to an inverse limit on* $[0, 1]$ *with only one bonding map*, Proc. Amer. Math Soc. **94** (1985), 167–172.

[31] J. D. McBryde Jr., *Inverse limits on arcs using certain logistic maps as bonding maps*, Master's Thesis, University of Houston (1987)

[32] S. B. Nadler Jr., *Continuum Theory: An Introduction*, Monographs and Textbooks in Pure and Applied Mathematics, 158. New York: Marcel Dekker, Inc. 1992.

[33] E. Pearl (ed.), *Open Problems in Topology, II*, Elsevier BV., Amsterdam, 2007

[34] B. Raines, *Orbits of turning points for maps of finite graphs and inverse limit spaces*, Proceedings of the Ninth Prague Topological Symposium (2001), 253–263

[35] D. Ryden, *Composant structure in inverse limits of intervals.*, Preprint

[36] R. M. Schori, *A universal snake-like continuum*, Proc. Amer. Math Soc. **16** (1965), 1313–1316.

[37] D. Singer, *Stable orbits and bifurcations of maps on the interval*, SIAM J. Appl. Math. **35** (1978), 260–267.

[38] R. W. Wardle, *On a property of J. L. Kelley*, Houston J. Math. **3** (1977), 291–299.

VITA

Robbie Allen Beane was born in Springfield, Missouri on July 31, 1979. He lived in West Plains, Missouri until graduating from West Plains High School in 1997, after which he moved to Rolla, Missouri to begin undergraduate work in mathematics at the University of Missouri - Rolla. He graduated from UMR in May 2002 with a Bachelor of Science degree in applied mathematics. He soon after moved to St. Louis where he worked as a temp for a year before returning to the University of Missouri - Rolla in the fall of 2003 to being graduate work in mathematics. He graduated with a Master of Science degree in May 2005, and began doctoral work. During his time as a graduate student, Mr. Beane was employed by the Department of Mathematics and Statistics as a Graduate Teaching Assistant. His doctoral work was further supported by a Chancellor's Fellowship granted by the University of Missouri - Rolla.