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On contests with complementarities

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ON CONTESTS WITH COMPLEMENTARITIES

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Abstract

In this dissertation, we consider the role of complementarities in contests. In most contests, there is either a single prize available, or multiple prizes whose joint value is simply the sum of the values of the individual prizes. We consider contests involving competitions for multiple objectives whose value depend on the combination of objectives won. These combinations of objectives are the basis for the complementarities we examine. We use contests consisting of multiple subcontests, with the subcontests determining the winner of each objective. The overall contest thus determines which combinations of objectives each player achieves, and thus the overall prize winner.

These complementarities are first established in a game between two players who have different complementarities, such that they must win combinations of subcontests to obtain a prize. Optimal decisions are determined for the players across different structures, with what structures benefit a particular player investigated. We then investigate the play of the game experimentally, seeing how actual players may deviate from optimal play. This game is then brought back a level, where the structure of the competition is itself determined by a game. Finally, we consider a different type of complementarity, where the complementarity occurs in the subcontests themselves, rather than for the overall prize.

Chapter 1: Introduction

Complementarities arise in several areas of economics. This is well established in consumption theory, with the archetypical example consisting of left and right shoes. Owning a number of left shoes provides virtually no benefit unless right shoes are also owned. This issue appears in many combinations of goods.

Such complementarities are often important in market structures. For example, Apple's iPod is virtually useless without a source of digital music. Their iTunes service is more generally useful, but gains a great deal from consumers also having iPods. The sum of the utilities of these two good independently is far less than the total benefit of having both. This complementarity explains much of the success Apple has enjoyed in recent years. Though consumers are able to choose alternatives to both the iPod and iTunes, no other firm is a serious challenger to both of these. Thus no other combination has become as deeply integrated, and thus consumers of competing firms enjoy fewer benefits from complementarity.

Apple is thus able to charge a premium relative to their competitors, with customers willingly paying this for the gains from the complementarity. The ability to charge higher prices has become even further established with the development of Apple's other products such as the iPhone and iPad, along with their App Store. All of these products provide significant complementarities, and thus a consumer who buys one of Apple's products is likely to buy several more.

Other firms, naturally, wish to be able to take advantage of similar complementarities. These complementarities appear to be especially prevalent in the technology fields, perhaps due to consumers being less familiar with the various products. Thus familiarity and interoperability provide greater complementarities to the consumers than occur in other fields. Though cars and tires have very little use without the other, the manufacturers of the two are separate.

We also see complementarities in manufacturing, most evidently with specialized tools. A single hobbyist woodworker will get little use out of a computer controlled lathe. A custom furnishings shop may get enough business to purchase such a specialized tool. An industrial scale wooden furniture manufacturer certainly will have such tools. The complementarity between making chair legs and table legs allows the larger manufacturer to lower their costs. This type of complementarity is also evident in large scale food processors. By separating out farm goods into various components, and using these components separately, they are able to cheaply make a wide variety of goods.[1] Such complementarities in production are a major component in firms making a variety of products, instead of only one type of good.

Complementarities also occur in other environments. For example, developers want to build connected towns, such that the value of a given development depends on neighboring areas. Preservationists on the other hand may desire wilderness corridors for wild animals to roam. These goals are incompatible, and so both sides may attempt to influence a zoning board to gain their preferred outcome. However, the zoning board does not simply award the zoning to the side which spent more time and money on lobbying efforts. As contests have an indivisible prize and often have probabilistic determination of the outcome, the efforts to lobby a zoning board may be usefully described as a contest.

In addition to the amounts spent by the sides on lobbying, the final outcome may depend on other factors. For example, the members of the zoning board may have preferences. These may be a general bias towards one side, or having a particular argument that may be particularly effective at swaying them. These factors are unknown to the sides, however, and thus from their point of view, this is probabilistic.

Similar situations may occur in network security, where one side is attempting to break the connection between two computers which have multiple possible paths, or in patent races, where different firms attempt to collect a set of patents which allow for the manufacturing of the firms' products. The cases share the essential elements of complementarity in collecting winning sets, probabilistic success functions, and exactly one winner.

In this paper we consider several game-theoretic problems involving complementarities. One of the major examples of this is a variant of the Colonel Blotto game. In the Blotto game an officer assigns troops to a number of battlefields, hoping to have a majority of the troops on a majority of battlefields. These games have been extensively studied, starting with Borel (1928). Further work by Gross (1950), along with Wagner (Gross and Wagner 1950), advanced the field, however a correct complete characterization of the equilibrium strategy was only recently discovered by Roberson (2006). Tullock (1980) considered single battlefields won not by simple majority, but rather probabilistically. The probability a force wins a given battlefield is proportional to the number of troops sent to the battle.

[1]Some of this gain comes from the fact that the waste products from one good, for example the cream from skimming milk, is then useable in making something else, perhaps ice cream.

In all the games described, the dependence between cells comes from the players having a fixed pool of resources to distribute across a number of subcontests. We take a different approach by introducing spatial complementarity, with combinations of the subcontests having complementarities. The most similar previous work in this direction comes from Clark and Konrad (2009), who examine a game in which one side needs to win all the subcontests, while the other side needs to win only a single subcontest. Such a game is useful as a model for issues such as terrorist attacks or guerrilla warfare, where a smaller force must fight asymmetric battles.

Golman and Page (2009) extend the original Colonel Blotto game to "General Blotto", in which each battlefield is worth some value, and there are also combinations of battlefields that are also worth various values. The side which obtains a greater total value from the individual and combination battlefields is declared the winner. Our work differs significantly from Golman and Page's in that in our model the different sides have different winning sets. Unlike Clark and Konrad, there is no assumption about one side starting out advantaged. Furthermore, in our game, unspent resources benefit the players, and thus the players have incentive to win with as little effort expended as possible.

In Chapter 2, we examine a model of such complementarities. Our model consists of a contest which is won by the player who is able to connect their two assigned sides of a gird. Connections are formed by winning subcontests to win the four cells of the grid. The grid is arranged such that exactly one player will be able to complete a winning path. This small model in based on the game of Hex. Hex was originally developed separately in the 1940s by Piet Hein and John Nash (Berlekamp, 1982). In the canonical version, play occurs on an 11 $\times$ 11 grid of hexagonal cells. Players alternate claiming ownership of cells, until one player has connected a path between their two opposite edges. The player who accomplishes this wins the game. Due to the nature of the hexagonal grid, this game can not end in a draw, as after all the cells are claimed, one player much have a winning path. An 11 $\times$ 11 grid is too large for our purposes. Such a game contains 121 cells, which makes for a computationally infeasible problem. Thus we will restrict ourselves to a 2 $\times$ 2 game with 4 cells.

In our model, the competition for cells does not merely consist of claiming them as it does in the canonical version. Instead, we use the Tullock Contest Success Function (Tullock, 1980) to determine the winner of each cell. This function, originally developed for a version of the Colonel Blotto problem, gives each player a probability of winning the contest proportional to the amount spent by the player on the contest. Such a function provides an appropriate model for our problem.

Investing time and money attempting to influence a zoning board is not simply a matter of which side spends more winning. More time and effort do, however, make the board more likely to vote in your favor. Thus the Tullock function is reasonable, and given that it is also fairly simple, it makes a good choice.

A contest across four cells may be arranged in several ways. All four cells may be contested simultaneously. Conversely, a sequential structure may be used, in which each cell is contested in a separate round, with the winners of previous rounds being known. There are also intermediate cases, where two or three rounds occur, some with multiple cells being contested simultaneously. All of these variations are considered, and optimal strategies for all computed. The optimal strategies and expected values are then compared. An explanation of what underlies the patterns of these strategies and expected values is then established.

It requires a significant amount of calculations to determine optimal strategies for this game. Thus, in Chapter 3 we ask if players are capable of finding these strategies, and if not, how closely they approximate the optimal solutions. This is done through a laboratory experiment. 72 students from the University of Arkansas were recruited into 12 sessions. In half the sessions the students played a series of path formation games, similar to those in Chapter 2. The other half of the sessions involved a series of similar games, however without any complementarities. The prizes for the games in the two types of sessions were designed such that optimal play would be identical in the four cell games. Each sessions' participants play three different games. First, they simply alternate claiming cells. The first game is done both to familiarize the players with the computer system and to provide a starting stake to the participants.

The focus of our work is in the second game, the 2×2 Hex game with the Tullock Contest Success Function. The players invest their endowment into cells, with the winner of each cell being determined by the Tullock function. Thus we are able to observe how well the participants are able to play the game, with the no complementarity case providing a useful baseline comparison. The final game is played on an expanded 4×4 version of the Hex game. Though computing optimal strategies for this is intractable, we do know how many winning paths go through each cell. We thus compare the rank ordering of average bids versus the rank ordering of the number of winning sets involving a cell to determine if the participants' plays are reasonable.

In Chapter 4, we examine competitions to establish the order in which the subcontests occur. For example, zoning boards may wish to schedule the hearings on the parcels of land they control so as to extract the maximal amount of lobbying expenditures from the developers and preservationists. The scheduling competition adds another possible level of complexity to the games.

There are many possible ways the cells may be divided. A single auctioneer has a simple decision problem, and simply has to choose which ordering provides the maximal expected payoffs. These payoffs are the expected spending by the contestants in the game analyzed in Chapter 2. What order the cells will be contested in becomes more complicated if there are two or three auctioneers. All possible divisions of the cells are examined, and the complete set of equlibria found. These take the form of both pure Nash equilibria and mixed strategy equilibria. We see that the number of auctioneers as well as the distribution of the cells between them impact not only the individual auctioneers' expected payoffs, but also the total expected payoff for all the auctioneers. We then examine the case with four separate auctioneers, which is simpler, as it only consists of a single case. A unique mixed strategy equilibrium is then established. Comparisons between the total expected values for the different distributions and numbers of auctioneers are made. These results are then compared to increasing the number of sellers in a market. Differences between these are noted, and an explanation provided.

There are, of course, other forms of possible complementarities. In Chapter 5 we examine one of these, with the competition being between political parties. In this example, the complementarities arise between campaign issues. Time and money spent campaigning on one issue will also benefit each party in another area. Which issues this applies to will vary depending on the parties' previously established platforms.

Budge (1982) and Petrocik (1996) both build issue based models, based on the 1979 British and 1980 U.S. Presidential elections respectively. Various issues are taken to be more closely identified with a particular party, and campaigning is done to emphasize the issues a party is strong on, while minimizing the importance of their relative weaknesses. Denter (2013) extends this idea into allowing campaigning to both improve how competent a party appears at a given issue, as well as the importance of the issue. Candidates are assumed to have individual strengths and weaknesses in various areas. These strengths and weaknesses determine which areas the candidates spend time and effort on.

Our model takes which pairs of issues have complementarities as given. The way a party discusses economic issues, for example, may also heavily influence how the party is seen on social issues. However, another party, given a different historical framing of these issues may instead see their campaigning on the economy making them seem strong on international affairs. We will treat these framing issues as giving each party a source issue and a destination issue. A portion of the spending on the source issue will also count as being effectively spent on the destination issue, as the way the candidate talks about the source issue also will contain statements about the destination issue. The complementarity involved is quite different from the previous chapters. Though we use the Tullock contest success function as before, the complementarities take the form of additional effective spending on issues. A portion of the spending on one area also counts as spending on another issue area. In the interests of solvability, we restrict our model to having only three issue areas, and to having a fixed ratio of half the spending on the source issue spill over to the destination issue. These restrictions still result in six distinct patterns of spillovers between the two parties.

Finally, in Chapter 6, we summarize the effects of complementarities on contests examined throughout this work. We also lay out a plan for future research on contests with complementarities.

Chapter 2: All-Pay Hex: A Contest with Complementarities

2.1 Introduction

We examine a competition comprised of multiple contests, combinations of which exhibit complementarity. In our game a benefit accrues to a player only by having won one of several winning combinations of contests. Players have different winning combinations, reflecting differing goals, but the combinations are such that exactly one player will always win. The complementarity between contests arises because success in a single contest or set of contests may yield the same payoff as losing every contest but combined with one more contest win may yield overall victory. These factors result in variations in the valuation of individual contests based on the identity and outcomes of contests already decided and the order of contests yet to be played.

The basic structure for this competition is given by the board game Hex. In the canonical form, Hex is played by two players on a 11 $\times$ 11 grid of hexagonal cells. The players are conventionally labeled Black and White. Each player alternates claiming an unclaimed cell of the board. Black's objective is to connect the two black sides of the board with a path of his pieces, while White's is to connect the two white sides of the board with her pieces. The figure below shows a game in progress.

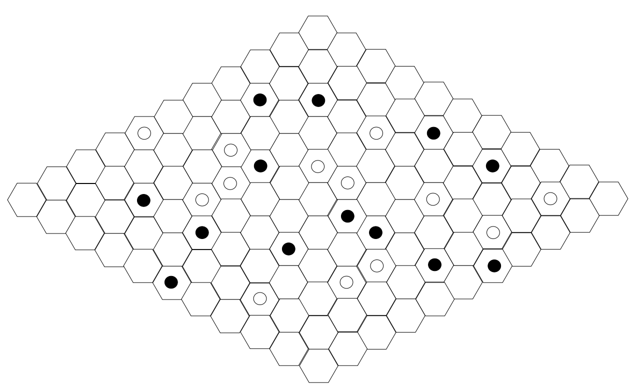

Figure 2.1: **Game of Hex in Progress**

As the game continues, the player who is able to connect her two assigned sides is declared the winner. Draws are impossible. If it becomes impossible for one player to make a connection, it means that all of their routes have been cut off. This in turn implies that there must be a continuous path that wins for the opposing player. We will use this basic structure. However, for the sake of tractability in the multibattle context, we will reduce the grid to being 2 hexagons on each side, as opposed to 11. Also, for simplicity, each player will value the prize that is being contested identically.

Network security provides a good practical example of this type of structure, because data can be routed around compromised servers as long as a connection exists between two nodes, thus avoiding servers which have been hacked or damaged. However, intermediate relay nodes hold no intrinsic value, since value is entirely due to the final connection. Thus we do not need to worry about the value of individual cells, only the completed path. These networks are often geographically dependent, such as wireless relay towers and fibre optic switching stations. Another example of a similar network is a cellular phone system, where towers relay signals, and interference limits the number of towers in an area.

Related work can be found in the literature on Colonel Blotto games (Borel 1921, Borel and Ville 1938, Gross 1950, Gross and Wagner 1950, Friedman 1958).[1] These games feature two players who simultaneously allocate their respective fixed budgets of a resource across n different contests, with the higher allocation in a given contest winning the contest. Players choose their allocations to maximize the expected number of battlefields won. In these early papers linkages between contests arise from the budget constraints; allocation of a unit of the resource to one contest reduces the availability of the resource to other contests. Recently there has been a resurgence of interest in these games with extensions to the cases of asymmetric budgets and a positive opportunity cost of the resource in games with both continuous and discrete strategy spaces (Hart 2008, Kvasov 2007, Laslier 2002, Laslier and Picard 2002, Roberson 2006, and Weinstein 2005). Colonel Blotto games have also been examined under the assumption that the winner of each contest is determined probabilistically by the players' respective allocations according to a Tullock contest success function (Tullock, 1980), with the success function itself having been introduced previously (Tullock, 1975). Contributions employing the Tullock contest success function include Friedman (1958) and Robson (2005).

In addition to linkages between contests that arise through the cost of resource allocation, such as the individual budget constraints of the Colonel Blotto game, there are also linkages that arise through the way in which individual battlefield outcomes are aggregated in determining the players'

[1]See Kovenock and Roberson 2010a for a survey of these and related games.

payoffs. Szentes and Rosenthal (2003 a,b) examine a game in which players simultaneously allocate a resource at constant unit cost to n different battlefields. A player earns a prize of common and known value if he is the higher bidder in m of those battlefields. The special case where n is odd and $m = \frac{n+1}{2}$ is the game in which the player who wins a majority of the contests is the winner. Szentes and Rosenthal solved this game for $n = 3$, but for $n > 3$ this remains an open problem. The corresponding n battlefield majoritarian problem with a generalization of the Tullock contest success function with exponent $\alpha \leq 1$ was examined by Snyder (1989) who obtained some partial results. Klumpp and Polborn (2006) solved the n battlefield game for a Tullock contest success function with exponent $\alpha \leq 1$ more generally, characterizing the nature of the nondegenerate mixed strategy equilibria that arise when there are sufficiently many battlefields that no pure strategy equilibrium exists.

More complex linkages that arise from the way in which battlefield outcomes are aggregated have also been examined. Clark and Konrad (2007) examine a game with n battlefields, a constant unit cost of expenditure, and a Tullock contest success function with exponent $\alpha = 1$, in which one player must win all of the contests in order to win a prize while the other needs only win at least one contest. Kovenock and Roberson (2010b) examine the corresponding game under the assumption that the high bidder in each contest wins the contest. Golman and Page (2009) examine a modified Colonel Blotto game, which they term "General Blotto," that takes the original budget-constrained Colonel Blotto game in which the high bidder wins each contest and adds compound contests formed by taking subsets of the sets of battlefields. In each of these added compound contests, a player's allocation is taken to be the product of the allocations in the battlefields defining the compound contest and the player with the higher such product wins. These contest wins are then added to those of the single battlefields to determine the number of contest won.

Our model is similar in spirit to the models with payoffs determined by a nonlinear aggregation of battlefield outcomes. In our model players allocate resources at a constant and identical unit cost to four battlefields. In the main text we analyze the polar cases with all cells being contested simultaneously and sequentially, with the sequential case done both when the order is known a priori and when it is random. Intermediate cases are also solved in the Appendices. Each battlefield outcome is determined by a Tullock contest success function with $r = 1$ (the lottery contest success function). The battlefields are spatially distributed to correspond with cells in a 2×2 game of Hex and the overall contest winner is the player who wins a configuration of contests that would win the game of Hex. Consequently, multiple combinations of individual contest wins may earn a player the overall prize, but exactly one player will be a winner. Players are risk neutral so that they maximize the expected prize winnings minus the cost of their allocation to the four battlefields.

Like the literature on the Colonel Blotto game, the 2×2 game of All-pay Hex that we examine is meant as a type of "toy model" designed to shed some insight into the strategic considerations that might arise when rivals compete in multiple contests that exhibit linkages. In All-pay Hex these linkages arise because the "prizes" contested exhibit complementarities. Moreover, players have asymmetric preferences over combinations of contest victories. Our examination of equilibrium under different assumptions governing the simultaneity and sequencing of contests aims to shed light on the impact of potential timing options for strategic behavior and payoffs.

The paper is laid out as follows. First, an example is described in Section 3, showing the general method of play. The model is then formally defined in Section 4, and specific illustrative examples detailed in Section 5. Finally, some robust conclusions about the game in general are presented in Section 6, with the mathematical details of the cases in the Appendices.

2.2 A Brief History of Hex

Hex occupies an unusual place in the history of game theory for several reasons. First, one of the two independent inventors of the game in the 1940's was John Nash, while the other was the Danish mathematician Piet Hein. Secondly, this game was proven to have a first mover advantage well before a winning strategy itself was found, even for smaller cases. The proof uses the notion of the strategy stealing argument and proceeds by contradiction. Note that in this game owning a space is always beneficial. Now if the second player had a winning strategy, the first player could copy this winning strategy with the advantage of already having a space. As ties have been ruled out, this guarantees a win for the first player (Berlekamp, 1982).

Hexagons are used instead of squares, as in a square gird, there are pairs of cells which adjoin only at corners. For example, in a checkerboard, the dark squares are only adjacent to light squares on the edges, however the dark squares touch other dark squares at their corners. A checkerboard also illustrates the problem with corner adjacency, as if touching corners are considered adjacent, light squares and dark squares both form paths across the board. However, if corners do not establish adjacency, neither light nor dark squares have a path across the checkerboard. As hexagonal cells never touch at only a corner, this issue does not arise for hexagonal grids, and thus in the end, exactly one player will complete a winning path.

Furthermore, the complexity of this game has proven very difficult (Evan et al., 1976), since determining an optimal strategy is PSPACE-Complete. PSPACE-Complete problems are those where obtaining a solution would require an ideal computer to have memory proportional to a polynomial function of the size of the problem.[2] The size of the problem in the case of Hex would be the size of the grid being played on. Thus the standard game of Hex would require a very large amount of memory to be certain that a perfect strategy has been found, and thus far such a solution remains unknown. Work continues on this game as in Anshelevich (2000) and Campbell (2004), including proposals for computer players employed in the absence of a solution for the general case.

2.3 All Pay Hex: An Illustrative Example

The structure of Hex provides a useful starting point for examining complementarities. Given each player's goal of connecting the opposite sides, winning combinations will vary, and the value of a cell will depend on the use of this cell in creating a winning path. Since the canonical version of Hex is computationally difficult, we will focus on a tractable 2×2 version with four cells being contested.

Our game differs from canonical Hex in that the players play simultaneously. Each player simultaneously commits a resource to the cells currently being contested. The winner of each cell is determined by the Tullock contest success function, given by $P(X) = \frac{X}{X+Y}$, where X and Y are the amounts committed to the cell being contested. After a set of cells has been contested, players observe whether either player has won the game, thereby receiving a prize V. If not, the outcomes are observed and a new subset of cells is contested, repeating this process until a winner of the overall contest is found. These subsets of cells come from taking an ordered partition of the set of cells, with the subset being contested in round n being the elements in the nth subset of the partition.

For the sake of illustration, let the prize for completing a connection be 100, with players simultaneously competing for all four cells. Suppose player X invests 20 in both the North and East cells, and only 1 in West and South, in an effort to have overwhelming force in 2 cells. Meanwhile, Player Y invests 10 in both the North and South, and 5 in the East and West.

We can now calculate the probabilities of X winning each cell. Based on the contest success function, in the North, she has a $\frac{2}{3}$ chance of victory, in the East $\frac{4}{5}$, in the West $\frac{1}{6}$, and in the South

[2]PSPACE-Complete problems are considered to be unsolvable in an amount of time equal to a polynomial function of the size of the problem, however this is not proven.

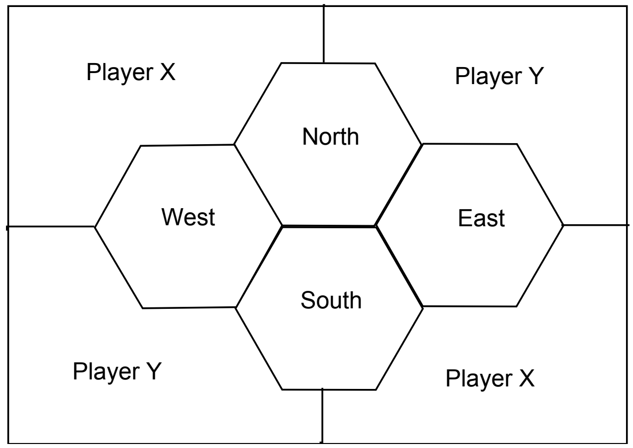

Figure 2.2: **The Cells Being Contested**

$\frac{1}{11}$. Thus player X has a chance of victory given by the sum of his probabilities of winning both the North and East $\frac{2}{3} \cdot \frac{4}{5}$, both the North and South but not the East $\frac{2}{3} \cdot \frac{1}{11} \cdot \frac{1}{5}$, and both the West and South but not the North $\frac{1}{6} \cdot \frac{1}{11} \cdot \frac{1}{3}$. This covers all winning sets for player X without double counting any sets, as all eight winning sets fall into exactly one of these three possibilities.

This gives a total chance of victory for player X of approximately 0.5505, i.e. his expected earnings are 55.05 at a cost of 42, for a net gain of 13.05. Player Y will have a 0.4495 probability of winning, and thus expected winnings of 44.95. However, player Y will have a net gain of 14.95, as she spent only 30.

2.4 The Model

We will now develop our formal model of the 2×2 case of All-Pay Hex. The game is played over the set of cells $A = \{N, S, E, W\}$.

Players: We denote the two risk neutral players in the game by X and Y. It is assumed that the players do not have a budget constraint.

Strategies: Since there are four cells, we will allow for the possibility that they can be contested at different points in time. Before the contest begins, the contest structure R is announced.[3] Let $R = \{R_1, R_2, R_3, R_4\}$ be an ordered partition of A, where R_r is the set of cells being put up for

[3]We also explore the implications of relaxing this assumption and having a random sequence later in the paper. We thank an anonymous referee for suggesting this.

contest in round r and let $|R_r| = C_r$. Since there are only four cells, we assume that the number of rounds cannot exceed this number.

In each round r, each player chooses an C_r-vector of bids with the bid for cell i by player $T \in \{X, Y\}$ being labeled T_i. We will require $T_i \geq 0$ and define $Z_i = \sum_{T=X,Y} T_i$.[4] The winner of each cell will be determined by the Tullock contest function, so that player T will win cell i with probability $\frac{T_i}{Z_i}$.

Payoffs: The payoff function of each player takes into account the expected benefits minus the costs. Each player obtains a identical benefit V from winning the game. We will require some additional notation for our calculations. Let $X^\star$ be the collection of winning sets of cells for player X, and $Y^\star$ for player Y. Specifically, from figure 2 it can be seen that

$$X^\star = \{\{N,E\},\{N,S\},\{S,W\},\{N,S,E\},\{N,S,W\},\{N,E,W\},\{S,E,W\},\{N,S,E,W\}\}$$
$$Y^\star = \{\{N,W\},\{N,S\},\{S,E\},\{N,S,E\},\{N,S,W\},\{N,E,W\},\{S,E,W\},\{N,S,E,W\}\}$$

Note that each player has three minimal winning sets consisting of two cells, along with the supersets of these. The set $\{N, S\}$ is the common minimal winning set for either player. Thus the probability that player X wins the prize is

$$\sum_{\alpha \in X^\star} \left(\prod_{i \in \alpha, j \in A \setminus \alpha} \frac{X_i}{Z_i} \frac{Y_j}{Z_j} \right)$$

Hence the payoff of player X can be written as

$$U_X \left(\{X_k, Y_k\}_{k \in N,S,E,W} \middle| R \right) = \sum_{\alpha \in X^\star} \left(\prod_{i \in \alpha, j \in A \setminus \alpha} \frac{X_i}{Z_i} \frac{Y_j}{Z_j} \right) V - \sum_{i \in A} X_i$$

where the second term is the amount of money that X spends. Note that the utility function for player X is a function of the amounts invested in each cell, which will be influenced by the round structure Σ. Thus we take the amounts invested by each player in the first round, the amounts invested in the second round, and so on until all cells have been accounted for. Similarly, Player Y's payoff function is given by:

$$U_Y \left(\{X_k, Y_k\}_{k \in N,S,E,W} \middle| R \right) = \sum_{\alpha \in Y^\star} \left(\prod_{i \in \alpha, j \in A \setminus \alpha} \frac{Y_i}{Z_i} \frac{X_j}{Z_j} \right) V - \sum_{i \in A} Y_i$$

[4] $T_i = 0$ will be limited to cases in which cell i has been made irrelevant by earlier rounds. This is a technical condition imposed by the Tullock contest success function.

2.5 Solving the Game

Before we analyze the game, we quickly discuss what happens when there are no complementarities between the cells. If there are no complementarities, the values of the cells are independent of one another. Then, if V_N is the common value to the players of winning the North cell, Player X's expected payoff for the contest occurring in the North will be $V_N\left(\frac{X_N}{Z_N}\right) - X_N$, and Player Y's expected payoff will be $V_N\left(\frac{Y_N}{Z_N}\right) - Y_N$. From the first-order conditions we have $V_N\left(\frac{Y_N}{Z_N^2}\right) = V_N\left(\frac{X_N}{Z_N^2}\right) = 0$, which gives us $X_N = Y_N = \frac{V_N}{4}$. Corresponding calculations apply to the East, South, and West cells. Thus each player will spend a quarter of the value of each cell, and have an expected payoff of a quarter of the value of the cell.[5]

From the previous section it should be clear that a complete analysis of the problem requires examining several different cases to identify the effects of the complementarity involved. The role of complementarity also varies depending on whether the winning sets are contested separately or simultaneously. Hence we will focus on two polar cases: those of all cells being contested simultaneously and the four cells being contested sequentially. The intermediate cases, where cells are contested over 2 or 3 rounds, will be examined briefly. Full solutions are provided in Appendix B and summarized in Table 2. The main distinction between the simultaneous and sequential cases is that the sequencing of cells allows for the introduction of asymmetries which are not otherwise possible. While there is only one case when all four cells are contested simultaneously, there are multiple distinct subcases for sequential contests, depending on the order of the cells. In the remainder of the paper we normalize V to 1.

I. Simultaneous Contests

To solve this game, we will simultaneously maximize the expected payoff of each player given by equations 2 and 3. Since the payoff functions are concave in each player's own allocation, this gives us a unique Nash equilibrium.[6] First, observe that there is a great deal of symmetry in the problem, with East for X being strategically equivalent to West for Y, North for X being strategically equivalent to South for Y. Moreover East is strategically equivalent to West for each player, as are North and South.[7] This creates symmetric payoff functions for the two players, leading

[5] Note that this differs from the typical Colonel Blotto problem in the sense that players do not have to win a majority of the cells, rather they would like to win as many cells as possible since each cell provides benefits for winning. Moreover, they do not have to distribute limited total resources amongst the cells.

[6] This is shown in Appendix A.

[7] See Appendix A for more on this.

to symmetric strategies in equilibrium. Thus, our problem only has two variables. The remark below summarizes the main findings for the simultaneous case, and the equilibrium calculations are provided in Appendix A.

Remark 1: When all four cells are contested simultaneously, in the unique Nash equilibrium both players will employ a symmetric strategy of spending $\frac{1}{8}$ on each of the North and South cells, and spending $\frac{1}{16}$ on each of the East and West cells, giving each player a probability of victory of $\frac{1}{2}$ and an expected payoff of $\frac{1}{8}$.

II. Sequential Contests with Pre-Specified Order

We now analyze sequential contests in which the order of play is known to both players before the contest begins. In order to solve for subgame perfect equilibrium strategies, we need to backward induct in the extensive form of the game, and obtain expected payoffs for all possible sequences in the final stage. These are then used in determining expected payoffs in the previous round, and so on to obtain the subgame perfect equilibrium. Given that the two players have differing winning sets, the ordering of cells can bias the game in favor of one player, as winning sets can become available to players at different times. In all cases where either North or South is the first cell contested, the game will still be symmetric between players X and Y. This is not necessarily true if East or West is the first cell contested. If East and West are the first two rounds contested, in either order, there will be no bias.

We now compare the two polar cases in terms of underdissipation. Underdissipation is the situation where the total aggregate expenditure of the players is less than the value of the prize being contested.[8]

Proposition 2.1. *All sequential structures have lower expected dissipation than the simultaneous case.*

Proof. See Appendix A.

The table below summarizes the payoffs of both players for the two polar cases. Detailed calculations of these results can be found in Appendix A.

Before proceeding further, some explanation of the table is in order. In the table, Type refers to the number of cells contested in each round, and Order denotes which cells are contested in each

[8]Similarly, overdissipation occurs when players spend more in the aggregate than the common value of the prize. The possibility of overdissipation in a game with Tullock contest functions has been explored, for instance in (Baye et al., 1999).

Table 2.1: Expected Payoffs under Simultaneous and Sequential Structures

Type	**Order**	$E[U_X]$	$E[U_Y]$
4	Simultaneous (NESW)	.1250	.1250
1-1-1-1	N or S as first round	.1797	.1797
1-1-1-1	E-W or W-E as first and second rounds	.1406	.1406
1-1-1-1	E-N or W-S as first and second rounds	.0731	.2315
1-1-1-1	W-N or E-S as first and second rounds	.2315	.0731

round. In each case, the rounds are separated by dashes. The third row consists of those sequential structures where North or South is the first cell, and the fourth row are those with East and West as the first two rounds, with North and South in either order in the third and fourth rounds. The last row can be understood in a similar manner. $E[U_X]$ and $E[U_Y]$ are the expected benefits of players X and Y respectively.

Remark 2: In a four round contest, the ordering of the *two* cells to be contested in the third and fourth rounds is irrelevant. There are three possible cases after the first two rounds: (i) either one player has won, or (ii) only one of the remaining cells is relevant, in which case the other cell can be ignored, or (iii) one player must win both remaining cells. In the first two cases, as we can ignore irrelevant cells the joint ordering cannot matter. In the third case, one player must win both remaining cells and the order in which the cells are contested is irrelevant. Thus, there is no situation under which the ordering of the third and fourth rounds matter. Note that an intermediate case with only 3 rounds, such as E-W-NS, where East is the first round, West the second round and North and South are to be contested simultaneously in the third round, is not equivalent.[9]

Intuitively, underdissipation occurs in the sequential case due to asymmetries between players in the number of winning subsets that have been covered by the contested cells at the end of each round. Using the knowledge of which cells have been won by which player, the remaining options are spread across a few possible outcomes of remaining cells, allowing for better decisions. The sequencing creates these asymmetries, resulting in underdissipation. Our next two results identify conditions under which expected payoffs in the sequential case are asymmetric and symmetric respectively.

[9]This is shown in Appendix B, and is due to the possibility that both North and South are still relevant after the second round.

Proposition 2.2. *In the sequential case, asymmetric expected payoffs for players X and Y require that the cells in $\{R_1, R_2\}$ form an element of exactly one of $\{X^\bigstar, Y^\bigstar\}$. If $\{R_1, R_2\}$ is an element of $X^\bigstar$, then EV[X] $\leq$ EV[Y] for the entire game, while if $\{R_1, R_2\}$ is an element of $Y^\bigstar$, EV[Y] $\leq$ EV[X] for the entire game.*

Proof. See Appendix A.

This proof proceeds by contradiction. While the formal proof is in Appendix A, here we provide a quick sketch. If North and South constitute the first two rounds, they form a winning set for *both* players, and thus no asymmetry exists. Similarly, if East and West are the first two rounds, they form a winning set for neither player, and again, no asymmetry exists. In the remaining cases, one of North and South, and one of East and West constitute the first two rounds. Each of these possible combinations form a minimal winning set for exactly one player.

We will now consider two distinct examples of sequential structures. This will illustrate the intuition behind this result, as well as demonstrate the fact that this Proposition provides a necessary but not a sufficient condition for payoffs to be asymmetric.

Since players have different winning sets, it may be possible for one player to obtain a winning set in a round in which the opposing player could not have done so. For example, consider the structure E-N-W-S. After the second round, the set of contested cells $\{E, N\}$ is an element of $X^\bigstar$. In this case player X has an expected payoff of approximately 0.0731 before the first round, while player Y has an expected payoff of approximately 0.2315, as shown in Appendix A.[10] Observe that if the player who could have completed a winning set has failed to do so, this player has fewer possible winning sets available to complete in subsequent rounds.

Following our example, if player X has not won after the second round he could not have won *both* the North and East, eliminating his winning sets $\{\{N, E\}, \{N, E, S\}, \{N, E, W\}, \{N, E, S, W\}\}$ from consideration, leaving $\{\{N, S\}, \{S, W\}, \{N, S, W\}, \{E, S, W\}\}$. No such eliminations are possible for player Y, as winning both the North and East does not comprise a winning set for Y. After a given round has been contested, if neither player has won, both players will be able to use the results of the contests up to that point for determining optimal strategies for the remaining rounds. The knowledge of what winning sets are still possible allows for recalculation of the value of each remaining cell by each player. If there is an asymmetry in the winning combinations that can still be possibly completed by the two players, they will have different opportunities to make use of this

[10]This expected payoff for X is the lowest value that exists for a single player in any structure R. The equivalent sequence W-N-E-S provides the lowest expected payoff for Y which is of identical value.

information. Thus, the knowledge of previous results may be more useful to one player, allowing her to obtain a greater expected payoff. Such asymmetries cannot occur in the simultaneous case, and so the expected payoffs must be identical.

Intuitively, two different factors are involved in the sequential case. Being able to form a winning set earlier provides a benefit from the greater chance of an early victory, thus ending the contest with spending in fewer rounds. In contrast, having a greater number of winning sets still available after a round allows a player to take advantage of the knowledge of the asymmetry in player strengths to increase his payoff.

In some situations these factors can balance each other out. For example, the structure N-W-S-E results in identical expected payoffs of approximately 0.1797 for each player, as shown in Appendix A. In this case, if after the second round, player Y has not yet won, then the winning sets $\{\{N,W\},\{N,S,W\},\{N,W,E\},\{N,S,E,W\}\}$ can no longer be obtained by player Y. However, player X does not have any winning sets necessarily eliminated after the second round. As player X would not win with winning both the North and West, all we know is that player X must have won at least one of the two cells contested. In spite of this difference, the expected payoffs for the entire game are the same for each player and we find that the two factors cancel each other out.

The fact that these two forces balance each other perfectly in this case may be due to the fact that although only Player Y has the possibility of victory after the second round, if Player X wins the North, the second round of West becomes irrelevant, and thus Player X can win after the third round (which is only the second round in which players actually expend effort). Thus both players have the ability to win after only two rounds of actual expenditures.

Thus Proposition 2 states only a necessary condition, not a sufficient one. In the sequential cases, the impact of this asymmetry favors the player that does not have a winning set available in the first two rounds. Note that under the intermediate cases, which are those lasting two or three rounds, the asymmetry can favor either player. In some of these cases, the player with the advantage is the one who has a greater number of winning sets which could be completed at the end of round r. For example, $NE-S-W$ favors player X. On the contrary, the sequence $N-ES-W$ favors player Y. This can occur because the advantage of winning early may be greater than the advantage of having more winning sets available after the winners of some subset of cells have been determined.

Proposition 2.3. *If a contest structure contains a round R_i, such that either $\{N, S\} \subseteq R_i$ or $\{E, W\} \subseteq R_i$, then the players have the same expected payoffs.*

Proof. See Table 2 and Appendices A and B.

We can see this result from the expected payoffs of the players shown in Table 2. Appendix B illustrates how these payoffs are computed. The forces that lead to Proposition 3 are similar to those driving Proposition 2. This proposition states that the players will exhibit symmetric behavior regardless of the round in which the $\{N, S\}$ or $\{E, W\}$ subset appears. Moreover, R_i prevents the existence of a winning set in another round for one player. Also note that this is a sufficient condition for identical expected payoffs, not a necessary one. For example, the order N-E-S-W results in identical expected payoffs for the players, despite lacking such a round.

2.5.1 Contests with Random Order

We will now consider a sequential contest in which the order of rounds is chosen randomly. Thus, unlike the previous section, in any given round the agents will not know the order in which the remaining cells will be contested in future rounds. First the cell to be contested in Round One is announced. After the players make their Round One decisions and the outcome is realized, the cell to be contested in Round Two is announced. Again, the players make their Round Two decisions and only after the realization occurs is the Round Three cell announced. This process continues until all the cells are announced. Of course, given that there are only 4 rounds, once Round Three is over both players will know which cell remains to be contested, so the uncertainty is over the first 3 rounds (Remark 2). Observe that there are 24 possible random permutations of North, South, East, and West. We begin by splitting these into cases based on the first round.

Case 1: North or South as Round One

In 12 of the 24 possible orders, North or South will be the randomly drawn in the first round. Without loss of generality let North be randomly chosen first. If player X wins North, player X needs to win one of South and East to form a winning set, while if Player Y wins North, player Y must win one of South and West to form a winning set. This means that the winner of North must win one of two cells to be contested sequentially, while the loser of North must win two cells contested sequentially.[11] Thus winning the North gives a player an expected payoff of $\frac{43}{64}$ for the

[11]The third remaining cell has no impact on the overall winner, and thus can be ignored.

subgame consisting of East, West, and South, while losing the North gives a player an expected payoff of $\frac{1}{64}$ for these remaining three rounds (as obtained by backwards induction in Appendix A). Thus in Round One we have the expected payoff equations $U_X = \left(\frac{X_N}{Z_N}\right)\frac{43}{64} + \left(\frac{Y_N}{Z_N}\right)\frac{1}{64} - X_N$ and $U_Y = \left(\frac{Y_N}{Z_N}\right)\frac{43}{64} + \left(\frac{X_N}{Z_N}\right)\frac{1}{64} - Y_N$. This gives us the first-order equations $\left(\frac{Y_N}{Z_N^2}\right)\frac{42}{64} = \left(\frac{X_N}{Z_N^2}\right)\frac{42}{64} = 1$, which yield $X_N = Y_N = \frac{21}{128}$, and thus $U_X = U_Y = \frac{23}{128} \approx .1797$.

Note that this is the same as the expected payoff in the sequential case where North or South is the first cell contested. Once the first round is resolved, one remaining cell becomes irrelevant, effectively leaving a two round game. The player who lost the first round must win both of the remaining relevant rounds, so the order of these two rounds does not matter. This holds true both when the sequence is known in advance, and when it is randomly drawn in each round, thus the identical expected payoffs follow directly.

Case 2: East or West as Round One

In 12 of the 24 possible orders, East or West will be the cell randomly drawn first. In these cases the ordering of the subsequent rounds matters. Thus, we must find expected payoffs for each player for all three possible second round contests. We then take the average across these three expected payoffs to find a total expected payoff. We then will then use these average expected payoffs to find the optimal strategy and the resulting expected payoff for the first round. Without loss of generality let East be randomly drawn in Round One.

Fourth Round: For the final round, either the cell will determine the overall winner, in which case each player will expend $\frac{1}{4}$, and thus have an expected payoff of $\frac{1}{4}$, or the cell is irrelevant and will not be contested.

Third Round: Moving backwards to the third round, once the cell being contested in the third round is announced, the cell to be contested in the fourth round is known by process of elimination. Thus the optimal strategies and expected payoffs for each player for the third round can be taken from the optimal strategies and expected payoffs found while obtaining the solutions for the sequential case (see Appendix A, case 2).[12]

[12] All expected payoffs used here are taken from the calculations contained in Appendix A.

Second Round: Assuming Player X won the East[13], whether the second round is West or South yields the same expected outcomes. This is because if X also wins the second round in either case, she will require one of the remaining two cells. As we saw in the sequential cases, at the start of this second round, X has an expected payoff of approximately 0.4245, and Y has an expected payoff of approximately 0.0455.

If the second round is North, the players have different expected payoffs, since X is victorious if he wins North. If X loses the North, X must win both the South and West. Again, taking the payoff from the fixed sequential case already solved, we see that at this stage X has an expected payoff of 0.2373, and Y of 0.1106.

First Round: Taking the average across all three possible second rounds gives a total expected payoff for X of $\frac{1}{3}(0.4245 + 0.4245 + 0.2373) = 0.3612$ and for Y of $\frac{1}{3}(0.0455 + 0.0455 + 0.1106 = 0.0672$. Since we assumed that X won the East, the expected payoff of winning the East is 0.3612, and the expected payoff of losing the East is 0.0672. Thus player X obtains an expected payoff of $\left(\frac{X_E}{Z_E}\right)(0.3612) + \left(\frac{Y_E}{Z_E}\right)(0.0672) - X_E$, generating a first-order condition $\left(\frac{Y_E}{Z_E^2}\right)(0.3612) - \left(\frac{Y_E}{Z_E^2}\right)(0.0672) = 1$. Because $X_E = Y_E$, due to symmetry, we can solve, giving us $X_E = Y_E = 0.0735$, and thus expected payoffs of 0.1407. Averaging this with the 0.1797 from the cases where North or South is the first round yields a total expected payoff to each player of 0.1602. This is lower than the overall average of all 24 sequential cases where the order is known, which is 0.1641. Since the total benefit to the players is the same, the lower expected payoff means the players must have greater expected expenditures. We summarize this as the following proposition.

Proposition 2.4. *Contests can be arranged in order of increasing dissipation, with sequential contests with specified order having the lowest expected dissipation, followed by sequential contests with random order and finally simultaneous contests.*

Since players have more difficulty determining which cells are likely to be important in the random case than they do in the pre-specified order case, resource expenditure generally becomes less efficient. This is especially true in the cases where East or West is drawn as the first cell to be contested, because future asymmetries are unknown. Since effort reducing asymmetries arising in the random order case are probabilistic, and are revealed only as the game progresses, the degree of dissipation is intermediate to sequential contests without random order and simultaneous contests.

[13] The analogous results hold true if player Y won the East.

The small differences in expected values between the random and fixed sequential cases are driven by the large number of sequences in which the strategies in the random draws end up being the same as in the sequential draw, such as if North or South is drawn in the first round. In the random order case, there is still some information available, unlike the simultaneous case, allowing for more informed decision making and the possibility of an early victory.

The random sequence contest helps explain why the sequential contests have lower resource expenditure in general. As the contest progresses, the relative importance of cell becomes clearer, with some cells becoming irrelevant due to the knowledge of the winners of previously contested cells. Thus the players know what can be safely ignored, and only expend effort in the cells that still matter. In the simultaneous case, the players lack this ability, and thus expend effort on cells that turn out to be irrelevant to the formation of the winning path, such as cases where the winner wins 3 or all 4 cells. This is also why North and South are more valued, as these appear in more winning sets for each player. At least one appears in all winning sets, and North and South form a winning set by themselves, a fact that is not true for the East and West cells. Thus, lacking other information, North and South are more likely to be relevant than East and West.

2.6 Discussion

In this section we discuss a number of possible extensions and identify some future research questions. We see that complementarity plays a major role in determining expected payoffs. The fact that different sets of cells constitute winning sets for different players means that there is an asymmetry in the order in which winning sets are contested. This creates an asymmetry in the expected payoffs of the players. Although omitted here in the interest of space, we also see similar results in the intermediate cases, shown in Appendix B, where the structure consists of 2 or 3 rounds. In these cases different expected payoffs for players can occur due to different winning sets. There are other variations of this game possible, some of which we will now discuss.

The most obvious variation would be to increase the size of the grid over which the competition takes place. Although tractability requires the number of cells to be limited, the logic of our results may be of use in structures with more cells. For example, in determining a structure for selling bandwidth on network routers, the owner should take into account the structure of connections the bidders wish to obtain. By identifying the routers that exhibit complementarity, the important routers for the structure may be found, thus allowing a reasonably good, though possibly suboptimal, solution to be found for the seller. We leave this as an open question for future research.

Throughout this paper we have assumed that the structure is externally imposed. If the previous owner of the cells is simply allowed to choose a structure, he will obviously choose one of the structures resulting in a total expected expenditure by the players of 0.75. However, if the structure is the result of decisions made by multiple owners, this may not hold.

If we use the generalized Tullock success function (Tullock, 1980), in which player X has a probability of winning cell i equal to $\frac{X_i^\alpha}{X_i^\alpha+Y_i^\alpha}$, where $0 < \alpha \leq 1$, we can obtain a generalized result for the simultaneous case, where each player spends $\frac{\alpha}{8}$ on the North and South, and $\frac{\alpha}{16}$ on each of the East and West. However, our preliminary investigations show that the sequential cases are quite problematic, as α terms appear as both exponents and as coefficients, giving non-linear behavior with respect to α. Thus, although interesting, the problem of generalized success functions is put aside for future work.

If instead of a costly, unlimited resource, we instead have a costless limited budget, in which unspent resources provide no benefit (Brams and Davis, 1974), we have a slightly different problem. In this case player X attempts to maximize the probability of victory given by

$$P_X\left(\{X_k, Y_k\}_{k\in\{N,S,E,W\}}\middle| R\right) = \sum_{\alpha\in X^\star}\left(\prod_{i\in\alpha, j\in A\setminus\alpha}\frac{X_i}{Z_i}\frac{Y_j}{Z_j}\right)V$$

subject to the budget constraint $X_N + X_S + X_E + X_W = B$.

Although the value of the prize does not matter, in the simultaneous case each player will spend $\frac{1}{3}$ of their budget on the North and South cells, and $\frac{1}{6}$ on the East and West. Therefore, if given a budget $B = \frac{3}{8}V$, the decisions made will be identical to those in the simultaneous case. Calculations are omitted in the interest of space, but proceed in a manner similar to the main case.

However, the sequential cases lead to different solutions. For example, consider the E-N-W-S, E-N-S-W, and E-N-SW cases. There is no advantage to not spending the entire budget, and thus if all four cells are contested, the entire budget will be spent. Players only would reserve resources for future rounds of competition if future rounds are possible. Obviously, once spending decisions for three cells are made, the spending decisions for the fourth cell become set. Thus all three of these structures are equivalent with budget constraints, because after the first two rounds, the only decision left is how to divide the remaining budget between the West and South cells. In contrast, with the costly resource the expected payoffs and expected expenditures vary between these cases.

Furthermore, in the costly resource case the identity of the player with the greater expected payoff differs depending on whether there is one of the two four round structures (E-N-W-S or E-N-S-W) listed versus the three round structure (E-N-SW). This obviously cannot be true in the constrained budget case. This difference between the costly resource and costless limited resource cases is likely due to there being no advantage to winning the contest quickly compared to winning it in the final round in the case of budget constraints. While this costless resource case is also an interesting problem, it lies outside the scope of this paper.

2.7 Mathematical Appendices

2.7.1 Appendix A: Polar Cases: Simultaneous versus Sequential Cases

For Proposition 1 we will start by computing the expected payoffs from the simultaneous and sequential cases. For the simultaneous case, we first prove two lemmas which show that in any Nash equilibrium, the players will play a symmetric strategy. In order to proceed we first need two definitions. A players strategy is called asymmetric if the strategy allocates an unequal amount to North and South or the strategy allocates an unequal amount to East and West. A strategy is called double asymmetric if the strategy allocates an unequal amount to North and South and the strategy allocates an unequal amount to East and West.

Case 1: All Four Cells Simultaneously

Lemma 1. *In any Nash equilibrium of the simultaneous contest game, neither player will play an asymmetric strategy.*

Proof. The proof is broken into three parts – one where only one player uses an asymmetric strategy and two cases where both use asymmetric strategies. Recall that symmetry here refers to players expending identical effort on North and South, as well as on East and West. In each case we will show that a player with an asymmetric strategy between North and South (or East and West) can deviate to a strategy with a higher probability of winning both cells and a lower probability of losing both cells while maintaining the same total expenditures. As the only relevant distinctions are winning zero, one, or two of these two cells, showing the existence of a strategy which increases the probability of winning both cells while decreasing the probability of losing both cells without any change in expenditure is enough to demonstrate the original strategy is not Nash. We demonstrate this first, before moving to the three different asymmetries.[14]

To see that increasing the probability of winning both cells and decreasing the probability of losing both is sufficient we will describe two cases of improvement. First, consider a Case J where the probability of winning both cells increases by P_2 while the probability of losing both decreases by $P_0 < P_2$. Thus the probability of winning exactly one cell P_1 must decrease by $P_2 - P_0$, as the sum of the changes $P_0 + P_1 + P_2$ must equal zero, as the sum of all probabilities must be 1. Now consider Case K, where the probability of winning two cells increases by P_0 while the probability of losing both cells decreases by P_0, leaving the probability of winning exactly one cell unchanged. Case K has a greater expected value than the starting case, as it consists solely of increasing the probability of winning both cells and decreasing the probability of losing both cells, a clear improvement. However, increasing the probability of winning both cells by the remaining $P_2 - P_0$ and decreasing the probability of winning exactly one cell by $P_2 - P_0$ while leaving the probability of winning zero cells the same is also certainly an improvement. Making these changes to Case K gives us Case J. Thus, by transitivity, the expected value must be greater for the new probabilities than the original probabilities, and thus the original strategy could not be a Nash equilibrium. Now we can move into the three different parts of the proof.

Part A. Without loss of generality, assume that we have a Nash equilibrium with player X is playing an asymmetric strategy. Assume player X spends $a < b$ on the North and b on the South, while player Y who is playing the symmetric strategy spends c on each. Recall that players are indifferent between North and South *a priori*, as well as between East and West. Player X will have a probability of winning both North and South of

$$\left(\frac{a}{a+c}\right)\left(\frac{b}{b+c}\right) = \frac{ab}{ab+ac+bc+c^2}$$

However, if player X deviates to a strategy of $a+\epsilon$ in the North and $b-\epsilon$ in the South for small $\epsilon > 0$, thus keeping total expenditures on the two cells constant, the probability of winning both becomes

$$\left(\frac{a+\epsilon}{a+\epsilon+c}\right)\left(\frac{b-\epsilon}{b-\epsilon+c}\right) = \frac{ab+b\epsilon-a\epsilon-\epsilon^2}{ab+ac+bc+c^2+b\epsilon-a\epsilon-\epsilon^2}$$

[14] We would like to thank an anonymous referee for raising this point.

which is greater than the value in equation (5) as we are adding the same positive value to both the numerator and denominator of a positive fraction, making the new fraction closer to 1. Similarly, the probability of losing both cells decreases from

$$\left(\frac{c}{a+c}\right)\left(\frac{c}{b+c}\right) = \frac{c^2}{ab+ac+bc+c^2} \Rightarrow$$
$$\left(\frac{c}{a+\epsilon+c}\right)\left(\frac{c}{b-\epsilon+c}\right) = \frac{c^2}{ab+ac+bc+c^2+b\epsilon-a\epsilon-\epsilon^2}$$

which must be smaller as the denominator has been increased with a constant numerator. As the probability of winning both cells has increased, and the probability of losing both cells has decreased, the original strategy could not have been a Nash equilibrium. If $a > b$, we simply reverse the direction of deviation, moving expenditure from a towards b.

Part B1. We assume player X spends $a < b$ on the North and b on the South, while player Y also has an asymmetric strategy where Y expends c in the North and d in the South. Further we assume $(d-c) < (b-a)$. [15] Thus we obtain a probability of winning both cells of

$$\left(\frac{a}{a+c}\right)\left(\frac{b}{b+d}\right) = \frac{ab}{ab+ac+bd+cd}$$

and if player X deviates to a strategy of $a+\epsilon$ in the North and $b-\epsilon$ in the South, the probability of winning both becomes

$$\left(\frac{a+\epsilon}{a+\epsilon+c}\right)\left(\frac{b-\epsilon}{b-\epsilon+d}\right) = \frac{ab+b\epsilon-a\epsilon-\epsilon^2}{ab+ac+bd+cd+b\epsilon-a\epsilon+d\epsilon-c\epsilon-\epsilon^2}$$

which again yields a greater probability of winning both cells. Similarly, the probability of losing both cells decreases from

$$\left(\frac{c}{a+c}\right)\left(\frac{d}{b+d}\right) = \frac{cd}{ab+ac+bd+cd} \Rightarrow$$
$$\left(\frac{c}{a+\epsilon+c}\right)\left(\frac{d}{b-\epsilon+d}\right) = \frac{cd}{ab+ac+bd+cd+b\epsilon-a\epsilon+d\epsilon-c\epsilon-\epsilon^2}$$

so again, the original strategy could not have been a Nash equilibrium. Again, we arbitrarily chose $a < b, (d-c) < (b-a)$, if $a > b, (d-c) < (b-a)$ we would instead subtract ϵ from a and add it to b.

[15] If we reverse this inequality, the equivalent results hold when we switch players X and Y.

Part B2. We assume player X spends $a < b$ on the North and b on the South, while player Y also has an asymmetric strategy where Y expends c in the North and d in the South, with $(d-c) < (b-a)$. We must break into two cases, depending on the relationship between $\frac{d-c}{b-a}$ and $\frac{cd}{ab}$.

First consider $\frac{d-c}{b-a} \leq \frac{cd}{ab}$. The probability of X winning both cells is

$$\left(\frac{a}{a+c}\right)\left(\frac{b}{b+d}\right) = \frac{ab}{ab+ac+bd+cd}$$

and if player X deviates to a strategy of $a+\epsilon$ in the North and $b-\epsilon$ in the South, the probability of winning both becomes

$$\left(\frac{a+\epsilon}{a+\epsilon+c}\right)\left(\frac{b-\epsilon}{b-\epsilon+d}\right) = \frac{ab+b\epsilon-a\epsilon-\epsilon^2}{ab+ac+bd+cd+b\epsilon-a\epsilon+d\epsilon-c\epsilon-\epsilon^2}$$

which yields a greater probability of winning both cells. Similarly, the probability of losing both cells decreases from

$$\left(\frac{c}{a+c}\right)\left(\frac{d}{b+d}\right) = \frac{cd}{ab+ac+bd+cd} \Rightarrow$$
$$\left(\frac{c}{a+\epsilon+c}\right)\left(\frac{d}{b-\epsilon+d}\right) = \frac{cd}{ab+ac+bd+cd+b\epsilon-a\epsilon+d\epsilon-c\epsilon-\epsilon^2}$$

so the original strategy could not have been a Nash equilibrium.

Next consider $\frac{d-c}{b-a} > \frac{cd}{ab}$. First, note that $d > c$, as we know a, b, c, d, and $(b-a)$ are all positive. We begin with a probability of Y winning both cells of

$$\left(\frac{c}{a+c}\right)\left(\frac{d}{b+d}\right) = \frac{cd}{ab+ac+bd+cd}$$

and if player Y deviates to a strategy of $c+\epsilon$ in the North and $d-\epsilon$ in the South, the probability of winning both becomes

$$\left(\frac{c+\epsilon}{c+\epsilon+a}\right)\left(\frac{d-\epsilon}{d-\epsilon+b}\right) = \frac{cd-c\epsilon+d\epsilon-\epsilon^2}{ab+ac+bd+cd-a\epsilon+b\epsilon-c\epsilon+d\epsilon-\epsilon^2}$$

which again yields a greater probability of winning both cells. Similarly, the probability of losing both cells decreases from

$$\left(\frac{a}{a+c}\right)\left(\frac{b}{b+d}\right) = \frac{ab}{ab+ac+bd+cd} \Rightarrow \left(\frac{a}{c+\epsilon+a}\right)\left(\frac{b}{d-\epsilon+b}\right) = \frac{ab}{ab+ac+bd+cd-a\epsilon+b\epsilon-c\epsilon+d\epsilon-\epsilon^2}$$

so again, the original strategy could not have been a Nash equilibrium. If $b < a$, we would have $d < c$, and again would switch the direction of the deviation. Thus though player X may not gain from deviating in the case, player Y will, and thus the original strategy could not have been Nash.

Thus at least one player will always benefit from deviating if a player is playing an asymmetric strategy, therefore the original strategies could not have been Nash. Thus we have eliminated asymmetric strategies from consideration, and need only consider symmetric strategies. □

Lemma 2. *In any Nash equilibrium of the simultaneous contest game, neither player will play a double asymmetric strategy.*

Proof. This proof proceeds by contradiction. Presume player Y spends $Y_N \neq Y_S$ and $Y_E \neq Y_W$. Now if player X spends $X_N = Y_N$, $X_S = Y_S$, and $X_E = X_W = \frac{Y_E+Y_W}{2}$, each of North and South will have a $\frac{1}{2}$ chance of being won by each player. Thus, each player will have a $\frac{1}{4}$ chance of winning both North and South, giving victory, and each of East and West will have a $\frac{1}{4}$ chance of being the deciding cell. Thus, player X will have a probability of victory of $\frac{1}{4} + \frac{1}{4}\frac{Y_E+Y_W 2}{Y_E+\frac{Y_E+Y_W}{2}} + \frac{1}{4}\frac{Y_E+Y_W 2}{Y_W+\frac{Y_E+Y_W}{2}}$, which simplifies to $\frac{1}{4} + \frac{1}{4}\frac{4(Y_E+Y_W)^2}{3Y_E^2+10Y_EY_W+3Y_W^2} = \frac{1}{4} + \frac{1}{4}\frac{4(Y_E+Y_W)^2}{4(Y_E+Y_W)^2-(Y_E-Y_W)^2} > \frac{1}{2}$. Thus, since player player X (who is playing the symmetric strategy) has expended the same resources and has a higher probability of winning, player Y's (double asymmetric) strategy could not be a part of a Nash equilibrium. □

Computing payoffs for the simultaneous case:

Now we can solve the optimization problem and thus obtain the Nash equilibrium. Without loss of generality, we will solve X's optimization problem, which is equivalent to maximizing equation 2. Moreover, due to Lemmas 1 and 2 $X_N = X_S$ and $X_E = X_W$. Hence, equation 2 can be written as:

$$\left(\frac{X_N}{Z_N}\right)\left(\frac{X_E}{Z_E}\right)^2 + 2\left(\frac{X_N}{Z_N}\right)\left(\frac{X_E}{Z_E}\right)\left(\frac{Y_E}{Z_E}\right) + \left(\frac{X_N}{Z_N}\right)^2\left(\frac{Y_E}{Z_E}\right) - 2X_N - 2X_E.$$

This gives us the first-order conditions with respect to X_N and X_E of:

$$\frac{\partial U_X}{\partial X_N} = \left(\frac{Y_N}{Z_N^2}\right)\left(\left(\frac{X_E}{Z_E}\right)\left(\frac{Y_S}{Z_S}\right) + \left(\frac{Y_W}{Z_W}\right)\left(\frac{X_S}{Z_S}\right)\right) - 1 = 0$$
$$\frac{\partial U_X}{\partial X_E} = \left(\frac{Y_E}{Z_E^2}\right)\left(\frac{X_N}{Z_N}\right)\left(\frac{Y_S}{Z_S}\right) - 1 = 0$$

As the equations for this case are symmetric for East versus West, North versus South and because of this, X versus Y, these are the only equations required. Making this substitution and simplifying gives us the system $\frac{1}{4X_N}\left(\frac{1}{4}+\frac{1}{4}\right) = 1$ and $\frac{1}{4X_E}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = 1$. Solving this system of equations, we obtain $X_N = \frac{1}{8}; X_E = \frac{1}{16}$.

Second order conditions are omitted, however intuitively as we have only one solution for the first-order conditions, equation 2 is continuous, and if a player expends ϵ arbitrarily close to zero on each cell they have an expected payoff arbitrarily close to zero, while expending greater than 1 on each cell leads to a negative expected payoff, the solution must be a maximum.

Case 2 : Four cells sequentially

We now will solve the sequential cases. Combining these with the simultaneous cases will give us Proposition 1, while comparing the sequential cases will prove Proposition 2. In order to obtain these results, we must work backwards from the terminal nodes. If only one cell remains to be contested, either the cell is irrelevant as there is already a winner, or the winner of this one cell will win the contest. Taking the remaining cell to be East without loss of generality, we have $U_X(., R_4) = \left(\frac{X_E}{Z_E}\right) - X_E$, and $U_Y(., R_4) = \left(\frac{Y_E}{Z_E}\right) - Y_E$. Taking derivatives gives us $\left(\frac{X_E}{Z_E^2}\right) = 1, \left(\frac{Y_E}{Z_E^2}\right) = 1$, so $X_E = Y_E = \frac{1}{4}$, giving $U_X(., R_4) = U_Y(., R_4) = \frac{1}{4}$.

Now we can work backwards to the previous stage. If there are two cells remaining, there are three possibilities. If the contest has already been won, they are both irrelevant, and we are done. However, if only one of the two remaining cells is relevant, we ignore the irrelevant cell, and this reduces to the above. Finally, there is the possibility that both are relevant, with one player needing to win both and the other needing to win either. We will assume without loss of generality that North and East remain to be contested sequentially, with Player X requiring both to win. Then $U_X(., R_3, R_4) = \left(\frac{X_N}{Z_N}\right)\frac{1}{4} - X_N$, $U_Y(., R_3, R_4) = \left(\frac{Y_N}{Z_N}\right) + \left(\frac{X_N}{Z_N}\right)\frac{1}{4} - Y_N$ gives the expected utility for each player in the North round. Taking derivatives gives us

$$\left(\frac{Y_N}{Z_N^2}\right)\frac{1}{4} = 1$$

$$\left(\frac{X_N}{Z_N^2}\right) - \left(\frac{X_N}{Z_N^2}\right)\frac{1}{4} = \left(\frac{X_N}{Z_N^2}\right)\frac{3}{4} = 1$$

Solving this gives $X_N = \frac{3}{64}$, while $Y_N = \frac{9}{64}$, and thus expected payoffs $U_X(., R_3, R_4) = \frac{1}{64}, U_Y(., R_3, R_4) = \frac{43}{64}$. These two solutions will be used extensively in the subcases below. However we cannot obtain a general set of solutions for the second round without having the specifics of the complementarity, so this is reserved for the subcases.

Case 2.a: *North or South as the first round*

After the first round, one of the remaining cells will become irrelevant, with the winner of the first round needing to win one of the remaining two cells, and the loser of the first round needing to win both. Thus, if North is the first cell, we have $U_X = \left(\frac{X_N}{Z_N}\right)\frac{43}{64} + \left(\frac{Y_N}{Z_N}\right)\frac{1}{64} - X_N$. Taking the derivative with respect to X_N gives $\left(\frac{X_N}{Z_N^2}\right)\frac{43}{64} - \left(\frac{X_N}{Z_N^2}\right)\frac{1}{64} = 1$. Due to symmetry $X_N = Y_N$, so solving gives us $X_N = Y_N = \frac{21}{128}$ and $U_X = U_Y = \frac{23}{128}$.

Case 2.b: *East and West in the first two rounds*

Without loss of generality, we consider the cases where East is the first round, results for West as opening round are symmetric. If player X wins the East, winning the West means he will need either the North or South, while losing the West means that the South is irrelevant, and the North will determine the overall winner. These have expected payoffs of $\frac{43}{64}$ and $\frac{1}{4}$ respectively for player X and $\frac{1}{64}$ and $\frac{1}{4}$ for player Y. Thus, if player X wins the East, the expected payoff functions for the second round are

$$U_X(., R_2, R_3, R_4) = \left(\frac{X_W}{Z_W}\right)\frac{43}{64} + \left(\frac{Y_W}{Z_W}\right)\frac{1}{4} - X_W$$
$$U_Y(., R_2, R_3, R_4) = \left(\frac{Y_W}{Z_W}\right)\frac{1}{4} + \left(\frac{X_W}{Z_W}\right)\frac{1}{64} - Y_W$$

This gives us first-order conditions of

$$\left(\frac{Y_W}{Z_W^2}\right)\frac{27}{64} = \left(\frac{X_W}{Z_W^2}\right)\frac{15}{64} = 1$$

and so $15X_W = 27Y_W$. Thus $X_W = \frac{27}{15}Y_W$, which when placed into the first-order conditions gives us $X_W \approx .0969$, $Y_W \approx .0538$. Using these payoffs in the expected payoff functions gives $U_X \approx 0.4243$ and $U_Y \approx 0.0455$. These payoffs will be reversed if player Y wins the East.

Thus, in the initial round, the expected payoff function for player X is $.4243\left(\frac{X_E}{Z_E}\right) + .0455\left(\frac{Y_E}{Z_E}\right) - X_E$, giving a first-order condition of $.3788\left(\frac{Y_E}{Z_E^2}\right) = 1$, with $X_E = Y_E$ due to symmetry. Thus, the optimal strategy is $X_E = Y_E = .0947$, which yields expected payoffs of 0.1406.

Case 2.c: *East or West in Round One and North or South in Round Two*

Without loss of generality, we consider the cases where East is the first round. If player X wins the East, winning the North means victory, while losing North means he must win both West and

South, for an expected payoff of $\frac{1}{64}$ for X and $\frac{43}{64}$ for Y. Thus the expected payoff functions in the second round if X won the East are

$$
\begin{aligned}
U_X(., R_2, R_3, R_4) &= \left(\frac{X_N}{Z_N}\right) + \left(\frac{Y_N}{Z_N}\right)\frac{1}{64} - X_N \\
U_Y(., R_2, R_3, R_4) &= \left(\frac{Y_N}{Z_N}\right)\frac{43}{64} - Y_N
\end{aligned}
$$

Thus we have $43X_N = 63Y_N$, which with our first-order condition gives $X_N = .2373$ and $Y_N = .1620$. Plugging these into the expected payoff equations give approximations of $U_X \approx 0.2373$, $U_Y \approx 0.1106$.

If player Y wins the East, winning either the South or North and West wins. This is the same as in the E-W-N-S case, which gives decimal approximations of $U_X \approx 0.0455$ and $U_Y \approx 0.4245$.

Thus for the first round, we have expected payoff functions of

$$
\begin{aligned}
U_X &= \left(\tfrac{X_E}{Z_E}\right).2373 + \left(\tfrac{Y_E}{Z_E}\right).0456 - X_E \\
U_Y &= \left(\tfrac{Y_E}{Z_E}\right).4245 + \left(\tfrac{X_E}{Z_E}\right).1106 - Y_E
\end{aligned}
$$

The first-order conditions yield $X_E \approx 0.0452$ and $Y_E \approx 0.0739$. Substituting these into the above expected payoff function gives $U_X \approx 0.0731$, $U_Y \approx 0.2315$. Combining cases 1 and 2 gives us Proposition 1, while comparing the expected payoffs for each player across case 2 gives us Proposition 2. □

2.7.2 Appendix B: The Other Sequential Contests

To explain the other types of sequential contests here we will go through the details of one case. The other cases are similar and details of these cases can be found in the working paper version.[16]

The case solved will be that of two rounds of selling, each round containing two cells. This case will require a solution to the expected payoffs of each player if only 2 cells remain, they are being auctioned simultaneously, and one player requires both cells for victory, while the other player requires either for victory. Without loss of generality, we will take player X as needing both the North and East cells. If player X needs both cells, his expected payoff is $\left(\frac{X_N}{Z_N}\right)\left(\frac{X_E}{Z_E}\right) - X_N - X_E$, while player Y has an expected payoff of $1 - \left(\frac{X_N}{Z_N}\right)\left(\frac{X_E}{Z_E}\right) - Y_N - Y_E$. Taking derivatives, we obtain the following set of first-order conditions

$$
\left(\frac{Y_N}{Z_N^2}\right)\left(\frac{X_E}{Z_E}\right) = \left(\frac{X_N}{Z_N}\right)\left(\frac{Y_E}{Z_E^2}\right) = \left(\frac{X_N}{Z_N^2}\right)\left(\frac{X_E}{Z_E}\right) = \left(\frac{X_N}{Z_N}\right)\left(\frac{X_E}{Z_E^2}\right) = 1
$$

[16] http://xythos.lsu.edu/users/mwiser1/Hex

After a bit of algebra, we obtain that $X_N = Y_N = X_E = Y_E = \frac{1}{8}$, so player X has an expected payoff of 0, while player Y has an expected payoff of $\frac{1}{2}$.

We now have enough information to write the expected payoff equations and thus solve this case. However, we will need to break this case into three subcases. The results for the other cases are summarized in the final table.

Case 3.a: East and West as the First Round

If player X wins both the East and West in the initial round, they will complete a winning set with either North or South, and thus X will have an expected payoff of $\frac{1}{2}$, while Y will have an expected payoff of 0. If players X and Y split East and West, only one of North and South will matter, and thus each player has an expected payoff of $\frac{1}{4}$. Thus, player X has an expected payoff of

$$\left(\frac{X_E}{Z_E}\right)\left(\frac{X_W}{Z_W}\right)\frac{1}{2} + \left(\frac{X_E}{Z_E}\right)\left(\frac{Y_W}{Z_W}\right)\frac{1}{4} + \left(\frac{Y_E}{Z_E}\right)\left(\frac{X_W}{Z_W}\right)\frac{1}{4} - X_E - X_W$$

As East and West are symmetric, as well as players X and Y, we know that $X_E = X_W = Y_E = Y_W$. Thus, taking the derivative with respect to X_E gives us the following

$$\begin{aligned} &\left(\frac{Y_E}{Z_E^2}\right)\left(\frac{X_W}{Z_W}\right)\frac{1}{2} + \left(\frac{Y_E}{Z_E^2}\right)\left(\frac{Y_W}{Z_W}\right)\frac{1}{4} - \left(\frac{Y_E}{Z_E^2}\right)\left(\frac{X_W}{Z_W}\right)\frac{1}{4} - 1 \\ &= \left(\frac{Y_E}{Z_E^2}\right)\left(\frac{Y_E}{Z_E}\right)\frac{1}{2} - 1 = \left(\frac{Y_E}{Z_E^2}\right)\frac{1}{4} - 1 = \frac{1}{4X_E}\frac{1}{4} - 1 \end{aligned}$$

Setting this equal to zero gives us first-order conditions of $X_E = X_W = Y_E = Y_W = \frac{1}{16}$, and using this gives us expected payoffs for each player of $\frac{1}{8}$.

Table 2.2: Expected Payoffs For Each Player and Seller Under All Structures

Type	**Order**	$E[U_X]$	$E[U_Y]$	EV[A]
4	NESW	.1250	.1250	.7500
3-1	NSW-E, NSE-W	.1300	.1300	.7400
3-1	EWN-S, EWS-N	.1563	.1563	.6875
2-2	NE-WS, WS-NE	.3396	.1793	.4811
2-2	SE-NW, NW-SE	.1793	.3396	.4811
2-2	NS-EW	.1250	.1250	.7500
2-2	EW-NS	.1250	.1250	.7500
1-3	N-ESW, S-ENW	.1250	.1250	.7500
1-3	E-NSW, W-NSE	.2292	.2292	.5417
2-1-1	EW-N-S, EW-S-N	.1328	.1328	.7344
2-1-1	NE-S-W, SW-N-E, NE-W-S, SW-E-N	.1583	.1471	.6947
2-1-1	NW-S-E, SE-N-W, NW-E-S, SE-W-N	.1471	.1583	.6947
2-1-1	NS-E-W, NS-W-E	.1250	.1250	.7500
1-2-1	N-EW-S, S-EW-N	.1797	.1797	.6719
1-2-1	N-ES-W, S-WN-E	.1006	.2269	.6725
1-2-1	S-EN-W, N-WS-E	.2269	.1006	.6725
1-2-1	E-NS-W, W-NS-E	.2368	.2368	.5265
1-2-1	E-NW-S, W-SE-N	.2040	.3372	.4588
1-2-1	W-NE-S, E-SW-N	.3372	.2040	.4588
1-1-2	N-S-EW, S-N-EW	.1797	.1797	.6719
1-1-2	N-E-SW, S-W-NE	.1006	.2269	.6725
1-1-2	N-W-SE, S-E-NW	.2269	.1006	.6725
1-1-2	E-S-NW, W-N-SE	.0933	.2003	.7064
1-1-2	E-N-SW, W-S-NE	.2003	.0933	.7064
1-1-2	E-W-NS, W-E-NS	.1250	.1250	.7500
1-1-1-1	N-(E,W,S), S-(E,W,N)	.1797	.1797	.6719
1-1-1-1	E-W-N-S, E-W-S-N, W-E-N-S, W-E-S-N	.1406	.1406	.7188
1-1-1-1	E-N-W-S, W-S-E-N, E-N-S-W, W-S-N-E	.0731	.2315	.6954
1-1-1-1	W-N-E-S, E-S-W-N, W-N-S-E, E-S-N-W	.2315	.0731	.6954

Chapter 3: An Experimental Investigation of Simultaneous Multi-battle Contests with Complementarities

3.1 Introduction

Contests have long been used to model competitive situations like lobbying (Krueger, 1974; Tullock, 1980; and Snyder 1989) and patent races (Fudenberg et al., 1983; Harris and Vickers, 1985, 1987). Recently, there has been renewed interest in contests as summarized in the recent survey of theoretical work by Kovenock and Roberson (2010a) and the book by Konrad (2009). A particular area of current interest involves contests that are composed of multiple battles. Security questions such as how to protect a pipeline, or a computer network that is only as strong as its weakest link are such issues. Similar problems arise in deciding how a state can assign its electoral college votes to influence the outcome of a US presidential election. However, concern about multi-battle contests goes back to discussion of the Colonel Blotto game (Borel, 1921; Borel and Ville, 1938; Gross, 1950; Gross and Wagner, 1950 and Friedman, 1958) in which two militaries allocate soldiers to a series of n battles and the and in the standard version of the game the winning side is the one that wins the most battles. In these games the outcome of a given battle depends on who has the larger army in that battle and the battlefields are linked by a budget constraint which captures the fact that the number of soldiers available is fixed.

The more recent work on multi-battle contests allows for asymmetric budgets and a positive opportunity cost of the resource in games with both continuous and discrete strategy spaces (Hart, 2008; Kvasov, 2007; Laslier, 2002; Laslier and Picard 2002; Roberson 2006 and Weinstein, 2005). Additionally, researchers have considered alternative forms of determining the overall winner based upon the outcomes of individual battles. Szentes and Rosenthal (2003a, b) examine a more general game in which one needs to win m battles to claim the prize (the majority rule is a special case where $m = \frac{n+1}{2}$). One example of such a contest would be the lobbying effort needed to achieve a supermajority. Clark and Konrad (2007) and Golman and Page (2009) consider a situation in which one party must win all of the battles while the other needs only a single victory. Such weak link games can be thought of as the defense and attack of a computer network or a pipeline. Deck and Sheremeta (2012) study a game of siege in which an attacker has repeated or multiple opportunities to achieve a single victory.

In a large leap, Kovenock, et al. (2013) move beyond the situation where the overall winner is determined solely by the number of victories to a setting where success depends on the specific combination of individual victories. This structure more realistically captures many strategic problems. For example, new products from electronics to pharmaceuticals are often not based on a single patent or innovation, but rather a combination of them. Apple holds over 1300 patents related to the iPhone (Thomson Reuters 2012). While not all of those patents may be in use, clearly a smartphone is of no use without a user interface, a power system and an antenna. Having three user interfaces and no power system or antenna would not result in a successful product. However, a portable video player requires an advanced screen and large storage capacity along with the power supply and user interface, but may not need an antenna. A developer seeking to acquire a large number of lots from separate homeowners cares about the number of contiguous pieces that can be claimed, not the total number.

The specific problem that Kovenock, et al. (2013) solve derives from the 2×2 game of Hex. In this game, there are two players who are attempting to complete a path through the game board connecting their two peripheral regions (see Figure 1) using a standard Tullock contest function. While there are multiple winning combinations for each player and the set of winning combinations is not identical for each player, there will always be a single winner in the game once every region is claimed. As described by Kovenock, et al. (2013) this setup mimics a telecommunications or internet network. There are multiple ways to navigate through the network and no particular relay is critical given the redundancies. Therefore, an attack on such a network has to successfully block all possible routes.[1] This situation is distinct from a weakest link game as no specific battle is decisive for overall victory, but some nodes are more important than others because of the number of potential winning paths of which it is a part.

In this paper we report the results of controlled laboratory experiments designed to test the theoretical predictions of Kovenock, et al. (2013). We also report the results of experiments involving a larger 4×4 game board. While the size of this larger game is still relatively small, the computations involved make developing theoretical predictions for it overly cumbersome. In fact, such scalability is a clear advantage of a behavioral approach to studying contests with complicated complementarities.

[1] In fact there is no weakest link in this game. However, this game still captures a situation where some nodes are more important than others because of the number of potential connecting paths in which it is included.

As a prelude to the results, we find that aggregate behavior in the 2 × 2 game is generally consistent with the theoretical predictions. However, we do observe substantial overbidding, a robust phenomenon in contest experiments (see Dechenaux, et al. 2012 for a review of contest experiments). Despite the overbidding, the relative bids between different regions are in line with the theoretical predictions. However, instead of bidding on each region as the theory predicts, individuals tend to focus their bids on a path through the game. This same behavioral pattern is observed in the larger game as well, with the result being that on average a larger amount is bid on those cells that have a greater number of winning paths running through them.

The remainder of the paper is organized as follows. The next sections presents the specific hypotheses to be tested based upon the theoretical model. The third section details the experimental design and the fourth section analyzes the results. Concluding remarks are given in the final section.

3.2 Model and Hypotheses

The model presented here is from Kovenock, et al. (2013). Consider the 2 × 2 game of Hex shown in Figure 1. There are two players, X and Y. X controls the top left and bottom right corners of the board, while Y controls the other two corners. The objective for each player is to form a contiguous path connecting their pair of corner regions. Thus player X wins a prize valued at V if he captures any of the following sets of regions {North, South, East, West}, {North, South, East}, {North, South, West}, {North, East, West}, {South, East, West}, {North, South}, {North, East}, {South, West}. There are also 8 winning combinations for Player Y, some of which are the same as those for player X and some that are different. For example, either player wins by capturing the set {North, South}, while only player X wins with the set {North, East} and only player Y wins with the set {South, East}. Let X* and Y* denote the set of winning sets for players X and Y respectively. (See Figure 2.2)

The players are assumed to be risk neutral, have a common value for winning the game, and do not face a budget constraint. Further, the winner of each region is determined by a standard Tullock contest success function. Letting $X_r(Y_r)$ denote the investment by player X (Y) in region $r \in R = \{North, South, East, West\}$, the probability that player X claims region r is $\frac{X_r}{X_r+Y_r}$. Hence the probability that X wins the overall prize is given by $P = \sum_{\alpha \in X^*} \left(\prod_{r \in \alpha} \left(\frac{X_r}{X_r+Y_r} \right) \prod_{r \notin \alpha} \left(\frac{X_r}{X_r+Y_r} \right) \right)$ with a similar calculation for the probability that Y wins the prize. As any investment is forgone regardless of the outcome, player X's profit function is given by $\pi_X = VP - \sum_{r \in R} X_r$ and similarly for player Y.

The unique Nash equilibrium of this game is $X_{North} = X_{South} = Y_{North} = Y_{South} = \frac{V}{8}$ and $X_{East} = X_{West} = Y_{East} = Y_{West} = \frac{V}{16}$. Notice, that each payer should invest a positive amount in each region. The equilibrium calculation is straightforward, but tedious involving the simultaneous solution to four first-order conditions of the profit maximization problem for each player. In equilibrium, each player has a 50% chance of winning and an expected payoff of $\frac{V}{8}$. Notice also that aggregate investment by the two players together is $4(\frac{V}{8}) + 4(\frac{V}{16}) = \frac{3V}{4}$.

For comparison, if there were no complementarities and each of the regions was valued at V_r then the standard result would hold for each region. Specifically, each bidder should bid $\frac{V_r}{4}$ for each region, resulting in a 50% chance of claiming any region and an expected profit of $\frac{V_r}{4}$ in each region. The total investment over R would be $\sum_r \frac{V_r}{2}$. If $\sum_r V_r = V$ so that the total prize was the same in the two games then without complementarities each player would invest a total of $\frac{V}{4}$, the expected profit per player would be $\frac{V}{4}$, and the total investment would be $\frac{V}{2}$. Therefore, for the same total prize the complementarity increases total investment, and the expected profits are lower.

The equilibrium investments serve as the basic hypotheses to be tested in the lab. However, given the robust result from previous experiments that people tend to overbid in simple contests we also investigate the following relative hypotheses that focus on the role of complementarities.

- Without complementarities, bids in a region are proportional to the value of that region and thus also proportional to the equilibrium bid for the region.
- With complementarities, bids in a region are proportional to the equilibrium bid for the region.

Before continuing to the experimental design, we briefly point out a few additional items. First, the theoretical problem described above can be extended to an n $\times$ n size game of hex. The equilibrium condition is determined by the simultaneous solution of $2n^2$ first-order conditions. Further, the profit function itself depends on the elements of X* and Y* which each contain 2^{n^2-1} entries. So for a 4 $\times$ 4 game there are 32768 winning combinations for player X and 32768 winning combinations for player Y and the equilibrium level of investment depends on 32 simultaneous equations based on those winning combinations. Clearly, this problem quickly becomes intractable, but despite the large number of winning combinations, it is trivial for a person to look at a 4 $\times$ 4 game and see who has won, a fact that we make use of when determining the winner in the experiments.

Finally, notice that in the 2 $\times$ 2 game every winning path for X is either (North, South), (North, East), (South, West), or some superset of one of these. These are our minimal winning sets. Minimal winning sets are the sets of cells which are sufficient for victory, but no proper subset of which is a winning set. There are also three minimal winning paths for player Y and again each involves 2 regions. For both players, the regions North and South are in two of their minimal wining sets and East and West are only in one of the minimal winning sets. Therefore, if a player were to invest uniformly along a minimal winning path and not bid off of that path, then on average twice as much would be invested in North and South than in East and West, the same aggregate pattern predicted by the equilibrium despite the difference in individual behavior. As the game becomes larger, $n > 2$, it is no longer the case that every minimal winning path is of length n, although each player does always have $\sum_{i=1}^{n} 2^{i-1}$ minimal wining paths of length n. Therefore, in addition to the optimal strategy that involves investing in all n^2 regions, players could pursue a strategy of investing only on some minimal winning path or further restrict investment only to wining paths of length n. Still, as noted earlier both the equilibrium strategy and either path investment strategy would lead to greater average investment in regions that are in more wining sets.

3.3 Experimental Design

To evaluate the theoretical predictions of the model, we conducted a series of contest experiments using a between subjects design. In one treatment subjects participated in a series of contests that did not involve complementarities and in the other treatment the contests involved complementarities.

Throughout the experiment, monetary amounts are denoted in Experimental Currency (EC), which would be converted to $US at the rate of $EC 25 = $US 1. This exchange rate was explained to the subjects when the experiment began. Unless otherwise noted, all monetary amounts below are in EC. Subjects also received a $US 5 payment for showing up to the lab on time for the one hour session, as is standard policy in the Behavioral Business Research Laboratory at the University of Arkansas where the experiments were conducted. Participants were 72 undergraduate students from the labs database of approximately 2000 volunteers. While some subjects had previously participated in other economics experiments, none had participated in any related studies.

Both treatments involved three phases as described below. Subjects read phase specific instructions just prior to the start of the phase. Subjects did not know how many periods were in any phase nor did they know of the existence of future phases.

Phase 1: Sequential Play in a 2 $\times$ 2 Game with No Investment

The first phase consisted of 10 games on a 2 $\times$ 2 board. For technical ease, the regions were actually squares instead of hexagons, but the arrangement was such that the winning combinations were the same as discussed in the previous section for the game with complementarities (see Figure 2). In each game one player moved first and was able to claim a region (at no cost) by simply clicking on their screen. The second player could observe the choice of the first player and then select a region to claim. This choice was revealed to the first mover who could then claim a second region leaving the final region for the second mover.

For the no complementarities treatment the North and South Regions were valued at 8 while the other two regions was valued at 4 each. Thus, the first mover should take either North or South and the second mover should take the other of these relatively high valued regions. For the complementarities treatment, the value of completing wining path was 48. In this case, the first mover should pick North or South initially. The second mover should then select whichever remains unclaimed of North or South in the hope that the first mover makes a subsequent mistake. The first mover should then make a winning move and hence the first player should always win. This first mover advantage holds in the sequential game of hex regardless of the board size, In fact, the board game from Parker Brothers is played sequentially on an 11 $\times$ 11 board.

Each player alternated between playing the first and second mover in these games. Further, there were 6 people in each session and players were randomly and anonymously matched each game to eliminate any issues with repeated play or reputation.

The purpose of phase 1 is twofold. First, it allows players to discover the strategic value of each region to both players in the game of Hex without explicitly being instructed about the importance of the North and South regions, which might bias bidding behavior. Second, it provides an opportunity for the subjects to earn money which can be used be as an endowment during the later phases of the experiment where one risks losing money. In fact, since both players forfeit their investment and only one player can claim the prize, there must be a financial loser in each contest. Therefore, to maintain control over subject incentives the standard procedure of providing an endowment from which losses can be deducted is used. Because the values in the two games are different, for reasons explained below, the subjects were also given a treatment specific initial endowment. Those in the complements treatment received an endowment of 160. Those in the no complements treatment received an endowment of 280. The end result is that after phase 1, each subject should have cumulative earnings of 400, assuming perfect play.

Phase 2: 2 $\times$ 2 Game with Regions Decided by Contests

The second phase of the experiment consisted of 20 games on a 2 $\times$ 2 board. Again, players were randomly and anonymously matched each period. In this phase the players privately and simultaneously submitted bids for each region as shown in Figures 3.1 and 3.2 with each region being awarded probabilistically according to the Tullock success function. For the games with complementarities, the prize for winning was again 48. Without complementarities, North and South were valued at 8 and East and West were valued at 4. Table 3.1 summarizes the equilibrium bids and profits for both treatments. As described in the model section, for the same total value available the expected profits differ between treatments. Therefore, the values in the no complements treatment were adjusted so that (1) the expected profit per bidder is held constant across treatments, and (2) the relative value of each region is held constant across treatments. The first point is important for ensuring that subjects have the same incentives in both treatments. The second point enables a clearer test to determine how the complementarity impacts investment as well as allowing for a test of the standard contest model as the value changes.

Table 3.1: Summary of Experiment Parameters and Predictions for 2 $\times$ 2 Games

	No Complementarities	With Complementarities
Value of Winning	North/South worth 8 East/West worth 4	Completed Path worth 48
Equilibrium Bid on North/South	2	6
Equilibrium Bid on East/West	1	3
Expected Profit per Player	6	6

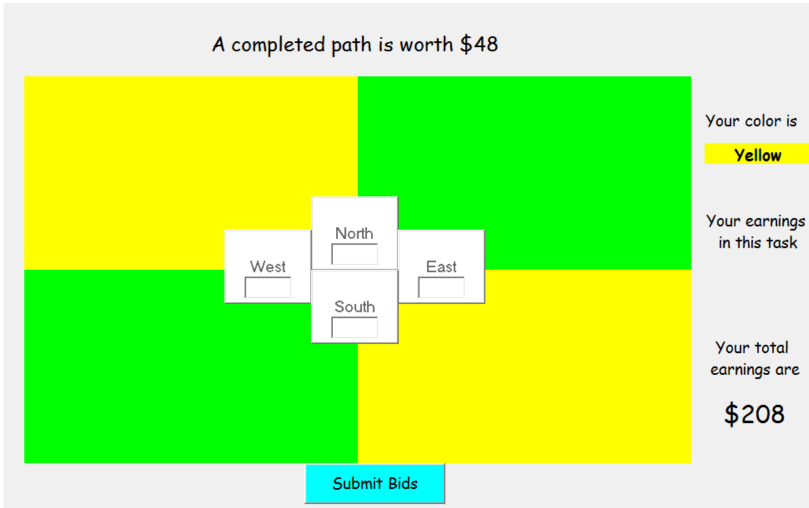

Figure 3.1: **Screenshot of 2 × 2 game with complementarities**

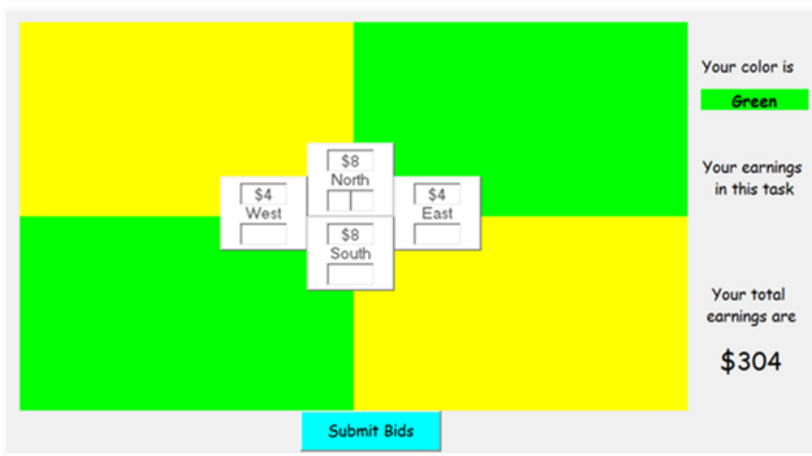

Figure 3.2: **Screenshot of 2 × 2 game with no complementarities**

Phase 3: 4 × 4 Game with Regions Decided by Contests

The procedures for this phase were identical to those in phase 2, except that the game board was increased to 4 × 4 and that subjects only played the game 5 times (and thus placed the same total number of bids as in phase 2 since there are four times as many regions in phase 3). Figure 3.6 shows a sample outcome of the 4 × 4 game with complementarities. The value of a winning path remained 48 for the treatment with complementarities. For the game without complements, each of the 16 regions was valued at 3 so that the total value was the same between the two treatments. This allows for an evaluation of the impact of incentives and additional variation in prize values for standard stand-alone contests. Given the exploratory nature of this sized game and the inherent increase in complexity when there are complementarities, this phase was always conducted last so as to 1) not influence behavior in the 2×2 game which is the main focus of the project and 2) provide subjects an opportunity to learn in the simpler game so that observed behavior in this game is meaningful. While these two concerns are not as important when there are no complementarities, the two treatments are kept parallel to make direct between treatment comparisons.

Figure 3.3: **Screenshot of 4 × 4 game with complementarities**

3.4 Behavioral Results

We begin our analysis with behavior in Phase 1 of the experiment. Overall, players made an optimal choice 98% of the time, where optimal is defined as claiming the higher valued region if it is available. For the more complicated sequential hex game, optimal behavior, defined as following the subgame perfect equilibrium strategy conditional on the decision point, is observed 95% of the time. Further, no suboptimal behavior was observed in the last three rounds for either treatment. This finding clearly indicates that subjects understood the strategic value of the different regions before beginning phase 2.

We now turn to analyzing behavior in phase 2. Given the symmetry in the game, regions are standardized around a vertical line drawn through the center of the game and reported with respect to the position of a region from the perspective of the Yellow player who is trying to complete a path from the top left to the bottom right. For the 2 2 game this means that for reporting purposes bids Green placed nominally for the West region are combined with Yellows bids for the East and vice versa. The average bid for each region is shown in Table 2.

We begin by considering the no complementarities treatment, the results of which are shown in the left column of Table 2. Several interesting features of the data are readily apparent. First, for all four regions players are bidding almost twice the equilibrium level on average. This is clear evidence that the subjects are not bidding according to the equilibrium predictions. Such overbidding is actually typical in simple contests experiments. The second main feature is that players are bidding basically the same amount for North and South, consistent with Hypothesis 1. The players are also bidding nearly identically on average for East and West, also consistent Hypothesis 1. Finally, players are bidding twice as much for North and South as for East and West, again consistent with Hypothesis 1.

Table 3.2: Average Bids by Region in the 2 × 2 Games

	No Complementarities		With Complementarities	
	Observed	Percent Overbid	Observed	Percent Overbid
North	3.81	91%	7.38	23%
South	3.75	88%	8.36	39%
East	1.89	89%	3.56	19%
West	1.85	85%	4.65	55%
$\frac{North}{South}$	1.02		0.89	
$\frac{East}{West}$	1.02		0.77	
$\frac{North+South}{East+West}$	2.03		1.92	

Table 3 reports the estimation results from regression analysis where the dependent variable is the amount bid on a region relative to the equilibrium bid for that region and South and West are dummy variables for those respective regions and Side is a dummy variable that takes the value of 1 for the East and West regions and is 0 otherwise. Thus the omitted case is the North region. Side captures the effect of being in the East region while West captures any differential effect between the East and West regions. Standard errors are clustered at the session level while each individual bidder is treated as a random effect. The joint lack of significance for the dummy variables in Table 3 provides statistical support in favor of Hypothesis 1. To test whether or not players are bidding according to the equilibrium predictions, involves comparing the constant term in Table 3 to the predicted value of 1. This hypothesis can be rejected in favor of systematic overbidding at even the 1% significance level.

Table 3.3: Region Game Ratio of actual bids to theoretical bids, regressed on region dummies

	Coefficient	Robust Std. Error
Constant	1.9070	0.0447
Side	-0.0535	0.0736
West	0.0238	0.0278
South	-0.0310	0.0385

Turning to the treatment with complementarities, the data presented in the right column of Table 3.2 reveal that subjects are again overbidding in each region, but not as dramatically as in the no complementarities case. Table 3.2 also provides strong evidence in support of Hypothesis 2. The average bids are very similar in North and South as predicted. Further, average bids are similar in the East and West, as predicted. Finally, bids are predicted to be twice as high in the North and South as in the east and West and this is what is observed. Statistical evidence is provided in Table 3.4, which reports a similar regression to that done for the no complements treatment. Hypothesis 2 is supported by the joint lack of significance on the three dummy variables. In this case however we cannot reject the hypothesis that the constant term equal 1 at the 5% significance level. This is clearly due to the lower overbidding that we find when complementarities are present.

Table 3.4: Path Formation Game Ratio of actual bids to theoretical bids, regressed on region dummies

	Coefficient	Robust Std. Error
Constant	1.2297	0.1503
Side	0.2409	0.2457
West	-0.2066	0.3342
South	0.1626	0.1318

The above analysis focuses on aggregate behavior and in general the results seem to support the relative theoretical predictions. However, this result is masking a distinct behavioral pattern. The theoretical prediction is that a player should bid on every region. This pattern is observed in only 35% of the realizations. Only 1% of the time does the set of regions on which a subject bid constitute a non-winning set. The missing majority of observations, 64%, are such that subjects bid on a path through the game such as {North, West}, or a set of three regions containing a winning set such as {North, South, West}. As discussed previously, if one randomly selects a minimum winning path and bids uniformly on it, the result would also be average bids that are twice as high for North and South as for East and West. Further, if the total bid equaled half of the value of winning, as occurred in the independent contests of the no complementarities treatment, then one would bid 12 on North and South with two-thirds probability and would bid 12 on East and West with one-third probability. The result would be an average bid of 8 for North and South and an average bid of 4 for East and West, the pattern revealed in Table 3.2. Hence it appears that theoretical model of Kovenock, et al. (2013) works in aggregate, but not for the right reason.

We now turn to behavior in the exploratory third phase of the experiment. In this third phase, we do not have established theory for the correct strategy in the path formation game. Thus we cannot compare the observed results to previously established optimal play, but rather simply wish to find what sort of strategies are actually employed. Figure 3.4 shows the average bid in each region for the no complementarities treatment. In this case, each region had a value of 3 and thus the equilibrium bid is 0.75. The average observed bid across all 16 independent regions was 1.09 and there are no major differences between regions. The average rate of overbidding was 45%, which is smaller than in phase 2. One reason for this drop in overbidding is that subjects only bid on every region 64% of the time with the other observations typically involving a few ignored regions.[2] Once non-bids are accounted for the rate of overbidding is similar to that in phase 2, suggesting that the level of incentives are not driving overbidding directly. Of course, failure to bid on every region could be due to fatigue or the large number of choices that had to be made, which may in turn be influenced by the level of the stakes.

In the 4×4 game with complementarities, the optimal bid for each region is related to the number of winning sets that contain the region. Figures 3.5 and 3.6 show both the number of winning sets and the average observed bid by region for this case. The correlation between the two measures is quite high at $\rho = 0.898$. As in the 2×2 game, it appears that on average relative bids between regions are in line with what the model would predict. However, this aggregate success masks individual heterogeneity that is distinctly out of equilibrium. Rather that bidding on every region as predicted, 36% of bids are for a 4 region path and only 31% of bids involve more than half of the regions.[3]

[2] 87% of the bids in this treatment involved more than half of the regions. For comparison, in the 2×2 game with complementarities, 81% of bids were for all four regions.

[3] 3% of bids are for only one, two or three regions and thus could not result in a winning path. In this treatment 4% of the bids are zero for all 16 regions. For comparison, 3% of bids in the no complementarities treatment were for zero in all 16 regions.

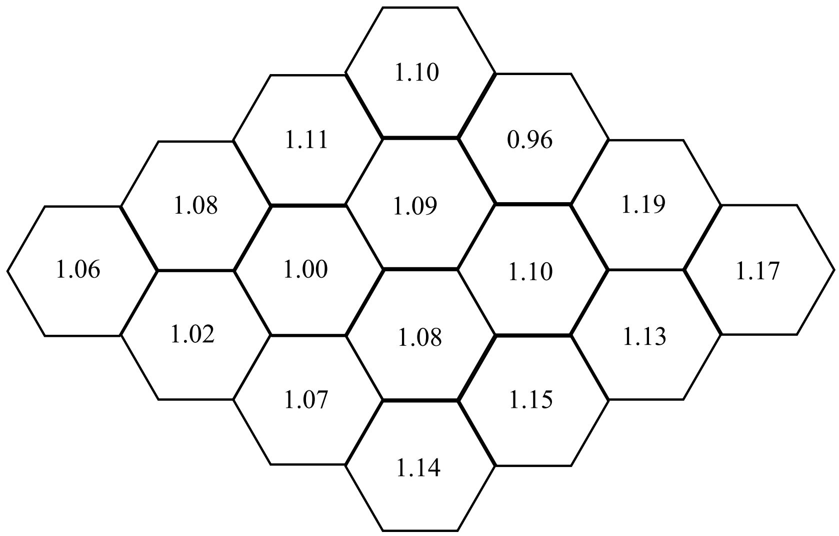

Figure 3.4: **Average Bid By Area**

3.5 Concluding Remarks

Contests have long been used to model a wide array of activity. Recently, researchers have begun to look at ever more complicated strategic situations where outcomes are based on a series of interconnected battles. One such example is the recent model by Kovenock, et al. (2013), which focuses on the game of Hex, but is representative of a wider class of games that involve complementarities in outcomes. This model is particularly relevant for the protection of networks that have built in redundancies, such as telecommunications or computer networks. Unfortunately, the tradeoff of additional complexity is often tractability. For example, Kovenock, et al. (2013) restrict attention to a 2×2 game board.

In this paper we report the results of controlled laboratory experiment designed to test the predictions of the Kovenock, et al. (2013) model specifically and also explore how players actually approach more complicated contests. After all, if people really solved the problem the way theorists do, there would not be much point in researchers considering such models as the solutions would already be apparent. What we find from the experiment is that in aggregate the models predictions accurately reflect the relative bids in the different regional battles. Overbidding is observed with and without complementarities, the later result being consistent with previous contest experiments. However, the models predictions are correct but for the wrong reason. Rather than fighting in every

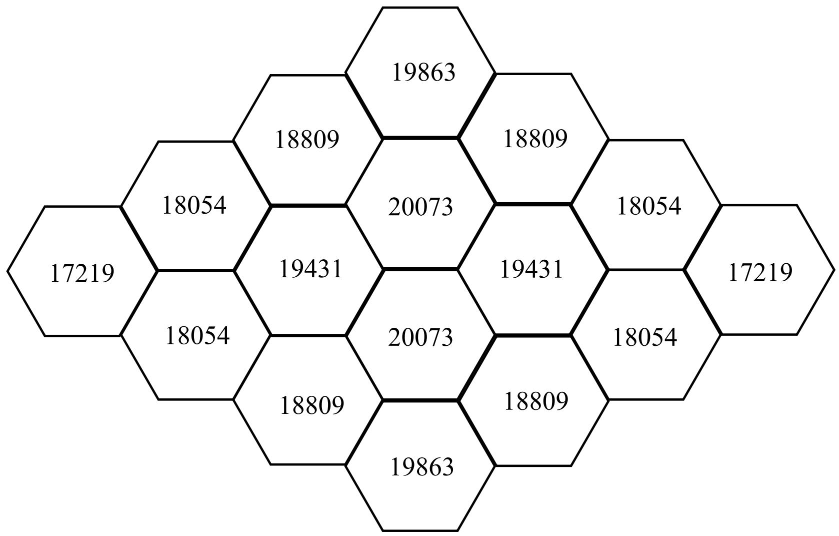

Figure 3.5: **Number of Winning Sets Containing Region**

region in accordance with the model, people actually concentrate on specific winning combinations and largely ignore the other battles. As in the case of independent values, people bid just less than half of the prize value, but in the Hex game they spread that amount along a winning path.

An advantage of a behavioral approach to investigating contest behavior is that one can investigate situations beyond what is computationally attractive. Here we also considered a more complex 4×4 that involves 32768 wining combinations for each player. Again aggregate behavior was consistent with theoretical play in that players bid more for regions that are in more winning combinations. However, this aggregate pattern is again explained by individuals focusing on some winning combination. This finding indicates that overbidding (which is lower in the presence of complementarities) and concentrated attacks are robust behavior. The implications of these patterns are potentially quite broad as researchers attempt to identify optimal strategies for attack and defense of networks in many naturally occurring applications such as cyber-security or supply chains.

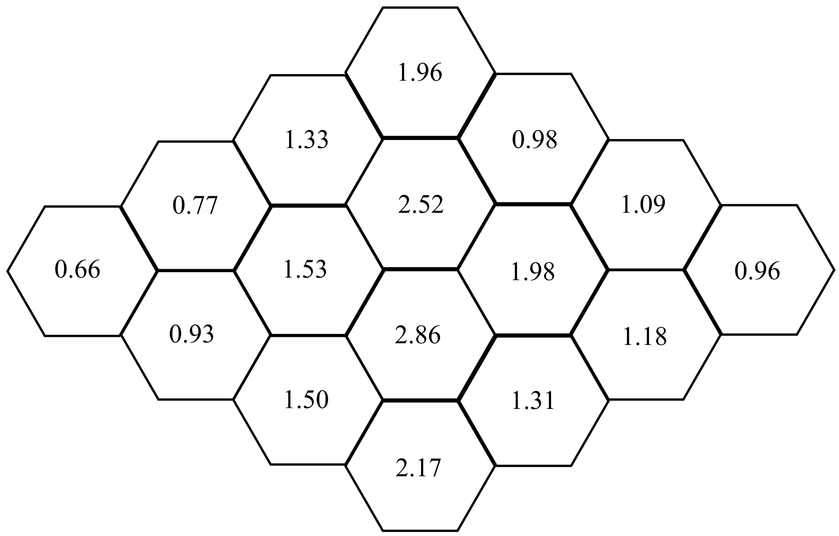

Figure 3.6: **Average Bid By Region**

3.6 Appendix: Subject Instructions

On the following pages there are two sets of instructions. The first set is for the treatment with no complementarities and the second set is for the treatment with complementarities. Text in [brackets] was not observed by subjects.

[No Complementarities Phase 1]

This is an experiment in the economics of decision making. You will be paid in cash at the end of the experiment based upon your decisions, so it is important that you understand the directions completely. Therefore, if you have a question at any point, please raise your hand and someone will assist you. Otherwise we ask that you do not talk or communicate in any other way with anyone else. If you do, you may be asked to leave the experiment and will forfeit any payment.

The experiment will proceed in three parts. You will receive the directions for part 2 after part 1 is completed and for part 3 after part 2 is completed. What happens in part 2 does not depend on what happens in part 1, and what happens in part 3 does not depend on what happens in part 1 or 2. But whatever money you earn in part 1 will carry over to part 2 and whatever you earn in part 2 will carry over to part 3.

Each part contains a series of decision tasks that require you to make choices. For each decision task, you will be randomly matched with another participant in the lab. None of the participants will ever learn the identity of the person they are matched with on any particular round.

In each task you have the opportunity to earn Lab Dollars. At the end of the experiment your lab dollars will be converted into US dollars at the rate $25 Lab Dollars = $US 1.

You have been randomly assigned to be either a "Green" or a "Yellow" participant. You will keep this color throughout the experiment. For every task you will be randomly matched with a person who has been assigned the other color. Your color is indicated in a box on the right side of your screen. This portion of your screen also shows you your earnings on a task once it is completed and your cumulative earnings in the experiment. You will start off with an earning balance of $280.

In the center of your screen you can see a map with four regions: North, South, East, and West. The North is worth $8. The South is worth $8. The East is worth $4. The West is worth $4. This information is displayed at the top of your screen above the map. In part 1 of the experiment, when it is your turn you can claim any one unclaimed region on the map by clicking on it. When you claim a region, that part of the map will be colored in with your color and your earnings will be increases by the value of the region you claimed. When the person you are matched with claims a region, it will be colored with that persons color and that persons earnings will be increased by the value of the claimed region.

On each task Green and Yellow alternate turns until all of the regions on a map are claimed. Once all of the regions are claimed the task is complete. At that point, you will be randomly rematched with another participant for the next task. Which color gets to go first alternates between tasks. You will go through this process several times.

[No Complementarities Phase 2]

Part 2 of this experiment is very similar to part 1. The map is the same, as are the values of each region. Your color will be the same and you will be randomly and anonymously rematched with someone in the opposite role for each task.

What is different is how the regions are claimed. Now you and the person you are matched up with have to bid for each of the four regions at the same time. However, any amount you bid on a region is deducted from your earnings regardless of whether or not you get to claim the region.

Since you have to pay what you bid, the sum of your bids for the four regions cannot exceed the amount of earnings you have when placing your bids.

Bidding for a region works in the following way. The chance that you claim a region is proportional to how much you bid relative to the total amount bid for that region. For example, suppose that Yellow bid \$6 for the North and Green bid \$2 for the North then the chance that Yellow would claim North is 6/(6+2) = 6/8 = 75%. The chance that Green would claim North is 2/(6+2) = 2/8 = 25%.

As another example, suppose that Yellow bid \$0 for the North and Green bid \$0.25 for the North then the chance that Yellow would claim North is 0/(0+0.25) = 0%. The chance that Green would claim North is 0.25/(0+0.25) = 100 %.

If both bidders bid \$0 for a region then each would claim the region with a 50% chance.

You and the person you are matched with will both privately and simultaneously place your bids for all four regions at one time. The computer will then determine who claims each region based upon the probabilities associated with the bids. Each region will turn Yellow or Green to indicate who claimed that region.

As before, whoever claims a region will receive the value for that region and have that value added to their earnings. Suppose Yellow bid \$5 for the North and Green bid \$2 for the North. If Yellow claims the North then Yellow will earn \$8 - \$5 = \$3 and Green will earn -\$2. Thus \$3 will be added to Yellows earnings and Green will have \$2 subtracted from their earnings. However, if Green claims the North then Yellow will earn -\$5 and Green will earn \$8 - \$2 = \$6. Earnings for the other three regions will be determined in the same fashion.

[No Complementarities Phase 3]

Part 3 of this experiment is just like part 2, except that there are now 16 regions on the map and the value of each region is \$3. Regions will be claimed in the same way, your color will be the same and you will be randomly and anonymously rematched with someone in the opposite role for each task.

[Complementarities Phase 1]

This is an experiment in the economics of decision making. You will be paid in cash at the end of the experiment based upon your decisions, so it is important that you understand the directions completely. Therefore, if you have a question at any point, please raise your hand and someone will assist you. Otherwise we ask that you do not talk or communicate in any other way with anyone else. If you do, you may be asked to leave the experiment and will forfeit any payment.

The experiment will proceed in three parts. You will receive the directions for part 2 after part 1 is completed and for part 3 after part 2 is completed. What happens in part 2 does not depend on what happens in part 1, and what happens in part 3 does not depend on what happens in part 1 or 2. But whatever money you earn in part 1 will carry over to part 2 and whatever you earn in part 2 will carry over to part 3.

Each part contains a series of decision tasks that require you to make choices. For each decision task, you will be randomly matched with another participant in the lab. None of the participants will ever learn the identity of the person they are matched with on any particular round.

In each task you have the opportunity to earn Lab Dollars. At the end of the experiment your lab dollars will be converted into US dollars at the rate $25 Lab Dollars = $US 1.

You have been randomly assigned to be either a “Green” or a “Yellow” participant. You will keep this color throughout the experiment. For every task you will be randomly matched with a person who has been assigned the other color. Your color is indicated in a box on the right side of your screen. This portion of your screen also shows you your earnings on a task once it is completed and your cumulative earnings in the experiment. You will start off with an earning balance of $160.

In the center of your screen you can see a map with four regions: North, South, East, and West. These regions are surrounded by large colored areas. The top left and bottom right areas are colored “Yellow”. The top right and bottom left areas are colored “Green”. In part 1 of the experiment, when it is your turn you can claim any one unclaimed region on the map by clicking on it. When you claim a region, that part of the map will be colored in with your color. If you can complete a continuous path of regions in your color connecting your two large colored areas you will earn $48. When the person you are matched with claims a region, that part of the map will be colored in with that persons color. If the person you are matched with completes a continuous path, that person will earn $48. Exactly one person can complete a path each time and the person that does not complete a path will earn $0.

On each task Green and Yellow alternate claiming regions until all four regions are claimed. At that point, you will be randomly rematched with another participant for the next task. Which color gets to go first alternates between tasks. You will go through this process several times.

[Complementarities Phase 2]

Part 2 of this experiment is very similar to part 1. The map is the same, as is the value of completing a path. Your color will be the same and you will be randomly and anonymously rematched with someone in the opposite role for each task.

What is different is how the regions are claimed. Now you and the person you are matched up with have to bid for each of the four regions at the same time. However, any amount you bid on a region is deducted from your earnings regardless of whether or not you get to claim the region. Since you have to pay what you bid, the sum of your bids for the four regions cannot exceed the amount of earnings you have when placing your bids.

Bidding for a region works in the following way. The chance that you claim a region is proportional to how much you bid relative to the total amount bid for that region. For example, suppose that Yellow bid \$6 for the North and Green bid \$2 for the North then the chance that Yellow would claim North is 6/(6+2) = 6/8 = 75%. The chance that Green would claim North is 2/(6+2) = 2/8 = 25%.

As another example, suppose that Yellow bid \$0 for the North and Green bid \$0.25 for the North then the chance that Yellow would claim North is 0/(0+0.25) = 0%. The chance that Green would claim North is 0.25/(0+0.25) = 100%.

If both bidders bid \$0 then each would claim the region with a 50% chance.

You and the person you are matched with will both privately and simultaneously place your bids for all four regions at one time. The computer will then determine who claims each region based upon the probabilities associated with the bids. Each region will turn Yellow or Green to indicate who claimed that region.

As before, whoever completes a path will receive the value for it and have that value added to their earnings. Suppose Yellow bid a total of \$23 on the four regions and Green bid a total of \$18 for the four regions. If Yellow completes a path then Yellow will earn \$48 - \$23 = \$25 and Green will earn -\$18. Thus \$25 will be added to Yellows earnings and Green will have \$18 subtracted from their earnings. However, if Green completes a path then Yellow will earn -$23 and Yellow will earn $48 - $18 = $30.

[Complementarities Phase 3]

Part 3 of this experiment is just like part 2, except that there are now 16 regions on the map. Regions will be claimed in the same way, a completed path is still worth $48, your color will be the same and you will be randomly and anonymously rematched with someone in the opposite role for each task.

Chapter 4: Strategic Timing in a Multiple Auctioneer Game of Hex

4.1 Introduction

Suppose a group of individuals who own cells decide to put the cells up for a competition. These individuals will be called auctioneers. These auctioneers decide how to hold this contest. The individuals competing in the contest organized by the auctioneers will be called contestants. The contestant each are attempting to obtain cell which will allow them to complete a path. The cells are arranged such that exactly one contestant will be able to complete a path. The competition will use the Tullock (1980) contest success function to determine which contestant wins each cell. Thus, we have a two stage game, with the first stage being between the auctioneers, and the second stage being between the contestants.

Kovenock et al. (2013) studied how the contestants would optimally behave in the second stage of this game. In this paper we assume that the contestants do behave optimally in the second stage, and use this to examine how the auctioneers would behave in the first stage.

The most similar previous paper is (Golman et al. 2009), in which a more general version of the Colonel Blotto game, dubbed General Blotto, is solved. In Colonel Blotto there are a number of subcontests, called battlefields, and two commanders who each have a limited pool of troops. Each commander assigns troops to the battlefields, and the side with more troops at a given battlefield wins that subcontest. The overall contest is won by the commander who wins the majority of these subcontests.

In General Blotto, several related structures are solved, such as games where there is a prize for each subcontest as well as an additional prize for winning the majority of subcontests, however the ordering of multi-round competitions is not investigated, which is the goal of this paper.

In this paper we use a probabilistic success function for determining the winners of our subcontests, which are the individual cells. This contest success function, previously used by Tullock (1980) in a Colonel Blotto game gives each contestant a chance of success proportional to the ratio of his spending on a contest relative to the total spent by all contestants. This success function will be used to determine spending on cells, which will be then used to determine expected values for the original owners. This is useful for models where the expenditures in the contests are modeling attempts to influence a decision made by an independent actor, rather than a simple rule. For example, a politician generally does not make decisions based solely on the who provided the largest contribution, but rather uses this as one of several factors.

4.2 Structure

We consider a 2×2 game of Hex with four cells, North, East, South, and West, and two contestants, labeled X and Y. Each contestants values making the connection across their two edges as being worth 1, and resources spent trying to win an cell are spent by both the winner and loser of the contest for that cell. (See Figure 2.2)

We now also introduce the idea of multiple auctioneers. Each auctioneer owns some subset of the cells that are to be contested, and places them up for competition. Once the contest structure R becomes known, the contestants then choose strategies for the first round, the winners of the first round's cells are determined and announced, and contestants then proceed to the second round if an overall winner has not been determined. Thus, we have added an initial stage, in which each auctioneer decides on the structure of the contesting of his cells, creating a two-stage game. These auctioneers are the contestants in the first stage, while contestants will refer to the contestants in the second stage.[1]

For simplicity, we will only consider a two round structure, where the auctioneers can put each cell up for contest in the early round, or hold onto the cell until the later round. Decisions about which round to sell each cell in are made privately by each auctioneer.

In order to find solutions, we must first find the expected amount spent on each cell. We find these by taking the sums of the optimal strategies for the contestants, given the result of any previous rounds. These sums are multiplied by the probability that the given result of any previous rounds occurred. We then sum these across all possible sets of results of previous rounds, giving us the total expected amount spent on each cell.

Thus we have a game consisting of a set of auctioneers $\mathbb{A}$. The set of cells $A = \{N, S, E, W\}$ is partitioned between the members of $\mathbb{A}$, such that $\cup_{\mathbb{A}_j \in \mathbb{A}} \mathbb{A}_j = A$ and $\cap_{\mathbb{A}_j \in \mathbb{A}} \mathbb{A}_j = \emptyset$. This partition forms the endowments of the auctioneers, the endowment being the cells belonging to each auctioneer $\mathbb{A}_j$. Each auctioneer's strategy consists of partitioning their endowment into $\mathbb{A}_{j1}$ and $\mathbb{A}_{j2}$, being the cells the auctioneers places in the first and in the second rounds respectively. The

[1] We use auctioneers and contestants to identify the players of the first and second stage to avoid confusion.

payoffs for the auctioneers are the sum of the spending by contestants X and Y on the cells in the auctioneers' endowment. The total payoff for the auctioneers is $V - EV[X] - EV[Y]$.

To summarize, each contestant in the second stage is attempting to collect a winning subset of cells. For both contestants any set of three cells, or a set containing both the North and South is a winning set, while contestant X also has winning sets comprised of North and East, or South and West, while contestant Y has winning sets comprised of East and South, or West and North. This gives us expected value functions of

$$U_X\left(\{X_k, Y_k\}_{k \in N,S,E,W} \middle| R\right) = \sum_{\alpha \in X^\star} \left(\prod_{i \in \alpha, j \in A \setminus \alpha} \frac{X_i}{Z_i} \frac{Y_j}{Z_j} \right) V - \sum_{i \in A} X_i$$

$$U_Y\left(\{X_k, Y_k\}_{k \in N,S,E,W} \middle| R\right) = \sum_{\alpha \in Y^\star} \left(\prod_{i \in \alpha, j \in A \setminus \alpha} \frac{Y_i}{Z_i} \frac{X_j}{Z_j} \right) V - \sum_{i \in A} Y_i$$

Here $X^\star$ is the collection of winning sets for contestant X, and $Y^\star$ is the collection of winning sets for contestant Y, as seen in this table

$$X^\star = \{\{N,E\},\{N,S\},\{S,W\},\{N,S,E\},\{N,S,W\},\{N,E,W\},\{S,E,W\},\{N,S,E,W\}\}$$
$$Y^\star = \{\{N,W\},\{N,S\},\{S,E\},\{N,S,E\},\{N,S,W\},\{N,E,W\},\{S,E,W\},\{N,S,E,W\}\}$$

We then solve this for the possible round structures, via backwards induction. For example, in the simultaneous case, our expected value equations are

$$\begin{aligned} EV[X] = & \left(\frac{X_N}{Z_N}\right)\left(\frac{X_S}{Z_S}\right)\left(\frac{X_E}{Z_E}\right)\left(\frac{X_W}{Z_W}\right) + \left(\frac{X_N}{Z_N}\right)\left(\frac{X_S}{Z_S}\right)\left(\frac{X_E}{Z_E}\right)\left(\frac{Y_W}{Z_W}\right) + \\ & \left(\frac{X_N}{Z_N}\right)\left(\frac{X_S}{Z_S}\right)\left(\frac{Y_E}{Z_E}\right)\left(\frac{X_W}{Z_W}\right) + \left(\frac{X_N}{Z_N}\right)\left(\frac{Y_S}{Z_S}\right)\left(\frac{X_E}{Z_E}\right)\left(\frac{X_W}{Z_W}\right) + \\ & \left(\frac{Y_N}{Z_N}\right)\left(\frac{X_S}{Z_S}\right)\left(\frac{X_E}{Z_E}\right)\left(\frac{X_W}{Z_W}\right) + \left(\frac{X_N}{Z_N}\right)\left(\frac{X_S}{Z_S}\right)\left(\frac{Y_E}{Z_E}\right)\left(\frac{Y_W}{Z_W}\right) + \\ & \left(\frac{X_N}{Z_N}\right)\left(\frac{Y_S}{Z_S}\right)\left(\frac{Y_W}{Z_W}\right)\left(\frac{X_E}{Z_E}\right) + \left(\frac{Y_N}{Z_N}\right)\left(\frac{X_S}{Z_S}\right)\left(\frac{X_W}{Z_W}\right)\left(\frac{Y_E}{Z_E}\right) - \\ & (X_N + X_S + X_E + X_W) \end{aligned}$$

$$
\begin{aligned}
EV[Y] = &\left(\frac{Y_N}{Z_N}\right)\left(\frac{Y_S}{Z_S}\right)\left(\frac{Y_E}{Z_E}\right)\left(\frac{Y_W}{Z_W}\right) + \left(\frac{Y_N}{Z_N}\right)\left(\frac{Y_S}{Z_S}\right)\left(\frac{Y_E}{Z_E}\right)\left(\frac{X_W}{Z_W}\right) + \\
&\left(\frac{Y_N}{Z_N}\right)\left(\frac{Y_S}{Z_S}\right)\left(\frac{X_E}{Z_E}\right)\left(\frac{Y_W}{Z_W}\right) + \left(\frac{Y_N}{Z_N}\right)\left(\frac{X_S}{Z_S}\right)\left(\frac{Y_E}{Z_E}\right)\left(\frac{Y_W}{Z_W}\right) + \\
&\left(\frac{X_N}{Z_N}\right)\left(\frac{Y_S}{Z_S}\right)\left(\frac{Y_E}{Z_E}\right)\left(\frac{Y_W}{Z_W}\right) + \left(\frac{Y_N}{Z_N}\right)\left(\frac{Y_S}{Z_S}\right)\left(\frac{X_E}{Z_E}\right)\left(\frac{X_W}{Z_W}\right) + \\
&\left(\frac{Y_N}{Z_N}\right)\left(\frac{X_S}{Z_S}\right)\left(\frac{Y_W}{Z_W}\right)\left(\frac{X_E}{Z_E}\right) + \left(\frac{X_N}{Z_N}\right)\left(\frac{Y_S}{Z_S}\right)\left(\frac{X_W}{Z_W}\right)\left(\frac{Y_E}{Z_E}\right) - \\
&(Y_N + Y_S + Y_E + Y_W)
\end{aligned}
$$

We then take first-order conditions, which for contestant X are

$$
\begin{aligned}
\frac{\partial U_X}{\partial X_N} &= \left(\frac{Y_N}{Z_N^2}\right)\left(\left(\frac{X_E}{Z_E}\right)\left(\frac{Y_S}{Z_S}\right) + \left(\frac{Y_W}{Z_W}\right)\left(\frac{X_S}{Z_S}\right)\right) - 1 = 0 \\
\frac{\partial U_X}{\partial X_E} &= \left(\frac{Y_E}{Z_E^2}\right)\left(\frac{X_N}{Z_N}\right)\left(\frac{Y_S}{Z_S}\right) - 1 = 0
\end{aligned}
$$

Some symmetry conditions allow us to solve these, and eventually we obtain $X_N = X_S = Y_N = Y_S = \frac{1}{8}$, $X_E = X_W = Y_E = Y_W = \frac{1}{16}$. For more details, and the solutions to other cases, see Kovenock et al. (2013) This gives us the expected spending on each cell, which yields the table of expected gains for the auctioneers. The Order column gives which cells are in the first and second rounds, with the first round before the hyphen and the second round after. The EV columns are the expected total spending by the contestants given the order. We will continue this hyphen notation throughout.

These calculations assume that there is no discounting factor for the future. The two rounds of contests can be taken to be close enough together that inflation and time inconsistent preferences can be reasonably ignored. Thus we have 15 distinct cases, as all 4 cells being in the first round will be identical to all 4 being in the second round. We will proceed by cases, starting with a single auctioneer. In all cases we will take the first listed example without loss of generality, as the structures listed in each case are the same up to symmetry.

We present Proposition 1 from Kovenock et al. (2013) here. The sequential structures referred to here are the four round structures, with one cell being contested in each round.

Proposition 4.1. *All sequential structures have lower expected dissipation than the simultaneous case.*

Table 4.1: Expected Total Contestant Spending on Each Cell Under Each Order of Rounds

Number	Order	EV[N]	EV[S]	EV[E]	EV[W]
1	NESW	.2500	.2500	.1250	.1250
2	NSW-E	.2400	.2400	.0800	.1800
3	NSE-W	.2400	.2400	.1800	.0800
4	NEW-S	.3750	.1250	.0938	.0938
5	EWS-N	.1250	.3750	.0938	.0938
6	WS-NE	.2967	.0930	.1042	.1069
7	NE-WS	.0930	.2967	.1069	.1042
8	NW-SE	.0930	.2967	.1042	.1069
9	SE-NW	.2967	.0930	.1069	.1042
10	NS-EW	.2500	.2500	.1250	.1250
11	EW-NS	.2500	.2500	.1250	.1250
12	N-SEW	.2500	.2500	.1250	.1250
13	S-NEW	.2500	.2500	.1250	.1250
14	E-NWS	.1458	.1458	.1250	.1250
15	W-NES	.1458	.1458	.1250	.1250

For proof of Proposition 4.1, see Kovenock et al. (2013). We mention it here to show that the ordering and number of rounds matter, even in a single auctioneer case. The lower expected dissipation means that the contestants are spending less in the sequential structures than in the simultaneous case, and thus the auctioneers have lower expected values. Thus for consistency, we state the following corollary

Corollary 1. *All sequential structures have lower expected values for the auctioneers than the simultaneous case.*

4.3 One Auctioneer

If we have a single auctioneer, they must clearly own all four cells to begin with, which makes it a simple decision-making problem. Thus they with to choose the structure that has the highest total expenditure. This is the same as the lowest expected payoff for the contestants, since between the auctioneers and contestants we have a zero-sum game. There are several options available to the auctioneer in this case, all of which have an expected expenditure of 0.75. Intuitively, these are the simultaneous case, along with the NS-EW, EW-NS, N-SEW, and S-NEW cases. In all these cases, the expected total expenditure on the North and South is 0.25, and on the East and West, 0.125. These are the cases in which the possibility of a cell in the second round being rendered irrelevant by the first round is exactly compensated for by the probability that the cell has become necessary due to the results of the first round.

4.4 Two Auctioneers

We now begin breaking into cases, starting with the symmetric case. We consider a total of four cases, (i) two where there is an equal cell distribution of two cells per auctioneer, and (ii) two where one auctioneer has three cells while the other has one cell. The equal cell distribution cases will be distinguished by each auctioneer having a winning set for a contestant, versus one auctioneer having a winning set for each contestant. The unequal cell distribution cases will be differentiated by if the auctioneer with one cell has the North or South, versus having the East or West. Throughout, we will mark each pure Nash Equilibria with an asterisk in the matrices.

4.4.1 Equal Cell Distribution

Here we must split into two cases. In one, each auctioneer has a winning set for one contestant, while in the second case, one auctioneer has a winning set for both contestants, while the other auctioneer has no winning set.

Case 2.1: NE,SW; SW,NE; NW,SE; SE,NW

First we will consider the case of 2 auctioneers, each owning a winning set for one contestant. Without loss of generality, we will take the example of one owning the North and East cells, and the other controlling the South and West cells. For each of the 15 distinct possible orderings of the cells into two rounds, we calculate the expected amount spent on the cells owned by the first auctioneer, who owns the North and East, and the second auctioneer, as shown in the table. Given the ownership of two cells, an auctioneer has four possible strategies, either auctioning them both in the first round, one in each of the first and second round, or both in the second round. Using the ownership of North and East as an example, these strategies would be labeled NE-, N-E, E-N, and -NE.

Thus, the choices of the first auctioneer, in order, are to sell both the North and East in the first round, sell North in the first and East in the second round, sell East in the first and North in the second round, or sell both in the second round. Combined with the choice of the second auctioneer, we have the round ordering for the contestants. For example, along the first row, we

Table 4.2: Expected Values for Case 2.1 (* indicates pure Nash equilibrium)

–	SW-	S-W	W-S	-SW
NE-	.375, .375	.320, .420	.219, .469	.401, .200
N-E	.420, .320	.375, .375*	.404, .197	.375, .375
E-N	.469, .219	.197, .404	.375, .375	.271, .271
-NE	.200, .401	.375, .375	.271, .271	.375, .375

have the round orders NSEW-, NSE-W, NEW-S, and NE-SW. We then take the expected spending on each cell from Table 1 above, for example for NSEW- we have expected spending on 0.25 on North and South, and 0.125 on East and West. We then sum the amounts for the North and East to find the first auctioneer's expected value of 0.375. Summing the amounts for South and West will similarly give us the second auctioneer's expected value of 0.375. We then repeat this for all the cells, giving us our table of expected values for the auctioneers. Looking at the table, we see that there is only one Nash equilibrium, N-E and S-W, giving both auctioneers an expected value of 0.375.

Case 2.2: NS,EW

The other possibility for two auctioneers each owning two cells is for one to own the North and South, with the other having the East and West. In this case, the payoff matrix is as follows:

Table 4.3: Expected Values for Case 2.2 (* indicates pure Nash equilibrium)

–	EW-	E-W	W-E	-EW
NS-	.500, .250	.480, .260*	.480, .260*	.500, .250
N-S	.500, .188	.389, .211	.389, .211	.500, .250*
S-N	.500, .188	.389, .211	.389, .211	.500, .250*
-NS	.500, .250*	.292, .250	.292, .250	.500, .250*

Similarly to Case 2.1, for each combination of choices by the two auctioneers, we find the corresponding order from Table 1, and take the expected values for each cell. For example, if NS- and E-W are the auctioneers' strategies, the cells will be contested as NSE-W, which Table 1 shows has an expected value of 0.24 for North and South, 0.18 for East, and 0.08 for West. This gives a total of 0.48 for the first auctioneer and 0.26 for the second auctioneer.

In addition to the NS-,E-W; NS-,W-E; -NS,EW-; N-S,-EW; S-N,-EW and -NS,-EW pure Nash equilibria, there are also an infinite number of mixed strategy equilibria consisting of assigning probability p to E-W and $1 - p$ to W-E, with the other auctioneer choosing NS-, the second auctioneer choosing EW- with probability q and -EW with probability $1-q$, with the first auctioneer choosing -NS, as well as the second auctioneer choosing -EW, with the first randomly choosing any strategy of N-S, S-N, or -NS. All of these mixed strategies use pure strategies with identical values and thus have the same expected value.

4.4.2 Unequal Cell Distribution

In these cases, one auctioneer owns three cells, while the other holds a single cell. In these we divide into cases based off if the single cell is one of the East and West, or of North and South.

Case 2.3: NSW,E; NES,W

Table 4.4: Expected Values for Case 2.3 (* indicates pure Nash equilibrium)

–	E-	-E
NWS-	.625, .125*	.660, .080
NS-W	.560, .180	.625, .125
NW-S, SW-N	.594, .094	.497, .104
N-WS, S-NW	.494, .107	.625, .125
W-NS	.625, .125*	.417, .125
-NWS	.417, .125	.625, .125

We obtain the values as before, For example if the auctioneers play NS-W, -E, there is an overall order of NS-EW. This has an expected value of 0.25 for North and South, and 0.125 for East and West. Thus the first auctioneer, who owns North, South, and West, has an expected value of 0.625 and the second auctioneer gains the value for East of 0.125.

There are the two pure equilibria NWS-,E- and W-NS,E-, along with mixed strategies consisting of the first auctioneer randomly choosing between the two. These all have expected values of 0.625 for the first auctioneer, with the second receiving 0.125. Note also the difference between the NS-W and W-NS rows. With NSE-W, both contestants have the possibility of winning after the first round, which is impossible in EW-NS. Thus the contestants have incentive to spend more on East in NSE-W than in EW-NS, as this increases the probability of winning the prize after the first round and thereby avoiding any spending on West.

Case 2.4: EWS,N; NEW,S

We can see from the table that ES-W, WS-E, E-WS, and W-ES are all at strictly dominated by EW-S, and thus can be eliminated. Similarly, S-EW and -EWS are weakly dominated by both

Table 4.5: Expected Values for Case 2.4

–	N-	-N
EWS-	.500, .250	.563, .125
EW-S	.563, .125	.500, .250
ES-W, WS-E	.500, .240	.304, .297
E-WS, W-ES	.508, .093	.396, .146
S-EW	.500, .250	.500, .250
-EWS	.500, .250	.500, .250

EWS- and EW-S, and strictly dominated by any (non-degenerate) mixed strategy between EWS- and EW-S, and thus can be eliminated. This leaves us a reduced matrix.

Table 4.6: Expected Values for Case 2.4 after elimination

–	N-	-N
EWS-	.500, .250	.563, .125
EW-S	.563, .125	.500, .250

The remaining choices behave as Matching Pennies, and thus both auctioneers will employ a mixed strategy, the first auctioneer choosing EWS- and EW-S both with .5 probability, and the second auctioneer choosing between N- and -N, again with both having .5 probability. This results in the first auctioneer having expected value 0.531 and the second 0.188. The lack of a pure strategy comes from the complementarity between North and South, with the owner of North attempting to make this complementarity the most valued as the expense of the complementarities involving the East and West, while the other owner benefits from spending by the contestants on all complementarities.

Proposition 4.2. *In the Nash equilibria, the total expected value for the auctioneers is maximized if East and West belong to different auctioneers.*

Cases 2.1 and 2.3, in which East and West belong to the two different auctioneers have total expected values of 0.75. Case 2.2 has pure strategy Nash equilibria with total expected value of 0.74, and Case 2.4 has only a mixed strategy equilibrium with a total expected value of 0.719.

In Case 2.1, each auctioneer has a winning set for one contestant, while in Case 2.3, one auctioneer has two winning sets for each contestant. Meanwhile, in Case 2.2, one auctioneer has one winning set for each contestant, while in Case 2.4, one auctioneer has a single winning set for each contestant. Thus the complementarities seen in the distribution of the cells between the auctioneers plays a role in the total expected expenditures.

4.5 Three Auctioneer Cases

Naturally, we can further extend the idea of multiple auctioneers to three auctioneers. Some such cases seem difficult to reconcile with the example being governments zoning land, however private groups could easily be the owners as well, where one owner of discontinuous parcels of land makes sense. Furthermore, quirks of geography and history may produce such situations. Again, we split into cases, with the cases differing by the auctioneer holding two cells having a winning set for both, one, or neither contestant.

Case 3.1: NS,E,W

With three auctioneers, we will list the first auctioneer's choices as the rows, the second auctioneer's choices as the columns, and the third auctioneer's choices as between the tables, with the third auctioneer's choice listed in bold.

Table 4.7: Expected Values for Case 3.1 (* indicates pure Nash equilibrium)

W-	E-	-E
NS-	.500, .125, .125*	.480, .080, .180
N-S	.500, .094, .094	.389, .104, .107
S-N	.500, .094, .094	.389, .107, .104
-NS	.500, .125, .125*	.297, .125, .125

-W	E-	-E
NS-	.480, .180, .080	.5, .125, .125
N-S	.389, .107, .104	.5, .125, .125
S-N	.389, .104, .107	.5, .125, .125
-NS	.292, .125, .125	.5, .125, .125*

We see that we have three pure Nash equilibria, NS-,E-,W-; -NS,E-,W-; and -NS,-E,-W. There is also a mixed strategy of the first auctioneer randomly choosing between the NS-,E-,W- and -NS,E-,W- pure equilibria, which all contestants are indifferent between. All of these equilibria have values of 0.5, 0.125, 0.125, for a total of 0.75. These are the simultaneous case and the EW-NS case, two of the options that a single auctioneer would choose.

Case 3.2: EW,N,S

We have 5 pure Nash equilibria, -EW,-N,S-; -EW,N-,-S; E-W,-N,-S; W-E,-N,-S; -EW,-N,-S; along with a mixed equilibrium of the first auctioneer randomly choosing between the last three.

Table 4.8: Expected Values for Case 3.2 (* indicates pure Nash equilibrium)

S-	N-	-N
EW-	.250, .250, .250	.188, .125, .375
E-W	.260, .240, .240	.211, .297, .093
W-E	.260, .240, .240	.211, .297, .093
-EW	.250, .250, .250	.250, .250, .250*
-S	N-	-N
EW-	.188, .375, .125	.250, .250, .250
E-W	.211, .093, .297	.250, .146, .146*
W-E	.211, .093, .297	.250, .146, .146*
-EW	.250, .250, .250*	.250, .250, .250*

These all result in an expected value for the first auctioneer of 0.25, with the values for the second and third ranging between 0.146 and 0.25. The only combination of strategies that result in a greater payoff for the first auctioneer are E-W, N-, S- and W-E, N-, S-, which cause both the second and third auctioneer to change strategies. Note that the second and third auctioneer will always have the same expected value as each other. Also note that -EW,-N,-S; -EW, N-, S-; and -EW, -N, -S are the only strong Nash equilibria, which are equilibria in which no coalition of contestants can all benefit from jointly deviating. If E-W or W-E are chosen, cooperation between the owners of N and S would allow the improvement to E-W, N-, S- or W-E, N-, S-, respectively. This is the only case we see in this paper where there exists a coalition that can all benefit from jointly deviating.

Case 3.3: NE,S,W; NW,S,E; SE,N,W; SW,N,E

Table 4.9: Expected Values for Case 3.3 (* indicates pure Nash equilibrium)

W-	S-	-S
NE-	.375, .250, .125	.469, .125, .094
N-E	.320, .240, .180	.197, .297, .104
E-N	.219, .375, .094	.375, .250, .125
-NE	.401, .093, .107	.271, .146, .125
-W	S-	-S
NE-	.420, .240, .080	.200, .297, .104
N-E	.375, .250, .125	.375, .250, .125*
E-N	.404, .093, .104	.271, .146, .125
-NE	.375, .250, .125	.375, .250, .125*

We see we have pure strategies of N-E,-S,-W, as well as -NE,-S,-W. Furthermore, looking at this table, we see that -NE weakly dominates N-E, however this is the only dominance that occurs. We will assign α to the probability of NE-, β to E-N, γ to S-, and δ to W-. Thus we can make a table of the probabilities for each combination being chosen.

Table 4.10: Probabilities of each Combination in Case 3.3

W-	S-	-S
NE-	$\alpha\gamma\delta$	$\alpha(1-\gamma)\delta$
N-E	0	0
E-N	$\beta\gamma\delta$	$\beta(1-\gamma)\delta$
-NE	$(1-\alpha-\beta)\gamma\delta$	$(1-\alpha-\beta)(1-\gamma)\delta$
-W	S-	-S
NE-	$\alpha\gamma(1-\delta)$	$\alpha(1-\gamma)(1-\delta)$
N-E	0	0
E-N	$\beta\gamma(1-\delta)$	$\beta(1-\gamma)(1-\delta)$
-NE	$(1-\alpha-\beta)\gamma(1-\delta)$	$(1-\alpha-\beta)(1-\gamma)(1-\delta)$

Multiplying these probabilities by the expected values for each cell gives us each player's expected value equation. Taking partial derivatives and solving gives us the optimal mixed strategies for the auctioneers, with the first auctioneer choosing NE- with probability .262, E-N with probability .297, and -NE with probability .441. The second auctioneer chooses S- with probability .795, and the third auctioneer chooses W- with probability .010. These probabilities result in approximate expected values of 0.374 for the first auctioneer, 0.207 for the second and 0.109 for the third, all worse than the pure strategies. If the third auctioneer could credibly commit to -W they would do so, as this would result in one of the two pure strategy equilibria, so the very low probability of choosing W- is reasonable.

In the two auctioneer cases, the total expected value for the auctioneers ranges from 0.719 in Case 2.4 to 0.75 in Cases 2.1 and 2.3. With three auctioneers, the total expected values range from 0.547 in two of the equilibria in Case 3.2 to 0.75, which occurs in at least one equilibrium of each of Cases 3.1, 3.2, and 3.3. Thus we can state the following results:

Proposition 4.3. *Increasing from one to two auctioneers decreases the minimum expected value for the auctioneers in equilibrium.*

Proposition 4.4. *Increasing from two to three auctioneers decreases the minimum expected value for the auctioneers in equilibrium.*

4.6 Four Auctioneers

None of the four auctioneers will have a pure strategy, as for any order seen in Table 1, there is always a better order for at least one auctioneer. Calculating expected values for each combination of strategies is thus required. These expected values are simply the sum of the expected values for each combination of strategies multiplied by the probability that the given strategy is chosen by the four auctioneers. Note that throughout Table 1, if North and South are in separate rounds, reversing the order of North and South gives us a reversal of the expected values for North and South. This also happens for East and West. Due to this symmetry of the outcomes, the optimal strategies for the North and South auctioneers should be identical, as should the East and West auctioneers. Thus, using ω for the probability that W- is chosen, and ν as the probability N- is chosen, the first-order conditions become

$$.0078\omega^2\nu - .0078\omega\nu + .2307\omega^2 - .1056\omega = 0$$
$$-.0254\nu^2\omega + .0154\nu\omega + .0913\nu^2 - .0363\nu = 0$$

There are five pairs of solutions to these equations, however only one non-degenerate solution, $\omega = .465, \nu = .366$. Three of the other pairs of solutions have a pure strategy for at least one auctioneer and create minimum expected values for the owners of the cells with degenerate probabilities. These degenerate strategies cause the benefits to accrue to the owners of the other cells. The fifth solution set involves probabilities greater than 1, which are impossible. The applicable mixed strategy results in an expected value of approximately 0.216 for the North and South auctioneers and 0.118 for the East and West. Thus the total expected value is approximately 0.667.

Proposition 4.5. *Increasing from three to four auctioneers increases the minimum expected value in equilibrium and decreases the maximum expected value.*

Note that increasing the number of auctioneers does not consistently increase or decrease the total expected values for the contestants.[2] This is in contrast to most market structures, where increasing the number of competitors drives down prices for the consumers. The difference is that winning the individual cells in our game does not provide benefits to the contestants, only combinations of cells carrying benefits. Thus the distribution of these complements between the various auctioneers plays a major role in the benefit to the contestants.

[2] Recall that the total excepted values for the auctioneers and contestants sum to 1.

4.7 Conclusions and Further Work

For the contestants, we saw that complementarity plays a major role in expected payoffs, both in the requirements of victory and in that this complementarity also occurs in the auction rounds. Differences in the order of possibly obtaining complementary sets create differences in the expected payoffs for the contestants.

Conversely, we see that the primary factor in the expected value for an auctioneer is owning the more valuable cells, North and South, and owning a larger number of cells. Little in the way of complementarity is seen in owning particular combinations, despite the contestants competing the cells having strong complementarity. The only seeming source of complementarity is owning both East and West results in some gains from complementarity, as in the three auctioneer case.

Furthermore, the original distribution of cells between auctioneers does not appear to have much impact on the contestants. The only case in which the original distribution between the auctioneers favors one contestant is when one auctioneer has a winning set for a contestant, in a three auctioneer case. This allows for a mixed strategy equilibrium which favors the contestant whose winning set is owned by an auctioneer.

Several extensions of this work are apparent. Many possible extensions are mentioned in our previous paper, consisting of variations in the structure of the competition between the two contestants attempting to connect the opposing sides. Such changes would change the expected value of the individual cells, and in some cases the number of cells, however the methods of solving the auctioneers' problem would remain the same.

Another possibility would be to link the contestants and auctioneers between games. Such a contest could consist of two individuals attempting to gain control of a winning set, with the winner receiving a prize, and then subsequently auctioning off their cells in the same contest method. This would obviously require a different analysis, as now individual cells do have a benefit, the value that they supply in the future contest. We leave this problem for future work.

Chapter 5: A Model of Political Competition

5.1 Introduction

Consider two politicians who are both running for the same Senate seat. Each politician, as well as the party they belong to as a whole, wishes to win the seat. However, if victory can easily be obtained, this politician's funds and time can be spent trying to secure the election of party members in other races. Thus each party wishes to win seats for the lowest possible cost.

Our objective in this paper is to focus on the political competition of this nature. We will assume that these politicians will compete across a variety of issues. The specifics of these issues are irrelevant, as long as they are all equally important to the voters. We will assume that each party has a pool of captive voters, and the remaining small number of decisive voters are relatively homogenous. The party who wins the majority of these areas will win the overall election.[1]

This paper will take the issues to be International, Social, and Economic.[2] Each party will allocate time and money, combined into a general resource, into convincing the voters that they are better than their opponent on each issue. By doing so, each party will attempt to convince the majority of the voting public that they are overall preferable and thereby win the election. We use the Tullock Contest Success Function (Tullock, 1980), as a party who spends more time campaigning on an issue is more likely to win over voters on this issue, however spending more time does not ensure winning the voters over on a given issue.

We add in the concept of spillover effects, where effort expended on one issue also benefits the party's standing in other areas. What spillover exists for the candidate from a party will be taken as exogenous, being determined by the party's historical platform and pre-existing voter beliefs. In examining these different links, we will see the impact they have on parties' behavior.

Much work has been done previously on elections as game theory problems. One of the foundations of this was Downs (1957), who modeled voters' and candidates' positions as belonging to a single dimensional spectrum. Each voter chooses to vote for the candidate that most closely matches the voter's position. If limited to two candidates, this leads directly to the Median Voter

[1] We use parties as the sides in this competition rather than candidates, so as to include the combined effects of the candidates' campaign along with national issue ads, endorsements from prominent national figures, and coverage by partisan media. These additional sources of influence are all important in determining the winner of an election, however are outside the control of the candidates' own decisions.

[2] There are, of course, several specific issues whose placement is open to debate. However, this makes for a reasonable approximation.

Theorem, which states that candidates will choose positions as close to the median voter as possible. This model does require several assumptions, weakening of which also produces interesting results.

One of these extensions involves candidates having ambiguous positions. As Shepsle (1972) notes, this is common among observed campaigns, with candidates often making vague statements instead of staking out clear positions. This is explained by a model in which there are two types of candidates: incumbents, whose positions are known with certainty, and challengers, whose positions are uncertain, though with some non-degenerate probability distribution. If voters are risk-averse, this uncertainty undermines the median voter, and gives an advantage to the incumbent. Furthermore, a challenger with a wide range of possible true positions will have a greater disadvantage.[3]

Budge (1982) takes an issue based approach, analyzing the policy statements of British parties and matching them to which issues a party has a comparative advantage in. For example, in 1979 the Conservatives clearly stated that they would cut government spending, while avoiding Labour's questions of where the cuts would come from. In this "saliency theory" approach, some portions of the electorate have a particular area they are focused on, which determines the party they then vote for. The parties compete for the remaining voters not by choosing particular positions on each issue area, but by emphasizing the areas which they are stronger than the opposing party. The areas which the party is weaker have their importance downplayed, thus making the party seem better on the important issues.

Petrocik (1996) extends this model by having both competence at various issues and relative importance of issues be variable. This model involves there being issues that are generally owned by one party or the other, however there may be short term fluctuations due to a party's failures at handling their issues. In this model both the party, which defines which issues are owned, and the candidate, which determines competence, are important. This is illustrated by the 1980 Presidential election, in which foreign policy, generally considered neutral between parties, greatly helped Reagan. This came about due to Carter's previous failures at foreign policy, thus making even neutral news reporting on foreign affairs reflect poorly on Carter. Much of Reagan's successful campaign is attributed by Petrocik to better focusing on advantageous issues, both those that were long term Republican advantages and the short term opportunities provided by Carter's record.

Another related paper is Denter (2013), which takes a different approach to campaign issues. In this paper, candidates spend effort to appear better in a given area, and also to make the issue

[3]Banks (1990) extends this to cases where both candidates have uncertain positions.

seem more important to voters. In this paper the candidates have inherent strengths in some issues, which then adjust with effort expended. This is similar to our work, however it treats each issue as completely distinct, as opposed to having interactions between areas. This does allow Denter to have both issue importance and competence vary with effort.

The basic model we use is related to the Colonel Blotto games. These were first proposed by Borel (Borel 1921, Borel and Ville 1938), and consist of allocating a fixed resource across several battlefields. The army with a greater number of troops at a given battle will win, and each side is attempting to win the majority of battlefields. This game was expanded upon by Gross and Wagner (1950) and Friedman (1958), however was not fully solved until Roberson (2006).

Sela and Erez (Forthcoming) examine a game with a similar concept of spillovers. In their game, a multi-stage contest is held in which each player has a budget constraint. In each stage, each player invests resources, with the winner of the stage determined by the Tullock contest success function. However, only a fraction of the resources spent are deducted from the budget available in the subsequent stage.

The paper is laid out as follows. First, in Section 2 we solve a base line case with no spillover effects for comparative purposes. This is followed in Section 3 by cases with common sources and common destinations of spillovers, and then the other possible combinations of spillovers in Section 4. Finally in Section 5 we have some concluding remarks and comments on future work.

5.2 The Model

We will label the two parties as R and D, and the issue areas as I, S, and E. R_J will indicate the *effective* spending by Party R on issue J, while r_J will indicate the actual spending. The effective spending includes both the amount spent on the area as well as any spillover, with the actual spending being only the amount directly spent on the area. Z_J will indicate the sum $R_J + D_J$. Thus, the Tullock success function gives us that the chance of Party R winning on issue E is $\left(\frac{R_E}{Z_E}\right)$. The probabilities of victory on each issue will be assumed to be independent. In the most basic version of the model, we assume that to win election, victory in two of the three issue areas is required and that effective and actual spending are identical. Thus Party R has a probability of victory of:

$$\left(\frac{R_E}{Z_E}\right)\left(\frac{R_S}{Z_S}\right)\left(\frac{R_I}{Z_I}\right)+\left(\frac{R_E}{Z_E}\right)\left(\frac{R_S}{Z_S}\right)\left(\frac{D_I}{Z_I}\right)+\left(\frac{R_E}{Z_E}\right)\left(\frac{D_S}{Z_S}\right)\left(\frac{R_I}{Z_I}\right)+\left(\frac{D_E}{Z_E}\right)\left(\frac{R_S}{Z_S}\right)\left(\frac{R_I}{Z_I}\right)$$

with Party D having the corresponding probability of

$\left(\frac{D_E}{Z_E}\right)\left(\frac{D_S}{Z_S}\right)\left(\frac{D_I}{Z_I}\right)+\left(\frac{D_E}{Z_E}\right)\left(\frac{D_S}{Z_S}\right)\left(\frac{R_I}{Z_I}\right)+\left(\frac{D_E}{Z_E}\right)\left(\frac{R_S}{Z_S}\right)\left(\frac{D_I}{Z_I}\right)+\left(\frac{R_E}{Z_E}\right)\left(\frac{D_S}{Z_S}\right)\left(\frac{D_I}{Z_I}\right).$

In each equation the first term is the probability of the party winning all three areas, the second term winning on issues E and S but losing I, the third term winning on E and I and losing on S, and the final term winning on S and I and losing on E.

Normalizing the value of winning the election to 1 thus gives us expected value equations of

$$\begin{aligned}
EV[R] = &\left(\frac{R_E}{Z_E}\right)\left(\frac{R_S}{Z_S}\right)\left(\frac{R_I}{Z_I}\right)+\left(\frac{R_E}{Z_E}\right)\left(\frac{R_S}{Z_S}\right)\left(\frac{D_I}{Z_I}\right)+ \\
&\left(\frac{R_E}{Z_E}\right)\left(\frac{D_S}{Z_S}\right)\left(\frac{R_I}{Z_I}\right)+\left(\frac{D_E}{Z_E}\right)\left(\frac{R_S}{Z_S}\right)\left(\frac{R_I}{Z_I}\right)-R_E-R_S-R_I \\
EV[D] = &\left(\frac{D_E}{Z_E}\right)\left(\frac{D_S}{Z_S}\right)\left(\frac{D_I}{Z_I}\right)+\left(\frac{D_E}{Z_E}\right)\left(\frac{D_S}{Z_S}\right)\left(\frac{R_I}{Z_I}\right)+ \\
&\left(\frac{D_E}{Z_E}\right)\left(\frac{R_S}{Z_S}\right)\left(\frac{D_I}{Z_I}\right)+\left(\frac{R_E}{Z_E}\right)\left(\frac{D_S}{Z_S}\right)\left(\frac{D_I}{Z_I}\right)-D_E-D_S-D_I
\end{aligned}$$

Taking partial derivatives gives us the first-order conditions shown in the Appendix. As for second order conditions, in this as well as in all the cases with spillovers, there is a single unique solution to the first-order conditions, which results in positive expected values for both parties. Expenditures of 0 on all areas leads to an expected value of 0, while spending 1 on each area gives a negative expected value. Thus our first-order conditions give the unique maximum. Solving these first-order conditions gives us that $R_E = R_S = R_I = D_E = D_S = D_I = \frac{1}{8}$ in equilibrium, and an expected value for each party of $\frac{1}{8}$. This gives us our first proposition.

Proposition 5.1. *If there are no spillovers, both parties spend equally on all three areas.*

5.3 Common Sources and Destinations of Spillovers

We will assume a spillover strength of $\frac{1}{2}$, such that half the amount spent on the source will also apply to the destination. It is important to note that this spillover is purely additional, all the spending on the source will apply to the competition for the source area, as opposed to some of the spending being transferred.

Common Source of Spillovers

We now consider the possibility of spillover effects, in which the spending on one area also impacts how the party is viewed in other areas. There are five distinct types of spillover patterns, as seen in this diagram.

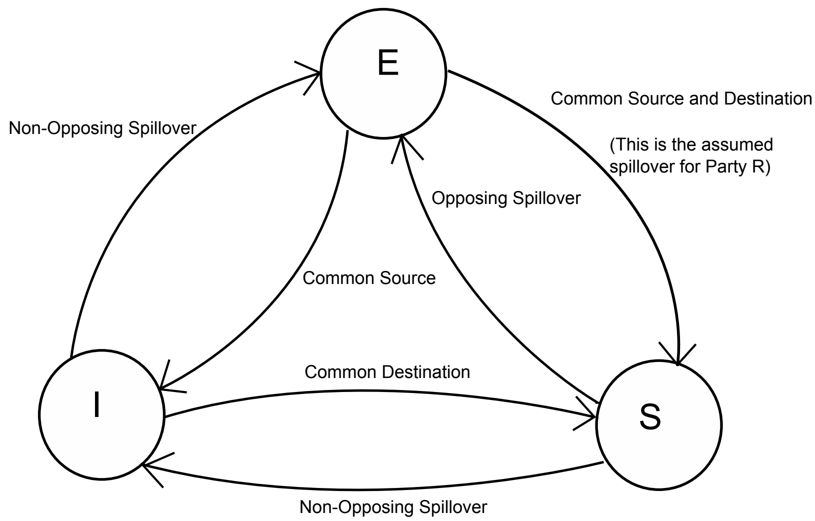

Figure 5.1: Types of Spillovers

Given a fixed spillover for party R, party D can have the same spillover, the same source but different destination, the same destination but different source, or an interchange of the source and destination. These all result in symmetrical games. The parties may also have asymmetric spillovers, such that the destination for one party is the source for the other party, which then spills over into the third issue.

For our first example, we will use a common source of spill overs, with spending on E spilling over to I for Party R, and S for Party D. We will take half the spending as having these spill over effects, so $R_I = r_I + \frac{1}{2}r_E$ and $D_S = d_S + \frac{1}{2}d_E$. Thus the expected value equations are

$$
\begin{aligned}
EV[R] = & \left(\frac{r_E}{(r_E+d_E)}\right)\left(\frac{r_S}{r_S+\frac{1}{2}d_E+d_S}\right)\left(\frac{\frac{1}{2}r_E+r_I}{\frac{1}{2}r_E+r_I+d_I}\right)+ \\
& \left(\frac{r_E}{(r_E+d_E)}\right)\left(\frac{r_S}{r_S+\frac{1}{2}d_E+d_S}\right)\left(\frac{d_I}{\frac{1}{2}r_E+r_I+d_I}\right)+ \\
& \left(\frac{r_E}{(r_E+d_E)}\right)\left(\frac{\frac{1}{2}d_E+d_S}{r_S+\frac{1}{2}d_E+d_S}\right)\left(\frac{\frac{1}{2}r_E+r_I}{\frac{1}{2}r_E+r_I+d_I}\right)+ \\
& \left(\frac{d_E}{(r_E+d_E)}\right)\left(\frac{r_S}{r_S+\frac{1}{2}d_E+d_S}\right)\left(\frac{\frac{1}{2}r_E+r_I}{\frac{1}{2}r_E+r_I+d_I}\right)-(r_E+r_S+r_I) \\
EV[D] = & \left(\frac{d_E}{(r_E+d_E)}\right)\left(\frac{\frac{1}{2}d_E+d_S}{r_S+\frac{1}{2}d_E+d_S}\right)\left(\frac{d_I}{\frac{1}{2}r_E+r_I+d_I}\right)+ \\
& \left(\frac{d_E}{(r_E+d_E)}\right)\left(\frac{\frac{1}{2}d_E+d_S}{r_S+\frac{1}{2}d_E+d_S}\right)\left(\frac{\frac{1}{2}r_E+r_I}{\frac{1}{2}r_E+r_I+d_I}\right)+ \\
& \left(\frac{d_E}{(r_E+d_E)}\right)\left(\frac{r_S}{r_S+\frac{1}{2}d_E+d_S}\right)\left(\frac{d_I}{\frac{1}{2}r_E+r_I+d_I}\right)+ \\
& \left(\frac{r_E}{(r_E+d_E)}\right)\left(\frac{\frac{1}{2}d_E+d_S}{r_S+\frac{1}{2}d_E+d_S}\right)\left(\frac{d_I}{\frac{1}{2}r_E+r_I+d_I}\right)-(d_E+d_S+d_I)
\end{aligned}
$$

As shown in the Appendix, solving the first-order conditions gives us $r_S = \frac{1}{8}$, $r_E = \frac{1}{6}$, and $r_I = \frac{1}{24}$. This solution involves using a symmetry argument, which is more fully explained in the Appendix with the solution. This involves the knowledge that in equilibrium, $r_E = d_E$, $r_S = d_I$, and $r_I = d_S$. We will use similar symmetry arguments for most of the cases considered. This solution makes sense, with the source of the spillovers having the highest spending, and the the destination having the lowest. Each player will spend $\frac{1}{3}$ in aggregate and thus have an expected value of $\frac{1}{6}$.

Common Destination for Spillovers

If instead we have a common destination for the spillovers, such that $R_E = r_E + \frac{1}{2}r_I$ and $D_E = d_E + \frac{1}{2}d_S$, our expected value equation for party R is

$$\begin{aligned} EV[R] = -2 &\left(\frac{r_E + \frac{1}{2}r_I}{r_E + \frac{1}{2}r_I + d_E + \frac{1}{2}d_S}\right)\left(\frac{r_S}{r_S + d_S}\right)\left(\frac{r_I}{r_I + d_I}\right) + \\ &\left(\frac{r_E + \frac{1}{2}r_I}{r_E + \frac{1}{2}r_I + d_E + \frac{1}{2}d_S}\right)\left(\frac{r_S}{r_S + d_S}\right) + \left(\frac{r_E + \frac{1}{2}r_I}{r_E + \frac{1}{2}r_I + d_E + \frac{1}{2}d_S}\right)\left(\frac{r_I}{r_I + d_I}\right) + \\ &\left(\frac{r_S}{r_S + d_S}\right)\left(\frac{r_I}{r_I + d_I}\right) - r_E - r_S - r_I \end{aligned}$$

The second, third, and fourth terms are the probabilities of party R winning areas E and S, E and I, and S and I, respectively. The first term in to account for party R winning all three areas, as this possibility is counted in each of the second, third, and fourth terms, when it should only be counted once. The expected value equation for party D is obtained in the same way. As before, we take the partial derivatives and then impose the symmetry conditions, in this case $r_E = d_E$, $r_S = d_I$, and $r_I = d_S$. As shown in the Appendix, these conditions combined with the three partial derivatives of $EV[R]$ are sufficient to solve for the unique maximum, allowing us to state the following proposition.

Proposition 5.2. *Having a common destination for spillovers results in greater expected values to the parties than having a common source of spillovers.*

In the common source case, we have $r_E = \frac{1}{6}$, $r_S = \frac{1}{8}$, and $r_I = \frac{1}{24}$, giving an expected value of $\frac{1}{6}$, while in the common destination case these are $r_E = \frac{19}{108}$, $r_S = \frac{1}{27}$, and $r_I = \frac{2}{27}$, giving an expected value of $\frac{23}{108}$. The greater expected value makes sense here, as in both cases, there is one area that is evenly contested, while each party has a distinct advantage in one other area. In the common destination case, the evenly contested area is receiving spillover benefits, and thus will already have some effective expenditure outside of that specifically directed towards the area. Combined with the diminishing returns to expenditure on an area, this leads to lower spending by each party.

Common Sources and Destinations of Spillovers

If we take the case when both parties have the same spillover, for example from E to S, we get an expected value equation for R of:

$$EV[R] = -2\left(\frac{r_E}{r_E+d_E}\right)\left(\frac{r_S+\frac{1}{2}r_E}{r_S+\frac{1}{2}r_E+d_S+\frac{1}{2}d_S}\right)\left(\frac{r_I}{r_I+d_I}\right)+$$
$$\left(\frac{r_E}{r_E+d_E}\right)\left(\frac{r_S+\frac{1}{2}r_E}{r_S+\frac{1}{2}r_E+d_S+\frac{1}{2}d_S}\right)+\left(\frac{r_E}{r_E+d_E}\right)\left(\frac{r_I}{r_I+d_I}\right)+$$
$$\left(\frac{r_S+\frac{1}{2}r_E}{r_S+\frac{1}{2}r_E+d_S+\frac{1}{2}d_S}\right)\left(\frac{r_I}{r_I+d_I}\right)-r_E-r_S-r_I$$

The expected value equation for party D is similar, with d_E interchanged with r_E, d_S interchanged with r_S, and d_I interchanged with r_I. The first-order equations are solved in the Appendix, with the appropriate symmetry conditions of $r_E = d_E$, $r_S = d_S$, and $r_I = d_I$, giving us the following proposition.

Proposition 5.3. *Common sources and destinations for spillovers have identical expected values for the parties as the no spillover case.*

As we see, each party spends $\frac{1}{4}$ on E and $\frac{1}{8}$ on I, with no spending on S. This gives the same expected value of $\frac{1}{8}$ that we saw in the no spillover case, with the spending simply shifting from the destination of the spillovers to the source. This makes sense, as we do not generally see parties having the same spillovers, which this shows us provide no benefit to the parties.

The overall pattern of spending we see here makes sense as well. When there is only a common source or common destination, this common area becomes the most heavily contested area as each party has an advantage is one of the two remaining areas. When there is both a common source and common destination, all three areas are competitive, and we simply see spending shift from the destination to the source.

Note that this differs from Budge's and Petrocik's results, where most efforts were expended on the areas where a party had an advantage. This is due to the different assumptions behind the models. Budge and Petrocik have effort being expended to increase the relative importance of an issue, which naturally drives campaigns to spend effort on the areas where they are seen as the more competent. Our model has the relative importance of all areas be fixed, with effort being spend on appearing more competent. If an area is seen as belonging to one side, there is little point in spending further effort on appearing competent at it, while areas where both sides begin equally competent provide large benefits for campaigning.

5.4 Other Spillovers

In this section we examine two other types of spillovers. With opposing spillovers, the source for one party is the destination for the other party, and vice versa. With non-opposing spillovers, the destination for one party is the source for the other party, however this second party's destination is the third area.

Opposing Spillovers

We can allow for the two parties to have symmetric spillovers, for example $R_S = r_S + \frac{1}{2}r_E$ and $D_E = d_E + \frac{1}{2}d_S$. This gives us the expected value equation

$$\begin{aligned} EV[R] = -2 &\left(\frac{r_E}{r_E + d_E + \frac{1}{2}d_S}\right)\left(\frac{r_S + \frac{1}{2}r_E}{r_S + \frac{1}{2}r_E + d_S}\right)\left(\frac{r_I}{r_I + d_I}\right) + \\ &\left(\frac{r_E}{r_E + d_E + \frac{1}{2}d_S}\right)\left(\frac{r_S + \frac{1}{2}r_E}{r_S + \frac{1}{2}r_E + d_S}\right)\left(\frac{r_E}{r_E + d_E + \frac{1}{2}d_S}\right)\left(\frac{r_I}{r_I + d_I}\right) + \\ &\left(\frac{r_S + \frac{1}{2}r_E}{r_S + \frac{1}{2}r_E + d_S}\right)\left(\frac{r_I}{r_I + d_I}\right) - r_E - r_S - r_I \end{aligned}$$

As above, we also have symmetry conditions, in this case $r_E = d_S$, $r_S = d_E$, and $r_I = d_I$. Making these symmetric substitutions gives us the similar expected value equation for party D. These symmetry conditions, combined with the partial derivatives of $EV[R]$ are sufficient to obtain the unique maximum. As the Appendix shows, this gives us $r_E = d_S = \frac{2}{9}$, $r_S = d_E = 0$, and $r_I = d_I = \frac{5}{36}$, and thus an expected value of $\frac{5}{36}$. Thus we see again each party having a major advantage in one area, and using their spillover from this area as the entire spending on the opponent's strength. However, unlike in the common source or common destination models, this does not drive the spending towards the contested area. The difference is that in this model, spending on the strong area spills over to the weak area, forcing a greater relative spending in the strong area.

Non-Opposing Spillovers

Finally, we can consider the case where there is neither a common source nor destination for the spillover effects. For example, we will take $R_I = r_I + \frac{1}{2}r_E$ and $D_E = d_E + \frac{1}{2}d_S$. Due to a lack of symmetry, we need to solve all six first-order conditions, giving us $r_E = \frac{1}{9}$, $r_S = \frac{1}{9}$, $r_I = \frac{1}{18}$,

$d_E = \frac{1}{9}$, $d_S = \frac{2}{9}$, and $d_I = \frac{1}{9}$. This leaves party R with a $\frac{1}{3}$ chance of victory, giving party R a total expected value of $\frac{1}{18}$ and party D a total expected value of $\frac{2}{9}$.

Proposition 5.4. *Non-opposing spillovers lead to the same total expected values for the parties as opposing spillovers.*

We see this directly from the expected values, which sum to $\frac{5}{18}$ in each case. Note that this happens despite the optimal strategies being completely different between the non-opposing and opposing cases. In the opposing spillover case we have symmetric strategies, and thus identical probability of overall victory. In the non-opposing case that parties have differing levels of spending, and differing probability of winning the election.

5.5 Conclusions

In this paper we compared the effects of various combinations of spillovers between issue areas for two parties. We see that when the spillover effects give each party an advantage in one area, the parties are content to concede the opponent's strength, and focus on the area where there is close competition. This is not surprising, as we would expect greater effort in close competitions in general. This does not always lead to the highest spending in the third area, as we saw in the common source and destination case. In that case the parties put effective effort into the opponent's strength due to the nature of the spillovers, which increased total spending.

By far the greatest total expected value comes in the case where both spillovers have a common destination. This makes sense, as both parties are free to concentrate on the common destination, which will already have significant effective spending from the spillovers. In this case, each party nearly as high of an expected value as the party with the advantage in the non-opposing spillover case.

The only case in which either party is disadvantaged by the introduction of spillovers is the non-opposing spillover case. This also makes sense, as the spillovers provide effectively free additional spending. This could lead to both parties establishing spillovers, and effectively agreeing to only campaign on one issue. The saving from doing so could give both parties sufficient funding to fight off any possible new third party challenger.

There are obviously a few ways to proceed from here. One would be to test this experimentally, to see if players realize the impacts of the spillovers. The asymmetric spillover case in particular has a non-obvious correct strategy, with the different players having markedly different optimal strategies.

Another possibility would be to extend the number of areas of competition. This would allow for new interactions between spillovers, and allow for different weighting of areas. For example, if there are five areas, it may be that two are more important than the other three, and a party that wins the two more important issues wins the overall competition, while if they split the other areas become meaningful.

Finally, we could allow for varying strengths of spillovers. By changing from the $\frac{1}{2}$ spillover term, we may see interesting differences in optimal strategies. We could also try having different strengths of spillovers for the two parties. This may simply bias the game towards the party with the stronger spillover effect, but it may also induce qualitative differences in strategies. The issue here is again tractability, but this may be solvable. It would even be possible to take this to negative spillovers, where campaigning on an issue directly hurts the party's efforts in another area. This would necessitate additional constraints on party expenditures to prevent negative effective spending.

5.6 Appendix

The first-order conditions for the no spillover case are

$$
\begin{aligned}
\frac{\partial EV[R]}{\partial R_E} &: \left(\frac{D_E}{Z_E^2}\right)\left(\frac{R_S}{Z_S}\right)\left(\frac{R_I}{Z_I}\right)+\left(\frac{D_E}{Z_E^2}\right)\left(\frac{R_S}{Z_S}\right)\left(\frac{D_I}{Z_I}\right)+\left(\frac{D_E}{Z_E^2}\right)\left(\frac{D_S}{Z_S}\right)\left(\frac{R_I}{Z_I}\right)-\left(\frac{D_E}{Z_E^2}\right)\left(\frac{R_S}{Z_S}\right)\left(\frac{R_I}{Z_I}\right)=1\\
\frac{\partial EV[R]}{\partial R_S} &: \left(\frac{R_E}{Z_E}\right)\left(\frac{D_S}{Z_S^2}\right)\left(\frac{R_I}{Z_I}\right)+\left(\frac{R_E}{Z_E}\right)\left(\frac{D_S}{Z_S^2}\right)\left(\frac{D_I}{Z_I}\right)-\left(\frac{R_E}{Z_E}\right)\left(\frac{D_S}{Z_S^2}\right)\left(\frac{R_I}{Z_I}\right)+\left(\frac{D_E}{Z_E}\right)\left(\frac{D_S}{Z_S^2}\right)\left(\frac{R_I}{Z_I}\right)=1\\
\frac{\partial EV[R]}{\partial R_I} &: \left(\frac{R_E}{Z_E}\right)\left(\frac{R_S}{Z_S}\right)\left(\frac{D_I}{Z_I^2}\right)-\left(\frac{R_E}{Z_E}\right)\left(\frac{R_S}{Z_S}\right)\left(\frac{D_I}{Z_I^2}\right)+\left(\frac{R_E}{Z_E}\right)\left(\frac{D_S}{Z_S}\right)\left(\frac{D_I}{Z_I^2}\right)+\left(\frac{D_E}{Z_E}\right)\left(\frac{R_S}{Z_S}\right)\left(\frac{D_I}{Z_I^2}\right)=1\\
\frac{\partial EV[D]}{\partial D_E} &: \left(\frac{R_E}{Z_E^2}\right)\left(\frac{D_S}{Z_S}\right)\left(\frac{D_I}{Z_I}\right)+\left(\frac{R_E}{Z_E^2}\right)\left(\frac{D_S}{Z_S}\right)\left(\frac{R_I}{Z_I}\right)+\left(\frac{R_E}{Z_E^2}\right)\left(\frac{R_S}{Z_S}\right)\left(\frac{D_I}{Z_I}\right)-\left(\frac{R_E}{Z_E^2}\right)\left(\frac{D_S}{Z_S}\right)\left(\frac{D_I}{Z_I}\right)=1\\
\frac{\partial EV[D]}{\partial D_S} &: \left(\frac{D_E}{Z_E}\right)\left(\frac{R_S}{Z_S^2}\right)\left(\frac{D_I}{Z_I}\right)+\left(\frac{D_E}{Z_E}\right)\left(\frac{R_S}{Z_S^2}\right)\left(\frac{R_I}{Z_I}\right)-\left(\frac{D_E}{Z_E}\right)\left(\frac{R_S}{Z_S^2}\right)\left(\frac{D_I}{Z_I}\right)+\left(\frac{R_E}{Z_E}\right)\left(\frac{R_S}{Z_S^2}\right)\left(\frac{D_I}{Z_I}\right)=1\\
\frac{\partial EV[D]}{\partial D_I} &: \left(\frac{D_E}{Z_E}\right)\left(\frac{D_S}{Z_S}\right)\left(\frac{R_I}{Z_I^2}\right)-\left(\frac{D_E}{Z_E}\right)\left(\frac{D_S}{Z_S}\right)\left(\frac{R_I}{Z_I^2}\right)+\left(\frac{D_E}{Z_E}\right)\left(\frac{R_S}{Z_S}\right)\left(\frac{R_I}{Z_I^2}\right)+\left(\frac{R_E}{Z_E}\right)\left(\frac{D_S}{Z_S}\right)\left(\frac{R_I}{Z_I^2}\right)=1
\end{aligned}
$$

Which solve to give us $R_E = R_S = R_I = D_E = D_S = D_I = \frac{1}{8}$.

5.6.1 Common Source for Spillovers

We have first-order conditions from party R of

$$\frac{\partial EV[R]}{\partial r_E}: \left(\frac{-d_E}{(r_E+d_E)^2}\right)\left(\frac{r_S}{r_S+\frac{1}{2}d_E+d_S}\right)\left(\frac{\frac{1}{2}r_E+r_I}{\frac{1}{2}r_E+r_I+d_I}\right)+$$
$$\left(\frac{r_E}{(r_E+d_E)}\right)\left(\frac{r_S}{r_S+\frac{1}{2}d_E+d_S}\right)\left(\frac{\frac{1}{2}d_I}{(r_I+\frac{1}{2}r_E+d_I)^2}\right)+$$
$$\left(\frac{-d_E}{(r_E+d_E)^2}\right)\left(\frac{r_S}{r_S+\frac{1}{2}d_E+d_S}\right)\left(\frac{d_I}{\frac{1}{2}r_E+r_I+d_I}\right)+$$
$$\left(\frac{r_E}{(r_E+d_E)}\right)\left(\frac{r_S}{r_S+\frac{1}{2}d_E+d_S}\right)\left(\frac{-\frac{1}{2}d_I}{(r_I+\frac{1}{2}r_E+d_I)^2}\right)+$$
$$\left(\frac{-d_E}{(r_E+d_E)^2}\right)\left(\frac{\frac{1}{2}d_E+d_S}{r_S+\frac{1}{2}d_E+d_S}\right)\left(\frac{\frac{1}{2}r_E+r_I}{\frac{1}{2}r_E+r_I+d_I}\right)+$$
$$\left(\frac{r_E}{(r_E+d_E)}\right)\left(\frac{\frac{1}{2}d_E+d_S}{r_S+\frac{1}{2}d_E+d_S}\right)\left(\frac{\frac{1}{2}d_I}{(r_I+\frac{1}{2}r_E+d_I)^2}\right)+$$
$$\left(\frac{-r_E-\frac{1}{2}r_S}{(r_E+d_E)^2}\right)\left(\frac{r_S}{r_S+\frac{1}{2}d_E+d_S}\right)\left(\frac{\frac{1}{2}r_E+r_I}{\frac{1}{2}r_E+r_I+d_I}\right)+$$
$$\left(\frac{d_E}{(r_E+d_E)}\right)\left(\frac{r_S}{r_S+\frac{1}{2}d_E+d_S}\right)\left(\frac{\frac{1}{2}d_I}{(r_I+\frac{1}{2}r_E+d_I)^2}\right)=1$$
$$\frac{\partial EV[R]}{\partial r_S}: \left(\frac{r_E}{(r_E+d_E)}\right)\left(\frac{d_S+\frac{1}{2}d_E}{(r_S+d_S+\frac{1}{2}d_E)^2}\right)\left(\frac{\frac{1}{2}r_E+r_I}{\frac{1}{2}r_E+r_I+d_I}\right)+$$
$$\left(\frac{r_E}{(r_E+d_E)}\right)\left(\frac{d_S+\frac{1}{2}d_E}{(r_S+d_S+\frac{1}{2}d_E)^2}\right)\left(\frac{d_I}{\frac{1}{2}r_E+r_I+d_I}\right)+$$
$$\left(\frac{r_E}{(r_E+d_E)}\right)\left(\frac{-d_S-\frac{1}{2}d_E}{(r_S+d_S+\frac{1}{2}d_E)^2}\right)\left(\frac{\frac{1}{2}r_E+r_I}{\frac{1}{2}r_E+r_I+d_I}\right)+$$
$$\left(\frac{d_E}{(r_E+d_E)}\right)\left(\frac{d_S+\frac{1}{2}d_E}{(r_S+d_S+\frac{1}{2}d_E)^2}\right)\left(\frac{\frac{1}{2}r_E+r_I}{\frac{1}{2}r_E+r_I+d_I}\right)=1$$
$$\frac{\partial EV[R]}{\partial r_I}: \left(\frac{r_E}{(r_E+d_E)}\right)\left(\frac{r_S}{r_S+\frac{1}{2}d_E+d_S}\right)\left(\frac{d_I}{(r_I+\frac{1}{2}r_E+d_I)^2}\right)+$$
$$\left(\frac{r_E}{(r_E+d_E)}\right)\left(\frac{r_S}{r_S+\frac{1}{2}d_E+d_S}\right)\left(\frac{-d_I}{(r_I+\frac{1}{2}r_E+d_I)^2}\right)+$$
$$\left(\frac{r_E}{(r_E+d_E)}\right)\left(\frac{\frac{1}{2}d_E+d_S}{r_S+\frac{1}{2}d_E+d_S}\right)\left(\frac{d_I}{(r_I+\frac{1}{2}r_E+d_I)^2}\right)+$$
$$\left(\frac{d_E}{(r_E+d_E)}\right)\left(\frac{r_S}{r_S+\frac{1}{2}d_E+d_S}\right)\left(\frac{d_I}{(r_I+\frac{1}{2}r_E+d_I)^2}\right)=1$$

These combined with the symmetry are sufficient to solve for optimal spending. We also know that due to symmetry, $r_E = d_E$, $r_I = d_S$, and $r_S = d_I$. This comes from E being the common source of the spillovers, with I being the destination for party R and S being the destination for party D. Thus the destinations have the same spending, as do the areas with no spillovers. Making these substitutions and simplifying gives us

$$\begin{aligned}
\frac{1}{4r_E} + \frac{1}{4}\frac{r_S}{(\frac{1}{2}r_E + r_I + r_S)^2} - \left(\frac{1}{2r_E}\right)\left(\frac{r_S}{r_S + \frac{1}{2}r_E + r_I}\right)\left(\frac{\frac{1}{2}r_E + r_I}{\frac{1}{2}r_E + r_I + r_S}\right) &= 1 \\
\frac{1}{2}\left(\frac{\frac{1}{2}r_E + r_I}{(r_S + \frac{1}{2}r_E + r_I)^2}\right) &= 1 \\
\frac{1}{2}\left(\frac{r_S}{(r_S + \frac{1}{2}r_E + r_I)^2}\right) &= 1
\end{aligned}$$

Thus clearly $r_S = \frac{1}{2}r_E + r_I$, which quickly gives us $r_S = \frac{1}{8}$, $r_E = \frac{1}{6}$, and $r_I = \frac{1}{24}$. Each player will spend $\frac{1}{3}$ in aggregate, and has a $\frac{1}{2}$ probability of winning, thus yielding an expected value for each player of $\frac{1}{6}$.

5.6.2 Common Destination of Spillovers

Our first-order conditions are

$$\begin{aligned}
\frac{\partial EV[R]}{\partial r_E} &: -2\left(\frac{d_E+\frac{1}{2}d_S}{(r_E+\frac{1}{2}r_S+d_E+\frac{1}{2}d_S)^2}\right)\left(\frac{r_S}{r_S+d_S}\right)\left(\frac{r_I}{r_I+d_I}\right)+\left(\frac{d_E+\frac{1}{2}d_S}{(r_E+\frac{1}{2}r_S+d_E+\frac{1}{2}d_S)^2}\right)+\\
&\quad \left(\frac{r_S}{r_S+d_S}\right)+\left(\frac{d_E+\frac{1}{2}d_S}{(r_E+\frac{1}{2}r_S+d_E+\frac{1}{2}d_S)^2}\right)\left(\frac{r_I}{r_I+d_I}\right)=1\\
\frac{\partial EV[R]}{\partial r_S} &: -2\left(\frac{r_E+\frac{1}{2}r_I}{r_E+\frac{1}{2}r_S+d_E+\frac{1}{2}d_S}\right)\left(\frac{r_S}{r_S+d_S}\right)\left(\frac{r_I}{r_I+d_I}\right)+\\
&\quad \left(\frac{r_E+\frac{1}{2}r_I}{r_E+\frac{1}{2}r_S+d_E+\frac{1}{2}d_S}\right)\left(\frac{d_S}{(r_S+d_S)^2}\right)+\left(\frac{d_S}{(r_S+d_S)^2}\right)\left(\frac{r_I}{r_I+d_I}\right)=1\\
\frac{\partial EV[R]}{\partial r_I} &: -2\left(\frac{d_E+\frac{1}{2}d_S}{(r_E+\frac{1}{2}r_S+d_E+\frac{1}{2}d_S)^2}\right)\left(\frac{r_S}{r_S+d_S}\right)\left(\frac{r_I}{r_I+d_I}\right)-\\
&\quad 2\left(\frac{r_E+\frac{1}{2}r_I}{r_E+\frac{1}{2}r_S+d_E+\frac{1}{2}d_S}\right)\left(\frac{r_S}{r_S+d_S}\right)\left(\frac{d_I}{(r_I+d_I)^2}\right)+\\
&\quad \frac{1}{2}\left(\frac{d_E+\frac{1}{2}d_S}{(r_E+\frac{1}{2}r_S+d_E+\frac{1}{2}d_S)^2}\right)\left(\frac{r_S}{r_S+d_S}\right)+\frac{1}{2}\left(\frac{d_E+\frac{1}{2}d_S}{(r_E+\frac{1}{2}r_S+d_E+\frac{1}{2}d_S)^2}\right)\left(\frac{r_I}{r_I+d_I}\right)-\\
&\quad \left(\frac{r_E+\frac{1}{2}r_I}{r_E+\frac{1}{2}r_S+d_E+\frac{1}{2}d_S}\right)\left(\frac{d_I}{(r_I+d_I)^2}\right)=1
\end{aligned}$$

By symmetry, we know that $r_E = d_E$, $r_I = d_S$, and $r_S = d_I$, as E plays the same role for both parties as the source of spillovers, with S being the destination for party R and I the destination for party D. These destination areas thus have matching spending, as do the areas with no spillovers. This allows us with a bit of algebra to obtain the equilibrium values of $r_E = d_E = \frac{19}{108}$, $r_S = \frac{1}{27}$, and $r_I = \frac{2}{27}$.

5.6.3 Common Source and Destination for Spillovers

We have first-order equations of

$$\begin{aligned}
\frac{\partial EV[R]}{\partial r_E} : & -2\left(\frac{d_E}{(r_E+d_E)^2}\right)\left(\frac{r_S+\frac{1}{2}r_E}{r_S+\frac{1}{2}r_E+d_S+\frac{1}{2}d_E}\right)\left(\frac{r_I}{r_I+d_I}\right)- \\
& 2\left(\frac{r_E}{r_E+d_E}\right)\left(\frac{\frac{1}{2}d_S+\frac{1}{4}d_E}{(r_S+\frac{1}{2}r_E+d_S+\frac{1}{2}d_E)^2}\right)\left(\frac{r_I}{r_I+d_I}\right)+ \\
& \left(\frac{d_E}{(r_E+d_E)^2}\right)\left(\frac{r_S+\frac{1}{2}r_E}{r_S+\frac{1}{2}r_E+d_S+\frac{1}{2}d_E}\right)+\left(\frac{r_E}{r_E+d_E}\right)\left(\frac{\frac{1}{2}d_S+\frac{1}{4}d_E}{(r_S+\frac{1}{2}r_E+d_S+\frac{1}{2}d_E)^2}\right)+ \\
& \left(\frac{d_E}{(r_E+d_E)^2}\right)\left(\frac{r_I}{r_I+d_I}\right)+\left(\frac{\frac{1}{2}d_S+\frac{1}{4}d_E}{(r_S+\frac{1}{2}r_E+d_S+\frac{1}{2}d_E)^2}\right)\left(\frac{r_I}{r_I+d_I}\right)=1 \\
\frac{\partial EV[R]}{\partial r_S} : & -2\left(\frac{r_E}{r_E+d_E}\right)\left(\frac{d_S+\frac{1}{2}d_E}{(r_S+\frac{1}{2}r_E+d_S+\frac{1}{2}d_S)^2}\right)\left(\frac{r_I}{r_I+d_I}\right)+ \\
& \left(\frac{r_E}{r_E+d_E}\right)\left(\frac{d_S+\frac{1}{2}d_E}{(r_S+\frac{1}{2}r_E+d_S+\frac{1}{2}d_S)^2}\right)+\left(\frac{d_S+\frac{1}{2}d_E}{(r_S+\frac{1}{2}r_E+d_S+\frac{1}{2}d_S)^2}\right)\left(\frac{r_I}{r_I+d_I}\right)=1 \\
\frac{\partial EV[R]}{\partial r_I} : & -2\left(\frac{r_E}{r_E+d_E}\right)\left(\frac{r_S+\frac{1}{2}r_E}{r_S+\frac{1}{2}r_E+d_S+\frac{1}{2}d_S}\right)\left(\frac{d_I}{(r_I+d_I)^2}\right)+\left(\frac{r_E}{r_E+d_E}\right)\left(\frac{d_I}{(r_I+d_I)^2}\right)+ \\
& \left(\frac{r_S+\frac{1}{2}r_E}{r_S+\frac{1}{2}r_E+d_S+\frac{1}{2}d_S}\right)\left(\frac{d_I}{(r_I+d_I)^2}\right)=1
\end{aligned}$$

Clearly $r_E = d_E$, $r_S = d_S$, and $r_I = d_I$. This gives us solutions of $r_E = d_E = \frac{1}{4}$, $r_S = d_S = 0$, and $r_I = d_I = \frac{1}{8}$.

5.6.4 Opposing Spillovers

$$\frac{\partial EV[R]}{\partial r_E} : -2\left(\frac{d_E+\frac{1}{2}d_S}{(r_E+d_E+\frac{1}{2}d_S)^2}\right)\left(\frac{r_S+\frac{1}{2}r_E}{r_S+\frac{1}{2}r_E+d_S}\right)\left(\frac{r_I}{r_I+d_I}\right)-$$
$$2\left(\frac{r_E}{r_E+d_E+\frac{1}{2}d_S}\right)\left(\frac{\frac{1}{2}d_S}{(r_S+\frac{1}{2}r_E+d_S)^2}\right)\left(\frac{r_I}{r_I+d_I}\right)+$$
$$\left(\frac{d_E+\frac{1}{2}d_S}{(r_E+d_E+\frac{1}{2}d_S)^2}\right)\left(\frac{r_S+\frac{1}{2}r_E}{r_S+\frac{1}{2}r_E+d_S}\right)+\left(\frac{r_E}{r_E+d_E+\frac{1}{2}d_S}\right)\left(\frac{\frac{1}{2}d_S}{(r_S+\frac{1}{2}r_E+d_S)^2}\right)+$$
$$\left(\frac{d_E+\frac{1}{2}d_S}{(r_E+d_E+\frac{1}{2}d_S)^2}\right)\left(\frac{r_I}{r_I+d_I}\right)+\left(\frac{\frac{1}{2}d_S}{(r_S+\frac{1}{2}r_E+d_S)^2}\right)\left(\frac{r_I}{r_I+d_I}\right)=1$$
$$\frac{\partial EV[R]}{\partial r_S} : -2\left(\frac{r_E}{r_E+d_E+\frac{1}{2}d_S}\right)\left(\frac{d_S}{(r_S+\frac{1}{2}r_E+d_S)^2}\right)\left(\frac{r_I}{r_I+d_I}\right)+$$
$$\left(\frac{r_E}{r_E+d_E+\frac{1}{2}d_S}\right)\left(\frac{d_S}{(r_S+\frac{1}{2}r_E+d_S)^2}\right)+\left(\frac{d_S}{(r_S+\frac{1}{2}r_E+d_S)^2}\right)\left(\frac{r_I}{r_I+d_I}\right)=1$$
$$\frac{\partial EV[R]}{\partial r_I} : -2\left(\frac{r_E}{r_E+d_E+\frac{1}{2}d_S}\right)\left(\frac{r_S+\frac{1}{2}r_E}{r_S+\frac{1}{2}r_E+d_S}\right)\left(\frac{d_I}{(r_I+d_I)^2}\right)+$$
$$\left(\frac{r_E}{r_E+d_E+\frac{1}{2}d_S}\right)\left(\frac{d_I}{(r_I+d_I)^2}\right)+\left(\frac{r_S+\frac{1}{2}r_E}{r_S+\frac{1}{2}r_E+d_S}\right)\left(\frac{d_I}{(r_I+d_I)^2}\right)=1$$

We know $r_E = d_S$, $r_S = d_E$, and $r_I = d_I$, which allows us to solve, getting $r_E = d_S = \frac{2}{9}$, $r_S = d_E = 0$, and $r_I = d_I = \frac{5}{36}$, giving an expected value of $\frac{5}{36}$.

5.6.5 Non-Opposing Spillovers

In this case we cannot use symmetry as in the other cases. Thus, we need six first-order conditions:

$$\frac{\partial EV[R]}{\partial r_E} : -2\left(\frac{d_S}{(r_s+d_S)^2}\right)\left(r_E r_E+\frac{1}{2}d_S+d_E\right)\left(\frac{r_I+\frac{1}{2}r_E}{r_I+\frac{1}{2}r_E+d_I}\right)+$$
$$\left(\frac{d_S}{(r_s+d_S)^2}\right)\left(\frac{r_E}{r_E+\frac{1}{2}d_S+d_E}\right)+\left(r_E r_E+\frac{1}{2}d_S+d_E\right)\left(\frac{r_I+\frac{1}{2}r_E}{r_I+\frac{1}{2}r_E+d_I}\right)+$$
$$\left(\frac{d_S}{(r_s+d_S)^2}\right)\left(\frac{r_I+\frac{1}{2}r_E}{r_I+\frac{1}{2}r_E+d_I}\right)=1$$

$$\frac{\partial EV[R]}{\partial r_S} : -2\left(\frac{r_S}{r_S+d_S}\right)\left(\frac{d_E\frac{1}{2}d_S}{(r_E+\frac{1}{2}d_S+d_E)^2}\right)\left(\frac{r_I+\frac{1}{2}r_E}{r_I+\frac{1}{2}r_E+d_I}\right)-$$

$$2\left(\frac{r_S}{r_S+d_S}\right)\left(\frac{r_E}{r_E+\frac{1}{2}d_S+d_E}\right)\left(\frac{\frac{1}{2}d_I}{(r_I\frac{1}{2}r_E+d_I)^2}\right)+$$

$$\left(\frac{r_S}{r_S+d_S}\right)\left(\frac{d_E\frac{1}{2}d_S}{(r_E+\frac{1}{2}d_S+d_E)^2}\right)+\left(\frac{r_S}{r_S+d_S}\right)\left(\frac{\frac{1}{2}d_I}{(r_I\frac{1}{2}r_E+d_I)^2}\right)+$$

$$\left(\frac{d_E\frac{1}{2}d_S}{(r_E+\frac{1}{2}d_S+d_E)^2}\right)\left(\frac{r_I+\frac{1}{2}r_E}{r_I+\frac{1}{2}r_E+d_I}\right)+\left(\frac{r_E}{r_E+\frac{1}{2}d_S+d_E}\right)\left(\frac{\frac{1}{2}d_I}{(r_I\frac{1}{2}r_E+d_I)^2}\right)=1$$

$$\frac{\partial EV[R]}{\partial r_I} : -2\left(\frac{r_S}{r_S+d_S}\right)\left(\frac{r_E}{r_E+\frac{1}{2}d_S+d_E}\right)\left(\frac{d_I}{(r_I+\frac{1}{2}r_E+d_I)^2}\right)+$$

$$\left(\frac{r_S}{r_S+d_S}\right)\left(\frac{d_I}{(r_I+\frac{1}{2}r_E+d_I)^2}\right)+\left(\frac{r_E}{r_E+\frac{1}{2}d_S+d_E}\right)\left(\frac{d_I}{(r_I+\frac{1}{2}r_E+d_I)^2}\right)=1$$

$$\frac{\partial EV[D]}{\partial d_E} : -2\left(\frac{r_S}{(r_S+d_S)^2}\right)\left(\frac{d_E+\frac{1}{2}d_S}{r_E+d_E+\frac{1}{2}d_S}\right)\left(\frac{d_I}{r_I+\frac{1}{2}r_E+d_I}\right)-$$

$$2\left(\frac{d_S}{r_S+d_S}\right)\left(\frac{\frac{1}{2}r_E}{(r_E+d_E+\frac{1}{2}d_S)^2}\right)\left(\frac{d_I}{r_I+\frac{1}{2}r_E+d_I}\right)+$$

$$\left(\frac{r_S}{(r_S+d_S)^2}\right)\left(\frac{d_I}{r_I+\frac{1}{2}r_E+d_I}\right)+\left(\frac{d_S}{r_S+d_S}\right)\left(\frac{\frac{1}{2}r_E}{(r_E+d_E+\frac{1}{2}d_S)^2}\right)-$$

$$2\left(\frac{r_S}{(r_S+d_S)^2}\right)\left(\frac{d_E+\frac{1}{2}d_S}{r_E+d_E+\frac{1}{2}d_S}\right)+\left(\frac{\frac{1}{2}r_E}{(r_E+d_E+\frac{1}{2}d_S)^2}\right)\left(\frac{d_I}{r_I+\frac{1}{2}r_E+d_I}\right)$$

$$\frac{\partial EV[D]}{\partial d_S} : -2\left(\frac{d_S}{r_S+d_S}\right)\left(\frac{r_E}{(r_E+d_E+\frac{1}{2}d_S)^2}\right)\left(\frac{d_I}{r_I+\frac{1}{2}r_E+d_I}\right)+$$

$$\left(\frac{d_S}{r_S+d_S}\right)\left(\frac{r_E}{(r_E+d_E+\frac{1}{2}d_S)^2}\right)+\left(\frac{r_E}{(r_E+d_E+\frac{1}{2}d_S)^2}\right)\left(\frac{d_I}{r_I+\frac{1}{2}r_E+d_I}\right)=1$$

$$\frac{\partial EV[D]}{\partial d_I} : -2\left(\frac{d_S}{r_S+d_S}\right)\left(\frac{d_E+\frac{1}{2}d_S}{r_E+d_E+\frac{1}{2}d_S}\right)\left(\frac{r_I+\frac{1}{2}r_E}{(r_I+\frac{1}{2}r_S+d_I)^2}\right)+$$

$$\left(\frac{d_S}{r_S+d_S}\right)\left(\frac{r_I+\frac{1}{2}r_E}{(r_I+\frac{1}{2}r_S+d_I)^2}\right)+\left(\frac{d_E+\frac{1}{2}d_S}{r_E+d_E+\frac{1}{2}d_S}\right)\left(\frac{r_I+\frac{1}{2}r_E}{(r_I+\frac{1}{2}r_S+d_I)^2}\right)=1$$

This solves to $r_S=\frac{1}{9}$, $r_E=\frac{1}{9}$, $r_I=\frac{1}{18}$, $d_S=\frac{2}{9}$, $d_E=\frac{1}{9}$, and $d_I=\frac{1}{9}$. This leaves R with a $\frac{1}{3}$ chance of victory, for an expected value of $\frac{1}{18}$, while D has a $\frac{2}{3}$ chance of victory and an expected value of $\frac{2}{9}$.

Chapter 6: Conclusions

We have established that complementarities transform contest problems in interesting ways. On the obvious level, the players must take these complementarities into consideration when choosing strategies. However, taking these complementarities into consideration creates some novel features. Adding in complementarities in general allows us to model some types of problems more accurately. Patent races and general research and development are clear cases where there are significant complementarities. A more accurate model of these situations may allow for the implementation of better policies to encourage further research development.

Notably, introducing complementarities changes the impact of having multiple auctioneers. As we see in Chapter 4, increasing the number of auctioneers does not have a consistent effect on the total expected value for the auctioneers. Normally having goods provided by an increasing number of sellers reduces the ability of the sellers to extract profits. However, this proves not to hold in our contests with complementarities. These multiple auctioneer game also are a case where finding the optimal strategy is difficult analytically, though individuals can still play the game fairly well. Players overbid for cells, as usually happens in contest experiments, but they do so consistently. This opens the possibility of using experiments to find initial values when attempting to find numeric solutions to otherwise intractable problems.

There are possible future research directions. One of these involves the ownership of cells to begin. The Multiple Auctioneer game in Chapter 4 involves each of the owners playing a game to determine the order the cells will be contested for by a different set of players. Linking these two games by having players own the cells they won for the subsequent Multiple Auctioneer game will add a new layer to the players' decision making. This can also be done by having the owners be the players in the ensuing path formation game, or combining both for an ongoing series of games alternating between the path formation and multiple auctioneer games.

Another possibility is to build models with different types of complementarity. in all of the models explored here are of games where there is exactly one winner with a fixed prize. We can easily envision other types of complementarities, however. For example, in patent races, it may be the case that various combinations of patents allow for products to be made, with many different products possible. In such a case, the number of products eventually manufactured, and thus the total prize available, will depend on the combinations of patents that the firms obtain.

Other types of complementarities also allow for games involving multiple players. Having more players allows for possibilities, such as coalition formation, that do not apply to two player games. Path formation is not suitable for these multi-player games, as a Hex grid may have no path with three colors, and due to the Four Color Theorem, any arbitrary planar graph may end with no winner with four colors, however something like the patent race complementarity described above may work well.

Another direction for further research would be further experimental work. Though we gained some insights from the experiment already run, we do not know what the effect of differing complexities of games would be. The path formation game may prove easier or more difficult for players than the political competition game or the patent races mentioned above. The multiple auctioneer game would also provide an interesting experiment, as determining optimal play in that game requires knowledge of optimal strategy in the path formation game.

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The Vita

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