Chiral 2-form actions and their applications to M5-brane(s)

KO, SHENG-LAN

Chiral 2-form actions and their applications to M5-brane(s)

Sheng-Lan Ko

A Thesis presented for the degree of
Doctor of Philosophy

Center for Particle Theory
Department of Mathematical Sciences
University of Durham
England

May 2015

Dedicated to

My Parents

Chiral p-form actions and their applications on M5-brane(s)

Sheng-Lan Ko

Submitted for the degree of Doctor of Philosophy
May 2015

Abstract

We study the symmetry and dynamics of M5-branes as well as chiral p-forms in this thesis. In the first part, we propose a model describing the gauge sector of multiple M5-branes. The model has modified six-dimensional Lorentz symmetry and its double dimensional reduction gives 5D Yang-Mills theory with higher derivative corrections. The non-abelian self-dual string solutions to this model are presented. In the second part of the thesis, we propose an alternative new action for the single M5-brane. The six-dimentional worldvolume space is covariantly split into 3+3. The relation of the new action to the conventional PST action as well as to the M2-brane action are studied. Finally, we briefly discuss the attempt to formulate the M5-brane action in a 2+4 splitting of worldvolume space and some duality properties and issues of chiral p-form actions.

Declaration

The work in this thesis is based on research carried out at the Centre for Particle Theory, the Department of Mathematical Sciences, England. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all my own work unless referenced to the contrary in the text.

The thesis is organised as follows. Chapter 1 is the introduction and review. Chapter 2 and 3 are based on works [1,2] with Chong-Sun Chu and Pichet Vanichchapongjaroen. Chapter 4 is based on the work [3] with Dmitri Sorokin and Pichet Vanichchapongjaroen. Chapter 5 is the ongoing work with Sorokin and Vanichchapongjaroen. We conclude the thesis on Chapter 6.

Acknowledgements

I would like to thank my supervisors, Chong-Sun Chu and Douglas Smith, and effective supervisors, Dmitri Sorokin and Pichet Vanichchapongjaroen, for their suggestions, discussions, guidance and collaboration.

In particular, I learned how to do research from Chong-Sun, and how to be creative, brave, and to enjoy the research from Pichet. Many thanks go to Douglas for giving me various valuable suggestions and clear explanations. I also enjoyed the insightful discussions with Dmitri, especially appreciate his specialty of formulations and resolving problems. They have made my academic journey in Durham fruitful, joyful, and warm. It is also delight to have many nice classmates in the department and flatmates in the Grey college, Anli, Yang, Chunlen, Alex, all office mates,...,etc. to discuss and have fun with.

I would also like to thank my lovely girlfriend, Bing-Jue, the thesis could not be completed without her. Moreover, the encouragement and accompany of my friends in Taiwan are also essential when I was an infant in the string theory society. Po-Hsiung, Kazu, Tomo, Keijiro, Isono,..., thank you so much. Last, but not least, I am indebted to my family for their constant love and support. They are always rooting for me and back me up no matter what.

Contents

List of Figures

Part I

Introduction & Basics

Chapter 1

Introduction

In this thesis, we study a fundamental building block of M-theory called M5-branes. They are non-perturbative extended objects propagating in eleven dimensional space-time. In particular, we will be dealing with a mathematical object — a self-dual 2-form gauge field living on the worldvolume of M5-branes. To this end, we will revisit the technique to write down a Lorentz covariant chiral p-form action, the Pasti-Sorokin-Tonin (PST) formulation, and apply this formalism on the M5-brane.

The goal of physicists is to find out the ultimate fundamental rules governing the world we are living in. Usually, the way people look at this world changes with the degree of understanding we have about the nature. Scientists need to develop new ideas and tackle with novel mathematical objects as they progress. We will eventually introduce the M5-brane and self-dual 2-form toward the end of this chapter.

Particles to Strings : Roughly speaking, we would like to answer the following two fundamental questions :

1. What are/is the elementary[1] degree(s) of freedom which constitute(s) everything in this world?

[1] What are elementary may depend on the regime of validity of a theory. For example, S-duality, which will be introduced later, exchanges fundamental excitations with solitonic excitations of a theory.

2. What are/is the interaction(s) between the elementary objects?

The Standard Model (SM) is the most successful theory so far, that has a more than reasonable agreement with experiments, to describe the interactions between the subatomic particles. The fundamental degrees of freedom in the SM are 'particles', including quarks and leptons that make up the matter, gauge bosons that mediate the interactions, and the famous Higgs boson that gives rise the masses to various particles. However, the SM is obviously incomplete as it describes only three of four discovered interactions; electromagnetism, weak and strong forces. The gravitational force is missing.

Interestingly, in the process to understand hadron dynamics, there is a failed attempt called the 'dual model'. It was not successful in the point of view of QCD as the consistency requires extra dimensions and the spectrum contains a massless spin 2 particle. In particular, in the generalisation of Veneziano's 4-point functions to higher multi-point ones, it was found that the factorisation could be described conveniently in terms of a set of harmonic oscillators [4] :

$$[a_m^M, a_n^N] = m\eta^{MN}\delta_{m+n}, \quad a_n^{M\dagger} = a_{-n}^M, \quad a_n^M|0\rangle = 0 \quad \forall n > 0, \tag{1.0.1}$$

where $\eta^{MN} = \text{diag}(-,+,+,+)^{MN}$[2], m, n are integers and $[\ ,\ ]$ denotes the commutation relation. In hindsight, these oscillator modes may be regarded naturally as the excitation modes of a one dimensional string. After more and more understanding of the properties of the dual model geometrically in terms of strings, it was realised that it is more appealing to promote the dual model as a fundamental string theory describing all known particles and interactions. The very unwanted massless spin 2 particle in the spectrum is then regarded as the graviton mediating the gravitational force.

Unlike the quantum field theory of particles, the consistency conditions in string theory are so strong that there are no free parameters, unlike in the SM. The only dimensionful parameter is the Regge slope α' which is related to the string tension

[2] The model was proposed to live in 4D at that time. The interesting story of the growth of dual model to string theory may be found in [5].

and string length by $T_{S1} = 1/(2\pi\alpha')$ and $\alpha' = l_s^2$. This string length also sets up an energy scale $1/l_s$. Strings exhibit particle behaviour if their energy is much less than $1/l_s$. The number of space-time dimension is also fixed by consistency. Moreover, the interaction of the string is dynamically generated by the background, there are no coupling constants that can be tuned by hand. Another advantage of string theory is that the ultra high energy behaviour of point particles in quantum field theory is nicely regulated, i.e. it is UV finite in string theory. The heuristic reason for this is that, the definite point-like interactions in Feynman diagrams are smeared out up to the string length scale by the corresponding stringy world-sheet diagram.

Another important ingredient to unification and to provide good control of quantum corrections is supersymmetry. Supersymmetry is a symmetry relating states of different spin statistics, it transforms a boson to a fermion, and vice versa. It is also a natural way to evade the Coleman-Mandula theorem [6] which states that the requirement of nontrivial S-matrix only allows for a trivial combination of space-time and internal symmetries, under some reasonable assumptions. However, the graded supersymmetry algebra allows for supercharges that transform under Lorentz group nontrivially, such that states with different spins/helicities are combined into a supermultiplet which transform into each other with the action of supercharges. The consideration of supersymmetry also makes the introduction of higher dimensions natural. When the supersymmetry is made local, it is necessary to include gravity and the resulting theories are called supergravity theories. The eleven dimensional supergravity turns out to be the low energy limit of M-theory, which will be introduced later, and one of the ingredients of M-theory, the M5-brane, has a six dimensional superconformal field theory living on its worldvolume. The largest number of dimensions for supergravity is eleven, while the superconformal field theory can only exist up to six dimensions [7].

There are in total five consistent superstring theories despite the consistency conditions strongly constraining the theories. Nevertheless, built on the results obtained in the early nineties, an eleven dimensional fundamental theory, which is called M-theory nowadays, was proposed in [8]. In this framework, the five superstring theories as well as eleven dimensional supergravity are unified as different lim-

its of the moduli space of the M-theory [9, 10]. Consideration of eleven dimensional supergravity suggests that there are M2-branes which couple to the background 3-form gauge field electrically as well as M5-branes which couple to the gauge field magnetically. These M2- and M5-branes are regarded as the fundamental degrees of freedom in M-theory although it might be possible that the M2- and M5- branes are just emergent objects. There are also other proposals to define M-theory as a matrix model [11].

The M5-branes worldvolume theory is described by the six dimensional superconformal field theory with (2,0) supersymmetry, in the low energy limit and the limit of decoupling of background gravity, when the renormalisation group flow is driven to the conformal infrared fixed point. Supercharges in six dimension can carry different chiralities, (2,0) means that the supersymmetry is generated by two charges with the same chirality. The only supermultiplet that contains no gravity is the tensor multiplet, which contains a chiral 2-form, five real scalars and symplectic Majorana spinors. The chiral 2-form is defined as a 2-form whose 3-form field strength is self-dual under the Hodge duality with respect to the six dimensional metric. This self-duality condition is required by the supersymmetry. The five scalars are naturally interpreted as the the transverse target space coordinates of the M5-branes. The chiral 2-form living on the worldvolume can couple to the boundary of the M2-branes ending on the M5-branes. The one-dimensional boundaries of M2-branes on the M5-branes are called self-dual strings, they appear to be the soliton solutions of the M5-brane worldvolume theory. The self-duality condition obscures the action formulation. Also, whenever there are multiple M5-branes on top of each other, the gauge symmetry is expected to be non-abelianized. It is widely believed that there is no action formulation for the non-abelian (2,0) field theory [12]. However, there indeed exists action formulations, at least for the single M5-brane case [13, 14]. We will propose an alternative action for the single M5-brane, and also put forward a simple non-abelian model to describe the bosonic gauge sector of the multiple M5 branes in this thesis.

The organisation of the thesis is as follows. Chapter 1 includes the general background introduction and the basics needed to understand the following chap-

ters of the thesis. The remainder of the thesis is divided into two parts; the first part contains the proposal of a non-abelian self-duality condition in Chapter 2 and the self-dual string solutions to it in Chapter 3, the second part describes the construction of a novel alternative single M5-brane action by splitting the worldvolume directions into 3+3 in Chapter 4, and the surprising result of the so-called 2+4 formulation in Chapter 5. The thesis is concluded in Chapter 6.

1.1 String theory

In the remaining part of Chapter 1, we review the general basics and background needed to understand the rest of the thesis. The reader may find more details in any standard textbooks, such as [15–19], and review articles [9, 10, 20–22] and original literature in the references therein.

1.1.1 Bosonic strings

The (perturbative) string theory describes one-dimensional extended objects, strings, propagating in a D-dimensional target space-time. These strings sweep out in time two-dimensional world-sheets. The dynamics of strings are described by the Polyakov action

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma\,\sqrt{-\gamma}\gamma^{mn}\partial_m X^M \partial_n X^N G_{MN}(X), \tag{1.1.2}$$

where G_{MN} is the target space metric, X^M, $M, N = 0, 1, \cdots, (D-1)$ are the target space coordinates, γ_{mn} are auxiliary fields on the world-sheet, and $\sigma^m = (\tau, \sigma)^m$, $m, n = 0, 1$ parametrize the world-surface. The parameter α' is related to the string tension and string length as $T_{S1} = 1/(2\pi\alpha')$, $\alpha' = l_s^2$ respectively. If one integrates out the auxiliary fields γ_{mn}, one obtains the Nambu-Goto action

$$S^{NG} = -T_{S1}\int d^2\sigma\sqrt{-\det\left(\partial_m X^M \partial_n X^N G_{MN}\right)}, \tag{1.1.3}$$

whose geometrical meaning is manifest. The Nambu-Goto action is the product of the area of the world-sheet with the string tension. It is convenient to work with the Polyakov action as it is quadratic in X. The Polyakov action enjoys the two-dimensional reparametrisation invariance as well as the Weyl symmetry, which is a

rescaling of the auxiliary field. The dynamics is described by the field equation of X^M, while the field equation of γ_{mn} imposes constraints. To proceed, it is clever to choose a convenient gauge after the variation of the action, for example, one can choose $\gamma_{mn} = e^{\phi(\sigma)}\eta_{mn}$ where $\eta_{mn} = \text{diag}(-1, 1)$ by making use the reparametrisation symmetry. For simplicity, let us consider the target space to be Minkowskian with $G_{MN} = \eta_{MN} = \text{diag}(-1, 1, \cdots, 1)$. In this gauge, the field equation and boundary condition for X are

$$\partial_m \partial^m X^M(\sigma) = 0, \tag{1.1.4}$$

$$\frac{\partial X_M}{\partial \sigma} \delta X^M \Big|_{\sigma=\sigma_0}^{\sigma=\pi} = 0, \tag{1.1.5}$$

where the two ends of strings are at σ_0 and π. Different ways to realise the boundary condition will result in different topologies for the strings. An obvious way to satisfy the boundary condition is to impose periodicity. In this case, we can choose $X^M(\tau, \sigma) = X^M(\tau, \sigma + 2\pi)$ and $\sigma_0 = -\pi$; this is known as the closed string. Alternatively, one can consider strings with open ends at $\sigma = \sigma_0 = 0$ and $\sigma = \pi$. One may impose $\partial X^M/\partial\sigma = 0$ at separated ends, known as Neumann boundary conditions. Finally, one may require $\delta X^M = 0$ at both ends, known as Dirichlet boundary condition. The last case is equivalent to $\partial X^M/\partial\tau = 0$ at the ends. Actually, for the case of open strings, one can have different choices of boundary conditions for different directions M, and for the two ends at $\sigma = \sigma_0$ and $\sigma = \pi$. The choice of Dirichlet boundary conditions freezes the motion of the end points in certain directions. This effectively defines a sub-manifold in the target space. The ends of strings satisfying Dirichlet boundary conditions actually attach to higher dimensional extended objects called D-branes. They are non-perturbative objects in string theory, which we will describe later.

The field equation of X^M is a wave equation. The solution to the field equation satisfying the boundary conditions can be written as

$$X^M(\tau, \sigma) = x^M + \sqrt{2\alpha'} a_0^M \tau + i\sqrt{\frac{\alpha'}{2}} \sum_{n\neq 0} \frac{1}{n} \left(a_n^M e^{-in(\tau+\sigma)} + a_n^M e^{-in(\tau-\sigma)} \right) \tag{1.1.6}$$

for Neumann open string boundary conditions, while for Dirichlet open string bound-

ary conditions the solution reads

$$X^M(\tau,\sigma) = c_1^M + \frac{1}{\pi}(c_2^M - c_1^M)\sigma + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{1}{n}a_n^M\left(e^{-in(\tau+\sigma)} - e^{-in(\tau-\sigma)}\right), \quad (1.1.7)$$

where the two ends of the open string are fixed at $X^M(\tau,0) = c_1^M$ and $X^M(\tau,\pi) = c_2^M$. For closed strings, the solution reads

$$X^M(\tau,\sigma) = X_L^M(\tau+\sigma) + X_R^M(\tau-\sigma), \quad (1.1.8)$$

where

$$X_L^M(\tau+\sigma) = \frac{1}{2}x^M + \alpha'\frac{p^M}{2}(\tau+\sigma) + i\sqrt{\frac{\alpha'}{2}}\sum_{n\neq 0}\frac{a_n^M}{n}e^{-in(\tau+\sigma)}, \quad (1.1.9)$$

and X_R^M can be obtained by the replacement $\sigma\to -\sigma$, $a_n^M\to\bar{a}_n^M$ via X_L^M. These Fourier coefficients $a_n^M, \bar{a}_n^M$ turn out to be the appropriate phase space variables to quantise the strings. Upon quantisation, the a_n^M satisfy exactly the algebra of (1.0.1) in the dual model. We now see that the infinite oscillators introduced in the dual model are naturally interpreted as the vibration modes of strings. If dealing with closed strings, one would have the other copy of algebra with the replacement $a_n^M\to\bar{a}_n^M$ in (1.0.1), and we have $[a_n^M, \bar{a}_m^N] = 0$.

The $M = N = 0$ of (1.0.1) give negative norm states. This is unacceptable as one would lose the probability interpretation of quantum mechanics. However, one should also consider the physical conditions required by the appropriate quantum mechanical version of constraints which are imposed classically by the vanishing of field equations of the auxiliary field γ_{mn}.[3] Appropriate implementation of the constraints shows that there will be no negative norm states in $D \leqslant 26$. Other consistency conditions of the theory would fix the critical dimension of space-time to be 26.

The algebra (1.0.1) generates a spectrum containing an infinite tower of states, with arbitrarily high spins and masses. However, in many cases, we are often interested in the low energy limit of the theory, in which only the massless states are

[3] We implicitly use the old covariant quantisation here. One can alternatively use the light-cone quantisation to solve the constraints.

accessible. For the open string with Neumann boundary conditions, the spectrum contains a tachyon and a massless 1-form gauge field and higher states. This may be seen by the mass formula $M_k^2 = (k-1)/\alpha'$, where $k = 0, 1, 2, \cdots$. For the closed string, we have, besides the tachyon, a massless scalar (dilaton), an anti-symmetric rank 2 tensor gauge field, a graviton, and massive states. For the closed string, the mass formula is $M_n^2 = \frac{4}{\alpha'}(N-1) = \frac{4}{\alpha'}(\bar{N}-1)$, where $N = \bar{N} = 0, 1, 2, \cdots$.

For the closed strings, one may impose the symmetry $\sigma \leftrightarrow -\sigma$ (hence $a_n^M \leftrightarrow \bar{a}_n^M$), so that one obtains "unoriented strings". In this case, the anti-symmetric 2-form gauge field drops from the massless spectrum. For superstrings, the symmetry under $\sigma \leftrightarrow -\sigma$ can only be applied alone on the IIB theory but not on IIA, as it will be discussed later that IIB is chiral but IIA is non-chiral. The existence of tachyons indicates the instability of the vacuum, however, we will see that they are absent in the case of superstrings.

1.1.2 Superstrings

The bosonic string theory can be made supersymmetric in the physical (target) space-time. There are two popular ways to achieve this. It turns out one can either embed a supersymmetric world-sheet into a bosonic target space, known as Ramond-Neveu-Schwarz (RNS) formulation; or embed a bosonic world-sheet into a target superspace, known as Green-Schwarz(GS) formulation. We will briefly review the RNS formalism in this subsection. The GS formulation requires additional technical complication and it is known to be hard to quantise strings Lorentz covariantly in GS formulation. Nevertheless, it turns out GS formalism can be straightforwardly generalised to describe D-branes, and we will come to this later. There are other formalisms for superstrings, for example, super-embedding and pure spinor formalisms. We will briefly describe the idea of the super-embedding formalism in Section 1.3.5. The pure spinor formalism allows one to quantise the superstring covariantly and allows one to compute scattering amplitudes more efficiently [23–25]. However, the pure spinor formalism is beyond the scope of this thesis.

We add in $X^M(\sigma)$ its fermionic partner $\chi_\alpha^M(\sigma)$ and an auxiliary field F^M to form a (1,1) supermultiplet in $2d$, where $\alpha = 1, 2$ is the Spin(2) index. The action which

enjoys the rigid supersymmetry is

$$\int d^2\sigma \left(-\frac{1}{2}\partial_m X^M \partial^m X^N - \frac{i}{2}\bar{\chi}^M \not{\partial}\chi^N + \frac{1}{2}F^M F^N\right)\eta_{MN}, \tag{1.1.10}$$

where $\bar{\chi} \equiv \chi^T C$, χ^M can be written as a column vector $(\chi_+^M \quad \chi_-^M)^T$, $\gamma^m = (i\sigma^2, \sigma^1)^m$, $C = -\gamma^0$ and $\sigma^1, \sigma^2, \sigma^3$ are Pauli matrices. Two Majorana-Weyl spinors with different chirality $\chi_\pm^M$ form a Majorana spinor χ^M. This action can be extended to couple to $2d$ supergravity for which the vielbein e_m^a and gravitino $\psi_{m\alpha}$ are added, $a = 0, 1$ is tangent space index. By a clever choice of the gauge, the dynamical field equations become

$$\partial_m \partial^m X^M = 0, \qquad \not{\partial}\chi^M = 0, \tag{1.1.11}$$

and the boundary terms are

$$\left(\chi_-^M \delta\chi_{-M} - \chi_+^M \delta\chi_{+M}\right)\Big|_{\sigma_0}^{\pi} = 0, \tag{1.1.12}$$

in addition to the bosonic ones (1.1.5).

R, NS sectors :

The vanishing of the boundary term (1.1.12) turns out to be subtle. Different boundary conditions lead to different sectors of superstrings that give space-time fermions or bosons.

Open string : For open strings, we take $\sigma_0 = 0$. As spinors satisfy first order differential equations, the general ansatz to realise (1.1.12) turns out to be

$$\chi_+^M(\tau, 0) = \chi_-^M(\tau, 0), \qquad \chi_+^M(\tau, \pi) = s\chi_-^M(\tau, \pi), \tag{1.1.13}$$

where $s = \pm$. One can freely choose the phase of the spinors at $\sigma = 0$, but then the sign s for the other end $\sigma = \pi$ becomes physical. The choice $s = 1$ is called Ramond sector (R) and the choice $s = -1$ is the Neveu-Schwarz sector (NS). A useful trick to realise the boundary condition and Fourier expansion is to extend the domain of definition by

$$X^M(\sigma) = \begin{cases} X^M(\sigma) & 0 \leqslant \sigma < \pi \\ X^M(-\sigma) & -\pi \leqslant \sigma < 0 \end{cases}, \Psi^M(\sigma) = \frac{1}{\alpha'} \begin{cases} \chi_+^M(\sigma) & 0 \leqslant \sigma < \pi \\ \chi_-^M(-\sigma) & -\pi \leqslant \sigma < 0 \end{cases}, \tag{1.1.14}$$

then we have

$$(R): \quad \Psi^M(\pi) = \Psi^M(-\pi), \qquad (NS): \quad \Psi^M(\pi) = -\Psi^M(-\pi), \tag{1.1.15}$$

so that we have the mode expansions

$$(R): \quad \Psi^M = \sum_{n\in\mathbb{Z}} d_n^M e^{-in\sigma^+}, \qquad (NS): \quad \Psi^M = \sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^M e^{-ir\sigma^+}, \tag{1.1.16}$$

where $\sigma^\pm \equiv \tau \pm \sigma$.

Closed string : We take $\sigma_0 = -\pi$. $\sigma = -\pi$ and $\sigma = \pi$ are at the same point. This means, as spinors, $\chi_\pm^M$ can be either periodic or anti-periodic for $+$ and $-$ independently,

$$(R): \quad \chi_\pm^M(\pi) = \chi_\pm^M(-\pi), \qquad (NS): \quad \chi_\pm^M(\pi) = -\chi_\pm^M(-\pi). \tag{1.1.17}$$

Combining the choices for left $(+)$ and right $(-)$ chiral parts, we then have four different sectors : (NS, NS), (NS, R), (R, NS) and (R, R). The Fourier expansion is then

$$(R): \quad \chi_+^M = \sqrt{\alpha'}\sum_{n\in\mathbb{Z}} d_n^M e^{-in\sigma^+}, \qquad (NS): \quad \chi_+^M = \sqrt{\alpha'}\sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^M e^{-in\sigma^+}, \tag{1.1.18}$$

where the expansion of χ_-^M can be obtained from the above by the substitution $\sigma^+ \to \sigma^-$, $d_n^M \to \bar{d}_n^M$, $b_r^M \to \bar{b}_r^M$.

Quantisation :

Open strings : Besides (1.0.1), we now also have the algebra

$$(R): \quad \left\{d_n^M, d_m^N\right\} = \delta_{n+m}\eta^{MN}, \qquad (NS): \quad \left\{b_r^M, b_s^N\right\} = \delta_{r+s}\eta^{MN}. \tag{1.1.19}$$

For the NS sector, we have a tachyon, a massless 1-form gauge field and massive bosons. For the R sector, notice that the zero modes form a Clifford algebra of space-time (up to a rescaling)

$$\left\{d_0^M, d_0^N\right\} = \eta^{MN}. \tag{1.1.20}$$

Quantum mechanically, states need to realise the representation of, in particular, this zero mode algebra. This means the space-time fermions are secretly encoded in

the choice of Ramond boundary conditions for the RNS formulation. The spectrum then contains a massless Majorana spinor and massive spinors.

For both sectors, we have a no-ghost theorem saying that there are no negative norm states if $D \leqslant 10$. Other consistency conditions fix the critical dimension to be $D = 10$ for superstrings.

GSO projection : To have a supersymmetric theory, a necessary condition is that the on-shell degrees of freedom between bosons and fermions must be matched. This is not the case for the raw superstring spectrum we just presented. However, the Gliozzi-Scherk-Olive (GSO) projection allows us to project out some unwanted states and finally construct the space-time supersymmetry of strings. This is like a refinement of the theory by a Z_2 grading. The projector is

$$(-)^F = \begin{cases} (-)^{\sum_{r=1/2}^{\infty} b_r^\dagger \cdot b_r - 1} & (NS) \\ \Gamma_{11}(-)^{\sum_{n=1}^{\infty} d_n^\dagger \cdot d_n} & (R), \end{cases} \tag{1.1.21}$$

where $\Gamma_{11} \equiv (\sqrt{2})^{10} d_0^0 \cdots d_0^9$. The inclusion of Γ_{11} is needed for the match of degrees of freedom. There is another copy $(-)^{\bar{F}}$ for the closed strings, with the oscillators replaced by their bar-ed partners.

For open strings, the projection condition is $(-)^F = 1$ for all states. The GSO projection kills the tachyon states and states at every second level in the NS sector. In the R sector, the projection would select a certain chirality for the Majorana fermions. Amazingly, the resulting theory turns out to be space-time supersymmetric.

For closed strings, there are two choices of projection conditions,

$$IIB: \quad (-)^F = (-)^{\bar{F}} = 1 \tag{1.1.22}$$

for all sectors, and

$$IIA: \quad \left((-)^F, (-)^{\bar{F}}\right) = \begin{cases} (1,-1) & (NS,R) \\ (1,1) & (R,NS) \\ (1,1) & (NS,NS) \\ (1,-1) & (R,R). \end{cases} \tag{1.1.23}$$

The massless field content for the two choices turn out to coincide with that of IIA and IIB supergravity in $D = 10$. IIA and IIB supergravity are the unique $10d$ theories with maximal 32 supercharges. In fact, the low energy limit of type IIA and IIB superstrings, or the point particle limit $\alpha' \to 0$, are just IIA and IIB supergravity respectively. IIA has $10d$ (1,1) supersymmetry, while IIB is chiral (0,2). Let us mention that both theories have the same field content in the NS-NS sector, dilaton Φ, anti-symmetric 2-form (Kalb-Ramond field) B_2 and the graviton G_{MN}.

Before ending this subsection, let us discuss a bit more about the (R,R) sector of closed strings. The bosonic states in this sector carry two spinor indices so they can be expanded in terms of the complete basis of Clifford algebra. If we apply the IIB projector condition on the expansion, we will see that the spectrum contains 1-form, 3-form and 5-form fields F_1, F_3 and F_5, where $F_5 = *F_5$, where $*$ denotes the Hodge duality. The physical conditions will further justify that these form fields are field strengths and satisfy second order field equations. However, we see that the 5-form field strength satisfies the first order self-duality equation. As they are all massless, they will survive in the low energy limit and hence the IIB supergravity involves a self-dual 5-form. The self-duality of the 5-form field strength makes the action formulation difficult, as the action principle usually gives a second order field equation but the self-duality equation is of first order. Usually, for many purposes, it suffices to impose the self-duality equations "by hand". However, we will see later that there is some trick allowing us to have an action formulation for the self-dual field strength. For IIA, the GSO projection gives the spectrum with field strength F_2, F_4. These differential form fields are usually called RR gauge fields and will be sourced by non-perturbative extended objects called D-branes introduced in the next section.

So far, we have been considering the string theories in a trivial background, i.e. in the eleven dimensional Minkowski space. Actually strings can be embedded in a nontrivial target background, for example, one can consider bosonic strings in a curved target space (with curvature R) with nontrivial dilaton Φ and Kalb-Ramond

2-form B_{MN},

$$S = -\frac{1}{4\pi\alpha'}\int d^2\sigma\,\sqrt{-\gamma}\left[(\gamma^{mn}+\epsilon^{mn})\partial_m X^M \partial_n X^N\,(G_{MN}+B_{MN})+\alpha'\Phi R\right]. \tag{1.1.24}$$

The Weyl invariance at the quantum level would require the background fields to satisfy certain differential equations. One of the solutions is the Minkowski vacuum : $B_{MN}=0$, $\Phi=\Phi_0$ is a constant, and $G_{MN}=\eta_{MN}$. In general, assuming Φ has the vacuum expectation value (vev) $\langle\Phi\rangle=\Phi_0$, the last term in the action then gives a topological invariant,

$$-\frac{\alpha'}{4\pi\alpha'}\langle\Phi\rangle\int_\Sigma d^2\sigma\,\sqrt{-\gamma}R=\Phi_0(2-2g)=\Phi_0\chi, \tag{1.1.25}$$

where g is the genus of the world-sheet and χ is the Euler characteristic. We thus realise that when we calculate amplitudes and sum over topologically inequivalent world-sheets in genus expansion, e^{Φ_0} serves as a coupling constant. Thus, the string interaction is determined dynamically from the background, rather than being specified by hand for the theory.

Moreover, there are also background gauge fields in the RR sector, we will see later that these RR fields could couple to D-branes worldvolume naturally.

1.2 D-branes

As something open strings can end on

D-branes are non-perturbative, dynamical objects in string theory. It was mentioned previously that Dirichlet boundary conditions of the open string effectively define a sub-manifold of target space-time. Actually, the directions on which the ends of the open string can freely move compose the worldvolume space of the D-branes. A D_p-brane is a D-brane which has a $(p+1)$-dimensional worldvolume space. Fields living on the worldvolume of D-branes can be determined by the quantisation of open strings. The tension of a D_p-brane can be calculated by a 1-loop open string amplitude [26] to be

$$T_{D_p}=\frac{1}{(2\pi)^p g_s l_s^{p+1}}. \tag{1.2.26}$$

The inverse proportionality to the coupling constant g_s suggests that D-branes are non-perturbative objects, and the factor of l_s^{p+1} gives the correct dimension for the tension.

Consider the cases that both ends of the bosonic open string satisfy the same type of boundary conditions. If both ends $\sigma = 0, \pi$ satisfy Neumann boundary condition in the direction M, we say M is of N−N type, if both ends satisfy Dirichlet boundary condition in the direction N, we say N is of D−D type. For a given coordinate X^M, it is allowed to have N−D or D−N boundary conditions on the end points, however, we ignore these possibilities here for simplicity. Thus, we only consider bosonic open strings ending on parallel D-branes (or on the same D-brane). The quantisation of open strings leads to the following wave functional

$$\Psi = \left(\phi(x^M) + \sum_{M \in N-N} a_{-1}^M A_M(x^N) + \sum_{M \in D-D} a_{-1}^M \Phi_M(x^N) + \cdots \right) |0\rangle, \qquad (1.2.27)$$

where x^M are the centre of mass coordinates in the N−N directions which can be identified as the coordinates of the D-branes upon the choice of static gauge, $|0\rangle$ is the vacuum state and the $\cdots$ denotes higher modes. ϕ is a tachyon, A_M is a vector field, while Φ_M are scalar fields. Assuming $M = 0, 1, \cdots, p$ are the N−N directions, the Lorentz symmetry of the target space is then broken as $SO(1, D-1) \to SO(1, p) \times SO(D-p-1)$. The factor $SO(1, p)$ is identified as the Lorentz symmetry of the branes while $SO(D-p-1)$ becomes an internal global symmetry. Consider the case of two parallel D-branes located at x_1^M and x_2^M for $M = p+1, \cdots, D-1$. The mass squared of the open string is given by

$$-\frac{1}{\alpha'} + \frac{1}{4\pi^2 \alpha'^2} \sum_{M,N \in D-D} (x_2^M - x_1^M) \eta_{MN} (x_2^N - x_1^N) + \cdots, \qquad (1.2.28)$$

where $\cdots$ denotes the contribution of oscillator modes. If we have a single D-brane, or the D-branes are on top of each other, $x_2^M = x_1^M$ and thus the second term vanishes. For the A_M and Φ_M states, the contribution to the mass-squared from the oscillator modes also vanish. Hence, we have a massless vector field and scalar fields living on the worldvolume of D-branes. The scalar fields can be identified as the Goldstone modes associated with the broken target space translational symmetry. For the case of superstrings, the tachyon will be GSO projected out and both A_M

and Φ_M will be preserved. When considering supersymmetry, there are $(p-1)+(10-p-1)=8$ on-shell bosonic and $(32/2)/2=8$ fermionic degrees of freedom respectively. The presence of the D-brane breaks half of space-time supersymmetries, this accounts for the first factor of 2. The second factor of 2 counts the fermionic degrees of freedom on-shell.

Considering N parallel D-branes. There are N^2 ways labelled by $[ij]$, called Chan-Paton factors, for open strings to connecting these parallel D-branes, with $i,j=1,\cdots,N$. Notice that $[ij]$ is considered to be inequivalent to $[ji]$ as we are interested in oriented strings. The $[ij]$ string has the $\sigma=0$ end attached to the brane i while the $[ji]$ string starts from the brane j. We have seen above that such $[ij]$ strings are massive if $i\neq j$, and these $[ii]$ strings contain a vector gauge field, actually they are abelian $U(1)$ gauge fields. Moreover, one can also consider the limit where the D-branes are on top of each other. In this limit, all the states $[ij]$ are massless, so that they are all relevant states in the low energy. This enhancement of number of massless states also suggests the enhancement of the gauge symmetry from $U(1)^N$ to $U(N)$. The centre of mass factor $U(1)$ of $U(N)$ is sometimes irrelevant in many discussions so that one takes the gauge group $SU(N)$. The non-abelianisation of the gauge symmetry for the coincident branes is expected to be a generic feature in both string theory and M-theory, though we don't have such a picture in terms of open strings in M-theory.

as solitonic solutions of supergravity

The low energy effective theories of IIA and IIB superstring theories are type IIA and IIB supergravity respectively. The bosonic parts of their actions are given by

$$\begin{aligned} S_{IIA} &= \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-g}\Big[e^{-2\phi}\left(R+4|\nabla\phi|^2-\frac{1}{12}H_3^2\right) \\ &\qquad +\frac{1}{2}dC_3\wedge dC_3\wedge B_2-\frac{1}{2}\sum_{n=2,4}\frac{1}{n!}F_n^2\Big], \end{aligned} \tag{1.2.29}$$

$$\begin{aligned} S_{IIB} &= \frac{1}{2\kappa_{10}^2}\int d^{10}x\sqrt{-g}\Big[e^{-2\phi}\left(R+4|\nabla\phi|^2-\frac{1}{12}H_3^2\right) \\ &\qquad -\frac{1}{2}C_3\wedge H_3\wedge F_3-\frac{1}{2}\sum_{n=1,3}F_n^2-\frac{1}{4}F_5^2\Big], \end{aligned} \tag{1.2.30}$$

where R is the Ricci scalar with respect to the string frame metric g_{MN}, H_3 is the field strength of the Kalb-Ramond gauge field B_2, and $F_2 = dC_1$, $F_4 = dC_3 + C_1 \wedge H_3$, $F_1 = dC_0$, $F_3 = dC_2 - C_0 dB_2$, $F_5 = dC_4 - \frac{1}{2} H_3 \wedge C_2 + \frac{1}{2} B_2 \wedge dC_2$. κ_{10} is related to 10-dimensional Newton's constant and string length by $2\kappa_{10}^2 = 16\pi G_{10} = (2\pi)^7 g_2^2 l_s^8$.

The Chern-Simons term and terms containing RR field strength can be written as having an overall factor $e^{-2\phi}$ in front by field redefinition. If one does this, it is clear from the overall factor $e^{-2\phi}$ of the Lagrangians that IIA and IIB supergravity are tree level approximation of IIA and IIB superstrings. The factor 1/4 (instead of 1/2) for the F_5^2 term accounts for the self-duality property of F_5. However, S_{IIB} is a pseudo action in the sense that the self-duality condition $F_5 = *F_5$ is imposed by hand. This means that the self-duality condition $F_5 = *F_5$ cannot be derived by varying the action S_{IIB} with respect to C_4. Instead, one would normally impose $F_5 = *F_5$, say, on the supersymmetry algebra, consistently with its second order field equation derived by varying S_{IIB} with respect to C_4.

D-branes are 1/2-BPS (Bogomol'nyi–Prasad–Sommerfield) objects that preserve one-half of the supersymmetries of the space-time. When the supercharges carry not only the spinor indices but also additional indices, such supersymmetries are said to be extended. For example, the supercharges would look like Q_α^i, where α is a spinor index and i labels the amount of supersymmetries. Typically, theories with extended supersymmetries and Majorana supercharges have superalgebras like this[4]

$$\left\{Q_\alpha^i, Q_\beta^j\right\} = -2\delta^{ij} \left(\Gamma^M C^{-1}\right)_{\alpha\beta} P_M + Z_{\alpha\beta}^{ij}, \tag{1.2.31}$$

where Γ^M is the gamma matrix, C is the charge conjugation matrix, P_M is the momentum generator, $Z_{\alpha\beta}^{ij}$ is called the central charge and M is the space-time index. In the rest frame of a massive particle, it can be shown that the expectation value of the operator Q^2, satisfies

$$\langle Q^2 \rangle \propto \left(m + \frac{z}{2}\right) c, \tag{1.2.32}$$

[4] Sometimes a supersymmetric theory which is not extended has also such forms of superalgebra schematically, for example, the M-theory superalgebra [27,28]. The discussion of properties of BPS states applies regardless of whether supercharges are Majorana spinors.

where $Q \equiv \bar{\epsilon}_i^\alpha Q_\alpha^i$, m is the mass of the particle and z is defined by $z\bar{\epsilon}_\alpha^i \equiv \bar{\epsilon}_j^\beta Z_{\beta\alpha}^{ij}$ and c is a positive number. Therefore, one would have

$$m \geqslant \frac{|z|}{2}, \tag{1.2.33}$$

where the inequality is saturated only when Q annihilates the state, that is, only when the state preserves some of the supersymmetries. The above inequality is called BPS bound. BPS states saturate the BPS bound, and they correspond to the representation of the short multiplet in the superalgebra. When the state saturates the BPS bound, the representation of the superalgebra effectively looks like the massless case and the representation becomes smaller because one has less raising and lowering operators formed by pairs of supercharges. BPS states are usually stable against quantum corrections and allow one to do extrapolation from weak to strong coupling of a given theory.

D-branes are solitonic solutions of IIA and IIB supergravity equations of motion, however, it is usually easier to obtain the solutions by solving the Killing spinor equation,

$$\delta_\epsilon \Psi = 0, \tag{1.2.34}$$

where $\delta\Psi$ is the supersymmetry transformation of fermions in the theory. The reason is that, we are interested in the classical solutions in the bosonic background. For these solutions preserving some supersymmetries, there exists Killing spinors ϵ such that $\delta_\epsilon \Psi = 0$. These transformations that vanish for the given BPS solution are those which are preserved by the states. Usually, the Killing spinor is in a form of certain function multiplying a constant spinor ϵ_0 that satisfies certain projection conditions, such as $\Gamma_{01\cdots p}\epsilon_0 = 1$, where $\Gamma_{01\cdots p}$ denotes the product of Gamma matrices. For example, if one solves the Killing spinor equation of eleven dimensional supergravity

$$0 = \delta_\epsilon \psi_M = D_M \epsilon + \frac{1}{288}\Gamma_M{}^{NPQR} F_{NPQR}\epsilon - \frac{1}{36}\Gamma^{PQR} F_{MPQR}\epsilon, \tag{1.2.35}$$

one could find the M5- and M2-branes solutions. For each case, the supersymmetry variation vanishes when ϵ is some specific function times a constant spinor ϵ_0, which satisfies $\Gamma_{012}\epsilon_0 = \epsilon_0$ for M2-branes and $\Gamma_{012345}\epsilon_0 = \epsilon_0$ for M5-branes.

The maximum number of dimensions allowed for supergravity is eleven. The eleven dimensional supergravity action [29] is

$$S_{11} = \frac{1}{2\kappa_{11}^2}\int d^{11}x\sqrt{-g}\left(R - \frac{1}{2\times 4!}F_4^2 + \frac{1}{6}F_4\wedge F_4\wedge C_3\right), \tag{1.2.36}$$

where F_4 is the field strength of the C_3 gauge field and $2\kappa_{11}^2 = 16\pi G_{11} = (2\pi l_p)^9/(2\pi)$. The solitonic solutions in $11d$ supergravity contain the M2- and M5- branes which have electrical and magnetical coupling to C_3 respectively. One can show that the dimensional reduction of $11d$ supergravity on a circle results in the type IIA supergravity. The explicit relation between eleven dimensional and IIA supergravity is

$$ds_{11}^2 = e^{-2\Phi/3}ds_{IIA}^2 + e^{4\Phi/3}\left(R_{11}d\psi + C_\mu dx^\mu\right)^2, \quad \tilde{F}_4 = F_4 + H_3\wedge dx^{10}, \tag{1.2.37}$$

where Φ is dilaton, C_μ is RR 1-form gauge field, F_4 is the RR 4-form field strength, H_3 is the field strength of Kalb-Ramond field, and $\tilde{F}_4$ is the 4-form field strength of 3-form gauge field in $11d$ supergravity. R_{11} is the radius of the compactified circle in the direction x^{10}. The connection between eleven dimensional supergravity and IIA supergravity has a deep relation with the duality between M-theory and type IIA superstring theory. Here we don't present the explicit brane solutions, interested readers may find them in the nice review [30].

In $10d$ supergravity theories, there are also solitonic solutions representing the fundamental strings and their magnetic dual, NS 5-branes. Unlike the D-branes, open strings cannot end on NS 5-branes. The tension of the NS 5-brane is given as

$$T_{NS5} = \frac{2\pi}{g_s^2 l_s^6}, \tag{1.2.38}$$

where the $1/g_s^2$ dependence suggests that the NS 5-brane is non-perturbative and is a closed string soliton.

Brane effective action

The low energy effective action of the string theory with the D-branes present can be written as

$$S \approx S_{\text{SUGRA}} + S_{D_p}, \tag{1.2.39}$$

where $S_{\rm SUGRA}$ is the supergravity action discussed just above and S_{D_p} is the world-volume action of the D-brane. Here we first consider the single D-brane case. We have also implicitly considered the probe brane limit. This is an approximation that the presence of the D-brane has no back reaction on the background geometry.

The action S_{D_p} can be formulated in terms of the Green-Schwarz formulation, in which we embed the bosonic worldvolume of the D-brane into a generic super target space. Thus the resulting theory has manifest space-time supersymmetry. To proceed, we accompany the bosonic coordinates $X^{\underline{M}}$ with the Grassmann odd coordinates $\theta^{\underline{\alpha}}$, so that we have $Z^{\underline{\mathcal{M}}} = (X^{\underline{M}}, \theta^{\underline{\alpha}})$, $\underline{M} = 0, 1, \cdots, D-1$ and $\underline{\alpha} = 1, 2, \cdots, 32$ in D-dimensional target space. The target space indices are denoted with underlines. The super target space geometry is described by the tangent space supervielbeins $E^{\underline{A}}(Z) = dZ^{\underline{\mathcal{M}}} E^{\underline{A}}_{\underline{\mathcal{M}}}(Z)$ and $E^{\underline{\alpha}}(Z) = dZ^{\underline{\mathcal{M}}} E^{\underline{\alpha}}_{\underline{\mathcal{M}}}(Z)$.

As the D-brane is a 1/2-BPS object which breaks half of space-time supersymmetries, there should be only 16 supercharges active in the worldvolume. However, the Green-Schwarz formulation naturally uses spinors that have 32 components. The way out for this puzzle is κ-symmetry. There is a local fermionic gauge symmetry on the worldvolume, called κ-symmetry, which allows one to gauge away one half of redundant Grassmann odd degrees of freedom. The transformation law is given by

$$\delta_\kappa Z^{\underline{\mathcal{M}}} E^{\underline{A}}_{\underline{\mathcal{M}}} = 0, \quad \delta_\kappa Z^{\underline{\mathcal{M}}} E^{\underline{\alpha}}_{\underline{\mathcal{M}}} = (1 + \bar{\Gamma})\kappa, \quad \delta_\kappa A_1 = \mathcal{P}^* i_\kappa B_2, \tag{1.2.40}$$

where $\mathcal{P}^*$ denotes the pull-back to the worldvolume, B_2 is the Kalb-Ramond field, A_1 is the worldvolume gauge field whose field strength is $F_2 = dA_1$, κ is a local parameter and $\bar{\Gamma}$ is a rank 16 matrix whose explicit form depends on the specific theory, and it satisfies $\bar{\Gamma}^2 = 1$, $\mathrm{tr}\bar{\Gamma} = 0$.

The form of the single D-brane effective action is (in string frame)

$$S_{D_p} = -T_{D_p} \int d^{p+1}\sigma \, e^{-\phi} \sqrt{-\det(g + \mathcal{F})} + \mu_{D_p} \int_{\mathcal{M}_{p+1}} \left(\sum_n C_n e^{\mathcal{F}} \sqrt{\frac{\hat{A}(R_T)}{\hat{A}(R_N)}} \right)_{p+1}, \tag{1.2.41}$$

where $\mathcal{F}_{\mu\nu} = 2\pi\alpha' F_{\mu\nu} - E^{\underline{\mathcal{A}}}_\mu E^{\underline{\mathcal{C}}}_\nu B_{\underline{\mathcal{A}}\underline{\mathcal{C}}}$, $E^{\underline{\mathcal{C}}}_\nu = \partial_\nu Z^{\underline{\mathcal{M}}} E^{\underline{\mathcal{C}}}_{\underline{\mathcal{M}}}$, $\mu, \nu = 0, 1, \cdots, p$ are world-volume indices, $g_{\mu\nu} = E^{\underline{A}}_\mu E^{\underline{B}}_\nu \eta_{\underline{A}\underline{B}}$, and C_r is the background RR gauge r-form field pull-back to worldvolume and ϕ is the dilaton. R_T and R_N are curvature 2-forms of tangent and normal bundles of the D-brane worldvolume, $\hat{A}$ is the Dirac A-roof

genus. The square root factor in the Wess-Zumino terms denote the gravitational interactions, which we will often ignore in the following discussions. The first term is called the Dirac-Born-Infeld (DBI) term and the second term is called the Wess-Zumino (WZ) term. The WZ term is presented with a succinct formal notation that only terms of $(p+1)$-form are kept to be integrated over the $(p+1)$-dimensional manifold $\mathcal{M}_{p+1}$. The exponential is a formal Taylor expansion with exterior product implicitly understood. Notice that the DBI term contains NS-NS fields while the WZ term involves RR fields. The κ-symmetry of S_{D_p} would impose the condition that the tension is equal to the charge

$$T_{D_p} = \mu_{D_p}. \tag{1.2.42}$$

This is expected as the D-brane is a stable BPS object, so that the force from the Kalb-Ramond field cancels exactly the force from the RR gauge fields. Also, the κ-symmetry puts the background supergravity on-shell. The variation of the action under κ-symmetry takes the form

$$\delta_\kappa S_{D_p} \propto (1-\bar{\Gamma})(1+\bar{\Gamma})\kappa, \tag{1.2.43}$$

so that the variation vanishes because of $\bar{\Gamma}^2 = 1$. More technical details in proving the invariance of D-brane actions under κ-symmetry can be found in [31] for single D-branes in $10d$ Minkowski space.

We have seen that when D-branes are coincident, the gauge symmetry gets non-abelianized. Thus, we expect the effective action for multiple D-branes to be the non-abelian DBI action. However, it is difficult to write down a non-abelian DBI action. The main difficulty is that the gauge fields are now matrices and their products are non-commutative, and thus the order of fields in a product of the determinant expansion of the DBI action matters. Usually, it is the symmetrised trace that defines the effective action [32]. Nevertheless, if one keeps only quadratic terms in field strength, the theory is just the super Yang-Mills (SYM) action. For example, the low energy effective action for N D4 branes is given by the SU(N) SYM action.

The effective action of type IIA NS 5-brane is given first in [33] in 2000. The late discovery of the effective action may be owing to the fact that the worldvolume

theory of IIA NS 5-brane is described by the (2,0) tensor multiplet.

gauge-fixing of κ-symmetry :

To demonstrate how the gauge-fixing of the κ-symmetry results in a supersymmetric worldvolume theory, let us consider a simple generic case with a flat target space. The κ-symmetry transformation is

$$\delta_\kappa \theta = \frac{1}{2}(1+\bar{\Gamma})\kappa, \tag{1.2.44}$$

while, in the bosonic background, the supersymmetry transformation is

$$\delta_\epsilon \theta = \epsilon. \tag{1.2.45}$$

Let us define the projection matrices $P_\pm = \frac{1}{2}(1 \pm \bar{\Gamma})$, we thus have

$$\delta_\kappa\left(P_-\theta\right) = 0, \quad \delta_\kappa\left(P_+\theta\right) = P_+\kappa; \qquad \delta_\epsilon\left(P_-\theta\right) = P_-\epsilon, \quad \delta_\epsilon\left(P_+\theta\right) = P_+\epsilon. \tag{1.2.46}$$

One can then consistently impose the gauge-fixing condition

$$(1+\bar{\Gamma})\theta = 0. \tag{1.2.47}$$

This condition is preserved under supersymmetry transformations by a compensating κ-symmetry with $\kappa = -\epsilon$. Therefore, only $P_-\theta$ is left and its supersymmetry transformation is

$$\delta_\epsilon(P_-\theta) = P_-\epsilon, \tag{1.2.48}$$

and the condition to preserve the worldvolume supersymmetry is then $\delta_\epsilon(P_-\theta) = 0$, or in the form of the projection condition

$$\Gamma\epsilon = \epsilon. \tag{1.2.49}$$

intersections and branes ending on branes

There are certain allowed configurations for D-branes that preserve some amount of supersymmetry. Here, we consider the cases of branes ending on branes orthogonally. For example, the following is a generic configuration for two intersecting branes,

$$\begin{array}{lcccc} & \overbrace{}^{(p+1)} & \overbrace{}^{q_1} & \overbrace{}^{q_2} & \overbrace{}^{d} \\ (p+q_1)\text{-brane} & e\cdots e & e\cdots e & -\cdots- & -\cdots- \\ (p+q_2)\text{-brane} & e\cdots e & -\cdots- & e\ldots e & -\cdots- \end{array} \tag{1.2.50}$$

where the symbol e denotes the directions the brane extends, while $-$ denotes the directions transverse to the brane. q_1-, q_2-dimensional subspaces contain relative transverse directions, and the d-dimensional subspace contains the overall transverse directions. It turns out that it is the orientations of the branes that determine the amount of supersymmetry preserved. For the generic case of (1.2.50), if $q_1 + q_2 = 0$ mod 4, the configuration preserves one quarter of the supersymmetry, i.e. has 8 supercharges. Actually, branes only intersect when they are at the same place of the overall transverse subspace, however, we still refer to configurations given by (1.2.50) as intersecting branes although they may be separated in the overall transverse space.

Let us briefly explain how the above conclusion is achieved. For simplicity, let us turn off the nontrivial worldvolume fields, then the projection conditions for the two branes take the form

$$\Gamma^{(1)}\epsilon = \epsilon, \qquad \Gamma^{(2)}\epsilon = \epsilon \tag{1.2.51}$$

respectively, where $\Gamma^{(1)}, \Gamma^{(2)}$ would be just products of gamma matrices. Thus, we have either

$$\left[\Gamma^{(1)}, \Gamma^{(2)}\right] = 0, \quad \text{or} \quad \left\{\Gamma^{(1)}, \Gamma^{(2)}\right\} = 0. \tag{1.2.52}$$

If the former relation is satisfied, 8 supercharges are preserved, while if the latter relation is satisfied, all supersymmetries are broken. It happens that if $q_1+q_2 = 0$ (mod 4), one would have $\left[\Gamma^{(1)}, \Gamma^{(2)}\right] = 0$. The above rule can be generalised easily for more than two sets of intersecting branes. For example, for the configuration containing three sets of orthogonally intersecting M5-branes which extend along directions $(X^0, X^1, X^2, X^3, X^4, X^5)$, $(X^0, X^1, X^2, X^3, X^6, X^7)$ and $(X^0, X^1, X^2, X^3, X^8, X^9)$, there are three projection gamma matrices $\Gamma^{(1)} = \Gamma_{012345}$, $\Gamma^{(2)} = \Gamma_{012367}$ and $\Gamma^{(3)} = \Gamma_{012389}$, and the projection conditions are

$$\Gamma_{012345}\epsilon = \epsilon, \quad \Gamma_{012367}\epsilon = \epsilon, \quad \Gamma_{012389}\epsilon = \epsilon. \tag{1.2.53}$$

It is straightforward to check that the above three projection conditions are independent and compatible so that such a configuration will preserve 1/8 of supersymmetry. In this example, the number of relative transverse directions among any pair of M5-

branes are four. It is also obvious that these gamma matrices commute with each other.

Branes can also end on branes. The locus is a co-dimension one object, for example, p-brane may end on another D-brane with a $(p-1)$-dimensional intersection. This $(p-1)$-dimensional objects on the worldvolume of the D-brane being ended on must have its charge carried away by an appropriate worldvolume gauge field. For example, D1 can end on D3, D1's charge is carried away by the worldvolume vector gauge field on the D3.

Consider the case with N D3-branes extended along X^0, X^1, X^2, X^3 and k D1-branes extended along X^4 and ending on the D3-branes. Obviously, we have $p = 0$ and $q_1 = 3, q_2 = 1$, so that the configuration preserves 8 supersymmetries. For the case of $N = 1 = k$, the end point of the D1-brane appears as a Dirac monopole on the D3. Another interesting case is $N = 2, k = 1$. In this case, on the worldvolume of the D3-branes, the D1 end point appears as a 't Hooft-Polyakov monopole. 't Hooft-Polyakov monopole solution is a solitonic solution in the SU(2) SYM, described by a non-abelian gauge field and a scalar field in the adjoint representation. The profile of the scalar field, in particular, the asymptotic vev of the scalar field represents the separation of the two D3-branes. The SU(2) symmetry is intact at the core but spontaneously broken to U(1) at infinity. The singularity of the abelian Dirac monopole is smeared out by the non-abelian symmetry to be smooth in the 't Hooft-Polyakov solution. From the point of view of the D1-brane, the monopole solution is described by the Nahm equation [34]. The Nahm transformation allows one to switch between the D1 and D3 descriptions of the monopoles.

The allowed configurations of branes ending on branes can also be obtained by dualities which will be introduced later. For example, we know that by definition the open strings can end on D3-branes, by performing an S-duality, one would find immediately that D1 can end on D3. By further performing T-dualities, one would find that D2 can end on D4, where the charges of the string-like objects on the D4 worldvolume are carried away by the dual of the worldvolume vector gauge field on the D4.

1.3 Dualities and introduction of M-theory

In this section, we describe the T and S dualities of string theory, then we will give a very basic introduction of M-theory and its ingredients that we will be interested in.

1.3.1 T and S dualities :

T duality is an exact symmetry of the string theory perturbatively, order by order in the loop expansion. In particular, IIA and IIB superstring theories are equivalent to each other via T duality. Let us compactify the space-time on a circle with radius R in the X^M direction, the mass-squared of closed string states is given by

$$\text{mass}^2 = \left(\frac{k}{R}\right)^2 + \left(\frac{wR}{\alpha'}\right)^2 + (\text{oscillator modes}), \tag{1.3.54}$$

where the first term is the Kaluza-Klein(KK) momenta and the second term is contributed from the winding modes. It is clear that the mass-squared is invariant under the exchange of KK and winding modes, i.e. $k \leftrightarrow w$, provided we also consider the interchange of the radii $R/\sqrt{\alpha'} \leftrightarrow \sqrt{\alpha'}/R$. In other words, one cannot distinguish the theory compactified on a circle with radius R, or on a circle with the dual radius $\tilde{R} = \alpha'/R$.

One can define the dual coordinates such that the T duality transforms the coordinates of the original space-time to

$$\tilde{X}^M(\tau,\sigma) = X_L^M(\sigma^+) - X_R^M(\sigma^-), \quad \tilde{\chi}_+(\sigma^+) = \chi_+^M(\sigma^+), \quad \tilde{\chi}_-^M(\sigma^-) = -\chi_-^M(\sigma^-). \tag{1.3.55}$$

Notice that there are sign flips for the right-moving modes. This shows that the T duality amounts to exchanging IIA and IIB superstring theories, as chiralities of right-moving Ramond states are changed.

As for the open strings, (1.3.55) implies that T duality swaps Dirichlet and Neumann boundary conditions. Thus, if one T dualizes along a longitudinal direction of a D_p-brane, one obtains a $D_{(p-1)}$-brane localised at the dual coordinate; if one T dualizes along a transverse direction of a D_p-brane, one obtains a $D_{(p+1)}$-brane extending along that transverse direction. One can also compactify the string theory on a torus T^n. In this case, the duality symmetry is generalised to be $O(n,n;\mathbb{Z})$.

S duality is a weak-strong duality, i.e. it maps a perturbative theory to its non-perturbative counterpart. This kind of duality was first observed on the gauge symmetry, say, in $\mathcal{N} = 4$ SYM in $4d$, the following complex coupling constant transforms as a modular parameter under SL(2,Z) duality transformations

$$\tau = \frac{\theta}{2\pi} + i\frac{4\pi}{g_{YM}^2}, \tag{1.3.56}$$

where θ is the theta angle of the topological term in the action, and g_{YM} is the Yang-Mills coupling constant. The SL(2,Z) transformations are generated by $\tau \to -1/\tau$ and $\tau \to \tau + 1$, in which $\tau \to -1/\tau$ is equivalent to the electromagnetic duality transformation.

It was also observed that there exists SL(2,R) symmetry in type IIB supergravity. This symmetry is reduced to SL(2,Z) for the IIB superstring theory by stringy and quantum effects. The identifications of gauge theory parameters and string theory ones are given by

$$C_0 = \frac{\theta}{2\pi}, \qquad g_s = \frac{g_{YM}^2}{4\pi} = e^{\Phi}, \tag{1.3.57}$$

where Φ is the dilaton field and C_0 is the axion field. The $SL(2,R) \to SL(2,Z)$ may be understood as the Dirac-Nepomechie-Teitelboim quantisation condition [35–37] of the (p,q) strings, where p, q are coprime. (p,q) strings are bound states composed of fundamental strings (F1) which are charged under Kalb-Ramond field B_2, and D1 strings which are charged under the Ramond-Ramond field C_2. The F1 strings are denoted as $(1,0)$ while D1 strings are denoted as $(0,1)$.

In particular, the D3-brane in the IIB superstring theory carries a charge that sources the Ramond-Ramond gauge field C_4 which is a $SL(2,Z)$ singlet. This means that the D3-brane action should be invariant under $SL(2,Z)$ duality. Also, it means that these (p,q) strings can end on the D3-branes, since the configuration of (p,q) strings on the D3-branes can be obtained via S duality from (1,0) strings on the D3-branes which is an allowed configuration by definition. The D3-brane action in Minkowski background with dilaton Φ and axion C_0 present is given as

$$S_{D3} = \int d^4\sigma \sqrt{-\det(\eta_{\mu\nu} + e^{-\Phi/2}F_{\mu\nu})} + \frac{i}{8}\epsilon^{\mu\nu\rho\sigma}C_0 F_{\mu\nu}F_{\rho\sigma}, \tag{1.3.58}$$

where the worldvolume induced metric is considered to be flat for simplicity. In [32], this D3-brane action was shown to be invariant under the $SL(2,Z)$ duality, in

particular, it is invariant under the electromagnetic duality transformation provided we also transform $\tau \to -1/\tau$ with $\tau = C_0 + ie^{-\Phi}$. Notice that the above D3-brane action is written in the Einstein frame, as S-duality is only manifest when IIB string theory is in Einstein frame. Notice also that the 2-form field strength $F_{\mu\nu}$ is not extended, and C_4 and C_2 terms in the Wess-Zumino terms are omitted for simplicity. This action can actually be obtained by performing the dimensional reduction of the M5-brane action [13] on a torus [38, 39]. Notice also that the low energy limit of the worldvolume theory for multiple D3-branes is $\mathcal{N} = 4$ super-Yang-Mills theory. $\mathcal{N} = 4$ SYM is not only invariant under S duality but also superconformal, and it is useful as a toy model both physically and mathematically.

For the recent development and point of view on dualities of string and M theory, see, for example, [40].

1.3.2 M theory and string theories

There are three more consistent superstring theories in additional to the type IIA and IIB theories that we have discussed. They are type I, heterotic $SO(32)$ and heterotic $E_8 \times E_8$. Closed strings in type I superstring theory can be obtained from type IIB theory by imposing the symmetry $\sigma \to -\sigma$, so that closed strings in type I are un-oriented. To be consistent, open strings in type I superstring theory are un-oriented as well, and therefore fundamental strings in type I do not carry Kalb-Ramond charge. As a result of gauging the symmetry $\sigma \to -\sigma$, the supersymmetry of type I string theory is $\mathcal{N} = 1$. The heterotic superstring theories are a hybrid combination of bosonic strings for the left movers and superstrings for the right movers. Specific gauge groups are required for the theory to be consistent. For example, the superconformal anomaly in type I superstring vanishes only if the gauge group is $SO(32)$ and heterotic superstring theories must have gauge groups $SO(32)$ or $E_8 \times E_8$ to have vanishing anomaly. Moreover, both heterotic superstring theories have $\mathcal{N} = 1$ as well. These superstrings theories are known to be connected by a web of duality relations [41]. For example, the T duality relates type IIA and IIB theories, as well as the two heterotic theories, and S duality maps type I theory to heterotic $SO(32)$, while type IIB is self-dual with respect to the S duality. Based

on these, a mother theory which has the low energy limit as the $11d$ supergravity and realises the five superstring theories as the different limits on its moduli space was proposed in [8]. This $11d$ fundamental theory is coined "M theory", whose complete quantum formulation is still unknown. However, the existence of M theory is quite promising as it is supported by various nontrivial examples. The relations between M theory and superstrings and $11d$ supergravity is summarised in figure 1.1. We review some relations between M theory and string theories in the following that will be relevant to us.

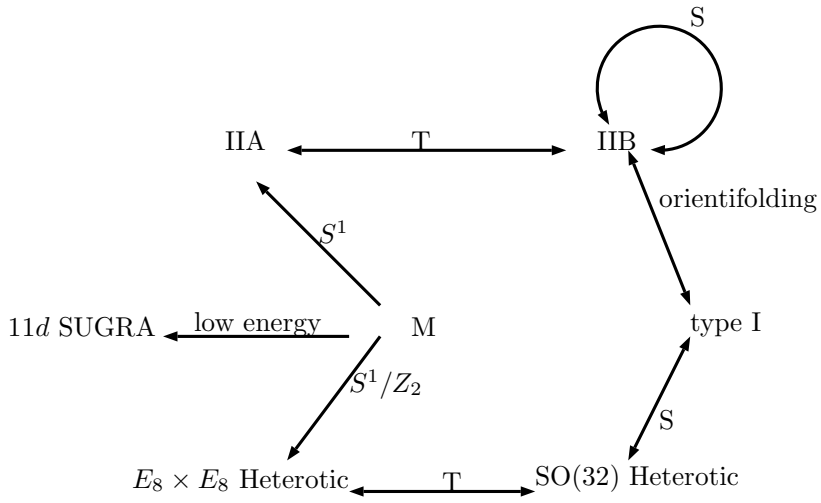

Figure 1.1: Five superstring theories and $11d$ supergravity (SUGRA) theory are realised as different limits in the moduli space of M theory. The theories are related by T-duality (denoted by T) and S-duality (denoted by S). IIA and heterotic $E_8 \times E_8$ theories are M theory on the circle or segment respectively. Although it is not shown in this diagram, it is also known that M-theory on K_3 is dual to SO(32) Heterotic string theory on T^3.

M theory type IIA duality

M theory is the strong coupling limit of type IIA superstring theory. This means, at strong coupling limit, an additional dimension is opened up. In other words, IIA

theory is equivalent to M theory on a circle of radius

$$R_{11} = l_s g_s. \tag{1.3.59}$$

This also explains why type IIA supergravity can be obtained from the $11d$ supergravity by dimensional reduction on a circle, as we have seen in (1.2.37). It also implies that there is no natural parameter on which we can do perturbation theory in $11d$. In this duality relation, the D0-branes spectrum is identified as the Kaluza-Klein excitations of the massless supergravity multiplet for M theory on a circle. The D0-branes have mass (in string frame) $1/(l_s g_s)$, while the Kaluza-Klein momenta are given by $p_{11} = N/R_{11}$, where N is an integer. Notice that both the D0-brane states and the Kaluza-Klein modes are BPS supermultiplets. By using (1.3.59), we see that the masses of the D0-brane and lowest KK mode can be identified.

Though the microscopic degrees of freedom in M theory are not clear a priori, there are two type of objects in M theory, M2-branes and M5-branes, whose tensions are given by

$$T_{M2} = \frac{2\pi}{(2\pi l_p)^3}, \qquad T_{M5} = \frac{2\pi}{(2\pi l_p)^6} \tag{1.3.60}$$

respectively. The fundamental string in type IIA theory can be obtained as the M2 brane on a circle R_{11}, while compactification of the M5 brane on a circle gives the D4-brane. Recall that the D-brane tension is given by (1.2.26) and that the tension of the fundamental string is $T_{F1} = 1/(2\pi l_s^2)$. One can easily check that

$$T_{F1} = 2\pi R_{11} T_{M2}, \qquad T_{D4} = 2\pi R_{11} T_{M5}, \tag{1.3.61}$$

by using the relation (1.3.59) and

$$l_p = g_s^{1/3} l_s, \tag{1.3.62}$$

which can be read off from (1.2.37). In particular, the D4-brane action can be obtained from the M5-brane action dimensionally reduced on a circle [13, 42]. On the other hand, if we compactify M theory on a circle that is transverse both to the M2- and M5- branes, the resulting objects in type IIA theory are D2-branes and NS 5-branes. One can also verify the relation between tensions by using $l_p = g_s^{1/3} l_s$, recalling the NS 5-brane tension is given by (1.2.38). While D0-branes are KK

modes along X^{11} circle, D6-branes are KK monopoles described by the Taub-NUT metric [43]. In particular, the $11d$ supergravity solution which reduces to a D6-brane in type IIA superstring theory is given by a geometric background which is a tensor product of $(6+1)$-dimensional Minkowski space-time with the Taub-NUT space. Unlike other objects in IIA, the D8 brane has no simple relation with M theory branes. The D8-brane is the domain wall in $10d$, the existence of D8-branes is predicted by T duality. Actually, the D8-brane is related to the M9-brane solution in massive $11d$ supergravity [44].

M theory on a torus is equivalent to type IIB superstring theory on a circle. This can be understood by the type IIA M-theory duality on a circle and T duality on another circle. One can directly verify this by matching the spectrum of zero modes of the BPS (p,q) strings with the KK excitations and winding modes of M2-branes, with the identification of the modular parameter of the torus, τ_M, and the complex coupling constant $\tau = C_0 + ie^{-\Phi}$. This identification of τ with the geometric properties of the M theory compactification τ_M is exactly what is needed to obtain the D3 brane action from the M5 action [38, 39]. In this way, the nonperturbative SL(2,Z) duality of type IIB string theory is realised as the action of the large diffeomorphism of the torus on which M theory is compactified.

M2- and M5-branes

The low energy limit of M theory is $11d$ supergravity, whose field content contains the 3-form gauge field C_3, graviton and fermions. The M2- and M5-branes are nonperturbative objects propagating in the $11d$ supergravity background. In the case of string theory, field contents of the nonperturbative D-branes can be read off from the quantisation spectrum of open strings. For M theory, there is no similar way to understand the field contents of M2- and M5-branes. However, one may view the fields on the M2- or M5-branes as the Goldstone/Goldstino modes for the broken background symmetries. For example, on the M2-brane, there are 8 scalars and 8 (on-shell) fermionic degrees of freedom which are associated with broken translational symmetry in the transverse directions and broken supersymmetry of the background respectively. For the M5-brane, there are a chiral 2-form

gauge field and 5 scalars as well as two symplectic Majorana spinors. The chiral 2-form has a 3-form field strength which is self-dual with respect to the $6d$ world-volume metric, therefore, it has 3 on-shell degrees of freedom. By choosing relevant gauge parameters appropriately for the broken background gauge fields, the Gold-stone mode analysis for M2- and M5-branes are carried out in [45]. In particular, the normalisability condition in the transverse directions (finite energy condition) requires that only self-duality modes of the 2-form survives. The existence of the chiral 2-form gauge field may also be understood from IIA M-theory duality as follows. The configuration of fundamental strings ending on the D4-branes is uplifted to be M2-branes ending on M5-branes. However, to realise the configuration of M2 ending on M5, there must be a gauge field carrying away the charges of the string like object on the intersection, and it is the chiral 2-form which does the job. The string-like object on the M5 worldvolume as the ending locus of the M2-branes is called a self-dual string [46]. We will study non-abelian self-dual string solutions [47] for our proposed multiple five branes model in chapter 3.

These M2- and M5-branes can also be realised as the solitonic solutions of the $11d$ supergravity. The thermodynamics for these solutions can be studied. Interestingly, the entropies of the multiple M2- and M5-branes scale as $N^{3/2}$ and N^3 respectively for N coincident branes. This is different from the case of D-branes in string theory. For the D-branes, the entropy for N coincident D-branes scales as N^2. This can be understood as the degrees of freedom of the $U(N)$ non-abelian gauge group for the Chan-Paton factor. The worldvolume field theories for coincident M2- and M5-branes are highly nontrivial, as the symmetry group is expected to be non-abelianized if there are multiple branes on top of each other. Moreover, the world-volume theories are expected to be strongly interacting, as M2- and M5-branes can be viewed as the strong coupling limits of D2- and D4-branes through IIA M-theory duality. There are already non-abelian theories for multiple M2-branes [48–52] that can produce the $N^{3/2}$ entropy behaviour [53]. However, it is fair to say that the non-abelian structure for multiple M5 branes is still an open question, though there are quite a few models on the market [54–64]. Nevertheless, we will try to propose an interesting model describing the gauge sector of multiple five branes [54] in chapter

2.

1.3.3 M2-branes

In this section, we review the action for a single M2-brane [65, 66] as well as models for multiple parallel M2-branes [48–52] that will be relevant for us.

While string theory predicts the existence of maximally supersymmetric Yang-Mills theories [67, 68], M-theory postulates the existence of strongly coupled $3d$ and $6d$ maximally supersymmetric conformal field theories [69–71], which are the worldvolume field theories of M2- and M5-branes respectively in the decoupling limit of gravity.

In the simple case of a single M2-brane, the action is given by

$$S_{M2} = -T_{M2}\int d^3\sigma\sqrt{-\det g_{\mu\nu}} + T_{M2}\int C_3, \tag{1.3.63}$$

where $g_{\mu\nu} = E^{\underline{A}}_{\mu}E^{\underline{B}}_{\nu}\eta_{\underline{AB}}$, $E^{\underline{A}}_{\nu} = \partial_\nu Z^{\underline{M}}E^{\underline{A}}_{\underline{M}}$, $C_3 = \frac{1}{3!}\epsilon^{\mu\nu\rho}E^{\underline{A}}_{\mu}E^{\underline{B}}_{\nu}E^{\underline{C}}_{\rho}C_{\underline{ABC}}$, and $\mu,\nu = 0,1,2$. The convention is essentially the same as in the section describing the D-brane action (1.2.41). There is no gauge field on the worldvolume, but 8 scalars and 8 (on-shell) fermionic degrees of freedom. The charge is fixed by kappa-symmetry to be equal to the tension. This means that the M2-brane is a BPS object. By semi-classically dualizing one of the scalar fields and dimensionally reducing one of the transverse directions, one can connect the above M2-brane action with the D2-brane action [9].

In the case of multiple M2 branes, two Lagrangians are constructed recently. A $3d$ maximally supersymmetric Chern-Simons-matter model was proposed [48–51], called Bagger-Lambert-Gustavsson (BLG) model. The discovery of the BLG model is remarkable because it is the first example of a maximally supersymmetric Lagrangian that is not of super-Yang-Mills type. There is a 1-form gauge field in the theory, however, it is not dynamical but purely topological described by Chern-Simons terms. This gauge field does not fit in the same representation of the gauge group as other fields, though they are related by supersymmetry.

Required by, among others, the conformal symmetry, one may deduce the fol-

lowing supersymmetry transformation rules :

$$\begin{aligned}\delta X^I_a &= i\bar{\epsilon}\Gamma^I\Psi_a \\ \delta\Psi_a &= D_\mu X^I_a\Gamma^\mu\Gamma^I\epsilon - \frac{1}{3!}X^I_b X^J_c X^K_d f^{bcd}{}_a\Gamma^{IJK}\epsilon \\ \delta\tilde{A}_\mu{}^b{}_a &= i\bar{\epsilon}\Gamma_\mu\Gamma_I X^I_c\Psi_d f^{cdb}{}_a,\end{aligned} \tag{1.3.64}$$

where $D_\mu X^I_d = \partial_\mu X^I_d - \tilde{A}_\mu{}^c{}_d X^I_c$, and $f^{bcd}{}_a = f^{[bcd]}{}_a$ is some coupling constant that will be related to a novel algebraic system. The supersymmetry algebra closes on-shell if we require the following "fundamental identity"

$$f^{[abc}{}_g f^{d]hg}{}_e = 0, \tag{1.3.65}$$

which is a generalisation of the Jacobi identity $f^{[bc}{}_d f^{a]d}{}_e = 0$ satisfied by the structure constants of usual Lie algebras, $\left[T^a, T^b\right] = f^{ab}{}_c T^c$, where T^a are generators of some Lie algebra. Thus, we see that M2-branes, in particular, the conformal symmetry ask for a notion of Lie 3-algebra,

$$\left[T^a, T^b, T^c\right] = f^{abc}{}_d T^d, \tag{1.3.66}$$

where $f^{abc}{}_d$ is by definition totally anti-symmetric for the upper indices. To define a Lagrangian, we would like to introduce an inner product on the Lie 3-algebra

$$\langle T^a, T^b\rangle = h^{ab}, \tag{1.3.67}$$

which can be used to move indices up or down. In particular, the invariance of the inner product under gauge transformations requires $f^{abcd} = h^{de} f^{abc}{}_e = f^{[abcd]}$. The Lagrangian for BLG model is given by

$$\mathcal{L} = -\frac{1}{2}D_\mu X^{aI} D^\mu X^I_a + \frac{i}{2}\bar{\Psi}^a\Gamma^\mu D_\mu\Psi_a + \frac{i}{4}\bar{\Psi}\Gamma_{IJ}X^I_c X^J_d\Psi_a f^{abcd} - V + \mathcal{L}_{CS}, \tag{1.3.68}$$

where

$$V = \frac{1}{12}X^I_a X^J_b X^K_c X^I_e X^J_f X^K_g f^{abcd} f^{efg}{}_d \tag{1.3.69}$$

is the Bagger-Lambert sextic potential, and

$$\mathcal{L}_{CS} = \frac{1}{2}\epsilon^{\mu\nu\lambda}\left(f^{abcd}A_{\mu ab}\partial_\nu A_{\lambda cd} + \frac{2}{3}f^{cda}{}_g f^{efgb}A_{\mu ab}A_{\nu cd}A_{\lambda ef}\right), \tag{1.3.70}$$

with the gauge field $A_{\mu ab}$ related to the physical gauge field $\tilde{A}_\mu{}^a{}_b$ via $\tilde{A}_\mu{}^b{}_a = A_{\mu cd} f^{cdb}{}_a$. The gauge transformation of a generic field X is defined as

$$\delta_{A,B}(X) = [A, B, X]\,, \tag{1.3.71}$$

where A, B are elements of the Lie 3-algebra. If we require the map $\delta_{A,B}$ to be a derivation,

$$\delta_{A,B}\left([X, Y, Z]\right) = [\delta_{A,B}(X), Y, Z] + [X, \delta_{A,B}(Y), Z] + [X, Y, \delta_{A,B}(Z)]\,, \tag{1.3.72}$$

the triple product (1.3.66) is then required to satisfy the fundamental identity,

$$\left[T^a, T^b, \left[T^c, T^d, T^e\right]\right] = \left[\left[T^a, T^b, T^c\right], T^d, T^e\right] + \left[T^c, \left[T^a, T^b, T^d\right], T^e\right] + \left[T^c, T^d, \left[T^a, T^b, T^e\right]\right], \tag{1.3.73}$$

which is equivalent to (1.3.65). By closing the supersymmetry algebra explicitly, one may find the equations of motion, and then construct a Lagrangian. The resulting theory has 16 supersymmetries and enjoys $SO(8)$ R-symmetry and conformal symmetry.

However, if one insists to have a positive definite metric and a finite dimensional representation, the fundamental identity is so strong that the only allowed nontrivial Lie 3-algebra is the so-called A_4 algebra with

$$f^{abcd} = \frac{2\pi}{k}\epsilon^{abcd}, \tag{1.3.74}$$

where $a, b, c, d = 1, 2, 3, 4$ and k is the level of the Chern-Simons term that will be quantised if one requires the path integral to be invariant under large gauge transformations. The BLG model is characterised by the choice of the Lie 3-algebra, however, we see that some reasonable assumptions fix the algebra to be A_4. By analysing the moduli space carefully using the sextic Bagger-Lambert potential (1.3.69), it was found in [72, 73] that the BLG model can at best describe two M2-branes. In particular, at Chern-Simons level $k = 1$ and $k = 2$, the classical moduli space was shown to coincide with the infrared limit of $SO(4)$ and $SO(5)$ super-Yang-Mills theory, and this means BLG model describes two M2-branes in the background of the orbifold R^8/Z_2, without and with discrete torsion respectively. Actually, there could be some relaxations on the assumptions for the 3-algebra. For example, indefinite metrics may make sense physically. Here, we are interested in another possibility, i.e.

infinite dimensional representations. In [74, 75], the Nambu-Poisson bracket, which is locally defined as

$$\left\{f^1, f^2, f^3\right\}_{NP} = \epsilon^{abc} \frac{\partial f^1}{\partial y^a} \frac{\partial f^2}{\partial y^b} \frac{\partial f^3}{\partial y^c}, \tag{1.3.75}$$

is used for the BLG model, where $a, b, c = 1, 2, 3$ and f^1, f^2, f^3 are three functions of y^a. The Nambu-Poisson bracket is a natural generalisation of the familiar Poisson bracket and it is associated with a three dimensional internal manifold $\mathcal{N}_3$ parametrized by the coordinates y^a. With the infinite dimensional basis $\chi^a(y)$, $a = 1, 2, 3, \cdots$, one can write

$$\left\{\chi^a, \chi^b, \chi^c\right\}_{NP} = f^{abc}{}_d \chi^d. \tag{1.3.76}$$

The Nambu-Poisson bracket also satisfies the fundamental identity. Moreover, one can also define a metric

$$\langle f, g \rangle = \int_{\mathcal{N}_3} d^3y \, f(y) g(y). \tag{1.3.77}$$

If $\mathcal{N}_3$ is not trivially R^3, in principle we need to cover $\mathcal{N}_3$ with patches, and there are therefore gauge transformations connecting different patches. These gauge transformations $y'^a = f^a(y)$ must satisfy

$$\left\{f^1, f^2, f^3\right\} = 1 \tag{1.3.78}$$

in order to keep the Nambu-Poisson bracket. This means that the gauge transformations in the BLG model with Nambu-Poisson bracket are volume-preserving diffeomorphisms. This particular BLG model describes an infinite number of M2-branes, as the Nambu-Poisson bracket is an infinite dimensional representation. What is interesting in [74–76] is that they view the internal manifold $\mathcal{N}_3$ as a sub-manifold of the worldvolume of a single M5-brane. Thus, the infinite M2-branes dissolve and form a single M5-brane through an M-theory uplifted Myers effect [77]. The resulting single M5-brane model is coined NP-M5 theory. However, this description is only good in the limit of strong C_3 background gauge field [78, 79]. Nevertheless, this NP-M5 model will play an important role in motivating our work in chapter 4.

To describe an arbitrary number of M2-branes, one needs to look for less restrictive conditions on the Lie 3-algebra. It turns out that the correct route is to look

for less (manifest) supersymmetries. This is what is done in the Aharony-Bergman-Jafferis-Maldacena (ABJM) model [52]. It is a $3d$ Chern-Simons-Matter theory with $\mathcal{N} = 6$ supersymmetries and $SO(6)$ R-symmetry. The symmetry group is a product group $U(N) \times U(N)$ with N arbitrary. The moduli space turns out to be

$$\frac{(\mathbb{C}^4/\mathbb{Z}_k)^N}{S_N}, \tag{1.3.79}$$

where S_N is the symmetric group acting on N objects, Z_k is the cyclic group and k is the Chern-Simons level. The analysis of the moduli space shows that the ABJM theory describes an arbitrary number N of M2 branes probing the orbifold $\mathbb{C}^4/\mathbb{Z}_k$ of the transverse space. When $N = 2$, one can show that the ABJM model is equivalent to the BLG model with A_4 Lie 3-algebra.

1.3.4 M5-branes

The worldvolume field theory of M5 branes predicted by M theory is likely to be the first example of a well-defined quantum field theory with a dimension higher than four. The worldvolume theory is given by the so-called (2,0) superconformal field theory in $6d$ (or simply (2,0) theory) whose field content includes a chiral 2-form, 5 scalars and 8 on-shell fermionic degrees of freedom. The theory is highly nontrivial as the nonabelian symmetry structure is unclear and the dynamics of self-duality conditions is difficult to deal with, especially at the action level. In particular, there are no-go theorems [80–82] saying that there is no nontrivial deformation of abelian chiral 2-form gauge theory if locality of the action and the transformation laws are assumed. Also, it is widely believed that there is no Lagrangian formulation for (2,0) theory. However, several models have been proposed to approach the final goal of complete (2,0) theory. The model which will be introduced in chapter 2 evades the no-go theorem by abandoning the locality condition. People have also tried to start from the five dimensional theory with (2,0) compactified on a circle [57, 59]. In [54, 58, 63, 64], they try to construct the (2,0) theory from the less supersymmetric (1,0) theories. There are also proposals based on the higher gauge theories and twistor approaches [60–62]. The conjecture that the (2,0) theory on a circle with a finite radius is equivalent to $5d$ super-Yang-Mills [55, 56] enables various calculations

and already has some nontrivial checks.

Nevertheless, even in the abelian case which corresponds to a single M5-brane, the complete formulation of a supersymmetric self-interacting worldvolume theory is already nontrivial. The field equations were first derived by the super-embedding formulation [83, 84]. It was later realised that they can also be derived from action principles [13, 14]. It was then found that these descriptions of the M5-brane are all equivalent [85, 86].

Moreover, according to Schwarz's "highly effective action" conjectured in [87], such actions might capture some non-abelian information, though they used to be thought of as low energy effective actions. In the next section, we will review the formulation for abelian (2,0) theory that will be relevant for our work in chapter 4 and 5.

1.3.5 Single M5 brane

Super-embedding description

Unlike the RNS or Green-Schwarz formulations for which only the worldsheet or the target space supersymmetries respectively are manifest, the super-embedding approach embeds a supersymmetric worldsheet into a supersymmetric target space [88]. It turns out that the super-embedding formulation can also be applied to higher dimensional branes. Moreover, the consistency condition is so strong that usually it determines the equations of motion, and this is the case for the M5-brane. Consider the embedding of the superspace M into the target superspace $\underline{M}$. M and $\underline{M}$ are parametrized by the coordinates $z^{\mathcal{M}} = (x^M, \theta^\alpha)$ and $Z^{\underline{\mathcal{M}}} = (X^{\underline{M}}, \Theta^{\underline{\alpha}})$ respectively, where θ and Θ are Grassmann odd coordinates. The geometries of M and $\underline{M}$ are described by the frame vector fields $E_{\mathcal{A}} = E_{\mathcal{A}}^{\mathcal{M}} \partial_{\mathcal{M}}$ and $E_{\underline{\mathcal{A}}} = E_{\underline{\mathcal{A}}}^{\underline{\mathcal{M}}} \partial_{\underline{\mathcal{M}}}$ respectively. Obviously, $E_{\mathcal{A}}$ can be expanded in terms of $E_{\underline{\mathcal{A}}}$, so that $E_{\mathcal{A}} = E_{\mathcal{A}}^{\underline{\mathcal{A}}} E_{\underline{\mathcal{A}}}$. The consistency condition is simply that the Grassmann odd part of the tangent space of M should lie in the Grassmann odd tangent space of $\underline{M}$, in other words, it is

$$E_\alpha{}^{\underline{A}} = 0. \tag{1.3.80}$$

The consistency condition in the super-embedding of the M5-brane gives the non-linear self-duality equations packed in a succinct algebraic form

$$\frac{1}{4}H_{\mu\nu\rho} = m_{\mu}^{-1\lambda}h_{\lambda\nu\rho}, \tag{1.3.81}$$

where $H_3 = dB_2$ is the field strength of the chiral 2-form and $h_3 = *h_3$ satisfying the linear self-duality condition may be viewed as auxiliary fields. m^{-1} is the inverse matrix of $m_{\mu}{}^{\lambda} = \delta_{\mu}{}^{\lambda} - 2k_{\mu}{}^{\lambda}$, where $k_{\mu}{}^{\lambda} = h_{\mu\nu\rho}h^{\lambda\nu\rho}$, $\mu, \nu, \lambda, \rho = 0, 1, \cdots, 5$. It is possible to eliminate h_3 and obtain the nonlinear self-duality equation in the following form [89]

$$*H_3 = \frac{\partial \mathcal{K}}{\partial H_3}, \tag{1.3.82}$$

with

$$\mathcal{K} = 2\sqrt{1 + \frac{1}{12}H^2 + \frac{1}{288}(H^2)^2 - \frac{1}{96}H_{\mu\nu\rho}H^{\nu\rho\lambda}H_{\lambda\tau\sigma}H^{\tau\sigma\mu}}. \tag{1.3.83}$$

However, it turns out that one can only construct a pseudo-action for this form of self-duality condition [89]. It is interesting that the covariant self-duality equation (1.3.82) can be rewritten in a non-manifestly covariant way,

$$H_{\hat{a}\hat{b}} = -\frac{\partial L}{\partial \tilde{H}_{\hat{a}\hat{b}}}, \tag{1.3.84}$$

with

$$L = \sqrt{\det(\delta_{\hat{a}}^{\hat{b}} + i\tilde{H}_{\hat{a}}{}^{\hat{b}})} = \sqrt{1 + \frac{1}{2}\mathrm{tr}\tilde{H}^2 + \frac{1}{8}\left(\mathrm{tr}\tilde{H}^2\right)^2 - \frac{1}{4}\mathrm{tr}\tilde{H}^4}, \tag{1.3.85}$$

where $H_{\hat{a}\hat{b}} = H_{\hat{a}\hat{b}\hat{5}}$, $\tilde{H}_{\hat{a}\hat{b}} = \tilde{H}_{\hat{a}\hat{b}\hat{5}}$ and $\tilde{H}_3 = *H_3$, and this form of nonlinear self-duality equation has an action principle [90]. Actually, it is possible to rewrite the covariant equation (1.3.82) in another closed form, and such a form also admits a Lagrangian description, as we will present in chapter 4. Notice that $\hat{a}, \hat{b} = 0, 1, 2, 3, 4$ so that the Lorentz symmetry is superficially lost. However, (1.3.84) is actually fully covariant. Moreover, the action leading to (1.3.84) can even be made manifestly covariant by introducing an additional auxiliary field [91]. We will present a simplest example illustrating how the action principle works for a chiral 2-form in $6d$ now.

Action formulation for chiral p-forms

The idea for the Lagrangian formulation of chiral p-forms is to sacrifice manifest Lorentz symmetry [90, 92–94]. Consider the following free 2-form action in $6d$ Minkowski space-time

$$S = \frac{1}{2}\int d^6x\, \tilde{H}^{\hat{a}\hat{b}\hat{5}}(H_{\hat{a}\hat{b}\hat{5}} - \tilde{H}_{\hat{a}\hat{b}\hat{5}}), \tag{1.3.86}$$

with $\hat{a}, \hat{b} = 0, 1, 2, 3, 4$ and $\tilde{H}$ denotes the Hodge dual of the field strength $H_3 = dB_2$. One may worry that this quadratic action also leads to a 2nd order field equation and hence it is still difficult to get the 1st order self-duality equation $H_3 = *H_3$. However, the punchline is that the 2nd order field equation will be equivalent to the 1st order self-duality conditions.

The field equations of $B_{\hat{a}\hat{5}}$ is trivially vanishing, but the field equation of $B_{\hat{a}\hat{b}}$ gives

$$\epsilon^{\hat{a}\hat{b}\hat{c}\hat{d}\hat{e}\hat{5}}\partial_{\hat{c}}\left(H_{\hat{d}\hat{e}\hat{5}} - \tilde{H}_{\hat{d}\hat{e}\hat{5}}\right) = 0, \tag{1.3.87}$$

which has the general solution (in topologically trivial space-time)

$$H_{\hat{d}\hat{e}\hat{5}} - \tilde{H}_{\hat{d}\hat{e}\hat{5}} = \partial_{\hat{d}}\Phi_{\hat{e}} - \partial_{\hat{e}}\Phi_{\hat{d}}, \tag{1.3.88}$$

for some arbitrary parameters $\Phi_{\hat{e}}$. The trivial equation of motion for $B_{\hat{a}\hat{5}}$ reflects the fact that the action enjoys the following gauge symmetry

$$\delta_\phi B_{\hat{a}\hat{5}} = \phi_{\hat{a}}, \qquad \delta_\phi B_{\hat{a}\hat{b}} = 0, \tag{1.3.89}$$

where $\phi_{\hat{a}}$ are arbitrary gauge parameters. In fact, components $B_{\hat{a}\hat{5}}$ show up in the action only through total derivative terms. By appropriately gauge fixing (1.3.89), we see that (1.3.88) is equivalent to the self-duality conditions

$$H_{\hat{a}\hat{b}\hat{5}} = \tilde{H}_{\hat{a}\hat{b}\hat{5}}. \tag{1.3.90}$$

As a special direction ∂_5 is singled out, the action (1.3.86) has only manifest SO(1,4) Lorentz symmetry of the full $SO(1,5)$ group. It is not clear if the action (1.3.86) could be invariant under the coset $SO(1,5)/SO(1,4)$ of the full Lorentz symmetry. Nevertheless, the action enjoys the following modified Lorentz symmetry $SO(1,5)/SO(1,4)$

$$\delta B_{\hat{a}\hat{b}} = \left[(\lambda\cdot x)\partial_5 - x^5(\lambda\cdot\partial)\right] B_{\hat{a}\hat{b}} - (x\cdot\lambda)(H_{\hat{a}\hat{b}\hat{5}} - \tilde{H}_{\hat{a}\hat{b}\hat{5}}), \tag{1.3.91}$$

where $\lambda^{\hat{a}} = \lambda^{\hat{a}}{}_5$ is the infinitesimal Lorentz parameter and $(\lambda \cdot x) = \lambda^{\hat{c}} x_{\hat{c}}, (\lambda \cdot \partial) = \lambda^{\hat{c}} \partial_{\hat{c}}$. Notice that we have implicitly worked in the gauge $B_{\hat{a}5} = 0$, so that we require $\delta B_{\hat{a}5} = 0$. Notice also that the second term vanishes if self-duality conditions are imposed, thus the transformation law is the same as the standard one on-shell.

The action (1.3.86) can be coupled to gravity [93, 95]. In the curved space, the Lagrangian becomes

$$\mathcal{L} = \frac{1}{4}\tilde{H}^{5\hat{a}\hat{b}} H_{5\hat{a}\hat{b}} + \frac{1}{8}\epsilon_{\hat{a}\hat{b}\hat{c}5\hat{x}\hat{y}} \tilde{H}^{5\hat{a}\hat{b}} \tilde{H}^{5\hat{x}\hat{y}} \frac{g^{5\hat{c}}}{g^{55}} - \frac{1}{4\sqrt{-g}} \tilde{H}^{5\hat{a}\hat{b}} \tilde{H}^{5\hat{c}\hat{d}} \frac{1}{g^{55}} g_{\hat{a}\hat{c}} g_{\hat{b}\hat{d}}, \tag{1.3.92}$$

where the action is given by $S = 2\int d^6x \mathcal{L}$, and the $\tilde{H}^{5mn}$ is now defined without any metric,

$$\tilde{H}^{5\hat{m}\hat{n}} \equiv \frac{1}{3!}\epsilon^{5\hat{m}\hat{n}\hat{p}\hat{q}\hat{r}} H_{\hat{p}\hat{q}\hat{r}}. \tag{1.3.93}$$

The action enjoys the following modified diffeomorphism

$$\delta B_{\hat{a}\hat{b}} = -\xi \frac{\partial \mathcal{I}}{\partial \tilde{H}^{\hat{a}\hat{b}5}}, \tag{1.3.94}$$

where $\mathcal{I}$ is the sum of the last two terms of the Lagrangian

$$\mathcal{I} \equiv \frac{1}{8}\epsilon_{\hat{a}\hat{b}\hat{c}5\hat{x}\hat{y}} \tilde{H}^{5\hat{a}\hat{b}} \tilde{H}^{5\hat{x}\hat{y}} \frac{g^{5\hat{c}}}{g^{55}} - \frac{1}{4\sqrt{-g}} \tilde{H}^{5\hat{a}\hat{b}} \tilde{H}^{5\hat{c}\hat{d}} \frac{1}{g^{55}} g_{\hat{a}\hat{c}} g_{\hat{b}\hat{d}}. \tag{1.3.95}$$

The diffeomorphism transformation of the metric is the standard one ($\mu, \nu = 0, 1, \cdots, 5$) :

$$\delta g_{\mu\nu} = (\mathcal{L}_\xi g)_{\mu\nu} = \xi \partial_5 g_{\mu\nu} + \partial_\mu \xi \, g_{5\nu} + \partial_\nu \xi \, g_{\mu 5}, \tag{1.3.96}$$

as a Lie derivative along the vector $\xi = \xi^5 \partial_5 \equiv \xi \partial_5$.

The action can be made manifestly covariant at the price of introducing an auxiliary scalar field $a(x)$[5] [96–98], this is known as Pasti-Sorokin-Tonin (PST) formulation. The resulting action is

$$S = \frac{1}{2}\int d^6x \, \tilde{H}^{\mu\nu\rho}(H_{\mu\nu\lambda} - \tilde{H}_{\mu\nu\lambda}) \frac{\partial_\rho a \partial^\lambda a}{(\partial a)^2}, \tag{1.3.97}$$

[5] One may instead introduce a closed auxiliary 1-form v, with $dv = 0$ and replace $\partial_\mu a$ in the Lagrangian with v_μ. This form of Lagrangian with auxiliary v may resolve the worry that $a(x)$ may not be well-defined in some topologically nontrivial space, e.g. if (1.3.97) were compactified on a circle, v is well-defined but a is not as a is now multi-valued.

where $(\partial a)^2 = \partial_\rho a \partial^\rho a$, $\mu, \nu, \lambda, \rho = 0, 1, \cdots, 5$. The auxiliary field forms a projection matrix through the combination $P_\mu{}^\nu = \partial_\mu a \partial^\nu a/(\partial a)^2$. The action (1.3.97) enjoys, apart from the tensor gauge symmetry $\delta B_2 = d\Lambda_1$, the following two gauge symmetries

$$\delta B_{\mu\nu} = 2\partial_{[\mu} a \phi_{\nu]}(x), \qquad \delta a(x) = 0, \tag{1.3.98}$$

$$\delta a = \varphi(x), \qquad \delta B_{\mu\nu} = \frac{\varphi(x)}{\sqrt{(\partial a)^2}} \left(H_{\mu\nu} - \tilde{H}_{\mu\nu} \right), \tag{1.3.99}$$

where

$$H_{\mu\nu} = H_{\mu\nu\rho} \frac{\partial^\rho a}{\sqrt{(\partial a)^2}}, \qquad \tilde{H}_{\mu\nu} = \tilde{H}_{\mu\nu\rho} \frac{\partial^\rho a}{\sqrt{(\partial a)^2}}. \tag{1.3.100}$$

(1.3.98) and (1.3.99) are called PST1 and PST2 gauge symmetry respectively. The PST1 gauge symmetry allows us to gauge fix the field equations to obtain the first order self-duality equations, while the PST2 gauge symmetry is responsible for the covariance of the action, and allows one to gauge fix the auxiliary field. For example, if one gauge fixes $a(x) = x^5$, one reduces the covariant PST action (1.3.97) to the non-manifestly covariant ones (1.3.86) or (1.3.92) depending on whether one is interested in flat or curved spaces. To get the modified Lorentz or diffeomorphism transformations, one considers a compensating PST2 gauge transformation to preserve the gauge choice $a = x^5$, the PST2 transformation combined with standard Lorentz or diffeomorphism transformations then give (1.3.91) or (1.3.94). The covariantisation is useful as it simplifies the construction of the consistent couplings to gravity and other fields significantly.

With the advent of the BLG model and in particular the construction of the NP-M5 action, it was realised that one can sacrifice the manifest Lorentz symmetry in different ways. For example, it was shown in [99] that one can construct manifestly $SO(1, D'-1) \times SO(D'')$ Lorentz symmetric chiral p-form Lagrangians in $(D' + D'')$-dimensional Minkowski spaces. We call such theories as $D' + D''$ formulations, for example, (1.3.86) is of 1+5 formulation in Minkowski space. The PST covariantisation technique can be applied, in particular, to the 3+3 case as shown in [76]. In this case, one needs to introduce a triplet of auxiliary scalar fields, $a^s(x)$, $s = 1, 2, 3$, transforming in the internal $GL(3, R)$ group. The explicit form of the

Lagrangian will be given in chapter 4.

Nonlinearisation and the M5-brane action

The free theories (1.3.92) or (1.3.97) can be nonlinearized [90] to obtain the actions describing the single M5-brane propagating in a trivial $11d$ space-time background [14], or even in a generic $11d$ supergravity background [13].

Here, we present the manifestly covariant version of the nonlinear action [90]

$$S = +2\int_{\mathcal{M}_6} d^6x \left[\sqrt{-\det(g_{\mu\nu}+i\tilde{H}_{\mu\nu})}+\frac{\sqrt{-g}}{4(\partial a)^2}\partial_\lambda a\tilde{H}^{\lambda\mu\nu}H_{\mu\nu\rho}\partial^\rho a\right] \quad (1.3.101)$$

with

$$\tilde{H}^{\rho\mu\nu}\equiv\frac{1}{6\sqrt{-g}}\epsilon^{\rho\mu\nu\lambda\sigma\tau}H_{\lambda\sigma\tau},\quad \tilde{H}_{\mu\nu}\equiv\frac{\partial^\rho a}{\sqrt{(\partial a)^2}}\tilde{H}_{\rho\mu\nu},\quad g=\det g_{\mu\nu}\,, \quad (1.3.102)$$

where

$$\epsilon^{0\cdots5}=-\epsilon_{0\cdots5}=1\,.$$

To promote this $6d$ action to be the M5-brane action propagating in a generic $11d$ supergravity background, one needs to work in the target superspace and more importantly, prove the existence of the kappa symmetry. The details of the proof are a bit technical but the way kappa symmetry works is in a similar way to the D-brane case (1.2.43). We refer the readers interested in completing the proof to the references [14, 100, 101].

Part II

non-abelian chiral 2-form proposal

Chapter 2

A proposal for the gauge sector of multiple M5-branes

In this chapter, we propose a model describing the gauge sector of multiple M-theory five branes. In particular, we will generalise the action of Perry and Schwarz [90] to a nonabelian one that gives rise to a nonabelian self-duality equation as its equation of motion. A gauge vector field which is auxiliary and non-propagating is introduced. The nonabelian action has a modified six dimensional Lorentz symmetry. Moreover, the double dimensional reduction of the nonabelian action produces the five dimensional super-Yang-Mills action plus some higher derivative corrections.

2.1 Introduction

The low energy theory of N coincident M5-branes is given by an interacting (2,0) superconformal theory in 6 dimensions [69, 102–104]. For a single M5-brane, the low energy theory is known [13, 14, 83–85, 90, 91, 98, 105]. So far very little is known about this theory for $N > 1$. There are a number of difficulties associated with this theory. First, the structure of (2,0) supersymmetry constrains the 2-form potential to have self-dual field strength. This makes it difficult to write down a Lorentz invariant action. This problem was solved in [13, 14, 85, 90, 91, 98] where an action principle was constructed with the self-duality equation obtained as the equation of motion. For the non-abelian case, there is an additional problem that an appropriate

generalization of the tensor gauge symmetry was not known. In particular, there are no-go theorems [79–82, 106–109] which state that there is no nontrivial deformation of the Abelian 2-form gauge theory if locality of the action and the transformation laws are assumed. The no-go theorems suggest an important direction to go is to give up locality.

Since M2-branes can end on M5-branes, one may wonder what one may learn by considering the intersecting M2-M5 branes system. In [110] and [111], a system of open N M2-branes described by the open ABJM theory [52] is considered. The gauge non-invariance of the boundary Chern-Simons action was shown [110] to imply the existence of a Kac-Moody current algebra on the worldsheet of multiple self-dual strings.[1] It was conjectured [58] that the Kac-Moody symmetry induces a $U(N) \times U(N)$ gauge symmetry in the theory of N coincident M5-branes. The precise nature of this gauge symmetry in the theory of M5-branes is however not known due to our little understanding of the self-dual strings. Motivated by this, in [58] a set of $U(N) \times U(N)$ gauge bosons was introduced and a version of non-abelian generalization of the tensor gauge symmetry of 2-form gauge potentials was constructed. This formulation has the advantage of having manifest Lorentz symmetry fully.

Generally, the non-abelian tensor gauge symmetry is linearly represented if the $U(N) \times U(N)$ gauge bosons are treated as independent fields. On the other hand, the (2,0) supersymmetry of M5-branes implies that no extra degrees of freedom is allowed and so these fields must be taken as auxiliary. This turns out to be very difficult for one of the auxiliary fields. So in this chapter we will consider a gauge fixed approach by given up manifest 6d Lorentz symmetry.

As a first step towards understanding the theory of multiple M5-branes, we will focus on the chiral tensor gauge fields in this chapter. Our action consists of a non-abelian generalization of the action of Perry and Schwarz [90] plus an additional term which sets the Yang-Mills gauge fields to become auxiliary. We emphasize that the action of Perry-Schwarz (PS) is of the same type as the action

[1] The ABJM theory with boundaries is also considered in [112], where boundary conditions instead of additional degrees of freedom are introduced.

originally introduced by Henneaux and Teitelboim (HT) [93] see also [113] for a recent discussion. The difference is that a time direction was separated from the rest in HT action as they were interested in a Hamiltonian description, while in the PS action a space direction was separated from the (5+1) dimensional spacetime, making it particularly suitable for discussing dimensional reduction of the system [2]. Since we will be interested in dimensional reduction of our action, so we will follow [90] in this chapter. As in Perry-Schwarz's construction, a direction x_5 is singled out and specially treated, so our theory is only manifestly 5d Lorentz invariant. Nevertheless, we manage to establish the existence of an additional non-manifest 6d Lorentz symmetry, generalizing the result of the abelian case [90, 93]. Moreover, on dimensional reduction on a circle, our action gives rise directly to the standard 5d Yang-Mills theory plus higher order corrections. Based on these properties, we propose that our action describes the gauge sector of a system of coincident M5-branes in flat space. The tensor gauge symmetry in our action turns out to be abelian, but highly nonlinear and nonlocal. In fact whether the tensor gauge symmetry is abelian or non-abelian is not constrained by any physical requirement we know of. The abelian nature of the tensor gauge symmetry is thus a prediction of our construction. The construction of a non-abelian tensor gauge symmetry is still an interesting mathematical question, but from our construction it seems not necessary for the non-covariant description of multiple M5-branes.

The plan of this chapter is as follows. In section 2.2, we review the construction of Perry and Schwarz [90]. In section 2.3, we present our construction of the action for non-abelian 2-form fields and establish the properties of self-duality, 6d Lorentz symmetry and dimensional reduction to 5d Yang-Mills action. Section 2.4 contains some further discussions. In particular we comment on the inclusion of fermions and scalar fields and supersymmetry in the discussion section. For completeness, an appendix reviewing the counting of the number of propagating degrees of freedom for the 5d Chern-Simons theory is included in the end of this chapter.

Recent related works on the subject includes: [55, 56] which proposed a funda-

[2] The covariant Pasti-Sorokin-Tonin (PST) formulation [13, 85, 91, 98] unifies both since one can gauge fix the auxiliary scalar to arrive at these different formulations.

mental definition of multiple M5-branes in terms of 5d supersymmetric Yang-Mills theory; [114] which constructed a non-abelian version of (2,0) supersymmetric equation of motion using Lie 3-algebra; [57] which constructed a compactified theory of non-abelian 2-form gauge potentials with a self-dual field strength; [64] which proposed a more general framework than [58] in utilizing a 3-form gauge potentials in addition to the 1-form gauge potentials; [74–76, 115–119] which studied the form of quantum geometry of M5-branes in a C-field background; [120] on amplitudes of multiple M5-branes theory; [121, 122] on the N^3 entropy counting of M5-branes; as well as other issues concerning multiple M5-branes [61, 123–128]. For a review on older results on M5-branes and superconformal theory in 6-dimensions, we suggest [21, 129].

2.2 Abelian Action of Perry-Schwarz

Let us start by reviewing the construction [90, 93] of an action for a self-dual tensor in 6-dimensions. A key feature of their construction is that a certain direction, x^0 in [93] or x^5 in [90], has to be singled out and so the formulation has only manifestly 5d rotational invariance or 5d Lorentz invariance. Nevertheless these theories do possess the full Lorentz symmetry. The existence of this modified Lorentz symmetry is a remarkable feature of these constructions.

We will be interested in the Lagrangian formulation of the chiral tensor gauge fields on multiple M5-branes and its dimensional reduction. Therefore let us follow the construction of Perry-Schwarz [90] in the following. Let us denote the 5d and 6d coordinates by $x^\mu = (x^0, x^1, \cdots, x^4)$ and $x^M = (x^\mu, x^5)$. We adopt the convention $\eta^{MN} = (-+++++)$ for the metric and

$$\epsilon^{01234} = -\epsilon_{01234} = 1, \quad \epsilon^{012345} = -\epsilon_{012345} = 1 \tag{2.2.1}$$

for the antisymmetric tensors. The Hodge dual of a 3-form G_{MNP} is defined by

$$\tilde{G}_{MNP} := -\frac{1}{6}\epsilon_{MNPQRS}\, G^{QRS}. \tag{2.2.2}$$

Note the minus sign in our definition of the Hodge dual follows from our convention of the antisymmetric tensor (2.2.1) which says that the 6d orientation is specified

by $dx^0 dx^1 \cdots dx^5$. The abelian field strength is given by

$$H_{MNP} = \partial_M B_{NP} + \partial_N B_{PM} + \partial_P B_{MN} := \partial_{[M} B_{NP]} \tag{2.2.3}$$

and the self-duality equation reads

$$\tilde{H}_{MNP} = H_{MNP}. \tag{2.2.4}$$

In the Perry-Schwarz formulation, the self-dual tensor gauge field is represented by a 5×5 antisymmetric tensor field $B_{\mu\nu}$. The action reads

$$S_0(B) = \frac{1}{2} \int d^6x \left(-\tilde{H}^{\mu\nu} \tilde{H}_{\mu\nu} + \tilde{H}^{\mu\nu} \partial_5 B_{\mu\nu} \right) \tag{2.2.5}$$

where

$$\tilde{H}^{\mu\nu} := \frac{1}{6} \epsilon^{\mu\nu\rho\lambda\sigma} H_{\rho\lambda\sigma}, \qquad H^{\mu\nu\rho} = -\frac{1}{2} \epsilon^{\mu\nu\rho\lambda\sigma} \tilde{H}_{\lambda\sigma}. \tag{2.2.6}$$

The action has the second order equation of motion

$$\epsilon^{\mu\nu\rho\lambda\sigma} \partial_\rho (\tilde{H}_{\lambda\sigma} - \partial_5 B_{\lambda\sigma}) = 0 \tag{2.2.7}$$

which has the general solution

$$\tilde{H}_{\lambda\sigma} - \partial_5 B_{\lambda\sigma} = \Phi_{\lambda\sigma} \tag{2.2.8}$$

for some function $\Phi_{\lambda\sigma}$ such that $\partial_{[\mu} \Phi_{\lambda\sigma]} = 0$. It is easy to check that the action (2.2.5) is invariant [3] under the gauge symmetry

$$\delta B_{\mu\nu} = \Sigma_{\mu\nu} \tag{2.2.9}$$

for arbitrary $\Sigma_{\mu\nu}$ such that $\partial_{[\mu} \Sigma_{\nu\lambda]} = 0$, or equivalently

$$\delta B_{\mu\nu} = \partial_\mu \varphi_\nu - \partial_\nu \varphi_\mu, \quad \text{for arbitrary } \varphi_\mu. \tag{2.2.10}$$

This is the tensor gauge symmetry of the model. An appropriate gauge fixing of this symmetry allows one to reduce the general solution (2.2.8) to the special form

$$\tilde{H}_{\mu\nu} = \partial_5 B_{\mu\nu}. \tag{2.2.11}$$

[3] This is under the usual assumption that fields, in this case $H_{\mu\nu\lambda}$, vanishes at infinity $|x^\mu| = \infty$.

This is the self-duality equation in this theory.

Let us digress a bit to give a pedagogical and explicit counting of the degrees of freedom in the Perry-Schwarz theory. The Perry-Schwarz theory initially has the equation of motion (2.2.7). Using the gauge symmetry (2.2.10) one can fix the equation of motion to the linear form (2.2.11). Doing so we are left with a x^5-independent residual symmetry. Now $\partial^\mu B_{\mu\nu}$ is x_5 independent as a result of (2.2.11). Using the residual symmetry, one can fix it to be zero

$$\partial^\mu B_{\mu\nu} = 0. \tag{2.2.12}$$

Differentiating (2.2.11) with respect to x_5 and use (2.2.12), we obtain that $B_{\mu\nu}$ is massless as expected, $\Box B_{\mu\nu} = 0$. Now (2.2.12) gives 4 independent conditions on the 10 components of $B_{\mu\nu}$. Using the self-duality condition, we have in total $(10-4)/2 = 3$ degrees of freedom.

The action is manifestly 5d Lorentz invariant. Nevertheless the action is indeed invariant under an additional Lorentz transformation mixing the μ directions with the 5 direction. The proposed modified Lorentz transformation is

$$\delta B_{\mu\nu} = (\Lambda \cdot x)\tilde{H}_{\mu\nu} - x_5(\Lambda \cdot \partial)B_{\mu\nu}, \tag{2.2.13}$$

where $\Lambda_\mu = \Lambda_{5\mu}$ denote the corresponding infinitesimal transformation parameters. One can check that

$$[\delta_{\Lambda_1}, \delta_{\Lambda_2}]B_{\mu\nu} = \delta^{(5d)}_{\Lambda_{\alpha\beta}} B_{\mu\nu} + \partial_\mu \varphi_\nu - \partial_\nu \varphi_\mu \tag{2.2.14}$$

gives, apart from terms that vanish on-shell (2.2.11), the expected 5d Lorentz transformation

$$\delta^{(5d)}_{\Lambda_{\alpha\beta}} B_{\mu\nu} = \Lambda_\mu{}^\lambda B_{\lambda\nu} - \Lambda_\nu{}^\lambda B_{\lambda\mu} + x_\lambda \Lambda^{\lambda\alpha}\partial_\alpha B_{\mu\nu} \tag{2.2.15}$$

plus the gauge transformation (2.2.10). The parameters are

$$\Lambda_{\mu\nu} = \Lambda_{1\mu}\Lambda_{2\nu} - \Lambda_{1\nu}\Lambda_{2\mu}, \qquad \varphi_\nu = x^\alpha \Lambda_{\alpha\lambda} B_\nu{}^\lambda. \tag{2.2.16}$$

Therefore the modified Lorentz transformation (2.2.13) does give rise to the desired 6d Lorentz group.

A couple of remarks follow concerning the Perry-Schwarz construction.

1. We note that in the proof [90] of the invariance of the action (2.2.5) under the Lorentz transformation (2.2.13), various total derivatives terms in the variation of the action were dropped under the natural assumption that

$$\partial_\lambda B_{\mu\nu} \to 0 \text{ as } |x^M| \to \infty \ . \tag{2.2.17}$$

Under the same assumption, the self-duality equation of motion (2.2.11) holds since $H_{\mu\nu\lambda} \to 0$ at infinity.

2. The Perry-Schwarz theory is based on the set of fields $B_{\mu\nu}$ which nevertheless is 6d Lorentz invariant. That it is possible to support the Lorentz symmetry without introducing the components $B_{\mu 5}$ is entirely due to the existence of the gauge symmetry (2.2.10) in the theory. In the manifestly Lorentz covariant formulation of PST [13, 85, 91, 98], the field $B_{\mu\nu}$ is extended to B_{MN}. In addition an auxiliary scalar field a is introduced with new gauge symmetries that allow one to choose the gauge $B_{\mu 5} = 0$ and $a = x_5$. In this gauge, the Perry-Schwarz action is obtained.

3. One may also combine the modified Lorentz transformation (2.2.13) with the gauge transformation (2.2.10) with a parameter $\varphi_\mu = -x_5 B_{\mu\kappa}\Lambda^\kappa$ and obtain an equivalent form of the modified Lorentz transformation

$$\delta B_{\mu\nu} = (\Lambda\cdot x)\tilde{H}_{\mu\nu} - x_5\Lambda^\kappa H_{\kappa\mu\nu}, \tag{2.2.18}$$

which is written entirely in terms of the field strength. It is instructive to show explicitly how the modified Lorentz symmetry works. The variation of the action can be written as

$$2\delta S_0 = \int \epsilon^{\mu\nu\rho\lambda\sigma}\Big[\big(\underbrace{(\Lambda\cdot x)\tilde{H}_{\mu\nu}}_{1} - \underbrace{x_5\Lambda^\kappa H_{\kappa\mu\nu}}_{2}\big)\big(\underbrace{\partial_\rho\tilde{H}_{\lambda\sigma}}_{a} - \underbrace{\partial_\rho\partial_5 B_{\lambda\sigma}}_{b}\big)\Big]. \tag{2.2.19}$$

The contributions are, respectively,

$$(1a) \quad = \quad -\frac{1}{2}\int(\epsilon^{\mu\nu\rho\lambda\sigma}\Lambda_\rho\tilde{H}_{\mu\nu}\tilde{H}_{\alpha\beta}) + \text{tot.}\ , \tag{2.2.20}$$

$$(2b) \quad = \quad -\int(\epsilon^{\mu\nu\lambda\alpha\beta}x_5\tilde{H}_{\alpha\beta}\partial_5\tilde{H}_{\mu\nu}\Lambda_\lambda) = \frac{1}{2}\int(\epsilon^{\mu\nu\rho\lambda\sigma}\Lambda_\rho\tilde{H}_{\mu\nu}\tilde{H}_{\alpha\beta}) + \text{tot.} \tag{2.2.21}$$

$$(1b) \quad = \quad -2\int(\Lambda\cdot x)(\tilde{H}_{\mu\nu}\partial_5\tilde{H}^{\mu\nu}) = \text{tot.}\ , \tag{2.2.22}$$

$$(2a) \quad = \quad \int 2x_5\Lambda^\kappa\,(H_{\kappa\mu\nu}\partial_\rho H^{\mu\nu\rho}) = \int 2x_5\Lambda^\kappa\,(\frac{1}{3}H^{\rho\mu\nu}\partial_{[\kappa}H_{\rho\mu\nu]}) + \text{tot.}\ , \tag{2.2.23}$$

where tot. stands for total derivative terms and we have used

$$\partial_{[\kappa}H_{\rho\mu\nu]} = \partial_\kappa H_{\rho\mu\nu} - \partial_{[\rho}H_{\mu\nu]\kappa} \tag{2.2.24}$$

in simplifying (2a). We see that (1a) cancels (2b). The term (2a) is zero due to the vanishing Bianchi identity $\partial_{[\kappa}H_{\rho\mu\nu]} = 0$, thus the variation vanishes up to total derivative terms.

2.3 Action for Non-Abelian Self-Dual Two-Form on M5-Branes

For simplicity, we will construct a theory of the 2-form potential without scalars and fermions. Supersymmetry is important and will be considered separately. For the gauge part, motivated by the construction of [58], we consider the addition of a set of 1-form gauge fields A^a_M for a gauge group G. The gauge group G is arbitrary for now. However, as the model will be applied to M5-branes and dimensionally reduced to get D4-branes, G will be taken to be $U(N)$ later.

2.3.1 Non-Abelian action

Following the above discussion, we will give up manifest 6d Lorentz symmetry and represent the self-dual tensor gauge field by a 5×5 antisymmetric field $B_{\mu\nu}$ in the adjoint. Since there is no room for extra degrees of freedom in the (2,0) tensor multiplets of M5-branes, therefore the gauge fields A_M must be determined in terms of the tensor gauge fields. It turns out we need to take the Yang-Mills gauge field to be a 5-dimensional field living in the 5d space x^μ, i.e. $A_\mu = A_\mu(x^\lambda)$ [4]. Let us introduce the following non-abelian generalization of the Perry-Schwarz action

$$S_0 = \frac{1}{2}\int d^6x \,\mathrm{tr}\left(-\tilde{H}^{\mu\nu}\tilde{H}_{\mu\nu} + \tilde{H}^{\mu\nu}\partial_5 B_{\mu\nu}\right), \tag{2.3.25}$$

[4] We note that a 5-dimensional gauge field was also employed in [57]. However our construction differs from theirs in essential ways: a compactified spacetime was considered in [57] and the gauge field was taken to be the zero mode of the tensor gauge field $B^{(0)}_{\mu 5}$. In our construction, we do not compactify the spacetime and A_μ is given by an integrated expression (2.3.36) on shell. We thank Pei-Ming Ho for a discussion on this point

where

$$H_{\mu\nu\lambda} = D_\mu B_{\nu\lambda} + D_\nu B_{\lambda\mu} + D_\lambda B_{\mu\nu} \tag{2.3.26}$$

and

$$\tilde{H}^{\mu\nu} = \frac{1}{6}\epsilon^{\mu\nu\rho\lambda\sigma} H_{\rho\lambda\sigma} \tag{2.3.27}$$

is the Hodge dual of $H_{\mu\nu\lambda}$. $H_{\mu\nu\lambda}$ obeys the modified Bianchi identity

$$D_{[\mu} H_{\nu\lambda\rho]} = \frac{3}{2}[F_{[\mu\nu}, B_{\lambda\rho]}]. \tag{2.3.28}$$

The action S_0 is invariant under the Yang-Mills gauge symmetry

$$\delta A_\mu = \partial_\mu \Lambda + [A_\mu, \Lambda], \quad \text{for arbitrary } \Lambda = \Lambda(x^\lambda), \tag{2.3.29}$$

$$\delta B_{\mu\nu} = [B_{\mu\nu}, \Lambda], \quad \delta H_{\mu\nu\lambda} = [H_{\mu\nu\lambda}, \Lambda] \tag{2.3.30}$$

and the following "tensor gauge symmetry" [5]:

$$\delta_T A_\mu = 0, \tag{2.3.32}$$

$$\delta_T B_{\mu\nu} = \Sigma_{\mu\nu}, \qquad \text{for arbitrary } \Sigma_{\mu\nu}(x^M) \text{ such that } D_{[\lambda}\Sigma_{\mu\nu]} = 0. \tag{2.3.33}$$

It is $[\delta_{T^{(1)}}, \delta_{T^{(2)}}] = 0$ and so the tensor gauge symmetry is abelian. Like the abelian case, we will consider field configurations with vanishing covariant derivatives at infinity:

$$D_\lambda B_{\mu\nu}\,,\ \partial_5 B_{\mu\nu} \to 0 \quad \text{as } |x^M| \to \infty. \tag{2.3.34}$$

It follows that $H_{\mu\nu\lambda}$ vanishes at infinity also.

An important observation is that the condition for the vanishing of field strength at infinity:

$$H_{\mu\nu\lambda} \to 0, \quad \text{at } x_5 \to \pm\infty \tag{2.3.35}$$

is equivalent to the Bianchi identity of the gauge field A_μ if $F_{\mu\nu}$ is identified with the boundary value of $B_{\mu\nu}$, e.g. $F_{\mu\nu} = B_{\mu\nu}(x_5 = \infty)$. With the anticipation of the

[5] Or equivalently

$$\delta_T B_{\mu\nu} = D_\mu \Lambda_\nu - D_\nu \Lambda_\mu \quad \text{for arbitrary } \Lambda_\mu(x^M) \text{ such that } [F_{[\mu\nu}\,,\,\Lambda_{\lambda]}] = 0. \tag{2.3.31}$$

self-duality equation of motion (2.3.51) in our theory, we will consider a different constraint

$$F_{\mu\nu} = c \int dx_5 \, \tilde{H}_{\mu\nu}, \tag{2.3.36}$$

where the constant c will be fixed by the quantisation condition of the self-dual strings solution of the theory. With the constraint (2.3.36), there is no new local degrees of freedom carried by A_μ[6] [7]. We will implement (2.3.36) in the action by introducing a 5-dimensional auxiliary field $E_{\mu\nu}(x^\mu)$ and add the action

$$S_E = \int d^5x \, \mathrm{tr} \left((F_{\mu\nu} - c \int dx_5 \, \tilde{H}_{\mu\nu}) E^{\mu\nu} \right). \tag{2.3.37}$$

The boundary condition of $E_{\mu\nu}$ will be taken as the trivial one

$$E_{\mu\nu} \to 0 \quad \text{as} \quad |x^\lambda| \to \infty. \tag{2.3.38}$$

$E_{\mu\nu}$ transforms under Yang-Mills and tensor gauge transformation as

$$\delta E_{\mu\nu} = [E_{\mu\nu}, \Lambda], \quad \delta_T E_{\mu\nu} = 0 \tag{2.3.39}$$

and so S_E is invariant under both transformations. The action is also invariant under the gauge symmetry

$$\delta E_{\mu\nu} = \alpha_{\mu\nu} \tag{2.3.40}$$

for arbitrary $\alpha(x^\lambda)$ such that

$$D_{[\mu}\alpha_{\nu\lambda]} = 0, \quad D^\mu \alpha_{\mu\lambda} = 0, \quad \text{and} \quad \alpha \to 0 \text{ as } |x^\lambda| \to \infty. \tag{2.3.41}$$

All in all, we propose the following action for a non-abelian theory of self-dual tensor

$$S = S_0 + S_E. \tag{2.3.42}$$

[6] As the constraint (2.3.36) constrains the field strength $F_{\mu\nu}$ (but not A_μ) by the tensor gauge field, the Wilson loops are actually allowed when the topology involves non-contractible cycles. Therefore, A_μ do carry new degrees of freedom albeit nonlocal ones. It will be interesting to further study this implication. The author thanks David Berman for pointing this out.

[7] One may be tempted to use a Chern-Simons action to enforce the gauge field to be auxiliary. However unlike the 3-dimensional case where a Chern-Simons gauge field is auxiliary and contains no local degrees of freedom, pure Chern-Simons gauge field in 5-dimension contains local degrees of freedom [130–132]. In the appendix 2.A, we review this argument as well as the extension for Chern-Simons coupled to a conserved source.

The action S is Yang-Mills gauge invariant and tensor gauge invariant. It is also invariant under the gauge symmetry (2.3.40) of $E_{\mu\nu}$. Five dimensional Lorentz symmetry is manifest. We will show below this action leads to a self-duality equation of motion. We will also demonstrate the existence of a non-manifest 6d Lorentz symmetry in our theory and the connection to 5d Yang-Mills theory of multiple D4-branes through dimensional reduction on a circle. The form of the constraint (2.3.36) is inspired by the analysis of this reduction.

2.3.2 Properties

Self-duality

The equation of motion of $E_{\mu\nu}$ gives the constraint

$$F_{\mu\nu} = c \int dx_5 \; \tilde{H}_{\mu\nu}. \tag{2.3.43}$$

This has to satisfy the Bianchi identity

$$\epsilon^{\mu\nu\rho\lambda\sigma} D_\rho F_{\lambda\sigma} = 0. \tag{2.3.44}$$

For $B_{\mu\nu}$, we have

$$\delta S_0 = \frac{1}{2} \int \epsilon^{\mu\nu\rho\lambda\sigma} \delta B_{\mu\nu} D_\rho (\tilde{H}_{\lambda\sigma} - \partial_5 B_{\lambda\sigma}) \tag{2.3.45}$$

and hence the equation of motion

$$\epsilon^{\mu\nu\rho\lambda\sigma} D_\rho (\tilde{H}_{\lambda\sigma} - \partial_5 B_{\lambda\sigma} + c E_{\lambda\sigma}) = 0, \tag{2.3.46}$$

Integrating it over x_5, we get

$$D_{[\rho} E_{\lambda\sigma]} = 0. \tag{2.3.47}$$

In fact $\int dx_5 \, \epsilon^{\mu\nu\rho\lambda\sigma} D_\rho (\tilde{H}_{\lambda\sigma} - \partial_5 B_{\lambda\sigma}) = 0$ where we have used (2.3.43) and the Bianchi identity of $F_{\mu\nu}$, and we have assumed that $H_{\mu\nu\lambda}$ vanishes at $x^5 = \pm\infty$. Our claim follows from the fact that $E_{\lambda\sigma}$ is independent of x_5. As a result, the equation (2.3.46) reads

$$\epsilon^{\mu\nu\rho\lambda\sigma} D_\rho (\tilde{H}_{\lambda\sigma} - \partial_5 B_{\lambda\sigma}) = 0 \tag{2.3.48}$$

and has the general solution

$$\tilde{H}_{\lambda\sigma} - \partial_5 B_{\lambda\sigma} = \Phi_{\lambda\sigma}, \tag{2.3.49}$$

where

$$D_{[\lambda}\Phi_{\mu\nu]} = 0. \tag{2.3.50}$$

Therefore with an appropriate fixing of the gauge symmetry (2.3.33), one can always reduce the second order equation (2.3.49) to the first order form

$$\tilde{H}_{\mu\nu} = \partial_5 B_{\mu\nu}. \tag{2.3.51}$$

This is the form of the self-duality equation in our theory.

The equation (2.3.51) implies that on-shell, $F_{\mu\nu}$ is simply given in terms of the boundary values of $B_{\mu\nu}$:

$$F_{\mu\nu} = c\left(B_{\mu\nu}(x_5 = \infty) - B_{\mu\nu}(x_5 = -\infty)\right), \tag{2.3.52}$$

and Bianchi identity is satisfied since the field strength vanishes at infinity. Finally, the equation of motion for A_μ gives

$$D^\mu E_{\mu\nu} - \frac{c}{4}\int dx_5\ \epsilon_\nu{}^{\alpha\beta\gamma\delta}[B_{\alpha\beta}, E_{\gamma\delta}] = -\frac{1}{2}\int dx_5\ \epsilon_\nu{}^{\alpha\beta\gamma\delta}[B_{\alpha\beta}, \partial_5 B_{\gamma\delta} - \frac{1}{2}\tilde{H}_{\gamma\delta}] := J_\nu. \tag{2.3.53}$$

We note that as a result of the self-duality equation of motion (2.3.51), the "current" is covariantly conserved $D_\lambda J^\lambda = 0$. Of course (2.3.53) is consistent with this.

Summarizing, the equations of motion in our theory are the auxiliary equation for A_μ (2.3.36), the self-duality equation (2.3.51) and the equations (2.3.47) and (2.3.53) for $E_{\mu\nu}$. Note that on eliminating A_μ using (2.3.36), the self-duality equation (2.3.51) is self-interacting and is completely independent of $E_{\mu\nu}$.

The counting of the degrees of freedom in our theory goes as follows. The equation of motion (2.3.43) says A_μ is auxiliary and is determined entirely in terms of $\tilde{H}_{\mu\nu}$. Using this, the action S can be written as a nonlocal action in terms of expansion in powers of $B_{\mu\nu}$. At the quadratic level, the action is simply given by dim$G$ copies of the Perry-Schwarz action, plus the action S_E. For small field strengths, we can take the higher order terms as small corrections and we can count the degrees of freedom using the linearized theory. In this limit, $A_\mu = 0$ and the tensor gauge symmetry and the self-duality equation of motion are precisely those of the original Perry-Schwarz theory. Thus we obtain $3 \times$ dim$G$ degrees of freedom

in $B_{\mu\nu}$. As for $E_{\mu\nu}$, the linearized equations of motion are

$$\partial_{[\mu}E_{\nu\lambda]} = 0, \quad \partial^{\mu}E_{\mu\nu} = 0, \tag{2.3.54}$$

and there is the gauge symmetry (2.3.40) with the parameters $\alpha_{\mu\nu}$ satisfying, in this case,

$$\partial_{[\mu}\alpha_{\nu\lambda]} = 0, \quad \partial^{\mu}\alpha_{\mu\nu} = 0. \tag{2.3.55}$$

Since $E_{\mu\nu}$ and $\alpha_{\mu\nu}$ also satisfy the same (vanishing) boundary condition at infinity, so we can use the gauge symmetry to remove the $E_{\mu\nu}$ field completely. This is compatible with the fact $E_{\mu\nu}$ was introduced as an auxiliary field to implement the constraint (2.3.36). All in all, our theory contains $3 \times \dim G$ degrees of freedom as required by (2,0) supersymmetry

We remark that when $B_{\mu\nu}$ is diagonal with distinct diagonal elements such that the gauge group is broken down to $U(1)^r$ (r is the rank of the gauge group), our action reduces to a sum of r copies of the abelian Perry-Schwarz theory and describes the gauge sector of r separated M5-branes. More generally, once the scalar and fermion fields are included in the theory, one can have a system of lumps of coincident M5-branes, BPS or non-BPS relative to each other; and as usual, the pattern of symmetry breaking as well as the interacting dynamics of M5-branes can be studied. In particular, in the subsequent chapter, we will activate one of the scalar fields to look for the BPS solution of self-dual strings.

Lorentz symmetry

Our action is manifestly 5d Lorentz invariant. It is straightforward to check that it is not invariant under the modified Lorentz transformation (2.2.13) or (2.2.18). Indeed, considering the natural generalisation,

$$\delta B_{\mu\nu} = (\Lambda \cdot x)\tilde{H}_{\mu\nu} - x_5\Lambda^{\kappa}H_{\kappa\mu\nu}, \tag{2.3.56}$$

$$\delta A_{\mu} = 0. \tag{2.3.57}$$

The calculation in the non-abelian case becomes

$$2\delta S_0 = \int \epsilon^{\mu\nu\rho\lambda\sigma}\Big[\big(\underbrace{(\Lambda\cdot x)\tilde{H}_{\mu\nu}}_{1} - \underbrace{x_5\Lambda^{\kappa}H_{\kappa\mu\nu}}_{2}\big)\big(\underbrace{D_{\rho}\tilde{H}_{\lambda\sigma}}_{a} - \underbrace{D_{\rho}\partial_5 B_{\lambda\sigma}}_{b}\big)\Big], \tag{2.3.58}$$

with

$$(1a) = -\frac{1}{2}\int \mathrm{tr}\,(\epsilon^{\mu\nu\rho\lambda\sigma}\Lambda_\rho \tilde{H}_{\mu\nu}\tilde{H}_{\alpha\beta}) + \text{tot.}\,, \tag{2.3.59}$$

$$(2b) = -\int \mathrm{tr}\,(\epsilon^{\mu\nu\lambda\alpha\beta}x_5\tilde{H}_{\alpha\beta}\partial_5\tilde{H}_{\mu\nu}\Lambda_\lambda) = \frac{1}{2}\int \mathrm{tr}\,(\epsilon^{\mu\nu\rho\lambda\sigma}\Lambda_\rho\tilde{H}_{\mu\nu}\tilde{H}_{\alpha\beta}) + \text{tot.} \tag{2.3.60}$$

$$(1b) = -2\int (\Lambda\cdot x)\mathrm{tr}(\tilde{H}_{\mu\nu}\partial_5\tilde{H}^{\mu\nu}) = \text{tot.}\,, \tag{2.3.61}$$

$$(2a) = \int 2x_5\Lambda^\kappa\,\mathrm{tr}\,(H_{\kappa\mu\nu}D_\rho H^{\mu\nu\rho}) = \int 2x_5\Lambda^\kappa\,\mathrm{tr}\,(\frac{1}{3}H^{\rho\mu\nu}D_{[\kappa}H_{\rho\mu\nu]}) + \text{tot.}\,, \tag{2.3.62}$$

where tot. stands for total derivative terms and we have used

$$D_{[\kappa}H_{\rho\mu\nu]} = D_\kappa H_{\rho\mu\nu} - D_{[\rho}H_{\mu\nu]\kappa} \tag{2.3.63}$$

in simplifying (2a).

In the abelian case, the term (2a) is zero due to the vanishing Bianchi identity $\partial_{[\kappa}H_{\rho\mu\nu]} = 0$. This is not so for the non-abelian case and so S_0 is not invariant under (2.3.56). It is also straightforward to see that S_0 is also not invariant under

$$\delta B_{\mu\nu} = (\Lambda\cdot x)\tilde{H}_{\mu\nu} - x_5(\Lambda\cdot D)B_{\mu\nu}. \tag{2.3.64}$$

Let us proceed by further modifying the Lorentz transformation. We observe that the equation (2.3.45) for the variation of S_0 under a general variation of $\delta B_{\mu\nu}$ can be rewritten as

$$\delta S_0 = \int d^6x\,\mathrm{tr}\left[\Delta B^{\mu\nu}\tilde{H}_{\mu\nu}\right], \tag{2.3.65}$$

where

$$\Delta B^{\mu\nu} := \partial_5(\delta B^{\mu\nu}) - \frac{1}{2}\epsilon^{\mu\nu\alpha\beta\gamma}D_\alpha(\delta B_{\beta\gamma}). \tag{2.3.66}$$

It is interesting to note that

$$\Delta B_{\mu\nu} = -\delta(\tilde{H}_{\mu\nu} - \partial_5 B_{\mu\nu}), \tag{2.3.67}$$

which is just the variation of the self-duality equation of motion.

Taking $\delta B_{\mu\nu}$ now as the 5-μ Lorentz transformation, it is clear that the action will be invariant if the variation satisfies $\Delta B_{\mu\nu} = 0$. This is a sufficient condition, but not necessary. In fact $\Delta B_{\mu\nu} \neq 0$ for the abelian case (2.2.18), nevertheless S_0 is invariant. So let us consider a general transformation of the form

$$\delta B_{\mu\nu} = (\Lambda\cdot x)\tilde{H}_{\mu\nu} - \lambda x_5\Lambda^\kappa H_{\kappa\mu\nu} + \Lambda^\kappa\phi_{\mu\nu\kappa} := \delta_{(1)}B_{\mu\nu} + \delta_{(2)}B_{\mu\nu}, \tag{2.3.68}$$

where λ is a constant and $\phi_{\mu\nu\kappa} = -\phi_{\nu\mu\kappa}$ is a quantity to be determined by demanding S_0 to be invariant. We have denoted the first two variation terms by $\delta_{(1)}B_{\mu\nu}$ and the third term by $\delta_{(2)}B_{\mu\nu}$. By redefining $\phi_{\mu\nu\kappa}$ with an appropriate shift, one can bring λ to any value one wants. This freedom will turn out to be convenient.

The variation of S_0 under $\delta_{(1)}B_{\mu\nu}$ is

$$\delta_{(1)}S_0 = \int \left[\frac{\lambda}{2}x_5\epsilon^{\mu\nu\alpha\beta\gamma}D_\alpha H_{\beta\gamma\kappa}\Lambda^\kappa + \frac{\lambda-1}{4}\Lambda_\rho\tilde{H}_{\alpha\beta}\epsilon^{\rho\alpha\beta\mu\nu}\right]\tilde{H}_{\mu\nu}. \tag{2.3.69}$$

For $\lambda = 1$, the result in the appendix is recovered. For the moment, let us keep λ arbitrary. Since (2.3.69) is of the form of (2.3.65), therefore it can be cancelled with $\delta_{(2)}B_{\mu\nu}$ if $\phi_{\mu\nu\kappa}$ satisfies

$$\partial_5\phi_{\mu\nu\kappa} - \frac{1}{2}\epsilon_{\mu\nu}{}^{\alpha\beta\gamma}D_\alpha\phi_{\beta\gamma\kappa} = -\frac{\lambda}{2}x_5\epsilon^{\mu\nu\alpha\beta\gamma}D_\alpha H_{\beta\gamma\kappa} - \frac{\lambda-1}{4}\tilde{H}^{\alpha\beta}\epsilon_{\kappa\alpha\beta\mu\nu} := J_{\mu\nu\kappa}. \tag{2.3.70}$$

In addition, we impose the boundary condition

$$\phi_{\mu\nu\kappa} \text{ vanishes as } |x_5| \to \infty. \tag{2.3.71}$$

A solution can always be written down using the Green function technique for general $J_{\mu\nu\kappa}$. Let $G^{ab}_{\mu\nu,\mu'\nu'}(x,y)$ be the Green function which satisfies

$$\partial_5 G^{ab\mu'\nu'}_{\mu\nu} - \frac{1}{2}\epsilon_{\mu\nu}{}^{\alpha\beta\gamma}(D^{(y)}_\alpha)^a{}_c G^{cb\mu'\nu'}_{\beta\gamma} = \delta^{\mu'\nu'}_{\mu\nu}\delta^{ab}\delta^{(6)}(x-y) \tag{2.3.72}$$

and the boundary condition

$$G^{ab\mu'\nu'}_{\mu\nu}(x,y) = 0, \quad |x_5| \to \infty. \tag{2.3.73}$$

Here $x = (x^M)$ and $(D_\alpha)^a{}_c = \partial_\alpha\delta^a{}_c + (\tilde{A}_\alpha)^a{}_c$ where $(\tilde{A}_\alpha)^{ac} := f^{abc}A^b_\alpha$. Then

$$\phi^a_{\mu\nu\kappa} = \int dy\, G^{ab\mu'\nu'}_{\mu\nu}(x,y)J^b_{\mu'\nu'\kappa}(y) \tag{2.3.74}$$

satisfies both (2.3.70) and (2.3.71). As a result, if also

$$\delta A_\mu = 0, \tag{2.3.75}$$

then S_0 is invariant. So far this works for any λ.

Next let us examine the action S_E. It follows from (2.3.68) that

$$\delta\tilde{H}_{\mu\nu} = \partial_5\phi_{\mu\nu\kappa}\Lambda^\kappa + \frac{\Lambda\cdot x}{2}\epsilon_{\mu\nu}{}^{\alpha\beta\gamma}D_\alpha\tilde{H}_{\beta\gamma} + \frac{\lambda+1}{4}\epsilon_{\mu\nu}{}^{\alpha\beta\gamma}\Lambda_\alpha\tilde{H}_{\beta\gamma}, \tag{2.3.76}$$

where we have used the differential equation (2.3.70). Therefore S_E is invariant if we take $\lambda = -1$ and if $E_{\mu\nu}$ transforms as

$$\delta E_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu}{}^{\alpha\beta\gamma} D_\alpha((\Lambda \cdot x)E_{\beta\gamma}). \tag{2.3.77}$$

All in all, our action is invariant under the transformation (2.3.68), (2.3.75) and (2.3.77).

In general the Lorentz invariance of the action implies that the equations of motion (i.e. (2.3.36), (2.3.48) (2.3.47) and (2.3.53)) are automatically Lorentz invariant, up to terms vanishes on shell and terms that can be interpreted as any other symmetry transformations of the theory. However since the self-duality equation (2.3.51) is obtained by a gauge fixing, it is not guaranteed to be Lorentz invariant. In fact, the transformation (2.3.68) implies that

$$\delta(\tilde{H}_{\mu\nu} - \partial_5 B_{\mu\nu}) = \frac{\Lambda \cdot x}{2}\epsilon_{\mu\nu}{}^{\alpha\beta\gamma} D_\alpha \tilde{H}_{\beta\gamma} - (\Lambda \cdot x)\partial_5 \tilde{H}_{\mu\nu} - \partial_5(x_5 H_{\mu\nu\kappa}\Lambda^\kappa). \tag{2.3.78}$$

This gives in (2.3.65) $\delta S_0 = 0$ as expected. Using the self-duality equation (2.3.51), the first and second term of (2.3.78) actually cancel and so

$$\delta(\tilde{H}_{\mu\nu} - \partial_5 B_{\mu\nu}) = -\partial_5(x_5 H_{\mu\nu\kappa}\Lambda^\kappa) + \text{EOM}, \tag{2.3.79}$$

where EOM denotes terms vanish when the equation of motion (2.3.51) is used. One can rewrite this further by using the equation of motion and obtains

$$\delta(\tilde{H}_{\mu\nu} - \partial_5 B_{\mu\nu}) = \frac{1}{2}\epsilon_{\mu\nu\kappa}{}^{\alpha\beta}\Lambda^\kappa(\tilde{H}_{\alpha\beta} + 2x_5\partial_5\tilde{H}_{\alpha\beta}) + x_5\Lambda^\kappa D_\kappa \tilde{H}_{\mu\nu} + D_{[\mu}\varphi_{\nu]} + \text{EOM}, \tag{2.3.80}$$

where $\varphi_\nu = x_5\tilde{H}_{\nu\kappa}\Lambda^\kappa$. Now the first and second term on the RHS of (2.3.80) respectively gives zero when substituted into (2.3.65) and so they corresponds to symmetry transformations of the action S_0 [8]. For the abelian case, the third term corresponds to the symmetry transformation $\delta B_{\mu\nu} = \partial_{[\mu}\alpha_{\nu]}$ of $B_{\mu\nu}$ and since S_E decouples from the theory, so we obtain that the self-duality equation is Lorentz invariant up to terms vanishes on shell and terms that correspond to a symmetry transformation of

[8] More specifically, the symmetry transformations are given by $\delta B_{\mu\nu} = \phi_{\mu\nu\kappa}\Lambda^\kappa$ where $\phi_{\mu\nu\kappa}$ is given by (2.3.74) with $J_{\mu\nu\kappa}$ specified by the first and second term of the RHS of (2.3.80) respectively.

the theory. However the above analysis breaks down in the non-abelian case and so we conclude that the self-duality equation of motion is not Lorentz invariant. We emphasize that the loss of Lorentz invariance in (2.3.51) is simply because it is a gauge fixed equation of motion. This is not surprising. For example, Yang-Mills equation of motion in the Coulomb gauge is not Lorentz invariant. The use of the self-duality equation is important for obtaining the correct counting on the degrees of freedom in the theory. However the use of the ungauge-fixed version (2.3.48) may be useful for some other purposes, for example, supersymmetry.

If we compute the algebra of commutator $[\delta(\Lambda_\mu^{(1)}), \delta(\Lambda_\mu^{(2)})]$ for the physical field $B_{\mu\nu}$, we get the standard 5d Lorentz transformation plus an additional transformation. This additional transformation is quite complicated but is a symmetry of the action since we know already the action is invariant under the 5d Lorentz transformation and is invariant under $[\delta(\Lambda_\mu^{(1)}), \delta(\Lambda_\mu^{(2)})]$. Therefore we can interpret (2.3.68) as a modified Lorentz symmetry. Note that the form of the transformation laws (2.3.75) and (2.3.77) are quite non-standard but they are compatible with the auxiliary nature of these fields.

We note that as $\phi_{\mu\nu\kappa}$ is determined explicitly as an integrated expression over the Green function, the transformation (2.3.68) is non-local in the fields. It is now clear that the different choices of λ simply correspond to different non-local form of the transformation (2.3.68). What we have shown is that one can make the action invariant by using a transformation law that has a nonlocal piece that is based on a local part with the particular choice of $\lambda = -1$. For the abelian case, we know the Lorentz transformation (2.2.18) is locally represented in terms of A_μ and $B_{\mu\nu}$; and corresponds to $\lambda = 1$ and $\phi_{\mu\nu\kappa} = 0$. Let us demonstrate that this is equivalent to having $\lambda = -1$ and a nontrivial $\phi_{\mu\nu\kappa}$ as determined above. To see this, the equation (2.3.70) reduces in the abelian case to

$$\partial_5 \phi_{\mu\nu\kappa} - \frac{1}{2}\epsilon_{\mu\nu}{}^{\alpha\beta\gamma}\partial_\alpha \phi_{\beta\gamma\kappa} = x_5 \partial_\kappa \tilde{H}_{\mu\nu} - H_{\mu\nu\kappa}. \tag{2.3.81}$$

Let us put $\phi_{\mu\nu\kappa} = -2x_5 H_{\mu\nu\kappa} + \varphi_{\mu\nu\kappa}$ and so

$$\partial_5 \varphi_{\mu\nu\kappa} - \frac{1}{2}\epsilon_{\mu\nu}{}^{\alpha\beta\gamma}\partial_\alpha \varphi_{\beta\gamma\kappa} = -\frac{1}{2}\epsilon_{\mu\nu\kappa}{}^{\alpha\beta}(\tilde{H}_{\alpha\beta} + 2x_5\partial_5 \tilde{H}_{\alpha\beta}) - x_5 \partial_k \tilde{H}_{\mu\nu}. \tag{2.3.82}$$

Now the right hand side of this equation when substituted into (2.3.65) actually

leaves S_0 invariant. Therefore as explained above, $\varphi_{\mu\nu\kappa}$ represents a symmetry and we recover (2.2.18) up to a symmetry transformation.

The Lorentz symmetry we proposed is nonlocal and is quite different from the usual representation of a symmetry in terms of local fields, but it seems this is what is needed for multiple M5-branes [9]. In fact, nonlocal symmetry is not uncommon in string theory. For example, the spacetime Lorentz symmetry in the light cone gauge string theory is nonlocal in the worldsheet coordinate [133]. There the nonlocality arises since a Lorentz transformation will generally bring one out of the lightcone gauge and so a worldsheet reparametrization (turns out to be nonlocal) is needed in order to restore the gauge condition. For us, we are in a formulation without the $B_{5\mu}$ fields. Since a standard 5-μ Lorentz transformation will turn $B_{\mu\nu}$ to $B_{5\mu}$, we suspect that the reason of having a modified Lorentz symmetry is similarly due to a compensating gauge transformation in a covariant formulation. In the abelian (free) case, the modification is not so drastic and the modified Lorentz transformation is still local. But this is not the case for the non-abelian case as we found here. To check our suspicion, it is needed to construct the covariantized theory. It is remarkable that for the abelian case, PST [13, 85, 91, 98] were able to provide a Lorentz covariant formulation by introducing additional auxiliary fields (scalar field a and the $B_{5\mu}$ components). It will be very interesting to covariantize our construction by following a similar construction of PST and it is possible that the employment of additional auxiliary fields would allow for a local representation of the Lorentz symmetry.

Reduction to D4-Branes

Let us consider a compactification of x_5 on a circle of radius R. The dimensional reduced action reads

$$S = \frac{2\pi R}{2} \int d^5x \, \mathrm{tr} \left(-\tilde{H}_{\mu\nu}^2 + (F_{\mu\nu} - 2\pi R c \tilde{H}_{\mu\nu}) E^{\mu\nu} \right) \tag{2.3.83}$$

[9] We thank Pei-Ming Ho and Yutaka Matsuo for emphasizing the nonlocal nature of our proposed Lorentz transformation and for a discussion on this point.

This form of action has been considered in [58] as a dual formulation of 5-dimensional Yang-Mills theory. In fact, if we integrate out $E_{\mu\nu}$, we obtain the expected relation

$$F_{\mu\nu} = 2\pi Rc\tilde{H}_{\mu\nu}. \tag{2.3.84}$$

Eliminate $\tilde{H}_{\mu\nu}$ using the constraint, we obtain the standard 5d Yang-Mills action

$$S_{YM} = -\frac{1}{4\pi Rc^2}\int d^5x \text{ tr } F_{\mu\nu}^2. \tag{2.3.85}$$

This is however not the complete answer. In fact if we look at the path integral and integrate out E first, we obtain

$$\int [DA][DB][DE]e^{-S} = \int [DA][DB]e^{-S_{YM}}\delta(F_{\mu\nu} - 2\pi Rc\tilde{H}_{\mu\nu}) = \int [DA]e^{-S_{YM}-S'}, \tag{2.3.86}$$

where $S' = S'(A)$ is a measure contribution obtained from integrating out the delta functional constraint and then rewritten in terms of A_μ. The direct determination of S' is nontrivial but it has to satisfy a consistency condition: the condition

$$D_\mu F^{\mu\nu} = -\frac{\pi Rc}{2}\epsilon^{\nu\alpha\beta\gamma\delta}[F_{\alpha\beta}, B_{\gamma\delta}] \tag{2.3.87}$$

which follows from (2.3.84) should be obtained as an equation of motion in the 5d theory. As a result, S' has to satisfy

$$\frac{\delta S'}{\delta A_\nu} = \frac{1}{2c}\epsilon^{\nu\alpha\beta\gamma\delta}[F_{\alpha\beta}, B_{\gamma\delta}] \tag{2.3.88}$$

with $B_{\mu\nu}$ understood to be a function of A_μ obtained by solving the duality relation (2.3.84).

The 5d theory is thus given by the action $S_{5d} = S_{YM} + S'$. The action S_{YM} corresponds to the expected form of the Yang-Mills coupling

$$g_{YM}^2 = \pi Rc^2 \tag{2.3.89}$$

and the gauge group in our construction is to be

$$G = U(N) \tag{2.3.90}$$

for a system of N M5-branes. The reproduction of the 5d Yang-Mills action gives further support that our construction gives a description of the gauge sector of a

system of multiple M5-branes. The action S' describes a correction term to the Yang-Mills theory which appears to be of high derivative in nature since $[F, B] \sim DDB$ and B is of the order of F from (2.3.84)). In the abelian case, Perry and Schwarz has also constructed the nonlinear five-brane action that gives the $U(1)$ DBI action of D4-brane upon dimensional reduction. It would be interesting to work out S' in more details and see whether it captures the non-abelian DBI action [32, 134, 135] in some way.

We remark that the necessity of non-locality in the M5-branes action has also been argued by Witten [12]. He observed that conformal invariance of the M5-branes theory implies that upon double dimensional reduction to five dimensions, the 5 dimensional action should be proportional to

$$\frac{1}{R}\int d^5x. \tag{2.3.91}$$

On the other hand, one should get

$$\int d^6x = 2\pi R \int d^5x \tag{2.3.92}$$

as a result of integrating over the x_5 direction for a standard reduction of a local action, In our analysis above, we see that both R-dependence are correct and the trick to arrive from (2.3.91) to (2.3.92) is due to the simple R dependence in the constraint (2.3.84).

In principle one could consider compactification in the other spacelike directions and one should get the same 5d YM action. However this is already non-trivial for the Perry-Schwarz action [90] (or the Henneaux-Teitelboim action [93]) and implies the existence of a symmetry of the D4-branes action which involves a non-local field redefinition. For a single M5-brane, this symmetry can be made explicit in a covariant PST-like formulation in which both, the vector field A_μ and the two-form field $B_{\mu\nu}$ are present and related to each other, on the mass-shell, by the duality condition which follows from the action. See for example [136] for the case of the duality-symmetric formulation of $D = 11$ supergravity with A_3 and A_6 gauge fields. The construction is completely generic and can be extended immediately to arbitrary D dimensional spacetime any pair of duality related fields of rank p and

$(D-p-2)$ whose field strengths are dual to each other on the mass shell [10]. It would be interesting to extend this construction to the non-abelian case.

2.4 Discussions

In this chapter, we have constructed a theory of non-abelian tensor fields with the properties that:

1. the action admits a self-duality equation of motion,
2. the action has manifest 5d Lorentz symmetry and a modified 6d Lorentz symmetry,
3. on dimensional reduction, the action gives the 5d Yang-Mills action plus certain higher derivative corrections.

Based on these properties, we propose our action to be the bosonic theory describing the gauge sector of coincident M5-branes in flat space. A special feature of our construction is that the tensor gauge symmetry is abelian although the theory is still fully interacting. This is an interesting difference between the self-interaction of Yang-Mills gauge fields and the self-interaction of 2-form gauge fields in our construction. It remains to be seen whether this is still the case in the Lorentz covariant formulation of the theory.

We note that conformal symmetry rules out the possibility of a Yang-Mills action, but a 5d Chern-Simons action is allowed for the gauge field A_μ:

$$S_{CS} = \frac{k}{24\pi^2}\int d^5x\, \epsilon^{\mu_1\cdots\mu_5}\mathrm{tr}\left(A_{\mu_1}\partial_{\mu_2}A_{\mu_3}\partial_{\mu_4}A_{\mu_5} + \frac{3}{2}A_{\mu_1}A_{\mu_2}A_{\mu_3}\partial_{\mu_4}A_{\mu_5} + \frac{3}{5}A_{\mu_1}A_{\mu_2}A_{\mu_3}A_{\mu_4}A_{\mu_5}\right). \quad (2.4.93)$$

However, the inclusion of a 5d Chern-Simons action is not enough to render the gauge field A_μ auxiliary, as a 5d Chern-Simons action allows the vector gauge field to carry propagating degrees of freedom. The counting of the number of propagating

[10] We thank Dmitri Sorokin for explaining this to us.

degrees of freedom for a 5d Chern-Simons gauge field is reviewed in the appendix 2.A.

Our construction is in principle only a low energy effective description for a system of coincident M5-branes. If one is lucky, the (2,0) supersymmetric completion may give a well-defined quantum theory as in the case of BLG [49–51] and ABJM theories [52] for multiple M2-branes and the $\mathcal{N} = 4$ SYM theory for multiple D3-branes. This is another strong reason to construct the supersymmetric completion.

To construct the supersymmetric theory, one needs to include scalar fields and fermions in the adjoint of $U(N)$. For (2,0) supersymmetry, all these fields are sitting in the tensor multiplet. Since there is no Yang-Mills multiplet in (2,0) supersymmetry, the Yang-Mills gauge field must be a supersymmetric singlet. This is rather difficult to implement. On the other hand, it is possible that only a fraction of the (2,0) supersymmetry, i.e. (1,0) supersymmetry, is visible in the classical action of multiple M5-branes, and full supersymmetry can be seen only nonperturbatively as in the ABJM theory [52]. With respect to (1,0) supersymmetry, the (2,0) tensor multiplet is simply the sum of a (1,0) tensor multiplet and a (1,0) hyper-multiplet. Moreover, one should employ a (1,0) Yang-Mills multiplet as an auxiliary multiplet. The recent results of (1,0) superconformal theories [64] should be useful in this regard.

However even before one enters into the details, a simple observation already indicates that the supersymmetric theory is going to be highly nontrivial. In six dimensions, scalar field has dimension 2. Conformal invariance plus locality imply that the potential term V for the scalar fields has to be cubic. However a nonvanishing cubic potential has no ground state and this is not compatible with supersymmetry. This means the potential term, if nonvanishing, will need to be nonlocal. For example, potential of the schematic form $V \sim \phi^4/|\phi|$ or $V \sim \int dx_5 \int dx_5\ \phi^4$ could avoid the problem of not having a ground state. It is amusing that the later form of the potential has a close resemblance with the scalar interaction term in [114] [11] if one exchanges $C_\mu \sim \delta^5_\mu \int dx_5$, both of which are of dimension -1. However, the above

[11] We thank Neil Lambert for pointing out this resemblance.

discussion is based on the canonical dimension of the scalar field, as the theory is strongly interacting, the mass dimensions of scalars might be anomalous.

It would be interesting to understand the connection between our description and the proposed SYM description of M5-branes [55, 56]. In particular an understanding of how a non-abelian 2-form gauge field would arise in the Yang-Mills description is needed. Incidentally, based on a fluctuation analysis of D1-branes around a large RR 3-form flux background, a matrix model description for M5-branes in a background C-field was suggested in [118] and there is the same question of how to extract a B-field from the matrix variables. This problem may be compared with the problem of extracting the spacetime fields and their dynamics, particularly the gravity field, from the matrix model [11, 137]. See for example [138–141]. Lessons drawn from those analysis may be useful here.

Our theory is based on fields in the adjoint of $U(N)$, i.e. taking N^2 values. Naively this is different from the N^3 counting from entropy argument [142]. To understand the counting, it will be important to understand the dynamics of the theory properly. See for example [121, 122] for some recent interesting analysis performed on the 5d SYM theory and a class of 6d SCFT in the Coulomb phase.

Appendix

2.A Counting of degrees of freedom for Chern-Simons theory

We will start with a review of the counting of degrees of freedom for pure Chern-Simons theory performed in [130, 131]. Then we extend the analysis to the case where the Chern-Simons theory is coupled to a covariantly conserved current. The details of the counting is not important for our results. They are included here for completeness.

2.A.1 Pure Non-Abelian Chern-Simons theory

Consider the five dimensional (dimension $D = 2n + 1$, $n = 2$ here) Chern-Simons action

$$S_{\rm CS} = \int_M \mathcal{L}_{\rm CS}, \qquad \text{with} \quad d\mathcal{L}_{\rm CS} = g_{abc} F^a \wedge F^b \wedge F^c \tag{2.1.94}$$

where g_{abc} is the symmetric invariant tensor of the gauge group and $a = 1, \cdots, \mathcal{N}$ with $\mathcal{N}$ being the dimension of the gauge group. The equation of motion

$$g_{aa_1a_2} F^{a_1}_{\mu_1\mu_2} F^{a_2}_{\mu_3\mu_4} \epsilon^{\mu_1\mu_2\mu_3\mu_4\lambda} = 0 \tag{2.1.95}$$

can be decomposed into

$$\begin{cases} k_a \equiv g_{aa_1a_2} F^{a_1}_{i_1i_2} F^{a_2}_{i_3i_4} \epsilon^{i_1i_2i_3i_4} = 0, \\ k^i_a \equiv 4 g_{aa_1a_2} F^{a_1}_{i_1i_2} F^{a_2}_{0i_3} \epsilon^{i_1i_2i_3i} = 0, \end{cases} \tag{2.1.96}$$

where $\mu = (0, i)$ and $i = 1, \cdots, 2n$. Introduce the "$2n\mathcal{N} \times 2n\mathcal{N}$ matrix" $\Omega^{ij}_{ab} \equiv 4\epsilon^{iji_1i_2} g_{abc} F^c_{i_1i_2}$ ((b, j) as a collective index), we can rewrite the equations of motion

in the compact form:

$$\begin{cases} k_a = \Omega^{ij}_{ab} F^b_{ij} = 0 \\ \Omega^{ij}_{ab} F^b_{0j} = 0 \end{cases} \tag{2.1.97}$$

A simple identity

$$\delta^i_{[k} g^{abc} \epsilon^{j\ell mn} F^b_{j\ell} F^c_{mn]} = 0, \quad \Rightarrow \quad \Omega^{ij}_{ab} F^b_{kj} = \delta^i_k k_a \tag{2.1.98}$$

shows that on the constraint surface $k_a = 0$, $(v_k)^b_j \equiv F^b_{kj}$ gives $2n$ null vectors to Ω^{ij}_{ab}. The non-invertibility of Ω is due to the existence of symmetry. In this case, the $2n$ null vectors F^b_{kj} generates the spatial diffeomorphism. In fact under diffeomorphism $\delta x^\mu = \eta^\mu$ of spacetime, the Chern-Simons theory is invariant with $\delta_\eta A^a_\mu = \mathcal{L}_\eta A^a_\mu$, or the improved diffeomorphism

$$\delta_\eta A^a_\mu = -\epsilon^\nu F^a_{\mu\nu}. \tag{2.1.99}$$

In general, the rank of Ω depends on the properties of the invariant tensor g_{abc}, and the phase space location of the system. For example, at $F^a_{\mu\nu} = 0$, $\Omega^{ij}_{ab} = 0$ and has zero rank. In [130, 131], a *generic condition* on g_{abc} was introduced. g_{abc} is said to be *generic* if there exists solution F^a_{ij} on the surface $k_a = 0$ such that:

(a) The matrix F^b_{kj} ((b, j) as row and k as column index) has the maximum rank $2n$ such that $\xi^k F^b_{kj} = 0$ implies $\xi^k = 0$, i.e. the $2n$ null vectors $(v_k)^b_j \equiv F^b_{kj}$ of Ω^{ij}_{ab} are linearly independent.

(b) The matrix Ω^{ij}_{ab} has maximum rank compatible with (a), i.e. Ω^{ij}_{ab} has no other null vectors except $(v_k)^b_j$ and so has rank $2n\mathcal{N} - 2n$

We remark that the presence of the null vectors of Ω on the surface $k_a = 0$ is due to the presence of spatial diffeomorphism $\delta x^i = \eta^i$, $i = 1, 2, 3, 4$. (under generic condition assumption, temporal diffeomorphism is not independent). If there were no such diffeomorphism, we would not expect the existence of such null vectors.

Now the equation of motion (2.1.97) together with the generic condition implies $F^b_{0j} = N^k F^b_{kj}$ for arbitrary $2n$ fields N^k, or

$$\dot{A}^a_i = D_i A^a_0 + N^k F^a_{ki} \tag{2.1.100}$$

Since (2.1.100) is invariant under

(a) Standard gauge transformation ($\mathcal{N}$ dimensional) :

$$\delta A_i^a = -D_i\lambda^a, \quad \delta_\lambda A_0^a = -\dot{\lambda}^a - [\lambda, A_0]^a, \quad \delta_\lambda N^k = 0 \tag{2.1.101}$$

(b) Spatial diffeomorphism ($2n$ dimensional) :

$$\delta_\xi A_i^a = -\xi^j F_{ij}^a, \quad \delta_\xi A_0^a = -\xi^j F_{0j}^a, \quad \delta_\xi N^k = \dot{\xi}^k + [\xi, N]^k \tag{2.1.102}$$

where $[\xi, N]^k$ is the Lie bracket of the vectors ξ and N,

we can use the above symmetries to go to the the time gauge

$$A_0 = 0, \qquad N^k = 0. \tag{2.1.103}$$

In this case, the equation of motion is equivalent to

$$k_a = 0, \qquad A_i^a = \text{time independent}. \tag{2.1.104}$$

In addition to the $\mathcal{N}$ constraints $k_a = 0$, the $2n\mathcal{N}$ functions $A_i^a(x_i)$ are subjected to the residual symmetry of the time gauge, these are $\mathcal{N}$ time-independent gauge symmetry (2.1.101) as well as the $2n$ time-independent spatial diffeomorphism (2.1.102), therefore the number of arbitrary functions in the solution to the equation of motion of Lagrange formulation is $2n\mathcal{N} - \mathcal{N} - (\mathcal{N} + 2n) = 2(n\mathcal{N} - \mathcal{N} - n)$. The local degrees of freedom is simply the half of it, therefore

$$\text{no. of local degrees of freedom of pure CS} \;=\; n\mathcal{N} - \mathcal{N} - n \tag{2.1.105}$$

with $n > 1$. In 5d, this would be $\mathcal{N} - 2$. We remark that the above analysis holds only for the non-abelian case. For the counting of local degrees of freedom in the abelian case, see [130, 131].

2.A.2 Chern-Simons theory coupled to conserved current

For the case that the Chern-Simons theory is coupled to a conserved current J^λ ($D_\lambda J^\lambda = 0$):

$$S = \int d^5x \ \mathrm{tr}\ A_\mu J_\mu + S_{\mathrm{CS}}, \tag{2.1.106}$$

the equation of motion of A_λ is

$$g_{aa_1a_2}F^{a_1}_{\mu\nu}F^{a_2}_{\lambda\sigma}\epsilon^{\mu\nu\lambda\sigma\rho} = c' J^a_\rho \tag{2.1.107}$$

where c' is some constant. In terms of the matrix $\Omega^{ij}_{ab} \equiv \epsilon^{iji_1i_2}g_{abc}F^c_{i_1i_2}$, the equation of motion can be written as

$$\begin{cases} \Omega^{ij}_{ab}F^b_{ij} = c' J^a_0 \\ 4\Omega^{ij}_{ab}F^b_{0j} = c' J^a_i \end{cases} \tag{2.1.108}$$

Generically, $J^a_i \neq 0$, this means that (2.1.98) can no longer be used to reduce the rank of Ω, so we have full rank $2n\mathcal{N}$ for Ω generically, i.e. Ω is invertible.

Now in the gauge $A^a_0 = 0$, the second line of the equation of motion (2.1.108) simply provides a first order partial differential equation in time:

$$\partial_0 A^b_j = c'(\Omega^{-1})^{ab}_{ji} J^a_i. \tag{2.1.109}$$

As for the first equation of motion of (2.1.108), it is indeed time-independent since

$$\begin{aligned} \partial_0(\Omega^{ij}_{ab}F^b_{ij} - c' J^a_0) &= \left(2g_{abc}\partial_0 F^b_{k\ell}F^c_{ij}\epsilon^{ijk\ell} - c'\partial_0 J^a_0\right) \\ = \; D_k[4g_{abc}F^b_{ij}F^c_{0\ell}\epsilon^{ijk\ell}] - c' D_i J^a_i &= c' D_k J^a_k - c' D_k J^a_k = 0 \end{aligned} \tag{2.1.110}$$

As a result, (2.1.108) simply provides a constraint on the initial values $A^b_j(x_i, t = 0)$. Therefore, in the time gauge, $A^b_j(x_i, t)$ are determined by (2.1.109) up to the initial conditions $A^b_j(x_i, t = 0)$. Both the time-independent gauge transformation and the time-independent constraints (2.1.108) remove $\mathcal{N}$ independent initial conditions, so we have local degrees of freedom

$$\frac{1}{2}(2n\mathcal{N} - \mathcal{N} - \mathcal{N}) = (n-1)\mathcal{N} \tag{2.1.111}$$

In 5d, it's $\mathcal{N}$.

Chapter 3

Non-abelian self-dual string solution

Having introduced the proposed model for the gauge sector of multiple M5-branes in chapter 2, we study their (BPS) self-dual string solutions in this chapter. We will find the solutions both in the uncompactified and compactified space-times. These self-dual string solutions are supported by Wu-Yang and 't-Hooft-Polyakov monopole solutions.

3.1 Introduction

The low energy theory of N coincident M5-branes is given by an interacting (2,0) superconformal theory in 6 dimensions [69, 102–104]. On the M5-brane worldvolume there are self-dual strings. For a single M5-brane, the low energy theory is known [13, 14, 83–85, 90, 91, 98, 105]. The self-dual string soliton has also been constructed [46, 90]. Much less is known about the theory of multiple M5-branes, as well as the properties of multiple self-dual strings.

Recently, a theory of non-abelian chiral 2-form in 6-dimensions was constructed [54]. The construction was motivated by the analysis in [58, 110] and a set of 5d Yang-Mills gauge fields was introduced in order to incorporate non-trivial interactions among the 2-form potential. The theory admits a self-duality equation on the field strength as the equation of motion. It has a modified 6d Lorentz symmetry.

On dimensional reduction on a circle, the action gives the standard 5d Yang-Mills action plus higher order corrections. Based on these properties, it was proposed that the theory describes the gauge sector of multiple M5-branes in flat space. An important feature of this theory is that the self-interaction of the two-form gauge field is mediated by a set of five-dimensional Yang-Mills gauge field $A_\mu, \mu = 0, 1, 2, 3, 4)$. The Yang-Mills gauge field is auxiliary and is constrained non-trivially to be given in terms of the non-abelian tensor gauge field and does not contain any propagating degrees of freedom. In the Abelian case, the 1-form gauge field is free and simply decouple. See also [114], [55, 56], [57], [118], [120], [121, 122, 143], [58, 64, 144–146], for some other more relevant recent developments.

In this chapter we give a further support of this proposal by constructing the non-abelian self-dual strings to the equation of motion of the non-abelian theory [54]. For simplicity, let us consider 1/2-BPS states, and so consider a $SU(2)$ gauge group which corresponds to a system of two M5-branes. A crucial observation in our construction is that the Perry-Schwarz solution is supported by a Dirac monopole $A_a, (a = 0, 1, 2, 3)$. As the solution is translational invariant along the direction (say x^4) of the string, this gauge field can be thought of as a five dimensional one with $A_4 = 0$ and be interpreted as the auxiliary 1-form gauge fields in the theory of [54]. This interpretation suggests that the non-abelian self-dual string solution may be constructed by taking the auxiliary Yang-Mills gauge field to be given by a non-abelian monopole. Quite remarkably this is indeed correct and we are able to construct a self-dual string solution both for uncompactified six dimensions as well as with one dimension compactified. Our solution is obtained by replacing the Dirac monopole in the Perry-Schwarz string, in the uncompactified case to the non-abelian Wu-Yang monopole; and in the compactified case to the 't Hooft-Polyakov monopole.

The plan of this chapter is as follows. In section 3.2, after reviewing the original Perry-Schwarz self-dual string solution, we present a new abelian self-dual string solution which is orientated in a different direction. The existence of the latter solution is guaranteed by the Lorentz symmetry of the Perry-Schwarz theory. Then we solve the non-abelian equation of motion of [54] and obtain an exact solution

describing a string. We then discuss how this solution can be lifted as a solution of the (2,0) supersymmetric theory. The resulting solution describes a non-abelian string with self-dual charges. In section 3.3, we consider the compactified case and construct the corresponding self-dual string solution. Finally it is concluded with some further comments and discussions in section 3.4.

3.2 Non-Abelian Self-Dual String Solution: Uncompactified Case

In this section, we construct a self-dual string solution that satisfies both (2.3.51) and (2.3.52). As mentioned above, a direct observation on the constraint (2.3.52) shows that the solution cannot be aligned in the x^5 direction since this would imply $F_{\mu\nu} = 0$ which is trivial. This does not imply the non-existence of a string solution in other directions, because the self-duality equation (2.3.51) has only 5d Lorentz symmetry as it's a gauge fixed equation of motion [54]. Therefore, as a preparation to constructing the more general non-abelian self-dual string solution, we will first construct an abelian self-dual string solution aligning in the x^4 direction and we will start by reviewing the original abelian self-dual string solution of Perry and Schwarz.

3.2.1 Self-dual string solution in the Perry-Schwarz Theory

In [90], a nonlinear theory of chiral 2-form gauge field which results in the Born-Infeld action for a $U(1)$ gauge field when reduced to 5 dimensions was constructed. The Perry-Schwarz non-linear field equation is given by

$$\tilde{H}_{\mu\nu} = \frac{(1-y_1)H_{\mu\nu5} + H_{\mu\rho5}H^{\rho\sigma5}H_{\sigma\nu5}}{\sqrt{1-y_1+\frac{1}{2}y_1^2-y_2}}, \tag{3.2.1}$$

where

$$y_1 := -\frac{1}{2}H_{\mu\nu5}H^{\mu\nu5}, \qquad y_2 := \frac{1}{4}H_{\mu\nu5}H^{\nu\rho5}H_{\rho\sigma5}H^{\sigma\mu5}. \tag{3.2.2}$$

As they demonstrated, the equation of motion (3.2.1) admits a solution describing a self-dual string soliton with finite tension aligning in the direction x^5. Since (3.2.1) is (non-manifest) 6d Lorentz covariant, it means there must also exist self-dual string

solution aligned in other directions. In the following, we review their construction in section 3.2.1. Then we construct new self-dual string solution aligned in a different direction in section 3.2.1.

Self-dual string in the x^5 direction

The ansatz Perry and Schwarz considered for their self-dual string solution is

$$B = \alpha(\rho) dt dx^5 + \frac{\beta}{8}(\pm 1 - \cos\tilde{\theta}) d\tilde{\phi} d\tilde{\psi}, \tag{3.2.3}$$

where the 6d metric is

$$ds^2 = -dt^2 + (dx^5)^2 + d\rho^2 + \rho^2 d\Omega_3^2, \tag{3.2.4}$$

with the three-sphere given in Euler coordinates

$$d\Omega_3^2 = \frac{1}{4}[(d\tilde{\psi} + \cos\tilde{\theta} d\tilde{\phi})^2 + (d\tilde{\theta}^2 + \sin^2\tilde{\theta} d\tilde{\phi}^2)], \tag{3.2.5}$$

where $0 \leq \tilde{\theta} \leq \pi, 0 \leq \tilde{\phi} \leq 2\pi, 0 \leq \tilde{\psi} \leq 4\pi$. For this ansatz, it is $y_1 = \alpha'^2$, $y_2 = \alpha'^4/2$ and the non-linear field equation (3.2.1) reads

$$\alpha'(\rho) = \frac{\beta}{\sqrt{\beta^2 + \rho^6}}. \tag{3.2.6}$$

This can be solved easily in terms of a hyper-geometric function. The solution is regular everywhere where $\alpha \sim \rho$ as $\rho \to 0$, while $\alpha \sim -\frac{\beta}{2\rho^2} +$ const. as $\rho \to \infty$. Note that the same ansatz also solves the linear self-duality equation, where in this case we have,

$$\alpha'(\rho) = \frac{\beta}{\rho^3} \tag{3.2.7}$$

and the solution is singular at $\rho = 0$. In other words, the non-linear terms in the field equation has smoothen out the singularity at $\rho = 0$.

The magnetic charge P and electric charge Q per unit length of the string are given by

$$P = \int_{S^3} H, \qquad Q = \int_{S^3} *H, \tag{3.2.8}$$

where $*$ denotes the Hodge dual operation and S^3 is a three sphere surrounding the string. It is straightforward to obtain that

$$P = 2\pi^2\beta, \quad \text{and} \quad Q = 2\pi^2\rho^3\alpha'(\rho)|_{\rho\to\infty} = 2\pi^2\beta, \tag{3.2.9}$$

hence the string is self-dual. This holds for both the nonlinear and the linear cases. Note that our answer is 1/8 of those in [90] as we have introduced the factor of 1/4 into the metric (3.2.5) in order to reproduce the correct volume $2\pi^2$ for a unit three sphere.

The charge quantization condition [147, 148]

$$PQ' + Q'P \in 2\pi\mathbf{Z} \tag{3.2.10}$$

for the self-dual string gives

$$\beta = \pm n\sqrt{\frac{1}{4\pi^3}}, \tag{3.2.11}$$

i.e.

$$P = Q = \pm n\sqrt{\pi}, \tag{3.2.12}$$

where n is a positive integer. Note that the charge quantization condition we used is different from the Dirac-Teitelboim-Nepomechie charge quantization condition [35–37] Perry and Schwarz used. The condition (3.2.10) is obtained with a self-dual string probing another self-dual string and the positive sign in the charge quantization condition is appropriate for dyonic branes in $D = 4k + 2$ spacetime dimensions [147, 148].

Perry and Schwarz have also computed the tension of their string solution. Since the solution is static, the energy can be identified with the Lagrangian and the energy per unit length is found to be

$$T = \tilde{c}\beta^{4/3}, \tag{3.2.13}$$

where $\tilde{c}$ is a numerical coefficient. We remark that for the self-dual string solution of the linearized theory, the tension is

$$T = 0 \tag{3.2.14}$$

since obviously the action vanishes on-shell. Since the charges and tension are well defined, it appears that the singularity at $\rho = 0$ is not harmful.

We also remark that the Perry-Schwarz self-dual string solution is non-BPS as there is no other matter field turned on to cancel the tensor field force. In the literature, there is also the 1/2 BPS self-dual string of Howe, Lambert and West [46].

In fact the Perry-Schwarz self-duality equation of motion can be embedded in the fully supersymmetric five-brane equation of motion of [83, 84] by setting all the matter fields to zero and hence the Perry-Schwarz self-dual string solution can be lifted to be a solution of the full five-brane equation of motion, albeit a nonsupersymmetric one. Unlike the nonlinear Perry-Schwarz self-dual string solution, the Howe-Lambert-West self-dual string solution is singular at the location of the string. In fact $B \sim 1/\rho^2$ near the string, which is exactly as in linearized Perry-Schwarz self-dual string solution.

Self-dual string soliton in the x^4 direction

The Perry-Schwarz solution is translationally invariant along x^5. One may want to generalize this solution directly and construct a non-Abelian self-dual string solution which is translationally invariant along x^5 but this is not possible. As reviewed above, the gauge field strength in the non-abelian theory is given on-shell by the boundary value of B-field as (2.3.52) Therefore, if the non-Abelian solution is translationally invariant along x^5, then $F_{\mu\nu} = 0$ which is trivial.

To get a non-trivial solution, we need to base our construction on Perry-Schwarz solitons which are translationally invariant along other direction, say x^4. Such a solution can be easily obtained by rotating the original Perry-Schwarz solution as Perry and Schwarz has proved that their theory and the non-linear equation (3.2.1) respect Lorentz symmetry. Therefore, a simple Lorentz transformation which swap $(x_4, x_5) \to (-x_5, x_4)$ can be applied on the original Perry-Schwarz solution (the minus sign is needed to preserve the orientation of spacetime) to obtain the desired solution.

To facilitate the discussion, it is more convenient to use the spherical polar coordinates which is related to the Euler coordinates by the change of coordinates

$$\tilde{\theta} = 2\theta, \qquad \tilde{\phi} = \psi - \phi, \qquad \tilde{\psi} = \psi + \phi. \tag{3.2.15}$$

With this coordinates, the three-sphere metric is given by

$$d\Omega_3^2 = d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\psi^2 \tag{3.2.16}$$

with the ranges $0 \leq \theta \leq \pi/2, 0 \leq \phi, \psi \leq 2\pi$, and the Perry-Schwarz ansatz (3.2.3) becomes

$$B = \alpha(\rho) dt dx^5 + \beta \left(\frac{1}{4} \pm \frac{1}{4} - \frac{1}{2} \cos^2\theta \right) d\phi d\psi. \tag{3.2.17}$$

Next change to Cartesian coordinates

$$x = \rho \sin\theta \cos\phi, \quad y = \rho \sin\theta \sin\phi, \quad z = \rho \cos\theta \cos\psi, \quad w = \rho \cos\theta \sin\psi, \tag{3.2.18}$$

where we have denoted $(x^1, x^2, x^3, x^4) = (x, y, z, w)$. The metric becomes

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 + dw^2 + d(x^5)^2, \tag{3.2.19}$$

and the Perry-Schwarz ansatz reads

$$B = \alpha(\rho) dt dx^5 + \beta \frac{\frac{1}{4} \pm \frac{1}{4} - \frac{1}{2}\frac{w^2+z^2}{\rho^2}}{(x^2+y^2)(z^2+w^2)} (xz dy dw - xw dy dz - yz dx dw + yw dx dz). \tag{3.2.20}$$

Keeping the orientation, we swap $(x_4, x_5) \to (-x_5, x_4)$ and obtain our ansatz for a string solution along the x^4 direction,

$$B = \alpha(\rho) dt dw - \beta \frac{\frac{1}{4} \pm \frac{1}{4} - \frac{1}{2}\frac{(x^5)^2+z^2}{\rho^2}}{(x^2+y^2)(z^2+(x^5)^2)} (xz dy dx^5 - x x^5 dy dz - yz dx dx^5 + y x^5 dx dz) \tag{3.2.21}$$

where now

$$\rho = \sqrt{(x^5)^2 + r^2}, \quad r := \sqrt{x^2+y^2+z^2}. \tag{3.2.22}$$

It follows that

$$H = \frac{\alpha'}{\rho} dt dw \big(x dx + y dy + z dz + x^5 dx^5 \big) + \frac{\beta}{\rho^4} \big(x^5 dx dy dz - x dy dz dx^5 + z dy dx dx^5 - y dz dx dx^5 \big), \tag{3.2.23}$$

$$*H = \frac{\alpha'}{\rho} \big(x^5 dx dy dz - x dy dz dx^5 + z dy dx dx^5 - y dz dx dx^5 \big) + \frac{\beta}{\rho^4} dt dw \big(x dx + y dy + z dz + x^5 dx^5 \big), \tag{3.2.24}$$

and

$$y_1 = \frac{(\alpha')^2 (x^5)^2}{\rho^2} - \frac{\beta^2 r^2}{\rho^8}, \qquad y_2 = \frac{\beta^4 r^4}{2\rho^{16}} + \frac{(\alpha')^4 (x^5)^4}{2\rho^4}. \tag{3.2.25}$$

Then the field equation (3.2.1) gives

$$\frac{\beta}{\rho^4} x^5 dt dw + \frac{\alpha'}{\rho} (-x dy dz + z dy dx - y dz dx) = \frac{\alpha' x^5}{\rho} G dt dw + \frac{1}{G} \frac{\beta}{\rho^4} (-x dy dz + z dy dx - y dz dx), \tag{3.2.26}$$

where

$$G = \sqrt{\frac{1+\beta^2 r^2 \rho^{-8}}{1-\alpha'^2 (x^5)^2 \rho^{-2}}}. \tag{3.2.27}$$

The equation (3.2.26) is equivalent to

$$\alpha' = \frac{\beta}{\sqrt{\beta^2+\rho^6}}, \tag{3.2.28}$$

which is the same equation as before. As a consistency check, we integrate over the S^3 transverses to x^4 and obtain the same charges

$$P = Q = 2\pi^2 \beta. \tag{3.2.29}$$

For the linearized case, $\alpha' = \beta/\rho^3$.

Self-dual string soliton in the x^4 direction in the $B_{\mu 5} = 0$ gauge

The potential B_{MN} in the solution (3.2.20) or (3.2.21) does not satisfy the condition $B_{\mu 5} = 0$ as needed in [54, 90]. However this is not a problem as they are indeed gauge equivalent to one which does. Instead of giving the gauge transformation, it is more instructive to construct directly the linearized self-dual string soliton in the x^4 direction in this gauge.

The starting point is (3.2.23) with $\alpha' = \beta/\rho^3$. Our strategy is to integrate the self-duality equation of motion

$$H_{\mu\nu 5} = \partial_5 B_{\mu\nu} \tag{3.2.30}$$

to get $B_{\mu\nu}$. Then we use $B_{\mu\nu}$ to compute the whole H_{MNP} and check its consistency with our ansatz. The components of H are

$$H_{twi} = \frac{\beta x^i}{\rho^4}, \qquad H_{ijk} = \frac{\epsilon_{ijk}\beta x^5}{\rho^4}, \tag{3.2.31}$$

$$H_{tw5} = \frac{\beta x^5}{\rho^4}, \qquad H_{ij5} = -\frac{\epsilon_{ijk}\beta x^k}{\rho^4}. \tag{3.2.32}$$

Integrating (3.2.32), we get the following components of $B_{\mu\nu}$:

$$B_{ij} = -\frac{1}{2}\frac{\beta \epsilon_{ijk} x_k}{r^3}\left(\frac{x^5 r}{\rho^2} + \tan^{-1}(x^5/r)\right), \qquad B_{tw} = -\frac{\beta}{2\rho^2}, \tag{3.2.33}$$

In principle, x^5 independent constants of integration can be added but we will not need them. It is now easy to check a consistent solution is obtained by setting all the other independent components of $B_{\mu\nu}$ to be zero.

Two remarks are in order:

1. We remark that if we apply the condition (2.3.43) to the Perry-Schwarz self-dual string solution, we obtain

$$F_{ij} = -\frac{c\beta\pi}{2}\frac{\epsilon_{ijk}x_k}{r^3}, \qquad F_{tw} = 0 \tag{3.2.34}$$

for the auxiliary gauge field. Certainly this $U(1)$ field decouples and play no role in the abelian case. However it is interesting to note that this is precisely the field strength of a Dirac monopole in the (x, y, z) subspace! The presence of a Dirac monopole was already apparent in the original solution of [90]. Here, we reveal that the same monopole configuration also appears as the auxiliary gauge field. It turns out the use of a non-abelian monopole in place of the Dirac monopole is precisely what is needed to construct the non-abelian self-dual string solution.

2. The solution in the form (3.2.33) will be our basis for the construction of the non-abelian self-dual string in the next subsection. We remark that it is also quite interesting that this form of the solution provides a link between linearized Perry-Schwarz self-dual string and Howe-Lambert-West self-dual string [46]. To explain this, let us first give a brief review on the key construction of Howe-Lambert-West self-dual string. In the (2,0) supersymmetric theory, there are two non-linearly related 3-forms which are called H and h. The 3-form H is exact but not necessarily self-dual while the 3-form h is self-dual but not necessarily exact. When constructing self-dual string, one of the scalar fields is also turned on. The equation of motion is non-linear. However, with an appropriate ansatz, it is possible to impose a BPS condition which eventually gives a linear differential relation between H and the scalar field. Writing in our notation, the BPS equations of motion read

$$H_{twi} = \partial_i\phi, \qquad H_{tw5} = \partial_5\phi, \tag{3.2.35}$$

$$H_{ijk} = \epsilon_{ijk}\partial_5\phi, \qquad H_{ij5} = -\epsilon_{ijk}\partial_k\phi, \tag{3.2.36}$$

where we have rescaled the scalar to absorb an inessential numerical factor. These conditions ensure the self-duality of H. Furthermore, they agree precisely with the Perry-Schwarz's equations of motion (3.2.30) if one identifies $B_{tw} = \phi$. In other words, the linearized Perry-Schwarz self-dual string solution could be lifted to a 1/2 BPS solution in the (2,0) supersymmetric theory by adding a scalar field that satisfies the 'BPS' condition (3.2.36) (due to self-duality, the condition (3.2.35) is not needed).

3.2.2 Non-abelian Wu-Yang string solution

Now we are ready for the non-abelian case. As noted above of the roles played by the Dirac monopole in the abelian Perry-Schwarz solution, it is natural to consider the non-abelian generalizations of the Dirac monopole in the construction of the non-abelian self-dual strings. Here we have two candidates: the Wu-Yang monopole and the 't Hooft-Polyakov monopole where the latter involves a Higgs scalar field while the former does not. See, for example, [149] for a review of these solutions. We will use these non-abelian configurations to construct non-abelian self-dual string solutions for both the uncompactified case (where the Wu-Yang solution will be used) and compactified case (where the 't Hooft-Polyakov monopole will be used).

Let us first briefly review the non-abelian Wu-Yang monopole. Without loss of generality, we will consider $SU(2)$ gauge group with Hermitian generators $T^a = \frac{\sigma^a}{2}$ satisfying

$$[T^a, T^b] = i\epsilon^{abc}T^c, \quad a, b, c = 1, 2, 3. \tag{3.2.37}$$

This corresponds to the relative gauge symmetry of a system of two five-branes. Our convention for the Lie algebra valued fields are: $F_{\mu\nu} = iF^a_{\mu\nu}T^a$, $A_\mu = iA^a_\mu T^a$ and $F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - \epsilon^{abc}A^b_\mu A^c_\nu$.

The non-abelian Wu-Yang monopole is given by

$$A^a_i = -\epsilon_{aik}\frac{x_k}{r^2}, \qquad F^a_{ij} = \epsilon_{ijm}\frac{x_m x_a}{r^4}, \tag{3.2.38}$$

where $i, j = 1, 2, 3$ and Note that the field strength for the Wu-Yang solution is related to the field strength $F_{ij}^{(\text{Dirac})} = \epsilon_{ijm} x_m / r^3$ of the Dirac monopole by a simple relation:

$$F_{ij}^a = F_{ij}^{(\text{Dirac})} \frac{x^a}{r}. \tag{3.2.39}$$

In fact by performing a (singular) gauge transformation

$$U = e^{i\sigma_3\varphi/2} e^{i\sigma_2\theta/2} e^{-i\sigma_3\varphi/2}, \tag{3.2.40}$$

one can go to an Abelian gauge where only the 3rd component of the gauge field survives. In this gauge

$$A_i^a = \delta_3^a \, A_i^{(\text{Dirac})}. \tag{3.2.41}$$

Despite its close connection with the Dirac monopole, the Wu-Yang solution is not a monopole since it does not source the non-abelian magnetic field. In fact the color magnetic charge vanishes

$$\int_{S^2} F^a = 0. \tag{3.2.42}$$

Nevertheless the Wu-Yang solution is a useful prototype for constructing a non-abelian monopole and we will follow the common practice of the literature to refer to it as the Wu-Yang monopole. In particular, a magnetic charge can be defined if there is also in presence a Higgs scalar field as in the 't Hooft-Polyakov monopole.

Inspired by the relation (3.2.39) of the Wu-Yang solution, we will try to solve the non-abelian self-duality equation (2.3.51) by adopting the following ansatz for the field strength,

$$H_{\mu\nu\lambda}^a = H_{\mu\nu\lambda}^{(\text{PS})} \frac{x^a}{r} \tag{3.2.43}$$

Here $r = \sqrt{x^2 + y^2 + z^2}$ and

$$\begin{aligned} H^{(\text{PS})} := \frac{\beta}{\rho^4} \Big[dtdw(xdx &+ ydy + zdz + x^5 dx^5) \\ &+ x^5 dxdydz - zdxdydx^5 - ydzdxdx^5 - xdydzdx^5 \Big] \end{aligned} \tag{3.2.44}$$

is the field strength for the linearized Perry-Schwarz solution in the x^4 direction (3.2.23). The self-duality of (3.2.43) follows immediately from the self-duality of the Perry-Schwarz solution. For the moment, we will allow β to be a free parameter.

Our strategy is again to integrate $H_{\mu\nu 5} = \partial_5 B_{\mu\nu}$ to get $B_{\mu\nu}$. Then we obtain $F_{\mu\nu}$ and A_μ from the boundary value of $B_{\mu\nu}$. Finally, we use $B_{\mu\nu}$ and A_μ to compute the whole H_{MNP} and check its consistency with our ansatz. Now the components of our ansatz are:

$$H^a_{twi} = \frac{\beta x^i x^a}{r\rho^4}, \qquad H^a_{ijk} = \frac{\epsilon_{ijk}\beta x^5 x^a}{r\rho^4}, \tag{3.2.45}$$

$$H^a_{tw5} = \frac{\beta x^5 x^a}{r\rho^4}, \qquad H^a_{ij5} = -\frac{\epsilon_{ijk}\beta x^k x^a}{r\rho^4}. \tag{3.2.46}$$

Integrating (3.2.46), we get the following components of $B_{\mu\nu}$:

$$B^a_{\mu\nu} = B^{(\mathrm{PS})}_{\mu\nu}\frac{x^a}{r}, \quad \mu\nu = ij \text{ or } tw, \tag{3.2.47}$$

where $B^{(\mathrm{PS})}_{ij}, B^{(\mathrm{PS})}_{tw}$ are the B-field components (3.2.33) for the Perry-Schwarz solution. In principle, x^5 independent constants of integration can be added but we will not need them.

A consistent solution can be obtained by setting all the other independent components of $B_{\mu\nu}$ to be zero. To see this, let us compute $F_{\mu\nu}$ from (2.3.52). It is remarkable that

$$F^a_{ij} = -\frac{c\beta\pi}{2}\frac{\epsilon_{ijm}x_m x_a}{r^4}, \qquad F^a_{tw} = 0, \tag{3.2.48}$$

which is precisely the form (3.2.38) of the Wu-Yang monopole if we take

$$c\beta = -\frac{2}{\pi}. \tag{3.2.49}$$

As a result, the non-vanishing component of the gauge field is given by

$$A^a_i = -\epsilon_{aik}\frac{x_k}{r^2}. \tag{3.2.50}$$

So far we have used only the field strength components H_{ij5}, H_{tw5} of (3.2.46). However since $D_\mu(x^a T^a/r) = 0$ for the Wu-Yang gauge field, therefore (3.2.45) is reproduced immediately and (3.2.43) is indeed satisfied.

Like the Wu-Yang monopole, the color magnetic charge of our Wu-Yang string solution vanishes. This is not a problem as we should not forget about the scalar fields as our ultimate aim is to construct the non-abelian self-dual string solution in the multiple M5-branes theory and so the inclusion of scalar fields is natural

from the point of view of (2,0) supersymmetry. Although we do not have the full (2,0) supersymmetric theory, one can argue that the self-duality equation of motion (2.3.51) is not modified by the presence of the scalar fields. This can be seen by a simple dimensional analysis since the dimension of a canonically normalized scalar field is two, and there is no local polynomial term one can write down which is consistent with conformal symmetry. That the self-duality equation is not modified by the scalar fields is also the case in the other proposed constructions [56, 64, 144]. As for the scalar field, first it is clear that due to R-symmetry, the self-interacting potential vanishes if there is only one scalar field turned on. As a result, the equation of motion of the scalar field is

$$D_M^2\phi = 0. \tag{3.2.51}$$

This is the general situation but for special cases, for example when a BPS condition is satisfied, the second order equation could be reduced to a first order equation. A reasonable form of the BPS equation is the non-abelian generalization of the BPS equation (3.2.35), (3.2.36)

$$H_{ijk} = \epsilon_{ijk}\partial_5\phi, \qquad H_{ij5} = -\epsilon_{ijk}D_k\phi. \tag{3.2.52}$$

We conjecture that (3.2.52) is indeed a BPS equation of the non-abelian (2,0) theory since first of all it implies the equation of motion (3.2.51). Moreover (3.2.52) would follow immediately from the supersymmetry transformation ($\Gamma^{012345}\epsilon = \epsilon$, $\Gamma^{012345}\psi = -\psi$)

$$\delta\psi = (\Gamma^M\Gamma^I D_M\phi^I + \frac{1}{3!2}\Gamma^{MNP}H_{MNP})\epsilon \tag{3.2.53}$$

(which is the most natural non-abelian generalization of the abelian (2,0) supersymmetry transformation) and the 1/2 BPS condition

$$\Gamma^{046}\epsilon = -\epsilon, \tag{3.2.54}$$

together with the condition that $\phi^6 := \phi = \phi(x^a), a = 1, 2, 3, 5$. Let us emphasise that we do not have supersymmetry in our model of Chapter 2. The supersymmetry transformation proposed above is just conjectural.

We note that (3.2.52) is compatible with the self-duality equation if the scalar

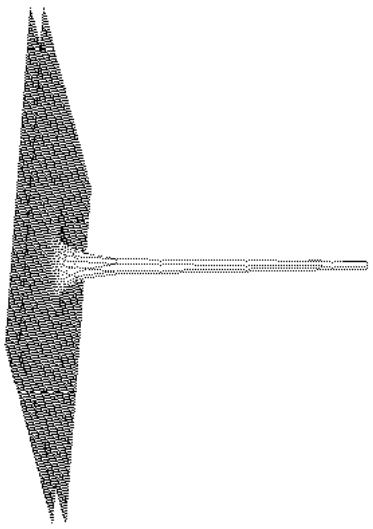

Figure 3.1: An M2 brane ending on a system of two parallel M5-branes separated by a distance.

field is equal to the B_{tw} component:

$$\phi^a = B^a_{tw} = -\frac{\beta}{2\rho^2}\frac{x^a}{r}, \tag{3.2.55}$$

or more generally,

$$\phi^a = -\left(u + \frac{\beta}{2\rho^2}\right)\frac{x^a}{r}, \tag{3.2.56}$$

where u is a constant and we will choose it to be of the same sign as β so that $|\phi|$ is never zero. To see the physical meaning of this solution, let us consider the transverse distance $|\phi|$ defined by $|\phi|^2 = \phi^a\phi^a$. This gives

$$|\phi| = |u + \frac{\beta}{2\rho^2}|. \tag{3.2.57}$$

This describes an M5-brane with a spike at $\rho = 0$ and level off to u as $\rho \to \infty$. Hence the physical interpretation of our self-dual string is that two M5-branes are separating by a distance u and with an M2-brane ending on them (see figure 1). With this interpretation, there is a symmetry breaking and one can identify a $U(1)$ B-field at the large distance ρ:

$$\mathcal{B}_{\mu\nu} \equiv \hat{\phi}^a B^a_{\mu\nu} = \pm B^{(\mathrm{PS})}_{\mu\nu} \tag{3.2.58}$$

where $\hat{\phi}^a := \phi^a/|\phi|$ and the $+$ $(-)$ sign in the second equation above corresponds to the case $c > 0$ $(c < 0)$. Since the field configuration approaches that of the abelian self-dual string at large distance, we immediately obtain the charges

$$P = Q = -2\pi^2|\beta| = -\frac{4\pi}{|c|}. \tag{3.2.59}$$

and charge quantization determines that

$$\beta = \mp n\sqrt{\frac{1}{4\pi^3}}, \quad c = \pm 4\frac{\sqrt{\pi}}{n} \tag{3.2.60}$$

and $P = Q = -n\sqrt{\pi}$. We require that the theory should admit solution with the minimal unit of charge and so the possible values of the constant c in the non-abelian action (**??**) is:

$$c = \pm 4\sqrt{\pi} \tag{3.2.61}$$

and the charges of our solution are $P = Q = -\sqrt{\pi}$.

Just as in the abelian case, the action for the gauge fields vanish on shell. Therefore the string gets its tension solely from the scalar field. In general, the kinetic term of scalar field is proportional to

$$\mathrm{tr}(D_M\phi D^M\phi). \tag{3.2.62}$$

Since the scalar field satisfies

$$D_M\phi \to 0, \qquad \rho \to \infty, \tag{3.2.63}$$

we see that at large distance $\rho \to \infty$ from the string, the kinetic term vanishes. However the singularity at the origin leads to an infinite tension. This is the same as the Howe-Lambert-West self-dual string solution [46].

3.3 Non-Abelian Self-Dual String Solution: Compactified Case

In this section, we consider the theory with x^5 compactified on a circle with radius R and construct the self-dual string solution. The constraint that the gauge field has to satisfy is now (3.3.64),

$$F_{\mu\nu} = 2\pi Rc\tilde{H}^{(0)}_{\mu\nu}, \tag{3.3.64}$$

Without loss of generality, let us assume that the string aligns in the $w = x^4$ direction.

In the compactified theory, the field strength can be expanded in terms of Fourier modes,

$$H_{MNP} = \sum_n e^{inx^5/R} H^{(n)}_{MNP}(r). \tag{3.3.65}$$

The gauge field $B_{\mu\nu}$ can be then obtained by integrating over the equation of motion $H_{\mu\nu 5} = \partial_5 B_{\mu\nu}$. It is

$$B_{\mu\nu} = \frac{x^5}{2\pi Rc} F_{\mu\nu}(r) + \sum_{n=-\infty}^{\infty} e^{inx^5/R} B^{(n)}_{\mu\nu}(r), \tag{3.3.66}$$

where we have used the boundary condition (3.3.64) to determine the first term and $B^{(0)}_{\mu\nu}(r)$ is an integration constant. The higher modes $B^{(n\neq 0)}_{\mu\nu}$ are given by:

$$H^{(n\neq 0)}_{\mu\nu 5}(r) = \frac{in}{R} B^{(n\neq 0)}_{\mu\nu}(r). \tag{3.3.67}$$

Notice that the first term on the right hand side has no contribution to $H_{\mu\nu\lambda}$ because of Bianchi identity and hence

$$H^{(n)}_{\mu\nu\lambda} = D_{[\lambda} B^{(n)}_{\mu\nu]} \tag{3.3.68}$$

for all n.

Let us consider an ansatz with the only nonzero components of gauge potential being B_{tw} and B_{ij}. The self-duality condition reads

$$H_{ijk} = \epsilon_{ijk} H_{tw5}, \quad H_{twk} = -\frac{1}{2}\epsilon_{ijk} H_{ij5}, \tag{3.3.69}$$

or, written in terms of modes,

$$D_{[i} B^{(0)}_{jk]} = \epsilon_{ijk} \frac{F_{tw}}{2\pi Rc}, \qquad D_k B^{(0)}_{tw} = -\frac{f_k}{2\pi Rc} \tag{3.3.70}$$

$$D_k b^{(n)}_k = \frac{in}{R} B^{(n)}_{tw}, \qquad D_k B^{(n)}_{tw} = -b^{(n)}_k \frac{in}{R}, \qquad n \neq 0, \tag{3.3.71}$$

where we have denoted

$$f_k(r) := \frac{1}{2}\epsilon_{ijk} F_{ij} \qquad \text{and} \qquad b^{(n)}_k(r) := \frac{1}{2}\epsilon_{ijk} B^{(n)}_{ij} \quad \text{for } n \neq 0. \tag{3.3.72}$$

Notice that the 2nd equation of (3.3.70) takes exactly the same form as the BPS equation for the 't Hooft-Polyakov magnetic monopole if we identify $-2\pi RcB^{(0)}_{tw}$ as the scalar field there. Indeed in the BPS limit, the equation of motion for the 't Hooft-Polyakov monopole reads

$$\frac{1}{2}\epsilon_{ijk}F_{ij} = D_k\phi, \tag{3.3.73}$$

where ϕ is an adjoint Higgs scalar field. The solution is given by

$$A^a_i = -\epsilon_{aik}\frac{x^k}{r^2}(1-k_v(r)), \qquad \phi^a = \frac{vx^a}{r}h_v(r), \tag{3.3.74}$$

where

$$k_v(r) := \frac{vr}{\sinh(vr)}, \qquad h_v(r) := \coth(vr) - \frac{1}{vr}. \tag{3.3.75}$$

Asymptotically $r \to \infty$, we have

$$A^a_i \to -\epsilon_{aik}\frac{x^k}{r^2}, \qquad \phi^a \to \frac{|v|x^a}{r} := \phi_\infty, \tag{3.3.76}$$

which coincides with Wu-Yang monopole. Note that the gauge symmetry is broken at infinity to $U(1)$, the little group of ϕ_∞. This may be identified as the electromagnetic gauge group and one could use this to define the magnetic monopole charge [150, 151]. The electromagnetic field strength can be defined as

$$\mathcal{F}_{ij} = F^a_{ij}\frac{\phi^a}{|v|} = \epsilon_{ijk}\frac{x^k}{r^3}, \qquad \text{for large } r. \tag{3.3.77}$$

The magnetic charge is given by $p = \int_{S^2}\mathcal{F} = 4\pi$, which corresponds to a magnetic monopole of unit charge. Note that at the core $r \to 0$, we have

$$A_i \to 0, \qquad \phi \to 0 \tag{3.3.78}$$

and hence the $SU(2)$ symmetry is unbroken at the monopole core.

The resemblance of our equation with the BPS equation of the 't Hooft-Polyakov monopole motivates us to take for A_μ the same ansatz as in the 't Hooft-Polyakov monopole,

$$A^a_i = -\epsilon_{aik}\frac{x^k}{r^2}(1-k_v(r)), \tag{3.3.79}$$

This implies $F_{tw} = 0$ and hence the 1st equation of (3.3.70) can be solved with

$$B^{(0)}_{ij} = c_0 F_{ij}, \tag{3.3.80}$$

where c_0 is an arbitrary constant. On the other hand, (3.3.71) gives

$$D_k D_k B_{tw}^{(n\neq 0)} = \frac{n^2}{R^2} B_{tw}^{(n\neq 0)}. \tag{3.3.81}$$

For zero mode, we have $D_k D_k B_{tw}^{(0)} = 0$, combine them together we can write

$$D_k D_k B_{tw}^{(n)} = \frac{n^2}{R^2} B_{tw}^{(n)}. \tag{3.3.82}$$

We take the ansatz for $B_{tw}^{(n)}$ as

$$B_{tw}^{(n)\,a} = a_n(r)\frac{vx^a}{r} \tag{3.3.83}$$

then the equation (3.3.82) is equivalent to

$$\frac{\partial_r(r^2\partial_r a_n(r))}{r^2} - \frac{2k_v(r)^2}{r^2}a_n(r) = \frac{n^2}{R^2}a_n(r). \tag{3.3.84}$$

The well-behaved physical solution is

$$a_0 = \alpha_0 h_v(r), \tag{3.3.85}$$

$$a_{n\neq 0}(r) = \alpha_n \frac{e^{-|n|r/R}}{vr}\left(1 + \frac{vR}{|n|}\coth(vr)\right), \tag{3.3.86}$$

where α_n are arbitrary constants. Here we have dropped the independent solutions which are exponentially increasing at large distance and hence not physical. As a result, we obtain for the gauge fields

$$B_{tw}^a = -\frac{h_v(r)}{2\pi Rc}\frac{vx^a}{r} + \sum_{n\neq 0}\alpha_n e^{inx^5/R}\frac{e^{-|n|r/R}}{vr}\left(1 + \frac{vR}{|n|}\coth(vr)\right)\frac{vx^a}{r}, \tag{3.3.87}$$

$$B_{ij}^a = \frac{x^5}{2\pi Rc}F_{ij}^a(r) + c_0 F_{ij}^a(r) + \sum_{n\neq 0} e^{inx^5/R}B_{ij}^{a\,(n)}(r). \tag{3.3.88}$$

where

$$b_k^{(n)\,a} = -v^3\frac{R}{in}(ra_n' - k_v(r)a_n)\frac{x^k x^a}{r} - \delta_k^a\frac{vR}{in}a_n k_v(r)\frac{1}{r}, \quad n\neq 0. \tag{3.3.89}$$

The proportionality factor for a_0 is determined by recalling that $-2\pi RcB_{tw}^{(0)}$ is the scalar of the 't Hooft-Polyakov monopole, while $\alpha_{n\neq 0}$ are left undetermined. Physically this corresponds to different excitations over the fundamental solution with all

$\alpha_{n\neq 0} = 0$. Note that there is a "winding mode" in B_{ij}, while there is no such mode in B_{tw} because $F_{tw} = 0$. Although this has no effect classically, we expect that this is observable quantum mechanically like the Berry phase. See, for example, [152–154] for a discussion of Berry phase associated with branes in string theory.

Next let us include a (2,0) scalar field ϕ. As above we assume that it satisfies the BPS equation (3.2.36), then the BPS equation is satisfied automatically if we identify $\phi^{(0)} = B^{(0)}_{tw}$. As a result, we have

$$\phi^{(0)\,a} = -u\left(\coth(vr) - \frac{1}{vr}\right)\frac{x^a}{r}. \tag{3.3.90}$$

where

$$u := \frac{v}{2\pi Rc} \tag{3.3.91}$$

set the scale of the vev of $\phi^{(0)}$ at large r since we can say $\phi^{(0)} \to -\frac{|v|}{2\pi Rc}x^a T^a/r$ as $r \to \infty$. In addition, one can define a $U(1)$ projection onto $\phi^{(0)}$. This allows us to define the charges

$$\begin{aligned} P = Q &= \int_{S^1\times S^2} H^a \hat{\phi}^a \\ &= \mp \int dx^5 dS_k \frac{1}{2}\epsilon_{ijk}\left(\frac{1}{2\pi Rc}F^a_{ij}\frac{x^a}{r} + (\text{KK})\right) \\ &= -\frac{4\pi}{|c|}, \end{aligned} \tag{3.3.92}$$

where the $-$ $(+)$ sign in the second equation above corresponds to the case $c > 0$ $(c < 0)$; and the term (KK) stands for the KK modes and their contribution to the charges is zero. Substituting (3.2.61), we find that the solution is self-dual and carries the charges $P = Q = -\sqrt{\pi}$. Physically one can identified this self-dual string with the uncompactified one obtained in the previous section and so they carry the same charges.

The scalar profile of (3.3.90) is plotted in figure 3.1, for two compactification radius $R = 1$ and $R = 4$ and a fixed vev $u = -0.5$. One may compare our results to the scalar profile in [155]. In this work, a modified Nahm's equation for the scalar field was conjectured. However unlike the ordinary Nahm's equation where one can obtain the non-abelian Yang-Mills gauge field at the same time, it is not clear how one might obtain the corresponding non-abelian tensor gauge field

from the modified Nahm's equation and the proposal still needed to be completed. Nevertheless, qualitatively their scalar profile is similar to ours.

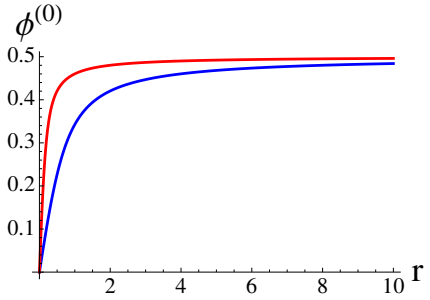

Figure 3.1: Scalar Profile. The red curve corresponds to $R = 4$ and the blue one to $R = 1$.

3.4 Discussions

In this chapter we have constructed the non-abelian string solutions of the non-abelian 5-brane theory constructed in [54], for both uncompactified and compactified spacetime. The string solution in non-compact spacetime is supported by a non-abelian Wu-Yang monopole, while the string solution in compact spacetime is supported by a non-abelian 't Hooft-Polyakov monopole. We showed how these solutions can be embedded in the (2,0) supersymmetric theory by including a single scalar field obeying a first order BPS equation. Although we don't have the full (2,0) supersymmetric construction yet, we argued that it is the correct BPS equation of the (2,0) theory since it solves the equation of motion, and moreover it can be derived from the most natural form of the supersymmetry transformation law in the non-abelian (2,0) theory. These string solutions carry self-dual charges and have infinite tension arising from the scalar profile which corresponds to having a M2-brane spike on the M5-branes system. These properties are consistent with what one expects for the non-abelian self-dual strings living on a system of two M5-branes. Hence the results we obtained provide further support that the non-abelian theory constructed in [54] describes the gauge sector of a system of multiple M5-branes. Needless to say, it is of utmost importance to obtain the supersymmetric completion

of the bosonic theory [54]. This is under investigation.

We have constructed a non-abelian self-dual string solution with unit charge. In the M-theory picture, it is possible to have non-abelian self-dual strings with higher charges. It would also be interesting to explore the possible loop space or twistor interpretation [62, 156, 157] of our self-dual string solution. In [158], the solution found in section 3.2.2 is generalised for arbitrary number N_5 of five-branes and arbitrary N_2 units of self-dual charges. The solution of section 3.2.2 corresponds to the case of $N_5 = 2$ and $N_2 = 1$. The generalisation constructed in [158] is based on the generalized non-abelian Wu-Yang monopole [149,159,160]. Remarkably, the radius–transverse relation describing the M2-branes spike in [158], in particular its N_2, N_5 dependence, agrees with the one in the supergravity description of [161,162]. Subsequently, a solution supported by the Yang-Mills instantons are found in [163], which corresponds to the M-wave propagating on the worldvolume of multiple M5-branes. Their result therefore provides further evidences that the non-abelian theory [54] does give a description for a system of multiple M5-branes. A more detailed further analysis of the nonlinear self-duality equations (2.3.51) and constraints (2.3.43) can be found in [164].

It is also hoped that the self-dual string solution constructed here could provide further insights into the understanding of the N^3 entropy growth of the multiple M5-branes system [142]. Recent progress on this problem has been achieved in [121, 122, 143]. It will be interesting to count the number of degrees of freedom by Goldstone mode analysis of our solution, and then compare the result with [165], where the number of degrees of freedom was counted by using anomaly cancellation.

As advocated in [110, 118], just as in the D-branes case where the Lie bracket which define the gauge symmetries for multiple D-branes captures the noncommutative geometry of a single D-brane in the presence of a large NSNS B-field, it is possible that the gauge symmetry for multiple M5-branes could also capture the structure of the quantum geometry of a single M5-branes in the presence of a large C-field. Given the dynamical evidence we presented in this chapter, we believe that the non-abelian tensor gauge theory of [54] does describe the gauge sector of multiple M5-branes. It is thus interesting to try to understand how the gauge symmetry

of the non-abelian theory [54] could describe the quantum Nambu geometry derived in [118] for a M5-brane in a large C-field. An encouraging sign is that both are described in terms of an ordinary commutator.

Part III

Single M5-brane action revisited

Chapter 4

The M5-brane action revisited

In this chapter, we move our focus to the single M5-brane case, and consider its action formulation describing the nonlinear self-interaction. In particular, we construct an alternative form of the M5–brane action in which the six–dimensional worldvolume is subject to a covariant split into 3+3 directions by a triplet of auxiliary fields.

4.1 Introduction

The construction of duality–symmetric actions has been an active topic of research since the 1970's [166, 167]. It has recently seen a revival of interest in relation with the discussion of possible finiteness of $N = 8, D = 4$ supergravity [168–175], and in connection with attempts of making progress in understanding the non–Abelian (2,0) $6d$ superconformal gauge theory [71] on the worldvolume of N coincident M5–branes [47, 54–57, 59, 60, 62, 114, 124, 158, 163, 176–184]. For a single M5–brane the complete set of equations of motion was derived in [83] and considered in detail in [84] using the superembedding approach put forward in [185] (see [186] and e.g. [88, 187] for review and a detailed list of references). A complete M5–brane action was constructed in [13, 14] as a result of a step–by–step generalization [90, 91, 94, 95, 98] of a self–dual action for a free chiral 2–form gauge field [92, 93]. It was then shown that the non–linear self–duality relation [86] and the complete set of the equations of motion [85] derived from the M5–brane action are equivalent to the manifestly

covariant equations obtained from superembedding.

It is well known that to lift the duality symmetry to the level of the action one should deal with the issue of space–time covariance of the theory. In the non–manifestly $SO(1,5)$ Lorentz–invariant construction of the $6d$ chiral 2–form action by Henneaux and Teitelboim [92, 93] only an $SO(5)$ or $SO(1,4)$ [90] subgroup of $SO(1,5)$ is manifest. The construction can be made space–time covariant (diffeomorphism invariant) by introducing into the action a normalized gradient of an auxiliary scalar field $a(x)$ [96, 97], [98]. The manifestly covariant formulation significantly simplifies the construction of the consistent couplings of the self-dual field action to gravity and other fields, and its non–linear deformations. Different gauge fixings of the value of $a(x)$ using an associated local symmetry (or its dualization [188]) results in different non–covariant forms of the self–dual action. On the other hand, the self-duality equations obtained from the action can be cast into the manifestly covariant form which does not contain the auxiliary field $a(x)$, thus the latter completely disappears on the mass–shell without imposing any gauge fixing condition.

With the advent of the Bagger–Lambert–Gustavsson (BLG) model [48, 49, 51], an alternative construction of a 5–brane action based on the BLG action with the gauge symmetry of volume preserving diffeomorphisms was put forward in [74, 75] (see [78, 189] for a review and references and [126, 190] for a related work). The space–time and duality symmetries of this construction were analyzed in detail in [76, 119]. The equivalence of this model to the M5–brane description of [13, 14] is still to be proved, though some steps have already been undertaken in [76, 191] and various checks via comparison of classical solutions on the both sides have been carried out (see [78] and references therein).

The relation between the two actions is not obvious, first, since the original non–linear M5–brane action is of a Dirac–Born–Infeld type whose chiral 2–form gauge field transforms under the usual Abelian gauge transformations, while the action of [74] is a polynomial of up to six order in the fields and has a Nambu–Poisson 3–algebra structure associated with an un–conventional gauge invariance under volume preserving diffeomorphisms. In [74, 75] it was conjectured that the Nambu–Poisson

(NP) M5–brane model is related to the conventional description of the M5–brane in a constant C_3–field background through a transformation analogous to the Seiberg–Witten map [192]. Such a map between the fields and gauge transformations of the two models was constructed in [78], however the relation between the two actions still remains to be established. The second reason which hampers the resolution of this issue is that in the NP M5–brane model the manifest $SO(1,5)$ $6d$ Lorentz symmetry is naturally broken by the presence of multiple M2–branes and the C_3–field to $SO(1,2) \times SO(3)$, which corresponds to a $3+3=6$ "splitting" of the six dimensions of the $M5$–brane worldvolume. In the original M5–brane action, as we mentioned above, the six dimensions split into 1+5. In [76] it was shown that, even when reduced to the second order in the fields, the two duality–symmetric actions are not equivalent off the mass shell, though both produce the same self-duality equation for the 2–form gauge field.

The M5–brane case exemplified the fact that the Lagrangian description of the self–dual fields and duality–symmetric fields in general is not unique (see also [193, 194]), and various free (quadratic) duality–symmetric actions in D dimensions with different splittings of $D = p + q + r + ...$ corresponding to various ways of breaking manifest space–time symmetry have been constructed [99, 195]. These different off–shell formulations may be useful for studying issues of the quantization of the self-dual fields in topologically non–trivial backgrounds [104, 193, 194, 196–199].

As far as the M5–brane is concerned, it is advisable for a better understanding of the relation between the original M5–brane descriptions and the NP 5–brane, to see whether the quadratic self–dual action of [75] with "3+3 splitting" can be extended to a full non–linear action which is invariant under the conventional gauge transformations of the gauge field and which would produce the same equations of motion as the ones obtained from the superembedding [83] and the action of [13, 14, 85]. This is the main goal of this chapter.

Our strategy to achieve this goal is as follows[1]. We will start with the covariant form [76] of the quadratic self–dual action of [75] for a 2–form chiral gauge field in

[1]For analogous procedures of getting manifestly duality–symmetric non–linear actions see e.g. [90, 169, 172].

six–dimensions. In addition to the conventional invariance under the gauge transformations of the chiral field, the covariant action possesses two more local symmetries. One of them ensures that the auxiliary fields, which make the action covariant, are non–dynamical and another one guarantees that the self–duality condition on the field strength of the chiral field is the general solution of its equations of motion. We will add to this quadratic action a generic non–linear function of components of the chiral field strength and derive conditions on the form of this function imposed by the two local symmetries. It is known that the conditions obtained in this way may have more than one solution (see e.g. [90, 169, 172, 200, 201]), so to single out the solution which describes the M5–brane we will look for the one which is equivalent to the non–linear self–duality relation of the superembedding approach. More concretely, we will check that the non–linearly self–dual field strength of the superembedding formulation satisfies the condition imposed on the non–linear part of the self–dual action and, as a result, will derive an explicit form of the M5–brane action in which the $6d$ diffeomorphism invariance is subject to "3+3 splitting".

As is known from an extensive literature (see e.g. [169–172, 174, 201] and references therein), in general, the functionals of gauge–field strengths which determine non–linear self–duality conditions are constructed order–by–order as perturbative series expansions in powers of the field strength and in general their explicit form is unknown except for the Born–Infeld–type actions and few other examples (see e.g. [175, 202]). Our construction is a new example of an explicit (closed) form of the non–linearly self–dual action which differs from the canonical form of the Born–Infeld–type actions by additional terms and factors.

This chapter is organized as follows. In Section 4.2 we introduce main notation and conventions. In Section 4.3 we review the original action and present the structure of the novel action for the M5–brane. The derivation of the new action is explained in Section 4.4. In the subsection 4.4.2, we also give details of the check of the form of the new M5–brane action by comparing the self–duality relations which follow from the action with those obtained in the superembedding description of the M5–brane. In Section 4.5 we show that on–shell values of the two actions are equal and in Section 4.6 briefly discuss the dimensional reduction of the novel M5–

brane action to that of the M2–brane. In Section 4.7, we show that putting our new nonlinear chiral 2-form action on a torus gives rise to the Schwarz-Sen type duality symmetric $4d$ theory [94]. The results are summarized in the Conclusion, where we also discuss open issues and possible directions of further research.

4.2 Notation and Conventions

The $6d$ and the $D = 11$ Minkowski metrics have the almost plus signature, x^μ ($\mu = 0, 1, \cdots, 5$) stand for the worldvolume coordinates of the M5–brane which carries the chiral gauge field $B_2(x) = \frac{1}{2} dx^\mu dx^\nu B_{\nu\mu}(x)$. The $D = 11$ bulk superspace is parametrized by $Z^{\mathcal{M}} = (X^M, \theta)$, where X^M are eleven bosonic coordinates and θ are 32 real fermionic coordinates. The geometry of the $D = 11$ supergravity are described by tangent–space vector supervielbeins $E^A(Z) = dZ^{\mathcal{M}} E_{\mathcal{M}}{}^A(Z)$ ($A = 0, 1, \cdots 10$) and Majorana–spinor supervielbeins $E^\alpha(Z) = dZ^{\mathcal{M}} E_{\mathcal{M}}{}^\alpha(Z)$ ($\alpha = 1, \cdots 32$).

The vector supervielbein satisfies the following essential torsion constraint, which is required for proving the kappa–symmetry of the $M5$–brane action,

$$T^A = DE^A = dE^A + E^B \Omega_B{}^A = -iE^\alpha \Gamma^A_{\alpha\beta} E^\beta \,, \tag{4.2.1}$$

where $\Omega_B{}^A(Z)$ is the one–form spin connection in $D = 11$, $\Gamma^A_{\alpha\beta} = \Gamma^A_{\beta\alpha}$ are real symmetric gamma–matrices and the external differential acts from the right.

The induced metric on the $M5$–brane worldvolume is constructed with the pull–backs of the vector supervielbeins $E^A(Z)$

$$g_{\mu\nu}(x) = E^A_\mu E^B_\nu \eta_{AB}, \qquad E^A_\mu = \partial_\mu Z^{\mathcal{N}} E_{\mathcal{N}}{}^A(Z(x)). \tag{4.2.2}$$

The $M5$–brane couples to the $D = 11$ supergravity 3–form gauge superfield $C_3(Z) = \frac{1}{3!} dZ^{\mathcal{M}_1} dZ^{\mathcal{M}_2} dZ^{\mathcal{M}_3} C_{\mathcal{M}_3\mathcal{M}_2\mathcal{M}_1}$ and its $C_6(Z)$ dual, their field strengths are constrained as follows

$$\begin{aligned}
dC_3 &= -\frac{i}{2} E^A E^B E^\alpha E^\beta (\Gamma_{BA})_{\alpha\beta} + \frac{1}{4!} E^A E^B E^C E^D F^{(4)}_{DCBA}(Z) \,, \\
dC_6 - C_3 dC_3 &= \frac{2i}{5!} E^{A_1} \cdots E^{A_5} E^\alpha E^\beta (\Gamma_{A_5 \cdots A_1})_{\alpha\beta} + \frac{1}{7!} E^{A_1} \cdots E^{A_7} F^{(7)}_{A_7 \cdots A_1}(Z), \\
F^{(7)\, A_1 \cdots A_7} &= \frac{1}{4!} \epsilon^{A_1 \cdots A_{11}} F^{(4)}_{A_8 \cdots A_{11}} \,, \qquad \epsilon^{0 \ldots 10} = -\epsilon_{0 \ldots 10} = 1.
\end{aligned} \tag{4.2.3}$$

The generalized field strengths of $B_2(x)$ which appears in the M5–brane action is

$$H_3 = dB_2 + C_3\,, \tag{4.2.4}$$

where $C_3(Z(x))$ is the pullback on the M5–brane worldvolume of the 3–form gauge field.

4.3 M5-brane actions

We start by briefly reviewing the original form of the M5–brane action and then will present our main result, namely, the alternative worldvolume action for the M5–brane in a generic $D = 11$ supergravity background.

4.3.1 Original M5–brane action

In this case to ensure the $6d$ worldvolume covariance of the $M5$–brane action one uses a normalized gradient of the auxiliary scalar field $a(x)$ which can be chosen to be time–like or space–like, e.g.

$$v_\mu(x) = \frac{\partial_\mu a}{\sqrt{\partial_\nu a\, g^{\nu\lambda}(x)\, \partial_\lambda a}}\,, \qquad v_\mu v^\mu = 1 \tag{4.3.1}$$

Both choices are equivalent since in the action $\partial_\mu a$ appears only in the projector of rank one

$$P_\mu{}^\nu(x) = \frac{\partial_\mu a \partial^\nu a}{(\partial a)^2}, \qquad PP = P, \qquad (\partial a)^2 \equiv \partial_\nu a\, g^{\nu\lambda}\, \partial_\lambda a = \partial_\nu a\, \partial^\nu a\,. \tag{4.3.2}$$

This projector singles out one worldvolume direction from the six, i.e. makes the 1+5 covariant splitting of the $6d$ worldvolume directions.

The $M5$–brane action in a generic $D = 11$ supergravity superbackground constructed in [13, 14, 91] has the following form:

$$\begin{aligned} S &= +2\int_{\mathcal{M}_6} d^6x \left[\sqrt{-\det(g_{\mu\nu} + i\tilde{H}_{\mu\nu})} + \frac{\sqrt{-g}}{4(\partial a)^2}\partial_\lambda a \tilde{H}^{\lambda\mu\nu} H_{\mu\nu\rho}\partial^\rho a\right] \\ &\quad - \int_{\mathcal{M}_6} (C_6 + H_3 \wedge C_3)\,, \end{aligned} \tag{4.3.3}$$

with

$$\tilde{H}^{\rho\mu\nu} \equiv \frac{1}{6\sqrt{-g}}\,\epsilon^{\rho\mu\nu\lambda\sigma\tau} H_{\lambda\sigma\tau}, \quad \tilde{H}_{\mu\nu} \equiv \frac{\partial^\rho a}{\sqrt{(\partial a)^2}}\tilde{H}_{\rho\mu\nu}, \quad g = \det g_{\mu\nu}\,, \tag{4.3.4}$$

where

$$\epsilon^{0\cdots 5} = -\epsilon_{0\cdots 5} = 1\,.$$

In addition to the conventional abelian gauge symmetry for the chiral 2-form, the action (4.3.3) has also the following two local gauge symmetries :

$$\delta B_{\mu\nu} = 2\partial_{[\mu} a\, \Phi_{\nu]}(x), \qquad \delta a(x) = 0, \tag{4.3.5}$$

as well as

$$\delta a = \varphi(x), \qquad \delta B_{\mu\nu} = \frac{\varphi(x)}{\sqrt{(\partial a)^2}}(H_{\mu\nu} - \mathcal{V}_{\mu\nu}), \tag{4.3.6}$$

where

$$\mathcal{V}^{\mu\nu}(\tilde{H}) \equiv -2\,\frac{\delta\sqrt{\det(\delta^\nu_\mu + i\tilde{H}_\mu{}^\nu)}}{\delta\tilde{H}_{\mu\nu}}, \quad H_{\mu\nu} \equiv H_{\mu\nu\rho}\frac{\partial^\rho a(x)}{\sqrt{(\partial a)^2}}, \tag{4.3.7}$$

with $\varphi(x)$ and $\Phi_\mu(x)$ being arbitrary local functions on the woldvolume. The first symmetry (4.3.5) ensures that the equation of motion of B_2 reduces to the non–linear self–duality condition

$$H_{\mu\nu} = \mathcal{V}_{\mu\nu}(\tilde{H})\,, \tag{4.3.8}$$

while the second symmetry (4.3.6) is responsible for the auxiliary nature of the scalar field $a(x)$ and the $6d$ covariance of the action.

The action (4.3.3) is also invariant under the local fermionic kappa–symmetry transformations with the parameter $\kappa^\alpha(x)$ which act on the pullbacks of the target–space supervilebeins and the B_2 field strength as follows

$$\begin{aligned} &i_\kappa E^\alpha \equiv \delta_\kappa Z^{\mathcal{M}} E^\alpha_{\mathcal{M}} = \frac{1}{2}(1+\bar{\Gamma})^\alpha{}_\beta \kappa^\beta, \quad i_\kappa E^A \equiv \delta_\kappa Z^{\mathcal{M}} E^A_{\mathcal{M}} = 0. \\ &\delta g_{\mu\nu} = -4i E^\alpha_{(\mu}(\Gamma_{\nu)})_{\alpha\beta}\, i_\kappa E^\beta, \quad \delta H^{(3)} = i_\kappa dC^{(3)}, \quad \delta_\kappa a(x) = 0\,, \end{aligned} \tag{4.3.9}$$

where $(1+\bar{\Gamma})/2$ is the projector of rank 16 with $\bar{\Gamma}$ having the following form

$$\begin{aligned} \sqrt{\det(\delta^\nu_\mu + i\tilde{H}_\mu{}^\nu)}\,\bar{\Gamma} &= \gamma^{(6)} - \frac{1}{2}\Gamma^{\mu\nu\lambda}P_\mu{}^\rho \tilde{H}_{\nu\lambda\rho} - \frac{1}{16\sqrt{-g}}\epsilon^{\mu_1\cdots\mu_6}\tilde{H}_{\mu_1\mu_2\lambda}\tilde{H}_{\mu_3\mu_4\rho}P^{\lambda\rho}\Gamma_{\mu_5\mu_6}, \\ \bar{\Gamma}^2 &= 1\,, \qquad \mathrm{tr}\bar{\Gamma} = 0, \end{aligned} \tag{4.3.10}$$

where

$$\Gamma_\mu = E_\mu{}^A\Gamma_A\,, \qquad \gamma^{(6)} = \frac{1}{6!\sqrt{-g}}\epsilon^{\mu_1\cdots\mu_6}\Gamma_{\mu_1\cdots\mu_6}\,. \tag{4.3.11}$$

4.3.2 New M5–brane action

For this case, to ensure worldvolume covariance of the construction, instead of the single scalar field we need to introduce a triplet of auxiliary scalar fields $a^s(x)$ with the index $(s = 1, 2, 3)$ labeling a 3-dimensional representation of $GL(3)$ which is an *internal* global symmetry of the action. The partial derivatives of the scalars are used to construct the projector matrices [76]

$$P_\mu{}^\nu = \partial_\mu a^r Y^{-1}_{rs} \partial^\nu a^s, \qquad \Pi_\mu{}^\nu = \delta^\nu_\mu - P_\mu{}^\nu, \qquad \Pi_\mu{}^\nu \, \partial_\nu a = 0 \tag{4.3.12}$$

with Y^{-1}_{rs} being the inverse matrix of

$$Y^{rs} \equiv \partial_\lambda a^r \partial_\rho a^s g^{\lambda\rho}. \tag{4.3.13}$$

The projectors identically satisfy the following differential condition

$$\Pi_{[\rho}{}^\lambda \Pi_{\kappa]}{}^\mu D_\lambda P^\nu_\mu = 0 = \Pi_{[\rho}{}^\lambda \Pi_{\kappa]}{}^\mu D_\lambda \Pi^\nu_\mu \tag{4.3.14}$$

where D_μ is the worldvolume covariant derivative with respect to the induced metric $g_{\mu\nu}$.

Note that the projectors (4.3.12) have rank 3 and thus effectively split the 6d directions into 3+3 ones orthogonal to each other.

The new M5–brane action coupled to a curved superbackground has the following form

$$S = \int_{\mathcal{M}_6} d^6x \left[-\frac{\sqrt{-g}}{6} (\tilde{G}^{\mu\nu\rho} G_{\mu\nu\rho} + 3\tilde{F}^{\mu\nu\rho} F_{\mu\nu\rho}) + 2\mathcal{L}_{M5}(F, G) \right] - \int_{\mathcal{M}_6} (C_6 + H \wedge C_3) \,, \tag{4.3.15}$$

where

$$\begin{aligned} \mathcal{L}_{M5} \;\; = \;\; & -\frac{1}{36(1+G^2)} \epsilon^{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6} G_{\mu_1\mu_2\mu_3} F_{\mu_4\nu\lambda} F_{\mu_5}{}^{\lambda\kappa} F_{\mu_6\kappa}{}^\nu \\ & + \frac{1}{1+G^2} \sqrt{-\det \left(g_{\mu\nu} + \frac{1}{2} (F+G)_{\mu\rho\sigma} (F+G)_\nu{}^{\rho\sigma} \right)}) \end{aligned} \tag{4.3.16}$$

and $F_{\mu\nu\rho}$ and $G_{\mu\nu\rho}$ are components of the field strength $H_{\mu\nu\rho}$ projected as follows

$$F_{\mu\nu\rho} \equiv H_{\tau\sigma\lambda} P^\tau_\mu \Pi^\sigma_\nu \Pi^\lambda_\rho, \quad G_{\mu\nu\rho} \equiv H_{\tau\sigma\lambda} \Pi^\tau_\mu \Pi^\sigma_\nu \Pi^\lambda_\rho, \quad G^2 \equiv \frac{1}{6} H_{\mu\nu\rho} \Pi^\mu_\tau \Pi^\nu_\sigma \Pi^\rho_\lambda H^{\tau\sigma\lambda}, \tag{4.3.17}$$

$$\tilde{F}_{\mu\nu\rho} \equiv \tilde{H}_{\tau\sigma\lambda} P^\tau_\mu \Pi^\sigma_\nu \Pi^\lambda_\rho, \qquad \tilde{G}_{\mu\nu\rho} \equiv \tilde{H}_{\tau\sigma\lambda} \Pi^\tau_\mu \Pi^\sigma_\nu \Pi^\lambda_\rho, \; . \tag{4.3.18}$$

The action enjoys the following two local gauge symmetries analogous to eqs. (4.3.5) and (4.3.6). The first one is

$$\delta B_{\mu\nu} = P^{\rho}_{\mu} P^{\sigma}_{\nu} \Phi_{\rho\sigma}(x), \qquad \delta a^s = 0, \tag{4.3.19}$$

where $\Phi_{\rho\sigma}(x)$ are arbitrary parameters. Note that in view of the conditions (4.3.14) it follows that the projected field strengths (4.3.17), and hence $\mathcal{L}_{M5}(G, F)$, are invariant under this symmetry

$$\delta_\Phi G_{\mu\nu\rho} = \delta_\Phi F_{\mu\nu\rho} = 0, \tag{4.3.20}$$

while their dual (4.3.18) are not.

The second symmetry ensures the triplet of the scalar fields $a^s(x)$ to be auxiliary

$$\delta a^s = \varphi^s(x), \quad \delta B_{\mu\nu} = \frac{1}{2}\varphi^r Y^{-1}_{rs} \partial^\rho a^s \, \epsilon_{\mu\nu\rho\tau\sigma\lambda} \left(\sqrt{-g} \tilde{F}^{\tau\sigma\lambda} - \frac{\partial \mathcal{L}_{M5}}{\partial F_{\tau\sigma\lambda}} \right), \tag{4.3.21}$$

where $\varphi^s(x)$ are local parameters[2].

This symmetry allows one to gauge fix $a^s(x)$ to coincide with three world–sheet coordinates, e.g. x^a $(a = 0, 1, 2)$ or x^i $(i = 3, 4, 5)$, thus getting a non–covariant but non–manifestly worldsheet diffeomorphism invariant M5–brane action. For instance, let us impose the gauge fixing condition

$$a^s = \delta^s_a \, x^a \,, \tag{4.3.22}$$

identifying a^s with x^a. Then the following combination of the worldvolume diffeomorphism $\delta x^\mu = \xi^\mu(x)$ and the local symmetry (4.3.21) leaves this gauge condition intact

$$\delta a^s(x) = \xi^\mu(x) \partial_\mu a^s + \varphi^s(x) = \xi^s(x) + \varphi^s(x) = 0, \quad \rightarrow \quad \varphi^s(x) = -\xi^s(x).$$

[2]In what follows we will use a normalization of the functional derivative, denoted by $\frac{\partial \mathcal{L}(F)}{\partial F_{\mu\nu\dots}}$, which differs from the one defined in (4.3.7). Namely, by definition the variation of a p–form $F_{\mu_1\cdots\mu_p}$ and the corresponding functional derivatives are defined as follows $\delta F_{\mu_1\cdots\mu_p} = \delta F_{\nu_1\cdots\nu_p} \frac{\delta F_{\mu_1\cdots\mu_p}}{\delta F_{\nu_1\cdots\nu_p}} = \delta F_{\nu_1\cdots\nu_p} \frac{1}{p!} \frac{\partial F_{\mu_1\cdots\mu_p}}{\partial F_{\nu_1\cdots\nu_p}}$. So that

$$\frac{\partial \mathcal{L}}{\partial F_{\nu_1\cdots\nu_p}} \equiv p! \, \frac{\delta \mathcal{L}}{\delta F_{\nu_1\cdots\nu_p}}.$$

Under the local transformation combined of the $6d$ diffeomorphism $\delta x^\mu = \xi^\mu(x)$ and the local variation (4.3.21) with $\varphi^a(x) = -\xi^a(x)$ the gauge field $B_{\mu\nu}$ transforms as follows

$$\Delta_{\xi^\mu} B_{\mu\nu} = \delta_{\xi^\mu} B_{\mu\nu} - \frac{1}{2}\xi^a(x)\,\epsilon_{a\mu\nu\tau\sigma\lambda}\left(\sqrt{-g}\tilde{F}^{\tau\sigma\lambda} - \frac{\partial\mathcal{L}_{M5}}{\partial F_{\tau\sigma\lambda}}\right),$$

while the other M5–brane fields $X^M(x)$ and $\theta^\alpha(x)$ being transformed in the conventional way as worldvolume scalars. In the gauge (4.3.22) the action (4.3.15), (4.3.16) is non–manifestly invariant under the modified worldvolume diffeomorphisms of the above form.

Upon tedious computations we have checked that the action is invariant under the kappa–symmetry transformations (4.3.9) but with a $\bar{\Gamma}$ projector which has the following form

$$\begin{aligned}\frac{1}{1+G^2}\sqrt{\det\left(\delta^\nu_\mu + \frac{1}{2}(F+G)_{\mu\rho\sigma}(F+G)^{\nu\rho\sigma}\right)}\,\bar{\Gamma} = \\ = \gamma^{(6)} + \frac{1}{6}\gamma^{(6)}(3F+G)^{\mu\nu\rho}\Gamma_{\mu\nu\rho} + \frac{1}{2(1+G^2)}\gamma^{(6)}F^{\mu\nu\tau}F^{\rho\lambda}{}_\tau\Gamma_{\mu\nu\rho\lambda} \\ + \frac{1}{6(1+G^2)}\gamma^{(6)}\Gamma^{\mu\nu\rho}\big(3(FFG)_{\mu\nu\rho} + (FFF)_{\mu\nu\rho}\big),\end{aligned} \tag{4.3.23}$$

where

$$(FFG)_{\mu\nu\rho} \equiv F_\mu{}^{\tau\sigma}F_{\nu\sigma\lambda}G_\rho{}^\lambda{}_\tau, \quad (FFF)_{\mu\nu\rho} \equiv F_\mu{}^{\tau\sigma}F_{\nu\sigma\lambda}F_\rho{}^\lambda{}_\tau. \tag{4.3.24}$$

Note that the term multiplying $\bar{\Gamma}$ on the left hand side of (4.3.23) is equal (modulo $\sqrt{-\det g_{\mu\nu}}$) to the last term of the non–linear part (4.3.16) of the M5–brane Lagrangian.

Finally, the non–linear self–duality condition which is obtained from action (4.3.15) as the consequence of the equations of motion of B_2 (see eq. (4.4.7) of the next Section) has the following form

$$\tilde{G}^{\mu\nu\rho} = \frac{1}{\sqrt{-g}}\left(\frac{\partial\mathcal{L}_{M5}}{\partial G}\right)^{\mu\nu\rho}, \qquad \tilde{F}^{[\mu\nu\rho]} = \frac{1}{\sqrt{-g}}\left(\frac{\partial\mathcal{L}_{M5}}{\partial F}\right)^{[\mu\nu\rho]}. \tag{4.3.25}$$

As we will show, this self–duality condition is related to eq. (4.3.8) via the manifestly covariant self–duality relation which comes from the superembedding approach [83].

4.4 Derivation and check of the new M5–brane action

To get the new M5–brane action (4.3.15) we start from the covariant form [76] of the quadratic action [75] for the 6d chiral field. It is obtained from (4.3.15) by truncating the latter to the second order in the chiral field strength H_3 [3]

$$\begin{aligned} S &= \frac{1}{6}\int d^6x\,\sqrt{-\det g_{\mu\nu}}\,(H-\tilde{H})_{\mu\nu\rho}(\Pi_\lambda{}^\mu\Pi_\kappa{}^\nu\Pi_\tau{}^\rho + 3\Pi_\lambda{}^\mu\Pi_\kappa{}^\nu P_\tau{}^\rho)H^{\lambda\kappa\tau} \\ &\equiv \frac{1}{6}\int d^6x\,\sqrt{-\det g_{\mu\nu}}\,[(G-\tilde{G})_{\mu\nu\rho}G^{\mu\nu\rho} + 3(F-\tilde{F})_{\mu\nu\rho}F^{\mu\nu\rho}]\,, \end{aligned} \tag{4.4.1}$$

The action is invariant under the symmetry (4.3.19) and under the linearized counterpart of (4.3.21)

$$\delta a^s = \varphi^s(x), \quad \delta B_{\mu\nu} = \frac{1}{2}\varphi^s Y^{-1}_{sr}\partial^\rho a^r\,\epsilon_{\mu\nu\rho\tau\sigma\lambda}\sqrt{-g}\left(\tilde{F}^{\tau\sigma\lambda} - F^{\tau\sigma\lambda}\right). \tag{4.4.2}$$

The quadratic action leads to the equation of motion

$$\partial_\rho\left(\sqrt{-g}(G-\tilde{G})^{\mu\nu\rho} + 3\sqrt{-g}(F-\tilde{F})^{[\mu\nu\rho]}\right) = 0, \tag{4.4.3}$$

which has the general solution

$$\sqrt{-g}(G-\tilde{G})^{\mu\nu\rho} + 3\sqrt{-g}(F-\tilde{F})^{[\mu\nu\rho]} = \frac{1}{2}\epsilon^{\mu\nu\rho\tau\sigma\lambda}\partial_\tau\left[\sqrt{-g}\tilde{\Phi}_{\eta\xi}P^\eta_\sigma P^\xi_\lambda\right], \tag{4.4.4}$$

for some arbitrary integration constant $\tilde{\Phi}_{\eta\xi}$. This integration constant can be compensated by a gauge transformation of the equation of motion under (4.3.19) with gauge parameter $-\tilde{\Phi}_{\xi\eta}$. Hence, in view of the definition of the projected components of the field strength (4.3.17), the solution of the dynamical equation is equivalent to the self-duality conditions

$$(G-\tilde{G})^{\mu\nu\rho} = 0, \quad (F-\tilde{F})^{[\mu\nu\rho]} = 0. \tag{4.4.5}$$

We are now looking for a non–linear generalization of the action (4.4.1) which would respect the both symmetries (4.3.19) and (4.3.21). Note that the second

[3] For simplicity, but without loss of generality, we consider (for a moment) the pullbacks of the $11D$ gauge fields be zero.

symmetry should be deformed by the non–linear terms, since the form of its transformation is associated with the form of the non–linear self–duality condition. In the case of the M5–brane these are (4.3.6)–(4.3.8), and (4.3.21) and (4.3.25).

Since the field strength components $F_{\mu\nu\rho}$ and $G_{\mu\nu\rho}$ are invariant under the transformations (4.3.19) (see eqs. (4.3.20)), while their dual (4.3.18) are not, the non–linear terms in the action should only depend on F and G. So the general form of the non–linear action which respects the symmetry (4.3.19) is obtained by replacing the quadratic terms FF and GG in (4.4.1) by an arbitrary function $\mathcal{L}(F,G)$

$$S=\int_{\mathcal{M}_6} d^6x\,\left(-\frac{\sqrt{-g}}{6}(\tilde{G}^{\mu\nu\rho}G_{\mu\nu\rho}+3\tilde{F}^{\mu\nu\rho}F_{\mu\nu\rho})+2\mathcal{L}(F,G)\right). \tag{4.4.6}$$

The variation of this action with respect to the gauge potential B_2 produces the equations of motion

$$\partial_\rho\left[\left(\frac{\partial\mathcal{L}}{\partial G}\right)^{\mu\nu\rho}-\sqrt{-g}\tilde{G}^{\mu\nu\rho}+3\left(\frac{\partial\mathcal{L}}{\partial F}\right)^{[\mu\nu\rho]}-3\sqrt{-g}\tilde{F}^{[\mu\nu\rho]}\right]=0. \tag{4.4.7}$$

In view of (4.3.20) and the fact that $\mathcal{L}$ only depends on F and G, we can integrate the above equation of motion with the help of the symmetry (4.3.19) along the same lines as in free theory. The integration produces the non–linear self–duality relations

$$\tilde{G}^{\mu\nu\rho}=\frac{1}{\sqrt{-g}}\left(\frac{\partial\mathcal{L}}{\partial G}\right)^{\mu\nu\rho},\qquad \tilde{F}^{[\mu\nu\rho]}=\frac{1}{\sqrt{-g}}\left(\frac{\partial\mathcal{L}}{\partial F}\right)^{[\mu\nu\rho]}. \tag{4.4.8}$$

We should now find conditions on the form of $\mathcal{L}(F,G)$ imposed by the requirement that the action is invariant under

$$\delta a^s=\varphi^s(x),\quad \delta B_{\mu\nu}=\frac{1}{2}\varphi^s Y^{-1}_{sr}\partial^\rho a^r\,\epsilon_{\mu\nu\rho\tau\sigma\lambda}\left(\sqrt{-g}\tilde{F}^{\tau\sigma\lambda}-\frac{\partial\mathcal{L}}{\partial F_{\tau\sigma\lambda}}\right). \tag{4.4.9}$$

Upon somewhat lengthy calculations using, in particular, the properties of the projectors (4.3.12)–(4.3.14) and the form of their variation under (4.4.9)

$$\delta_\varphi P_{\mu\nu}=2\Pi_{\rho(\mu}\partial^\rho\varphi^r Y^{-1}_{rs}\partial_{\nu)}a^s \tag{4.4.10}$$

we get the following condition on $\mathcal{L}(F,G)$

$$\partial_\mu\left[Y^{-1}_{rs}\partial^\nu a^s\left(\sqrt{-g}\left(\frac{\partial\mathcal{L}}{\partial G}\right)^{\mu\tau\sigma}F_{\nu\tau\sigma}-\sqrt{-g}G^{\mu\tau\sigma}\left(\frac{\partial\mathcal{L}}{\partial F}\right)_{\nu\tau\sigma}\right.\right.$$
$$\left.\left.-\frac{g}{2}\epsilon_{\nu\tau\sigma\lambda\xi\eta}F^{\lambda\xi\eta}F^{\tau\sigma\mu}-\frac{1}{2}\epsilon_{\nu\tau\sigma\lambda\xi\eta}\left(\frac{\partial\mathcal{L}}{\partial F}\right)^{\lambda\xi\eta}\left(\frac{\partial\mathcal{L}}{\partial F}\right)^{\tau\sigma\mu}\right)\right]=0. \tag{4.4.11}$$

This condition is analogous to those found in other instances of models with non–linear (twisted) self–duality, e.g in $D = 6$ [90] and $D = 4$ [169,172]. It is well known that these conditions may have different solutions leading to different non–linear generalizations of quadratic duality–symmetric actions (see e.g. [90, 169, 172, 200, 201]). We are interested in a particular solution of the above equation, i.e. in the form of $\mathcal{L}(F, G)$ which describes the $M5$–brane. To find this form we assume that, as in the case of the self–duality condition (4.3.8) obtained from the original M5–brane action, also the self–duality conditions (4.3.25) (or (4.4.8)) should be equivalent to the self–duality conditions appearing in the superembedding formulation of the M5–brane [83]. Exploring these conditions we shall derive the form (4.3.16) of the non–linear M5–brane Lagrangian.

4.4.1 Non–linear self–duality of the M5–brane in the superembedding approach

In the superembedding description of the $M5$–brane [83,84] the field strength H_3 of the chiral field B_2 is expressed in terms of a self–dual tensor $h_3 = *h_3$ as follows[4]

$$\frac{1}{4}H_{\mu\nu\rho} = m_\mu^{-1\lambda} h_{\lambda\nu\rho}\,, \qquad \frac{1}{4}\tilde{H}^{\mu_1\nu_1\rho_1} = \frac{1}{6}\epsilon^{\mu_1\nu_1\rho_1\mu\nu\rho} m_\mu^{-1\lambda} h_{\lambda\nu\rho} = Q^{-1} m^{\mu_1\lambda} h_\lambda{}^{\nu_1\rho_1} \tag{4.4.12}$$

where $m_\mu^{-1\lambda}$ is the inverse matrix of

$$m_\mu{}^\lambda = \delta_\mu{}^\lambda - 2k_\mu{}^\lambda\,, \qquad m_\mu^{-1\lambda} = Q^{-1}(2\delta_\mu{}^\lambda - m_\mu{}^\lambda), \qquad k_\mu{}^\lambda = h_{\mu\nu\rho}h^{\lambda\nu\rho} \tag{4.4.13}$$

and

$$Q = 1 - \frac{2}{3}\mathrm{tr}\, k^2\,. \tag{4.4.14}$$

As was shown in [86], by splitting the indices in eqs. (4.4.12) into 1+5 and expressing components of h_3 in terms of $\tilde{H}_{\mu\nu 5}$ one gets the duality relation (4.3.8)[5].

[4] Our normalization of the field strength differs from that in [86] by the factor of $\frac{1}{4}$ in front of H_3.

[5] This splitting is amount to projecting the tensor fields along the direction of $\partial_\mu a$ and orthogonal to it.

We shall now carry out a similar procedure, but splitting the 6d indices into 3+3, and upon a somewhat lengthy algebra will arrive at the self–duality condition in the form of (4.3.25), thus getting the non–linear function $\mathcal{L}_{M5}(F,G)$ (4.3.16) which enters the M5–brane action (4.3.15).

The 3+3 splitting can be performed with the use of the projectors (4.3.12), but for computational purposes we have found it more convenient to pass to a local tangent–space frame using $6d$ vielbeins e_μ^m ($e_\mu^m \eta_{mn} e_\nu^n = g_{\mu\nu}$) and to write the 3+3 tangent space indices explicitly. So the three directions singled out by the projector $P_m{}^n \equiv e_m^\mu P_\mu{}^\nu e_\nu^n$, which we assume to contain the time direction, will be labeled by the indices a, b, c, and the three spacial directions singled out by $\Pi_m{}^n \equiv e_m^\mu \Pi_\mu{}^\nu e_\nu^n$ will be labeled by i, j, k:

$$P_m{}^n \to \delta_a^b\,, \qquad \Pi_m{}^n \to \delta_i^j\,, \qquad a,b,c = 0,1,2; \qquad i,j,k = 3,4,5\,, \tag{4.4.15}$$

while the $6d$ Levi–Civita tensor splits as follows

$$\epsilon^{\mu\nu\rho\lambda\tau\kappa} \;\Rightarrow \epsilon^{abc}\epsilon^{ijk}, \qquad \epsilon^{012} = -\epsilon_{012} = 1, \qquad \epsilon^{345} = 1\,. \tag{4.4.16}$$

We are now ready to split the indices of H_3 and h_3 in (4.4.12).

3+3 splitting

As h_3 is self–dual, we pick its 10 independent components in the local Lorentz frame as follows

$$h_{ija}, \quad h_{ijk} \tag{4.4.17}$$

and define[6]

$$f_a^k \equiv \frac{1}{2}\epsilon^{ijk} h_{ija}, \quad g \equiv \frac{1}{6}\epsilon^{ijk} h_{ijk}. \tag{4.4.18}$$

In view of the self–duality

$$h^{mnp} = \frac{1}{3!}\epsilon^{mnpl_1l_2l_3} h_{l_1l_2l_3}, \tag{4.4.19}$$

we have

$$h^{jab} = -\epsilon^{abc} f_c^j, \quad h^{abc} = g\epsilon^{abc}, \tag{4.4.20}$$

[6]One should not confuse the field $g(x)$ with the determinant of the induced metric $g_{\mu\nu}$.

or

$$f_{ic} = \frac{1}{2}\epsilon_{abc}h_i{}^{ab} \qquad g = -\frac{1}{6}\epsilon_{abc}h^{abc}. \tag{4.4.21}$$

The corresponding components of H_3 are defined as

$$F_a^k \equiv \frac{1}{2}\epsilon^{ijk}H_{ija}, \quad G \equiv \frac{1}{6}\epsilon^{ijk}H_{ijk}. \tag{4.4.22}$$

The duals of F and G are

$$\tilde{F}_{ic} \equiv \frac{1}{2}\epsilon_{abc}H_i{}^{ab}, \quad \tilde{G} \equiv -\frac{1}{6}\epsilon_{abc}H^{abc}. \tag{4.4.23}$$

Note that the tensors (4.4.22) and (4.4.23) are counterparts of (4.3.17) and (4.3.18) in the local Lorentz frame (4.4.15).

Our final goal is to write $\tilde{F}, \tilde{G}$ in terms of F, G using the relations (4.4.12). To this end, using (4.4.12) we first find the expressions for $F, G, \tilde{F}$ and $\tilde{G}$ in terms of g and f_i^a

$$\begin{aligned}\frac{1}{4}F_i^a &= Q^{-1}\left(f(1+4g^2-4\mathrm{tr}f^2)+8f^3-8gf^{-1}\det f\right)_i^a \\ &= Q^{-1}\frac{\partial}{\partial f_a^i}\left(\frac{1}{2}(g^2+\mathrm{tr}f^2)-\frac{1}{16}Q\right),\end{aligned} \tag{4.4.24}$$

$$\begin{aligned}\frac{1}{4}G &= Q^{-1}\left(g+4g^3+4g\mathrm{tr}f^2-8\det f\right) \\ &= Q^{-1}\frac{\partial}{\partial g}\left(\frac{1}{2}(g^2+\mathrm{tr}f^2)-\frac{1}{16}Q\right),\end{aligned} \tag{4.4.25}$$

and

$$\begin{aligned}\frac{1}{4}\tilde{F}_i^a &= Q^{-1}\left(f(1-4g^2+4\mathrm{tr}f^2)-8f^3+8gf^{-1}\det f\right)_i^a \\ &= Q^{-1}\frac{\partial}{\partial f_a^i}\left(\frac{1}{2}(g^2+\mathrm{tr}f^2)+\frac{1}{16}Q\right),\end{aligned} \tag{4.4.26}$$

$$\begin{aligned}\frac{1}{4}\tilde{G} &= Q^{-1}\left(g-4g^3-4g\mathrm{tr}f^2+8\det f\right) \\ &= Q^{-1}\frac{\partial}{\partial g}\left(\frac{1}{2}(g^2+\mathrm{tr}f^2)+\frac{1}{16}Q\right),\end{aligned} \tag{4.4.27}$$

where

$$Q = 1-16g^4+16(\mathrm{tr}f^2)^2-32g^2\mathrm{tr}f^2-32\mathrm{tr}f^4+128\,g\det f, \tag{4.4.28}$$

$$\mathrm{tr}f^2 \equiv f_i^a f_j^b \delta^{ij}\eta_{ab}\,, \qquad \det f \equiv \frac{1}{6}\epsilon_{ijk}\epsilon^{abc}f_a^i f_b^j f_c^k\,, \qquad (f^{-1})_i^a \det f \equiv \frac{1}{2}\epsilon_{ijk}\epsilon^{abc}f_b^j f_c^k\,. \tag{4.4.29}$$

The M5–brane action in terms of G and F_a^i

For the fields (4.4.22) and (4.4.23) the M5–brane action (4.3.15) takes the following form

$$S_{3+3} = -\int d^6x \left(\sqrt{-\det g_{\mu\nu}}(F_a^i \tilde{F}_i^a + G\tilde{G}) - 2\mathcal{L}_{M5}\right) - \int_{\mathcal{M}_6} (C_6 + H \wedge C_3)\,, \quad (4.4.30)$$

where the term $\mathcal{L}_{M5}$ is

$$\mathcal{L}_{M5} = \sqrt{-\det g_{\mu\nu}} \left(\frac{G \det F}{1+G^2} + \frac{\sqrt{\det\left(\delta_i^j(1+G^2) + F_i^a F_a^j\right)}}{1+G^2} \right), \quad (4.4.31)$$

and the non–linear self–duality relations (4.3.25) become

$$\tilde{G} = \frac{1}{\sqrt{-\det g_{\mu\nu}}} \frac{\partial \mathcal{L}_{M5}}{\partial G}\,, \qquad \tilde{F}_i^a = \frac{1}{\sqrt{-\det g_{\mu\nu}}} \frac{\partial \mathcal{L}_{M5}}{\partial F_a^i}. \quad (4.4.32)$$

Self–duality relations in particular cases

To guess the form (4.4.31) of the function $\mathcal{L}_{M5}$ in the M5-brane action we first considered a number of simple cases.

$f = 0$ case

The relations (4.4.24)-(4.4.28) reduce to

$$F_i^a = \tilde{F}_i^a = 0\,, \qquad Q = 1 - 16g^4,$$

$$\frac{1}{4}G = \frac{g + 4g^3}{1 - 16g^4} = \frac{g}{1 - 4g^2}, \quad (4.4.33)$$

$$\frac{1}{4}\tilde{G} = \frac{g - 4g^3}{1 - 16g^4} = \frac{g}{1 + 4g^2}. \quad (4.4.34)$$

We now solve eq. (4.4.33) for g

$$g = \frac{\pm\sqrt{1+G^2} - 1}{2G}\,. \quad (4.4.35)$$

Since, due to (4.4.33), in the linear approximation $G/4 = g$, we should pick up only the solution with the upper sign. Substituting this solution into (4.4.34) we get the relation between $\tilde{G}$ and G

$$\tilde{G} = \frac{G}{\sqrt{1+G^2}} = \frac{\partial\sqrt{1+G^2}}{\partial G}\,. \quad (4.4.36)$$

We see that eq. (4.4.36) is exactly the same as (4.4.32) when in (4.4.31) we put $F_i^a = 0 = F_{\mu\nu\rho}$. This demonstrates how the (square root of) factor $1 + G^2$ appears in the function $\mathcal{L}_{M5}(F, G)$ (4.3.16) or (4.4.31) of the M5–brane action (4.3.15).

$g = \det f = 0$ **case**

Now the relations (4.4.24)–(4.4.28) reduce to

$$G = \tilde{G} = 0\,, \qquad Q = 1 + 16(\mathrm{tr}f^2)^2 - 32\mathrm{tr}f^4,$$

$$\frac{1}{4}F_i^a = Q^{-1}\left(f(1 - 4\mathrm{tr}f^2) + 8f^3\right)_i^a = Q^{-1}\frac{\partial}{\partial f_a^i}\left(\frac{1}{2}\mathrm{tr}f^2 - \frac{1}{16}Q\right), \quad (4.4.37)$$

$$\frac{1}{4}\tilde{F}_i^a = Q^{-1}\left(f(1 + 4\mathrm{tr}f^2) - 8f^3\right)_i^a = Q^{-1}\frac{\partial}{\partial f_a^i}\left(\frac{1}{2}\mathrm{tr}f^2 + \frac{1}{16}Q\right), \quad (4.4.38)$$

Let us simplify things even further by considering a solution of the non–linear self–duality equation such that the only non–zero components of f_i^a are f_i^1. Then the above equations further reduce to

$$G = \tilde{G} = 0\,, \qquad Q = 1 - 16(f^2)^2, \qquad f^2 \equiv f_i^1 f_i^1\,,$$

$$\frac{1}{4}F_i^1 = Q^{-1}\left(1 + 4f^2\right) f_i^1 = \frac{f_i^1}{1 - 4f^2}, \quad (4.4.39)$$

$$\frac{1}{4}\tilde{F}_i^1 = \frac{f_i^1}{1 + 4f^2}, \quad (4.4.40)$$

From these equations we find that

$$1 - 4f^2 = -\frac{2(1 \mp \sqrt{1 + F^2})}{F^2}\,, \qquad 1 + 4f^2 = \frac{2\sqrt{1 + F^2}}{F^2}(\sqrt{1 + F^2} \mp 1)\,,$$

$$f_i^1 = \frac{F_i^1}{2F^2}(\pm\sqrt{1 + F^2} - 1).$$

Since, due to (4.4.39), in the linear approximation $F_i^a/4 = f_i^a$, in the above relation we should pick the upper sign and upon substituting it into (4.4.38) we get the duality relation

$$\tilde{F}_i^1 = \frac{F_i^1}{\sqrt{1 + F^2}} = \frac{\partial\sqrt{1 + F^2}}{\partial F_1^i}\,. \quad (4.4.41)$$

We see that this relation coincides with (4.4.32) for $G = 0$ and F_i^a having only the non–zero components F_i^1.

Self–dual string soliton ($g \neq 0, \det f = 0$ case)

Let us now consider a more complicated particular case of a string soliton solution of [90]. A similar consideration is applicable to the BPS self–dual string of [46]. For the string aligned along the x^2–coordinate, in terms of fields (4.4.22) and (4.4.23) the string soliton solution of [90] has the following form:

$$G = -\frac{\beta x^1}{\rho^4}, \qquad F_1^k = -\frac{\beta x^k}{\rho^4}, \tag{4.4.42}$$

$$\tilde{G} = -\frac{\alpha' x^1}{\rho}, \qquad \tilde{F}_1^k = -\frac{\alpha' x^k}{\rho}. \tag{4.4.43}$$

where $k = 3, 4, 5$, $\rho := \sqrt{x_1^2 + x_3^2 + x_4^2 + x_5^2}$, β is a constant and

$$\alpha'(\rho) = \frac{\beta}{\sqrt{\beta^2 + \rho^6}}. \tag{4.4.44}$$

In this form the string soliton solution was considered in [47]. It naturally splits the $6d$ worldvolume into 3+3 directions.

The form (4.4.42) of G and F suggests that in (4.4.24) and (4.4.25) $g \neq 0$ and the non–zero components of f_i^a are f_i^1. So the equations (4.4.24)–(4.4.28) reduce to

$$Q = 1 - 16(g^2 + f^2)^2. \tag{4.4.45}$$

$$\frac{1}{4}F_i^1 = \frac{f_i^1}{1 - 4(g^2 + f^2)}, \qquad \frac{1}{4}G = \frac{g}{1 - 4(g^2 + f^2)}, \tag{4.4.46}$$

and

$$\frac{1}{4}\tilde{F}_i^1 = \frac{f_i^1}{1 + 4(g^2 + f^2)}, \qquad \frac{1}{4}\tilde{G} = \frac{g}{1 + 4(g^2 + f^2)}. \tag{4.4.47}$$

Carrying out the same analysis as in the previous examples, from (4.4.45)–(4.4.47) we get the duality relations

$$\tilde{F}_i^1 = \frac{F_i^1}{\sqrt{1 + G^2 + F^2}} = \frac{\partial\sqrt{1 + G^2 + F^2}}{\partial F_1^i}, \qquad \tilde{G} = \frac{G}{\sqrt{1 + G^2 + F^2}} = \frac{\partial\sqrt{1 + G^2 + F^2}}{\partial G} \tag{4.4.48}$$

which are again a particular case of (4.4.32). One can then guess that in the manifestly covariant formulation the expression under the square root combines into the determinant of the matrix formed by the bilinear combinations of $G_{\mu\nu\rho}$ and $F_{\mu\nu\rho}$ as in eq. (4.3.16) or (4.4.31).

To see that this is indeed so and that (4.4.31) should also contain the term $G \det F$ let us consider the case in which $G = 0$ while F_i^a is (otherwise) generic.

$G = 0$ case

We have

$$G = 0 = \left(g + 4g^3 + 4g\mathrm{tr}f^2 - 8\det f\right) , \tag{4.4.49}$$

$$\frac{1}{4}\tilde{G} = 2Q^{-1}g\,, \tag{4.4.50}$$

$$\begin{aligned} Q &= 1 - 16g^4 + 16(\mathrm{tr}f^2)^2 - 32g^2\mathrm{tr}f^2 - 32\mathrm{tr}f^4 + 128g\det f \\ &= 1 + 16g^2 + 16(\mathrm{tr}f^2)^2 + 48g^4 + 32g^2\mathrm{tr}f^2 - 32\mathrm{tr}f^4, \end{aligned} \tag{4.4.51}$$

$$\frac{1}{4}F_i^a = Q^{-1}\left(f(1 + 4g^2 - 4\mathrm{tr}f^2) + 8f^3 - 8gf^{-1}\det f\right)_i^a \tag{4.4.52}$$

and

$$\frac{1}{4}\tilde{F}_i^a = Q^{-1}\left(f(1 - 4g^2 + 4\mathrm{tr}f^2) - 8f^3 + 8gf^{-1}\det f\right)_i^a . \tag{4.4.53}$$

Now, the direct computation of $\det F$ using (4.4.52) and (4.4.49) gives (see also eq. (4.4.70) of the Section 4.4.2)

$$\det F = 8Q^{-1}g\,. \tag{4.4.54}$$

Comparing this equation with (4.4.50) we get

$$\tilde{G} = \det F\,, \tag{4.4.55}$$

which is exactly the relation that we get by varying the term (4.4.31) of the M5-brane action (4.3.15) or (4.4.30) with respect to G and setting $G = 0$ afterwards. This explains the appearance of the term $G \det F$ in the M5–brane action.

On the other hand, upon expressing the right–hand side of (4.4.53) in terms of F_i^a and performing somewhat lengthy computations using *Mathematica* one gets the duality relation for $\tilde{F}$ which coincides with eq. (4.4.32) evaluated at $G = 0$.

Finally, by a direct check using *Mathematica* one can verify that also in the generic case the components F, $\tilde{F}$, G and $\tilde{G}$ of the field strength H_3 determined

by the superembedding relations (4.4.24)–(4.4.28) satisfy the non–linear duality relations (4.4.32) which follow from the M5–brane action (4.3.15). Main steps of the calculation will be described in the subsequent subsection 4.4.2.

The last point that one should check is that the function (4.3.16) satisfies eq. (4.4.11) which insures the invariance of the M5–brane action under the local transformations (4.3.21). The direct calculation shows that this is indeed so. Actually, (4.3.16) satisfies even stronger relation, namely, it makes to vanish the expression under the derivative in (4.4.11).

4.4.2 Exact check of the M5–brane action non–linear self–duality from superembedding

To check the form of (4.3.16) (or, equivalently, (4.4.31)), using the superembedding relations (4.4.24)–(4.4.27) we should verify that

$$\tilde{G}(f,g) = 4Q^{-1}\left(g - 4g^3 - 4g\mathrm{tr}f^2 + 8\det f\right) = \frac{1}{\sqrt{-\det g_{\mu\nu}}}\frac{\partial \mathcal{L}_{M5}}{\partial G}\Big(F(f,g), G(f,g)\Big) \quad (4.4.56)$$

and

$$\begin{aligned}\tilde{F}(f,g) &= 4Q^{-1}\left(f(1 - 4g^2 + 4\mathrm{tr}f^2) - 8f^3 + 8gf^{-1}\det f\right)\\ &= \frac{1}{\sqrt{-\det g_{\mu\nu}}}\frac{\partial \mathcal{L}_{M5}}{\partial F}\Big(F(f,g), G(f,g)\Big)\,. \end{aligned} \quad (4.4.57)$$

To verify the above relations, on their right hand sides we should take G– and F–derivatives of $\mathcal{L}_{M5}$ in the form (4.4.31), substitute into the results the expressions (4.4.24) and (4.4.25) for F and G in terms of f and g, and to see that they coincide with the left hand sides of (4.4.56) and (4.4.57), i.e. with $\tilde{G}$ and $\tilde{F}$ expressed in terms of f and g. In particular, we will need to express $\mathrm{tr}(F^2)$, $\mathrm{tr}(F^4)$ and $\det(F)$ in terms of f and g.

The algebra is very involved but it is manageable systematically by *Mathematica*. To this end we used NCAlgebra package [203] which is found in http://math.ucsd.edu/∼ncalg/.

Matrix Notation

To use Mathematica we should properly define the matrices we deal with. Let $F_a{}^i$ be the components of the matrix F, η_{ab} or η^{ab} be the components of the matrix η

and δ_{ij} or δ^{ij} be the component of the matrix δ. It will be clear from the context whether the indices of η and δ are up or down. To simplify the notation, we drop δ from all the matrix expressions.

For example, $F_a{}^j\delta_{jk}F^k{}_b\eta^{bc}F_c{}^i$ is denoted as $F\delta F^T\eta F$ or just $FF^T\eta F$. This expression is what in previous sections we simply referred to as F^3.

The inverse matrix F^{-1} has the components $(F^{-1})_i{}^a$. We will, actually, encounter the adjugate matrix $\mathrm{adj}(F)$ and the cofactor matrix $\mathrm{co}(F) \equiv \mathrm{adj}(F)^T$ more often than F^{-1} and $(F^{-1})^T$. The definition of $\mathrm{adj}(F)$ is

$$\mathrm{adj}(F)_i{}^a \equiv (F^{-1})_i{}^a \det F = \frac{1}{2}\epsilon_{ijk}\epsilon^{abc}F_b{}^jF_c{}^k, \tag{4.4.58}$$

where

$$\det F \equiv \frac{1}{6}\epsilon_{ijk}\epsilon^{abc}F_a{}^iF_b{}^jF_c{}^k. \tag{4.4.59}$$

In the matrix form, the equation (4.4.24) reads

$$F = 4Q^{-1}\left(f(1+4g^2-4\mathrm{tr}(f^T\eta f)) + 8ff^T\eta f - 8g\,\eta\,\mathrm{co}(f)\right) \tag{4.4.60}$$

its transpose is given by

$$F^T = 4Q^{-1}\left(f^T(1+4g^2-4\mathrm{tr}(f^T\eta f)) + 8f^T\eta ff^T - 8g\,\mathrm{adj}(f)\,\eta\right). \tag{4.4.61}$$

and

$$Q = 1 - 16g^4 + 128g\det(f) - 32g^2tr(f^T\eta f) + 16(tr(f^T\eta f))^2 - 32tr(f^T\eta ff^T\eta f). \tag{4.4.62}$$

We are ready to discuss the computation of the expressions $\mathrm{tr}(F^2) \equiv \mathrm{tr}F^T\eta F$, $\mathrm{tr}(F^4) \equiv \mathrm{tr}F^T\eta FF^T\eta F$ and $\det(F)$ in terms of f and g.

Outline of computation

To compute $F^T\eta F$, the following identities are useful to simplify the results:

$$\eta^2 = 1, \tag{4.4.63}$$

$$f\mathrm{adj}(f) = \mathrm{adj}(f)f = \det f, \qquad f^T\mathrm{co}(f) = \mathrm{co}(f)f^T = \det f, \tag{4.4.64}$$

$$\begin{aligned}\mathrm{adj}(f)\eta\mathrm{co}(f) &= -(f^T\eta f)^{-1}\det(f^T\eta f) = -\mathrm{adj}(f^T\eta f)\\ &= -(f^T\eta f)^2 + \mathrm{tr}(f^T\eta f)f^T\eta f - \frac{1}{2}[(\mathrm{tr}(f^T\eta f))^2 - \mathrm{tr}((f^T\eta f)^2)],\end{aligned} \tag{4.4.65}$$

where in the last equality we used the Cayley–Hamilton formula for 3×3 matrices. We also need the Cayley-Hamilton formula of the form

$$(f^T\eta f)^3 = \mathrm{tr}(f^T\eta f)(f^T\eta f)^2 - \frac{1}{2}[(\mathrm{tr}(f^T\eta f))^2 - \mathrm{tr}((f^T\eta f)^2)]f^T\eta f - (\det f)^2. \tag{4.4.66}$$

Using these formulas one can see that each term in the expression for $F^T\eta F$ is proportional to either

$$1, \quad \text{or} \quad f^T\eta f, \quad \text{or} \quad (f^T\eta f)^2. \tag{4.4.67}$$

Therefore,

$$\begin{aligned}\frac{\mathrm{tr}(F^2)}{16Q^{-2}} =& \mathrm{tr}\left(f^2\right) + \left(-48g\det(f) + 16\mathrm{tr}\left(f^4\right) + 8g^2\mathrm{tr}\left(f^2\right) - 8\left(\mathrm{tr}f^2\right)^2\right)\\ &+ \left(64g\det(f)\mathrm{tr}\left(f^2\right) - 192g^3\det(f) - 192(\det f)^2 + 96g^2\mathrm{tr}\left(f^4\right) + 16g^4\mathrm{tr}\left(f^2\right)\right.\\ &\left. -64g^2\left(\mathrm{tr}f^2\right)^2 - 16\left(\mathrm{tr}f^2\right)^3 + 32\mathrm{tr}\left(f^2\right)\mathrm{tr}\left(f^4\right)\right),\end{aligned} \tag{4.4.68}$$

where $\mathrm{tr}(f^2)$ and $\mathrm{tr}(f^4)$ are shorthand for $\mathrm{tr}(f^T\eta f)$ and $\mathrm{tr}((f^T\eta f)^2)$.

We compute $\mathrm{tr}(F^4)$ and $\mathrm{tr}(F^6) \equiv \mathrm{tr}((F^T\eta F)^3)$ using the same method. We finally trade $\mathrm{tr}(F^6)$ with $\det F$ using the Cayley–Hamilton formula

$$\det F = \sqrt{-\frac{1}{6}(\mathrm{tr}(F^2)^3 - 3\mathrm{tr}(F^4)\mathrm{tr}(F^2) + 2\mathrm{tr}(F^6))} \tag{4.4.69}$$

The explicit expression for $\det F$ in terms of f and g looks as follows

$$\begin{aligned}\frac{1}{64}Q^3\det(F) \;=\;& \det(f) + 12g^2\det(f) + 48g^4\det(f) + 64g^6\det(f) + 192g\det(f)^2\\ &+1280g^3\det(f)^2 - 512\det(f)^3 - 4\det(f)\mathrm{tr}(f^2) - 96g^2\det(f)\mathrm{tr}(f^2)\\ &-320g^4\det(f)\mathrm{tr}(f^2) + 256g\det(f)^2\mathrm{tr}(f^2) + 4g(\mathrm{tr}f^2)^2 + 32g^3(\mathrm{tr}f^2)^2\\ &+64g^5(\mathrm{tr}f^2)^2 + 16\det(f)(\mathrm{tr}f^2)^2 + 320g^2\det(f)(\mathrm{tr}f^2)^2 - 64\det(f)(\mathrm{tr}f^2)^3\\ &+64g(\mathrm{tr}f^2)^4 - 4g\mathrm{tr}(f^4) - 32g^3\mathrm{tr}(f^4) - 64g^5\mathrm{tr}(f^4) - 32\det(f)\mathrm{tr}(f^4)\\ &-640g^2\det(f)\mathrm{tr}(f^4) + 128\det(f)\mathrm{tr}(f^2)\mathrm{tr}(f^4) - 192g\,(\mathrm{tr}f^2)^2\mathrm{tr}(f^4)\\ &+128g(\mathrm{tr}f^4)^2,\end{aligned} \tag{4.4.70}$$

We can now compute the expression in terms of f and g of the term in (4.4.31) containing the square root

$$\begin{aligned}&\sqrt{1-\frac{\det(F^2)}{(1+G^2)^2}+(G^2+\mathrm{tr}\,(F^2))+\frac{1}{2}\frac{\left((\mathrm{tr}F^2)^2-\mathrm{tr}\,(F^4)\right)}{1+G^2}}\\ =\;& Q^{-3}(1+G^2)^{-1}\sqrt{\left(\sum_{n=0}^{12}a_n(f)g^n\right)^2}\,,\end{aligned}\tag{4.4.71}$$

where the argument of the square root in the last line, which turns out to form a perfect square, is a polynomial in g with coefficients $a_n(f)$ depending on $\mathrm{tr}(f^2)$, $\mathrm{tr}(f^4)$ and $\det(f)$. The form of these coefficients is rather cumbersome, and we do not give it here. Using the above expressions we can then check that (4.4.56) indeed holds.

We now pass to the check of (4.4.57). In the matrix form it reads

$$\tilde{F}=4Q^{-1}\left(f(1-4g^2+4\mathrm{tr}f^2)-8ff^T\eta f+8g\,\eta\,\mathrm{co}(f)\right)=\frac{1}{\sqrt{-\det g_{\mu\nu}}}\eta\frac{\partial\mathcal{L}_{M5}}{\partial F^T}\,.\tag{4.4.72}$$

This is a matrix equation, and we need to compute F, $FF^T\eta F$, and $\eta\,\mathrm{co}(F)$. To do this, we proceed as above and compute F, $FF^T\eta F$ and $FF^T\eta FF^T\eta F$ and then trade $FF^T\eta FF^T\eta F$ with $\eta\mathrm{co}(F)$ using the relation

$$\eta\mathrm{co}(F)=-\frac{\left(FF^T\eta FF^T\eta F+\frac{1}{2}F\left((\mathrm{tr}F^2)^2-\mathrm{tr}(F^4)\right)-\mathrm{tr}(F^2)FF^T\eta F\right)}{\det F}.\tag{4.4.73}$$

In the final result, the matrices F, $FF^T\eta F$ and $\eta\,\mathrm{co}(F)$ are expressed in terms of g, f, $ff^T\eta f$, and $\eta\,\mathrm{co}(f)$. We can then substitute these into $\partial\mathcal{L}_{M5}/(\sqrt{-detg_{\mu\nu}}\partial F)$ which is given by

$$\frac{1}{\sqrt{-detg_{\mu\nu}}}\eta\frac{\partial\mathcal{L}_{M5}}{\partial F^T}=\frac{G\eta\mathrm{co}(F)}{(1+G^2)}+\frac{-\frac{\det(F)\,\eta\,\mathrm{co}(F)}{(1+G^2)^2}+\frac{\left(F\mathrm{tr}\left(F^2\right)-FF^T\eta F\right)}{1+G^2}+F}{\sqrt{1-\frac{\det\left(F^2\right)}{(1+G^2)^2}+(G^2+\mathrm{tr}\,(F^2))+\frac{1}{2}\frac{\left((\mathrm{tr}F^2)^2-\mathrm{tr}(F^4)\right)}{1+G^2}}}\,,\tag{4.4.74}$$

and check that eq. (4.4.57) does hold.

4.5 Comparison of the two M5–brane actions

As was discussed in [76] duality symmetric actions corresponding to different splittings of space–time differ from each other by terms that vanish on–shell, *i.e.* when

(an appropriate part of) the self–duality relations is satisfied. In [76] this was discussed for the free chiral 2–form in $6d$.

We shall now confront the two M5–brane actions (4.3.3) and (4.3.15) by comparing their values for the 3-form field strength satisfying the non–linear self–duality equation. As we have seen, the non–linear self–duality relations that follow from these actions are similar and are equivalent to the self–duality condition that follows from the superembedding formulation. Therefore, to compute the on–shell values of the M5–brane actions we will substitute into them the expressions of the components of H_3 and $\tilde{H}_3$ in terms of the components of the self–dual tensor h_3.

In the case of the novel action these are eqs. (4.4.24)–(4.4.27). Substituting them into the action (4.3.15) (or (4.4.30)) and using Mathematica we find that the on–shell value of the self–dual M5–brane action is

$$S_{M5}^{\text{on-shell}} = 4\int d^6x\sqrt{-\det g_{\mu\nu}}\; Q^{-1} - \int_{\mathcal{M}_6}(C_6 + H\wedge C_3)\,. \tag{4.5.1}$$

Notice that the Lagrangian of this action is the functional of $Q(h)$ defined in (4.4.14). We thus see that the on–shell action is manifestly $6d$ covariant and does not depend on the auxiliary fields $a^r(x)$ (4.3.12).

To compute the corresponding on–shell value of the original M5–brane action (4.3.3) we perform the 1+5 splitting of the duality relations (4.4.12) which take the following form

$$\tilde{H}_{\hat{a}\hat{b}\bar{5}} = 4Q^{-1}\left((1-2\mathrm{tr}\overline{f}^2)\overline{f} + 8\overline{f}^3\right)_{\hat{a}\hat{b}\bar{5}}\,,\quad H_{\hat{a}\hat{b}\bar{5}} = 4Q^{-1}\left((1+2\mathrm{tr}\overline{f}^2)\overline{f} - 8\overline{f}^3\right)_{\hat{a}\hat{b}\bar{5}}\,,$$

where $\overline{f}_{\hat{a}\hat{b}} = h_{\hat{a}\hat{b}\bar{5}}$ and $\hat{a},\hat{b} = 0,1,2,3,4$. Upon substituting the above expressions into the action (4.3.3) we find that its value is again given by eq. (4.5.1). Thus the two forms of the M5–brane action give rise to the same equations of motion and their on–shell values are equal and are given by the superembedding scalar function $Q(h)$. For the self–dual string soliton considered in Section 4.4.1, the value of the action determines the tension of the string, as was discussed in [90].

An interesting open problem that may have important consequences for the issue of quantization of the self–dual fields is the understanding of the off–shell relationship between the different self–dual actions.

4.6 Relation to $M2$–branes

The new form of the $M5$–brane action can be useful for studying its relation to the Nambu–Poisson description of the M5–brane in a constant C_3 field which is originated from the $3d$ BLG model with the gauge group of volume preserving diffeomoprhisms [74, 75]. The BLG model invariant under the volume preserving diffeomorphisms describes a condensate of M2–branes which via a Myers effect may grow into an M5–brane. In [74, 75] it was conjectured that the Nambu–Poisson M5–brane model is related to the conventional description of the M5–brane in a constant C–field background through a transformation analogous to the Seiberg–Witten map [192]. Such a map between the fields and gauge transformations of the two models was constructed in [78], however the relation between the two actions still remains to be established. We leave the study of this issue for future and will only show that in a flat background without C–field the worldvolume dimensional reduction of the bosonic M5–brane action (4.3.15) (or (4.4.30)) directly results in the membrane action. To this end we fix the $6d$ worldvolume diffeomorphisms by imposing the static gauge

$$x^\mu = X^\mu, \qquad X^I(x^\mu) \qquad I = 6, 7, 8, 9, 10$$

where $X^I(x)$ are five physical scalar fields corresponding to the target–space directions transversal to the M5–brane worldvolume. We perform the dimensional reduction of three worldvolume directions x^i $(i = 3, 4, 5)$ assuming that the scalar fields X^I and the chiral tensor field $B_{\mu\nu}$ only depend on the three un–compactified coordinates x^a and not on x^i. Then the induced worldvolume metric takes the form

$$g_{\mu\nu} = (\eta_{ab} + \partial_a X^I \partial_b X^I, \, \delta_{ij}) \,, \qquad g_{ai} = 0 \,. \tag{4.6.2}$$

We use the local gauge symmetry (4.3.21) to fix the values of the three auxiliary scalars $a^r(x)$ in such a way that the projectors (4.3.12) take the form

$$P_\mu{}^\nu = \delta^a_\mu \delta^\nu_a \,, \qquad \Pi_\mu{}^\nu = \delta^i_\mu \delta^\nu_i \,. \tag{4.6.3}$$

Then the components $G_{\mu\nu\rho}$ (4.3.17) of the gauge field strength vanish and $F_{\mu\nu\rho}$ reduce to

$$F_{aij} = \partial_a B_{ij} \quad \Rightarrow \quad F^i_a = \frac{1}{2}\epsilon^{ijk}F_{ajk} = \partial_a \tilde{X}^i \,, \tag{4.6.4}$$

where the dualized components of the gauge field $B_{ij}(x^a)$,

$$\tilde{X}^i \equiv \frac{1}{2}\epsilon^{ijk}B_{jk}$$

play the role of the additional three scalar fluctuations of the membrane associated with $D = 11$ target–space directions orthogonal to the membrane worldvolume. Indeed, upon dimensional reduction the M5 brane action (4.4.30) becomes

$$S_{M2} = \int d^3x \sqrt{-\det(\eta_{ab} + \partial_a X^I \partial_b X^I + \partial_a \tilde{X}^i \partial_b \tilde{X}^i)} \,, \tag{4.6.5}$$

which is the action for a membrane in flat $D = 11$ space–time in the static gauge.

4.7 to get 4d 1+3 duality symmetric action

In this section, we present another consistency check that the dimensional reduction of our action (4.3.15) on a torus[7] gives the 1+3 duality-symmetric action of Schwarz and Sen type (1+3 Schwarz-Sen action) [94].

In the work [38], the author shows that the dimensional reduction of the 6d PST nonlinear action (4.3.3) on a torus gives either the self-dual DBI action (1.3.58) or the 1+3 Schwarz-Sen type action [94] depending on how we gauge fix the auxiliary field $a(x)$. Let us notice that, in [204, 205], a complete comparison for PST M5-brane action on a torus and the D3-brane action on a circle is done. Here, after reviewing [38], we follow the lines of Berman and show that the same 1+3 Schwarz-Sen type action can be obtained from our new action (4.3.15). This may be served as another consistency check of our new 3+3 action.

[7] For simplicity, we don't consider the action as embedded in a generic 11d supergravity background, but just view it as a 6d nonlinear theory for a chiral 2-form. That is, we are just interested in the resulting 4d theory when we put the 6d nonlinear 3+3 chiral 2-form theory on a torus. For the complete analysis of the supersymmetric PST M5-brane action on a torus and the D3-brane on a circle, see [204, 205].

4.7.1 review of PST (1+5) to self-dual DBI (0+4) and Schwarz-Sen (1+3)

Consider the compactification of the 6d space as $\mathcal{M}_6 \to \mathcal{M}_4 \oplus T_2$,

$$\text{metric}: \qquad g = \eta \oplus \pi, \tag{4.7.6}$$

$$\text{gauge potential}: \qquad B_{a\dot{a}} = A^I_a(x^b)\gamma_{I\dot{a}}(x^{\dot{b}}), \tag{4.7.7}$$

where η and π denote the metric of the resulting 4d space and the torus respectively, x^a, $a, b = 0, 1, 2, 3$[8], are coordinates on the 4d space and $x^{\dot{a}}$, $\dot{a}, \dot{b} = 4, 5$ are coordinates on the torus. γ_I are canonical 1-forms associated with nontrivial homology one cycles on the torus, they form basis of $H^1(T^2, Z)$ and satisfy $d\gamma_I = d\,{}^{*_2}\gamma_I = 0$, with $*_2$ being the Hodge dual on the torus. The explicit form of the γ_I will be given soon later. The reduction of the field strength is therefore,

$$H_{ab\dot{a}} = F^I_{ab}(x^c)\gamma_{I\dot{a}}(x^{\dot{b}}), \tag{4.7.8}$$

other components are zero.

to self-dual DBI action

Let us align da in the direction of the torus, or

$$\partial_{\dot{a}} a = \gamma_{I\dot{a}}, \qquad \text{for } I = 1 \text{ or } 2, \tag{4.7.9}$$

say, $I = 2$ for definiteness. Recall that the PST action has the gauge symmetry (4.3.5), which reduces to $\delta B_{a\dot{a}} = -\psi_a \partial_{\dot{a}} a = -\psi_a \gamma_{2\dot{a}}$ in the dimensional reduction. This observation tells us that the A^2_a component of the reduced gauge field (4.7.7) can be gauged away, if we choose $\psi_a = A^2_a$. Therefore, we are left with a 4d covariant gauge field $A_a \equiv A^1_a$.

The homology $H^1(T^2, Z)$ basis can be chosen to satisfy

$$\gamma_I \wedge {}^{*_2}\gamma_J = \frac{M_{IJ}}{\mathcal{V}}\Omega, \qquad \gamma_I \wedge \gamma_J = \frac{L_{IJ}}{\mathcal{V}}\Omega, \tag{4.7.10}$$

[8] Let us emphasise that the a, b indices in this subsection run from 0 to 3. Don't be confused with the convention of the splitting in (4.4.15).

where $\Omega = \sqrt{\pi} d^2x$ is the volume form on the torus, $\mathcal{V} = \int_{T^2} \Omega$ is the volume, and M_{IJ} and L_{IJ} are period and intersection matrices defined as

$$M = \int_{T_2} \gamma_I \wedge {}^{*_2}\gamma_J = \frac{1}{\tau_2} \begin{pmatrix} 1 & \tau_1 \\ \tau_1 & |\tau|^2 \end{pmatrix}, \quad L = \int_{T^2} \begin{pmatrix} \gamma_1 \wedge \gamma_1 & \gamma_1 \wedge \gamma_2 \\ \gamma_2 \wedge \gamma_1 & \gamma_2 \wedge \gamma_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{4.7.11}$$

The $\tau \equiv \tau_1 + i\tau_2$ is the complex structure of the torus. Performing the reduction on the action (4.3.3) (with trivial background) we will get nothing but the DBI action of the D3-brane (1.3.58) after some trivial field redefinition with the complex structure of the torus being identified with the dilaton and axion, $\tau = C_0 + ie^{-\Phi}$.

to Schwarz-Sen type 1+3 action

Instead of aligning da in the torus, one can choose da to lie in the direction of the 4d space. In this way, the manifest 4d covariance will be broken and we will see that the resulting theory corresponds to the Schwarz-Sen type 1+3 action.

If da is chosen to be dx^0, $\tilde{H}_{\mu\nu}$ $(\mu, \nu = 0, 1, \cdots, 5)$ appearing in the determinant has only nonzero components $\tilde{H}_{a\dot{a}}$, the matrix being taken the determinant is therefore not block diagonal. However, it can be calculated by

$$\det(g_{\mu\nu} + i\tilde{H}_{\mu\nu}) = \det g \left(1 + \frac{1}{2}\mathrm{tr}\tilde{H}^2 + \frac{1}{8}(\mathrm{tr}\tilde{H}^2)^2 - \frac{1}{4}\mathrm{tr}\tilde{H}^4\right), \tag{4.7.12}$$

where $\mathrm{tr}\tilde{H}^2 = \tilde{H}^{\mu\nu}\tilde{H}_{\nu\mu}$ and $\mathrm{tr}\tilde{H}^4 = \tilde{H}^{\mu\nu}\tilde{H}_{\nu\rho}\tilde{H}^{\rho\sigma}\tilde{H}_{\sigma\mu}$. The resulting action is :

$$S = \int_{\mathcal{M}_4} d^4x \sqrt{-\eta'}\sqrt{P} + \frac{1}{2}\sum_{a=1}^{3} \left(E^1_a B^{2a} - E^2_a B^{1a}\right), \tag{4.7.13}$$

where $\eta'_{ab} = \sqrt{\mathcal{V}}\eta_{ab}$, and

$$P \equiv -(1+(B\cdot B)^{IJ})M_{IJ} + \frac{1}{2}\left[(B\cdot B)^{IJ}M_{IJ}\right]^2 - \frac{1}{2}(B\cdot B)^{IL}(B\cdot B)^{JK}M_{IJ}M_{KL}, \tag{4.7.14}$$

with $(B \cdot B)^{IJ} \equiv \sum_{a=1}^{3} B^I_a B^{Ja}$. This is precisely a nonlinear generalisation of the 1+3 duality symmetric action constructed by Schwarz and Sen [94].

If instead, we choose da to be space-like, say $da = dx^1$, then we will still get the Schwarz-Sen type theory but with electric fields showing up in (4.7.14).

4.7.2 new (3+3) action to Schwarz-Sen type (1+3)

In this subsection, we again use the index convention :

$$\begin{aligned} \dot{a}, \dot{b}, \dot{c} &= 4, 5 \\ a, b, c &= 0, \underbrace{1, 2, 3}_{=i,j,k}, \end{aligned} \tag{4.7.15}$$

with the 3+3 splitting being

$$\alpha, \beta, \gamma = 0, 4, 5, \qquad i, j, k = 1, 2, 3, \tag{4.7.16}$$

We also follow Berman's reduction ansatz [38] :

$$g = \eta \oplus \pi, \qquad B_{a\dot{a}} = A^I_a \gamma_{I\dot{a}}, \qquad (B_{ab} = B_{\dot{a}\dot{b}} = 0), \tag{4.7.17}$$

where η is now chosen as the Minkowski metric for simplicity.

The nonzero components of the gauge potential are

$$B_{\alpha i} \neq 0, \qquad B_{0\alpha} \neq 0, \qquad (\alpha \neq 0). \tag{4.7.18}$$

This simplifies the action a lot, for example, $H_{ijk} = 3\partial_{[k} B_{ij]} = 0$, so that $G^2 = 0$ in the 3+3 action. The nonzero components of the field strengths are :

$$H_{\alpha ij} = \partial_i (A^I_j \gamma_{I\alpha}) - \partial_j (A^I_i \gamma_{I\alpha}) = F^I_{ij} \gamma_{I\alpha}, \qquad (\alpha \neq 0), \tag{4.7.19}$$

$$H_{0\beta k} = \partial_0 B_{\beta k} + \partial_k B_{0\beta} = F^I_{k0} \gamma_{I\beta}, \qquad (\beta \neq 0). \tag{4.7.20}$$

Hence, the quadratic piece reduces as

$$\begin{aligned} & -\frac{1}{2}\sqrt{-g}\tilde{H}^{\alpha ij} H_{\alpha ij}, \qquad (\alpha \neq 0) \\ = \; & -\frac{1}{2}\frac{2}{2!}\epsilon^{\alpha ij \beta 0k} H_{\beta 0k} H_{\alpha ij} = -\frac{1}{2}\epsilon^{\alpha\beta}\epsilon^{ijk} F^I_{0k}\gamma_{I\beta} F^J_{ij}\gamma_{J\alpha} \\ = \; & -\sqrt{\pi} E^I_k B^{Jk} \tilde{\epsilon}^{\alpha\beta}\gamma_{J\alpha}\gamma_{I\beta} = -\sqrt{\pi} E^I_k B^{Jk} \frac{1}{\mathcal{V}} \epsilon_{JI}, \end{aligned} \tag{4.7.21}$$

where $\epsilon_{12} = \epsilon^{12} = 1$, $F^I_{0k} = E^I_k$, $\frac{1}{2}\epsilon^{ijk} F^J_{ij} = B^{Jk}$. Therefore, after integration, we have

$$\int_{\mathcal{M}_4} (E^1_k B^{2k} - E^2_k B^{1k}). \tag{4.7.22}$$

For the determinant part, the $2\mathcal{L}_{M5}$ in this ansatz becomes ($\mu,\nu,\rho,\sigma = 0,1,\cdots,5$)

$$2\sqrt{-\det\left(g_{\mu\nu}+\frac{1}{2}F_{\mu\rho\sigma}F_{\nu}{}^{\rho\sigma}\right)}. \tag{4.7.23}$$

We can expand the determinant according to

$$\det(1+F^2) = 1+\mathrm{tr}F^2+\frac{1}{2}\left((\mathrm{tr}F^2)^2-\mathrm{tr}(F^4)\right)+\det F^2, \tag{4.7.24}$$

where F is the matrix $F_{i\alpha}$ defined as $F_{i\alpha} = \epsilon^{ijk}H_{jk\alpha}$ and it becomes $B^I_i\gamma_{I\alpha}$ with $\alpha \neq 0$ by the ansatz. Since one column vanishes in the matrix $B^I_i\gamma_{I\alpha} \equiv (B^I\gamma^I)_{i\alpha}$, the determinant $\det F^2$ simply vanishes and hence there will be no order 6 term on the expansion of $\det(1+F^2)$. For the 2nd order and 4th order terms :

$$\mathrm{tr}F^2 = F_{i\alpha}F^{i\alpha} = B^I_i\gamma_{I\alpha}B^{Ji}\gamma^{\alpha}_J = (B\cdot B)^{IJ}M_{IJ}\frac{1}{\mathcal{V}}, \tag{4.7.25}$$

$$\frac{1}{2}(\mathrm{tr}F^2)^2 = \frac{1}{2}\left((B\cdot B)^{IJ}M_{IJ}\frac{1}{\mathcal{V}}\right)^2, \tag{4.7.26}$$

$$\begin{aligned} -\frac{1}{2}\mathrm{tr}F^4 &= -\frac{1}{2}F_{i\alpha}F^{\alpha j}F_{j\beta}F^{\beta i} = -\frac{1}{2}B^I_i\gamma_{I\alpha}B^{Ji}\gamma^{\alpha}_J B^K_j\gamma_{K\beta}B^{Li}\gamma^{\beta}_L \\ &= -\frac{1}{2}(B\cdot B)^{IL}(B\cdot B)^{JK}M_{IJ}M_{KL}\frac{1}{\mathcal{V}^2}. \end{aligned} \tag{4.7.27}$$

The $1/\mathcal{V}$ area factors will be absorbed after rescaling the 4D metric : $\eta^{ab}\to\eta'^{ab} = \frac{1}{\sqrt{\mathcal{V}}}\eta^{ab}$. Putting everything together, we therefore reproduced the nonlinear duality symmetric action (4.7.13).

Another ansatz

We may try another possible 1+3 splitting :

$$\begin{aligned} \dot{a},\dot{b} &= 4,5 \\ a,b &= 0,1,2,3, \end{aligned} \tag{4.7.28}$$

with the 3+3 splitting being

$$\alpha,\beta,\gamma = 0,1,2, \qquad i,j,k = 3,4,5. \tag{4.7.29}$$

In this case, the G^2 variable in 3+3 also vanishes :

$$H_{345} = \partial_4 B_{53}+\partial_5 B_{34} = -\partial_4(A^I_3\gamma_{I5})+\partial_5(A^I_3\gamma_{I4}) = A^I_3(\partial_5\gamma_{I4}-\partial_4\gamma_{I5}) = 0, \tag{4.7.30}$$

since $d\gamma_I = 0$. Performing the dimensional reduction in almost the same lines as the previous steps, one recovers the action (4.7.13), but with the electric field appearing in determinant expansion. This is not surprising as the ansatz (4.7.15) selects out a temporal special direction x^0 while the ansatz (4.7.28) selects out a spatial direction x^3, from the $4d$ point of view.

4.8 Conclusion

Using the non–linear self–duality equation for the 3–form gauge field strength arising in the superembedding description of the M5–brane we have derived a novel form of the kappa–symmetric M5–brane action with a covariant 3+3 splitting of its $6d$ worldvolume.

The value of this action on the mass–shell of the non–linear self–dual gauge field coincides with the on–shell value of the original M5–brane action expressed in terms of the $6d$ scalar function Q of the self–dual chiral field h_3 appearing in the superembedding description of the M5–brane. It would be interesting and important to better understand the off–shell relation between the two actions.

Having at hand the M5–brane action in the form (4.3.15), (4.3.16) one can repeat the steps of [191] towards understanding the link of this action to the Nambu–Poisson 5–brane of [74, 75] by restricting the worldvolume pullback of the 11D gauge field C_3 to be constant and by partial gauge fixing local symmetries of (4.3.15), (4.3.16) to a group of $3d$ volume preserving diffeomorphisms. The Seiberg–Witten–like map constructed in [78] may be required to relate the fields of the two models. It would be also of interest to relate our construction to a noncommutative M5–brane of [206].

The novel form of the action is also naturally suitable for studying the effective theory of the M5–brane wrapping a $3d$ compact Riemann–manifold.

As another direction of study, one may try, using the superembedding form of the self–duality relation, to construct an M5–brane action in the form which exhibits 2+4 splitting of the $6d$ worldvolume which may be useful for studying $M5$–branes wrapping $2d$ and $4d$ manifolds, and M5–brane istantons wrapping $4d$ divisors of Calabi–Yau 4–folds in $M_3 \times CY_4$ compactifications of M–theory as discussed e.g.

in [207–212].

Chapter 5

$\mathbf{6} \approx 2 + 4$

In this chapter we briefly summarise an ongoing project which will be reported elsewhere soon [213].

Having realised that the M5-brane action can be formulated by covariant splittings of worldvolume space into 1+5 and 3+3 directions, it is natural to ask if it is possible to derive the "interpolation", i.e. 2+4, and hopefully to gain some new insights into M-theory. In this chapter, we explore this possibility and present some open questions.

5.1 Introduction & summary of the results

Recently, an alternative M5-brane action in a generic eleven-dimensional supergravity background was constructed in [3] with the aim of better understanding the connection of the original M5-brane action [13, 14] to the 5-brane proposal of [74, 75] based on the three-dimensional Bagger-Lambert-Gustavsson model [48, 49, 51] with the gauge symmetry of a $3d$ volume preserving diffeomorphism. In [3] it was shown that the field equations derived from the new action are equivalent to the ones deduced from the superembedding approach [83, 84] and hence to the equations of motion which follow from the original action [85].

The difference between the two M5-brane actions is that in the original action of [13, 14] the 6-dimensional M5-brane worldvolume gets split into 1+5 directions and the manifest $6d$ space-time invariance is maintained by the presence of a single

auxiliary scalar field, while in the action of [3] the $6d$ worldvolume is effectively split into 3+3 directions and the manifest $6d$ space-time invariance is maintained by the introduction of a triplet of auxiliary scalar fields [76].

Different formulations of the theory may allow one to gain different insights into its structure. The action of [214], for instance, in addition to its relation to the BLG model, can also be useful for studying M2-M5 bound states discussed e.g. in [215, 216].

The Lagrangian formulation of self-dual or duality-symmetric fields is essentially not unique, which is related to different possible ways of tackling the issue of (non-manifest) space-time invariance of the duality-symmetric actions (see e.g. [75, 90, 92–94, 166, 167, 188, 193]). Various possible ways of constructing actions which produce the (self)-duality relations as (a consequence of) equations of motion by effectively splitting d-dimensional space-time into p- and q-dimensional subspaces, with $d = p + q$, were explored for free theories and without the coupling to $6d$ gravity in [99, 195]. The actions with different space-time splitting are generically inequivalent off-shell, as was shown for the $6 = 1+5$ and $6 = 3+3$ cases in [76, 214]. Different off-shell inequivalent formulations may be useful for studying the dynamics of duality-symmetric fields in topologically non-trivial backgrounds [193, 217–219] and their quantization [104, 193, 196–199, 220].

The above reasoning has motivated us to complete the list of different Lagrangian formulations of the M5-brane by constructing its action with an effective 2+4 splitting of the $6d$ worldvolume. Another motivation is that this form of the action for the Abelian $\mathcal{N} = (2, 0)$ $d = 6$ theory provides us with an appropriate off–shell starting point for its topological twisting considered recently in [221, 222]. However, we will show that the effective 2+4 splitting of the $6d$ space brings generic difficulties in coupling the theories (both free and nonlinear ones) to $6d$ gravity, and hence we are not able to construct the M5-brane action in a 2+4 splitting at this moment. Nevertheless, we are able to deform the free non-covariant 2+4 theory [99] to a nonlinear one as well as obtaining the supersymmetric extension of [99], without coupling to $6d$ gravity for both cases though. As the nonlinear theory we discovered cannot be coupled to $6d$ gravity, one cannot embed the theory in $11d$ target space

generically. In other words, the best we can do for the potential M5-brane action with 2+4 splitting of worldvolume space is to construct an M5-brane action in a trivial $11d$ target space-time and with all its transverse fluctuations frozen.

The strategy we used to deform the free 2+4 action is to follow the same lines as that of the 3+3 action [214]. As is well known, the non-linear generalization of a self-dual system is not unique. With the aim to apply the resulting 2+4 theory to the M5-brane, we shall look for such a form of the action that produces the same non-linear self-duality equations as those of the superembedding description of the M5-brane.

In comparison with its previous counterparts, the 2+4 Lagrangian formulation of the M5-brane has several new features, complications and difficulties. Namely, some of the gauge symmetries of the action become semi-local[1]. A semi-local symmetry is a symmetry with parameters that are not totally arbitrary local functions but constrained. For these semi-local symmetries to be *gauge* symmetries, the time direction of the $d = 2 + 4$ worldvolume should be in the two-dimensional subspace, thus breaking $6d$ space-time democracy, though the action does possess a (modified) $6d$ worldvolume invariance. In other words, to split $6d$ one can choose any $2d$ subspace of Lorentz signature. The structure of the nonlinear action with 2+4 splitting is much more complicated in comparison with a Born-Infeld-like structures of the actions of [13, 14] and [214]. A defining function of components of the chiral tensor field strength which enters the action should satisfy an algebraic equation of sextic order which can only be solved perturbatively.

To at least make the idea clear about the new features of the 2+4 formulation, let us conclude this chapter with a review of free 2+4 theory in flat $6d$ space in the next section.

[1]Semi-local symmetries have previously appeared also in other formulations of duality-symmetric fields in different dimensions (see e.g. [188, 223, 224]) and topologically non-trivial backgrounds [217–219].

5.2 Linear theory with non-manifest $6d$ Lorentz-invariance

We will review the non-manifestly $6d$ Lorentz invariant quadratic chiral 2-form action in $6d$ Minkowski space.

5.2.1 review of free theory

In this section we review the derivation of the self-duality condition for the $6d$ chiral 2-form gauge potential B_2 with field strength $H_3 = dB_2$,

$$H_{\mu\nu\rho} = \frac{1}{6}\varepsilon_{\mu\nu\rho\lambda_1\lambda_2\lambda_3}\, H^{\lambda_1\lambda_2\lambda_3} = \tilde{H}_{\mu\nu\rho} \tag{5.2.1}$$

from a $6d$ Lagrangian with a 2+4 splitting of six-dimensional tensor indices [99]. $\epsilon^{012345} = -\epsilon_{012345} = 1$.

Let us perform the following 2+4 splitting of $H_{\mu\nu\rho}$

$$H_{\mu\nu\rho} = (H_{abj}, H_{ijk}, H_{aij}), \qquad a, b, c, \cdots = 0, 5; \qquad i, j, k, \cdots = 1, 2, 3, 4. \tag{5.2.2}$$

Then, the Hodge-dual field-strength $\tilde{H}_{\mu\nu\rho}$ splits as follows

$$\varepsilon_{\mu_1\cdots\mu_6} \quad \Rightarrow \quad \epsilon_{abijkl} = \epsilon_{ab}\epsilon_{ijkl}, \tag{5.2.3}$$

$$\tilde{H}_{abi} = \frac{1}{3!}\epsilon_{ab}\varepsilon_{ijkl}H^{jkl}, \quad \tilde{H}_{aij} = \frac{1}{2}\epsilon_{ab}\varepsilon_{ijkl}H^{bkl}, \quad \tilde{H}_{ijk} = \frac{1}{2}\epsilon_{ijkl}\varepsilon_{ab}H^{abl}, \tag{5.2.4}$$

The quadratic action which produces (5.2.1) has the following form [99]

$$S = -\int d^6x \left(\frac{1}{2}\tilde{H}_{abi}H^{abi} + \frac{1}{4}H_{aij}H^{aij} + \frac{1}{6}H_{ijk}H^{ijk}\right). \tag{5.2.5}$$

The action has the local gauge symmetry

$$\delta B_{ab} = \Omega_{ab}(x^\mu), \tag{5.2.6}$$

where $\Omega_{ab}(x^\mu)$ are arbitrary functions. In addition, we found that the action is also invariant under the following semi-local transformations

$$\delta B_{ai} = \Phi_{ai}(x^b, x^j) \tag{5.2.7}$$

whose parameters Φ_{ai} are restricted to satisfy the anti-self-duality condition

$$\partial^{[i}\Phi^{k]a} = -\frac{1}{2}\epsilon^{ab}\epsilon^{ikjl}\partial_j\Phi_{bl}, \quad \text{so that} \quad \partial_k\partial^{[k}\Phi^{i]a} = 0, \tag{5.2.8}$$

i.e. Φ_{ai} obey the differential equation in the four-dimensional subspace parametrized by the coordinates x^i.

We should check that, though being semi-local, the transformations (5.2.7) form a genuine gauge symmetry which will allow us to get rid of redundant degrees of freedom [2].

A semi-local symmetry is a fully-fledged *gauge symmetry* if its associated Noether charge vanishes (at least) on the mass shell [225]. The conserved Noether current associated with (5.2.7) is

$$j^\mu = \delta^\mu_j(H^{jai} - \tilde{H}^{jai})\Phi_{ai}, \qquad \mu = 0, 1, \cdots, 5. \tag{5.2.9}$$

It is clear from the structure of (5.2.9) that the Noether charge $Q = \int d^5x\, j^0$ is identically zero off-shell, if the temporal direction is in the 2d subspace of the '2+4' split six-dimentional space. Therefore, in this formulation we lose the freedom to place the time direction in the 4d subspace. This makes the 2+4 splitting different from the 1+5 and 3+3 splittings of the previous formulations of the $6d$ chiral 2-form action.

The field equations which one obtains by varying (5.2.5) are

$$\partial_k\left(-\tilde{H}^{aki} + H^{aki}\right) = 0, \tag{5.2.10}$$

$$\partial_k\left(-\tilde{H}^{ijk} + 2H^{ijk}\right) + \partial_a H^{aij} = 0. \tag{5.2.11}$$

Equation (5.2.10) has the general solution

$$-\tilde{H}^{aik} + H^{aik} = \epsilon^{ab}\epsilon^{ikjl}\partial_j\tilde{\Phi}_{bl}, \tag{5.2.12}$$

where $\tilde{\Phi}_{bl}$ satisfy the condition (5.2.8), as the left-hand-side of the above equation is anti-self-dual. Hence, we can obtain the self-duality equation

$$H_{aij} = \tilde{H}_{aij}, \tag{5.2.13}$$

[2] The presence of this semi-local gauge symmetry is effectively translated into the choice of appropriate boundary conditions for integration functions considered in [99].

by fixing the semi-local gauge symmetry (5.2.7) appropriately. Substituting (5.2.13) into (5.2.11), and using the Bianchi identity, we get

$$\partial_k \left(-\tilde{H}^{ijk} + H^{ijk} \right) = 0, \tag{5.2.14}$$

which has the general solution

$$-\tilde{H}_{ijk} + H_{ijk} = \frac{1}{2} \epsilon_{ab} \epsilon_{ijkl} \partial^l \tilde{\Omega}^{ab}, \tag{5.2.15}$$

where $\tilde{\Omega}_{ab}$ are arbitrary functions which can be put to zero with the use of the local gauge transformations (5.2.6). We thus arrive at another set of self-duality equations

$$H_{ijk} = \tilde{H}_{ijk}. \tag{5.2.16}$$

Eqs. (5.2.13) and (5.2.16) together are equivalent to (5.2.1).

The action (5.2.5) is manifestly invariant under an $SO(1,1) \times SO(4)$ subgroup of the Lorentz group. However, it is less obvious that the action also enjoys the modified Lorentz symmetry with parameters $\lambda^a{}_j \equiv \lambda^a_j$ ($\lambda_a{}^i \equiv \lambda^i_a$) associated with the coset transformations $SO(1,5)/[SO(1,1) \times SO(4)]$. For simplicity, we present the modified part of the $SO(1,5)$ Lorentz symmetry in the gauge $B_{ab} = 0$

$$\delta B_{ai} = \delta_1 B_{ai} + \delta_2 B_{ai}, \qquad \delta B_{ij} = \delta_1 B_{ij} + \delta_2 B_{ij}, \tag{5.2.17}$$

with

$$\begin{aligned} \delta_1 B_{ai} &= \lambda^j_a B_{ji} + \lambda^b_j (x_b \partial^j - x^j \partial_b) B_{ai}, \\ \delta_1 B_{ij} &= -\lambda^b_i B_{bj} + \lambda^b_j B_{bi} + \lambda^b_k (x_b \partial^k - x^k \partial_b) B_{ij}, \end{aligned} \tag{5.2.18}$$

being the standard Lorentz transformation and[3]

$$\delta_2 B_{ai} = \lambda^b_j x^j (H - \tilde{H})_{bai}, \qquad \delta_2 B_{ij} = \frac{1}{2} \lambda^b_k x^k (H - \tilde{H})_{bij} \tag{5.2.21}$$

[3]There is the freedom to add

$$\delta_3 B_{ai} = \lambda^j_b x^b (H - \tilde{H})_{aij} \tag{5.2.19}$$

to the transformation rules. One may check that the Lagrangian is invariant up to a total derivative term

$$\delta_3 S = \frac{1}{2} \int d^6 x \partial_k (\lambda^j_b x^b \mathcal{H}_{aij} \mathcal{H}^{aik}), \tag{5.2.20}$$

under δ_3.

vanish on the mass shell. Thus, the modified $SO(1,5)$ Lorentz symmetry reduces to the standard one when the field strength of the 2-form B_2 is self-dual.

Thus, even for the free theory without coupling to $6d$ gravity, the 2+4 formulation is already special compared to its counter parts formulations of 1+5 and 3+3. One of the new feature is the appearance of semi-local symmetry. In order for this semi-local symmetry to be a genuine gauge symmetry, it is required to put the temporal direction in the two dimensional subspace of the 2+4 splitting. Moreover, one would find it difficult to follow [93,95] to couple the free 2+4 theory to $6d$ gravity like what we reviewed in (1.3.92) for 1+5 formulation. Equivalently, one would find that the PST covariantisation [96,97], [98] of the free theory 2+4 formulation is reluctant to be completed. Nevertheless, it is interesting that the nonlinear deformation of the free 2+4 formulation is possible and the nonlinear self-duality equation in a form of 2+4 splitting is already encoded in the super-embedding algebraic formula. Physical reasons why 2+4 formulation is so special are under investigation and we hope to report the progress or to resolve the issue elsewhere.

Chapter 6

Conclusions

In this thesis, we have studied various aspects of the action formulations for chiral 2-forms and their applications to M-theory five-brane(s).

In Part II, we proposed a simple model [1] in Chapter 2 to describe the bosonic gauge sector of the multiple M5-branes as a generalisation of the Perry-Schwarz [90] abelian theory. The double dimensional reduction of the model leads to five dimensional super-Yang-Mills theory with higher derivative corrections. Moreover, the action enjoys a modified Lorentz symmetry. In Chapter 3, we presented supporting evidence for our model by constructing explicitly the non-abelian self-dual string solutions [2] to the equations proposed in [1]. These non-abelian self-dual string solutions are supported by the monopole solutions of Wu-Yang and 't Hooft-Polyakov. These solutions can be viewed as a non-abelian generalisation of the Perry-Schwarz abelian string soliton, which is supported by the Dirac monopole.

In Part III of the thesis, we have successfully rewritten the known single M5-brane action [13] in a form in Chapter 4 which reveals different aspects of the M-theory branes. The worldvolume space of our new action is subject to a covariant splitting into 3+3 directions by a triplet of ancillary scalars. The dimensional reductions as well as the relation to the original M5-brane action were also studied. The new theory [3] shares the same on-shell value of the action with the old one [13]. The dimensional reduction on T^3 gives the single membrane action while the reduction on the torus T^2 gives the duality-symmetric action of Schwarz and Sen. Finally, in Chapter 5 we summarised the attempt to write the M5-brane action in a 2+4

splitting of worldvolume space and mentioned various new features and difficulties of the action formulation for the chiral 2-form in terms of 2+4 splitting.

We have approached the M-theory five branes by exploring their possible action formulations in terms of the techniques developed for chiral p-forms put forward in [90, 93] and [76, 98, 172, 226, 227]. Chiral p-forms are important objects as they show up in fundamental theories like M-theory and string theory. The low energy effective theory of the M5-branes in the decoupling limit of gravity, the so-called (2,0) theory, may also be a Holy Grail for future theoretical physics. In particular, the (2,0) theory demands a creative breakthrough to uncover its mysterious phases. In this thesis, we tried to move forward a little bit towards the understanding of single and multiple M5-branes by the conventional action principle.

Bibliography

[1] C.-S. Chu and S.-L. Ko, "Non-abelian Action for Multiple Five-Branes with Self-Dual Tensors," *JHEP* **1205** (2012) 028, arXiv:1203.4224 [hep-th].

[2] C.-S. Chu, S.-L. Ko, and P. Vanichchapongjaroen, "Non-Abelian Self-Dual String Solutions," arXiv:1207.1095 [hep-th].

[3] S.-L. Ko, D. Sorokin, and P. Vanichchapongjaroen, "The M5-brane action revisited," *JHEP* **1311** (2013) 072, arXiv:1308.2231 [hep-th].

[4] S. Fubini, D. Gordon, and G. Veneziano, "A general treatment of factorization in dual resonance models," *Phys.Lett.* **B29** (1969) 679–682.

[5] P. Goddard, "From Dual Models to String Theory," arXiv:0802.3249 [physics.hist-ph].

[6] S. R. Coleman and J. Mandula, "All possible symmetries of the S matrix," *Phys. Rev.* **159** (1967) 1251–1256.

[7] W. Nahm, "Supersymmetries and their Representations," *Nucl.Phys.* **B135** (1978) 149.

[8] E. Witten, "String theory dynamics in various dimensions," *Nucl.Phys.* **B443** (1995) 85–126, arXiv:hep-th/9503124 [hep-th].

[9] P. K. Townsend, "Four lectures on M-theory," arXiv:hep-th/9612121.

[10] J. H. Schwarz, "Lectures on superstring and M theory dualities: Given at ICTP Spring School and at TASI Summer School," *Nucl.Phys.Proc.Suppl.* **55B** (1997) 1–32, arXiv:hep-th/9607201 [hep-th].

[11] T. Banks, W. Fischler, S. Shenker, and L. Susskind, "M theory as a matrix model: A Conjecture," *Phys.Rev.* **D55** (1997) 5112–5128, arXiv:hep-th/9610043 [hep-th].

[12] E. Witten, "Conformal Field Theory In Four And Six Dimensions," arXiv:0712.0157 [math.RT].

[13] I. A. Bandos, K. Lechner, A. Nurmagambetov, P. Pasti, D. P. Sorokin, *et al.*, "Covariant action for the superfive-brane of M theory," *Phys.Rev.Lett.* **78** (1997) 4332–4334, arXiv:hep-th/9701149 [hep-th].

[14] M. Aganagic, J. Park, C. Popescu, and J. H. Schwarz, "World volume action of the M theory five-brane," *Nucl.Phys.* **B496** (1997) 191–214, arXiv:hep-th/9701166 [hep-th].

[15] J. Polchinski, "SUPERSTRING THEORY," *Annals N.Y.Acad.Sci.* **571** (1989) 9–15.

[16] J. Polchinski, "String theory. Vol. 2: Superstring theory and beyond,".

[17] B. Zwiebach, "A first course in string theory,".

[18] P. West, "Introduction to strings and branes,".

[19] K. Becker, M. Becker, and J. Schwarz, "String theory and M-theory: A modern introduction,".

[20] D. J. Smith, "Intersecting brane solutions in string and M theory," *Class.Quant.Grav.* **20** (2003) R233, arXiv:hep-th/0210157 [hep-th].

[21] D. S. Berman, "M-theory branes and their interactions," *Phys. Rept.* **456** (2008) 89–126, arXiv:0710.1707 [hep-th].

[22] N. Lambert, "M-Theory and Maximally Supersymmetric Gauge Theories," *Ann.Rev.Nucl.Part.Sci.* **62** (2012) 285–313, arXiv:1203.4244 [hep-th].

[23] N. Berkovits, "Super Poincare covariant quantization of the superstring," *JHEP* **0004** (2000) 018, arXiv:hep-th/0001035 [hep-th].

[24] N. Berkovits, "ICTP lectures on covariant quantization of the superstring," arXiv:hep-th/0209059 [hep-th].

[25] O. A. Bedoya and N. Berkovits, "GGI Lectures on the Pure Spinor Formalism of the Superstring," arXiv:0910.2254 [hep-th].

[26] J. Polchinski, "Dirichlet Branes and Ramond-Ramond charges," *Phys.Rev.Lett.* **75** (1995) 4724–4727, arXiv:hep-th/9510017 [hep-th].

[27] P. Townsend, "M theory from its superalgebra," arXiv:hep-th/9712004 [hep-th].

[28] D. P. Sorokin and P. K. Townsend, "M Theory superalgebra from the M five-brane," *Phys.Lett.* **B412** (1997) 265–273, arXiv:hep-th/9708003 [hep-th].

[29] E. Cremmer, B. Julia, and J. Scherk, "Supergravity Theory in Eleven-Dimensions," *Phys.Lett.* **B76** (1978) 409–412.

[30] J. Simon, "Brane Effective Actions, Kappa-Symmetry and Applications," *Living Rev.Rel.* **15** (2012) 3, arXiv:1110.2422 [hep-th].

[31] M. Aganagic, C. Popescu, and J. H. Schwarz, "D-brane actions with local kappa symmetry," *Phys. Lett.* **B393** (1997) 311–315, arXiv:hep-th/9610249.

[32] A. A. Tseytlin, "On nonAbelian generalization of Born-Infeld action in string theory," *Nucl.Phys.* **B501** (1997) 41–52, arXiv:hep-th/9701125 [hep-th].

[33] I. A. Bandos, A. Nurmagambetov, and D. P. Sorokin, "The Type IIA NS5-brane," *Nucl.Phys.* **B586** (2000) 315–330, arXiv:hep-th/0003169 [hep-th].

[34] D.-E. Diaconescu, "D-branes, monopoles and Nahm equations," *Nucl.Phys.* **B503** (1997) 220–238, arXiv:hep-th/9608163 [hep-th].

[35] P. A. Dirac, "Quantized Singularities in the Electromagnetic Field," *Proc.Roy.Soc.Lond.* **A133** (1931) 60–72.

[36] R. I. Nepomechie, "Magnetic Monopoles from Antisymmetric Tensor Gauge Fields," *Phys.Rev.* **D31** (1985) 1921.

[37] C. Teitelboim, "Gauge Invariance for Extended Objects," *Phys.Lett.* **B167** (1986) 63–68.

[38] D. Berman, "SL(2,Z) duality of Born-Infeld theory from nonlinear selfdual electrodynamics in six-dimensions," *Phys.Lett.* **B409** (1997) 153–159, arXiv:hep-th/9706208 [hep-th].

[39] A. Nurmagambetov, "Duality symmetric three-brane and its coupling to type IIB supergravity," *Phys.Lett.* **B436** (1998) 289–297, arXiv:hep-th/9804157 [hep-th].

[40] D. S. Berman and D. C. Thompson, "Duality Symmetric String and M-Theory," arXiv:1306.2643 [hep-th].

[41] C. Hull and P. Townsend, "Unity of superstring dualities," *Nucl.Phys.* **B438** (1995) 109–137, arXiv:hep-th/9410167 [hep-th].

[42] M. Aganagic, J. Park, C. Popescu, and J. H. Schwarz, "Dual D-brane actions," *Nucl.Phys.* **B496** (1997) 215–230, arXiv:hep-th/9702133 [hep-th].

[43] R. Sorkin, "Kaluza-Klein Monopole," *Phys.Rev.Lett.* **51** (1983) 87–90.

[44] E. Bergshoeff, Y. Lozano, and T. Ortin, "Massive branes," *Nucl.Phys.* **B518** (1998) 363–423, arXiv:hep-th/9712115 [hep-th].

[45] T. Adawi, M. Cederwall, U. Gran, B. E. Nilsson, and B. Razaznejad, "Goldstone tensor modes," *JHEP* **9902** (1999) 001, arXiv:hep-th/9811145 [hep-th].

[46] P. S. Howe, N. Lambert, and P. C. West, "The Selfdual string soliton," *Nucl.Phys.* **B515** (1998) 203–216, arXiv:hep-th/9709014 [hep-th].

[47] C.-S. Chu, S.-L. Ko, and P. Vanichchapongjaroen, "Non-Abelian Self-Dual String Solutions," *JHEP* **1209** (2012) 018, arXiv:1207.1095 [hep-th].

[48] J. Bagger and N. Lambert, "Modeling multiple M2's," *Phys. Rev.* **D75** (2007) 045020, arXiv:hep-th/0611108.

[49] J. Bagger and N. Lambert, "Gauge Symmetry and Supersymmetry of Multiple M2-Branes," *Phys. Rev.* **D77** (2008) 065008, arXiv:0711.0955 [hep-th].

[50] J. Bagger and N. Lambert, "Comments On Multiple M2-branes," *JHEP* **02** (2008) 105, arXiv:0712.3738 [hep-th].

[51] A. Gustavsson, "Algebraic structures on parallel M2-branes," *Nucl. Phys.* **B811** (2009) 66–76, arXiv:0709.1260 [hep-th].

[52] O. Aharony, O. Bergman, D. L. Jafferis, and J. Maldacena, "N=6 superconformal Chern-Simons-matter theories, M2-branes and their gravity duals," *JHEP* **10** (2008) 091, arXiv:0806.1218 [hep-th].

[53] N. Drukker, M. Mariño, and P. Putrov, "From weak to strong coupling in ABJM theory," arXiv:1007.3837 [hep-th].

[54] C.-S. Chu and S.-L. Ko, "Non-abelian Action for Multiple Five-Branes with Self-Dual Tensors," *JHEP* **1205** (2012) 028, arXiv:1203.4224 [hep-th].

[55] M. R. Douglas, "On D=5 super Yang-Mills theory and (2,0) theory," *JHEP* **1102** (2011) 011, arXiv:1012.2880 [hep-th].

[56] N. Lambert, C. Papageorgakis, and M. Schmidt-Sommerfeld, "M5-Branes, D4-Branes and Quantum 5D super-Yang-Mills," *JHEP* **1101** (2011) 083, arXiv:1012.2882 [hep-th].

[57] P.-M. Ho, K.-W. Huang, and Y. Matsuo, "A Non-Abelian Self-Dual Gauge Theory in 5+1 Dimensions," *JHEP* **1107** (2011) 021, arXiv:1104.4040 [hep-th].

[58] C.-S. Chu, "A Theory of Non-Abelian Tensor Gauge Field with Non-Abelian Gauge Symmetry G x G," arXiv:1108.5131 [hep-th].

[59] F. Bonetti, T. W. Grimm, and S. Hohenegger, "Non-Abelian Tensor Towers and (2,0) Superconformal Theories," arXiv:1209.3017 [hep-th].

[60] C. Saemann and M. Wolf, "Six-Dimensional Superconformal Field Theories from Principal 3-Bundles over Twistor Space," arXiv:1305.4870 [hep-th].

[61] C. Saemann and M. Wolf, "On Twistors and Conformal Field Theories from Six Dimensions," *J.Math.Phys.* **54** (2013) 013507, arXiv:1111.2539 [hep-th].

[62] C. Saemann and M. Wolf, "Non-Abelian Tensor Multiplet Equations from Twistor Space," arXiv:1205.3108 [hep-th].

[63] I. Bandos, H. Samtleben, and D. Sorokin, "Duality-symmetric actions for non-Abelian tensor fields," *Phys.Rev.* **D88** (2013) 025024, arXiv:1305.1304 [hep-th].

[64] H. Samtleben, E. Sezgin, and R. Wimmer, "(1,0) superconformal models in six dimensions," *JHEP* **1112** (2011) 062, arXiv:1108.4060 [hep-th].

[65] M. Duff, P. S. Howe, T. Inami, and K. Stelle, "Superstrings in D=10 from Supermembranes in D=11," *Phys.Lett.* **B191** (1987) 70.

[66] E. Bergshoeff, E. Sezgin, and P. Townsend, "Supermembranes and Eleven-Dimensional Supergravity," *Phys.Lett.* **B189** (1987) 75–78.

[67] F. Gliozzi, J. Scherk, and D. I. Olive, "Supersymmetry, Supergravity Theories and the Dual Spinor Model," *Nucl.Phys.* **B122** (1977) 253–290.

[68] L. Brink, J. H. Schwarz, and J. Scherk, "Supersymmetric Yang-Mills Theories," *Nucl.Phys.* **B121** (1977) 77.

[69] A. Strominger, "Open p-branes," *Phys.Lett.* **B383** (1996) 44–47, arXiv:hep-th/9512059 [hep-th].

[70] E. Witten, "Five-branes and M theory on an orbifold," *Nucl.Phys.* **B463** (1996) 383–397, arXiv:hep-th/9512219 [hep-th].

[71] E. Witten, "Some comments on string dynamics," arXiv:hep-th/9507121 [hep-th].

[72] N. Lambert and D. Tong, "Membranes on an Orbifold," *Phys.Rev.Lett.* **101** (2008) 041602, arXiv:0804.1114 [hep-th].

[73] J. Distler, S. Mukhi, C. Papageorgakis, and M. Van Raamsdonk, "M2-branes on M-folds," *JHEP* **0805** (2008) 038, arXiv:0804.1256 [hep-th].

[74] P.-M. Ho, Y. Imamura, Y. Matsuo, and S. Shiba, "M5-brane in three-form flux and multiple M2-branes," *JHEP* **0808** (2008) 014, arXiv:0805.2898 [hep-th].

[75] P.-M. Ho and Y. Matsuo, "M5 from M2," *JHEP* **0806** (2008) 105, arXiv:0804.3629 [hep-th].

[76] P. Pasti, I. Samsonov, D. Sorokin, and M. Tonin, "BLG-motivated Lagrangian formulation for the chiral two-form gauge field in D=6 and M5-branes," *Phys.Rev.* **D80** (2009) 086008, arXiv:0907.4596 [hep-th].

[77] R. C. Myers, "Dielectric branes," *JHEP* **9912** (1999) 022, arXiv:hep-th/9910053 [hep-th].

[78] C.-H. Chen, K. Furuuchi, P.-M. Ho, and T. Takimi, "More on the Nambu-Poisson M5-brane Theory: Scaling limit, background independence and an all order solution to the Seiberg-Witten map," *JHEP* **1010** (2010) 100, arXiv:1006.5291 [hep-th].

[79] C.-H. Chen, P.-M. Ho, and T. Takimi, "A No-Go Theorem for M5-brane Theory," *JHEP* **1003** (2010) 104, arXiv:1001.3244 [hep-th].

[80] X. Bekaert, M. Henneaux, and A. Sevrin, "Deformations of chiral two forms in six-dimensions," *Phys.Lett.* **B468** (1999) 228–232, arXiv:hep-th/9909094 [hep-th].

[81] X. Bekaert, M. Henneaux, and A. Sevrin, "Chiral forms and their deformations," *Commun.Math.Phys.* **224** (2001) 683–703, arXiv:hep-th/0004049 [hep-th].

[82] X. Bekaert and S. Cucu, "Deformations of duality symmetric theories," *Nucl.Phys.* **B610** (2001) 433–460, arXiv:hep-th/0104048 [hep-th].

[83] P. S. Howe and E. Sezgin, "D = 11, p = 5," *Phys.Lett.* **B394** (1997) 62–66, arXiv:hep-th/9611008 [hep-th].

[84] P. S. Howe, E. Sezgin, and P. C. West, "Covariant field equations of the M theory five-brane," *Phys.Lett.* **B399** (1997) 49–59, arXiv:hep-th/9702008 [hep-th].

[85] I. A. Bandos, K. Lechner, A. Nurmagambetov, P. Pasti, D. P. Sorokin, *et al.*, "On the equivalence of different formulations of the M theory five-brane," *Phys.Lett.* **B408** (1997) 135–141, arXiv:hep-th/9703127 [hep-th].

[86] P. S. Howe, E. Sezgin, and P. C. West, "The six-dimensional self-dual tensor," *Phys. Lett.* **B400** (1997) 255–259, arXiv:hep-th/9702111.

[87] J. H. Schwarz, "Highly Effective Actions," *JHEP* **1401** (2014) 088, arXiv:1311.0305 [hep-th].

[88] D. P. Sorokin, "Superbranes and superembeddings," *Phys. Rept.* **329** (2000) 1–101, arXiv:hep-th/9906142.

[89] E. Sezgin and P. Sundell, "Aspects of the M5-brane," arXiv:hep-th/9902171 [hep-th].

[90] M. Perry and J. H. Schwarz, "Interacting chiral gauge fields in six-dimensions and Born-Infeld theory," *Nucl.Phys.* **B489** (1997) 47–64, arXiv:hep-th/9611065 [hep-th].

[91] P. Pasti, D. P. Sorokin, and M. Tonin, "Covariant action for a D = 11 five-brane with the chiral field," *Phys.Lett.* **B398** (1997) 41–46, arXiv:hep-th/9701037 [hep-th].

[92] M. Henneaux and C. Teitelboim, "Consistent quantum mechanics of chiral p forms," *In Santiago 1987, Proceedings, Quantum mechanics of fundamental systems V2, 113-152* (1987) .

[93] M. Henneaux and C. Teitelboim, "DYNAMICS OF CHIRAL (SELFDUAL) P FORMS," *Phys.Lett.* **B206** (1988) 650.

[94] J. H. Schwarz and A. Sen, "Duality symmetric actions," *Nucl.Phys.* **B411** (1994) 35–63, arXiv:hep-th/9304154 [hep-th].

[95] J. H. Schwarz, "Coupling a self-dual tensor to gravity in six dimensions," *Phys. Lett.* **B395** (1997) 191–195, arXiv:hep-th/9701008.

[96] P. Pasti, D. P. Sorokin, and M. Tonin, "Note on manifest Lorentz and general coordinate invariance in duality symmetric models," *Phys.Lett.* **B352** (1995) 59–63, arXiv:hep-th/9503182 [hep-th].

[97] P. Pasti, D. P. Sorokin, and M. Tonin, "Duality symmetric actions with manifest space-time symmetries," *Phys.Rev.* **D52** (1995) 4277–4281, arXiv:hep-th/9506109 [hep-th].

[98] P. Pasti, D. P. Sorokin, and M. Tonin, "On Lorentz invariant actions for chiral p forms," *Phys.Rev.* **D55** (1997) 6292–6298, arXiv:hep-th/9611100 [hep-th].

[99] W.-M. Chen and P.-M. Ho, "Lagrangian Formulations of Self-dual Gauge Theories in Diverse Dimensions," *Nucl.Phys.* **B837** (2010) 1–21, arXiv:1001.3608 [hep-th].

[100] P. Vanichchapongjaroen, "Branes and applications in string theory and M-theory,".

[101] P. Claus, R. Kallosh, and A. Van Proeyen, "M five-brane and superconformal (0,2) tensor multiplet in six-dimensions," *Nucl.Phys.* **B518** (1998) 117–150, arXiv:hep-th/9711161 [hep-th].

[102] G. Gibbons and P. Townsend, "Vacuum interpolation in supergravity via super p-branes," *Phys.Rev.Lett.* **71** (1993) 3754–3757, arXiv:hep-th/9307049 [hep-th].

[103] D. M. Kaplan and J. Michelson, "Zero modes for the D = 11 membrane and five-brane," *Phys.Rev.* **D53** (1996) 3474–3476, arXiv:hep-th/9510053 [hep-th].

[104] E. Witten, "Five-brane effective action in M theory," *J.Geom.Phys.* **22** (1997) 103–133, arXiv:hep-th/9610234 [hep-th].

[105] M. Cederwall, B. E. Nilsson, and P. Sundell, "An Action for the superfive-brane in D = 11 supergravity," *JHEP* **9804** (1998) 007, arXiv:hep-th/9712059 [hep-th].

[106] M. Henneaux and B. Knaepen, "All consistent interactions for exterior form gauge fields," *Phys.Rev.* **D56** (1997) 6076–6080, arXiv:hep-th/9706119 [hep-th].

[107] M. Henneaux, "Uniqueness of the Freedman-Townsend interaction vertex for two form gauge fields," *Phys.Lett.* **B368** (1996) 83–88, arXiv:hep-th/9511145 [hep-th].

[108] M. Henneaux and B. Knaepen, "A Theorem on first order interaction vertices for free p form gauge fields," *Int.J.Mod.Phys.* **A15** (2000) 3535–3548, arXiv:hep-th/9912052 [hep-th].

[109] R. I. Nepomechie, "APPROACHES TO A NONABELIAN ANTISYMMETRIC TENSOR GAUGE FIELD THEORY," *Nucl.Phys.* **B212** (1983) 301.

[110] C.-S. Chu and D. J. Smith, "Multiple Self-Dual Strings on M5-Branes," *JHEP* **1001** (2010) 001, arXiv:0909.2333 [hep-th].

[111] D. S. Berman and D. C. Thompson, "Membranes with a boundary," *Nucl.Phys.* **B820** (2009) 503–533, arXiv:0904.0241 [hep-th].

[112] D. S. Berman, M. J. Perry, E. Sezgin, and D. C. Thompson, "Boundary Conditions for Interacting Membranes," *JHEP* **1004** (2010) 025, arXiv:0912.3504 [hep-th].

[113] C. Bunster and M. Henneaux, "The Action for Twisted Self-Duality," *Phys.Rev.* **D83** (2011) 125015, arXiv:1103.3621 [hep-th].

[114] N. Lambert and C. Papageorgakis, "Nonabelian (2,0) Tensor Multiplets and 3-algebras," *JHEP* **1008** (2010) 083, arXiv:1007.2982 [hep-th].

[115] C.-S. Chu and D. J. Smith, "Towards the Quantum Geometry of the M5-brane in a Constant C-Field from Multiple Membranes," *JHEP* **0904** (2009) 097, arXiv:0901.1847 [hep-th].

[116] J. DeBellis, C. Saemann, and R. J. Szabo, "Quantized Nambu-Poisson Manifolds and n-Lie Algebras," *J.Math.Phys.* **51** (2010) 122303, arXiv:1001.3275 [hep-th].

[117] J. DeBellis, C. Saemann, and R. J. Szabo, "Quantized Nambu-Poisson Manifolds in a 3-Lie Algebra Reduced Model," *JHEP* **1104** (2011) 075, arXiv:1012.2236 [hep-th].

[118] C.-S. Chu and G. S. Sehmbi, "D1-Strings in Large RR 3-Form Flux, Quantum Nambu Geometry and M5-Branes in C-Field," *J.Phys.A* **A45** (2012) 055401, arXiv:1110.2687 [hep-th].

[119] K. Furuuchi, "Non-Linearly Extended Self-Dual Relations From The Nambu-Bracket Description Of M5-Brane In A Constant C-Field Background," *JHEP* **1003** (2010) 127, arXiv:1001.2300 [hep-th].

[120] B. Czech, Y.-t. Huang, and M. Rozali, "Amplitudes for Multiple M5 Branes," arXiv:1110.2791 [hep-th].

[121] S. Bolognesi and K. Lee, "1/4 BPS String Junctions and N^3 Problem in 6-dim (2,0) Superconformal Theories," *Phys.Rev.* **D84** (2011) 126018, arXiv:1105.5073 [hep-th].

[122] H.-C. Kim, S. Kim, E. Koh, K. Lee, and S. Lee, "On instantons as Kaluza-Klein modes of M5-branes," *JHEP* **1112** (2011) 031, arXiv:1110.2175 [hep-th].

[123] S. Terashima and F. Yagi, "On Effective Action of Multiple M5-branes and ABJM Action," *JHEP* **1103** (2011) 036, arXiv:1012.3961 [hep-th].

[124] H. Singh, "Super-Yang-Mills and M5-branes," *JHEP* **1108** (2011) 136, arXiv:1107.3408 [hep-th].

[125] Y. Tachikawa, "On S-duality of 5d super Yang-Mills on S^1," *JHEP* **1111** (2011) 123, arXiv:1110.0531 [hep-th].

[126] A. Gustavsson, "M5 brane on $R^{1,2} \times S^3$," *JHEP* **1201** (2012) 057, arXiv:1111.5392 [hep-th].

[127] N. Lambert, H. Nastase, and C. Papageorgakis, "5D Yang-Mills instantons from ABJM Monopoles," *Phys.Rev.* **D85** (2012) 066002, arXiv:1111.5619 [hep-th].

[128] A. Gustavsson, "A preliminary test of Abelian D4-M5 duality," *Phys.Lett.* **B706** (2011) 225–227, arXiv:1111.6339 [hep-th].

[129] P. Arvidsson, "Superconformal Theories in Six Dimensions," arXiv:hep-th/0608014 [hep-th].

[130] M. Banados, L. J. Garay, and M. Henneaux, "The Local degrees of freedom of higher dimensional pure Chern-Simons theories," *Phys.Rev.* **D53** (1996) 593–596, arXiv:hep-th/9506187 [hep-th].

[131] M. Banados, L. J. Garay, and M. Henneaux, "The Dynamical structure of higher dimensional Chern-Simons theory," *Nucl.Phys.* **B476** (1996) 611–635, arXiv:hep-th/9605159 [hep-th].

[132] O. Miskovic, R. Troncoso, and J. Zanelli, "Canonical sectors of five-dimensional Chern-Simons theories," *Phys.Lett.* **B615** (2005) 277–284, arXiv:hep-th/0504055 [hep-th].

[133] M. B. Green, J. Schwarz, and E. Witten, "SUPERSTRING THEORY. VOL. 1: INTRODUCTION," *Cambridge Monogr.Math.Phys.* (1987) .

[134] A. A. Tseytlin, "Born-Infeld action, supersymmetry and string theory," arXiv:hep-th/9908105 [hep-th].

[135] P. Koerber and A. Sevrin, "The NonAbelian D-brane effective action through order alpha-prime**4," *JHEP* **0210** (2002) 046, arXiv:hep-th/0208044 [hep-th].

[136] I. A. Bandos, N. Berkovits, and D. P. Sorokin, "Duality symmetric eleven-dimensional supergravity and its coupling to M-branes," *Nucl.Phys.* **B522** (1998) 214–233, arXiv:hep-th/9711055 [hep-th].

[137] N. Ishibashi, H. Kawai, Y. Kitazawa, and A. Tsuchiya, "A Large N reduced model as superstring," *Nucl.Phys.* **B498** (1997) 467–491, arXiv:hep-th/9612115 [hep-th].

[138] W. Taylor, "M(atrix) theory: Matrix quantum mechanics as a fundamental theory," *Rev.Mod.Phys.* **73** (2001) 419–462, arXiv:hep-th/0101126 [hep-th].

[139] M. R. Douglas, "D-branes and matrix theory in curved space," *Nucl.Phys.Proc.Suppl.* **68** (1998) 381–393, arXiv:hep-th/9707228 [hep-th].

[140] M. R. Douglas, A. Kato, and H. Ooguri, "D-brane actions on Kahler manifolds," *Adv.Theor.Math.Phys.* **1** (1998) 237–258, arXiv:hep-th/9708012 [hep-th].

[141] M. Hanada, H. Kawai, and Y. Kimura, "Describing curved spaces by matrices," *Prog.Theor.Phys.* **114** (2006) 1295–1316, arXiv:hep-th/0508211 [hep-th].

[142] I. R. Klebanov and A. A. Tseytlin, "Entropy of near extremal black p-branes," *Nucl.Phys.* **B475** (1996) 164–178, arXiv:hep-th/9604089 [hep-th].

[143] T. Maxfield and S. Sethi, "The Conformal Anomaly of M5-Branes," *JHEP* **1206** (2012) 075, arXiv:1204.2002 [hep-th].

[144] H. Samtleben, E. Sezgin, R. Wimmer, and L. Wulff, "New superconformal models in six dimensions: Gauge group and representation structure," *PoS* **CORFU2011** (2011) 071, arXiv:1204.0542 [hep-th].

[145] M. Akyol and G. Papadopoulos, "(1,0) superconformal theories in six dimensions and Killing spinor equations," *JHEP* **1207** (2012) 070, arXiv:1204.2167 [hep-th].

[146] H. Nishino and S. Rajpoot, "N=1 Non-Abelian Tensor Multiplet in Four Dimensions," *Phys.Rev.* **D85** (2012) 105017, arXiv:1204.1379 [hep-th].

[147] S. Deser, A. Gomberoff, M. Henneaux, and C. Teitelboim, "Duality, selfduality, sources and charge quantization in Abelian N form theories," *Phys.Lett.* **B400** (1997) 80–86, arXiv:hep-th/9702184 [hep-th].

[148] S. Deser, A. Gomberoff, M. Henneaux, and C. Teitelboim, "P-brane dyons and electric magnetic duality," *Nucl.Phys.* **B520** (1998) 179–204, arXiv:hep-th/9712189 [hep-th].

[149] Y. M. Shnir, "Magnetic Monopoles,".

[150] G. 't Hooft, "Magnetic Monopoles in Unified Gauge Theories," *Nucl.Phys.* **B79** (1974) 276–284.

[151] A. M. Polyakov, "Particle Spectrum in the Quantum Field Theory," *JETP Lett.* **20** (1974) 194–195.

[152] B. Chen, H. Itoyama, and H. Kihara, "NonAbelian Berry phase, Yang-Mills instanton and USp(2k) matrix model," *Mod.Phys.Lett.* **A14** (1999) 869–879, arXiv:hep-th/9810237 [hep-th].

[153] B. Chen, H. Itoyama, and H. Kihara, "NonAbelian monopoles from matrices: Seeds of the space-time structure," *Nucl.Phys.* **B577** (2000) 23–46, arXiv:hep-th/9909075 [hep-th].

[154] C. Pedder, J. Sonner, and D. Tong, "The Geometric Phase and Gravitational Precession of D-Branes," *Phys.Rev.* **D76** (2007) 126014, arXiv:0709.2136 [hep-th].

[155] V. L. Campos, G. Ferretti, and P. Salomonson, "The NonAbelian self dual string on the light cone," *JHEP* **0012** (2000) 011, arXiv:hep-th/0011271 [hep-th].

[156] C. Saemann, "Constructing Self-Dual Strings," *Commun.Math.Phys.* **305** (2011) 513–532, arXiv:1007.3301 [hep-th].

[157] S. Palmer and C. Saemann, "Constructing Generalized Self-Dual Strings," *JHEP* **1110** (2011) 008, arXiv:1105.3904 [hep-th].

[158] C.-S. Chu and P. Vanichchapongjaroen, "Non-abelian Self-Dual String and M2-M5 Branes Intersection in Supergravity," arXiv:1304.4322 [hep-th].

[159] M. Atiyah and N. J. Hitchin, "THE GEOMETRY AND DYNAMICS OF MAGNETIC MONOPOLES. M.B. PORTER LECTURES,".

[160] N. Manton and P. Sutcliffe, "Topological solitons,".

[161] V. Niarchos and K. Siampos, "M2-M5 blackfold funnels," *JHEP* **1206** (2012) 175, arXiv:1205.1535 [hep-th].

[162] V. Niarchos and K. Siampos, "The black M2-M5 ring intersection spins," *PoS* **Corfu2012** (2013) 088, arXiv:1302.0854 [hep-th].

[163] C.-S. Chu and H. Isono, "Instanton String and M-Wave in Multiple M5-Branes System," arXiv:1305.6808 [hep-th].

[164] C.-S. Chu, "Non-Abelian Self-Dual Strings in Six Dimensions from Four Dimensional 1/2-BPS Monopoles," *Nucl.Phys.* **B882** (2014) 289–302, arXiv:1310.7710 [hep-th].

[165] D. S. Berman and J. A. Harvey, "The Self-dual string and anomalies in the M5-brane," *JHEP* **0411** (2004) 015, arXiv:hep-th/0408198 [hep-th].

[166] D. Zwanziger, "Local Lagrangian quantum field theory of electric and magnetic charges," *Phys.Rev.* **D3** (1971) 880.

[167] S. Deser and C. Teitelboim, "Duality Transformations of Abelian and Nonabelian Gauge Fields," *Phys.Rev.* **D13** (1976) 1592–1597.

[168] R. Kallosh, "N=8 Counterterms and $E_{7(7)}$ Current Conservation," *JHEP* **1106** (2011) 073, arXiv:1104.5480 [hep-th].

[169] G. Bossard and H. Nicolai, "Counterterms vs. Dualities," *JHEP* **1108** (2011) 074, arXiv:1105.1273 [hep-th].

[170] J. J. M. Carrasco, R. Kallosh, and R. Roiban, "Covariant procedures for perturbative non-linear deformation of duality-invariant theories," *Phys.Rev.* **D85** (2012) 025007, arXiv:1108.4390 [hep-th].

[171] W. Chemissany, R. Kallosh, and T. Ortin, "Born-Infeld with Higher Derivatives," *Phys.Rev.* **D85** (2012) 046002, arXiv:1112.0332 [hep-th].

[172] P. Pasti, D. Sorokin, and M. Tonin, "Covariant actions for models with non-linear twisted self-duality," *Phys.Rev.* **D86** (2012) 045013, arXiv:1205.4243 [hep-th].

[173] R. Roiban and A. Tseytlin, "On duality symmetry in perturbative quantum theory," arXiv:1205.0176 [hep-th].

[174] S. M. Kuzenko, "Nonlinear self-duality in N = 2 supergravity," *JHEP* **1206** (2012) 012, arXiv:1202.0126 [hep-th].

[175] E. Ivanov and B. Zupnik, "Bispinor Auxiliary Fields in Duality-Invariant Electrodynamics Revisited," *Phys.Rev.* **D87** (2013) 065023, arXiv:1212.6637.

[176] N. Lambert and P. Richmond, "(2,0) Supersymmetry and the Light-Cone Description of M5-branes," *JHEP* **1202** (2012) 013, arXiv:1109.6454 [hep-th].

[177] N. Lambert, C. Papageorgakis, and M. Schmidt-Sommerfeld, "Deconstructing (2,0) Proposals," arXiv:1212.3337 [hep-th].

[178] K.-W. Huang, "Non-Abelian Chiral 2-Form and M5-Branes," arXiv:1206.3983 [hep-th].

[179] F. Bonetti, T. W. Grimm, and S. Hohenegger, "A Kaluza-Klein inspired action for chiral p-forms and their anomalies," *Phys.Lett.* **B720** (2013) 424–427, arXiv:1206.1600 [hep-th].

[180] H. Singh, "The Yang-Mills and chiral fields in six dimensions," *JHEP* **1302** (2013) 056, arXiv:1211.3281 [hep-th].

[181] H.-C. Kim and K. Lee, "Supersymmetric M5 Brane Theories on R x CP2," arXiv:1210.0853 [hep-th].

[182] D. Fiorenza, H. Sati, and U. Schreiber, "Multiple M5-branes, String 2-connections, and 7d nonabelian Chern-Simons theory," arXiv:1201.5277 [hep-th].

[183] C. Saemann, "M-Brane Models and Loop Spaces," *Mod.Phys.Lett.* **A27** (2012) 1230019, arXiv:1206.0432 [hep-th].

[184] S. Palmer and C. Saemann, "M-brane Models from Non-Abelian Gerbes," *JHEP* **1207** (2012) 010, arXiv:1203.5757 [hep-th].

[185] D. P. Sorokin, V. I. Tkach, and D. V. Volkov, "Superparticles, Twistors and Siegel Symmetry," Preprint KIPT-31 04/04/1988, *Mod. Phys. Lett.* **A4** (1989) 901–908.

[186] I. A. Bandos, D. P. Sorokin, M. Tonin, P. Pasti, and D. V. Volkov, "Superstrings and supermembranes in the doubly supersymmetric geometrical approach," *Nucl. Phys.* **B446** (1995) 79–118, arXiv:hep-th/9501113.

[187] I. A. Bandos, "Superembedding approach to Dp-branes, M-branes and multiple D(0)-brane systems," *Phys.Part.Nucl.Lett.* **8** (2011) 149–172, arXiv:0912.2530 [hep-th].

[188] A. Maznytsia, C. R. Preitschopf, and D. P. Sorokin, "Duality of self-dual actions," *Nucl. Phys.* **B539** (1999) 438–452, arXiv:hep-th/9805110.

[189] P.-M. Ho, "A Concise Review on M5-brane in Large C-Field Background," *Chin.J.Phys.* **48** (2010) 1, arXiv:0912.0445 [hep-th].

[190] A. Gustavsson, "M5 brane from mass deformed BLG theory," *JHEP* **0911** (2009) 071, arXiv:0909.2518 [hep-th].

[191] I. A. Bandos and P. K. Townsend, "Light-cone M5 and multiple M2-branes," *Class.Quant.Grav.* **25** (2008) 245003, arXiv:0806.4777 [hep-th].

[192] N. Seiberg and E. Witten, "String theory and noncommutative geometry," *JHEP* **9909** (1999) 032, arXiv:hep-th/9908142 [hep-th].

[193] D. Belov and G. W. Moore, "Holographic Action for the Self-Dual Field," arXiv:hep-th/0605038 [hep-th].

[194] S. Monnier, "The Anomaly line bundle of the self-dual field theory," arXiv:1109.2904 [hep-th].

[195] W.-H. Huang, "Lagrangian of Self-dual Gauge Fields in Various Formulations," *Nucl.Phys.* **B861** (2012) 403–423, arXiv:1111.5118 [hep-th].

[196] L. Dolan and C. R. Nappi, "A modular invariant partition function for the fivebrane," *Nucl. Phys.* **B530** (1998) 683–700, arXiv:hep-th/9806016.

[197] M. Henningson, B. E. Nilsson, and P. Salomonson, "Holomorphic factorization of correlation functions in (4k+2)-dimensional (2k) form gauge theory," *JHEP* **9909** (1999) 008, arXiv:hep-th/9908107 [hep-th].

[198] E. Witten, "Duality relations among topological effects in string theory," *JHEP* **05** (2000) 031, arXiv:hep-th/9912086.

[199] W.-M. Chen, P.-M. Ho, H.-c. Kao, F. S. Khoo, and Y. Matsuo, "Partition Function of Chiral Boson on 2-Torus from Floreanini-Jackiw Lagrangian," arXiv:1307.2172 [hep-th].

[200] M. K. Gaillard and B. Zumino, "Duality Rotations for Interacting Fields," *Nucl.Phys.* **B193** (1981) 221.

[201] G. Gibbons and D. Rasheed, "Electric - magnetic duality rotations in nonlinear electrodynamics," *Nucl.Phys.* **B454** (1995) 185–206, arXiv:hep-th/9506035 [hep-th].

[202] M. Hatsuda, K. Kamimura, and S. Sekiya, "Electric magnetic duality invariant Lagrangians," *Nucl.Phys.* **B561** (1999) 341–353, arXiv:hep-th/9906103 [hep-th].

[203] J. W. Helton *et al.*, "Non Commutative Algebra Packages run under *Mathematica* http://www.math.ucsd.edu/ ncalg/," Department of Mathematics UCSB.

[204] D. Berman, "M5 on a torus and the three-brane," *Nucl.Phys.* **B533** (1998) 317–332, arXiv:hep-th/9804115 [hep-th].

[205] D. Berman, "The M five-brane on a torus," *Lect.Notes Phys.* **525** (1999) 398, arXiv:hep-th/9812053 [hep-th].

[206] E. Bergshoeff, D. Berman, J. van der Schaar, and P. Sundell, "A Noncommutative M theory five-brane," *Nucl.Phys.* **B590** (2000) 173–197, arXiv:hep-th/0005026 [hep-th].

[207] E. Witten, "Nonperturbative superpotentials in string theory," *Nucl.Phys.* **B474** (1996) 343–360, arXiv:hep-th/9604030 [hep-th].

[208] R. Kallosh and D. Sorokin, "Dirac action on M5 and M2 branes with bulk fluxes," *JHEP* **0505** (2005) 005, arXiv:hep-th/0501081 [hep-th].

[209] L. Anguelova and K. Zoubos, "Five-brane Instantons vs Flux-induced Gauging of Isometries," *JHEP* **0610** (2006) 071, arXiv:hep-th/0606271 [hep-th].

[210] D. Tsimpis, "Fivebrane instantons and Calabi-Yau fourfolds with flux," *JHEP* **0703** (2007) 099, arXiv:hep-th/0701287 [hep-th].

[211] M. Kerstan and T. Weigand, "Fluxed M5-instantons in F-theory," *Nucl.Phys.* **B864** (2012) 597–639, arXiv:1205.4720 [hep-th].

[212] M. Bianchi, G. Inverso, and L. Martucci, "Brane instantons and fluxes in F-theory," *JHEP* **1307** (2013) 037, arXiv:1212.0024 [hep-th].

[213] S.-L. Ko, D. Sorokin, and P. Vanichchapongjaroen, "to appear,".

[214] S.-L. Ko, D. Sorokin, and P. Vanichchapongjaroen, "The M5-brane action revisited," *JHEP* **1311** (2013) 072, arXiv:1308.2231 [hep-th].

[215] V. Niarchos, "Supersymmetric Perturbations of the M5 brane," *JHEP* **1405** (2014) 023, arXiv:1402.4132 [hep-th].

[216] H. Mori and S. Yamaguchi, "M5-branes and Wilson Surfaces in AdS_7/CFT_6 Correspondence," arXiv:1404.0930 [hep-th].

[217] X. Bekaert and M. Henneaux, "Comments on chiral p forms," *Int.J.Theor.Phys.* **38** (1999) 1161–1172, arXiv:hep-th/9806062 [hep-th].

[218] I. Bandos, "On Lagrangian approach to self-dual gauge fields in spacetime of nontrivial topology," arXiv:1406.5185 [hep-th].

[219] H. Isono, "Note on the self-duality of gauge fields in topologically nontrivial spacetime," arXiv:1406.6023 [hep-th].

[220] X. Bekaert and S. Cucu, "Antifield BRST quantization of duality-symmetric Maxwell theory," *JHEP* **01** (2001) 015, arXiv:hep-th/0010266.

[221] L. Anderson and H. Linander, "The trouble with twisting (2,0) theory," *JHEP* **1403** (2014) 062, arXiv:1311.3300 [hep-th].

[222] U. Gran, H. Linander, and B. E. W. Nilsson, "Off-shell structure of twisted (2,0) theory," arXiv:1406.4499 [hep-th].

[223] S. A. Cherkis and J. H. Schwarz, "Wrapping the M theory five-brane on K3," *Phys.Lett.* **B403** (1997) 225–232, arXiv:hep-th/9703062 [hep-th].

[224] A. Sevrin and D. C. Thompson, "A Note on Supersymmetric Chiral Bosons," *JHEP* **1307** (2013) 086, arXiv:1305.4848 [hep-th].

[225] M. Henneaux and C. Teitelboim, "Quantization of gauge systems," (1992) .

[226] P. Pasti, D. P. Sorokin, and M. Tonin, "Comment on covariant duality symmetric actions," *Phys. Rev.* **D56** (1997) 2473–2474, arXiv:hep-th/9607171.

[227] P. Pasti, D. P. Sorokin, and M. Tonin, "Space-time symmetries in duality symmetric models," arXiv:hep-th/9509052 [hep-th].