Handle Cancellation in Flow Categories and the Khovanov Stable Homotopy Type

JONES, DANIEL

Handle Cancellation in Flow Categories and the Khovanov Stable Homotopy Type

Daniel Jones

A Thesis presented for the degree of
Doctor of Philosophy

Durham University

The Pure Mathematics Group
Department of Mathematical Sciences
University of Durham
England

2015

Declaration

The work in this thesis is based on research carried out at the Department of Mathematical Sciences, Durham University, England. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all my own work unless referenced to the contrary in the text.

Acknowledgements

I would like to collectively thank anybody who has blessed me with great company throughout the entire process of producing this thesis; there are many of you! A special thanks is due to my inspiration and life partner, Gina. You have helped me fill many dull moments with joy and happiness, and have played a huge part in the completion of this thesis. Without you, I would be lost.

Both Tomas 'hermano' Andrade and Cynthia Meersohn-Schmidt have provided a home away from home during my time in Durham. Thank you for many long nights full of delicious food, wine and dancing. It is comforting to know that the future will be full of such nights with you. In addition to my Durham family, I was glad to have Karin Sasaki play the role of my academic sister. It is such an advantage to have somebody looking out for you who was in your position a couple of years ago. Thanks for always throwing tips and encouragement my way. When it came to making a decision about my future career, your advice was invaluable.

Aside from making many new friends in the last few years, I have also grown closer to existing ones. Of them, I would particularly like to thank Whitney Arostegui for being an amazing friend, and for filling me with lots of tacos during my visit to Austin. Without them, I would probably not have had the energy to burn through this thesis.

I would also like to thank my supervisor, Dirk Schütz, for always being available, and for pointing a finger (sometimes literally) in the right direction during some of the most difficult stages of my PhD. Additionally, I was very fortunate in having Andrew Lobb play the role of my second supervisor. I would like to thank him for introducing me to such an interesting topic in topology, but also for accompanying me during many fun moments in Durham, whether it be in the gym, office, restaurant or pub.

The environment created for me as a PhD student could not have been better. The Department of Mathematical Sciences has provided me with an abundance of resources and support in the last few years. In particular, the ladies in the main office have been absolutely amazing. Without them, the department would fall apart. So thank you for answering all of my questions and making everything so easy and comfortable (and full of cake!). The department also have their very own Saint in the form of John Parker, and I would like to thank him for being such a kind human being and always being available for a chat.

As a PhD student, it was not uncommon for me to spend 12 hours a day in my office. Fortunately, I was always surrounded by great office mates who helped the time pass by like a breeze. I would therefore like to thank all of the Johns and Luke Stanbra for putting up with my attempts to rap my favourite verses, and in general for being a massive distraction. In particular, streaming live sports events with Lewis Paton and late night sessions with DJJC radio kept me motivated.

During the process of writing up this thesis, there were many technical complications. I would like to give a special mention to tech-whizzes Paul Jennings, Andy Iskauskas and John Lawson, who usually solved these complications in less than five minutes. Thank you also to Lauren Scanlon for helping me navigate around KnotPlot. A big thank you to Ben Thorpe for being the sole proof-reader for the original version of this thesis. Your comments were extremely helpful and your conversations were extremely inappropriate, but I enjoyed all of them.

Finally, a special thanks to both John Hunton and Liam Watson. You could not have been more accommodating during my viva, and I am so grateful to you for providing recommendations that have made this thesis feel complete and something that I can be even more proud of.

Contents

Chapter 1

Introduction

Back in 2000, Khovanov produced a link invariant in the form of a bi-graded homology theory now known as *Khovanov homology* and whose graded Euler characteristic recovers the Jones polynomial. To be more precise, if D is a diagram of a link L, then $Kh^{i,j}(L)$ is a bi-graded abelian group which is obtained by taking homology of a chain complex $KhC^{i,j}(D)$ with a differential

$$d : KhC^{i,j}(D) \to KhC^{i+1,j}(D).$$

The fundamental property of this chain complex is that two link diagrams of the same link differing by a sequence of Reidemeister moves give rise to chain complexes that are homotopy equivalent, and this makes Khovanov homology a link invariant. The property of 'categorifying' the Jones polynomial is

$$\chi(Kh^{i,j}(L)) := \sum_{i,j}(-1)^i q^j \ \mathrm{rk} Kh^{i,j}(L) = (q+q^{-1})J(L)$$

where $J(L)$ is the normalised Jones polynomial, with $J(U) = 1$ for the unknot U, and χ is the graded Euler characteristic. Since then, a plethora of invariants has been defined. For example, Khovanov homology can be thought of as a single member of a whole family of link homology theories for each integer $n \geq 2$ known as Khovanov-Rozansky homologies, which are defined in [KR08a] and categorify the quantum $sl(n)$ link polynomials. Other invariants have a much more geometric flavour, providing, for example, slice genus bounds arising from perturbations of Khovanov homology (see [Ras04]) and further, Khovanov-Rozansky homologies (see [Lob09], [Lob12] [Wu09]).

Of particular interest to this thesis is a more recent invariant arising from Khovanov homology known as the *Khovanov stable homotopy type*, which was constructed by Lipshitz-Sarkar in [LS14a] as a space-level refinement of Khovanov homology. In their paper, they construct a family of spaces $\mathcal{X}^j_{Kh}(L)$, for each $j \in \mathbb{Z}$, whose stable homotopy type is an invariant of L and whose reduced singular cohomology recovers the Khovanov homology of L. That is,

$$\widetilde{H}^i(\mathcal{X}^j_{Kh}(L)) = Kh^{i,j}(L).$$

In fact, this thesis focuses more directly on a construction that predates and directly influences the work of Lipshitz-Sarkar. What inspired their work was a paper written by Cohen-Jones-Segal [CJS95a] which asks whether such a space-level refinement exists that would provide a stable homotopy type for Floer homology. Although such a homotopy-theoretic description of Floer homology was not produced in [CJS95a], Cohen-Jones-Segal did set up a construction using higher-dimensional moduli spaces from Floer theory, and this was successfully implemented in [LS14a]. The fact that a machine designed for Floer theory can be applied successfully to Khovanov homology should be of no surprise, since the two theories have been shown to be related via spectral sequences (see [Bal10], [OS05]) and further relations between the two is actively studied. The information of such higher-dimensional moduli spaces is encompassed in what Cohen-Jones-Segal call a *framed flow category.* A framed flow category is a category $\mathscr{C}$ whose objects are $\mathbb{Z}$-graded, and whose morphisms are manifolds with corners satisfying certain compatibility conditions, which are outlined in Chapter 2. Cohen-Jones-Segal in [CJS95a] provides a machine whose input is a framed flow category $\mathscr{C}$ and whose output is a CW-complex $|\mathscr{C}|$ called the *realisation* of $\mathscr{C}$. Thus, in order to produce a Khovanov stable homotopy type, Lipshitz-Sarkar in [LS14a] produce a *Khovanov flow category*, denoted $\mathscr{C}_{Kh}$, and a fair amount of work goes into producing a framing for this category. To do this, a relatively simple flow category is used to model $\mathscr{C}_{Kh}$, and this is called the *cube flow category.* In their sequel paper [LS14b], Lipshitz-Sarkar make use of stable cohomology operations induced by the stable homotopy type $\mathcal{X}_{Kh}(L)$. In particular, they develop a combinatorial definition for computing the second Steenrod square

$$\mathrm{Sq}^2 : Kh^{i,j}(L, \mathbb{Z}/2) \to Kh^{i+2,j}(L, \mathbb{Z}/2).$$

When the link L has relatively simple Khovanov homology, Baues [Bau95] shows that the operation Sq^2, along with Sq^1 (which is determined by integral Khovanov homology), is enough to determine the stable homotopy type of $\mathcal{X}_{Kh}(L)$. This includes all links with up to 11 crossings, and [LS14b] provides a list of the stable homotopy types for all such links. Moreover, it was later shown by Seed in [See12] that there are several pairs of knots and links with isomorphic Khovanov homologies that can be distinguished by their associated homotopy type, making the stable Khovanov homotopy type a stronger invariant than Khovanov homology.

Even more recently, the author of this thesis along with Lobb and Schütz has looked at providing a more compact description of the Khovanov stable homotopy type for knots and links that admit a special kind of diagram. In [JLS15], Jones-Lobb-Schütz consider knots and links that admit a diagram which can be constructed by gluing together a collection of elementary tangles (see Definition 5.3.1, Figure 5.3). Such diagrams are referred to as *matched diagrams* and are usually denoted $D_{\mathbf{r}}$, where $\mathbf{r} = (r_1, \ldots, r_m) \in (\mathbb{Z} \setminus \{0\})^m$ for an underlying diagram D that can be constructed by gluing together m elementary tangles, each with $|r_i|$ crossings for $i = 1, \ldots, m$. An observation by Bar-Natan in [BN07] (see also [Kho00, Subsection 6.2]) describes how Gaussian elimination can be used at the level of the Khovanov chain complex of such links, resulting in a description of the complex as a tensor product of 'flattened' complexes. Jones-Lobb-Schütz consider this simplification under Gaussian elimination and construct a framed flow category $\mathscr{L}(D_{\mathbf{r}})$ whose objects correspond to the standard generators of the cancelled complex, and whose morphisms are modified accordingly (this is described in Chapter 5.3). In [JLS15], difficulties similar to that in [LS14a] arise in providing a framing of the flow category $\mathscr{L}(D_{\mathbf{r}})$. A flow category known as the *sock flow category* is constructed which is analogous to the cube flow category in [LS14a], and models $\mathscr{L}(D_{\mathbf{r}})$. The ultimate outcome of the construction in [JLS15] is that if a link L has a matched diagram $D_{\mathbf{r}}$, then $|\mathscr{L}(D_{\mathbf{r}})| \simeq |\mathscr{C}_{Kh}(L)|$. Khovanov homology is a special homology theory ($n = 2$) that occurs as a generalisation ($n \geq 2$) of a family of homology theories defined by Khovanov and Rozansky in [KR08b]. Additional motivation for studying matched diagrams is the realisation of Krasner in [Kra09] that by performing Gaussian elimination on matched diagrams, the Khovanov-Rozansky chain complex can be described using a TQFT similar to the one above, but with $V = \mathbb{Z}[x]/x^n$. Therefore, by producing a homotopy type associated to such diagrams one may be able to produce a conjectural description of a Khovanov-Rozansky homotopy type. This is precisely the approach taken in [JLS15] where the authors describe in detail how the generalisation for all $n > 2$ works. For the purpose of this thesis, we shall only be focusing on the case $n = 2$, where we give a simpler description of the Khovanov homotopy type associated to matched diagrams.

The original work of this thesis can be summarised as follows. Firstly, the main theorem (Theorem 3.1.1) extends classical handle-cancellation occurring in Morse theory to framed flow categories. That is, by assuming that two objects of a framed flow category $\mathscr{C}$ have gradings that differ by one and have a moduli space between them that is a single point, one may cancel these two objects to give a new framed flow category $\mathscr{C}_H$ with two fewer objects, whose realisation $|\mathscr{C}_H|$ is homotopy equivalent to the realisation $|\mathscr{C}|$. The theorem is stated in Chapter 3 as follows.

Theorem 3.1.1 Let $(\mathscr{C}, \imath, \varphi)$ be a framed flow category containing two objects x and y with $\mathcal{M}(x, y) = *$. The realisation $|\mathscr{C}|$ is stably homotopy equivalent to the realisation $|\mathscr{C}_H|$, where $\mathrm{Ob}(\mathscr{C}_H) = \mathrm{Ob}(\mathscr{C}) \setminus \{x, y\}$ and the morphisms of $\mathscr{C}_H$ are defined in Definition 3.1.1.

This is a completely general result in the language of flow categories, and a direct application of the theorem is exhibited in the Khovanov set-up that is described above. In particular, the calculations of Steenrod squares can be simplified so that computations that once required the use of computer-implementation can be cancelled down and computed by hand. Chapter 6.4 does this explicitly for the first knot ($8_{19} = T(3,4)$) which gives rise to a non-trivial Steenrod square, and thus provides the non-triviality result of [LS14b, Theorem1]; $\mathcal{X}^{11}_{Kh}(8_{19})$ is not a wedge sum of Moore spaces. In this example, a sequence of handle-cancellations is provided that results in a final flow category containing only 3 objects. The stable homotopy type can then be determined directly.

This thesis is organised as follows. Chapter 2 introduces the general language of flow categories, and describes what is meant by a manifold with corners. It goes on to describe how such manifolds may be embedded into cornered Euclidean space in a nice way, particularly for morphisms of flow categories that are themselves manifolds with corners. The chapter then describes (in the language of [LS14a]) the Cohen-Jones-Segal construction of a CW-complex from a framed flow category. Continuing in the general setting of flow categories, Chapter 3 provides the main theorem on handle-cancellation in framed flow categories. Khovanov homology is not mentioned until Chapter 4, where the entire set-up of [LS14a] is described. This includes a diagrammatic definition of Khovanov homology using *resolution configurations* and the construction of the framed flow category $\mathscr{C}_{Kh}$, whose output from the Cohen-Jones-Segal machine gives the Khovanov stable homotopy type. Chapter 5 provides an overview of the work in [JLS15], namely the effect of Gaussian elimination on the Khovanov stable homotopy type. Then finally, Chapter 6 highlights how second Steenrod squares can be described for framed flow categories in general, and outlines the procedure of [JLS15] for computing the second Steenrod square of a cancelled category. The thesis concludes by highlighting how the cancellation theorem (Theorem 3.1.1) can be used to simplify drastically the computations of second Steenrod squares.

Chapter 2

Framed Flow Categories

2.1 Manifolds With Corners

In order to define a (framed) flow category, a fair amount of technical language has to be introduced, such as manifolds with corners and neat immersions/embeddings. We begin this chapter by setting up this language before proceeding to define the notion of a flow category, and further, a framed flow category. A smooth n-dimensional manifold-with-corners is similar to an ordinary smooth n-manifold, except that the differential structure is such that neighbourhoods of points in the manifold are homeomorphic to open subsets of $[0,\infty)^n = \mathbb{R}^n_+$, as opposed to $\mathbb{R}^n$. The language here follows [Lau00] and [Jän68].

Definition 2.1.1 A smooth n-dimensional *manifold-with-corners* is a topological space X along with a collection of charts $\{(U_\alpha, h_\alpha)\}_{\alpha\in A}$ where the U_α cover X and each h_α is a homeomorphism from U_α to an open subset of $\mathbb{R}^n_+$ such that the transition maps

$$h_{\alpha\beta} := h_\beta \circ h_\alpha^{-1}|_{h_\alpha\left(U_\alpha \cap U_\beta\right)} : h_\alpha\big(U_\alpha \cap U_\beta\big) \to h_\beta\big(U_\alpha \cap U_\beta\big)$$

extend smoothly for each $\alpha, \beta \in A$.

For each $x \in X$, let $c(x)$ be the number of 0 coordinates in $h_\alpha(x)$ for any $\alpha \in A$. Then the *codimension-i* boundary of X is denoted

$$\partial^i X = \{x \in X | c(x) = i\}.$$

Each $x \in X$ belongs to at most $c(x)$ connected components of $\partial^1 X$. A *boundary hypersurface* is the closure of a connected component of $\partial^1 X$ and a *face* of X is a union of disjoint (possibly empty) boundary hypersurfaces of X.

If X is a smooth manifold-with-corners and each $x \in X$ belongs to precisely $c(x)$ connected components of $\partial^1 X$, then X is said to be a *smooth manifold-with-faces.* The ordinary boundary ∂X is defined as the closure of the codimension-1 boundary $\partial^1 X$.

It should be clear that every face of a manifold-with-faces is itself a manifold-with-faces. In this way, the notion of a face can be extended to higher codimensions of a manifold-with-faces. For example, if Y is a face of a manifold-with-faces X, then any face of Y can also be considered a face of X (one codimension higher).

For non-negative integers n and k, a smooth n-dimensional manifold-with-faces X can be endowed with a *k-face structure* $(\partial_1 X, \ldots, \partial_k X)$, which is an ordered k-tuple of faces of X satisfying:

1. $\bigcup_i \partial_i X = \partial X$.

2. $\partial_i X \cap \partial_j X$ is a face of both $\partial_i X$ and $\partial_j X$ for all $i \neq j$.

A smooth manifold-with-faces X along with a k-face structure is called an *n-dimensional smooth $\langle k \rangle$-manifold.*

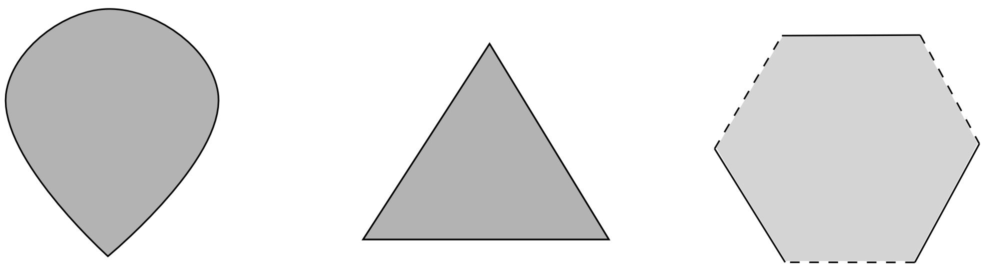
If $a = (a_1, \ldots, a_k) \in \{0,1\}^k$ then define

$$X(a) := \begin{cases} X & \text{if } a = (1, \ldots, 1) \\ \displaystyle\bigcap_{\{i|a_i=0\}} \partial_i X & \text{otherwise.} \end{cases}$$

For each $a \leq b$ in $\{0,1\}^n$ there is an obvious inclusion $X(a) \hookrightarrow X(b)$.

Definition 2.1.2 Define $\underline{2}^n$ to be the category with one object for every element of $\{0,1\}^n$ and a unique morphism from b to a if $b_i \leq a_i$ for $i = 1, \ldots, n$. Let $|a|$ be the sum of the entries of $a \in \{0,1\}^n$, which is called the *weight* of a. Denote by $\underline{0}$ the zero vector $(0,0,\ldots,0)$ and by $\underline{1}$ the vector $(1,1,\ldots,1)$.

The category $\underline{2}^n$ is used to define an *n-diagram*, which is a functor from $\underline{2}^n$ to the category of topological spaces. An $\langle n \rangle$-manifold can be viewed as an n-diagram



(a) A manifold-with-corners, but not a manifold-with-faces. (b) A manifold-with-faces, but not a $\langle 2\rangle$-manifold. (c) A $\langle 2\rangle$-manifold with ∂_1 solid and ∂_2 dashed.

Figure 2.1: Examples of manifolds with corners, faces and $\langle n\rangle$-manifolds.

by considering the $\langle|a|\rangle$-manifold $X(a)$ defined above. A product of an $\langle n\rangle$-manifold X with an $\langle m\rangle$-manifold Y is defined as the $\langle n+m\rangle$-manifold $X \times Y$ obtained via the isomorphism $\underline{2}^{n+m} \cong \underline{2}^n \times \underline{2}^m$.

Let Y be an $\langle n\rangle$-manifold. Then the inclusion maps $Y(a) \to Y(\underline{1})$ are all injective, and $Y = Y(\underline{1})$ is the direct limit of Y, considered as an n-diagram. Thus, it is possible to describe the boundary of an $\langle n\rangle$-manifold Y without mentioning $Y(\underline{1})$. This is the purpose of the following definition and, along with the proposition that follows (which is [LS14a, Proposition 3.9]), is used in the inductive procedure of defining moduli spaces between decorated resolution configurations (see Proposition 4.3.2, particularly (B2)).

Definition 2.1.3 Consider $\underline{2}^n \setminus \underline{1}$, the full subcategory of $\underline{2}^n$ consisting of all objects except $\underline{1}$. A *truncated n-diagram* is a functor from $\underline{2}^n \setminus \underline{1}$ to the category of topological spaces. If the restriction of the functor for a truncated n-diagram to each maximal dimensional face of $\underline{2}^n \setminus \underline{1}$ produces a k-dimensional $\langle n-1\rangle$-manifold, then the truncated n-diagram is called a *k-dimensional $\langle n\rangle$-boundary.*

For an $\langle n\rangle$-boundary Y, there is a continuous map

$$\phi_a : Y(a) \to \mathrm{colim} Y$$

for each $a \in \{0,1\}^n$ aside from $\underline{1}$, where colim$Y$ is the pushout for Y. This means that for any k-dimensional $\langle n \rangle$-manifold X, the restriction to $\underline{2}^n \setminus \underline{1}$ produces a $(k-1)$-dimensional $\langle n \rangle$-boundary Y such that colim$Y = \partial X$. Moreover, the map ϕ_a is injective.

Proposition 2.1.4 Let X and Y be two n-boundaries, so that $X(a)$ and $Y(a)$ are both compact for each object a in $\underline{2}^n \setminus \{\underline{1}\}$. Consider a natrual transformation $F : Y \to X$ that has the property that F restricted to each face of Y is an $(n-1)$-map. If $F(a) : Y(a) \to X(a)$ is a covering map for every object a of $\underline{2}^n \setminus \{\underline{1}\}$, then

$$G = \operatorname{colim}(F) : \operatorname{colim}(Y) \to \operatorname{colim}(X)$$

is a covering map.

Definition 2.1.5 Given an $(n+1)$-tuple $\mathbf{d} = (d_0, \dots, d_n) \in \mathbb{N}^{n+1}$ define the *Euclidean space with corners*

$$\mathbb{E}_n^{\mathbf{d}} = \mathbb{R}^{d_0} \times \mathbb{R}_+ \times \mathbb{R}^{d_1} \times \mathbb{R}_+ \times \cdots \times \mathbb{R}_+ \times \mathbb{R}^{d_n}.$$

This can be given an n-face structure by defining

$$\partial_i \mathbb{E}_n^{\mathbf{d}} = \mathbb{R}^{d_0} \times \mathbb{R}_+ \times \cdots \times \mathbb{R}^{d_{i-1}} \times \{0\} \times \mathbb{R}^{d_i} \times \cdots \times \mathbb{R}_+ \times \mathbb{R}^{d_n}$$

which will be referred to as the *i-boundary* or *i-face* of $\mathbb{E}_n^{\mathbf{d}}$.

For $0 \le B < A \le n+1$ let $\mathbb{E}_{\mathbf{d}}[B:A]$ denote the Euclidean space with corners $\mathbb{E}_{A-B-1}^{(d_B,\dots,d_{A-1})}$.

Definition 2.1.6 A *neat immersion* $\imath : X \looparrowright \mathbb{E}_n^{\mathbf{d}}$ of an $\langle n \rangle$-manifold X is a smooth immersion (for some $\mathbf{d} \in \mathbb{N}^{n+1}$) satisfying:

1. $\imath^{-1}(\partial_i \mathbb{E}_n^{\mathbf{d}}) = \partial_i X$ for each i.

2. The intersection of $X(a)$ and $\mathbb{E}_n^{\mathbf{d}}(b)$ is perpendicular for each $b < a$ in $\{0,1\}^n$.

Condition (1) is that $\imath$ is an *$\langle n \rangle$-map* which is defined in general as a smooth map between $\langle n \rangle$-manifolds which sends i-faces to i-faces.

A *neat embedding* is a neat immersion that is also an embedding.

A given neat immersion $\imath : X \looparrowright \mathbb{E}_n^{\mathbf{d}}$ with $\mathbf{d} \in \mathbb{N}^{n+1}$ can be stabilised in the sense that there is an induced neat immersion $\imath[\mathbf{d}'] : X \looparrowright \mathbb{E}_n^{\mathbf{d}+\mathbf{d}'}$ for each $\mathbf{d}' \in \mathbb{N}^{n+1}$.

Given a neat immersion $\imath : X \looparrowright \mathbb{E}_n^{\mathbf{d}}$ the *normal bundle* $\nu_\imath$ is defined as the normal bundle $\nu_{\imath(a)}$ for each immersion $\imath(a) : X(a) \looparrowright \mathbb{E}_n^{\mathbf{d}}(a)$ given $a \in \{0,1\}^n$.

The fact that neat embeddings exist is provided by the following variant of the Whitney embedding theorem for manifolds with corners from [Lau00, Proposition 2.1.7].

Lemma 2.1.7 Let X be a compact $\langle n \rangle$-manifold and $\imath : \partial X \hookrightarrow \mathbb{E}_n^{\mathbf{d}}$ be an embedding satisfying the properties

- $\imath^{-1}\big(\partial_i \mathbb{E}_n^{\mathbf{d}}\big) = \partial_i X$ for each i, and
- $X(a)$ intersects $\mathbb{E}_n^{\mathbf{d}}(b)$ perpendicularly for each $b < a$ in $\{0,1\}^n \setminus \{\underline{1}\}$.

Then $\imath[\mathbf{d}' - \mathbf{d}]$ can be extended to a neat embedding of X in $\mathbb{E}_n^{\mathbf{d}'}$ for some $\mathbf{d}' \geq \mathbf{d}$.

2.2 Flow Categories

Definition 2.2.1 A *flow category* is a pair $(\mathscr{C}, |\cdot|)$ where $\mathscr{C}$ is a category with finitely many objects $\mathrm{Ob}(\mathscr{C})$ and $|\cdot| : \mathrm{Ob}(\mathscr{C}) \to \mathbb{Z}$ is a function called the *grading* satisfying the following conditions:

1. $\mathrm{Hom}(x,x) = \{\mathrm{Id}\}$ for each object $x \in \mathrm{Ob}(\mathscr{C})$ and for $x \neq y \in \mathrm{Ob}(\mathscr{C})$, $\mathrm{Hom}(x,y)$ is a smooth compact $(|x|-|y|-1)$-dimensional $\langle |x|-|y|-1 \rangle$-manifold which will be denoted by $\mathcal{M}(x,y)$. Note that $\mathcal{M}(x,y) = \emptyset$ for $|y| \geq |x|$ (a negative dimensional manifold). In particular, set $\mathcal{M}(x,x) = \emptyset$.

2. For $x, y, z \in \mathrm{Ob}(\mathscr{C})$ with $|z| - |y| = m$, the composition map

$$\circ : \mathcal{M}(z,y) \times \mathcal{M}(x,z) \to \mathcal{M}(x,y)$$

is an embedding into $\partial_m \mathcal{M}(x,y)$. Further,

$$\circ^{-1}\big(\partial_i \mathcal{M}(x,y)\big) = \begin{cases} \partial_i \mathcal{M}(z,y) \times \mathcal{M}(x,z) & \text{for } i < m \\ \mathcal{M}(z,y) \times \partial_{i-m} \mathcal{M}(x,z) & \text{for } i > m. \end{cases}$$

3. For $x \neq y \in \mathrm{Ob}(\mathscr{C})$ the composition map $\circ$ induces a diffeomorphism

$$\partial_i \mathcal{M}(x,y) \cong \coprod_{\{z:|z|=|y|+i\}} \mathcal{M}(z,y) \times \mathcal{M}(x,z)$$

The morphism space $\mathcal{M}(x,y)$ defined in the first condition is called the *moduli space from x to y.*

In light of this definition and the discussion in the previous subchapter, let $\mathfrak{D}$ be the diagram with vertices given by the spaces

$$\mathcal{M}(z_m,y) \times \mathcal{M}(z_{m-1},z_m) \times \cdots \times \mathcal{M}(x,z_1) \tag{2.1}$$

for a distinct collection of objects $z_1,\ldots,z_m$ in $\mathrm{Ob}(\mathscr{C}) \setminus \{x,y\}$, and with arrows corresponding to a composition of moduli spaces. Then,

$$\partial\mathcal{M}(x,y) \cong \bigcup_i \partial_i \mathcal{M}(x,y) \cong \mathrm{colim}\mathfrak{D}. \tag{2.2}$$

Definition 2.2.2 Let $f : M \to \mathbb{R}$ be a Morse-Smale function on a closed manifold M (that is, f is a Morse function with a gradient flow such that the stable and unstable manifolds intersect transversally). The *Morse flow category* $\mathscr{C}_f$ is the flow category whose objects are the critical points of f (graded by their index), and whose moduli spaces are defined as follows. Let $p \in \mathrm{Ob}(\mathscr{C}_f)$ be a critical point of f. The stable manifold of p, with respect to the positive gradient flow ∇f, is given by

$$W^s(p) = \left\{ x \in M \;\middle|\; \lim_{t\to\infty} \gamma_x(t) = p \right\}$$

where $\gamma_x : \mathbb{R} \to M$ is the smooth flow given by ∇f and $\gamma_x(0) = x$. The unstable manifold of p with respect to this flow is

$$W^u(p) = \left\{ x \in M \;\middle|\; \lim_{t\to-\infty} \gamma_x(t) = p \right\}.$$

For a pair of critical points $p, q \in \mathrm{Ob}(\mathscr{C}_f)$ denote by

$$\tilde{\mathcal{M}}(p,q) = W^s(p) \cap W^u(q)/\mathbb{R}$$

the quotient on this intersection by the action of the gradient flow. The transversality condition ensures this is a smooth $(|p|-|q|-1)$-dimensional manifold, and for each

$a \in \big(f(q), f(p)\big) \subset \mathbb{R}$, this manifold $\tilde{\mathcal{M}}(p,q)$ can be embedded into $W^s(p) \cap f^{-1}(\{a\})$. The moduli space $\mathcal{M}(p,q)$ can then be defined as the compactification of $\tilde{\mathcal{M}}(p,q)$ by adding broken flow lines between the critical points p and q, as described by Austin and Braam in [AB95, Lemma 2.6].

The following definition describes a useful convention that was introduced in [LS14a].

Definition 2.2.3 Let $\mathscr{C}$ be a flow category and $\mathrm{Ob}_i(\mathscr{C})$ be all the objects of $\mathscr{C}$ with grading i, for some integer i. For pairs of integers i and j, let

$$\mathcal{M}(i,j) = \coprod_{x \in \mathrm{Ob}_i(\mathscr{C}), y \in \mathrm{Ob}_j(\mathscr{C})} \mathcal{M}(x,y).$$

For all m, n, i with $1 \leq i \leq m - n$, the diffeomorphism from part (3) of Definition 2.2.1 induces a diffeomorphism

$$\partial_i \mathcal{M}(m,n) \cong \mathcal{M}(n+i,n) \times_{\mathrm{Ob}_{n+i}(\mathscr{C})} \mathcal{M}(m, n+i).$$

Definition 2.2.4 Let $\mathscr{C}$ be a flow category with each object $x \in \mathrm{Ob}(\mathscr{C})$ satisfying $B \leq |x| \leq A$ for some integers $A, B \in \mathbb{Z}$. Then for a fixed $\mathbf{d} = (d_B, \ldots, d_{A-1}) \in \mathbb{N}^{A-B}$ a *neat immersion* $\imath$ of $\mathscr{C}$ relative $\mathbf{d}$ is a collection of neat immersions $\imath_{x,y} : \mathcal{M}(x,y) \looparrowright \mathbb{E}_{\mathbf{d}}[|y| : |x|]$ for each pair of objects $x, y \in \mathrm{Ob}(\mathscr{C})$ such that:

1. For all i and j with $B \leq j < i \leq A$, $\imath$ induces a neat immersion

$$\imath_{i,j} : \coprod_{x,y : |x|=i, |y|=j} \mathcal{M}(x,y) \to \mathbb{E}_{\mathbf{d}}[j : i].$$

2. For all objects $x, y, z \in \mathrm{Ob}(\mathscr{C})$ and points $(p,q) \in \mathcal{M}(z,y) \times \mathcal{M}(x,z)$

$$\imath_{x,y}(p \circ q) = \big(\imath_{z,y}(p), 0, \imath_{x,z}(q)\big).$$

A neat immersion $\imath$ is called a *neat embedding* if the induced maps $\imath_{i,j}$ in (1) are all embeddings. Given a neat immersion $\imath$ relative $\mathbf{d}$ and some other $\mathbf{d}' \in \mathbb{N}^{n+1}$, denote by $\imath[\mathbf{d}']$ the neat immersion relative $\mathbf{d} + \mathbf{d}'$ which is induced by the inclusions $\mathbb{E}_{\mathbf{d}}[B : A] \hookrightarrow \mathbb{E}_{\mathbf{d}+\mathbf{d}'}[B : A]$, which themselves are induced by natural inclusions $\mathbb{R}^{d_i} \cong \mathbb{R}^{d_i} \times \{0\} \hookrightarrow \mathbb{R}^{d_i + d'_i}$.

As highlighted in [LS14a], the Cohen-Jones-Segal machine (Section 2.3) requires the input of a framed neat embedding, which is defined below. However, much of the language constructed in this section regards the more general notion of a neat immersion, which will be necessary later when discussing covering spaces. Lemmata 2.2.5, 2.2.6, 2.2.8 and 2.2.11 will ensure that any neat immersion may be perturbed (in a unique way) to a neat embedding to satisfy the Cohen-Jones-Segal prerequisite.

Lemma 2.2.5 There is a neat embedding for any flow category, relative to some $\mathbf{d}$.

Proof: This is essentially a corollary of Lemma 2.1.7, where a choice of $\mathbf{d}$ can be made to ensure each $\imath_{i,j}$ is a neat embedding for each $B \leq j < i \leq A$. □

Lemma 2.2.6 Let $\imath(\theta)$ be a smooth S^{k-1}-parameter family of neat immersions of a flow category $\mathscr{C}$ relative some (fixed) $\mathbf{d}$. Then for a sufficiently large choice of $\mathbf{d}'$, the family of neat immersions $\imath(\theta)[\mathbf{d}']$ can be extended to a smooth $\mathbb{D}^k$-parameter family of neat immersions $\imath'(r,\theta)$ of the flow category $\mathscr{C}$, where $S^{k-1} = \partial\mathbb{D}^k$. Furthermore, if $\imath(\theta_0)$ is a neat embedding for a single point $\theta_0 \in S^{k-1}$ then $\imath'(r,\theta)$ can be made into a neat embedding for all interior points (r,θ) of $\mathbb{D}^k$. In particular, two neat embeddings of $\mathscr{C}$ relative $\mathbf{d}$ are isotopic through neat embeddings relative $\mathbf{d}'$.

Proof: The interpretation in [LS14a, Lemma 3.18] of [Lau00, Lemma 2.2.3] gives the sufficient choice of $\mathbf{d}'$ as $d'_i = 2d_i + 1$. This enables them to define an isotopy of neat immersions/embeddings as follows. Assume that $\mathbb{D}^k$ is the unit disk in $\mathbb{R}^k$ with points in $\mathbb{D}^k$ given by (r,θ) where r is the distance of the point from the origin, and $\theta \in S^{k-1} = \partial\mathbb{D}^k$. It suffices to define $\imath'_{x,y}(r,\theta)$ for each pair of objects $x, y \in \mathrm{Ob}(\mathscr{C})$. Let $|x| = m$ and $|y| = n$. Then $\imath_{x,y}(\theta)[\mathbf{d}'] : \mathcal{M}(x,y) \looparrowright \mathbb{E}_{\mathbf{d}'}[n : m]$ has coordinates defined by

$$\imath_{x,y}(\theta)[\mathbf{d}'] = (\imath_n(\theta), 0, 0, \bar{\imath}_{n+1}(\theta), \ldots, \bar{\imath}_{m-1}(\theta), \imath_{m-1}(\theta), 0, 0).$$

Now consider the function $f : [0,1] \to \mathbb{R}_+$ defined as

$$f(r) = \begin{cases} e^{1/(r-r^2)} & \text{if } r \neq 0,1 \\ 0 & \text{otherwise.} \end{cases}$$

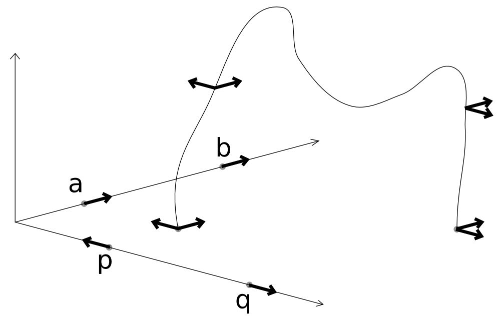


Figure 2.2: A framed neat embedding of the flow category in Example 2.2.1.

For a choice of point $\theta_0 \in S^{k-1}$ (that possibly gives a neat embedding $\imath(\theta_0)$), the isotopy defined in [LS14a, Lemma 3.18] is

$$\begin{aligned}\imath'_{x,y}(r,\theta) = \;& \big((1-r)\imath_n(\theta_0) + r\imath_n(\theta), f(r)\imath_n(\theta_0), f(r)\bar{\imath}_{n+1}(\theta_0),\\ & (1-r)\bar{\imath}_{n+1}(\theta_0) + r\bar{\imath}_{n+1}(\theta), \ldots (1-r)\bar{\imath}_{m-1}(\theta_0) + r\bar{\imath}_{m-1}(\theta),\\ & (1-r)\imath_{m-1}(\theta_0) + r\imath_{m-1}(\theta), f(r)\imath_{m-1}(\theta_0), 0\big).\end{aligned}$$

□

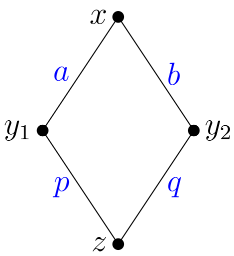
Definition 2.2.7 Let $\mathscr{C}$ be a flow category with a neat immersion $\imath$ relative $\mathbf{d}$. A *coherent framing* φ for $\imath$ is a framing for $\nu_{\imath_{x,y}}$ for each $x, y \in \mathrm{Ob}(\mathscr{C})$ such that the product framing $\nu_{\imath_{z,y}} \times \nu_{\imath_{x,z}}$ is equal to the pullback framing $\circ^*\nu_{\imath_{x,y}}$ for all $x, y, z \in \mathrm{Ob}(\mathscr{C})$ (see Figure 2.2).

A framing φ of an immersion $\imath : \mathcal{M}(a,b) \looparrowright \mathbb{E}_{\mathbf{d}}[|b| : |a|]$ may be interpreted as an immersion

$$\varphi : \mathcal{M}(a,b) \times [-\varepsilon, \varepsilon]^{d_{|b|}+\cdots+d_{|a|-1}} \to \mathbb{E}_{\mathbf{d}}[|b| : |a|]$$

such that $\varphi(x, 0) = \imath(x)$, and we shall adopt this interpretation throughout.

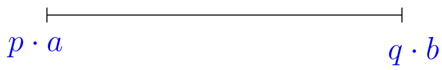
Example 2.2.1 Let $\mathscr{C}$ be the flow category whose objects are $\mathrm{Ob}(\mathscr{C}) = \{x, y_1, y_2, z\}$ such that $|x| = 2$, $|y_1| = |y_2| = 1$ and $|z| = 0$, depicted as follows



where the 0-dimensional moduli spaces are labelled in blue. The moduli space $\mathcal{M}(x, z)$ is therefore an interval whose boundary is given by the product of the 0-dimensional moduli spaces as

$$\partial\mathcal{M}(x, z) = \big(\mathcal{M}(y_1, z) \times \mathcal{M}(x, y_1)\big) \cup \big(\mathcal{M}(y_2, z) \times \mathcal{M}(x, y_2)\big).$$

This interval can be depicted as

$\mathcal{M}(x, z) =$ 

A neat embedding (relative $\mathbf{d} = (d_0, d_1) = (1, 1)$) for $\mathscr{C}$ is depicted in Figure 2.2, where the points p and q are embedded into $\mathbb{R}^{d_0} = \mathbb{R}$, and the points a and b are embedded into $\mathbb{R}^{d_1} = \mathbb{R}$, along with their framings. The coherent framing condition of Definition 2.2.7 is also illustrated in Figure 2.2 where the interval $\mathcal{M}(x, z)$ is embedded into $\mathbb{R}^{d_0} \times \mathbb{R}_+ \times \mathbb{R}^{d_1}$.

Lemma 2.2.8 Let $\mathbb{D}^k$ be a k-dimensional disk with $\partial\mathbb{D}^k = \mathbb{D}^{k-1}_1 \cup_{S^{k-2}} \mathbb{D}^{k-1}_2$. Consider a smooth $\mathbb{D}^k$-parameter family of neat immersions $\imath(t)$ of a flow category $\mathscr{C}$ along with a smooth $\mathbb{D}^{k-1}$-parameter family of coherent framings $\varphi(t)$ for $\imath(t)|_{\mathbb{D}^{k-1}}$ (for either of the hemispherical boundary disks). Then $\varphi(t)$ can be extended to a smooth $\mathbb{D}^k$-parameter family of coherent framings for $\imath(t)$.

Proof: By considering the normal bundle of the collection of neat immersions

$$\imath_{x,y}(t) : \mathcal{M}(x, y) \looparrowright \mathbb{E}[|y| : |x|]$$

for each pair of objects $x, y \in \mathrm{Ob}(\mathscr{C})$ and $t \in \mathbb{D}^k$, [LS14a] provide the dashed map which extends the following commutative diagram

$$\begin{tikzcd}
\mathcal{M}(x,y) \times \mathbb{D}^{k-1} \arrow[r] \arrow[d, hook] & EO(N) \arrow[d] \\
\mathcal{M}(x,y) \times \mathbb{D}^{k} \arrow[r] \arrow[ur, dashed] & BO(N)
\end{tikzcd}$$

where $N = d_{|y|} + \cdots + d_{|x|-1}$ and $EO(N) \to BO(N)$ is the principal bundle of the classifying space of $O(N)$. They do this inductively on $|x| - |y|$ using the collar neighbourhood theorem (see [Jän68, §1]) and the fact that boundary points of moduli spaces are given by products of points in lower-dimensional moduli spaces. $\square$

Definition 2.2.9 A *framed flow category* is a triple $(\mathscr{C}, \imath, \varphi)$ where $\mathscr{C}$ is a flow category which is neatly embedded with $\imath$ relative $\mathbf{d}$, along with a coherent framing φ for $\imath$.

A chain complex $C_*(\mathscr{C}, \imath, \varphi)$ can be associated to a framed flow category in the following way. Each $C_n(\mathscr{C}, \imath, \varphi)$ is the free abelian group generated by objects $x \in \mathrm{Ob}_n(\mathscr{C})$ with grading n. If $x, y \in \mathrm{Ob}(\mathscr{C})$ are objects with $|x| = |y| + 1$, then the moduli space $\mathcal{M}(x, y)$ is a 0-dimensional $\langle 0 \rangle$-manifold (i.e. a disjoint union of points). These points are assigned a framing in $\{\pm 1\}$, and the boundary map between x and y is the signed sum of points in $\mathcal{M}(x, y)$. Dually, a cochain complex $C^*(\mathscr{C}, \imath, \varphi)$ can also be associated to $(\mathscr{C}, \imath, \varphi)$. Following the terminology of [LS14a], we say that the framed flow category *refines* its associated (co)chain complex.

Definition 2.2.10 Let φ be a coherent framing for a neat immersion $\imath$ of some flow category $\mathscr{C}$. By Lemma 2.2.5, one can choose a neat embedding $\imath'$ of $\mathscr{C}$. By Lemma 2.2.6, $\imath$ and $\imath'$ can be stabilised so that $\imath[\mathbf{d}]$ and $\imath'[\mathbf{d}']$ are isotopic for some choice of $\mathbf{d}, \mathbf{d}'$. Further, φ induces a coherent framing for $\imath[\mathbf{d}]$; call this φ too. Finally, Lemma 2.2.8 allows one to produce a coherent framing φ' for $\imath'[\mathbf{d}']$. Then the framed flow category $(\mathscr{C}, \imath'[\mathbf{d}], \varphi')$ is said to be a *perturbation* of the framed flow category $(\mathscr{C}, \imath, \varphi)$.

The previous two lemmata then combine to give the following.

Lemma 2.2.11 Let $(\mathscr{C}, \imath_0, \varphi_0)$ and $(\mathscr{C}, \imath_1, \varphi_1)$ be two perturbations of a framed flow category $(\mathscr{C}, \imath, \varphi)$. Then there exists a smooth 1-parameter family of framings of $\mathscr{C}$ that connects $(\imath_0[\mathbf{d}_0], \varphi_0)$ to $(\imath_1[\mathbf{d}_1], \varphi_1)$ for some $\mathbf{d}_0, \mathbf{d}_1$.

Proof: By definition of a perturbation, there exist some $\mathbf{d}'_0, \mathbf{d}'_1$ for which there exists a 1-parameter family of neat immersions $\imath_i(t)$ connecting $\imath[\mathbf{d}'_i]$ to $\imath_i$ as well as a 1-parameter family of coherent framings $\varphi_i(t)$ for $\imath_i(t)$ for each $i \in \{0, 1\}$. Moreover, Lemma 2.2.6 ensures that, for a sufficiently large choice of $\mathbf{d}''_i$, there is a 1-parameter family of neat embeddings $\imath(t)$ connecting $\imath_0[\mathbf{d}''_0]$ to $\imath_1[\mathbf{d}''_1]$.

The following set-up is from [LS14a, Lemma 3.23]. There is a $\partial\triangle$-parameter family of neat immersions of $\mathscr{C}$ where $\triangle$ is a triangle whose edges correspond to the neat immersions $\imath_i(t)[\mathbf{d}''_i]$ for $i = 0, 1$ and $\imath(t)$. By Lemma 2.2.6 a choice of large $\mathbf{d}$ can be made so that there exists a $\triangle$-parameter family $\bar{\imath}$ of neat immersions of $\mathscr{C}$, such that $\bar{\imath}$ restricted to the corresponding edges of $\partial\triangle$ gives $\imath_i(t)[\mathbf{d}''_i + \mathbf{d}]$ for $i = 0, 1$ and $\imath(t)[\mathbf{d}]$, respectively. A framing $\bar{\varphi}$ for $\bar{\imath}$ can be produced using Lemma 2.2.8 by extending the framing on the edges of $\partial\triangle$ which is induced by $\varphi_0(t)$ and $\varphi_1(t)$. A 1-parameter family of framings connecting $\imath_0[\mathbf{d}_0]$ to $\imath_1[\mathbf{d}_1]$ is then given by $(\bar{\imath}|_e, \bar{\varphi}|_e)$ where e is the edge of $\triangle$ corresponding to the 1-parameter family of neat immersions $\imath(t)$ of $\mathscr{C}$. $\square$

2.3 Cohen-Jones-Segal construction via Lipshitz-Sarkar

As was briefly alluded to in the introduction, there is a process that allows one to construct, from a given framed flow category $\mathscr{C}$, a CW complex $|\mathscr{C}|$. This CW complex is constructed in such a way that its cellular chain complex $C^*(|\mathscr{C}|)$ is isomorphic (after some grading shift) to the chain complex $C^*(\mathscr{C})$ obtained from $\mathscr{C}$. An outline of the construction of $|\mathscr{C}|$ was first given by Cohen-Jones-Segal (inspired by Franks [Fra79]) in attempt to acheive a spectrum (or space-level refinement) for Floer homology. As expressed in [CJS95a], their attempt was not entirely successful but they do outline a detailed recipe for constructing a CW complex from any given framed flow category; a recipe that was implemented successfully in [LS14a] to produce such a spectrum for Khovanov homology, namely a Khovanov stable homotopy type. The immediate output of the Cohen-Jones-Segal machine is a CW complex that we shall define in this section. It should be noted that the Khovanov stable homotopy type is defined as a (de-)suspension of this output where the input is a particular Khovanov flow category, constructed in [LS14a]. We shall handle this special case in later sections.

Definition 2.3.1 Let $(\mathscr{C}, \imath, \varphi)$ be a framed flow category relative $\mathbf{d}$. For an arbitrary object a in $\mathrm{Ob}(\mathscr{C})$ of degree m, recall that for each object b in $\mathrm{Ob}(\mathscr{C})$ of degree $n < m$, we have a framed neat embedding

$$\imath_{a,b} : \mathcal{M}(a, b) \times [-\varepsilon, \varepsilon]^{d_n+\cdots+d_{m-1}} \to [-R, R]^{d_n} \times [0, R] \times \cdots \times [0, R] \times [-R, R]^{d_{m-1}}$$

where R is chosen to be large enough that all moduli spaces $\mathcal{M}(a, b)$ can be embedded in this way. Moreover, choose $B < A \in \mathbb{Z}$ as in Definition 2.2.4 so that every object $a \in \mathrm{Ob}(\mathscr{C})$ satisfies $B \leq |a| \leq A$. The CW complex $|\mathscr{C}|$ consists of one 0-cell (the basepoint) and one $(d_B + \cdots + d_{A-1} - B + m)$-cell $\mathcal{C}(a)$ for every object a of $\mathscr{C}$ defined as

$$[0, R] \times [-R, R]^{d_B} \times \cdots \times [-R, R]^{d_{m-1}} \times \{0\} \times [-\varepsilon, \varepsilon]^{d_m} \times \{0\} \times \cdots \times \{0\} \times [-\varepsilon, \varepsilon]^{d_{A-1}}.$$

Each cell $\mathcal{C}(a)$ is considered a subset of a different copy of the ambient space $\mathbb{R}_+ \times \mathbb{R}^{d_B} \times \cdots \times \mathbb{R}_+ \times \mathbb{R}^{d_{A-1}}$. The neat embedding $\imath$ can be used to identify particular subsets

$$\mathcal{M}(a, b) \times \mathcal{C}(b) \cong \mathcal{C}_b(a) \subset \partial_n \mathcal{C}(a) \tag{2.3}$$

in the following way:

$$\begin{aligned}\mathcal{C}_b(a) =& [0, R] \times [-R, R]^{d_B} \times \cdots \times [-R, R]^{d_{n-1}} \times \{0\} \times \\ & \imath_{a,b}\big(\mathcal{M}(a, b) \times [-\varepsilon, \varepsilon]^{d_n+\cdots+d_{m-1}}\big) \times \{0\} \times [-\varepsilon, \varepsilon]^{d_m} \times \cdots \times \{0\} \times [-\varepsilon, \varepsilon]^{d_{A-1}} \\ & \subset \partial \mathcal{C}(a).\end{aligned}$$

It will be useful to introduce notation for this identification by letting

$$\Gamma_{a,b} : \mathcal{M}(a, b) \times \mathcal{C}(b) \to \partial_n \mathcal{C}(a) \tag{2.4}$$

be the identification $\mathcal{M}(a, b) \times \mathcal{C}(b) \cong \mathcal{C}_b(a)$. Let

$$C = d_B + \cdots + d_{A-1} - B. \tag{2.5}$$

Then the attaching map for each cell $\partial\mathcal{C}(a) \to |\mathscr{C}|^{(C+m-1)}$ is defined via the Thom construction for each embedding into $\partial\mathcal{C}(a)$ simultaneously. That is, for each subset $\mathcal{M}(a,b) \times \mathcal{C}(b) \cong \mathcal{C}_b(a) \subset \partial\mathcal{C}(a)$, the attaching map projects to $\mathcal{C}(b)$ (which carries trivialisation information), and sends the rest of the boundary $\partial\mathcal{C}(a) \setminus \bigcup_b \mathcal{C}_b(a)$ to the basepoint.

The fact that this construction is well-defined is shown in [LS14a, Lemma 3.25] which also describes how the attaching maps give a natural isomorphism of chain complexes. The lemma is re-stated below.

Lemma 2.3.2 The CW complex $|\mathscr{C}|$ defined in Definition 2.3.1 is well-defined. Moreover, the cochain complex $C^*(\mathscr{C}, \imath, \varphi)$ associated to $\mathscr{C}$ is isomorphic to the reduced cellular cochain complex $\widetilde{C}^*(|\mathscr{C}_{\imath,\varphi}|)[-C]$, where C in the degree shift is also defined in Equation 2.5 of Definition 2.3.1.

The isomorphism type of $|\mathscr{C}|$ is shown to be independent of the choice of real numbers R and ε in [LS14a, Lemma 3.25] and, by considering a one-parameter family of framed neat embeddings between two perturbations $(\imath_0, \varphi_0)$ and $(\imath_1, \varphi_1)$ of $(\imath, \varphi)$, it can be shown that the CW complexes $|\mathscr{C}|_{\imath_0,\varphi_0}$ and $|\mathscr{C}|_{\imath_1,\varphi_1}$ are isomorphic (also [LS14a, Lemma 3.25]). A choice of different A, B, and $\mathbf{d}$ gives rise to a stably homotopy equivalent CW complex (see [LS14a, Lemma 3.26]) that is a suspension of the original CW complex a number of times. These results are re-stated below for later reference.

Lemma 2.3.3 The isomorphism type of $|\mathscr{C}_{\imath,\varphi}|$ is independent of the choice of real numbers R and ε and different choices of R and ε give homeomorphic spaces. Moreover, by considering a one-parameter family of framed neat embeddings between two perturbations $(\imath_0, \varphi_0)$ and $(\imath_1, \varphi_1)$ of $(\imath, \varphi)$, it can be shown that the CW complexes $|\mathscr{C}|_{\imath_0,\varphi_0}$ and $|\mathscr{C}|_{\imath_1,\varphi_1}$ are isomorphic.

Lemma 2.3.4 Let $(\mathscr{C}, \imath, \varphi)$ be a framed flow category, with a neat embedding $\imath$ relative $\mathbf{d}$. Choose integers $A, B \in \mathbb{Z}$ as in Definition 2.3.1 and consider the CW complex $|\mathscr{C}|_{\imath,\varphi}$ constructed with this choice.

For $\mathbf{d}' \geq \mathbf{d}$, let $\imath' = \imath[\mathbf{d}' - \mathbf{d}]$ denote the induced neat embedding relative $\mathbf{d}'$ and let φ' be the induced coherent framing for $\imath'$. By using $\mathbf{d}'$ if necessary, choose integers $A' \geq A$ and $B' \leq B$. Recall from Definition 2.3.1 that $C = d_B + \cdots + d_{A-1} - B$, so let $C' = d_{B'} + \cdots + d_{A'-1} - B'$. Then there is a homotopy equivalence

$$|\mathscr{C}|_{\imath',\varphi',B',A'} \simeq \Sigma^{C'-C}|\mathscr{C}|_{\imath,\varphi,B,A}.$$

This construction of $|\mathscr{C}|$ is shown to agree with the construction of [CJS95a] in [LS14a, Prop3.27], and is referred to as the *realisation* of $\mathscr{C}$. Since this thesis is based on the language of [LS14a], this equivalence is omitted.

2.4 Covers of flow categories, sub-complexes and quotient-complexes

This subchapter follows [LS14a, Subsection 3.4] and outlines important methods for proving the invariance of the Khovanov homotopy type. These methods may also be used in [JLS15] (see Subchapter 5.5) to prove equivalence of stable homotopy types arising from different choices of tangle decompositions of a link diagram. Since the methods are of particular importance, we shall also include the proofs from [LS14a].

Definition 2.4.1 A *cover* is a functor $\mathcal{F} : \mathscr{C}_1 \to \mathscr{C}_2$ between two flow categories which preserves gradings such that for all $x, y \in \mathrm{Ob}(\mathscr{C}_1)$ either

1. $\mathcal{M}_{\mathscr{C}_1}(x,y) = \emptyset$ or
2. $\mathcal{F} : \mathcal{M}_{\mathscr{C}_1}(x,y) \to \mathcal{M}_{\mathscr{C}_2}\big(\mathcal{F}(x), \mathcal{F}(y)\big)$ is a $(|x| - |y| - 1)$-map, local diffeomorphism and a covering map.

A cover $\mathcal{F}$ is *trivial* if for all $x, y \in \mathrm{Ob}(\mathscr{C}_1)$ the covering map $\mathcal{F} : \mathcal{M}_{\mathscr{C}_1}(x,y) \to \mathcal{M}_{\mathscr{C}_2}\big(\mathcal{F}(x), \mathcal{F}(y)\big)$ restricted to each component of $\mathcal{M}_{\mathscr{C}_1}(x,y)$ is a homeomorphism onto its image.

Given a neat immersion $\imath$ of $\mathscr{C}_2$ relative $\mathbf{d}$, a cover $\mathcal{F} : \mathscr{C}_1 \to \mathscr{C}_2$ provides a neat immersion of $\mathscr{C}_1$ through a composition $\imath \circ \mathcal{F}$. A coherent framing for $\imath \circ \mathcal{F}$ is induced by a coherent framing for $\imath$, and then the flow category $\mathscr{C}_1$ can be framed using a perturbation as in Definition 2.2.10.

Definition 2.4.2 Let $\mathscr{C}'$ be a full subcategory of a flow category $\mathscr{C}$. Then $\mathscr{C}'$ is a *downward closed subcategory* if for all $x, y \in \mathrm{Ob}(\mathscr{C})$ such that $\mathcal{M}(x, y) \neq \emptyset$, $x \in \mathrm{Ob}(\mathscr{C}')$ implies that $y \in \mathrm{Ob}(\mathscr{C}')$. If for all such pairs of objects, $y \in \mathrm{Ob}(\mathscr{C}')$ implies that $x \in \mathrm{Ob}(\mathscr{C}')$ then $\mathscr{C}'$ is a *upward closed subcategory.* Upward and downward subcategories of flow categories are themselves flow categories.

Definition 2.4.3 Let $\mathscr{C}'$ be a downward (respectively upward) closed subcategory of a flow category $\mathscr{C}$, and let $\imath$ be a neat embedding of $\mathscr{C}$ with a coherent framing φ. Let $\imath'$ and φ' be the induced/inherited neat embedding and coherent framing for $\mathscr{C}'$. Define $|\mathscr{C}|'_{(\imath,\varphi)}$ to be the subcomplex (respectively, quotient complex) of $|\mathscr{C}|_{(\imath,\varphi)}$ which is constructed using precisely the cells corresponding to objects of $\mathscr{C}'$.

Remark: It is important in the previous definition that the subcategory $\mathscr{C}'$ is a downward (respectively upward) closed subcategory, so that $|\mathscr{C}|'_{(\imath,\varphi)}$ is precisely the CW complex $|\mathscr{C}'|_{(\imath',\varphi')}$ constructed from $\mathscr{C}'$.

The following lemma will be the key to describing the Khovanov stable homotopy type as a wedge sum decomposition over the quantum gradings (see Definition 4.3.6).

Lemma 2.4.4 If $\mathscr{C}'$ and $\mathscr{C}''$ are two downward closed subcategories of a framed flow category $\mathscr{C}$ satisfying the property that every object of $\mathscr{C}$ is in precisely one of $\mathscr{C}'$ or $\mathscr{C}''$, then the CW complex $|\mathscr{C}|$ decomposes as wedge sum of $|\mathscr{C}'|$ with $|\mathscr{C}''|$ where both downward flow categories are framed with the induced framing. That is, $|\mathscr{C}| = |\mathscr{C}'| \vee |\mathscr{C}''|$.

Proof: This follows by considering the attaching maps $\mathcal{C}_b(a) \to \mathcal{C}(b)$ from the construction of $|\mathscr{C}|$ in Definition 2.3.1. Let $|\mathscr{C}|'$ (respectively, $|\mathscr{C}|''$) be the subcomplex of $|\mathscr{C}|$ whose cells correspond precisely to the objects of $\mathscr{C}'$ (respectively, $\mathscr{C}''$). Then since the subcategories $\mathscr{C}'$ and $\mathscr{C}''$ are both downward closed subcategories that bisect $\mathscr{C}$, any object a of $\mathscr{C}$ is in either of these categories. Moreover, $\mathcal{M}(a, b) \times \mathcal{C}(b) \cong \mathcal{C}_b(a)$ is attached to $\mathcal{C}(b)$ where b is forced to be an object of the same category as a. The rest of $\partial\mathcal{C}(a)$ is sent to the basepoint and so $|\mathscr{C}| = |\mathscr{C}|' \vee |\mathscr{C}|''$. Moreover, $|\mathscr{C}|' = |\mathscr{C}'|$ and $|\mathscr{C}|'' = |\mathscr{C}''|$ since both subcategories are downward closed. $\square$

Lemma 2.4.5 Let $\mathscr{C}'$ be a downward closed subcategory of a framed flow category $\mathscr{C}$, and let $\mathscr{C}''$ be the complementary upward closed subcategory (both framed with the induced framings). The following statements about the associated chain complexes of these framed flow categories hold.

1. The inclusion $|\mathscr{C}'| \hookrightarrow |\mathscr{C}|$ induces a homotopy equivalence when $C^*(\mathscr{C}'')$ is acyclic.

2. The quotient map $|\mathscr{C}| \twoheadrightarrow |\mathscr{C}''|$ induces a homotopy equivalence when $C^*(\mathscr{C}')$ is acyclic.

3. The Puppe map $|\mathscr{C}''| \to \Sigma|\mathscr{C}'|$ induces a homotopy equivalence when $C^*(\mathscr{C})$ is acyclic.

Proof: The subcomplex $|\mathscr{C}'|$ of $|\mathscr{C}|$ has a corresponding quotient complex $|\mathscr{C}''| = |\mathscr{C}|/|\mathscr{C}'|$. Consequently, there is a short exact sequence of reduced cellular cochains

$$0 \to \widetilde{C}^*(|\mathscr{C}''|) \to \widetilde{C}^*(|\mathscr{C}|) \to \widetilde{C}^*(|\mathscr{C}'|) \to 0$$

which gives rise to a long exact sequence of reduced cohomology groups

$$\cdots \to \widetilde{H}^{i-1}(|\mathscr{C}'|) \to \widetilde{H}^i(|\mathscr{C}''|) \to \widetilde{H}^i(|\mathscr{C}|) \to \widetilde{H}^i(|\mathscr{C}'|) \to \widetilde{H}^{i+1}(|\mathscr{C}''|) \to \cdots.$$

The maps $\widetilde{H}^i(|\mathscr{C}|) \to \widetilde{H}^i(|\mathscr{C}'|)$, $\widetilde{H}^i(|\mathscr{C}''|) \to \widetilde{H}^i(|\mathscr{C}|)$ and $\widetilde{H}^{i-1}(|\mathscr{C}'|) \to \widetilde{H}^i(|\mathscr{C}''|)$ for each i, are induced by the maps highlighted in the statement of the lemma; namely, the inclusion $|\mathscr{C}'| \hookrightarrow |\mathscr{C}|$, the quotient $|\mathscr{C}| \twoheadrightarrow |\mathscr{C}''|$ and the Puppe map $|\mathscr{C}''| \to \Sigma|\mathscr{C}'|$, respectively.

Using Lemma 2.3.2, if $C^*(\hat{\mathscr{C}})$ is acyclic, then $\widetilde{H}^i(|\hat{\mathscr{C}}|) = 0$ for all i where $\hat{\mathscr{C}}$, in turn, is given by the framed flow category $\mathscr{C}''$, $\mathscr{C}'$ and $\mathscr{C}$. In Case (1), this gives rise to an isomorphism of cohomology groups $\widetilde{H}^i(|\mathscr{C}|) \cong \widetilde{H}^i(|\mathscr{C}'|)$. By the version of Whitehead's Theorem obtained as a corollary of the Hurewicz Theorem (see [Hat02, Th.4.32, Cor.4.33]), a map between two simply-connected CW complexes that induces isomorphisms on homology is a homotopy equivalence. Since each $|\hat{\mathscr{C}}|$ is well-defined up to stable homotopy, each of these CW complexes can be assumed to be simply connected, (de)suspending if necessary. Here, the inclusion map $|\mathscr{C}'| \hookrightarrow |\mathscr{C}|$ induces a homotopy equivalence. Case (2) is similar. Case (3) is slightly different in the sense that the isomorphisms $\widetilde{H}^{i-1}(|\mathscr{C}'|) \cong \widetilde{H}^{i}(|\mathscr{C}''|)$ are induced by the Puppe map $|\mathscr{C}''| \to \Sigma|\mathscr{C}'|$ and so the suspension is required to shift the degrees of the cohomology before arguing as above. $\square$

Chapter 3

Handle Cancellation in Framed Flow Categories

3.1 A cancellation theorem for framed flow categories

In this subchapter, we show that the space arising from a handle-cancelled category is stably equivalent to the space arising from the original category. Much of the argument relies heavily on a particular homotopy (or deformation retract) which takes place in the cells of $|\mathscr{C}|$. Throughout this subchapter, let $\mathscr{C}$ denote a framed flow category $(\mathscr{C}, \imath, \varphi)$ with two of its objects having a one-point moduli space between them, $\mathcal{M}(x, y) = *$. Let $|x| = i$ and $|y| = i - 1$.

Definition 3.1.1 Denote by $\mathscr{C}_H$ the flow category whose object set is identical to that of $\mathscr{C}$, minus x and y, and whose moduli spaces[1] are given by

$$\mathcal{M}(\bar{a}, \bar{b}) = \mathcal{M}(a, b) \cup_f \big(\mathcal{M}(x, b) \times \mathcal{M}(a, y)\big)$$

where f identifies the subsets

$$\mathcal{M}(x, b) \times \mathcal{M}(a, x) \cup \mathcal{M}(y, b) \times \mathcal{M}(a, y) \subset \mathcal{M}(a, b)$$

[1] In order to distinguish the two categories, we write $\bar{a}$ for the object of $\mathscr{C}_H$ if a is an object of $\mathscr{C}$ different from x or y.

and

$$\mathcal{M}(x,b) \times (\mathcal{M}(x,y) \times \mathcal{M}(a,x)) \cup (\mathcal{M}(y,b) \times \mathcal{M}(x,y)) \times \mathcal{M}(a,y) \\ \subset \mathcal{M}(x,b) \times \mathcal{M}(a,y).$$

We call $\mathscr{C}_H$ the *cancelled category* (relative to x and y) of $\mathscr{C}$.

It follows from the existence of collar neighbourhoods for $\langle n\rangle$-manifolds, see [Lau00, Lemma 2.1.6], that $\mathcal{M}(\bar{a},\bar{b})$ is a $(|a|-|b|-1)$-dimensional $\langle |a|-|b|-1\rangle$-manifold, and $\mathscr{C}_H$ is a flow category, with grading obtained by restriction. This chapter is dedicated to the proof of the following theorem about handle cancellation in general framed flow categories.

Theorem 3.1.1 Let $(\mathscr{C},\imath,\varphi)$ be a framed flow category containing two objects x and y with $\mathcal{M}(x,y) = *$. The realisation $|\mathscr{C}|$ is stably homotopy equivalent to the realisation $|\mathscr{C}_H|$ of the cancelled category, where $\mathrm{Ob}(\mathscr{C}_H) = \mathrm{Ob}(\mathscr{C}) \setminus \{x,y\}$ and the morphisms of $\mathscr{C}_H$ are defined in Definition 3.1.1.

In their definition of a realisation of a framed flow category in [CJS95a], Cohen-Jones-Segal were trying to achieve a description of a Floer homotopy type. Moreover, Floer theory is an infinite-dimensional analogue of Morse theory (see [BH04, §9], for example), and the realisation of a Morse flow category $\mathscr{C}_f$ recovers the underlying manifold (see main theorem of [CJS95b] and [CJS95a, §5]). Therefore, Theorem 3.1.1 can be thought of as a generalisation of handle-cancellation in Morse theory (see [Mil65, Theorem 5.4]) and we shall highlight how the Morse theoretic cancellation can be described in our language in the following example.

Example 3.1.1 Let M be the deformed 2-sphere depicted in Figure 3.1a, and let $f_M : M \to \mathbb{R}$ be a self-indexing Morse-Smale function on M. Consider the Morse flow category $\mathscr{C}_{f_M}$, whose object set is given by the critical points of f as

$$\mathrm{Ob}(\mathscr{C}_{f_M}) = \mathrm{Crit}(f_M) = \{a,x,y,b\}$$

where $|a| = |x| = 2$, $|y| = 1$ and $|b| = 0$. With respect to the positive gradient flow ∇f_M, the (transverse) intersection of the stable and unstable manifolds of x and y,

(a) A deformed 2-sphere with two maximal critical points.

(b) A 2-sphere with a single maximal and single minimal critical point.

Figure 3.1: An example of handle cancellation.

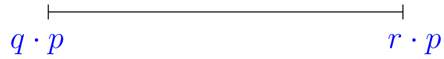
respectively, is given by the single point

$$\tilde{\mathcal{M}}(x, y) = W^s(x) \cap W^u(y)/\mathbb{R} = \{*\}$$

as highlighted in red in Figure 3.1a. This corresponds to the 0-dimensional moduli space $\mathcal{M}(x, y)$ of $\mathscr{C}_{f_M}$ being a single point, $*$, and will be the point about which we will cancel. The remaining intersections of stable and unstable manifolds of critical points with relative index 1 are also highlighted in red in Figure 3.1a. The corresponding 0-dimensional moduli spaces are given by $\mathcal{M}(a, y) = \{p\}$ and $\mathcal{M}(y, b) = \{q, r\}$. Handle cancellation in Morse theory allows us to alter the gradient-like vector field on an arbitrarily small neighbourhood of the single trajectory from x to y, producing a new gradient-like vector field for the Morse function f_{S^2} on the 2-sphere. In particular, this provides a diffeomorphism between the two manifolds M and S^2.

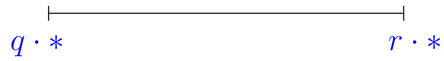
To see how this is described in terms of a cancelled category, consider the 1-dimensional moduli spaces of $\mathscr{C}_{f_M}$. Recall from Definition 2.2.2 that the moduli spaces of a Morse flow category are defined as the compactification of the intersection of their corrseponding stable and unstable manifolds by adding broken flow lines. Here, this means that the moduli spaces are intervals whose boundaries are given by products of the 0-dimensional moduli spaces. Firstly, $\mathcal{M}(a, b)$ is a single interval with $\partial\mathcal{M}(a, b) = \mathcal{M}(y, b) \times \mathcal{M}(a, y) = \{q \cdot p, r \cdot p\}$ given by

$$\mathcal{M}(a,b) =$$

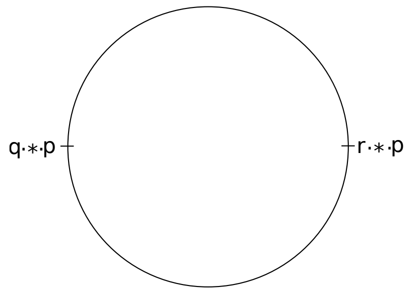


and secondly, $\mathcal{M}(x,b)$ is a single interval with $\partial\mathcal{M}(x,b) = \mathcal{M}(y,b) \times \mathcal{M}(x,y) = \{q \cdot *, r \cdot *\}$ given by

$$\mathcal{M}(x,b) =$$



By choosing the single-point moduli space $\mathcal{M}(x,y) = \{*\}$ to cancel, we can produce a cancelled flow category $\mathscr{C}_H = \mathscr{C}_{f_{S^2}}$ whose objects are $\mathrm{Ob}(\mathscr{C}_H) = \{\bar{a}, \bar{b}\}$ and whose only moduli space is given by $\mathcal{M}(\bar{a}, \bar{b}) = \mathcal{M}(a,b) \cup \big(\mathcal{M}(x,b) \times \mathcal{M}(a,y)\big)$. Since $|a| = |x|$ and $\mathcal{M}(a,x) = \emptyset$, the gluing takes place only along $\mathcal{M}(y,b) \times \mathcal{M}(a,y)$. The boundary points $\{q \cdot *, r \cdot *\} = \partial\mathcal{M}(x,b)$ become $\{q \cdot * \cdot p, r \cdot * \cdot p\}$ in the product $\mathcal{M}(x,b) \times \mathcal{M}(a,y)$, and so the two intervals are glued together to form a single circle of the form



This circle is highlighted in red in Figure 3.1b as the equator of the 2-sphere S^2, and is precisely the intersection of the stable and unstable manifolds of a and b, respectively, with respect to the positive gradient flow ∇f_{S^2}. Moreover, Theorem 3.1.1 provides a homotopy equivalence between $|\mathscr{C}_{f_M}|$ and $|\mathscr{C}_{f_{S^2}}|$.

The previous example highlights how the main theorem (Theorem 3.1.1) contains handle cancellation occurring in Morse theory. For a general framed flow category $\mathscr{C}$, we must provide framed neat embeddings of the new moduli spaces of $\mathscr{C}_H$ described in Definition 3.1.1, so that we can exhibit a homotopy between the realisation of $\mathscr{C}$ and the realisation of $\mathscr{C}_H$. A somewhat simple deformation, which is defined on the cell $\mathcal{C}(x)$ of $|\mathscr{C}|$, will provide an intuitive description of how to embed the moduli spaces of $\mathscr{C}_H$. Recall that $\mathcal{C}(x)$ is a $(C+i)$-cell and a single copy of $\mathcal{C}(y) \cong \mathcal{M}(x,y) \times \mathcal{C}(y)$ is identified on its boundary (particularly on the $(i-1)$-face) via

$$\begin{aligned}\mathcal{C}_y(x) =& [0,R] \times [-R,R]^{d_B} \times \cdots \times [-R,R]^{d_{i-2}} \times \{0\} \times \imath_{x,y}\big(\mathcal{M}(x,y) \times [-\varepsilon,\varepsilon]^{d_{i-1}}\big) \times \\ & \{0\} \times [-\varepsilon,\varepsilon]^{d_i} \times \cdots \times \{0\} \times [-\varepsilon,\varepsilon]^{d_{A-1}} \subset \partial_{i-1}\mathcal{C}(x).\end{aligned}$$

Note that we can assume $\imath_{x,y}$ embeds the point $\mathcal{M}(x,y) = *$ as $\imath_{x,y}(*) = 0$ in $\mathbb{R}^{d_{i-1}}$ and hence we can think of the framing of that point as the restriction of an isomorphism of $\mathbb{R}^{d_{i-1}}$. This gives a homeomorphism between $\mathcal{C}_y(x)$ and the $(C+i-1)$-cell $\mathcal{C}(y)$.

Definition 3.1.2 Let $\Psi_t : \partial\mathcal{C}(x) \setminus \mathcal{C}_y(x) \to \mathcal{C}(x)$ (with parameter $t \in [0,1]$) define a deformation of $\mathcal{C}(x)$ which takes each face of the $(C+i)$-cube $\mathcal{C}(x)$ (including the $(i-1)$-face away from $\mathcal{C}_y(x)$) through the interior of $\mathcal{C}(x)$, so that eventually when $t=1$, each face lies in $\mathcal{C}_y(x) \cong \mathcal{C}(y)$. This deformation can be characterised by the following properties:

1. Each $\Psi_t : \partial\mathcal{C}(x) \setminus \mathcal{C}_y(x) \to \mathcal{C}(x)$ is a deformation of $\partial\mathcal{C}(x) \setminus \mathcal{C}_y(x)$ that is the identity on $\partial\mathcal{C}_y(x)$.

2. The map $\Psi_0 : \partial\mathcal{C}(x) \setminus \mathcal{C}_y(x) \hookrightarrow \mathcal{C}(x)$ is the inclusion map.

3. The map $\Psi_1 : \partial\mathcal{C}(x) \setminus \mathcal{C}_y(x) \to \mathcal{C}_y(x)$ takes the union of all the faces of $\mathcal{C}(x)$ to $\mathcal{C}_y(x)$ on the $(i-1)$-face.

It is necessary to outline some required features regarding the smoothness of these maps Ψ_t. It will be required that these maps are smooth on subsets $\mathcal{C}_b(x)$ of $\partial\mathcal{C}(x) \setminus \mathcal{C}_y(x)$ in order to keep track of their framings. Since each of the subsets $\mathcal{C}_b(x)$ is identified on the n-face of $\mathcal{C}(x)$, a stronger condition that can be assumed is that Ψ_t, for each t, is piecewise-smooth in the sense that it is smooth on each face $\partial_k\mathcal{C}(x)$ for $k = B, \ldots, i-1$ (see Figure 3.4). Each Ψ_t will also have to behave well on particular corners $\partial_{j,k}\mathcal{C}(x) := \partial_j\mathcal{C}(x) \cap \partial_k\mathcal{C}(x)$ for $B \leq j < k \leq i-1$. Here, it is enough to allow the corners to be flattened as long as the piecewise smooth condition is preserved on the mutual faces.

In addition, the face-structure of $\partial\mathcal{C}(x) = \bigcup\limits_{k=B}^{i-1} \partial_k\mathcal{C}(x)$ is altered by the deformation Ψ_t. Each k-face is collapsed to $\Psi_1|_{\partial_k}\big(\partial_k\mathcal{C}(x)\big)$ in $\mathcal{C}_y(x)$ of the $(i-1)$-face. It will be useful to keep track of these faces in order to define neat embeddings for $\mathscr{C}_H$, since each $\mathcal{C}_b(x)$ is identified with the n-face ($|b| = n$) and we need this to remain true, even after collapsing.

As we run the parameter t through $[0,1]$, we are in essence taking an interval product with $\mathcal{C}_b(x)$, and thus there is an inherent framing on these newly embedded moduli spaces $\mathcal{M}(x,b)$, for each t. Since each Ψ_t fixes $\partial\mathcal{C}_y(x)$, subspaces of the form

$$
\begin{aligned}
&[0,R] \times [-R,R]^{d_B} \times \cdots \times [-R,R]^{d_{n-1}} \times \{0\} \times \\
&\imath_{y,b}\big(\mathcal{M}(y,b) \times [-\varepsilon,\varepsilon]^{d_n+\cdots+d_{i-2}} \times \big) \times \{0\} \times \imath_{x,y}\big(\mathcal{M}(x,y) \times [-\varepsilon,\varepsilon]^{d_{i-1}}\big) \times \\
&\{0\} \times [-\varepsilon,\varepsilon]^{d_i} \times \cdots \times \{0\} \times [-\varepsilon,\varepsilon]^{d_{A-1}} \subset \partial\mathcal{C}_y(x) \subset \partial\mathcal{C}(x)
\end{aligned}
$$

which have a product framing given by the framings of $\mathcal{M}(y,b)$ and $\mathcal{M}(x,y)$, are also fixed. This framing is coherent with the inherent framing from the interval product of $\Psi_t|_{\mathcal{C}_b(x)}$ for each t in $[0,1]$. The deformation $\Psi_t : \partial\mathcal{C}(x) \setminus \mathcal{C}_y(x) \to \mathcal{C}(x)$ will assist in describing embeddings of the moduli spaces $\mathcal{M}(\bar{a},\bar{b})$ of $\mathscr{C}_H$, and subsequently providing a CW complex $|\mathscr{C}_H|$.

Recall from Definition 3.1.1 that $\mathcal{M}(\bar{a},\bar{b})$ is formed by gluing the two moduli spaces $\mathcal{M}(a,b)$ and $\mathcal{M}(x,b) \times \mathcal{M}(a,y)$ along their common boundaries. Thus we shall define an embedding for each of these moduli spaces separately, and emphasise how the gluing works; the former of the two is embedded with its original embedding from $(\mathscr{C},\imath,\varphi)$. A framed embedding $\Gamma_{x,b\times a,y}$ of the product moduli spaces $\mathcal{M}(x,b) \times \mathcal{M}(a,y)$ is described in Lemma 3.1.3 along with a description of how the two moduli spaces are glued together. A framed embedding $\Gamma_{\bar{a},\bar{b}}$ of the moduli spaces $\mathcal{M}(\bar{a},\bar{b})$ is then described in Lemma 3.1.4. Some alterations are needed to ensure that these embeddings are neat embeddings, and this is described in Lemma 3.1.5. The embeddings defined in this section follow the notation of Equation 2.4 of Definition 2.3.1, where identifications

$$\Gamma_{a,b} : \mathcal{M}(a,b) \times \mathcal{C}(b) \cong \mathcal{C}_b(a) \hookrightarrow \partial\mathcal{C}(a)$$

were defined.

Lemma 3.1.3 There is an embedding

$$\Gamma_{x,b\times a,y} : \mathcal{M}(x,b) \times \mathcal{M}(a,y) \times \mathcal{C}(b) \to \partial\mathcal{C}(a).$$

Moreover, this embedding can be defined to agree with $\Gamma_{a,b}$ on the boundary components $\big(\mathcal{M}(y,b) \times \mathcal{M}(x,y)\big) \times \mathcal{M}(a,y) \cup \mathcal{M}(x,b) \times \big(\mathcal{M}(x,y) \times \mathcal{M}(a,x)\big)$.

Proof: Consider each $\mathcal{M}(x,b)$ embedded into Euclidean space as

$$\imath_{x,b} : \mathcal{M}(x,b) \times [-\varepsilon,\varepsilon]^{d_n+\cdots d_{i-1}} \to [-R,R]^{d_n} \times [0,R] \times \cdots \times [0,R] \times [-R,R]^{d_{i-1}}.$$

As this embedding provides an identification $\Gamma_{x,b}$ of $\mathcal{M}(x,b) \times \mathcal{C}(b)$ with the subset $\mathcal{C}_b(x)$ of $\partial\mathcal{C}(x)\backslash\mathcal{C}_y(x)$, it is affected by the deformation Ψ_t. We know that varying t in $[0,1]$ provides an interval product of framed embedded subspaces $\Psi_t|_{\mathcal{C}_b(x)}(\mathcal{C}_b(x))$ inside $\mathcal{C}(x)$, so in particular we can look at the framed embedded subspace $\Psi_1|_{\mathcal{C}_b(x)}(\mathcal{C}_b(x))$ inside $\mathcal{C}_y(x)$ (see Figure 3.4). This provides an embedding of $\mathcal{M}(x,b) \times \mathcal{C}(b)$ into

$$\begin{aligned}&[0,R] \times [-R,R]^{d_B} \times \cdots \times [-R,R]^{d_{i-2}} \times \{0\} \times \imath_{x,y}\big(\mathcal{M}(x,y) \times [-\varepsilon,\varepsilon]^{d_{i-1}}\big)\\ &\times \{0\} \times [-\varepsilon,\varepsilon]^{d_i} \times \cdots \times \{0\} \times [-\varepsilon,\varepsilon]^{d_{A-1}} = \mathcal{C}_y(x).\end{aligned}$$

given by

$$\Psi_1|_{\mathcal{C}_b(x)} \circ \Gamma_{x,b} : \mathcal{M}(x,b) \times \mathcal{C}(b) \to \mathcal{C}_y(x).$$

Now by abusing notation slightly, let $\Gamma_{x,y}^{-1} : \mathcal{C}_y(x) \to \mathcal{M}(x,y) \times \mathcal{C}(y)$ be the obvious homeomorphism, so that

$$\Gamma_{x,y}^{-1} \circ \Psi_1|_{\mathcal{C}_b(x)} \circ \Gamma_{x,b} : \mathcal{M}(x,b) \times \mathcal{C}(b) \to \mathcal{C}(y)$$

provides an embedding of $\mathcal{M}(x,b)\times\mathcal{C}(b)$ into a one-point product of $\mathcal{C}(y)$. To embed the product moduli space $\mathcal{M}(x,b) \times \mathcal{M}(a,y) \times \mathcal{C}(b)$, consider the identification

$$\Gamma_{a,y} : \mathcal{M}(a,y) \times \mathcal{C}(y) \to \mathcal{C}_y(a) \subset \partial_{i-1}\mathcal{C}(a)$$

with $\mathcal{M}(x,b) \times \mathcal{C}(b)$ embedded into the $\mathcal{C}(y)$ component via $\Gamma_{x,y}^{-1} \circ \Psi_1|_{\mathcal{C}_b(x)} \circ \Gamma_{x,b}$. Then define the embedding

$$\Gamma'_{x,b\times a,y} : \mathcal{M}(x,b) \times \mathcal{M}(a,y) \times \mathcal{C}(b) \to \partial\mathcal{C}(a) \tag{3.1}$$

by $\Gamma'_{x,b\times a,y}(p,q,\delta) = \Gamma_{a,y}\big(q, \Gamma_{x,y}^{-1} \circ \Psi_1|_{\mathcal{C}_b(x)} \circ \Gamma_{x,b}(p,\delta)\big)$ (see Figure 3.5).

Notice that the embeddings $\Gamma'_{x,b\times a,y}$ and $\Gamma_{a,b}$ agree on $\big(\mathcal{M}(y,b)\times\mathcal{M}(x,y)\big)\times\mathcal{M}(a,y)$ since Ψ_1 is the identity on those points, but on $\mathcal{M}(x,b)\times\big(\mathcal{M}(x,y)\times\mathcal{M}(a,x)\big)$ they disagree. We fix this by altering $\Gamma'_{x,b\times a,y}$ in the collar neighbourhood

$$\mathcal{M}(x,b)\times\mathcal{M}(a,x)\times[0,1]\times\mathcal{C}(b)$$

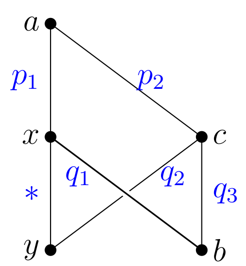
of $\mathcal{M}(x,b)\times\mathcal{M}(a,y)\times\mathcal{C}(b)$. Now consider the embedding

$$\Gamma_{x,b\times a,y}:\mathcal{M}(x,b)\times\mathcal{M}(a,y)\times\mathcal{C}(b)\to\partial\mathcal{C}(a) \tag{3.2}$$

that is defined as $\Gamma'_{x,b\times a,y}$ away from this collar neighbourhood, and defined as $\Gamma_{x,b\times a,y}(p,q,t,\delta)=\Gamma_{a,x}\big(q,\Psi_t|_{\mathcal{C}_b(x)}(\Gamma_{x,b}(p,\delta))\big)$ on points (p,q,t,δ) in the collar neighbourhood. Altering t from 0 to 1 has the effect of tracing from the embedding $\Gamma_{a,b}$ (when $t=0$) to the embedding $\Gamma'_{x,b\times a,y}$ (when $t=1$). We need to check that this alteration in the collar neighbourhood $\mathcal{M}(x,b)\times\mathcal{M}(a,x)\times[0,1]$ does not effect the intersection with the collar neighbourhood $\mathcal{M}(y,b)\times\mathcal{M}(a,y)\times[0,1]$, which is $\mathcal{M}(y,b)\times\mathcal{M}(a,x)\times[0,1]^2$. However, since $\Gamma_{x,b}$ sends the boundary component $\mathcal{M}(y,b)\times\mathcal{M}(x,y)\times\mathcal{C}(b)$ to $\mathcal{C}_b(x)\cap\mathcal{C}_y(x)\subset\partial\mathcal{C}_y(x)$, then each Ψ_t has no effect on this particular collar. Hence, $\Gamma_{x,b\times a,y}$ provides an embedding satisfying the required properties. □

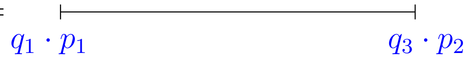
Since the embeddings defined in the proof of the previous lemma may seem a little abstract, let us consider an example.

Example 3.1.2 Let $\mathscr{C}_\bigstar$ be a framed flow category with $\mathrm{Ob}(\mathscr{C}_\bigstar)=\{a,x,c,y,b\}$ such that $|a|=2$, $|x|=|c|=1$, and $|y|=|b|=0$. Consider the following illustration of $\mathscr{C}_\bigstar$

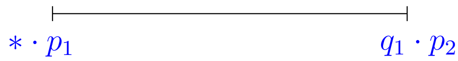


where the 0-dimensional moduli spaces are all single points that are labelled in blue. The 1-dimensional moduli spaces can be drawn as

$\mathcal{M}(a,b) =$



$\mathcal{M}(a,y) =$



Now assume that $\imath$ is a neat embedding of $\mathscr{C}_\bigstar$ relative $\mathbf{d} = (d_0, d_1) = (1, 1)$. The cells needed to construct the CW complex $|\mathscr{C}_\bigstar|$ are:

$$\mathcal{C}(a) = [0, R] \times [-R, R] \times [0, R] \times [-R, R]$$
$$\mathcal{C}(x) = [0, R] \times [-R, R] \times \{0\} \times [-\varepsilon, \varepsilon] = \mathcal{C}(c)$$
$$\mathcal{C}(y) = \{0\} \times [-\varepsilon, \varepsilon] \times \{0\} \times [-\varepsilon, \varepsilon] = \mathcal{C}(b).$$

All of these cells are subsets of $\mathbb{E} = \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}$. Moreover,

$$\partial\mathbb{E} = \big(\{0\} \times \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}\big) \bigcup \big(\mathbb{R}_+ \times \mathbb{R} \times \{0\} \times \mathbb{R}\big)$$

is a 3-dimensional $\langle 2 \rangle$-manifold, which can be illustrated by flattening out the corner-plane $\{0\} \times \mathbb{R} \times \{0\} \times \mathbb{R}$ to give a homeomorphism $\partial\mathbb{E} \cong \mathbb{R}^3$ (c.f. [LS14a, Figure 3.3]). Under this homeomorphism, the boundary $\partial\mathcal{C}(a)$ has subsets identified with certain moduli spaces that are depicted in Figure 3.2.

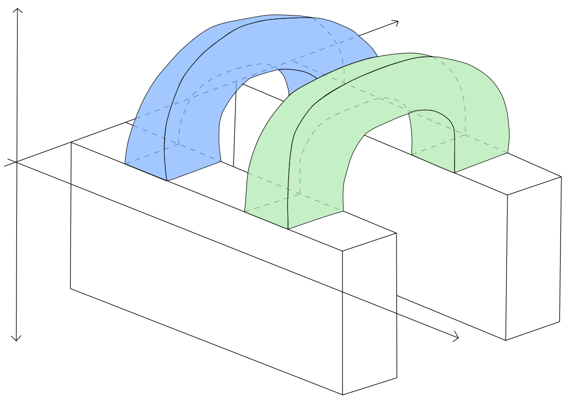


Figure 3.2: Example: $\mathscr{C}_\bigstar$. Identifications of cells in $\partial\mathcal{C}(a)$.

In blue is the embedding $\Gamma_{a,b}$ and in green the embedding $\Gamma'_{a,y}$ of $\mathcal{M}(a,b) \times \mathcal{C}(b)$ and $\mathcal{M}(a,y) \times \mathcal{C}(y)$, respectively. The rightmost (white) cube is $\mathcal{C}_c(a) \cong \mathcal{C}(c)$ and the leftmost (white) cube is $\mathcal{C}_x(a) \cong \mathcal{C}(x)$. The cell $\mathcal{C}(x)$ has parts of its boundary identified with both $\mathcal{C}_y(x)$ and $\mathcal{C}_b(x)$, and can be depicted on its own as in Figure 3.3.

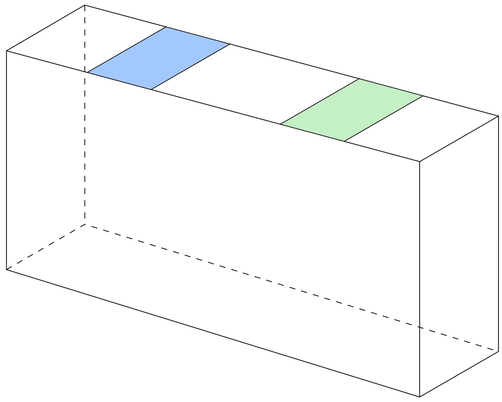


Figure 3.3: Example: $\mathscr{C}_\bigstar$. The cell $\mathcal{C}(x)$.

The cell $\mathcal{C}(y)$ is green and the cell $\mathcal{C}(b)$ is blue. Further, the deformation Ψ_t on $\mathcal{C}(x)$ (which is piecewise smooth on faces) sends $\partial\mathcal{C}(x)\backslash\mathcal{C}_y(x)$ through $\mathcal{C}(x)$ to $\mathcal{C}_y(x)$. This results in an embedding $\Gamma^{-1}_{x,y}\circ\Psi_1|_{\mathcal{C}_b(x)}\circ\Gamma_{x,b}$ of $\mathcal{M}(x,b)\times\mathcal{C}(b)$ in the cell $\mathcal{C}_y(x)\cong\mathcal{C}(y)$. This embedding is highlighted in blue in Figure 3.4, where the image of the faces are outlined.

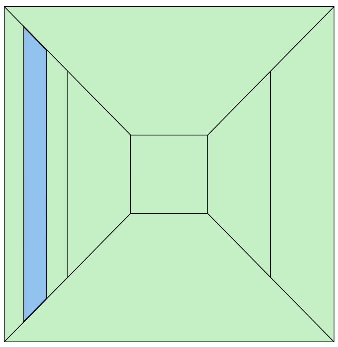


Figure 3.4: Example: $\mathscr{C}_\bigstar$. The result of collapsing $\mathcal{C}(x)$ using Ψ_1.

Now recall that the embedding $\Gamma'_{x,b\times a,y}$ is defined in Equation 3.1 as the embedding $\Gamma_{a,y}$ of $\mathcal{M}(a,y)\times\mathcal{C}(y)$ with $\mathcal{C}_b(x)$ embedded into $\mathcal{C}(y)$ as above. This embedding, together with the embedding $\Gamma_{a,b}$ is depicted in Figure 3.5.

The cells $\mathcal{C}_x(a)$ and $\mathcal{C}_y(a)$ are dashed lines because they correspond to the objects that are being cancelled. The embedding $\Gamma_{a,b}$ is highlighted blue as before, and the

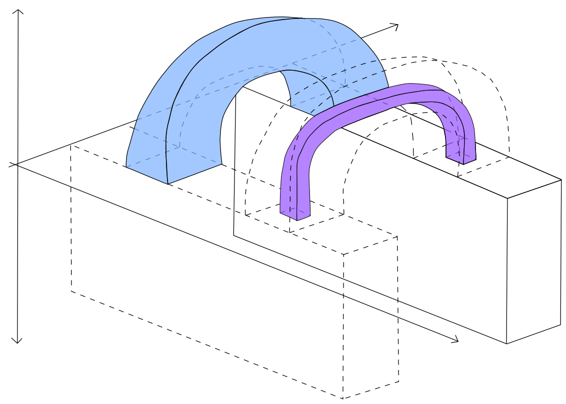


Figure 3.5: Example: $\mathscr{C}_\bigstar$. The embedding $\Gamma_{a,b}$ (in blue) and $\Gamma'_{x,b\times a,y}$ (in purple).

embedding $\Gamma'_{x,b\times a,y}$ is highlighted purple. Notice that the two framed intervals do not agree on their boundaries corresponding to

$$\mathcal{M}(x,b)\times\mathcal{M}(a,x) \text{ and } \mathcal{M}(x,b)\times\big(\mathcal{M}(x,y)\times\mathcal{M}(a,x)\big).$$

This is the purpose of the alteration of $\tilde{\Gamma}_{x,b\times a,y}$ in the proof of Lemma 3.1.3. The embedding $\Gamma_{x,b\times a,y}$ is defined in Equation 3.2 by altering the embedding $\Gamma'_{x,y\times a,y}$ in a collar neighbourhood of $\mathcal{M}(x,b)\times\mathcal{M}(a,y)\times\mathcal{C}(b)$. The alteration uses the deformation Ψ_t and takes place inside $\mathcal{C}_x(a)$; it is highlighted red in Figure 3.6.

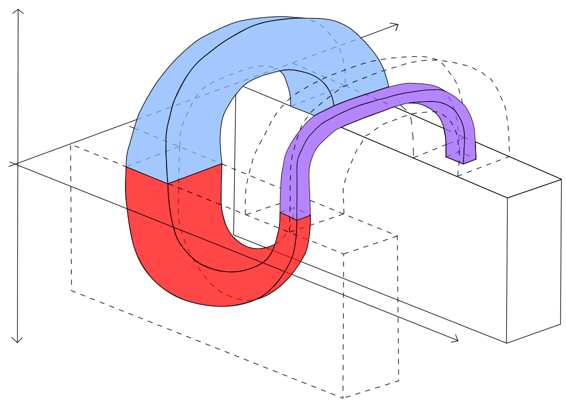


Figure 3.6: Example: $\mathscr{C}_\bigstar$. The embeddings $\Gamma_{a,b}$ and $\Gamma_{x,b\times a,y}$.

In this figure, the embedding $\Gamma_{x,b\times a,y}$ of $\mathcal{M}(x,b)\times\mathcal{M}(a,y)\times\mathcal{C}(b)$ is depicted as the concatenation of both the red and purple framed embedded intervals.

Lemma 3.1.4 There is an embedding $\Gamma_{\bar{a},\bar{b}} : \mathcal{M}(\bar{a},\bar{b}) \times \mathcal{C}(b) \to \partial\mathcal{C}(a)$.

Proof: Since $\mathcal{M}(\bar{a},\bar{b}) = \mathcal{M}(a,b) \cup_f \mathcal{M}(x,b) \times \mathcal{M}(a,y)$ (Definition 3.1.1), we may use $\Gamma_{a,b}$ and $\Gamma_{x,b\times a,y}$ to embed the respective components of $\mathcal{M}(\bar{a},\bar{b})$ into $\partial\mathcal{C}(a)$. The fact that these embeddings agree on the gluings defined by f in Definition 3.1.1 is given by Lemma 3.1.3, and we can define an embedding as $\Gamma_{\bar{a},\bar{b}} = \Gamma_{a,b} \cup \Gamma_{x,b\times a,y}$. □

Example 3.1.3 Consider again the framed flow category $\mathscr{C}_\bigstar$ from Example 3.1.2. In Figure 3.6, the embeddings $\Gamma_{a,b}$ and $\Gamma_{x,b\times a,y}$ are depicted. The embedding $\Gamma_{x,b\times a,y}$ agrees with $\Gamma_{a,b}$ on the boundary component $\big(\mathcal{M}(x,b)\times\mathcal{M}(a,x)\big)\times\mathcal{C}(b)$ of $\mathcal{M}(a,b)\times \mathcal{C}(b)$. Moreover, the embedding $\Gamma_{\bar{a},\bar{b}} = \Gamma_{a,b} \cup \Gamma_{x,b\times a,y}$ can be seen in Figure 3.6 as the concatenation of the blue, red and purple framed embedded intervals.

If we let $\Pi[b:a] : \mathcal{C}(a) \to \mathbb{E}_{\mathbf{d}}[b:a]$ denote the obvious projection map, and

$$\tilde{\Gamma}_{a,b} : \mathcal{M}(a,b) \times [-\varepsilon,\varepsilon]^{d_n+\cdots+d_{m-1}} \to \partial\mathcal{C}(a)$$

denote the embedding $\Gamma_{a,b}$ restricted to the necessary framing coordinates of $\mathcal{C}(b)$, then the original embeddings $\imath_{a,b} : \mathcal{M}(a,b) \times [-\varepsilon,\varepsilon]^{d_n+\cdots+d_{m-1}} \to \mathbb{E}_{\mathbf{d}}[b:a]$ can be recovered from the composition $\Pi[b:a]|_\partial \circ \tilde{\Gamma}_{a,b}$. Therefore, one would expect to be able to define embeddings of the moduli spaces $\mathcal{M}(\bar{a},\bar{b})$ in a similar way. However, these embeddings would not be neat since boundary components of both $\mathcal{M}(a,b)$ and $\mathcal{M}(x,b)\times\mathcal{M}(a,y)$ that are defined as interior points of $\mathcal{M}(\bar{a},\bar{b})$ need to be identified with part of the interior of $\partial_n\mathcal{C}(a)$. This is the case in Examples 3.1.2 and 3.1.3 of $\mathscr{C}_\bigstar$, where the embedding $\Gamma_{\bar{a},\bar{b}}$ protrudes into the 1-face of $\mathcal{C}(a)$ (the red framed interval in Figure 3.6). Therefore, relevant information about the moduli spaces is lost after projecting via the analogous projection map $\Pi[\bar{b}:\bar{a}] : \mathcal{C}(\bar{a}) \to \mathbb{E}_{\mathbf{d}}[\bar{b}:\bar{a}]$. Instead, we alter these embeddings slightly as outlined in the proof of the following lemma.

Lemma 3.1.5 For each $\bar{a},\bar{b}$ in $\mathrm{Ob}(\mathscr{C}_H)$, there are framed neat embeddings

$$\bar{\imath}_{\bar{a},\bar{b}} : \mathcal{M}(\bar{a},\bar{b}) \times [-\varepsilon,\varepsilon]^{d_n+\cdots d_{m-1}} \to \mathbb{E}_{\mathbf{d}}[b:a]$$

Proof: The framed neat embedding $\bar{\imath}_{\bar{a},\bar{b}}$ will be defined on each component of $\mathcal{M}(\bar{a},\bar{b}) = \mathcal{M}(a,b) \cup \big(\mathcal{M}(x,b)\times\mathcal{M}(a,y)\big)$ individually. The original embedding

$\imath_{a,b} = \Pi[\bar{b} : \bar{a}]|_\partial \circ \tilde{\Gamma}_{\bar{a},\bar{b}}$ is already neat on the $\mathcal{M}(a,b)$ component of $\mathcal{M}(\bar{a},\bar{b})$, however recall that there are boundary components of $\mathcal{M}(a,b)$ in $\mathscr{C}$ that are interior points in the corresponding component of $\mathcal{M}(\bar{a},\bar{b})$ in $\mathscr{C}_H$. This also holds for the framed embedding

$$\tilde{\Gamma}_{x,b\times a,y} : \mathcal{M}(x,b) \times \mathcal{M}(a,y) \times [-\varepsilon,\varepsilon]^{d_n+\cdots d_{m-1}} \to \partial\mathcal{C}(a)$$

which is defined as the restriction of $\Gamma_{x,b\times a,y}$ to the framing coordinates $[-\varepsilon,\varepsilon]^{d_n+\cdots+d_{m-1}}$ of $\mathcal{C}(b)$. The embedding $\bar{\imath}_{\bar{a},\bar{b}}$ should reflect this accordingly, and will therefore be defined as a perturbation of the embedding $\tilde{\Gamma}_{a,b} \cup \tilde{\Gamma}_{x,b\times a,y}$.

Recall that in Lemma 3.1.3 the embedding $\Gamma_{x,b\times a,y}$ was defined on the collar

$$\mathcal{M}(x,b) \times \mathcal{M}(a,x) \times [0,1] \times \mathcal{C}(b)$$

by $\Gamma_{x,b\times a,y}(p,q,t,\delta) = \Gamma_{a,x}\big(q, \Psi_t|_{\mathcal{C}_b(x)}(\Gamma_{x,b}(p,\delta))\big)$. For $t = 0$, the framed moduli spaces $\mathcal{M}(x,b) \times \mathcal{M}(a,x) \times \mathcal{C}(b)$ are embedded into the corner $\partial_{n,i}\mathcal{C}(a)$. As the parameter t runs through $[0,1]$, the embeddings of these framed moduli spaces protrude into the i-face $\partial_i\mathcal{C}(a) \subset \partial\mathcal{C}(a)$ and in particular, lie slightly in the interior of $\mathcal{C}_x(a) \cong \mathcal{M}(a,x) \times \mathcal{C}(x)$. Since Ψ_t is a deformation retraction of the cell $\mathcal{C}(x)$ to $\mathcal{C}(y)$, there is a neighbourhood of the corner $\partial_{n,i}\mathcal{C}(a)$ that contains this deformation. Moreover, this neighbourhood is contained in the $\mathcal{C}(x)$ component of $\mathcal{C}_x(a)$ and is homotopically a disk. Let $N_\epsilon\big(\partial_{n,i}\mathcal{C}(a)\big)$ denote this neighbourhood, where $\epsilon > 0$ is such that

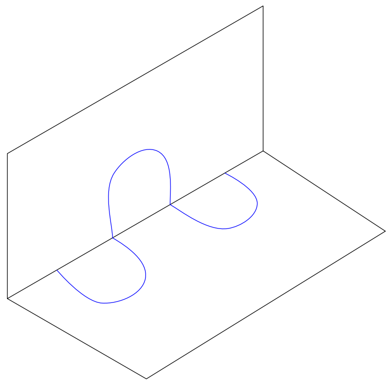
$$\Gamma_{x,b\times a,y} : \mathcal{M}(x,b) \times \mathcal{M}(a,y) \times \mathcal{C}(b) \to \partial_n\mathcal{C}(a) \cup N_\epsilon\big(\partial_{n,i}\mathcal{C}(a)\big)$$

is a framed embedding (see Figure 3.7), where the coordinates of $N_\epsilon\big(\partial_{n,i}\mathcal{C}(a)\big)$ are identical to $\partial_{n,i}\mathcal{C}(a)$ away from $\partial_n\mathcal{C}_x(a) \subset \partial_{n,i}\mathcal{C}(a)$ and are contained in

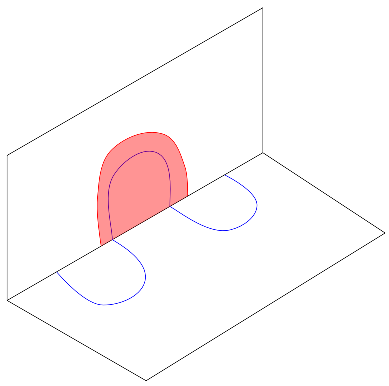
$$\begin{aligned}
&[0,R] \times [-R,R]^{d_B} \times \cdots \times [-R,R]^{d_{n-1}} \times [0,\epsilon] \times [-R,R]^{d_n} \times \cdots \\
&\times [-R,R]^{d_{i-1}} \times \{0\} \times [-R,R]^{d_i} \times \cdots \times [-R,R]^{d_{m-1}} \times \\
&\{0\} \times [-\varepsilon,\varepsilon]^{d_m} \times \cdots \times \{0\} \times [-\varepsilon,\varepsilon]^{d_{A-1}} \subset \partial_i\mathcal{C}(a).
\end{aligned}$$

The necessary perturbation then comes in the form of a continuous map

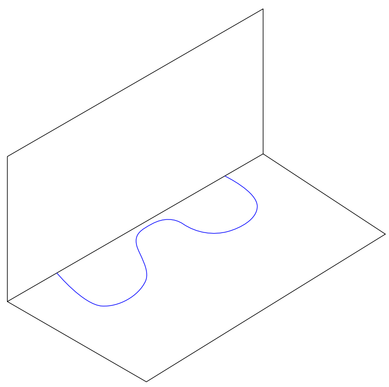
$$\overline{P} : \partial_n\mathcal{C}(a) \cup N_\epsilon\big(\partial_{n,i}\mathcal{C}(a)\big) \to \partial_n\mathcal{C}(a) \tag{3.3}$$



(a) Two faces of a cube $\mathcal{C}(a)$; vertically is the i-face, horizontally is the n-face. The blue interval protrudes into the vertical i-face.



(b) The neighbourhood $N_\epsilon\big(\partial_{n,i}\mathcal{C}(a)\big)$ of the 'corner' line, highlighted in red.



(c) The result of pushing the neighbourhood $N_\epsilon\big(\partial_{n,i}\mathcal{C}(a)\big)$ using $\overline{P}$.

Figure 3.7: A small illustration of the definitions of $N_\epsilon\big(\partial_{n,i}\mathcal{C}(a)\big)$ and $\overline{P}$.

which will have the effect of pushing this disk into the n-face, from the i-face, of $\partial\mathcal{C}(a)$. To describe $\overline{P}$, let $\overline{N}_\epsilon\big(\partial_{n,i}\mathcal{C}(a)\big) \subset \partial_n\mathcal{C}(a)$ be a disk identical to $N_\epsilon\big(\partial_{n,i}\mathcal{C}(a)\big)$, but lying in the n-face of $\mathcal{C}(a)$ so that both disks intersect on the (n,i)-corner (in Figure 3.7, $\overline{N}_\epsilon\big(\partial_{n,i}\mathcal{C}(a)\big)$ would be the horizontal equivalent of $N_\epsilon\big(\partial_{n,i}\mathcal{C}(a)\big)$). $\overline{P}$ is defined to

1. send $N_\epsilon\big(\partial_{n,i}\mathcal{C}(a)\big) \cap \overline{N}_\epsilon\big(\partial_{n,i}\mathcal{C}(a)\big)$ to $\partial\overline{N}_\epsilon\big(\partial_{n,i}\mathcal{C}(a)\big) \setminus N_\epsilon\big(\partial_{n,i}\mathcal{C}(a)\big) \cap \overline{N}_\epsilon\big(\partial_{n,i}\mathcal{C}(a)\big)$

2. send $\partial N_\epsilon\big(\partial_{n,i}\mathcal{C}(a)\big) \setminus N_\epsilon\big(\partial_{n,i}\mathcal{C}(a)\big) \cap \overline{N}_\epsilon\big(\partial_{n,i}\mathcal{C}(a)\big)$ to $N_\epsilon\big(\partial_{n,i}\mathcal{C}(a)\big) \cap \overline{N}_\epsilon\big(\partial_{n,i}\mathcal{C}(a)\big)$

and be continuous on the rest of $\partial_n\mathcal{C}(a) \cup N_\epsilon\big(\partial_{n,i}\mathcal{C}(a)\big)$. Now let

$$P : \partial\mathcal{C}(a) \to \partial\mathcal{C}(a) \tag{3.4}$$

be a continuous map obtained by extending

$$\overline{P} : \partial_n\mathcal{C}(a) \cup N_\epsilon\big(\partial_{n,i}\mathcal{C}(a)\big) \to \partial_n\mathcal{C}(a)$$

to the entire boundary of $\mathcal{C}(a)$. The '*push map*' P ensures that the embeddings of $\mathcal{M}(\bar{a},\bar{b}) \times \mathcal{C}(b)$ can be pushed to lie entirely in the n-face of $\partial\mathcal{C}(a)$.

The remaining hurdle in providing a neat embedding is to ensure that boundary components of $\mathcal{M}(\bar{a},\bar{b})$ that come from boundary components of $\mathcal{M}(x,b) \times \mathcal{M}(a,y)$ are identified with parts of the n-face of $\mathcal{C}(a)$. Currently, $\Gamma_{x,b\times a,y}$ identifies such components with the interior of $\mathcal{C}(y)$ in $\mathcal{C}_y(a) \cong \mathcal{M}(a,y) \times \mathcal{C}(y)$ (see Figure 3.4, where $\mathcal{C}_b(x)$ is embedded into $\mathcal{C}(y)$). But recall that the deformation Ψ_t alters the faces of $\mathcal{C}(x)$ in the sense that each $\partial_k\mathcal{C}(x)$ is mapped (smoothly) to $\Psi_1|_{\partial_k}\big(\partial_k\mathcal{C}(x)\big)$ inside $\mathcal{C}_y(x) \cong \mathcal{M}(x,y) \times \mathcal{C}(y)$. This alteration is therefore apparent in $\partial\mathcal{C}(a)$, particularly in $\mathcal{C}_y(a)$. By redefining the corners $\partial_{k,i-1}\mathcal{C}(a)$ precisely on the faces $\partial_k\mathcal{C}_y(a) \subset \partial_{k,i-1}\mathcal{C}(a)$, we can ensure that the embeddings $\Gamma_{x,b\times a,y}$ send the boundaries of $\mathcal{M}(x,b) \times \mathcal{M}(a,y) \times \mathcal{C}(b)$ to the correct faces of $\mathcal{C}(a)$. So let the k-face of $\mathcal{C}_y(a)$ be replaced by $\Psi_1|_{\partial_k}\big(\partial_k\mathcal{C}(x)\big) \subset \mathcal{C}(y)$. The boundary components of $\mathcal{M}(x,b) \times \mathcal{M}(a,y)$ are then embedded to the correct boundaries as inherited by $\mathcal{C}_b(x) \cong \mathcal{M}(x,b) \times \mathcal{C}(b) \subset \partial_n\mathcal{C}(x)$.

Now define the embedding

$$\bar{\imath}_{\bar{a},\bar{b}} : \mathcal{M}(\bar{a},\bar{b}) \times [-\varepsilon,\varepsilon]^{d_n+\cdots d_{m-1}} \to \mathbb{E}_{\mathbf{d}}[b:a]$$

by $\bar{\imath}_{\bar{a},\bar{b}} = \Pi[\bar{b},\bar{a}] \circ P \circ (\tilde{\Gamma}_{a,b} \cup \tilde{\Gamma}_{x,b\times a,y})$. $\square$

Example 3.1.4 Consider the recurring example of the framed flow category $\mathscr{C}_{\bigstar}$. The embedding $\Gamma_{\bar{a},\bar{b}}$, as seen in Figure 3.6, embeds $\mathcal{M}(\bar{a},\bar{b})\times\mathcal{C}(b)$ in $\partial\mathcal{C}(a)$. However, the embedding protrudes into the 1-face (which is the i-face) of $\partial\mathcal{C}(a)$. Applying the map

$$P : \partial\mathcal{C}(a) \to \partial\mathcal{C}(a)$$

will push this interval into the 0-face only, and the result is illustrated in Figure 3.8. The neighbourhood $N_\epsilon\big(\partial_{0,1}\mathcal{C}(a)\big)$ is a small 3-disk inside $\mathcal{C}_x(a) \cong \mathcal{C}(x)$ (the

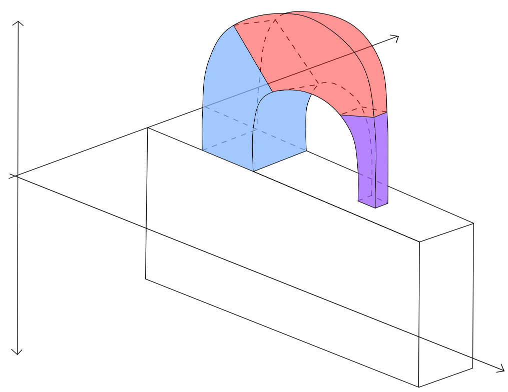


Figure 3.8: Example: $\mathscr{C}_{\bigstar}$. The result of applying the push map P to $\partial\mathcal{C}(a)$.

dashed box in Figure 3.6) that contains the framed (red) interval which corresponds to the alteration $\Gamma_{x,b\times a,y}(p,q,t,\delta) = \Gamma_{a,x}\big(q, \Psi_t|_{\mathcal{C}_b(x)}(\Gamma_{x,b}(p,\delta))\big)$ for $t \in (0,1)$, which is defined in Equation 3.2.

Corollary 3.1.6 Let $(\mathscr{C},\imath,\varphi)$ be a framed flow category containing two objects x and y with $\mathcal{M}(x,y) = *$. Then there is a framed flow category $(\mathscr{C}_H,\bar{\imath},\bar{\varphi})$ whose objects and moduli spaces are defined in Definition 3.1.1.

Proof: The flow category $\mathscr{C}_H$ itself is defined in Definition 3.1.1. The framed neat embeddings $\bar{\imath}_{\bar{a},\bar{b}}$ are constructed in Lemmata 3.1.3, 3.1.4 and 3.1.5. $\square$

Proof of Theorem 3.1.1: The realisation $|\mathscr{C}_H|$ is built up inductively from cells $\mathcal{C}(\bar{a})$ for objects $\bar{a}$ of $\mathscr{C}_H$ with increasing indices, $|\bar{a}|$, as prescribed by the Cohen-Jones-Segal construction in Definition 2.3.1. The way in which these cells are attached correspond to the framed neat embeddings $\bar{\imath}_{\bar{a},\bar{b}}$ defined in Lemma 3.1.5. By attaching all cells $\mathcal{C}(\bar{a})$ with $|\bar{a}| < |y| = i-1$, the pair of skeleta $|\mathscr{C}|^{(i-2)}$ and $|\mathscr{C}_H|^{(i-2)}$ are identical. Further attaching all cells $\mathcal{C}(\bar{a})$ for objects $\bar{a}$ with index equal to $i-1$, we have

$$|\mathscr{C}|^{(i-1)} = |\mathscr{C}_H|^{(i-1)} \cup \mathcal{C}(y) \hookleftarrow |\mathscr{C}_H|^{(i-1)}.$$

For objects $\bar{a}$ of $\mathscr{C}_H$ of index $m \geq |x| = i$ the cells $\mathcal{C}(a)$ are attached to $|\mathscr{C}|^{(m-1)}$ inductively in the usual way, corresponding to the identifications

$$\Gamma_{a,b} : \mathcal{M}(a,b) \times \mathcal{C}(b) \to \partial\mathcal{C}(a)$$

defined in Equation 2.4. However, the cells $\mathcal{C}(\bar{a})$ are attached to $|\mathscr{C}_H|^{(m-1)}$ corresponding to different identifications that come from the embeddings $\bar{\imath}_{\bar{a},\bar{b}}$ constructed in this subchapter. To show that the CW complexes produced as a result of these methods are homotopy equivalent, consider the CW complex that is obtained from $|\mathscr{C}|$ by collapsing the cell $\mathcal{C}(x)$ via Ψ_t (Definition 3.1.2). Denote this collapsed CW complex by X, so that X is homotopy equivalent to $|\mathscr{C}|$.

As a result of collapsing $\mathcal{C}(x)$ in $|\mathscr{C}|$, cells $\mathcal{C}(a)$ that were once attached to $\mathcal{C}(x)$ in $|\mathscr{C}|^{(m-1)}$ using the identifications

$$\Gamma_{a,x} : \mathcal{M}(a,x) \times \mathcal{C}(x) \to \partial\mathcal{C}(a)$$

are now attached to $X^{(m-1)}$ in a slightly different way. In particular, the deformation $\Psi_t : \partial\mathcal{C}(x) \setminus \mathcal{C}_y(x) \to \mathcal{C}_y(x)$ ensures that the identifications of $\mathcal{C}_x(a) \subset \partial\mathcal{C}(a)$ are pushed to $\mathcal{C}_x(a) \cap \mathcal{C}_y(a)$ and thus will occur in the normal fibre $\mathcal{C}(y)$ which comes from the identification

$$\Gamma_{a,y} : \mathcal{M}(a,y) \times \mathcal{C}(y) \to \mathcal{C}(y).$$

The attachment corresponding to identifications of $\partial\mathcal{C}(a)$ that do not involve $\mathcal{C}(x)$ or $\mathcal{C}(y)$ remain the same. Recall, from Lemma 3.1.3, that this was precisely how the embeddings

$$\Gamma_{x,b\times a,y} : \mathcal{M}(x,b) \times \mathcal{M}(a,y) \to \partial\mathcal{C}(a)$$

were defined. Therefore, the way in which the cell $\mathcal{C}(a)$ is attached to $X^{(m-1)}$ is given by the embeddings $\Gamma_{\bar{a},\bar{b}} = \Gamma_{a,b} \cup \Gamma_{x,b\times a,y}$ defined in Lemma 3.1.4. The collapsed CW complex X is therefore the Thom space of the normal bundle coming from the embeddings $\Gamma_{\bar{a},\bar{b}}$ that describe how each $\mathcal{C}(a)$ is attached to $X^{(m-1)}$. Recall that

$$\tilde{\Gamma}_{\bar{a},\bar{b}} : \mathcal{M}(\bar{a},\bar{b}) \times [-\varepsilon,\varepsilon]^{d_n+\cdots+d_{m-1}} \to \partial\mathcal{C}(a)$$

is defined as the embedding $\Gamma_{\bar{a},\bar{b}}$ restricted to the framing coordinates

$$[-\varepsilon,\varepsilon]^{d_n} \times \{0\} \times \cdots \times \{0\} \times [-\varepsilon,\varepsilon]^{d_{m-1}}$$

of $\mathcal{C}(b)$. The fact that X does not give the realisation $|\mathscr{C}_H|$ immediately is due to the embeddings $\tilde{\Gamma}_{\bar{a},\bar{b}}$ not coinciding with the coordinates of $\partial\mathcal{C}(a)$ which are identified with the embeddings $\bar{\imath}_{\bar{a},\bar{b}}$. However, in Lemma 3.1.5 the neat embeddings $\bar{\imath}_{\bar{a},\bar{b}}$ of the framed flow category $\mathscr{C}_H$ are defined via perturbations of the embeddings $\tilde{\Gamma}_{\bar{a},\bar{b}}$ inside $\partial\mathcal{C}(a)$. In particular, $\bar{\imath}_{\bar{a},\bar{b}} = \Pi[\bar{b},\bar{a}] \circ P \circ (\tilde{\Gamma}_{a,b} \cup \tilde{\Gamma}_{x,b\times a,y})$, where $\Pi[\bar{b},\bar{a}]$ is the projection to the coordinates $\mathbb{E}_{\mathbf{d}}[b:a]$ and P is the push map defined in Equation 3.4 in the proof of Lemma 3.1.5.

Since both $\tilde{\Gamma}_{\bar{a},\bar{b}}$ and $P \circ \tilde{\Gamma}_{\bar{a},\bar{b}}$ are embeddings of $\mathcal{M}(\bar{a},\bar{b}) \times [-\varepsilon,\varepsilon]^{d_n+\cdots+d_{m-1}}$ into $\partial\mathcal{C}(a)$, and the push map P is an isotopy inside $\partial\mathcal{C}(a)$, the Isotopy Extension Theorem (which is stated after this proof) ensures that there is a global isotopy in the Euclidean space $\mathbb{E}_{\mathbf{d}}[b:a]$ between the two embeddings which extends the isotopy in $\partial\mathcal{C}(a)$. In particular, it can be assumed that the cells $\mathcal{C}(a)$ are attached to $X^{(m-1)}$ in the same way the cells $\mathcal{C}(\bar{a}) = \mathcal{C}(a)$ are attached to $|\mathscr{C}_H|^{(m-1)}$. This gives a homotopy equivalence $|\mathscr{C}| \simeq X \simeq |\mathscr{C}_H|$ of CW complexes. Since these CW complexes are well-defined up to stable homotopy type, the homotopy equivalence described above gives rise to an equivalence of spectra when passing to suspensions. □

The following version of the Isotopy Extension Theorem has been stated in terms of the language used in this thesis. An original statement and proof can be found in [Hir94, Theorem 1.3, §8].

Theorem 3.1.2 (Isotopy Extension Theorem) Let $A, B \subset \mathbb{E}$ be compact subsets of a Euclidean space $\mathbb{E}$, and let $T : A \to B$ be an isotopy. Then there is a global isotopy $\Phi : \mathbb{E} \to \mathbb{E}$ such that $T = \Phi|_A$.

Chapter 4

Khovanov Stable Homotopy Type

In [LS14a], Lipshitz and Sarkar define a particular framed flow category $\mathscr{C}_{Kh}$ which refines the Khovanov chain complex. They define this category as a cover (in the sense of Definition 2.4.1) of another, simple flow category $\mathscr{C}_C(n)$ known as the cube flow category. Once $\mathscr{C}_{Kh}$ is defined, a CW spectrum $\mathcal{X}_{Kh}$ can be defined which is the result of plugging $\mathscr{C}_{Kh}$ into the Cohen-Jones-Segal machine (where it is necessary to construct a coherent framing for a neat embedding of $\mathscr{C}_{Kh}$). Since the objects of the Khovanov flow category are in one-to-one correspondence with the generators of the Khovanov chain complex, we shall outline a definition of Khovanov homology beforehand, following the approach in [LS14a] to use a diagrammatic definition using resolution configurations.

4.1 The cube flow category and its framing

This subchapter is an overview of the construction in [LS14a, Section 4] of a local model for the Khovanov flow category; the cube flow category. Certain results are of particular use in the description of Steenrod squares (see [LS14a, Subsection 3.3]) so we shall focus a little more on the details of their proofs.

Definition 4.1.1 Let $f_1 : \mathbb{R} \to \mathbb{R}$ be a self-indexing Morse function with a single critical point of index 0 and a single critical point of index 1. Then denote by

$f_n : \mathbb{R}^n \to \mathbb{R}$ the Morse function

$$f_n(x_1, \ldots, x_n) = f_1(x_1) + \cdots + f_1(x_n)$$

which has critical points in $\{0,1\}^n$. Then define the *n-dimensional cube flow category* $\mathscr{C}_C(n)$ as the Morse flow category $\mathscr{C}_{f_n}$ of f_n. As noted in [LS14a, Definition 4.1], a concrete choice for f_1 could be $f_1(x) = 3x^2 - 2x^3$.

Denote by $\mathcal{C}(n)$ the n-cube $[0,1]^n$ considered as a CW complex. For a vertex $v = (v_1, \ldots, v_n) \in \{0,1\}^n$ of $\mathcal{C}(n)$, let $|v| = v_1 + \cdots + v_n$, and write $v \leq_i u$ if $v \leq u$ on the cube with $|u| - |v| = i$. The following observation about $\mathscr{C}_C(n)$ is [LS14a, Lemma 4.2].

Lemma 4.1.2 There is a one-to-one correspondence between objects of $\mathscr{C}_C(n)$ and vertices of $\mathcal{C}(n)$ that preserves the grading. Moreover,

1. The moduli space $\mathcal{M}_{\mathscr{C}_C(n)}(u,v) = \emptyset$ for all $u \leq v$ in $\{0,1\}^n$.
2. Whenever $u > v$, $\mathcal{M}_{\mathscr{C}_C(n)}(u,v)$ is naturally diffeomorphic to $\mathcal{M}_{\mathscr{C}_C(|u|-|v|)}(\underline{1}, \underline{0})$ where $\underline{1} = (1, \ldots, 1)$ and $\underline{0} = (0, \ldots, 0)$.

One particular result that is important in Lipshitz-Sarkar's description of a Steenrod square on Khovanov homology is that $\mathcal{M}_{\mathscr{C}_C(n)}(\underline{1}, \underline{0})$ has a simple form for $n = 1, 2, 3$. This is contained in the following lemma.

Lemma 4.1.3 The moduli spaces $\mathcal{M}_{\mathscr{C}_C(n+1)}(\underline{1}, \underline{0})$ are difeomorphic to $\mathbb{D}^n$ for each non-negative integer n, with the boundary $\partial\mathcal{M}_{\mathscr{C}_C(n+1)}(\underline{1}, \underline{0})$ homeomorphic to S^{n-1}. In particular, $\mathcal{M}_{\mathscr{C}_C(n)}(\underline{1}, \underline{0})$ is diffeomorphic to a single point, an interval, and a hexagon for $n = 1, 2, 3$, respectively.

Proof: We shall only give the diffeomorphism type of $\mathcal{M}_{\mathscr{C}_C(n)}(\underline{1}, \underline{0})$ for $n = 1, 2, 3$. The inductive argument for general n is given in [LS14a, Lemma 4.3] and uses the properties of the Morse function f_n.

For $n = 1$, the moduli space $\mathcal{M}_{\mathscr{C}_C(1)}(1, 0)$ is clearly a point by characteristics of the Morse function $f_1(x)$ (there is only one flow line between 1 and 0). When $n = 2$, the 'square' moduli space for $\mathcal{C}(2) = [0,1]^2$ has flow lines from $(1,1)$ to $(0,0)$.

The moduli space $\mathcal{M}_{\mathscr{C}_C(2)}((1,1),(0,0))$ can be seen to be an interval classified by its two boundary points which correspond to the products of the single-point moduli spaces $\mathcal{M}_{\mathscr{C}_C(2)}((0,1),(0,0)) \times \mathcal{M}_{\mathscr{C}_C(2)}((1,1),(0,1))$ and $\mathcal{M}_{\mathscr{C}_C(2)}((1,0),(0,0)) \times \mathcal{M}_{\mathscr{C}_C(1)}((1,1),(1,0))$, which are both canonically diffeomorphic to $\mathcal{M}_{\mathscr{C}_C(1)}(1,0) \times \mathcal{M}_{\mathscr{C}_C(1)}(1,0)$ by Lemma 4.1.2. Note that $\mathcal{M}_{\mathscr{C}_C(2)}(u,v) =$ for $u,v \in \{(0,1),(1,0)\}$. There is an inductive procedure for the $n=3$ case that is similar to the one from the $n=2$ case. Firstly, note that the moduli space $\mathcal{M}_{\mathscr{C}_C(3)}((1,0,1),(0,1,0)) = \emptyset$ since $(1,0,1) \not\prec (0,1,0)$. For all $u,v \in \{0,1\}^3$ with $u \prec_1 v$, $\mathcal{M}_{\mathscr{C}_C(3)}(u,v)$ is a single point, and for all $u,v \in \{0,1\}^3$ with $u \prec_2 v$, $\mathcal{M}_{\mathscr{C}_C(3)}(u,v)$ is an interval. The only new information required for the 3-dimensional case is contained in the moduli space $\mathcal{M}_{\mathscr{C}_C(3)}((1,1,1),(0,0,0))$ since this is the only pair with $(0,0,0) \prec_3 (1,1,1)$. Clearly this is a hexagon, with vertices made up of moduli spaces corresponding to twice-broken flow lines (that is, a product of codimension-2 moduli spaces which are all single points), and edges corrsponding to once-broken flow lines (that is, a product of an interval with a point). $\square$

Given two vertices $v \leq_m u$ of the cube $\mathcal{C}(n)$, define a cellular structure on it by letting the associated m-cell be

$$\mathcal{C}_{u,v} = \{\, x \in [0,1]^n \mid v_i \leq x_i \leq u_i \forall i \,\} .$$

Definition 4.1.4 Let $\mathcal{C}_{u,v}$ be an m-cell of $\mathcal{C}(n)$. Define the corresponding inclusion functor

$$\mathcal{I}_{u,v} : \mathcal{C}_C(m) \hookrightarrow \mathcal{C}_C(n)$$

as follows. Let u and v differ in the coordinates with indices $j_1 < j_2 < \cdots < j_m$ and consider an object $x \in \{0,1\}^m = \mathrm{Ob}(\mathcal{C}_C(m))$. Then define a new object $x' \in \{0,1\}^n$ by

$$x'_i = \begin{cases} 1 & \text{if } v_i = 1 \\ 0 & \text{if } u_i = 0 \\ x_i & \text{if } i = j_i. \end{cases}$$

The full subcategory of $\mathcal{C}_C(n)$ with objects given by

$$\{\, x' \mid x \in \mathrm{Ob}(\mathcal{C}_C(m)) \,\}$$

is naturally isomorphic to $\mathcal{C}_C(m)$ (since by Lemma 4.1.2, all moduli spaces are naturally diffeomorphic). The functor $\mathcal{I}_{u,v}$ is defined to be this isomorphism.

4.1.1 A particular framing of the cube flow category

Throughout we shall be interested in the cellular cochain complex $C^*(\mathcal{C}(n), \mathbb{Z})$ of $\mathcal{C}(n)$. The generators will be thought of as the cochains $\mathcal{C}^{u,v}$ given by

$$\begin{cases} \mathcal{C}_{u,v} \mapsto 1 \\ \mathcal{C}_{u',v'} \mapsto 0 \quad \text{if } (u',v') \neq (u,v). \end{cases}$$

With $\mathbb{Z}/2$-coefficients, the differential on $C^*(\mathcal{C}(n), \mathbb{Z}/2)$ can be written easily as

$$\delta \mathcal{C}^{u,v} = \sum_{u' : u' \geq_1 u} \mathcal{C}^{u',v} + \sum_{v' : v' \leq_1 v} \mathcal{C}^{u,v'}.$$

Definition 4.1.5 Let $1^i \in C^i(\mathcal{C}(n), \mathbb{Z}/2)$ be the i-dimensional cochain which evaluates each i-cell to 1. A *sign assignment* is a 1-cochain $s \in C^1(\mathcal{C}(n), \mathbb{Z}/2)$ such that $\delta s = 1^2$.

Note that $\delta 1^2 = 0$ so 1^2 is a 2-cocycle. Moreover, since $H^2(\mathcal{C}(n), \mathbb{Z}/2) = 0$, then such a sign assignment exists. If s and s' are two sign assignments, then $\delta(s - s') = 0$ so $s - s'$ is a cocycle. Since $H^1(\mathcal{C}(n), \mathbb{Z}/2) = 0$, $s - s'$ is a coboundary so that $s = s' + \delta p$ for some $p \in C^0(\mathcal{C}(n), \mathbb{Z}/2)$.

The *standard sign assignment* s_0 is the sign assignment given by

$$s_0(\mathcal{C}_{(\varepsilon_1,\ldots,\varepsilon_r,1,\varepsilon_{r+1},\ldots,\varepsilon_n),(\varepsilon_1,\ldots,\varepsilon_r,0,\varepsilon_{r+1},\ldots,\varepsilon_n)}) = \varepsilon_1 + \cdots \varepsilon_r \pmod 2 .$$

Note that we can think of a sign-assignment in the following way. Let $v \leq_1 u$ so that $\mathcal{C}_{u,v}$ corresponds to an edge of the cube. If $s(\mathcal{C}_{u,v}) = 0$, then $\mathcal{C}_{u,v}$ is decorated with a $+$ in the cube, and if $s(\mathcal{C}_{u,v}) = 1$, then $\mathcal{C}_{u,v}$ is decorated with a $-$ in the cube. The condition that $\delta s = 1^2$ guarantees that there are an odd number of $-$ signs on each face of the cube.

Definition 4.1.6 Any sign assignment s gives a chain complex $C_s^*(n)$ generated by vertices $\{0,1\}^n$ over $\mathbb{Z}$. The chain complex $C_s^*(n)$ is called the *n-dimensional cube complex*. Moreover, if $|v| = k$ in the cube, then v is a generator for $C_s^k(n)$ and

$$\delta v = \sum_{u \geq_1 v} (-1)^{s(\mathcal{C}_{u,v})} u$$

is the differential.

Remark: This sign assignment corresponds to Khovanov's original sign assignment, and the n-dimensional cube complex corresponds to Khovanov's skew-commutative cube (see [Kho00, §3.3])

The procedure in [LS14a, Subsection 4.2] provides a framing for the cube flow category in such a way that it refines the cube complex $C^*_s(n)$. By orienting the cells of $\mathcal{C}(n)$ by the product orientation, the differential in $C^*(\mathcal{C}(n), \mathbb{Z})$ can be written as

$$\delta\mathcal{C}^{u,v} = \sum_{v':v'\leq_1 v} (-1)^{s_0(\mathcal{C}_{u-v',u-v})}\mathcal{C}^{u,v'} - \sum_{u':u'\geq_1 u} (-1)^{s_0(\mathcal{C}_{u'-v,u-v})}\mathcal{C}^{u',v}$$

given by the standard sign assignment s_0.

Definition 4.1.7 A *coherent orientation* for $\mathcal{C}_C(n)$ is a choice of orientation for all moduli spaces $\mathcal{M}(u,v)$ in $\mathcal{C}_C(n)$ such that the following conditions are satisfied:

1. For $u \leq_1 u'$, consider $\mathcal{M}(u,v)$ is a subspace of

$$\mathcal{M}(u,v) \cong \mathcal{M}(u,v) \times \mathcal{M}(u',u) \subset \partial\mathcal{M}(u',v).$$

   Then the orientation of $\mathcal{M}(u,v)$ has a subspace orientation which differs from the boundary orientation by $-(-1)^{s_0(\mathcal{C}_{u'-v,u-v})}$.

2. For $v' \leq_1 v$, then the difference of the subspace orientation from the boundary orientation is $(-1)^{s_0(\mathcal{C}_{u-v',u-v})}$ where $\mathcal{M}(u,v)$ is considered as the subspace

$$\mathcal{M}(u,v) \cong \mathcal{M}(v,v') \times \mathcal{M}(u,v) \subset \partial\mathcal{M}(u,v').$$

By using the standard sign assignment, [LS14a, Lemma 4.8] shows the following.

Lemma 4.1.8 A coherent orientation for $\mathcal{C}_C(n)$ exists.

Consider a neat embedding $\imath$ of $\mathscr{C}_C(n)$ relative some $\mathbf{d}$ with a framing on all moduli spaces of dimension strictly less than k. In [LS14a, Subsection 4.2], Lipshitz and Sarkar discuss the question of when one can extend this over moduli spaces of dimension equal to k. The answer is given using obstruction theory. Since this thesis focuses on the application of handle-cancellation in framed flow categories (the key application of handle-cancellation in framed flow categories here being at the level of Steenrod square computations), and handle-cancellation is not necessary when producing a cancelled version of the Khovanov flow category (see [JLS15]), the details of this obstruction theory argument are omitted. An overview of the argument is given as follows.

By considering some $(k+1)$-cell $\mathcal{C}_{u,v}$ of the cube $\mathcal{C}(n)$, the boundary of embedded moduli spaces $\partial\imath_{u,v}\big(\mathcal{M}(u,v)\big)$ is an oriented $(k-1)$-sphere S^{k-1} which is framed by the product framing $\varphi_{u,v}$ of the boundary components, and these are already framed by assumption since they are of lower dimension. Now taking some framing φ_0 of the normal bundle to $\imath_{u,v}\big(\mathcal{M}(u,v)\big) \cong \mathbb{D}^k$ gives an element of $\pi_{k-1}(O(N))$, for some sufficiently large N, in the following way. At each point p on S^{k-1}, chose the element of $O(N)$ that sends the frame $\varphi_{u,v}(p)$ to the null-cobordant frame $\varphi_0(p)$. This element of $\pi_{k-1}(O(N))$ is independent of φ_0 since two different null-cobordant frames produce a map from S^{k-1} to the orthogonal group and their null-cobordance produces a null-homotopy of the map.

Moreover, since $N > k-1$, $\pi_{k-1}(O(N))$ can be identified with $\pi_{k-1}(\mathbf{O})$, and this gives an element $\sigma \in C^{k+1}(\mathcal{C}(n), \pi_{k-1}(\mathbf{O}))$. The element σ is the obstruction class, and by definition $\sigma = 0$ if and only if the framing extends over k-dimensional moduli spaces. This obstruction class is used to show that different sign assignments will give rise to homotopy equivalent stable homotopy types, and therefore ensuring that we can choose (and fix) the standard sign assignment. The necessary results from [LS14a] are now listed below.

Lemma 4.1.9 The obstruction class σ defined above is a cocycle.

Additionally, the obstruction class σ is also a coboundary since $\mathcal{C}(n)$ is acyclic. Then the following proposition is a result of showing that one can modify framings in the interior of the $(k-1)$-dimensional moduli spaces using a cocycle $\alpha \in C^k(\mathcal{C}(n), \pi_{k-1}(\mathbf{O}))$ which produces a new obstruction class $\sigma' = \sigma + \delta\alpha$. The proof of the proposition focuses on providing such a modification so that $\delta\alpha = -\sigma$, and hence shows that the resulting obstruction class σ' is trivial.

Proposition 4.1.10 Given a sign assignment s for $\mathcal{C}(n)$, the cube flow category

$\mathscr{C}_C(n)$ can be framed in such a way that it refines the n-dimensional cube complex $C_s^*(n)$ associated to the given sign assignment.

Finally, the following lemma ensures that neat embeddings and framings for the cube flow category $\mathscr{C}_C(n)$ yield equivalent stable homotopy types (see Choice (5) of Proposition 4.4.1).

Lemma 4.1.11 Given a sign assignment s for $\mathcal{C}(n)$ and two framed embeddings $(\imath_0, \varphi_0)$ and $(\imath_1, \varphi_1)$ of $\mathscr{C}_C(n)$ that refine the cube complex $C_s^*(n)$, there is a smooth 1-parameter family of framed embeddings connecting $(\imath_0[\mathbf{d}_0], \varphi_0)$ and $(\imath_0[\mathbf{d}_1], \varphi_1)$ for some $\mathbf{d}_0$ and $\mathbf{d}_1$.

Proof: The existence of a 1-parameter family of neat embeddings $\imath(t)$ of $\mathscr{C}_C(n)$ (for $t \in [0,1]$) with $\imath(0) = \imath_0[\mathbf{d}_0]$ and $\imath(1) = \imath_1[\mathbf{d}_1]$ for sufficiently large $\mathbf{d}_0$ and $\mathbf{d}_1$, is given by Lemma 2.2.6. The framings φ_0 and φ_1 will be extended to a 1-parameter family of framings φ_t of $\imath(t)$ by framing the moduli spaces $\imath_{u,v}(t)\big(\mathcal{M}(u,v)\big)$ inductively on $|u-v|$. To begin with, the sign assignment ensures that when $v \leq_1 u$, the framings of the moduli spaces $\imath_{u,v}(0)\big(\mathcal{M}(u,v)\big)$ and $\imath_{u,v}(1)\big(\mathcal{M}(u,v)\big)$ (which are both points) are the same. In particular, this framing is positive when $s(\mathcal{C}_{u,v}) = 0$ and negative when $s(\mathcal{C}_{u,v}) = 1$, and there does exist a 1-parameter family of framings connecting the two endpoints.

Now assume that for some $k \geq 1$, such a 1-parameter family of framings φ_t for $\imath_{u,v}(t)$ exists for all $|u - v| \leq k$. An extension of φ_t to all k-dimensional moduli spaces is produced up to some modification, if necessary, for the $(k-1)$-dimensional moduli spaces (fixing the endpoints). So fix a coherent orientation of $\mathscr{C}_C(n)$ and look at the boundary $\partial\big(\mathcal{M}(u,v) \times [0,1]\big)$ oriented as the boundary of a product. Since $\mathcal{M}(u,v)$ is a $(k+1)$-disk, $\partial\big(\mathcal{M}(u,v) \times [0,1]\big)$ is a k-sphere S^k. The framing on $\mathcal{M}(u,v) \times \{i\}$ is the pullback of the framing φ_i on $\imath_{u,v}$ for $i = 0, 1$. The framing on $\partial\mathcal{M}(u,v) \times [0,1]$ is the pullback of the product of the framings of all the lower dimensional moduli spaces that the boundary consists of. The question of extending the framing over the $(k+1)$-disk $\mathbb{D}^{k+1}$ with $\partial\mathbb{D}^{k+1} = S^k$ again reduces to an obstruction theory argument by comparing null-concordant framings of S^k. Lipshitz and Sarkar then use a modification argument similar to the one above to produce a

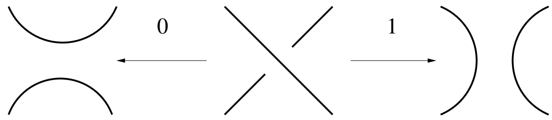


Figure 4.1: The 0- and 1-resolutions of a crossing.

vanishing obstruction class (c.f. [LS14a, Lemma 4.13]). □

4.2 Khovanov homology using resolution configurations

This subchapter is an overview of the definition of Khovanov homology using resolution configurations, as described in [LS14a, Section 2]. In Subchapter 5.3, an analogous definition from [JLS15] will be given for a cancelled complex associated to a matched diagram. The original appearance of resolution configurations predates [LS14a]; see in particular [ORS13], which contains some diagrams that will be referred to later in this chapter. Note, however, that [ORS13] focuses on odd Khovanov homology and in this thesis we are solely concerned with ordinary Khovanov homology.

Definition 4.2.1 A *resolution configuration* D is a pair $(Z(D), A(D))$ where $Z(D)$ is a disjoint union of embedded circles in S^2, and $A(D)$ is a totally ordered collection of embedded arcs in S^2 such that $Z(D) \cap A(D) = \partial A(D)$.

The *index* of the resolution configuration D is the number of arcs in $A(D)$, denoted $\operatorname{ind}(D)$.

Definition 4.2.2 Let L be a link diagram with n ordered crossings. For each crossing, a *resolution* is a choice of either a 0- or 1-resolution which replaces the crossing with a non-crossing diagram using the convention of Figure 4.1. For $v = (v_1, \ldots, v_n) \in \{0,1\}^n$, the *$v$-resolution* is the collection of circles obtained as result of taking the v_i-resolution at the i^{th} crossing for $i = 1, \ldots, n$.

Definition 4.2.3 For any link diagram L with n crossings, a resolution configuration $D_L(v)$ can be associated to each $v \in \{0,1\}^n$ by taking the v-resolution of the diagram. If the 0-resolution was taken for the i^{th} crossing (i.e. if $v_i = 0$), then an arc is placed between the two arcs of the resolution. Clearly, $|v| = \Sigma v_i = n - \text{ind}(D_L(v))$.

The arcs in the previous definition will signify where a 1-resolution is still to be made in the Khovanov cube.

Definition 4.2.4 Given two resolution configurations D and E, a third resolution configuration $D \setminus E$ can be defined by

$$Z(D \setminus E) = Z(D) \setminus Z(E) \qquad A(D \setminus E) = \{ A \in A(D) \mid \forall Z \in Z(E), \partial A \cap Z = \emptyset \} .$$

Let the intersection resolution configuration be given by $D \cap E = D \setminus (D \setminus E)$. Although the total orders on $A(D \cap E)$ and $A(E \cap D)$ may be different, it is still true that $A(D \cap E) = A(E \cap D)$ and also $Z(D \cap E) = Z(E \cap D)$.

Definition 4.2.5 A resolution configuration D is said to be *basic* if every circle in $Z(D)$ intersects an arc in $A(D)$. That is, there are no circles in $Z(D)$ that are disjoint from all of the arcs in $A(D)$.

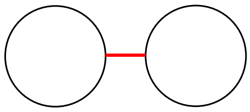
Definition 4.2.6 Given a collection of arcs $A' \subset A(D)$ in a resolution configuration D, it is possible to perform surgery along these arcs by deleting a neighbourhood of $\partial A'$ in $Z(D)$ and connecting the resulting endpoints up along the boundary of the neighbourhood that is outside $Z(D)$. Doing this results in another resolution configuration $s_{A'}(D)$ known as the *surgery of D along A'*. The circles $Z(s_{A'}(D))$ are the ones obtained by performing surgery along all arcs in A', and the arcs of $s_{A'}(D)$ are the arcs of D that are not in A'. The maximal surgery on D is the one obtained by performing surgery along all arcs of $A(D)$, and is denoted $s(D) = s_{A(D)}(D)$.

The following lemma follows immediately from the previous definition.

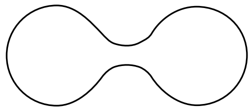
Lemma 4.2.7 Let E be a resolution configuration obtained from a resolution configuration D by a surgery. Then E has the following properties:

1. $D \setminus E$ is a basic resolution configuration.

2. $E \setminus D = s(D \setminus E)$.



(a) An index 1 resolution configuration D.



(b) The resolution configuration $s_A(D)$.

Figure 4.2: Surgery on an index 1 resolution configuration D, consisting of a single circle Z and single arc A.

3. $D \cap E = E \cap D$.

Definition 4.2.8 A resolution configuration D admits a *dual resolution configuration* D^* in the following way. The circles $Z(D^*)$ are obtained from $Z(D)$ by performing maximal surgery on D, so that $Z(D^*) = Z(s(D))$. The arcs $A(D^*)$ are dual to the arcs $A(D)$ in the sense that if A_i is an arc of $A(D)$, then A_i^* is an arc of $A(D^*)$ with boundary on $Z(D^*)$ that intersects A_i once.

By collapsing the circles of a resolution configuration to points/nodes, one can obtain an associated graph which can be thought of as the result of a forgetful map. This language will be useful when talking about the 2-dimensional moduli spaces of the Khovanov flow category.

Definition 4.2.9 Let D be a resolution configuration. A graph $G(D)$ can be associated to D which consists of a vertex $G(Z')$ for each circle Z' in $Z(D)$, and an edge $G(A')$ for each arc A' of $A(D)$. A *leaf* of a resolution configuration D is a circle Z' in $Z(D)$ such that $G(Z')$ is a leaf of $G(Z)$ (that is, a vertex of degree 1). A *co-leaf* of D is an arc A_i of $A(D)$ such that its dual A_i^* has an endpoint which is a leaf in the dual resolution configuration D^*.

Definition 4.2.10 A *labelled resolution configuration* is a pair (D, x), where D is a resolution configuration and x is a choice of labelling for each circle in $Z(D)$ with either x_+ or x_-.

Definition 4.2.11 Let (D, x) and (E, y) be two labelled resolution configurations. Define a partial order $\prec$ on such objects so that $(E, y) \prec (D, x)$ when

1. D is obtained from E by performing surgery along a single arc in $A(E)$. In this case, either:

   (a) $Z(E \setminus D)$ contains exactly one circle Z_1, and $Z(D \setminus E)$ contains exactly two circles Z_2 and Z_3, or

   (b) $Z(E \setminus D)$ contains exactly two circles Z_1 and Z_2, and $Z(D \setminus E)$ contains exactly one circle Z_3.

2. Both labellings x and y induce the same labelling on $D \cap E = E \cap D$.

3. In case (1a), either $y(Z_1) = x(Z_2) = x(Z_3) = x_-$ or $y(Z_1) = x_+$ and $\{x(Z_2), x(Z_3)\} = \{x_+, x_-\}$. In case (1b), either $y(Z_1) = y(Z_2) = x(Z_3) = x_+$ or $\{y(Z_1), y(Z_2)\} = \{x_-, x_+\}$ and $x(Z_3) = x_-$.

The partial order $\prec$ is then defined for all resolution configurations as the transitive closure of this relation between two resolution configurations.

Definition 4.2.12 A *decorated resolution configuration* is a triple (D, x, y) such that $(s(D), x)$ and (D, y) are both labelled resolution configurations satisfying the property that $(D, y) \preceq (s(D), x)$.

Given a decorated resolution configuration (D, x, y), a partially ordered set $P(D, x, y)$ can be associated to it, consisting of all labelled resolution configurations (E, z) such that $(D, y) \preceq (E, z) \preceq (s(D), x)$.

Definition 4.2.13 Given a decorated resolution configuration (D, x, y), the *dual* is defined as the decorated resolution configuration (D^*, x^*, y^*), where D^* is the dual of D, x^* is the dual of x in the sense that it disagrees on every circle in $Z(s(D)) = Z(D^*)$, and y^* is the dual of y in the sense that it disagrees on every circle in $Z(D) = Z(s(D^*))$.

A case-by-case check for all cases in Definition 4.2.11 where $ind(D) = 1$ gives the following.

Lemma 4.2.14 Let (D, x, y) be a decorated resolution configuration. The partially ordered set $P(D^*, y^*, x^*)$ is the reverse of the partially ordered set $P(D, x, y)$.

The following lemma, which is [LS14a, Lemma 2.14], describes how to associate a resolution configuration to an already existent resolution configuration which contains a leaf. We state it here since it is used later when describing the 2-dimensional moduli spaces of the Khovanov flow category (see Lemmata 4.3.9 and 4.3.17).

Lemma 4.2.15 Let (D, x, y) be a decorated resolution configuration with a leaf $Z_1 \in Z(D)$ and an arc $A_1 \in A(D)$ with one point of ∂A_1 lying on Z_1. Further, let $Z_2 \in Z(D)$ be the circle containing the other point in ∂A_1, $Z_1^* \in Z(s(D))$ be the circle containing both points of ∂A_1^*, and finally let

$$Z_2^* \in Z\big(s_{A(D)\setminus A_1}(D)\big) \setminus \{Z_1\}$$

be the unique circle containing a point of ∂A_1. Now consider the resolution configuration D' obtained from D by deleting Z_1 and A_1. The label y on $Z(D)$ induces a label y' on $Z(D')$, and the label x on $Z\big(s(D)\big) \setminus \{Z_1^*\}$ induces a label x' on $Z\big(s(D')\big) \setminus \{Z_2^*\}$ so that (D', x', y') is also a decorated resolution configuration. Set $x'(Z_2^*) = x(Z_1^*)$ if $y(Z_1) = x_+$, or set $x'(Z_2^*) = x_+$ if not. Then,

$$P(D, x, y) = P(D', x', y') \times \{0, 1\}$$

where $\{0, 1\}$ is the partially ordered set with two elements and $0 \prec 1$.

Proof: The isomorphism of partially ordered sets

$$P(D, x, y) \to P(D', x', y') \times \{0, 1\}$$

presented in the proof of [LS14a, Lemma 2.14] produces, for each $(E, z) \in P(D, x, y)$ with $E = s_A(D)$ for some $A \subset A(D)$, an object $\big((E', z'), i\big) \in P(D', x', y') \times \{0, 1\}$. This task is separated into two cases corresponding to whether the arc A_1 is in A or not. When $A_1 \in A$, i is set as 0 and $E' = s_A(D')$. In this case, a labelling z' on $Z(s_A(D'))$ is induced by the labelling z on $Z(s_A(D)) = Z(s_A(D')) \amalg \{Z_1\}$. When $A_1 \notin A$, i is set as 1 and $E' = s_{A\setminus\{A_1\}}(D')$. Then denoting $\{Z_A\}$ as the collection of circles $Z(s_A(D)) \setminus Z\big(s_{A\setminus\{A_1\}}(D')\big)$, and letting $\{Z'_A\}$ denote the collection of circles $Z\big(s_{A\setminus\{A_1\}})(D')\big) \setminus Z\big(s_A(D)\big)$, a labelling z' on $Z\big(s_{A\setminus A_1}(D')\big) \cap Z(s_A(D))$ is also induced by the labelling z. Finally, z' labels Z'_A in the following way: if $y(Z_1) = x_+$, set $z'(Z'_A) = z(Z_A)$, otherwise set $z'(Z'_A) = x_+$. $\square$

Definition 4.2.16 Let L be an oriented link diagram with n crossings that are ordered. Then the *Khovanov chain complex* $\{KhC^{i,j}(L),\delta\}$ is defined in the following way.

The bigraded chain group $KhC(L)$ is a freely generated $\mathbb{Z}$-module with generators given by labelled resolution configurations $(D_L(u),x)$ for $u\in\{0,1\}^n$. There are two gradings on $KhC(L)$, an i-grading known as the *homological grading* denoted gr_h and a j-grading known as the *quantum grading* denoted gr_q. They are defined as:

$$\begin{aligned}
&\mathrm{gr}_h\big((D_L(u),x)\big) = |u| - n_- \text{ and,}\\
&\mathrm{gr}_q\big((D_L(u),x)\big) = n_+ - 2n_- + \#\{\, Z\in Z(D_L(u)\mid x(Z)=x_+\,\}\\
&-\#\{\, Z\in Z(D_L(u)\mid x(Z)=x_-\,\}
\end{aligned}$$

where n_+ denotes the number of positive crossings of L and $n_- = n - n_+$ denotes the number of negative crossings. The differential δ preserves the quantum grading and increases the homological grading by 1. It is defined by

$$\delta(D_L(v),y) = \sum_{\substack{(D_L(u),x)\\ |u|=|v|+1\\ (D_L(v),y)\prec(D_L(u),x)}} (-1)^{s_0(\mathcal{C}_{u,v})}(D_L(u),x)$$

where s_0 is the standard sign assignment from Definition 4.1.5.

4.3 The Khovanov flow category and stable homotopy type

Up to this point, it has only been mentioned briefly that the Khovanov flow category $\mathscr{C}_{Kh}(D)$ is a flow category associated to a link diagram D, which covers the cube flow category $\mathscr{C}_C(n)$ where n is the number of crossings in D. Here, we follow Section 5 of [LS14a] closely, where they construct the moduli spaces of this flow category inductively. Using the Cohen-Jones-Segal construction with $\mathscr{C}_{Kh}$ will give the Khovanov stable homotopy type. Firstly, we introduce some language for moduli spaces of resolution configurations.

Let (D, x, y) be a basic decorated resolution configuration of index n. Then we shall associate to (D, x, y) an $(n-1)$-dimensional $\langle n-1\rangle$-manifold $\mathcal{M}(D, x, y)$ together with an $(n-1)$-map

$$\mathcal{F} : \mathcal{M}(D, x, y) \to \mathcal{M}_{\mathscr{C}_C(n)}(\underline{1}, \underline{0}).$$

These spaces are built inductively with the inductive hypothesis that the following four conditions are satisfied, the first three of which are analogous to the second and third conditions of Definition 2.2.1:

(A1): For an index n basic decorated resolution configuration (D, x, y), let $(E, z) \in P(D, x, y)$ be of index m. Denote by $x|$ and $y|$ the induced labellings on $s(E \setminus s(D)) = s(D) \setminus E$ and $D \setminus E$, respectively. Denote by $z|$ the induced labellings on both (for simplicity of notation) $s(D \setminus E) = E \setminus D$ and $E \setminus s(D)$. Then there is a composition map

$$\circ : \mathcal{M}(D \setminus E, z|, y|) \times \mathcal{M}(E \setminus s(D), x|, z|) \to \mathcal{M}(D, x, y)$$

that respects the map $\mathcal{F}$ in the sense of the following commutative diagram:

$$\begin{tikzcd}
\mathcal{M}(D \setminus E, z|, y|) \times \mathcal{M}(E \setminus s(D), x|, z|) \arrow[r, "\circ"] \arrow[d, "\mathcal{F}\times\mathcal{F}"] & \mathcal{M}(D, x, y) \arrow[dd, "\mathcal{F}"] \\
\mathcal{M}_{\mathscr{C}_C(n-m)}(\underline{1}, \underline{0}) \arrow[d, "\mathcal{I}_{v,\underline{0}}\times\mathcal{I}_{\underline{1},v}"] & \\
\mathcal{M}_{\mathscr{C}_C(n)}(v, \underline{0}) \times \mathcal{M}_{\mathscr{C}_C(n)}(\underline{1}, v) \arrow[r, "\circ"] & \mathcal{M}_{\mathscr{C}_C(n)}(\underline{1}, \underline{0})
\end{tikzcd}$$

where the vector $v = (v_1, \ldots, v_n)$ is defined so that $v_i = 0$ if the i^{th} arc of $A(D)$ is also in $A(E)$, and $v_i = 1$ otherwise.

(A2): The faces of $\mathcal{M}(D, x, y)$ are defined by

$$\begin{aligned}\partial_i \mathcal{M}(D, x, y) &= \partial_{exp,i} \mathcal{M}(D, x, y) \\ &:= \coprod_{\substack{(E,z)\in P(D,x,y) \\ \operatorname{ind}(D\setminus E)=i}} \circ\big(\mathcal{M}(D \setminus E, z|, y|) \times \mathcal{M}(E \setminus s(D), x|, z|)\big).\end{aligned}$$

(A3): $\mathcal{F}$ is a covering map and local diffeomorphism with:

(A4): The covering map $\mathcal{F}$ is trivial on each component of $\mathcal{M}(D, x, y)$.

Recall from the discussion in Subchapter 2.2 after Definition 2.2.1 that one can define a diagram $\mathfrak{D}$ with vertices (c.f. Equation 2.1) given by the spaces

$$\mathcal{M}(D \setminus E_k, z_k|, y|) \times \mathcal{M}(E_k \setminus E_{k-1}, z_{k-1}|, z_k|) \times \cdots \times \mathcal{M}(E_1 \setminus s(D), x|, z_1|)$$

for a sequence of $(D, y) \prec (E_k, z_k) \prec \cdots \prec (E_1, z_1) \prec (s(D), x)$ in $P(D, x, y)$, and edges corresponding to compositions using $\circ$ above. Denote by $\partial_{exp}\mathcal{M}(D, x, y)$ the colimit of $\mathfrak{D}$ so that

$$\partial_{exp}\mathcal{M}(D, x, y) = \operatorname{colim}\mathfrak{D} = \bigcup_i \partial_{exp,i}\mathcal{M}(D, x, y)$$

is an $\langle n-1 \rangle$-boundary whose top-dimensional faces are equal to $\partial_{exp,i}\mathcal{M}(D, x, y)$ (c.f. Equation 2.2). Then (A2) above gives the following.

Lemma 4.3.1 $\partial_{exp}\mathcal{M}(D, x, y) = \partial\mathcal{M}(D, x, y)$

Lipshitz-Sarkar describe the moduli spaces $\mathcal{M}(D, x, y)$ inductively, assuming that for $\operatorname{ind}(D) = 1$, $\mathcal{M}(D, x, y)$ is defined as a single point. Assuming that each $\mathcal{M}(D, x, y)$ has been defined whenever $\operatorname{ind}(D) < k$, the following proposition, which is [LS14a, Proposition 5.2], describes the inductive argument of producing $\mathcal{M}(D, x, y)$ whenever $\operatorname{ind}(D) = k$.

Proposition 4.3.2 For each basic decorated resolution configuration (D, x, y) with $\operatorname{ind}(D) \leq n$, let $\mathcal{M}(D, x, y)$ and $\mathcal{F} : \mathcal{M}(D, x, y) \to \mathcal{M}_{\mathscr{C}_C(\operatorname{ind}(D))}(\underline{1}, \underline{0})$ satisfy conditions (A1)-(A4). Then, for any index $(n+1)$ basic decorated resolution configuration (D, x, y), the following conditions are satisfied:

(B1): The previously defined $\mathcal{F}$ together define a continuous map

$$\mathcal{F}|_\partial : \partial_{exp}\mathcal{M}(D, x, y) \to \partial\mathcal{M}_{\mathscr{C}_C(n+1)}(\underline{1}, \underline{0})$$

which preserves the structure of the $\langle n \rangle$-boundary on both sides. This means that $\mathcal{F}|_\partial^{-1}\big(\partial_i\mathcal{M}(\underline{1}, \underline{0})\big) = \partial_{exp,i}\mathcal{M}(D, x, y)$, or that $\mathcal{F}|_{\partial_{exp,i}\mathcal{M}(D,x,y)}$ is an n-map.

(B2): Moreover, the map $\mathcal{F}|_\partial$ is a covering map.

(B3): The existence of an $\langle n \rangle$-manifold $\mathcal{M}(D,x,y)$ along with an n-map

$$\mathcal{F} : \mathcal{M}(D,x,y) \to \mathcal{M}_{\mathscr{C}_C(n+1)}(\underline{1},\underline{0})$$

that satisfy conditions (A1)-(A4) is equivalent to the condition that $\mathcal{F}|_\partial$ is a trivial covering map. In particular, for $n \geq 3$ the moduli space $\mathcal{M}(D,x,y)$ and map $\mathcal{F}$ necessarily exist.

(B4): If $\mathcal{M}(D,x,y)$ and $\mathcal{F} : \mathcal{M}(D,x,y) \to \mathcal{M}_{\mathscr{C}_C(n+1)}(\underline{1},\underline{0})$ exist and $n \geq 2$, then they are both unique up to diffeomorphisms that fix the boundaries.

Proof: We describe the proof, following [LS14a].

(B1): Consider the diagram $\mathfrak{C}$ with vertices given by the spaces

$$\mathcal{M}_{\mathscr{C}_C(n+1)}(v_k,\underline{0}) \times \mathcal{M}_{\mathscr{C}_C(n+1)}(v_{k-1},v_k) \times \cdots \times \mathcal{M}_{\mathscr{C}_C(n+1)}(\underline{1},v_1)$$

for $v_i \in \{0,1\}^{n+1}$ such that $\underline{0} < v_k < \cdots < v_1 < \underline{1}$, and edges corresponding to compositions $\circ$ for $\mathscr{C}_C(n+1)$. This composition induces a homeomporphism

$$\operatorname{colim}\mathfrak{C} \cong \partial\mathcal{M}_{\mathscr{C}_C(n+1)}(\underline{1},\underline{0})$$

by definition of $\mathscr{C}_C(n+1)$ being a flow category (Definition 2.2.1). The maps $\mathcal{F}$ (assumed to exist by induction) induce a map of diagrams $\mathcal{G} : \mathfrak{D} \to \mathfrak{C}$ which is defined on $\mathcal{M}(D \setminus E_k, z_k|, y|) \times \mathcal{M}(E_k \setminus E_{k-1}, z_{k-1}|, z_k|) \times \cdots \times \mathcal{M}(E_1 \setminus s(D), x|, z_1|)$ to be

$$(\mathcal{I}_{v_k,\underline{0}} \times \mathcal{I}_{v_{k-1},v_k} \times \cdots \times \mathcal{I}_{\underline{1},v_1}) \circ (\mathcal{F} \times \cdots \times \mathcal{F})$$

where v_m has its i^{th} entry equal to 0 if the i^{th} arc of $A(D)$ is in $A(E_m)$, and is equal to 1 otherwise (c.f. the commutative diagram in (A1), which ensures $\mathcal{G}$ defines a map of diagrams). Setting $\mathcal{F}|_\partial = \operatorname{colim}\mathcal{G}$ is enough.

(B2): The inductive hypothesis is that $\mathcal{F} : \mathcal{M}(D',x',y') \to \mathcal{M}_{\mathscr{C}_C(\operatorname{ind}(D'))}(\underline{1},\underline{0})$ is a covering map whenever $\operatorname{ind}(D') \leq n$. To see that this is also true for $\mathcal{M}(D,x,y)$ with $\operatorname{ind}(D) = (n+1)$, consider $\mathcal{F}|_\partial = \operatorname{colim}\mathcal{G}$ defined in the proof of (B1). The result then follows from Proposition 2.1.4 (see also [LS14a, Proposition 3.9]).

(B3): By Lemma 4.1.3, each $\mathcal{M}_{\mathscr{C}_C(n+1)}(\underline{1},\underline{0})$ is topologically an n-disk, with boundary $\partial\mathcal{M}_{\mathscr{C}_C(n+1)}(\underline{1},\underline{0})$ topologically an $(n-1)$-sphere. Then (B1) along with

(B2) means that $\partial_{exp}\mathcal{M}(D,x,y)$ (which has to cover these spheres) is itself a disjoint union of $(n-1)$-spheres, so the covering map is

$$\mathcal{F}|_\partial : \coprod_{i=1}^{m} S^{n-1} \to S^{n-1}$$

and the result follows. The statement that $\mathcal{M}(D,x,y)$ and $\mathcal{F}$ exist necessarily when $n \geq 3$ comes from the fact that any cover of S^{n-1} is trivial for such n.

(B4): Finally, this condition follows since Lemma 4.3.1 ensures that any other $\mathcal{M}'(D,x,y)$ satisfying conditions (A1)-(A4) has the same boundary as $\mathcal{M}(D,x,y)$. Then both are disjoint unions of disks, as argued in the proof of (B3). Therefore, they are diffeomorphic as long as the boundary is fixed and the diffeomorphism commuted with $\mathcal{F}$. $\square$

Thanks to the previous proposition, we can assume that spaces $\mathcal{M}(D,x,y)$ and maps $\mathcal{F}$ exist satisfying conditions (A1)-(A4) for every index n basic decorated resolution configuration (D,x,y). In particular, when n is equal to the number of crossings of a link diagram L, we can define the Khovanov flow category $\mathscr{C}_{Kh}(L)$ as a cover of the cube flow category $\mathscr{C}_C(n)$.

Definition 4.3.3 The *Khovanov flow category* $\mathscr{C}_{Kh}(L)$ is the flow category whose objects are in one-to-one correspondence with the standard generators of Khovanov homology, as defined in Definition 4.2.16. That is to say, $\mathrm{Ob}\big(\mathscr{C}_{Kh}(L)\big)$ consists of labelled resolution configurations $\mathbf{x} = (D_L(u), x)$ for $u \in \{0,1\}^n$. The grading $|\mathbf{x}|$ is the homological grading $\mathrm{gr}_h(\mathbf{x})$ from Definition 4.2.16. Note that an orientation of L is needed to define the gradings, but the rest of $\mathscr{C}_{Kh}(L)$ is defined independently. For objects

$$\mathbf{x} = (D_L(u), x) \text{ and } \mathbf{y} = (D_L(v), y)$$

of $\mathscr{C}_{Kh}(L)$, define $\mathcal{M}_{\mathscr{C}_{Kh}(L)}(\mathbf{x},\mathbf{y})$ to be empty when $\mathbf{x} \prec \mathbf{y}$ (with respect to the partial order of Definition 4.2.11). When $y \prec x$, the moduli space $\mathcal{M}_{\mathscr{C}_{Kh}(L)}(\mathbf{x},\mathbf{y})$ can be defined as follows. Denote by $x|$ the restriction of the labelling x on $s\big(D_L(v)\backslash D_L(u)\big) = D_L(u)\backslash D_L(v)$, and denote by $y|$ the restriction of the labelling y on $D_L(v)\backslash D_L(u)$. Then $(D_L(v)\backslash D_L(u), x|, y|)$ is a basic decorated resolution configuration and the moduli spaces of the Khovanov flow category are defined as

$$\mathcal{M}_{\mathscr{C}_{Kh}(L)}(\mathbf{x}, \mathbf{y}) = \mathcal{M}(D_L(v) \setminus D_L(u), x|, y|).$$

Moreover, condition (A1) on moduli spaces of resolution configurations provides composition maps that induce composition maps

$$\mathcal{M}_{\mathscr{C}_{Kh}(L)}(\mathbf{z}, \mathbf{y}) \times \mathcal{M}_{\mathscr{C}_{Kh}(L)}(\mathbf{x}, \mathbf{z}) \to \mathcal{M}_{\mathscr{C}_{Kh}(L)}(\mathbf{x}, \mathbf{y})$$

for the Khovanov flow category.

Definition 4.3.4 A functor $\mathscr{F} : \mathscr{C}_{Kh}(L) \to \mathscr{C}_C(n)[-n_-]$ is defined for the Khovanov flow category as a cover of the cube flow category (where $[-n_-]$ denotes a shift in homological gradings of the category, and is omitted unless it is needed). The functor is defined on objects by

$$\mathscr{F}(D_L(u), x) = u.$$

The associated morphisms

$$\mathscr{F} : \mathcal{M}_{\mathscr{C}_{Kh}(L)}(D_L(u), x), (D_L(v), y)) \to \mathcal{M}_{\mathscr{C}_C(n)}(u, v)$$

are given by the composition

$$\mathcal{M}(D_L(v) \setminus D_L(u), x|, y|) \xrightarrow{\mathcal{F}} \mathcal{M}_{\mathscr{C}_C(|u|-|v|)}(\underline{1}, \underline{0}) \xrightarrow{\mathcal{I}_{u,v}} \mathcal{M}_{\mathscr{C}_C(n)}(u, v).$$

Lemma 4.3.5 The moduli spaces $\mathcal{M}(\mathbf{x}, \mathbf{y})$ are only non-empty when $\mathrm{gr}_q(\mathbf{x}) = \mathrm{gr}_q(\mathbf{y})$ and $\mathrm{gr}_h(\mathbf{y}) < \mathrm{gr}_h(\mathbf{x})$.

Proof: The moduli spaces are only non-empty when $\mathbf{y} \prec \mathbf{x}$, and from the definition of the partial order (Definition 4.2.11) along with the definition of the Khovanov chain complex (Definition 4.2.16), this happens only if $\mathrm{gr}_q(\mathbf{x}) = \mathrm{gr}_q(\mathbf{y})$ and $\mathrm{gr}_h(\mathbf{y}) < \mathrm{gr}_h(\mathbf{x})$. $\square$

Lemma 4.3.5 ensures that objects of $\mathscr{C}_{Kh}(L)$ related by moduli spaces split up in quantum gradings. Therefore, if $\mathscr{C}^q_{Kh}(L)$ denotes the full subcategory of $\mathscr{C}_{Kh}(L)$

with all objects of quantum grading q, then the Khovanov flow category $\mathscr{C}_{Kh}(L)$ can be described as the disjoint union of all such full subcategories,

$$\mathscr{C}_{Kh}(L) = \coprod_{q\in\mathbb{Z}} \mathscr{C}^q_{Kh}(L).$$

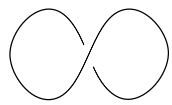
Using Lemma 2.4.4, the Khovanov space $\mathcal{X}_{Kh}(L)$ (defined in the following definition) therefore decomposes as a wedge sum over such splittings of quantum gradings. If the oriented link diagram L has n crossings, the Khovanov flow category is defined as the cover of the cube flow category $\mathscr{C}_C(n)$, so one can choose a sign assignment s for $\mathscr{C}_C(n)$ and a framing of $\mathscr{C}_C(n)$ relative s. Then a neat immersion of $\mathscr{C}_{Kh}(L)$ together with a coherent framing is produced using the cover $\mathscr{F} : \mathscr{C}_{Kh}(L) \to \mathscr{C}_C(n)$ and the framing of $\mathscr{C}_C(n)$ relative s (as argued in Subchapter 2.4). Finally, a choice of perturbation can be made so that one can obtain a framing of $\mathscr{C}_{Kh}(L)$. Recall that by Lemma 2.3.2, if we define the Khovanov space as the output of the Cohen-Jones-Segal machine, then the Khovanov homology of the link is recovered from the reduced cohomology of the Khovanov space shifted by $(-C)$ for the positive integer C defined in Equation 2.5 of Definition 2.3.1. In particular, we have the following definition.

Definition 4.3.6 The *Khovanov space* is the CW complex $|\mathscr{C}_{Kh}(L)|$ obtained as a result of the Cohen-Jones-Segal construction. A framed neat embedding for $\mathscr{C}_{Kh}(L)$ can be provided using the cover $\mathscr{F}$ as described above. The *Khovanov spectrum* $\mathcal{X}_{Kh}$ is defined as the suspension spectrum of the Khovanov space which is de-suspended C times (in order to give the correct Khovanov homology). The Khovanov spectrum decomposes as a wedge sum over the quantum gradings as

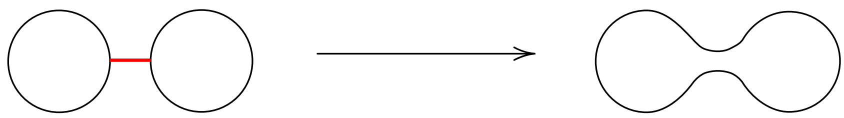
$$\mathcal{X}_{Kh}(L) = \bigvee_j \mathcal{X}^j_{Kh}(L).$$

The following example describes how the previous definition ties together with the material described throughout this chapter. It is similar to the example of [LS14a, §9.1].

Example 4.3.1 Consider the 1-crossing unknot U as:



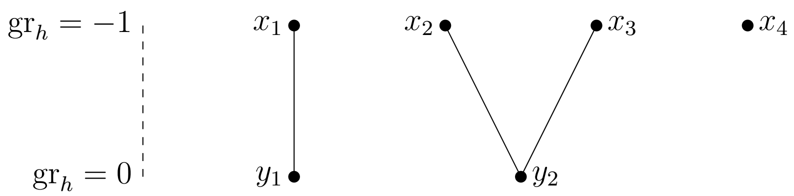
There are two resolutions of this diagram, and they are the resolution configurations depicted in Figure 4.2. The *resolution interval* is an interval whose boundary points consist of the 0-resolution and the 1-resolution as



The 0-resolution $D_U(0)$ consists of two circles, and there are four possible labellings. Let

$$x_1 = (D_U(0), v_+v_+), x_2 = (D_U(0), v_+v_-), x_3 = (D_U(0), v_-v_+), x_4 = (D_U(0), v_-v_-)$$

be the four corresponding objects in $\mathscr{C}_{Kh}(U)$. The homological degrees are $\mathrm{gr}_h(x_i) = -1$ for $i = 1, 2, 3, 4$. The 1-resolution $D_U(1)$ consists of a single circle with two possible labellings. Let $y_1 = (D_U(1), v_+)$ and $y_2 = (D_U(1), v_-)$ be the two corresponding objects in $\mathscr{C}_{Kh}(U)$. The homological degrees are $\mathrm{gr}_h(y_i) = 0$ for $i = 1, 2$. The two objects x_1 and y_1 have the same quantum grading $\mathrm{gr}_q(x_1) = \mathrm{gr}_q(y_1) = 0$ and a single moduli space between them. The objects x_2, x_3 and y_2 all have the same quantum grading $\mathrm{gr}_q(x_2) = \mathrm{gr}_q(x_3) = \mathrm{gr}_q(y_2) = -2$. There is a single moduli space between x_2 and y_2, and a single moduli space between x_3 and y_2. The only object with quantum degree -4 is x_4. The Khovanov flow category $\mathscr{C}_{Kh}(U)$ is:



The only moduli spaces are single points. In particular, a choice of framed embedding of $\mathscr{C}_{Kh}(U)$ relative $\mathbf{d} = (d_{-1}, d_0) = (-1, 0)$ can be made so that the disjoint union

$$\mathcal{M}(x_1, y_1) \amalg \mathcal{M}(x_2, y_2) \amalg \mathcal{M}(x_3, y_2)$$

of three points can be embedded into $\mathbb{R}^{d-1} = \mathbb{R}$, with a positive framing coming from the standard sign assignment. Now $|\mathscr{C}_{Kh}(U)|$ is constructed using Definition 2.3.1 as follows. The two cells $\mathcal{C}(y_1)$ and $\mathcal{C}(y_2)$ are given by

$$\{0\} \times [-\varepsilon, \varepsilon]/\partial\{0\} \times [-\varepsilon, \varepsilon] \cong S^1.$$

The 1-skeleton of $|\mathscr{C}_{Kh}(U)|^{(1)}$ is therefore $S^1 \vee S^1$. The cells $\mathcal{C}(x_i)$ are given by

$$[0, R] \times [-R, R]/ \sim_i$$

for $i = 1, 2, 3, 4$ where $\sim_i$ are identifications defined as follows. The identification $\sim_1$ identifies

$$\{0\} \times \varphi_{x_1,y_1}(\mathcal{M}(x_1, y_1) \times [-\varepsilon, \varepsilon]) \subset \partial\mathcal{C}(x_1)$$

with $\mathcal{C}(y_1)$, and the rest of $\partial\mathcal{C}(x_1)$ to the basepoint. Therefore, $\mathcal{C}(x_1)$ is a 2-cell whose boundary is attached to the first wedge-summand of $|\mathscr{C}_{Kh}(U)|^{(1)} = S^1 \vee S^1$ by a degree-one map, yielding $\mathbb{D}^2 \vee S^1$. The identifications $\sim_j$ for $j = 2, 3$ identify

$$\{0\} \times \varphi_{x_j,y_2}(\mathcal{M}(x_j, y_2) \times [-\varepsilon, \varepsilon]) \subset \partial\mathcal{C}(x_j)$$

with $\mathcal{C}(y_2)$, and the rest of $\partial\mathcal{C}(x_j)$ to the basepoint. Therefore, $\mathcal{C}(x_j)$ is a 2-cell whose boundary is attached to the second wedge-summand of $|\mathscr{C}_{Kh}(U)|^{(1)} = S^1 \vee S^1$ by a degree-one map, for $j = 2, 3$. This produces an S^2 component, and the result of attaching the first three 2-cells to $|\mathscr{C}_{Kh}(U)|^{(1)}$ is $\mathbb{D}^2 \vee S^2$. Finally, $\sim_4$ identifies the entire $\partial\mathcal{C}(x_4)$ to the basepoint, providing another S^2 wedge-summand. The CW complex $|\mathscr{C}_{Kh}(U)|$ is given by

$$|\mathscr{C}_{Kh}(U)| = \mathbb{D}^2 \vee S^2 \vee S^2$$

and so the Khovanov spectrum $\mathcal{X}_{Kh}(U)$ is given by

$$\mathcal{X}_{Kh}(U) = \Sigma^{-2}(S^2 \vee S^2) = S^0 \vee S^0.$$

4.3.1 The 0-dimensional moduli spaces

Defining the 0-dimensional moduli spaces $\mathcal{M}(D, x, y)$ as a single point is all that is required. Since $\mathcal{M}_{\mathscr{C}_C(1)}(\underline{1}, \underline{0})$ is also a single point, the map

$$\mathcal{F} : \mathcal{M}(D, x, y) \to \mathcal{M}_{\mathscr{C}_C(1)}(\underline{1}, \underline{0})$$

is defined for such basic decorated resolution configurations to send point to point (and conditions (A1)-(A4) are satisfied trivially).

Remark: Note that the simplicity of this definition is not a luxury that can be assumed of the sock flow category defined later in Subchapter 5.2 since there exist two-point moduli spaces after cancellation.

4.3.2 The 1-dimensional moduli spaces

Let (D, x, y) be a decorated resolution configuration of index 2. The definition of the 1-dimensional moduli spaces $\mathcal{M}(D, x, y)$ actually depends on a global choice due to an occurrence of what Lipshitz-Sarkar refer to as *ladybugs* in [LS14a]. Before defining a ladybug, it will be useful to define the following notion which is used throughout this subsection.

Definition 4.3.7 Let (D, x, y) be an index 2 basic decorated resolution configuration. A *maximal chain* consists of a labelled resolution configuration (D', z) such that $(D, y) \prec (D', z) \prec (s(D), x)$. For an index n basic decorated resolution configuration, a maximal chain would require a sequence of $n-1$ intermediate labelled resolution configurations (D'_i, z_i) for $i = 1, \ldots, n-1$ with

$$(D, y) \prec (D'_1, z_1) \prec \cdots \prec (D'_{n-1}, z_{n-1}) \prec (s(D), x).$$

Definition 4.3.8 A *ladybug configuration* is a decorated resolution configuration (D, x, y) of index 2 satisfying the properties that:

1. $Z(D)$ is just a single circle Z, and

2. The two arcs Z_1 and Z_2 of $A(D)$ intersect Z in such a way that their boundary points alternate around Z.

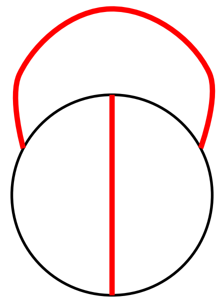


Figure 4.3: An index 2 resolution configuration that is a ladybug.

Due to a restriction of gradings that occur in this way, the labels are $y(Z) = x_+$ and $x(s(Z)) = x_-$, where $s(Z)$ is the single circle in $Z(s(D))$. In certain contexts, D may also be referred to as a ladybug configuration. See Figure 4.3

Lemma 4.3.9 Consider a decorated resolution configuration (D, x, y) of index 2 and write $A(D) = \{A_1, A_2\}$ where A_1 precedes A_2 in the total ordering of arcs in D. The number m of labelled resolution configurations (D', z) such that $(D, y) \prec (D', z) \prec (s(D), x)$ is either 2 or 4. The case $m = 4$ occurs if and only if (D, x, y) is a ladybug configuration.

Proof: If D admits a leaf or a co-leaf in the sense of Definition 4.2.9, then Lemmata 4.2.14 and 4.2.15 imply that $P(D, x, y)$ is isomorphic to $\{0, 1\}$, and hence $m = 2$. All possible index 2 decorated resolution configurations are listed in [ORS13, Figure 2] and as highlighted in [LS14a], the only ones that do not admit leaves or co-leaves are ladybug configurations where $m = 4$. □

As a result of the previous lemma, $\partial_{exp}\mathcal{M}(D, x, y)$ consists of either 2 or 4 points. Firstly, consider the case when $m = 2$. Then the map

$$\mathcal{F}|_{\partial} : \partial_{exp}\mathcal{M}(D, x, y) \to \partial\mathcal{M}_{\mathscr{C}_C(2)}(\underline{1}, \underline{0})$$

is a trivial covering space and condition (B3) of Proposition 4.3.2 ensures that $\mathcal{M}(D, x, y)$ can be defined with a map $\mathcal{F} : \mathcal{M}(D, x, y) \to \mathcal{M}_{\mathscr{C}_C(2)}(\underline{1}, \underline{0})$ that satisfy conditions (A1)-(A4). To be explicit, $\mathcal{M}(D, x, y)$ is a single interval with boundary

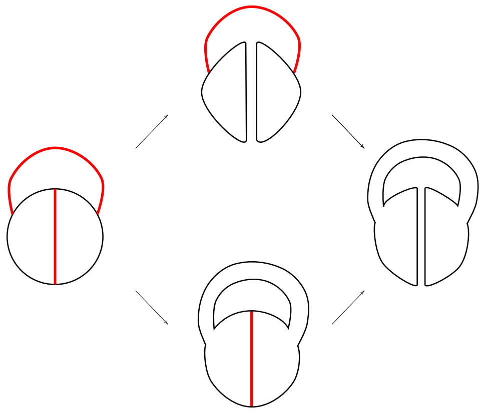


Figure 4.4: The various surgeries on a ladybug configuration. Each of the two intermediate resolution configurations have two different labellings.

$\partial_{exp}\mathcal{M}(D, x, y)$, and $\mathcal{F}$ is a diffeomorphism that is fixed on the boundary corresponding to the two diagrams D_1 and D_2 that contain A_1 and A_2 in the set up of Lemma 4.3.9.

For the case $m = 4$, there is a choice to be made about which of the 4 points are paired off to bound an interval. This choice was made explicit in [LS14a, 5.4.2] in the following way. There are two ways to deconstruct the maximal surgery on the single circle Z of $Z(D)$; by first performing surgery along A_1 followed by surgery along A_2, or by first performing surgery along A_2 followed by surgery along A_1. In each case, the first surgery yields two circles, which we denote as $\{Z_{i,1}, Z_{i,2}\} = Z(s_{A_i}(D))$ for $i = 1, 2$. The choice is then a bijection

$$\{Z_{1,1}, Z_{1,2}\} \leftrightarrow \{Z_{2,1}, Z_{2,2}\}.$$

In order to explain this bijection, some notation is needed.

Definition 4.3.10 Let (D, x, y) be the index 2 basic decorated resolution configuration in question. Assume that $\infty \in S^2$ is not part of $D \subset S^2 \setminus \{\infty\} \cong \mathbb{R}^2$. Then the two arcs A_1 and A_2 lie inside this plane with one outside of Z (denoted A_{out} and one inside Z (denoted A_{in}). Let Z be oriented with its inherited orientation from the disk is bounds in $\mathbb{R}^2$. There are four arcs of $Z \setminus (\partial A_1 \cup \partial A_2)$, so with respect to this orientation, define the *right pair* of components to be the arcs which direct A_{out} to A_{in}. The remaining pair is called the *left pair*.

This definition is invariant under isotopy by the following lemma, which can be verified by considering a right pair as the pair of arcs obtained by traversing along A_1 or A_2 and then turning right.

Lemma 4.3.11 Any isotopy of $Z \cup A_1 \cup A_2$ in S^2 takes the right pair to the right pair.

The convention of [LS14a] is to choose the right pair to construct $\mathcal{M}(D, x, y)$, and we shall follow this convention for convenience. Thus, if $\{P, Q\}$ is the right pair, assume that $P \subset Z_{i,1}$ and $Q \subset Z_{i,2}$ for $i = 1, 2$. This determines a bijection

$$Z_{1,1} \leftrightarrow Z_{2,1} \qquad Z_{1,2} \leftrightarrow Z_{2,2}$$

which will be called the *ladybug matching*. The left pair will result in different moduli spaces, so denote them by $\mathcal{M}_*(D, x, y)$. They will be of use later. There are four maximal chains in $P(D, x, y)$ that give the four points of $\partial\mathcal{M}(D, x, y)$. They are:

$$\begin{aligned}
a &= [(D, x_+) \prec (s_{A_1}(D), x_-x_+) \prec (s(D), x_-)] \\
b &= [(D, x_+) \prec (s_{A_1}(D), x_+x_-) \prec (s(D), x_-)] \\
c &= [(D, x_+) \prec (s_{A_2}(D), x_-x_+) \prec (s(D), x_-)] \\
d &= [(D, x_+) \prec (s_{A_2}(D), x_+x_-) \prec (s(D), x_-)]
\end{aligned}$$

where the labellings are listed in order for $Z_{i,1}$ and $Z_{i,2}$, respectively, for $i = 1, 2$. Then the two intervals of $\mathcal{M}(D, x, y)$ are the ones whose boundaries are $a \sqcup c$ and $b \sqcup d$. The existence (and uniqueness up to isotopy) of the map

$$\mathcal{F} : \mathcal{M}(D, x, y) \to \mathcal{M}_{\mathscr{C}_C(2)}(\underline{1}, \underline{0})$$

is given since $\mathcal{M}_{\mathscr{C}_C(2)}(\underline{1}, \underline{0})$ is a single interval I, and $\mathcal{F}|_\partial$ sends a and b to one endpoint of I, and send c and d to the other endpoint of I. This results in the following proposition.

Proposition 4.3.12 For an index $k \leq 2$ basic decorated resolution configuration (D, x, y), there are spaces $\mathcal{M}(D, x, y)$ and maps $\mathcal{F}$ satisfying the conditions (A1)-(A4).

4.3.3 The 2-dimensional moduli spaces

By (B3) of Proposition 4.3.2, in order to provide $\mathcal{M}(D, x, y)$ for $\operatorname{ind}(D) = 3$, it is only necessary to show that

$$\mathcal{F}|_{\partial} : \partial_{exp}\mathcal{M}(D, x, y) \to \partial\mathcal{M}_{\mathscr{C}_C(3)}(\underline{1}, \underline{0})$$

which is a covering map (by (B2) of Proposition 4.3.2), is a trivial covering map on each component of $\partial_{exp}\mathcal{M}(D, x, y)$. Both $\partial_{exp}\mathcal{M}(D, x, y)$ and $\partial\mathcal{M}_{\mathscr{C}_C(3)}(\underline{1}, \underline{0})$ are 2-boundaries, and in particular $\partial\mathcal{M}_{\mathscr{C}_C(3)}(\underline{1}, \underline{0})$ is a 6-cycle (by Lemma 4.1.3). So consider both as graphs, so that $\mathcal{F}|_{\partial}$ respects the graph structure and it is enough to show that $\partial_{exp}\mathcal{M}(D, x, y)$ is a disjoint union of 6-cycles. Following [ORS13] and the definition of dual resolution configurations (Definitions 4.2.8 and 4.2.13), proving that $\partial_{exp}\mathcal{M}(D, x, y)$ is a disjoint union of 6-cycles reduces to checking a number of cases, and this is precisely the method of [LS14a, Subsection 5.5]. These cases are illustrated in Figure 4.5, and the following three lemmata from [LS14a, Subsection 5.5] are needed to ensure that checking the cases is sufficient. The first follows from the fact that $\partial_{exp}\mathcal{M}(D, x, y)$ covers a 6-cycle.

Lemma 4.3.13 The graph $\partial_{exp}\mathcal{M}(D, x, y)$ consists of a disjoint union of cycles, each of which has a number of vertices that is divisible by 6. Moreover, if the graph $\partial_{exp}\mathcal{M}(D, x, y)$ has exactly 6 vertices, then it is a 6-cycle.

Lemma 4.3.14 The boundary of the right-pair moduli space $\partial_{exp}\mathcal{M}(D, x, y)$ is isomorphic to the left-pair moduli space of the dual $\partial_{exp}\mathcal{M}_*(D^*, x^*, y^*)$.

Lemma 4.3.15 Both graphs $\partial_{exp}\mathcal{M}(D, x, y)$ and $\partial_{exp}\mathcal{M}_*(D^*, x^*, y^*)$ have the same number of vertices and the same parity of the number of cycles. Consequently, the graphs are isomorphic if $\partial_{exp}\mathcal{M}(D, x, y)$ has at most 12 vertices.

The last two lemmata result in the following:

Corollary 4.3.16 The graph $\partial_{exp}\mathcal{M}(D, x, y)$ is a 6-cycle (respectively, a disjoint union of 6-cycles) if and only if $\partial_{exp}\mathcal{M}_*(D^*, x^*, y^*)$ is a 6-cycle (respectively, a disjoint union of 6-cycles).

In [ORS13, Figure 4] and [LS14a, Figure 5.3], seven resolution configurations are illustrated that are of fundamental importance for the discussion of this section. These are listed in Figure 4.5.

The resolution configurations D_a, D_b and D_c are the only basic index 3 resolution configurations with leaves and ladybugs up to isotopy in S^2 and reordering of arcs. The resolution configurations D_d, D_e, D_f, D_g are the only basic index 3 resolution configurations without leaves or co-leaves up to isotopy in S^2 and reordering of arcs. Moreover, the configurations D_f and D_g are dual to configurations D_d and D_e, respectively.

Lemma 4.3.17 If D has a leaf, then $\partial_{exp}\mathcal{M}(D, x, y)$ is either a 6-cycle or a disjoint union of 6-cycles.

Proof: The proof of this lemma uses the naturally associated index 2 resolution configuration (D', x', y') constructed in Lemma 4.2.15. The key property is that $P(D, x, y)$ is naturally isomorphic to $P(D', x', y') \times \{0, 1\}$. When (D', x', y') is not a ladybug configuration, $P(D', x', y')$ is isomorphic to $\{0, 1\}^2$. Therefore, $P(D, x, y)$ has exactly six maximal chains in this case, and Lemma 4.3.13 implies that $\partial_{exp}\mathcal{M}(D, x, y)$ is a 6-cycle.

When (D', x', y') is a ladybug configuration, then by reordering arcs if necessary, D is isotopic to either D_a, D_b or D_c. Grading restrictions mean that the only possible labelling y of the leaf in D_a and D_b is x_+, and y has to label one of the leaves of D_c with x_+.

The four maximal chains in $P(D', x', y')$ have the form

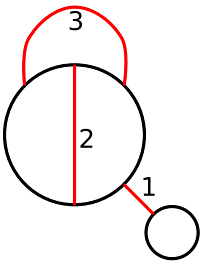
$$d_i = [\beta \prec c_i \prec \alpha]$$

for $i = 1, 2, 3, 4$. Let the ladybug matching be $d_1 \leftrightarrow d_2$ and $d_3 \leftrightarrow d_4$. Then the 12 vertices in $P(D, x, y) = P(D', x', y') \times \{0, 1\}$ are represented by the following maximal chains

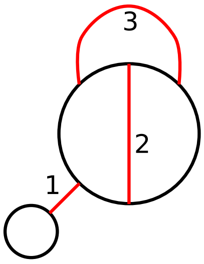
$$u_i = [(\beta, 0) \prec (c_i, 0) \prec (\alpha, 0)(\alpha, 1)]$$

$$v_i = [(\beta, 0) \prec (c_i, 0) \prec (c_i, 1)(\alpha, 1)]$$

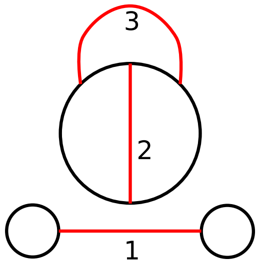
$$w_i = [(\beta, 0) \prec (\beta, 1) \prec (c_i, 1)(\alpha, 1)]$$



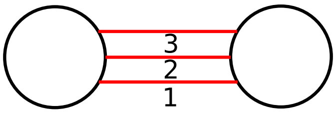
(a) D_a.



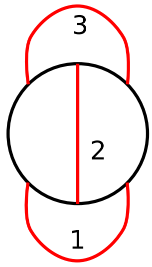
(b) D_b.



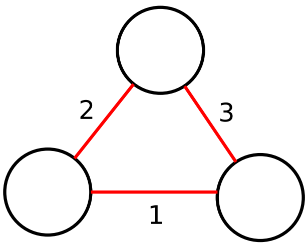
(c) D_c.



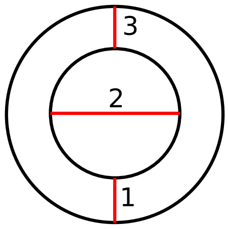
(d) D_d.



(e) D_e.



(f) D_f.



(g) D_g.

Figure 4.5: The only basic index 3 resolution configurations with leaves and ladybugs are D_a, D_b and D_c, up to isotopy in S^2. The only basic index 3 resolution configurations without leaves or coleaves are D_d, D_e, D_f and D_g, up to isotopy in S^2.

for $i = 1, 2, 3, 4$. Then [LS14a, Lemma 5.14] shows that the corresponding components of $\partial_{exp}\mathcal{M}(D, x, y)$ are

$$v_1 \_\_ u_1 \_\_ u_2 \_\_ v_2 \_\_ w_2 \_\_ w_1 \_\_ v_1 \qquad \text{and} \qquad v_3 \_\_ u_3 \_\_ u_4 \_\_ v_4 \_\_ w_4 \_\_ w_3 \_\_ v_3$$

where $a \_\_ b$ denotes an edge joining a to b. □

The previous lemma and corollary then summarise to give the following:

Corollary 4.3.18 If D is a co-leaf, then $\partial_{exp}\mathcal{M}(D, x, y)$ is a disjoint union of 6-cycles.

Lemma 4.3.19

1. Let D be a resolution configuration isotopic to D_d from Figure 4.5 (perhaps reordering the arcs). Then $\partial_{exp}\mathcal{M}(D_a, x, y)$ is a 6-cycle.
2. Let D be a resolution configuration isotopic to D_e from Figure 4.5 (perhaps reordering the arcs). Then $\partial_{exp}\mathcal{M}(D, x, y)$ is a disjoint union of two 6-cycles.

Proof: Part (1): Clearly $P(D, x, y)$ has 6 maximal chains, each of the form

$$v = [(D, y) \prec (E_2, z_2) \prec (E_1, z_1) \prec (s(D), x)]$$

where E_2 is obtained from D_e by merging Z_1 and Z_2. Thus, the labelling z_2 is determined completely by D, E_2 and y (c.f. Definition 4.2.11). For each of the three choices for (E_2, z_2) (given by each arc), $E_2 \setminus s(D)$ is not a ladybug configuration and so there are precisely 2 maximal chains in $P(E_2 \setminus s(D), x, z_2)$. This means $\partial_{exp}\mathcal{M}(D, x, y)$ has exactly 6 vertices and is a 6-cycle by Lemma 4.3.13.

Part (2): The only possible compatible labellings is $x = x_-x_-$ and $y = x_+$. The vertices of the cube of resolutions $\partial_{exp}\mathcal{M}(D, x, y)$ are given by the following maximal chains of $P(D, x, y)$:

$$
\begin{aligned}
v_1 &= [(D, x_+) \prec (s_{A_1}(D), x_+x_-) \prec (s_{A_1,A_2}(D), x_-) \prec (s(D), x_-x_-)] \\
v_2 &= [(D, x_+) \prec (s_{A_1}(D), x_+x_-) \prec (s_{A_1,A_3}(D), x_-x_-x_+) \prec (s(D), x_-x_-)] \\
v_3 &= [(D, x_+) \prec (s_{A_1}(D), x_-x_+) \prec (s_{A_1,A_2}(D), x_-) \prec (s(D), x_-x_-)] \\
v_4 &= [(D, x_+) \prec (s_{A_1}(D), x_-x_+) \prec (s_{A_1,A_3}(D), x_-x_+x_-) \prec (s(D), x_-x_-)] \\
v_5 &= [(D, x_+) \prec (s_{A_2}(D), x_+x_-) \prec (s_{A_1,A_2}(D), x_-) \prec (s(D), x_-x_-)] \\
v_6 &= [(D, x_+) \prec (s_{A_2}(D), x_+x_-) \prec (s_{A_2,A_3}(D), x_-) \prec (s(D), x_-x_-)] \\
v_7 &= [(D, x_+) \prec (s_{A_2}(D), x_-x_+) \prec (s_{A_1,A_2}(D), x_-) \prec (s(D), x_-x_-)] \\
v_8 &= [(D, x_+) \prec (s_{A_2}(D), x_-x_+) \prec (s_{A_2,A_3}(D), x_-) \prec (s(D), x_-x_-)] \\
v_9 &= [(D, x_+) \prec (s_{A_3}(D), x_+x_-) \prec (s_{A_2,A_3}(D), x_-) \prec (s(D), x_-x_-)] \\
v_{10} &= [(D, x_+) \prec (s_{A_3}(D), x_+x_-) \prec (s_{A_1,A_3}(D), x_-x_+x_-) \prec (s(D), x_-x_-)] \\
v_{11} &= [(D, x_+) \prec (s_{A_3}(D), x_-x_+) \prec (s_{A_2,A_3}(D), x_-) \prec (s(D), x_-x_-)] \\
v_{12} &= [(D, x_+) \prec (s_{A_3}(D), x_-x_+) \prec (s_{A_1,A_3}(D), x_-x_-x_+) \prec (s(D), x_-x_-)].
\end{aligned}
$$

After making a choice of ladybug matchings, [LS14a, Lemma 5.17] provides the resulting components of $\partial_{exp}\mathcal{M}(D, x, y)$ as

$$
v_1 \_\_ v_2 \_\_ v_{12} \_\_ v_{11} \_\_ v_8 \_\_ v_7 \_\_ v_1 \qquad \text{and} \qquad v_3 \_\_ v_4 \_\_ v_{10} \_\_ v_9 \_\_ v_6 \_\_ v_5 \_\_ v_3
$$

where, again, $a \_\_ b$ denotes an edge joining a to b. □

Since the resolution configurations D_f and D_g from Figure 4.5 are dual to the resolution configurations D_d and D_e from Figure 4.5, respectively, then the following corollary follows immediately from Corollary 4.3.16.

Corollary 4.3.20 If a resolution configuration D is isotopic to D_f or D_g from Figure 4.5, then $\partial_{exp}\mathcal{M}(D, x, y)$ is a disjoint union of 6-cycles.

Proposition 4.3.21 For an index $k \leq 3$ basic decorated resolution configuration (D, x, y), there are spaces $\mathcal{M}(D, x, y)$ and maps $\mathcal{F}$ satisfying the conditions (A1)-(A4).

Proof: As discussed at the beginning of this subchapter, it was enough to show that $\partial_{exp}\mathcal{M}(D, x, y)$ is a disjoint union of 6-cycles. Using Proposition 4.3.12 proves the case $k \leq 2$. By Proposition 4.3.2, in order to prove the case $k = 3$, it suffices to verify the map

$$\partial_{exp}\mathcal{M}(D, x, y) \to \partial\mathcal{M}_{\mathscr{C}_C(n)}(\underline{1}, \underline{0})$$

is a trivial covering map which, as discussed, was equivalent to verifying that each component of $\partial_{exp}\mathcal{M}(D, x, y)$ is a 6-cycle. This problem was broken up into cases in [LS14a, Subsection 5.5]. When D has a leaf or a co-leaf, Lemma 4.3.17 and Corollary 4.3.18 ensure the 6-cycle condition is satisfied. Since the remaining four cases are isotopic to one of D_d,D_e,D_f or D_g, Lemma 4.3.19 and Corollary 4.3.20 collectively deal with the remaining cases. □

4.3.4 The n-dimensional moduli spaces, $n \geq 3$

Proposition 4.3.22 For an index n basic decorated resolution configuration (D, x, y), there are spaces $\mathcal{M}(D, x, y)$ and maps $\mathcal{F}$ satisfying the conditions (A1)-(A4).

Proof: This follows from Proposition 4.3.21 when $n \leq 3$, and Condition (B3) of Proposition 4.3.2 when $n \geq 4$. □

4.4 Invariance of the Khovanov Spectrum

This subchapter makes a note on the invariance of the Khovanov spectrum, which gives the main theorem of [LS14a]:

Theorem 4.4.1 For an oriented link diagram L, let $\mathcal{X}_{Kh}(L) = \vee_j \mathcal{X}^j_{Kh}(L)$ be the Khovanov spectrum of L from Definition 4.3.6 and let $Kh(L) = \bigoplus_{i,j} Kh^{i,j}(L)$ be the Khovanov homology of L from Definition 4.2.16. Then $\mathcal{X}_{Kh}(L)$ satisfies the following two properties:

1. The Khovanov homology $Kh^{*,j}(L) = \bigoplus_i Kh^{i,j}(L)$ can be recovered from the reduced cohomology of $\mathcal{X}^j_{Kh}$, with

$$\widetilde{H}^i(\mathcal{X}^j_{Kh}(L)) = Kh^{i,j}(L).$$

2. The stable homotopy type of $\mathcal{X}^j_{Kh}(L)$ is an invariant of the isotopy class of the link corresponding to L, and is independent of all the choices made in its construction.

A collection of choices are made in the construction of $\mathcal{X}_{Kh}$, and [LS14a, Section 6] proves that the Khovanov spectrum is invariant under these choices. They are listed as follows:

1. A convention of a left or a right ladybug matching.
2. An n-crossing diagram L of the link.
3. An ordering of the crossings of L.
4. A particular sign assignment s of the cube $\mathcal{C}(n)$.
5. A framed neat embedding $\imath$, and a framing φ of that embedding, for $\mathscr{C}_C(n)$ relative the sign assignment s.
6. A framed neat embedding of the Khovanov flow category $\mathscr{C}_{Kh}(L)$ relative some $\mathbf{d}$, which is a perturbation of the framed embedding $(\imath, \varphi)$.
7. The numbers used in the Cohen-Jones-Segal construction of the CW complex. In particular, the integers A and B, and real numbers ε and R.

Recall from Lemma 2.3.3 that the independence of Choice (7) has already been shown. The independence of choices (3)-(7) is shown in [LS14a, Proposition 6.1], assuming a fixed choice of ladybug matching.

Proposition 4.4.1 Given a fixed choice of ladybug matching, the stable homotopy type of $\mathcal{X}_{Kh}(L)$ is an invariant of the chosen link diagram (that is, invariant of the choices (3)-(7) above).

In order to show that the construction of $\mathcal{X}^j_{Kh}(L)$ is invariant of any choice of link diagram (i.e. choice (2)), it suffices to show that it is invariant under all three Reidemeister moves RMI, RMII and RMIII, and [LS14a, Propositions 6.2-6.4] deals with them independently. It is worth noting that the arguments there are inspired by

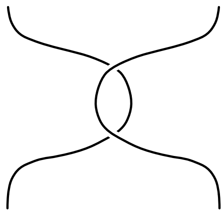


Figure 4.6: The 2-crossing tangle where Reidemeister Move II takes place.

the chain homotopy equivalences (see [BN02], for example) for Khovanov homology under RMI-RMIII, where particular subcomplexes of the Khovanov complex are quotiented out yielding homotopy equivalent chains. Let us show the RMII case in order to highlight the use of upward and downward closed subcategories (see Subchapter 2.4).

Proposition 4.4.2 Let L be a link diagram and L' be a link diagram obtained from L by performing a Reidemeister-II move. Then the two Khovanov spaces $\mathcal{X}^j_{Kh}(L)$ and $\mathcal{X}^j_{Kh}(L')$ are homotopy equivalent.

Proof: Since the RMII move takes place locally on a 2-crossing tangle (see Figure 4.6), consider the resolution 'square' with vertices (00),(10),(01) and (11) corresponding to the resolutions taken at each crossing. Only one of these vertices yields a diagram with a circle in it and depending on the ordering of crossings, it is either (10) or (01). Let us assume that this vertex is (01). Let C_1 be the acyclic subcomplex of $KhC(L')$ that is generated by the (01)-resolutions that label the circle with x_+ along with all (11) resolutions. Then C_1 corresponds to an upward closed subcategory $\mathscr{C}_1$ or $\mathscr{C}_{Kh}(L')$, with a complementary downward-closed subcategory $\mathscr{C}_2$ that corresponds to the quotient complex $C_2 = KhC(L')/C_1$ generated by all objects not in C_1 (that is, all (00)-resolutions, all (10)-resolutions and the (01)-resolutions that label the circle with x_-). Further, the chain complex C_2 can be decomposed into an acyclic quotient complex C_3 and a complementary complex C_4. The chain complex C_3 consists of all (00) resolutions and the (01) resolution with an x_- labelling of the circle. Thus, C_4 is the remaining (10)-resolution, which is clearly isomorphic to $KhC(L)$. Since the category $\mathscr{C}_3$ corresponding to C_3 is a downward closed subcategory of $\mathscr{C}_2$, and the complementary subcategory $\mathscr{C}_4$ corresponding to

C_4 is isomorphic to $\mathscr{C}_{Kh}(L)$, applying Lemma 2.4.5 twice yields the result. □

The one choice remaining (choice (1)) that is dealt with last in [LS14a] is the ladybug matching.

Proposition 4.4.3 The stable homotopy type of $\mathcal{X}_{Kh}(L)$ is independent of the ladybug matching.

The proof of the independence of ladybug matching is given in [LS14a, Proposition 6.5] by an argument which describes a rotation by π of a link diagram L around the y-axis, whose key characteristic is that is exchanges right and left pairs in each ladybug configuration. The independence of ladybug matching then reduces to independence of Reidemeister moves since the resulting link diagram is also a diagram of the underlying link.

Let us conclude this chapter with a note on handle cancellation (Theorem 3.1.1) in the Khovanov flow category. The invariance of the Khovanov homotopy type under Reidemeister moves I-III is shown in [LS14a] using arguments similar to those in [BN02]. In particular, the existence of the upward and downward closed subcategories that are described in this chapter are essential. In the Khovanov flow category, all 0-dimensional moduli spaces are single points and so cancelling these special subcategories to exhibit invariance under Reidemeister moves can be described in terms of handle cancellation, where the new moduli spaces are unaffected by the choice of moduli space to cancel (see Definition 3.1.1). So whilst the arguments in this chapter can be reformulated using handle cancellation, they are not simplified in doing so. Handle cancellation in flow categories allows further cancellations to be made with no restriction to upward or downward closed subcategories and we will see how this can be taken further in later chapters. In particular, the worked example in Subchapter 6.4 describes how various cancellations are made to significantly simplify the Khovanov flow category associated to the knot 8_{19}.

Chapter 5

Gaussian Elimination and its effect on the Khovanov space

It has been known for some time that the Khovanov complex for knots/links that can be represented by a particular type of diagram can be simplified (or 'flattened' in a sense) via Gaussian elimination. The links of interest are those with a diagram that can be built up of positive and negative elementary (2-crossing) tangles (see Definition 5.3.1 and Figure 5.3). Such a diagram will be known as a *matched* diagram, as in [JLS15]. The observation that the Khovanov chain complex can be simplified was made by Khovanov in his original paper on the subject [Kho00, Subsec.6.2], which computes the Khovanov homology of $(2, n)$-torus links. Bar-Natan (in [BN05]) later extended Khovanov homology to tangles by studying a local version of the Khovanov complex which looks at chain complexes of tangles up to chain homotopy equivalence. In Chapter 3, the notion of handle cancellation was extended to general framed flow categories and (as in [JLS15]) this allows for a simpler description of the Khovanov homotopy type of associated to a matched diagram.

We shall overview how Gaussian elimination works in Khovanov homology in Subchapter 5.1, and then proceed by outlining the description in [JLS15] of a more compact version of the Khovanov homotopy type associated to matched diagrams. In order to describe the original Khovanov homotopy type, it was necessary to construct the Khovanov flow category which satisfies the nice property that it covers the cube flow category; one thing that allowed for this was that all 0-dimensional moduli spaces are single points. By performing Gaussian elimination, however, one allows for 0-dimensional moduli spaces to be more than just single points and therefore we require a framed flow category, the *sock flow category*, that is analogous to the cube flow category, but whose 0-dimensional moduli spaces consist of either 1 or 2 points. The sock flow category was recently constructed by Jones-Lobb-Schütz in [JLS15] for the purpose of describing a compact version of the Khovanov homotopy type. We will highlight the construction of the sock flow category in Subchapter 5.2 and then describe the flow category that covers the sock flow category in Subchapter 5.3.

5.1 Gaussian elimination in Khovanov homology

Gaussian elimination is an extremely useful method for simplifying the Khovanov chain complex associated to matched diagrams. We shall reserve this subchapter to giving a brief overview of how Gaussian elimination can be applied to this set-up, which should illustrate the motivation of [JLS15] and the rest of this chapter. The preliminaries are quite algebraic and as such, it will be useful to consider the Topological Quantum Field Theory (TQFT) version of Khovanov homology. There are two main reasons for shifting to this algebraic definition of Khovanov homology. Firstly, the TQFT definition allows us to describe Gaussian elimination algebraically in terms of linear maps of vector spaces, as it was described in [BN07]. Secondly, Krasner in [Kra09] describes the generalised Khovanov-Rozansky homologies for matched diagrams in terms of a general TQFT that contains Khovanov homology as a special case; this reason is also the motivation in [JLS15]. Here, we shall only describe the Khovanov ($n = 2$) case.

The Khovanov complex is constructed in the usual way, by taking a diagram D of a knot or link with n ordered crossings. At each crossing, either the 0- or 1-resolution can be taken, giving a choice of 2^n different resolutions (see Figure 5.1) which all yield collections of circles. These are presented on the vertices of the cube $[0,1]^n$ according to the corresponding resolutions taken. The vertices of the cube are given the obvious partial order $((0,\dots,0)$ being the minimum and

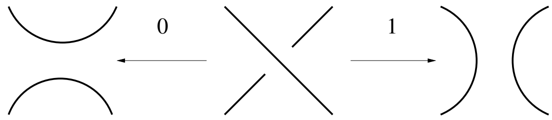


Figure 5.1: The 0- and 1-resolutions of a crossing.

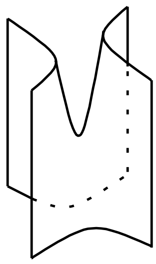


Figure 5.2: The saddle cobordism between a 0-resolution and a 1-resolution of a crossing.

$(1, \ldots, 1)$ being the maximum), and travelling along an edge of the cube involves changing exactly one coordinate from 0 to 1 on the vertices that bound that edge. The corresponding change in the resolution diagrams is from a 0-resolution to a 1-resolution, and therefore one can present the saddle cobordism of Figure 5.2 on this edge.

Definition 5.1.1 Let V be a graded vector space spanned by v_+ and v_-. Then a $(1{+}1)$-*dimensional TQFT* $\mathcal{A}$ is a functor from 1-manifolds together with cobordisms to vector spaces together with linear maps between them. To each circle in the resolution cube, $\mathcal{A}$ assigns the vector space V. Each additional circle corresponds to a tensor factor of V. Moreover, $\mathcal{A}$ assigns to a cobordism of circles, a map between vector spaces. Since the cobordisms on edges change one crossing from a 0-smoothing to a 1-smoothing, we will either be merging circles, or splitting them. The corresponding maps are:

$$\begin{aligned} m : &V \otimes V \to V & \qquad \Delta : &V \to V \otimes V \\ &v_+ \otimes v_+ \mapsto v_+ & &v_+ \mapsto v_+ \otimes v_- + v_- \otimes v_+ \\ &v_+ \otimes v_- \mapsto v_- & &v_- \mapsto v_- \otimes v_- \\ &v_- \otimes v_+ \mapsto v_- & & \\ &v_- \otimes v_- \mapsto 0 & & \end{aligned}$$

The vector space $V = \langle v_+, v_- \rangle$ can be thought of as the Frobenius algebra $\mathbb{Z}[x]/x^2$ by an identification taking $v_+ \mapsto 1$ and $v_- \mapsto x$.

By applying $\mathcal{A}$ to the cube of resolutions for a diagram D, one obtains a cube whose vertices are presented by tensor products of V and whose edges correspond to the linear maps m and Δ on the necessary tensor products. The Khovanov chain complex is obtained from this cube by considering the following definitions.

Definition 5.1.2 Let v be a vertex of the resolution cube of a diagram D. It corresponds to a collection of k circles, or equivalently the k-tensor product $V^{\otimes k}$. Define a *homological grading* as $h(v) = |v| - n_-$, where $|v|$ is the sum of the coordinates and n_- is the number of negative crossings in D. Denote by w the image of v under an edge-cobordism, then $h(w) = h(v) + 1$ (hinting at a cohomology theory). Define an intermediate grading α by setting $\alpha(v_+) = 1$ and $\alpha(v_-) = -1$. This is extended to tensors as $\alpha(v_1 \otimes \cdots \otimes v_k) = \alpha(v_1) + \cdots + \alpha(v_k)$. Notice that $\alpha(w) = \alpha(v) - 1$. This is used to define a *quantum grading* q as $q(v) = \alpha(v) + h(v) + n_+ - n_-$, where n_+ is the number of positive crossings in D. Notice that $q(w) = q(v)$.

The Khovanov complex is defined by taking chain groups as a direct sum of all tensor products on vertices with the same homological grading. Since the quantum grading does not change passing over edges, this complex splits as a direct sum of complexes; one for each quantum grading. This gives a bi-graded chain complex. To ensure that this does give a chain complex, one can just sprinkle the cube with $(-)$-signs so that every face anti-commutes, enforcing $d^2 = 0$. A coherent way to do this is by considering a basis of the exterior algebra in n generators and using the exterior multiplication $\wedge dx_i$ to label each edge which changes the i^{th} coordinate by dx_i.

By considering the local picture of tangles, we can apply the following form of Gaussian elimination (as described in [BN07]) to the Khovanov chain complex of matched diagrams.

Lemma 5.1.3 If $\phi : B \to D$ is an isomorphism (in some additive category), then the four term complex segment given by

$$\cdots [A] \xrightarrow{\begin{pmatrix}\alpha\\ \beta\end{pmatrix}} \begin{bmatrix}B\\ C\end{bmatrix} \xrightarrow{\begin{pmatrix}\phi & \delta\\ \gamma & \epsilon\end{pmatrix}} \begin{bmatrix}D\\ E\end{bmatrix} \xrightarrow{\begin{pmatrix}\mu & \nu\end{pmatrix}} \begin{bmatrix}F\end{bmatrix} \cdots$$

is isomorphic to the (direct sum) complex segment

$$\cdots [A] \xrightarrow{\begin{pmatrix}0\\ \beta\end{pmatrix}} \begin{bmatrix}B\\ C\end{bmatrix} \xrightarrow{\begin{pmatrix}\phi & 0\\ 0 & \epsilon-\gamma\phi^{-1}\delta\end{pmatrix}} \begin{bmatrix}D\\ E\end{bmatrix} \xrightarrow{\begin{pmatrix}0 & \nu\end{pmatrix}} \begin{bmatrix}F\end{bmatrix} \cdots$$

and both of which are homotopy equivalent to the (flattened) complex segment

$$\cdots [A] \xrightarrow{\begin{pmatrix}\beta\end{pmatrix}} \begin{bmatrix}C\end{bmatrix} \xrightarrow{\begin{pmatrix}\epsilon-\gamma\phi^{-1}\delta\end{pmatrix}} \begin{bmatrix}E\end{bmatrix} \xrightarrow{\begin{pmatrix}\nu\end{pmatrix}} \begin{bmatrix}F\end{bmatrix} \cdots.$$

Here, A, C, E and F are all arbitrary columns of objects in the same additive category as B and D. Moreover, the Greek letters are all arbitrary matrices of morphisms in that category (with appropriate dimensions, domains and ranges).

Proof: Since the matrices in the first and second complexes differ by invertible row operations (or a change of basis), the complexes are isomorphic. Moreover,

$$\begin{pmatrix}\mu-\nu\gamma\phi^{-1} & \nu\end{pmatrix} \text{ and } \begin{pmatrix}\alpha-\phi^{-1}\delta\beta\\ \beta\end{pmatrix}$$

are produced by the column and row operations on

$$\begin{pmatrix}\mu & \nu\end{pmatrix} \text{ and } \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

respectively. From the $d^2 = 0$ property of the original complex, $\mu\phi - \nu\gamma = 0$ and $\phi\alpha - \delta\beta = 0$ and so

$$\begin{pmatrix}\mu-\nu\gamma\phi^{-1} & \nu\end{pmatrix} = \begin{pmatrix}0 & \nu\end{pmatrix} \text{ and } \begin{pmatrix}\alpha-\phi^{-1}\delta\beta\\ \beta\end{pmatrix} = \begin{pmatrix}0\\ \beta\end{pmatrix}.$$

Since $\phi : B \to D$ is an isomorphism, the last two complexes differ by the removal of a contractible direct summand and hence are both homotopy equivalent. $\square$

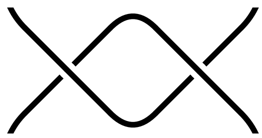


Figure 5.3: The 2-crossing elementary tangle T_2.

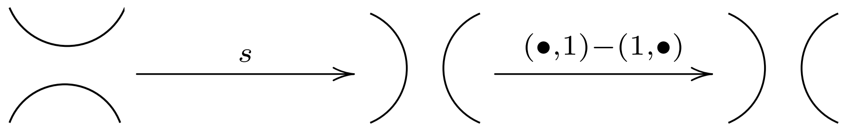


Figure 5.4: The resultant chain complex of T_2 after Gaussian elimination.

In order to highlight the use of this lemma, consider the positive 2-crossing elementary tangle T_2 in Figure 5.3. Then the resolution square $[0,1]^2$ consists of two horizontal crossingless strands at $(0,0)$, two vertical crossingless strands at $(0,1)$ and $(1,0)$, and a circle sandwiched by two crossingless vertical strands at $(1,1)$ (c.f. Definition 5.3.7). The TQFT assigns the vector space V to the circle at $(1,1)$, and so this local complex is isomorphic to one which replaces the resolution at the vertex $(1,1)$ by a direct summand of two copies of a vertical smoothing; one tensored with v_+ and one tensored with v_-. Since the homological grading increases by one along each edge, there is an isomorphism between the direct summand which is tensored with v_- and both of the vertical smoothings at $(0,1)$ and $(1,0)$. Making a choice of which isomorphism to use (here, we shall use the one with $(1,0)$), one can apply Lemma 5.1.3 to produce a 'flattened' three-term complex which consists of all resolutions of $(0,0)$ and $(0,1)$, but only the resolutions of $(1,1)$ that have a circle decorated with v_+. The differential from $(0,0)$ to $(0,1)$ remains the same, but the differential from $(0,1)$ to $(1,1)$ obtains an additional component (see Figure 5.4 where the maps are defined below).

Definition 5.1.4 Consider a resolution consisting of two vertical crossingless strands that are assumed to lie on separate circles. Applying the TQFT $\mathcal{A}$ to these circles provides a tensor product $V \otimes V$. A map

$$(\bullet,1) \pm (1,\bullet) : V \otimes V \to V \otimes V$$

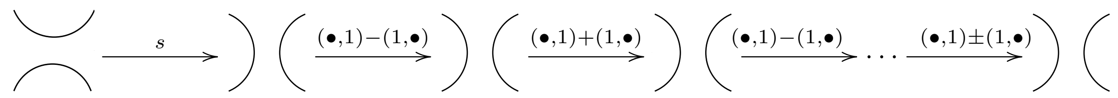


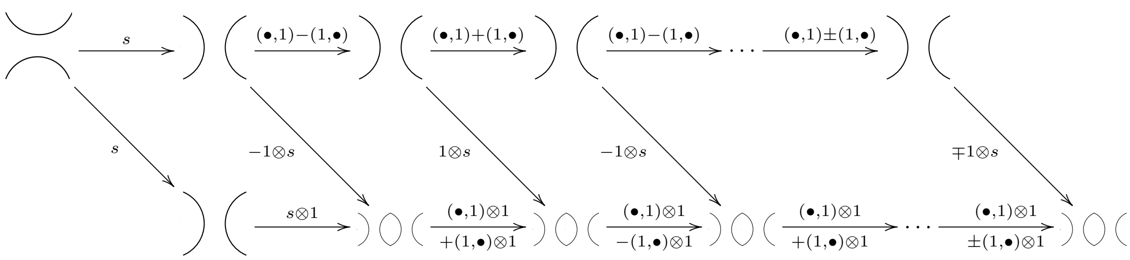
Figure 5.5: The chain complex L_n obtained from the chain complex of T_n by Gaussian elimination.

can be defined by $((\bullet,1) \pm (1,\bullet))(v,w) = (v_- \otimes v, w) \pm (v, v_- \otimes w)$. The map $\bullet : V \to V$ can also be defined as a map on a single vertical strand which multiplies the decoration of that strand by v_- (or x in $\mathbb{Z}[x]/x^2$). This means that composing $\bullet$ twice results in the trivial map.

Note that if both strands of the vertical resolution are part of the same circle, then $(\bullet,1) \pm (1,\bullet)$ labels the circle with $0 = x^2 \in \mathbb{Z}[x]/x^2$. Therefore, we can assume that the maps $(\bullet,1) \pm (1,\bullet)$ are defined only on separate circles.

Proposition 5.1.5 Let T_n be the positive n-crossing elementary tangle. The chain homotopy class $[T_n]$ can be represented by the flattened complex L_n given in Figure 5.5.

Proof: The map s in Figure 5.4 denotes the saddle map between resolutions (see Figure 5.2). The proof follows by induction. When $n = 1$, the result is obvious by taking the single 0-resolution and 1-resolution with the saddle map in between (the $n = 2$ case is also described above). So assume that the result follows for T_{n-1} for some $n > 2$. To show that L_n does indeed give a chain complex representing T_n, consider the decomposition of T_n into the two tangles T_1 and T_{n-1}. The complex $[T_n]$ can therefore be represented by the tensor product $L_1 \otimes L_{n-1}$



where each of the top (respectively, the bottom) diagrams are given by the 0-resolution (respectively, the 1-resolution) of T_1 glued to the corresponding diagram of T_{n-1}. Notice that gluing the 0-resolution produces identical diagrams (up to homotopy). Since each circle corresponds to $V = \mathbb{Z}[x]/x^2$, and each decomposes into a direct sum of a diagram where the circle is either labelled with v_+ or v_-. Consider the right-most diagonal map $\mp 1 \otimes s$. This map splits into two components via the direct sum decomposition described. In the same way as the T_2 case described immediately before this proposition, this is an isomorphism on the diagram in which the circle is labelled with v_-. Therefore, using Gaussian elimination cancels these two isomorphic components, leaving the top complex one diagram short, and the bottom-right given by a single vertical smoothing labelled with an extra v_+. This process can be repeated with the new right-most diagram in the top complex, and so on, until all that is left of the top complex is the 0-smoothing. This complex is L_n. $\square$

5.2 The sock flow category

The sock flow category is obtained as a quotient subcategory of the Morse flow category $\mathscr{C}_{\mathbb{R}\mathbf{P}^n}$ of the real projective space with its standard Morse function. Consider $\mathbb{R}\mathbf{P}^n$ as

$$\mathbb{R}\mathbf{P}^n = \left\{ [x_0 : x_1 : \cdots : x_n] \;\middle|\; \underline{0} \neq (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \right\}$$

where $[x_0 : x_1 : \cdots : x_n] = [\lambda x_0 : \lambda x_1 : \cdots : \lambda x_n]$ for $0 \neq \lambda \in \mathbb{R}$. Then consider the self-indexing Morse function $f \colon \mathbb{R}\mathbf{P}^n \to \mathbb{R}$ defined by

$$f([x_0 : \cdots : x_n]) = \sum_{i=1}^{n} i x_i^2 \Bigg/ \sum_{i=0}^{n} x_i^2 \,.$$

This Morse function has $(n+1)$ critical points given by

$$p_i = [0 : \cdots 0 : 1 : 0 : \cdots : 0] \in \mathbb{R}\mathbf{P}^n$$

where the i^{th} coordinate is non-zero, for $i = 0, \ldots, n$. Let v be the gradient for f such that the gradient flow lines through $\underline{x} = [x_0 : \cdots : x_n]$ are given by

$$\gamma_{\underline{x}}(t) = [x_0 : e^t x_1 : \cdots : e^{nt} x_n].$$

With respect to the positive gradient flow, the stable manifold of a critical point p_i is given by

$$W^s(p_i) = \{\, [x_0 : x_1 : \cdots : x_{i-1} : 1 : 0 : \cdots : 0] \mid x_0, \ldots, x_{i-1} \in \mathbb{R} \,\}$$

and the unstable manifold of a critical point p_i is given by

$$W^u(p_i) = \{\, [0 : 0 : \cdots : 0 : 1 : x_{i+1} : \cdots : x_n] \mid x_{i+1}, \ldots, x_n \in \mathbb{R} \,\}.$$

Therefore, given two critical points p_i and p_j of f with $j < i$, the intersection of the stable and unstable manifolds can be realised as

$$W^u(p_j) \cap W^s(p_i) = \{\, [0 : \cdots : 0 : x_j : \cdots : x_{i-1} : 1 : 0 : \cdots : 0] \mid x_j \neq 0 \,\}$$

which gives the uncompactified moduli space

$$\tilde{\mathcal{M}}(p_i, p_j) = \{[0 : \cdots : 0 : \varepsilon : x_{j+1} : \cdots : x_{i-1} : 1 : 0 : \cdots : 0] \,|\, \varepsilon \in \{\pm 1\}, x_k \in \mathbb{R}\}.$$

The compactification of this moduli space can be written as $\mathcal{M}(p_i, p_j) = S^0 \times [-\infty, \infty]^{i-j-1}$. The composition maps are given as

$$\begin{aligned}&[\varepsilon : x_{j+1} : \cdots : x_{k-1} : 1] \circ [\varepsilon' : x_{k+1} : \cdots : x_{i-1} : 1] \\ &= [\varepsilon\varepsilon' : \varepsilon' x_{j+1} : \cdots : \varepsilon' x_{k-1} : \varepsilon'\infty : x_{k+1} : \cdots : x_{i-1} : 1]\end{aligned}$$

for $\varepsilon, \varepsilon' \in \{\pm 1\}$, where the obvious string of zeros are omitted for compact notation.

Definition 5.2.1 The *n-sock flow category* $\mathcal{S}_n$ is the flow category whose objects are $\mathrm{Ob}(\mathcal{S}_n) = \{0, 1, \ldots, n\}$. The moduli spaces are obtained from the Morse flow category $\mathscr{C}_{\mathbb{RP}^n}$ in the following way. Let

$$\jmath_{i,j} : \tilde{\mathcal{M}}(p_i, p_j) \to \mathbb{R}^{i-2}$$

denote the composition of the embedding $\imath_{i,j} : \tilde{\mathcal{M}}(p_i, p_j) \to W^s(p_i)$ with the projection to the last $(i-2)$ coordinates of $W^s(p_i)$. Then let $\tilde{\mathcal{M}}(i, j)$ be the quotient space of $\tilde{\mathcal{M}}(p_{i+1}, p_{j+1})$ which identifies points that have the same image under $\jmath_{i+1,j+1}$. This is necessary since $\jmath_{i,1}$ is a $2:1$ immersion, but $\jmath_{i,j}$ for $j \geq 2$ remains an embedding.

The flow category $\mathcal{S}_n$ is then obtained as the suspension of a quotient subcategory of $\mathscr{C}_{\mathbb{RP}^n}$. In particular, the moduli spaces are

$$\begin{aligned}\mathcal{M}_{\mathcal{S}_n}(i,j) &= \mathcal{M}_{\mathscr{C}_{\mathbb{RP}^n}}(p_{i+1}, p_{j+1}) \\ &\cong S^0 \times [-1,1]^{i-j-1}\end{aligned}$$

whenever $j \geq 1$, and

$$\mathcal{M}_{\mathcal{S}_n}(i,0) \cong [-1,1]^{i-1}.$$

Consider the following illustration of $\mathcal{S}_n$:

$$\cdots \xrightarrow{++} 4 \xrightarrow{+-} 3 \xrightarrow{++} 2 \xrightarrow{+-} 1 \xrightarrow{+} 0$$

where each sign above an arrow corresponds to a single point moduli space between those objects framed by the given sign. By writing $\mathcal{M}(i+1,i) = \{L, R\} = \{+,-\}$ for $i \geq 1$, the 1-dimensional moduli spaces $\mathcal{M}(i+2,i)$ are the two disjoint intervals I_1, I_2 whose boundaries can be described as $\partial I_1 = \{LL, RR\}$ and $\partial I_2 = \{LR, RL\}$. This assignment of boundary components is inherited from the Morse flow category of $\mathscr{C}_{\mathbb{RP}^n}$ and applies here since $i \geq 1$. The moduli space $\mathcal{M}(2,0)$ is a single interval with the obvious boundary components.

This entire construction can be dualised to obtain the *dual n-sock flow category* $\mathcal{S}_n^*$ by setting the dual moduli spaces as $\mathcal{M}^*(i,j) = \mathcal{M}(-j,-i)$, and can be illustrated in a similar way as:

$$0 \xrightarrow{+} -1 \xrightarrow{+-} -2 \xrightarrow{++} -3 \xrightarrow{+-} -4 \xrightarrow{++} \cdots$$

Definition 5.2.2 For a k-tuple $\mathbf{n} = (n_1, \dots, n_k) \in \mathbb{Z}^k$ with each $n_i \geq 1$, define the Morse function

$$f_{\mathbf{n}} : \mathbb{RP}^{n_1+1} \times \cdots \times \mathbb{RP}^{n_k+1} \to \mathbb{R}$$

by $f_{\mathbf{n}}(x_1, \dots, x_k) = f_1(x_1) + \cdots + f_k(x_k)$, where each $f_i : \mathbb{RP}^{n_i+1} \to \mathbb{R}$ is the standard Morse function defined at the beginning of this section. A Morse-Smale gradient for $f_{\mathbf{n}}$ is given as the sum of the gradients for each f_i. In particular, $Crit(f_{\mathbf{n}}) = \left\{ (p_{i_1}, \dots, p_{i_k}) \;\middle|\; p_{i_j} \in Crit(f_j) \text{ for } 1 \leq j \leq k \right\}$ with $ind((p_{i_1}, \dots, p_{i_k})) = ind(p_{i_1}) + \cdots + ind(p_{i_k})$.

The open moduli spaces $\tilde{\mathcal{M}}((p_{i_1}, \ldots, p_{i_k}), (p_{j_1}, \ldots, p_{j_k}))$ are embedded in

$$W^s((p_{i_1}, \ldots, p_{i_k})) \cong W^s(p_{i_1}) \times \cdots \times W^s(p_{i_k}).$$

Definition 5.2.3 Let $\mathscr{C}_{f_{\mathbf{r}}}$ be the Morse flow category of the function described in the previous definition with $\mathbf{r} = (r_1, \ldots, r_k) \in (\mathbb{Z} \setminus \{0\})^k$. Repeating the quotient construction of $\mathcal{S}_n$ in Definition 5.2.1 for n equal to each r_j, yields a framed flow category $\mathcal{S}_{(r_1,\ldots,r_k)}$ called the $\mathbf{r}$-*sock flow category* whose objects are the k-tuples $(i_1, \ldots, i_k)$ with grading given by the sum of the entries. Each $\mathcal{S}_{r_j}$ is a subcategory in $\mathcal{S}_{(n_1,\ldots,n_k)}$. When $r_j < 0$, the j^{th} coordinate is dualised using $\mathcal{S}^*_{r_j}$.

Note that if $\mathbf{r} = (\pm 1, \ldots, \pm 1)$, then the flow category $\mathcal{S}_{\mathbf{r}}$ reduces to the k-cube flow category of [LS14a] in Definition 4.1.1. In this case, the different framings produce stably equivalent homotopy types (see Sections 4.4 and 4.1.1). The analogous argument for $\mathcal{S}_{\mathbf{r}}$ is given in Section 5.4 where, in fact, the moduli spaces of each $\mathcal{S}_n$ of dimension greater than 1 need not be framed explicitly.

Proposition 5.2.4 Let $\mathbf{r} \in (\mathbb{Z} - \{0\})^k$. Then every moduli space $\mathcal{M}(a, b)$ is a disjoint union of discs where $a, b \in \mathrm{Ob}(\mathcal{S}_{\mathbf{r}})$.

Proof: Since the flow category $\mathcal{S}_{\mathbf{r}}$ is constructed as a quotient of the Morse flow category $\mathscr{C}_{f_{\mathbf{r}}}$, it is enough to show that the same holds for the Morse flow category. When $k = 1$, this follows from the description of the Morse flow category given above, where all moduli spaces are disjoint unions of cubes. The inductive argument then follows from the following lemma (Lemma 5.2.5). $\square$

Lemma 5.2.5 If $f\colon M \to \mathbb{R}$ and $g\colon N \to \mathbb{R}$ are Morse functions on closed, smooth manifolds M and N, let v_M, v_N be their Morse-Smale gradients. Let $\mathscr{C}_F$ be the Morse flow category of the Morse function $F\colon M \times N \to \mathbb{R}$ on the product given by $F(x, y) = f(x) + g(y)$, and let $v(x, y) = (v_M(x), v_N(y))$ be the Morse-Smale gradient for F. For $a, b \in Crit(f)$ with $\mathrm{ind}(a) \geq \mathrm{ind}(b)$ and $p, q \in Crit(g)$ with $\mathrm{ind}(p) \geq \mathrm{ind}(q)$, the moduli space $\mathcal{M}_{\mathscr{C}_F}((a, p), (b, q))$ is PL-homeomorphic to

1. $\mathcal{M}_{\mathscr{C}_f}(a, b) \times \mathcal{M}_{\mathscr{C}_g}(p, q) \times [0, 1]$ if $\mathrm{ind}(a) > \mathrm{ind}(b)$ and $\mathrm{ind}(p) > \mathrm{ind}(q)$.
2. $\mathcal{M}_{\mathscr{C}_f}(a, b)$ if $\mathrm{ind}(a) > \mathrm{ind}(b)$ and $p = q$.

3. $\mathcal{M}_{\mathscr{C}_g}(p,q)$ if $a=b$ and $\mathrm{ind}(p)>\mathrm{ind}(q)$.

Note that the statement of the lemma is for PL-homeomorphisms because for $\mathrm{ind}(a)>\mathrm{ind}(b)$ and $\mathrm{ind}(p)>\mathrm{ind}(q)$ the moduli space $\mathcal{M}_{\mathscr{C}_F}((a,p),(b,q))$ will, in general, produce more corners than $\mathcal{M}_{\mathscr{C}_f}(a,b)\times\mathcal{M}_{\mathscr{C}_g}(p,q)\times[0,1]$ and the two will not be diffeomorphic.

Proof: The cases $a=b$ and $p=q$ are easy to see, so we will focus on the case where $\mathrm{ind}(a)>\mathrm{ind}(b)$ and $\mathrm{ind}(p)>\mathrm{ind}(q)$. In this case, $k=\mathrm{ind}(a,p)-\mathrm{ind}(b,q)\geq 2$ and the proof is by induction on k. The base case is trivial since when $k=2$, both $\mathcal{M}_{\mathscr{C}_f}(a,b)$ and $\mathcal{M}_{\mathscr{C}_g}(p,q)$ are disjoint unions of points. The products of these points give the boundaries for the intervals in $\mathcal{M}_{\mathscr{C}_F}((a,p),(b,q))$. So assume true for all $\mathcal{M}_{\mathscr{C}_F}((a,p),(b,q))$ with $\mathrm{ind}(a,p)-\mathrm{ind}(b,q)\leq k-1$. Throughout the proof it should be obvious which category the moduli spaces are from, and so we shall omit the category from the notation.

Let c be a critical point of f with $\mathrm{ind}(a)>\mathrm{ind}(c)>\mathrm{ind}(b)$ and r a critical point of g with $\mathrm{ind}(p)>\mathrm{ind}(r)>\mathrm{ind}(q)$. Since $\mathscr{C}_F$ is a flow category, the boundary of $\mathcal{M}(ap,bq)$ is

$$\begin{aligned}\partial\mathcal{M}_{\mathscr{C}_F}(ap,bq)=&\mathcal{M}(bp,bq)\times\mathcal{M}(ap,bp)\cup\mathcal{M}(aq,bq)\times\mathcal{M}(ap,aq)\\ &\bigcup_{(c,r)}\mathcal{M}(cp,bq)\times\mathcal{M}(ap,cp)\cup\mathcal{M}(cq,bq)\times\mathcal{M}(ap,cq)\\ &\bigcup\qquad\mathcal{M}(br,bq)\times\mathcal{M}(ap,br)\cup\mathcal{M}(ar,bq)\times\mathcal{M}(ap,ar)\\ &\bigcup\qquad\mathcal{M}(cr,bq)\times\mathcal{M}(ap,cr).\end{aligned}$$

These boundary components are all simplified when $\mathrm{ind}(a)-\mathrm{ind}(b)=1$ or $\mathrm{ind}(p)-\mathrm{ind}(q)=1$. Note that

$$\begin{aligned}\mathcal{M}(bp,bq)\times\mathcal{M}(ap,bp)&\cong\mathcal{M}(p,q)\times\mathcal{M}(a,b)\\ &\cong\mathcal{M}(a,b)\times\mathcal{M}(p,q)\cong\mathcal{M}(aq,bq)\times\mathcal{M}(ap,aq)\end{aligned}$$

To distinguish, write

$$\mathcal{M}(a,b)\boxtimes\mathcal{M}(p,q)\subset\mathcal{M}(ap,bq)$$

for $\mathcal{M}(bp, bq) \times \mathcal{M}(ap, bp)$ to indicate that these are the broken flow lines that first go from ap to bp, and then from bp to bq. Similarly, write

$$\mathcal{M}(p, q) \boxtimes \mathcal{M}(a, b)$$

for $\mathcal{M}(aq, bq) \times \mathcal{M}(ap, aq)$ indicating that these broken flow lines first go from ap to aq, and then from aq to bq. It is sufficient to show that $\mathcal{M}(ap, bq)$ is a cylinder between these two boundary components. By induction hypothesis, we have

$$\mathcal{M}(cp, bq) \times \mathcal{M}(ap, cp) \cong (\mathcal{M}(c, b) \times \mathcal{M}(p, q) \times [0, 1]) \times \mathcal{M}(a, c) \text{ and}$$
$$\mathcal{M}(cq, bq) \times \mathcal{M}(ap, cq) \cong \mathcal{M}(c, b) \times (\mathcal{M}(a, c) \times \mathcal{M}(p, q) \times [0, 1]).$$

These two boundary components can be combined to give a cylinder C_1 with the two boundaries

$$\mathcal{M}(a, c) \boxtimes \mathcal{M}(c, b) \boxtimes \mathcal{M}(p, q) \subset \mathcal{M}(a, b) \boxtimes \mathcal{M}(p, q) \text{ and}$$
$$\mathcal{M}(p, q) \boxtimes \mathcal{M}(a, c) \boxtimes \mathcal{M}(c, b) \subset \mathcal{M}(p, q) \boxtimes \mathcal{M}(a, b)$$

sandwiching $\mathcal{M}(a, c) \boxtimes \mathcal{M}(p, q) \boxtimes \mathcal{M}(c, b)$. Similarly,

$$\mathcal{M}(br, bq) \times \mathcal{M}(ap, br) \cup \mathcal{M}(ar, bq) \times \mathcal{M}(ap, ar)$$

is a cylinder C_2 between

$$\mathcal{M}(a, b) \boxtimes \mathcal{M}(p, r) \boxtimes \mathcal{M}(r, q) \subset \mathcal{M}(a, b) \boxtimes \mathcal{M}(p, q) \text{ and}$$
$$\mathcal{M}(p, r) \boxtimes \mathcal{M}(r, q) \boxtimes \mathcal{M}(a, b) \subset \mathcal{M}(p, q) \boxtimes \mathcal{M}(a, b)$$

sandwiching $\mathcal{M}(p, r) \boxtimes \mathcal{M}(a, b) \boxtimes \mathcal{M}(r, q)$. The two cylinders C_1 and C_2 intersect in two disjoint cylinders

$$\begin{aligned}\mathcal{M}(r, q) \times (\mathcal{M}(c, b) \times \mathcal{M}(p, r) \times [0, 1]) \times \mathcal{M}(a, c) \sqcup \\ \mathcal{M}(c, b) \times (\mathcal{M}(a, c) \times \mathcal{M}(r, q) \times [0, 1]) \times \mathcal{M}(p, r).\end{aligned}$$

Finally,

$$\begin{aligned}\mathcal{M}(cr, bq) \times \mathcal{M}(ap, cr) \cong & (\mathcal{M}(c, b) \times \mathcal{M}(r, q) \times [0, 1]) \times \\ & (\mathcal{M}(a, c) \times \mathcal{M}(p, r) \times [0, 1]),\end{aligned}$$

and this can be thought of as a square between the following four products

$$\mathcal{M}(a,c) \boxtimes \mathcal{M}(p,r) \boxtimes \mathcal{M}(c,b) \boxtimes \mathcal{M}(r,q),$$
$$\mathcal{M}(a,c) \boxtimes \mathcal{M}(p,r) \boxtimes \mathcal{M}(r,q) \boxtimes \mathcal{M}(c,b),$$
$$\mathcal{M}(p,r) \boxtimes \mathcal{M}(a,c) \boxtimes \mathcal{M}(c,b) \boxtimes \mathcal{M}(r,q) \text{ and}$$
$$\mathcal{M}(p,r) \boxtimes \mathcal{M}(a,c) \boxtimes \mathcal{M}(r,q) \boxtimes \mathcal{M}(c,b)$$

and this is bounded precisely by the two cylinders C_1 and C_2. By considering their collar neighbourhoods, the boundary components

$$\begin{aligned}\mathcal{M}(cp,bq) \times \mathcal{M}(ap,cp) \cup \mathcal{M}(cq,bq) \times \mathcal{M}(ap,cq) \cup \mathcal{M}(cr,bq) \times \mathcal{M}(ap,cr)\\ \cup \mathcal{M}(br,bq) \times \mathcal{M}(ap,br) \cup \mathcal{M}(ar,bq) \times \mathcal{M}(ap,ar)\end{aligned}$$

combine together to give a cylinder between

$$\mathcal{M}(a,c) \boxtimes \mathcal{M}(c,b) \boxtimes \mathcal{M}(p,q) \cup \mathcal{M}(a,b) \boxtimes \mathcal{M}(p,r) \boxtimes \mathcal{M}(r,q) \text{ and}$$
$$\mathcal{M}(p,q) \boxtimes \mathcal{M}(a,c) \boxtimes \mathcal{M}(c,b) \cup \mathcal{M}(p,r) \boxtimes \mathcal{M}(r,q) \boxtimes \mathcal{M}(a,b).$$

This can be repeated for every pair (c, r), providing such a cylinder between

$$\partial(\mathcal{M}(a,b) \boxtimes \mathcal{M}(p,q)) \text{ and } \partial(\mathcal{M}(p,q) \boxtimes \mathcal{M}(a,b)).$$

The interior of $\mathcal{M}(ap, bq)$ can be used to fill this boundary-cylinder to a cylinder of $\mathcal{M}(a,b) \times \mathcal{M}(p,q)$. This is permissible by the existence of collar neighborhoods in flow categories (see [Lau00] and [AB95]). □

5.3 A flow category associated to a matched diagram

In this subchapter, we follow the construction of a flow category $\mathscr{L}(D_{\mathbf{r}})$ associated to a matched diagram $D_{\mathbf{r}}$ from [JLS15]. Recall that a matched diagram is a diagram consisting of a number of copies of elementary tangles T_n (see Figure 5.6).

Definition 5.3.1 For an integer $r \in \mathbb{Z} \setminus \{0\}$, an *elementary tangle* T_r of index r is a 2-strand tangle of $|r|$ crossings. The sign of r determines the sign of all the

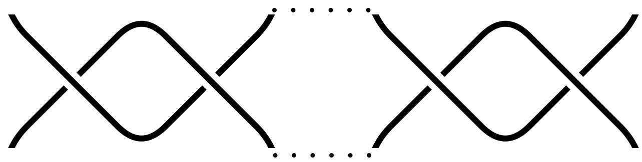


Figure 5.6: The n-crossing elementary tangle T_n, for positive n. The mirror image would be the tangle T_{-n}.

crossings in the tangle. Moreover, there is a choice of a pair of *left* and a pair of *right* endpoints of the tangle.

Definition 5.3.2 A *weighted matched configuration* is a special resolution configuration (D, w_D) (see Definition 4.2.1). In particular, a choice of the endpoints of each arc $A \in A(D)$ is given as the *L-endpoint* and *R-endpoint*, so that $\partial A = \{p_L(A), p_R(A)\}$. Further, each arc A carries a *weighting* which is a pair of integers $w_D(A) = (r, s)$ such that either $0 < s \leq r$ or $r \leq s < 0$.

Definition 5.3.3 A *labelled weighted matched configuration* is a triple (D, w_D, x) such that (D, x) is a labelled resolution configuration and (D, w_D) is a weighted matched configuration.

Definition 5.3.4 Let (D, w_D, x) and (E, w_E, y) be two labelled weighted matched configurations. Define a partial order $\prec$ on such objects so that $(E, w_E, y) \prec (D, w_D, x)$ whenever one of the following are satisfied:

1. D and E are the same as resolution configurations (that is, they have the same circles and arcs), but the weightings differ on a single arc A where $w_E(A) = (r, s)$ and $w_D(A) = (r, s+1)$. In this case, the circles that are disjoint from ∂A are labelled identically by x and y. The circle(s) that intersect ∂A are labelled as follows:

   (a) If the endpoints of A lie on the same circle, Z, then $y(Z) = x_+$ and $x(Z) = x_-$.

   (b) If the endpoints of A lie on different circles, then denote them as Z_L and Z_R where $p_L(A) \in Z_L$ and $p_R(A) \in Z_R$. Then either $y(Z_L) = y(Z_R) =$

x_+ and $\{x(z_L), x(Z_R)\} = \{x_+, x_-\}$; or $\{y(Z_L), y(Z_R)\} = \{x_+, x_-\}$ and $x(Z_L) = x(Z_R) = x_-$.

2. (D, x) is obtained from (E, y) by performing surgery along an arc A where both labellings x and y (respectively, the weights w_D,w_E) induce identical labellings (respectively, an identical weight w) on $D \cap E = E \cap D$. In this case, the weighting $w(A) = (r, s)$ is restricted to $s = -1$. Alternatively, (E, y) may be obtained from (D, y) by performing surgery with both labellings x and y (respectively, the weights w) inducing identical labellings on $D \cap E = E \cap D$. In this case, the weighting $w(A) = (r, s)$ is restricted to $s = 1$. Since both of these situations are traits of Definition 4.2.11, one of the following situations must occur:

    (a) $Z(E \setminus D)$ contains exactly one circle Z_1, and $Z(D \setminus E)$ contains exactly two circles Z_2 and Z_3. Then either $y(Z_1) = x(Z_2) = x(Z_3) = x_-$; or $y(Z_1) = x_+$ and $\{x(Z_2), x(Z_3)\} = \{x_+, x_-\}$.

    (b) $Z(E \setminus D)$ contains exactly two circles Z_1 and Z_2, and $Z(D \setminus E)$ contains exactly one circle Z_3. Then either $y(Z_1) = y(Z_2) = x(Z_3) = x_+$; or $\{y(Z_1), y(Z_2)\} = \{x_-, x_+\}$ and $x(Z_3) = x_-$.

The partial order $\prec$ is defined for all labelled weighted matched configurations as the transitive closure of this relation between two labelled weighted matched configurations.

If $(E, w_E, y) \prec (D, w_D, x)$ and there is a chain of precisely i such relations between pairs of labelled weighted matched configurations connecting (E, w_E, y) to (D, w_D, x), then write $(E, w_E, y) \prec_i (D, w_D, x)$.

The following definition is an adaptation of Definition 2.4.1.

Definition 5.3.5 A *morphism* $\mathcal{F} : \mathscr{C}_1 \to \mathscr{C}_2$ of two flow categories is a grading-preserving functor such that

$$\mathcal{F} : \mathrm{Hom}(x, y) \to \mathrm{Hom}(\mathcal{F}(x), \mathcal{F}(y))$$

is a local diffeomorphism. A morphism $\mathcal{F}$ is called a *disc cover* if the morphism spaces of $\mathscr{C}_2$ are all homeomorphic to a disc. In this case, $\mathscr{C}_1$ is said to be a *disc cover* of $\mathscr{C}_2$.

A matched link diagram D gives rise to a flow category $\mathscr{L}(D_{\mathbf{r}})$ which will be a disc cover of the $\mathbf{r}$-sock flow category $\mathcal{S}_{\mathbf{r}}$, where $\mathbf{r} = (r_1, r_2, \ldots, r_m)$ are the indices of the m elementary tangles that make up D. Following the influence of Gaussian elimination, the idea is to take a single 0-smoothing or 1-smoothing for an entire elementary tangle to form weighted matched configurations. The rest of this subchapter is devoted to the construction of $\mathscr{L}(D_{\mathbf{r}})$. As a result of the following proposition from [JLS15], it may only be necessary to describe the objects of $\mathscr{L}(D_{\mathbf{r}})$ along with the moduli spaces of dimension less than 2.

Proposition 5.3.6 Consider a finite $\mathbb{Z}$-graded set $\widetilde{\mathrm{Ob}}_1$ and a flow category $\widetilde{\mathscr{C}}_{1,i}$ with $\mathrm{Ob}(\widetilde{\mathscr{C}}_{1,i})$ given by the elements of $\widetilde{\mathrm{Ob}}_1$ with gradings i, $i+1$, and $i+2$. Let $\widetilde{\mathscr{C}}_1$ be the category (which is not a flow category) with $\mathrm{Ob}(\widetilde{\mathscr{C}}_1) = \widetilde{\mathrm{Ob}}_1$ and with the smallest morphism sets such that each $\widetilde{\mathscr{C}}_{1,i}$ is a full subcategory (that is to say that the morphisms of $\widetilde{\mathscr{C}}_1$ are determined only by the 0- and 1-dimensional spaces at the minimum).

Further, suppose $\mathscr{C}_2$ is a flow category whose morphism spaces are all discs. Lastly, suppose that $\widetilde{\mathcal{F}} : \widetilde{\mathscr{C}}_1 \to \mathscr{C}_2$ is a grading-preserving functor whose restriction to each $\widetilde{\mathscr{C}}_{1,i}$ is a morphism of flow categories. In this case

$$\partial \mathrm{Hom}_{\widetilde{\mathscr{C}}_1}(b,a) = \bigcup_{c:|a|<|c|<|b|} \mathrm{Hom}_{\widetilde{\mathscr{C}}_1}(c,a) \times \mathrm{Hom}_{\widetilde{\mathscr{C}}_1}(b,c)$$

is topologically a disjoint union of circles (since it only maps to the boundary of the 2-disc morphism spaces of $\mathscr{C}_2$). If the induced covering maps

$$\widetilde{\mathcal{F}}_{b,a} \colon \mathrm{Hom}_{\widetilde{\mathscr{C}}_1}(b,a) \to \mathrm{Hom}_{\mathscr{C}_2}(\widetilde{\mathcal{F}}(b), \widetilde{\mathcal{F}}(a))$$

are trivial (on each component of $\mathrm{Hom}_{\widetilde{\mathscr{C}}_1}(b,a)$), then there is a unique flow category $\mathscr{C}_1$ and a disc cover $\mathcal{F} : \mathscr{C}_1 \to \mathscr{C}_2$ satisfying:

1. $\mathrm{Ob}(\mathscr{C}_1) = \widetilde{\mathrm{Ob}}_1$.

2. The flow category $\mathscr{C}_1$, restricted to objects of grading i, $i+1$ and $i+2$, gives the flow category $\widetilde{\mathscr{C}}_{1,i}$.

3. $\widetilde{\mathscr{C}_1}$ is a subcategory of $\mathscr{C}_1$.

4. $\widetilde{\mathcal{F}}$ is the restriction of the functor $\mathcal{F}$.

Proof: The proof determines $\mathscr{C}_1$ and $\mathcal{F}$ inductively. On the level of objects and 0- and 1-dimensional moduli spaces, the conditions are vacuously satisfied. So assume that $a, b \in \mathrm{Ob}(\mathscr{C}_1) = \widetilde{\mathrm{Ob}}_1$ with $|b| - |a| = 3$. In this case, whenever $\mathcal{M}(b,a) \neq \emptyset$, $\partial\mathcal{M}(b,a)$ is also determined as a disjoint union of circles. Since $\mathcal{F}$ is required to be a disc cover, $\mathcal{M}(b,a)$ must be obtained by filling each circle with a disc. As a result of doing so, a map $\mathcal{F}$ can be found which is defined on each 2-dimensional moduli space and this follows from the trivial covering map hypothesis on $\widetilde{\mathcal{F}}_{b,a}$.

Now assume that $a, b \in \mathrm{Ob}(\mathscr{C}_1) = \widetilde{\mathrm{Ob}}_1$ with $|b| - |a| = 4$. Then the map

$$\mathcal{F}_{b,a}|_{\partial\mathcal{M}(b,a)} : \partial\mathcal{M}(b,a) \to \partial\mathcal{M}(\mathcal{F}(b), \mathcal{F}(a))$$

is determined (as long as $\mathcal{M}(b,a)$ and $\mathcal{F}_{b,a}$ are defined consistently) and is a local covering map of a 2-sphere, hence a trivial covering. Therefore, the moduli space $\mathcal{M}(b,a)$ must be obtained by filling each sphere with a 3-ball that it bounds. This inductive step continues for each relative index $|b| - |a| = k$ until all $\mathcal{M}(b,a)$ are determined for all objects $a, b \in \mathrm{Ob}(\mathscr{C}_1)$. □

Therefore, in order to describe $\mathscr{L}(D_{\mathbf{r}})$, it suffices to describe the objects and 0- and 1-dimensional moduli spaces and then Proposition 5.3.6 prescribes a way to simultaneously construct a disc cover $\mathcal{F} : \mathscr{L}(D_{\mathbf{r}}) \to \mathcal{S}_{\mathbf{r}}$ providing the 1-dimensional trivial cover condition is satisfied.

Definition 5.3.7 Let $D_{\mathbf{r}}$ be a matched diagram with m tangle summands. At each of the tangles, define a *vertical* (respectively, *horizontal*) smoothing as the diagram obtained by replacing the tangle with a non-crossing 2-strand diagram that connects the four endpoints in two pairs vertically (respectively, horizontally).

Definition 5.3.8 Let $D_{\mathbf{r}}$ be a matched diagram with m tangle summands. By adding an arc that connects the two strands of every vertical smoothing of the i^{th} tangle for $i = 1, \ldots, m$, one can decorate that arc with (r_i, s_i) for a permissible choice of s_i according to Definition 5.3.2. In this way, a weighted matched configuration (D, w_D) can be obtained from any matched diagram $D_{\mathbf{r}}$. In this case, (D, w_D) is called the *weighted matched configuration associated to* $D_{\mathbf{r}}$.

The *flow category associated to a matched diagram* $D_{\mathbf{r}}$ is the flow category $\mathscr{L}(D_{\mathbf{r}})$ defined as follows. The objects of $\mathscr{L}(D_{\mathbf{r}})$ are all possible labelled weighted matched configurations associated to $D_{\mathbf{r}}$. The morphisms of $\mathscr{L}(D_{\mathbf{r}})$ are described below using a cover map and Proposition 5.3.6.

Recall that the objects of the $\mathbf{r}$-sock flow category $\mathcal{S}_{\mathbf{r}}$ (Definition 5.2.3) are of the form $(s_1, \ldots, s_m)$ where $|s_i| \leq |r_i|$ and each s_i is 0 or agrees in sign with r_i for $i = 1, \ldots, m$. The disc cover $\mathcal{F} : \mathscr{L}(D_{\mathbf{r}}) \to \mathcal{S}_{\mathbf{r}}$ is defined on the objects, 0-dimensional and 1-dimensional moduli spaces, respectively, in the following series of definitions. There will be a non-identity morphism between (E, w_E, y) and (D, w_D, x) if and only if $(E, w_E, y) \prec (D, w_D, x)$.

Definition 5.3.9 Let $(D, w_D, x) \in \mathrm{Ob}(\mathscr{L}(D_{\mathbf{r}}))$ and $(s_1, \ldots, s_m)$ be as in Definition 5.3.2. Assuming that $s_i = 0$ corresponds to the horizontal smoothing being chosen at the i^{th} tangle. Then $\mathcal{F} : \mathscr{L}(D_{\mathbf{r}}) \to \mathcal{S}_{\mathbf{r}}$ is defined on objects to be the composition of

$$(D, w_D, x) \mapsto (s_1, \ldots, s_m)$$

with the forgetful map that ignores labellings.

Definition 5.3.10 Suppose $(E, w_E, y) \prec_1 (D, w_D, x)$ for two objects of $\mathscr{L}(D_{\mathbf{r}})$. The 0-dimensional moduli space $\mathcal{M}((E, w_E, y), (D, w_D, x))$, which is necessarily a disjoint union of points, consists of

1. Two points if $(E, w_E, y) \prec_1 (D, w_D, x)$ as in case (1a) of Definition 5.3.4.

2. A single point if $(E, w_E, y) \prec_1 (D, w_D, x)$ as in each of the other cases (1b),(2a) or (2b) of Definition 5.3.4.

The morphism of flow categories $\mathcal{F}$ is defined on the 0-dimensional moduli spaces as follows. Let $p = (i_1, \ldots, i_\alpha, \ldots, i_m)$ and $q = (i_1, \ldots, i_\alpha + 1, \ldots, i_m)$ be two objects of $\mathcal{S}_{\mathbf{r}}$. If $0 \in \{i_\alpha, i_\alpha + 1\}$, then the one-point moduli space $\mathcal{M}_{\mathcal{S}_{\mathbf{r}}}(q, p)$ will be denoted by P_0 (to avoiding confusion of using $L = R$). If $0 \notin \{i_\alpha, i_\alpha + 1\}$, then the moduli space $\mathcal{M}_{\mathcal{S}_{\mathbf{r}}}(q, p)$ consists of two points. As described in Section 5.2, there is an identification of moduli spaces

$$\mathcal{M}_{\mathcal{S}_{\mathbf{r}}}(q, p) = \mathcal{M}_{\mathcal{S}_{r_1}}(i_1, i_1) \times \cdots \times \mathcal{M}_{\mathcal{S}_{r_\alpha}}(i_\alpha + 1, i_\alpha) \times \cdots \times \mathcal{M}_{\mathcal{S}_{r_m}}(i_m, i_m)$$

where the two points of $\mathcal{M}_{\mathcal{S}_{r_\alpha}}(i_\alpha + 1, i_\alpha)$ were described by their preimages in $\mathscr{C}_{\mathbb{R}\mathbf{P}^{r_\alpha+1}}$. These points will be denoted by L and R.

To determine the morphism $\mathcal{F}$ completely on the 0-dimensional moduli spaces, it is left to describe which points of $\mathcal{M}((E, w_E, y), (D, w_D, x))$ get sent to L and which get sent to R.

Definition 5.3.11 Suppose $(E, w_E, y) \prec_1 (D, w_D, x)$ for two objects of $\mathscr{L}(D_{\mathbf{r}})$ corresponding to one of the four cases in Definition 5.3.4. In each of the corresponding cases, the points described in Definition 5.3.10 are mapped under $\mathcal{F}$ as follows:

1. (a) Both points are mapped projectively to $\{L, R\}$.

   (b) One circle in $\{Z_L, Z_R\}$ has a decoration that changes from x_+ to x_-. If it is Z_L, the point is sent to L. If it is Z_R, the point is sent to R.

2. (a) The single point is sent to P_0.

   (b) The single point is sent to P_0.

The 1-dimensional moduli spaces of $\mathscr{L}(D_{\mathbf{r}})$ are defined as follows, along with how they are mapped under $\mathcal{F}$.

Definition 5.3.12 Suppose $(E, w_E, y) \prec_2 (D, w_D, x)$ for two objects of $\mathscr{L}(D_{\mathbf{r}})$. Then one of two cases occurs. Either the two objects are related by a double surgery in the sense of a ladybug configuration (Definition 4.3.8), or they are not. In the ladybug case, there are exactly four points and a choice of pairs needs to be made. This choice is the same as in Subchapter 4.3.2 and in particular, $\mathcal{M}((E, w_E, y), (D, w_D, x))$

consists of two intervals with the chosen pairs as boundary points. This, in turn, determines the morphism $\mathcal{F}$ on $\mathcal{M}((E, w_E, y), (D, w_D, x))$.

For the non-ladybug case, there is a unique choice (that is essentially forced) for the moduli space $\mathcal{M}((E, w_E, y), (D, w_D, x))$ and a morphism $\mathcal{F}|_{\mathcal{M}((E,w_E,y),(D,w_D,x))}$ which is consistent with the existence of a disc-cover $\mathcal{F}$ that maps the 0-dimensional moduli spaces as prescribed in Definition 5.3.11.

5.4 Framing the sock flow category

Since the matched flow category $\mathscr{L}(D_{\mathbf{r}})$ is described as a cover of the sock flow category $\mathcal{S}_{\mathbf{r}}$, a framing of $\mathcal{S}_{\mathbf{r}}$ will induce a framing of $\mathscr{L}(D_{\mathbf{r}})$. As with the cube flow category $\mathcal{C}_C(n)$, which the Khovanov flow category $\mathscr{C}_{Kh}(L)$ covers (see Subchapter 4.3), there is a natural question which asks whether different framings of $\mathcal{S}_{\mathbf{r}}$ give rise to stable homotopy types that are equivalent. This section outlines the generalised argument of [LS14a, Subsection 4.2] which is given in [JLS15]. Since not every moduli space of $\mathcal{S}_{\mathbf{r}}$ is in the image of the covering map $\mathcal{F} : \mathscr{L}(D_{\mathbf{r}}) \to \mathcal{S}_{\mathbf{r}}$, a complete framing on $\mathscr{L}(D_{\mathbf{r}})$ may be given from a partial framing on $\mathcal{S}_{\mathbf{r}}$. Moreover, we shall use properties of a general family of flow categories that the sock flow category lives in; a disc flow category.

Definition 5.4.1 A *disc flow category* $\mathscr{D}$ is a flow category whose morphism spaces are all homeomorphic to a disjoint union of discs.

Associated to a disc flow category $\mathscr{D}$ is a CW complex $\mathcal{C}(\mathscr{D})$ that is described in the following definition. Note that this is not the CW complex $|\mathscr{D}|$ from the Cohen-Jones-Segal construction, since to define which would require a framed embedding.

Definition 5.4.2 Let $\mathscr{D}$ be a disc flow category. A CW complex $\mathcal{C}(\mathscr{D})$ is associated to $\mathscr{D}$ which is defined as follows. $\mathcal{C}(\mathscr{D})$ has a 0-cell for each object of $\mathscr{D}$, and an $(r+1)$-cell $C(M)$ for every component M of the r-dimensional moduli spaces of $\mathscr{D}$. The cellular structure of $\mathcal{C}(\mathscr{D})$ is built up inductively. Assume that the i-skeleton $\mathcal{C}(\mathscr{D})^{(i)}$ has been constructed. Then for $a, b \in \mathrm{Ob}(\mathscr{D})$ with $|a| = r$ and $|a| - |b| - 1 = i$, let $M \subset \mathcal{M}(b, a)$ be a component of the moduli space (for all such pairs of objects where this is non-empty). Since M is a manifold with corners which is homeomorphic to $\mathbb{D}^i$, the cell defined as $C(M) = M \times [r, r+i+1]$ is an $(i+1)$-cell. It is attached via

$$\chi_M : \partial C(M) \to \mathcal{C}(\mathscr{D})$$

with the property that $\chi_M(M \times \{r\}) = \{a\}$ and $\chi_M(M \times \{r+i+1\}) = \{b\}$. Boundary components of $\partial \mathcal{M}(b, a)$ are given by products $\mathcal{M}(c, a) \times \mathcal{M}(b, c)$ for objects c of $\mathscr{D}$ with index $r < |c| = s < r + i + 1$. So if M_{ca} is a component of $\mathcal{M}(c, a)$ and M_{bc} is a component of $\mathcal{M}(b, c)$ such that $M_{ca} \times M_{bc} \subset \partial M$, then define

$$\chi_M|_{M_{ca} \times M_{bc} \times [r,s]} : M_{ca} \times M_{bc} \times [r, s] \twoheadrightarrow M_{ca} \times [r, s] = C(M_{ca}) \text{ and}$$
$$\chi_M|_{M_{ca} \times M_{bc} \times [s,r+i+1]} : M_{ca} \times M_{bc} \times [s, r + i + 1] \twoheadrightarrow M_{bc} \times [s, r + i + 1] = C(M_{bc})$$

as the projection maps.

Definition 5.4.3 A flow category $\widetilde{\mathscr{F}}$ is a *partial flow category* of the flow category $\mathscr{F}$ if $\widetilde{\mathscr{F}}$ is a wide subcategory of $\mathscr{F}$ and for all objects $a, b \in \mathrm{Ob}(\mathscr{F}) = \mathrm{Ob}(\widetilde{\mathscr{F}})$ either one of the following conditions holds:

1. If M is a component of $\mathcal{M}_{\mathscr{F}}(b, a)$ then $M \subset \mathcal{M}_{\widetilde{\mathscr{F}}}(b, a)$.
2. If M is a component of $\mathcal{M}_{\mathscr{F}}(b, a)$, then no interior point of M is in $\mathcal{M}_{\widetilde{\mathscr{F}}}(b, a)$.

Note that if $\mathscr{F}$ is a disc flow category and $\widetilde{\mathscr{F}}$ is a partial flow category of $\mathscr{F}$, then $\widetilde{\mathscr{F}}$ determines a subcomplex $\mathcal{C}(\widetilde{\mathscr{F}})$ of $\mathcal{C}(\mathscr{F})$ which consists of all cells of $\mathcal{C}(\mathscr{F})$ that correspond to components of moduli spaces that have full dimension.

Suppose that $\widetilde{\mathscr{F}}$ is a partial flow category of $\mathscr{F}$, where $\mathscr{F}$ comes with an embedding, and the 0-dimensional moduli spaces in $\widetilde{\mathscr{F}}$ are framed so that the framing can be extended coherently to the 1-dimensional moduli space components of $\widetilde{\mathscr{F}}$ that are of full-dimension. This determines a sign assignment s of $\widetilde{\mathscr{F}}$ as a cochain in $C^1(\mathcal{C}(\widetilde{\mathscr{F}}), \mathbb{Z}/2)$ in the same way as the sign assignment for the cube flow category in Definition 4.1.5. Moreover, consider the question of whether one can extend this framing of 0-dimensional moduli spaces of $\widetilde{\mathscr{F}}$ to a coherent framing (see Definition 2.2.7) of all full-dimensional moduli space components of $\widetilde{\mathscr{F}}$. As with the cube flow category, the answer can be reduced to an argument in obstruction theory. Again, since the main focus of this thesis is to highlight an application of handle-cancellation in flow categories, we shall omit the full details of this obstruction argument and simply provide an overview. The key point will be that there exists a particular framing for the sock flow category, and that the resultant homotopy type does not depend on this choice of framing.

The argument is set-up as follows (c.f. Subchapter 4.1.1 throughout). For all full-dimensional components $N \subset \mathcal{M}_{\widetilde{\mathscr{F}}}(b,a)$, fix a choice of orientation which orients all 0-dimensional moduli spaces positively. The interior $\mathrm{int}(C(N))$ can be identified with $\mathrm{int}(N) \times \mathbb{R}$, and thus can be given the corresponding product orientation. For $a < c <_1 b$, consider each full-dimensional component $M \subset \mathcal{M}_{\widetilde{\mathscr{F}}}(c,a)$ as

$$\coprod_i M \times \{p_i\} \cong M \times \mathcal{M}_{\widetilde{\mathscr{F}}}(b,c) \subset \partial N.$$

For such triples, associate $(M, N, p_i) \mapsto t(M, N, p_i) \in \{0,1\}$ where $t(M,N,p_i) = 0$ if the chosen orientation of M agrees with the orientation induced on $M \cong M \times \{p_i\}$ as the boundary component of N, and $t(M, N, p_i) = 1$ if the orientations differ. Similarly, for $a <_1 c < b$ each full-dimensional component $M \subset \mathcal{M}_{\widetilde{\mathscr{F}}}(b,c)$ can be considered as

$$\coprod_i \{q_i\} \times M \cong \mathcal{M}_{\widetilde{\mathscr{F}}}(c,a) \times M \subset \partial N.$$

Then associate $(M, N, q_i) \mapsto t(M, N, q_i) \in \{0,1\}$ to such triples in the same way. Then, for a full-dimensional component $M \subset \mathcal{M}_{\widetilde{\mathscr{F}}}(b,a)$ the differential in the cochain complex $C^*(\mathcal{C}(\widetilde{\mathscr{F}}), \mathbb{Z})$ can be written as

$$\delta(C(M)') = \sum_{\substack{a<b<_1 c \\ N\subset \mathcal{M}_{\widetilde{\mathscr{F}}}(c,a) \\ p_i \in \mathcal{M}_{\widetilde{\mathscr{F}}}(c,a) \\ M\times\{p_i\}\subset \partial N}} (-1)^{t(M,N,p_i)} C(N)' + \sum_{\substack{c<_1 a <_b \\ q_i\in \mathcal{M}_{\widetilde{\mathscr{F}}}(a,c) \\ N\subset \mathcal{M}_{\widetilde{\mathscr{F}}}(b,c) \\ \{q_i\}\times M \subset \partial N}} (-1)^{t(M,N,q_i)} C(N)'$$

where $C(M)'$ corresponds to the element whose dual is $C(M)$ (i.e. the cochain which assigns 1 to $C(M)$ and 0 otherwise). By induction, assuming that all full-dimensional moduli space components of $\widetilde{\mathscr{F}}$ of dimension less than k have been coherently framed, [JLS15] proceeds in a similar manner to [LS14a] by producing a cochain $\sigma \in C^{k+1}(\mathcal{C}(\widetilde{\mathscr{F}}), \pi_{k-1}(\mathbf{O}))$. In particular, on each full-dimensional component $M \subset \mathcal{M}_{\widetilde{\mathscr{F}}}(b,a)$ of dimension k, ∂M is a $(k-1)$-sphere S^{k-1} which has a product framing from lower dimensional moduli spaces. Comparing this framing with a null-cobordant framing produces an element $\pi_{k-1}(\mathbf{O})$, and by definition $\sigma = 0$ if and only if the coherent framing of $\widetilde{\mathscr{F}}$ can be extended to all full-dimensional moduli space components of $\widetilde{\mathscr{F}}$ that are of dimension k. The important results from [JLS15] are listed below.

Proposition 5.4.4 The obstruction class σ is a cocycle.

An analogous modification argument (as in [LS14a, Lemma 4.11]) provides for a cochain $\tau \in C^k(\mathcal{C}(\widetilde{\mathscr{F}}), \pi_{k-1}(\mathbf{O}))$ a modification of the framings for the $(k-1)$-dimensional moduli space components. This produces a new cocycle $\sigma' = \sigma + \delta\tau$ along with the following proposition.

Proposition 5.4.5 If the cocycle σ satisfies

$$[\sigma] \in H^{k+1}(\mathcal{C}(\widetilde{\mathscr{F}}), \pi_{k-1}(\mathbf{O})) = 0$$

then the coherent framing of the moduli space components of $\widetilde{\mathscr{F}}$ can be extended to the full-dimensional k-dimensional moduli space components, where one may have to modify the framing on the $(k-1)$-dimensional components as described above.

In a similar fashion to [LS14a], the inductive argument in [JLS15] takes a 1-parameter family $\{\imath_t\}_{t\in[0,1]}$ of embeddings of $\mathscr{F}$ that connects two framed embeddings $\imath_0$ and $\imath_1$ of $\widetilde{\mathscr{F}}$. In particular, if it is assumed that $\imath_0$ and $\imath_1$ are both supplied with coherent framings for the full-dimensional moduli spaces of $\widetilde{\mathscr{F}}$ that give identical sign assignments, then the framings can be extended to frame each 0-dimensional moduli space $\imath_{b,a}(t)\big(\mathcal{M}_{\widetilde{\mathscr{F}}}(b,a)\big)$ for $0 < t < 1$. So taking the inductive hypothesis to be that the framings of the full-dimensional i-dimensional moduli spaces $\imath_{b,a}(0)\big(\mathcal{M}_{\widetilde{\mathscr{F}}}(b,a)\big)$ and $\imath_{b,a}(1)\big(\mathcal{M}_{\widetilde{\mathscr{F}}}(b,a)\big)$ can be extended to the corresponding moduli spaces $\imath_{b,a}(t)\big(\mathcal{M}_{\widetilde{\mathscr{F}}}(b,a)\big)$ for $0 < t < 1$ and $i < k$ for some $k \geq 1$. The following sphere is defined using these moduli spaces (c.f. the proof of Lemma 4.1.11) whose inherited framing is compared with a null-homotopic framing to give the final obstruction class $\tilde{\sigma}$ of the inductive procedure.

Definition 5.4.6 For each k-dimensional full-dimensional moduli space component $M \subset \mathcal{M}_{\widetilde{\mathscr{F}}}(b,a)$, define the *obstruction sphere of* M as the k-sphere

$$S^k_M = \imath_{b,a}(0)(M) \cup \imath_{b,a}(1)(M) \cup \bigcup_{i \in (0,1)} \imath_{b,a}(t)(\partial M).$$

The aforementioned obstruction class $\tilde{\sigma}$ is shown, through a similar argument to its predecessor σ, to be trivial in demand of the following proposition.

Proposition 5.4.7 The obstruction class $\tilde{\sigma}$ is a cocycle and if

$$[\tilde{\sigma}] \in H^{k+1}(\mathcal{C}(\widetilde{\mathscr{F}}), \pi_k(\mathbf{O})) = 0$$

then the coherent framing of the full-dimensional k-dimensional moduli space components of $\imath_{b,a}(0)\big(\mathcal{M}_{\widetilde{\mathscr{F}}}(b,a)\big)$ and $\imath_{b,a}(1)\big(\mathcal{M}_{\widetilde{\mathscr{F}}}(b,a)\big)$ can be extended to the family of framed embedded moduli spaces $\imath_{b,a}(t)\big(\mathcal{M}_{\widetilde{\mathscr{F}}}(b,a)\big)$, where one may have to make modifications of the framing on the $(k-1)$-dimensional components similar in fashion to Proposition 5.4.5.

With this overview of the obstruction theory argument, recall that the goal of this section is to provide a framing of the matched flow category $\mathscr{L}(D_{\mathbf{r}})$ associated to the matched diagram $D_{\mathbf{r}}$ through the disc cover $\mathcal{F} : \mathscr{L}(D_{\mathbf{r}}) \to \mathcal{S}_{\mathbf{r}}$. All of the above work, which deals with disc flow categories, becomes useful if the covering morphism $\mathcal{F}$ factors through a partial flow category; which it does. The partial flow category we are interesred in is denoted $\widetilde{\mathcal{S}}_{\mathbf{r}}$ and defined below.

Definition 5.4.8 Let $\mathbf{r} = (r_1, \ldots, r_k)$. To define the partial flow category $\widetilde{\mathcal{S}}_{\mathbf{r}}$ of $\mathcal{S}_{\mathbf{r}}$, it is sufficient to describe which full-dimensional moduli spaces of the latter are included in the former. All 0-dimensional moduli spaces of $\mathcal{S}_{\mathbf{r}}$ are included in $\widetilde{\mathcal{S}}_{\mathbf{r}}$. Recall that these are denoted as R and L. If $a = (i_1, \ldots, i_\alpha, \ldots, i_k)$ and $b = (i_1, \ldots, i_\alpha + 2, \ldots, i_k)$ are two objects of $\mathcal{S}_{\mathbf{r}}$, then $\mathcal{M}(b,a)$ is 1-dimensional. In the case that $0 \in \{i_\alpha, i_\alpha + 2\}$, this 1-manifold is a single interval. The interior of these intervals are not included in $\widetilde{\mathcal{S}}_{\mathbf{r}}$. All other 1-dimensional moduli spaces are included in $\widetilde{\mathcal{S}}_{\mathbf{r}}$. In particular, when $0 \notin \{i_\alpha, i_\alpha + 2\}$, the 1-manifold is a disjoint union of two intervals; one has boundary $\{LL, RR\}$ and the other has boundary $\{LR, RL\}$.

The partial flow category $\widetilde{\mathcal{S}}_{\mathbf{r}}$ is then defined to be the largest partial flow category of $\mathcal{S}_{\mathbf{r}}$ satisfying these conditions.

Proposition 5.4.9 The covering morphism $\mathcal{F} : \mathscr{L}(D_{\mathbf{r}}) \to \mathcal{S}_{\mathbf{r}}$ factors through the partial flow category $\widetilde{\mathcal{S}}_{\mathbf{r}}$.

Proof: Since the moduli spaces in the image of $\mathcal{F}$ form a partial flow category of $\mathcal{S}_{\mathbf{r}}$, this result follows if it can be shown that the intervals that are not included in the definition of $\widetilde{\mathcal{S}}_{\mathbf{r}}$ in Definition 5.4.8 are not in this image. However, if a, b, c are three objects of $\mathscr{L}_{\mathbf{r}}$ that are mapped to

$$\begin{aligned}
&\mathcal{F}(a) = (i_1, \ldots, i_\alpha, \ldots, i_k),\\
&\mathcal{F}(b) = (i_1, \ldots, i_\alpha + 1, \ldots, i_k) \text{ and}\\
&\mathcal{F}(c) = (i_1, \ldots, i_\alpha + 2, \ldots, i_k)
\end{aligned}$$

such that $0 \notin \{i_\alpha, i_\alpha + 2\}$, then it cannot be possible that $L \in \mathcal{M}(b, a)$ and $L \in \mathcal{M}(c, b)$. The corresponding statement in the Khovanov chain complex is that multiplying by x^2 in the Frobenius algebra $\mathbb{Z}[x]/x^2$ is the trivial map. $\square$

Using this, [JLS15] applies the obstruction theory arguments to the partial flow category $\widetilde{\mathcal{S}}_{\mathbf{r}}$ and its cover $\mathscr{L}(D_{\mathbf{r}})$. A particularly useful result is the following.

Proposition 5.4.10 The CW complex $\mathcal{C}(\widetilde{\mathcal{S}}_{\mathbf{r}})$ (which is a subcomplex of $\mathcal{C}(\mathcal{S}_{\mathbf{r}})$) is contractible.

To prove this proposition, the following result concerning particular Morse flow categories that are disc flow categories is useful.

Lemma 5.4.11 Let $f : M \to \mathbb{R}$ and $g : N \to \mathbb{R}$ be two Morse functions on closed smooth manifolds M and N with Morse-Smale gradients v_f and v_g, respectively. Consider the Morse flow category $\mathscr{C}_F$ of the Morse function $F : M \times N \to \mathbb{R}$ defined by $F(x, y) = f(x) + g(y)$, with the Morse-Smale gradient $v_F = (v_f, v_g)$. If $\mathscr{C}_f$ and $\mathscr{C}_g$ are disc flow categories, then it follows (c.f. Lemma 5.2.5) that $\mathscr{C}_F$ is too. In this case,

$$\mathcal{C}(\mathscr{C}_F) = \mathcal{C}(\mathscr{C}_f) \times \mathcal{C}(\mathscr{C}_g).$$

Proof: As in the set-up of the proof of Lemma 5.2.5, consider $a, b \in \mathrm{Crit}(f)$ and $p, q \in \mathrm{Crit}(g)$ with $\mathrm{ind}(a) > \mathrm{ind}(b)$ and $\mathrm{ind}(p) > \mathrm{ind}(q)$. Then the proof is by induction on $k = \mathrm{ind}(a,p) - \mathrm{ind}(b,q) \geq 2$ and $\mathcal{M}_{\mathscr{C}_F}((a,p),(p,q)) \cong \mathcal{M}_{\mathscr{C}_f}(a,b) \times \mathcal{M}_{\mathscr{C}_g}(p,q) \times [0,1]$. When $k = 2$, both moduli spaces $\mathcal{M}_{\mathscr{C}_f}(a,b)$ and $\mathcal{M}_{\mathscr{C}_g}(p,q)$ are disjoint unions of points. Moreover, if $\mathcal{M}_{\mathscr{C}_f}(a,b) = \sqcup \alpha_i$ (respectively, $\mathcal{M}_{\mathscr{C}_g}(p,q) = \sqcup \beta_j$), then $C(\alpha_i) = \{\alpha_i\} \times [|b|, |a|]$ (respectively, $C(\beta_j) = \{\beta_j\} \times [|q|, |p|]$) is an interval which is attached to the 0-cells corresponding to a and b (respectively, p and q). The corresponding moduli space in $\mathscr{C}_F$ is an interval given by $I = \{\alpha_i\} \times \{\beta_j\} \times [0,1]$ with $C(I) = I \times [|(a,p)|, |(b,q)|]$. This clearly gives the product of the two intervals above attached to the corresponding intervals whose boundaries are the points $\{\alpha_i\} \times \{\beta_j\}$. This deals with the base case.

For the inductive argument, it is helpful to highlight an observation about any general disc flow category $\mathscr{C}$. If $M \subset \mathcal{M}_{\mathscr{C}}(x,y)$ is a component with $|x| = T$ and $|y| = 0$, then the attaching map χ_M factors through the map

$$\phi_M : \partial(C(M)) = \partial(M \times [0,1]) \to S^{T-1}_M \cong S^{T-1}$$

which is defined as a composition of the following maps. First, collapse each boundary disc $M \times \{0\}$ and $M \times \{1\}$ to a point. Then by considering each sequence of components

$$N_1 \subset \mathcal{M}_{\mathscr{C}}(z_1, z_0), \ldots, N_k \subset \mathcal{M}_{\mathscr{C}}(z_k, z_{k-1})$$

which satisfy $z_0 = y$, $z_k = x$ and $0 < |z_1| < \cdots < |c_{k-1}| < T$ with $N = N_1 \times \cdots \times N_k \subset \partial M$, collapse $N \times [0,T]$ to

$$(N_1 \times [0, |z_1|) \cup \cdots \cup (N_k \times [|c_{k-1}, T]).$$

Since each N_i is a disc, it should be clear that the overall result of this composition gives a $(T-1)$-sphere.

Since k is a relative degree, it can be assumed without loss of generality that $\mathrm{ind}(b) = \mathrm{ind}(q) = 0$. Then the argument proceeds as follows. For a component $M_{(ap,bq)} \subset \mathcal{M}_{\mathscr{C}_F}(ap, bq)$, let $M_{(a,b)} \subset \mathcal{M}_{\mathscr{C}_f}(a,b)$ and $M_{(p,q)} \subset \mathcal{M}_{\mathscr{C}_g}(p,q)$ be the corresponding components such that $M_{(ap,bq)} = M_{(a,b)} \times M_{(p,q)} \times [0,1]$. Now consider the cell $C(M_{(ap,bq)}) = M_{(ap,bq)} \times [0,k]$ of $C(\mathscr{C}_F)$. Then $\chi_{M_{(ap,bq)}}$ factors through the quotient map

$$\phi_{M_{(ap,bq)}} : \partial(C(M_{(ap,bq)})) = \partial(M_{(ap,bq)} \times [0,1]) \to S^{k-1}_{M_{(ap,bq)}} \cong S^{k-1}.$$

If $n = \mathrm{ind}(a)$ and $m = \mathrm{ind}(p)$ (recall that $\mathrm{ind}(a) + \mathrm{ind}(p) = k$), it suffices to show that

$$S^{k-1}_{M_{(ap,bq)}} = C(M_{(a,b)}) \times S^{n-1}_{M_{(p,q)}} \cup S^{m-1}_{M_{(a,b)}} \times C(M_{(p,q)})$$

since this implies that $\mathcal{C}(\mathscr{C}_F)$ is homeomorphic to $\mathcal{C}(\mathscr{C}_f) \times \mathcal{C}(\mathscr{C}_g)$. For each object c of $\mathscr{C}_f$ with the property that $M_{(a,b)} \cap \big(\mathcal{M}_{\mathscr{C}_f}(c,b) \times \mathcal{M}_{\mathscr{C}_f}(a,c)\big) \neq \emptyset$, it follows that $\mathcal{M}_{\mathscr{C}_F}(cp,cq) \cap M_{(ap,bq)} \cong M_{(p,q)}$. In light of the above, the obstruction sphere of $M_{(ap,bq)}$ is given as

$$S_{M_{(ap,bq)}} = \big(C_1(N) \cup C_2(N)\big) \cup \big(C_3(N) \cup C_4(N)\big)$$

where $C_1(N)$, $C_2(N)$, $C_3(N)$ and $C_4(N)$ are defined as the unions

$$C_1(N) = \bigcup_{\substack{c<_1 a \\ N\subset \mathcal{M}_{\mathscr{C}_F}((cp,bq)) \\ x\in \mathcal{M}_{\mathscr{C}_F}((ap,cp)) \\ N\times\{x\}\subset \partial M_{(ap,bq)}}} C(N), \qquad C_2(N) = \bigcup_{\substack{b<_1 c \\ N\subset \mathcal{M}_{\mathscr{C}_F}((ap,cq)) \\ x\in \mathcal{M}_{\mathscr{C}_F}((cq,bq)) \\ \{x\}\times N\subset \partial M_{(ap,bq)}}} C(N),$$

$$C_3(N) = \bigcup_{\substack{r<_1 p \\ N\subset \mathcal{M}_{\mathscr{C}_F}((ar,bq)) \\ x\in \mathcal{M}_{\mathscr{C}_F}((ap,ar)) \\ N\times\{x\}\subset \partial M_{(ap,bq)}}} C(N), \qquad C_4(N) = \bigcup_{\substack{q<_1 r \\ N\subset \mathcal{M}_{\mathscr{C}_F}((ap,br)) \\ x\in \mathcal{M}_{\mathscr{C}_F}((br,bq)) \\ \{x\}\times N\subset \partial M_{(ap,bq)}}} C(N).$$

The induction hypothesis then implies that $C_1(N) \cup C_2(N) = S^{m-1}_{(a,b)} \times C(M_{(p,q)})$ and $C_3(N) \cup C_4(N) = C(M_{(a,b)}) \times S^{n-1}_{M_{(p,q)}}$. Moreover, the intersection is

$$\begin{aligned}\big(C_1(N) \cup C_2(N)\big) \cap \big(C_3(N) \cup C_4(N)\big) &= \bigcup_{\substack{b\le c_2 <_{m-1} c_1 \le a \\ q \le d_2 <_{n-1} d_1 \le p \\ N\subset \mathcal{M}_{\mathscr{C}_F}((c_1d_1, c_2d_2)) \\ \left\{ x \,\middle|\, \{x\}\times N \subset \partial M_{(ap,bq)} \right\}}} C(N) \\ &= S^{m-1}_{M_{(a,b)}} \times S^{n-1}_{M_{(p,q)}}.\end{aligned}$$

□

Now since the flow category $\mathcal{S}_{\mathbf{r}}$ is constructed as a Morse flow category of products of real projective spaces, it follows from Lemma 5.4.11 that the moduli spaces of $\mathcal{S}_{(r_1,\ldots,r_k)}$ can be constructed as manifolds with corners from the moduli spaces of both $\mathcal{S}_{(r_1,\ldots,r_{k-1})}$ and $\mathcal{S}_{r_k}$ yielding

$$\mathcal{C}(\mathcal{S}_{(r_1,\ldots,r_k)}) = \prod_{i=1}^{k} \mathcal{C}(\mathcal{S}_{r_i}).$$

Moreover, since the partial flow category $\widetilde{\mathcal{S}}_{\mathbf{r}}$ of $\mathcal{S}_{\mathbf{r}}$ is the maximal such that includes interiors of only particular 1-dimensional moduli spaces of $\mathcal{S}_{\mathbf{r}}$ (Definition 5.4.8), the subcomplex $\mathcal{C}(\widetilde{\mathcal{S}}_{\mathbf{r}})$ of $\mathcal{C}(\mathcal{S}_{\mathbf{r}})$ is the maximal such that includes the corresponding 2-cells. Considering the above decomposition, these 2-cells are products of the 2-cells and 0-cells of various factors. Consequently, it is also true that

$$\mathcal{C}(\widetilde{\mathcal{S}}_{(r_1,\ldots,r_k)}) = \prod_{i=1}^{k} \mathcal{C}(\widetilde{\mathcal{S}}_{r_i}).$$

Proposition 5.4.10 then follows from the lemma below.

Lemma 5.4.12 The CW complex $\mathcal{C}(\mathcal{S}_r)$ is contractible.

Proof: Assume without loss that $r > 0$. The dual case when $r < 0$ is similar. First note that the associated complex $\mathcal{C}(\widetilde{\mathcal{S}}_r)$ is a 2-complex since it only has full-dimensional moduli spaces of dimensions 0 and 1. In particular, since $\mathrm{Ob}(\widetilde{\mathcal{S}}_r) = \{0, 1, \ldots, r\}$, there are $r + 1$ 0-cells of $\mathcal{C}(\widetilde{\mathcal{S}}_r)$. Further, there are $2r - 1$ 1-cells given by a single interval $\mathcal{M}_{\widetilde{\mathcal{S}}_r}(1, 0)$ and two intervals $\mathcal{M}_{\widetilde{\mathcal{S}}_r}(i+1, i)$ for each $i = 1, \ldots, r-1$. Finally, the 2-cells $\mathbb{D}^2$ are attached corresponding only to those intervals that are included as full-dimensional moduli space components. Each $\mathbb{D}^2_i$ for $i = 0, \ldots, r-2$ is attached as follows. $\mathbb{D}^2_0$ is attached by letting its boundary circle S^1_0 trace along the interval from 0 to 1, followed by one of the intervals from 1 to 2, and reversing by tracing along the other interval from 1 to 2 and again along the interval from 0 to 1. For the other $\mathbb{D}^2_i$ where $i = 1, \ldots, r-2$, the boundary circles S^1_i trace along the corresponding pairs of intervals. The important characteristic here is that a deformation retraction can be obtained sequentially by collapsing $\mathbb{D}^2_{r-2}$ to produce $\mathcal{C}(\widetilde{\mathcal{S}}_{r-1})$ from $\mathcal{C}(\widetilde{\mathcal{S}}_r)$, then collapsing $\mathbb{D}^2_{r-3}$ producing $\mathcal{C}(\widetilde{\mathcal{S}}_{r-2})$ from $\mathcal{C}(\widetilde{\mathcal{S}}_{r-1})$, and further until one obtains $\mathcal{C}(\widetilde{\mathcal{S}}_1)$. Since $\mathcal{C}(\widetilde{\mathcal{S}}_1)$ is an interval, all these spaces are homotopy equivalent to a point. $\square$

Corollary 5.4.13 The obstruction class $[\tilde{\sigma}]$ vanishes.

Which leads to the climax of this subchapter.

Theorem 5.4.1 If $\tilde{\imath}_0$ and $\tilde{\imath}_1$ are two framed embeddings of the sock flow category $\widetilde{\mathscr{S}}_{\mathbf{r}}$ that extend a particular sign assignment, then there are two corresponding framed embeddings $\imath_0$ and $\imath_1$ of the matched flow category $\mathscr{L}(D_{\mathbf{r}})$ obtained as perturbed lifts. The CW complexes $|\mathscr{L}(D_{\mathbf{r}})|_{\imath_0}$ and $|\mathscr{L}(D_{\mathbf{r}})|_{\imath_1}$ obtained from the Cohen-Jones-Segal construction are stably homotopy equivalent.

5.5 Inductive handle cancellation

This subchapter follows [JLS15] in showing that the stable homotopy types arising from different choices of tangle decompositions are stably homotopy equivalent. Consider a matched diagram $D_{\mathbf{r}}$ of a link which consists of m elementary tangles, so that $\mathbf{r} = (r_1, \ldots, r_m) \in (\mathbb{Z} \setminus \{0\})^m$. Subchapter 5.3 describes how to associate a flow category $\mathscr{L}(D_{\mathbf{r}})$ to this diagram, and Subchapter 5.4 ensures that this is a framed flow category, so one can apply to Cohen-Jones-Segal construction and produce a stable homotopy type $|\mathscr{L}(D_{\mathbf{r}})|$. The matched diagram $D_{\mathbf{r}}$ can be described alternatively as a diagram consisting of $|r_1| + \cdots + |r_m|$ tangles, each with a single crossing (which is equivalent to evaluating a 0- or 1-smoothing at each crossing individually). Let $\mathscr{L}(D_{(\pm 1,\ldots,\pm 1)})$ denote the corresponding framed flow category, which reduces to the Lipshitz-Sarkar construction of the underlying diagram D. Again, this produces a stable homotopy type $|\mathscr{L}(D_{(\pm 1,\ldots,\pm 1)})|$ by the Cohen-Jones-Segal construction. The two stable homotopy types just described are shown to be stably homotopy equivalent in [JLS15] and we shall give an overview of the proof here, which is by induction.

Consider a matched diagram $D_{\mathbf{r}}$ of an underlying link diagram D, where $\mathbf{r} = (r_1 + 1, r_2, \ldots, r_m)$ and r_1 is assumed to be at least 1 (the case for negative r_1 is symmetrical). Then the first tangle T_{r_1+1} in the construction of $D_{\mathbf{r}}$ as a concatenation of elementary tangles, can be decomposed into the two elementary tangles T_{r_1} and T_1. Let $D_{\tilde{\mathbf{r}}}$ denote the matched diagram of this new decomposition, where $\tilde{\mathbf{r}} = (1, r_1, r_2, \ldots, r_m)$.

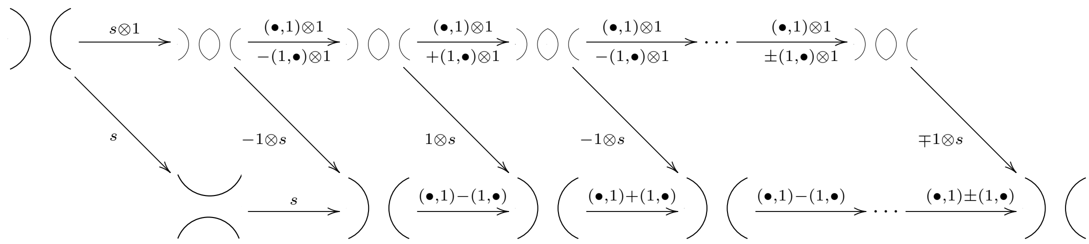


Figure 5.7: The first two coordinates of the cochain complex associated to the flow category $\mathscr{L}(D_{\tilde{\mathbf{s}}})$.

Proposition 5.5.1 Consider the matched diagrams $D_{\mathbf{r}}$ and $D_{\tilde{\mathbf{r}}}$ described above, and assume that their respective associated framed flow categories $\mathscr{L}(D_{\mathbf{r}})$ and $\mathscr{L}(D_{\tilde{\mathbf{r}}})$ are disc flow covers of the relevant sock flow categories. Then,

$$|\mathscr{L}(D_{\mathbf{r}})| \simeq |\mathscr{L}(D_{\tilde{\mathbf{r}}})|.$$

The proof of the above proposition becomes almost obvious when considering the following elegant change of perspective. Consider an underlying link diagram D that admits a matched diagram $D_{\tilde{\mathbf{s}}}$ where $\tilde{\mathbf{s}} = (-1, s_1, s_2, \ldots, s_m)$. Then D also admits the matched diagrams $D_{\mathbf{s}}$ where $\mathbf{s} = (s_1 - 1, s_2, \ldots, s_m)$ and $D_{\bar{\mathbf{s}}}$ where $\bar{\mathbf{s}} = (s_1, -1, s_2, \ldots, s_m)$. Then Proposition 5.5.1 follows from the following proposition.

Proposition 5.5.2 Assume that the associated framed flow categories $\mathscr{L}(D_{\tilde{\mathbf{s}}})$ and $\mathscr{L}(D_{\mathbf{r}})$ and $\mathscr{L}(D_{\bar{\mathbf{s}}})$ of the matched diagrams described above are disc flow covers of the relevant sock flow categories. Then,

$$|\mathscr{L}(D_{\tilde{\mathbf{s}}})| \simeq |\mathscr{L}(D_{\mathbf{r}})| \simeq |\mathscr{L}(D_{\bar{\mathbf{s}}})|.$$

Proof: We shall only highlight the first equivalence, the second is similar. In order to describe the proof, consider the cochain complex associated to $\mathscr{L}(D_{\tilde{\mathbf{s}}})$ and, in particular, consider the first two coordinates of the cochain complex as illustrated in Figure 5.7. The standard generators of this cochain complex are in one-to-one correspondence with the objects of $\mathscr{L}(D_{\tilde{\mathbf{s}}})$. That is, the standard generators are given on each vertex by an $(m+1)$-tuple of integers $(i_0, \ldots, i_m)$ where each i_j lies between 0 and r_m (so has the same sign) and every circle in the resolution on that vertex is decorated with either x_+ or x_-.

Figure 5.7 depicts the first two coordinates where the top-left coordinate corresponds to $(i_0, i_1) = (-1, 0)$ and the bottom-right coordinate corresponds to $(i_0, i_1) = (0, s_1)$. The argument from [JLS15] then proceeds as follows (c.f. Proposition 5.1.5). By taking a direct sum decomposition of every smoothing on the top row (that is, for all $i_0 = -1$) depicted in Figure 5.7 into circles labelled with x_+ and circles labelled with x_-, two collections of objects in $\mathrm{Ob}(\mathscr{L}(D_{\tilde{\mathbf{s}}}))$ can be defined as follows. Let Y denote all of the objects with bigrading $(-1, 0)$ and all the objects with bigrading $(-1, i_1)$ for $i_1 = 1, \ldots, s_1$ with the circle labelled with x_+. Let X denote all of the objects with bigrading $(0, s_1)$ and all the objects with bigrading $(-1, i_1)$ for $i_1 = 1, \ldots, s_1$ with the circle labelled with x_-.

A bijection

$$\alpha : Y \to X$$

can be defined by $\alpha(y) = x$ if $\mathcal{M}(x, y) = *$, a single point, and this bijection assigns pairs that will be cancelled with each other using handle-cancellation. Proceeding to do so with pairs of objects in an order with non-decreasing cohomological degree, leaves the first $s_1 - 1$ of s_1 vertices of the bottom row of the cochain complex in Figure 5.7.

The observation that for any object $a \in \mathrm{Ob}(\mathscr{L}(D_{\tilde{\mathbf{s}}}))$ that is not in $X \cup Y$ we have $\mathcal{M}(a, y) = \emptyset$ or $\mathcal{M}(\alpha(y), a) = \emptyset$, ensures that no moduli spaces change during these cancellations. Clearly, the remaining full-subcategory can be identified with $\mathscr{L}(D_{\mathbf{s}})$ with a framing induced from the pre-cancelled set-up which is necessarily a disc flow cover of the sock flow category $\mathcal{S}_{\mathbf{r}}$. □

It should be noted that the above proof can also be described in the language of upward and downward closed subcategories (see Subchapter 2.4) from [LS14a], and so handle-cancellation does not become necessary for this to work. Proposition 5.5.1 now follows from Proposition 5.5.2, and these results give the following.

Theorem 5.5.1 For the matched diagrams $D_{\mathbf{r}}$ and $D_{(\pm 1, \ldots, \pm 1)}$ described at the beginning of this subchapter, let $\mathscr{L}(D_{\mathbf{r}})$ and $\mathscr{L}(D_{(\pm 1, \ldots, \pm 1)})$ be their associated framed flow categories, the latter of which agrees with the construction of [LS14a]. Then,

$$|\mathscr{L}(D_{\mathbf{r}})| \simeq |\mathscr{L}(D_{(\pm 1,\ldots,\pm 1)})|$$

are stably homotopy equivalent.

Chapter 6

A Combinatorial Steenrod Square on Khovanov Homology

In [LS14b], Lipshitz-Sarkar make use of stable cohomology operations induced on Khovanov homology by the stable homotopy type $\mathcal{X}_{Kh}(L)$. In particular, they study the second Steenrod square

$$\mathrm{Sq}^2 : Kh^{i,j}(L, \mathbb{Z}/2) \to Kh^{i+2,j}(L, \mathbb{Z}/2)$$

which is induced by the Khovanov spectrum. Lipshitz-Sarkar give a combinatorial description of Sq^2 and this is shown to agree with the one induced by the Khovanov spectrum (see [LS14b, Theorem 2]). The significance of the Steenrod square was highlighted at the end of their paper, where they determine the Khovanov homotopy type $\mathcal{X}_{Kh}(L)$ for all links L with up to 11 crossings, many of which yield non-trivial Steenrod squares. This also exhibits the strength of the homotopy type as an invariant since, as Seed shows in [See12], there are several pairs of knots and links with isomorphic Khovanov homologies that can be distinguished by their associated homotopy type. In this chapter, we shall define the Steenrod square using the description in [LS14b] relating to the Khovanov flow category and also extend this definition (as in [JLS15]) to a general framed flow category. Once the necessary language has been set-up, we conclude by highlighting how handle cancellation in flow categories (Theorem 3.1.1) can be used to simplify computations, and work through an example that computes a non-trivial Steenrod square for the first knot for which this occurs. This example will emphasise how the entire content of this thesis ties together.

6.1 The second Steenrod square of a framed flow category

This subchapter follows the modification in [JLS15] of the constructions in [LS14b] to describe the second Steenrod square for a general framed flow category.

Denote by $M_m = M(\mathbb{Z}/2, m)$ the m^{th} Moore space for $\mathbb{Z}/2$, which is defined as the space obtained from S^m by attaching one $(m+1)$-cell via a degree-2 attaching map. Throughout, assume that m is sufficiently large ($m \geq 3$ will do).

Proposition 6.1.1 For the Moore space M_m, $\pi_{m+1}(M_m) \cong \mathbb{Z}/2$.

Proof: Recall that K_m is a space that, by definition, requires $\pi_m(K_m) = \mathbb{Z}/2$ and $\pi_i(K_m) = 0$ for $i \neq m$. The m-skeleton $K_m^{(m)}$ can be chosen to be a single m-cell with its entire boundary attached to the basepoint; so $K_m^{(m)} = S^m$. Moreover, $\pi_m(K_m) = \mathbb{Z}/2$ can be forced by attaching an $(m+1)$-cell e^{m+1} via a degree-2 map $\partial e^{m+1} \to K_m^{(m)} = S^m$. This is the same as M_m and so M_m can be thought of as the $(m+1)$-skeleton of the Eilenberg-MacLane space $K_m = K(\mathbb{Z}/2, m)$. It suffices to show that $\pi_{m+1}(K_m^{(m+1)}) \cong \mathbb{Z}/2$. The pair $(K_m^{(m+1)}, S^m)$ produces the exact sequence

$$\pi_{m+2}(K_m^{(m+1)}, S^m) \to \pi_{m+1}(S^m) \to \pi_{m+1}(K_m^{(m+1)}) \to \pi_{m+1}(K_m^{(m+1)}, S^m) \to \pi_m(S^m)$$

and by using the excision theorem,

$$\pi_{m+1}(K_m^{(m+1)}, S^m) = \pi_{m+1}(K_m^{(m+1)}/S^m) = \pi_{m+1}(S^{m+1}) = \mathbb{Z} \text{ and}$$
$$\pi_{m+2}(K_m^{(m+1)}, S^m) = \pi_{m+2}(K_m^{(m+1)}/S^m) = \pi_{m+2}(S^{m+1}) = \mathbb{Z}/2.$$

The maps

$$\pi_{i+1}(K_m^{(m+1)}, S^m) = \pi_{i+1}(S^{m+1}) \to \pi_i(S^m)$$

are therefore given by twice the Freudenthal isomorphisms, so that the sequence becomes

$$\mathbb{Z}/2 \xrightarrow{2} \mathbb{Z}/2 \to \pi_{m+1}(K_m^{(m+1)}) \to \mathbb{Z} \xrightarrow{2} \mathbb{Z}.$$

Then it follows that $\pi_{m+1}(K_m^{(m+1)}) \cong \mathbb{Z}/2$ as represented by the Hopf map $S^{m+1} \to S^m = K_m^{(m)} \hookrightarrow K_m^{(m+1)}$. □

It follows from the previous proposition that in order to construct an Eilenberg-MacLane space K_m from M_m, it suffices to attach one $(m+2)$-cell τ via a degree 0 map in order to kill the $\mathbb{Z}/2$. The resultant cohomology of the $(m+2)$-skeleton $K_m^{(m+2)}$ is

$$H^m(K_m^{(m+2)}, \mathbb{Z}) = 0, \qquad H^{m+1}(K_m^{(m+2)}, \mathbb{Z}) = \mathbb{Z}/2, \qquad H^{m+2}(K_m^{(m+2)}, \mathbb{Z}) = \mathbb{Z},$$
$$H^m(K_m^{(m+2)}, \mathbb{Z}/2) = \mathbb{Z}/2, \qquad H^{m+1}(K_m^{(m+2)}, \mathbb{Z}) = \mathbb{Z}/2, \qquad H^{m+2}(K_m^{(m+2)}, \mathbb{Z}) = \mathbb{Z}/2.$$

By considering the generator $\iota \in H^m(K_m, \mathbb{Z}/2)$, a Theorem originally due to Serre (see [McC01, Theorem 6.19]) ensures that $H^{m+2}(K_m, \mathbb{Z}/2) = \mathbb{Z}/2$ is generated by $\mathrm{Sq}^2(\iota)$.

Now let X be a CW-complex with a cohomology class $u \in H^m(X, \mathbb{Z}/2)$. Consider a cellular map $f : X \to K_m$ with $f^*\iota = u$ so that u is classified by f. Then $\mathrm{Sq}^2(u) = f^*\mathrm{Sq}^2(\iota) \in H^{m+2}(X, \mathbb{Z}/2)$ is determined by its restriction to $H^{m+2}(X^{(m+2)}, \mathbb{Z}/2)$. Therefore, it suffices to provide a cellular map $X^{(m+2)} \to K_m^{(m+2)}$ for which ι pulls back to u. That is, it suffices to determine $f^*\tau'$, where $\tau' \in C^{m+2}(K_m, \mathbb{Z}/2)$ is necessarily the cocycle representing $\mathrm{Sq}^2(\iota)$ and the dual of τ. If σ is an $(m+2)$-cell in X with an attaching map $\chi_\sigma\colon S^{m+1} \to X^{(m+1)}$, then the restriction $f|\colon X^{(m+1)} \to M_m$ composed with χ_σ gives a representative in

$$\pi_{m+1}(M_m) = \pi_{m+1}(K_m^{(m+1)}) = \pi_{m+1}(S^m) \cong \mathbb{Z}/2$$

and by the construction of f this element also represents the evaluation of the cocycle $f^*\tau'$ on σ. Essentially, $\mathrm{Sq}^2(u)$ is the obstruction of being able to produce a homotopy of f that sends $X^{(m+2)}$ to $K_m^{(m+1)}$. A cocycle representative $\mathrm{sq}^2(u) \in C^{m+2}(X, \mathbb{Z}/2)$ for $\mathrm{Sq}^2(u)$ is therefore determined by the elements $[f| \circ \chi_\sigma] \in \pi_{m+1}(M_m)$.

Since $K_m^{(m+2)}$ has no cells of dimension less than m, a map $X^{(m+2)} \to K_m^{(m+2)}$ factors through the quotient $X^{(m+2)}/X^{(m-1)}$. Therefore, to understand Sq^2 it is sufficient to describe f explicitly on $X^{(m+2)}/X^{(m-1)}$. Indeed, for a general framed flow category $(\mathscr{C}, \imath, \varphi)$ where $\imath$ is a neat embedding of $\mathscr{C}$ relative some $\mathbf{d} = (d_i, d_{i+1})$, [LS14b, Proposition 3.6] ensures that the objects that are used to build X can all be assumed to have gradings in $\{i, i+1, i+2\}$. Write $m = d_i + d_{i+1}$. The proposition states a relation between the CW complex X constructed under this assumption, and the skeleta of the CW complex obtained using the Cohen-Jones-Segal construction. In particular,

$$\Sigma^{C+i-m} X = |\mathscr{C}|^{(C+i+2)}/|\mathscr{C}^{(C+i-1)}$$

(where C is from Definition 2.3.1) so that $H^i(\mathscr{C}, \mathbb{Z}/2) \cong \tilde{H}^m(|\mathscr{C}|, \mathbb{Z}/2)$.

Now consider a cocycle $c \in C^i(\mathscr{C}; \mathbb{Z}/2)$ representing u. Then

$$c = \sum_{x \in \mathrm{Ob}_i(\mathscr{C})} n_x x$$

for some $n_x \in \mathbb{Z}/2$. The skeleta are built up inductively, and $|\mathscr{C}|^{(m)} = \bigvee_{x \in \mathrm{Ob}_i(\mathscr{C})} S^m$ is a wedge of m-spheres. Define a map $f^{(m)} \colon |\mathscr{C}|^{(m)} \to M_m$ whose restriction to each cell $\mathcal{C}(x)$ is a degree one map onto $M_m^{(m)}$ whenever $n_x = 1$, and the constant map to the basepoint whenever $n_x = 0$. By considering objects one grading higher, say $y \in \mathrm{Ob}(\mathscr{C})$ with $|y| = i+1$, there is framed embedding of all points in $\mathcal{M}(y, x)$ as

$$\imath_y \colon \coprod_{x \in \mathrm{Ob}_i(\mathscr{C})} \mathcal{M}(y, x) \times [-\varepsilon, \varepsilon]^{d_i} \to \mathbb{R}^{d_i}.$$

The fact that c is a cocycle forces $\coprod \mathcal{M}(y, x)$ to be an even number of points for each y. Let $n(y)$ denote half of the number of such points (so $2n(y)$ is the cardinality of this set of points).

Definition 6.1.2 If c is the cocycle described above, then a *topological boundary matching for c* is a collection of disjoint, embedded, framed arcs

$$\eta_y^j \colon [0,1] \times [-\varepsilon, \varepsilon]^{d_i} \to [0, \infty) \times \mathbb{R}^{d_i}$$

for each $y \in \mathrm{Ob}_{i+1}(\mathscr{C})$ and $j = 1, \ldots, n(y)$, satisfying

$$(\eta_y^j)^{-1}(\{0\} \times \mathbb{R}^{d_i}) = \{0,1\} \times [-\varepsilon, \varepsilon]^{d_i}$$

with

$$\bigcup_j \eta_y^j(\{0,1\} \times \{0\}) = \{0\} \times \imath_y \left(\coprod_x \mathcal{M}(y, x) \right).$$

Consider pairs of points $p_0, p_1 \in \coprod_x \mathcal{M}(y,x)$ that correspond to endpoints of an arc η_y^j so that $\eta_y^j(0,0) = (0, \imath_y(p_0, 0))$ and $\eta_y^j(1,0) = (0, \imath_y(p_1, 0))$. The points p_0 and p_1 come with framings (signs) and if these differ, then the framing of $\eta_y^j|\{0,1\}$ can be assumed to agree with the framings of p_0 and p_1. If this happens, the arc is said to be *boundary-coherent*.

If, on the other hand, the framings of p_0 and p_1 agree (that is, they have the same sign), then the framing of η_y^j can only be assumed to agree with the framing at one endpoint; at the other endpoint it will agree with the framing at that endpoint after reflecting in the first coordinate via $R\colon \mathbb{R}^{d_i} \to \mathbb{R}^{d_i}$ given by $R(x_1, \ldots, x_{d_i}) = (-x_1, x_2, \ldots, x_{d_i})$. Quite naturally, arcs of this type are called *boundary-incoherent*. This arc is endowed with an orientation from the endpoint at which the framings agree to the endpoint at which the framings disagree.

For all objects y with grading $i+1$, the topological boundary matchings provide an embedding

$$\eta_y\colon \coprod_{j=1}^{n(y)} [0,1] \times [-\varepsilon, \varepsilon]^{d_i + d_{i+1}} \to \mathcal{C}(y)$$

defined as $\eta_y(t, x_1, \ldots, x_{d_i}, y_1, \ldots, y_{d_{i+1}}) = (\eta_y^j(t, x_1, \ldots, x_{d_i}), y_1, \ldots, y_{d_{i+1}})$ on the j^{th} disjoint union. Using these embeddings, the map $f^{(m)}$ can be extended to a map on the $(m+1)$-skeleton $f^{(m+1)} : |\mathscr{C}|^{(m+1)} \to M_m$ by attaching all cells $\mathcal{C}(y)$ with $y \in \mathrm{Ob}_{i+1}(\mathscr{C})$ in the following way. Away from the image of η_y, the cell is sent to the basepoint. For boundary-coherent arcs, the attaching map is given by the projection

$$[0,1] \times [-\varepsilon, \varepsilon]^{d_i + d_{i+1}} \to [-\varepsilon, \varepsilon]^{d_i + d_{i+1}} \to S^{d_i + d_{i+1}} = M_m^{(m)}$$

where $[-\varepsilon, \varepsilon]^{d_i + d_{i+1}} \to S^{d_i + d_{i+1}}$ is the quotient map on the boundary of $[-\varepsilon, \varepsilon]^{d_i + d_{i+1}}$. For boundary-incoherent arcs, the attaching map is given by a degree 1 map over the single $(m+1)$-cell of M_m. The procedure described above for obtaining a cocycle representative for $\mathrm{Sq}^2(u)$ can now be carried out. However, the computable descriptions that are outlined in both [LS14b] and [JLS15] require an amount of care to be taken regarding framings. We shall outline what this entails and then go on to state the necessary results (this set-up follows [LS14b, §3.2] and [JLS15]).

Any framing of a path embedded in $\mathbb{R}^{m+1}$ gives rise to a path of m orthonormal vectors in $\mathbb{R}^{m+1}$, and this collection of orthonormal vectors gives an element of $SO(m+1)$. That is, a framed path in $\mathbb{R}^{m+1}$ gives rise to a path in $SO(m+1)$. There is a one-to-one correspondence between isotopy classes of framed paths (with fixed endpoints) in $\mathbb{R}^{m+1}$ and homotopy classes of paths (with fixed endpoints) in $SO(m+1)$ provided $m \geq 3$ where both sets have two elements (hence our assumption for large m at the beginning of the section). Thus, to specify an isotopy class of framed paths for such m, it is enough to specify a homotopy class of paths in $SO(m+1)$.

Now consider objects $z \in \mathrm{Ob}_{i+2}(\mathscr{C})$ and the one-dimensional moduli spaces $\mathcal{M}(z,x)$. These are disjoint unions of intervals and circles and are embedded with framings into $\mathbb{R}^{d_i} \times [0,\infty) \times \mathbb{R}^{d_{i+1}}$ in such a way that the endpoints of the intervals are embedded into $\mathbb{R}^{d_i} \times \{0\} \times \mathbb{R}^{d_{i+1}}$. There are four possible framings at the endpoints and these are

$$(e_1,\dots,e_{d_i},e_{d_i+1},\dots,e_{e_i+e_{i+1}}),(-e_1,e_2,\dots,e_{d_i},e_{d_i+1},\dots,e_{e_i+e_{i+1}}),$$
$$(e_1,\dots,e_{d_i},-e_{d_i+1},e_{d_i+2}\dots,e_{e_i+e_{i+1}}),(-e_1,e_2,\dots,e_{d_i},-e_{d_i+1},e_{d_i+2}\dots,e_{e_i+e_{i+1}})$$

denoted in compact notation as ${}^{+}_{+}$, ${}^{+}_{-}$, ${}^{-}_{+}$, ${}^{-}_{-}$, respectively.

Definition 6.1.3 A *coherent system of paths joining* ${}^{+}_{+}$*,* ${}^{+}_{-}$*,* ${}^{-}_{+}$*,* ${}^{-}_{-}$ is a choice of path $\overline{\varphi_1\varphi_2}$ in $\mathrm{SO}(m+1)$ from φ_1 to φ_2 for each pair of frames $\varphi_1,\varphi_2 \in \{{}^{+}_{+},{}^{+}_{-},{}^{-}_{+},{}^{-}_{-}\}$ satisfying the following cocycle conditions:

1. For all $\varphi \in \{{}^{+}_{+},{}^{+}_{-},{}^{-}_{+},{}^{-}_{-}\}$ the loop $\overline{\varphi\varphi}$ is null-homotopic;
2. For all $\varphi_1,\varphi_2,\varphi_3 \in \{{}^{+}_{+},{}^{+}_{-},{}^{-}_{+},{}^{-}_{-}\}$ the path $\overline{\varphi_1\varphi_2} \cdot \overline{\varphi_2\varphi_3}$ is homotopic (relative endpoints) to $\overline{\varphi_1\varphi_3}$.

A coherent system of paths is shown to exist in [LS14b, Lemma 3.1], which is described as follows. Firstly, denote by e_1 the first coordinate of $\mathbb{R}^{d_i}$, denote by e_2 the first coordinate of $\mathbb{R}^{d_{i+1}}$ and denote by $\bar{e}$ the coordinate of $[0,\infty)$. Then for $\varphi_1,\varphi_2 \in \{{}^{+}_{+},{}^{+}_{-},{}^{-}_{+},{}^{-}_{-}\}$, the coherent system of paths chosen in [LS14b] gives $\overline{\varphi_1\varphi_2}$ as follows:

1. $\overline{\substack{++\\+-}}, \overline{\substack{++\\-+}}, \overline{\substack{--\\+-}}, \overline{\substack{--\\-+}}$: Rotate 180° around the e_2-axis, such that the first vector equals $\bar{e}$ halfway through.

2. $\overline{\substack{+-\\++}}, \overline{\substack{-+\\++}}$: Rotate 180° around the e_1-axis, such that the second vector equals $\bar{e}$ halfway through.

3. $\overline{\substack{+-\\--}}, \overline{\substack{-+\\--}}$: Rotate 180° around the e_1-axis, such that the second vector equals $-\bar{e}$ halfway through.

4. $\overline{\substack{+-\\+-}}, \overline{\substack{-+\\-+}}, \overline{\substack{+-\\-+}}, \overline{\substack{-+\\+-}}$: Rotate 180° around the $\bar{e}$-axis, such that the second vector equals $-e_1$ halfway through.

Consider a framed path in $\mathbb{R}^{d_i} \times [0, \infty) \times \mathbb{R}^{d_{i+1}}$ that has endpoints in $\mathbb{R}^{d_i} \times \{0\} \times \mathbb{R}^{d_{i+1}}$ given by two elements of $\{\substack{+\\+}, \substack{+\\-}, \substack{-\\+}, \substack{-\\-}\}$. If the framed path has a homotopy class in $\mathrm{SO}(m+1)$ that is one of the classes described in (1)-(4), then it is called a *standard frame path*. Otherwise, the framed path is a *non-standard frame path*.

Any framed flow category $\mathscr{C}$ whose 0-dimensional moduli spaces are all framed using the standard $+$ or $-$ framings has intervals as 1-dimensional moduli spaces. These intervals are either standard or a non-standard frame paths. All other 1-dimensional moduli spaces are circles and represent an element of $\pi_1(\mathrm{SO}(m+1)) \cong H_1(\mathrm{SO}(m+1)) \cong \mathbb{Z}/2$. Therefore, a function

$$fr\colon \pi_0(\mathcal{M}(z, x)) \to \mathbb{Z}/2$$

can be defined for all 1-dimensional moduli spaces $\mathcal{M}(z, x)$ to encode the framing, where $fr(I) = 0 \in \mathbb{Z}/2$ means that an interval I being the standard frame path; the trivial element of $\pi_1(SO(m+1))$.

Definition 6.1.4 Framed interval components of a 1-dimensional moduli space are called *Pontryagin-Thom arcs*, and the framed circle components are called *Pontryagin-Thom circles*.

Recall that for a given cohomology class $u \in H^i(\mathscr{C}, \mathbb{Z}/2)$, the cellular map $f^{(m+1)}\colon |\mathscr{C}|^{(m+1)} \to M_m$ has been constructed satisfying $f^{m+1*}\iota = u$. Let $z \in \mathrm{Ob}_{i+2}(\mathscr{C})$ and denote by

$$\chi_z\colon \partial\mathcal{C}(z) \cong S^{m+1} \to |\mathscr{C}|^{(m+1)}$$

the attaching map of $\mathcal{C}(z)$. In particular,

$$S^{m+1} \cong \partial\mathcal{C}(z) = \partial([0,R] \times [-R,R]^{d_i} \times [0,R] \times [-R,R]^{d_{i+1}}).$$

Recall also that the inclusion $S^m \subset M_m$ induces an isomorphism on π_{m+1} and composing χ_z with $f^{(m+1)}$ produces an element in $\pi_{m+1}(M_m) \cong \mathbb{Z}/2$. Let $f_z\colon S^{m+1} \to M_m$ denote this composition. Moreover, each $y \in \mathrm{Ob}_{i+1}(\mathscr{C})$ identifies a set on the boundary of $\mathcal{C}(z)$ as

$$C_y(z) = [0,R] \times [-R,R]^{d_i} \times \{0\} \times \mathcal{M}(z,y) \times [-\varepsilon,\varepsilon]^{d_{i+1}} \subset \partial\mathcal{C}(z)$$

and this set contains the framed arcs $\gamma_j \times [-\varepsilon,\varepsilon]^{d_i} \times \{0\} \times \mathcal{M}(z,y) \times [-\varepsilon,\varepsilon]^{d_{i+1}}$ that come from the topological boundary matchings. Since the Pontryagin-Thom arcs also sit in the boundary sets $C_x(z) \subset \partial\mathcal{C}(z)$ identified for each $x \in \mathrm{Ob}_i(\mathscr{C})$ with $\mathcal{M}(z,x) \neq \emptyset$, these framed arcs can be concatenated with the Pontryagin-Thom arcs to produce a collection of framed circles in $\partial\mathcal{C}(z)$ (the Pontryagin-Thom circles obviously already produce framed circles in $\partial\mathcal{C}(z)$). In particular, such objects x form the linear combination that represents u. The map f_z sends anything outside of a neighbourhood of these framed circles to the basepoint. By construction, if all topological boundary matching arcs are boundary-coherent, then the image of f_z is already contained in $M_m^{(m)} \cong S^m$. Otherwise, there exist boundary-incoherent topological boundary matching arcs and it is shown in [LS14a, Lemma 3.9] that there are always an even number of them. In this case, [LS14b, Proposition 3.10] provides an alteration of the maps f_z that represent the same element in $\pi_{m+1}(M_m)$. The key point to note is that the construction does not focus on any particular framed flow category and can therefore be used here. The framed circles obtained in this way are denoted (K,Φ). The finite collection of them together determine the value of the cocycle c_f representing $\mathrm{Sq}^2(u)$. The work of [LS14b] then provides a compact combinatorial way to read off the component corresponding to each (K,Φ) as an element of the first framed bordism group $\Omega_1^{fr} = \pi_{m+1}(S^m) \cong \mathbb{Z}/2$. The relevant proposition and its extension to Pontryagin-Thom circles from [JLS15] is stated below.

Proposition 6.1.5 Let (K, Φ) be a framed circle of the form described above. If K is a concatenation of Pontryagin-Thom arcs and topological boundary matchings, then its value in $\Omega_1^{fr} \cong \mathbb{Z}/2$ is given as the sum of

1. The number 1.
2. The number of Pontryagin-Thom arcs in K with the non-standard framing.
3. The number of arrows on K which point in a fixed direction.

If K is a Pontryagin-Thom circle, its value in $\Omega_1^{fr} \cong \mathbb{Z}/2$ is given as the sum of

1. The number 1.
2. The value of the map $fr\colon \pi_0(\mathcal{M}(z, x)) \to \mathbb{Z}/2$ on the component corresponding to K.

   The value of $c_f \in C^{i+2}(\mathscr{C}, \mathbb{Z}/2)$ corresponding to z is then the total sum of these numbers for all such framed circles (K, Φ).

6.2 A particular frame assignment for the sock flow category

This subchapter defines a particular frame assignment for the sock flow category $\mathcal{S}_{\mathbf{r}}$ for $\mathbf{r} \in (\mathbb{Z} \setminus \{0\})^k$ by providing a particular sign-assignment and frame-assignment of the 0- and 1-dimensional moduli spaces of $\widetilde{\mathcal{S}}_{\mathbf{r}}$. In light of the previous subchapter, providing framing of the 1-dimensional moduli spaces is the key to computing Steenrod squares of links admitting matched diagrams. The fact that this can be extended to a framing for the entire flow category $\widetilde{\mathcal{S}}_{\mathbf{r}}$ follows from some results of [JLS15] (which we shall state) and the obstruction theory arguments highlighted in subchapter 5.4. To begin with, let us recall the definition of $\widetilde{\mathcal{S}}_{\mathbf{r}}$ and set up some concrete notation for the 0- and 1-dimensional moduli spaces.

Consider the CW-complex $\mathcal{C}(\widetilde{\mathcal{S}}_{\mathbf{r}})$ defined in Subchapter 5.4 where $\mathbf{r} \in (\mathbb{Z} \setminus \{0\})^k$ is denoted $\mathbf{r} = (r_1, \ldots, r_k)$. The CW-complex is given by the product

$$\mathcal{C}(\widetilde{\mathcal{S}}_{\mathbf{r}}) = \mathcal{C}(\widetilde{\mathcal{S}}_{r_1}) \times \cdots \times \mathcal{C}(\widetilde{\mathcal{S}}_{r_k})$$

and its cells are products of the cells from each $\mathcal{C}(\widetilde{\mathcal{S}}_{r_i})$. The following notation is consistent with [JLS15]. The 0-cells of $\mathcal{C}(\widetilde{\mathcal{S}}_r)$ $(r \neq 0)$ are precisely the objects of $\mathcal{S}_r$, so let us denote them by

$$\mathrm{Ob}(\mathcal{S}_r) = \{\, a \in \mathbb{Z} \mid 0 \leq a \leq r \text{ if } r > 0, \text{ and } r \leq a \leq 0 \text{ if } r < 0 \,\}.$$

The 1-cells of $\mathcal{C}(\widetilde{\mathcal{S}}_r)$ are in correspondence with the 0-dimensional moduli spaces and so these cells can be denoted

$$R_0 \ldots, R_{r-1}, L_0, \ldots, L_{r-1} \text{ for } r > 0, \text{ and } R_r, \ldots, R_{-1}, L_r, \ldots, L_{-1} \text{ for } r < 0.$$

Recall that $R_0 = L_0$ are identified and $R_{-1} = L_{-1}$ is the corresponding identification for negative r. If $|r| = 1$, then the 2-cells will be non-existent. Otherwise, assume $|r| > 1$ and denote the 2-cells by

$$M_0 \ldots, M_{r-2} \text{ for } r > 1, \text{ and } M_{r+1}, \ldots, M_{-1} \text{ for } r < -1.$$

Let $\overline{a} \in C_0(\mathcal{C}(\widetilde{\mathcal{S}}_r))$ be the generators corresponding to the objects a. Let the generators corresponding to higher-dimensional cells keep the same notation above. Then

$$\partial R_j = \partial L_j = \overline{j+1} - \overline{j}$$
$$\partial M_j = L_j + R_{j+1} - L_{j+1} - R_j.$$

This notation can be extended to the product complex $\mathcal{C}(\widetilde{\mathcal{S}}_{\mathbf{r}})$ in the following way. Let $a = (a_1, \ldots, a_k) \in \mathbb{Z}^k$ be the 0-cells, and let the 1-cells be denoted by

$$R_a^j = a_1 \times \cdots \times a_{j-1} \times R_{a_j} \times a_{j+1} \times \cdots \times a_k$$
$$L_a^j = a_1 \times \cdots \times a_{j-1} \times L_{a_j} \times a_{j+1} \times \cdots \times a_k.$$

The 2-cells M_a^j can be written in a similar way, but notice that there are additional 2-cells arising as products of the 1-cells. Denote these by

$$(R_a^j, R_a^i),\ (R_a^j, L_a^i),\ (L_a^j, R_a^i),\ (L_a^j, L_a^i)$$

where $1 \leq j < i \leq k$. In particular,

$$(R_a^j, L_a^i) = a_1 \times \cdots \times a_{j-1} \times R_{a_j} \times a_{j+1} \times \cdots \times a_{i-1} \times L_{a_i} \times a_{i+1} \times \cdots \times a_k$$

and similarly for the others. Higher dimensional cells admit a similar notation. Continuing to follow the notation of [JLS15], a cell is said to be *based at a* if it has a subscript a (since this really implies that the cells corresponds to a component of a moduli space $\mathcal{M}(b,a)$ for some other object b). To define the particular sign assignment of [JLS15] for $\mathcal{C}(\widetilde{\mathcal{S}}_{\mathbf{r}})$, a slight addition is needed to ensure that the assignment is compatible with (possibly) negative values of r_i. By defining $\delta = (\delta_1, \dots, \delta_k) \in \{0,1\}^k$ to be

$$\delta_i = \begin{cases} 0 & if\ r_i > 0 \\ 1 & if\ r_i < 0 \end{cases}$$

then for every $a = (a_1, \dots, a_k) \in \mathbb{Z}^k$ let

$$a'_i = a_i + \delta_i$$

for each $i = 1, \dots, k$.

Definition 6.2.1 The *standard sign assignment $s \in C^1(\mathcal{C}(\tilde{\mathscr{S}}_{\mathbf{r}}), \mathbb{Z}/2)$* is the cochain defined by

$$s(R^j_a) = a_1 + \cdots + a_{j-1}$$
$$s(L^j_a) = a_1 + \cdots + a_{j-1} + a'_j.$$

Both of these sums are elements of $\mathbb{Z}/2$, and whenever an empty sum occurs, it is treated as 0.

It should be noted that the standard sign assignment gives the framing of the 0-dimensional moduli spaces of $\mathcal{S}_{\mathbf{r}}$, where sign assignments of products in the chain complex $C_*(\mathcal{S}_{\mathbf{r}})$ are given by products of sign assignments in each of the chain complexes $C_*(\mathcal{S}_{r_i})$ whose product gives $C_*(\mathcal{S}_{\mathbf{r}})$. The addition of a' ensures that $s(L_{-1}) = s(R_{-1})$ in the negative case ($s(L_0) = s(R_0)$ in the positive case already). The addition of δ defined above is also needed to define a frame assignment of the 1-dimensional moduli spaces.

Definition 6.2.2 The *standard frame assignment* $f \in C^2(\mathcal{C}(\tilde{\mathscr{S}_{\mathbf{r}}}), \mathbb{Z}/2)$ is the 2-cochain defined as

$$\begin{aligned}
f(R_a^j, R_a^i) &= (a_1 + \cdots + a_{j-1})(a_j + \cdots + a_{i-1}) \\
f(R_a^j, L_a^i) &= (a_1 + \cdots + a_{j-1})(a_j + \cdots + a_{i-1} + a_i') \\
f(L_a^j, R_a^i) &= (a_1 + \cdots + a_{j-1} + a_j')(\delta_j + a_{j+1} + \cdots + a_{i-1}) \\
f(L_a^j, L_a^i) &= (a_1 + \cdots + a_{j-1} + a_j')(\delta_j + a_{j+1} + \cdots + a_{i-1} + a_i') \\
f(M_a^j) &= (a_1 + \cdots + a_{j-1})a_j'
\end{aligned}$$

where again these sums are elements in $\mathbb{Z}/2$, and empty sums are treated as 0.

A very important property of the above standard sign and frame assignments from [JLS15] is that when $|r_i| = 1$ for all $i = 1, \ldots, k$ only the frame assignments $f(R_a^j, R_a^i)$ are required and these recover the sign and frame assignments of [LS14b], which is necessary since this is the result of evaluating resolutions at each crossing (which corresponds to [LS14a]) rather than at a collection of crossings that form an elementary tangle (which corresponds to [JLS15]).

The standard sign and frame assignments defined above provide a framing of the 0- and 1-dimensional moduli spaces of $\widetilde{\mathscr{S}_{\mathbf{r}}}$. Following the arguments of [LS14b, Lemma 3.5] which ensures that the partial framing defined there can be extended to the entire cube flow category $\mathscr{C}_C(n)$, [JLS15] also show that the partial framing defined above can be extended to a framing of the entire category $\widetilde{\mathscr{S}_{\mathbf{r}}}$. To give an overview, both of these papers proceed by showing that the image of the 2-cochain f under the differential δ, once applied to a 3-cell in the corresponding set-up, is equal to a sum of sign-assignments on lower dimensional cells that form the boundary components of the 3-cell. This then provides assistance in showing that the framings of the 0- and 1-dimensional moduli spaces can be extended to a framing of the 2-dimensional moduli space. Then the inductive framing of the cube flow category in [LS14a] which is described in Subchapter 4.1.1 (respectively, the sock flow category in [JLS15] desribed in Subchapter 5.4) provides a way to extend the partial framing from [LS14b, Definition 3.4] (respectively, Definitions 6.2.1 and 6.2.2) to the entire cube flow category (respectively, sock flow category). The following result is all that we need to take away from these arguments.

Proposition 6.2.3 The partial framing from Definitions 6.2.1 and 6.2.2 can be extended to a framing of the entire category $\tilde{\mathscr{S}}_{\mathbf{r}}$.

6.3 Framing formulae for gluing moduli spaces

The ultimate goal of this chapter is to illustrate how the cancellation theorem of Chapter 3 (Theorem 3.1.1) can be applied to simplify computations of Steenrod squares. In particular, if $\mathscr{C}$ is a framed flow category that contains two objects x, y with $\mathcal{M}(x, y) = \{*\}$, then the cancelled framed flow category $\mathscr{C}_H$ (see Definition 3.1.1) has moduli spaces that are obtained as a result of gluing moduli spaces from $\mathscr{C}$. Since the resulting spaces are stably homotopy equivalent, their Steenrod squares coincide. This subchapter describes a compact collection of gluing formulae which were developed in [JLS15] to frame the moduli spaces of the cancelled category. Moreover, these formulae give the framings of the new moduli spaces in terms of the framings of the original moduli spaces. For objects $a, b \in \mathrm{Ob}(\mathscr{C})$, recall that the new moduli spaces are defined as

$$\mathcal{M}(\bar{a}, \bar{b}) = \mathcal{M}(a, b) \cup \big(\mathcal{M}(x, b) \times \mathcal{M}(a, y)\big)$$

where the gluing takes place along $\mathcal{M}(x, b) \times \mathcal{M}(a, x)$ and along $\mathcal{M}(y, b) \times \mathcal{M}(a, y)$. The framed embeddings of these moduli spaces were defined in Chapter 3 via embeddings $\Gamma_{\bar{a},\bar{b}} = \Gamma_{a,b} \cup \Gamma_{x,b\times a,y}$, and providing a formula for a framing of two glued intervals (which have pre-assigned framings) involves analysing the map $\Gamma_{x,b\times a,y}$. Let us highlight how this works when $\mathcal{M}(x, b) \times \mathcal{M}(a, y)$ is a newly embedded disjoint union of points; that is, when $|a| = |x| = i + 1$ and $|b| = |y| = i$. Consider a product of points $(B, A) \in \mathcal{M}(x, b) \times \mathcal{M}(a, y)$. The relevant cells corresponding to these objects are

$$\begin{aligned}
\mathcal{C}(x) &= [0, R] \times [-R, R]^{d_i} \\
\mathcal{C}(a) &= [0, R] \times [-R, R]^{d_i} \\
\mathcal{C}(y) &= \{0\} \times [-\varepsilon, \varepsilon]^{d_i} \text{ and} \\
\mathcal{C}(b) &= \{0\} \times [-\varepsilon, \varepsilon]^{d_i}.
\end{aligned}$$

Then $\Gamma_{x,b\times a,y} : \mathcal{M}(x,b)\times\mathcal{M}(a,y)\times\{0\}\times[-\varepsilon,\varepsilon]^{d_i} \to \{0\}\times[-R,R]^{d_i}$, which is defined as the composition of $\imath_{x,b}$, Ψ_1, $\imath_{x,y}^{-1}$ and $\imath_{a,y}$ (see Lemma 3.1.3). In particular, only Ψ_1 may produce a non-standard framing, which can be assumed to be an inversion of $\mathbb{R}^{d_i}$ of a small sphere centered at 0 (or more specifically, a small sphere centered around $\imath_{x,y}(*)$ with its framing). In fact, by assuming that points $B \in \mathcal{M}(x,b)$ are only embedded into the first coordinate of $\mathbb{R}^{d_i}$, then it can be seen that Ψ_1 has the effect of flipping the first coordinate of the framing of B. Therefore, if $\varepsilon_p \in \{\pm 1\}$ denotes the framing of a point p, then $\varepsilon_{(B,A)} = -\varepsilon_B\varepsilon_*\varepsilon_A$, where the negative sign arises from the map Ψ_1 as described above.

For 1-dimensional moduli spaces, assume that the objects a and b have gradings $|a| = i+1$ and $|b| = i-1$. The analysis of this set-up breaks down into two cases, each of which contains a number of subcases. The first case is when $|x| = i$, and the second is when $|x| = i+1$, recalling that y ought to always have one degree less than x. Each of these cases have been considered in detail in [JLS15] and consists of an analysis similar to that above. Here, we shall only state the final formulae that occur from the gluings of interval moduli spaces (and, perhaps, embeddings of circles).

Let us first consider the case when $|x| = i$. Let $I \subset \mathcal{M}(a,b)$ be an interval whose endpoints are given by $(C,D) \in \mathcal{M}(c,b)\times\mathcal{M}(a,c)$ and $(C',D') \in \mathcal{M}(c',b)\times\mathcal{M}(a,c')$ for some $c,c' \in \mathrm{Ob}(\mathscr{C})$ with $|c| = i = |c'|$ and $x \notin \{c,c'\}$. Then I will remain an interval in $\mathcal{M}(\bar{a},\bar{b})$ with the same framed embedding, and so its framing $fr(I)$ does not change. This is also true for any components of 1-dimensional moduli spaces that are circles. However, if $x \in \{c,c'\}$ then I is glued to an interval of the form $\{B\}\times J$ for $B \in \mathcal{M}(x,b)$ and $J \subset \mathcal{M}(a,y)$. Such intervals $\{B\}\times J$ are embedded using $\Gamma_{x,b\times a,y}$ into $\partial_{i-1}\mathcal{C}(a)$ and their framings are

$$fr(\{B\}\times J) = fr(J) + 1 + \varepsilon_* + \varepsilon_B. \tag{6.1}$$

In combination, [JLS15] provides the following:

Proposition 6.3.1 In the case that $|x| = i$, let $a,b \neq y$ be objects with $|a| = i+1 = |b|+2$ and $K \subset \mathcal{M}(\bar{a},\bar{b})$.

1. If K is a circle then either

(a) $K \subset \mathcal{M}(a,b)$, in which case $fr(K)$ is the same as before.

(b) $K = \{B\} \times K' \subset \mathcal{M}(x,b) \times \mathcal{M}(a,y)$ in which case $fr(K) = fr(K')$.

(c) K is the result of gluing intervals $I_1, \ldots, I_k \subset \mathcal{M}(a,b)$ with intervals $\{B_1\} \times J_1, \ldots, \{B_k\} \times J_k \subset \mathcal{M}(x,b) \times \mathcal{M}(a,y)$ in which case

$$fr(K) = k(1+\varepsilon_*) + \sum_{i=1}^{k}(fr(I_i) + fr(J_i) + \varepsilon_{B_i}).$$

2. If K is an interval, then either

(a) The endpoints of K are of the form $(B,A) \in \mathcal{M}(c,b) \times \mathcal{M}(a,c)$, $(B',A') \in \mathcal{M}(c',b) \times \mathcal{M}(a,c')$ with $x \notin \{c,c'\}$. Then K is the result of gluing intervals $I_1, \ldots, I_{k+1} \subset \mathcal{M}(a,b)$ with intervals $\{B_1\} \times J_1, \ldots, \{B_k\} \times J_k \subset \mathcal{M}(x,b) \times \mathcal{M}(a,y)$ in which case

$$fr(K) = k(1+\varepsilon_*) + \sum_{i=1}^{k}(fr(I_i) + fr(J_i) + \varepsilon_{B_i}) + fr(I_{k+1}).$$

(b) The endpoints of K are of the form $(B,A) \in \mathcal{M}(c,b) \times \mathcal{M}(a,c)$, and $(B_k, C, D) \in \mathcal{M}(x,b) \times \mathcal{M}(c',y) \times \mathcal{M}(a,c')$ with $c, c' \neq x$. Then K is the result of gluing intervals $I_1, \ldots, I_k \subset \mathcal{M}(a,b)$ with intervals $\{B_1\} \times J_1, \ldots, \{B_k\} \times J_k \subset \mathcal{M}(x,b) \times \mathcal{M}(a,y)$ in which case

$$fr(K) = k(1+\varepsilon_*) + \varepsilon_{B_k} + \sum_{i=1}^{k}(fr(I_i) + fr(J_i) + \varepsilon_{B_i}).$$

(c) The endpoints of K are of the form $(B_1, C, D) \in \mathcal{M}(x,b) \times \mathcal{M}(c,y) \times \mathcal{M}(a,c)$ and $(B_{k+1}, C', D') \in \mathcal{M}(x,b) \times \mathcal{M}(c',y) \times \mathcal{M}(a,c')$ with $c, c' \neq x$. Then K is the result of gluing intervals $I_1, \ldots, I_k \subset \mathcal{M}(a,b)$ with intervals $\{B_1\} \times J_1, \ldots, \{B_{k+1}\} \times J_{k+1} \subset \mathcal{M}(x,b) \times \mathcal{M}(a,y)$ in which case

$$\begin{aligned} fr(K) = &(k+1)(1+\varepsilon_*) + \varepsilon_{B_1} + \varepsilon_{B_{k+1}} + \\ &\sum_{i=1}^{k+1}(fr(J_i) + \varepsilon_{B_i}) + \sum_{i=1}^{k} fr(I_i). \end{aligned}$$

Now consider the case when $|x| = i+1$, and let $J \subset \mathcal{M}(x,b)$ whose endpoints are given by $(C,D) \in \mathcal{M}(c,b) \times \mathcal{M}(a,c)$ and $(C',D') \in \mathcal{M}(c',b) \times \mathcal{M}(x,c')$ with

$y \notin \{c, c'\}$. For a point $A \in \mathcal{M}(a, y)$, the interval $J \times \{A\}$ is embedded into $\mathbb{R}^{d_{i-1}} \times [0, \infty) \times \mathbb{R}^{d_i}$ using $\Gamma_{x,b\times a,y}$, and its framing is given by

$$fr(J \times \{A\}) = fr(J) \tag{6.2}$$

for any $A \in \mathcal{M}(a, y)$. If $y \in \{c, c'\}$ then the interval is glued to another and [JLS15] provides the following:

Proposition 6.3.2 In the case that $|x| = i + 1$, let $a \neq x$ and b be objects with $|a| = i + 1 = |b| + 2$ and $K \subset \mathcal{M}(\bar{a}, \bar{b})$.

1. If K is a circle, then either

   (a) $K \subset \mathcal{M}(a, b)$, in which case $fr(K)$ is the same as before.

   (b) $K = K' \times \{A\} \subset \mathcal{M}(x, b) \times \mathcal{M}(a, y)$ in which case $fr(K) = fr(K')$.

   (c) K is the result of gluing intervals $I_1, \ldots, I_k \subset \mathcal{M}(a, b)$ with intervals $J_1 \times \{A_1\}, \ldots, J_k \times \{A_k\} \subset \mathcal{M}(x, b) \times \mathcal{M}(a, y)$ in which case

$$fr(K) = k + \sum_{i=1}^{k} (fr(I_i) + fr(J_i)).$$

2. If K is an interval, then either

   (a) The endpoints of K are of the form $(C, A) \in \mathcal{M}(c, b) \times \mathcal{M}(a, c)$, $(C', A') \in \mathcal{M}(c', b) \times \mathcal{M}(a, c')$ with $c, c' \neq x$. Then K is the result of gluing intervals $I_1, \ldots, I_{k+1} \subset \mathcal{M}(a, b)$ with intervals $J_1 \times \{A_1\}, \ldots, J_k \times \{A_k\} \subset \mathcal{M}(x, b) \times \mathcal{M}(a, y)$ in which case

$$fr(K) = k + \sum_{i=1}^{k} (fr(I_i) + fr(J_i)) + fr(I_{k+1}).$$

   (b) The endpoints of K are of the form $(C, A) \in \mathcal{M}(c, b) \times \mathcal{M}(a, c)$, and $(C', D', A_k) \in \mathcal{M}(c', b) \times \mathcal{M}(x, c') \times \mathcal{M}(a, y)$ with $c, c' \neq x$. Then K is the result of gluing intervals $I_1, \ldots, I_k \subset \mathcal{M}(a, b)$ with intervals $J_1 \times \{A_1\}, \ldots, J_k \times \{A_k\} \subset \mathcal{M}(x, b) \times \mathcal{M}(a, y)$ in which case

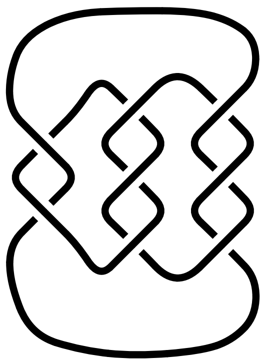
$$fr(K) = k + \varepsilon_{C'} + \varepsilon_{D'} + \sum_{i=1}^{k} (fr(I_i) + fr(J_i)).$$

(c) The endpoints of K are of the form $(C, D, A_1) \in \mathcal{M}(c,b) \times \mathcal{M}(x,c) \times \mathcal{M}(a,y)$ and $(C', D', A_{k+1}) \in \mathcal{M}(c',b) \times \mathcal{M}(x,c') \times \mathcal{M}(a,y)$ with $c, c' \neq x$. Then K is the result of gluing intervals $I_1, \ldots, I_k \subset \mathcal{M}(a,b)$ with intervals $J_1 \times \{A_1\}, \ldots, J_{k+1} \times \{A_{k+1}\} \subset \mathcal{M}(x,b) \times \mathcal{M}(a,y)$ in which case

$$fr(K) = 1 + k + \varepsilon_{C'} + \varepsilon_{D'} + \varepsilon_C + \varepsilon_D + \sum_{i=1}^{k}(fr(I_i) + fr(J_i)) + fr(J_{k+1}).$$

6.4 Example: The $P(-2,3,3)$ Pretzel Knot

In this section, we consider the knot 8_{19} in the form of the pretzel knot $P(-2,3,3)$ and highlight the power of Theorem 3.1.1 by computing its Steenrod square, as outlined in [LS14b], after cancellation of numerous objects in the Khovanov Flow Category. The significance of the knot 8_{19} is that it is the first knot for which the Khovanov homotopy type is not a wedge sum of Moore spaces, yielding the nontriviality result in [LS14b, Theorem 1]. In particular, this occurs in quantum degree 11, and $\mathcal{X}^{11}_{Kh}(8_{19})$ is not a wedge sum of Moore spaces. This is not, however, the smallest link which has a non-trivial Khovanov homotopy type. In [LS14b], the link with the fewest crossings satisfying this property is shown to be the 6-crossing link $L6n1$ which is also the $P(-2,2,2)$ pretzel knot. In fact, we will show that the 8_{19} example contains the $L6n1$ example, and this will be highlighted at the appropriate time. Consider the following diagram of $8_{19} = P(-2,3,3)$:



The process of performing Gaussian elimination was described in Subchapter 5.1, and applying this process to the Khovanov complex of $P(-2,3,3)$ in quantum degree 11 results in a chain complex that has 31 generators. These generators are in one-to-one correspondence with the objects of quantum degree 11 in the Khovanov

Table 6.1: Objects of $\mathscr{L}(D_{(-2,3,3)})$

Object	Generator	Grading	Object	Generator	Grading
α_1	$(D_{(-1,0,0)}, x_+x_+)$	1	α_{17}	$(D_{(-1,0,2)}, x_-)$	3
α_2	$(D_{(0,0,0)}, x_+x_+x_-)$	2	α_{18}	$(D_{(-2,3,0)}, x_-)$	3
α_3	$(D_{(0,0,0)}, x_+x_-x_+)$	2	α_{19}	$(D_{(-2,0,3)}, x_-)$	3
α_4	$(D_{(0,0,0)}, x_-x_+x_+)$	2	α_{20}	$(D_{(-2,1,2)}, x_+x_-)$	3
α_5	$(D_{(-2,2,0)}, x_+)$	2	α_{21}	$(D_{(-2,1,2)}, x_-x_+)$	3
α_6	$(D_{(-2,0,2)}, x_+)$	2	α_{22}	$(D_{(-2,2,1)}, x_+x_-)$	3
α_7	$(D_{(-1,1,0)}, x_+)$	2	α_{23}	$(D_{(-2,2,1)}, x_-x_+)$	3
α_8	$(D_{(-1,0,1)}, x_+)$	2	α_{24}	$(D_{(0,1,1)}, x_-)$	4
α_9	$(D_{(-2,1,1)}, x_+x_+)$	2	α_{25}	$(D_{(0,2,0)}, x_-x_-)$	4
α_{10}	$(D_{(0,1,0)}, x_+x_-)$	3	α_{26}	$(D_{(0,0,2)}, x_-x_-)$	4
α_{11}	$(D_{(0,1,0)}, x_-x_+)$	3	α_{27}	$(D_{(-1,2,1)}, x_-x_-)$	4
α_{12}	$(D_{(0,0,1)}, x_+x_-)$	3	α_{28}	$(D_{(-1,1,2)}, x_-x_-)$	4
α_{13}	$(D_{(0,0,1)}, x_-x_+)$	3	α_{29}	$(D_{(-2,1,3)}, x_-x_-)$	4
α_{14}	$(D_{(-1,1,1)}, x_+x_-)$	3	α_{30}	$(D_{(-2,2,2)}, x_-x_-)$	4
α_{15}	$(D_{(-1,1,1)}, x_-x_+)$	3	α_{31}	$(D_{(-2,3,1)}, x_-x_-)$	4
α_{16}	$(D_{(-1,2,0)}, x_-)$	3			

flow category and their homological degress lie between 1 and 4. They are listed in Table 6.1 (for a simpler example describing the notation used in the table, see Example 4.3.1). The flow category associated to this matched diagram, along with its 0-dimensional moduli spaces, is displayed in Figure 6.1. The signs are sprinkled as prescribed by the standard sign assignment of Definition 6.2.1 and the notation for the 0-dimensional moduli spaces is described in the following definition.

Definition 6.4.1 Suppose that $(E, y) <_1 (D, x)$ as in Definition 5.3.10. Let $j = 1, 2, 3$ where for each j, (D, x) differs from (E, y) by taking the $(s_j + 1)^{\text{th}}$ resolution instead of the s_j^{th} resolution. Let $R^j_{s_j} = +1$ and $L^j_{s_j} = (-1)^{s_j}$ denote the $\bullet$ maps from Definition 5.1.4 of the chain complex obtained after Gauss elimination, so that

$L_0^j = R_0^j$ are the moduli spaces that correspond to the saddle map s on the j^{th} elementary tangle, and $R_{s_j}^j + L_{s_j}^j$ is the two-point moduli space between the two resolutions on the j^{th} elementary tangle. Here, $R_{s_j}^j$ corresponds to an application of $\bullet$ on the right vertical strand of the resolution, and $L_{s_j}^j$ corresponds to an application of $\bullet$ on the left vertical strand. The saddle maps are the single point moduli spaces from [LS14a] corresponding to maps from 0-resolutions to 1-resolutions (see Figure 5.2).

Remark: Whilst the addition notation $R_{s_j}^j + L_{s_j}^j$ comes from an operation on the chain complex, we will use this notation to mean that there are two single point moduli spaces between the corresponding objects. Occasionally, this may be written as $p \sqcup q$, where p and q are both single point moduli spaces. This ambiguity should be understood throughout the worked example.

The higher dimensional moduli spaces are described in Subchapter 5.3 and are omitted from Figure 6.1 since the first six pairs of objects that we cancel can be chosen in such a way that the remaining moduli spaces are intact and all 1-dimensional. To see how this works, recall that in Definition 3.1.1, moduli spaces $\mathcal{M}(a,b)$ in the original category become $\mathcal{M}(\bar{a},\bar{b}) = \mathcal{M}(a,b) \times \big(\mathcal{M}(x,b) \times \mathcal{M}(a,y)\big)$ in the cancelled category. So let $\mathcal{M}(\alpha_i,\alpha_j) = *$ denote the pair being cancelled in $\mathscr{L}(D_{(-2,3,3)})$. The first six pairs can be cancelled in the following order, where $\mathcal{M}(\alpha_i,\alpha_l) \times \mathcal{M}(\alpha_k,\alpha_j) = \emptyset$ so $\mathcal{M}(\bar{\alpha}_k,\bar{\alpha}_l) = \mathcal{M}(\alpha_k,\alpha_l)$ for each $k,l \notin \{i,j\}$ (see Definition 3.1.1 and Figure 6.1, where the moduli spaces being cancelled are thickened):

1. $\mathcal{M}(\alpha_8,\alpha_1) = -R_0^3$
2. $\mathcal{M}(\alpha_{12},\alpha_3) = R_0^3$
3. $\mathcal{M}(\alpha_{10},\alpha_2) = R_0^2$
4. $\mathcal{M}(\alpha_{24},\alpha_{13}) = R_0^2$
5. $\mathcal{M}(\alpha_{31},\alpha_{18}) = -R_0^3$
6. $\mathcal{M}(\alpha_{29},\alpha_{19}) = R_0^2$

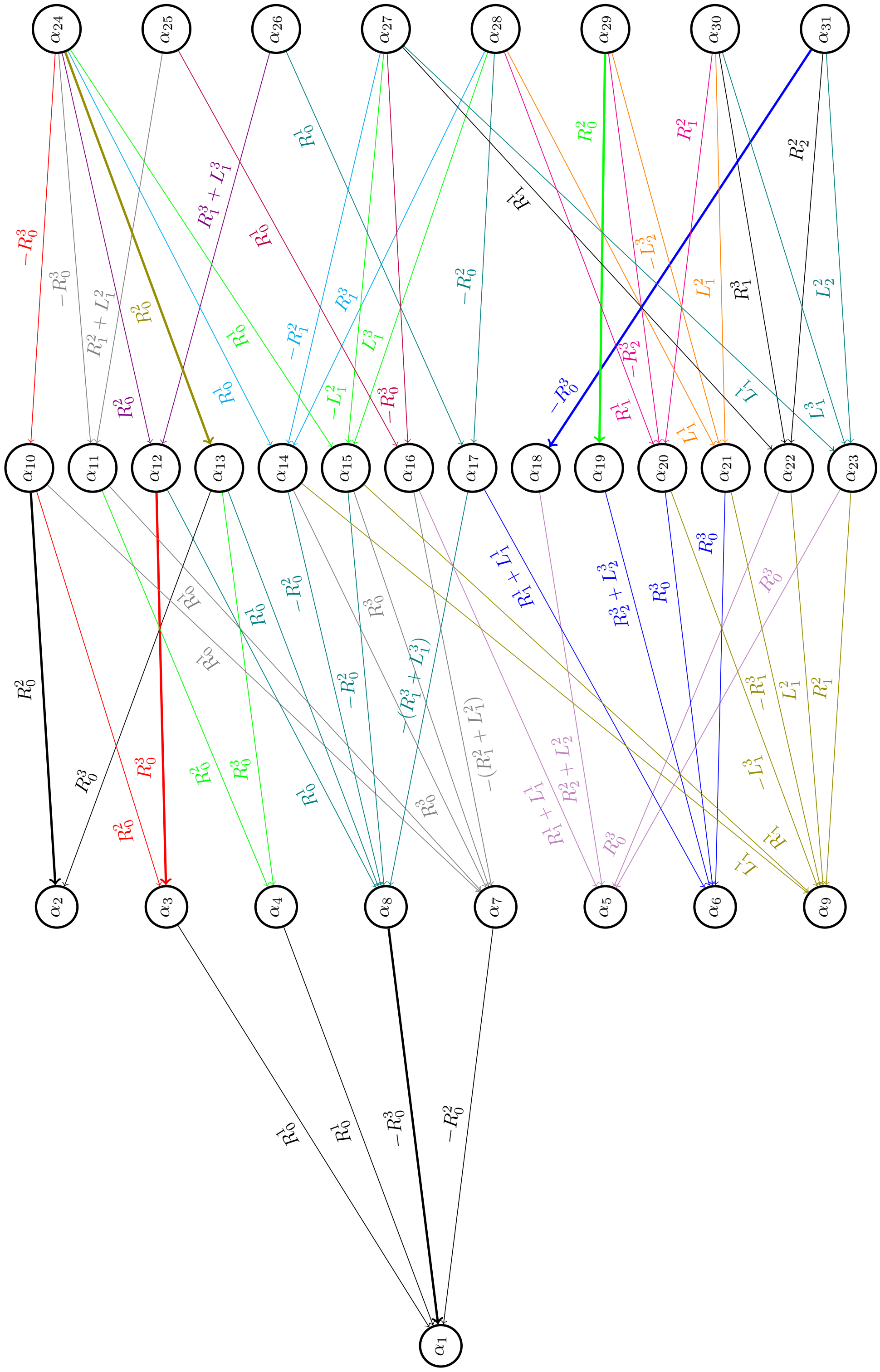


Figure 6.1: The flow category $\mathscr{L}(D_{(-2,3,3)})$ of $8_{19} = P(-2,3,3)$.

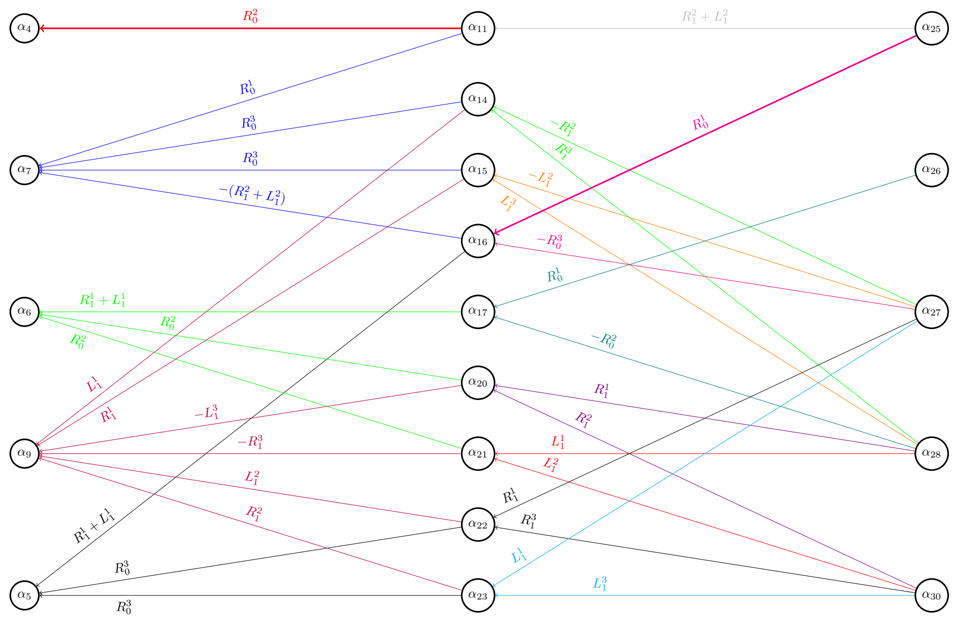


Figure 6.2: The first cancelled flow category $\mathscr{C}_1$ for $P(-2,3,3)$ (and $P(-2,2,2)$ by Proposition 6.4.2).

Proposition 6.4.2 The 6-crossing link $P(-2,2,2)$ has the same stable homotopy type as the 8-crossing knot $P(-2,3,3)$.

Proof: The 3-component pretzel link $P(-2,2,2)$ admits a matched diagram $D_{(-2,2,2)}$. All of the objects of the flow category $\mathscr{L}(D_{(-2,2,2)})$ are also objects of $\mathscr{L}(D_{(-2,3,3)})$, where the latter has four additional objects corresponding to all coordinates of the form $(*,*,3)$ and $(*,3,*)$. These objects are α_{18}, α_{19}, α_{29} and α_{31}; all of which get cancelled immediately with no effect on the remaining moduli spaces. This means that the resulting stable homotopy types of $P(-2,2,2)$ and $P(-2,3,3)$ must be equivalent. $\square$

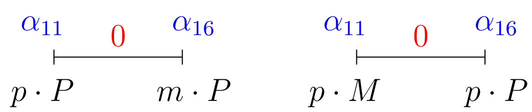
After cancelling the six pairs listed above, the resulting flow category, which we shall denote by $\mathscr{C}_1$, contains 19 objects, and is displayed together with its 0-dimensional moduli spaces in Figure 6.2. The 1-dimensional moduli spaces have frame assignments as prescribed in Definition 6.2.2 and are listed in Figure 6.3. In order to calculate the framings of each of the intervals, one must refer to the R^i_j, L^i_j notation of the 0-dimensional moduli spaces in Figure 6.2. However, once these framings have been calculated, this notation is excessive and we simply refer to a plus (respectively, a minus) using either a p or P (respectively, using m or M) where lower case corresponds to a moduli space between homological degree 2 and 3, and upper case corresponds to a moduli space between 3 and 4. The boundary points of the 1-dimensional moduli spaces (which are listed in Figure 6.3) are labelled using this notation. The objects of the flow category that correspond to each particular boundary are also highlighted (in blue).

6.4.1 Cancelling moduli spaces

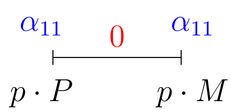
Continuing the process of cancellation further, we shall cancel the remaining single-point moduli spaces a couple of pairs at a time; one pair with cohomological degrees 4 and 3, the other with cohomological degrees 3 and 2. The pairs being cancelled are all highlighted in their corresponding flow category by a thickened line. Thus, to obtain the subsequent cancelled flow category $\mathscr{C}_2$ of Figure 6.4, let us cancel $\mathcal{M}(\alpha_{11}, \alpha_4) = R_0^2$ followed by $\mathcal{M}(\alpha_{25}, \alpha_{16}) = R_0^1$. By cancelling the moduli space $\mathcal{M}(\alpha_{11}, \alpha_4)$, the moduli space $\mathcal{M}(\alpha_{25}, \alpha_7)$ is altered, and is now given by

$$\mathcal{M}(\bar{\alpha}_{25}, \bar{\alpha}_7) = \mathcal{M}(\alpha_{25}, \alpha_7) \cup \mathcal{M}(\alpha_{11}, \alpha_7) \times \mathcal{M}(\alpha_{25}, \alpha_4).$$

Originally, $\mathcal{M}(\alpha_{25}, \alpha_7)$ consists of the two intervals



and the product moduli space $\mathcal{M}(\alpha_{11}, \alpha_7) \times \mathcal{M}(\alpha_{25}, \alpha_4)$ is the single interval

$\{p\}\times$ 

where, again, the framings (computed using Definition 6.2.2) of the intervals are labelled in red and the boundaries of the intervals are given by the product of the lower-dimensional moduli spaces, as illustrated. The gluing of these two moduli spaces takes place along the point being cancelled, α_{11}, and the result is the single interval $\mathcal{M}(\bar{\alpha}_{25}, \bar{\alpha}_7)$ given by

$$\mathcal{M}(\alpha_{25},\alpha_4) = \begin{array}{ccc} \alpha_{11} & 0 & \alpha_{11} \\ \vdash & \rule[0.5ex]{3em}{0.4pt} & \dashv \\ p\cdot P & & p\cdot M \end{array}$$

$$\mathcal{M}(\alpha_{25},\alpha_5) = \begin{array}{ccc} \alpha_{16} & 0 & \alpha_{16} \\ \vdash & \rule[0.5ex]{3em}{0.4pt} & \dashv \\ p\cdot P & & m\cdot P \end{array}$$

$$\mathcal{M}(\alpha_{25},\alpha_7) = \begin{array}{ccc} \alpha_{11} & 0 & \alpha_{16} \\ \vdash & \rule[0.5ex]{3em}{0.4pt} & \dashv \\ p\cdot P & & m\cdot P \end{array} \qquad \begin{array}{ccc} \alpha_{11} & 0 & \alpha_{16} \\ \vdash & \rule[0.5ex]{3em}{0.4pt} & \dashv \\ p\cdot M & & p\cdot P \end{array}$$

$$\mathcal{M}(\alpha_{26},\alpha_6) = \begin{array}{ccc} \alpha_{17} & 0 & \alpha_{17} \\ \vdash & \rule[0.5ex]{3em}{0.4pt} & \dashv \\ p\cdot P & & m\cdot P \end{array}$$

$$\mathcal{M}(\alpha_{27},\alpha_5) = \begin{array}{ccc} \alpha_{16} & 0 & \alpha_{22} \\ \vdash & \rule[0.5ex]{3em}{0.4pt} & \dashv \\ p\cdot M & & p\cdot P \end{array} \qquad \begin{array}{ccc} \alpha_{16} & 1 & \alpha_{23} \\ \vdash & \rule[0.5ex]{3em}{0.4pt} & \dashv \\ m\cdot M & & p\cdot M \end{array}$$

$$\mathcal{M}(\alpha_{27},\alpha_7) = \begin{array}{ccc} \alpha_{14} & 1 & \alpha_{16} \\ \vdash & \rule[0.5ex]{3em}{0.4pt} & \dashv \\ p\cdot M & & m\cdot M \end{array} \qquad \begin{array}{ccc} \alpha_{15} & 0 & \alpha_{16} \\ \vdash & \rule[0.5ex]{3em}{0.4pt} & \dashv \\ p\cdot P & & p\cdot M \end{array}$$

$$\mathcal{M}(\alpha_{27},\alpha_9) = \begin{array}{ccc} \alpha_{14} & 1 & \alpha_{23} \\ \vdash & \rule[0.5ex]{3em}{0.4pt} & \dashv \\ m\cdot M & & p\cdot M \end{array} \qquad \begin{array}{ccc} \alpha_{15} & 0 & \alpha_{22} \\ \vdash & \rule[0.5ex]{3em}{0.4pt} & \dashv \\ p\cdot P & & m\cdot P \end{array}$$

$$\mathcal{M}(\alpha_{28},\alpha_6) = \begin{array}{ccc} \alpha_{17} & 0 & \alpha_{20} \\ \vdash & \rule[0.5ex]{3em}{0.4pt} & \dashv \\ p\cdot M & & p\cdot P \end{array} \qquad \begin{array}{ccc} \alpha_{17} & 1 & \alpha_{21} \\ \vdash & \rule[0.5ex]{3em}{0.4pt} & \dashv \\ m\cdot M & & p\cdot M \end{array}$$

$$\mathcal{M}(\alpha_{28},\alpha_7) = \begin{array}{ccc} \alpha_{14} & 0 & \alpha_{15} \\ \vdash & \rule[0.5ex]{3em}{0.4pt} & \dashv \\ p\cdot P & & p\cdot M \end{array}$$

$$\mathcal{M}(\alpha_{28},\alpha_9) = \begin{array}{ccc} \alpha_{14} & 0 & \alpha_{21} \\ \vdash & \rule[0.5ex]{3em}{0.4pt} & \dashv \\ m\cdot P & & m\cdot M \end{array} \qquad \begin{array}{ccc} \alpha_{15} & 0 & \alpha_{20} \\ \vdash & \rule[0.5ex]{3em}{0.4pt} & \dashv \\ p\cdot M & & p\cdot P \end{array}$$

$$\mathcal{M}(\alpha_{30},\alpha_5) = \begin{array}{ccc} \alpha_{22} & 0 & \alpha_{23} \\ \vdash & \rule[0.5ex]{3em}{0.4pt} & \dashv \\ p\cdot P & & p\cdot M \end{array}$$

$$\mathcal{M}(\alpha_{30},\alpha_6) = \begin{array}{ccc} \alpha_{20} & 0 & \alpha_{21} \\ \vdash & \rule[0.5ex]{3em}{0.4pt} & \dashv \\ p\cdot P & & p\cdot M \end{array}$$

$$\mathcal{M}(\alpha_{30},\alpha_9) = \begin{array}{ccc} \alpha_{20} & 0 & \alpha_{23} \\ \vdash & \rule[0.5ex]{3em}{0.4pt} & \dashv \\ p\cdot P & & p\cdot M \end{array} \qquad \begin{array}{ccc} \alpha_{21} & 0 & \alpha_{22} \\ \vdash & \rule[0.5ex]{3em}{0.4pt} & \dashv \\ m\cdot M & & m\cdot P \end{array}$$

Figure 6.3: 1-dimensional moduli spaces for the first cancelled flow category $\mathscr{C}_1$.

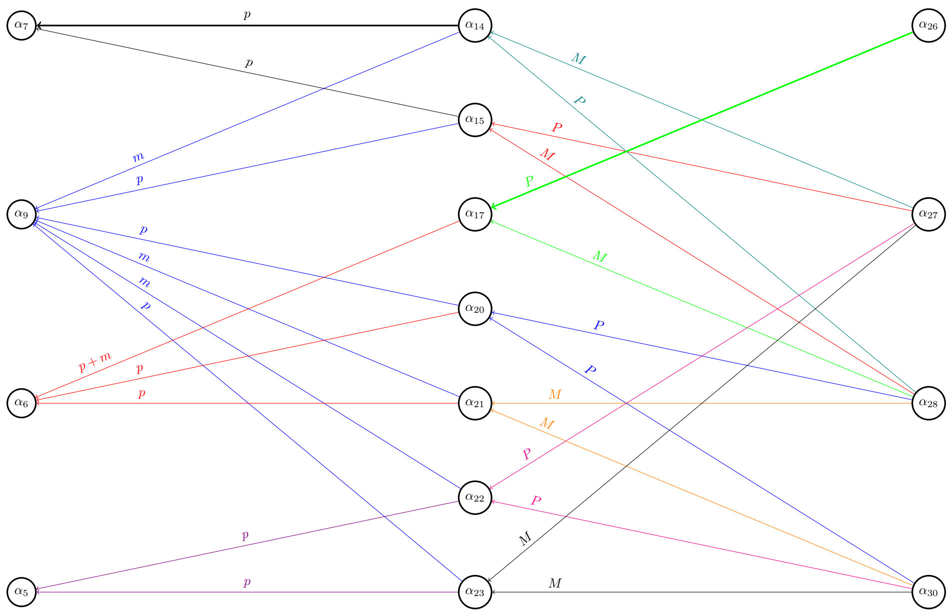
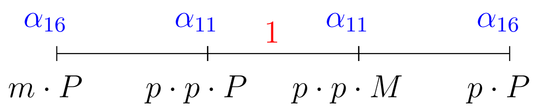


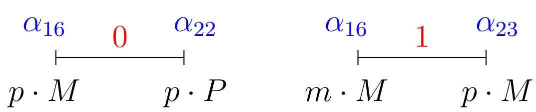
Figure 6.4: The second cancelled flow category $\mathscr{C}_2$.



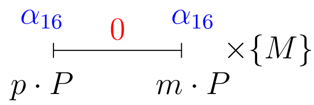
whose framing is calculated using part (2a) of Proposition 6.3.1 as follows. Firstly, part (2a) is used since the cancelled pair lie in degrees 2 and 3. The single interval $K = \mathcal{M}(\bar{\alpha}_{25}, \bar{\alpha}_7)$ is obtained by gluing the two intervals $I_1, I_2 \subset \mathcal{M}(\alpha_{25}, \alpha_7)$ to the single interval $\{B_1\} \times J_1 = \mathcal{M}(\alpha_{11}, \alpha_7) \times \mathcal{M}(\alpha_{25}, \alpha_4)$. Moreover, $k = 1$, $\varepsilon_* = 0$, and $\varepsilon_{B_1} = 0$ and

$$\begin{aligned} fr(K) &= 1(1+\varepsilon_*) + (fr(I_1) + fr(J_1) + \varepsilon_{B_1}) + fr(I_2) \\ &= 1(1+0) + (0+0+0) + 0 \equiv 1 \pmod 2. \end{aligned}$$

Secondly, cancelling the moduli space $\mathcal{M}(\alpha_{25}, \alpha_{16})$ effects the two 1-dimensional moduli spaces $\mathcal{M}(\alpha_{27}, \alpha_5)$ and $\mathcal{M}(\alpha_{27}, \alpha_7)$. These moduli spaces both become single intervals as follows. The moduli space $\mathcal{M}(\alpha_{27}, \alpha_5)$ originally consists of the two intervals



whose framings are highlighted in red. The product moduli space $\mathcal{M}(\alpha_{25}, \alpha_5) \times \mathcal{M}(\alpha_{27}, \alpha_{16})$ is the single interval

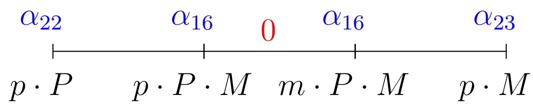


and these two moduli spaces are glued together, along the moduli spaces that break at α_{16}, to form

$$\mathcal{M}(\bar{\alpha}_{27},\bar{\alpha}_5) = \mathcal{M}(\alpha_{27},\alpha_5) \cup \mathcal{M}(\alpha_{25},\alpha_5) \times \mathcal{M}(\alpha_{27},\alpha_{16})$$

which is the single interval

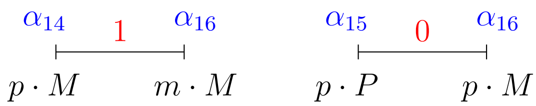
$\mathcal{M}(\bar{\alpha}_{27},\bar{\alpha}_5) =$



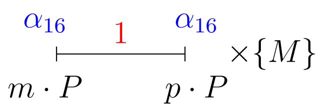
whose framing is calculated using part (2a) of Proposition 6.3.2 as follows. Firstly, the cancelled pair lie in degrees 4 and 3. The single interval $K = \mathcal{M}(\bar{\alpha}_{27},\bar{\alpha}_5)$ is obtained by gluing two intervals $I_1, I_2 \subset \mathcal{M}(\alpha_{27},\alpha_5)$ to the single interval $J_1 \times \{A_1\} = \mathcal{M}(\alpha_{25},\alpha_5) \times \mathcal{M}(\alpha_{27},\alpha_{16})$. Moreover, $k = 1$ and

$$\begin{aligned} fr(K) &= 1(fr(I_1) + fr(J_1)) + fr(I_2) \\ &= 1 + (0+0) + 1 \equiv 0 \pmod 2. \end{aligned}$$

The second moduli space that is altered as a result of cancelling the moduli space $\mathcal{M}(\alpha_{25},\alpha_{16})$ is $\mathcal{M}(\alpha_{27},\alpha_7)$, which originally consists of the two intervals



This moduli space is glued to the product moduli space $\mathcal{M}(\alpha_{25},\alpha_7) \times \mathcal{M}(\alpha_{27},\alpha_{16})$, which is the single interval



These two moduli spaces are also glued together along the moduli spaces that break at α_{16} to form

$$\mathcal{M}(\bar{\alpha}_{27},\bar{\alpha}_7) = \mathcal{M}(\alpha_{27},\alpha_7) \cup \mathcal{M}(\alpha_{25},\alpha_7) \times \mathcal{M}(\alpha_{27},\alpha_{16})$$

which is the single interval

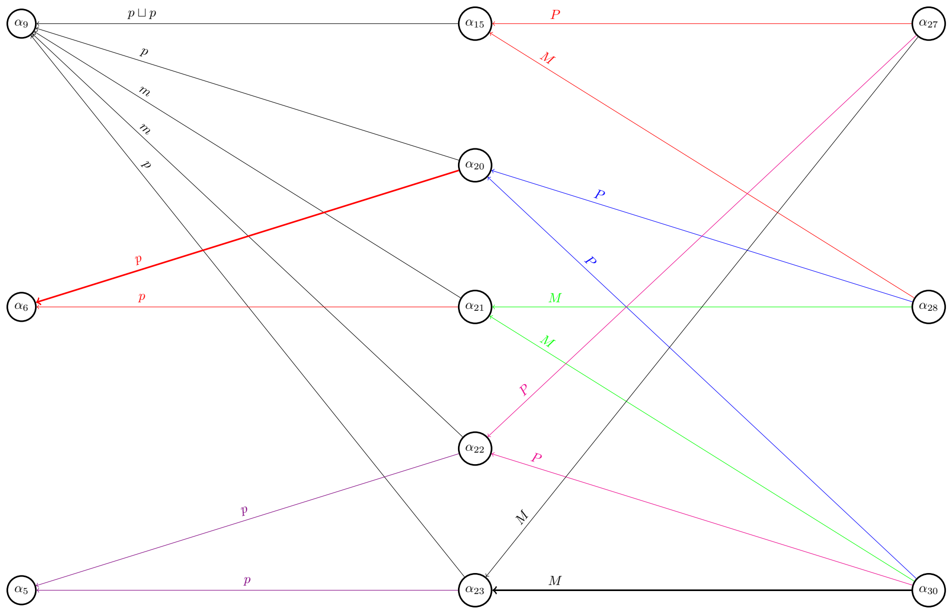
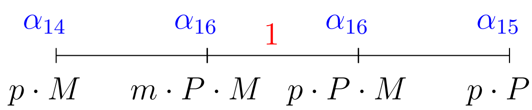


Figure 6.5: The third cancelled flow category $\mathscr{C}_3$.



The framing is also computed using part (2a) of Proposition 6.3.2 since the cancelled pair is still the same, and the interval $K = \mathcal{M}(\bar{\alpha}_{27}, \bar{\alpha}_7)$ is formed by gluing two intervals $I_1, I_2 \subset \mathcal{M}(\alpha_{27}, \alpha_7)$ to the single interval $J_1 \times \{A_1\} = \mathcal{M}(\alpha_{25}, \alpha_7) \times \mathcal{M}(\alpha_{27}, \alpha_{16})$. Moreover, $k = 1$ and

$$\begin{aligned} fr(K) &= 1(fr(I_1) + fr(J_1)) + fr(I_2) \\ &= 1 + (1+1) + 0 \equiv 1 \pmod 2. \end{aligned}$$

Further calculation of the following 1-dimensional moduli spaces along with their framings is left to the reader, and we proceed as follows. To obtain the next cancelled flow category $\mathscr{C}_3$ (Figure 6.5), we cancel the moduli space $\mathcal{M}(\alpha_{26}, \alpha_{17}) = P$ followed by the moduli space $\mathcal{M}(\alpha_{14}, \alpha_7) = p$. There is only one 1-dimensional moduli space affected by cancelling $\mathcal{M}(\alpha_{26}, \alpha_{17})$, which is $\mathcal{M}(\alpha_{28}, \alpha_6)$. The new moduli space

$$\mathcal{M}(\bar{\alpha}_{28}, \bar{\alpha}_6) = \mathcal{M}(\alpha_{28}, \alpha_6) \cup \times \mathcal{M}(\alpha_{26}, \alpha_6) \times \mathcal{M}(\alpha_{28}, \alpha_{17})$$

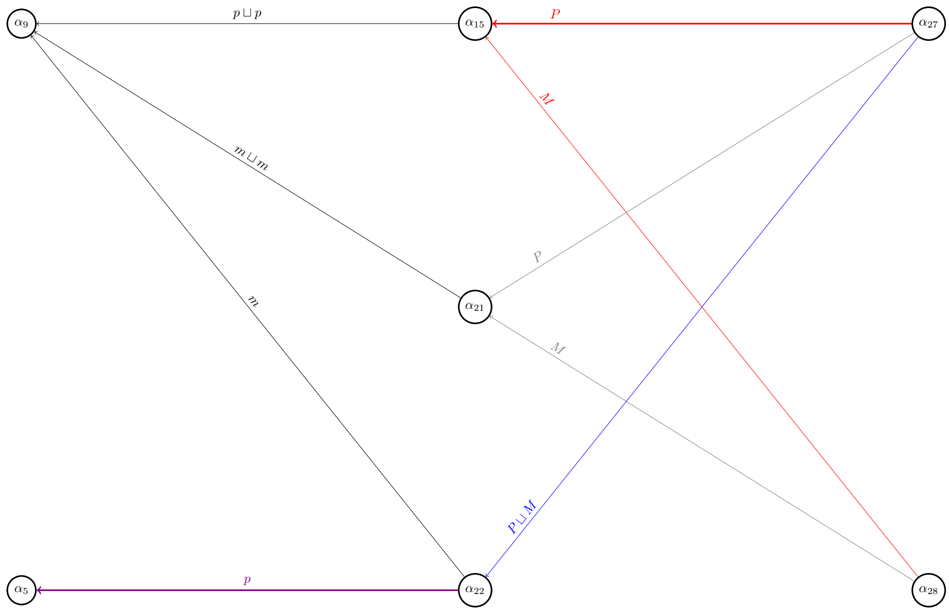


Figure 6.6: The fourth cancelled flow category $\mathscr{C}_4$.

is once again an interval obtained by gluing one interval between two others (as above). Cancelling $\mathcal{M}(\alpha_{14}, \alpha_7)$ alters two 1-dimensional moduli spaces. They are

$$\mathcal{M}(\bar{\alpha}_{27}, \bar{\alpha}_9) = \mathcal{M}(\alpha_{27}, \alpha_9) \cup \mathcal{M}(\alpha_{14}, \alpha_9) \times \mathcal{M}(\alpha_{27}, \alpha_7)$$

$$\mathcal{M}(\bar{\alpha}_{28}, \bar{\alpha}_9) = \mathcal{M}(\alpha_{28}, \alpha_9) \cup \mathcal{M}(\alpha_{14}, \alpha_9) \times \mathcal{M}(\alpha_{28}, \alpha_7).$$

They way in which both of these two new moduli spaces are formed is by gluing two intervals together along a single point, whilst a separate interval remains unchanged. These examples require the use of the gluing formula from part (2b) of Proposition 6.3.1.

The fourth cancelled flow category $\mathscr{C}_4$ (Figure 6.6) is obtained by cancelling the moduli space $\mathcal{M}(\alpha_{20}, \alpha_6) = p$ followed by the moduli space $\mathcal{M}(\alpha_{30}, \alpha_{23}) = M$. As a result of cancelling $\mathcal{M}(\alpha_{20}, \alpha_6)$, two 1-dimensional moduli spaces are altered. These are:

$$\mathcal{M}(\bar{\alpha}_{28}, \bar{\alpha}_9) = \mathcal{M}(\alpha_{28}, \alpha_9) \cup \mathcal{M}(\alpha_{20}, \alpha_9) \times \mathcal{M}(\alpha_{28}, \alpha_6)$$

$$\mathcal{M}(\bar{\alpha}_{30}, \bar{\alpha}_9) = \mathcal{M}(\alpha_{30}, \alpha_9) \cup \mathcal{M}(\alpha_{20}, \alpha_9) \times \mathcal{M}(\alpha_{30}, \alpha_6).$$

Cancelling $\mathcal{M}(\alpha_{30}, \alpha_{23})$ also alters two 1-dimensional moduli spaces:

$$\mathcal{M}(\bar{\alpha}_{27}, \bar{\alpha}_9) = \mathcal{M}(\alpha_{27}, \alpha_9) \cup \mathcal{M}(\alpha_{30}, \alpha_9) \times \mathcal{M}(\alpha_{27}, \alpha_{23})$$

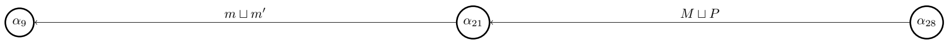


Figure 6.7: The fifth cancelled flow category $\mathscr{C}_{\mathrm{Fin}}$.

$$\mathcal{M}(\bar{\alpha}_{27}, \bar{\alpha}_5) = \mathcal{M}(\alpha_{27}, \alpha_5) \cup \mathcal{M}(\alpha_{30}, \alpha_5) \times \mathcal{M}(\alpha_{27}, \alpha_{23}).$$

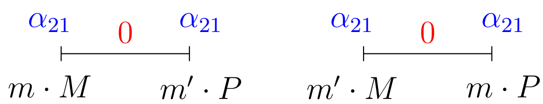
Finally, we obtain the fifth (and final) cancelled flow category $\mathscr{C}_{\mathrm{Fin}}$ (Figure 6.7) by cancelling the moduli space $\mathcal{M}(\alpha_{22}, \alpha_5) = p$ followed by the moduli space $\mathcal{M}(\alpha_{27}, \alpha_{15}) = P$. After the first of these cancellations, there is one new 1-dimensional moduli space

$$\mathcal{M}(\bar{\alpha}_{27}, \bar{\alpha}_9) = \mathcal{M}(\alpha_{27}, \alpha_9) \cup \mathcal{M}(\alpha_{22}, \alpha_9) \times \mathcal{M}(\alpha_{27}, \alpha_5).$$

The final sole 1-dimensional moduli space is

$$\mathcal{M}(\bar{\alpha}_{28}, \bar{\alpha}_9) = \mathcal{M}(\alpha_{28}, \alpha_9) \cup \mathcal{M}(\alpha_{27}, \alpha_9) \times \mathcal{M}(\alpha_{28}, \alpha_{15})$$

which consists of the two intervals



whose framings are calculated using Propositions 6.3.1 and 6.3.2 in the sequence described above.

6.4.2 Computing Steenrod squares

In this subsection, we shall exhibit the consistency of the Steenrod square calculations after the series of cancellations described above. We shall first highlight how the Steenrod square can be calculated from the first cancelled flow category $\mathscr{C}_1$ (Figure 6.2). It is the non-triviality of this Steenrod square that exhibits the non-triviality result in [LS14b, Theorem 1]. Firstly, a choice of topological boundary matchings has to be made and these are highlighted in Figure 6.8. Notice in particular that there are three pairs of choices for η_{16} (since α_7 and α_5 both have two-point moduli spaces from α_{16}), and we have chosen a boundary-coherent matching of the two points corresponding to α_5, one with a positive and the other with a negative (similarly for α_7). According to these choices of topological boundary matchings, a

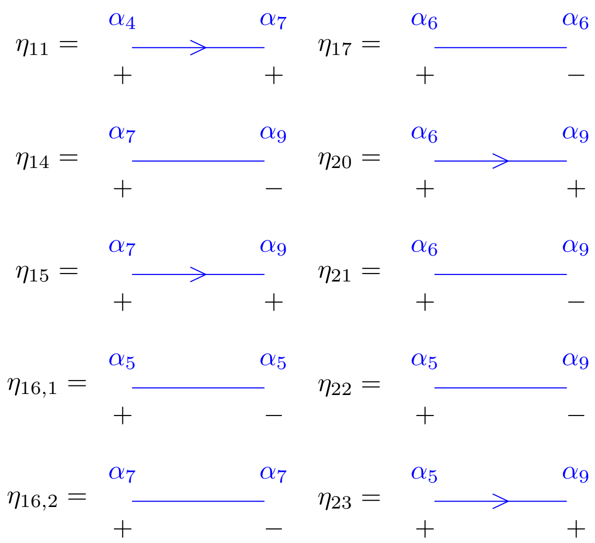


Figure 6.8: Choice of topological boundary matchings from the first cancelled flow category $\mathscr{C}_1$.

Steenrod square can be computed using Proposition 6.1.5 in the following way. Let $u \in H^2(\mathscr{C}_1; \mathbb{Z}/2)$ be the cohomology class represented by the cocycle $c \in C^2(\mathscr{C}_1; \mathbb{Z}/2)$ given by

$$c = \alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_9.$$

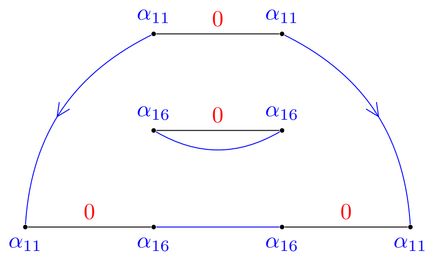
Following the construction of Subchapter 6.1, the Steenrod square $\mathrm{Sq}^2(u)$ is given by the sum of the values assigned to each framed circle (K, Φ) inside $\partial\mathcal{C}(z)$ for each $z \in \{\alpha_{25}, \alpha_{26}, \alpha_{27}, \alpha_{28}, \alpha_{30}\}$. The computation of the Steenrod square is illustrated in Figure 6.9 and the individual components of the sum are listed below (where we count arrows in the clockwise direction and sum in the order corresponding to Proposition 6.1.5):

- The component from $\partial\mathcal{C}(\alpha_{25})$ is $1 + 0 + 1 \equiv 0 \pmod 2$ for the outside circle, and $1 + 0 + 0 \equiv 1 \pmod 2$ from the inside circle, giving a total of $1 \pmod 2$.
- The component from $\partial\mathcal{C}(\alpha_{26})$ is $1 + 0 + 0 \equiv 1 \pmod 2$.
- The component from $\partial\mathcal{C}(\alpha_{27})$ is $1 + 3 + 2 \equiv 0 \pmod 2$.
- The component from $\partial\mathcal{C}(\alpha_{28})$ is $1 + 1 + 1 \equiv 1 \pmod 2$.
- The component from $\partial\mathcal{C}(\alpha_{30})$ is $1 + 0 + 1 \equiv 0 \pmod 2$.

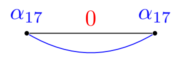
Thus giving a non-trivial Steenrod square $\mathrm{Sq}^2(u) = 1 \in \mathbb{Z}/2$.

In the previous section, the first cancellation that produced new moduli spaces

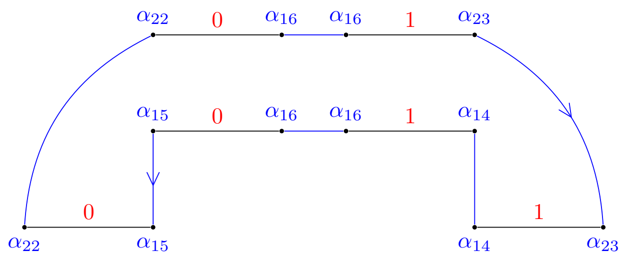
$\partial\mathcal{C}(\alpha_{25})$:



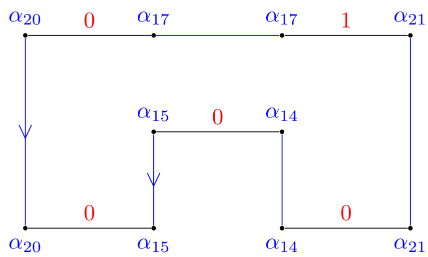
$\partial\mathcal{C}(\alpha_{26})$:



$\partial\mathcal{C}(\alpha_{27})$:



$\partial\mathcal{C}(\alpha_{28})$:



$\partial\mathcal{C}(\alpha_{30})$:

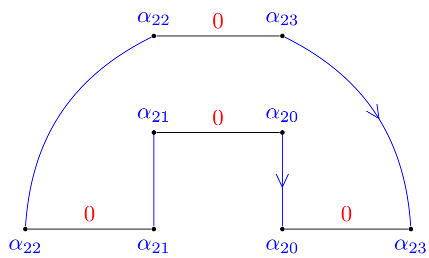
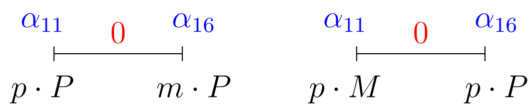
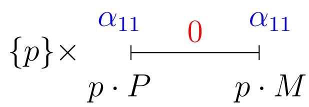


Figure 6.9: Computation of the Steenrod square from the first cancelled flow category.

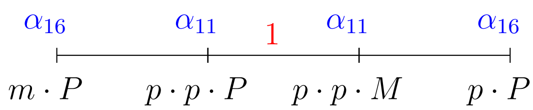
was the moduli space $\mathcal{M}(\alpha_{11},\alpha_4)$. Recall that the moduli space $\mathcal{M}(\alpha_{25},\alpha_7)$ was originally given by the two intervals



and the product moduli space $\mathcal{M}(\alpha_{11},\alpha_7)\times\mathcal{M}(\alpha_{25},\alpha_4)$ was a single interval



As a result of cancelling $\mathcal{M}(\alpha_{11},\alpha_4)$, these two moduli spaces are glued together to form $\mathcal{M}(\bar{\alpha}_{25},\bar{\alpha}_7)$, the single interval



which was shown to have a non-trivial framing using Proposition 6.3.1. This is a nice example that highlights the necessity of a gluing formula. Without it, the framing would be 0 and the Steenrod square would be trivial; so the framing must necessarily change.

In the previous subsection, a series of subsequent cancellations were made resulting in the flow category $\mathscr{C}_{Fin}$ (see Figure 6.7), which has no single-point moduli spaces. To emphasise that this (relatively small) framed flow category contains all the information that was essential in the original framed flow category in Figure 6.1, we can compute the Steenrod square from here. The single topological boundary matching and computation is illustrated in Figure 6.10. The only component for the Steenrod square (where we count arrows in the clockwise direction and sum in the order corresponding to Proposition 6.1.5) is

$$\mathrm{Sq}^2(u) = 1 + 0 + 2 \equiv 1 \bmod 2.$$

The Khovanov stable homotopy type $\mathcal{X}^{11}_{Kh}(8_{19})$ can then be determined immediately as $\Sigma^{-1}\mathbb{R}\mathbf{P}^5/\mathbb{R}\mathbf{P}^2$, where it is necessary to desuspend once since the Steenrod square is being computed from cohomological degree $i = 2$ (see [LS14b, Corollary

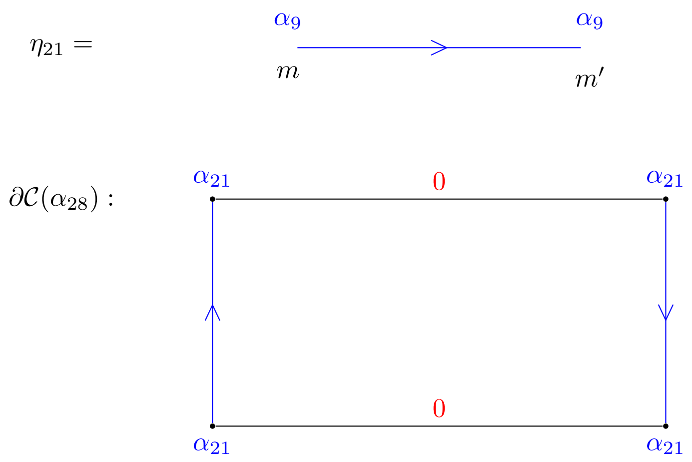


Figure 6.10: Computation of the Steenrod square from the final cancelled framed flow category $\mathscr{C}_{\text{Fin}}$.

4.4]). Consequently, this is also the same stable homotopy type of the pretzel link $P(-2,2,2)$.

Finally, it should be noted that in [LS14b, Table 1], the stable homotopy type of $L6n1$ is computed as $\Sigma^{-2}\mathbb{R}\mathbf{P}^4/\mathbb{R}\mathbf{P}^1$, but this arises since Lipshitz-Sarkar are really computing the Khovanov stable homotopy type of the mirror image of the pretzel link $P(2,-2,-2) = \bar{P}(-2,2,2)$. This provides evidence for the conjecture in [LS14a, Conjecture 10.1] that Khovanov spectrum of the mirror image of a link is given by the Spanier-Whitehead dual. In particular, the two truncated real projective spaces are Spanier-Whitehead duals and we have the following.

Proposition 6.4.3 Let $\mathcal{X}_{Kh}^{\vee}$ denote the Spanier-Whitehead dual of the Khovanov spectrum $\mathcal{X}_{Kh}$. Then

$$\mathcal{X}_{Kh}(P(2,-2,-2)) = \mathcal{X}_{Kh}(P(-2,2,2))^{\vee}.$$

Bibliography

[AB95] D.M. Austin and P.J. Braam. Morse-Bott theory and equivariant cohomology. In *The Floer memorial volume*, volume 133 of *Progr. Math.*, pages 123–183. Birkhäuser, 1995.

[Bal10] J.A. Baldwin. On the spectral sequence from Khovanov homology to Heegaard Floer homology. *Int. Math. Res. Not.*, 2010, 2010.

[Bau95] Hans-Joachim Baues. Morse-Bott theory and equivariant cohomology. In *Handbook of Algebraic Topology*, pages 1–72. North-Holland, Amsterdam, 1995.

[BH04] Augustin Banyaga and David Hurtubise. *Lectures on Morse homology*, volume 29 of *Kluwer Texts in the Mathematical Sciences*. Kluwer Academic Publishers Group, Dordrecht, 2004.

[BN02] Dror Bar-Natan. On Khovanov's categorification of the Jones polynomial. *Algebr. Geom. Topol.*, 2:337, 2002.

[BN05] Dror Bar-Natan. Khovanov's homology for tangles and cobordisms. *Geom. Topol.*, 9:1443–1499, 2005.

[BN07] Dror Bar-Natan. Fast Khovanov homology computations. *J. Knot Theory Ramifications*, 16(3):243–255, 2007.

[CJS95a] R. Cohen, J.D.S. Jones, and G.B. Segal. Floer's infinite dimensional Morse theory and homotopy theory. In *The Floer Memorial Volume*, volume 133 of *Prog. in Math.*, pages 297–325. Birkhauser Verlag, 1995.

[CJS95b] Ralph L. Cohen, John D. S. Jones, and Graeme B. Segal. Morse theory and classifying spaces, 1995.

[Fra79] John M. Franks. Morse-smale flows and homotopy theory. *Topology*, 18(3):199 – 215, 1979.

[Hat02] Allen Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002.

[Hir94] Morris W. Hirsch. *Differential topology*, volume 33 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994. Corrected reprint of the 1976 original.

[Jän68] K. Jänich. On the classification of O(n)-manifolds. *Math. Ann.*, 176:53–76, 1968.

[JLS15] D. Jones, A. Lobb, and D. Schütz. Milnor cancellation in flow categories, the Khovanov homotopy type, and an $\mathfrak{sl}_n$ stable homotopy type for bipartite knots. *preprint*, 2015.

[Kho00] M. Khovanov. A categorification of the Jones polynomial. *Duke Math. J.*, 101(3):359–426, 2000.

[KR08a] M. Khovanov and L. Rozansky. Matrix factorizations and link homology. *Fund. Math.*, 199(1):1–91, 2008.

[KR08b] Mikhail Khovanov and Lev Rozansky. Matrix factorizations and link homology. *Fund. Math.*, 199(1):1–91, 2008.

[Kra09] Daniel Krasner. A computation in Khovanov-Rozansky homology. *Fund. Math.*, 203(1):75–95, 2009.

[Lau00] G. Laures. On cobordism of manifolds with corners. *Trans. Amer. Math. Soc.*, 352(12):5667–5688, 2000.

[Lob09] Andrew Lobb. A slice genus lower bound from sl(n) Khovanov-Rozansky homology. *Adv. Math.*, 222(4):1220–1276, 2009.

[Lob12] Andrew Lobb. A note on Gornik's perturbation of Khovanov-Rozansky homology. *Algebr. Geom. Topol.*, 12(1):293–305, 2012.

[LS14a] R. Lipshitz and S. Sarkar. A Khovanov stable homotopy type. *J. Amer. Math. Soc.*, 27:983–1042, 2014.

[LS14b] R. Lipshitz and S. Sarkar. A steenrod square on Khovanov homology. *J. Topol.*, 2014.

[McC01] John McCleary. *A user's guide to spectral sequences*, volume 58 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2001.

[Mil65] John Milnor. *Lectures on the h-cobordism theorem*. Notes by L. Siebenmann and J. Sondow. Princeton University Press, Princeton, N.J., 1965.

[ORS13] P. Ozsváth, J. Rasmussen, and Z. Szabó. Odd Khovanov homology. *Alg. Geom. Top.*, 13:1465–1488, 2013.

[OS05] P. Ozsváth and Z. Szabó. On the Heegaard Floer homology of branched double-covers. *Adv. Math.*, 194(1):1–33, 2005.

[Ras04] J. Rasmussen. Khovanov homology and the slice genus. *Inventiones Mathematicae*, 2004.

[See12] C. Seed. Computations of the Lipshitz-Sarkar Steenrod Square on Khovanov Homology. 2012.

[Wu09] Hao Wu. On the quantum filtration of the Khovanov-Rozansky cohomology. *Adv. Math.*, 221(1):54–139, 2009.