The Field of Norms Functor and the Hilbert Symbol

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Abstract

The classical Hilbert symbol of a higher local field F containing a primitive p^M-th root of unity ζ_M is a pairing $F^*/(F^*)^{p^M} \times K_N(F)/p^M \to \mu_{p^M}$, describing Kummer extensions of exponent p^M. In this thesis we define a generalised Hilbert symbol and prove a formula for it. Our approach has several ingredients.

The field of norms functor of Scholl associates to any strictly deeply ramified tower $F_\bullet$ a field $\mathcal{F}$ of characteristic p. Separable extensions of $\mathcal{F}$ correspond functorially to extensions of $F_\bullet$, giving rise to $\Gamma_{\mathcal{F}} \cong \Gamma_{F_\infty} \subset \Gamma_F$.

We define morphisms $\mathcal{N}_{\mathcal{F}/F_n} : K_N^t(\mathcal{F})/p^M \to K_N^t(F_n)/p^M$ which are compatible with the norms N_{F_{n+m}/F_n} for every m. Using these, we show that field of norms functor commutes with the reciprocity maps $\Psi_{\mathcal{F}} : K_N^t(\mathcal{F}) \to \Gamma_{\mathcal{F}}^{ab}$ and $\Psi_{F_n} : K_N^t(F_n) \to \Gamma_{F_n}^{ab}$ constructed by Fesenko.

Imitating Fontaine's approach, we obtain an invariant form of Parshin's formula for the Witt pairing in characteristic p. The 'main lemma' from [1] relates Kummer extensions of F and Witt extensions of $\mathcal{F}$, allowing us to derive a formula for the generalised Hilbert symbol $\widehat{F}_\infty^* \times K_N(\mathcal{F}) \to \mu_{p^M}$, where $\widehat{F}_\infty$ is the p-adic completion of $\varinjlim_n F_n$.

Declaration

The work in this thesis is based on research carried out at the Pure Maths group, Department of Mathematical Sciences, University of Durham, England, United Kingdom. No part of this thesis has been submitted elsewhere for any other degree or qualification and it all my own work unless referenced to the contrary in the text.

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Contents

Chapter 0

Introduction

Abelian p-extensions of fields are explicitly described in two cases. If the field F contains some primitive p^M-th root of unity, Kummer-theory states that any abelian extension of exponent dividing p^M is obtained by joining p^M-th roots of elements of F^* and gives a non-degenerate pairing

$$F^*/(F^*)^{p^M} \times \Gamma_F^{ab}/p^M \longrightarrow \mu_{p^M}, \quad (x, \gamma) \mapsto \frac{\gamma(\xi)}{\xi},$$

where Γ_F^{ab} is the Galois group of the maximal abelian extension of F and $\xi^{p^M} = x$. On the other hand, if $\mathcal{F}$ is of finite characteristic p, abelian p-extensions are described by the Witt-pairing

$$W_M(\mathcal{F})/\wp \times \Gamma_{\mathcal{F}}^{ab}/p^M \longrightarrow \mathbb{Z}/p^M, \quad (b, \gamma) \mapsto \gamma(B) - B,$$

where $B \in W_M(\mathcal{F}^{sep})$ is such that $\wp(B) = \sigma(B) - B = b$.

This thesis is concerned with higher local fields whose first residue field is of characteristic $p > 2$. We use the field of norms functor [35] and class field theory [11, 12] to deduce a formula for a generalised Hilbert symbol from an invariant formula for Parshin's pairing.

In chapter 1, we give an overview over the theory of higher local fields. By definition, an N-dimensional local field is a complete discrete valuation field F whose (first) residue field $F^{(1)}$ is of dimension $(N-1)$, where 0-dimensional fields are defined to be finite fields.

The first four sections of chapter 2 deal with Milnor K-groups. After mentioning some basic properties, we describe the definition of a topology on Milnor K-groups of higher local fields. The advantage of the topological Milnor K-groups K_n^t is that they admit explicit topological generators. For details on K_n^t, see e.g. [4, 11, 12, 14, 28, 29, 32, 43]. We go on to define the valuation $\mathbf{v} : K_N^t(F) \to \mathbb{Z}$ for any N-dimensional local field F in section 2.3. In section 2.4, we outline the definition of a norm map $N_{L/F} : K_N^t(L) \to K_N^t(F)$ for finite field-extensions L/F.

Milnor K-groups were used by Kato [23, 24, 25] and Parshin [32, 33], and later Fesenko [11, 12, 13, 14, 15, 16] to define class field theories for higher local fields. Section 3.1 treats the construction of the norm-residue symbol

$$r_{L/F} : \mathrm{Gal}(L/F)^{ab} \longrightarrow K_N^t(F)/N_{L/F}K_N^t(L)$$

for Galois extensions L/F, see [11, 12]. Taking projective limits over all finite abelian extensions L, the inverses of all $r_{L/F}$ gives rise to the reciprocity map

$$\Psi_F : K_N^t(F) \longrightarrow \Gamma_F^{ab}.$$

In [19, 41], Fontaine-Wintenberger defined the field of norms functor for local fields. Their construction has been generalised amongst others by Abrashkin [3] and Scholl [35]. Section 3.2 gives a description of the construction from [35] in the special case of N-dimensional local fields. A tower $F_\bullet = \{F_n\}_{n\geqslant 0}$ is said to be strictly deeply ramified (SDR) with parameters (n_0, c) if all F_n have the same last residue field k, and if there exists a system of local parameters $\pi_1^{(n)}, \ldots, \pi_N^{(n)}$ of F_n such that $(\pi_i^{(n)})^p \equiv \pi_i^{(n-1)} \mod \mathfrak{p}_c$ for all $n > n_0$. Here $\mathfrak{p}_c$ is the ideal $\{x \in \mathcal{O}_F \mid v_F(x) > c\}$, where v_F is normalised by $v_F(\pi_1^{(0)}) = 1$. The field of norms functor X from [35] attaches to each (equivalence class of towers) $F_\bullet$ an N-dimensional local field $X(F_\bullet) = \mathcal{F}$ of characteristic p. Its first valuation ring is obtained as $\mathcal{O}_\mathcal{F} = \varprojlim \mathcal{O}_{F_n}/\mathfrak{p}_{c,F_n}$, with local parameters $\bar{t}_i = (\pi_i^{(n)})_n$ and last residue field k. Furthermore, the field of norms functor provides us with a one to one correspondence between separable extensions of $\mathcal{F}$ and extensions of $F_\infty = \varinjlim_n F_n$, inducing an identification $\Gamma_\mathcal{F} \cong \Gamma_{F_\infty} \subset \Gamma_F$ of absolute Galois groups.

The rest of chapter 3 concerns the interaction between class field theory and the field of norms functor. For special SDR towers $F_\bullet$, section 3.3 shows the existence of canonical maps

$$\mathcal{N}_{\mathcal{F}/F_n} : K_N^t(\mathcal{F}) \longrightarrow K_N^t(F_n)$$

which are compatible with the norms N_{F_{n+m}/F_n} and induce an isomorphism $K_N^t(\mathcal{F}) \xrightarrow{\sim} \varprojlim K_N^t(F_n)$. Section 3.4 defines analogous maps, modulo quotients by p^M, for arbitrary SDR towers, assuming that F_∞ contains a primitive p^M-th root of unity. Compatibility of class field theory and the field of norms is proved in section 3.5.

Theorem *Let $F_\bullet$ be a special SDR tower and $L_\bullet$ the special SDR tower given by $L_n = LF_n$ for some finite Galois extension L/F_0. Let $\mathcal{L}/\mathcal{F}$ be the corresponding extensions of their fields of norms. Then the diagram*

$$\begin{tikzcd}
\mathrm{Gal}(\mathcal{L}/\mathcal{F}) \arrow[r, "r_{\mathcal{L}/\mathcal{F}}"] \arrow[d] & K_N^t(\mathcal{F})/N_{\mathcal{L}/\mathcal{F}}K_N^t(\mathcal{L}) \arrow[d, "\mathcal{N}_{\mathcal{F}/F_n}"] \\
\mathrm{Gal}(L_n/F_n) \arrow[r, "r_{L_n/F_n}"] & K_N^t(F_n)/N_{L_n/F_n}K_N^t(L_n)
\end{tikzcd}$$

is commutative.

For arbitrary SDR towers, the above statement holds after taking quotients modulo p^M. In particular, we always have

$$\begin{tikzcd}
K_N(\mathcal{F})/p^M \arrow[r, "\Psi_{\mathcal{F}}"] \arrow[d, "\mathcal{N}_{\mathcal{F}/F_n}"'] & \Gamma_{\mathcal{F}}^{ab}/p^M \arrow[d] \\
K_N(F_n)/p^M \arrow[r, "\Psi_{F_n}"] & \Gamma_{F_n}^{ab}/p^M.
\end{tikzcd}$$

Chapter 4 treats abelian p-extensions of N-dimensional fields $\mathcal{F}$ of finite characteristic. After a section on differential forms, section 4.2 treats Parshin's pairing $W_M(\mathcal{F}) \times K_N^t(\mathcal{F})/p^M \to \mathbb{Z}/p^M$ for fields $\mathcal{F}$ of characteristic p (see [32, 33]). We first show that it is equivalent to a pairing

$$\mathcal{O}_M(\mathcal{F}) \times K_N^t(\mathcal{F})/p^M \to \mathbb{Z}/p^M,$$

where $\mathcal{O}_M(\mathcal{F})$ is the flat $\mathbb{Z}/p^M$-lift of $\mathcal{F}$ from [6]. We use this form to prove that the composition of Parshin's pairing with the reciprocity map $\Psi_{\mathcal{F}} : K_n^t(\mathcal{F}) \to \Gamma_{\mathcal{F}}^{ab}$ yields the Witt pairing. In particular, this shows that the class field theories from [12] and [32] coincide for p-extensions of fields of finite characteristic.

Section 4.3 requires the use of Milnor K-groups of rings, which were defined in section 2.5. Following the approach taken in [17], we show that there is a special section $Col : K_N^t(\mathcal{F}) \to K_N^t(\mathcal{O}(\mathcal{F}))$ of the canonical projection $K_N^t(\mathcal{O}(\mathcal{F})) \to K_N^t(\mathcal{F})$ which allows us to find an invariant formula for Parshin's pairing

Theorem *Parshin's pairing is induced, for each $M \geqslant 1$, by*

$$[-,-) : \mathcal{O}(\mathcal{F}) \times K_N^t(\mathcal{F}) \longrightarrow \mathbb{Z}_p,$$
$$\big[b, \{x_1, \dots, x_N\}\big) = \mathrm{Tr}_{\,W(k)/\mathbb{Z}_p} \circ \mathrm{Res}_{\,\mathcal{O}(\mathcal{F})} \big(b\, d_{\log} Col\{x_1, \dots, x_N\}\big).$$

Chapter 5 is concerned with Kummer-extensions of higher local fields of characteristic zero. Let $F_\bullet$ be an SDR tower such that F_∞ contains some primitive p^M-th roof of unity ζ_M, and let $\mathcal{F}$ be its field of norms.

Consider the subring $A = \big\{ \sum_{\underline{a}} \alpha_{\underline{a}} p^{a_0} t_1^{a_1} \cdots t_N^{a_n} \,\big|\, (a_1, \dots, a_N) \geqslant (0, \dots, 0) \big\}$ of the flat $\mathbb{Z}_p$-lift $\mathcal{O}(\mathcal{F})$ and its prime ideal $\mathfrak{m}_A$ of all series with $(a_1, \dots, a_N) > (0, \dots, 0)$. The Artin-Hasse-Shafarevich exponential induces an isomorphism $e : \mathfrak{m}_A \to 1 + \mathfrak{m}_A$, $f \mapsto \exp\big(\sum \frac{\sigma^n}{p^n} f\big)$. Let $\theta : \mathfrak{m}_A \to \widehat{F}_\infty^*$ be its composition with the map induced by $t_i \mapsto \lim_{n\to\infty} (\pi_i^{(n)})^{p^n}$ which takes values in the p-adic completion $\widehat{F}_\infty$ of F_∞.

Section 5.1 gives a slightly modified version of the 'main lemma' from [1], relating Kummer extensions of $\widehat{F}_\infty$ and Witt-extensions of $\mathcal{F}$. In section 5.2, we define the generalised Hilbert symbol

$$(-,-)_M^{F_\bullet} : \widehat{F}_\infty^* \times \Gamma_{F_\infty} \longrightarrow \mu_{p^M}, \quad (u, \gamma)_M^{F_\bullet} = \frac{\gamma(\sqrt[p^M]{u})}{\sqrt[p^M]{u}}.$$

Let $F_\bullet$ be an SDR tower with parameters $(0, c)$. Suppose that $cp^m \geqslant \frac{2v_F(p)}{p-1}$ for some $m \in \mathbb{N}$ such that F_m contains a primitive p^{M+m}-th root of unity ζ_{M+m}. For $H'_{M+m} \in \mathcal{O}_\mathcal{F}$ such that $H'_{M+m} \mod \mathfrak{p}_{cp^m, \mathcal{F}} = \zeta_{M+m} \mod \mathfrak{p}_{c, F_m}$, let H_{M+m} be a lift of H'_{M+m} to $\mathcal{O}(\mathcal{F})$ and set $H = H_{M+m}^{p^{M+m}} - 1$. **Theorem** *The generalised Hilbert symbol is given by*

$$\big(\theta(f), \mathcal{N}_{\mathcal{F}/F}(\beta)\big)_M^{F_\bullet} = \zeta_{M+m}^{p^m \phi}, \quad \phi = \mathrm{Tr} \circ \mathrm{Res} \Big(\frac{f}{H} d_{\log} Col(\beta)\Big),$$

for $f \in \mathfrak{m}_A$ and $\beta \in K_N^t(\mathcal{F})$. Noting that θ takes values in F^* if $F_\bullet$ is of the form $F_n = F(\sqrt[p^n]{\pi_1}, \dots, \sqrt[p^n]{\pi_N})$ for some local parameters $\pi_1, \dots, \pi_N$ of F, we also obtain a (partial) formula for the classical Hilbert symbol. In section 5.3 we consider Vostokov's symbol

$$(-,-)_M : (F^*)^{N+1} \longrightarrow \mu_{p^M}, \quad \big(u_0, \{u_1, \dots, u_N\}\big)_M = \zeta_M^{\mathrm{Tr} \circ \mathrm{Res}\,\Phi}, \quad \text{where}$$
$$\Phi = \sum_{0 \leqslant i \leqslant N} \frac{(-1)^i}{H} l(u_i) \frac{\sigma}{p} d_{\log} u_1 \wedge \dots \wedge \frac{\sigma}{p} d_{\log} u_{i-1} \wedge d_{\log} u_{i+1} \wedge \dots \wedge d_{\log} u_N.$$

It was first proved in [39] that this coincides with the Hilbert pairing. Kato [26] obtained the formula as a special case of his approach to Fontaine-Messing theory. Recently Zerbes [42] applied the theory of (φ, Γ)-modules to prove it for fields F having a first local parameters π_1 for which $\mathbb{Q}_p\{\{\pi_1\}\}$ coincides with the algebraic closure of $\mathbb{Q}_p$ in F. We give a proof by first showing that it agrees with our formula for $u_0 \in V_F$ and $\{u_1, \dots, u_N\} \in \mathrm{Im}(\mathcal{N}_{\mathcal{F}/F})$ coming from $\Gamma_{\mathcal{F}}^{ab}$ and then reducing the remaining cases to this one.

A word on notation. Unless otherwise stated, F is an N-dimensional local field and $\pi_1, \dots, \pi_N$ a system of local parameters. We assume that the first residue field is of finite odd characteristic p. k always denotes the last residue field, which is a finite extension of $\mathbb{F}_p$. Where a statement is made about fields of either mixed or equal characteristic, the notation F is used. When treating mixed and equal characteristic separately, F is used for mixed characteristic and $\mathcal{F}$ for fields of equal characteristic. The local parameters of $\mathcal{F}$ are denoted $\bar{t}_1, \dots, \bar{t}_N$, reserving $t_1, \dots, t_N$ for their Teichmüller representatives. The absolute Frobenius on any ring of characteristic p as well as any endomorphism induced by it on rings of Witt vectors and flat $\mathbb{Z}_p$-lifts will be denoted by σ. On the other hand, $\varphi = \varphi_F$ is used for the automorphism of higher local field induced by the Frobenius of the last residue field k, so that if $[k : \mathbb{F}_p] = f$, $\varphi_F(\alpha) = \sigma^f(\alpha)$ for every $\alpha \in k^*$ or $\alpha \in W(k)^*$.

Chapter 1

Higher Local Fields

In this chapter we introduce higher local fields, paying special attention to those properties needed in later chapters.

1.1 Basic Properties

Recall that a classical local field is a complete discrete valuation field with finite residue field, that is, a field F equipped with a valuation $v : F^* \to \mathbb{Z}$ such that any sequence x_n of elements in F with $v(x_m - x_{n+m}) \to \infty$ as $n \to \infty$ has a limit in F. N-dimensional local fields are generalisations of classical local fields in the following sense.

Definition 1.1 *An N-dimensional local field F is defined inductively to be a complete discrete valuation field, with valuation $v_F^{(1)}$ and residue field $F^{(1)}$ of dimension $(N-1)$. A 0-dimensional local field is a finite field.*

We will only consider higher local fields whose first residue field is of odd characteristic p. We write $k = k_F$ for the last residue field $F^{(N)}$ of F. k is a finite extension of $\mathbb{F}_p$.

A system of *local parameters* is a set of elements $\pi_1, \ldots, \pi_N$ such that π_1 is a uniformiser of F for $v_F^{(1)}$ and $\pi_2, \ldots, \pi_N$ are units for $v_F^{(1)}$ whose residues $\overline{\pi}_2, \ldots, \overline{\pi}_N$ are local parameters for $F^{(1)}$. One defines on F a rank N valuation $\underline{v} = (v^{(1)}, \ldots, v^{(N)}) :$

$F^* \to \mathbb{Z}^N$, where $v_F^{(1)}$ is the usual valuation on the complete discrete valuation field F^*, and for $i \geqslant 1$, $v^{(i+1)}(x) = v_{F^{(i)}}(x\pi_1^{-v^{(1)}(x)} \cdots \pi_i^{-v^{(i)}(x)})$.

Remark It should be noted that most authors use different notation. For the numbering of local parameters, the correspondence is given by $n \leftrightarrow N+1-n$. The sequence of residue fields $F, F^{(1)}, \ldots, F^{(N)}$ is also denoted $F = F_N, F_{N-1}, \ldots, F_0$. Finally, for the valuation $(v^{(1)}, \ldots, v^{(N)}) : F^* \to \mathbb{Z}^N$, the ordering on $\mathbb{Z}^N$ prioritises the last coordinate, with $v^{(N)}$ being the discrete valuation on F.

The valuation $\underline{v}$ is unique up to multiplication on the right by an upper triangular matrix with diagonal entries equal to 1.

Define the lexicographic ordering on $\mathbb{Z}^n$ by setting $(a_1, \ldots, a_n) < (b_1, \ldots, b_n)$ if $a_1 = b_1, \ldots, a_i = b_i$, and $a_{i+1} < b_{i+1}$ for some $0 \leqslant i \leqslant n$. For simplicity, we often write $\underline{a}$ for the vector $(a_1, \ldots, a_n) \in \mathbb{Z}^n$. Using this, we define the *total valuation ring* to be $O_F = \{x \in F \,|\, (v^{(1)}, \ldots, v^{(N)})(x) \geqslant (0, \ldots, 0)\}$. It can also be defined recursively by setting $O_{F^{(N)}} = F^{(N)}$ and

$$O_{F^{(i)}} = \big\{x \in \mathcal{O}_{F^{(i)}}, |, \overline{x} \in O_{F^{(i+1)}}\big\}.$$

For $1 \leqslant n \leqslant N$ and $(c_1, \ldots, c_n) \in \mathbb{Z}^n$, put

$$\mathfrak{p}_{(c_1,\ldots,c_n)} = \big\{x \in O_F \,|\, (v^{(1)}, \ldots, v^{(n)})(x) \geqslant (c_1, \ldots, c_n)\big\}.$$

We denote by $U^{(c_1,\ldots,c_n)} = 1 + \mathfrak{p}_{(c_1,\ldots,c_n)}$ the corresponding subgroup of principal units in F^*. In the special case $c = (0, \ldots, 0, 1)$, write $\mathfrak{p}_{(0,\ldots,0,1)} = \mathfrak{m}$ and $1 + \mathfrak{m} = V_F$. $\mathfrak{m}$ is the maximal ideal of O_F with residue field $F^{(N)}$. Note that, in general, the ideals $\mathfrak{p}_{(c_1,\ldots,c_n)}$ depend on the choice of uniformisers.

Example $\mathbb{F}_q((t_N)) \cdots ((t_1))$ is an N-dimensional local field with local parameters $t_1, \ldots, t_N$ and first valuation ring $\mathbb{F}_q((t_N)) \cdots ((t_2))[[t_1]]$. Its first residue field is $\mathbb{F}_q((\bar{t}_N)) \cdots ((\bar{t}_2))$.

Another important class of examples of higher local fields is obtained as follows. If F is a (complete) discrete valuation field with valuation v, $F\{\{X\}\}$ is the field of formal power series $\sum_{i\in\mathbb{Z}} a_i X^i$ with $v(a_i) \to \infty$ as $i \to -\infty$ and $\inf v(a_i) > -\infty$. $F\{\{X\}\}$ is again a complete discrete valuation field, with valuation

$$v_{F\{\{X\}\}}\big(\sum a_i X^i\big) = \min_i v_F(a_i)$$

and residue field $F^{(1)}((\overline{X}))$. To any local parameters $\pi_1, \ldots, \pi_N$ of F there correspond local parameters $\pi_1, X, \pi_2, \ldots, \pi_N$ of $F\{\{X\}\}$. Any element $\sum a_i X^i$ of $F\{\{X\}\}$ can be re-written as a convergent sum

$$\sum_{j \geqslant J} \Big(\sum_{i \geqslant I_j} a_{ij} X^i \Big) \pi^j,$$

which emphasises the fact that any uniformiser π of F is also a uniformiser of $F\{\{X\}\}$.

To formalise the analogy, note that

$$F((X)) = \big(\varprojlim_n F[X]/(X^n) \big)[X^{-1}] = \varprojlim_n \Big(\varprojlim_m (\mathcal{O}_F/(\pi^m))[\pi^{-1}] \Big)[X^{-1}],$$

with first local parameter X and second local parameter π, while

$$F\{\{X\}\} = \Big[\varprojlim_n \big(\varprojlim_m (\mathcal{O}_F[X]/(X^m))[X^{-1}] \big)/(\pi^n) \Big][\pi^{-1}]$$

has first local parameter π and second local parameter X.

Example The field $\mathbb{Q}_p\{\{t\}\} = \big\{ \sum_{j \geqslant J} (p^j \sum_{i \geqslant I(j)} \alpha_{ij} t^i) \big\}$ has first valuation ring $\mathbb{Z}_p\{\{t\}\}$, the ring of power series with $J = 0$. Notice that it is isomorphic to $\mathcal{O}(\mathbb{F}_p((t)))$, the flat $\mathbb{Z}_p$-lift of the one-dimensional field $\mathbb{F}_p((t))$ defined in appendix A.2. Its total valuation ring is $p\mathbb{Z}_p\{\{t\}\} + \mathbb{Z}_p[[t]] \subset \mathbb{Q}_p\{\{t\}\}$.

More generally it follows from the construction that $\mathcal{O}(\mathcal{F})$ is the first valuation ring of a mixed characteristic $(N+1)$-dimensional field whenever $\mathcal{F}$ is an N-dimensional local field of characteristic p.

The following result due to Zhukov is taken from [28]

Theorem 1.2 (Classification) *If F is an N-dimensional local field of equal characteristic p, then $F \cong F^{(1)}((t)) \cong k((t_N)) \cdots ((t_1))$ for any set of local parameters $t_1, \ldots, t_N$. If F is of mixed characteristic, then F is a finite extension of $F'\{\{t_N\}\} \cdots \{\{t_2\}\}$ for $F' = Frac(W(k))$ finite over $\mathbb{Q}_p$. Furthermore, there exists a finite extension F_1 of F which is again of the form $F''\{\{t'_N\}\} \cdots \{\{t'_1\}\}$*

1.2 Topology

For a classical local field F with uniformiser π, the valuation $v : F \to \mathbb{Z} \cup \{\infty\}$ defines a metric $|x|_v = r^{v(x)}$ for any fixed $r \in \mathbb{R}$, $0 < r < 1$. With respect to this metric, any element can be written as a convergent sum $x = a_v\pi^v + a_{v+1}\pi^{v+1} + \cdots$, where $v(a_i) = 0$ and the a_i may be taken from some fixed set of coset representatives of the residue field. This analytic point of view underlines the analogy with the real numbers. Viewing the situation from an algebraic perspective, we start with the ring of integers $\mathcal{O}_F$ with maximal ideal $\mathfrak{p}$. The natural map $\mathcal{O}_F \to \varprojlim \mathcal{O}_F/\mathfrak{p}^n$ is surjective iff $\mathcal{O}_F$ is complete with respect to the valuation topology, and the valuation topology is discrete iff it is injective (see, e.g. [20]). If the valuation is discrete, $\mathfrak{p} = (\pi)$ is a principal ideal. The valuation topology on $\mathcal{O}_F$ is then identical to the topology induced from the product topology of $\prod_n \mathcal{O}_F/\mathfrak{p}^n$ via $\varprojlim \mathcal{O}_F/\mathfrak{p}^n \subset \prod_n \mathcal{O}_F/\mathfrak{p}^n$, where $\mathcal{O}_F/\mathfrak{p}^n$ carries the discrete topology. Using the isomorphism $\mathcal{O}_F \cong \pi^{-n}\mathcal{O}_F$, the valuation topology on F is induced by the coproduct topology via $F \cong \varinjlim_n \pi^{-n}\mathcal{O}_F \subset \coprod_n \pi^{-n}\mathcal{O}_F$.

If F is a higher-dimensional local field with first valuation ring $\mathcal{O}_F$ and uniformiser π_1, we still have $\mathcal{O}_F \cong \varprojlim_n \mathcal{O}_F/(\pi_1^n)$ as abstract rings. Using the (first) valuation topology, i.e. the metric derived from the first valuation would correspond to using the discrete topology on all quotients $\mathcal{O}_F/(\pi_1^n)$. However, $\mathcal{O}_F/(\pi_1) = F^{(1)}$ is itself a complete discrete valuation field. To avoid this problem, one defines a finer topology on higher local fields, the so-called canonical topology.

Example In the equal characteristic case $F = F^{(1)}((t))$, the canonical topology is constructed inductively as follows. Let $\{U_i\}_{i\in\mathbb{Z}}$ be a system of neighbourhoods of zero in $F^{(1)}$ with $U_i = F^{(1)}$ if $i \gg 0$. Then $\mathcal{U} = \{\sum a_i t_i \mid a_i \in U_i\}$ is a neighbourhood of 0 in F. If F is of mixed characteristic, the construction uses sections of the projection $\mathcal{O}_F \to F^{(1)}$.

The canonical topology has the following properties (see, e.g. [28, 29])

(i) The canonical topology is independent of the choice of local parameters,

(ii) multiplication is sequentially continuous

(iii) the topology is compatible with finite extensions.

Let now F be any N-dimensional local field with local parameters $\pi_1, \dots, \pi_N$. It follows inductively that any element $x \in F$ can be written as

$$\begin{aligned} x &= \sum_{i_1 \geqslant I_1} x_{i_1} \pi_1^{i_1} = \sum_{i_1 \geqslant I_1} \Big(\sum_{i_2 \geqslant I_2(i_1)} x_{i_1 i_2} \pi_2^{i_2} \Big) \pi_1^{i_1} = \cdots \\ &= \sum_{i_1 \geqslant I_1} \sum_{i_2 \geqslant I_2(i_1)} \cdots \sum_{i_N \geqslant I_N(i_1, \dots, i_{N_1})} [\alpha_{(i_1, \dots, i_N)}] \pi_1^{i_1} \cdots \pi_N^{i_N}, \end{aligned} \qquad (*)$$

where the x_{i_1} are in some fixed set of coset representatives of $F^{(1)}$, the $x_{i_1 i_2}$ in some fixed lift of a coset representatives of $F^{(2)} \leftarrow \mathcal{O}_{F^{(1)}}$ to $\mathcal{O}_F$, etc. The $[\alpha_{\underline{i}}]$ are lifts of elements from the last residue field k which, by definition of the total valuation ring, lie in O_F. If $\mathrm{char}(F) = 0$, it is usually assumed that the elements $[\alpha]$ are the images of the Teichmüller representatives in some unramified extension of $\mathbb{Q}_p$, while in the equal characteristic case one uses the canonical inclusion $k \to F$.

The canonical topology is such that an N-tuple formal series converges if and only if it comes from an element of F as above. A subset $A \subset \mathbb{Z}^N$ is called *admissible* if, for every $i_1, \dots, i_n \in \mathbb{Z}$ there exists $I_{n+1}(i_1, \dots, i_n) \in \mathbb{Z}$ satisfying the condition that if $(a_1, \dots, a_N) \in A$ and $a_1 = i_1, \dots, a_n = i_n$, then $a_{n+1} \geqslant I_{n+1}(i_1, \dots, i_n)$.

For a family $\{A_i\}_{i \in I}$ of admissible sets, $A_i \subset \mathbb{Z}^N_{>\underline{0}}$, consider the conditions

(A1) $A = \bigcup_{i \in I} A_i$ is again admissible

(A2) $\bigcap_{j \in J} A_j = \emptyset$ for any infinite subset $J \subset I$

Thm. 1 in [28] implies

Theorem 1.3 *For every α in some fixed set of coset representatives of the last residue field k^* in O_F and for every $\underline{a} \in \mathbb{Z}^N$ fix an element*

$$x_{\underline{a},\alpha} = \alpha \underline{\pi}^{\underline{a}} + \sum_{\substack{\underline{b} > \underline{a} \\ \underline{b} \in A_{\underline{a},\alpha}}} \beta \pi_1^{b_1} \cdots \pi_N^{b_N},$$

for some family of admissible sets $A_{\underline{a},\alpha}$ satisfying (A1) and (A2). Then every $x \in F$ can be uniquely written as $x = \sum_{\underline{a} \in A_x} x_{\underline{a},\alpha(\underline{a})}$ for some admissible $A_x \in \mathbb{Z}^N$.

The structure of the multiplicative group of a higher local field is similar to the one-dimensional case. The following follows from the additive expansion $(*)$ of an element of F.

Lemma 1.4 *For any set of local parameters $\pi_1, \ldots, \pi_N$, the group of non-zero elements of F is*

$$F^* \cong \langle \pi_1 \rangle \times \cdots \times \langle \pi_N \rangle \times k^* \times V_F,$$

where $V_F = 1 + \mathfrak{m}$ is the group of principal units (section 1.1).

The Parshin-topology or P-topology on F^* is defined to be the product topology of the discrete topology on $\langle \pi_1 \rangle \times \cdots \times \langle \pi_N \rangle$ and k^*, and the subset-topology induced on V_F by F. Thm. 2 from [28] describes convergent expansions in F^*:

Theorem 1.5 *Let$x_{\underline{a},\alpha} \in F^*$ be as in the previous theorem. Then any $x \in F^*$ can be uniquely written as*

$$x = \theta \pi_1^{n_1} \cdots \pi_N^{n_N} \prod_{\underline{a} \in A_x} \left(1 + x_{\underline{a},\alpha(\underline{a})}\right)$$

for some admissible set $A_x \subset \mathbb{Z}^N_{>\underline{0}}$ and any such product converges.

1.3 Principal Units

In the decomposition $F^* \cong \langle \pi_1 \rangle \times \cdots \times \langle \pi_N \rangle \times k^* \times V_F$, the first N factors are infinite cyclic while k^* is a cyclic group of order $|k| - 1$. In this section, we study the group of principal units $V_F = 1 + \mathfrak{m} \subset O_F^*$.

From [43] §1.6, we need the following

Lemma 1.6 *For any neighbourhood $\mathcal{U}$ of 1 in F, there exists $m \in \mathbb{N}$ such that the group of p^m-th powers $V_F^{(p^m)} \subset \mathcal{U}$*

Corollary 1.7 *V_F has a natural structure of $\mathbb{Z}_p$-module*

Proof Let $\alpha \in \mathbb{Z}_p$, write $\alpha = \sum \alpha_i p^i$ for $\alpha_i \in \mathbb{N}$. Given $u \in V_F$ and a neighbourhood $\mathcal{U}$ of 1 in F, the above lemma implies that $u^{\alpha_m p^m} \in \mathcal{U}$ for $m \geqslant m_{\mathcal{U}}$, thus the sequence $u_n = u^{\alpha_0 + \alpha_1 p + \cdots + \alpha_n p^n}$ converges in F. $\square$

Since $p \nmid l$, $l \in \mathbb{Z}$, implies $l \in \mathbb{Z}_p^*$, this also implies the following

Corollary 1.8 *The group V_F is l-divisible for any $p \nmid l$.*

Remark The second corollary can also be proved formally by noting that for $p \nmid l$ there exists $f_l(X) \in \mathbb{Z}_p[[X]]$ such that $(f_l(X))^l = 1 + X$ as formal power series. It then suffices to note that for $x \in \mathfrak{m}_F$, $f_l(x)$ converges in F.

The structure of V_F as $\mathbb{Z}_p$-module depends primarily on the characteristic of F.

Proposition 1.9 *If $\mathcal{F}$ is of characteristic p with local parameters $\bar{t}_1, \dots, \bar{t}_N$ then $V_{\mathcal{F}}$ is generated topologically by all $1 + \alpha \bar{t}_1^{a_1} \cdots \bar{t}_N^{a_N}$ for α running through a basis of $k/\mathbb{F}_p$, $p \nmid \underline{a}$, and $\underline{0} < \underline{a}$.*

Proposition 1.10 *If F is of characteristic 0 with local parameters $\pi_1, \dots, \pi_N$, then V_F admits topological generators $1 + \alpha \pi_1^{a_1} \cdots \pi_N^{a_N}$ for α running through a basis of $W(k)/\mathbb{Z}_p$ and $\underline{0} < \underline{a} < \underline{e}p/(p-1)$, $p \nmid \underline{a}$, where $\underline{e} = \underline{v}(p)$ is the absolute ramification index of F. If $p-1 \mid \underline{e}$, an additional element in $1+\mathfrak{p}_{\underline{e}/(p-1)}$ is needed. If $\zeta_p \in F$, this can be taken to be $\varepsilon(\alpha_0) = 1-\alpha_0(1-\zeta_p)^p$, for some $\alpha_0 \in W(k)$ with $Tr_{W(k)/\mathbb{Z}_p}(\alpha_0) \in \mathbb{Z}_p^*$.*

For proofs, see e.g. [28], theorems 2.1 and 2.2.

It can be convenient to use a different set of generators, given by the Shafarevich basis of $F^*/(F^*)^{p^M}$. We shall use them in chapter 5.

Lemma 1.11 *The Artin-Hasse exponential map*

$$\mathcal{E}(X) = \exp\left(X + \frac{X^p}{p} + \cdots + \frac{X^{p^n}}{p^n} + \cdots\right) = \prod_{p \nmid i} (1 - X^i)^{-\mu(i)/i}$$

lies in $\mathbb{Z}_{(p)}[[X]] \subset \mathbb{Z}_p[[X]]$ and satisfies $\mathcal{E}(X) \equiv 1 + X \mod X^2\mathbb{Z}_p[[X]]$. Here μ is the Möbius function, $\mu(i) = (-1)^r$ if i has r distinct prime factors and $\mu(i) = 0$ otherwise.

For a proof, see, e.g. [16] I, (9.1). Using $\sum_{d|n} \mu(d) = 0$ if $n > 1$ and $= 1$ if $n = 1$, we obtain $1 - X = \prod_{p \nmid i} \mathcal{E}(X^i)^{-1/i}$.

For higher local fields we need to generalise this slightly: For a ring R, consider the subring

$$R[[\underline{X}]] = \Big\{ \sum r_{\underline{a}} X_1^{a_1} \cdots X_N^{a_N} \mid \underline{a} \geqslant \underline{0} \Big\} \subset R((X_N)) \cdots ((X_1))$$

and its ideal $\mathfrak{m}_{R[[\underline{X}]]}$ consisting of all series with $\underline{a} > \underline{0}$. Notice that, by definition, the exponents with non-zero coefficients will always lie in some admissible subset of $\mathbb{Z}^N$. With this notation, we see that $\mathcal{E}(\underline{X}^{\underline{a}}) \in \mathbb{Z}_p[[\underline{X}]]$ for any $\underline{a} > \underline{0}$, and $\mathcal{E}(\underline{X}^{\underline{a}}) \equiv 1 + \underline{X}^{\underline{a}}$ mod $1 + \underline{X}^{\underline{a}} \mathfrak{m}_{R[[\underline{X}]]}$ as congruence of elements in the unit group $R[[\underline{X}]]^*$.

The Artin-Hasse exponential $\mathcal{E}(X)$ has been generalised by Shafarevich to arguments in $W(k)[[X]]$. For higher-dimensional local fields, we need to instead work with $W(k)[[\underline{X}]] \subset W(k)((X_N)) \cdots ((X_1))$. Extend $\sigma : W(k) \to W(k)$ to $\sigma : W(k)[[\underline{X}]] \to W(k)[[\underline{X}]]$ by $X_i \mapsto X_i^p$.

Lemma 1.12 *The Artin-Hasse-Shafarevich exponential*

$$E_{\underline{X}}(f(\underline{X})) = \exp\Big(f(\underline{X}) + \frac{\sigma}{p} f(\underline{X}) + \cdots + \frac{\sigma^n}{p^n} f(\underline{X}) + \cdots \Big)$$

defines an isomorphism $\mathfrak{m}_{W(k)[[\underline{X}]]} \longrightarrow 1 + \mathfrak{m}_{W(k)[[\underline{X}]]}$ *with inverse*

$$l_{\underline{X}}(u(\underline{X})) = \frac{1}{p} \log \Big(\frac{u(\underline{X})^p}{\sigma u(\underline{X})} \Big).$$

If $f(\underline{X}) \equiv \alpha \underline{X}^{\underline{a}} \mod p^k W(k)[[\underline{X}]] + X^{\underline{a}} W(k)[[\underline{X}]]$ *with* $\alpha \in W(k)$, *and* $\underline{a} > \underline{0}$, *then* $E_{\underline{X}}(f(\underline{X})) \equiv (1 + \alpha \underline{X}^{\underline{a}})(1 + g(\underline{X}))^{p^k} \mod \underline{X}^{\underline{a}} \mathfrak{m}_{W(k)[[\underline{X}]]}$ *for some* $g(X) \in X\, W(k)[[X]]$.

The proof is a direct but tedious generalisation of the arguments in [16], VI, sections (2.2) through (2.4). Convergence of all series follows from theorem 1.5 by carefully keeping track of admissible sets. In the special case where $f = f(X) = \alpha \underline{X}^{\underline{a}}$, for $\alpha \in W(k)$ and $\underline{a} > \underline{0}$, convergence follows from the obvious inclusion $W(k)[[f]] \subset W(k)((X_N)) \cdots ((X_1))$, where $W(k)[[f]]$ is the usual formal power series ring in the variable f. If F is any local field with local parameters $\pi_1, \ldots, \pi_N$, the result of substituting $X_i = \pi_i$ into $E_X(\alpha X_1^{a_1} \cdots X_N^{a_N})$ is denoted by $E(\alpha, \pi_1^{a_1} \cdots \pi_N^{a_N})$.

Corollary 1.13 *If* $\mathrm{char}(\mathcal{F}) = p$, $V_{\mathcal{F}}$ *is topologically generated by all* $E(\alpha, \underline{\bar{t}}^{\underline{a}})$, *for* $p \nmid \underline{a}$, $\underline{a} > \underline{0}$, *and* α *running through a basis of* $k/\mathbb{F}_p$. *If* $\mathrm{char}(F) = 0$, V_F *is* *topologically generated by all $E(\alpha, \underline{\pi}^{\underline{a}})$, for $p \nmid \underline{a}$, $\underline{0} < \underline{a} < \underline{e}p/(p-1)$, and α running through a basis of $W(k)/\mathbb{Z}_p$, together with an additional element if $(p-1) \mid \underline{e}$.*

If F contains some primitive p^M-th root of unity ζ_M, we shall want to replace the generator $\varepsilon(\alpha_0) = 1 - \alpha_0(1-\zeta_p)^p$ by the element $\omega(\alpha_0)$ constructed as follows. Let $\widehat{\zeta} \in W(k)((X_N)) \cdots ((X_1))$ be such that $\widehat{\zeta}|_{\underline{X}=\underline{\pi}} = \zeta$, and put $H = \widehat{\zeta}_M^{p^M} - 1$. Then for $\alpha_0 \in W(k)$ with $\mathrm{Tr}_{W(k)/\mathbb{Z}_p}(\alpha_0) \in \mathbb{Z}_p^*$, let

$$\omega(\alpha_0) = E_X(\alpha_0 H)\big|_{\underline{X}=\underline{\pi}}.$$

We show that $\omega(\alpha_0)$ may be used as generator of V_F instead of $\varepsilon(\alpha_0)$:

Lemma 1.14 *$\omega(\alpha_0) \equiv \varepsilon(\alpha_0) \mod V_F^{(p)}$, where $V_F^{(p)}$ is the subgroup of p-th powers of V_F. In particular, we may use $\omega(\alpha_0)$ as a generator of V_F in place $\varepsilon(\alpha_0)$.*

Proof In O_F, $1 - \zeta_p = 1 - \zeta_M^{p^{M-1}} \sim \underline{\pi}^{\underline{e}/(p-1)}$. Thus there exists $u \in O_F$ with $1 - \zeta_M^{p^{M-1}} = u\underline{\pi}^{\underline{e}/(p-1)}$, and hence $H = \widehat{\zeta}_M^{p^M} - 1$ satisfies

$$H = (1 - \widehat{u}X^{\underline{e}/(p-1)})^p - 1 \equiv -X^{\underline{e}p/(p-1)}\widehat{u}^p \mod pW(k)[[\underline{X}]] + \underline{X}^{\underline{e}p/(p-1)}\mathfrak{m}_{W(k)[[\underline{X}]]}.$$

Substituting $\underline{X} = \underline{\pi}$, we obtain

$$\omega(\alpha_0) = E_X(\alpha_0 H)\big|_{X=\underline{\pi}} = (1 - \alpha_0\underline{\pi}^{\underline{e}p/(p-1)}u^p)(1 + g(X)|_{X=\underline{\pi}})^p \mod \mathfrak{m}_F\mathfrak{p}_{\underline{e}p/(p-1)}.$$

But a congruence of units in a ring modulo $\mathfrak{m}_F\,\mathfrak{p}_{\underline{e}p/(p-1)}$ becomes a congruence as elements of the unit group modulo $1 + \mathfrak{m}\,\mathfrak{p}_{\underline{e}p/(p-1)}$, which is contained in $V_F^{(p)}$. Thus $\omega(\alpha_0) \equiv 1 - \alpha_0\pi^{\underline{e}p/(p-1)}u^p = 1 - \alpha_0(1-\zeta_p)^p \mod V_F^{(p)}$, as desired. □

The importance of $\omega(\alpha_0)$ lies in the fact that its p^M-th root generates an unramified extension of F. This will follow from the main lemma in section 5.2, see lemma 5.18. This property means that $\omega(\alpha_0)$ is a so-called p^M-*primary* element.

1.4 Extensions

We consider extensions L/F of higher local fields. Let $\pi_1, \ldots, \pi_N$ be local parameters of F and $\pi'_1, \ldots, \pi'_N$ local parameters of L with associated valuation $\underline{v}_L : L^* \to \mathbb{Z}^N$.

The ramification matrix $(e^{(ij)}_{L/F})$ is defined by $e^{(ij)} = v^{(j)}_L \pi'_i$. It is upper-triangular and its diagonal entries satisfy

$$[L : F] = f e_{11} \cdots e_{NN}$$

for $f = [L^{(N)} : F^{(N)}]$ the degree of the last residue extension. An extension L/F is called *purely unramified* if $[L : F] = f_{L/F} = [L^{(N)} : F^{(N)}]$, it is *tamely ramified* if $p \nmid e_{11} \cdots e_{NN} \neq 0$, and *wildly ramified* otherwise.

Any extension L/F has a maximal purely unramified extension L_0/F corresponding to the extension of last residue fields, so any purely unramified extension is obtained by joining roots of unity coprime to p.

The following shows that in certain special cases, there exists an analogue of this for maximal sub-extension with ramification restricted to certain local parameters.

Lemma 1.15 *If L/F is an extension of N-dimensional local fields with $e_{ii} = 1$ for $i > s$ such that L/F and $L^{(s)}/F^{(s)}$ are separable then there exists a sub-extension $F \subset E \subset F$ with $[E : F] = e_{ss} f$ and $E^{(s)} = L^{(s)}$.*

Proof Let L_{nc} be a normal closure of L/F, $G = \mathrm{Gal}(L_{nc}/F)$ and $G' = \mathrm{Gal}(L_{nc}/L)$. G acts on the s-th residue field $L^{(s)}_{nc}$, fixing $F^{(s)}$ pointwise, so there exists $H \subset G$ with $G/H \cong \mathrm{Gal}(L^{(s)}_{nc}/F^{(s)})$. Similarly, G' acts on $L^{(s)}_{nc}$ fixing $L^{(s)}$ pointwise, thus there is $H' \supset H$, such that $L^{(s)} = (L^{(s)}_{nc})^{H'/H}$ is the fixed field of $H'/H \subset G/H$. By construction, $H' \supset G'$. Furthermore, the index of H'/H in G/H satisfies $(G/H : H'/H) = [L^{(s)} : F^{(s)}] = e_{ss} f$. Then the fixed field $E = L^{H'}_{nc}$ satisfies the claim. □

There is no analogous result for extensions of non-perfect intermediate residue fields

Example If $F = \mathbb{Q}_p\{\{t\}\}$, $E = F(\pi)$ for some first uniformiser $\pi \not\sim p$, e.g. $\pi = \sqrt[n]{p}$, and $L = E(T)$ with $T^p = t + \pi$. Then $L^{(1)} = E^{(1)}(\overline{T})$ with $\overline{T}^p = \overline{t}$ is an inseparable extension of $E^{(1)} = F^{(1)} = \mathbb{F}_p((\overline{t}))$. Taking as uniformisers of L the elements π, T, we obtain $e_{ij} = \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix}$ but there does not exist any sub-extension E_1 with $[E' : F] = p$ and $E^{(1)}_1 = L^{(1)}$, i.e. which only comes from the π_2-ramified part.

Any Galois extension of higher local field has a maximal tamely ramified sub-extension, given by the fixed field of any Sylow-p-subgroup.

Proposition 1.16 *Let F be a higher local field with local parameters $\pi_1, \ldots, \pi_N$ and F^{sep} be a separable closure. Let F_{ur} be the maximal purely unramified extension of F, and put $F_{tr} = \varinjlim_{p \nmid n} F_{ur}(\sqrt[n]{\pi_1}, \ldots, \sqrt[n]{\pi_N})$. Then any tamely ramified extension L/F is contained in F_{tr}.*

Proof Let $e = e^{(11)}_{L/F} \cdots e^{(NN)}_{L/F}$. Let $\widetilde{k}/F^{(N)}$ be the extension of degree $[L : F]$. For a generator $\alpha \in \widetilde{k}$ of $\widetilde{k}^*$, let $E_0 = F([\alpha]) \subset F_{ur}$. Next, for a system of local parameters $\pi_1, \ldots, \pi_N$ of F, set $E = E_0(\sqrt[e]{[\alpha]}, \sqrt[e]{\pi_1}, \ldots, \sqrt[e]{\pi_N})$. In the composite EL, the local parameters $\pi'_1, \ldots, \pi'_N$ of L are related to those of F by In $\mathcal{O}_{(LE)^{(N-1)}}$, we have $\pi'^{e(NN)}_N \sim \pi_N$, in $\mathcal{O}_{(LE)^{(N-2)}}$, $\pi'^{e(N-1,N-1)}_{N-1} \sim \pi_{N-1}$, etc, and in $\mathcal{O}_{LE}$, $\pi'^{e(11)}_1 \sim \pi_1$. Working in the absolute valuation ring O_{LE}, this translates as

$$
\begin{aligned}
(\pi'_N)^{e(NN)} &= \pi_N \alpha_N v_N \\
(\pi'_{N-1})^{e(N-1,N-1)} &= \pi'^{a(N,N-1)}_N \pi_{N-1} \alpha_{N-1} v_{N-1} \\
&\vdots \\
(\pi'_n)^{e(nn)} &= \pi'^{a(N,n)}_N \cdots \pi'^{a(n+1,n)}_{n+1} \pi_n \alpha_n v_n \\
&\vdots \\
(\pi'_1)^{e(11)} &= \pi'^{a(N,1)}_N \cdots \pi'^{a(2,1)}_2 \pi_1 \alpha_1 v_1
\end{aligned}
$$

for $\alpha_i \in k^*$ (or Teichmüller representatives), principal units $v_i \in V_{LE}$, and integers $a(i,j)$. But L/F is tamely ramified so $p \nmid e$ and hence V_{LE} is e-divisible. It follows by working backwards that $\pi'_1, \ldots, \pi'_N \in E$. Since F_{tr} also contains the maximal purely unramified sub-extension of L/F, this implies that $F_{tr} \supset L$. □

Definition 1.17 *For a higher local field F and $n = p^m d$ with $p \nmid d$, let $\widetilde{k}/F^{(N)}$ be of degree n and let $\alpha \in \widetilde{k}$ generate $\widetilde{k}^*$. Set $F(n) = F(\sqrt[d]{[\alpha]}, \sqrt[d]{\pi_1}, \ldots, \sqrt[d]{\pi_N})$ for any set of local parameters $\pi_1, \ldots, \pi_N$ of F.*

With this definition, we have

Corollary 1.18 *Any tamely ramified extension of F of degree dividing n is contained in $F(n)$.*

Chapter 2

Milnor K-groups

2.1 Definitions

Definition 2.1 *The n-th Milnor K-group of a field F is defined to be*

$$K_n(F) = (F^*)^{\otimes n}/St_n(F),$$

where $St_n(F)$ is the subgroup generated by all elements $x_1 \otimes \cdots \otimes x_n$ with $x_i + x_j = 1$ for $i \neq j$. The class of $x_1 \otimes \cdots \otimes x_n$ is denoted $\{x_1, \ldots, x_n\}$. In dimension 0, one defines $K_0(F) = \mathbb{Z}$.

Note that $K_1(F) = F^*$ is just the multiplicative group of the field since there are no relations in dimension 1. The canonical map $(F^*)^{\otimes n} \times (F^*)^{\otimes m} \to (F^*)^{\otimes m+n}$ induces a multiplication of K-groups $K_n(F) \times K_m(F) \to K_{n+m}(F)$ which makes $K_*(F) = \bigoplus_n K_n(F)$ into a graded ring.

K_n is functorial: to any inclusion $F \subset L$ it associates a map $j = j_{F/L} : K_n(F) \to K_n(L)$

The subgroups $U^{(\underline{c})} = 1 + \mathfrak{p}_{\underline{c}}$ and $V_F = 1 + \mathfrak{m}$ of the multiplicative group F^* give rise to the subgroups $U^{(\underline{c})}K_n(F)$ and $VK_n(F)$ of $K_n(F)$. They are, by definition, the subgroups generated by all symbols having at least one entry in $U_F^{(\underline{c})}$ (resp. in V_F). We shall need the case where $\underline{c} = (c) \in \mathbb{Z}^1$.

We give some useful identities in $K_*(F)$ for future reference.

Lemma 2.2 *For any $a, b \in F^*$ such that $a+b \in F^*$, $\{a, b\} = \{a+b, -b/a\} \in K_2(F)$. For any $x \in F^*$, $\{x, -x\} = 0$ and $\{-, -\}$ is skew-symmetric.*

Proof If $x = 1$ then clearly $\{x, -x\} = 0$. If $x \neq 0, 1$ then $-x = (1-x)/(1-1/x)$, thus $\{x, -x\} = \{x, 1-x\} - \{x, 1-1/x\} = 0$. Skew-symmetry follows since $\{x, y\} + \{y, x\} + \{x, -x\} + \{y, -y\} = \{xy, -xy\} = 0$ for any $x, y \in F$. Finally note that $\{a, b\} = \{a, b\} + \{a, -a\} + \{1 + \frac{b}{a}, \frac{-b}{a}\} = \{a, -ab\} + \{a+b, -b\} - \{a+b, a\} - \{a, -b\} - \{a, a\} = \{a+b, \frac{-b}{a}\}$. $\square$

Lemma 2.2 is used to prove the following

Lemma 2.3 *The image of $U^{(c)} \times U^{(d)}$ in $K_2(F)$ lies in $U^{(c+d)}K_2(F)$*

Proof This follows from

$$\begin{aligned}
\{1 + x\pi^{c+d}, -1 - y\pi^d\} &= \{x\pi^{c+d} - y\pi^d, (1 + y\pi^d)/(1 + x\pi^{c+d})\} \\
&= \{-y\pi^d, (1 + y\pi^d)/(1 + x\pi^{c+d})\} + \{1 - x/y\pi^c, (1 + y\pi^d)/(1 + x\pi^{c+d})\} \\
&\equiv \{-y\pi^d, 1 + y\pi^d\} + \{1 - x/y\pi^c 1 + y\pi^d\} \mod U^{(c+d)} \\
&\equiv \{1 - x/y\pi^c, 1 + y\pi^d\} \mod U^{(c+d)}
\end{aligned}$$

for any $x, y \in \mathcal{O}_F$. $\square$

Remark The same holds for $\underline{c}, \underline{d} \in \mathbb{Z}^n$ with $1 \leqslant n \leqslant N$ and x, y in the pre-image of $\mathcal{O}_{F^{(N-n)}}$ in $\mathcal{O}_F$, but we shall only need the case $n = 1$.

Lemma 2.4 *For any l coprime to p, $VK_n(F)$ is l-divisible.*

This follows from the l-divisibility of V_F (corollary 1.8). In fact by [4], prop. 1.2, $VK_n(F)$ is uniquely l-divisible for $n \geqslant 2$.

Lemma 2.5 *If x, y are roots of unity in a higher local field F with $\mathrm{char}(F^{(N)}) = p > 2$, then $\{x, y\} = 0$. If $\mathrm{char}(F^{(N)}) = 2$, the statement is true only if x, y are of odd order.*

Proof Suppose $p > 2$ and $x = \zeta^a, y = \zeta^b \in \mu_n$, so that $\{x, y\} = ab\{\zeta, \zeta\}$. It follows from $\{\zeta, -\zeta\} = 0$ that $2\{\zeta, \zeta\} = 0$. Now if $n = p^M$, then $\zeta^a = \zeta^{(p^M+1)a}$, so, replacing

a with $(p^M+1)a$ if necessary, we may assume that ab is even, hence $\{\zeta^a, \zeta^b\} = 0$. If $p \nmid n$ and $\zeta_n \in F$, then also $\zeta_n \in k$, so we may assume $n = q-1$, $q = |k|$. We use the trick from [16], IX, prop. (1.3) to prove that K_2 of a finite field is trivial. k^* has $(q-1)/2$ squares and $(q-1)/2$ non-squares. Since 1 is a square, the map $k \setminus \{0,1\} \to k \setminus \{0,1\}$, $\alpha \mapsto 1-\alpha$ cannot map all non-squares to squares. This means that there exist odd k, l with $\zeta_n^k = 1 - \zeta_n^l$ in k. In F, this means that there exists $z \in \mathfrak{m}_F$ such that $\zeta^k = (1-\zeta^l)(1+z)$, hence $lk\{\zeta, \zeta\} = \{\zeta^l, \zeta^k\} = \{\zeta^l, 1+z\}$. But $1+z \in V_F$ is $(q-1)$-divisible, so $\{\zeta^l, 1+z\} = 0$. Since lk is odd, we again get $\{\zeta, \zeta\} = 0$. Finally, any root of unity ζ is of the form $\zeta_n^i \zeta_{p^M}^j$ for some M and $p \nmid n$, and

$$\{\zeta, \zeta\} = i^2\{\zeta_n, \zeta_n\} + j^2\{\zeta_{p^M}, \zeta_{p^M}\} = 0,$$

since the cross-terms cancel.

If $\mathrm{char}(F^{(N)}) = 2$ and $x, y \in \mu_n$ for $2 \nmid n$, then $n\{x,y\} = 0$, but $2\{x,y\} = 0$ since $2\{\zeta, \zeta\} = 0$ still holds. So again $\{x, y\} = 0$. $\square$

Example Notice that if $\mathrm{char}(F^{(N)}) = 2$, $\{-1,-1\} \neq 0$ in general. However if, e.g. $F \supset \mathbb{Q}_3$, we have $-1 = 1-(-1)$ in $\mathbb{F}_3$ which lifts to

$$-1 = (1-(-1))(1+3+3^2+\cdots), \quad \text{with} \quad 3+3^2+\cdots \in \mathfrak{m}_F.$$

So $\{-1,-1\} = \{-1, 1-(-1)\} + \{-1, 1+3+3^2+\cdots\} = 0$ because $1+3+3^2+\cdots$ and 2 are squares in $\mathbb{Q}_3$.

Using this, we can describe the structure of $K_n(F)$. See, e.g. [43], prop. 1.2.

Proposition 2.6 *Let F be an N-dimensional local field and $\pi_1, \ldots, \pi_N$ a set of local parameters. Then*

$$K_n(F) \cong \bigoplus_{i_1 < \cdots < i_n} \langle \{\pi_{i_1}, \ldots, \pi_{i_n}\} \rangle \oplus \bigoplus_{i_1 < \cdots < i_{n-1}} \langle \{\varrho, \pi_{i_1}, \ldots, \pi_{i_{n-1}}\} \oplus VK_n(F),$$

where ϱ is a generator of the multiplicative group k^* if $\mathrm{char}(F) = p$ (resp. the Teichmüller representative of a generator of k^* in $W(k)$ if $\mathrm{char}(F) = 0$). In particular,

$$K_N(F) = \big\langle \{\pi_1, \ldots, \pi_N\} \big\rangle \oplus \bigoplus_{1 \leqslant i \leqslant N} \big\langle \{\varrho, \pi_1, \ldots, \pi_{i-1}, \pi_{i+1}, \ldots, \pi_N\} \big\rangle \oplus VK_n(F)$$

Proof Because $F^* = \langle\pi_1\rangle \times \cdots \langle\pi_N\rangle \times k^* \times V_F$, any n-symbol can be written as a sum of symbols whose entries are local parameters, principal units, or in k^* (resp. Teichmüller representatives). If a symbol contains two elements from k^*, it is zero by lem. 2.5. If a symbol contains an element of k^* and a principal unit, it is again zero since $\alpha^{q-1} = 1$ for $\alpha \in k^*$ and $q = |k|$, whereas V_F is $(q-1)$-divisible. The result follows because the intersection of any two of the above subgroups is clearly trivial. □

As with the multiplicative group F^*, the first factor in this decomposition is a direct sum of infinite cyclic groups, while the second one is a direct sum of cyclic groups of order $|k^*|$, so it remains to study $VK_n(F)$.

2.2 Topological K-groups

In this section we define a topology on $K_n(F)$ in such a way that its maximal Hausdorff quotient admits generators for $VK_n(F)$. The definition of topological K-groups can be motivated by the following description, due to Fesenko, taken from [43].

Proposition 2.7 *Let $\pi_1, \ldots, \pi_N$ be local parameters of F, and r any positive integer. Then for given $u_1 \in V_F$, $u_2, \ldots, u_n \in F^*$ there exist $v_i \in V_F$ such that*

$$\{u_1, \ldots, u_N\} \equiv \sum_{1 \leqslant i \leqslant N} \{v_i, \pi_{i_1}, \ldots, \pi_{i_{n-1}}\} \mod p^r VK_N(F).$$

This indicates that the groups $K_n'(F) = K_n(F)/(\bigcap_m p^m K_n(F))$ are of interest.

We introduce a topology on $K_n(F)$ with respect to which $VK_n(F)$ admits topological generators (see [32, 12] for the equal characteristic case and [11] for the mixed characteristic case, as well as [14, 43]). Let V_F and F^* be equipped with the P-topology. $VK_n(F)$ is given the strongest topology satisfying

(i) The map induced by multiplication $V_F \times (F^*)^{n-1} \to VK_n(F)$ is sequentially continuous, and

(ii) Addition and subtraction of symbols in $VK_n(F)$ is sequentially continuous.

The factors $\langle\{\pi_{i_1},\dots,\pi_{i_n}\}\rangle$ and $\langle\{\varrho,\pi_{i_1},\dots,\pi_{i_{n-1}}\}\rangle$ of the decomposition of $K_n(F)$ from prop. 2.6 are given the discrete topology.

Definition 2.8 *The topological Milnor K-groups are $K_n^t(F) = K_n(F)/\Lambda_n$, where Λ_n is the intersection of all neighbourhoods of zero, with the induced topology.*

By [14], prop. 2.6, $\Lambda_n = \bigcap_{n\geqslant 1} nVK_n(F)$. Since V_F is l-divisible for $p \nmid l$, this implies $\Lambda_n = \bigcap_{m\geqslant 1} p^m VK_n(F)$ so that, as abstract groups, $K_n'(F) = K_n^t(F)$ are equal.

The structure theorem clearly holds for $K_n^t(F)$ in the same way as it does for $K_n(F)$. Moreover, prop. 2.7 implies the following

Corollary 2.9 *Every element $x \in VK_n^t(F)$ can be written as a sum of elements $\{v_{\underline{i}},\pi_{i_1},\dots,\pi_{i_{n-1}}\}$ with $v_{(i_1,\dots,i_{n-1})} \in V_F$ and $1 \leqslant i_1 < \cdots < i_{n-1} \leqslant N$.*

The relation from lemma 2.2, is used in the proofs (see [32, 11, 43]) of the following results.

Proposition 2.10 *If* $\mathrm{char}(\mathcal{F}) = p$*, with local parameters $t_1,\dots,t_N$ then $VK_N^t(\mathcal{F})$ is generated by all elements $\{1+\alpha\underline{\bar{t}}^{\underline{a}},\bar{t}_1,\dots,\bar{t}_{i-1},\bar{t}_{i+1},\dots,\bar{t}_N\}$, for α running through a basis of $k/\mathbb{F}_p$, $\underline{a} > 0$, and i maximal with $p \nmid a_i$. $K_N^t(\mathcal{F})$ is free on those generators.*

The second part is proved using the non-degeneracy of Parshin's pairing ([32], see chapter 4).

Proposition 2.11 *If* $\mathrm{char}(F) = 0$ *then $VK_N^t(F)$ has topological generators $\{1+\alpha\underline{\pi}^{\underline{a}},\pi_1,\dots,\pi_{i-1},\pi_{i+1},\dots,\pi_N\}$ for α running through a basis of $W(k)/\mathbb{Z}_p$, $0 < \underline{a} < \underline{e}p/(p-1)$, and i is maximal (or minimal) subject to $p \nmid a_i$. If $\zeta_p \in F^*$ then one also needs $\{\varepsilon,\pi_1,\dots,\pi_{j-1},\pi_{j+1},\dots,\pi_N\}$ for $1 \leqslant j \leqslant N$ and ε as in prop. 1.10.*

Using Vostokov's symbol, it is shown ([11, 39]) that if $\zeta_p \in F$, these topological generators are minimal for $K_N^t(F)/p$. Furthermore, if M is maximal such that $\zeta_{p^M} \in F$, then $K_N^t(F)/p^M$ is free on those generators.

Remark It follows from the proofs of the above two propositions that the condition $p \mid a_j$ for all $j \leqslant i$, $p \nmid a_i$ may be replaced with an analogous statement for any chosen numbering of the local parameter. We will make use of this in section 4.

2.3 The morphism ∂

In this section, we define the boundary morphism of Milnor K-groups for fields with a discrete valuation. In order to simplify the exposition, we only consider ordinary Milnor K-groups in this section and 2.4. All statements hold for topological Milnor K-groups by continuity.

For a discrete valuation field F with valuation v, uniformiser π, and residue field $F^{(1)}$, define a map $\partial : K_n(F) \to K_{n-1}(F^{(1)})$ by

$$\partial\{x_1, \ldots, x_n\} = \sum(-1)^{r_1+\cdots+r_s}\partial^{\underline{r}}\{x_1, \ldots, x_n\},$$

where, for any $\underline{r} = (r_1, \ldots, r_s)$ with $r_1 < \cdots < r_s$,

$$\partial^{\underline{r}}\{x_1, \ldots, x_N\} = v(x_{r_1}) \cdots v(x_{r_s})\ x\ \{-1, \ldots, -1\},$$

$x \in K_{n-s}(F^{(1)})$ is the symbol consisting of the residues of $x_i\pi^{-v_1(x_i)}$ with the r_i-th places omitted, and $\{-1, \ldots, -1\} \in K_{s-1}(F^{(1)})$. For the straightforward verification that ∂, defined on $(F^*)^n$, does indeed factor through $K_n(F)$ see, e.g. [16], IX, (2.1). Given $x \in F^*$, write it as $x = \pi^{v(x)}u$ for some unit u. Using $\{\pi, \pi\} = \{\pi, -1\}$ we see that any n-symbol can be written as a linear combination of two types of symbols, namely $\{\pi, v_1, \ldots, v_{n-1}\}$ and $\{v'_1, \ldots, v'_n\}$ for π-units v_i, v'_i. This shows that ∂ is independent of the choice of uniformiser. If $\pi' = v\pi$, with v a 1-unit, then $\{\pi', u_1, \ldots, u_{N-1}, \pi'\} = \{\pi, u_1, \ldots, u_{N-1}\} + \{v, u_1, \ldots, u_{N-1}\}$ has the same image under ∂_π and $\partial_{\pi'}$. The following can be used as an alternative definition of ∂

Lemma 2.12 *For units $u_1, \ldots, u_n$, we have $\partial\{\pi, u_1, \ldots, u_{n-1},\} = \{\overline{u}_1, \ldots, \overline{v}_{n-1}\} \in K_{n-1}(F^{(1)})$, and $\partial\{u_1, \ldots, u_{n-1}, u_n\} = 0$.*

Proof Since $v(u_i) = 0$, $\partial^{(r_1,\ldots,r_s)}(\{\pi, u_1, \ldots, u_{n-1}\}) = 0$ unless $\underline{r} = (n)$, in which case $\partial^{(n)}\{\pi, u_1, \ldots, u_{n-1}\} = \{\overline{u}_1, \ldots, \overline{u}_{n-1}\}$. Clearly $\partial^{\underline{r}}\{u_1, \ldots, u_n\} = 0$ for all $\underline{r}$. □

Let now F be a higher local field. For an intermediate residue field $F^{(n-1)}$ with uniformiser $\overline{\pi}_{N-n}$ denote the corresponding map ∂ by ∂_n.

Definition 2.13 *The valuation* $\mathbf{v}$ *on* $K_N(F)$ *is defined to be the composite*

$$\mathbf{v} : K_N(F) \xrightarrow{\partial_1} K_{N-1}(F^{(1)}) \xrightarrow{\partial_2} \cdots \xrightarrow{\partial_{N-1}} K_1(F^{(N-1)}) \xrightarrow{\partial_N} K_0(F^{(N)}) = \mathbb{Z}.$$

A 'uniformiser' with respect to this valuation is a symbol consisting of any complete set of local parameters $\{\pi_1, \dots, \pi_N\}$.

Note that ∂_N is the usual discrete valuation on the 1-dimensional local field $F^{(N-1)}$.

Lemma 2.14 *For a finite extension L/F of discrete valuation fields, the diagram*

$$\begin{array}{ccc} K_n(F) & \xrightarrow{j} & K_n(L) \\ {\scriptstyle \partial_F}\downarrow & & \downarrow{\scriptstyle \partial_L} \\ K_{n-1}(F^{(1)}) & \xrightarrow{e\,j} & K_{n-1}(L) \end{array}$$

is commutative, with $e = v_L(\pi_F)$. In particular, if $L \supset F$ are N-dimensional local fields, the valuation $\mathbf{v}$ satisfies

$$\mathbf{v}_L(j_{F/L}(x)) = e^{(11)} \cdots e^{(NN)}\, \mathbf{v}_F(x)$$

for any $x \in K_N(F)$

Proof For a uniformiser π_F of F, $\partial_F\{\pi_F, x_1, \dots, x_{N-1}\} = \{\overline{x}_1, \dots, \overline{x}_{N-1}\}$ and $\partial_L\{\pi_F, x_1, \dots, x_{N-1}\} = e\,\{\overline{x}_1, \dots, \overline{x}_{N-1}\}$ since $\pi_F \sim \pi_E^e$ in $\mathcal{O}_L$ $\square$

In the following section, we shall consider ∂ on a function field $F(X)$ in one variable, where F may be any field, although we are only interested in higher local fields. The discrete valuations on $F(X)$ are in one to one correspondence with the monic irreducible polynomials of $F[X]$, with one additional valuation corresponding to $\frac{1}{X}$. We write $v_{a(X)}$ for the valuation corresponding to $a(X) \in F[X]$, and v_∞ to the one corresponding to $\frac{1}{X}$. Following [4], we denote the residue field of $v_{a(X)}$ by $F(v) = F[X]/(a(X))$. If $v = v_\infty$, the residue field is $F\left[\frac{1}{X}\right]/\left(\frac{1}{X}\right) \cong F$.

Any element in $K_n(F(X))$ can be written as a linear combination of elementary symbols consisting of irreducible monic polynomials in $F[X]$ and elements of F^*. The following two explicit formulae will be used throughout the following section.

Lemma 2.15 *If $a_1, \dots, a_{m-1} \in F^*$ and $a_m(X), \dots, a_n(X) \in F[X]$ with $m < n$ then $\partial_v(\{a_1, \dots, a_{m-1}, a_m(X), \dots, a_n(X)\}) = \{a_1, \dots, a_{m-1}\}\, \partial_v(\{a_m(X), \dots, a_n(X)\})$ for any v.*

This follows directly from the definition of ∂. We will often tacitly make use of it by assuming $m = 1$ for simplicity.

Lemma 2.16 *Let $A = \{a_1(X), \ldots, a_n(X)\} \in K_n(F(X))$ with $a_i(X)$ monic and irreducible of degree d_i, and let $\alpha_i \in F^{alg}$ be a fixed root of $a_i(X)$. Then*

$$\partial_v(A) = \begin{cases} (-1)^{n(n+1)/2} d_1 \cdots d_n \{-1, \ldots, -1\} & \textit{if } v = v_\infty \\ (-1)^i \{a_1(\alpha_i), \ldots, a_{i-1}(\alpha_i), a_{i+1}(\alpha_i), \ldots, a_n(\alpha_i)\}, & \textit{if } v = v_{a_i(X)} \\ 0 & \textit{otherwise,} \end{cases}$$

where the image lies in the respective residue field $F(v) \cong F(\alpha_i)$ for each $v = v_{a_i(X)}$.

Proof For the case $v = v_\infty$, we use the original definition of ∂ as sum over all $\partial^{(r_1,\ldots,r_s)}$. Since the $a_i(X)$ are monic with $v_\infty(a_i(X)) = -d_i$, we have

$$a_i(X)\pi_\infty^{-v_\infty(a_i(X))} = \left(\tfrac{1}{X}\right)^{d_i} a_i(X) \in 1 + \tfrac{1}{X} F\left[\tfrac{1}{X}\right],$$

which has residue 1 in $F(v_\infty) = F$. Thus the only $\partial^{(r_1,\ldots,r_s)}$ which does not vanish is for $(r_1, \ldots, r_s) = (1, \ldots, n)$, with

$$\partial^{(1,\ldots,n)}\{a_1(X), \ldots, a_n(X)\} = (-1)^{1+\cdots+n}(-d_1)\cdots(-d_n)\{-1, \ldots, -1\}.$$

If $v = v_{a_j(X)}$, all $a_i(X)$ with $i \neq j$ are v-units, and the claim follows from lemma 2.12, as does the last case. $\square$

2.4 The Norm map

We outline the definition of a norm map $K_N(L) \to K_N(F)$ for finite extensions L/F. We begin by considering simple extensions $L = F(\alpha) = F[X]/(m(X))$ for some irreducible polynomial $m(X) \in F[X]$. Bass-Tate proved the following ([4]).

Theorem 2.17 *The sequence*

$$0 \longrightarrow K_n(F) \xrightarrow{j} K_n(F(X)) \xrightarrow{\oplus \partial_v} \bigoplus_{v \neq v_\infty} K_{n-1}(F(v)) \longrightarrow 0$$

is exact and splits

The norm maps N_v are defined simultaneously for all $F(v)/F$ by requiring that the extended sequence

$$0 \longrightarrow K_{n+1}(F) \xrightarrow{j} K_{n+1}(F(X)) \xrightarrow{\oplus\partial} \bigoplus_{\text{all } v} K_n(F(v)) \xrightarrow{\oplus N_v} K_n(F) \longrightarrow 0$$

be exact, where $F(v_\infty) = F$ and N_{v_∞} is the identity map. This means that the composite

$$K_{n+1}(F(X)) \xrightarrow[v\neq v_\infty]{\oplus\partial_v} \bigoplus\nolimits_{v\neq v_\infty} K_n(F(v)) \xrightarrow{\oplus N_v} K_n(F)$$

equals $-\partial_{v_\infty}$ Since the first map is surjective and $\mathrm{Hom}\big(\bigoplus A_v, B\big) = \bigoplus \mathrm{Hom}\big(A_v, B\big)$ for any objects A_v and B, this uniquely defines the maps N_v for $v \neq \infty$. Moreover, $\partial_\infty = \mathrm{id}$ is $K_*(F)$-linear, so again by surjectivity of $\oplus\partial_v$, we have

Lemma 2.18 *The norm $N_v : K_*(F(v)) \to K_*(F)$ is $K_*(F)$-linear in the sense that $N_v(j_{F/F(v)}(x)\, y) = x\, N_v(y) \in K_{n+m}(F)$ for $x \in K_n(F)$, $y \in K_m(F)$.*

Lemma 2.19 *For $n = 1$, $N_v : F(v)^* \to F^*$ is the usual norm of fields.*

Proof Since the N_v are uniquely defined by $\sum N_v \circ \partial_v = -\partial_\infty$, it suffices to show that the usual norm satisfies this property. Noting that lemma 2.15 implies $K_*(F)$-linearity of ∂_v, it suffices to consider $A = \{a(X), b(X)\}$ with $a(X), b(X)$ monic, of degrees n, m and with roots α of $a(X)$ and β of $b(X)$. Then

$$\partial_{v_{a(X)}}(A) = b(\alpha),\;\; \partial_{v_{b(X)}}(A) = a(\beta)^{-1},\;\; \partial_\infty(A) = (-1)^{(-1)^{1+2}\, nm} = (-1)^{nm}.$$

If $v_{a(X)}$ and $v_{b(X)}$ are non-equivalent, then the extensions $F(\alpha)$ and $F(\beta)$ are linearly disjoint and $a(\beta)$ splits in $F(\alpha, \beta)^{nc}$ as $a(\beta) = \prod_{i=1}^n(\beta - \alpha_i)$. Now

$$N_{F(\alpha_i,\beta)/F(\alpha_i)}(\beta - \alpha_i) = \prod_{j=1}^{m}(\beta_j - \alpha_i) = (-1)^m \prod_j(\alpha_i - \beta_j) = (-1)^m b(\alpha_i).$$

Thus $N_{F(\beta)/F}(a(\beta)) = (-1)^{nm}\prod_i b(\alpha_i)$. Clearly also $N_{F(\alpha)/F}(b(\alpha)) = \prod_i b(\alpha_i)$, hence $N_{F(\alpha)/F}(b(\alpha))\, N_{F(\beta)/F}(a(\beta)^{-1}) = (-1)^{nm}$, as required. $\square$

In analogy to the case of norms on fields we also have

Lemma 2.20 *The composite $K_n(F) \xrightarrow{j} K_n(F(v)) \xrightarrow{N_v} K_n(F)$ is equal to multiplication by $[F(v) : F]$.*

Proof Let v correspond to the irreducible polynomial $m(X) \in F[X]$ and consider the symbol $A = \{m(X), a_1, \ldots, a_n\} \in K_{n+1}(F(X))$, for $a_i \in F^*$. Then $\partial_{v_{m(X)}}(A) = \{a_1, \ldots, a_n\}$ and $\partial_v(A) = 0$ for all other $v \neq v_\infty$. Also,

$$\partial_\infty\{m(X), a_1, \ldots, a_n\} = \partial_\infty(\{m(X)\})\ \{a_1, \ldots, a_n\} = -d\{a_1, \ldots, a_n\},$$

for $d = \deg(m(X)) = [F(v) : F]$. The claim follows $\square$

The following is weaker than prop. 2.22 below, but can be proved by explicit manipulation.

Proposition 2.21 *Given $\{a_1(X), \ldots, a_n(X)\} \in K_n(F(X))$, where the $a_i(X)$ are irreducible polynomials, of degree d_i, with root α_i. Let $E = F(\alpha_1, \ldots, \alpha_n)^{nc}$ be the composite of the normal closures of all $F(\alpha_i)/F$. Then the norms $F(\alpha_i)/F$, for all i, satisfy*

$$j_{F/E}N_{F(\alpha_i)/F}\big(\{a_1(\alpha_i), \ldots, \widehat{a}_i, \ldots, a_n(\alpha_i)\}\big) = \sum_{\gamma_i} \big\{\gamma_i(a_1(\alpha_i)), \ldots, \widehat{a}_i, \ldots, \gamma_i(a_n(\alpha_i))\big\},$$

where γ runs through set of F-embeddings of $F(\alpha_i)$ into $F(\alpha_i)^{nc}$, with multiplicities if the extension is not separable, and $\widehat{a}_i$ means the i-th place is omitted.

Proof For a fixed root α_i of $a_i(X)$ in F^{alg}, let $\alpha_i^{(r_i)}$ be its conjugates, $1 \leqslant r_i \leqslant d_i$, counted with multiplicities if the extension is inseparable.

By lemma 2.16,

$$\partial_v(\{a_1(X), \ldots, a_n(X)\}) = (-1)^i\{a_1(\alpha_i), \ldots, a_{i-1}(\alpha_i), a_{i+1}(\alpha_i), \ldots, a_n(\alpha_i)\},$$

if $v = v_{a_i(X)}$, and 0 otherwise. Working in E, we see that $a_j(\alpha_i) = \prod_{r_j}(\alpha_i - \alpha_j^{(r_j)})$, for $1 \leqslant r_j \leqslant d_j$, and therefore

$$j_{F(v_i)/E}\,\partial_{v_i}\{a_1(X), \ldots, a_n(X)\} = \sum_{\substack{j\neq i\\ 1\leqslant r_j\leqslant d_j}} (-1)^i\big\{\alpha_i - \alpha_1^{(r_1)}, \ldots, \alpha_i - \alpha_n^{(r_n)}\big\},$$

where the α_i-term is missing. Denoting by M_i the sum over all conjugates of α_i, we have

$$\begin{aligned} &M_i \circ j_{F(v_i)/E} \circ \partial_{v_i}(\{a_1(X), \ldots, a_n(X)\}) \\ &= \sum_{1\leqslant r_i\leqslant d_i} \sum_{\substack{j\neq i\\ 1\leqslant r_j\leqslant d_j}} (-1)^i\big\{\alpha_i^{(r_i)} - \alpha_1^{(r_1)}, \ldots, \alpha_i^{(r_i)} - \alpha_{i-1}^{(r_{i-1})}, \ldots, \alpha_i^{(r_i)} - \alpha_n^{(r_n)}\big\}. \end{aligned}$$

Then the image of $\{a_1(X),\ldots,a_n(X)\}$ under the composition of maps

$$K_n(F(X)) \xrightarrow{\oplus\partial_v} \bigoplus K_{n-1}(F(v)) \xrightarrow{\oplus j_{F(v_i)/E}} \bigoplus K_{n-1}(E) \xrightarrow{\oplus M_i} K_n(E)$$

is equal to

$$\sum_{1\leqslant i\leqslant d_i} \sum_{\substack{\text{all } j\\ 1\leqslant r_j\leqslant d_j}} (-1)^i \big\{\alpha_i^{(r_i)}-\alpha_1^{(r_1)},\ldots,\alpha_i^{(r_i)}-\alpha_{i-1}^{(r_{i-1})},\ldots,\alpha_i^{(r_i)}-\alpha_n^{(r_n)}\big\}. \qquad (\star)$$

We shall show that this equals

$$-\partial_{v_\infty}\{a_1(X),\ldots,a_{n+1}(X)\} = (-1)^m d_1\cdots d_n\{-1,\ldots,-1\}\in K_{n-1}(F),$$

for $m=n(n+1)/2+1$, that is, that the maps M_i satisfy the defining equation of the N_{v_i} after going up to $K_{n-1}(E)$.

Suppose for the moment that all $d_i=1$, i.e. $a_i(X)=X-\alpha_i$ for $\alpha_i\in F$. Then $F(v_{a_i(X)})=F$ and $N_{v_{a_i(X)}}: K_n(F)\to K_n(F)$ is the identity. In this case the definition of the norm becomes

$$-\partial_\infty\{X-\alpha_1,\ldots,X-\alpha_n\} = \sum_{v\neq\infty}(N_v\circ\partial_v)\big(\{X-\alpha_1,\ldots,X-\alpha_n\}\big) \quad \text{i.e.}$$

$$(-1)^m\{-1,\ldots,-1\} = \sum_{1\leqslant i\leqslant n}(-1)^i\{\alpha_i-\alpha_1,\ldots,\alpha_i-\alpha_{i-1},\alpha_i-\alpha_{i+1},\ldots,\alpha_i-\alpha_n\}.$$

Returning to $(\star)$, fix any j and any r_j. Then the above implies that the sum over i equals $(-1)^m\{-1,\ldots,-1\}$. Since there are d_j of the r_j and n of the j, this means that

$$(*) = d_1\cdots d_n(-1)^m\{-1,\ldots,-1\} = -\partial_{v_\infty}\{a_1(X),\ldots,a_n(X)\},$$

so $j_{F/E}\circ\sum N_{v_i}\circ\partial_{v_i} = \sum_i M_i$, as required. $\square$

A stronger statement follows from the following result taken from [16], IX, prop. 3.3.

Proposition 2.22 *The diagram*

$$\begin{tikzcd}
0 \arrow[r] & K_{n+1}(F) \arrow[r] & K_{n+1}(F(X)) \arrow[r, "\oplus\partial_v"] \arrow[d, "j"] & \bigoplus_v K_n(F(v)) \arrow[r, "\oplus N_v"] \arrow[d, "\oplus e_{w/v} j"] & K_n(F) \arrow[d, "j"] \\
0 \arrow[r] & K_{n+1}(F') \arrow[r] & K_{n+1}(F'(X)) \arrow[r, "\oplus\partial_w"'] & \bigoplus_w K_n(F'(w)) \arrow[r, "\oplus N_w"] & K_n(F')
\end{tikzcd}$$

is commutative

Corollary 2.23 *If $L = F(v)$ with $v = v_{a(X)}$ for some monic irreducible $a(X) \in F[X]$, and L' is the normal closure, then $j_{F/L'} \circ N_{L/F} : K_n(L) \to K_n(L')$ is equal to $p^s \sum_i \gamma_i$, where p^s is the degree of inseparability and γ_i runs through a set of F-embeddings of L into L'.*

We shall also need the following corollary

Corollary 2.24 *If $L = F(v)$ for $v = v_{a(X)}$ and F' is such that $L \cap F' = F$, let w be such that $L = F(w)$. Then $N_w \circ j_{L/LF'} = j_{F/F'} \circ N_v$*

In order to define the norm for extensions rather than elements generating simple extensions, one starts by showing that $N_v = N_\alpha$ is independent of the choice of element generating it, i.e. that $N_\alpha = N_{\alpha'}$ if $F(\alpha) = F(\alpha')$. Then one generalises this to extensions $L = F(\alpha_1, \dots, \alpha_r)$ obtained by joining more than one element. As a last step, one needs to prove that defined for a string $(\alpha_1, \dots, \alpha_r)$ is independent of the choice of elements α_i generating the extension. This is then defined to be the norm $N_{L/F} : K_n(L) \to K_n(F)$. The following is taken from [16], IX,(3.8).

Theorem 2.25 (Bass-Tate-Kato) *Let L/F be a finite extension, then there exists a norm map $N_{L/F} K_*(L) \to K_*(F)$ which is $K_*(F)$-linear and satisfies*

(1) $N_{L/F}$ coincides with $N_{\alpha_1,\dots,\alpha_l}$ for any $\alpha_i \in L$ such that $L(\alpha_1, \dots, \alpha_l)$

(2) For any $F \subset M \subset L$, $N_{L/F} = N_{M/F} \circ N_{L/M}$

(3) $N_{L/F}$ acts on $K_0(L) = \mathbb{Z} = K_0(F)$ as multiplication by $[L : F]$ and on $K_1(L)$ as the usual norm.

(4) $N_{L/F} \circ j_{F/L}$ is multiplication by $[L : F]$

(5) If $L' \supset L \supset F$, then $j_{L/L'} \circ N_{L/F} = p^s \sum \gamma_i$ where p^s is the degree of inseparability and γ_i runs through a set of distinct F-embeddings of L into L'

(6) $N_{L/F} \circ \sigma = N_{L/F}$ for any F-automorphism σ of L.

We will make ample use of (2) and (5), as well as the following corollary of (4).

Corollary 2.26 *The kernel of $j_{F/L}$ is contained in the $[L : F]$-torsion subgroup of $K_n(F)$.*

Note that for simple extensions, this follows from lemma 2.20.

Lemma 2.27 *The valuation $\mathbf{v}_F$ on $K_N(F)$ satisfies $\mathbf{v}_F \circ N_{L/F} = f_{L/F}\mathbf{v}_L$ where $f_{L/F} = [L^{(N)} : F^{(N)}]$ is the last residue degree of the extension L/F.*

Proof For any set $\pi_1, \ldots, \pi_N$ of local parameters of F, $\mathbf{v}(K_N^t(E)) = \mathbb{Z}$ is generated by $\mathbf{v}(\{\pi_1, \ldots, \pi_N\}) = 1$. Then

$$\mathbf{v}_F \circ N_{E/F} \circ j_{F/E}(\{\pi_1, \ldots, \pi_N\}) = [E : F]\, \mathbf{v}_F(\{\pi_1, \ldots, \pi_N\}) = [E : F].$$

On the other hand, $\mathbf{v}_E \circ j_{F/E}(\{\pi_1, \ldots, \pi_N\}) = e^{(11)} \cdots e^{(NN)}$ by iterating lemma 2.14. Since $[E : F] = f\, e^{(11)} \cdots e^{(NN)} \neq 0$ and $\mathbb{Z}$ is free, the lemma follows. □

2.5 K-groups of rings

In section 4.2, we will need a generalisation of Milnor K-groups to rings. We propose two possible constructions, each having its advantages and disadvantages.

For rings with 'sufficiently many' units such as (complete) discrete valuation rings, Milnor K-groups are defined, e.g. in [10]

Definition 2.28 *The Milnor K-groups $K_n(A)$ are defined to be*

$$K_n(A) = (A^*)^{\otimes n}/St_n(A),$$

where $St_n(A)$ is generated by all elements $a_1 \otimes \cdots \otimes a_n$ with $a_i + a_j = 0$ or $a_i + a_j = 1$ for $i \neq j$. The image of $a_1 \otimes \cdots \otimes a_n$ in $K_n(A)$ is denoted $\{a_1, \ldots, a_n\}$.

Because $x \neq 0, 1$ in the ring A need not imply $1 - x \in A^*$, the relation $\{x, -x\}$ which holds in $K_2(F)$ for any field F has to be enforced in the case of rings.

As in the case of fields, K_n is functorial: to any ring-homomorphism $f : A \to B$ it associates $K_n(f) : K_n(A) \to K_n(B)$, satisfying the usual properties. We shall need the special case where $f : A \to A/\mathfrak{p}$ is the projection of a discrete valuation ring onto its residue field.

In [10] it is proved that if A is a semi-local PID with field of fractions F, then $K_n(A) \to K_n(F)$ is injective. In particular, if $\mathcal{O}$ is the first valuation ring of a higher local field Q then $j : K_n(\mathcal{O}) \hookrightarrow K_n(Q)$. One may define the topological Milnor K-groups to be $K'_n(\mathcal{O}) = \mathrm{Im}\big(K_n(\mathcal{O}) \hookrightarrow K_n(Q) \twoheadrightarrow K^t_n(Q)\big)$ with the induced topology.

While this definition of K_n of rings is very natural, it can not be used to determine a set of generators small enough to be of any use. In the special case of valuation rings of higher local fields, the following turns out to be more appropriate. In view of the applications (section 4.3), we consider $(N+1)$-dimensional local fields.

Definition 2.29 *For a higher local field Q with local parameters $\pi = \pi_0, \pi_1, \dots, \pi_N$ and first valuation ring $\mathcal{O}$ define the subgroup of $K^t_n(Q)$ corresponding to $\mathcal{O}$ to be the closure $K^t_n(\mathcal{O})$ of the subgroup generated by all elements*

$$\{1+\pi_0 x, \pi_{j_1}, \dots, \pi_{j_{n-1}}\}, \text{ for } x \in \mathcal{O},\ 0 \leqslant j_1 < \dots < j_{n-1} \leqslant N;$$
$$\text{and } \{1+\alpha\pi_1^{a_1}\cdots\pi_N^{a_N}, \pi_{i_1}, \dots, \pi_{i_{n-1}}\},\ \{\pi_{i_1}, \dots, \pi_{i_n}\},\ \{\alpha, \pi_{i_1}, \dots, \pi_{i_{n-1}}\}$$

for $\alpha \in k^*$, $1 \leqslant i_1 < \dots \leqslant N$, and $(a_1, \dots, a_N) > (0, \dots, 0)$.

By cor. 2.9 or prop. 2.11 on generators of $K^t_n(Q)$, we may assume that $1+\pi_0 x = 1+\beta\pi_0^{b_0}\pi_1^{b_1}\cdots\pi_N^{b_N}$ for $(b_0, b_1, \dots, b_N) > (0, \dots, 0)$, $p \nmid \underline{b}$. Notice $K^t_n(Q)$ is generated by $K^t_n(\mathcal{O})$ together with three types of generators, namely

$$\{1+\alpha\pi_1^{a_1}\cdots\pi_N^{a_N}, \pi_0, \pi_{i_1}, \dots, \pi_{i_{n-2}}\},\ \{\pi_0, \pi_{i_1}, \dots, \pi_{i_{n-1}}\},\ \{\alpha, \pi_0, \pi_{i_1}, \dots, \pi_{i_{n-2}}\},$$

for $1 \leqslant i_1 \leqslant \dots \leqslant N$, $\underline{a} > \underline{0}$ and $\alpha \in k^*$. Using this, we can prove the following implicit description of $K^t_n(\mathcal{O})$.

Lemma 2.30 *For any uniformiser π of Q, the sequence*

$$0 \longrightarrow K^t_n(\mathcal{O}) \longrightarrow K^t_n(Q) \xrightarrow{\partial} K^t_{n-1}(\mathcal{F}) \longrightarrow 0$$

is exact, i.e. $K^t_n(\mathcal{O})$ may be defined independently of generators as $K^t_n(\mathcal{O}) = \ker(\partial)$.

Proof $K_n^t(\mathcal{O}) \to K_n^t(Q)$ is injective by definition. If $\mathrm{char}(Q) = p$, surjectivity of ∂ is clear. If $\mathrm{char}(Q) = 0$, surjectivity follows since the multi-index $(a_1, \ldots, a_N)$ needed for generators of $V_{\mathcal{F}}$ corresponds to $(0, a_1, \ldots, a_N)$, and since $\mathrm{char}(\mathcal{F}) = p$, the absolute ramification index $\underline{e} = (e_0, \ldots, e_N)$ of Q satisfies $e_0 > 0$, thus $(0, a_1, \ldots, a_N) < \underline{e}p/(p-1)$ for all $(a_1, \ldots, a_N) > \underline{0} \in \mathbb{Z}^N$. Thus ∂ is always surjective. Considering the generators of $K_n^t(\mathcal{O})$ from def. 2.29, it follows that $K_n^t(\mathcal{O}) \subset \ker(\partial)$. Finally notice that the images of the above complementary generators of $K_n^t(Q)$ are free generators of $K_n^t(\mathcal{F})$, thus no linear combination of them lies in the kernel. □

Corollary 2.31 *The groups $K_n'(\mathcal{O})$ and $K_n^t(\mathcal{O})$ are related by $j(K_n'(\mathcal{O})) \subset K_n^t(\mathcal{O})$, where $j : K_n'(\mathcal{O}) \subset K_n^t(Q)$.*

Proof Consider the alternative definition of ∂ given by lemma 2.12 for the two types of elements $\{v_1, \ldots, v_n\}$ and $\{v_1', \ldots, v_{n-1}', \pi\}$ of $K_n^t(Q)$, with π-units v_i, v_j'. Elements coming from $K_n^t(\mathcal{O})$ are of the first type, hence $\partial(K_n^t(\mathcal{O})) = 0$. □

Remark Working in $K_n^t(Q)$, elements coming from $K_n'(\mathcal{O})$ may be presented as linear combinations of symbols having entries outside $\mathcal{O}^*$. For example, in $K_2^t(Q)$ we have $n\{1 + \pi^n v, \pi\} = -\{1 + \pi^n v, -v\}$, and $\pi \notin \mathcal{O}^*$. This also shows that the inclusion $K_n'(\mathcal{O}) \subset K_n^t(\mathcal{O})$ is, in general, strict: If $p \mid n$, $1 + \pi\mathcal{O}$ is not n-divisible, so $\{1 + \pi^n v, \pi\} \in K_2^t(\mathcal{O}) \setminus K_2'(\mathcal{O})$.

The subgroup of $K_n^t(\mathcal{O})$ corresponding to $1 + \pi_0\mathcal{O}$ is defined to be the subgroup generated by the first type of generators, it is denoted $U^{(1)}K_n^t(\mathcal{O})$. For a fixed uniformiser π_0 of Q, define a map $\delta : (Q^*)^{\otimes n} \to K_{n-1}(\mathcal{F})$, where $\mathcal{F}$ is the first residue field of Q, by $\delta(x_1 \otimes \cdots \otimes x_n) = \{\overline{u}_1, \ldots, \overline{u}_n\}$, where $u_i = x_i \pi_i^{-v(x_i)}$. To see that δ induces a map on $K_n(Q)$ note that if $x = \pi_0^i u, y = \pi_0^j v$ then $x + y = 1$ can only happen if $i \neq j$, say $i < j$, and moreover $i = 0$, but then $u = 1 - \pi_0^j v$ so $\overline{u} = 1$ and $\{\overline{u}, \overline{v}\} = 0$.

Lemma 2.32 *The sequence*

$$0 \longrightarrow U^{(1)}K_n^t(\mathcal{O}) \longrightarrow K_n^t(\mathcal{O}) \xrightarrow{\delta} K_n^t(\mathcal{F}) \longrightarrow 0$$

is exact.

PROOF Surjectivity of δ uses the same argument as in the above proof of the surjectivity of ∂, together with the fact that lifts of elements of $\mathcal{F}$ may be taken in $\mathcal{O}^*$. Also, $\delta(U^{(1)}K_n^t(\mathcal{O})) = 0$ since $\overline{1+\pi x} = \overline{1}$ for any $x \in \mathcal{O}$. For the converse, note again that the images of the generators of $K_n^t(\mathcal{O})$ which are not generators of $U^{(1)}K_n^t(\mathcal{O})$ are free generators of $K_n^t(\mathcal{F})$. $\square$

δ can be extended to $K_n^t(Q) \to K_n^t(\mathcal{F})$, but this depends on the choice of uniformiser since for $\pi' = \pi u$, $\delta_\pi\{\pi', v\} = \{\overline{u}, \overline{v}\} \neq 0$ for units u, v, whereas $\delta_{\pi'}\{\pi', v\} = \{1, \overline{v}\} = 0$.

Lemma 2.33 *The restriction* $\delta\big|_{K_n^t(\mathcal{O})}$ *is independent of the choice of uniformiser* π_0*. In particular,* $U^{(1)}K_n^t(\mathcal{O}) = \ker(\delta)$ *is independent of the choice of* π_0*.*

PROOF Let $\pi' = v\pi$ for $v \in \mathcal{O}^*$. The only generators of $K_n^t(\mathcal{O})$ affected are the first two types: They become $\{1 + x\pi', \dots\} = \{1 + xv\pi, \dots\}$ and $\{1 + x\pi', \pi', \dots\} = \{1 + xv\pi, \pi, \dots\} + \{1 + xv\pi, v, \dots\}$, thus they are in the kernel of both δ_π and $\delta_{\pi'}$.

Corollary 2.34 *The composite* $K_n'(\mathcal{O}) \subset K_n^t(\mathcal{O}) \xrightarrow{\delta} K_n^t(\mathcal{F})$ *is equal to the map induced by the natural projection* $\mathcal{O}^* \to \mathcal{F}^*$*.*

Chapter 3

Class-Field Theory and Field of Norms

3.1 Class-Field Theory

For classical one-dimensional local fields, Class-Field theory gives an explicit description of abelian Galois groups. More precisely, for any finite Galois extension L/F, the norm-residue symbol is an isomorphism $r_{L/F} : \mathrm{Gal}(L/F)^{ab} \to F^*/N_{L/F}L^*$. For varying abelian extensions L, this yields the reciprocity map

$$\Psi_F : F^* \longrightarrow \varprojlim_L F^*/N_{L/F}(L^*) \longrightarrow \varprojlim \mathrm{Gal}(L/F) \xrightarrow{\sim} \Gamma_F^{ab}.$$

Neukirch's construction (see [30, 31]) of the norm-residue symbol was generalised by Fesenko in [11, 12] as follows. Let L/F be a finite extension of N-dimensional local fields with Galois group $G = \mathrm{Gal}(L/F)$. Let L_{ur} and F_{ur} be the maximal purely unramified extensions of L and F. $\mathrm{Gal}(F_{ur}/F) \cong \widehat{\mathbb{Z}}$ is pro-cyclic, generated topologically by the Frobenius φ_F of F. If the extension of last residue fields $L^{(N)}/F^{(N)}$ is of degree $f = [L^{(N)} : F^{(N)}]$, then $\varphi_F^f = \varphi_L$. The isomorphism $\mathrm{Gal}(F_{ur}/F) \cong \widehat{\mathbb{Z}}$ induces $\deg_F : \mathrm{Gal}(L_{ur}/F) \to \widehat{\mathbb{Z}}$ defined by $\deg(\widetilde{\gamma}) = \alpha$ if $\widetilde{\gamma}|_{F_{ur}} = \varphi_F^{\alpha}$. Setting $\varnothing(L_{ur}/F) = \{\widetilde{\gamma} \in \mathrm{Gal}(L_{ur}/F) \,|\, \deg(\widetilde{\gamma}) \in \mathbb{N}\}$, it is shown that the restriction map $\varnothing(L_{ur}/F) \to \mathrm{Gal}(L/F)$ is surjective.

Given $\gamma \in \mathrm{Gal}(L/F)$, let $\widetilde{\gamma} \in \varnothing(L_{ur}/F)$ be a lift with $\widetilde{\gamma}|_{F_{ur}} = \varphi_F^n$, $n \in \mathbb{N}$, and let

$S = L_{ur}^{\langle\widetilde{\gamma}\rangle}$ be the fixed field of the closed subgroup generated by $\widetilde{\gamma}$, as in the diagram

$$\begin{tikzcd}
F_{ur} \arrow[rr, dash, "\gamma"] & & L_{ur} \arrow[dl, dash, "\widetilde{\gamma}"'] \arrow[dd, dash, "\varphi_L"] \\
& S & \\
F \arrow[uu, dash, "\varphi_F"] \arrow[ur, dash] \arrow[rr, dash] & & L
\end{tikzcd}$$

It is shown that $[S : F]$ is finite, with last residue extension of degree $[S^{(N)} : F^{(N)}] = n$. Furthermore, $S_{ur} = L_{ur}$ and $\widetilde{\gamma} = \varphi_S$ is the Frobenius of S. By [11, 12], we have

Theorem 3.1 *For any $\Pi_S \in K_N(S)$ with $\mathbf{v}_S(\Pi_S) = 1$, the element*

$$r_{L/F}(\gamma) = N_{S/F}(\Pi_S) + N_{L/F}K_N^t(L) \in K_N^t(F)/N_{L/F}K_N^t(L)$$

is independent of the choice of $\widetilde{\gamma}$ and Π_S. $r_{L/F}$ induces an isomorphism

$$r_{L/F} : \mathrm{Gal}(L/F)^{ab} \longrightarrow K_N^t(F)/N_{L/F}K_N^t(L).$$

Taking the projective limit over all finite abelian extensions L of F, the inverses of these maps gives rise to the reciprocity map

$$\Psi_F : K_N^t(F) \longrightarrow \varprojlim K_N^t(F)/N_{L/F}K_N^t(L) \longrightarrow \varprojlim \mathrm{Gal}(L/F) \cong \Gamma_F^{ab}.$$

The norm-residue symbol in dimension N has analogous properties to the classical case. In particular, if L/F and L'/F' are finite Galois extensions, with $F \subset F'$ and $L \subset L'$. Then ([12])

$$\begin{tikzcd}
\mathrm{Gal}(L'/F') \arrow[r, "r_{L'/F'}"] \arrow[d] & K_N^t(F')/N_{L'/F'}K_N^t(L') \arrow[d, "N_{F'/F}"] \\
\mathrm{Gal}(L/F) \arrow[r, "r_{L/F}"] & K_N^t(F)/N_{L/F}K_N^t(L)
\end{tikzcd}$$

is commutative, where the right-hand vertical morphism is induced by the norm

We compute $r_{L/F}$ in a few explicit cases

Example Suppose L/F is unramified of finite degree f. Then $\mathrm{Gal}(L/F)$ is cyclic, generated by the restriction $\sigma = \varphi_F|_L$ of the Frobenius of F. Thus all admissible lifts $\widetilde{\sigma}$ are of the form φ_F^{1+nf} for $n \in \mathbb{N}$ and the corresponding fixed fields S_n

are the unramified extensions of F of degree $1+nf$. Therefore we may choose $\Pi_{S_n} = \{\pi_1, \ldots, \pi_N\} \in K_N^t(S_n)$, where $\pi_1, \ldots, \pi_N$ are local parameters of F. Then $N_{S_n/F}(\Pi_{S_n}) = (1+nf)\{\pi_1, \ldots, \pi_N\}$. But $f\{\pi_1, \ldots, \pi_N\} \in N_{L/F}K_N^t(L)$, thus all $N_{S_n/F}(\Pi_{S_n})$ are congruent modulo $N_{L/F}K_N^t(L)$, and $r_{L/F}(\sigma) = \{\pi_1, \ldots, \pi_N\} + N_{L/F}K_N^t(L)$.

Example If F contains a primitive p^M-th root of unity ζ, let ε be a p^M-primary element. For a set of local parameters $\pi_1, \ldots, \pi_N$, let $L = F(\sqrt[p^M]{\pi_j})$ for some j. Then $\mathrm{Gal}(L/F)$ is cyclic of order p^M with generator $\sigma : \sqrt[p^M]{\pi_j} \mapsto \zeta \sqrt[p^M]{\pi_j}$. Let φ_F be the absolute Frobenius of F and let $-p^M < a < 0$, $p \nmid a$, be such that Frobenius acts on $\sqrt[p^M]{\varepsilon}$ as $\varphi_F(\sqrt[p^M]{\varepsilon}) = \zeta^a(\sqrt[p^M]{\varepsilon})$. Pick $0 < b < p^M$ such that $ab \equiv 1 \mod p^M$ and pick a lift $\widetilde{\sigma}$ of σ such that $\widetilde{\sigma}|_{F_{ur}} = \varphi^b$. This is possible because F_{ur} and L are linearly disjoint. Then the fixed field S of $\widetilde{\sigma}$ is $F(\sqrt[p^M]{\varepsilon\pi_j})$, with local parameters $\pi_1, \ldots, \pi_{j-1}, \sqrt[p^M]{\varepsilon\pi_j}, \pi_{j+1}, \ldots, \pi_N$, and $N_{S/F}\{\pi_1, \ldots, \pi_{j-1}, \sqrt[p^M]{\varepsilon\pi_j}, \pi_{j+1}, \ldots, \pi_N\} = \{\pi_1, \ldots, \varepsilon\pi_j, \ldots, \pi_N\}$. Since $\{\pi_1, \ldots, \pi_N\} \in N_{L/F}K_N^t(L)$, this shows that $r_{L/F}(\sigma) = \{\pi_1, \ldots, \pi_{j-1}, \varepsilon, \pi_{j+1}, \ldots, \pi_N\} + N_{L/F}K_N^t(L)$.

3.2 The Field of Norms Functor

In [19], Fontaine-Wintenberger developed a way of relating local fields of mixed characteristic to those of equal characteristic. To any so-called arithmetically profinite extension F_∞/F of local fields (with perfect residue field) of characteristic 0 their field of norms functor associates a field of characteristic $\mathcal{F} := X_F(F_\infty)$ which induces an equivalence of the category of separable extensions of F_∞ with that of separable extensions of $X_F(F_\infty)$. In particular, it provides us with

$$\Gamma_{\mathcal{F}} \xrightarrow{\sim} \Gamma_{F_\infty} \subset \Gamma_F.$$

Suppose an arithmetically profinite extension F_∞ is obtained as $F_\infty = \varinjlim_n F_n$ for some tower of extensions $F_\bullet$. Then the field of norms is constructed as follows. Its multiplicative group is $\mathcal{F}^* = \varprojlim_n F_n^*$, where the limit is taken with respect to norms. Arithmetic profiniteness of F_∞/F implies that $N_{F_{n+m}/F_m}(x_{n+m}+y_{n+m})$ converges in F_m as $n \to \infty$, and addition in $\mathcal{F}^*$ is defined via $(x^{(m)})_m + (y^{(m)})_m = (z^{(m)})_m$ with

$z^{(m)} = \lim_{n\to\infty} N_{F_{n+m}/F_m}(x^{(n+m)} + y^{(n+m)})$. Since the subgroup $1 + p\mathcal{O}_{F_m}$ of $\mathcal{O}^*_{F_m}$ satisfies $\bigcap_n N_{F_{n+m}/F_m}\big(1 + p\mathcal{O}_{F_{n+m}}\big) = \{1\}$, one sees that $\mathcal{F}^* = \varprojlim_n F_n^*/\big(1 + p\mathcal{O}_{F_n}\big)$.

[19] provides an alternative definition. Let $\mathbb{C}_p$ be the p-adic completion of a fixed algebraic closure of $\mathbb{Q}_p$ and let $\mathcal{O}_{\mathbb{C}_p}$ be its ring of integers. Define the ring $R = \varprojlim \mathcal{O}_{\mathbb{C}_p}$, where the projective limit is taken with respect to p-th power maps, and addition is defined via $(a^{(m)})_m + (b^{(m)})_m = (c^{(m)})_m$ with $c^{(m)} = \lim_{n\to\infty}(a^{(m+n)} + b^{(n+m)})^{p^n}$. R is of characteristic p, with valuation $v_R : R^* \to \mathbb{Q}$ defined by $v_R((x^{(m)})_m) = v_p(x^{(0)})$, maximal ideal $\mathfrak{p} = \{x \,|\, v_R(x) > 0\}$ and residue field $\mathbb{F}_p^{alg}$. The projection $\mathcal{O}_{\mathbb{C}_p} \to \mathcal{O}_{\mathbb{C}_p}/p$ induces an isomorphism $R \to \varprojlim \mathcal{O}_{\mathbb{C}_p}/p$. In particular the unit group of R is $R^* \cong \varprojlim \mathcal{O}^*_{\mathbb{C}_p}/(1 + p\mathcal{O}_{\mathbb{C}_p})$.

Fontaine-Wintenberger go on to prove that the inclusion $F_n^* \to \mathbb{C}_p^*$ induces

$$\mathcal{F}^* = \varprojlim_n F_n^*/\big(1 + p\mathcal{O}_{F_n}\big) \hookrightarrow \mathbb{C}_p^*/(1 + p\mathcal{O}_{\mathbb{C}_p}) \cong \big(\mathrm{Frac}(R)\big)^*,$$

where the projective limit on the left-hand side is taken with respect to norms, for $n \geqslant n_0$, some n_0, and the one on the right-hand side with respect to p-th powers.

Example If $F_0 \supset \mathbb{Q}_p(\zeta_p)$ with uniformiser π and last residue field k, set $F_n = F(\pi^{(n)})$ for $\pi^{(n)} = \sqrt[p^n]{\pi}$, then $F_n^* \cong \langle \pi^{(n)} \rangle \times k^* \times (1 + \pi^{(n)}\mathcal{O}_{F_n})$. Taking quotients by $1 + \pi\mathcal{O}_{F_n}$ instead of $1 + p\mathcal{O}_{F_n}$ does not change the limit, and so

$$\mathcal{F}^* \cong \varprojlim F_n^*/(1 + \pi\mathcal{O}_{F_n}) \cong \langle t \rangle \times k^* \times \varprojlim(1 + \pi^{(n)}\mathcal{O}_{F_n})/(1 + \pi\mathcal{O}_{F_n}),$$

with $t = (\pi^{(n)})_n$. Using that $\gamma(x) \equiv x \mod (1 + (1 - \zeta_p)\mathcal{O}_{F_n})$ for every $x \in \mathcal{O}_{F_n}$ and $\gamma \in \mathrm{Gal}(F_n/F_{n-1})$, we see that

$$N_{F_n/F_{n-1}}\Big(1+\sum[\alpha_i]\pi^{(n)i}\Big) \equiv \Big(1+\sum[\alpha_i]\pi^{(n)i}\Big)^p \equiv 1+\sum[\alpha_i]\pi^{(n-1)i} \mod \big(1+\pi\mathcal{O}_{F_n}\big)$$

for Teichmüller representatives $[\alpha_i]$. It follows that

$$1 + t\,k[[t]] \longrightarrow \varprojlim(1 + \pi^{(n)}\mathcal{O}_{F_n})/(1 + \pi\mathcal{O}_{F_n}), \quad 1 + \sum_{i\geqslant 1}\alpha_i t^i \mapsto \Big(1 + \sum_{i\geqslant 1}[\alpha_i](\pi^{(n)})^i\Big)_n$$

is an isomorphism. Thus $\mathcal{F}^* \cong k((t))^*$. By the definition of addition in the field of norms, this map is also additive, and therefore $\mathcal{F} \cong k((t))$.

In the case of higher-dimensional local fields the construction involving norms does not generalise naturally: If, e.g. $F_n = F(\pi_1^{(n)}, \ldots, \pi_N^{(n)})$ with $(\pi_i^{(n)})^{p^n} = \pi_i \in F$, then

$N_{F_n/F_{n-1}}(\pi_i^{(n)}) = (\pi_i^{(n-1)})^{p^{N-1}}$ since $[F_n : F_{n-1}] = p^N$. Taking p-th powers, on the other hand, behaves well.

This approach has been adopted by Scholl ([35]) to define a generalisation of the field of norms functor. We describe his construction in the special case of N-dimensional local fields, which are special cases of so-called d-big fields, for $d = N - 1$. The main ideas of this construction are as follows.

Let $v_F : F \to \mathbb{Z} \cup \{\infty\}$ be the first valuation of F and extend it (uniquely) to an algebraic closure F^{alg}. For $c > 0$ and for any algebraic extension E/F, define the ideals

$$\mathfrak{p}_{c,E} = \left\{ x \in \mathcal{O}_E \,\middle|\, v_F(x) \geqslant c \right\} \subset \mathcal{O}_E.$$

If the field E is clear from the context we may simply write $\mathfrak{p}_c$.

Suppose $F_\bullet = \{F_n\}_{n \geqslant 0}$ is a tower of N-dimensional local fields. Scholl calls $F_\bullet$ strictly deeply ramified (SDR) with parameters (n_0, c) if there exists an index $n_0 \geqslant 0$ and $c > 0$ such that $[F_{n+1} : F_n] = p^N$ for all $n \geqslant n_0$ and if there is a surjective map

$$\Omega^1_{\mathcal{O}_{F_{n+1}}/\mathcal{O}_{F_n}} \longrightarrow (\mathcal{O}_{F_{n+1}}/\mathfrak{p}_c)^d.$$

By [35], prop. 1.2.1, this implies that for $n \geqslant n_0$, the first ramification index is $e_{F_{n+1}/F_n} = p$, the extension of first residue fields is the inseparable extension $F_{n+1}^{(1)} = (F_n^{(1)})^{1/p}$, and the p-th power map induces a surjection $\sigma : \mathcal{O}_{F_{n+1}}/\mathfrak{p}_c \to \mathcal{O}_{F_n}/\mathfrak{p}_c$.

It follows that for $n \geqslant n_0$, all F_n have the same last residue field $k = F_{n_0}^{(N)}$ and there exist local parameters $\pi_1^{(n)}, \dots, \pi_N^{(n)}$ of F_n such that $(\pi_i^{(n+1)})^p \equiv \pi_i^{(n)} \mod \mathfrak{p}_c$.

Define two towers $F_\bullet \sim F'_\bullet$ to be equivalent whenever there exists $r \in \mathbb{Z}$ and $n_2 \in \mathbb{N}$ with $F'_n = F_{n+r}$ for all $n \geqslant n_2$. Set $X^+(F_\bullet) = \varprojlim_{n \geqslant n_0} \mathcal{O}_{F_n}/\mathfrak{p}_c$, where the projective limit is taken with respect to the p-th power map. By thm. 1.3.2 of [35], $X^+(F_\bullet, c, n_0)$ is a complete discrete valuation ring of characteristic p and residue field canonically isomorphic to $F_n^{(1)}$ for any $n \geqslant n_0$. Up to isomorphism, it only depends on the equivalence class of the tower $F_\bullet$ and is independent of c and n_0.

Going to equivalent towers, we may therefore assume $n_0 = 0$ and denote the field of fractions of $X^+(F_\bullet, c, n_0)$ by $X(F_\bullet) = \mathcal{F}$. It is an N-dimensional local field with local parameters $t_i = (\pi_i^{(n)})_n$ and first residue field $\mathcal{F}^{(1)} \cong F_0^{(1)}$.

By construction, $\mathcal{O}_{\mathcal{F}} = \varprojlim \mathcal{O}_{F_n}/\mathfrak{p}_c$, and there is a canonical isomorphism

$$\mathcal{O}_{\mathcal{F}}/\mathfrak{p}_{c,\mathcal{F}} \xrightarrow{\sim} \mathcal{O}_{F_n}/\mathfrak{p}_c,$$

given by $\sum \alpha_{\underline{a}} \underline{t}^{\underline{a}} \mapsto \sum [\alpha^{\sigma^{-n}}](\underline{\pi}^{(n)})^{\underline{a}}$ for all $n \geqslant n_0$.

Theorem 1.3.5. of [35] states that the Field of Norms defines an equivalence between finite extensions of $F_\infty = \varinjlim_n F_n$ and finite separable extensions of $\mathcal{F}$. In particular, any separable extension $\mathcal{L}/\mathcal{F}$ of $\mathcal{F}$ is the field of norms of some strictly deeply ramified tower $L_\bullet$ with $L_n = L_0 F_n$ for some finite extension L_0/F_0. This defines $\Gamma_{\mathcal{F}} \cong \Gamma_{F_\infty} \subset \Gamma_{F_0}$.

3.3 Special towers

The aim of this section is to construct canonical projections $\mathcal{N}_{\mathcal{F}/F_n} : K_N^t(\mathcal{F}) \to K_N^t(F_n)$ which are compatible with the norms N_{F_{n+m}/F_n} for every $m \geqslant 0$.

Definition 3.2 *We call a strictly deeply ramified (SDR) tower $F_\bullet$ with parameters (n_0, c) a* special *SDR tower if every extension F_n/F_{n-1} appears as a tower of N p-extensions*

$$F_{n-1} = {}^0F_n \subset {}^1F_n \subset \cdots \subset {}^NF_n = F_n$$

for all $n \geqslant n_0$. $F_\bullet$ will be called very special *if $F_n = F(\sqrt[p^n]{\pi_1}, \ldots, \sqrt[p^n]{\pi_N})$ for some system of local parameters $\pi_1, \ldots, \pi_N$ of $F = F_0$.*

Lemma 3.3 *For any SDR tower, there exists $n_1 \geqslant n_0$ such that for $n \geqslant n_1$, there is a canonical projection*

$$\overline{\mathcal{N}}_{\mathcal{F}/F_n} : K_N^t(\mathcal{F}) \longrightarrow K_N^t(F_n)/U^{(c_1)}K_N^t(F_n),$$

for $c_1 = c - v_F(\pi_1^{(n_1)})$. Furthermore, $\overline{\mathcal{N}}_{\mathcal{F}/F_n}$ is given on topological generators of $K_N^t(\mathcal{F})$ by $\{\bar{t}_1, \ldots, \bar{t}_N\} \mapsto \{\pi_1^{(n)}, \ldots, \pi_N^{(n)}\}$ and $\{1+\alpha\underline{\bar{t}}^{\underline{a}}, \bar{t}_1, \ldots, \bar{t}_{i-1}, \bar{t}_{i+1}, \ldots, \bar{t}_N\} \mapsto \{1 + [\alpha^{\sigma^{-n}}](\underline{\pi}^{(n)})^{\underline{a}}, \pi_1^{(n)}, \ldots \pi_{i-1}^{(n)}, \pi_{i+1}^{(n)}, \ldots, \bar{t}_N^{(n)}\}$.

Proof Since the tower $F_\bullet$ is strictly deeply ramified, $v_F(\pi_1^{(n)}) \to 0$ as $n \to \infty$, thus there exists n_1 such that $c_1 = c - v_F(\pi_1^{(n_1)}) > 0$. The projection $pr : \mathcal{O}_{\mathcal{F}} \to$

$\mathcal{O}_{F_n}/\mathfrak{p}_c \to \mathcal{O}_{F_n}/\mathfrak{p}_{c_1}$ induces projections of multiplicative groups $\mathcal{O}_{\mathcal{F}}^* \to \mathcal{O}_{F_n}^*/U_{F_n}^{(c_1)}$ and maps $t_1 \mapsto \pi_1^{(n)}$. Using $\mathcal{F}^* = \mathcal{O}_{\mathcal{F}}^* \times \langle t_1 \rangle$ and $F_n^* = \mathcal{O}_{F_n}^* \times \langle \pi_1^{(n)} \rangle$, we define $\mathcal{F}^* \to F_n^*/U_{F_n}^{(c_1)}$ by $t_1 \mapsto \pi_1^{(n)}$. By the choice of c_1 this is well-defined. By construction, it is multiplicative. To see that it respects Steinberg relations, let $x, y \in \mathcal{F}$ with $x+y=1$. Let r, s be such that $t^r x, t^s y \in \mathcal{O}_{\mathcal{F}}^*$, then $pr(x) = pr(t^{-r}(t^r x)) = (\pi_1^{(n)})^{-r}\, pr(t^r x)$ and $pr(y) = (\pi_1^{(n)})^{-s}\, pr(t^s y)$. If $r = s$ then $pr(t^r x) + pr(t^r y) = pr(t^r x + t^r y)$ since both summands are in $\mathcal{O}_{\mathcal{F}}^*$. If $r < s$, say, then $r = 0$ and $x = 1 - t^s y \in \mathcal{O}_{\mathcal{F}}^*$, thus again $pr(x) = 1 - pr(t^s y)$. It follows that pr induces $\overline{\mathcal{N}}_{\mathcal{F}/F_n}$ as required.

The explicit description of $\overline{\mathcal{N}}_{\mathcal{F}/F_n}$ is obtained by noting that $t_i \mapsto \pi_i^{(n)}$ and $\alpha \mapsto [\alpha^{\sigma^{-n}}]$ under the projection $\mathcal{O}_{\mathcal{F}} \to \mathcal{O}_{F_n}/\mathfrak{p}_c$. □

Our next aim is to lift $\overline{\mathcal{N}}_{\mathcal{F}/F_n} : K_N^t(\mathcal{F}) \to K_N^t(F)/U^{(c_1)}K_N^t(F_n)$ to $\mathcal{N}_{\mathcal{F}/F_n} : K_N^t(\mathcal{F}) \to K_N^t(F_n)$. We illustrate our approach in the case of a very special SDR tower $F_n = F(\pi_1^{(n)}, \ldots, \pi_N^{(n)})$ and $(\pi_i^{(n)})^p = \pi_i^{(n-1)}$.

Lemma 3.4 *In the very special case $F_n = F(\pi_1^{(n)}, \ldots, \pi_N^{(n)})$ and $(\pi_i^{(n)})^p = \pi_i^{(n-1)}$, the projections $K_N^t(\mathcal{F}) \to K_N^t(F_n)/U^{(c_1)}K_N^t(F_n)$ are compatible with the norm maps $K_N^t(F_n) \to K_N^t(F_{n-1})$.*

Proof $\mathcal{O}_{\mathcal{F}} \to \mathcal{O}_{F_n}/\mathfrak{p}_c$ maps $\bar{t}_i \mapsto \pi_i^{(n)} \mod \mathfrak{p}_c$ and $\alpha \mapsto [\alpha^{\sigma^{-n}}] \mod \mathfrak{p}_c$. Thus, on generators of $K_N^t(\mathcal{F})$, the projection is given by $\{\bar{t}_1, \ldots, \bar{t}_N\} \mapsto \{\pi_1^{(n)}, \ldots, \pi_N^{(n)}\}$ and

$$\{1 + \alpha \underline{\bar{t}}^{\underline{a}}, \bar{t}_1, \ldots, \bar{t}_{i-1}, \bar{t}_{i+1}, \ldots, \bar{t}_N\} \mapsto \{1 + [\alpha^{\sigma^{-n}}]\underline{\pi}^{(n)\underline{a}}, \pi_1^{(n)}, \ldots, \pi_{i-1}^{(n)}, \pi_{i+1}^{(n)}, \ldots, \pi_N^{(n)}\}$$

for all n. Since the extensions $F_{n-1}(\pi_j^{(n)})$ for $j \neq i$ and $F_{n-1}(\underline{\pi}^{(n)\underline{a}})$ $(p \nmid a_i)$ are pairwise linearly disjoint over F_{n-1}, the norm in this case can be decomposed as

$$N_{F_n/F_{n-1}} = N_N \circ \cdots N_{i+1} \circ N_{i-1} \circ \cdots N_1 \circ N_{\underline{a}},$$

corresponding to the tower of sub-extensions obtained by first joining $\pi_N^{(n)}, \ldots, \pi_{i+1}^{(n)}$, skipping $\pi_i^{(n)}$, continuing with $\pi_{i-1}^{(n)}, \ldots, \pi_1^{(n)}$, and finally adding $(\underline{\pi}^{(n)})^{\underline{a}}$. But for the above generators of $K_N^t(F_n)$, the norm only acts on one entry, and it remains to note that $N_{\underline{a}}(1 + [\alpha^{\sigma^{-n}}]\underline{\pi}^{(n)\underline{a}}) = 1 + [\alpha^{\sigma^{-n+1}}]\underline{\pi}^{(n-1)\underline{a}}$, and $N_j \pi_j^{(n)} = \pi_j^{(n-1)}$. □

For those very special towers, this gives $K_N(\mathcal{F}) \to \varprojlim K_N(F_n)/U^{(c_1)}K_N(F_n)$, where the projective limit is taken with respect to norm maps.

Lemma 3.5 *In the very special case $F_n = F(\sqrt[p^n]{\pi_1}, \ldots, \sqrt[p^n]{\pi_N})$, the norm $N_{F_n/F_{n-1}}$: $K_N^t(F_n) \to K_N^t(F_{n-1})$ satisfies $N_{n/n-1}(U^{(d)}K_N^t(F_n)) \subset U^{(pd)}K_N(F_{n-1})$ for any $d >$ 0.*

Proof Note that $U^{(d)}K_N^t(F_n) \subset VK_N^t(F_n)$ is generated topologically by the elements $\{1 + \alpha\underline{\pi}^{(n)\underline{a}}, \pi_1^{(n)}, \ldots, \pi_{i-1}^{(n)}, \pi_{i+1}^{(n)}, \ldots, \pi_N^{(n)}\}$, where $a_1 \geqslant d$. Since $v_F(\pi_1^{(n-1)}) = pv_F(\pi_1^{(n)})$, the claim follows from the explicit formulae for the norm from the previous proof. $\square$

Corollary 3.6 *$\varprojlim U^{(c_1)}K_N^t(F_n) = 0$, i.e. $\varprojlim K_N^t(F_n) \to \varprojlim K_N^t(F_n)/U^{(c_1)}K_N^t(F_n)$ is an isomorphism.*

Using this, $\mathcal{N}_{\mathcal{F}/F_m}$ is defined to be the composite of $\varprojlim \overline{\mathcal{N}}_{\mathcal{F}/F_N}$ with the projection to $K_N^t(F_m)$,

$$\mathcal{N}_{\mathcal{F}/F_n} : K_N^t(\mathcal{F}) \longrightarrow \varprojlim K_N^t(F_n)/U^{(c_1)}K_N^t(F_n) \cong \varprojlim_n K_N^t(F_n) \longrightarrow K_N^t(F_m).$$

In particular, $\mathcal{N}_{\mathcal{F}/F_m}(x) = \lim_{n\to\infty} N_{F_{n+m}/F_m}\left(\overline{\mathcal{N}}_{\mathcal{F}/F_{n+m}}(x)\right)$ for every $x \in K_N^t(\mathcal{F})$.

The approach in this very special case can be generalised to special SDR towers. Let $F_\bullet$ be a special SDR tower with parameters $(0, c)$. For each $n \geqslant 1$, the ramification index is $\underline{e}_{F_n/F_{n-1}} = (p, \ldots, p)$, thus there exist local parameters $\pi_1^{(n)}, \ldots, \pi_N^{(N)}$ and a permutation $i = \binom{1\ 2\ \cdots\ N}{i_1\ i_2\ \cdots\ i_N} \in S_n$ such that the r-th subextension ${}^rF_n/{}^{r-1}F_n$ is of the form ${}^rF_n = {}^{r-1}F_n(\pi_{i_r}^{(n)})$ for all r.

Proposition 3.7 *If $F_\bullet$ is a special tower with parameters $(0, c)$, let $\pi_1^{(n)}, \ldots, \pi_N^{(n)}$ be local parameters of F_n satisfying $(\pi_i^{(n)})^p \equiv \pi_i^{(n-1)} \mod \mathfrak{p}_c$ for each i. Let $n_1 \geqslant 0$ be fixed such that $c_1 = c - v_F(\pi_1^{(n_1)}) > 0$, and set $c_2 = c_1/p > 0$. Then the norm $N_{n/n-1} : K_N^t(F_n) \to K_N^t(F_{n-1})/U^{(c_2)}K_N^t(F_{n-1})$ is given on topological generators by $\{\pi_1^{(n)}, \ldots, \pi_N^{(n)}\} \mapsto \{\pi_1^{(n-1)}, \ldots, \pi_N^{(n-1)}\}$, and $\{1+\alpha\underline{\pi}^{(n)\underline{a}}, \pi_1^{(n)}, \ldots, \pi_{i-1}^{(n)}, \pi_{i+1}^{(n)}, \ldots, \pi_N^{(n)}\}$ $\mapsto \{1 + \sigma(\alpha)\underline{\pi}^{(n-1)\underline{a}}, \pi_1^{(n-1)}, \ldots, \pi_{i-1}^{(n-1)}, \pi_{i+1}^{(n-1)}, \ldots, \pi_N^{(n-1)}\}$.*

Proof Using the above decomposition of F_n/F_{n-1} as a power of N simple p-extensions, it suffices to consider extensions F'/F with $[F' : F] = p$, $F' = F(\pi'_j)$ for some j, and $\pi_j'^p \equiv \pi_j \mod \mathfrak{p}_c$. Also, it follows from the linearity of the norm-map and the special structure of the generators that it suffices to consider three cases: the one-symbols $\{\pi'_j\}$, $\{1+x\pi_j'^a\}$ and the two-symbol $\{1+x\pi_j'^a, \pi'_j\}$, for $x \in F$. Here x takes account of α and the π_i for $i \neq j$. Furthermore, using local parameters of F if $p \mid a$, we may assume that $p \nmid a$. But then $\{1+x\pi_j'^a, \pi'_j\} = \frac{1}{a}\{1+x\pi'_j, -x\}$, so this reduces to the second case.

Note that the congruence $\pi_j'^p \equiv \pi_j \mod \mathfrak{p}_c$ in $\mathcal{O}_{F'}$ implies that $\pi_j'^p \equiv \pi_j \mod U^{(c_1)}$ as congruence in F'^*. So for any $\gamma \in \mathrm{Hom}_F(F, F'^{nc})$, $\gamma\pi'_j = u_\gamma \pi'_j$ for some u_γ with $u_\gamma^p \in U_{F'}^{(c_1)}$. But this means that $u_\gamma \in U_{F'}^{(c_2)}$, with $c_2 = c_1/p$. Therefore $N_{F'/F}\pi'_j \equiv \pi_j'^p \equiv \pi_j \mod U^{(c_2)}$ and similarly $N_{F'/F}(1+x\pi'_j) \equiv 1 + x^p\pi_j \mod U^{(c_2)}$. $\square$

Corollary 3.8 *The projections $\overline{\mathcal{N}}_{\mathcal{F}/F_n}$ are compatible with the norms N_{F_{n+1}/F_n} for $n \geqslant n_1$ and induce $\varprojlim \overline{\mathcal{N}}_{\mathcal{F}/F_n} : K_N^t(\mathcal{F}) \to \varprojlim K_N^t(F_n)/U^{(c_2)}K_N^t(F_n)$, where the projective limit is taken with respect to norms.*

Proposition 3.9 *If $F_\bullet$ is a special SDR tower with parameters $(0, c)$, the norm $N_{n/n-1} : K_N^t(F_n) \to K_N^t(F_{n-1})$ satisfies $N_{n/n-1}U^{(d)}K_n^t(F_n) \subset U^{(d+\delta)}K_n^t(F_{n-1})$ for every $d > 0$ and $n \geqslant n_1$, where $\delta = \min\{d, c_2\}$.*

Proof To ease notation, set $F = F_{n-1}$ and $F' = F_n$, and write $\pi'_1, \ldots, \pi'_N$ (resp. $\pi_1, \ldots, \pi_N$) for local parameters of F' (resp. of F). Let $F = {}^0F \subset \cdots \subset {}^NF = F'$ be the tower of sub-extensions of degree p with ${}^rF = {}^{r-1}F(\pi'_{i(r)})$ for $1 \leqslant r \leqslant N$. Using the remark after prop. 2.11, we consider the special topological generators

$$u = \{1 + \alpha\underline{\pi}'^{\underline{a}}, \pi'_{i(1)}, \ldots, \pi'_{i(s-1)}, \pi'_{i(s+1)}, \ldots, \pi'_{i(N)}\}$$

of $U^{(d)}K_N^t(F')$, where $i \in S_n$ is such that $F_r = F_{r-1}(\pi'_{i(r)})$, and $j = i(s)$ is such that $p \,|\, a_{i(r)}$ for $s < r \leqslant N$ and $p \nmid a_j$ (i.e. s is maximal such that $p \nmid a_{i(s)}$). By using local parameters of F whenever $a_i > p$, we may assume that $0 \leqslant a_i < p$ for each i, and replace α with $\alpha\underline{\pi}^{\underline{b}} \in F$ if necessary. Thus we have $a_{i(r)} = 0$ for $s < r \leqslant N$.

Now any fixed generator u of $U^{(d)}K_N^t(F')$ of the above type can be written as a product of two symbols

$$u = \{1 + \alpha\underline{\pi}'^{\underline{a}}, \pi'_{i(1)}, \ldots, \pi'_{i(s-1)}\}\,\{\pi'_{i(s+1)}, \ldots, \pi'_{i(N)}\} = u'_1 u_2,$$

with $u_1' = j_{F_s/F_N} u_1$ for $u_1 \in U^{(d)} K_s(F_s)$, and $u_2 \in K'_{N-s}(F_N)$.

The proof is in three steps.

Firstly, by the linearity of the norm map,

$$\begin{aligned} N_{^N F/^s F}(u_1' u_2) &= u_1 \, N_{^N F/^s F}(u_2) \equiv u_1 \left\{ (\pi'_{i(s+1)})^p, \ldots, (\pi'_{i(N)})^p \right\} \\ &\equiv \{1 + \alpha \underline{\pi}'^{\underline{a}}, \pi'_{i(1)}, \ldots, \pi'_{i(s-1)}\} \{\pi_{i(s+1)}, \ldots, \pi_{i(N)}\} \mod U^{(d)} K_N^t({}^s F). \end{aligned}$$

Since the second factor is in $j_{^0 F/^s F} K^t_{N-s}({}^0 F)$, we may ignore it by linearity.

The second step is $N_{^s F/^{s-1}F}$. Here we need to consider $N_{^s F/^{s-1}F}(1 + x\pi_j'^{a_j})$, for $x \in {}^{s-1}F$ such that $x\pi_j'^{a_j} = \alpha \underline{\pi}'^{\underline{a}}$, and for $p \nmid a_j$, $j = i(s)$. Using $p \nmid a_j$, we see that ${}^s F = {}^{s-1}F(\pi_j') = {}^{s-1}F(\pi_j'^{a_j})$. As before, all conjugates of π_j' over ${}^{s-1}F$ are congruent modulo $U^{(c_2)}$. Thus for $x\pi_j'^{a_j} \in \mathfrak{p}_d$, we obtain

$$N_{^s F/^{s-1}F}(1 + x\pi_j'^{a_j}) \equiv (1 + x\pi_j'^{a_j})^p \mod U^{(c_2+d)}.$$

Thus $N_{^s F/^{s-1}F}(1 + x\pi_j'^{a_j}) \in U^{(d+\delta)}_{^s F}$ and therefore $N_{^N F/^{s-1}F}(u) \in U^{(d+\delta)} K_N^t({}^{s-1}F)$, for $\delta = \min\{d, c_2\}$.

The third step is to show that for any $r < s$,

$$N_{^r F/^{r-1}F} U^{(d+\delta)} K_N^t({}^r F) \subset U^{(d+\delta)} K_N^t({}^{r-1}F).$$

If a generator of $U^{(d+\delta)} K_N^t({}^r F)$ only has $\pi'_{i(r)}$ in one entry, the arguments of the first two steps apply. Otherwise, it is of the form $\{1 + x\pi'^a_{i(r)}, \pi'_{i(r)}\} j_{F_{r-1}/F_r}(y)$ for $x \in {}^{r-1}F$ and $y \in K^t_{N-2}({}^{r-1}F)$. But $\{1 + x\pi'^a_{i(r)}, \pi'_{i(r)}\} = \{(1 + x\pi'^a_{i(r)})^{1/a}, -x\}$, so this is again the same as the second step. □

Corollary 3.10 *If $F_\bullet$ is a special SDR tower, $\varprojlim U^{(c_2)} K_N^t(F_n) = 0$, i.e. the canonical map $\varprojlim K_N^t(F_n) \to \varprojlim K_N^t(F_n)/U^{(c_1)} K_N^t(F_n)$ is an isomorphism, where again the projective limit is taken with respect to norm maps.*

We define $\mathcal{N}_{\mathcal{F}/F_n} : K_N^t(\mathcal{F}) \to K_N^t(F_n)$ to be the composite

$$K_N^t(\mathcal{F}) \to \varprojlim K_N^t(F_n)/U^{(c_1)} K_N^t(F_n) \cong \varprojlim K_N^t(F_n) \to K_N^t(F_n).$$

In particular, $\mathcal{N}_{\mathcal{F}/F_n}(x) = \lim_{m\to\infty} N_{F_{n+m}/F_n}(\overline{\mathcal{N}}_{\mathcal{F}/F_{n+m}}(x))$ for $x \in K_N^t(\mathcal{F})$.

Corollary 3.11 *$\mathcal{N}_{\mathcal{F}/F_n}$ commutes with the valuation $\mathbf{v}$ on N-th K groups in the sense that $\mathbf{v}_{F_n} \circ \mathcal{N}_{\mathcal{F}/F_n}(x) = \mathbf{v}_{\mathcal{F}}(x)$ for any $x \in K_N^t(\mathcal{F})$. In particular, $\mathbf{v}_{F_n} \circ \mathcal{N}_{\mathcal{F}/F_n}(\{\bar{t}_N, \ldots, \bar{t}_1\}) = 1$.*

Proposition 3.12 *For a special SDR tower $F_\bullet$ with associated field of norms $\mathcal{F}$, the map induced by all $\mathcal{N}_{\mathcal{F}/F_n}$ yields an isomorphism*

$$\varprojlim \mathcal{N}_{\mathcal{F}/F_n} : \ K_N^t(\mathcal{F}) \xrightarrow{\sim} \varprojlim K_N^t(F_n)/U^{(c_2)} K_N^t(F_n) \cong \varprojlim K_N^t(F_n).$$

Proof To prove injectivity, consider a set of topological generators $\{\bar{t}_1, \ldots, \bar{t}_N\}$ and $\left(\{1 + \alpha \underline{t}^{\underline{a}}, \bar{t}_1, \ldots, \bar{t}_{i-1}, \bar{t}_{i+1}, \ldots, \bar{t}_N\}\right)_{\underline{a}}$ of $K_N^t(\mathcal{F})$, say $\underline{a} < \underline{A}$ for some $\underline{A}$. Since the F_n are of mixed characteristic, their absolute ramification indices $\underline{e}_{F_n}$ have first coordinate $e_{F_n}^{(1)} > 0$. Thus $\underline{A} < \underline{e}_{F_n} p/(p-1)$ for all n sufficiently large. For such n, the above topological generators mapped to a basis of $K_N^t(F_n)/p$. This shows that for fixed $\underline{A}$ and all $\underline{a} < \underline{A}$, the kernel is trivial. By the definition of the topology on $V_{\mathcal{F}}$ (and therefore $V_{\mathcal{F}} K_N^t(\mathcal{F})$), every element is a limit of a finite sum of elements with $\underline{a} < \underline{A}$ for $\underline{A}$ fixed, so $\varprojlim \mathcal{N}_{\mathcal{F}/F_n}$ is injective.

To prove surjectivity, we may without loss of generality assume that $c > 0$ is such that $1 - \zeta_p \in \mathfrak{p}_c$ if $\zeta_p \in F_\infty$. Then $K_N^t(F_n)/U^{(c_2)} K_N^t(F_n)$ is topologically generated by the symbols $\{\pi_1^{(n)}, \ldots, \pi_N^{(n)}\}$ and $\{1 + \alpha(\underline{\pi}^{(n)})^{\underline{a}}, \pi_1, \ldots, \pi_{i-1}^{(n)}, \pi_{i+1}^{(n)}, \ldots, \pi_N^{(n)}\}$ for $\underline{a} < \underline{e}p/(p-1)$, which lie in the image of $\mathcal{N}_{\mathcal{F}/F_n}$. □

3.4 Arbitrary Towers

In this section we consider arbitrary SDR towers $F_\bullet$, with parameters $(0, c)$. The idea is to find a finite extension $E_\bullet$ which is a special SDR tower. Using the valuation induced from F on both $F_\bullet$ and $E_\bullet$ to simplify notation, one has $j_{F_m/E_m} U^{(d)} K_N^t(F_m) \subset U^{(d)} K_N^t(E_m)$ for each $d > 0$, and $\varprojlim_n U^{(d)} K_N^t(E_n) = \{1\}$ by cor. 3.10. The main difficulty is to control the kernel of j_{F_n/E_n}.

Lemma 3.13 *Let F'/F be a totally ramified separable extension of degree $[F' : F] = p^n$, i.e. $F'^{(N)} = F^{(N)}$. Let $m, d \in \mathbb{N}$ be such that $(p^n)! = p^m d$ and $p \nmid d$. Then there* *exists a tower* $E_0 \subset \cdots \subset E_n$ *with* $E_0 \supset F$ *such that* $[E_i : E_{i-1}] = p$ *for* $1 \leqslant i \leqslant N$, E_0/F *is tamely ramified of degree dividing* d, *and* $F'E_0 = E_n$.

Proof Let F^{nc} be the Galois closure of F'/F, so that $[F^{nc} : F] \,\big|\, (p^n)! = p^m d$. As in the proof of prop. 1.16, let $\widetilde{k}/F^{(N)}$ be of degree $p^m d$ and let $\alpha \in \widetilde{k}$ be a generator of $\widetilde{k}^*$. For a system $\pi_1, \ldots, \pi_N$ of local parameters of F, let $E' = F(\sqrt[d]{\alpha}, \sqrt[d]{\pi_1}, \ldots, \sqrt[d]{\pi_N})$. Then $E_0 := E' \cap F^{nc}$ is the maximal tamely ramified sub-extension of F^{nc}/F, hence of degree dividing d and $G = \mathrm{Gal}(F^{nc}/E_0)$ is a p-group. Let $H = \mathrm{Gal}(F^{nc}/F'E_0)$ be the subgroup corresponding to the sub-extension $E_0 \subset F'E_0 \subset F^{nc}$. By group-theory, there exists a tower $H = H_N \leqslant H_{N-1} \leqslant \cdots \leqslant H_1 \leqslant H_0 = G$ of subgroups with $(H_{i-1} : H_i) = p$ for each i. The fixed fields $E_i = (EF^{nc})^{H_i}$ satisfy the claims of the lemma. $\square$

Corollary 3.14 *Let* $F_\bullet$ *be an arbitrary SDR tower with parameters* (n_0, c). *Then there exists a tamely ramified extension* E *of* F_{n_0} *such that the tower* $E_\bullet$ *with* $E_n = EF_n$ *for* $n \geqslant n_0$ *is a special SDR tower.*

The case of special SDR towers and $j_{F_{n-1}/LF_{n-1}} \circ N_{F_n/F_{n-1}} = N_{EF_n/EF_{n-1}} \circ j_{F_n/EF_n}$ imply the following

Lemma 3.15 *If* $F_\bullet$ *is a SDR tower with associated special SDR tower* $E_\bullet$ *and field of norms* $\mathcal{F}$ *then the composite*

$$K_N^t(\mathcal{F}) \longrightarrow K_N^t(F_n)/U^{(c_1)}K_N^t(F_n) \xrightarrow{j_{F/E}} K_N^t(EF_n)/U^{(c_1)}K_N^t(EF_n)$$

is compatible with norms N_{E_{n+1}/E_n} *for different* $n \geqslant n_1$.

For arbitrary SDR towers, we obtain a weaker result.

Proposition 3.16 *Let* $F_\bullet$ *be an SDR tower such that* F_∞ *contains a primitive* p^M*-th root of unity* ζ_M. *Then* $\varprojlim U^{(c_2)}K_N(F_n)/p^M = 0$

Proof Without loss of generality, assume $F_\bullet$ has parameters $(0, c)$ and $\zeta_M \in F_0$. Let E/F_0 be the associated tamely ramified extension such that $E_\bullet$, $E_n = EF_n$ is a special SDR tower. Let

$$C_n = \ker\left(j_{F_n/E_n} : U^{(c_2)}K_N(F_n)/p^M \longrightarrow U^{(c_2)}K_N(E_n)/p^M\right)$$

be the kernel of j_{F_n/E_n}. By cor. 3.10, $\varprojlim_n U^{(c)} K_N(E_n)/p^M = 0$, thus it remains to show that $\varprojlim_n C_n = 0$. Let $\widetilde{E}_n$ be the maximal unramified p-subextension of E_n/F_n. Then $p \nmid [E_n : \widetilde{E}_n]$ implies $\ker\big(j_{\widetilde{E}_n/E_n} : K_N(\widetilde{E}_n)/p^M \to K_N(E_n)/p^M\big) = 0$ by cor. 2.26, so it suffices to consider

$$\widetilde{C}_n = \ker\big(j_{F_n/\widetilde{E}_n} : U^{(c_2)} K_N(F_n)/p^M \longrightarrow K_N(\widetilde{E}_n)/p^M\big).$$

Since $[\widetilde{E}_n : F_n] = [\widetilde{E}_n^{(N)} : F_n^{(N)}]$, there is an $\mathbb{F}_p$-basis of $\widetilde{E}_n^{(N)}$ containing an $\mathbb{F}_p$-basis $\mathcal{B}_n$ of $F_n^{(N)}$. Using this, we may take as Shafarevich basis of $K_N(F_n)_M$ the elements

(i) $\{\pi_1, \dots, \pi_N\}$, for a system of local parameters $\pi_1, \dots, \pi_N$ of F_n,

(ii) $\{E(\alpha, \underline{\pi}^{\underline{a}}), \pi_1, \dots, \pi_{i-1}, \pi_{i+1}, \dots, \pi_N\}$, for $\alpha \in \mathcal{B}_n$, i minimal with $p \nmid a_i$,

(iii) $\{\varepsilon, \pi_1, \dots, \pi_{i-1}, \pi_{i+1}, \dots, \pi_N\}$, for some p^M-primary element ε and $1 \leqslant i \leqslant N$.

A Shafarevich basis for $K_N(\widetilde{E}_n)_M$ can be chosen to contain the elements of (i) and (ii). Thus $\widetilde{C}_n$ is contained in the subgroup of $K_N(F_n)/p^M$ generated by the elements of type (iii). Since $\varepsilon \in F_0$, this reduces the problem to showing that $\varprojlim D_n = 0$, where D_n is the subgroup of $K_{N-1}(F_n)$ generated by $\{\pi_1^{(n)}, \dots, \pi_{i-1}^{(n)}, \pi_{i+1}^{(n)}, \dots, \pi_N^{(n)}\}$ for $1 \leqslant i \leqslant N$. We prove this by iterating the above approach and reducing it to $N_{F_n/F_{n-1}} : K_1(F_n) \ni \varepsilon \mapsto \varepsilon^p \in pK_1(F_{n-1})$, which is clear.

To prove $\varprojlim D_n = 0$, consider again the associated special SDR tower $E_\bullet$ from above. Since E_n/E_{n-1} breaks up into a tower of $N-1$ extensions of degree p, each obtained by joining one local parameter, we clearly have $N_{E_n/E_{n-1}} K_{N-1}(E_n) \subset pK_{N-1}(E_{n-1})$, hence $\varprojlim j_{F_n/E_n} D_n = 0$. By the same argument as before, it thus suffices to consider the kernel $\widetilde{D}_n = \ker\big(j_{F_n/\widetilde{E}_n} : K_{N-1}(F_n)/p^M \to K_{N-1}(\widetilde{E}_n)/p^M\big)$. Using the analogous Shafarevich basis elements of $K_{N-1}(F_n)$, we may iterate this argument as indicated. □

Corollary 3.17 *For an SDR tower $F_\bullet$ with $\zeta_M \in F_\infty$, there exist canonical maps*

$$\mathcal{N}_{\mathcal{F}/F_n} : K_N(\mathcal{F})/p^M \longrightarrow K_N(F_n)/p^M$$

for each $n \geqslant 0$, such that $\mathcal{N}_{\mathcal{F}/F_n} = N_{F_{n+m}/F_n} \circ \mathcal{N}_{\mathcal{F}/F_{n+m}}$ for each m, n. They commute with the valuation $\mathbf{v}$ on K_N in the sense that $\mathbf{v}_{F_n} \circ \mathcal{N}_{\mathcal{F}/F_n} = \mathbf{v}_{\mathcal{F}}$. In particular,

$\mathbf{v}_{F_n}\big(\mathcal{N}_{\mathcal{F}/F_n}\{\bar{t}_N,\dots,\bar{t}_1\}\big) \equiv 1 \mod p^M$. *Furthermore, the induced map*

$$\varprojlim_n \mathcal{N}_{\mathcal{F}/F_n} : K_N(\mathcal{F})/p^M \longrightarrow \varprojlim_n K_N(F_n)/p^M$$

is an isomorphism.

3.5 Compatibility

We are now ready to prove the compatibility of class field theory and the field of norms functor.

Theorem 3.18 *Let $F_\bullet$ be an SDR tower and let $L_\bullet$ be given by $L_n = LF_n$ where L/F_0 is a finite abelian Galois extension. Let $\mathcal{L}/\mathcal{F}$ be the corresponding extension of their fields of norms.*

Suppose either that $F_\bullet$ is a special SDR tower, or that $F_\infty \ni \zeta_M$ and $\mathrm{Gal}(L/F_0)$ is of exponent dividing p^M. Then diagram

$$\begin{tikzcd}
\mathrm{Gal}(\mathcal{L}/\mathcal{F}) \arrow[r, "r_{\mathcal{L}/\mathcal{F}}"] \arrow[d] & K_N^t(\mathcal{F})/N_{\mathcal{L}/\mathcal{F}}K_N^t(\mathcal{L}) \arrow[d, "\mathcal{N}_{\mathcal{F}/F_n}"] \\
\mathrm{Gal}(L_n/F_n) \arrow[r, "r_{L_n/F_n}"] & K_N^t(F_n)/N_{L_n/F_n}K_N^t(L_n)
\end{tikzcd}$$

is commutative.

Proof The proof is identical for special and arbitrary powers. We treat the case of special towers. Dealing with arbitrary towers requires taking quotients by p^M everywhere.

The groups $\mathrm{Gal}(LF_n/F_n)$ are canonically isomorphic, denote them by G. Consider the following commutative diagram

$$\begin{tikzcd}
& L_n^{ur} \arrow[rr, dash] \arrow[dd, dash] & & L_{n+1}^{ur} \arrow[dd, dash] \\
L_n \arrow[ur, dash] \arrow[rr, dash, crossing over] \arrow[dd, dash, "G"'] & & L_{n+1} \arrow[ur, dash] & \\
& F_n^{ur} \arrow[rr, dash] & & F_{n+1}^{ur} \\
F_n \arrow[ur, dash] \arrow[rr, dash] & & F_{n+1}. \arrow[ur, dash] \arrow[from=uu, dash, crossing over, "G"'] &
\end{tikzcd}$$

For $\sigma \in G$, pick lifts $\widetilde{\sigma}_n \in \mathrm{Gal}(L_n^{ur}/F_n)$ and $\widetilde{\sigma}_{n+1} \in \mathrm{Gal}(L_{n+1}^{ur}/F_{n+1})$ such that their restrictions satisfy $\widetilde{\sigma}_n|_{F_n^{ur}} = \varphi_{F_n}^m$ and $\widetilde{\sigma}_{n+1}|_{F_{n+1}^{ur}} = \varphi_{F_{n+1}}^m$ for the same $m \in \mathbb{N}$.

Let S_{n+1} and S_n be their respective fixed fields. Then

$$S_{n+1} = (L_{n+1}^{ur})^{\widetilde{\sigma}_{n+1}} = (F_{n+1}L_n^{ur})^{\widetilde{\sigma}_{n+1}} = F_{n+1}S_n,$$

so the tower $S_\bullet$ is also strictly deeply ramified and is a finite extension of $F_\bullet$, with $[S_n : F_n] = m$ for n sufficiently large.

The reciprocity map for L_n/F_n is

$$r_{L_n/F_n}(\sigma) = N_{S_n/F_n}(\Pi_{S_n}) + N_{L_n/F_n}K_N^t(L_n),$$

where $\Pi_{S_n} \in K_N^t(S_n)$ is any element satisfying $\mathbf{v}_{S_n}(\Pi_{S_n}) = 1$. Since the extension F_{n+1}/F_n has no unramified part, the same holds for S_{n+1}/S_n, so by lemma 2.27

$$\mathbf{v}_{S_n} \circ N_{S_{n+1}/S_n}(\Pi_{S_{n+1}}) = \mathbf{v}_{S_{n+1}}(\Pi_{S_{n+1}}) = 1$$

so there exists a system $(\Pi_{S_n})_n$ of $\Pi_{S_n} \in K_N^t(S_n)$ satisfying $N_{S_n/S_{n-1}}(\Pi_{S_n}) = \Pi_{S_{n-1}}$ and $\mathbf{v}_{S_n}(\Pi_{S_n}) = 1$

On the level of fields of norms, pick a lift $\widetilde{\sigma}$ satisfying $\widetilde{\sigma}|_{\mathcal{F}^{ur}} = \varphi_{\mathcal{F}}^m$ for the same m as previously. If $\mathcal{S}$ is the fixed field of this $\widetilde{\sigma}$, take $\Pi_{\mathcal{S}} \in K_N^t(\mathcal{S})$ such that $\mathcal{N}_{\mathcal{S}/S_n}(\Pi_{\mathcal{S}}) = \Pi_{S_n}$ for each n. Then $\mathbf{v}_{\mathcal{S}}(\Pi_{\mathcal{S}}) = 1$, so

$$r_{\mathcal{L}/\mathcal{F}}(\sigma) = N_{\mathcal{S}/\mathcal{F}}(\Pi_{\mathcal{S}}) + N_{\mathcal{L}/\mathcal{F}}K_N^t(\mathcal{L}).$$

To finish the proof, note that

$$\begin{tikzcd}
K_N^t(\mathcal{S}) \arrow[r, "\mathcal{N}_{\mathcal{S}/S_n}"] \arrow[d, "N_{\mathcal{S}/\mathcal{F}}"'] & K_N^t(S_n) \arrow[d, "N_{S_n/F_n}"] \\
K_N^t(\mathcal{F}) \arrow[r, "\mathcal{N}_{\mathcal{F}/F_n}"] & K_N^t(F_n).
\end{tikzcd}$$

is commutative by construction, so for $\sigma \in \mathrm{Gal}(\mathcal{L}/\mathcal{F})$,

$$\begin{aligned}
\mathcal{N}_{\mathcal{F}/F_n} \circ r_{\mathcal{L}/\mathcal{F}}(\sigma) &= \mathcal{N}_{\mathcal{F}/F_n} \circ N_{\mathcal{S}/\mathcal{F}}(\Pi_{\mathcal{S}}) \mod \mathcal{N}_{\mathcal{F}/F_n}(N_{\mathcal{L}/\mathcal{F}}K_N^t(\mathcal{L})) \\
&= N_{S_n/F_n}(\Pi_{S_n}) \mod N_{L_n/F_n}K_N^t(L_n) = r_{L_n/F_n}(\sigma),
\end{aligned}$$

identifying σ with its image in $\mathrm{Gal}(L_n/F_n)$. $\square$

Corollary 3.19 *If $F_\bullet$ is a special SDR tower, the total diagram*

$$\begin{tikzcd} K_N^t(\mathcal{F}) \arrow[r, "\Psi_{\mathcal{F}}"] \arrow[d, "\mathcal{N}_{\mathcal{F}/F_n}"'] & \Gamma_{\mathcal{F}}^{ab} \arrow[d] \\ K_N^t(F_n) \arrow[r, "\Psi_{F_n}"] & \Gamma_{F_n}^{ab}, \end{tikzcd}$$

is commutative, where the right-hand vertical map is the composite of the isomorphism $\Gamma_{\mathcal{F}}^{ab} \cong \Gamma_{F_\infty}^{ab}$ given by the field of norms functor, and the inclusion $\Gamma_{F_\infty}^{ab} \subset \Gamma_{F_n}^{ab}$.

Corollary 3.20 *If $F_\bullet$ is an SDR tower with $\zeta_M \in F_\infty$, then*

$$\begin{tikzcd} K_N^t(\mathcal{F})/p^M \arrow[r, "\Psi_{\mathcal{F}}"] \arrow[d, "\mathcal{N}_{\mathcal{F}/F_n}"'] & \Gamma_{\mathcal{F}}^{ab}/p^M \arrow[d] \\ K_N^t(F_n)/p^M \arrow[r, "\Psi_{F_n}"] & \Gamma_{F_n}^{ab}/p^M, \end{tikzcd}$$

is commutative where $\Gamma_{\mathcal{F}}^{ab}/p^M \hookrightarrow \Gamma_{F_n}^{ab}/p^M$ is induced by the field of norms functor.

Chapter 4

The Witt-Artin-Schreier Pairing

In this chapter, we describe abelian p-extensions of higher local fields of equal characteristic p.

4.1 Differential Forms

Let $\mathcal{F}$ be a higher local field of equal characteristic p, with system of local parameters $\bar{t}_n, \ldots, \bar{t}_1$ and last residue field k. Consider its flat $\mathbb{Z}/p^M$-lift $\mathcal{O}_M(\mathcal{F}) = W_M(k)((t_N)) \cdots ((t_1))$, where $t_i = [\bar{t}_i] \in W_M(\mathcal{F})$ are Teichmüller representatives of the local parameters (see appendix A.2). Since $\mathcal{O}_M(\mathcal{F})$ is obtained from $W(k)$ by a succession of steps involving taking polynomial algebras, completions, and localisations, its module of continuous differential forms over $\mathbb{Z}_p$, $\Omega_{W_M(k)((t_N))\cdots((t_1))}$ is free with basis $dt_1, \ldots, dt_N$. For the same reason, $\mathcal{O}(\mathcal{F}) = \varprojlim \mathcal{O}_M(\mathcal{F})$, its field of fractions $Q(\mathcal{F})$ and the $W(k)$-subalgebra $Q_0(\mathcal{F}) = W(k)((t_N)) \cdots ((t_1))$ of $Q(\mathcal{F})$ all have the property that their module of differential forms over $\mathbb{Z}_p$, resp. $\mathbb{Q}_p$, is free of rank N.

To ease notation later on, put $d_{\log}x = \mathrm{d}x/x$. Then $\Omega^N_{Q(\mathcal{F})}$ is free over $\mathbb{Q}_p$ and the residue of an N-form is

$$\operatorname{Res}_{Q(\mathcal{F})}\Big(\sum a_{\underline{i}} t_1^{i_1} \cdots t_N^{i_N} d_{\log}t_1 \wedge \cdots \wedge d_{\log}t_N\Big) = a_{(0,\ldots,0)} \in \operatorname{Frac}(W(k)),$$

and similarly for $\Omega^N_{\mathcal{O}(\mathcal{F})}$, $\Omega^N_{\mathcal{O}_M(\mathcal{F})}$, and $\Omega^N_{Q_0(\mathcal{F})}$. The residue has the following standard properties

(i) If $\omega \in \Omega^{N-1}_{Q(\mathcal{F})}$ then $\mathrm{Res}_{Q(\mathcal{F})} d\omega = 0$,

(ii) if $\bar{t}'_1, \dots, \bar{t}'_N$ is another system of local parameters of $\mathcal{F}$ and $t'_1, \dots, t'_N$ are lifts to $\mathcal{O}(\mathcal{F})$, then $\mathrm{Res}_{Q(\mathcal{F})}\big(d_{\log} t'_1 \wedge \cdots \wedge d_{\log} t'_N\big) = 1$,

(iii) if $\mathrm{Res}\,(\omega) = \alpha$, then there exists $\omega' \in \Omega^{N-1}_{Q(\mathcal{F})}$ such that $\omega = d\omega' + \alpha d_{\log} t'_1 \wedge \cdots \wedge d_{\log} t'_N$.

By construction, $\mathcal{O}(\mathcal{F})$ depends on the choice of local parameters $\bar{t}_1, \dots, \bar{t}_N$ of $\mathcal{F}$ used to construct the flat lifts. We illustrate an alternative approach to residues which is independent of local parameters in $\mathcal{F}$. For $n \geqslant 0$, and a fixed choice of local parameters $\bar{t}_i$, let $\mathcal{O}_M(\sigma^n \mathcal{F})$ be the flat $\mathbb{Z}/p^M$-lift constructed using the local parameters $\bar{t}_1^{p^n}, \dots, \bar{t}_N^{p^n}$ of $\sigma^n(\mathcal{F})$. Also, let $\sigma^{-n}\mathcal{F}$ be the inseparable extension obtained by joining $\bar{t}_i^{1/p}$ for $1 \leqslant i \leqslant N$ and denote by σ^{-n} the isomorphism $\mathcal{F} \xrightarrow{\sim} \sigma^{-n}\mathcal{F}$. Then

$$W_M(\sigma^{M-1}\mathcal{F}) \subset \mathcal{O}_M(\mathcal{F}) \subset W_M(\mathcal{F}) \subset \mathcal{O}_M(\sigma^{1-M}\mathcal{F}),$$
$$W_M(\mathcal{F}) = \mathcal{O}_M(\mathcal{F}) + p\mathcal{O}_M(\sigma^{-1}\mathcal{F}) + \cdots + p^{M-1}\mathcal{O}_M(\sigma^{1-M}\mathcal{F}).$$

Define $\widetilde{\Omega}(\mathcal{F}, M)$ to be the submodule of $\Omega^N_{W_M(\mathcal{F})}$ generated as $\mathbb{Z}_p$-module by all forms $\omega = y d_{\log} x_1 \wedge \cdots \wedge d_{\log} x_N$ for all $y \in W_M(\sigma^{M-1}\mathcal{F})$ and $x_i \in W_M(\mathcal{F})^*$. Since $y \in \mathcal{O}_M(\mathcal{F})$ and $x_i \in \mathcal{O}_M(\sigma^{1-M}\mathcal{F})$, ω can be written as $\omega = w\, d_{\log} t_1 \wedge \cdots \wedge d_{\log} t_N$, for $w \in \mathcal{O}_M(\sigma^{1-M}\mathcal{F})$. This induces a natural embedding

$$\iota_{\mathcal{O}_M(\mathcal{F})} : \widetilde{\Omega}(\mathcal{F}, M) \to \mathcal{O}_M(\sigma^{1-M}\mathcal{F}) \otimes_{\mathcal{O}_M(\mathcal{F})} \Omega^N_{\mathcal{O}_M(\mathcal{F})}, \quad \omega \mapsto w \otimes d_{\log} t_1 \wedge \cdots \wedge d_{\log} t_N.$$

Note that $w \in \mathcal{O}_M(\sigma^{1-M}\mathcal{F})$ may in turn be written as $w = \sum \alpha_{\underline{a}} t_1^{a_1} \cdots t_N^{a_N}$ for $\alpha \in W(k)$ and $(a_1, \dots, a_N) \subset p^{1-M}\mathbb{Z}^N$ running through some admissible set. Using this, we define the residue $\mathrm{Res}_{W_M(\mathcal{F})}$ on $\widetilde{\Omega}(\mathcal{F}, M)$ to be $\mathrm{Res}_{W_M(\mathcal{F})}(\omega) = \alpha_{(0,\dots,0)}$. Using the canonical inclusion $\Omega^N_{\mathcal{O}_M(\mathcal{F})} \subset \widetilde{\Omega}(\mathcal{F}, M)$, it can be seen that $\mathrm{Res}_{\mathcal{O}_M(\mathcal{F})}(\omega) = \mathrm{Res}_{W_M(\mathcal{F})}(\omega)$ for any N-form $\omega \in \Omega^N_{\mathcal{O}_M(\mathcal{F})}$.

We want to show that $\mathrm{Res}_{W_M(\mathcal{F})}$ is independent of the choice of local parameters of $\mathcal{F}$. Let $\bar{t}'_1, \dots, \bar{t}'_N$ be a different set of local parameters of $\mathcal{F}$. Let $\mathcal{O}'_M(\sigma^n\mathcal{F})$ be the flat $\mathbb{Z}/p^M$-lifts constructed using the elements $\bar{t}_i'^{p^n}$ and let $\mathrm{Res}'_{W_M(\mathcal{F})}$ be the residue defined using $\iota_{O'_M(\mathcal{F})}$.

Proposition 4.1 *For any* $\omega \in \widetilde{\Omega}(\mathcal{F}, M)$, $\mathrm{Res}_{W_M(\mathcal{F})}(\omega) = \mathrm{Res}'_{W_M(\mathcal{F})}$.

Proof Any $x \in W_M(\mathcal{F})^*$ can be written as $x = \alpha\, t_1^{a_1} \cdots t_N^{a_N}\, \epsilon\eta$ with $\alpha \in W(k)^*$, $(a_1, \ldots, a_N) \in \mathbb{Z}^N$, $\epsilon \in (1 + \mathfrak{m}_{Q(\mathcal{F})}) \bmod p^M \mathcal{O}(\mathcal{F})$ and

$$\eta \in 1 + p\mathcal{O}_M(\sigma^{-1}\mathcal{F}) + \cdots + p^{M-1}\mathcal{O}_M(\sigma^{1-M}\mathcal{F}) = 1 + pW_M(\sigma^{-1}\mathcal{F}).$$

Using $pW_M(\sigma^{-1}\mathcal{F}) = VW_{M-1}(\mathcal{F}) \subset p\mathcal{O}_M(\sigma^{1-M}\mathcal{F})$, we see that log converges on $1 + pW_M(\sigma^{-1}\mathcal{F})$. Letting $\eta' = \log(\eta) \in pW_{M-1}(\sigma^{-1}\mathcal{F})$, it follows that $d_{\log}x$ can be written

$$d_{\log}x = a_1 d_{\log}t_1 + \cdots + a_N d_{\log}t_N + d_{\log}\epsilon + d\eta'.$$

Writing ϵ as a convergent product $\epsilon = \prod_{\underline{b}} \left(1 - \beta_{\underline{b}} \underline{t}^{\underline{b}}\right)$, we see furthermore that $d_{\log}\epsilon = -\sum (\beta_{\underline{a}} \underline{t}^{\underline{b}})^n d_{\log}(\underline{t}^{\underline{b}})$ for $\underline{b}$ in some admissible set in $\mathbb{Z}^N_{>\underline{0}}$ and $\beta_{\underline{a}} \in W_M(k)$. Now note that $\underline{t}^{\underline{b}n} d_{\log}\underline{t}^{\underline{b}} \wedge d\eta = \left(b_1 d_{\log}t_1 + \cdots + b_N d_{\log}t_N\right) \wedge d(\underline{t}^{\underline{b}n}\eta)$, for $\underline{b} \geqslant 0$ and $\eta \in pW_M(\sigma^{-1}\mathcal{F})$ since this reduces to $d\underline{t}^{\underline{b}} \wedge d\underline{t}^{\underline{b}} = 0$. Then we see that any $\omega \in \widetilde{\Omega}(\mathcal{F}, M)$ can be written as a sum of the three types of elements

(i) $\alpha_\omega d_{\log}t_1 \wedge \cdots \wedge d_{\log}t_N$, for $\alpha_\omega \in W_M(k)$,

(ii) $m d_{\log}t_1 \wedge \cdots \wedge d_{\log}t_N$, with $m \in \mathfrak{m}_{Q(\mathcal{F})} \bmod p^M$, and

(iii) $d_{\log}t_{i_1} \wedge \cdots \wedge d_{\log}t_{i_s} \wedge d\eta_1 \wedge \cdots \wedge d\eta_{N-s}$, for $\eta_j \in pW_M(\sigma^{-1}\mathcal{F})$.

Because $\mathrm{Res}_{W_M(\mathcal{F})} = 0$ for all elements from (ii) and (iii), one has $\mathrm{Res}_{W_M(\mathcal{F})}(\omega) = \alpha_\omega \in W_M(\mathcal{F})$. Thus we need to check that $\mathrm{Res}_{W_M(\mathcal{F})}\left(d_{\log}t'_1 \wedge \cdots \wedge d_{\log}t'_N\right) = 1$. To see this, note that $t'_i = [\alpha_i]\, t_i t_{i+1}^{a_{i+1}^{(i)}} \cdots t_N^{a_N^{(i)}}\, \epsilon_i\eta_i$ as above, where $a_1^{(i)} = \cdots = a_{i-1}^{(i)} = 0$ and $a_i^{(i)} = 1$. Then the claim follows from the above manipulations. □

4.2 Parshin's Pairing

If $\mathcal{F}$ is any field of characteristic p, any abelian extension of exponent p^M is obtained by joining all coefficients of $\wp^{-1}X \subset W_M(\mathcal{F}^{sep})$ for some subgroup $X \subset$

$W_M(\mathcal{F})/\wp W_M(\mathcal{F})$. Witt-Artin-Schreier theory provides a perfect pairing

$$W_M(\mathcal{F})/\wp W_M(\mathcal{F}) \times \Gamma_{\mathcal{F}}^{ab}/p^M \to W_M(\mathbb{F}_p)$$
$$((b_0, \dots, b_{M-1}), \gamma) \longmapsto (\gamma(\beta_0), \dots, \gamma(\beta_{M-1})) - (\beta_0, \dots, \beta_{M-1})$$

where $(\beta_0, \dots, \beta_{M-1}) \in W_M(\mathcal{F}^{sep})$ is any element satisfying $\wp(\beta_0, \dots, \beta_{M-1}) = (b_0, \dots, b_{M-1})$. Here, as usual, $\wp(w) = \sigma(w) - w$ for any $w \in W_M(\mathcal{F})$.

We shall consider the case where $\mathcal{F}$ is an N-dimensional local field of characteristic p. In [32], the Witt-Artin-Schreier pairing is used to construct the p-part of class-field theory for higher local fields of characteristic p by defining the pairing

$$[-,-\}_M : W_M(\mathcal{F}) \times K_N(\mathcal{F})/p^M K_N(\mathcal{F}) \longrightarrow W_M(k).$$

We start by clarifying the construction of $[-,-\}_M$.

Let $\widetilde{b}_i, \widetilde{x}_j$ be lifts of $b_i, c_i \in \mathcal{F}$ with respect to the map $W(k)((t_N)) \cdots ((t_1)) \to \mathcal{F}$ induced by $W(k) \to k$, for $0 \leqslant i \leqslant M-1$, $1 \leqslant j \leqslant N$. Parshin's pairing is

$$\big[(b_0, \dots, b_{M-1}), \{x_1, \dots, x_N\}\big)_M = (y_0, \dots, y_{M-1}) \in W_M(k).$$

$(y_0, \dots, y_{M-1})$ is the unique Witt-vector with ghost-components

$$y^{(i)} = \operatorname{Res}\ \big(\widetilde{b}^{(i)} d_{\log}\widetilde{x}_1 \wedge \dots \wedge d_{\log}\widetilde{x}_N\big),$$

where the residue is taken in $\Omega^N_{Q_0(\mathcal{F})}$. By [32], lemma 3.1, the residue is integral, i.e. lies in $W(k)$, so $y_i \in k$ are well-defined. Instead of taking ghost-components in characteristic zero, taking the residue there, and going back to $W_M(k)$ using the inverse operation to taking ghost-components, we work in $W_M(\mathcal{F})$.

Notice that any $b = (b_0, \dots, b_{M-1}) \in W_M(\mathcal{F})$ can be written as

$$b = [b_0] + V[b_1] + \dots + V^{M-1}[b_{M-1}] \in W_M(\mathcal{F}).$$

Taking as lifts of b_i the Teichmüller representatives $[b_i] \in W_M(\mathcal{F})$, it follows that the $(M-1)$-st ghost-component of b is

$$\begin{aligned} b^{(M-1)} &= [b_0]^{p^{M-1}} + \dots + p^i[b_i]^{p^{M-i-1}} + \dots + p^{M-1}[b_{M-1}] \\ &= [\sigma^M b_0] + \dots + V^i[\sigma^{M-1} b_i] + \dots + V^{M-1}[\sigma^{M-1} b_{M-1}] = \sigma^{M-1} b. \end{aligned}$$

In particular, this shows that $b^{(M-1)} \in \mathcal{O}_M(\mathcal{F})$. Thus $[-,-\}_M$ may be defined as

$$\big[b, \{x_1, \dots, x_N\}\big\}_M = \operatorname{Res}_{W_M(\mathcal{F})}\big(\sigma^{M-1}(b)\, d_{\log}\widetilde{x}_1 \wedge \dots \wedge d_{\log}\widetilde{x}_N\big),$$

where $\widetilde{x}_i \in W_M(\mathcal{F})^*$ is any lift of $x_i \in \mathcal{F}$.

Lemma 4.2 *The value of Parshin's symbol*

$$\big[(b_0, \dots, b_{M-1}), \{x_1, \dots, x_N\}\big)_M = \operatorname{Res}_{W_M(\mathcal{F})} \sigma^{M-1}(b)\, d_{\log}\widetilde{x}_1 \wedge \dots \wedge d_{\log}\widetilde{x}_N,$$

is independent of the choice of lifts $\widetilde{x}_i \in \mathcal{O}_M(\mathcal{F})^*$.

Proof For $x \in \mathcal{F}^*$, let $\widetilde{x}, \widetilde{x}'$ be two different lifts to $\mathcal{O}_M(\mathcal{F})^*$. Then $\widetilde{x} - \widetilde{x}' \in p\mathcal{O}_M(\mathcal{F})$, so there exists $a \in \mathcal{O}_M(\mathcal{F})$ with $\widetilde{x}' = \widetilde{x}(1+pa)$. Now $\mathcal{O}_M(\mathcal{F})$ is a p-adic ring, so the logarithm $\log(1+pa)$ converges in $p\mathcal{O}_M(\mathcal{F})$. Thus $d_{\log}\widetilde{x}' = d_{\log}\widetilde{x} + d\big(\log(1+pa)\big)$ and $\log(1+pa) = py$ for some $y \in \mathcal{O}_M(\mathcal{F})$. We need to show that

$$\operatorname{Res}_{\mathcal{O}_M(\mathcal{F})}\big(b_0^{p^{M-1}} + p b_1^{p^{M-2}} + \dots + p^{M-2} b_{M-2}^{p} + p^{M-1} b_{M-1}\big) p\, dy \equiv 0 \mod p^M.$$

But $b^{p^i} dy \equiv d(b^{p^i} y) \mod p^i$ implies that $\operatorname{Res}_{\mathcal{O}_M(\mathcal{F})}\big(b^{p^i} dy\big) \equiv 0 \mod p^i$ for each i, which proves the claim. $\square$

It would be nice to generalise this result to lifts in $W_M(\mathcal{F})^*$. However, the element $\widetilde{x} - \widetilde{x}'$ above would then lie in $VW_M(\mathcal{F}) = pW_M(\sigma^{-1}(\mathcal{F}))$ and hence $a, y \in W_M(\sigma^{-1}(\mathcal{F}))$, and we no longer get the extra factor of p in the above expression.

Lemma 4.3 *We have $\sigma\,[b,x\}_M = [\sigma(b), x\}_M$ for any $b \in W_M(\mathcal{F})$ and $x \in K_N(\mathcal{F})$.*

Proof $K_N^t(\mathcal{F})$ is generated by all symbols $\{\bar{t}'_1, \dots, \bar{t}'_N\}$, for varying local parameters $\bar{t}'_1, \dots, \bar{t}'_N$. By prop. 4.1, $\operatorname{Res}_{W_M(\mathcal{F})}$ is independent of the choice of local parameters, thus we may assume $x = \{\bar{t}_1, \dots, \bar{t}_N\}$. Writing $\sigma^{M-1}b = \sum \alpha_{\underline{a}} t_1^{a_1} \cdots t_N^{a_N} \in \mathcal{O}_M(\mathcal{F})$, we obtain $[b,x\}_M = \alpha_{(0,\dots,0)}$. Also, $\sigma^{M-1}(\sigma b) = \sum \sigma(\alpha_{\underline{a}}) t_1^{pa_1} \cdots t_N^{pa_N}$, and hence $[\sigma(b), x\} = \sigma(\alpha_{(0,\dots,0)}) = \sigma[b,x\}_M$, as required. $\square$

Using this, we obtain Parshin's pairing

$$[-,-)_M : W_M(\mathcal{F})/\wp \times K_N(\mathcal{F}) \longrightarrow W_M(\mathbb{F}_p) \cong \mathbb{Z}/p^M, \quad [b,x)_M = \operatorname{Tr}\,[b,x\}_M,$$

where $\mathrm{Tr} : W_M(k) \to W_M(\mathbb{F}_p)$ is induced by the trace of fields $k \to \mathbb{F}_p$ and the identification $W_M(\mathbb{F}_p) \cong \mathbb{Z}/p^M$ is given by $(M-1)$-st ghost-components.

The chain of inclusions $W_M(\sigma^{M-1}\mathcal{F}) \subset \mathcal{O}_M(\mathcal{F}) \subset W_M(\mathcal{F})$ shows that $b \mapsto \sigma^{M-1}b$ induces $W_M(\mathcal{F})/\wp \xrightarrow{\sim} \mathcal{O}_M(\mathcal{F})/\wp$. Since $[b,x)_M = [\sigma b, x) = \cdots = [\sigma^{M-1}b, x)$ for any $b \in W_M(\mathcal{F})$ and $x \in K_N^t(\mathcal{F})$, this shows that Parshin's pairing is equivalent to

$$[-,-)_M : \mathcal{O}_M(\mathcal{F})/\wp \times K_N^t(\mathcal{F}) \to \mathbb{Z}/p^M,$$
$$\big[b, \{x_1, \ldots, x_N\}\big)_M = \mathrm{Tr} \circ \mathrm{Res}\ \big(\sigma^{M-1}(b)\, d_{\log}\widetilde{x}_1 \wedge \cdots \wedge d_{\log}\widetilde{x}_N\big),$$

where the lifts $\widetilde{x}_i$ are in $\mathcal{O}_M(\mathcal{F}) \subset W_M(\mathcal{F})$, and the residue is $\mathrm{Res}_{\mathcal{O}_M(\mathcal{F})}$.

In [32, 33], Parshin proves that this pairing is non-degenerate and thus can be used to define the p-part of class field theory $\Psi_{\mathcal{F}}^P : K_N^t(\mathcal{F})/p^M \to \Gamma_{\mathcal{F}}^{ab}/p^M$. To prove that $\Psi_{\mathcal{F}}^P$ coincides with the construction from [12], it suffices to show that Parshin's pairing, composed with the reciprocity map $\Psi_{\mathcal{F}}^F : K_N^t(\mathcal{F}) \to \Gamma_{\mathcal{F}}^{ab}$ due to Fesenko induces the Witt pairing. We give details of the outlined proof from [12], §2.

Theorem 4.4 *For an N-dimensional local field $\mathcal{F}$ of characteristic p and a finite abelian p-extension $\mathcal{L}/\mathcal{F}$, the class field theories constructed by Parshin ([32]) and Fesenko ([12]) agree.*

Proof Let M be the exponent of $\mathrm{Gal}(\mathcal{L}/\mathcal{F})$ so that $\mathcal{L}$ is contained in the composite of finitely many linearly disjoint cyclic extensions of degree p^M. Therefore we may without loss of generality assume that $\mathcal{L}/\mathcal{F}$ is cyclic, $\mathcal{L} = \mathcal{F}(X)$ for $X \in \mathcal{O}_M(\mathcal{F}^{sep})$ with $\wp X = x \in \mathcal{O}_M(\mathcal{F})$.

We need to show that for $\{y_1, \ldots, y_N\} \in K_N^t(\mathcal{F})$,

$$\big[x, \{y_1, \ldots, y_N\}\big)_M = \gamma(X) - X,$$

where $[-,-)_M$ is Parshin's pairing, $\gamma = r_{\mathcal{L}/\mathcal{F}}^{-1}(\{y_1, \ldots, y_N\}) \in \mathrm{Gal}(\mathcal{L}/\mathcal{F})$ corresponds to $\{y_1, \ldots, y_N\}$ under Fesenko's reciprocity map, and $\wp(X) = x$.

Notice first that $K_N^t(\mathcal{F})$ is generated by all symbols $\{\bar{t}_1, \ldots, \bar{t}_N\}$ for various sets of local parameters $\bar{t}_1, \ldots, \bar{t}_N$. Thus it suffices to prove the theorem for $\{\bar{t}_1, \ldots, \bar{t}_N\} \in K_N^t(\mathcal{F})$ where the $\bar{t}_i$ are any fixed set of local parameters.

Also, $\mathcal{O}_M(\mathcal{F})/\wp$ is generated as $\mathbb{Z}/p^M$-module by two types of elements. On the one hand elements $\sum \alpha_{\underline{a}} t_1^{a_1} \cdots t_N^{a_N}$, where the sum is over some admissible set with $\underline{A} < \underline{a} < \underline{0}$ for some fixed $\underline{A}$, and, on the other hand, $\alpha_0 \in W_M(k)$ of trace $\mathrm{Tr}\,(\alpha_0) = 1 \in \mathbb{Z}/p^M$. So we may furthermore assume that x (with $\mathcal{L} = \mathcal{F}(X), \wp(X) = x$) is of either form.

In the second case, Parshin's symbol yields

$$\left[\alpha_0, \{\bar{t}_1, \ldots, \bar{t}_N\}\right)_M = \mathrm{Tr} \circ \mathrm{Res}\left(\alpha_0 d_{\log} t_1 \wedge \cdots \wedge d_{\log} t_N\right) = \mathrm{Tr}\,(\alpha_0) = 1,$$

by the choice of α_0. Using Fesenko's construction, we note that $\mathcal{L}/\mathcal{F}$ is totally unramified, with Galois group $\mathrm{Gal}(\mathcal{L}/\mathcal{F}) = \langle \varphi_F|_L \rangle$ generated by the restriction of the Frobenius of $\mathcal{F}$ to $\mathcal{L}$. By the first example in section 3.2, $r^F_{\mathcal{L}/\mathcal{F}}(\varphi_F|_L) = \{\bar{t}_1, \ldots, \bar{t}_N\}$ mod $N_{\mathcal{L}/\mathcal{F}} K^t_N(\mathcal{L})$. But $\wp(X) = \alpha_0$ just means that the absolute Frobenius φ_F acts as $\varphi_F(X) = X + \alpha_0$. Now if $[F^{(N)} : \mathbb{F}_p] = f$ then $\varphi_F\big|_k = \sigma^f$ where σ is the absolute Frobenius. Thus

$$\varphi(X) = X + \alpha_0 + \sigma(\alpha_0) + \cdots + \sigma^{f-1}\alpha_0 = X + \mathrm{Tr}\,(\alpha_0) = X + 1$$

and consequently $\varphi(X) - X = 1$, as required.

In the first case, for $x = \sum \alpha_{\underline{a}} t_1^{a_1} \cdots t_N^{a_N}$ as above and $\mathcal{L} = \mathcal{F}(X)$ with $\wp(X) = x$, Parshin's pairing gives

$$\left[x, \{\bar{t}_1, \ldots, \bar{t}_N\}\right)_M = \mathrm{Tr} \circ \mathrm{Res}\left(\sum \alpha_{\underline{a}} t_1^{a_1} \cdots t_N^{a_N} d_{\log} t_1 \wedge \cdots \wedge d_{\log} t_N\right) = 0$$

since $\underline{a} < \underline{0}$ for all $\underline{a}$. By [33], prop. 2, this implies that $\{\bar{t}_1, \ldots, \bar{t}_N\} \in N_{\mathcal{L}/\mathcal{F}} K^t_N(\mathcal{L})$, so $\{\bar{t}_1, \ldots, \bar{t}_N\}$ mod $N_{\mathcal{L}/\mathcal{F}} K^t_N(\mathcal{L}) = r_{\mathcal{L}/\mathcal{F}}(\mathrm{id})$ corresponds to the trivial element of the Galois group, so $\mathrm{id}(X) - X = 0$, too. □

4.3 An Invariant Formula

The pairings $\mathcal{O}_M(\mathcal{F})/\wp \times K_N(\mathcal{F})/p^M \to \mathbb{Z}/p^M$ are not a priori compatible with the projections modulo p^{M-1}. For classical local fields, Fontaine [17] proves an invariant formula using special lifts of $\mathcal{F}$ to $\mathcal{O}(\mathcal{F})$. We adapt his method to higher dimensions.

Given $\mathcal{F}$, fix a set of local parameters $\bar{t}_1, \dots, \bar{t}_N$. They provide a p-basis of $\mathcal{F}$. Consider its corresponding flat $\mathbb{Z}_p$-lift $\mathcal{O}(\mathcal{F}) = W_M(k)\{\{t_N\}\}\cdots\{\{t_1\}\}$ (see appendix A.2), with field of fractions $Q(\mathcal{F})$. $Q(\mathcal{F})$ is an $(N+1)$-dimensional local field with parameters $p, t_1, \dots, t_N$.

Consider the inseparable extension $\mathcal{F}' = \sigma^{-1}\mathcal{F} = \mathcal{F}(\overline{T}_1, \dots, \overline{T}_N)$, where $\overline{T}_i^p = \bar{t}_i$. Using $\overline{T}_i$ as p-basis of $\mathcal{F}'$, we obtain a corresponding extension of fields of fractions $Q(\mathcal{F}') = Q(\mathcal{F})(T_1, \dots, T_N)$ and an isomorphism $\sigma : Q(\mathcal{F}') \to Q(\mathcal{F})$ which maps $T_i \mapsto t_i$ and is equal to the frobenius on $W(k)$. Denote by σ^{-1} its inverse.

Finally, denote by N_σ the composite

$$N_\sigma = N_{Q(\mathcal{F}')/Q(\mathcal{F})} \circ \sigma^{-1} : K_N^t(Q(\mathcal{F})) \longrightarrow K_N^t(Q(\mathcal{F})).$$

Note that N_σ induces $N_\sigma : K_N^t(\mathcal{O}(\mathcal{F})) \to K_N^t(\mathcal{O}(\mathcal{F}))$. This can be seen by considering topological generators and noting that $Q(\mathcal{F}')/Q(\mathcal{F})$ breaks up into a tower of N sub-extensions of degree p in such a way that the norms of the N sub-extensions act at most on one entry of those generators.

Working with the groups $K_n^t(\mathcal{O}(\mathcal{F}))$ defined in section 3.5, we shall find a special section of reduction modulo p: $K_N^t(\mathcal{O}(\mathcal{F})) \to K_N^t(\mathcal{F})$. Start with the exact sequence

$$0 \longrightarrow U^{(1)}K_N^t(\mathcal{O}(\mathcal{F})) \longrightarrow K_N^t(\mathcal{O}(\mathcal{F})) \longrightarrow K_N^t(\mathcal{F}) \longrightarrow 0,$$

and apply $N_\sigma - 1$ to each group. Since $\mathcal{F}'/\mathcal{F}$ is inseparable, $N_\sigma = 1$ on $\mathcal{F}$. The snake lemma yields

$$\begin{aligned} \big(U^{(1)}K_N^t(\mathcal{O}(\mathcal{F}))\big)_{N_\sigma=1} &\longrightarrow \big(K_N^t(\mathcal{O}(\mathcal{F}))\big)_{N_\sigma=1} \longrightarrow K_N^t(\mathcal{F}) \\ &\longrightarrow U^{(1)}K_N^t(\mathcal{O}(\mathcal{F}))/(N_\sigma - 1)U^{(1)}K_N^t(\mathcal{O}(\mathcal{F})). \end{aligned}$$

Lemma 4.5 *The middle morphism of the above diagram is an isomorphism.*

PROOF $U^{(1)}K_N^t(\mathcal{O}(\mathcal{F}))$ is generated by two types of generators. On the one hand, $u = \{1+[\alpha]p^{a_0}\underline{t}^{\underline{a}}, t_{i_1}, \dots, t_{i_{N-1}}\}$, and we see that $N_\sigma u = \{1+[\alpha^\sigma]p^{pa_0}\underline{t}^{\underline{a}}, t_{i_1}, \dots, t_{i_{N-1}}\}$. Similarly, for the second type $v = \{1+[\beta]p^{b_0}\underline{t}^{\underline{b}}, p, t_{j_1}, \dots, t_{j_{N-2}}\}$ of generators, we have $N_\sigma v = \{(1+[\beta^\sigma]p^{pb_0})^p, p, t_{j_1}, \dots, t_{j_{N-2}}\}$. Notice that

$$\lim_{n\to\infty} \left(1+[\alpha^{\sigma^n}]p^{p^n a_0}\underline{t}^{\underline{a}}\right) = 1 \quad \text{and} \quad \lim_{n\to\infty} \left(1+[\beta^{\sigma^n}]p^{p^n b_0}\underline{t}^{\underline{b}}\right)^{p^n} = 1.$$

Now by the definition of the topology on $K_N^t(Q(\mathcal{F}))$, $V_{Q(\mathcal{F})} \times (Q(\mathcal{F})^*)^{\otimes(N-1)} \to K_N^t(Q(\mathcal{F}))$ is sequentially continuous and therefore $N_\sigma^n(u) \to 0$ and $N_\sigma^n(v) \to 0$ as $n \to \infty$. Since $\mathcal{O}(\mathcal{F})$ is absolutely unramified, it follows from [43], prop. 2.1 that $K_N^t(\mathcal{O}(\mathcal{F}))$ is topologically free, so we conclude that $\big(U^{(1)}K_N^t(\mathcal{O}(\mathcal{F}))\big)_{N_\sigma=1} = 0$ and the middle morphism is injective.

To see that it also surjective, note that $\{1+[\alpha]\underline{t}^{\underline{a}}, t_{i_1}, \ldots, t_{i_{N-1}}\} \in \big(K_N^t(\mathcal{O}(\mathcal{F}))\big)_{N_\sigma=1}$ for all $\alpha \in k^*$ and $\underline{a} > \underline{0}$, and $\{t_1, \ldots, t_N\} \in \big(K_N^t(\mathcal{O}(\mathcal{F}))\big)_{N_\sigma=1}$ and that their images in $K_N^t(\mathcal{F})$ topologically generate it. Alternatively, notice that by the explicit description of N_σ on generators of $U^{(1)}K_N^t(\mathcal{O}(\mathcal{F}))$, $(1+N_\sigma+N_\sigma^2+\cdots)(u) = \{u', t_{i_1}, \ldots, t_{i_{N-1}}\}$ converges in $K_N^t(\mathcal{O}(\mathcal{F}))$ because $u' = \prod\big(1+[\alpha^{\sigma^n}]p^{p^n a_0}\underline{t}^{\underline{a}}\big)$ converges in F^*, and similarly for v. But $(1-N_\sigma)(1+N_\sigma+N_\sigma^2+\cdots)(u) = 1$ so all generators of $U^{(1)}K_N^t(\mathcal{O}(\mathcal{F}))$ are also in $(N_\sigma - 1)U^{(1)}K_N^t(\mathcal{O}(\mathcal{F}))$ and it follows that the last quotient in the above long exact sequence is trivial. □

For the groups $K_N'(\mathcal{O}(\mathcal{F})) \subset K_N^t(\mathcal{O}(\mathcal{F}))$, one sees that the map induced by the projection $\mathcal{O}(\mathcal{F}) \to \mathcal{F}$ again induces an isomorphism $(K_N'(\mathcal{O}(\mathcal{F})))_{N_\sigma} \to K_N^t(\mathcal{F})$ by considering that the lifts $\{t_1, \ldots, t_N\}$ and $\{1+[\alpha]\underline{t}^{\underline{a}}, t_{i_1}, \ldots, t_{i_{N-1}}\}$ of generators of $K_N^t(\mathcal{F})$ lie in $\big(K_N'(\mathcal{O}(\mathcal{F}))\big)_{N_\sigma=1}$. This indicates that the example of an element in $K_N^t(\mathcal{O}) \setminus K_N'(\mathcal{O})$ given in the remark after cor. 2.31 was typical. If $\mathcal{O} = \mathcal{O}(\mathcal{F})$, $n\{1+\pi^n v, \pi\} = -\{1+\pi^n v, -v\} \notin \big(K_N^t(\mathcal{O}(\mathcal{F}))\big)_{N_\sigma=1} = \big(K_N'(\mathcal{O}(\mathcal{F}))\big)_{N_\sigma=1}$.

We denote by $Col : K_N^t(\mathcal{F}) \to \big(K_N^t(\mathcal{O}(\mathcal{F}))\big)_{N_\sigma=1} \subset K_N^t(\mathcal{O}(\mathcal{F}))$ ('Coleman lifts') the inverse map.

Corollary 4.6 *$Col : K_N^t(\mathcal{F}) \to K_N^t(\mathcal{O}(\mathcal{F}))$ is continuous. On the basis of $K_N^t(\mathcal{F})$ from prop. 2.10, Col is given by*

$$Col\big(\{\bar{t}_1, \ldots, \bar{t}_N\}\big) = \{t_1, \ldots, t_N\}$$
$$Col\big(E(\alpha, \underline{\bar{t}}^{\underline{a}}), \bar{t}_1, \ldots, \bar{t}_{i-1}, \bar{t}_{i+1}, \ldots, \bar{t}_N\}\big) = \{E([\alpha], \underline{t}^{\underline{a}}), t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_N\}.$$

Proof The explicit formulae for Col on the level of generators follows from the fact that the elements on the right-hand side lie in $\big(K_N^t(\mathcal{O}(\mathcal{F}))\big)_{N_\sigma=1}$ and are lifts of those on the left-hand side. To see that Col is continuous, note that for $\underline{a} = (a_1, \ldots, a_N)$ running through an admissible set in $\mathbb{Z}^N_{>\underline{0}}$, $(0, a_1, \ldots, a_N)$ runs through an admissible set of $\mathbb{Z}^{N+1}_{>\underline{0}}$. Thus if $\prod \big(E(\alpha_{\underline{a}}, \underline{\overline{t}}^{\underline{a}})\big)$ converges in $\mathcal{F}$, so does $\prod \big(E([\alpha_{\underline{a}}], \underline{t}^{\underline{a}}\big)$ in $\mathcal{O}(\mathcal{F}) \subset Q(\mathcal{F})$ (since the first local parameter p of $Q(\mathcal{F})$ appears with exponent 0). □

In what follows, we shall need to work in $\Omega^N_{Q(\mathcal{F})} = \Omega^N_{\mathcal{O}(\mathcal{F})} \otimes Q(\mathcal{F})$. The morphism $Q(\mathcal{F})^* \to \Omega_{Q(\mathcal{F})}$ given by $x \mapsto d_{\log}x = \frac{\mathrm{d}x}{x}$ induces

$$K^t_N(Q(\mathcal{F})) \to \Omega^N_{Q(\mathcal{F})}, \quad \{x_1, \dots, x_N\} \mapsto d_{\log}x_1 \wedge \cdots \wedge d_{\log}x_N,$$

which we shall also denote by $d_{\log}$.

Lemma 4.7 *For $x \in \mathcal{O}(\mathcal{F})$ and $u \in K^t_N(\mathcal{F})$, we have*

$$\sigma\big(\mathrm{Res}\,(x\, d_{\log}Col(u))\big) = \mathrm{Res}\,(\sigma(x)\, d_{\log}Col(u))$$

Proof It suffices to consider generators $u_{\alpha,i} := \{E(\alpha, \underline{\overline{t}}^{\underline{a}}), \overline{t}_1, \dots, \overline{t}_{i-1}, \overline{t}_{i+1}, \dots, \overline{t}_N\}$ and $u_0 := \{\overline{t}_1, \dots, \overline{t}_N\}$ of $K^t_N(\mathcal{F})$. Writing $x = \sum_{\underline{b}>\underline{0}} w_{\underline{b}}\underline{t}^{\underline{b}}$, then $\sigma(x) = \sum \sigma(w_{\underline{a}})\underline{t}^{\underline{a}p}$. For the first type of generators, we have

$$d_{\log}Col(u_{\alpha,i}) = \sum [\alpha^{\sigma^n}]\underline{t}^{\underline{a}p^n}(-1)^i a_i d_{\log}t_1 \wedge \cdots \wedge d_{\log}t_N$$

and $\mathrm{Res}\,\big(x\, d_{\log}Col(u_{\alpha,i})\big) = \sum_n [\alpha^{\sigma^n}]a_i w_{-\underline{a}p^n}$, where the sum is taken over all (finitely many) n such that $w_{-\underline{a}p^n} \neq 0$. On the other hand, $\sigma(x) = \sum \sigma(w_{\underline{b}})\underline{t}^{\underline{b}p}$ and thus $\mathrm{Res}\,\big(\sigma(x)\, d_{\log}Col(u_{\alpha,i})\big) = \sum_n [\alpha^{\sigma^{n+1}}]a_i\sigma(w_{-\underline{a}p^n}) = \sigma\big(\mathrm{Res}\,(x\, d_{\log}Col(u_{\alpha,i}))\big)$. Also, $\sigma\big(\mathrm{Res}\,(xd_{\log}Col(u_0))\big) = \sigma(w_{\underline{0}}) = \mathrm{Res}\,\big(\sigma(x)d_{\log}Col(u_0)\big)$, as required. □

Following the argument in [17], this can be obtained more naturally as a consequence of the defining property of Col, the N_σ-invariance, as follows.

With $Q(\mathcal{F}') = Q(\sigma^{-1}\mathcal{F})$ as before, we have $\Omega^N_{Q(\mathcal{F})} = Q(\mathcal{F})\, d_{\log}t_1 \wedge \cdots \wedge d_{\log}t_N$ and

$$\Omega^N_{Q(\mathcal{F}')} = Q(\mathcal{F}')\, d_{\log}T_1 \wedge \cdots \wedge d_{\log}T_N = Q(\mathcal{F}')\, d_{\log}t_1 \wedge \cdots \wedge d_{\log}t_1,$$

since $d_{\log}t_i = p\, d_{\log}T_i$ for all i and p is invertible in $Q(\mathcal{F}')$. Again $t_i \mapsto T_i$ induces $\sigma^{-1} : \Omega^N_{Q(\mathcal{F})} \to \Omega^N_{Q(\mathcal{F}')}$ given by

$$\sigma^{-1}\big(xd_{\log}t_1 \wedge \cdots \wedge d_{\log}t_N\big) = \sigma^{-1}(x)d_{\log}T_1 \wedge \cdots \wedge d_{\log}T_N = p^{-N}\sigma^{-1}(x)d_{\log}t_1 \wedge \cdots \wedge t_N$$

Define the trace map $tr : \Omega^N_{Q(\mathcal{F}')} \longrightarrow \Omega_{Q(\mathcal{F})}$ to be the usual trace on $Q(\mathcal{F}') \to Q(\mathcal{F})$, and the identity on $d_{\log}t_1 \wedge \cdots \wedge d_{\log}t_N$. It is $Q(\mathcal{F})$-linear and after 'going up' to $Q(\mathcal{F}')(\zeta_p)/Q(\mathcal{F})(\zeta_p)$, it coincides with taking the sum over all Galois conjugates.

Using the fact that the composite of the norm $N = N_{Q(\mathcal{F}')/Q(\mathcal{F})}$ with the map $j : K_N^t(Q(\mathcal{F})) \to K_N^t(Q(\mathcal{F}')(\zeta_p))$ is also equal to the sum over all Galois conjugates, it follows that the outer diagram in

$$\begin{tikzcd}
\Omega^N_{Q(\mathcal{F}')} \arrow[r, "tr"] & \Omega^N_{Q(\mathcal{F})} \arrow[r, "i"] & \Omega^N_{Q(\mathcal{F}')(\zeta_p)} \\
K_N^t(Q(\mathcal{F}')) \arrow[u, "d_{\log}"] \arrow[r, "N"'] & K_N^t(Q(\mathcal{F})) \arrow[u, "d_{\log}"] \arrow[r, "j"'] & K_N^t(Q(\mathcal{F}')(\zeta_p)), \arrow[u, "d_{\log}"]
\end{tikzcd}$$

is commutative. Since i is injective, so is the left-hand diagram.

Noting that $\sigma^{-1} \circ d_{\log} = d_{\log} \circ \sigma^{-1} : K_N^t(Q(\mathcal{F})) \to \Omega^N_{Q(\mathcal{F}')}$, this implies that $tr\,\sigma^{-1} d_{\log} Col(x) = d_{\log} Col(x)$, which is analogous to the property $N_\sigma Col(x) = Col(x)$ on the level of K-groups.

Lemma 4.8 *For any $\omega \in \Omega^N_{Q(\mathcal{F})}$, $\mathrm{Res} \circ tr(\sigma^{-1}(\omega)) = \sigma^{-1}(\mathrm{Res}\,(\omega))$.*

Proof Write ω as $\omega = \sum [\alpha_{\underline{a}}] p^{a_0} t_1^{a_1} \cdots t_N^{a_N}\ d_{\log} t_1 \wedge \cdots \wedge d_{\log} t_N$ for $\alpha \in k^*$, and $(a_0, \ldots, a_N)$ running through some admissible set. Then

$$\sigma^{-1}(\omega) = \sum_{\underline{a}} [\alpha_{\underline{a}}^{\sigma^{-1}}] p^{a_0} T_1^{a_1} \cdots T_N^{a_N}\, p^{-N}\, d_{\log} t_1 \wedge \cdots \wedge d_{\log} t_N.$$

If $p \mid a_i$ for all $1 \leqslant i \leqslant N$, then tr acts as multiplication by p^N on this term. If there is some $i \geqslant 1$ with $p \nmid a_i$, then $tr(T_1^{a_1} \cdots T_N^{a_N}) = 0$. Thus

$$Res \circ tr\big(\sigma^{-1}(\omega)\big) = \sum_{\underline{a}=(a_0,0,\ldots,0)} [\alpha_{\underline{a}}^{\sigma^{-1}}] p^{a_0} = \sigma^{-1}\Big(\sum_{\underline{a}=(a_0,0,\ldots,0)} [\alpha_{\underline{a}}] p^{a_0}\Big) = \sigma^{-1}(Res(\omega)),$$

as required. $\square$

Noting that tr is $Q(\mathcal{F})$-linear, we have

$$tr\,\sigma^{-1}(\sigma(x).d_{\log} Col(u)) = x.tr\,\sigma^{-1}(d_{\log} Col(u)) = x\, d_{\log} Col(u).$$

Together with the lemma, this implies $\sigma\,\mathrm{Res}\,(x\, d_{\log} Col(u)) = Res(\sigma(x)\, d_{\log} Col(u))$ more generally for $x \in Q(\mathcal{F})$ and without needing to consider generators.

We are now ready to prove the following invariant formula for Parshin's pairing.

Theorem 4.9 *The Witt-pairing $\mathcal{O}(\mathcal{F}) \times K_N^t(\mathcal{F}) \to \mathbb{Z}_p$ is given by*

$$\big[b\big|\{u_1, \ldots, u_N\}\big) = Tr_{W(k)/\mathbb{Z}_p} \circ Res(b\, d_{\log} Col\{u_1, \ldots, u_N\}) \ \in \mathbb{Z}_p.$$

Proof We need to prove that for each M,

$$\mathrm{Tr} \circ \mathrm{Res}\left(b.d_{\log} Col\{u_1, \ldots, u_N\}\right) \mod p^M = [b \mod p^M, \{u_1, \ldots, u_N\})_M$$

is Parshin's formula. Since $[-,-)_M$ is independent of the choice of lifts $\widetilde{u}_i \in \mathcal{O}(\mathcal{F})$ of $u_i \in \mathcal{F}$, we may assume that the lifts are chosen such that $\{\widehat{u}_1, \ldots, \widehat{u}_N\} = Col(\{u_1, \ldots, u_N\})$. Then the identity $\sigma \,\mathrm{Res}\,(x \, d_{\log} Col(u)) = \mathrm{Res}\,(\sigma(x) \, d_{\log} Col(u))$ implies

$$\begin{aligned}
\left[b \mod p^M, \{u_1, \ldots, u_N\}\right)_M &= \mathrm{Tr} \circ \mathrm{Res}\,(\sigma^{M-1}(b) \, d_{\log} Col\{u_1, \ldots, u_N\}) \mod p^M \\
&= \mathrm{Tr} \circ \sigma^{M-1} \circ \mathrm{Res}\,(b \, d_{\log} Col\{u_1, \ldots, u_N\}) \mod p^M \\
&= \mathrm{Tr} \circ \mathrm{Res}\,(b \, d_{\log} Col\{u_1, \ldots, u_N\}) \mod p^M,
\end{aligned}$$

since $\mathrm{Tr} \circ \sigma = \mathrm{Tr} : W(k) \to \mathbb{Z}_p$. $\square$

Chapter 5

The Hilbert Pairing

In this chapter we use the field of norms functor to derive formulae for the Hilbert symbol in characteristic zero from the invariant formula of Parshin's pairing in characteristic p.

5.1 Relating Kummer and Witt extensions

Consider an SDR tower $F_\bullet$ with parameters $(0, c)$, $F_\infty = \varinjlim_n F_n$ and associated field of norms $\mathcal{F}$.

Definition 5.1 *An SDR $F_\bullet$ tower is called m-*admissible*, for $m \in \mathbb{N}$, if $F_\bullet$ has parameters $(0, c)$ with $c \geqslant \frac{2e_F}{p^m(p-1)} = \frac{2e_{F_m}}{p-1}$ and if F_m contains some primitive p^{M+m}-th root of unity ζ_{M+m}. Here $e_F = v_F(p)$ is the (first) absolute ramification index of F.*

Following [3], define an N-dimensional analogue of Fontaine's ring R as follows. Let $\mathbb{C}(N)_p$ be the completion of an algebraic closure of $\mathbb{Q}_p\{\{\pi_N\}\}\cdots\{\{\pi_2\}\}$ and let $\mathcal{O}_{\mathbb{C}(N)_p}$ be the integral closure of its first valuation ring in $\mathbb{C}(N)_p$. Then $R(N) = \varprojlim \mathcal{O}_{\mathbb{C}(N)_p}/\mathfrak{p}_c$, where the projective limit is taken with respect to p-th powers and $\mathfrak{p}_c = \{x \in \mathbb{C}(N)_p \,|\, v_p(x) \geqslant c\}$ for $c > 0$. As sets, one has $R(N) \cong \varprojlim \mathcal{O}_{\mathbb{C}(N)_p}$ given by $(x^{(n)})_n \mapsto (\widetilde{x}^{(n)})_n$ with $\widetilde{x}^{(n)} = \lim_{m\to\infty}(\widehat{x}^{(n+m)})^{p^m}$ for any lift $\widehat{x}^{(n+m)}$ of $x^{(n+m)}$ to $\mathcal{O}_{\mathbb{C}(N)_p}$. $R(N)$ is a perfect ring of characteristic p with valuation $v_R(x) = v_p(\widetilde{x}^{(0)})$.

Its field of fractions is denoted $R(N)_0$. Let $W(R(N))$ be the ring of Witt vectors of $R(N)$, and define

$$\eta : W(R(N)) \longrightarrow \mathcal{O}_{\mathbb{C}(N)_p}, \quad \sum p^i[x_i] \mapsto \sum p^i \widetilde{x}_i^{(0)},$$

for $\widetilde{x}_i^{(0)} \in \mathcal{O}_{\mathbb{C}(N)_p}$ as before.

To see that η is a ring homomorphism, consider $a, b \in R(N)$. Then $[a] + [b] = [c_0] + p[c_1^{\sigma^{-1}}] + \cdots + p^n[c_n^{\sigma^{-n}}] + \cdots$ for some $c_i \in R(N)$. If $S_i(X_0, \ldots, X_i; Y_0, \ldots, Y_i)$ are the polynomials defining addition of Witt-vectors, we have, for each M and i,

$$(c_i^{\sigma^{-i}})^{(0)} \equiv S_i(a^{(0)}, 0, \ldots; b^{(0)}, 0, \ldots)^{p^{M-i}} \mod p^{M-i}$$

since $(\sigma^{-i}a)^{(0)} = a^{(i)}$ for $a \in R(N)$. Using this, $c_0^{(0)} + \cdots + p^M(c_M^{\sigma^{-M}})^{(0)} \equiv a^{(0)} + b^{(0)}$ mod p^{M+1} by the definition of addition in $W_{M+1}(R(N))$. The claim follows since this holds for all any M.

As in [18], let $\varepsilon \in R(N)$ be such that $\varepsilon^{(0)} = 1$ and $\varepsilon^{(1)} = \zeta_p \neq 1$. Then $\ker(\eta) = sW(R(N))$ is the principal ideal generated by $s = \frac{[\varepsilon]-1}{[\varepsilon^{\sigma^{-1}}]-1}$.

If $e_F = v_F(p)$ is the first ramification index of F, $v_p(x) = v_F(x)/e_F$ for every $x \in F$. This shows the inclusion $F_n \subset \mathbb{C}(N)_p$ induces $\mathcal{O}_{F_n}/\mathfrak{p}_c \subset \mathcal{O}_{\mathbb{C}(N)_p}/\mathfrak{p}_{c/e_F}$ and thus $\mathcal{O}_{\mathcal{F}} \subset R(N)$

Let $\mathcal{O}(\mathcal{F}) = W(k)\{\{t_N\}\} \cdots \{\{t_1\}\}$ be the flat $\mathbb{Z}_p$-lift constructed using as $\mathbb{Z}_p$-basis the local parameters $\bar{t}_1, \ldots, \bar{t}_N$ of $\mathcal{F}$ with $\bar{t}_i = (\pi_i^{(n)})_n$ for $\pi_i^{(n)} \in F_n$. Any $x \in \mathcal{O}(\mathcal{F})$ can be written as a convergent sum

$$x = \sum_{(a_0, \ldots, a_N)} [\alpha_{\underline{a}}]\, p^{a_0} \bar{t}_1^{a_1} \cdots \bar{t}_N^{a_N},$$

for $(a_0, \ldots, a_N) \in \mathbb{Z}^{N+1}$ subject to the conditions $a_0 \geqslant 0, a_1 \geqslant I_1(a_0), \ldots, a_N \geqslant I_N(a_0, \ldots, a_{N-1})$ for some $I_1, \ldots, I_N$. Let $A \subset \mathcal{O}(\mathcal{F})$ be the $W(k)$-subalgebra

$$A = \big\{ x \in \mathcal{O}(\mathcal{F}) \;\big|\; (I_1(a_0), \ldots, I_N(a_0, \ldots, a_{N-1})) \geqslant (0, \ldots, 0) \big\}$$

of t-integral elements and let $\mathfrak{m}_A$ be the prime ideal of all $x \in A$ with $(a_1, \ldots, a_N) > (0, \ldots, 0)$. Taking as p-basis of the absolute valuation ring $O_{\mathcal{F}}$ the same set of local parameters $\bar{t}_1, \ldots, \bar{t}_N$ and letting $t_i = [\bar{t}_i]$ be their Teichmüller representatives in

$W_M(O_{\mathcal{F}})$, it can be seen that A is the flat $\mathbb{Z}_p$-lift of $O(\mathcal{F})$. The absolute Frobenius σ on $\mathcal{F}$ induces $\sigma : A \to A$.

Denote the restriction of η to A again by $\eta : A \to \widehat{F}_\infty$, where $F_\infty = \varinjlim F_n$ and $\widehat{F}_\infty$ is its p-adic completion. By construction, η is the identity on $W(k) \subset A$, and $t_i \mapsto \lim_{m\to\infty}(\pi_i^{(m)})^{p^m}$.

In order to translate between (additive) Witt-theory and (multiplicative) Kummer theory, we let $e : \mathfrak{m}_A \to 1 + \mathfrak{m}_A$ be the map induced by the Artin-Hasse Shafarevich exponential, $e(f) = \exp\big(\sum \frac{\sigma^n}{p^n}(f)\big)$. It is a group isomorphism with inverse $l : 1 + \mathfrak{m}_A \to \mathfrak{m}_A$ given by $l(u) = \frac{1}{p}\log\big(\frac{u^p}{\sigma u}\big)$. Denote by θ the composite group homomorphism

$$\theta = \eta \circ e : \mathfrak{m}_A \to \widehat{F}^*_\infty.$$

Suppose now that $F_\bullet$ is m-admissible and fix a primitive p^{M+m}-th root of unity $\zeta_{M+m} \in F_m$. Consider the identification $\mathcal{O}_{\mathcal{F}}/\mathfrak{p}_{cp^m,\mathcal{F}} = \mathcal{O}_{F_m}/\mathfrak{p}_c$ (the valuation on F_m being the induced valuation from F) from the definition of the field of norms, and let $H'_{M+m} \in \mathcal{O}_{\mathcal{F}}$ be such that

$$H'_{M+m} \mod \mathfrak{p}_{cp^m,\mathcal{F}} = \zeta_{M+m} \mod \mathfrak{p}_c.$$

For any lift $H_{M+m} \in A$ of H'_{M+m}, i.e. $H_{M+m} \mod p = H'_{M+m}$, set $H = H_{M+m}^{p^{M+m}} - 1$. For $f \in \mathfrak{m}_A$, pick $T \in W(\mathcal{F}^{sep})$ such that $\wp(T) = \frac{f}{H} \in \mathcal{O}(\mathcal{F})$. For $\gamma \in \Gamma_{\mathcal{F}}$, define $a_\gamma(f) \in \mathbb{Z}_p$ by $a_\gamma(f) = \gamma(T) - T$.

On the level of Kummer theory, the canonical isomorphism $\Gamma_{\mathcal{F}} \cong \Gamma_{F_\infty}$ means we may view γ as element of Γ_F. For $x \in \widehat{F}_\infty$, pick $\xi \in (\widehat{F}_\infty)^{sep}$ such that $\xi^{p^M} = x$, and define $b_\gamma(x) \in \mathbb{Z}/p^M$ by $\frac{\gamma(\xi)}{\xi} = \zeta_{M+m}^{p^m b_\gamma(x)}$.

They are related by the following result (see [1])

Lemma 5.2 (Main Lemma) *For $\gamma \in \Gamma_{\mathcal{F}}^{ab}$ and $f \in \mathfrak{m}_A$,*

$$a_\gamma(f) \equiv b_\gamma\big(\theta(f)\big) \mod p^M.$$

The proof in [1] deals with the case of very special towers which are 0-admissible and have $c = e_F$. The first step of the proof needs to be modified for this context.

Since $F_\bullet$ is m-admissible, c satisfies $cp^m \geqslant \frac{2e_F}{p-1}$. Then $H'_{M+m} \bmod \mathfrak{p}_{cp^m,\mathcal{F}} = \zeta_{M+m} \bmod \mathfrak{p}_{c,F}$ implies

$$H'_{M+m} \equiv \sigma^{1-M-m}(\varepsilon) \mod \mathfrak{p}_{2/(p-1),R},$$

since v_R is defined using the valuation v_p on $\mathbb{C}(N)_p$. For $\varepsilon^{\sigma^{-1}} \in R$ we have

$$v_R(\varepsilon^{\sigma^{-1}}-1) = v_p\big[(\varepsilon^{\sigma^{-1}}-1)^{(0)}\big] = v_p\big[\lim_{n\to\infty}(\zeta_{p^n}-1)^{p^{n-1}}\big] = v_p(\zeta_p-1) = \tfrac{1}{p-1}.$$

Thus $H'_{M+m} \equiv \sigma^{-M-m}(\varepsilon) \mod (\varepsilon^{\sigma^{-1}}-1)^2R$. Applying σ to both sides, we obtain $\sigma(H'_{M+m}) \equiv \sigma^{1-M-m}(\varepsilon) \mod (\varepsilon-1)^2R$.

On the level of lifts, $H_{M+m} \in A$ satisfies $\sigma H_{M+m} \equiv H^p_{M+m} \mod p$. Combining this with the previous congruence, we see that there exist $w_1 \in W(R(N))$ and $w'_1 \in W(R(N)_0)$ such that

$$H^p_{M+m} = \sigma^{1-M-m}[\varepsilon] + ([\varepsilon]-1)^2 w_1 + p w'_1.$$

Taking p^{M+m-1}-th powers, it follows that

$$H = H^{p^{M+m}}_{M+m} - 1 = [\varepsilon] - 1 + ([\varepsilon]-1)^2 w_2 + p^{M+m} w'_2$$

for some $w_2 \in W(R(N))$ and $w'_2 \in W(R(N)_0)$. Finally, diving through by $H([\varepsilon]-1)$, we obtain

$$\frac{1}{H} \equiv \Big(\frac{1}{[\varepsilon]-1} + w\Big) \mod p^M W(R(N)_0)$$

for some $w \in W(R(N))$.

Now let $T' \in W(R(N)_0)$ be such that $\wp(T') = \frac{f}{[\varepsilon]-1}$ and set $a'_\gamma(f) = \gamma(T') - T'$ for $\gamma \in \Gamma_{\mathcal{F}}$. Since $\lim_{m\to\infty} \sigma^m(fw) = 0$, we have $a_\gamma(f) \equiv a'_\gamma(f) \mod p^M$.

We outline the approach taken in [1] to complete the proof, which generalises easily to higher dimensions. The ultimate aim is to translate the additive Witt equation to a multiplicative Kummer extension. This is achieved by first constructing a solution of a Witt-equation in the ideal $sW(R(N)) \subset W(R(N))$.

Set $T_1 = T'([\varepsilon]^{\sigma^{-1}}-1)$, then $\sigma(T_1) - sT_1 = f$. Modulo p, this becomes $T_1^p - sT_1 \equiv f \mod pW(R(N)_0)$ which is monic. It follows from $s, f \in R(N)$ and induction that $T_1 \in W(R(N))$. Thus $X = T'([\varepsilon]-1) = sT_1 \in W^1(R(N)) = sW(R(N))$, and X is a solution of $\frac{\sigma X}{\sigma s} - X = f$ in $W^1(R(N))$.

After 'going up' to an N-dimensional analogue of Fontaine's ring A_{cris}, one can make use of the property

$$\sigma s = p s_1, \qquad \text{for } s_1 \equiv 1 \mod \left([\varepsilon]-1, \tfrac{([\varepsilon]-1)^{p-1}}{p}\right)$$

in A_{cris} to conclude $\frac{\sigma X}{p} - X \equiv f \mod S$ where $S \subset A_{\mathrm{cris}}$ is an ideal on which $\frac{\sigma}{p}$ is topologically nilpotent. This means that there exists an exact solution m with $\sigma(m) - pm = pf$, $X \equiv m \mod S$.

One then puts $Y = \exp(m)$ to obtain $\sigma(Y)\, Y^{-1} = \exp(pf)$ and proves that such $Y \in A_{\mathrm{cris}}$ correspond bijectively to solutions $Y \in 1 + sW(R(N))$. Finally an explicit description of $\gamma(m) - m$ is used to show that the element $u = \eta\big(\sigma^{-M}(Y\, e(f))\big) \in \mathcal{O}_{\mathbb{C}(N)_p}$ satisfies $u^{p^M} = \theta(f)$ and $\frac{\gamma(u)}{u} = \zeta^{a''_\gamma}$.

5.2 The Generalised Hilbert Symbol

In this section we define a generalised Hilbert symbol and use the 'main lemma' to deduce a formula from the invariant formula for Parshin's pairing.

Definition 5.3 *Let $F_\bullet$ be an SDR tower with associated field of norms $\mathcal{F}$. If $F_\infty \ni \zeta_M$ for some primitive p^M-th root of unity ζ_M, define the generalised Hilbert symbol to be*

$$(-,-)_M^{F_\bullet} \;:\; \widehat{F}_\infty^* \times K_N(\mathcal{F})/p^M \longrightarrow \mu_{p^M}, \qquad \big(u,b\big)_M^{F_\bullet} = \frac{\gamma(U)}{U},$$

where $U \in (\widehat{F}_\infty)^{sep}$ satisfies $U^{p^M} = u$ and $\gamma = \Psi_{\mathcal{F}}(b) \in \Gamma_{\mathcal{F}}^{ab}$ is viewed as an element of $\Gamma_{F_\infty}^{ab}$ via the identification given by the field of norms functor.

Using the projection $\mathcal{N}_{\mathcal{F}/F} : K_N(\mathcal{F})/p^M \to K_N(F)/p^M$ from section 3.4, we give a partial description of this pairing.

Theorem 5.4 *Suppose $F_\bullet$ is an m-admissible SDR tower. For $f \in \mathfrak{m}_A$ and $\beta \in K_N^t(\mathcal{F})$, the generalised Hilbert symbol is given by*

$$\big(\theta(f), \mathcal{N}_{\mathcal{F}/F}(\beta)\big)_M^{F_\bullet} = \zeta_{M+m}^{p^m\phi}, \qquad \phi = \mathrm{Tr} \circ \mathrm{Res}\left(\frac{f}{H} d_{\log} Col(\beta)\right).$$

Proof Let $\gamma = \Psi_{\mathcal{F}}(\beta)$ for $\beta \in K_N^t(\mathcal{F})$, and $a_\gamma(f) = \gamma(T) - T$ for $\wp(T) = \frac{f}{H}$. By thm.4.9,

$$a_\gamma(f) = \Big[\frac{f}{H}, \beta\Big) = \mathrm{Tr} \circ \mathrm{Res}\Big(\frac{f}{H}\, d_{\log} Col(\beta)\Big).$$

On the other hand, the compatibility of class field theory and the field of norms for arbitrary towers shows that under the identification

$$\Gamma^{ab}_{\mathcal{F}}/p^M \cong \Gamma^{ab}_{F_\infty}/p^M \subset \Gamma^{ab}_F/p^M,$$

$\gamma = \Psi_{\mathcal{F}}(\beta)$ is identified with $\Psi_F(\mathcal{N}_{\mathcal{F}/F}(\beta)) \in K_N(F)/p^M$. By the main lemma, $b_\gamma(\theta(f)) = a_\gamma(f)$ and the formula follows. □

We indicate how this formula can be obtained from the case of 0-admissible SDR towers. Let $F'_\bullet$ be the 0-admissible SDR tower defined by $F'_n = F_{n+m}$. Then $F'_\bullet \sim F_\bullet$ as towers (see [35]) and the identification $\mathcal{F}' \cong \mathcal{F}$ is given by taking p^m-th powers, as can be seen from

$$\begin{array}{ccccccc}
\mathcal{O}_{\mathcal{F}} = \varprojlim_n \mathcal{O}_{F_n}/\mathfrak{p}_c & \longrightarrow & \varprojlim \mathcal{O}_{\mathbb{C}(N)_p}/\mathfrak{p}_c & \xrightarrow{\sim} & \varprojlim \mathcal{O}_{\mathbb{C}(N)_p} & \longrightarrow & \mathcal{O}_{\mathbb{C}(N)_p} \\
\uparrow & & \uparrow & & \uparrow & & \\
\mathcal{O}_{\mathcal{F}'} = \varprojlim_n \mathcal{O}_{F_{m+n}}/\mathfrak{p}_c & \longrightarrow & \varprojlim \mathcal{O}_{\mathbb{C}(N)_p}/\mathfrak{p}_c & \xrightarrow{\sim} & \varprojlim \mathcal{O}_{\mathbb{C}(N)_p} & \longrightarrow & \mathcal{O}_{\mathbb{C}(N)_p}.
\end{array}$$

An element $(x^{(n)})_n \in \mathcal{O}_{\mathcal{F}}$ is mapped, along the top row, to $\widetilde{x}^{(0)} = \lim_{i\to\infty}(x^{(i)})^{p^i}$. Similarly, $(x'^{(n)}) \in \mathcal{O}_{\mathcal{L}}$ is mapped to $\widetilde{x}'^{(0)} = \lim_{i\to\infty}(x'^{(i)})^{p^i}$. But $F'_n = F_{m+n}$, so $x'^{(i)} = x^{(m+i)} = (x^{(m+i)})^{p^m} \in \mathcal{O}_{\mathbb{C}(N)_p}/\mathfrak{p}_c$ and therefore $(\widetilde{x}'^{(0)})^{p^m} = \widetilde{x}^{(0)}$.

Let $\theta : \mathfrak{m}_A \to \widehat{F}^*_\infty$ be the map corresponding to the tower $F_\bullet$ and $\theta' : \mathfrak{m}_A \to \widehat{F}^*_\infty$ the one corresponding to $F'_\bullet$. Then θ is defined by $[(x^{(n)})_n] \mapsto \widetilde{x}^{(0)}$, and therefore, using the identification $\mathcal{F}' \cong \mathcal{F}$, we obtain $\theta(f) = \theta'(f)^{p^m}$ for any $f \in \mathfrak{m}_A$.

Using the commutative diagram,

$$\begin{array}{ccccc}
K_N(\mathcal{F})/p^M & \xrightarrow{\mathcal{N}_{\mathcal{F}/F_m}} & K_N(F_m)/p^M & \xrightarrow{N_{F_m/F}} & K_N(F)/p^M \\
\downarrow{\scriptstyle \Psi_{\mathcal{F}}} & & \downarrow{\scriptstyle \Psi_{F_m}} & & \downarrow{\scriptstyle \Psi_F} \\
\Gamma^{ab}_{\mathcal{F}}/p^M & \longrightarrow & \Gamma^{ab}_{F_m}/p^M & \longrightarrow & \Gamma^{ab}_F/p^M,
\end{array}$$

it follows that we need to identify $\gamma' = \Psi_{F_m}(\mathcal{N}_{\mathcal{F}'/F_m}(\beta))$ with $\gamma = \Psi_F(\mathcal{N}_{\mathcal{F}/F}(\beta))$ for any $\beta \in K_N(\mathcal{F})/p^M \cong K_N(\mathcal{F}')/p^M$. By the previous theorem for $F'_\bullet$ and $M+m$,

$$\zeta_{M+m}^{\mathrm{Tr} \circ \mathrm{Res}\, \phi} = \big(\theta'(f), \mathcal{N}_{\mathcal{F}'/F'}(\beta)\big)^{F'_\bullet}_{M+m} = \frac{\gamma'(U)}{U},$$

with $U^{p^{M+m}} = \theta'(f)$. This shows that

$$\big(\theta_F(f), \mathcal{N}_{\mathcal{F}/F}(\beta)\big)_M^{F_\bullet} = \frac{\gamma_F(U^{p^m})}{U^{p^m}} = \zeta_{M+m}^{p^m \mathrm{Tr} \circ \mathrm{Res}\, \phi},$$

and the formula for $F_\bullet$ follows from $(U^{p^m})^{p^M} = (\theta'(f))^{p^{m+M}} = (\theta(f))^{p^M}$ and the formula for $F'_\bullet$

As an application of this, we give a formula for the classical Hilbert symbol. Suppose $F \ni \zeta_M$. Let $\pi_1, \dots, \pi_N$ be a system of local parameters of F and set $F_n = F(\sqrt[p^n]{\pi_1}, \dots, \sqrt[p^n]{\pi_N})$. Then the tower $F_\bullet$ is very special SDR, with field of norms $\mathcal{F} = k((\bar{t}_N))\cdots((\bar{t}_1))$ for $\bar{t}_i = (\pi_i^{(n)})_n \in \varprojlim \mathcal{O}_{F_n}/\mathfrak{p}_c$. For this very special tower, $\eta : A \to \widehat{F}_\infty$ takes values in F. Since η is defined on Teichmüller representatives, this follows from $\eta(t_i) = \lim_{m\to\infty}(\pi_i^{(m)})^{p^m} = \pi_i^{(0)} \in F$ for each i.

Let $\mathcal{R} \subset \mathcal{O}(\mathcal{F})^*$ be the subgroup

$$\mathcal{R} = \langle t_1 \rangle \times \cdots \times \langle t_N \rangle \times k^* \times (1 + \mathfrak{m}_A),$$

where k^* is identified with the groups of its Teichmüller representatives. Note that $\eta(\mathcal{R}) = F^*$ is all of F^*.

The classical Hilbert symbol h is defined by

$$F^*/(F^*)^{p^M} \times K_N(F)/p^M \longrightarrow \mu_{p^M}, \quad (u_0, \{u_1, \dots, u_N\})_M = \zeta_M^{h(u_0,\dots,u_N)},$$

for $h(u_0, \dots, u_N) \in \mathbb{Z}/p^M$ and some fixed primitive p^M-th root of unity ζ_M.

Then we have

Corollary 5.5 *If $u_0 \in V_F$ and $\{u_1, \dots, u_N\} \in \mathrm{Im}\big(\mathcal{N}_{\mathcal{F}/F} : K_N^t(\mathcal{F}) \to K_N^t(F)\big)$, then the classical Hilbert symbol is given by*

$$h(u_0, \dots, u_N) = \mathrm{Tr} \circ \mathrm{Res}\left(\frac{l(\widehat{u}_0)}{H} d_{\log}\widehat{u}_1 \wedge \cdots \wedge d_{\log}\widehat{u}_N\right),$$

for some $\widehat{u}_i \in \mathcal{R}$ with $\eta(\widehat{u}_i) = u_i$.

Proof For $u_0 \in V_F$, pick any lift $\widehat{u}_0 \in 1 + \mathfrak{m}_A$. By the explicit description of *Col* and $\mathcal{N}_{\mathcal{F}/F}$, the composite

$$K_N^t(\mathcal{O}(\mathcal{F})) \supset Col(K_N(\mathcal{F})) \xrightarrow{\sim} K_N^t(\mathcal{F}) \xrightarrow{\mathcal{N}_{\mathcal{F}/F}} K_N(F)$$

is induced by $\widetilde{t_i} \mapsto \pi_i \in F$ for $1 \leqslant i \leqslant N$. So we may pick $\widehat{u}_i \in \mathcal{R}$ such that $\{\widehat{u}_1, \ldots, \widehat{u}_N\} = Col(\{g_1, \ldots, g_N\})$ for $g = \{g_1, \ldots, g_N\} \in K_N^t(\mathcal{F})$ with

$$\mathcal{N}_{\mathcal{F}/F}\{g_1, \ldots, g_N\} = \{u_1, \ldots, u_N\}.$$

Then $d_{\log}Col(g) = d_{\log}\widehat{u}_1 \wedge \cdots \wedge d_{\log}\widehat{u}_N$, as required. □

5.3 Vostokov's Symbol

We start by defining a multilinear form $\widehat{V} : (Q_0(\mathcal{F})^*)^{N+1} \to \mathbb{Z}/p^M$, for $Q_0(\mathcal{F}) = W(k)((t_N))\cdots((t_1))$ as before, by

$$\widehat{V}(\widehat{u}_0, \ldots, \widehat{u}_N) = \mathrm{Tr} \circ \mathrm{Res}\left(\sum_{0 \leqslant i \leqslant N} \Phi_i\right)$$

$$\Phi_i = \frac{(-1)^i}{H} l(\widehat{u}_i) \tfrac{\sigma}{p} d_{\log}\widehat{u}_1 \wedge \cdots \wedge \tfrac{\sigma}{p} d_{\log}\widehat{u}_{i-1} \wedge d_{\log}\widehat{u}_{i+1} \wedge \cdots \wedge d_{\log}\widehat{u}_N.$$

Here $H = \widehat{\zeta}_M^{p^M} - 1$. We put $\Phi = \sum_{0 \leqslant i \leqslant N} \Phi_i \in \Omega^N_{Q_0(\mathcal{F})}$.

Remark If $F_\bullet$ is a very special tower, we may assume that it has parameters $(0, e_F)$. Then for $H'_M \in \mathcal{F}$ with $H'_M \bmod p\mathcal{O}_{\mathcal{F}} \equiv \zeta_M \bmod p\mathcal{O}_F$ and $H_M \in A$ a lift of H'_M, we see that $H_M^{p^M} - 1 \equiv \widehat{\zeta}_M^{p^M} - 1 \mod p^M$, so in this case the two constructions of H coincide.

Proposition 5.6 *$\widehat{V}$ is skew-symmetric.*

Proof To prove $V(\widehat{u}_0, \ldots, \widehat{u}_i, \ldots, \widehat{u}_j, \ldots, \widehat{u}_N) = -V(\widehat{u}_0, \ldots, \widehat{u}_j, \ldots, \widehat{u}_i, \ldots, \widehat{u}_N)$, we may assume that $j = i + 1$. Since $\wedge$ is skew-symmetric, all but two terms of $\Phi(\widehat{u}_0, \ldots, \widehat{u}_N)$ cancel and we are left with

$$\begin{aligned}
&(-1)^i\big(\Phi(\ldots, \widehat{u}_i, \widehat{u}_{i+1}, \ldots) + \Phi(\ldots, \widehat{u}_{i+1}, \widehat{u}_i, \ldots)\big) = \\
&= \tfrac{1}{H} l(\widehat{u}_i) \tfrac{\sigma}{p} d_{\log}\widehat{u}_0 \wedge \cdots \wedge \tfrac{\sigma}{p} d_{\log}\widehat{u}_{i-1} \wedge d_{\log}\widehat{u}_{i+1} \wedge \cdots \wedge d_{\log}\widehat{u}_N \\
&-\tfrac{1}{H} l(\widehat{u}_{i+1}) \tfrac{\sigma}{p} d_{\log}\widehat{u}_0 \wedge \cdots \wedge \tfrac{\sigma}{p} d_{\log}\widehat{u}_i \wedge d_{\log}\widehat{u}_{i+2} \wedge \cdots \wedge d_{\log}\widehat{u}_N \\
&+\tfrac{1}{H} l(\widehat{u}_{i+1}) \tfrac{\sigma}{p} d_{\log}\widehat{u}_0 \wedge \cdots \wedge \tfrac{\sigma}{p} d_{\log}\widehat{u}_{i-1} \wedge d_{log}\widehat{u}_i \wedge \cdots \wedge d_{\log}\widehat{u}_N \\
&-\tfrac{1}{H} l(\widehat{u}_i) \tfrac{\sigma}{p} d_{\log}\widehat{u}_0 \wedge \cdots \wedge \tfrac{\sigma}{p} d_{\log}\widehat{u}_{i+1} \wedge d_{\log}\widehat{u}_{i+2} \wedge \cdots \wedge d_{\log}\widehat{u}_N \\
&= \tfrac{\sigma}{p} d_{\log}\widehat{u}_0 \wedge \cdots \wedge \tfrac{\sigma}{p} d_{\log}\widehat{u}_{i-1} \wedge \big[\tfrac{1}{H} l(\widehat{u}_i)(d_{\log}\widehat{u}_{i+1} - \tfrac{\sigma}{p} d_{\log}\widehat{u}_{i+1}) + \\
&+\tfrac{1}{H} l(\widehat{u}_{i+1})(d_{\log}\widehat{u}_i - \tfrac{\sigma}{p} d_{\log}\widehat{u}_i)\big] \wedge d_{\log}\widehat{u}_{i+2} \wedge \cdots \wedge d_{\log}\widehat{u}_N. \qquad (\dagger)
\end{aligned}$$

Note that

$$\begin{aligned} & d\big[l(\widehat{u}_i)l(\widehat{u}_{i+1})\tfrac{1}{H}\big] - l(\widehat{u}_i)l(\widehat{u}_{i+1})d\big(\tfrac{1}{H}\big) \\ = & \big[d_{\log}\widehat{u}_i - \tfrac{\sigma}{p}d_{\log}\widehat{u}_i\big]l(\widehat{u}_{i+1})\tfrac{1}{H} + l(\widehat{u}_i)\big[d_{log}\widehat{u}_{i+1} - \tfrac{\sigma}{p}d_{\log}\widehat{u}_{i+1}\big]\tfrac{1}{H}, \end{aligned}$$

which is the middle term in (†) above. Now $d(H^{-1}) = H^{-2}p^M\zeta^{p^M-1}\,\mathrm{d}(\widehat{\zeta})$, so

$$\operatorname{Res}\big(l(\widehat{u}_i)l(\widehat{u}_j)d(\tfrac{1}{H}) \wedge \tfrac{\sigma}{p}d_{\log}\widehat{u}_0 \wedge \cdots \wedge d_{\log}\widehat{u}_N\big) \equiv 0 \mod p^M A,$$

hence $\widehat{V}$ is skew-symmetric. □

Let $\underline{e} = \underline{v}_F(p) \in \mathbb{Z}^N$ be the absolute ramification index. In analogy with [1], define the rings

$$\mathcal{A}^0 = A\big[\big[\tfrac{p}{\underline{t}^{\underline{e}(p-1)}}, \tfrac{\underline{t}^{\underline{e}p}}{p}\big]\big], \qquad \text{and} \qquad \mathcal{A} = \mathcal{A}^0 \otimes Q_0(\mathcal{F}),$$

so $\mathcal{A} = \varinjlim_{\underline{a}>\underline{0}} \underline{t}^{-\underline{a}}\mathcal{A}^0$. Elements of $a \in \mathcal{A}$ may be viewed as formal Laurent power series $a = w_{\underline{a}}\underline{t}^{\underline{a}}$, for $\underline{a} \in \mathbb{Z}^N$ and $w_{\underline{a}} \in W(k)$ and $a \in \mathcal{A}^0$ if and only if for every $n \geqslant 0$, $v_p(w_{\underline{a}}) \geqslant -n$ whenever $\underline{a} \geqslant \underline{e}pn$, and $v_p(w_{\underline{b}}) \geqslant n$ whenever $b \geqslant -\underline{e}p(n-1)$.

Using this expansion, we define the residue Res ω of any $\omega \in \Omega^N_{\mathcal{A}}$ to be the coefficient $w_{\underline{0}}$ of $\underline{t}^{\underline{0}}$ if $\omega = \sum w_{\underline{a}}\underline{t}^{\underline{a}} d_{\log}t_1 \wedge \cdots \wedge d_{\log}t_N$.

Finally let $\mathcal{A}^{-1} \subset \mathcal{A}$ be the subalgebra

$$\mathcal{A}^{-1} = \Big\{ x = \sum w_{\underline{a}}\underline{t}^{\underline{a}} \;\Big|\; \sigma(x) = \sum w_{\underline{a}}^{\sigma}\underline{t}^{p\underline{a}} \in \mathcal{A}^0 \Big\}.$$

Notice that $\mathcal{A}^{-1} \supset A\big[\big[\frac{\underline{t}^{\underline{e}}}{p}\big]\big]$ and σ defines a morphism $\mathcal{A}^{-1} \to \mathcal{A}^0$.

Lemma 5.7 *Let $\lambda \in O^*_{\mathcal{F}}$ be such that $p = \lambda\pi_1^{e_1}\cdots\pi_N^{e_N}$, and let $\widehat{\lambda} \in A$ be such that $\eta(\widehat{\lambda}) = \lambda$. Then the kernel of $\eta : A \to O_F$ is generated by $p - \widehat{\lambda}\underline{t}^{\underline{e}}$.*

Proof By construction, $\eta(\widehat{\lambda}\underline{t}^{\underline{e}} - p) = 0$. Suppose now that $x = \sum[\alpha_{a_0,\underline{a}}]p^{a_0}\underline{t}^{\underline{a}} \in \ker(\eta)$. Since $A/(\widehat{\lambda}\underline{t}^{\underline{e}} - p, p) = A/(p, \underline{t}^{\underline{e}})$, η induces $A/(p, \underline{t}^{\underline{e}}) \cong O_F/p$. Thus for $x \in \ker(\eta)$, we conclude that $[\alpha_{0,\underline{a}}] = 0$ if $\underline{a} < \underline{e}$. For $y_1 = \sum[\alpha_{a_0,\underline{a}}]\big(\widehat{\lambda}^{-1}p - \underline{t}^{\underline{e}}\big)\underline{t}^{\underline{a}-\underline{e}}$, where the sum is over $a_0 \geqslant 0$ and $\underline{a} > \underline{e}$, set $x_1' = x - y_1$. Then $x_1' \in pA$, so $x_1' = px_1$ for some $x_1 \in A$ and $x_1 \in \ker(\eta)$ by construction. Iterating this argument, we obtain elements $y_n \in (p - \widehat{\lambda}\underline{t}^{\underline{e}})A$ and $x_n \in \ker(\eta)$ such that $x = y_1 + px_1 =$

$y_1 + p(y_2 + px_2) = \cdots = y_1 + py_2 + \cdots + p^{n-1}y_n + p^n x_n$ for each n. Since A is a p-adic ring, $y_1 + \cdots + p^{n-1}y_n + \cdots$ converges, hence $\ker(\eta) = (p - \lambda \underline{t}^{\underline{e}})$. $\square$

We state a few estimates that will be needed below.

Lemma 5.8 *For a lift $\widehat{\zeta} \in \mathcal{R}$ of $\zeta \in F$, the element $H = \widehat{\zeta}^{p^M} - 1$ satisfies*

(a) $H = a_1 \underline{t}^{\underline{e}p/(p-1)} + pa_2 \underline{t}^{\underline{e}/(p-1)}$ *for* $a_1 \in A^*$, $a_2 \in A$.

(b) $\dfrac{1}{H} = a_1^{-1} \underline{t}^{-\underline{e}p/(p-1)} \Big(1 + a_4 \dfrac{p}{\underline{t}^{\underline{e}}}\Big)$ *for* $a_4 \in A\big[\big[\frac{p}{\underline{t}^{\underline{e}}}\big]\big] \subset \mathcal{A}$,

(c) $\frac{1}{p} H^{p-1} = a_3 \frac{\underline{t}^{\underline{e}p}}{p} + a_4 \in A\big[\big[\frac{\underline{t}^{\underline{e}p}}{p}\big]\big]$ *for* $a_3 \in A^*$ *and* $a_4 \in \mathfrak{m}_A$.

(d) $H = w \underline{t}^{\underline{e}/(p-1)} \big(\lambda \underline{t}^{\underline{e}} - p\big)$ *for* $w \in A^*$

Proof In F, $\zeta^{p^{M-1}} - 1 = \zeta_p - 1 = v \underline{\pi}^{\underline{e}/(p-1)}$ for some unit v. Thus $\widehat{\zeta}^{p^{M-1}} = 1 + \widehat{v} \underline{t}^{\underline{e}/(p-1)} + a(p - \widehat{\lambda} \underline{t}^{\underline{e}}) = 1 + \widehat{v}' \underline{t}^{\underline{e}/(p-1)}$ for $\widehat{v}, \widehat{v}' \in A$. Thus

$$\begin{aligned} H &= \big(1 + \widehat{v}' \underline{t}^{\underline{e}/(p-1)}\big)^p - 1 = \widehat{v}'^p \underline{t}^{\underline{e}p/(p-1)} + p\widehat{v}'^{p-1} \underline{t}^{\underline{e}} + \cdots + p\widehat{v}' \underline{t}^{\underline{e}/(p-1)} \\ &= a_1 \underline{t}^{\underline{e}p/(p-1)} + pa_2 \underline{t}^{\underline{e}/(p-1)} = \widehat{a}_1 \underline{t}^{\underline{e}p/(p-1)} \Big(1 + a_1^{-1} a_2 \frac{p}{\underline{t}^{\underline{e}}}\Big). \end{aligned}$$

(a) and (b) follow. For (c), one obtains

$$\frac{1}{p} H^{p-1} = \frac{1}{p} \Big[(\widehat{v} t^{e/(p-1)} + 1)^p - 1\Big]^{p-1} = \frac{1}{p} \Big[\widehat{v}^p \underline{t}^{\underline{e}p/(p-1)} + p\widehat{v}^{p-1} \underline{t}^{\underline{e}} + \cdots + p\widehat{v} \underline{t}^{\underline{e}/(p-1)}\Big]^{p-1}.$$

To verify (d), (a) implies that $H = a_1' \underline{t}^{\underline{e}/(p-1)} \big(\widehat{\lambda} \underline{t}^{\underline{e}} + a_2' p\big)$ with $a_1' \in A^*$ and $a_2' \in A$ Using $\eta(H) = 0$, we see that $\eta(a_2') = -1$, so $a_2' = -1 + a_2''(\lambda \underline{t}^{\underline{e}} - p)$ for $a_2'' \in A$. Therefore $H = a_1' \underline{t}^{\underline{e}/(p-1)} \big(\widehat{\lambda} \underline{t}^{\underline{e}} - p)(1 + a_2'' p)$ is of the required form. $\square$

Proposition 5.9 *If $\eta(\widehat{u}_i) = 1$ for some i, then $\widehat{V}(\widehat{u}_0, \ldots, \widehat{u}_N) \equiv 0 \mod p^M$.*

Proof We may assume that $i = 0$. By the lemma, this implies that $\widehat{u}_0 = a(p - \widehat{\lambda} \underline{t}^{\underline{e}})$ for some $a \in A$, hence $\widehat{u}_0 = 1 + a(p - \widehat{\lambda} \underline{t}^{\underline{e}})$. It follows that

$$\log(\widehat{u}_0),\ \tfrac{\sigma}{p} \log(\widehat{u}_0) \ \in \ A\big[\big[\tfrac{\underline{t}^{\underline{e}p}}{p}\big]\big] \subset \mathcal{A}^0$$

converge.

Let $f_i = l(\widehat{u}_i) = \frac{1}{p}\log\frac{\widehat{u}_i^p}{\sigma\widehat{u}_i}$. Consider the exact differential

$$d\Big(\frac{f_i}{H}\,\tfrac{\sigma}{p}(\log(\widehat{u}_0))\tfrac{\sigma}{p}d_{\log}\widehat{u}_1\wedge\cdots\wedge\tfrac{\sigma}{p}d_{\log}\widehat{u}_{i-1}\wedge d_{\log}d_{log}\widehat{u}_{i+1}\wedge\cdots\wedge d_{\log}\widehat{u}_N\Big)$$
$$=\Bigg[\frac{df_i}{H}\,\tfrac{\sigma}{p}\big(\log(\widehat{u}_0)\big)+\frac{f_i}{H}\,\tfrac{\sigma}{p}d_{\log}\widehat{u}_0+f_i\tfrac{\sigma}{p}\big(\log(\widehat{u}_0)\big)d\Big(\frac{1}{H}\Big)\Bigg]\wedge\ldots \qquad (\star)$$

The second term of $(\star)$ is the i-th term Φ_i of Φ, up to a factor of $(-1)^i$.The following lemma shows that the third term of $(\star)$ has zero residue modulo p^M, thus we may replace the i-th term in Φ with the first term of $(\star)$.

Lemma 5.10 *For* $1\leqslant i\leqslant N$,

$$Res\Big(f\tfrac{\sigma}{p}(\log\widehat{u}_0)d\big(\tfrac{1}{H}\big)\wedge\tfrac{\sigma}{p}d_{\log}\widehat{u}_1\wedge\cdots\wedge d_{\log}\widehat{u}_N\Big)\equiv 0 \mod p^M A,$$

where the $\widehat{u}_i$*-term between* $\frac{\sigma}{p}d_{\log}\widehat{u}_{i-1}$ *and* $d_{\log}\widehat{u}_{i+1}$ *is missing.*

PROOF Note that $d\big(\frac{1}{H}\big)=H^{-2}p^M\widehat{\zeta}^{p^M-1}d(\widehat{\zeta})$. Also, $\frac{\sigma}{p}\log\widehat{u}_0\in A\big[\big[\frac{\underline{t}^{\underline{e}p}}{p}\big]\big]$, $f_i\in A$, and $\frac{1}{H^2}\in\underline{t}^{-2\underline{e}p/(p-1)}A\big[\big[\frac{p}{\underline{t}^{\underline{e}}}\big]\big]$, so

$$f\ \tfrac{\sigma}{p}(\log\widehat{u}_0)\ d\big(\tfrac{1}{H}\big)\ \in\ \tfrac{p^M}{\underline{t}^{-2\underline{e}p/(p-1)}}\ A\big[\big[\tfrac{p}{\underline{t}^{\underline{e}}},\tfrac{\underline{t}^{\underline{e}p}}{p}\big]\big]\ d(\widehat{\zeta}).$$

The residue occurs in a generic term $p^M\underline{t}^{-2\underline{e}p/(p-1)}\frac{p^i}{\underline{t}^{\underline{e}i}}\frac{\underline{t}^{\underline{e}jp}}{p^j}$ with $\frac{2\underline{e}p}{p-1}+\underline{e}i-\underline{e}pj\geqslant(1,\ldots,1)$, but $\frac{2\underline{e}p}{p-1}\leqslant\underline{e}p$, so this implies that the exponent of p is $M+i-j\geqslant M$. $\square$

Let Φ' be obtained from Φ by replacing the i-th term $\Phi_i=(-1)^i\frac{f_i}{H}\frac{\sigma}{p}d_{\log}\widehat{u}_0\wedge\ldots$ with $\frac{(-1)^{i+1}}{H}\frac{\sigma}{p}(\log\widehat{u}_0)\,df_i$ for $1\leqslant i\leqslant N$. By the above argument and the lemma, $\mathrm{Res}\,(\Phi')\equiv\mathrm{Res}\,(\Phi)\mod p^M$. Since $df=d_{\log}\widehat{u}-\frac{\sigma}{p}d_{\log}\widehat{u}$, the i-th term of Φ' is then

$$\Phi'_i=\frac{(-1)^i}{H}\ \tfrac{\sigma}{p}\ \log(\widehat{u}_0)\big(\tfrac{\sigma}{p}d_{\log}\widehat{u}_i-d_{\log}\widehat{u}_i\big)\wedge\tfrac{\sigma}{p}d_{\log}\widehat{u}_1\wedge\cdots\cdots\wedge d_{\log}\widehat{u}_N.$$

Substituting $\frac{1}{H}l(\widehat{u}_0)=\frac{1}{H}(\log(\widehat{u}_0)-\frac{\sigma}{p}\log(\widehat{u}_0))$ in the 0-th term $\Phi_0=\Phi'_0$, we obtain

$$
\begin{aligned}
H\Phi' &= \big(\log \widehat{u}_0 - \tfrac{\sigma}{p}\log \widehat{u}_0\big) d_{\log}\widehat{u}_1 \wedge \cdots \wedge d_{\log}\widehat{u}_N \\
&- \tfrac{\sigma}{p}\log \widehat{u}_0 \big(\tfrac{\sigma}{p} d_{\log}\widehat{u}_1 - d_{\log}\widehat{u}_1\big) \wedge d_{\log}\widehat{u}_2 \wedge \cdots \wedge d_{\log}\widehat{u}_N \\
&+ \tfrac{\sigma}{p}\log \widehat{u}_0 \big(\tfrac{\sigma}{p} d_{\log}\widehat{u}_2 - d_{\log}\widehat{u}_2\big) \wedge \tfrac{\sigma}{p} d_{\log}\widehat{u}_1 \wedge d_{\log}\widehat{u}_3 \wedge \cdots \wedge d_{\log}\widehat{u}_N \\
&\ \ \vdots \\
&+ (-1)^N \tfrac{\sigma}{p}\log \widehat{u}_0 \big(\tfrac{\sigma}{p} d_{\log}\widehat{u}_N - d_{\log}\widehat{u}_N\big) \wedge \tfrac{\sigma}{p} d_{\log}\widehat{u}_1 \wedge \cdots \wedge \tfrac{\sigma}{p} d_{\log}\widehat{u}_{N-1} \\
&= \big(\log \widehat{u}_0 - \tfrac{\sigma}{p}\log \widehat{u}_0\big) d_{\log}\widehat{u}_1 \wedge \cdots \wedge d_{\log}\widehat{u}_N \\
&+ \tfrac{\sigma}{p}\log \widehat{u}_0 \big(d_{\log}\widehat{u}_1 - \tfrac{\sigma}{p} d_{\log}\widehat{u}_1\big) \wedge d_{\log}\widehat{u}_2 \wedge \cdots \wedge d_{\log}\widehat{u}_N \\
&\ \ \vdots \\
&+ \tfrac{\sigma}{p}\log \widehat{u}_0 \tfrac{\sigma}{p} d_{\log}\widehat{u}_1 \wedge \cdots \wedge \tfrac{\sigma}{p} d_{\log}\widehat{u}_{i-1} \wedge \big(d_{\log}\widehat{u}_i - \tfrac{\sigma}{p} d_{\log}\widehat{u}_i\big) \wedge d_{\log}\widehat{u}_{i+1} \wedge \cdots \wedge d_{\log}\widehat{u}_N \\
&\ \ \vdots \\
&+ \tfrac{\sigma}{p}\log \widehat{u}_0 \tfrac{\sigma}{p} d_{\log}\widehat{u}_1 \wedge \cdots \wedge \tfrac{\sigma}{p} d_{\log}\widehat{u}_{N-1} \wedge \big(d_{\log}\widehat{u}_N - \tfrac{\sigma}{p} d_{\log}\widehat{u}_N\big) \\
&= (\log \widehat{u}_0)\, d_{\log}\widehat{u}_1 \wedge \cdots \wedge d_{\log}\widehat{u}_1 - \tfrac{\sigma}{p}(\log \widehat{u}_0)\, \tfrac{\sigma}{p} d_{\log}\widehat{u}_1 \wedge \cdots \wedge \tfrac{\sigma}{p} d_{\log}\widehat{u}_N
\end{aligned}
$$

Notice that if $d_{\log}\widehat{u} = \sum_i a_i d_{\log}t_i$ (for $a_i \in A$), then $\frac{\sigma}{p} d_{\log}\widehat{u} = \sum \sigma(a_i) d_{\log}t_i$. Therefore Φ' is of the form

$$\Phi' = \tfrac{1}{H}\big(\tfrac{\sigma}{p}(\log(\widehat{u}_0))\sigma(x) - \log(\widehat{u}_0)x\big) d_{\log}t_1 \wedge \cdots \wedge d_{\log}t_N,$$

for $x \in A$ and $\log(\widehat{u}_0)$, $\frac{\sigma}{p}\log(\widehat{u}_0) \in A\big[\big[\frac{\underline{t}^{ep}}{p}\big]\big]$. We need the following result, which we shall prove below.

Lemma 5.11 *For any* $y \in A\big[\big[\frac{\underline{t}^{ep}}{p}\big]\big]$,

$$\operatorname{Res}\left(\frac{y}{H}\, d_{\log}t_1 \wedge \cdots \wedge d_{\log}t_N\right) \equiv \operatorname{Res}\left(\frac{y}{\frac{\sigma}{p}H}\, d_{\log}t_1 \wedge \cdots \wedge d_{\log}t_N\right) \mod p^M A.$$

For $y = \frac{\sigma}{p}\log(\widehat{u}_0)\, x$, this shows that

$$\operatorname{Res}(\Phi') = \operatorname{Res}\left(\frac{\sigma(x)\,\frac{\sigma}{p}\log(\widehat{u}_0)}{\frac{\sigma}{p}H} - \frac{x\,\log(\widehat{u}_0)}{H}\right) d_{\log}t_1 \wedge \cdots \wedge d_{\log}t_N.$$

By lemma 5.8 (d), $H = w\underline{t}^{\underline{e}/(p-1)}\big(\lambda\underline{t}^{\underline{e}} - p\big)$ for $w \in A^*$. Also, $\widehat{u}_0 = a(p - \widehat{\lambda}\underline{t}^{\underline{e}})$ for some $a \in A$ since $\eta(\widehat{u}_0) = 1$, and therefore $\log(\widehat{u}_0) = \log(1 + a(p - \widehat{\lambda}\underline{t}^{\underline{e}}))$. It follows that

$z := H^{-1}\log(\widehat{u}_0)x \in \mathcal{A}^{-1}$ and therefore $\sigma\left(\frac{x\log(\widehat{u}_0)}{H}\right) = \frac{\sigma}{p}(x\log(\widehat{u}_0))/\frac{\sigma}{p}H$. Finally, we have

$$\mathrm{Tr}\circ\mathrm{Res}\,(\Phi') = \mathrm{Tr}\circ\mathrm{Res}\,(\sigma(z)-z)d_{\log}t_1\wedge\cdots\wedge d_{\log}t_N = 0$$

and thus $\widehat{V}(\widehat{u}_0,\ldots,\widehat{u}_N)\equiv 0 \mod p^M$. □

Proof [of lemma 5.11] To start with, it follows from $\sigma(\widehat{\zeta})\equiv\widehat{\zeta}^p \bmod pA$ and $H = \widehat{\zeta}^{p^M}-1$ that $\sigma H \equiv (H+1)^p - 1 \bmod p^{M+1}A$, hence $\sigma H = pH(1+bH)+H^p$ for some $b\in A$. Thus we can write $\frac{\sigma}{p}H = H(1+bH+\frac{H^{p-1}}{p}+c\frac{p^M}{H})$, for $c\in A$. Considering the expansion

$$\frac{y}{\frac{\sigma}{p}H}-\frac{y}{H} = \frac{y}{H}\Big(-\big(bH+\frac{H^{p-1}}{p}+c\frac{p^M}{H}\big)+\big(bH+\frac{H^{p-1}}{p}+c\frac{p^M}{H}\big)^2+\cdots\Big)$$

in $\mathcal{A}^0$, the right-hand side is a sum of terms $\frac{x}{H}H^r\left(\frac{H^{p-1}}{p}\right)^s\left(\frac{p^M}{H}\right)^n$ with coefficients in A and $r+s+n\geqslant 1$. We shall show that for each of them, the coefficient of $\underline{t}^{\underline{0}}$ is congruent to 0 mod p^M. Since $y' := y\,H^r\left(\frac{H^{p-1}}{p}\right)^s \in A\left[\left[\frac{\underline{t}^{\underline{e}p}}{p}\right]\right]$ again, it is sufficient to consider $r=s=0$ and $n\geqslant 1$, noting that, if $n=0$, there clearly is no residue.

Write

$$x = \sum v_i\frac{\underline{t}^{\underline{e}pi}}{p^i} \quad\text{and}\quad \frac{1}{H^{n+1}} = \left(\frac{1}{\underline{t}^{\underline{e}p/(p-1)}}\right)^{n+1}\sum w_j\frac{p^j}{\underline{t}^{\underline{e}j}},$$

for $v_i, w_j\in A$. The coefficient of $\underline{t}^{\underline{0}}$ occurs when $i\underline{e}p-\underline{e}j-(n+1)\frac{\underline{e}p}{(p-1)}\leqslant\underline{0}$, it remains to show that then the exponents of p satisfy $j-i+Mn\geqslant M$. Since $i,j\geqslant 0$ it suffices to consider $i\geqslant M(n-1)$. If $i=M(n-1)$ the condition becomes $j\geqslant 0$ which is always satisfied, thus we may assume $i\geqslant M(n-1)+1$ or $i\geqslant n$, since $M\geqslant 1$. Using $j(p-1)\geqslant ip(p-1)-(n+1)p$, we have

$$\begin{aligned}(p-1)(j-i+M(n-1)) &\geqslant \big[ip(p-1)-(n+1)p\big]-i(p-1)+M(n-1)(p-1)\\ &\geqslant n(p-1)^2-(n+1)p+(n-1)(p-1) = np(p-2)-2p.\end{aligned}$$

If $p\geqslant 5$, or if $p=3$ and $n\geqslant 2$, this is $\geqslant 0$, i.e. $j-i+Mn\geqslant M$. If $p=3$ and $n=1$ then the condition coming from the coefficients of t^0 gives $j\geqslant ip-\frac{2}{p-1}=3i-1$. Since $i,j\geqslant 0$ by assumption, we again get $j-i+M\geqslant M$. □

Remark The analogous result in [1], lemma 3.1.3, is obtained by replacing $d_{\log}t_i$ by dt_i in the statement of the lemma. The proof found there can be used for our statement in almost all cases: Noting that

$$\frac{x'}{H}\left(\frac{p^M}{H}\right)^n \in \frac{p^M}{t^{2\underline{e}p/(p-1)}}\mathcal{A}^0,$$

one sees that the only way the coefficient of $\underline{t}^0$ can be non-divisible by p^M is if $2ep/(p-1) = ep$ and $n = 1$, i.e. $p = 3$ and $n = 1$. In this case, taking e.g. $y' = \frac{\underline{t}^{ep}}{p} \in A[[\frac{\underline{t}^{ep}}{p}]]$ yields the non-trivial residue p^{M-1}.

We define Vostokov's symbol

$$V : (F^*)^{N+1} \longrightarrow \mathbb{Z}/p^M, \quad V(u_0, \ldots, u_N) = \widehat{V}(\widehat{u}_0, \ldots, \widehat{u}_N),$$

where $\widehat{u}_i \in \mathcal{R}$ are such that $\eta(\widehat{u}_i) = u_i$.

Corollary 5.12 *The value of $V \mod p^M$ is independent of the choice of lifts $\widehat{u}_i$ of $u_i \in F^*$.*

Proof Let $\widehat{u}_1, \ldots, \widehat{u}_N$ be lifts of the elements $u_0, \ldots, u_N$. Any other lift of u_j is of the form $\widehat{u}'_j = \widehat{u}_j\widehat{v}$ for $\widehat{v}$ with $\eta(\widehat{v}) = 1$. Thus

$$\Phi(\widehat{u}_0, \ldots, \widehat{u}_j, \ldots, \widehat{u}_N) = \Phi(\widehat{u}_0, \ldots, \widehat{u}_j, \ldots, \widehat{u}_N) + \Phi(\widehat{u}_0, \ldots, \widehat{v}, \ldots, \widehat{u}_N),$$

and the residue of the second term is divisible by p^M. □

Proposition 5.13 *V is symbolic, i.e. $V(u_0, \ldots, u_N) = 0$ if $u_i + u_j = 1$ for $i \neq j$.*

Proof By skew-symmetry, we may assume that $i = 0, j = 1$. Also, by cor. 5.12, we may choose lifts in $\mathcal{R}$ such that $\widehat{u}_0 + \widehat{u}_1 = 1$ again. Then

$$\Phi(\widehat{u}_0, \ldots, \widehat{u}_N) = \left[l(\widehat{u}_0)\tfrac{\sigma}{p}d_{\log}\widehat{u}_1 - l(\widehat{u}_1)d_{\log}\widehat{u}_0\right] \wedge d_{\log}\widehat{u}_2 \wedge \cdots \wedge d_{\log}\widehat{u}_N.$$

We need to distinguish three cases. Assume first that one of $\widehat{u}_0, \widehat{u}_1 \in \mathfrak{m}_A$, say $x = \widehat{u}_0 \in \mathfrak{m}_A$. We show that $l(x)d_{\log}(1-x) - l(1-x)d_{\log}x$ is an exact differential. Working in $Q(\mathcal{F})$, set

$$F = \mathrm{Li}_2(x) + \tfrac{1}{p^2}\mathrm{Li}_2(\sigma x) + \log(1-x)l(x),$$

for the dilogarithm $\mathrm{Li}_2(X) = \sum \frac{X^n}{n^2}$. Then $dF = l(x)d_{\log}(1-x) - l(1-x)\frac{\sigma}{p}d_{\log}x$ and it remains to show that $F \in \mathfrak{m}_A$. To verify the claim, write

$$\begin{aligned} F &= \sum_{n\geqslant 1} \frac{x^n}{n^2} - \frac{\sigma(x)^n}{p^2n^2} - \frac{x^n\, l(x)}{n} \\ &= \sum_{\substack{m\geqslant 1\\ p\nmid m}} x^m\Big(\frac{1}{m^2} - \frac{l(x)}{m}\Big) + \sum_{k\geqslant 1}\sum_{\substack{m\geqslant 1\\ p\nmid m}} x^{mp^k}\Big[\frac{1}{m^2p^{2k}}\big(1 - \frac{\sigma x^{mp^{k-1}}}{x^{mp^k}}\big) - \frac{l(x)}{mp^k}\Big]. \end{aligned}$$

The first sum is clearly in $\mathfrak{m}_A$. To see that the terms of the double sum are integral, note that the coefficients of x^{mp^k} are

$$\frac{1}{m^2p^{2k}}\Big(1 - \frac{\sigma x^{mp^{k-1}}}{x^{mp^k}}\Big) - \frac{l(x)}{mp^k} = \Big[\frac{1}{p^{2k}X^2}\big(1 + p^kX - \exp(p^kX)\big)\Big]\Big|_{X=-ml(x)},$$

so $F \in \mathfrak{m}_A$, as required.

Using this, we obtain

$$\Phi(x, 1-x, \widehat{u}_2, \ldots, \widehat{u}_N) = \Big[d\big(\tfrac{F}{H}\big) - F\,d\big(\tfrac{1}{H}\big)\Big] \wedge d_{\log}\widehat{u}_2 \wedge \cdots \wedge d_{\log}\widehat{u}_N.$$

Since $d\big(\frac{1}{H}\big) = H^{-2}p^M\widehat{\zeta}^{p^M-1}d(\widehat{\zeta})$, we have

$$F\,d\Big(\frac{1}{H}\Big) = \tfrac{-F}{H^2}dH \in p^M\underline{t}^{-2ep/(p-1)}A\big[\big[\tfrac{p}{\underline{t}^e}\big]\big]d(\widehat{\zeta}),$$

and so again $\mathrm{Res}\big(\Phi(x, 1-x, \widehat{u}_2, \ldots, \widehat{u}_N)\big) = 0$ for $x \in \mathfrak{m}_A$.

To deduce the last two cases from the first one, we follow [5]. Since we only consider odd primes p, the computation simplifies slightly. To ease notation, we write $[\widehat{u}_0, \widehat{u}_1] = \phi(\widehat{u}_0, \widehat{u}_1, \ldots, \widehat{u}_N)$ for arbitrary but fixed $\widehat{u}_2, \ldots, \widehat{u}_N$.

Suppose now that $\widehat{u}_0^{-1} \in \mathfrak{m}_A$ or $\widehat{u}_1^{-1} = (1-\widehat{u}_0)^{-1} \in \mathfrak{m}_A$. The relation used in lemma 2.2 to prove that the 2-symbol $\{x, -x\}$ vanishes allows us to deduce this case from the previous one as follows. Writing $-x = (1-x)/(1-\frac{1}{x})$, we obtain

$$[x, 1-x] = -[x^{-1}, 1-x] = -[x^{-1}, 1-x] - [x^{-1}, -x^{-1}] = -[x^{-1}, 1-x^{-1}] = 0$$

if $x^{-1} \in \mathfrak{m}$.

If none of $\widehat{u}_0, \widehat{u}_0^{-1}, 1-\widehat{u}_0, (1-\widehat{u}_0)^{-1}$ is in $\mathfrak{m}$ then one of the four is $a(1+x)$ for $a \in W(k)^*$, $a \neq 1$, and $x \in \mathfrak{m}$. Let $y = xa^{-1} \in 1 + \mathfrak{m}_A$ so that $\widehat{u}_0 = ay$.

Since $\frac{a(1-y)}{a-1} \in \mathfrak{m}$, we have

$$\begin{aligned}
0 &= [\tfrac{a(1-y)}{a-1}, 1 - \tfrac{a(1-y)}{a-1}] = [\tfrac{a(1-y)}{a-1}, -\tfrac{1-ay}{a-1}] \\
&= [1-y, ay-1] - [1-y, a-1] + [\tfrac{a}{a-1}, 1-ay] - [a, 1-a] + [a-1, 1-a] \\
&= [1-y, ay-1] - [1-y, a-1] + [\tfrac{a}{a-1}, 1-ay], \qquad (*)
\end{aligned}$$

noting that $[a, 1-a] = 0$ since $da = 0 = d(1-a)$ for $a \in W(k)^*$.

Also, $\frac{1-y}{1-ay} \in \mathfrak{m}$, thus

$$\begin{aligned}
0 &= [\tfrac{1-y}{1-ay}, 1 - \tfrac{1-y}{1-ay}] = [\tfrac{1-y}{1-ay}, \tfrac{(1-a)y}{1-ay}] \\
&= [1-y, 1-a] - [1-y, 1-ay] + [1-y, y] - [1-ay, (1-a)y] + [1-ay, 1-ay] \\
&= [1-y, 1-a] - [1-y, ay-1] + 0 - ([1-ay, \tfrac{1-a}{a}] + [1-ay, ay]) + 0 \\
&\overset{(\triangle)}{=} 0 - [1-ay, ay] + 0 = [x, 1-x],
\end{aligned}$$

where $(\triangle)$ follows by substituting 0 for the three terms of $(*)$ above. □

Corollary 5.14 *V induces $V : K_N^t(F) \to \mathbb{Z}/p^M$.*

Consider now $h : (F^*)^{N+1}$ defined by the Hilbert symbol $(u_0, \{u_1, \ldots, u_N\}) = \zeta^{h(u_0,\ldots,u_N)}$.

Lemma 5.15 *h is skew-symmetric.*

Proof Consider $h(u_0, \ldots, u_i, \ldots, u_j, \ldots, u_N) + h(u_i, \ldots, u_j, \ldots, u_i, \ldots, u_N)$. If both $i, j > 0$ then this is 0 because $K_N(F)$ is skew-symmetric. If $i = 0$, suppose $u_0 = u_j$ and let $L = F(\sqrt[p^M]{u_0})$. Then $\{u_1, \ldots, u_N\} = N_{L/F}\{u_1, \ldots, \sqrt[p^M]{u_0}, \ldots, u_N\} \in N_{L/F}K_N(L)$ and thus $\Psi_F(\{u_1, \ldots, u_N\}) = 0$ by the definition of the reciprocity map, hence $h = 0$. Skew-symmetry follows. □

Corollary 5.16 *h induces $h : K_{N+1}(F) \to \mathbb{Z}/p^M$.*

Theorem 5.17 *The Vostokov pairing coincides with the Hilbert symbol, i.e.*

$$h(u_0, \{u_1, \ldots, u_N\}) \equiv V(u_0, u_1, \ldots, u_N) \mod p^M$$

for any $u_i \in F^*$ and lifts $\widehat{u}_i \in \mathcal{R}$.

PROOF By cor. 5.5,

$$h(u_0,\dots,u_N) = \tfrac{1}{H} l(u_0) d_{\log}\widehat{u}_1 \wedge \dots \wedge \widehat{u}_N = \Phi_0(u_0,\dots,u_N)$$

is the first term of V, so it remains to prove that

$$\mathrm{Tr} \circ \mathrm{Res}\Big(\sum_{1\leqslant i\leqslant N} \frac{(-1)^i}{H} l(u_i) \tfrac{\sigma}{p} d_{\log}u_0 \wedge \dots \wedge \tfrac{\sigma}{p} d_{\log}u_{i-1} \wedge d_{\log}u_{i+1} \wedge \dots \wedge d_{\log}u_N\Big) = 0.$$

It suffices to consider Coleman lifts of the topological generators $\{\bar{t}_1,\dots,\bar{t}_N\}$ and $\{E(\alpha,\underline{\bar{t}}^{\underline{a}}),\bar{t}_1,\dots,\bar{t}_{i-1},\bar{t}_{i+1},\dots,\bar{t}_N\}$ of $K_N^t(\mathcal{F})$.

If $\{u_1,\dots,u_N\} = \{t_1,\dots,t_N\}$, then $l(u_i) = 0$ for $1 \leqslant i \leqslant N$, so the remaining N terms vanish and hence $\phi(u_0,t_1,\dots,t_N) = h(u_0,t_1,\dots,t_N)$.

If $\{u_1,\dots,u_N\} = \{E([\alpha],\underline{t}^{\underline{a}}),t_1,\dots,t_{i-1},t_{i+1},\dots,t_N\}$ then the first two terms of Φ are non-zero. Because $l(E([\alpha],\underline{t}^{\underline{a}})) = [\alpha]\underline{t}^{\underline{a}}$, it remains to show that

$$Tr \circ Res \frac{[\alpha]}{H} \underline{t}^{\underline{a}} \tfrac{\sigma}{p} d_{\log}u_0 \wedge d_{\log}t_1 \wedge \dots \wedge d_{\log}t_{i-1} \wedge d_{\log}t_{i+1} \wedge \dots \wedge d_{\log}t_N \equiv 0 \mod p^M.$$

Since $u_0 \in 1+\mathfrak{m}$, the $d_{\log}t_i$-component of $d_{\log}u_0$ is equal to $y d_{\log}t_i$ for y in $\mathfrak{m}$. By lemma 5.8 (a), $\frac{1}{H} = \underline{t}^{-\underline{e}p/(p-1)} \sum_{n\geqslant 0} a_n \frac{p^n}{\underline{t}^{\underline{e}n}}$ for some $a_n \in A$. It follows that the above residue is the coefficient of $\underline{t}^{\underline{0}}$ in

$$[\alpha]\underline{t}^{\underline{a}}\sigma(y)\underline{t}^{\underline{e}p/(p-1)} \sum_{n\geqslant 0} a_n \frac{p^n}{\underline{t}^{\underline{e}n}}.$$

This happens when $\underline{a} + p\underline{b} - \frac{\underline{e}p}{p-1} - \underline{e}n = \underline{0}$, where $p\underline{b}$ is the contribution from $\sigma(y)$. This implies that $p \mid n\underline{e} + \underline{a}$, but $p \nmid \underline{a}$ by assumption, thus also $p \nmid n\underline{e}$, hence $p \nmid \underline{e}$. Since $\zeta_M \in F$, this means that $M = 1$, but n is the exponent of p so for $M = 1$, the only interesting case is $n = 0$, in which case $p \mid \underline{a}$ is a contradiction. Thus the residue of the second summand is $\equiv 0 \mod p^M$, and again $h(u_0,\dots,u_N) \equiv V(u_0,\dots,u_N)$ mod p^M in this case.

Considering topological generators of $K_N^t(F)$, it follows that the only remaining cases are

(1) $\phi(v,\{\omega(\alpha_0),\pi_1,\dots,\pi_{i-1},\pi_{i+1},\dots,\pi_N\})$ for $1 \leqslant i \leqslant N$.

(2) $\phi(\pi_i,\{\omega(\alpha_0),\pi_1,\dots,\pi_{i-1},\pi_{i+1},\dots,\pi_N\}) = (-1)^i$

(3) $\phi(\pi_i, \{E(\alpha, \underline{\pi}^{\underline{a}}), \pi_1, \ldots, \pi_{i-1}, \pi_{i+1}, \ldots, \pi_N\}) = 0$,

For $\omega(\alpha_0) = E_{\underline{X}}(\alpha_0 H)\big|_{\underline{X}=\underline{\pi}}$ as in lemma 1.14. By skew-symmetry, they can be reduced to the first case. $\square$

Remark Considering that $K_{N+1}(F)/p^M \cong \mu_{p^M}$ is generated by $\{\omega(\alpha_0), \pi_1, \ldots, \pi_N\}$, one can further reduce the proof to the case $u_0 = \omega(\alpha_0)$, $\widehat{u}_i = t_i$.

Lemma 5.18 *The element $\omega(\alpha_0)$ is p^M-primary*

Proof Let $F_\bullet$ and $\mathcal{F}$ be as above. For $\alpha_0 \in W(k) \subset W(\mathcal{F})$, the extension $\mathcal{L} = \mathcal{F}(A_M)$ of $\mathcal{F}$ obtained by joining all coefficients of $A_M \in W_M(k^{sep})$ with $\wp(A_M) = \alpha_0 \mod p^M$ is unramified. If $\mathrm{Tr}_{W(k)/\mathbb{Z}_p}(\alpha_0) \in \mathbb{Z}_p^*$, it is of degree p^M by Witt theory. The Kummer-extension L/F corresponding to $\mathcal{L}/\mathcal{F}$ is given by joining a p^M-th root of $\theta(\alpha_0 H) = \omega(\alpha_0)$. Since the field of norms preserves unramified extensions by construction, we see that $F(\sqrt[p^M]{\omega(\alpha_0)})/F$ is unramified of degree p^M. $\square$

Corollary 5.19 $h(\omega(\alpha_0), \pi_1, \ldots, \pi_N) = \mathrm{Tr}_{W(k)/\mathbb{Z}_p}(\alpha_0) = V(\omega(\alpha_0), \pi_1, \ldots, \pi_N)$.

Proof For V, this follows by taking lifts t_i of π_i and noting that $l(t_i) = 0$, hence $\Phi = \Phi_0$. For h, the lemma shows that $L = F(\sqrt[p^M]{\omega(\alpha_0)})$ is unramified of degree p^M over F, thus $\mathrm{Gal}(L/F) = \langle \varphi_F|_L \rangle$ is generated by a restriction of the Frobenius of F. By class field theory, $r_{L/F}(\varphi_F|_L) = \{\pi_1, \ldots, \pi_N\}$. Thus $h(\omega(\alpha_0), \pi_1, \ldots, \pi_N) = \varphi_F(\xi)/\xi$ where $\xi^{p^M} = \omega(\alpha_0)$. Again by the main lemma, $\varphi_F(\xi)/\xi = \varphi_F(A_M) - A_M$ for $A_M \in W_M(k^{sep})$ such that $\wp(A_M) = \alpha_0$. But if $[F^{(n)} : \mathbb{F}_p] = f$, then $\varphi_F = \sigma^f$ acting on $W_M(k^{sep})$. Thus

$$\begin{aligned}\varphi_F(A_M) &= \sigma^f(A_M) = \sigma^{f-1}(A_M) + \alpha_0 = \sigma^{f-2}(A_M) + \sigma(\alpha_0) + \alpha_0 = \cdots \\ &= A_M + \sigma^{f-1}(\alpha_0) + \cdots + \sigma(\alpha_0) + \alpha_0 = A_M + \mathrm{Tr}_{W(k)/\mathbb{Z}_p}(\alpha_0),\end{aligned}$$

and $\varphi_F(A_M) - A_M = \mathrm{Tr}_{W(k)/\mathbb{Z}_p}(\alpha_0)$, as required. $\square$

Appendix A

Lifts

In this appendix we give two constructions of lifts of lifts of rings of characteristic p to characteristic p^M or 0. They agree in the case of perfect rings.

A.1 Witt vectors

Let A be a ring of characteristic p and $n \geq 0$ an integer. The ring of Witt-vectors of length n, $W_n(A)$, is given as a set by the product of n copies of A, A^n. Addition and multiplication are defined as follows. Consider the polynomials

$$w_i(X_0, \ldots, X_{i-1}) = X_0^{p^i} + pX_1^{p^{i-1}} + \cdots + p^{i-1}X_{i-1} \in \mathbb{Z}[X_0, \ldots, X_{i-1}].$$

It can be shown that there exist unique $S_{i-1}, P_{i-1} \in \mathbb{Z}[X_0, \ldots, X_{i-1}; Y_0, \ldots, Y_{i-1}]$ such that

$$w_i(S_0, \ldots, S_{i-1}) = w_i(X_0, \ldots, X_{i-1}) + w_i(Y_0, \ldots, Y_{i-1})$$
$$w_i(P_0, \ldots, P_{i-1}) = w_i(X_0, \ldots, X_{i-1})\, w_i(Y_0, \ldots, Y_{i-1})$$

for each $i \geq 0$. Now for Witt-vectors $a = (a_0, \ldots, a_{n-1}), b = (b_0, \ldots, b_{n-1}) \in W_n(A)$, define addition and multiplication by

$$a + b = \big(S_0(a_0, b_0), S_1(a_0, a_1; b_0, b_1), \ldots, S_{n-1}(a_0, \ldots, a_{n-1}; b_0, \ldots, b_{n-1})\big)$$
$$a\, b = \big(P_0(a_0, b_0), P_1(a_0, a_1; b_0, b_1), \ldots, P_{n-1}(a_0, \ldots, a_{n-1}; b_0, \ldots, b_{n-1})\big).$$

It follows from this definition that $p^n = 0$ in $W_n(A)$. By construction, if A_n is any ring in which $p^n = 0$, then any ring-homomorphism $\alpha : A \to A_n/p$ induces a ring-homomorphism $W_n(A) \to A_n$ given by $(a_0, \ldots, a_{n-1}) \mapsto w_n(\alpha(a_0), \ldots, \alpha(a_{n-1}))$. w_{n-1} is called the $(n-1)$-st *ghost component* of $a = (a_0, \ldots, a_{n-1})$ and is denoted $a^{(n-1)} = w_{n-1}(a)$.

It can be seen that the projection to the first n coordinates defines a surjective homomorphism $W_{m+n}(A) \to W_n(A)$ for any m. The (total) Witt ring of A is defined to be $W(A) = \varprojlim_n W_n(A)$ with respect to these projections. $W(A)$ is the set of sequences $(a_0, \ldots, a_n, \ldots)$ of $a_i \in A$ with addition and multiplication given by $(S_0, \ldots, S_n, \ldots)$ and $(P_0, \ldots, P_n, \ldots)$, respectively.

The map $A \to W(A)$, $a \mapsto (a, 0, \ldots)$ is multiplicative but not additive. If $a \neq 0$, $(a, 0, \ldots)$ is usually denoted $[a]$ and is called the Teichmüller representative of A. Taking Teichmüller representatives defines an injection of multiplicative groups $[-] : A^* \to W(A)^*$ and we shall identify $a \in A^*$ with its image in $W(A)^*$ when there is no risk of confusion.

W and W_n are functorial in that to any homomorphism $f : A \to B$ (of rings) there corresponds a homomorphism

$$W(f) : W(A) \to W(B) : \quad W(f)((a_0, \ldots, a_n, \ldots)) = (f(a_0), \ldots, f(a_n), \ldots)$$

which respects composition of morphisms and the identity morphism. In particular, the absolute Frobenius $\sigma : a \mapsto a^p$ of A induces the Frobenius (usually denoted F)

$$\sigma : W(A) \to W(A) : \quad (a_0, \ldots, a_n, \ldots) \mapsto (a_0^p, \ldots, a_n^p, \ldots)$$

on Witt-vectors (and similarly for W_n).

The *Verschiebung* $V : W(A) \to W(A)$ (resp. $W_n(A) \to W_n(A)$) is given by $V((a_0, a_1, \ldots, a_n, \ldots)) = (0, a_0, \ldots, a_n, \ldots)$. V is additive and satisfies $V^i(a)\, V^j(b) = V^{i+j}(\sigma^j(a)\, \sigma^i(b))$. Any Witt-vector can be written as

$$(a_0, \ldots, a_n, \ldots) = [a_0] + V([a_1]) + \cdots + V^n([a_n]) + V^{n+1}((a_{n+1}, \ldots))$$

for any n.

σ and V are related by $\sigma V = V\sigma = p$. If A is a perfect field k of characteristic p, the absolute Frobenius is an isomorphism, hence so is σ, and any Witt-vector can be written as

$$(a_0, \dots, a_n, \dots) = [a_0] + p[a_1^{\sigma^{-1}}] + \dots + p^n[a_n^{\sigma^{-n}}] + p^{n+1}\sigma^{-n-1}(a_{n+1}, a_{n+2}, \dots).$$

This shows in particular that if k is perfect, $W(k)$ is a p-adic complete discrete valuation ring with valuation $v(0, \dots, 0, a_i, \dots,) = i$ (if $a_i \neq 0$), and residue field k.

Example If $k = \mathbb{F}_p$, $W_n(\mathbb{F}_p) \cong \mathbb{Z}/p^n\mathbb{Z}$ via $w_n : (\overline{a}_0, \dots, \overline{a}_{n-1}) \mapsto a_0^{p^n} + pa_1^{p^{n-1}} + \dots + p^{n-1}a_{n-1}$, where $a_i \in \mathbb{Z}/p^n\mathbb{Z}$ are any lifts with of $\overline{a}_i$. Taking the projective limit, this induces $W(\mathbb{F}_p) \cong \mathbb{Z}_p$ given by $(a_0, \dots, a_n, \dots) \mapsto [a_0]+p[a_1]+\dots+p^n[a_n]+\dots$, where $[a_i] = \lim_{m\to\infty} a_i^{p^m}$ is the usual Teichmüller representative in $\mathbb{Z}_p$. More generally, $W(\mathbb{F}_{p^m})$ is the ring of integers of the unramified extension of $\mathbb{Q}_p$ of degree m.

We remark that the functor Witt-vectors can be defined for arbitrary rings, together with an additive Verschiebung and a multiplicative Frobenius (see, e.g. [21])

A.2 Flat Lifts

If A is a non-perfect ring of characteristic p, we still have a canonical isomorphism $W(A)/VW(A) \cong A$, but $VW(A) \neq pW(A)$ since σ is not surjective. This indicates that $W(A)$ is in a way "too big". In [6], a *flat lift* of A to $\mathbb{Z}_p$ is defined to be a flat $\mathbb{Z}_p$-module $\mathcal{O}(A)$ such that $\mathcal{O}(A)/p\mathcal{O}(A) \cong A$. This is equivalent to giving, for every $n \geq 1$, a flat $\mathbb{Z}/p^n\mathbb{Z}$-module $\mathcal{O}_n(A)$ such that the sequence

$$0 \longrightarrow \mathcal{O}_m(A) \xrightarrow{p^n} \mathcal{O}_{n+m}(A) \longrightarrow \mathcal{O}_{n+m}(A)/p^n = \mathcal{O}_n(A) \longrightarrow 0$$

is exact for every n, m. The equivalence is given by $\mathcal{O}_n(A) = \mathcal{O}(A)/p^n$ and $\mathcal{O}(A) = \varprojlim \mathcal{O}_n(A)$.

We describe the construction of lifts in the special case of N-dimensional local fields $\mathcal{F} = k((t_N))\cdots((t_1))$. In this case, $\sigma(\mathcal{F}) = k((t_N^p))\cdots((t_1^p))$ and we see that $\mathcal{F}$ is a vector space over $\sigma(\mathcal{F})$ with basis consisting of all monomials $t_1^{a_1}\cdots t_N^{a_N}$ with $0 \leq a_i < p$ for all i. This means that $t_1, \dots, t_N$ is a so-called p-basis for $\mathcal{F}$, and by prop. 1.1.7 of [6], a lift $\mathcal{O}_n(\mathcal{F})$ exists and is equal to the subring of $W_n(\mathcal{F})$ generated by all elements of the form $p^j[x^{p^{n-j}}][t_1]^{a_1}\cdots[t_1]^{a_N}$, for $x \in \mathcal{F}$ and $0 \le a_i < p^n$ for all i.

Lemma A.1 *For any fixed set of local parameters $\bar{t}_1, \dots, \bar{t}_N$, the lift $\mathcal{O}_M(\mathcal{F})$ constructed by using them as p-basis is canonically isomorphic to $W_M(k)((t_N))\cdots((t_1))$, where $t_i = [\bar{t}_i]$ are Teichmüller representatives.*

Proof For any $x \in \mathcal{F}$, $p^j[x^{p^{n-j}}] = (0, \dots, 0, x^{p^n}, 0, \dots, 0) \in W_M(\mathcal{F})$, where the x^{p^n} is at the j-th place. It follows that $W_M(\sigma^{M-1}(\mathcal{F}))[t_1, \dots, t_N] \subset \mathcal{O}_M(\mathcal{F})$. The inclusion $W_M(k)[t_N] \subset W_M(\sigma^{M-1}(\mathcal{F}))[t_1, \dots, t_N]$ extends to an inclusion $W_M(k)[[t_N]] \subset W_M(\sigma^{M-1}(\mathcal{F}))[t_1, \dots, t_N]$ since $\bar{t}_N^{p^{M-1}} \in \sigma^{M-1}(\mathcal{F})$. Also, $t_N^{-1} = (t_N^{p^{M-1}})^{-1} t_N^{p^{M-1}-1}$, so we obtain $W_M(k)((t_N)) \subset W_M(\sigma^{M-1}(\mathcal{F}))[t_1, \dots, t_N]$. Continuing inductively, we deduce that

$$W_M(k)((t_N))\cdots((t_1)) \subset W_M(\sigma^{M-1}(\mathcal{F}))[t_1, \dots, t_N] \subset \mathcal{O}_M(\mathcal{F}).$$

But $W_M(k)((t_N))\cdots((t_1))$ is flat over $\mathbb{Z}/p^M\mathbb{Z}$ since it is obtained from $W_M(k)$ by a sequence of steps involving taking polynomial rings, completions, and localisations, and it satisfies $W_M(k)((t_N))\cdots((t_1))/(p) \cong k((\bar{t}_N))\cdots((\bar{t}_1)) = \mathcal{F}$, and it follows that all inclusions are equalities. $\square$

Taking projective limits, we see that $\mathcal{O}(\mathcal{F}) = W(k)\{\{t_N\}\}\cdots\{\{t_1\}\}$ is the p-adic completion of $W(k)((t_N))\cdots((\widetilde{t}_N))$. By construction, $\mathcal{O}(\mathcal{F}) = \varprojlim \mathcal{O}(\mathcal{F})/p^n$ and we see that it is a complete discrete valuation ring with uniformiser p and residue field $\mathcal{F}$.

We denote by $Q(\mathcal{F})$ the field of fractions $Q(\mathcal{F}) = \mathrm{Frac}(\mathcal{O}(\mathcal{F}))$. It is an $(N+1)$-dimensional local field of characteristic 0, with local parameters $p, \widetilde{t}_1, \dots, \widetilde{t}_N$, first valuation ring $\mathcal{O}(\mathcal{F})$ and first residue field $\mathcal{F}$. We denote by $Q_0(\mathcal{F})$ the subring $W(k)((t_N))\cdots((t_1)) \subset \mathcal{O}(\mathcal{F})$.

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