Heterotic Braneworld Gravity

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Abstract

In the past decade, there has been considerable interest in braneworld scenarios where the universe lives on a brane in a higher-dimensional bulk and gravity is modified. The heterotic braneworld scenario of Lukas, Ovrut, Stelle and Waldram (LOSW) is derived from Horava-Witten M-theory, where six of the eleven dimensions have been compactified on a Calabi-Yau manifold. The solution consists of two parallel three-branes separated by the 11th dimension with a scalar field in the bulk. In this dissertation we review some of the alternative theories of gravity, including the Kaluza-Klein model as an early example of a theory featuring extra dimensions, and the more recent braneworld models, in particular the models proposed by Randall and Sundrum, based on which many braneworld techniques were developed. We use these techniques to study gravity in the LOSW model, and explore the possibilities for a black hole solution. Using perturbation theory, we find that the zero mode sector consists of the graviton and the radion which is coupled to the bulk scalar field, and there is a continuum of massive states. The brane gravity is scalar-tensor with a Brans-Dicke parameter of $\omega = 0.5$. Then we show that although it is possible to have a black string between the two branes, it suffers from a Gregory-Laflamme instability. We also show that it is not possible to obtain spherically symmetric solutions, so we solve the coupled brane and bulk Einstein equations for an axisymmetric metric. We obtain a solution which asymptotes the LOSW vacuum and resembles the black string. The solution looks like the Schwarzshild solution at a large distance, but the interbrane distance is not constant and the string becomes infinite as it reaches the Schwarzshild radius.

Declaration

The work in this thesis is based on research carried out at the Centre for Particle Theory, the Department of Mathematical Sciences, Durham University, United Kingdom. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all my own work unless referenced to the contrary in the text.

Chapter 2 of this dissertation is a review of the relevant background material. Chapters 3 and 4 contain original work done in collaboration with my supervisor Prof. Ruth Gregory and Bina Mistry [1].

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Introduction

1.1 The state of modern particle physics

Since the time of the Ancient Greeks, humans have been seeking answers for simple questions such as "What is matter?" and "what is time?" In many ancient civilizations, there existed ideas that a few fundamental elements are the building blocks of the entire universe. This idea has persisted over several thousand years and eventually gave birth to the modern field of particle physics.

Particle physics is the systematic study of the elementary particles that build our universe and their interactions. The Standard Model (SM) of particle physics describes the electromagnetic, weak and strong interactions. A fourth interaction, gravity, is not yet integrated into SM. The General Theory of Relativity (GR), published by Einstein in 1916 [2], is widely accepted as a theory for gravitational interactions. It is the dream of many physicists to find a theory unifying SM and GR that will be able to describe quantum gravitational phenomena.

In 1803, Dalton introduced the idea of atoms (from the Greek word atomos meaning indivisible) as the fundamental building blocks of the universe, and that there are many different types of atoms. Atoms of the same type can combine to make a chemical element, and different types of atoms combine to make more complex molecules. Dalton was eventually proven wrong, when the electron was discovered in 1897 by Thomson. This, along with the discoveries of the proton and the neutron, showed that the atom is composed by the aforementioned three particles. The protons and neutrons combine to make the nucleus at the centre of the atom, bound by the strong force, while the electrons are located far away from the nucleus.

In 1931, Anderson *et al.* discovered a particle which behaved like the electron but with a positive charge. Four years earlier Dirac found that the equation for free electrons now bearing his name had two solutions, one with positive energy and another with negative energy [3]. Dirac explained the negative energy states as holes in a sea of electrons, in practice behaving as a normal electron with positive energy but with positive charge. Anderson's particle fit the bill for Dirac's particle, and is known as the positron, or anti-electron. Thus the first antimatter particle was found.

Gradually, more particles were found and a periodic table ala Mendeleev's was desired. It was Gell-Mann who proposed the Eightfold way as a scheme to organise the particles [4, 5]. He later expanded his idea by proposing that baryons and mesons consisted of fundamental particles called quarks. Although nobody has seen a free quark, there is strong evidence for their existence in deep inelastic scattering experiments. Now there are three big groups of particles: hadrons, which are all composite particles made of quarks and undergo the strong interaction; leptons which are elementary particles and undergo the weak interaction; and mediator particles which facilitate interactions between particles.

In the 19th century, Maxwell published his work on electricity and magnetism [6]. He stated that the two forces are closely related to each other, in fact they are two aspects of a single force, thus providing an early example of unification. With the emergence of quantum mechanics and the theory of relativity [7] in the beginning of the 20th century, many physicists worked to find a theory of electromagnetism obeying the principles of the above two theories. This effort culminated in a theory known as Quantum Electrodynamics (QED) by Tomonaga [8, 9], Schwinger [10, 11] and Feynman [12]. QED is a relativistic, perturbative, quantum field theory (abelian gauge theory with the symmetry group U(1)) describing interactions between electrons, positrons, and photons (the mediator particle of QED). It shows the importance of the principle of local gauge invariance in demonstrating how the gauge field

A_μ gives rise to the field strength tensor $F_{\mu\nu}$. QED has been tested in experiments to successfully predict the anomalous magnetic moment of the electron, and the Lamb shift of the energy levels of hydrogen.

The theory for strong interactions was initially proposed by Yukawa in 1935 [13]. This was later regarded as a basic theory insufficient to explain the strong interaction in detail. In the 1960's two competing theories were popular, the approach based on QED, and a more radical one known as S-matrix theory [14–16]. Eventually the first one won, and became known as Quantum Chromodynamics (QCD), a non-abelian gauge theory with SU(3) gauge group. In QCD, the colour charge is the equivalent of the electric charge in QED. The term colour was chosen because the similarities between the properties of this charge with the theory of colour in art. Interactions in QCD are a lot more complicated than in QED because the gluons (the mediator particle in QCD) may interact with themselves. In addition to that, the coupling constant is not actually a constant, but is often referred to as a running coupling constant. In practice, this means that the quarks are confined so that particles are "colourless" and there has never been a free quark observed.

Weak interactions are responsible for interactions involving quarks changing flavours. The first example of weak interactions was found in nuclear beta decays. There are two types of interactions: charged (common) and neutral (rare, example: neutrino scattering). An important concept in the theory of weak interactions is the quark mixing matrix by Cabibbo [17], Kobayashi and Maskawa [18]. Fermi was the first physicist to propose a theory of weak interactions. His theory was not renormalizable, and a good theory was not obtained until electroweak unification.

It was later realised by Glashow [19], Weinberg [20] and Salam [21] that the electromagnetic and weak interactions were two faces of the same thing. Finally particle physicists were able to unify two of the fundamental forces, into the electroweak interaction. This unification happens at energy scales on the order of 100 GeV. The GWS theory has $SU(2) \times U(1)$ gauge group. This symmetry is broken (from $SU(2) \times U(1)_Y$ to $U(1)_{em}$), and causes the theory to split into electromagnetic and weak interactions. In this process, the W and Z bosons of the weak theory gain mass, while the photon remains massless. This is the Higgs mechanism, and it predicts the existence of another boson necessary to complete the process. Even though masses of W and Z bosons were successfully predicted by this theory, the Higgs boson itself has not yet been found in experiments.

We will now discuss Einstein's theory of gravitational interactions. The action for GR is given by the Einstein-Hilbert action

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}R. \tag{1.1}$$

Using the variational principle gives the equations of motion (also known as the Einstein field equations)

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi G T_{\mu\nu}, \tag{1.2}$$

where $G_{\mu\nu}$ is the Einstein tensor, $R_{\mu\nu}$ is the Ricci tensor obtained by contracting the Riemann curvature tensor $R^{\lambda}{}_{\mu\kappa\nu} = \partial_\kappa\Gamma^\lambda_{\mu\nu} - \partial_\nu\Gamma^\lambda_{\mu\kappa} + \Gamma^\lambda_{\kappa\sigma}\Gamma^\sigma_{\mu\nu} - \Gamma^\lambda_{\nu\sigma}\Gamma^\sigma_{\mu\kappa}$ on the first and third indices, R is the Ricci scalar obtained by contracting the Ricci tensor, and the Christoffel symbol is defined by the expression $\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\kappa}\left(\partial_\nu g_{\mu\kappa} + \partial_\mu g_{\nu\kappa} - \partial_\kappa g_{\mu\nu}\right)$. The expression for the energy-momentum tensor $T_{\mu\nu}$ will depend on the nature of the matter considered. We work in units where $c = 1$.

GR field equations are hard to solve because of nonlinearity, but applying symmetries can greatly simplify the problem. Solutions to these equations have been found: Schwarzschild, Nordstrom, Kerr, FRW to name a few, each describing a different geometry.

The Schwarzschild solution describes a static uncharged mass with a spherical symmetry in vacuum. It is suitable for describing the gravitational field around a planet, a star or even a black hole. The approximation is valid as long as the object is rotating slowly. The presence of an event horizon leads to the black hole interpretation.

Reissner and Nordström discovered a solution describing a spherically symmetric non-rotating charged mass. This solution is mainly of theoretical interest because it has been argued that a charged black hole would quickly have its charge neutralised. Furthermore, the universe as a whole seems to be electrically neutral so it is likely that black holes in this universe are also electrically neutral.

The Kerr solution describes a rotating uncharged mass. Instead of spherical symmetry, this solution relied on axial symmetry. A remarkable property of the Kerr metric is the double event horizon it possesses. It is also predicted to exhibit frame dragging (Lense-Thirring effect) [22,23]. The generalization of the Kerr metric to include electric charge is known as the Kerr-Newman metric.

The Friedmann-Lemaitre-Robertson-Walker solution (FLRW) describes a universe that is expanding in time. It is based on two principles: (spatial) homogeneity and isotropy. Simply stated, it means that there is no preferred location in the universe, and no preferred direction. The above requirement allows for time evolution. The form of the energy-momentum tensor is determined by the content of the universe, and this will in turn describe the time evolution of the universe.

GR has been confirmed through various experiments to be a successful theory of gravity [24, 25]. Einstein calculated the precession of the perihelion of Mercury using his newly formulated theory and found that it accounted for the observed anomalous value of roughly 43 arcseconds per century [26]. A team of astronomers led by Eddington announced that they were able to observe the bending of light predicted by GR at the solar eclipse of 1919 [27].

A more elegant form of the bending of light experiment is gravitational lensing [28]. Light from distant objects such as galaxies or quasars is "bent" by a very massive object on the way to the observer. This produces multiple images of the same object. In a perfect alignment, the lensed image can form a complete ring, known as an Einstein ring.

On three of the Apollo missions (Apollo 11, 14 and 15), astronauts installed retroreflectors which allowed scientists to measure the distance of the Moon to the Earth [29]. The results from three decades of Lunar laser ranging experiments have found that Newton's gravitational constant is very stable and confirmed the predictions of GR about the orbit of the Moon [30, 31]. There has not been any evidence for the Nordtvedt effect, in which the Moon and the Earth are observed to have different rates of acceleration and thus violating the Strong Equivalence Principle [32–35].

An indirect confirmation of GR is the success of the Big Bang model. The

Big Bang is based on the FLRW metric, and by choosing a suitable form of energy-momentum tensor, it describes a universe that was initially very hot and very dense, continuously expanding in time. The Big Bang model is supported by the observed expansion of the universe (Hubble's law), the abundance of primordial elements in correct proportions [36], the observed cosmic microwave background radiation [37] and the formation of large-scale structures [38].

In 1961 Brans-Dicke [39] theory emerged as an alternative to GR. It claims to incorporate Mach's principle more completely than GR. In this theory, Newton's constant is a function of spacetime. This theory contains a scalar field, and is often called a scalar-tensor theory for this reason. Some of the models from string theory predict a scalar-tensor type gravity, so we briefly review the basics of Brans-Dicke gravity.

The action for Brans-Dicke gravity is given by

$$S = \int d^4x \sqrt{-g} \left(\Phi R - \omega \frac{\partial_\sigma \Phi \partial^\sigma \Phi}{\Phi} \right) + S_M, \tag{1.3}$$

where Φ is a massless scalar field, ω is called the Brans-Dicke parameter and S_M is the matter piece of the action. The gravitational equation of motion is given by

$$G_{\mu\nu} = \frac{8\pi}{\Phi} T^{(M)}_{\mu\nu} + \frac{\omega}{\Phi^2} \left(\partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} g_{\mu\nu} \partial_\sigma \Phi \partial^\sigma \Phi \right) + \frac{1}{\Phi} \left(\nabla_\mu \partial_\nu \Phi - g_{\mu\nu} \Box \Phi \right), \tag{1.4}$$

where the energy-momentum tensor of other matter fields is given by

$$T^{(M)}_{\mu\nu} = \frac{2}{\sqrt{-g}} \frac{\delta S_M}{\delta g^{\mu\nu}}, \tag{1.5}$$

and the middle term comes from the energy-momentum tensor of the scalar field. The final term is new and shows the contribution of the scalar field in Brans-Dicke theory. The equation of motion for the scalar field is

$$2\omega \frac{\Box \Phi}{\Phi} - \omega \frac{\partial_\sigma \Phi \partial^\sigma \Phi}{\Phi^2} + R = 0. \tag{1.6}$$

which can be rewritten using (1.4) to give

$$\Box \Phi = \frac{8\pi}{3 + 2\omega} T^{(M)}. \tag{1.7}$$

Brans-Dicke gravity has not been ruled out by experiments. Data from the Cassini probe have given a lower bound for the value of ω to be $\omega > 10^4$ [40].

1.2 Outstanding questions

After reviewing the successes of particle physics theories, in this section we would like to look at the problems that are yet unsolved.

Recalling Maxwell's success in unifying electricity and magnetism, and the electroweak unification by GWS, physicists aim to discover a Theory of Everything (TOE) describing all four interactions as different aspects of a single theory. Although the strong, electromagnetic and weak interactions have been successfully formulated as gauge theories, the theory for gravitational interactions is still a classical one. The first obvious step in constructing this TOE was to treat GR as a quantum field theory. Unfortunately, it was found that GR is non-renormalizable at high energies and ceases to give meaningful results to calculations, although it may be possible to have an effective field theory at low energies [41].

Another problem that may be an indication that the Standard Model is not a complete theory, but may only be a low-energy approximation of a more fundamental theory is the (gauge) hierarchy problem. There is an energy scale associated with the electroweak unification, called the weak scale. There is another energy scale associated with gravity, called the Planck scale. It is not obvious why there should be two energy scales, and why there is such a big difference (17 orders of magnitude) between the two. Viewed another way, the hierarchy problem is the problem of why gravity is so much weaker than the other interactions.

There have been several proposed solutions to the hierarchy problem. It has been proposed that entirely new physics will be discovered in the range between the two hierarchies. This new physics would come from superpartner particles, particles which have the exact same properties as existing ones, but with their spin differing by half a unit. This idea, called supersymmetry or SUSY (reviewed here [42–44]), is now widely accepted although there is no direct evidence for it in experiments.

Another solution was proposed by Arkani-Hamed *et al.* [45,46], using large extra dimensions. This model (also known as ADD) was also criticised for not really solving the problem but merely shifting it. However it introduced a new idea, that there could be large extra dimensions in nature, but undetectable to us because this extra dimension is only accessible to gravity. A similar model to ADD was proposed by Randall and Sundrum (RS) [47, 48], where there is also an extra dimension accessible only to gravity but this dimension is now warped.

The hierarchy problem is an example of a fine tuning problem in particle physics. Another example of a fine tuning problem is the cosmological constant problem. It refers to the discrepancy between the observed value of Λ and the value expected by the Standard Model by 120 orders of magnitude. At first the connection between particle physics and cosmology may not seem obvious, as the the first deals with objects that are very small, and the latter deals with objects that are very large. However, with the advent of precision cosmology in the 20th century, it became apparent that the two subjects are complementary. In order to study how the universe evolves, it is necessary to understand its contents, which is done through particle physics. On the other hand, some of the concepts in particle physics, for example the unification of all gauge forces, is believed to happen at very high energies and can not be tested using current colliders, but may be tested indirectly through cosmology because these conditions resemble the early universe.

Current results from cosmology indicate that the make-up of the universe is 5% baryonic matter, 23% dark matter and 72% vacuum energy [49, 50]. This will be discussed sequentially. From Dirac's equation, we know that a particle has a partner with the opposite charge, called an anti-particle. However, aside from observations of cosmic rays, anti-particles are almost not observed in the universe. The mechanism causing this asymmetry is not yet understood, but it may require an extension of SM.

Dark matter is matter that is not visible to us but can be detected from its gravitational effect. Its existence was suggested to explain the problem found in galaxy rotation curves, where the predicted orbital velocity calculated from the amount of matter seen did not correspond to the observed orbital velocity.

Modified Newtonian Dynamics (MOND) by Milgrom [51–54] and Tensor-Vector-Scalar gravity (TeVeS) by Bekenstein [55] are theories that have been proposed to solve the problem of galaxy rotation curves without invoking dark matter. However, there has been strong evidence for dark matter in astronomical data (for example from the Bullet cluster [56]). The remaining problem then is to find out what dark matter is made of. On this question, the Standard Model gives us no clue.

Supernova observations provide evidence that the universe is slightly accelerating [57–59], and the cause is still unknown. For the time being, the cause of the acceleration is called "dark energy" The Standard Model of particle physics also does not provide a clue what this mysterious dark energy might be.

So far, solutions proposed to solve accelerating universe problem have been similar to the strategy employed in solving the dark matter problem. The theory for gravity needs to be modified, or the contents of the universe have to change.

Even though in the previous section it was stated that the Big Bang model is supported by astrophysical data, there are still issues that are not resolved by the model. The first one is the flatness problem. It questions the observed flatness of the universe which in turn implies a fine-tuning of initial conditions. The second is known as the horizon problem. It states that there are regions of the observed universe which have very similar temperatures, even though they should not have had any contact in the history of the universe.

Cosmological inflation, proposed independently by Guth [60] and Starobinsky [61, 62], has been proposed to tackle the above two issues. In this scenario, the universe underwent a period of very rapid expansion very early in its history. This super-rapid inflation is offered as the solution to both the flatness and horizon problems. Inflation has been accepted into the mainstream of cosmological theory but it has not been satisfactorily confirmed through experiments. Furthermore, it still lacks strong theoretical motivation. Many models of inflation require one or two scalar fields, but these scalar fields are not identified in current theories of particle physics.

Even though there have been many models and theories proposed to solve the problems listed above (which is not an exhaustive list), the ultimate test will come from experiments and observations. Currently, the most anticipated results will come from the LHC. In addition, there are several more particle physics experiments around the world (Fermilab, DESY, KEK, etc). Many astrophysical/physical cosmology experiments will also be of interest to particle physicists (WMAP, Planck, GLAST, PAMELA, etc)

1.3 Extra dimensions and the original Kaluza Klein idea

GR does not require the universe to have four dimensions, so extra dimensions are not excluded in theory. This fact has been used by many physicists to construct theories with extra dimensions. In reality, however, only four are observed, so the extra dimensions must be hidden somehow. There are two known ways to hide the extra dimensions.

1. The extra dimensions are very small and compactified. This was done in the original Kaluza-Klein model and will be discussed shortly.

2. The extra dimensions are large but have not been observed because they can only be sensed through gravitational interactions and existing probes are not sensitive enough. This will be discussed in the next chapter.

After Nordstrom's attempt to unify Einstein's special relativity with EM [63], a more successful early attempt at unification was offered by Kaluza in 1921 [64]. By solving the Einstein field equations in five dimensions, he was able to derive the equations for gravitational and electromagnetic interactions. Kaluza's idea was later refined by Klein [65, 66], and the theory became known as Kaluza-Klein theory (reviewed here [67, 68]).

The original model by Kaluza had a five dimensional metric, consisting of one temporal dimension and four spatial. The action was given by the usual Einstein-Hilbert action in five dimensions

$$S = -\frac{1}{16\pi\hat{G}} \int dy d^4x \sqrt{-\hat{g}} \hat{R}, \tag{1.8}$$

where y is the coordinate of the extra dimension. The hat denotes five dimensional quantities, and $\hat{G}$ is the five-dimensional analog to Newton's gravitational constant.

The key point in Kaluza's theory is that matter in four dimensions is a manifestation of pure geometry in five dimensions. Following Einstein, Kaluza assumed an empty universe, $\hat{G}_{ab} = 0$. The gravitational equation of motion derived from the above action is simply Einstein's equation in 5D

$$\hat{G}_{ab} \equiv \hat{R}_{ab} - \frac{1}{2}\hat{g}_{ab}\hat{R} = 0, \tag{1.9}$$

where the relevant quantities are defined exactly the same as in 4 dimensions: G_{ab} is the 5D Einstein tensor, $\hat{R}_{ab}$ is the 5D Ricci tensor obtained by contracting the 5D Riemann curvature tensor $\hat{R}^{c}{}_{adb} = \partial_d \hat{\Gamma}^c_{ab} - \partial_b \hat{\Gamma}^c_{ad} + \hat{\Gamma}^c_{dm}\hat{\Gamma}^m_{ab} - \hat{\Gamma}^c_{bm}\hat{\Gamma}^m_{ad}$ on the first and third indices, $\hat{R}$ is the 5D Ricci scalar obtained by contracting the 5D Ricci tensor, and the 5D Christoffel symbol is defined by the expression $\hat{\Gamma}^c_{ab} = \frac{1}{2}\hat{g}^{cd}\left(\partial_a \hat{g}_{bd} + \partial_b \hat{g}_{ad} - \partial_d \hat{g}_{ab}\right)$.

The metric proposed by Kaluza took the form

$$ds^2 = \left(g_{\mu\nu}(x) + \kappa^2\,\psi^2\,A_\mu A_\nu\right) dx^\mu dx^\nu + \kappa\,\psi^2\,A_\mu(x)\,dx^\mu dy + \psi^2\,dy^2, \tag{1.10}$$

where the metric signature of $g_{\mu\nu}$ is $(+\,-\,-\,-)$, A_μ is a vector field, ψ is a scalar field, and κ is a scaling parameter related to the the four-dimensional gravitational constant G by:

$$\kappa \equiv 4\sqrt{\pi G}. \tag{1.11}$$

The equations obtained from plugging in the metric into the Einstein equation are

$$G_{\mu\nu} = \frac{1}{2}\kappa^2\psi^2 T^{EM}_{\mu\nu} - \frac{1}{\psi}\left[\nabla_\mu(\partial_\nu\psi) - g_{\mu\nu}\Box\psi\right], \tag{1.12}$$

$$\nabla^\mu F_{\mu\nu} = -3\,\frac{\partial^\mu\psi}{\psi}\,F_{\mu\nu}, \tag{1.13}$$

$$\Box\psi = \frac{1}{4}\kappa^2\psi^3\,F_{\mu\nu}F^{\mu\nu}, \tag{1.14}$$

where $T^{EM}_{\mu\nu} \equiv g_{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}/4 - F^\rho_\mu F_{\nu\rho}$ is the electromagnetic energy-momentum tensor, and $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$. By setting $\psi =$ constant then the third equation vanishes and we are left with the Einstein and Maxwell equations:

$$G_{\mu\nu} = 8\pi G\psi^2 T^{EM}_{v}, \tag{1.15}$$

$$\nabla^\mu F_{\mu\nu} = 0. \tag{1.16}$$

Unfortunately, it was later shown that if $\psi =$ constant then equation (1.14) will be inconsistent unless $F_{\mu\nu}F^{\mu\nu} = 0$.

Using the above metric and Kaluza's cylinder condition which will be explained shortly, it is possible to rewrite the action to give

$$S = -\int d^4x\,\sqrt{-g}\,\psi\left(\frac{R}{16\pi G} + \frac{1}{4}\psi^2 F_{\mu\nu}F^{\mu\nu} + \frac{2}{3\kappa^2}\frac{\partial^\mu\psi\,\partial_\mu\psi}{\psi^2}\right), \tag{1.17}$$

where the four-dimensional and five-dimensional gravitational constants are related by

$$G \equiv \hat{G}/\int dy. \tag{1.18}$$

This action will also yield the familiar Einstein and Maxwell equations, and a third equation for the scalar field. We note that it is possible to obtain Brans-Dicke gravity by setting the vector field $A_\mu = 0$.

Having shown that it was possible to unify GR and EM, Kaluza now faced the problem of hiding the extra spatial dimension. He added the cylinder condition which means identifying the points $y = 0$ with $y = 2\pi$ and setting all derivatives with respect to the fifth dimension to zero. Klein showed that Kaluza's cylinder condition could be obtained naturally using two assumptions. The first is that the extra dimension is spatial with a circular topology (S^1). From a modern point of view, the S^1 topology introduced a U(1) gauge group. This explains how it is possible to obtain electromagnetism from KK theory. He also set $\psi = 1$. The usual spatial coordinates are now periodic, and any field can be expressed using $f(x^\mu, y) = f(x^\mu, y + 2\pi r)$ where r is the radius of the extra dimension. This periodicity means they can be Fourier expanded to give

$$g_{\mu\nu}(x^\mu, y) = \sum_{n=-\infty}^{n=\infty} g_{\mu\nu}^{(n)}(x^\mu)e^{iny/r}, \tag{1.19}$$

$$A_\mu(x^\mu, y) = \sum_{n=-\infty}^{n=\infty} A_\mu^{(n)}(x^\mu)e^{iny/r}, \tag{1.20}$$

$$\psi(x^\mu, y) = \sum_{n=-\infty}^{n=\infty} \psi^{(n)}e^{iny/r}, \tag{1.21}$$

where the Fourier modes are indicated by the superscript $^{(n)}$. The fields now have a momentum in the y-direction of the order n/r, sometimes referred to as the Kaluza-Klein tower. Klein's second assumption is that r is small enough, so that only the ground state modes $n = 0$ will be observable because they are independent of y. The momenta of the $n \geq 1$ modes will be too large and undetectable.

One feature of KK theory was the quantization of charge. If a massless five-dimensional scalar field $\hat{\Psi}$ is added onto the action in the form

$$S_{\hat{\Psi}} = -\int d^4x dy \sqrt{-\hat{g}}\, \partial^a \hat{\Psi}\, \partial_a \hat{\Psi}, \tag{1.22}$$

then this field can also be Fourier-expanded, yielding

$$\hat{\Psi}(x^\mu, y) = \sum_{n=-\infty}^{n=\infty} \hat{\Psi}^{(n)} e^{iny/r}. \tag{1.23}$$

This expansion is substituted into the new action to give

$$\begin{aligned} S_{\hat{\Psi}} = -\left(\int dy\right) \sum_n \int d^4x \sqrt{-g}\, & \left[\left(\partial^\mu + \frac{in\kappa A^\mu}{r}\right) \hat{\Psi}^{(n)} \left(\partial_\mu + \frac{in\kappa A_\mu}{r}\right) \hat{\Psi}^{(n)} \right. \\ & \left. - \frac{n^2}{\Psi\, r^2}\, \hat{\Psi}^{(n)\,2}\right], \end{aligned} \tag{1.24}$$

from which it can be shown that the nth Fourier mode has a quantized charge and mass given by

$$q_n = \frac{n\sqrt{16\pi G}}{r\sqrt{\psi}}, \tag{1.25}$$

$$m_n = \frac{|n|}{r\sqrt{\psi}}. \tag{1.26}$$

Unfortunately, using the above expressions to check the electron mass would result in a discrepancy of 22 orders of magnitude. This is a sign that the theory is not a good approximation of nature.

It is possible to test KK gravity, treating it as a higher dimensional extension of Einstein gravity. A discussion of the subject can be found in [69]. However, due to other, more fundamental problems with the theory, at best KK gravity can be thought of as an inspiration for modern theories containing extra dimensions. Although GR has been tested to scales as small as $10^{-6}m$ [70–72], there is still hope that there may be evidence of very small extra dimensions under $R \leq 44\,\mu m$ [72].

1.4 Motivation and outline

In this dissertation, we will focus on theoretical models that attempt to answer several of the above questions by using extra dimensions. To provide the reader with an idea of what lies ahead, we present an outline of the dissertation.

In this chapter we have reviewed the state of modern particle physics and the outstanding problems within. We focus on theories of gravitation and see that there have been alternatives proposed to Einstein's General Relativity. We are particularly interested in theories which have extra dimensions and modify physics using the extra dimensions.

In chapter 2 we introduce the braneworld scenario. Then we will review several braneworld models, in particular the one by Randall and Sundrum. We then review a braneworld model directly derived from M-theory, "The Universe as a Domain Wall" solution by Lukas, Ovrut, Stelle and Waldram. Hereafter, this solution will be referred to as the LOSW model.

In chapter 3 we discuss the LOSW model in great detail. We use perturbation theory to study the brane gravity of this model. We see that the brane gravity is of scalar-tensor type, and the radion is coupled to the bulk scalar field.

In chapter 4 we attempt to find a black hole solution. We first see that although a black string solution was permitted, it was unstable. We also see that it was not possible to have a solution with spherical symmetry. With these difficulties in mind, we attempt to construct a solution using an axisymmetric metric. The solution we found looks like a Schwarzschild solution from afar, and resembles a black string solution, but the string became infinite in length as we approach the singularity.

We conclude in chapter 5.

Chapter 2

Braneworld models

2.1 Overview

The Kaluza-Klein model is not an accurate theory to describe our universe, but it has motivated many physicists to pursue the goal of unification using extra dimensions. For a long time it was thought that extra dimensions had to be compactified in order to be hidden from observation, but some models have been proposed in which the extra dimensions can be large or even infinite. In many of these models, our universe is imagined to be a 3-brane embedded in a higher-dimensional bulk, thus the name braneworld scenarios. The Standard Model particles are confined to the hypersurface, while gravity is allowed to propagate in the bulk. This setup allows gravity to behave differently from the standard prescription from GR in 4D. Depending on the model, gravity may appear to be five-dimensional (or more) in either the short range or the long range.

Although the name braneworld is a recent coinage, the idea that particles can be confined on a submanifold embedded in higher-dimensional space has been proposed in several early models. The models by Akama [73], and independently by Rubakov and Shaposhnikov [74, 75] proposed that the universe is a topological defect and particles are bound to it. Visser [76] and Squires [77] offered ways of trapping particles gravitationally. Gibbons and Wiltshire [78] presented a model more similar to the original KK model, where higher KK modes go undetected because of a large mass gap which they argue could arise naturally if the membrane universe has curvature or negative higher-dimensional cosmological term.

The model proposed by Arkani-Hamed *et al.* [45, 46, 79], known as the ADD or Large Extra Dimensions model, received much attention when it was published. Initially proposed as a solution to the hierarchy problem, it stated that the four-dimensional Planck scale M_{Pl} is not fundamental, but instead derived from the Planck scale $M_{Pl(4+n)}$ in $(4+n)$-dimensions

$$M_{Pl}^2 = M_{Pl(4+n)}^{n+2} R^n, \tag{2.1}$$

where R is the radius of the extra dimension and n is the number of the extra dimensions. The case $n = 1$ is excluded because it would contradict Newton's law at solar system distances, but it would be possible to have unification of gravity and standard model interactions at the weak scale with $n \geq 2$. For $n = 2$ this model predicts that Newton's law would change from r^{-2} to r^{-4} Even more exciting at the time, this model predicted that this effect could be detected at the $100\mu\mathrm{m} - 1\mathrm{mm}$ range. With the most recent results, however, it seems that the extra dimensions must be smaller.

In the ADD model, gravity is weak because it leaks into the extra dimensions. The standard model particles, on the other hand, are confined to the brane at energies below the weak scale. Thus, the model predicts for particles that have high enough energies, it is possible that they may escape into the extra dimensions, and an observer on the brane would see the energy simply vanishing. This is also something that can be tested in future experiments.

The drawback of the ADD model is that it does not actually solve the hierarchy problem, but merely shifts it to the hierarchy between the weak scale and the compactification radius.

The Randall-Sundrum model [47] offered a different solution to the hierarchy problem without introducing a new hierarchy using a warped extra dimension. The problem of obtaining 4D gravity on the brane was tackled in a second paper [48] by removing one of the branes, although this model no longer addressed the hierarchy problem. The Randall-Sundrum models will be discussed in more detail in the next section.

Lukas, Ovrut, Stelle and Waldram [82], working from eleven-dimensional Horava-Witten theory [80,81], found a solution to the effective five-dimensional theory which corresponds to two parallel 3-branes. These two branes are separated by the eleventh dimension containing a scalar field which arose from the deformation of the Calabi-Yau background metric. This LOSW model appears similar to the RS model, with the addition of a scalar field in the bulk, but has the added advantage of being well motivated from string theory. The LOSW model will be discussed in the section following the next, and is the main topic of this dissertation.

The braneworld models discussed so far have predicted that gravity will be modified at short distances. There are other models in which the universe appears four-dimensional at small scales, but five-dimensional at large scales, or even at both extremes. An example is the model by Kogan *et al.* [83] with negative tension branes that are free to move.

A similar model was proposed by Gregory, Rubakov and Sibiryakov (GRS) [84] where gravity is effectively higher dimensional at both small and very large distances. In a later model [85], it was even argued that at very large distances, the interaction became anti-gravity. It was shown in [86] that the Kogan *et al.* model and the GRS model are related. Unfortunately the GRS model was shown to violate the weak energy condition and would likely be unstable [87].

The brane induced gravity model was proposed by Dvali, Gabadadze and Porrati (DGP) [88]. In this model, gravity is 4D at short distances, but 5D at large distances. The model consists of a 3-brane in a 5D flat bulk of infinite size. By including an explicit Ricci scalar curvature term in the brane action, it is possible to obtain 4D scalar-tensor gravity on a brane embedded in 5D space. In a later paper [89], 4D tensor gravity was obtained.

The most remarkable aspect of the DGP model is that it allows solutions in which the universe is accelerating when the brane tension is zero [90, 91]. This makes the model extremely appealing to cosmologists who are looking for an alternative to dark energy to explain the observed acceleration of the universe.

Although the DGP model looked promising as an alternative to modifying the universe content using dark energy, and there even is a way to test the model using

Lunar laser ranging [92], it is not possible to embed it in string theory. If, however, there emerges experimental evidence to support the DGP model, then it will also be a way to falsify local quantum field theory and perturbative string theory [93]. Other drawbacks to the DGP model include ghosts in the self-accelerating branch [94, 95], pressure singularities and other problems [96]. Fortunately, the DGP model is not the only one capable of producing accelerating universes, and other models have been proposed [97, 98].

In addition to the above list, more detailed introductions to the braneworld scenario, including their applications to cosmology can be found in [99–104]. Having gone through the many braneworld proposals and the wide variety of their consequences, we will now focus on the Randall-Sundrum and the LOSW models.

2.2 Randall-Sundrum braneworlds

Out of the many braneworld proposals, the one by Randall and Sundrum is the most popular. It is also the most important because many braneworld techniques were developed using this model. It is codimension 1, meaning it has only one extra dimension. The model consists of two parts, which we will call RS1 and RS2 here.

2.2.1 RS1

The first RS model [47] (RS1) attempted to solve the hierarchy problem with the simple idea that gravity is weak because of a warped extra dimension. It consisted of two parallel three branes separated by a fifth dimension that was large but finite. The two branes are located at the orbifold fixed points.

The action describing the RS1 model is

$$\begin{aligned} S = & \int d^4x \int_{-\pi}^{\pi} dz \sqrt{-g}\left(-\Lambda + 2M^3 R\right) + \int d^4x \sqrt{-g^{vis}}\left(\mathcal{L}_{vis} - V_{vis}\right) \\ & + \int d^4x \sqrt{-g^{hid}}\left(\mathcal{L}_{hid} - V_{hid}\right), \end{aligned} \tag{2.2}$$

where M is the fundamental five-dimensional Planck scale, g_{ab}, is the five-dimensional metric, z is the coordinate of the extra dimension from $-\pi$ to π and the brane metrics are given by

$$g_{\mu\nu}^{vis}(x^\mu) \equiv g_{\mu\nu}(x^\mu, z=\pi),$$
$$g_{\mu\nu}^{hid}(x^\mu) \equiv g_{\mu\nu}(x^\mu, z=0), \tag{2.3}$$

The convention we use is that Latin indices denote five-dimensional quantities, and Greek indices denote four-dimensional ones.

From the above action we get the Einstein equation

$$\sqrt{-g}\left(R_{ab} - \frac{1}{2}g_{ab}R\right) = -\frac{1}{4M^3}[\Lambda\sqrt{-g}\ g_{ab} + V_{vis}\sqrt{-g_{vis}}\ g_{\mu\nu}^{vis}\ \delta_a^\mu \delta_b^\nu\ \delta(z-\pi)$$
$$+ V_{hid}\sqrt{-g_{hid}}\ g_{\mu\nu}^{hid}\ \delta_a^\mu \delta_b^\nu\ \delta(z)]. \tag{2.4}$$

The metric that solves the above equations is given by

$$ds^2 = e^{-2kr_c|z|}\eta_{\mu\nu}dx^\mu dx^\nu + r_c^2 dz^2, \tag{2.5}$$

where $a(z) = e^{-2kr_c|z|}$ is called the warp factor shown in Figure 2.1, r_c is the compactification radius and is independent of x. In order to have a solution that respects four-dimensional Poincare invariance, the relations between the boundary and bulk cosmological terms must satisfy

$$V_{hid} = -V_{vis} = 24M^3k,$$
$$\Lambda = -24M^3k^2. \tag{2.6}$$

The bulk geometry is a slice of AdS_5 and $\Lambda < 0$.

To understand the physical implications of the RS1 model, we study the four-dimensional effective theory. The zero modes of the classical solution can be expressed by

$$ds^2 = e^{-2kT(x)|z|}[\eta_{\mu\nu} + h_{\mu\nu}(x)]dx^\mu dx^\nu + T^2(x)dz^2, \tag{2.7}$$

where $h_{\mu\nu}$ is the physical graviton of the four-dimensional effective theory and the massless mode in the Kaluza-Klein decomposition of $g_{\mu\nu}$, and $T(x)$ is a modulus field. By substituting (2.7) into (2.2) we obtain the four-dimensional effective action

$$S_{eff} \supset \int d^4x \int_{-\pi}^{\pi} dz\ 2M^3 r_c e^{-2kr_c|z|}\sqrt{-g^{(4)}}\ R^{(4)}, \tag{2.8}$$

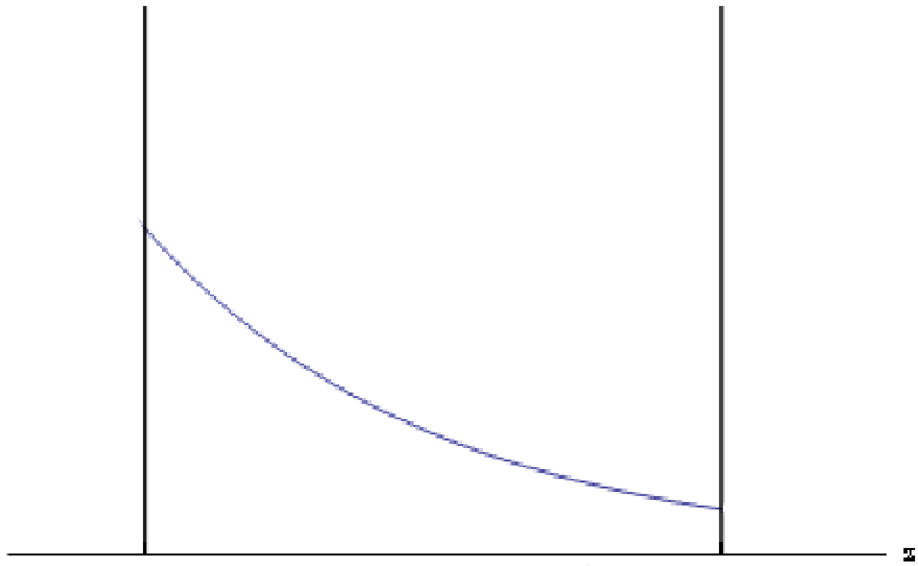

Figure 2.1: A plot of the RS1 warp factor. The two branes are shown as vertical lines.

with $g^{(4)}_{\mu\nu}(x) \equiv \eta_{\mu\nu} + h_{\mu\nu}(x)$. This can be integrated over z, from which we get a relation between the four-dimensional Planck mass M_{Pl} and the five-dimensional Planck scale M

$$M_{Pl}^2 = \frac{M^3}{k}[1 - e^{-2kr_c\pi}]. \tag{2.9}$$

From this relation we see that M_{Pl} depends only weakly on r_c in the large kr_c limit and this relation on its own does not solve the hierarchy problem.

On the other hand, the physical mass m that is observed on the brane is derived from a five-dimensional mass parameter m_0 using the relation

$$m \equiv e^{-kr_c\pi} m_0. \tag{2.10}$$

From the above equation, if $e^{kr_c\pi} \sim 10^{15}$ then it is possible to have an observed Higgs mass of TeV order from a five-dimensional mass parameter in the order of the Planck scale. In other words, the exponential function allows the two scales to be related by a relatively small number.

There is one more aspect of the RS1 to discuss, namely the extra degree of freedom associated with the separation between the two branes. This corresponds to a massless four-dimensional scalar called the radion. Initially the radion appeared in the original RS1 model as the modulus $T(x)$. However it was argued in [105] that this method of inserting the radion into the model would imply that it would have no interaction with matter on the positive tension brane. It was then shown that the RS1 metric can be expressed as

$$ds^2 = e^{-2k(z+f(x)e^{2kz})} g_{\mu\nu}(x) dx^\mu dx^\nu + \left(1 + 2kf(x)e^{2kz}\right)^2 dz^2, \tag{2.11}$$

where f is the radion mode that correctly solves the linearised equation of motion. This new form of the metric should then describe the full long distance dynamics of the RS model.

It is well known from previous work on higher dimensional theories that the modulus field must be stabilised [106, 107]. If the radion fluctuates, then the mechanism to solve the hierarchy problem as described above will not work. It is then clear that the radion must be stabilised, for example by adding a bulk scalar [108–110], or by using gaugino condensates in the bulk and on a brane [111]. Tanaka and Montes [112] showed that by using the Goldberger and Wise mechanism [108, 109], the radion mode would vanish.

2.2.2 RS2

The second RS model [48] (RS2), is obtained by removing one of the branes from the first RS model to infinity. In effect, the model consisted of one brane and an extra dimension which was infinite. This model no longer addressed the question of solving the hierarchy problem, but it provided a way to obtain Einstein gravity on the brane.

Because the setup is basically the same as RS1, the expressions derived in the previous subsection will still be relevant here. The warp factor for RS2 is shown in Figure 2.2. The second brane is not placed at infinity until later.

To study the brane gravity we use perturbation theory as originally discussed by Garriga and Tanaka [113]. It is easier to work with the metric in Gaussian Normal gauge so the y-coordinate measures the proper distance from the brane [128]

$$ds^2 = g_{ab} dx^a dx^b = \gamma_{\mu\nu} dx^\mu dx^\nu + dz^2 = a^2(z) \eta_{\mu\nu} dx^\mu dx^\nu + dz^2, \tag{2.12}$$

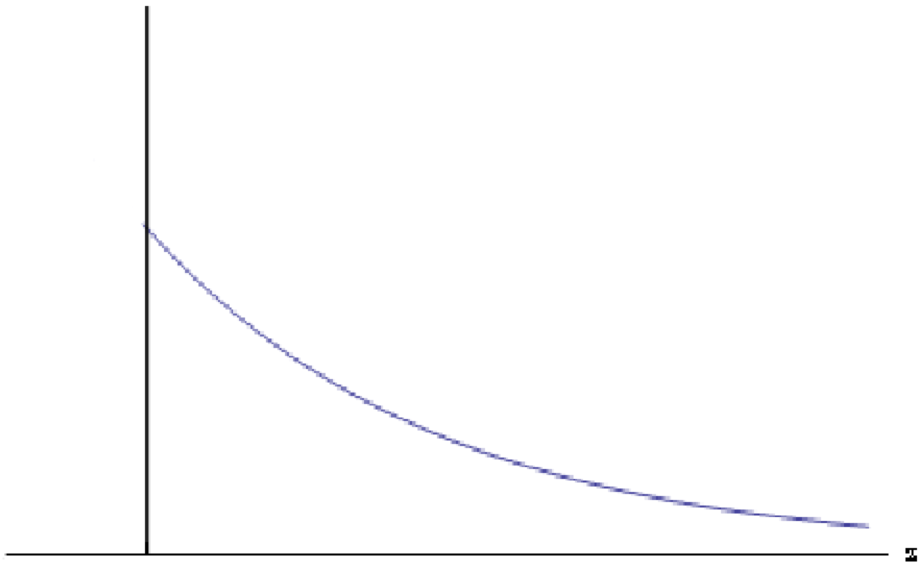

Figure 2.2: A plot of the RS2 warp factor. The single brane is shown as a vertical line.

where the warp factor is now $a(z) = e^{-|z|/\ell}$. We denote the perturbed metric by

$$\tilde{g}_{ab} = g_{ab} + h_{ab}. \tag{2.13}$$

In order to simplify the calculation, the following gauge is chosen

$$h_{zz} = h_{\mu z} = 0, \quad h_{\mu}{}^{\lambda}{}_{,\lambda} = 0, \quad h^{\mu}{}_{\mu} = 0, \tag{2.14}$$

where commas denote partial derivatives. However this choice of gauge means that the brane will not be located at $z = 0$. To work in Gaussian Normal coordinates, we consider diffeomorphisms of the form

$$z \longrightarrow \tilde{z} = z + \xi^z(x^\mu, z), \tag{2.15}$$

$$x^\mu \longrightarrow \tilde{x}^\mu = x^\mu + \xi^\mu(x^\mu, z), \tag{2.16}$$

which will cause the metric perturbations to transform according to $\tilde{h}_{ab} = h_{ab} - \nabla_a \xi_b - \nabla_b \xi_a$, so we get

$$\tilde{h}_{\mu\nu} = h_{\mu\nu} - (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu + 2aa'\eta_{\mu\nu}\xi_z), \tag{2.17}$$

$$\tilde{h}_{\mu z} = h_{\mu z} - (\partial_\mu \xi_z + \partial_z \xi_\mu - 2\frac{a'}{a}\xi_\mu), \tag{2.18}$$

$$\tilde{h}_{zz} = h_{zz} - 2\partial_z \xi_z, \tag{2.19}$$

Applying the requirement $\tilde{h}_{\mu z} = \tilde{h}_{zz} = 0$ and $h_{\mu z} = h_{zz} = 0$, we get the transformations

$$\begin{aligned} \xi^z &= \hat{\xi}^z(x^\mu), \\ \xi^\mu &= \frac{-\ell}{2}\gamma^{\mu\nu}\hat{\xi}^z(x^\rho)_{,\nu} + \hat{\xi}^\mu(x^\rho). \end{aligned} \tag{2.20}$$

Using the above two relations, (2.17) becomes

$$h_{\mu\nu} = \tilde{h}_{\mu\nu} - \ell\hat{\xi}^5{}_{,\mu\nu} - 2\ell^{-1}\gamma_{\mu\nu}\hat{\xi}^5 + \gamma_{\rho(\mu}\hat{\xi}^\rho{}_{,\nu)}. \tag{2.21}$$

The equation of motion for the perturbed metric is

$$\delta R_{ab} = -\frac{1}{2}\Delta_L h_{ab}, \tag{2.22}$$

where $\Delta_L h_{ab}$ is called the Lichnerowicz operator defined by

$$\Delta_L h_{ab} = \Box h_{ab} + 2R_a{}^c{}_b{}^d h_{cd} - 2\nabla_{(a}\nabla_{|c|}\bar{h}^c_{b)} - 2R_{c(a}h_{b)}{}^c, \tag{2.23}$$

and $\bar{h}_{ab} = h_{ab} - \frac{1}{2}hg_{ab}$ is the trace-reversed metric perturbation. The junction conditions at $z = 0+$ give us

$$(\partial_z + 2\ell^{-1})\tilde{h}_{\mu\nu} = -\kappa\left(T_{\mu\nu} - \frac{1}{3}\gamma_{\mu\nu}T\right), \tag{2.24}$$

$T = T^\mu{}_\mu$, and $\kappa = 8\pi G_5$.

Using the RS metric and the above definitions, the equation of motion is given by

$$\left[a^{-2}\Box^{(4)} + \partial_z^2 - 4\ell^{-2}\right] h_{\mu\nu} = 0. \tag{2.25}$$

Using (2.21), the junction conditions at $z = 0+$ (2.24) becomes

$$(\partial_y + 2\ell^{-1})h_{\mu\nu} = -\kappa\Sigma_{\mu\nu}, \tag{2.26}$$

where we have introduced the combination

$$\Sigma_{\mu\nu} = \left(T_{\mu\nu} - \frac{1}{3}\gamma_{\mu\nu}T\right) + 2\kappa^{-1}\hat{\xi}^5_{,\mu\nu}. \tag{2.27}$$

The equation of motion can be combined with the junctrion condition to give

$$\left[a^{-2}\Box^{(4)} + \partial_z^2 - 4\ell^{-2} + 4\ell^{-1}\delta(z)\right] h_{\mu\nu} = -2\kappa\Sigma_{\mu\nu}\delta(z). \tag{2.28}$$

The 5D retarded Green's function satisfies

$$\left[a^{-2}\Box^{(4)} + \partial_z^2 - 4\ell^{-2} + 4\ell^{-1}\delta(z)\right] G_R(x,x') = \delta^{(5)}(x-x'). \tag{2.29}$$

The solution to (2.28) is then given by

$$h_{\mu\nu}(x) = -2\kappa \int d^4x' G_R(x,x')\Sigma_{\mu\nu}(x'). \tag{2.30}$$

The condition $h^\mu{}_\mu = 0$ implies $\Sigma^\mu{}_\mu = 0$ and we have

$$\Box^{(4)}\hat{\xi}^5 = \frac{\kappa}{6}T. \tag{2.31}$$

The Green's function itself is given by

$$G_R(x,x') = -\int \frac{d^4k}{(2\pi)^4} e^{ik_\mu(x^\mu - x'^\mu)} \left[\frac{a(z)^2 a(z')^2 \ell^{-1}}{\mathbf{k}^2 - (\omega + i\epsilon)^2} + \int_0^\infty dm \, \frac{u_m(z)u_m(z')}{m^2 + \mathbf{k}^2 - (\omega+i\epsilon)^2}\right], \tag{2.32}$$

where

$$u_m(z) = \sqrt{m\ell/2}\{J_1(m\ell)Y_2(m\ell/a) - Y_1(m\ell)J_2(m\ell/a)\}/\sqrt{J_1(m\ell)^2 + Y_1(m\ell)^2} \tag{2.33}$$

is the continuum of KK modes.

To find the metric on the wall, we use (2.21):

$$\tilde{h}_{\mu\nu} = h^{(m)}_{\mu\nu} + h^{(\xi)}_{,\mu\nu} + \ell\hat{\xi}^5{}_{,\mu\nu} + \frac{2}{\ell}\gamma_{\mu\nu}\hat{\xi}^5 - \hat{\xi}_{(\mu,\nu)}, \tag{2.34}$$

with the matter and brane-bending components given respectively by

$$h^{(m)}_{\mu\nu} = -2\kappa \int d^4x' G_R(x,x')\left(T_{\mu\nu} - \frac{1}{3}\gamma_{\mu\nu}T\right)(x'), \tag{2.35}$$

$$h^{(\xi)} = -4\int d^4x' G_R(x,x')\hat{\xi}^5(x'). \tag{2.36}$$

On the wall $z=0$ and $\hat{\xi}_\mu$ can be chosen such that the result is a simple expression

$$\tilde{h}_{\mu\nu} = h^{(m)}_{\mu\nu} + \frac{2}{\ell}\gamma_{\mu\nu}\hat{\xi}^5. \tag{2.37}$$

Finally, if both arguments of the two-point function are on the wall, then $G_R(x,x')$ is dominated by the zero mode contribution, $G_R(x,x') \approx \delta^{(4)}(x^\mu - x^{\mu\prime})/\ell\,\Box^{(4)}$. Using this result we find the induced metric on the wall is given by the expression

$$\tilde{h}_{\mu\nu} = -16\pi G \frac{1}{\Box^{(4)}}\left(T_{\mu\nu} - \frac{1}{2}\gamma_{\mu\nu}T\right), \tag{2.38}$$

where $G = G_5/\ell$ is the four dimensional Newton's constant. The $\hat{\xi}^5$ term has been absorbed by choosing a more appropriate gauge and we have recovered the linearised Einstein equation.

An active area of research within the braneworld framework is the search for black hole solutions [114]. Black holes have strong gravitational fields so they are good for testing new theories of gravity. A black hole on the brane is in an accelerating frame because geodesics in the bulk curve away from the brane, so the C-metric description is suitable. The original C-metric described two pairs of accelerating black holes [115]. Although so far there exists no five-dimensional counterpart of the C-metric, this idea has been tested in a lower dimensional setting by Emparan, Horowitz and Myers [116].

An attempt to find a black hole solution within the RS braneworld model was first performed by Chamblin, Hawking and Reall [117]. Unfortunately there were two problems with their solution. The first was that the adS horizon was singular, although this singularity is removed in the case of a black string in the RS1 model because the singularity lies beyond the positive tension brane. The second problem was that the solution was unstable [118].

A second approach to finding a black hole solution is by using the method of Shiromizu, Maeda and Sasaki [119] to derive a four-dimensional equation reminiscent of the Tolman-Oppenheimer-Volkoff (TOV) equation, that is the Einstein equation for a spherically symmetric metric with a perfect fluid energy-momentum tensor. The TOV equation is also known as the equation of hydrostatic equilibrium and is used to describe the interior of stars. It is then possible to define relations between the unknown parameters in order to solve the equations. An example of a solution found using this method is described in [120].

The brane approach described above has the drawback of not being able to describe the bulk behaviour in full. To remedy this issue, the bulk approach was applied to find a black hole solution on the brane. As the name implies, this method starts by taking a known bulk solution and uses the Israel junction conditions to find the brane trajectories. By adding matter that corresponds to a homogeneous and isotropic fluid to a spherically symmetric brane, the brane trajectories will then yield the equivalent of the TOV equations. Creek *et al.* used this method to find brane star solutions, but they did not find black hole solutions because their solutions were completely nonsingular [121].

Other methods that have been tried to find black hole solutions include a perturbative approach, for example in [122–125], and numerical methods, for example in [126].

There has been considerable success in applying the braneworld models (RS) to cosmology. Binetruy, Deffayet and Langlois used the brane based approach of Shiromizu *et al.* [119] to show that braneworld cosmology will be different from standard FLRW cosmology [127]. This problem can be fixed by adding a bulk cosmological constant, and the resulting model will be more compatible with standard cosmology [128].

The bulk based approach for braneworld cosmology was described by Ida [129]. This work is a generalization of an earlier paper by Kraus [130] describing moving domain walls in the RS bulk. It was shown that in [131] that a fixed brane in a non-static bulk is equivalent to a moving brane in a static bulk. Thus above two methods are equivalent to each other.

One way of testing the RS model is by using gravitational lensing. It was found that strong gravitational lensing from braneworld black holes may have different observational signatures from regular four-dimensional black holes [132]. Alternatively, if braneworld black holes acted as dark matter, this can be tested with gravitational atto-lensing that will produce interference patterns of gamma rays [133].

It is also possible to test the RS model using collider experiments. If miniature black holes are produced through high-energy particle collisions at the LHC, then they may be detected by the emission of Hawking radiation when the black hole evaporates or the missing energy when particles escape into the bulk [134–137].

Another collider experiment involves the search for the radion. If the radion in the RS1 model is stabilised then it should acquire a mass that would be within the reach of future experiments at the LHC [109]. More details on the topic can be found in [138].

2.3 Heterotic M-theory

Modern attempts at unification have been mostly under the banner of String Theory. There are other models and theories being developed (for example Loop Quantum Gravity), but we will not discuss them further. Although it has yet to be verified by experiments, string theory is appealing because of its mathematical consistency. Initially string theory contained only bosons and required 26 dimensions to be mathematically consistent. By imposing supersymmetry, the number of dimensions can be reduced to 10. Before string dualities were discovered, it was thought that there were five versions of string theory. Witten proposed that these five different theories were merely different aspects of a single theory in 11 dimensions and this became known as M-theory [139–141].

Furthermore, the low energy limit of the new theory is the well studied supergravity in $D = 11$ [142]. Horava and Witten then showed that the strongly coupled $E_8 \times E_8$ heterotic string can be identified as the eleven-dimensional limit of M-theory compactified on an S^1/Z_2orbifold with a set of E_8 gauge fields on each orbifold fixed plane [80, 81]. This M-theory limit can be compactified to four dimensions on a deformed Calabi-Yau threefold [143]. However, matching the phenomenological gravitational and grand-unified couplings [143, 144] shows the orbifold must be larger than the Calabi-Yau radius. This suggests that at energies below the unification scale there is a regime where the universe appears five-dimensional.

Lukas *et al.* showed that it was possible to construct a five-dimensional effective theory of Horava-Witten heterotic M-theory by compactifying six spatial dimensions on a Calabi-Yau manifold. This effective theory allows a solution in which our universe is one of two four-dimensional domain walls separated by an extra dimension containing a scalar field/modulus V which encodes the variation of the Calabi-Yau volume along the orbifold (the Calabi-Yau breathing mode) [82]. The setup is similar to RS1 although now there is an additional scalar field in the bulk, and it is appealing because it is well motivated from string theory.

The action for heterotic M-theory (we have dropped some terms from the original

LOSW action) is given by

$$S_5 = S_{\text{grav}} + S_{\text{hyper}} + S_{\text{bound}}, \tag{2.39}$$

where

$$S_{\text{grav}} = -\frac{1}{2\kappa_5^2}\int_{M_5}\sqrt{-g}R, \tag{2.40}$$

$$S_{\text{hyper}} = -\frac{1}{2\kappa_5^2}\int_{M_5}\sqrt{-g}\left[\frac{1}{2}V^{-2}\partial_m V\partial^m V + \frac{1}{3}V^{-2}\alpha^2\right], \tag{2.41}$$

$$S_{\text{bound}} = -\frac{1}{2\kappa_5^2}\left\{-2\sqrt{2}\int_{M_4^{(1)}}\sqrt{-g}\,V^{-1}\alpha + 2\sqrt{2}\int_{M_4^{(2)}}\sqrt{-g}\,V^{-1}\alpha\right\}. \tag{2.42}$$

where α is a constant parametrizing the compactified Calabi–Yau, $\kappa_5^2 = 8\pi G_5$ and G_5 is the 5D Newtonian constant.

We obtain the equations of motion by varying the action

$$G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R = \frac{1}{4}g_{ab}V^{-2}\partial_m V\partial^m V - \frac{1}{2}V^{-2}\partial_a V\partial_b V + \frac{\alpha}{6}g_{ab}V^{-2}$$
$$- \sqrt{2}\alpha V^{-1}g_{\rho\sigma}\delta_a^\rho\delta_b^\sigma(\delta(y) - \delta(y-\pi\rho))g_{yy}^{-1/2}, \tag{2.43}$$

$$V^{-1}\partial_m V\partial^m V + V^{-2}\partial_m V\partial^m V = -\frac{2}{3}\alpha V^{-2} - 2\sqrt{2}\alpha V^{-1}(\delta(y) - \delta(y-\pi\rho))g_{yy}^{-1/2}. \tag{2.44}$$

The solutions to the equations of motion are given by

$$ds_5^2 = a(y)^2 dx^\mu dx^\nu \eta_{\mu\nu} + b(y)^2 dy^2, \tag{2.45}$$

$$V = V(y) \tag{2.46}$$

where

$$a = a_0 H^{1/2}, \tag{2.47}$$

$$b = b_0 H^2, \tag{2.48}$$

$$V = b_0 H^3, \tag{2.49}$$

$$H = \frac{\sqrt{2}}{3}\alpha|y| + c_0, \tag{2.50}$$

with a_0, b_0 and c_0 constants. The boundary sources at $y = 0$ and $y = \pi\rho$ have to be matched, so we glue together the two pieces $y \in [0, \pi\rho]$ and $y \in [-\pi\rho, 0]$ and obtain

$$\partial_y^2 H = \frac{2\sqrt{2}}{3}\alpha(\delta(y) - \delta(y-\pi\rho)). \tag{2.51}$$

From this we see that the solution consists of two parallel three-branes located at the orbifold planes.

Cosmological solutions of heterotic M-theory were found by Lukas *et al.* by assuming separability [145]. Their work was generalised by Reall in [146]. Lukas *et al.* also found inflationary solutions using the vacuum energy of the boundary [147]. Chamblin and Reall focused on the case of a static bulk and were able to find inflating cosmological solutions [148]. The work of Chamblin and Reall is similar to the work of Kraus [130] mentioned in the previous section, in that the solutions they found were of branes moving in a static bulk. Ellwanger argued that by adding matter on the brane, it was possible to recover the standard cosmological evolution [149].

Khoury *et al.* argued that a collision of a brane with a bounding orbifold plane would produce the Big Bang [150, 151]. This scenario later became known as the ekpyrotic universe. Steinhardt and Turok proposed a universe that cycles endlessly from a big bang to a big crunch [152, 153].

Chen *et al.* concluded that the cosmological solution of Horava-Witten theory would evolve to a singularity that will annihilate the universe [154]. However, Lehners, McFadden and Turok argued that the singularity is mild and the branes will simply bounce [155, 156]

Arnowitt, Dent and Dutta showed that it is possible to obtain FRW cosmology for relativistic matter on the branes, assuming a static volume modulis and a static fifth dimension. However, it was not possible to do the same for non-relativistic matter [157].

So far, we have seen that there are many varieties of the braneworld scenario. We have discussed the Randall-Sundrum model in great detail and some of the techniques used to find braneworld black hole and cosmological solutions. We have also discussed the heterotic braneworld model derived from Horava-Witten M-theory, and we will analyze the brane gravity and look for a black hole solution in the following two chapters.

Chapter 3

Heterotic braneworld gravity

3.1 Perturbations of the heterotic braneworld

We will now study the brane gravity and the behaviour of the scalar field in the heterotic braneworld model. We work with the domain wall solution to heterotic M-theory proposed by Lukas *et al.* in [82]. In that paper, they compactified 6 dimensions out of the 11-dimensional theory, and found a solution consisting of two parallel three-branes separated by the remaining dimension. The compactified 6 dimensions are represented by a scalar field. This setup reminds us of the first Randall-Sundrum model consisting of two parallel branes (RS1) [47], but with the addition of a bulk scalar field.

The heterotic model has been used as a basis for interesting cosmological solutions, notably the ekpyrotic universe and the cyclic universe models. However, a full description of the brane gravity has not been found in the literature. In particular, because the heterotic braneworld model has a scalar field in the bulk, it is important to study how this scalar field behaves. This chapter is an expanded version of the work published in [1].

We choose to rescale the value of α

$$\alpha_{LOSW} \to 3\sqrt{2}\alpha, \tag{3.1}$$

and to parameterise the modulus V

$$V(y) = e^{\phi(y)}, \tag{3.2}$$

so that the heterotic action is

$$
\begin{aligned}
S =& \frac{1}{2\kappa_5^2}\int d^5x\sqrt{-g}\left[R-\frac{1}{2}(\partial\phi)^2-6\alpha^2e^{-2\phi}\right]+\int_{y=\mp y_0} d^4x\sqrt{-g^{\mp}}\mathcal{L}_{\text{matter}} \\
&+\frac{6\alpha}{\kappa_5^2}\left[\int_{y=-y_0} d^4x\sqrt{-g^-}e^{-\phi}-\int_{y=+y_0} d^4x\sqrt{-g^+}e^{-\phi}\right]. \qquad (3.3)
\end{aligned}
$$

The branes are located at $y=-y_0$ and $y=+y_0$. Varying the action with respect to the inverse metric and the scalar field ϕ yields

$$
\begin{aligned}
\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{ab}} = G_{ab} &- \frac{1}{2}\phi_{,a}\phi_{,b}+\frac{1}{4}g_{ab}\phi_{,}{}^{c}\phi_{,c}+3\alpha^2e^{-2\phi}g_{ab} \\
&-6\alpha e^{-\phi}g_{\mu\nu}\delta_a^\mu\delta_b^\nu\frac{[\mathcal{D}]}{\sqrt{g_{yy}}}-\kappa_5^2T_{ab}^{\mp}=0, \qquad (3.4)
\end{aligned}
$$

$$
\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta\phi}=\Box\phi+12\alpha^2e^{-2\phi}-12\alpha e^{-\phi}\frac{[\mathcal{D}]}{\sqrt{g_{yy}}}=0, \qquad (3.5)
$$

where we have introduced the notation $[\mathcal{D}]=[\delta(y+y_0)-\delta(y-y_0)]$ for brevity. The energy-momentum tensor of matter on the brane is

$$
T_{ab}^{\mp}=\delta_a^\mu\delta_b^\nu T_{\mu\nu}^{\mp}\frac{\delta(y\pm y_0)}{\sqrt{g_{yy}}}, \qquad (3.6)
$$

with

$$
T_{\mu\nu}^{\mp}=\frac{-2}{\sqrt{-g^{\mp}}}\frac{\delta(\sqrt{-g^{\mp}}\mathcal{L}_{matter})}{\delta g^{\mu\nu}}. \qquad (3.7)
$$

The Einstein equation is given by

$$
G_{ab}=\frac{1}{2}\partial_a\phi\partial_b\phi-\frac{1}{4}g_{ab}\partial_c\phi\partial^c\phi-3\alpha^2e^{-2\phi}g_{ab}+6\alpha e^{-\phi}g_{\mu\nu}\delta_a^\mu\delta_b^\nu\frac{[\mathcal{D}]}{\sqrt{g_{yy}}}+\kappa_5^2T_{ab}^{\mp}, \qquad (3.8)
$$

and the scalar field equation is

$$
\Box\phi=-12\alpha^2e^{-2\phi}+12\alpha e^{-\phi}\frac{[\mathcal{D}]}{\sqrt{g_{yy}}}, \qquad (3.9)
$$

where $\Box\phi=\nabla_a\nabla^a\phi$ is the d'Alembertian operator.

We make the following gauge choice that preserves the Gaussian normal (GN) coordinate system to simplify the perturbation analysis we perform in the next section

$$
g_{yy}=1, \qquad g_{\mu y}=0, \qquad (3.10)
$$

so we begin with the metric ansatz

$$
ds^2=a^2(y)\eta_{\mu\nu}dx^\mu dx^\nu+dy^2, \qquad (3.11)
$$

where the signature of $\eta_{\mu\nu}$ is $(- + ++)$. This metric is similar to the RS metric, however the equations of motion will be different because of the additional scalar field. Following the discussion by Garriga and Tanaka [113], this choice of gauge will usually mean that the branes will not be located at fixed points. The brane bending terms will be analyzed later.

The non-zero Christoffel symbols are:

$$\begin{aligned} \Gamma^{y}_{\mu\nu} &= -aa'\eta_{\mu\nu}, \\ \Gamma^{\lambda}_{\mu y} &= \frac{a'}{a}\delta^{\lambda}_{\mu}, \end{aligned} \tag{3.12}$$

where the prime denotes differentiation with respect to y. This enables us to calculate the elements of the Riemann curvature tensor:

$$\begin{aligned} R^{\lambda}_{y\kappa y} &= -\frac{a''}{a}\delta^{\lambda}_{\kappa}, \\ R^{y}_{\mu y\nu} &= -aa''\eta_{\mu\nu}, \\ R^{\lambda}_{\mu\kappa\nu} &= -(a')^2\delta^{\lambda}_{\kappa}\eta_{\mu\nu} + (a')^2\delta^{\lambda}_{\nu}\eta_{\mu\kappa}, \end{aligned} \tag{3.13}$$

the elements of the Ricci tensor:

$$\begin{aligned} R_{yy} &= -4\frac{a''}{a}, \\ R_{\mu\nu} &= -(3a'^2 + aa'')\eta_{\mu\nu}, \end{aligned} \tag{3.14}$$

and the Ricci scalar is given by:

$$R = -4\left(3\frac{a'^2}{a^2} + 2\frac{a''}{a}\right). \tag{3.15}$$

Finally we have the elements of the Einstein tensor:

$$\begin{aligned} G_{\mu\nu} &= 3(a'^2 + a''a)\eta_{\mu\nu}, \\ G_{yy} &= 6\frac{a'^2}{a^2}, \end{aligned} \tag{3.16}$$

and the d'Alembertian operator:

$$\Box\phi = \phi'' + 4\frac{a'}{a}\phi'. \tag{3.17}$$

Plugging the above into the equations of motion, we find the solution is given by

$$a(y) = (6\alpha|y| + c_0)^{1/6}\,, \tag{3.18}$$

$$\phi(y) = \ln(6\alpha|y| + c_0)\,. \tag{3.19}$$

Now we will study the brane gravity of the heterotic model using perturbation theory.

Consider the following linearised metric and scalar field perturbations

$$g_{ab} \to g_{ab} + h_{ab}, \tag{3.20}$$

$$\phi(y) \to \phi(y) + \delta\phi(x^\mu, y). \tag{3.21}$$

Rewriting the Einstein equation (3.8) in terms of the Ricci tensor

$$\begin{aligned} R_{ab} = {}& \frac{1}{2}\partial\phi\partial\phi + 2\alpha^2 e^{-2\phi} g_{ab} + \left(-8g_{ab} + 6\delta^\mu_a\delta^\nu_b g_{\mu\nu}\right)\alpha e^{-\phi}[\mathcal{D}] \\ & + \kappa_5^2\left[T^{\mp}_{ab} - \frac{1}{3}g_{ab}T^{\mp c}{}_c\right], \end{aligned} \tag{3.22}$$

enables us to compute the linearised Einstein equation $\Delta_L h_{ab} = -2\delta R_{ab}$ where

$$\begin{aligned} \Delta_L h_{ab} = {}& \Box h_{ab} - 2\nabla_{(a}\nabla_{|c|}\bar{h}^c_{b)} + 2R_{acbd}h^{cd} - \partial_c\phi\partial_{(a}h^c_{b)} \\ & - 4\alpha^2 e^{-2\phi}h_{ab} + \left[16h_{ab} - 12\delta^\mu_a\delta^\nu_b h_{\mu\nu}\right]\alpha e^{-\phi}[\mathcal{D}], \end{aligned} \tag{3.23}$$

$$\begin{aligned} -2\delta R_{ab} = {}& -2\partial_{(a}\phi\partial_{b)}(\delta\phi) - 4\alpha^2 e^{-2\phi}(h_{ab} - 2g_{ab}\delta\phi) \\ & + \left[16(h_{ab} - g_{ab}\delta\phi) - 12\delta^\mu_a\delta^\nu_b(h_{\mu\nu} - g_{\mu\nu}\delta\phi)\right]\alpha e^{-\phi}[\mathcal{D}] \\ & - 2\kappa_5^2\left[\delta^\mu_a\delta^\nu_b T^{\mp}_{\mu\nu} - \frac{1}{3}g_{ab}T^{\mp\lambda}{}_\lambda\right]\delta(y \pm y_0), \end{aligned} \tag{3.24}$$

eventually giving us

$$\begin{aligned} & \Box h_{ab} - 2\nabla_{(a}\nabla_{|c|}\bar{h}^c_{b)} + 2R_{acbd}h^{cd} - \partial_c\phi\partial_{(a}h^c_{b)} \\ & \quad = -2\partial_{(a}\phi\partial_{b)}(\delta\phi) - \left(16g_{ab} - 12\delta^\mu_a\delta^\nu_b g_{\mu\nu}\right)\alpha e^{-\phi}\delta\phi[\mathcal{D}] \\ & \qquad - 2\kappa_5^2\left[\delta^\mu_a\delta^\nu_b T^{\mp}_{\mu\nu} - \frac{1}{3}g_{ab}T^{\mp\lambda}{}_\lambda\right]\delta(y \pm y_0). \end{aligned} \tag{3.25}$$

The linearised scalar field equation is

$$-h^{cd}\nabla_c\nabla_d\phi + \Box^{(5)}\delta\phi - \nabla_c\phi\nabla_d\bar{h}^{cd} = 24\alpha^2 e^{-2\phi}\delta\phi - 12\alpha e^{-\phi}\delta\phi[\mathcal{D}], \tag{3.26}$$

where in the above two equations, $\bar{h}_{ab} = h_{ab} - \frac{1}{2}hg_{ab}$ is the trace-reversed metric perturbation.

We impose the following gauge on the metric perturbation

$$h_{\mu y} = h_{yy} = 0. \tag{3.27}$$

The five-dimensional trace of the metric perturbation is related to the four-dimensional one by

$$h^{(5)a}{}_{a} = g^{ab}h_{ab} = \frac{h}{a^2}, \tag{3.28}$$

and we will always use the notation $h = h^{(4)\lambda}{}_{\lambda}$.

We can calculate the components of the Lichnerowicz operator

$$\Delta_L h_{yy} = \frac{h''}{a^2} - 2\frac{a'h'}{a^3} + 2\left(\frac{a'^2}{a^4} - \frac{a''}{a^3}\right)h = \frac{1}{a^2}\left[a^2\left(\frac{h}{a^2}\right)'\right]', \tag{3.29}$$

$$\Delta_L h_{\mu y} = \left(\frac{h_{,\mu} - h_{\mu\rho}{}^{,\rho}}{a^2}\right)', \tag{3.30}$$

$$\begin{aligned}\Delta_L h_{\mu\nu} = {} & \frac{1}{a^2}\left(\Box^{(4)}h_{\mu\nu} + h_{,\mu\nu} - h_{\nu\rho,}{}^{\rho}{}_{\mu} - h_{\mu\rho,}{}^{\rho}{}_{\nu}\right) + h''_{\mu\nu} - 2\left(\frac{a''}{a} + \frac{a'^2}{a^2}\right)h_{\mu\nu} \\ & + \eta_{\mu\nu}\left(\frac{a'}{a}h' - 2\frac{a'^2}{a^2}h\right) - 4\alpha^2 e^{-2\phi}h_{\mu\nu} + 4h_{\mu\nu}\alpha e^{-\phi}[\mathcal{D}]\,.\end{aligned} \tag{3.31}$$

Plugging in the components of the Lichnerowicz operator, we get the linearised Einstein equations

•(yy)

$$\begin{aligned}\frac{1}{a^2}\left[a^2\left(\frac{h}{a^2}\right)'\right]' &= -2\phi'(\delta\phi)' + 8\alpha^2 e^{-2\phi}\delta\phi - 16\alpha e^{-2\phi}\delta\phi[\mathcal{D}] + \frac{2}{3}\kappa_5^2\frac{T^{\mp\lambda}{}_{\lambda}}{a^2}\delta(y\pm y_0) \\ &= -12\frac{a'}{a}\left(\delta\phi\right)' + 8\left(\frac{a'}{a}\right)^2\delta\phi - 16\frac{a'}{a}\delta\phi[\mathcal{D}] + \frac{2}{3}\kappa_5^2\frac{T^{\mp\lambda}{}_{\lambda}}{a^2}\delta(y\pm y_0),\end{aligned} \tag{3.32}$$

•(μy)

$$\left[\frac{\left(h_{\mu\lambda}{}^{,\lambda} - h_{,\mu}\right)}{a^2}\right]' = \phi'(\delta\phi)_{,\mu} = 6\frac{a'}{a}(\delta\phi)_{,\mu}, \tag{3.33}$$

•$(\mu\nu)$

$$\begin{aligned}&\frac{\Box h_{\mu\nu} + h_{,\mu\nu} - 2h_{\lambda(\mu,\nu)}{}^{\lambda}}{a^2} + \frac{1}{a^2}\left[a^4\left(\frac{h_{\mu\nu}}{a^2}\right)'\right]' + a'a\left[\frac{h}{a^2}\right]'\eta_{\mu\nu} \\ &= 8\alpha^2 e^{-2\phi}\delta\phi a^2\eta_{\mu\nu} - 4\alpha e^{-\phi}\delta\phi[\mathcal{D}]a^2\eta_{\mu\nu} - 2\kappa_5^2\left(T^{\mp}_{\mu\nu} - \frac{1}{3}\eta_{\mu\nu}T^{\mp\lambda}{}_{\lambda}\right)\delta(y\pm y_0) \\ &= 8\left(\frac{a'}{a}\right)^2\delta\phi a^2\eta_{\mu\nu} - 4\frac{a'}{a}\delta\phi[\mathcal{D}]a^2\eta_{\mu\nu} - 2\kappa_5^2\left(T^{\mp}_{\mu\nu} - \frac{1}{3}\eta_{\mu\nu}T^{\mp\lambda}{}_{\lambda}\right)\delta(y\pm y_0),\end{aligned} \tag{3.34}$$

and the linearised scalar field equation is

$$\begin{aligned}\frac{1}{2}\phi'\left(\frac{h}{a^2}\right)' + \frac{\Box\delta\phi}{a^2} + (\delta\phi)'' + 4\frac{a'}{a}(\delta\phi)' &= 24\alpha^2 e^{-2\phi}\delta\phi - 12\alpha e^{-\phi}\delta\phi[\mathcal{D}] \\ &= 24\left(\frac{a'}{a}\right)^2\delta\phi - 12\frac{a'}{a}\delta\phi[\mathcal{D}].\end{aligned} \tag{3.35}$$

In the above equations we have substituted $\phi' = 6a'/a$ and $\alpha e^{-\phi} = a'/a$.

It is beneficial to introduce conformal coordinates defined by

$$dz = \frac{dy}{a}. \tag{3.36}$$

In this new variable, the warp factor is given by $a(z) = (5\alpha z)^{1/5}$. We can now rewrite the components of the linearised Einstein equations in the z-variable:

$$\begin{aligned}\frac{1}{z^{1/5}}\frac{d}{dz}\left[z^{1/5}\frac{d}{dz}\left(\frac{h}{(5\alpha z)^{2/5}}\right)\right] &= -\frac{12}{5z}\frac{d}{dz}(\delta\phi) + \frac{8}{25}\frac{\delta\phi}{z^2} \\ &\quad + \frac{2}{3}\kappa_5^2\frac{T^{\mp\lambda}{}_\lambda}{(5\alpha z)^{2/5}}\delta(y\pm y_0),\end{aligned} \tag{3.37}$$

$$\frac{d}{dz}\left[\frac{h_{,\mu} - \left(h_{\mu\lambda}{}^{,\lambda}\right)}{(5\alpha z)^{2/5}}\right] = -\frac{6}{5}\frac{(\delta\phi)_{,\mu}}{z}, \tag{3.38}$$

$$\begin{aligned}\frac{\Box h_{\mu\nu} + h_{,\mu\nu} - 2h_{\lambda(\mu,\nu)}{}^{\lambda}}{(5\alpha z)^{2/5}} + \frac{1}{z^{3/5}}\frac{d}{dz}\left[z^{3/5}\frac{d}{dz}\left(\frac{h_{\mu\nu}}{(5\alpha z)^{2/5}}\right)\right] + \frac{1}{5z}\frac{d}{dz}\left(\frac{h}{(5\alpha z)^{2/5}}\right)\eta_{\mu\nu} \\ = \frac{8}{25}\frac{\delta\phi}{z^2}\eta_{\mu\nu} - 2\kappa_5^2\left(T^{\mp}_{\mu\nu} - \frac{1}{3}\eta_{\mu\nu}T^{\mp\lambda}{}_\lambda\right)\delta(y\pm y_0),\end{aligned} \tag{3.39}$$

and the linearised scalar field equation

$$\frac{3}{5z}\frac{d}{dz}\left(\frac{h}{(5\alpha z)^{2/5}}\right) = -\frac{d^2}{dz^2}(\delta\phi) - \frac{3}{5z}\frac{d}{dz}(\delta\phi) + \frac{24}{25}\frac{\delta\phi}{z^2} - \Box\delta\phi. \tag{3.40}$$

Matching the second order derivatives with the terms containing $[\mathcal{D}]$ will give us the boundary conditions:

$$\frac{1}{a^2}\left[a^2\left(\frac{h}{a^2}\right)'\right]' = -16\frac{a'}{a}\delta\phi[\mathcal{D}] + \frac{2}{3}\kappa_5^2\frac{T^{\mp\lambda}{}_\lambda}{a^2}\delta(y\pm y_0), \tag{3.41}$$

$$\frac{1}{a^2}\left[a^4\left(\frac{h_{\mu\nu}}{a^2}\right)'\right]' = -4aa'\delta\phi[\mathcal{D}]\eta_{\mu\nu}, -2\kappa_5^2\left(T^{\mp}_{\mu\nu} - \frac{1}{3}\eta_{\mu\nu}T^{\mp\lambda}{}_\lambda\right)\delta(y\pm y_0). \tag{3.42}$$

$$(\delta\phi)'' = -12\frac{a'}{a}\delta\phi[\mathcal{D}] \tag{3.43}$$

The boundary conditions given above are equivalent to the Israel junction conditions [158] which can be also be derived using the Gauss-Codazzi formalism (for example see [159]). Integrating the boundary conditions and applying Z_2 symmetry, we find that they are given by

$$\dot{h} - \frac{2}{5}\frac{h}{z} = -\frac{8}{5}(5\alpha)^{2/5}\frac{\delta\phi}{z^{3/5}} + \frac{1}{3}\kappa_5^2(5\alpha z)^{1/5}T^{\mp\lambda}{}_{\lambda}, \tag{3.44}$$

$$\dot{h}_{\mu\nu} - \frac{2}{5}\frac{h_{\mu\nu}}{z} = -\frac{2}{5}(5\alpha)^{2/5}\frac{\delta\phi}{z^{3/5}}\eta_{\mu\nu} - \kappa_5^2(5\alpha z)^{1/5}\left(T^{\mp}_{\mu\nu} - \frac{1}{3}\eta_{\mu\nu}T^{\mp\lambda}{}_{\lambda}\right), \tag{3.45}$$

$$\dot{\delta\phi} = -\frac{6}{5}\frac{\delta\phi}{z}, \tag{3.46}$$

where we have used overdots to denote differentiation with respect to z.

3.2 Solutions to the perturbation equations

In this section, we assume there is no matter on the branes so we drop the terms containing T_{ab}. First we will find solutions assuming the transverse-tracefree gauge because it is a common choice. After seeing the restrictions, we use another approach, which is to decompose the metric perturbation into irreducible components. We then use brane-based coordinates to see the effect of the relative motion of the two branes in the heterotic model.

3.2.1 Transverse tracefree solutions

In our first attempt at solving the Einstein equations, we may assume the transverse tracefree (TTF) gauge in empty space. Explicitly, this is done by setting $h_{\mu\lambda},^{\lambda} = h = 0$. The gravitational perturbation equations become:

$$0 = -\frac{12}{5z}\frac{d}{dz}(\delta\phi) + \frac{8}{25}\frac{\delta\phi}{z^2}, \tag{3.47}$$

$$0 = -\frac{6}{5}\frac{(\delta\phi)_{,\mu}}{z}, \tag{3.48}$$

$$\frac{\Box h_{\mu\nu}}{(5\alpha z)^{2/5}} + \frac{1}{z^{3/5}}\frac{d}{dz}\left[z^{3/5}\frac{d}{dz}\left(\frac{h_{\mu\nu}}{(5\alpha z)^{2/5}}\right)\right] = \frac{8}{25}\frac{\delta\phi}{z^2}\eta_{\mu\nu}, \tag{3.49}$$

and the scalar field perturbation equation is now

$$0 = -\frac{d^2}{dz^2}(\delta\phi) - \frac{3}{5z}\frac{d}{dz}(\delta\phi) + \frac{24}{25}\frac{\delta\phi}{z^2} - \Box\delta\phi. \tag{3.50}$$

Taking the trace of (3.49) gives

$$0 = \frac{32}{25}\frac{\delta\phi}{z^2}. \tag{3.51}$$

Equations (3.47) and (3.50) tell us that $\delta\phi$ is a function of z only, but (3.48) indicates that $\delta\phi$ is a function of x^μ only. These equations, along with the trace of the $(\mu\nu)$ equation suggest that $\delta\phi = 0$ for the TTF gauge to be consistent. We are left with a simplified version of (3.49)

$$\frac{\Box h_{\mu\nu}}{(5\alpha z)^{2/5}} + \frac{1}{z^{3/5}}\frac{d}{dz}\left[z^{3/5}\frac{d}{dz}\left(\frac{h_{\mu\nu}}{(5\alpha z)^{2/5}}\right)\right] = 0. \tag{3.52}$$

This homogeneous differential equation has the solution

$$h_{\mu\nu} = z^{3/5}\left[\zeta_{\mu\nu}J_{1/5}(mz) + \chi_{\mu\nu}J_{-1/5}(mz)\right], \tag{3.53}$$

where J is a Bessel function of the first kind and we have used the approximation $N_p(z) \sim J_{-p}(z)$ to transform the Bessel function of the second kind present in the original solution. If we set $m = 0$, we get the solution

$$h_{\mu\nu} = \chi_{\mu\nu}z^{2/5} + \zeta_{\mu\nu}z^{4/5}. \tag{3.54}$$

The boundary condition is given by

$$\left(\dot{h}_{\mu\nu} - \frac{2}{5}\frac{h_{\mu\nu}}{z}\right)\Big|_{z_+=(5\alpha)^{-1}} = 0. \tag{3.55}$$

For the zero mode, this gives

$$\frac{2}{5}(5\alpha)^{1/5}\zeta_{\mu\nu} = 0, \tag{3.56}$$

and for the massive mode, this gives

$$\zeta_{\mu\nu}J_{-4/5}(\tfrac{m}{5\alpha}) - \chi_{\mu\nu}J_{4/5}(\tfrac{m}{5\alpha}) = 0, \tag{3.57}$$

from which we get the normalization

$$\zeta_{\mu\nu} = \chi_{\mu\nu}\frac{J_{4/5}(\frac{m}{5\alpha})}{J_{-4/5}(\frac{m}{5\alpha})}. \tag{3.58}$$

The solution for the TTF zero mode is

$$h_{\mu\nu} = \chi_{\mu\nu}z^{2/5}, \tag{3.59}$$

which we identify as the graviton. The TTF massive mode solution is given by $h_{\mu\nu} = u_m(y)\chi_{\mu\nu}(x^\mu)$ with

$$u_m(y) = z^{3/5}\left[\frac{J_{4/5}(\frac{m}{5\alpha})}{J_{-4/5}(\frac{m}{5\alpha})}J_{1/5}(mz) + J_{-1/5}(mz)\right]. \tag{3.60}$$

Using the delta-function normalization

$$\int \frac{dy}{a^2} u_m(y) u_{m'}(y) = \delta(m - m') \tag{3.61}$$

and the identity

$$\int x J(mx) J(m'x) dx = \frac{1}{m}\delta(m - m') \tag{3.62}$$

we have an approximation for u_m that is valid in the limit $\alpha \to 0$, $y_0 \to \infty$

$$u_m(y) = z^{3/5}\sqrt{\frac{m}{5\alpha}}\frac{\left[J_{4/5}(\frac{m}{5\alpha})J_{1/5}(mz) + J_{-4/5}(\frac{m}{5\alpha})J_{-1/5}(mz)\right]}{\sqrt{J^2_{-4/5}(\frac{m}{5\alpha}) + J^2_{4/5}(\frac{m}{5\alpha})}}. \tag{3.63}$$

3.2.2 The zero mode and the massive KK tower

We now wish to drop the assumption of the TTF gauge because it is too restrictive and look for zero mode solutions of the perturbation equations. The metric perturbation can be decomposed into irreducible representations of the diffeomorphism group, following the example in [95], yielding

$$h_{\mu\nu} = h^{TT}_{\mu\nu} + 2A_{(\mu,\nu)} + \psi_{,\mu\nu} + \frac{1}{4}\eta_{\mu\nu}\left(h - \Box\psi\right) \tag{3.64}$$

where $h^{TT}_{\mu\nu}$ is the transverse trace-free metric perturbation satisfying $h^{TT,\nu}_{\mu\nu} = h^{TT\lambda}_{\lambda} = 0$, A_μ is a Lorentz-gauge vector satisfying the condition $A_{\mu,}{}^{\mu} = 0$, and ψ and $h = h_\mu{}^\mu$ are two scalar fields. The (μy) equation (3.38) then takes the form

$$\frac{d}{dz}\left[\frac{\Box A_\mu - \frac{3}{4}(h - \Box\psi)_{,\mu}}{(5\alpha z)^{2/5}}\right] = \frac{6}{5}\frac{(\delta\phi)_{,\mu}}{z}, \tag{3.65}$$

and the $(\mu\nu)$ equation (3.39)

$$\frac{\Box h^{TT}_{\mu\nu}}{(5\alpha z)^{2/5}} + \frac{(h - \Box\psi)_{,\mu\nu}}{2(5\alpha z)^{2/5}} + \frac{1}{z^{3/5}}\frac{d}{dz}\left[z^{3/5}\frac{d}{dz}\left(\frac{h^{TT}_{\mu\nu} + \psi_{,\mu\nu} + 2A_{(\mu,\nu)}}{(5\alpha z)^{2/5}}\right)\right]$$
$$+ \eta_{\mu\nu}\left\{\frac{\Box(h - \Box\psi)}{4(5\alpha z)^{2/5}} + \frac{1}{4z^{3/5}}\frac{d}{dz}\left[z^{3/5}\frac{d}{dz}\left(\frac{h - \Box\psi}{(5\alpha z)^{2/5}}\right)\right] + \frac{1}{5z}\frac{d}{dz}\left(\frac{h}{(5\alpha z)^{2/5}}\right)\right\} \tag{3.66}$$

$$= \frac{8}{25}\frac{\delta\phi}{z^2}\eta_{\mu\nu}, \tag{3.67}$$

and trace of (3.67) is

$$\frac{3\Box(h - \Box\psi)}{2(5\alpha z)^{2/5}} + \frac{1}{z^{3/5}}\frac{d}{dz}\left[z^{3/5}\frac{d}{dz}\left(\frac{h}{(5\alpha z)^{2/5}}\right)\right] + \frac{4}{5z}\frac{d}{dz}\left(\frac{h}{(5\alpha z)^{2/5}}\right) = \frac{32}{25}\frac{\delta\phi}{z^2}. \tag{3.68}$$

The (yy) equation (3.37) and the scalar field equation (3.40) remain the same. Differentiating (3.40) and plugging it into (3.37) yields a third-order differential equation

$$\delta\phi^{(3)} + \frac{9}{5}\frac{\delta\phi^{(2)}}{z} + \left[m^2 z^2 - \frac{57}{25}\right]\frac{(\dot{\delta\phi})}{z^2} + \left[\frac{6}{5}m^2 z^2 + \frac{24}{25}\right]\frac{\delta\phi}{z^3} = 0, \tag{3.69}$$

where we have used $\Box\delta\phi = m^2\delta\phi$. By setting $m = 0$ in the combined (yy)–ϕ equation (3.69) we obtain

$$\delta\phi(z) = \frac{C_1}{z^{6/5}} + C_2 z^{2/5} + C_3 z^2. \tag{3.70}$$

We can find h from the linearised scalar field equation (3.40)

$$\frac{d}{dz}\left(\frac{h}{(5\alpha z)^{2/5}}\right) = -\frac{5}{3}z\frac{d^2}{dz^2}(\delta\phi) - \frac{d}{dz}(\delta\phi) + \frac{8}{5}\frac{\delta\phi}{z} - \frac{5}{3}m^2 z\delta\phi. \tag{3.71}$$

For the massless case, the trace is given by

$$\frac{h}{a^2} = \frac{h}{(5\alpha z)^{2/5}} = \frac{4}{3}\frac{C_1}{z^{6/5}} + 4C_2 z^{2/5} - \frac{28}{15}C_3 z^2, \tag{3.72}$$

with C_1, C_2 and C_3 constants. We can check the consistency of the massless mode solutions using the trace of the $(\mu\nu)$ equation (3.68) but with $m^2 h = m^2\psi = 0$. We get

$$-\frac{512}{75}C_3 = 0. \tag{3.73}$$

Plugging the above result into the (μy) equation (3.65) with $m^2\psi = 0$ we get

$$\frac{16}{5z^{3/5}}C_2 = 0. \tag{3.74}$$

We have shown that the constants C_2 and C_3 are zero.

For the massless case, ψ is tied to $h^{TT}_{\mu\nu}$ so we do not have an explicit expression for it. Next we look at the gauge field A_μ. From the (μy) equation (3.65)

$$\frac{d}{dz}\left[\frac{\Box A_\mu}{(5\alpha z)^{2/5}}\right] = \frac{3}{4}\frac{d}{dz}\left[\frac{(h - \Box\psi)_{,\mu}}{(5\alpha z)^{2/5}}\right] + \frac{6}{5}\frac{(\delta\phi)_{,\mu}}{z}. \tag{3.75}$$

The RHS is exactly zero, so we get $A_\mu = A_\mu(x^\mu)$ only. Thus, in the next equation, equation (3.67) without the $\eta_{\mu\nu}$ pieces, we can eliminate A_μ, giving

$$\frac{h_{,\mu\nu}}{2(5\alpha z)^{2/5}} + \frac{1}{z^{3/5}}\frac{d}{dz}\left[z^{3/5}\frac{d}{dz}\left(\frac{h^{TT}_{\mu\nu} + \psi_{,\mu\nu}}{(5\alpha z)^{2/5}}\right)\right] = 0. \tag{3.76}$$

This has the solution

$$h_{\mu\nu}^{TT} + \psi_{,\mu\nu} = (5\alpha)^{2/5}\left[-\frac{25}{12}C_{1,\mu\nu}z^{6/5} + \frac{5}{2}\Pi_{\mu\nu}z^{4/5} + \chi_{\mu\nu}z^{2/5}\right]. \tag{3.77}$$

We note that the $\chi_{\mu\nu}$ and $\Pi_{\mu\nu}$ terms are the solutions we get for the massless case in TTF gauge.

Finally, by plugging in the relevant expressions into (3.64), our full metric perturbation for the massless case is given by

$$h_{\mu\nu} = (5\alpha)^{2/5}\left[\frac{5}{2}\Pi_{\mu\nu}z^{4/5} + \chi_{\mu\nu}z^{2/5} - \frac{25}{12}C_{1,\mu\nu}z^{6/5} + \frac{1}{3}\frac{C_1}{z^{4/5}}\eta_{\mu\nu}\right], \tag{3.78}$$

and the scalar field perturbation is

$$\delta\phi(z) = \frac{C_1}{z^{6/5}}. \tag{3.79}$$

The next step is to apply the boundary conditions to the above results. Equation (3.46) tells us that the solution for $\delta\phi$ is consistent. The (yy) boundary condition also checks the consistency of the solution, and the $(\mu\nu)$ boundary condition gives us the relation

$$C_{1,\mu\nu} = \frac{3}{5}\frac{\Pi_{\mu\nu}}{z^{2/5}} \tag{3.80}$$

Having found the zero modes, we now wish to look for massive mode solutions. The combined (yy)–ϕ equation (3.69) has the solution

$$\begin{aligned}\delta\phi = {} & \frac{C_1}{z^{6/5}} - C_3\frac{5\Gamma\left(\frac{4}{5}\right)}{4\times 2^{1/5}m^{4/5}\pi}\,{}_1F_2\left(\frac{4}{5};\frac{1}{5},\frac{9}{5};-\frac{m^2z^2}{4}\right)z^{2/5} \\ & - \frac{25m^{4/5}\Gamma\left(\frac{1}{5}\right)}{256\times 2^{4/5}\pi}\left(-\sqrt{10-2\sqrt{5}}C_2 + \left(1+\sqrt{5}\right)C_3\right){}_1F_2\left(\frac{8}{5};\frac{9}{5},\frac{13}{5};-\frac{m^2z^2}{4}\right)z^2.\end{aligned} \tag{3.81}$$

Integrating the dilaton equation (3.40) gives the trace

$$\begin{aligned}\frac{h}{a^2} = {} & C_1\frac{(16-25m^2z^2)}{12z^{6/5}} \\ & + \frac{C_2}{12}z^{2/5}\left(48\,{}_0F_1\left(\frac{1}{5},-\frac{m^2z^2}{4}\right) + 4\left(12+25m^2z^2\right){}_0F_1\left(\frac{6}{5},-\frac{m^2z^2}{4}\right)\right. \\ & \qquad \left. -25m^2z^2\,{}_1F_2\left(\frac{4}{5};\frac{1}{5},\frac{9}{5};-\frac{m^2z^2}{4}\right) - 48\,{}_1F_2\left(\frac{4}{5};\frac{6}{5},\frac{9}{5};-\frac{m^2z^2}{4}\right)\right) \\ & + C_3\frac{128}{75m^2}\left(-9 + 6\,{}_0F_1\left(-\frac{6}{5},-\frac{m^2z^2}{4}\right) - 3\,{}_0F_1\left(-\frac{1}{5},-\frac{m^2z^2}{4}\right) - 2\,{}_0F_1\left(\frac{4}{5},-\frac{m^2z^2}{4}\right)\right) \\ & + \frac{C_3}{12}z^2\left(16-25m^2z^2\right){}_1F_2\left(\frac{8}{5};\frac{9}{5},\frac{13}{5};-\frac{m^2z^2}{4}\right).\end{aligned}$$

The notations ${}_0F_1$ and ${}_1F_2$ denote generalised hypergeometric functions. To find ψ we use the (μy) equation (3.65), differentiating with respect to x^μ once ($\partial^\mu A_\mu = 0$)

$$m^4 \frac{d}{dz}\left[\frac{\psi}{(5\alpha z)^{2/5}}\right] = m^2 \frac{d}{dz}\left[\frac{h}{(5\alpha z)^{2/5}}\right] + m^2 \frac{8}{5}\frac{\delta\phi}{z}. \tag{3.82}$$

The solution to the above differential equation is given by

$$\begin{aligned}
\frac{\psi}{a^2} &= -\frac{25}{12} z^{4/5} C_1 \\
&+ \frac{(mz)^{1/5} C_2}{10 \times 2^{7/10}\sqrt{5-\sqrt{5}}\, m^5 z^{9/5} \Gamma\left(-\frac{4}{5}\right)\Gamma\left(\frac{6}{5}\right)\Gamma\left(\frac{11}{5}\right)} \\
&\times \left[4\Gamma\left(\frac{6}{5}\right)^2 \left(4(12+25m^2z^2)\,{}_0F_1\left(-\frac{4}{5}, -\frac{m^2z^2}{4}\right) - (48+25m^2z^2)\,{}_0F_1(\frac{1}{5}, -\frac{m^2z^2}{4})\right)\right. \\
&\left. +5m^4z^4\Gamma\left(-\frac{4}{5}\right)\Gamma\left(\frac{1}{5}\right){}_1F_2\left(\frac{4}{5}; \frac{1}{5}, \frac{9}{5}; -\frac{m^2z^2}{4}\right)\right] \\
&+ \frac{z^{1/5} C_3 \Gamma\left(\frac{8}{5}\right)}{1200 \times 2(3/10)\sqrt{5-\sqrt{5}}\, m^3 (mz)^{1/5} \Gamma\left(\frac{9}{5}\right)\Gamma\left(\frac{13}{5}\right)} \\
&\times \left[128\left(-12(1+{}_0F_1(-\frac{1}{5}, -\frac{m^2z^2}{4})) + (24+25m^2z^2)\,{}_0F_1\left(\frac{4}{5}; -\frac{m^2z^2}{4}\right)\right)\right. \\
&\left. +625m^4z^4\,{}_1F_2\left(\frac{8}{5}; \frac{9}{5}, \frac{13}{5}; -\frac{m^2z^2}{4}\right)\right].
\end{aligned} \tag{3.83}$$

Checking the consistency of the massive solution using equation (3.68), we get

$$\begin{aligned}
&C_2 \frac{4 \times 2^{3/10} m z^{1/5} (mz)^{1/5}}{5\sqrt{5-\sqrt{5}}\,\Gamma\left(\frac{11}{5}\right)} \\
&\times \left(4\,{}_0F_1\left(\frac{6}{5}; -\frac{m^2z^2}{4}\right) - {}_1F_2\left(\frac{4}{5}; \frac{1}{5}, \frac{9}{5}; -\frac{m^2z^2}{4}\right) - 3\,{}_1F_2\left(\frac{4}{5}; \frac{6}{5}, \frac{9}{5}; -\frac{m^2z^2}{4}\right)\right) = 0,
\end{aligned} \tag{3.84}$$

and

$$-C_3 \frac{8 \times 2^{7/10} (mz)^{4/5}}{15\sqrt{5-\sqrt{5}}\, z^{4/5} \Gamma\left(\frac{9}{5}\right)} = 0. \tag{3.85}$$

Thus we conclude that C_2 and C_3 are zero. After eliminating the inconsistent solutions we have only

$$\delta\phi = C_1 \frac{1}{z^{6/5}}, \tag{3.86}$$

$$\frac{h}{a^2} = \frac{h}{(5\alpha z)^{2/5}} = \frac{(16-25m^2z^2)\, C_1}{12 z^{6/5}}, \tag{3.87}$$

$$\frac{\psi}{a^2} = \frac{\psi}{(5\alpha z)^{2/5}} = -\frac{25}{12} z^{4/5} C_1. \tag{3.88}$$

The expression for $\delta\phi$ is now exactly the same as for the massless case, and the trace will reduce to the trace for the massless case if we set $m = 0$.

Now we need to find A_μ and $h^{TT}_{\mu\nu}$. Plugging in (3.87) and (3.88) into (3.65) gives

$$\frac{d}{dz}\left[\frac{\Box A_\mu}{(5\alpha z)^{2/5}}\right] = \frac{3}{4}\frac{d}{dz}\left[\frac{(h-\Box\psi)_{,\mu}}{(5\alpha z)^{2/5}}\right] + \frac{6}{5}\frac{(\delta\phi)_{,\mu}}{z}, \tag{3.89}$$

we see that the RHS is exactly zero, so we get $A_\mu = A_\mu(x^\mu)$ only. Thus, in the next equation, equation (3.67) without the $\eta_{\mu\nu}$ pieces, we can eliminate A_μ, giving

$$\frac{\Box h^{TT}_{\mu\nu}}{(5\alpha z)^{2/5}} + \frac{(h-\Box\psi)_{,\mu\nu}}{2(5\alpha z)^{2/5}} + \frac{1}{z^{3/5}}\frac{d}{dz}\left[z^{3/5}\frac{d}{dz}\left(\frac{h^{TT}_{\mu\nu}+\psi_{,\mu\nu}}{(5\alpha z)^{2/5}}\right)\right] = 0. \tag{3.90}$$

Plugging in the expressions for h and ψ, we recover equation (3.52)

$$\ddot{h}^{TT}_{\mu\nu} - \frac{1}{5}\frac{\dot{h}^{TT}_{\mu\nu}}{z} + \left(\frac{8}{25} + m^2 z^2\right)\frac{h^{TT}_{\mu\nu}}{z^2} = 0. \tag{3.91}$$

The solution to this is given by $h^{TT}_{\mu\nu} = u_m(y)\chi_{\mu\nu}(x^\mu)$ with $u_m(y)$ given by (3.63).

Now that we have all the components of the decomposition, we can write our full metric perturbation for the massive case

$$h_{\mu\nu} = (5\alpha)^{2/5}\left[h^{TT}_{\mu\nu} - \frac{25}{12}C1_{,\mu\nu}z^{6/5} + \frac{1}{3}\frac{C_1}{z^{4/5}}\eta_{\mu\nu}\right]. \tag{3.92}$$

where we have ignored the A_μ terms because we have shown that they do not contribute to the perturbation equations.

3.2.3 The radion mode

In order to simplify the perturbation analysis, we have chosen the bulk GN gauge where the branes are allowed to flutter and their position will be given by $y = F$. It is also possible to choose a brane GN gauge, with brane-based coordinates, where the brane position will be fixed and the perturbations will have explicit terms to account for the fluctuation of the brane. Following the analysis in [105], when we have two branes we will need two coordinate patches to study the relative motion associated with the interbrane distance. First, we consider diffeomorphisms of the form

$$y \to \tilde{y} = y + \xi^y(x^\mu, y), \tag{3.93}$$

$$x^\mu \to \tilde{x}^\mu = x^\mu + \xi^\mu(x^\mu, y). \tag{3.94}$$

The metric perturbations transform according to $\tilde{h}_{ab} = h_{ab} - \nabla_a \xi_b - \nabla_b \xi_a$ so we get the following equations

$$\tilde{h}_{\mu\nu} = h_{\mu\nu} - (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu + 2aa'\eta_{\mu\nu}\xi_y), \tag{3.95}$$

$$\tilde{h}_{\mu y} = h_{\mu y} - (\partial_\mu \xi_y + \partial_y \xi_\mu - 2\frac{a'}{a}\xi_\mu), \tag{3.96}$$

$$\tilde{h}_{yy} = h_{yy} - 2\partial_y \xi_y, \tag{3.97}$$

and the scalar field perturbation transforms according to $\tilde{\delta\phi} = \delta\phi - \partial^a \phi \xi_a$ giving us

$$\tilde{\delta\phi} = \delta\phi - \phi' \xi_y. \tag{3.98}$$

Applying the Gaussian Normal gauge ($\tilde{h}_{\mu y} = \tilde{h}_{yy} = 0$ and $h_{\mu y} = h_{yy} = 0$) yields the following expressions

$$\xi_y = F(x^\mu), \tag{3.99}$$

$$\xi_\mu = -a^2 \partial_\mu F(x^\mu) \int \frac{dy}{a^2}. \tag{3.100}$$

Substituting (3.100) into (3.97) and (3.98), the transformations are given by

$$\tilde{h}_{\mu\nu} = h_{\mu\nu} + 2a^2 \partial_\mu \partial_\nu F \left(\int \frac{dy}{a^2} \right) - 2aa'\eta_{\mu\nu} F, \tag{3.101}$$

$$\tilde{\delta\phi} = \delta\phi - 6\frac{a'}{a} F, \tag{3.102}$$

where F is a function of x^μ only. In the z-variable, our perturbations are then given by:

$$\tilde{h}_{\mu\nu} = h_{\mu\nu} + (5\alpha)^{1/5} \left[\frac{5}{2} F_{,\mu\nu} z^{6/5} - \frac{2}{5} \frac{F}{z^{4/5}} \eta_{\mu\nu} \right], \tag{3.103}$$

$$\tilde{\delta\phi} = \delta\phi - \frac{6}{5} \frac{1}{(5\alpha)^{1/5}} \frac{F}{z^{6/5}}. \tag{3.104}$$

The expressions for $h_{\mu\nu}$ and $\delta\phi$ can be found from equations (3.78) and (3.79) for the zero mode case, and from equations (3.92) and (3.86) for the massive modes.

The dilaton boundary condition remains unchanged because the extra F terms cancel each other precisely. The (yy) boundary condition for the zero mode tells us that the radion F is massless because at $z = 1/(5\alpha)$

$$2m^2 F(5\alpha z)^{1/5} = 0, \tag{3.105}$$

and for the massive mode we get the expression

$$F = \frac{5}{6}(5\alpha)^{1/5} C_1. \tag{3.106}$$

From this expression we see that C_1 is proportional to F so it can be gauged away. The $(\mu\nu)$ boundary condition for the zero mode gives the relation

$$F_{,\mu\nu} = \frac{(5\alpha)^{1/5}}{2}\left(\frac{5}{3}C_{1,\mu\nu} - \frac{\Pi_{\mu\nu}}{z^{2/5}}\right). \tag{3.107}$$

Taking this expression with the scalar field perturbation implies that the scalar field perturbation is tied to the radion. This result is in agreement with the results of [160, 161].

The heterotic braneworld model contains two branes, and thus it is necessary to have two coordinate patches, one that GN to each brane although it may not be GN with respect to the other brane. Using the equation (3.107)

$$\Pi_{\mu\nu} = -\frac{2}{(5\alpha)^{3/5}} a_\pm^2 F_{,\mu\nu}, \tag{3.108}$$

and by requiring $2a_+^2 F_+ = 2a_-^2 F_- \equiv F$, we may write the zero mode metric perturbation (3.78) in each patch

$$h_{\pm\mu\nu} = a^2\chi_{\mu\nu} + \frac{F_{,\mu\nu}}{2\alpha}\left[\frac{a^6}{2a_\pm^2} - a^4\right] - \frac{\alpha}{a_\pm^2 a^4} F\eta_{\mu\nu}. \tag{3.109}$$

This clearly shows the zero mode perturbation consists only of the spin-2 graviton $\chi_{\mu\nu}$ and the massless scalar radion F. Similarly, we obtain for the scalar field

$$\delta\phi = -\frac{3\alpha}{a_\pm^2 a^6} F, \tag{3.110}$$

which shows that the radion does not give rise to an extra scalar degree of freedom. The transformation on the overlap is given by

$$y \rightarrow y + \frac{f}{2a_+^2} - \frac{f}{2a_-^2}. \tag{3.111}$$

3.3 Gravity in the heterotic braneworld

Having found the solutions of the Einstein equations in vacuum, we now bring back the T_{ab} terms to investigate the effect of matter on the brane. We restrict our analysis for the positive tension brane, denoted by the superscript $^+$.

The (yy) boundary condition gives

$$\partial^2 F^+ = \frac{\kappa_5^2}{6} T^{+\lambda}{}_{\lambda}, \tag{3.112}$$

for both the zero and massive modes. The perturbations for the metric and the scalar field are given by

$$h^+_{\mu\nu} = h^{TT}_{\mu\nu} + \frac{a^6}{2\alpha} F^+_{,\mu\nu} - \frac{2\alpha}{a^4} F^+ \eta_{\mu\nu} \tag{3.113}$$

$$\delta\phi^+ = -\frac{6\alpha}{a^6} F^+ \tag{3.114}$$

From (3.112) we get

$$F^+ = \int D_0(x - x') \frac{\kappa_5^2}{6} T^{+\lambda}{}_{\lambda}. \tag{3.115}$$

The Green's function relevant to the TTF part of the metric perturbation is given by

$$G_R(x, x') = \frac{4\alpha a^2(z) a^2(z')}{a_+^8} D_0(x - x') + \int_0^\infty dm\ u_m(z) u_m(z') D_m(x - x'). \tag{3.116}$$

From there we can write the metric perturbation

$$\begin{aligned} h^+_{\mu\nu} = &- 8\alpha\kappa_5^2 \int d^4x'\ D_0(x - x') \left[T^+_{\mu\nu} - \frac{3}{8} T^+ \eta_{\mu\nu} \right] \\ &- \kappa_5^2 \int d^4x' \int dm\ u_m^2(y_0) D_m(x - x') \left[T^+_{\mu\nu} - \frac{T^+}{3} \eta_{\mu\nu} \right], \end{aligned} \tag{3.117}$$

where

$$u_m^2(y_0) = \frac{m}{5\alpha} \frac{\left[J_{4/5}(\frac{m}{5\alpha}) J_{1/5}(\frac{m}{5\alpha}) + J_{-4/5}(\frac{m}{5\alpha}) J_{-1/5}(\frac{m}{5\alpha}) \right]^2}{\left[J^2_{4/5}(\frac{m}{5\alpha}) + J^2_{-4/5}(\frac{m}{5\alpha}) \right]}, \tag{3.118}$$

and the dilaton

$$\delta\phi^+ = \alpha\kappa_5^2 \int d^4x'\ D_0(x - x')\ T^+. \tag{3.119}$$

The brane gravity is Brans-Dicke type with $\omega = 0.5$.

We would like to summarise the results in this chapter. We started with the LOSW action and derived the perturbation equations. We solved the perturbation equations by decomposing the metric perturbation into irreducible representations. We found that the zero mode solutions consisted of a tensor mode which we identified as the graviton, and a scalar mode which we identified as the radion. Moreover, the radion is coupled to the bulk scalar field, so there is only one scalar degree of freedom. The other solution consisted of a massive KK tower of tensor modes. We also found that the brane gravity is Brans-Dicke type with $\omega = 0.5$. Unfortunately, this value has been ruled out by experimental data, so we conclude that the heterotic braneworld is not a suitable description of our universe.

Chapter 4

Black holes in heterotic M-theory

4.1 The black string

In this chapter we attempt to find a black hole solution for the heterotic braneworld. We start by considering the black string and the bulk approach that have been applied to the RS braneworld. Then we derive a solution for an axisymmetric bulk setup by assuming separability. This chapter is also an extended version of the work published in [1], with an updated calculation but eventually leading to the same result.

Since it has been shown that it is possible to construct a black string within the RS model, we check if we can do the same in heterotic M-theory. However because the RS black string was found to have an instability [118], it is likely that the heterotic black string will suffer the same fate.

We start with the black string metric

$$ds^2 = a^2\left[-\left(1-2\frac{G_N M}{r}\right)dt^2 + \left(1-2\frac{G_N M}{r}\right)^{-1} dr^2 + r^2 d\Omega_{II}^2\right] + dy^2, \qquad (4.1)$$

and perform a perturbation analysis similar to the one in the last chapter. In fact the perturbation equations are the same, the only difference is in the form of the d'Alembertian operator. We also impose the transverse-tracefree (TTF) gauge. The relevant equation is

$$\frac{\Box h_{\mu\nu} - 2h_{\lambda(\mu,\nu)}{}^{\lambda}}{a^2} + \frac{1}{a^2}\left[a^4\left(\frac{h_{\mu\nu}}{a^2}\right)'\right]' = 0, \qquad (4.2)$$

where a is the warp factor for the heterotic braneworld model given in the previous chapter. The unstable tensor mode is given by

$$u_m(y)h_{\mu\nu}^{(GL)}(t,r), \tag{4.3}$$

with $h_{\mu\nu}^{(GL)}(t,r)$ is:

$$h_{\mu\nu}^{(GL)} = e^{\Omega t}\begin{bmatrix} h_0 & h_1 & 0 & 0 \\ h_1 & h_2 & 0 & 0 \\ 0 & 0 & K & 0 \\ 0 & 0 & 0 & K\sin^2\theta \end{bmatrix}, \tag{4.4}$$

and h_0, h_1, h_2 and K can be found in [162, 163]. For the 4D Schwarzschild metric the parameter Ω can be approximated by:

$$\Omega(m) = \frac{m}{2} - m^2 G_N M. \tag{4.5}$$

If the mass of the black string is less than $1/2G_N m_0$, where m_0 is the minimum eigenvalue permitted for the massive tensor tower, then the unstable mode will exist.

It is possible to obtain a far-field approximation of the black hole metric using the linearised theory. Starting with a point source on the brane

$$T_{\mu\nu} = M\delta(\mathbf{r})\delta(y-y_0)\delta_\mu^0\delta_\nu^0, \tag{4.6}$$

we can calculate the linearised solution for the dilaton

$$\phi(r) = \frac{2\alpha G_5 M}{r}. \tag{4.7}$$

Repeating for the tensor component, and expanding the Bessel functions in $u_m(0)$ at small m, we obtain an unusual Newtonian potential

$$V(r) = -\frac{10\alpha G_5 M}{r}\left(1 + \frac{2^{7/5}\Gamma[\frac{8}{5}]}{(5\alpha r)^{8/5}3\Gamma[\frac{4}{5}]^2}\right). \tag{4.8}$$

We see that the correction to the Newtonian potential has a fractional power coming from the fractional order of the Bessel functions. However this is only valid for small values of α and large values of y_0 because we used the continuum approximation.

4.2 Axisymmetric solutions

To avoid the instability of the black string, we would like to try a different approach to finding a black hole solution. A reasonable choice is to apply the bulk approach described in the previous chapter, where a known bulk solution is used to calculate possible brane trajectories and finding trajectories that correspond to the brane TOV equations. Unfortunately, it was not possible to find a spherically symmetric solution to the Einstein equations of the LOSW model. It was shown by Chan, Horne and Mann [164] that only certain values of α^2 would give spherically symmetric solutions, and have unusual asymptotics. Thus, we consider an axisymmetric bulk metric as in [165]

$$ds^2 = e^{2\sigma}dt^2 - \frac{e^{2\chi-\sigma}}{\sqrt{B}}(dr^2 + dz^2) - Be^{-\sigma}d\Omega_{II}^2, \tag{4.9}$$

where B, σ, and χ only depend on r and z.

The non-vanishing Christoffel symbols are given by

$$\begin{aligned}
\Gamma^t_{tr} &= \sigma', & \Gamma^k_{ir} &= \frac{B' - B\sigma'}{2B}\delta^k_i,\\
\Gamma^t_{tz} &= \dot{\sigma}, & \Gamma^k_{iz} &= \frac{\dot{B} - B\dot{\sigma}}{2B}\delta^k_i,\\
\Gamma^r_{tt} &= e^{3\sigma-2\chi}\sqrt{B}\sigma', & \Gamma^z_{tt} &= e^{3\sigma-2\chi}\sqrt{B}\dot{\sigma},\\
\Gamma^r_{rr} &= -\frac{B'}{4B} - \frac{1}{2}\sigma' + \chi', & \Gamma^z_{rr} &= \frac{1}{4}\frac{\dot{B}}{B} + \frac{1}{2}\dot{\sigma} - \dot{\chi},\\
\Gamma^r_{rz} &= -\frac{\dot{B}}{4B} - \frac{1}{2}\dot{\sigma} + \dot{\chi}, & \Gamma^z_{rz} &= -\frac{B'}{4B} - \frac{1}{2}\sigma' + \chi',\\
\Gamma^r_{zz} &= \frac{1}{4}\frac{B'}{B} + \frac{1}{2}\sigma' - \chi', & \Gamma^z_{zz} &= -\frac{\dot{B}}{4B} - \frac{1}{2}\dot{\sigma} + \dot{\chi},\\
\Gamma^r_{ij} &= \frac{1}{2}e^{-2\chi}\sqrt{B}\left(-B' + B\sigma'\right)\gamma_{ij}, & \Gamma^z_{ij} &= \frac{1}{2}e^{-2\chi}\sqrt{B}\left(-\dot{B} + B\dot{\sigma}\right)\gamma_{ij},
\end{aligned}$$

where we have used primes to denote differentiation with respect to r and overdots to denote differentiation with respect to z. Using the notation $\nabla = \partial_r + \partial_z$ for the 2D gradient on (r, z) space and $\Delta = \partial_r^2 + \partial_z^2$ for the 2D Laplacian, the elements of the Ricci tensor are given by

$$R^t_t = e^{\sigma-2\chi}\sqrt{B}\left(\frac{\nabla B}{B}\nabla\sigma + \Delta\sigma\right), \tag{4.10}$$

$$R^r_r = \frac{e^{\sigma-2\chi}\sqrt{B}}{2}\left(-\frac{\nabla B}{B}\nabla\sigma - 2\frac{B'}{B}\chi' + 2\frac{\dot{B}}{B}\dot{\chi} + \frac{3}{2}\frac{B''}{B} - \frac{1}{2}\frac{\ddot{B}}{B} + 3\sigma'^2 - \Delta\sigma + 2\Delta\chi\right), \tag{4.11}$$

$$R^z_z = \frac{e^{\sigma-2\chi}\sqrt{B}}{2}\left(-\frac{\nabla B}{B}\nabla\sigma + 2\frac{B'}{B}\chi' - 2\frac{\dot{B}}{B}\dot{\chi} - \frac{1}{2}\frac{B''}{B} + \frac{3}{2}\frac{\ddot{B}}{B} + 3\dot{\sigma}^2 - \Delta\sigma + 2\Delta\chi\right), \tag{4.12}$$

$$R^k_i = -\frac{e^{\sigma-2\chi}\sqrt{B}}{2}\left(\frac{\nabla B}{B}\nabla\sigma - \frac{\Delta B}{B} + \Delta\sigma\right)\delta^k_i - \frac{e^\sigma}{B}R^{k\,(2D)}_i, \tag{4.13}$$

$$R_{rz} = \frac{B'}{B}\dot{\chi} + \frac{\dot{B}}{B}\chi' - \frac{\dot{B}'}{B} - \frac{3}{2}\sigma'\dot{\sigma}, \tag{4.14}$$

and the d'Alembertian operator is given by

$$\Box\phi = -e^{\sigma-2\chi}\sqrt{B}\left(\Delta\phi + \frac{\nabla B}{B}\nabla\phi\right). \tag{4.15}$$

The Einstein equations are given by

$$R^t_t = e^{\sigma-2\chi}\sqrt{B}\left(\frac{\nabla B}{B}\nabla\sigma + \Delta\sigma\right) = -2\alpha^2 e^{-2\phi}, \tag{4.16}$$

$$\begin{aligned} R^r_r + R^z_z &= \frac{1}{2}e^{\sigma-2\chi}\sqrt{B}\left(-2\frac{\nabla B}{B}\nabla\sigma + \frac{\Delta B}{B} + 3(\nabla\sigma)^2 - 2\Delta\sigma + 4\Delta\chi\right) \\ &= -\frac{1}{2}e^{-2\chi+\sigma}\sqrt{B}(\nabla\phi)^2 - 4\alpha^2 e^{-2\phi}, \end{aligned} \tag{4.17}$$

$$R^\theta_\theta = -\frac{1}{2}e^{\sigma-2\chi}\sqrt{B}\left(\frac{\nabla B}{B}\nabla\sigma - \frac{\Delta B}{B} + \Delta\sigma\right) - \frac{e^\sigma}{B} = -2\alpha^2 e^{-2\phi}, \tag{4.18}$$

$$R_{rz} = \frac{B'}{B}\dot{\chi} + \frac{\dot{B}}{B}\chi' - \frac{\dot{B}'}{B} - \frac{3}{2}\sigma'\dot{\sigma} = \frac{1}{2}\phi'\dot{\phi}, \tag{4.19}$$

$$\begin{aligned} R_{rr} - R_{zz} &= \left(2\frac{B'}{B}\chi' - 2\frac{\dot{B}}{B}\dot{\chi} - \frac{B''}{B} + \frac{\ddot{B}}{B} - \frac{3}{2}\left(\sigma'^2 - \dot{\sigma}^2\right)\right) \\ &= \frac{1}{2}\left(\phi'^2 - \phi^{\circ 2}\right), \end{aligned} \tag{4.20}$$

and the scalar field equation is:

$$\Box\phi = -e^{\sigma-2\chi}\sqrt{B}\left(\Delta\phi + \frac{\nabla B}{B}\nabla\phi\right) = 12\alpha^2 e^{-2\phi}. \tag{4.21}$$

Some of the above equations may be combined to give the following equations:

$$\frac{\Delta B}{B} = \left(2\frac{e^{2\chi}}{B} - 6\alpha^2 e^{-\sigma-2\phi+2\chi}\right)\frac{1}{\sqrt{B}} \quad \text{from } R^t_t + 2R^\theta_\theta, \tag{4.22}$$

$$\Delta\chi + \frac{3}{4}(\nabla\sigma)^2 + \frac{1}{4}(\nabla\phi)^2 = -\frac{e^{2\chi}}{2B^{3/2}} - \frac{3e^{-\sigma-2\phi+2\chi}\alpha^2}{2\sqrt{B}} \quad \text{from } R^t_t + R^r_r + R^z_z, \tag{4.23}$$

$$2\left(\frac{\partial_\pm B}{B}\chi\right) - \left(\frac{\partial^2{}_\pm B}{B}\right) - \frac{3}{2}\left(\partial_\pm\sigma\right)^2 - \frac{1}{2}\left(\partial_\pm\phi\right)^2 = 0 \quad \text{from } R_{rr} - R_{zz} \pm 2\mathrm{i}R_{rz}, \tag{4.24}$$

where $\partial_\pm = \partial_r \pm i\partial_z$.

The next step is to use the method of separation of variables by writing

$$B = b_1(r)b_2(z), \tag{4.25}$$

$$\sigma = \sigma_0 + \sigma_1(r) + \sigma_2(z), \tag{4.26}$$

$$\chi = \chi_0 + \chi_1(r) + \chi_2(z), \tag{4.27}$$

$$\phi = \phi_0 + \phi_1(r) + \phi_2(z). \tag{4.28}$$

Upon examining the $R^t_t + 2R^\theta_\theta$ equation, we may assume that the RHS has the form $\text{function}(r) + \text{function}(z)$. The LOSW vacuum can be obtained by setting $e^{2\chi}B^{-1/2}$ as a function of r and $e^{2\chi-\sigma-2\phi}B^{-1/2}$ as a function of z. We then obtain the two equations:

$$\frac{b_1''}{b_1} = 2\frac{e^{2\chi_0}e^{2\chi_1}e^{2\chi_2}}{b_1^{3/2}b_2^{3/2}}, \tag{4.29}$$

$$\frac{\ddot{b}_2}{b_2} = -6\alpha^2\frac{e^{-2\phi_0}e^{-2\phi_1}e^{-2\phi_2}e^{2\chi_0}e^{2\chi_1}e^{2\chi_2}e^{-\sigma_0}e^{-\sigma_1}e^{-\sigma_2}}{b_1^{1/2}b_2^{1/2}}. \tag{4.30}$$

This means that both $2\frac{e^{2\chi_0}e^{2\chi_2}}{b_2^{3/2}}$ and $-6\alpha^2\frac{e^{-2\phi_0}e^{-2\phi_1}e^{2\chi_0}e^{2\chi_1}e^{-\sigma_0}e^{-\sigma_1}}{b_1^{1/2}}$ are constants, giving

$$\chi_2(z) = \frac{3}{4}\ln b_2(z), \tag{4.31}$$

$$\chi_1(r) = \frac{1}{4}\ln b_1(r) + \phi_1(r) + \frac{1}{2}\sigma_1(r). \tag{4.32}$$

We can use this to substitute for χ_1 and χ_2 in the $R^t_t + R^r_r + R^z_z$ equation

$$\frac{1}{4}\frac{b_1''}{b_1} + \frac{3}{4}\frac{\ddot{b}_2}{b_2} - \frac{1}{4}\frac{b_1'^2}{b_1^2} - \frac{3}{4}\frac{\dot{b}_2^{\,2}}{b_2^2} + \frac{1}{2}\sigma_1'' + \frac{3}{4}\left(\sigma_1'^2 + \sigma_2'^2\right) + \phi_1'' + \frac{1}{4}\left(\phi_1'^2 + \phi_2'^2\right) =$$
$$-\frac{1}{2}\frac{e^{2\chi_0}e^{2\chi_1}e^{2\chi_2}}{b_1^{3/2}b_2^{3/2}} - \frac{3}{2}\alpha^2\frac{e^{-2\phi_0}e^{-2\phi_1}e^{-2\phi_2}e^{2\chi_0}e^{2\chi_1}e^{2\chi_2}e^{-\sigma_0}e^{-\sigma_1}e^{-\sigma_2}}{b_1^{1/2}b_2^{1/2}}, \tag{4.33}$$

and in the $R_{rr}-R_{zz}\pm 2\mathrm{i}R_{rz}$ equation, which we are splitting into an imaginary part:

$$\frac{1}{2}\frac{\dot{b}_2}{b_2}\sigma_1' + \frac{\dot{b}_2}{b_2}\phi_1' - \frac{3}{2}\sigma_1'\sigma_2' - \frac{1}{2}\phi_1'\phi_2' = 0, \tag{4.34}$$

where we have divided the whole line by 2i, and a real part:

$$\frac{b_1''}{b_1} - \frac{1}{2}\frac{b_1'^2}{b_1^2} - \frac{b_1'}{b_1}\sigma_1' - 2\frac{b_1'}{b_1}\phi_1' + \frac{3}{2}\sigma_1'^2 + \frac{1}{2}\phi_1'^2 = \mathrm{constrz} = \frac{\ddot{b}_2}{b_2} - \frac{3}{2}\frac{\dot{b}_2^2}{b_2^2} + \frac{3}{2}\sigma_2'^2 + \frac{1}{2}\phi_2'^2. \tag{4.35}$$

Here we define a new function $g(r)$ obeying the relation

$$\sigma(r,z) = g(r) + \frac{\phi(r,z)}{6}, \tag{4.36}$$

so that σ and χ can be expressed as

$$\sigma(r,z) = \frac{\phi_0}{6} + g(r) + \frac{\phi_1(r)}{6} + \frac{\phi_2(z)}{6}, \tag{4.37}$$

$$\chi(r,z) = \chi_0 + \frac{1}{4}\ln b_1(r) + \frac{1}{2}g(r) + \frac{13}{12}\phi_1(r) + \frac{3}{4}\ln b_2(z). \tag{4.38}$$

We plug in these expressions into the z-component of the $R_t^t + 2R_\theta^\theta$ and $\Box\phi$ equations to get

$$2\frac{\ddot{b}_2}{b_2} = \frac{\dot{b}_2}{b_2}\phi_2' + \phi_2''. \tag{4.39}$$

We also plug in these expressions into the $R_{rr}-R_{zz}\pm 2\mathrm{i}R_{rz}$ equation, split into the imaginary part, the r and the z-components of the real part

$$\frac{\dot{b}_2}{b_2}\left(\frac{1}{2}g' + \frac{13}{12}\phi_1'\right) - \frac{1}{4}g'\phi_2' - \frac{13}{24}\phi_1'\phi_2' = 0, \tag{4.40}$$

$$\frac{b_1''}{b_1} - \frac{1}{2}\frac{b_1'^2}{b_1^2} - \frac{b_1'}{b_1}\left(g' + \frac{13}{6}\phi_1'\right) + \frac{3}{2}g'^2 + \frac{13}{24}\phi_1'^2 + \frac{1}{2}g'\phi_1' = \mathrm{constrz} \tag{4.41}$$

$$\frac{\ddot{b}_2}{b_2} - \frac{3}{2}\frac{\dot{b}_2^2}{b_2^2} + \frac{13}{24}\phi_2'^2 = \vartheta, \tag{4.42}$$

with ϑ a constant.

Next we require that the derivative of g

$$g' = \frac{\nu}{b_1} \Leftrightarrow b_1 = \frac{\nu}{g'}, \tag{4.43}$$

and assume

$$\phi_1' = \frac{\lambda}{b_1} \Leftrightarrow \phi_1 = \frac{\lambda}{\nu}g, \tag{4.44}$$

so that separation of variables definitions are now

$$B = \frac{\nu}{g'(r)} b_2(z), \tag{4.45}$$

$$\sigma = \frac{\phi_0}{6} + \left(1 + \frac{1}{6}\frac{\lambda}{\nu}\right) g(r) + \frac{\phi_2(z)}{6}, \tag{4.46}$$

$$\chi = \chi_0 + \frac{1}{4}\ln\left(\frac{\nu}{g'(r)}\right) + \left(\frac{13}{12}\frac{\lambda}{\nu} + \frac{1}{2}\right) g(r) + \frac{3}{4}\ln b_2(z), \tag{4.47}$$

$$\phi = \phi_0 + \frac{\lambda}{\nu} g(r) + \phi_2(z). \tag{4.48}$$

Substituting ϕ_1', g' into the imaginary part of the $R_{rr} - R_{zz} \pm 2iR_{rz}$ equation gives

$$\frac{\dot{b}_2}{b_2} = \frac{1}{2}\phi_2', \tag{4.49}$$

which can be integrated to get

$$\phi_2 = 2\ln b_2. \tag{4.50}$$

The consistency of this can be checked using (4.39). Next we substitute this expression into the z-component of the $\left(R_t^t + 2R_\theta^\theta\right)$ equation to get a differential equation

$$\frac{\ddot{b}_2}{b_2} = -(\text{const}) b_2^{-\frac{10}{3}}, \tag{4.51}$$

which can be solved by

$$b_2 = (\text{const}) z^{\frac{3}{5}}. \tag{4.52}$$

This solution is plugged into the z-component of the $R_{rr} - R_{zz} \pm 2iR_{rz}$ equation to determine $\vartheta = 0$.

Differentiating the r-component of the $\left(R_t^t + 2R_\theta^\theta\right)$ equation yields

$$b_1''' = \left(\nu + \frac{13}{6}\lambda\right)\frac{b_1''}{b_1}, \tag{4.53}$$

and having set $\vartheta = 0$, differentiating the r-component of the real part of the $R_{rr} - R_{zz} \pm 2iR_{rz}$ equation gives

$$b_1''' = \frac{(13\lambda^2 + 12\lambda\nu + 36\nu^2)}{24}\frac{b_1'}{b_1^2} + \frac{(6\nu + 13\lambda)}{6}\left(\frac{b_1''}{b_1} - \frac{b_1'^2}{b_1^2}\right) + \frac{b_1' b_1''}{b_1} - \frac{1}{2}\frac{b_1'^3}{b_1^2}, \tag{4.54}$$

The above two equations can be used to check the consistency of the assumptions so far.

Finally, by rescaling/renaming variables using

$$\frac{g(r)}{\nu} \to g(r), \qquad b_2(z) \to f(z), \tag{4.55}$$

$$\nu \to \left(a - \frac{ab}{6}\right), \qquad \lambda \to ab, \tag{4.56}$$

the assumptions for the method of separation of variables become

$$B = \frac{f(z)}{g'(r)}, \tag{4.57}$$

$$\sigma = \frac{\phi_0}{6} + ag(r) + \frac{1}{3}\ln f(z), \tag{4.58}$$

$$\chi = \chi_0 + \left(\left(\frac{a}{2} + ab\right) g(r) - \frac{1}{4}\ln g'(r)\right) + \frac{3}{4}\ln f(z), \tag{4.59}$$

$$\phi = \phi_0 + abg(r) + 2\ln f(z). \tag{4.60}$$

We note that this result is different from the one given in [1] in that there are now no terms containing c. Fortunately, it was stated in the paper that a vanishing c is in fact required for the two-brane solution as we shall see in the next sub-section.

Using the equations of motion, we get a set of equations for the variables f and g, all of which must be consistent.

•$R_t^t + 2R_\theta^\theta$

$$\left(\frac{1}{g'}\right)'' = 2e^{2\chi_0}e^{2\left(\frac{a}{2}+ab\right)g}, \tag{4.61}$$

$$\frac{\ddot{f}}{f} = -6\alpha^2 e^{-\frac{13\phi_0}{6}+2\chi_0}\frac{1}{f^{10/3}}, \tag{4.62}$$

•$R_t^t + R_r^r + R_z^z$

$$\frac{1}{4}g'\left(\frac{1}{g'}\right)'' - \frac{1}{4}\frac{g''^2}{g'^2} + \left(3+b^2\right)\frac{a^2g'^2}{4} + \left(\frac{1}{2}+b\right)ag'' = -\frac{1}{2}e^{2\chi_0}e^{2\left(\frac{a}{2}+ab\right)g}g', \tag{4.63}$$

$$\frac{3\ddot{f}}{4f} + \frac{\dot{f}^2}{3f^2} = -\frac{3}{2}\alpha^2 e^{-\frac{13\phi_0}{6}+2\chi_0}\frac{1}{f^{10/3}}, \tag{4.64}$$

•$R_{rr} - R_{zz} \pm 2iR_{rz}$

$$\left(\frac{1}{g'}\right)'' = \frac{g'}{2}\left(\frac{1}{g'}\right)'^2 + (1+2b)ag'\left(\frac{1}{g'}\right)' - \left(3+b^2\right)\frac{a^2}{2}g', \tag{4.65}$$

$$\frac{\ddot{f}}{f} = -\frac{2\dot{f}^2}{3f^2}, \tag{4.66}$$

We also find that the scalar field equation gives exactly the same equation as the (z)-component of $R_t^t + 2R_\theta^\theta$.

We can combine the $R_t^t + 2R_\theta^\theta$ and the $R_{rr} - R_{zz} \pm 2iR_{rz}$ equations to give a set of nonlinear differential equations:

$$\begin{aligned}
\frac{\ddot{f}}{f} &= -\frac{2\dot{f}^2}{3f^2} \\
&= -6\alpha^2 e^{-\frac{13\phi_0}{6}+2\chi_0} \frac{e^{\frac{1}{3}(-6+b)c\zeta}}{f^{10/3}}, && (4.67) \\
\left(\frac{1}{g'}\right)'' &= \frac{g'}{2}\left(\frac{1}{g'}\right)'^2 + (1+2b)ag'\left(\frac{1}{g'}\right)' - \left(3+b^2\right)\frac{a^2}{2}g' \\
&= 2e^{2\chi_0}e^{2\left(\frac{a}{2}+ab\right)g}. && (4.68)
\end{aligned}$$

The solutions are given by

$$f = \left(\frac{5}{3}\right)^{3/5}\left(\pm 3\alpha e^{\chi 0} e^{-\frac{13\phi 0}{12}} z + \text{const}\right)^{3/5} \tag{4.69}$$

$$g = \frac{1}{2E}\ln V_s(\rho), \tag{4.70}$$

where $E^2 = a^2(1 + b + 5b^2/4)$, V_s is the standard 4D Schwarzschild potential

$$V_s(\rho) = \left[1 - \frac{2E}{\rho}\right], \tag{4.71}$$

and we have introduced a new coordinate

$$\rho = \int e^{(2b+1)ag}. \tag{4.72}$$

We have found a solution assuming separability, and the metric is given by

$$ds^2 = f^{\frac{2}{3}}\left[V_s(\rho)^{\frac{a}{E}}dt^2 - V_s(\rho)^{-\frac{(1+b)a}{E}}[d\rho^2 + \rho(\rho - 2E)d\Omega^2] - V_s(\rho)^{\frac{ab}{E}}dz^2\right], \tag{4.73}$$

while the scalar field is given by

$$e^{2\phi} = V_s(\rho)^{\frac{ab}{E}}f^4. \tag{4.74}$$

By taking $a = 0$, we get $E = 0$ and the functions now only depend on z and the metric becomes

$$ds^2 = f(z)^2\left(\eta_{\mu\nu}dx^\mu dx^\nu - dz^2\right). \tag{4.75}$$

This corresponds to the LOSW vacuum.

Another possibility is to set $b = 0$ but with $a = E \neq 0$, which gives the uniform black string solution. If $b \neq 0$, then $a \neq E$ and the metric becomes

$$\begin{aligned} ds^2 = a^2(y) & \left[\left(1 - \frac{2E}{\rho}\right)^{\frac{a}{E}} dt^2 - \left(1 - \frac{2E}{\rho}\right)^{-\frac{a(1+b)}{E}} [d\rho^2 + \rho(\rho - 2E)d\Omega^2] \right] \\ & - \left(1 - \frac{2E}{\rho}\right)^{\frac{ab}{E}} dy^2. \end{aligned} \tag{4.76}$$

We note that this metric is not Gaussian normal. The scalar field in this case is given by

$$e^{2\phi} = \left(1 - \frac{2E}{\rho}\right)^{\frac{ab}{E}} a^{12}(y). \tag{4.77}$$

The function $a = a(y)$ is the warp factor introduced at the beginning of Chapter 3.

4.2.1 A braneworld from the axisymmetric solution

For a braneworld solution, we introduce branes at $z = z_\pm$ with the normal given by

$$n_z = f^{1/3} V^{\frac{ab}{2E}} dz, \tag{4.78}$$

and compute the extrinsic curvature using

$$K_{\mu\nu} = \nabla_\mu n_\nu, \tag{4.79}$$

so that we end up with

$$K_{tt} = -e^{-\phi} \left(\frac{\dot{f}}{3f^{1/3}} \right) g_{tt}, \tag{4.80}$$

$$K_{\rho\rho} = -e^{-\phi} \left(\frac{\dot{f}}{3f^{1/3}} \right) g_{\rho\rho}, \tag{4.81}$$

$$K_{\theta\theta} = -e^{-\phi} \left(\frac{\dot{f}}{3f^{1/3}} \right) g_{\theta\theta}. \tag{4.82}$$

It is clear that the requirement for a brane solution (where energy equals tension) is satisfied. The Israel junction condition is given by

$$K_{\mu\nu} = -\frac{1}{2} \left(T_{\mu\nu} - \frac{1}{3} g_{\mu\nu} T \right) \tag{4.83}$$

and is easily satisfied. From this we conclude that the two-brane solution is given by (4.76) and (4.77). The branes can be set at any fixed y-coordinate, so to recover the LOSW vacuum we take $y = \pm y_0$.

On the + brane, the solution is given by

$$ds^2 = \left(1 - \frac{2E}{\rho}\right)^{\frac{a}{E}} dt^2 - \left(1 - \frac{2E}{\rho}\right)^{-\frac{a(1+b)}{E}} [d\rho^2 + \rho(\rho - 2E)d\Omega^2], \tag{4.84}$$

$$e^{2\phi} = \left(1 - \frac{2E}{\rho}\right)^{\frac{ab}{E}}, \tag{4.85}$$

and the interbrane distance depends on ρ

$$D = \int_{-y_0}^{y_0} dy|g_{yy}|^{1/2} = 2y_0 V_s(\rho)^{\frac{ab}{2E}}. \tag{4.86}$$

From this we see there are two possibilities: For $ab > 0$, as ρ decreases D also decreases so the branes move closer together, and at $\rho = 2E$, $D = 0$ and the extra dimension is closed. For $ab < 0$, we get the opposite behaviour, with the distance between the two branes becoming larger until at $\rho = 2E$ it becomes infinite.

The two possibilities outlined above both give spherically symmetric braneworld solutions. The linearised solution obtained in the previous chapter is given by

$$h_{tt}^+ = -\frac{10\alpha G_5 M}{r}, \tag{4.87}$$

$$\delta\phi^+ = \frac{2\alpha G_5 M}{r}. \tag{4.88}$$

Expanding (4.84) and (4.85) at large ρ gives:

$$g_{tt} \simeq 1 - \frac{2a}{\rho}, \tag{4.89}$$

$$\phi \simeq -\frac{ab}{\rho}, \tag{4.90}$$

and finally

$$a = 5\alpha G_5 M, \tag{4.91}$$

$$b = -2/5. \tag{4.92}$$

So the linearised solution is the case where $ab < 0$. The branes move apart until the distance becomes infinite.

We conclude this chapter by summarising our findings. Starting with an axisymmetric bulk bounded by two branes, we assume that the metric is separable and derived the general brane solution which asymptotes the LOSW vacuum. This solution is singular at $\rho = 2E$, but looks like the Schwarzschild solution at large ρ.

Although it has the appearance of a string solution, we notice that as ρ decreases and the solution approaches singularity, the interbrane distance (and thus the length of the string) approaches infinity.

Chapter 5

Discussion

The present time is certainly an interesting time in particle physics. We have an established theory describing three of the four fundamental forces. The Standard Model of particle physics describes the electromagnetic, weak and strong interaction. It has been tested to a high degree of accuracy in many experiments. The fourth fundamental force is described by Einstein's General Theory of Relativity which has also been successfully tested in experiments.

Despite the experimental successes, there are many outstanding questions in modern particle physics. Firstly, the lack of gravity in SM motivates many physicists to look for a unified theory, an idea that has been around since Einstein's time, something Einstein himself had been working on. A first succesful example of a unified theory was offered by Kaluza and Klein, a minimal extension of Einstein's GR which included electromagnetism at the price of introducing one extra dimension. The Kaluza-Klein theory also offered an explanation for the quantization of charge, but this aspect was less successful because it failed to give the correct value for the mass of the electron. However, the spirit of unification lives on in theories like string theory and loop quantum gravity.

Despite this problem, the Standard Model is regarded as the most successful theories of physics, supported by a wealth of experimental data. Unfortunately, there is one missing piece in this picture. In SM, the Higgs mechanism explains how the W and Z boson gain masses while leaving the photon massless through spontaneous symmetry breaking. The mechanism predicts the existence of a massive scalar particle, called the Higgs boson. It has not been found, and discovery of this particle will put the Standard Model at an even stronger footing.

Another problem in SM is the hierarchy problem. It is the question of why the Planck scale is so much larger than the electroweak scale. Phrased a different way, it questions why gravity is so weak. It may be an indication that SM is not a complete theory, or that some fine tuning may be necessary to make things work. Another problem involving fine tuning in SM is the cosmological constant problem. The value of the the cosmological constant predicted by SM is in fact 10^{120} bigger than the observed value, a nonsensical result.

The Standard Model provides a very good description for all known particles and their interactions. However, astrophysical and cosmological observations indicate that there may be a new kind of matter which is not described in SM called dark matter. There is significant evidence that dark matter exists but because dark matter only interacts gravitationally, it is difficult to study in Earth-based experiments, and so far there is no consensus on what it is. To further complicate matters, there is strong evidence that the expansion of the universe is accelerating. The cause of this is also yet unknown, and so far has been dubbed dark energy. These problems show that particle physics and cosmology are closely related, and further illustrates the need for unification.

Finally, there is still no satisfactory answer to the question of why there are four dimensions. Other than the lack of observational evidence for extra dimensions, or assuming there is a consistent way to hide them, there is no reason to say that they do not exist, because the laws of physics do not exclude them.

There have been many ideas proposed to solve the outstanding questions. Some of them are well motivated from existing theories, while some are toy models which are nevertheless helpful in developing ideas. Some of the approaches we have seen involve adding new matter to existing theories, such as the dark matter and dark energy proposals, and other approaches involve modifying the existing theories, for example the Kaluza-Klein model, scalar-tensor/Brans-Dicke gravity, Modified Newtonian Dynamics (MOND), and finally the braneworld scenario, which can be regarded as the modern day incarnation of the KK model.

The braneworld scenario offer an interesting way to modify gravity. Some models modify gravity in the short range, others feature large scale modification. In recent years, it was shown that the extra dimensions need not be compactified, in contrast to the original Kaluza-Klein theory. The most well-known of these so called braneworld scenarios, the RS model, showed that it was possible to hide the extra dimension by making it warped. This warped extra dimension has a significant effect on the gravity of the model, making gravity appear four-dimensional on the brane where we live, but five-dimensional in the bulk. Another model which also received much attention was the one put forward by Dvali, Gabadadze and Poratti. The DGP model has a flat bulk, as opposed to the warped bulk of the RS model, and admitted cosmological solutions in which the universe is accelerating even when the brane tension is zero, commonly known as the self-accelerating solution. Although the RS model may be embedded into string theory, this is not possible to do for the DGP model.

Fortunately, Lukas, Ovrut, Stelle and Waldram showed that it was possible to obtain a solution of heterotic M-theory consisting of two parallel three branes separated by a bulk containing a scalar field. This scalar field arose from the compactification of six of the eleven dimensions in the original heterotic model. This setup reminds us of the first RS model, only this time there is a scalar field in the bulk. Cosmological solutions of the heterotic braneworld model have been studied by many physicists and have given the ekpyrotic and the cyclic universe models. However, there has not been a complete description of the heterotic braneworld gravity in the literature. As black holes are considered a good environment to test gravity, it is also important to find a black hole solution in the context of the heterotic M-theory braneworld. These are the two topics discussed in this dissertation.

We saw that the heterotic M-theory braneworld, although similar to the RS braneworlds, has several different properties. Using perturbation theory, we were able to study the brane gravity. We saw that the solutions correspond to a graviton and a massive KK tower, and that the radion is coupled to the bulk scalar field. The brane gravity is Brans-Dicke type with the BD parameter $\omega = 0.5$, while current solar system data requires $\omega > 10^4$, so this indicates that the heterotic braneworld model is not supported by experimental data.

Nevertheless, we moved on to look for black hole solutions in this setup. The first step we took was to construct a black string between the two branes. This black string was seen to be unstable and has unusual asymptotics. Due to these problems, we look further for our black hole solutions. We could not find spherically symmetric bulk solutions because the LOSW vacuum is anisotropic. It is not possible to construct a black hole perturbatively, because even a small perturbation will interact with the scalar field in the LOSW bulk.

The next step was to use an axisymmetric bulk setup, and to use the method of separation of variables to find solutions to the equations of motion. We saw that it was possible to obtain a solution that approximates the LOSW vacuum which looks like the Schwarzschild solution at large distances. Unfortunately, it was found that the interbrane distance is not a constant, and becomes infinite at the Schwarzschild radius, stretching the black string solution along with it.

We conclude that the heterotic braneworld model of LOSW is still not an adequate model of the universe. It predicts a scalar-tensor type gravity with a value of the BD parameter which has been excluded from astronomical data. There is still some debate as to the correct cosmological solution for LOSW, with one interpretation arguing for the ekpyrotic model and another for the cyclic universe one. One interpretation predicts a doomed universe with the brane crashing into a singularity, while another is more optimistic and argues that the singularity is mild.

However, research in particle physics goes on, and new models will appear that hopefully will better approximate the real universe. New experiments are conducted, and we gain better data to test the models. With the start of the LHC, it is possible within the next couple of years we can see the Higgs boson, evidence for SUSY, and evidence for extra dimensions.

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