Asmptotically Plane Wave Spacetimes

LE-WITT, JULIAN,ALEXANDER

Asymptotically plane wave spacetimes

Julian Le Witt

A Thesis presented for the degree of
Doctor of Philosophy

Centre for Particle Theory
Department of Mathematical Sciences
University of Durham
England

December 2009

Dedicated to

My family

Asymptotically plane wave spacetimes

Julian Le Witt

Submitted for the degree of Doctor of Philosophy
December 2009

Abstract

In this thesis we study aspects of plane wave spacetimes in the hope of shedding light on the nature of holography for plane waves. In particular, we would like to understand better the space of asymptotically plane wave solutions. We first review the necessary background on plane waves, variational principles for gravity and black holes in higher dimensions. We then propose a definition of asymptotically plane wave spacetimes in vacuum gravity in terms of the asymptotic fall-off of the metric and discuss the relation to previously constructed exact solutions. We construct a well-behaved action principle for such spacetimes, using the formalism developed by Mann and Marolf. We show that this action is finite on-shell and that the variational principle is well-defined for solutions of vacuum gravity satisfying our asymptotically plane wave fall-off conditions.

Next we investigate the construction of black holes and black strings in vacuum plane wave spacetimes using the method of matched asymptotic expansions. We find solutions of the linearised equations of motion in the asymptotic region for a general source on a plane wave background. We observe that these solutions have some unusual properties and do not satisfy our previously defined conditions for being asymptotically plane wave. Hence, the space of asymptotically plane wave solutions is restricted. We consider the solution in the near horizon region, treating the plane wave as a perturbation of a black object, and find that there is a regular black string solution. We find that no regular black hole solution exists, which is a counter-example to a conjecture of Emparan et. al. We end with a discussion of our results and suggest possible directions for future work.

Declaration

The work in this thesis is based on research carried out at the Centre for Particle Theory, Department of Mathematical Sciences, University of Durham, England. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all my own work unless referenced to the contrary in the text.

Chapters 1 and 2 serve as motivation and background for the rest of the thesis, and consist of a review of known results. The remainder of the thesis is original work. Chapters 3, 4 and 5 are based on published work [1, 2], done in collaboration with Simon Ross.

Acknowledgements

I would like to thank my supervisor, Simon Ross, for his excellent guidance during the course of my studies. I am very grateful for his generosity with his time and have benefited greatly from our many discussions. I also gratefully acknowledge the Science and Technology Facilities Council (STFC) for supporting my research. Furthermore I would like to thank my family, especially my parents and grandma, for their support and encouragement. Finally, many thanks to Jamie Parsons and my dad for reading and commenting on a draft of my thesis.

Contents

Chapter 1

Introduction

1.1 Plane waves

Plane wave spacetimes were first introduced by Hans Brinkmann in 1925 in a paper on "Einstein spaces which are mapped conformally on each other" [3]. Interest in plane waves was revived by Rosen in 1937 and they were comprehensively studied by Jordan, Ehlers and Kundt during the late 1950s and early 1960s [4–6]. Though the plane wave metric does describe the propagation of waves, it is not meant to be a realistic model of gravitational waves. Far from the source of a realistic gravitational wave the gravitational field is weak and well described by the linearised Einstein equations. The strong gravitational fields which produce the waves (for example, produced by a system of two orbiting black holes) will require solutions of the full Einstein equations; these solutions, however, will be of a much more complex form than the plane wave metric [7].

Plane waves are interesting from a variety of different points of view. One of their most intriguing properties is that they can be thought of as arising from any spacetime in a certain limit. This is known as the Penrose limit [8] and essentially consists of choosing any null geodesic in the spacetime and zooming onto it; the spacetime in the neighbourhood of the null geodesic is a plane wave. Moreover, plane wave spacetimes are not globally hyperbolic, so there is no Cauchy hypersurface from which a causal evolution would cover the whole spacetime [9]. This means that their causal structure is very different to that of flat spacetime where any spacelike hypersurface is a Cauchy hypersurface. Plane waves also have unusual boundary dimensionality. Most familiar examples such as Minkowski, Anti-de Sitter (AdS) and de Sitter (dS) spacetimes in d dimensions have a $d-1$ dimensional boundary. However, a large class of plane wave solutions has been found that have a one-dimensional boundary [10, 11].

Plane waves also provide a rich class of exact solutions to Einstein's equations, including some maximally supersymmetric solutions of supergravity. Supergravities arise as low energy effective theories of strings and can, in general, receive α' corrections involving higher powers of the curvature. As we describe in section 2.1, plane waves admit a covariantly constant null Killing vector and their curvature is null, so they receive no α' corrections [12]. Thus plane waves are exact α' solutions of supergravity on which the string worldsheet theory is exactly solvable [13]. This property makes plane waves a particularly interesting background for the study of holography. The holographic principle states that all the physics of a quantum gravity theory in some spacetime can be exactly described in terms of some non-gravitational quantum theory on the boundary of the spacetime [14, 15]. This principle can be motivated by the Bekenstein bound which states that the maximum entropy of a given region of spacetime is proportional to the area of the spacetime [16]. An important realisation of holography is the AdS/CFT correspondence [17], which relates a gravitational theory on the bulk of AdS to a conformal field theory living on its boundary. More specifically, the conjecture states that type IIB string theory on $\text{AdS}_5 \times S^5$ is equivalent to $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in four dimensions. However, despite much effort, string theory on AdS is still poorly understood and computations must be performed in the low energy limit where supergravity is a good approximation. An exciting development was the seminal work of [18] in which the Penrose limit of $\text{AdS}_5 \times S^5$ was shown to be the maximally supersymmetric plane wave of [19]. Since then, string theory on this background has been of intense interest as an example of holography [20]. The spectrum of strings on the plane wave is related to the spectrum of a quantum mechanical system obtained from the dual CFT on the boundary of the AdS_5 space. Since string theory on plane waves is solvable, this connection provides stringy tests of the AdS/CFT correspondence and has significantly deepened our understanding of this duality.

However, our understanding of holography for the plane wave is still incomplete; the duality is more indirect than AdS/CFT since the dual quantum mechanics is obtained from the theory on the boundary of AdS, whereas the Penrose limit which gives rise to the plane wave focuses on a region at the centre of AdS. Although a well-defined notion of the boundary of the maximally supersymmetric plane was obtained by conformal compactification in [10], and this boundary turns out to be one-dimensional, a direct connection between the string theory on this plane wave and a theory living in some sense on its asymptotic boundary has not yet been constructed. As a result, it has not been possible to extend the results of [20] to discuss a holographic duality for general plane waves.

Another interesting issue is whether plane waves admit event horizons. If they did then we would have black hole spacetimes with a covariantly constant null Killing field which, as discussed above, would correspond to α' exact solutions of supergravity. Unfortunately, it was shown in [21] that plane waves cannot admit event horizons. Every point in the spacetime can communicate "out to infinity" and since black holes are regions bounded by a horizon, there can be no black holes in plane waves. This does not mean, however, that we cannot look for solutions that are asymptotically plane wave. Indeed, a useful approach to deepening our understanding of the duality for plane waves is to construct asymptotically plane wave spacetimes and to look for interpretations of these spacetimes in field theory terms. In particular, it is clearly interesting to construct asymptotically plane wave black holes and black strings. The construction of such solutions has been discussed in [21–26]. The asymptotic structure of plane waves has also been discussed from a general point of view in [11, 27, 28], using the causal completion of the spacetime.

1.2 Variational principles for gravity

Variational principles play an important role in theoretical physics; most fundamental physical theories can be described in terms of an action and the equations of motion derived from a variational principle. The action provides a link between classical and quantum theories and a well-behaved action principle is essential for the treatment of semi-classical issues. The classical limit of a quantum partition function is obtained as a saddle point approximation where stationary points of the action dominate the path integral. It is well known, however, that the familiar Einstein-Hilbert and Gibbons-Hawking action does not fulfil the necessary conditions for a well-behaved action principle for non-compact spaces [29], namely being finite on-shell and vanishing under all variations that preserve the boundary conditions. The Einstein-Hilbert action is constructed from the Ricci scalar R which contains terms which are linear in second derivatives of the metric. Gibbons and Hawking [29] found these second derivative terms could be eliminated by the addition of a boundary term constructed from the extrinsic curvature of the boundary. This resulted in an action depending only on first derivatives of the metric, as required by path integral approaches to quantum gravity. The action, however, remains divergent for non-compact solutions of the field equations. These divergences may be removed by a procedure known as background subtraction [29]. Given any spacetime, the prescription involves isometrically embedding a regulating boundary into a suitable reference spacetime. A finite action may then be obtained by taking the difference in the regulated actions between the original spacetime and the reference spacetimes in the limit that the regulating boundary goes to infinity. One can think of this new action as a description of the spacetime properties that were not already present in the reference background. This technique has produced physically reasonable results; however, it has significant limitations. Firstly, the choice of reference background is not unique and this can lead to inconsistent results. Secondly, in $D \geq 4$ spacetime dimensions, Weyl's embedding theorem states that an isometric embedding of a regulating boundary in a reference background may not exist [30]. Hence, there is no guarantee that this procedure will work for a given spacetime. This problem is not restricted to pathological spacetimes as even simple solutions like the Kerr spacetime suffers from ambiguities [31].

Inspired by holography and in particular the AdS/CFT correspondence, a new approach to cancelling the large volume divergences in the gravitational action has been formulated [32,33]. This approach, known as the counterterm method, involves introducing an additional boundary term to the usual action, chosen to cancel any divergences. The counterterms are functionals only of the curvature invariants of the induced metric on the boundary and so they do not contribute to the bulk field equations. This procedure is intrinsic to the spacetime of interest, unlike background subtraction, and gives unique results once the counterterm has been specified. This was originally developed for asymptotically AdS spacetimes [32, 33], but counterterms have since been developed for special classes of asymptotically flat spacetimes [34, 35]. An exciting recent development was the construction of a well-behaved action principle for any asymptotically flat spacetime in [30] (see also [36–38]), which was argued in [39] to provide an approach to defining a holographic dual to asymptotically flat space. This was extended to study holography for linear dilaton spacetimes in [40,41]. In chapter 3, we will use the Mann-Marolf counterterm introduced in [30] to construct a well-behaved action principle for asymptotically plane wave spacetimes.

1.3 Black holes

Black holes are the most basic objects of general relativity and have revealed much about the nature of gravity and indeed, quantum gravity. Classical, four dimensional, asymptotically flat black holes are well understood and have been found to exhibit a number of remarkable properties, namely uniqueness, stability, rigidity, spherical topology and the laws of black hole thermodynamics [42]. Recently, there has been a great deal of interest in the study of black holes in higher dimensions [43–46] and also with non-flat asymptotics [21–23, 47, 48]. There are a number of motivations for this interest. Firstly, string theory contains gravity and requires more than four dimensions, as do brane world models. Secondly, one might expect the study of black holes in higher dimensions to lead to a better understanding of gravity in general.

In higher dimensions the spectrum of solutions becomes much more complicated; for example, in addition to the higher dimensional analogues of the Schwarzschild and Kerr solutions, i.e. the Schwarzschild-Tangherlini and Myers-Perry solutions

[43], there are also black p-branes, black rings [44] and multi-black hole solutions such as black saturns [46], in which a Myers-Perry black hole is surrounded by a spinning black ring. Higher dimensional solutions are in general no longer unique or stable and may have non-spherical topology and extended event horizons [42]. It seems that attempting to find exact solutions for all possible higher dimensional black holes may not be the best approach and that instead it may be more fruitful to develop some general framework for the approximate construction and classification of black hole solutions. The presence of extended event horizons is an important new feature of black hole solutions in higher dimensions as it results in two or more widely separated length scales. We can take advantage of this separation of scales by integrating out the short-distance physics to obtain a long-distance effective theory. This can be implemented in practice using either the method of matched asymptotic expansions [49] or classical effective field theory [50]. For black holes with two widely separated length scales, a general effective theory describing the dynamics at scales much larger than the small scale has recently been developed [51, 52]. In this approach the black hole is viewed as a blackfold, that is a *black* brane which is embedded into a curved submani*fold* of the spacetime. The theory describes which embeddings are allowed and hence can be used to classify the spectrum of black holes.

1.4 Overview of thesis

In chapter 2, we discuss the essential background material required for chapters 3 and 4. We start by defining plane waves and pp-waves in terms of both Brinkmann and Rosen coordinates. We then consider some special cases of plane waves, in particular maximally symmetric plane waves and vacuum plane waves in four and five dimensions. Finally, we describe the steps required in taking the Penrose limit.

In section 2.2, we examine the Einstein-Hilbert and Hawking-Gibbons action in the context of asymptotically flat spacetimes and show that the variational principle is not well-defined for non-compact spacetimes. Specifically, we demonstrate that the action is neither finite on-shell nor is it stationary under all variations of the metric preserving the boundary conditions. Different approaches to creating a well-defined variational principle are then considered, namely background subtraction methods and the addition of boundary counterterms. The Mann-Marolf counterterm is then introduced and its form is motivated by consideration of the Gauss-Codazzi equations. We then show that the addition of the Mann-Marolf counterterm to the Einstein-Hilbert and Gibbons-Hawking action results in a well-defined action principle for asymptotically flat spacetimes.

In section 2.3, we turn to the description of neutral, vacuum black holes in higher dimensions as background to the work in chapter 4 where we attempt the construction of black holes in plane waves. We first consider a new general description of higher dimensional black holes in terms of a blackfold and then the construction of an approximate solution for an asymptotically flat, neutral, thin rotating black ring in some detail as a particular realisation of this method. We introduce the conjecture which states that satisfying the blackfold equations (2.87) guarantees the existence of a regular horizon and we consider evidence in support of it. Later, in chapter 4 we find a counter-example to this conjecture.

In chapter 3, we construct an action principle for asymptotically plane wave spacetimes. To discuss the action for asymptotically plane wave spacetimes, we first need a suitable notion of what it means for a spacetime to be asymptotically plane wave. In section 3.1, we propose a definition in terms of a set of fall-off conditions on the metric at large spatial distances in directions orthogonal to the wave. We then need to determine the behaviour of the components of the metric with indices parallel to the wave; we use the linearised equations of motion to relate the fall-off conditions of different components, by assuming that all components make contributions of the same order to each term in the Einstein equations. This fixes the fall-off of the other components of the metric. We show that the known solutions which asymptotically approach a vacuum plane wave [21–23] satisfy our fall-off conditions.

In section 3.2, we show that the definition of the action for vacuum gravity introduced in [30] can be applied to asymptotically plane wave spacetimes with our fall-off conditions without significant modification. We demonstrate that the on-shell action is finite and that the variational principle is well-defined. This provides confirmation that this is a useful definition of an asymptotically plane wave, and provides another example where the counter-term approach of [30] is useful, suggesting that this approach to defining the gravitational action should have a broad applicability.

In chapter 4, we adopt the method of matched asymptotic expansions to find approximate solutions when the horizon size r_+ of the black hole or black string is small compared to the curvature scale μ^{-1} of the plane wave. This gives a separation of scales which can be exploited to solve the equations of motion in the linearised approximation in separate regions, matching the solutions in an overlap region.

We proceed in a similar way to the earlier example in chapter 2, first finding the metric far from the source (for $r \gg r_+$) by studying the linearised approximation to gravity with an appropriate delta-function source. The wave equation in the plane wave background is rather complicated, so we focus on solving this problem in an intermediate region $r_+ \ll r \ll \mu^{-1}$ where the deviations from flat space due to both the source and the plane wave are small.

Solving the equation in this regime, we find that simple dimensional analysis indicates that the solutions will violate the asymptotic boundary conditions proposed in chapter 3 as a definition of asymptotically plane wave spacetimes. In fact, the perturbation due to the delta-function source becomes large relative to the background metric at large distances. An explicit analysis in four and five dimensions shows that the terms violating these boundary conditions are indeed non-zero.

We then obtain the near horizon metric in the region $r \ll \mu^{-1}$ by solving the linearised Einstein equations on the background of the black object, treating the plane wave as a perturbation. For a black hole, we find that there is no linearised solution which is regular on the horizon. For the black string, we obtain a regular solution in the near horizon region and verify that it matches on to the solution in the intermediate region.

The calculation in the region $r \gg r_+$ is described in section 4.1, and the calculation in the region $r \ll \mu^{-1}$ is described in section 4.2.

In chapter 5, we conclude the thesis with some remarks on the interpretation and implications of our results.

Chapter 2

Background

2.1 Plane waves

In this section, we introduce the plane wave and pp-wave metrics. Brinkmann and Rosen coordinates for plane waves are discussed and the transformation between them is given. We then consider some special cases of plane waves: homogeneous plane waves, maximally symmetric plane waves and vacuum plane waves in four and five dimensions. Finally, we describe the process of recovering a plane wave from any spacetime, the Penrose limit.

Generally, when considering gravitational plane waves in flat space far from their source, one assumes a metric of the form [53]

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \tag{2.1}$$

where $\eta_{\mu\nu}$ is the Minkowski background and $h_{\mu\nu}$ is a small perturbation. When working to linear order in the perturbation, Einstein's equations reduce to a wave equation whose solutions are gravitational waves. A solution representing a gravitational wave travelling in the (t, z) direction is given by

$$ds^2 = -dt^2 + dz^2 + (\delta_{IJ} + h_{IJ}(t + z))dx^I dx^J, \tag{2.2}$$

where x^I are Cartesian coordinates for the directions transverse to the wave. Introducing light-cone coordinates $x^+ = t + z, x^- = (t - z)/2$ the metric becomes

$$ds^2 = -2dx^+dx^- + (\delta_{IJ} + h_{IJ}(x^+))dx^I dx^J. \tag{2.3}$$

We can now define plane waves by this metric, dropping the assumption that the perturbation is small, i.e. a plane wave has a metric of the form [7]

$$ds^2 = -2dx^+dx^- + g_{IJ}(x^+)dx^Idx^J. \tag{2.4}$$

These are Rosen coordinates for the plane wave.

An alternative approach to defining plane waves is to first consider the more general class of pp-waves. These are defined as spacetimes that support a covariantly constant null Killing vector field; i.e. a Killing vector field v^μ that satisfies

$$\nabla_\mu v_\nu = 0, \quad v_\mu v^\mu = 0. \tag{2.5}$$

The most general metric satisfying these conditions takes the form [13]

$$ds^2 = -2dx^+dx^- - F(x^+,x^I)(dx^+)^2 + 2A_J(x^+,x^I)dx^+dx^J + g_{JK}(x^+,x^I)dx^Jdx^K \tag{2.6}$$

where $g_{JK}(x^+,x^I)$ is the metric on the space transverse to light-cone directions x^+, x^- and the coefficients $F(x^+,x^I)$, $A_J(x^+,x^I)$ and $g_{JK}(x^+,x^I)$ are constrained by Einstein's equations. It is clear that the above metric has a null Killing vector field $\left(\frac{\partial}{\partial x^-}\right)^\mu$ which is covariantly constant due to the vanishing of the Γ^-_{-+} component of the Christoffel symbol. The most commonly considered waves have $A_J = 0$ and are flat in the transverse direction[1]

$$ds^2 = -2dx^+dx^- - F(x^+,x^I)(dx^+)^2 + \delta_{IJ}dx^Idx^J. \tag{2.7}$$

In this thesis we are interested in the sub-class of pp-waves known as plane waves. By definition, plane waves have $F(x^+,x^I)$ quadratic in the transverse coordinates[2] but an arbitrary function of x^+. The metric for a plane wave then takes the form

$$ds^2 = -2dx^+dx^- - \mu_{IJ}(x^+)x^Ix^J(dx^+)^2 + \delta_{IJ}dx^Idx^J, \tag{2.8}$$

with $\mu_{IJ}(x^+)$ symmetric. Notice that in the limit $\mu_{IJ} \to 0$ we recover flat space. This metric has a "plane" symmetry given by translations of the transverse coordinates

[1] In fact, we can set $A_J = 0$ by a choice of coordinates so long as g_{JK} is non-degenerate.

[2] Constant and linear terms in x^I can be removed by a coordinate transformation.

on the wavefronts $x^+ =$ constant, $x^- =$ constant. This is most easily seen in Rosen coordinates

$$ds^2 = -2dx^+dx^- + g_{IJ}(x^+)dx^I dx^J. \tag{2.9}$$

The transformation back to the Brinkmann form (2.8) is given by the change of coordinates

$$x^I \to h^I{}_J(x^+)x^J, \quad x^- \to x^- + \frac{1}{2}g_{IJ}(x^+)h'^I{}_K(x^+)h^J{}_L(x^+)x^K x^L \tag{2.10}$$

with $h_{IK}g^{IJ}h_{JL} = \delta_{KL}$ and $\mu_{KL} = g_{IJ}h''^I{}_K h^J{}_L$, where the prime denotes differentiation with respect to x^+. Throughout this thesis, we write plane waves in the Brinkmann coordinate system. Brinkmann coordinates are more useful as they are globally well-defined, whilst Rosen coordinates are not unique and the metric can exhibit unphysical coordinate singularities.

It is easy to show that the only non-zero component of the Riemann tensor for the plane wave (2.8) is

$$R_{+I+J} = -\mu_{IJ} \tag{2.11}$$

and because of the null Killing vector $\left(\frac{\partial}{\partial x^-}\right)^\mu$ the only non-zero component of the Ricci tensor is

$$R_{++} = -\delta^{IJ}\mu_{IJ}, \tag{2.12}$$

and the Ricci scalar is zero,

$$R = 0. \tag{2.13}$$

Hence, for vacuum gravity Einstein's equations imply that $\mu_{IJ}(x^+)$ must be traceless. If μ_{IJ} is constant then we have what are known as homogeneous plane waves

$$ds^2 = -2dx^+dx^- - \mu_{IJ}x^I x^J(dx^+)^2 + \delta_{IJ}dx^I dx^J. \tag{2.14}$$

Finally if we take $\mu_{IJ} = \mu^2\delta_{IJ}$ we have the maximally symmetric plane wave

$$ds^2 = -2dx^+dx^- - \mu^2\delta_{IJ}x^I x^J(dx^+)^2 + \delta_{IJ}dx^I dx^J. \tag{2.15}$$

Note that in this case μ_{IJ} is not traceless, so this is not a vacuum solution and there must be some matter support. This plane wave has been the subject of intense study since the discovery that this metric in ten dimensions, supported by a self-dual five-form flux is a maximally supersymmetric solution of type-IIB supergravity [19] and can be obtained by taking the Penrose limit of $AdS_5 \times S^5$ [18].

Consider the plane wave metric (2.8). We can always diagonalise μ_{IJ} by a rotation of the transverse coordinates x^I to put the metric into the form

$$ds^2 = -2dx^+dx^- - \mu_{II}(x^+)x^I x^I (dx^+)^2 + \delta_{IJ}dx^I dx^J. \tag{2.16}$$

It is clear that for vacuum gravity in $d = 2$ transverse dimensions the metric can be written as

$$ds^2 = -2dx^+dx^- - a(x^2 - y^2)(dx^+)^2 + dx^2 + dy^2, \tag{2.17}$$

where a is an arbitrary function of x^+ and we have used $Tr\mu = 0$. In $d = 3$ transverse dimensions the constraint $Tr\mu = 0$ defines a plane and so the metric can be written in terms of a two parameter family

$$ds^2 = -2dx^+dx^- - [\alpha(x^2 - y^2) + \beta(x^2 + y^2 - 2w^2)](dx^+)^2 + dx^2 + dy^2 + dw^2, \tag{2.18}$$

with α and β arbitrary functions of x^+.

2.1.1 The Penrose limit

As discussed in the introduction it is possible to generate a plane wave from any spacetime through a process known as the Penrose limit [8]. The Penrose limit may be successfully applied to any Lorentzian spacetime; however, if the initial spacetime is a solution of Einstein's equations then so too will be the resulting plane wave after taking the Penrose limit. In this way the Penrose limit can be used to generate new solutions.

The required steps for taking the Penrose limit are [13]:

- Find a null geodesic in the initial spacetime
- Then choose a coordinate system such that the metric takes the form

$$ds^2 = R^2 \left[-2dx^+d\widetilde{x}^- + d\widetilde{x}^- \left(d\widetilde{x}^- + A_J(x^+, \widetilde{x}^-, \widetilde{x}^I)d\widetilde{x}^J\right) + g_{JK}(x^+, \widetilde{x}^-, \widetilde{x}^I)d\widetilde{x}^J d\widetilde{x}^K\right] \tag{2.19}$$

where x^+ is an affine parameter for the null geodesic, the distance between such geodesics is parametrised by $\widetilde{x}^-$ and $\widetilde{x}^I$ parametrises the remaining coordinates. Any metric may be written in this form in the neighbourhood of the null geodesic.

- Finally, take the limit $R \to \infty$ with

$$\widetilde{x}^- = \frac{x^-}{R^2}, \quad \widetilde{x}^I = \frac{x^I}{R}, \quad with\ x^+, x^-, x^I\ fixed. \tag{2.20}$$

Taking this limit, it is easy to see that the A_J term drops out and $g_{JK}(x^+, \widetilde{x}^-, \widetilde{x}^I)$ becomes only a function of x^+. The resulting metric is

$$ds^2 = -2dx^+dx^- + g_{IJ}(x^+)dx^I dx^J, \tag{2.21}$$

which is simply the plane wave metric in Rosen coordinates.

2.2 Gravitational counterterms

In this section, we consider the construction of a well-defined variational principle for gravity. We first discuss the standard gravitational action, the Einstein-Hilbert action, and show that the variational principle is not well-defined, even for compact manifolds with boundary, due to a non-vanishing boundary term. We then consider the addition of the Gibbons-Hawking term and show that the resulting variational principle is well-defined on compact manifolds. However, in general, the action will not be finite for non-compact manifolds . Two different approaches to making the action finite are then considered: background subtraction methods and holographic renormalization, and some specific examples are provided. The more general Mann-Marolf counterterm is then introduced and its form is motivated by consideration of the Gauss-Codazzi equations [53]. It is then shown that the addition of the Mann-Marolf counterterm to the Einstein-Hilbert and Gibbons-Hawking action gives a well-defined action principle for asymptotically flat spacetimes.

2.2.1 The action of general relativity

A variational approach to general relativity was proposed independently by Einstein and Hilbert. Their action is unique given the requirements that it contains no higher than second derivatives of the metric[3] (or a cosmological constant) and is a scalar under Lorentz transformations. The Einstein-Hilbert action is given by

$$S_{EH} = -\frac{1}{16\pi G}\int_{\mathcal{M}} \sqrt{-g}Rd^dx \tag{2.22}$$

where $R = g^{\mu\nu}R_{\mu\nu}$ is the Ricci scalar, $g^{\mu\nu}$ the inverse metric, $R_{\mu\nu}$ the Ricci tensor and $g = detg_{\mu\nu}$, and where $\mathcal{M}$ is the manifold of interest.

In general relativity, when we are interested in the behaviour of some non-compact manifold, we can manage the resulting infinities by performing calculations on a finite subspace $\mathcal{M}_r \subset \mathcal{M}$ by introducing a regulating boundary $\partial\mathcal{M}_r$ to cut-off the spacetime at finite "radius" and then remove the cut-off by taking the limit $\partial\mathcal{M}_r \to \infty$, such that $\mathcal{M}_r$ converges to $\mathcal{M}$. Throughout this thesis we will be primarily interested in non-compact manifolds and will have the above procedure in mind when we perform computations. There are two essential properties that a well-defined variational principle must possess [30]:

1. Requiring the action to be stationary, when considering all variations which preserve the boundary conditions, should result in precisely the classical equations of motion. In particular, any resulting boundary terms must vanish for any allowed variation.

2. The action is finite on-shell, i.e. when the classical equations of motion are satisfied.

In general when one is interested in some non-gravitational field theory one considers an action constructed from first derivatives in the fields. In this case a boundary condition which fixes the fields on the boundary will give a well-defined variational principle. When we consider gravity we will similarly impose the condition that the metric is fixed on the boundary. If we now consider variations of the Einstein-Hilbert action, we find that extremising the action does indeed yield Einstein's field equations on the bulk spacetime $\mathcal{M}$. However, since the Einstein-Hibert action is constructed from second derivatives in the metric a non-zero boundary term which

[3]First derivatives of the metric can always be set to zero locally and so any non-trivial scalar must involve at least second derivatives of the metric.

depends not only on variations of the metric but also on variations of derivatives of the metric results. Since we have not fixed derivatives of the metric on the boundary[4] the variational principle is not well-defined. Let us now demonstrate the above result. It is convenient to vary the action with respect to the inverse metric $g^{\mu\nu}$ where variations of the metric and its inverse are related by $\delta g_{\mu\nu} = -g_{\mu\lambda}g_{\nu\rho}\delta g^{\lambda\rho}$. We find

$$\delta S_{EH} = -\frac{1}{16\pi G}\left[\int_{\mathcal{M}} d^d x\sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}R g_{\mu\nu}\right)\delta g^{\mu\nu} + \int_{\mathcal{M}} d^d x\sqrt{-g}\nabla^\sigma v_\sigma\right] \quad (2.23)$$

with $v_\sigma = \nabla^\rho(\delta g_{\sigma\rho}) - g^{\kappa\lambda}\nabla_\sigma(\delta g_{\kappa\lambda})$. Notice that the second term is a total derivative so, using Stokes' theorem [53] can be written as a boundary term

$$-\frac{1}{16\pi G}\int_{\partial\mathcal{M}} d^{d-1}x\sqrt{-h}v_\sigma n^\sigma, \quad (2.24)$$

where h is the determinant of the induced metric on the boundary $h_{\mu\nu}$ given by the pullback of $g_{\mu\nu}$ to $\partial\mathcal{M}$ and n^μ is the unit vector normal to the boundary. This boundary term clearly does not vanish when only the metric is fixed on the boundary. Gibbons and Hawking [29] found that, if a suitable boundary term was added to the Einstein-Hilbert action, the second derivative terms could be removed so that the resulting boundary term depends only on variations of the metric and not its derivatives. The required boundary term is

$$S_{GH} = -\frac{1}{8\pi G}\int_{\partial\mathcal{M}} \sqrt{-h}K d^{d-1}x \quad (2.25)$$

where $K = h^{\mu\nu}K_{\mu\nu}$ is the trace of the extrinsic curvature of the boundary defined by the covariant derivative of the unit normal vector n^μ of the boundary

$$K_{\mu\nu} = h^\sigma{}_\mu \nabla_\sigma n_\nu. \quad (2.26)$$

We now show that addition of the Gibbons-Hawking term does indeed give a well-defined variational principle for compact manifolds, resulting in variations of the action that depend only on the metric on the boundary. In particular, we show that

$$\delta S_{EH+GH} = -\frac{1}{16\pi G}\int_{\partial\mathcal{M}} d^{d-1}x\sqrt{-h}\pi^{\mu\nu}\delta h_{\mu\nu}, \quad (2.27)$$

[4]Fixing first derivatives of the metric as well as the metric itself would overly restrict the space of solutions.

where $\pi^{\mu\nu} = K^{\mu\nu} - Kh^{\mu\nu}$. To do this it is convenient to make use of the Hamiltonian formulation of gravity in the ADM formalism [53] in which a global time function t is used to foliate the spacetime into spacelike hypersurfaces of constant t. In our case we are interested in the boundary at large r so we proceed analogously and introduce a family of timelike hypersurfaces $\partial\mathcal{M}$ of constant radius r with normal vector n^μ pointing in the direction of increasing r and satisfying $n_\mu n^\mu = 1$. This radial analogue of Hamiltonian formulation was used in the context of the AdS/CFT correspondence in [54, 55]; we will follow their approach.

The induced metric on the hypersurface $\partial\mathcal{M}$ is given by the pullback of the bulk metric. It can be written in terms of $g_{\mu\nu}$ and the normal vector to the surface

$$h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu. \tag{2.28}$$

We raise and lower indices using the bulk metric and its inverse and define a radial vector field r^μ by $r^\mu \partial_\mu r = 1$, a lapse function

$$N = r^\mu n_\mu, \tag{2.29}$$

and a shift vector

$$N^\mu = r^\mu - N n^\mu, \tag{2.30}$$

which are simply decompositions of r^μ into its normal and tangential components with respect to the hypersurface. The metric can now be decomposed in terms of the lapse function, the shift vector and the induced metric

$$ds^2 = (N^2 + N_\mu N^\mu)dr^2 + 2N_\mu dx^\mu dr + h_{\mu\nu}dx^\mu dx^\nu. \tag{2.31}$$

The extrinsic curvature can also be written as

$$K_{\mu\nu} = \frac{1}{2}\pounds_n h_{\mu\nu}, \tag{2.32}$$

where $\pounds_n$ is the Lie derivative along n^μ. In this form we see that the extrinsic curvature encodes the radial evolution of the induced metric and describes the geometry of the hypersurface $\partial\mathcal{M}$ relative to the bulk $\mathcal{M}$. The bulk curvature pulled back to $\partial\mathcal{M}$ can be related to the intrinsic and extrinsic curvature on the boundary hypersurface $\partial\mathcal{M}$ by Gauss's equation

$$h^{\mu'}{}_\mu h^{\nu'}{}_\nu h^{\rho'}{}_\rho h^{\sigma'}{}_\sigma R_{\mu'\nu'\rho'\sigma'} = \mathcal{R}_{\mu\nu\rho\sigma} + K_{\mu\sigma}K_{\nu\rho} - K_{\mu\rho}K_{\nu\sigma}, \tag{2.33}$$

and Codazzi's equation

$$h^{\rho}{}_{\nu} n^{\sigma} R_{\rho\sigma} = D_{\mu} K^{\mu}{}_{\nu} - D_{\nu} K, \tag{2.34}$$

where D_μ is the covariant derivative on $\partial\mathcal{M}$ compatible with $h_{\mu\nu}$ and $\mathcal{R}_{\mu\nu\rho\sigma}$ is the Riemann tensor of the boundary metric $h_{\mu\nu}$ on $\partial\mathcal{M}$. We will also need contractions of Gauss's equation

$$K^2 - K_{\mu\nu} K^{\mu\nu} = \mathcal{R} + 2 G_{\mu\nu} n^{\mu} n^{\nu}, \tag{2.35}$$

$$\pounds_n K_{\mu\nu} + K K_{\mu\nu} - 2 K_{\mu}{}^{\sigma} K_{\sigma\nu} = \mathcal{R}_{\mu\nu} - h_{\mu}{}^{\sigma} h_{\nu}{}^{\rho} R_{\sigma\rho}, \tag{2.36}$$

where $G_{\mu\nu}$ is the bulk Einstein tensor. Note that these equations are purely geometrical as we have not yet imposed the equations of motion.

Substituting the definition of the Ricci tensor $R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu}$ where the Riemann tensor is given by $R^{\rho}{}_{\mu\sigma\nu} n^{\mu} = [\nabla_{\sigma}, \nabla_{\nu}]\, n^{\rho}$ into equation (2.35) allows us to write the bulk Ricci scalar as

$$R = \mathcal{R} + K^2 - K_{\mu\nu} K^{\mu\nu} - 2\nabla_{\mu}(n^{\mu}\nabla_{\nu} n^{\nu}) + 2\nabla_{\nu}(n^{\mu}\nabla_{\mu} n^{\nu}). \tag{2.37}$$

We now substitute (2.37) into the Einstein-Hilbert and Gibbons-Hawking action. Having applied Stokes' theorem to the divergence terms in (2.37) we find the resulting boundary terms are precisely cancelled by the Gibbons-Hawking term and we are left with

$$S_{EH+GH} = -\frac{1}{16\pi G} \int_{\mathcal{M}} d^d x \sqrt{-g} \left(\mathcal{R} + K^2 - K_{\mu\nu} K^{\mu\nu} \right). \tag{2.38}$$

By writing the extrinsic curvature in terms of the lapse and shift functions

$$K_{\mu\nu} = \frac{1}{2N} \left(\partial_r h_{\mu\nu} - 2 D_{(\mu} N_{\nu)} \right) \tag{2.39}$$

we see that the action depends only on $(h_{\mu\nu}, \partial_r h_{\mu\nu}, N, N^{\mu})$. We can now find the conjugate momenta densities to these fields given by

$$\pi^{\mu\nu} \equiv \frac{\delta\mathcal{L}}{\delta(\partial_r h_{\mu\nu})}, \quad \pi^{\mu}_{N} \equiv \frac{\delta\mathcal{L}}{\delta(\partial_r N_{\mu})}, \quad \pi_N \equiv \frac{\delta\mathcal{L}}{\delta(\partial_r N)}, \tag{2.40}$$

where $\mathcal{L}$ is the Lagrangian density. The Lagrangian density does not contain any radial derivatives of the lapse or shift functions so their conjugate momenta vanish identically. This tells us that the lapse and shift functions are not dynamical variables and can be fixed by a choice of gauge. A convenient choice are Gaussian normal coordinates for which $N = 1, N^{\mu} = 0$ and

$$ds^2 = dr^2 + h_{ij}dx^i dx^j, \tag{2.41}$$

$$n^{\mu} = \delta^{\mu}_{r}, \quad K_{ij} = \frac{1}{2}h'_{ij}. \tag{2.42}$$

The momentum conjugate to the boundary metric is readily shown to be

$$\pi^{ij} = K^{ij} - Kh^{ij}. \tag{2.43}$$

Let us now consider arbitrary variations of the Einstein-Hilbert action about a solution of the classical equations of motion. We have reduced the Einstein-Hilbert action to a function of the metric and its radial derivative. It is convenient to write the Einstein-Hilbert action in terms of a Lagrangian

$$S_{EH+GH} = -\frac{1}{16\pi G}\int_{\mathcal{M}} d^d x L(h_{ij}, \partial_r h_{ij}), \tag{2.44}$$

where the Lagrangian is related to the Lagrangian density by $L = \sqrt{-h}\mathcal{L}$. We now find the variation of the action with respect to the metric and its radial derivative

$$\delta S_{EH+GH} = -\frac{1}{16\pi G}\int_{\mathcal{M}} d^d x \left(\frac{\partial L}{\partial h_{ij}}\delta h_{ij} + \frac{\partial L}{\partial(\partial_r h_{ij})}\delta(\partial_r h_{ij})\right). \tag{2.45}$$

Integrating the second term by parts with respect to r yields

$$\begin{aligned}\delta S_{EH+GH} \quad = \quad & -\frac{1}{16\pi G}\int_{\mathcal{M}} d^d x \left[\frac{\partial L}{\partial h_{ij}} - \partial_r\left(\frac{\partial L}{\partial(\partial_r h_{ij})}\right)\right]\delta h_{ij} \\ & -\frac{1}{16\pi G}\int_{\partial\mathcal{M}} d^{d-1}x \frac{\partial L}{\partial(\partial_r h_{ij})}\delta h_{ij}.\end{aligned} \tag{2.46}$$

The integrand of the first term is simply the Euler-Lagrange equation, which vanishes on-shell, whilst the integrand of the second term is just $\sqrt{-h}\pi^{ij}$ so we have shown

$$\delta S_{EH+GH} = -\frac{1}{16\pi G}\int_{\partial\mathcal{M}} d^{d-1}x\sqrt{-h}\pi^{ij}\delta h_{ij}. \tag{2.47}$$

Thus the Einstein-Hilbert action, supplemented with the Gibbons-Hawking term gives a well-defined Dirichlet problem where only the metric and not its normal derivatives need to be fixed on the boundary. For compact spacetimes the action principle is indeed well-defined; however, as we will see, the action is not stationary for asymptotically flat space. Let us consider the definition of asymptotic flatness given in [30] where the defining metric in d dimensions is taken to admit a radial foliation of the form

$$ds^2 = \left(1 + \mathcal{O}(r^{3-d)})\right) dr^2 + r^2 \left(h^0_{ij} + \mathcal{O}(r^{3-d})\right) d\eta^i d\eta^j + r\mathcal{O}(r^{3-d}) dr d\eta^j \qquad (2.48)$$

where h^0_{ij} and η^i are the metric and coordinates on the unit $(d-2,1)$ hyperboloid $\mathcal{H}^{d-1}$, and the notation $\mathcal{O}(r^{3-d})$ means that any perturbations to the flat space metric must fall off at least as fast as r^{3-d}. We are primarily interested in the large volume divergences associated with taking the boundary at constant r to infinity, with η fixed; however, we also need to cut off the spacetime $\mathcal{M}$ in time. We consider the set-up where $\mathcal{M}$ is the region between two Cauchy hypersurfaces related by an asymptotic translation. In this case the volume of $\partial\mathcal{M}$ scales like r^{d-2} [30].

Let us now calculate the variation of the action for the above class of spacetimes. To first order in the perturbations to flat space we find the extrinsic curvature is given by

$$K_{ij} = rh^0_{ij} + \mathcal{O}(r^{4-d}). \qquad (2.49)$$

Hence, we have $\pi^{ij} \sim \mathcal{O}(r^{-3}) + \mathcal{O}(r^{-d})$ and, using $\delta h_{ij} \sim \mathcal{O}(r^{5-d})$ we find

$$\delta S_{EH+GH} \sim \mathcal{O}(r^0). \qquad (2.50)$$

Thus the action is not stationary, its variation generically approaches a non-zero constant as the boundary is taken to spacelike infinity, $r \to \infty$ with η fixed. Note that the action itself is also not finite as the boundary is taken to infinity. We are considering Ricci flat spacetimes so the Einstein-Hilbert term vanishes and we need only consider the Gibbons-Hawking term. Since $K \sim \mathcal{O}(r^{-1})$ to leading order we have

$$S_{GH} = -\frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^{d-1}x\sqrt{-h}K \sim \mathcal{O}(r^{d-3}), \qquad (2.51)$$

which diverges for $d \geq 4$. Hence we require some modification of the action so that it is finite on solutions and vanishes under all variations preserving the boundary conditions, as the regulating boundary is taken to infinity.

2.2.2 Counterterms

The background subtraction approach [29] attempts to make the action finite with the addition of a new term to the action

$$S_{Ref} = \frac{1}{8\pi G} \int_{\partial\mathcal{M}} d^{d-1}x\sqrt{-h}K_{Ref}, \tag{2.52}$$

where K_{Ref} is the extrinsic curvature of the boundary $\partial\mathcal{M}$ when it is isometrically embedded into some reference spacetime. For asymptotically flat gravity the appropriate reference background is Minkowski space; however, as discussed in the introduction, for spacetimes with other asymptotics there is in general some ambiguity with regard to choice of reference background. Furthermore, in higher dimensions the required embeddings will not always exist and so a well-defined variational principle cannot be constructed.

Boundary counterterms intrinsic to the spacetime of interest initially arose from a study of gauge/gravity duality [32, 33]. The method, known as holographic renormalization involves the addition of terms defined locally on the boundary, chosen to make the action finite. In [33] the authors construct counterterms that are functionals of the intrinsic geometry of the boundary of asymptotically anti de-Sitter space (AAdS) and are able to reproduce the conserved quantities for various AAdS spacetimes. In [34] it was shown that a counterterm proportional to $\sqrt{\mathcal{R}}$ yields a finite action for Schwarzschild $d = 4$ spacetimes and this result was extended to arbitrary dimensions in [35]. These counterterms, though successful in rendering the action finite and reproducing conserved quantities calculated by reference subtraction, are somewhat case specific.

2.2.3 The Mann-Marolf counterterm

Recently, a new covariant counterterm was proposed by Mann and Marolf [30] for asymptotically flat spacetimes with dimension $d \geq 4$. The authors show that when the Einstein-Hilbert and Gibbons-Hawking terms are supplemented with the Mann-Marolf counterterm, the resulting action is both finite on-shell and stationary under all variations preserving asymptotic flatness. We now consider the Mann-Marolf counterterm and its origins in detail, as we will use this counterterm to construct an action principle for asymptotically plane wave spacetimes in chapter 3. The Mann-Marolf counterterm is given by

$$S_{MM} = \frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^{d-1}x\sqrt{-h}\hat{K}, \tag{2.53}$$

where $\hat{K} = h^{\alpha\beta}\hat{K}_{\alpha\beta}$ is defined implicitly by solving

$$\mathcal{R}_{\alpha\beta} = \hat{K}_{\alpha\beta}\hat{K} - h^{\gamma\delta}\hat{K}_{\alpha\gamma}\hat{K}_{\delta\beta}, \tag{2.54}$$

where $\mathcal{R}_{\alpha\beta}$ is the Ricci tensor of the metric $h_{\alpha\beta}$ induced on the boundary $\partial\mathcal{M}$. Note that $\hat{K}$ is a locally determined function of the boundary metric $h_{\alpha\beta}$ and is defined for any asymptotically flat spacetime as required for a well-defined variational principle. Let us consider Gauss's equation (2.33) written in terms of boundary coordinates α, β for some reference background $(\mathcal{M}^{Ref}, g^{Ref})$

$$R^{Ref}_{\alpha\beta\gamma\delta} = \mathcal{R}^{Ref}_{\alpha\beta\gamma\delta} + K^{Ref}_{\alpha\delta}K^{Ref}_{\beta\gamma} - K^{Ref}_{\alpha\gamma}K^{Ref}_{\beta\delta.}, \tag{2.55}$$

where the bulk Riemann tensor $R^{Ref}_{\alpha\beta\gamma\delta}$ has been pulled back to $\partial\mathcal{M}$. That is to say, if we had embedded our boundary spacetime $(\partial\mathcal{M}, h)$ into some reference background $(\mathcal{M}^{Ref}, g^{Ref})$ then the above equation would hold and we could solve it to find a counterterm K_{Ref}. For asymptotically flat spacetime the natural choice of reference background is Minkowski space. For any spacetime that is asymptotically Ricci flat we will have $R^{Ref}_{\alpha\beta} = 0$, hence the trace of (2.55) reduces to (2.54). We discussed in section 1.2 that an embedding into a reference spacetime is not always possible; however, when it is, the extrinsic curvature of the embedding will be given by (2.54). Even when such an embedding is not possible we may still use (2.54) to define a counterterm. Indeed, we can see it has just the properties we require; firstly, it does not contain normal derivatives of the metric since it is related only to the intrinsic curvature of the boundary so it won't upset the cancellation between the Einstein-Hilbert and Gibbons-Hawking terms. Secondly, it agrees with K to leading order (with differences being sourced by the bulk Riemann tensor) and so will cancel the divergences coming from K. We will now show that the addition of this new counterterm does indeed give a well-defined action principle for asymptotically flat spacetimes as defined by (2.48). This will serve as useful background for the similar but more complicated analysis for asymptotically plane wave spacetimes presented in chapter 3. We closely follow the approach of [40]. We first consider the finiteness of the action before looking at variations of the action.

Finiteness of the action

The action for asymptotically flat spacetimes in the case of vacuum gravity is given by just the boundary terms

$$S = -\frac{1}{8\pi G}\int_{\partial\mathcal{M}} d^{d-1}x\sqrt{-h}(K-\hat{K}). \tag{2.56}$$

Let us first consider the value of this action for the zeroth order case; that is, for Minkowski space

$$ds^2 = dr^2 + r^2 h^0_{ij} d\eta^i d\eta^j. \tag{2.57}$$

The extrinsic curvature is $K_{ij} = rh^0_{ij}$, so $K = \frac{d-1}{r}$. The Ricci tensor on the boundary is $\mathcal{R}_{ij} = (d-2)h^0_{ij}$. Substituting this into (2.54) we find $\hat{K}_{ij} = rh^0_{ij}$, so $\hat{K} = \frac{d-1}{r}$. Thus the on-shell action for Minkowski space is zero. Let us now consider the action for asymptotically flat spacetimes as defined by (2.48). We can write the linear order contribution to the action as

$$K^{(1)} - \hat{K}^{(1)} = K^{(1)}_{ij} h^{(0)ij} - \hat{K}^{(1)}_{ij} h^{(0)ij}. \tag{2.58}$$

Since $\sqrt{-h} \sim \mathcal{O}(r^{d-2})$, we need $K^{(1)} - \hat{K}^{(1)} \sim \mathcal{O}(r^{2-d})$ to have a finite action. It is easy to show $K^{(1)}_{ij} \sim \mathcal{O}(r^{4-d})$, so $K^{(1)} \sim \mathcal{O}(r^{2-d})$. To evaluate $\hat{K}^{(1)}_{ij}$, we can linearise (2.54) to give

$$\mathcal{R}^{(1)}_{ij} = \hat{K}^{(1)}_{mn} L^{(0)mn}_{ij} + \left(\hat{K}^{(0)}_{ij}\hat{K}^{(0)}_{mn} - \hat{K}^{(0)}_{im}\hat{K}^{(0)}_{jn}\right) h^{(1)mn}, \tag{2.59}$$

where[5]

$$L^{(0)mn}_{ij} = h^{mn}\hat{K}_{ij} + \frac{1}{2}\left(\delta^m_i\delta^n_j\hat{K} + \delta^n_j\delta^m_i\hat{K}\right) - \frac{1}{2}\left(\delta^m_i\hat{K}^n_j + \delta^m_j\hat{K}^n_i + \delta^n_i\hat{K}^m_j + \delta^n_j\hat{K}^m_i\right). \tag{2.60}$$

[5] Note that we define $L^{(0)\gamma\delta}_{\alpha\beta}$ so that it is symmetric in both pairs of indices, so this is slightly different from the corresponding expression in [40].

We can now invert this to give us an expression for $\hat{K}^{(1)}_{ij}$,

$$h^{(0)ij}\hat{K}^{(1)}_{ij} = M^{(0)ij}\mathcal{R}^{(1)}_{ij} - M^{(0)ij}(\hat{K}^{(0)}_{ij}\hat{K}^{(0)}_{mn} - \hat{K}^{(0)}_{im}\hat{K}^{(0)}_{jn})h^{(1)mn}. \tag{2.61}$$

where $M^{ij} = h^{mn}\left(L^{-1}\right)^{ij}_{mn}$. The operator $L^{(0)mn}_{ij}$ is generically invertible and of the same order as $K^{(0)}$, that is $\mathcal{O}(r^{-1})$. Therefore we have $\left(L^{-1}\right)^{ij}_{mn} \sim \mathcal{O}(r)$ and $M^{ij} \sim \mathcal{O}(r^{-1})$. For the second term in (2.61) we have $\hat{K}^{(0)}_{mn} \sim \mathcal{O}(r)$, and $h^{(1)mn} \sim \mathcal{O}(r^{1-d})$, so this term is $\mathcal{O}(r^{2-d})$. To evaluate the first term in (2.61), we express $\mathcal{R}^{(1)}_{ij}$ by [53]

$$\mathcal{R}^{(1)}_{ij} = -\frac{1}{2}h^{(0)mn}D^{(0)}_iD^{(0)}_jh^{(1)}_{mn} - \frac{1}{2}h^{(0)mn}D^{(0)}_mD^{(0)}_nh^{(1)}_{ij} + h^{(0)mn}D^{(0)}_mD^{(0)}_{(i}h^{(1)}_{j)n}, \tag{2.62}$$

where D_i is the covariant derivative compatible with h_{ij}. Using this expression we can see that $\mathcal{R}^{(1)}_{ij} \sim \mathcal{O}(r^{3-d})$, so this term also makes a finite contribution. Hence we find that the on-shell action is finite for asymptotically flat spacetimes.

Variations of the action

We would also like to see that the action is stationary under arbitrary variations of h_{ij} about a solution of the equations of motion. We have

$$\delta S_{EH+GH} = -\frac{1}{16\pi G}\int d^{d-1}x\sqrt{-h}\pi^{ij}\delta h_{ij}, \tag{2.63}$$

and since $\sqrt{-h} \sim \mathcal{O}(r^{d-2})$, $\pi^{(0)ij} \sim \mathcal{O}(r^{-3})$ and $\delta h_{ij} \sim \mathcal{O}(r^{5-d})$, this gives a non-vanishing r^0 term. We need this term to be cancelled with a corresponding term coming from δS_{MM}. Since

$$\delta\left(\sqrt{-h}\hat{K}\right) = -\frac{1}{2}\hat{K}h^{ij}\delta h_{ij} + \hat{K}_{ij}\delta h^{ij} + h^{ij}\delta\hat{K}_{ij}, \tag{2.64}$$

we can write the variation of the Mann-Marolf term as

$$\delta S_{MM} = \frac{1}{8\pi G}\int d^{d-1}x\sqrt{-h}\left(\frac{1}{2}\hat{\pi}^{ij}\delta h_{ij} + \frac{1}{2}\hat{K}^{ij}\delta h_{ij} + h^{ij}\delta\hat{K}_{ij}\right), \tag{2.65}$$

where $\hat{\pi}^{ij} = \hat{K}^{ij} - h^{ij}\hat{K}$. To zeroth order, $\hat{\pi}^{(0)ij} = \pi^{(0)ij}$, so the first term in (2.65) cancels the non-zero contribution from (2.63). However, the second term in (2.65) also has a non-zero leading order part, so this must be cancelled by a contribution from the final term. We find,

$$h^{ij}\delta\hat{K}_{ij} = M^{(0)ij}\delta\mathcal{R}^{(0)}_{ij} - M^{(0)ij}\left(\hat{K}^{(0)}_{ij}\hat{K}^{(0)}_{mn} - \hat{K}^{(0)}_{im}\hat{K}^{(0)}_{jn}\right)\delta h^{mn}. \tag{2.66}$$

Let us consider the first term above. We have $\delta\mathcal{R}^{(0)}_{ij} \sim \mathcal{O}(r^{3-d})$ which involves covariant derivatives with respect to the unit metric on $\mathcal{H}^{d-1}$, h^0_{ij}. Since the only η^i dependence in the terms multiplying $\delta\mathcal{R}^{(0)}_{ij}$ is through the covariantly constant metric h^0_{ij}, this term is a total derivative. Higher-order contributions from this term will not be total derivatives, but they are suppressed by further powers of r so their contribution to the action vanishes in the large r limit. Finally, we evaluate the last term in (2.66). By explicit computation we find $M^{(0)ij} = \frac{1}{2(d-2)r} h^0_{ij}$ and, using $\hat{K}^{(0)}_{ij} = r h^0_{ij}$, we see that

$$h^{ij}\delta\hat{K}_{ij} \to -M^{(0)ij}\left(\hat{K}^{(0)}_{ij}\hat{K}^{(0)}_{mn} - \hat{K}^{(0)}_{im}\hat{K}^{(0)}_{jn}\right)\delta h^{mn} = -\frac{1}{2}r(h^o)^{ij}\delta h_{ij} = -\frac{1}{2}\hat{K}^{(0)ij}\delta h_{ij}. \tag{2.67}$$

This will indeed cancel with the leading order part of the second term in (2.65) leaving no finite contributions to the variation of the action in the large r limit. So the addition of the Mann-Marolf term to the Einstein-Hilbert and Gibbons-Hawking action gives a well-defined variational principle for asymptotically flat spacetimes.

2.3 Black holes in higher dimensions

In this section, we consider the general description of neutral, vacuum black holes in higher dimensions as background to the work in chapter 4 where we attempt the construction of black holes in plane wave spacetimes. We first set out the new effective theory of [51,52] in which a black hole is described by a *black* brane curved into a submani*fold* of a background spacetime - a blackfold. We then look at the construction of an approximate solution for an asymptotically flat, neutral, thin rotating black ring in some detail as a particular realisation of this method. We encounter a conjecture which states that satisfying the blackfold equations (2.87) guarantees the existence of a regular horizon [51] and we consider an example in support of it [56]. We see, however, in chapter 4 that a counter-example exists.

An important new feature of higher dimensional black holes is the existence of event horizons with two length scales of very different magnitude $r_0 \ll R$, where r_0 and R are the length scales associated with the mass of the black hole and its angular momentum respectively. In four spacetime dimensions, the angular momentum is bounded $J \leq GM^2$, so the Kerr black hole is always approximately round with $r_0 \sim GM$. However, the competition between the gravitational and centrifugal terms changes with dimensionality. The gravitational potential falls off as r^{3-D}, whereas the centrifugal barrier in a particular direction only depends on the rotation in the plane, so will fall off as r^{-2} in each plane of rotation. Thus, for $D \geq 5$, regimes exist where the length scales $r_0 \sim (GM)^{1/(D-3)}$ and $R \sim J/M$ can be widely separated. Indeed, five dimensional black rings are known to exist in ultra-spinning regimes where the ring's radius R is much larger than its thickness r_0 [44]. Similarly, Myers-Perry black holes in $D \geq 6$ have ultra-spinning regimes in which the horizon flattens and approaches a thin black brane with thickness r_0 and large radius R in the plane of rotation [43]. This suggests organising black holes in a hierarchy of scales.

1. $R \lesssim r_0$ - black holes behave qualitatively similarly to the Kerr black hole in four dimensions.

2. $R \approx r_0$ - threshold of emergence of new phenomena.

3. $R \gg r_0$ - regime of new dynamics, very different to four dimensions. Separation of scales suggests approximation with a long wavelength effective theory.

The first and second regimes are described by the full Einstein equations and, in general, finding solutions will be challenging as known solution generating techniques do not extend to higher dimensions. The third regime, thanks to the existence of a small parameter r_0/R, is well described by approximate analytical methods. Fortunately, there is much of interest in this regime with new dynamics not seen in four dimensions appearing.

Let us first set out some of the notation we use in this section. Following [52], we introduce

$$n = D - p - 3, \tag{2.68}$$

for a blackfold with p spatial dimensions embedded in D-dimensional spacetime. We denote spacetime coordinates by X^μ, with indices $\mu, \nu ... = (0, ..., D-1)$, and spacetime metric, connection and covariant derivative by $g_{\mu\nu}, \Gamma^\rho_{\mu\nu}$, and ∇_μ, respectively. Worldvolume coordinates are denoted by σ^a, with indices $a, b ... = (0, ..., p)$, and the worldvolume metric connection and covariant derivative are given by γ_{ab}, $\bar{\Gamma}^a_{bc}$, and D_a. Indices μ, ν are raised and lowered with $g_{\mu\nu}$ and its inverse and indices a, b are raised and lowered with γ_{ab} and its inverse.

2.3.1 Blackfolds

We are interested in studying the large distance dynamics of higher dimensional black hole horizons with some effective theory. To construct such a theory the gravitational degrees of freedom are split into near horizon and far region components

$$g_{\mu\nu} = \{g^{(near)}_{\mu\nu}, g^{(far)}_{\mu\nu}\}, \tag{2.69}$$

and the Einstein-Hilbert action is approximated by

$$S_{EH} \approx -\frac{1}{16\pi G}\int d^d x\sqrt{-g^{(far)}}R^{(far)} - S_{eff}[g^{(far)}_{\mu\nu}, \phi], \tag{2.70}$$

where $S_{eff}[g^{(far)}_{\mu\nu}, \phi]$ is an effective action resulting from integrating out the short distance degrees of freedom. What is meant by this is that Einstein's equations are solved for $r \ll R$ and the effects of this solution at large distances $r \gg r_0$ are encoded in an effective action. The coupling of these short wavelength degrees of freedom to the long wavelength components occurs via some effective fields ϕ, which we now identify.

We are guided by the result that known black holes approach flat black branes in the limit $r_0/R \to 0$. Therefore we take the effective theory to describe the dynamics of black $p-$branes, with geometry in $D = 3 + p + n$ spacetime dimensions given by

$$ds^2_{p-brane} = -\left(1 - \frac{r_0^n}{r^n}\right)dt^2 + \sum_{i=1}^{p}(dz^i)^2 + \frac{dr^2}{1 - \frac{r_0^n}{r^n}} + r^2 d\Omega^2_{n+1}. \tag{2.71}$$

We obtain a more general form of the metric by boosting the worldvolume coordinates $\sigma^a = (t, z^i)$. If the velocity field is given by u^a, with $u^a u^b \eta_{ab} = -1$, then

$$ds^2_{p-brane} = \left(\eta_{ab} + \frac{r_0^n}{r^n}u_a u_b\right)d\sigma^a d\sigma^b + \frac{dr^2}{1 - \frac{r_0^n}{r^n}} + r^2 d\Omega^2_{n+1}. \tag{2.72}$$

The parameters describing this black $p-$brane are the p independent components of the velocity u, the horizon thickness r_0 and the $D - p - 1$ coordinates parametrizing the position of the brane in directions transverse to the worldvolume, $X^{\perp}$. Thus the effective fields of (2.70) can be written

$$\phi(\sigma^a) = \{X^{\perp}(\sigma^a), r_0(\sigma^a), u^i(\sigma^a)\}. \tag{2.73}$$

Let us now consider embedding the black brane worldvolume $\mathcal{W}_{p+1}$ into the spacetime. It is useful to enlarge the set of embedding coordinates to include all the spacetime coordinates $X^\mu(\sigma^a)$ in order to preserve manifest diffeomorphism invariance. The pullback of the bulk metric results in an induced metric on the brane worldvolume

$$\gamma_{ab} = g_{\mu\nu}\partial_a X^\mu \partial_b X^\nu. \tag{2.74}$$

Given the induced metric γ_{ab} on the worldvolume $\mathcal{W}_{p+1}$, we define the first fundamental form of the submanifold

$$h^{\mu\nu} = \gamma^{ab}\partial_a X^\mu \partial_b X^\nu. \tag{2.75}$$

The metric can be decomposed into a projector onto $\mathcal{W}_{p+1}$ and a projector onto directions perpendicular to $\mathcal{W}_{p+1}$ respectively

$$g_{\mu\nu} = h_{\mu\nu} + \perp_{\mu\nu}. \tag{2.76}$$

The tangential projection tensor $h^\mu{}_\nu$ satisfies the relations

$$h^\mu{}_\nu \partial_a X^\nu = \partial_a X^\mu, \tag{2.77}$$

and

$$h^\mu{}_\nu h^\nu{}_\rho = h^\mu{}_\rho, \tag{2.78}$$

whilst the orthogonal projection tensor $\perp_{\mu\nu}$ satisfies

$$\perp_{\mu\nu} \partial_a X^\mu = 0, \tag{2.79}$$

and

$$\perp_\mu{}^\nu \perp_\nu{}^\rho = \perp_\mu{}^\rho. \tag{2.80}$$

The extrinsic curvature tensor can be defined as [6]

$$K_{\mu\nu}{}^\rho = h^\lambda{}_\mu h^\sigma{}_\nu \nabla_\lambda h^\rho{}_\sigma, \tag{2.81}$$

[6] This is the generalisation of the extrinsic curvature defined in (2.26) for submanifolds with codimension greater than one.

and is tangent to $\mathcal{W}_{p+1}$ along its lower indices and orthogonal to $\mathcal{W}_{p+1}$ along its upper index. It is useful to introduce the tangential covariant derivative

$$\bar{\nabla}_\mu = h_\mu{}^\nu \nabla_\nu, \tag{2.82}$$

which we can use to rewrite the extrinsic curvature

$$K_{\mu\nu}{}^\rho = h_\nu{}^\sigma \bar{\nabla}_\mu h_\sigma{}^\rho \tag{2.83}$$

We are now in a position to write down the dynamical equations for a brane embedded in a background spacetime. These equations were first derived by Carter in [57]; however, when applied to black branes we refer to them as the blackfold equations. The equations are formulated in terms of an effective stress tensor on $\mathcal{W}_{p+1}$ satisfying the tangentiality condition

$$\perp^\rho{}_\mu T^{\mu\nu} = 0. \tag{2.84}$$

The effective stress tensor is derived from solving Einstein's equations in the near region. Since general relativity is a conservative and diffeomorphism invariant theory, the stress tensor must obey the conservation equations

$$\bar{\nabla}_\mu T^{\mu\rho} = 0. \tag{2.85}$$

These equations can be decomposed along directions parallel and orthogonal to $\mathcal{W}_{p+1}$

$$\begin{aligned} \bar{\nabla}_\mu T^{\mu\rho} &= \bar{\nabla}_\mu (T^{\mu\nu} h_\nu{}^\rho) \\ &= T^{\mu\nu} \bar{\nabla}_\mu h_\nu{}^\rho + h_\nu{}^\rho \bar{\nabla}_\mu T^{\mu\nu} \\ &= T^{\mu\nu} h_\nu{}^\sigma \bar{\nabla}_\mu h_\sigma{}^\rho + h_\nu{}^\rho \bar{\nabla}_\mu T^{\mu\nu} \\ &= T^{\mu\nu} K_{\mu\nu}{}^\rho + \partial_a X^\rho D_b T^{ab} \end{aligned} \tag{2.86}$$

Hence we have separated the D equations into $D-p-1$ equations orthogonal to the brane worldvolume $\mathcal{W}_{p+1}$ and $p+1$ equations parallel to it,

$$T^{\mu\nu} K_{\mu\nu}{}^\rho = 0 \quad (\textit{extrinsic equations}), \tag{2.87}$$

$$D_b T^{ab} = 0 \quad (\textit{intrinsic equations}). \tag{2.88}$$

We can rewrite the extrinsic equations (2.87) in terms of the embedding $X^\mu(\sigma^a)$

$$T^{ab}(D_a\partial_b X^\rho + \Gamma^\rho_{\mu\nu}\partial_a X^\mu \partial_b X^\nu) = 0. \tag{2.89}$$

In this form, we see that the extrinsic equations are generalisations of the geodesic equation for free particles to p-branes. Blackfolds differ from other branes in that they have event horizons and we need to consider regularity on the horizon as the black brane is bent. It has been conjectured in [52] that regularity on the horizon is preserved under such large scale perturbations when the blackfold equations (2.87) are satisfied (we will refer to this as the blackfolds regularity conjecture). There is some evidence in support of this conjecture, for example the analysis of [56], reviewed in the next subsection. In [56], a black string is bent into a circle and it is shown that satisfying the extrinsic equations (2.87) avoids naked singularities on or outside the horizon. However, in chapter 4, when we consider the construction of black holes in plane wave spacetimes, we find that our black hole solution is not regular on the horizon even when the blackfold equations for a 0-brane are (trivially) satisfied. This implies that although satisfying the blackfold equations is a necessary condition for the existence of a regular black hole horizon, it is not always a sufficient condition.

So far we have described an effective theory for the dynamics of black holes for the far region $r \gg r_0$. However, in general, we are also interested in finding a solution for the near horizon region $r \ll R$. We may construct solutions in a perturbative expansion in r_0/R using the method of matched asymptotic expansions [49, 58]. To zeroth order in r_0/R, the near region solution is simply the black p-brane (2.72) and the far region solution is the background metric $g_{\mu\nu}$; this is the test brane approximation. To go to the next order we must take account of the gravitational backreaction of the black brane. We solve the linearised Einstein equations in the far region with an effective stress tensor derived from the near region and with appropriate asymptotics at infinity. This results in a correction to the far region metric of order $(r_0/R)^n$ and provides boundary conditions in the intermediate region $r_0 \ll r \ll R$ for the near region solution. The first order correction to the near region solution can be found by linearly perturbing the near region metric (2.72) and solving with boundary conditions given by the requirements that the near region solution matches the far region solution in the overlap region and that the horizon remains regular. This procedure can be thought of as a dialogue of multipoles [58] and can be iterated order by order, where the solution in one region is used to provide the boundary conditions for the solution in the other region, by matching solutions in the intermediate region where both expansions are valid.

2.3.2 Construction of an approximate black ring solution

We now consider in some detail the construction in [56] of an approximate solution for an asymptotically flat, neutral, thin rotating black ring in any dimension $D \geq 5$. This will be instructive as an example of how the blackfolds procedure works in practice and as background to the construction of black hole solutions in plane wave backgrounds in chapter 4. The appropriate blackfold for this problem is a black 1-brane; that is, a black string with thickness r_0 corresponding to the horizon radius which is bent into a large circle of radius R. The required steps to construct an approximate black ring solution (up to first order) are:

1. We first consider the near horizon region $r \ll R$, to zeroth order in $1/R$, that is, we take a boosted black string of infinite radius $R \to \infty$. Implementing the blackfold equations $T^{\mu\nu} K_{\mu\nu}{}^{\rho} = 0$, fixes the boost parameter α. This can be interpreted as balancing the string tension and centrifugal repulsion such that the string is in mechanical equilibrium.

2. We now solve the linearised Einstein equations to first order in r_0^n, in the far region $r \gg r_0$, for some appropriate source. The source we require is that of the energy-momentum tensor of an infinitely thin rotating ring.

3. Finally, we solve the linearised Einstein equations in the near region for the first order corrections to the black string. We are finding the geometry of a black string that has been bent into a circle with some large radius R. The matching of this solution to the far region solution in the intermediate region $r_0 \ll r \ll R$, together with the requirement of regularity at the horizon, provide the boundary conditions.

2.3.3 Far region solution

In order to find the far region solution of a rotating, neutral, thin black ring we must first determine the energy-momentum tensor of a thin black ring of radius R. Consider the metric of a straight boosted black string

$$\begin{aligned} ds^2 = &-\left(1-\frac{r_0^n}{r^n}\cosh^2\alpha\right)dt^2 - 2\frac{r_0^n}{r^n}\cosh\alpha\sinh\alpha dt dz \\ &+\left(1+\frac{r_0^n}{r^n}\sinh^2\alpha\right)dz^2 + \frac{dr^2}{1-\frac{r_0^n}{r^n}} + r^2 d\Omega_{n+1}^2 \end{aligned} \tag{2.90}$$

with α the boost parameter. We take the z direction to be along an S^1 with circumference $2\pi R$ and introduce an angular coordinate ψ defined by

$$\psi = \frac{z}{R}, \quad 0 \leq \psi < 2\pi. \tag{2.91}$$

We must find a source such that the metric it produces in the far region is the same as that of the full solution. Since the thin black ring locally approaches the solution for a boosted black string, we choose an energy-momentum tensor that reproduces (2.90) in the weak field limit

$$T_{tt} = \frac{r_0^n}{16\pi G}(n\cosh^2\alpha + 1)\delta^{(n+2)}(r), \tag{2.92}$$

$$T_{tz} = \frac{r_0^n}{16\pi G} n\cosh\alpha\sinh\alpha\delta^{(n+2)}(r), \tag{2.93}$$

$$T_{zz} = \frac{r_0^n}{16\pi G}(n\sinh^2\alpha - 1)\delta^{(n+2)}(r), \tag{2.94}$$

where $r = 0$ corresponds to a circle of radius R in $(n+3)$-dimensional Euclidean flat space. The black ring is described by the parameters r_0, R and α or equivalently by its mass, radius and angular momentum, M, R and J. In mechanical equilibrium, however, given particular values of mass and radius, the angular momentum will be fixed so the solution will only depend on two parameters.

Let us now solve the equations of motion for the string probe

$$T^{\mu\nu}K_{\mu\nu}{}^{\rho} = 0. \tag{2.95}$$

The extrinsic curvature of the circle is $1/R$, so our equilibrium condition is

$$T_{\psi\psi} = 0; \tag{2.96}$$

that is, the pressure tangential to the ring must vanish. This equilibrium condition fixes the boost parameter

$$\sinh^2\alpha = \frac{1}{n}. \tag{2.97}$$

We will demonstrate shortly that regularity of the near-horizon region requires this choice of boost parameter. Now that we have the appropriate energy-momentum tensor we can solve the linearised Einstein equations in the far region to get an approximate solution for a thin black ring. In [56], these are solved in the transverse gauge in which the linearised Einstein equations are

$$\Box\bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}, \tag{2.98}$$

with $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}hg_{\mu\nu}$ and $\nabla_\mu\bar{h}^{\mu\nu} = 0$. We write the $(n+3)$-dimensional Euclidean flat space metric in bi-polar coordinates

$$ds^2(\mathbb{E}^{n+3}) = dr_1^2 + r_1^2 d\Omega_n^2 + dr_2^2 + r_2^2 d\psi^2 \tag{2.99}$$

and take the ring source to lie at $r_1 = 0$ and $r_2 = R$. When the equilibrium condition (2.96) is satisfied the general solution is

$$ds^2 = (-1+2\Phi)dt^2 - 2A\,dt\,d\psi + \left(1 + \frac{2}{n+1}\Phi\right) ds^2(\mathbb{E}^{n+3}), \tag{2.100}$$

where away from the source Φ satisfies the Laplace equation and A the Maxwell equation for a gauge potential $Ad\psi$. The solutions are given by

$$\Phi = \frac{4GM}{(n+2)\Omega_{n+2}} \int_0^{2\pi} d\psi \frac{1}{(r_1^2 + (R\cos\psi - r_2)^2 + R^2\sin^2\psi)^{(n+1)/2}}, \tag{2.101}$$

and

$$A = \frac{8GJ}{(n+1)\Omega_{n+2}R} \int_0^{2\pi} d\psi \frac{r_2\cos\psi}{(r_1^2 + (R\cos\psi - r_2)^2 + R^2\sin^2\psi)^{(n+1)/2}}. \tag{2.102}$$

These integrals can be approximated for the far region $r_1, r_2 \gg R$ and the intermediate region $r_1, r_2 - R \ll R$ to give a linearised equilibrium solution, i.e. a solution for arbitrary mass and angular momentum but with zero tension. We would now like to provide an example in which the blackfolds regularity conjecture of [52] is satisfied. In order to do this, we need to solve the linearised Einstein equations for a source with tension $T_{zz} \neq 0$ and show that a regular solution results only if in fact $T_{zz} = 0$. This is a much harder problem but it simplifies considerably if we work in the intermediate region $r_0 \ll r_1, r_2 - R \ll R$, where we are interested in the local effects of bending a black string into a large circle of radius R.

As is often the case, it will simplify matters considerably if we find a coordinate system adapted to the problem. We would like coordinates for flat space such that $r = 0$ is a section of the bent black string of radius R. We require a metric for which

- The Riemann tensor vanishes up to first order in $1/R$.
- The curve $r = 0$ on the plane of the ring $\theta = 0, \pi$ has constant extrinsic curvature.
- Surfaces of constant r are equipotential surfaces of the Laplace equation $\nabla^2 r^{-n} = 0$, for a delta-function source at $r = 0$.

The metric which satisfies these requirements is

$$ds^2 = \left(1 + \frac{2r\cos\theta}{R}\right) dz^2 + \left(1 - \frac{2r\cos\theta}{nR}\right) \left(dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\Omega_n^2\right). \quad (2.103)$$

Our analysis is made clearer if we choose a more general energy-momentum source given by

$$T_{tt} = \frac{n(n+2)}{n+1}\mu \frac{r_0^n}{16\pi G}\delta^{(n+2)}(r), \quad (2.104)$$

$$T_{tz} = np\frac{r_0^n}{16\pi G}\delta^{(n+2)}(r), \quad (2.105)$$

$$T_{zz} = \frac{n(n+2)}{n+1}\tau \frac{r_0^n}{16\pi G}\delta^{(n+2)}(r). \quad (2.106)$$

We recover the source of the boosted black string by the identification

$$\frac{n(n+2)}{n+1}\mu = n\cosh^2\alpha + 1, \quad (2.107)$$

$$p = \cosh\alpha \sinh, \quad (2.108)$$

$$\frac{n(n+2)}{n+1}\tau = n\sinh^2\alpha - 1. \quad (2.109)$$

Using the symmetry of the solution and by choice of gauge, the perturbations can be written in the form

$$h_{tt} = f_1(r,\theta), \tag{2.110}$$

$$h_{tz} = f_2(r,\theta), \tag{2.111}$$

$$h_{zz} = f_3(r,\theta)\gamma_{zz}, \tag{2.112}$$

$$h_{rr} = f_4(r,\theta)\gamma_{rr}, \tag{2.113}$$

$$h_{\theta\theta} = f_5(r,\theta)\gamma_{\theta\theta}, \tag{2.114}$$

$$h_{\Omega\Omega} = f_6(r,\theta)\gamma_{\Omega\Omega}, \tag{2.115}$$

where $\gamma_{\mu\nu}$ is the flat space metric (2.103) and the indices $\Omega\Omega$ are coordinates on S^n. However, not all of these functions f_i are independent and it may be shown that one of the relations that these functions satisfy is given by (see appendix B of [56])

$$f_1 - f_3 - f_4 - f_5 - (n-2)\, f_6 = 0. \tag{2.116}$$

The radial dependence of the functions f_i can be fixed by dimensional analysis and we can write up to first order in $1/R$

$$f_i(r,\theta) = \frac{r_0^n}{r^n}\left(f_i^{(0)} + \frac{r}{R} f_i^{(1)}(\theta)\right). \tag{2.117}$$

The functions $f_i^{(0)}$ must simply be constants, as to zeroth order in $1/R$ we have a straight energy-momentum source so the $SO(n+2)$ symmetry of the S^{n+1} spheres is unbroken. It is easy to show

$$f_1^{(0)} = \mu + \frac{\tau}{n+1}, \tag{2.118}$$

$$f_2^{(0)} = -p, \tag{2.119}$$

$$f_3^{(0)} = \tau + \frac{\mu}{n+1}, \tag{2.120}$$

$$f_4^{(0)} = f_5^{(0)} = f_6^{(0)} = \frac{\mu - \tau}{n+1}. \tag{2.121}$$

Both the R_{tt} and the R_{zz} equations are of the form

$$f^{''} + n\cot\theta f^{'} - (n-1)\, f = 0, \tag{2.122}$$

where the prime denotes differentiation with respect to θ. This equation can be transformed into an associated Legendre equation by the substitution $f = (\sin\theta)^{\frac{1-n}{2}}\, y$

and it can be shown that the only regular solution is $f = 0$, hence we have

$$f_1^{(1)} = f_3^{(1)} = 0. \tag{2.123}$$

Using these solutions and (2.116) the $R_{r\theta}$ Einstein equation takes the form

$$f' + (n-1)\cot\theta f - B\sin\theta = 0, \tag{2.124}$$

with $f = f_6^{(1)} - f_5^{(1)}$ and $B = \frac{n+2}{n+1}\tau$. In order to prevent a singularity at the poles of the sphere we must have

$$f(0) = f(\pi) = 0. \tag{2.125}$$

Making the substitution $f = (\sin\theta)^{1-n}\, w$, (2.124) takes the form

$$w' - B\sin^n\theta = 0, \tag{2.126}$$

which can be solved using a hypergeometric function

$$w = k - B\cos\theta\, {}_2F_1\left(\frac{1}{2}, \frac{1-n}{2}; \frac{3}{2}; \cos^2\theta\right), \tag{2.127}$$

with k a constant of integration. The hypergeometric function takes the same finite value at the poles of the sphere but because of the $\cos\theta$ pre-factor k cannot be chosen so w vanishes on both poles. Thus the only way to satisfy (2.125) is for both k and B (and hence τ) to vanish. So we find a regular solution only if

$$T_{zz} = 0; \tag{2.128}$$

that is, if the blackfold equations (2.95) are satisfied, which is the result we set out to show. It is now simple to solve the remaining equations; we have

$$f_4^{(1)} = f_5^{(1)} = f_6^{(1)} = 0, \tag{2.129}$$

and

$$f_2^{(1)} = -p\cos\theta \tag{2.130}$$

as the remaining regular solutions. Using the identifications (2.107), (2.108), (2.109), and the equilibrium boost (2.97), the solution in the intermediate region $r_0 \ll r \ll$

R, written in the Schwarzschild gauge, [7] takes the form

$$
\begin{aligned}
g_{tt} &= -1 + \frac{n+1}{n}\frac{r_0^n}{r^n}, \\
g_{tz} &= -\frac{\sqrt{n+1}}{n}\frac{r_0^n}{r^n}\left(1 + \frac{r\cos\theta}{R}\right), \\
g_{zz} &= 1 + \frac{1}{n}\frac{r_0^n}{r^n}\left(1 + \frac{r\cos\theta}{R}\right) + \frac{2r\cos\theta}{R}, \\
g_{rr} &= 1 + \frac{r_0^n}{r^n}\left(1 - \frac{2n-1}{n^2}\frac{r\cos\theta}{R}\right) - \frac{2}{n}\frac{r\cos\theta}{R}, \\
g_{ij} &= \hat{g}_{ij}\left(1 + \frac{1}{n^2}\frac{r_0^n}{r^{n-1}R}\cos\theta - \frac{2}{n}\frac{r\cos\theta}{R}\right),
\end{aligned}
\tag{2.131}
$$

with

$$
\hat{g}_{ij}dx^i dx^j = r^2\left(d\theta^2 + \sin^2\theta d\Omega_n^2\right), \tag{2.132}
$$

the metric of S^{n+1} of radius r.

2.3.4 Near horizon analysis

We now complete the construction by finding the first order perturbations of the boosted black string in the near horizon region. The perturbations arise from bending the straight boosted black string into a circle of large radius R such that the metric asymptotes to (2.131) at large r. The zeroth order metric is simply given by the boosted black string (2.90) with the equilibrium boost parameter (2.97)

$$
\begin{aligned}
&g_{tt}^{(0)} = -1 + \frac{n+1}{n}\frac{r_0^n}{r^n}, \quad g_{tz}^{(0)} = -\frac{\sqrt{n+1}}{n}\frac{r_0^n}{r^n}, \quad g_{zz}^{(0)} = 1 + \frac{1}{n}\frac{r_0^n}{r^n}, \\
&g_{rr} = \left(1 + \frac{r_0^n}{r^n}\right)^{-1}, \quad g_{\theta\theta}^{(0)} = \hat{g}_{\theta\theta}, \quad g_{\Omega\Omega}^{(0)} = \hat{g}_{\Omega\Omega},
\end{aligned}
\tag{2.133}
$$

with $\hat{g}_{\theta\theta}, \hat{g}_{\Omega\Omega}$ given by (2.132). We now need to put the first order perturbations into the simplest possible form to facilitate solving Einstein's equations. It is convenient to decompose the perturbations into scalar, vector and tensor modes under coordinate transformations of S^{n+1}. The boundary conditions (2.131) are invariant under simultaneously taking $t \to -t, z \to -z$ and have an unbroken symmetry group $SO(n+1)$ of S^n, so the only perturbations we need to consider are the

[7] This gauge choice will be convenient for the near region analysis. We change from transverse gauge to Schwarzschild gauge by the transformation $r \to \frac{r_0^n}{2nr^{n-1}}$.

scalars $h_{tt}, h_{tz}, h_{zz}, h_{rr}$, the vector $h_{r\theta}$, and the tensors $h_{\theta\theta}, h_{\Omega\Omega}$. In fact, it has been shown [58] that there is a choice of gauge for which the vector perturbations vanish, $h_{r\theta} = 0$. We also note that the boundary conditions (2.131) imply a deviation from the zeroth order solution that is proportional to $\cos\theta$, so in the linearised theory we can assume that the perturbations only have this ($l = 1$) mode turned on, since equations for different modes decouple. It has also been shown [58] that for $l = 1$ modes, for scalar-derived tensors, we must have

$$h_{\theta\theta} = h_{\Omega\Omega}, \tag{2.134}$$

which tells us that there is only a longitudinal mode on the sphere. This allows the perturbations to be reduced to the form

$$g_{tt} = -1 + \frac{n+1}{n}\frac{r_0^n}{r^n} + \frac{\cos\theta}{R}a(r), \tag{2.135}$$

$$g_{tz} = -\frac{\sqrt{n+1}}{n}\left(\frac{r_0^n}{r^n} + \frac{\cos\theta}{R}b(r)\right), \tag{2.136}$$

$$g_{zz} = 1 + \frac{1}{n}\frac{r_0^n}{r^n} + \frac{\cos\theta}{R}c(r), \tag{2.137}$$

$$g_{rr} = \left(1 + \frac{r_0^n}{r^n}\right)^{-1}\left(1 + \frac{\cos\theta}{R}f(r)\right), \tag{2.138}$$

$$g_{ij} = \hat{g}_{ij}\left(1 + \frac{\cos\theta}{R}g(r)\right). \tag{2.139}$$

There is remaining coordinate freedom; under

$$r \to r + \gamma(r)\frac{r_0}{R}\cos\theta, \quad \theta \to \theta + \beta(r)\frac{r_0}{R}\sin\theta, \tag{2.140}$$

with

$$\beta'(r) = \frac{\gamma(r)}{r^2\left(1 - \frac{r_0^n}{r^n}\right)}, \tag{2.141}$$

the metric above is unchanged to first order in $1/R$. For the horizon to remain fixed we also require

$$\gamma(r_0) = 0. \tag{2.142}$$

These coordinate transformations produce shifts in the perturbation variables

$$a(r) \to a(r) - (n+1)\,\frac{r_0^{n+1}}{r^{n+1}}\gamma(r), \tag{2.143}$$

$$b(r) \to b(r) - n\frac{r_0^{n+1}}{r^{n+1}}\gamma(r), \tag{2.144}$$

$$c(r) \to c(r) - \frac{r_0^{n+1}}{r^{n+1}}\gamma(r), \tag{2.145}$$

$$f(r) \to f(r) + r_0\left(2\gamma'(r) - n\frac{r_0^n}{r^n}\frac{\gamma(r)}{r\left(1-\frac{r_0^n}{r^n}\right)}\right), \tag{2.146}$$

$$g'(r) \to g'(r) + 2\frac{r_0}{r}\left(\gamma'(r) + \frac{r_0^n}{r^n}\frac{\gamma(r)}{r\left(1-\frac{r_0^n}{r^n}\right)}\right). \tag{2.147}$$

We could, at this stage, fix the gauge; however, it is more useful to construct combinations which are invariant under these coordinate transformations and work with them. Suitable combinations are given by

$$A(r) = a(r) - (n+1)c(r), \tag{2.148}$$

$$B(r) = b(r) - nc(r), \tag{2.149}$$

$$F(r) = f(r) + 2r_0\left(\frac{r_0^{n+1}}{r^{n+1}}c(r)\right)' - \frac{n}{\left(1-\frac{r_0^n}{r^n}\right)}c(r), \tag{2.150}$$

$$G'(r) = g'(r) + 2\frac{r_0}{r}\left(\frac{r_0^{n+1}}{r^{n+1}}c(r)\right)' + \frac{2}{r\left(1-\frac{r_0^n}{r^n}\right)}c(r). \tag{2.151}$$

The goal now is to solve Einstein equations for these functions. This can be done by deriving a master equation for a single gauge invariant variable from which the rest of the solution can be obtained. We will not consider the derivation here which is very similar to our analysis for black strings in plane waves in chapter 4 (see [56] for details). The master equation that results is a fourth order ordinary differential equation which can be solved in terms of hypergeometric functions. It is shown that regular solution exists and is fully specified (up to choice of gauge) by requiring regularity on the horizon and by the asymptotic boundary conditions. For our analysis of black strings in chapter 4 the master equation will be a second order ordinary differential equation and will have solutions in terms of ordinary functions.

Chapter 3

Asymptotically plane wave spacetimes and their actions

Our aim in this chapter is to construct an action principle for asymptotically plane wave spacetimes, in the hope that this will shed light on the issue of holography for plane waves. Our results may also be useful for other investigations of asymptotically plane wave spacetimes; for example, these methods can be used to calculate conserved quantities.

To discuss the action for asymptotically plane wave spacetimes, we first need a suitable notion of what it means for a spacetime to be asymptotically plane wave. In section 3.1, we propose a definition in terms of a set of fall-off conditions on the metric at large spatial distances in directions orthogonal to the wave. We start by assuming that the components of the metric with indices along the spatial directions orthogonal to the wave fall off as $\mathcal{O}(r^{2-d})$, where r is a radial coordinate and d is the number of spatial directions orthogonal to the wave, corresponding to the influence of a localised source being spread over a $(d-1)$-sphere at large distances[1]. We then need to determine the behaviour of the components of the metric with indices parallel to the wave; we use the linearised equations of motion to relate the fall-off conditions of different components by assuming that all components make contributions of the same order to each term in the Einstein equations. This fixes

[1]We focus on the case $d \geq 3$.

the fall-off of the other components of the metric. We will show that the known solutions which asymptotically approach a vacuum plane wave [21–23] satisfy our fall-off conditions.

We only study solutions of the vacuum Einstein equations; it would clearly be interesting to extend this to include matter and, in particular, to supergravity. We will see that the black string solution of [25] which asymptotically approaches a plane wave solution in supergravity does not satisfy our fall-off conditions.

In section 3.2, we show that the definition of the action for vacuum gravity introduced in [30] can be applied to asymptotically plane wave spacetimes with our fall-off conditions without significant modification. We demonstrate that the on-shell action is finite and that the variational principle is well-defined. This provides confirmation that this is a useful definition of asymptotically plane wave and provides another example where the counter-term method of [30] is useful, suggesting that this approach to defining the gravitational action should have a broad applicability.

3.1 Asymptotically plane wave fall-off conditions

We consider asymptotically plane wave solutions in vacuum gravity. The plane wave solutions in $d+2$-dimensional vacuum gravity can be written in Brinkmann coordinates as[2]

$$ds^{2(0)} = -2dx^+dx^- - \mu_{IJ}(x^+)x^I x^J \left(dx^+\right)^2 + \delta_{IJ}dx^I dx^J, \tag{3.1}$$

where $I, J = 1, \ldots, d$, and $\mu_{IJ}(x^+)$ are arbitrary functions subject only to $\delta^{IJ}\mu_{IJ}(x^+) = 0$, which ensures that the solution satisfies the vacuum equations of motion. The coordinates in the plane wave solution split into two coordinates $x^\pm$ along the direction of the wave and the spatial coordinates x^I in the directions orthogonal to the wave. In the spatial directions, we will use both Cartesian coordinates x^I, and polar coordinates r, θ^i, $i = 1, \ldots (d-1)$:

$$\delta_{IJ}dx^I dx^J = dr^2 + r^2\hat{h}_{ij}d\theta^i d\theta^j, \tag{3.2}$$

[2]In this chapter we use the superscript $^{(0)}$ to denote the plane wave (3.1) and $^{(1)}$ to denote some perturbation of the wave.

where $\hat{h}_{ij}$ is the metric and θ^i are the coordinates on the unit $(d-1)$-sphere S^{d-1}.

A general asymptotically plane wave spacetime will have a metric $g = g^{(0)} + g^{(1)}$, where $g^{(1)}$ will have some suitable fall-off conditions at large distance. We will focus on studying the fall-off conditions at large radial distance in the directions orthogonal to the wave. In the spatial direction that the wave is travelling in, we will consider either perturbations which are independent of x^-, like the wave itself, or perturbations which fall off at large x^-, but we will not explicitly specify the fall-off conditions in this direction.[3]

Considering first metrics which are independent of x^-, we specify the fall-off conditions at large r by making two assumptions. First, we assume that the spatial components (in the above Cartesian coordinate system) $g^{(1)}_{IJ} \sim \mathcal{O}\left(r^{2-d}\right)$. These are the same fall-off conditions as for the spatial components of an asymptotically flat metric in $d+1$ dimensions. This seems appropriate because we would expect a perturbation which is independent of x^- to correspond to the effect of a source which is extended along the direction of the wave, but localised in the transverse spatial directions, so its effect at large r should be diluted by spreading on the S^{d-1}.

To fix the fall-offs of $g_{\pm\pm}$, $g_{\pm I}$, we make a second assumption, namely that all components make contributions of the same order to each term in the Einstein equations.[4] This is essentially a genericity assumption, so it should be appropriate for finding the general fall-off conditions on metric components. In vacuum gravity, the linearised equations of motion are $R^{(1)}_{\mu\nu} = 0$, where [53]

$$R^{(1)}_{\mu\nu} = -\frac{1}{2} g^{(0)\rho\sigma}\, \nabla^{(0)}_{\rho}\, \nabla^{(0)}_{\sigma} g^{(1)}_{\mu\nu} - \frac{1}{2} g^{(0)\rho\sigma}\, \nabla^{(0)}_{\mu}\, \nabla^{(0)}_{\nu} g^{(1)}_{\rho\sigma} + g^{(0)\rho\sigma}\, \nabla^{(0)}_{\rho}\, \nabla^{(0)}_{(\mu} g^{(1)}_{\nu)\sigma}. \quad (3.3)$$

The idea of our assumption is that the cancellations which give $R^{(1)}_{\mu\nu} = 0$ should generically involve all the terms in $R^{(1)}_{\mu\nu}$. The contribution of $g^{(1)}_{IJ}$ to (3.3) gives

$$R^{(1)}_{IJ} \sim \mathcal{O}\left(r^{-d}\right), \quad R^{(1)}_{+I} \sim \mathcal{O}\left(r^{1-d}\right), \quad R^{(1)}_{++} \sim \mathcal{O}\left(r^{2-d}\right). \quad (3.4)$$

[3]This is similar to the treatment of linear dilaton spacetimes in [40] where the fall-offs in the directions along the brane were not explicitly treated.

[4]We will not attempt to fully exploit the information in the asymptotic Einstein equations; we just use them to determine a set of fall-off conditions. The consistency of our fall-off conditions with the dynamical equations of motion is demonstrated by verifying that the solutions we consider in the next subsection satisfy our fall-off conditions.

Because of the assumption that $g^{(1)}_{IJ}$ is constant in x^-, it does not make any contribution to $R^{(1)}_{-I}, R^{(1)}_{+-}$ and $R^{(1)}_{--}$. Assuming the other terms in $g^{(1)}_{\mu\nu}$ produce effects at the same order determines

$$g^{(1)}_{++} \sim \mathcal{O}\left(r^{4-d}\right), \quad g^{(1)}_{+-} \sim \mathcal{O}\left(r^{2-d}\right), \quad g^{(1)}_{--} \sim \mathcal{O}\left(r^{-d}\right), \tag{3.5}$$

$$g^{(1)}_{+I} \sim \mathcal{O}\left(r^{3-d}\right), \quad g^{(1)}_{-I} \sim \mathcal{O}\left(r^{1-d}\right). \tag{3.6}$$

With these fall-offs, all terms also give

$$R^{(1)}_{-I} \sim \mathcal{O}\left(r^{-d-1}\right), \quad R^{(1)}_{+-} \sim \mathcal{O}\left(r^{-d}\right), \quad R^{(1)}_{--} \sim \mathcal{O}\left(r^{-d-2}\right). \tag{3.7}$$

The faster fall-off conditions required for metric components with an x^- index arise because $g^{(0)--} \sim r^2$, so terms in a given component of $R^{(1)}_{IJ}$ coming from $g^{(1)}_{--}$ have an extra factor of r^2 compared to terms coming from $g^{(1)}_{IJ}$. Similarly, the less restrictive conditions on components with an x^+ index are due to the vanishing of $g^{(0)++}$.

If we consider the more general case, allowing the perturbation to depend on x^-, there will be additional terms in $R^{(1)}_{\mu\nu}$ involving derivatives ∂_-. These terms will also come with extra powers of r coming from $g^{(0)--}$. As a result, if we think of a general perturbation as composed of a part which is independent of x^- and a part which depends on x^-, the part which depends on x^- will be required to fall off more quickly than the constant part.[5] We find

$$\partial_- g^{(1)}_{IJ} \sim \mathcal{O}\left(r^{-d}\right), \quad \partial_- g^{(1)}_{+J} \sim \mathcal{O}\left(r^{1-d}\right), \quad \partial_- g^{(1)}_{-J} \sim \mathcal{O}\left(r^{-d-1}\right), \tag{3.8}$$

$$\partial_- g^{(1)}_{++} \sim \mathcal{O}\left(r^{2-d}\right), \quad \partial_- g^{(1)}_{+-} \sim \mathcal{O}\left(r^{-d}\right), \quad \partial_- g^{(1)}_{--} \sim \mathcal{O}\left(r^{-d-2}\right), \tag{3.9}$$

and

$$\partial_-\partial_- g^{(1)}_{IJ} \sim \mathcal{O}\left(r^{-d-2}\right), \quad \partial_-\partial_- g^{(1)}_{+J} \sim \mathcal{O}\left(r^{-d-1}\right), \quad \partial_-\partial_- g^{(1)}_{-J} \sim \mathcal{O}\left(r^{-d-3}\right), \tag{3.10}$$

[5]Even without this additional factor, the x^- dependent parts would be required to fall off faster than the constant parts. The situation is analogous to the solution for a localised source described in a cylindrical coordinate system, which involves

$$\frac{1}{(r^2+z^2)^{\frac{d-2}{2}}} \approx \frac{1}{r^{d-2}} - \frac{(d-2)z^2}{2r^d} + \ldots,$$

so the z-dependent term falls off faster than the constant term at large r. The effect of $g^{(0)--}$ is to make these contributions fall off even more quickly in the plane wave background.

$$\partial_-\partial_- g^{(1)}_{++} \sim \mathcal{O}\left(r^{-d}\right), \quad \partial_-\partial_- g^{(1)}_{+-} \sim \mathcal{O}\left(r^{-d-2}\right), \quad \partial_-\partial_- g^{(1)}_{--} \sim \mathcal{O}\left(r^{-d-4}\right). \quad (3.11)$$

We take the above constraints on the asymptotic fall-off of the metric to define a class of asymptotically plane wave spacetimes.

Not all of these components of the metric carry independent physical information; by an appropriate diffeomorphism, we can set some of the components $g^{(1)}_{\mu\nu}$ to zero at large distance. In [40], this diffeomorphism freedom was fixed by choosing a Gaussian normal gauge in which the components of $g^{(1)}_{\mu\nu}$ with radial indices are set to zero. In the present case, because the directions $x^{\pm}$ are singled out as special, it seems more convenient to us to choose a gauge in which

$$g^{(1)}_{+-} = g^{(1)}_{--} = g^{(1)}_{-I} = 0. \quad (3.12)$$

Because of the faster fall-off conditions on the x^- components, the diffeomorphism which sets these components to zero will not modify the asymptotic fall-off of the other components.

3.1.1 Comparison to known solutions

There have been a few papers on exact solutions of the Einstein equations which asymptotically approach a plane wave. These provide a useful check of our analysis; if we have an appropriate set of fall-off conditions, they should be satisfied by these solutions. The first such solution was constructed in [21, 22], where a Garfinkle-Vachaspati transform was applied to a black string solution with a non-trivial scalar field to obtain an asymptotically plane wave black string,

$$ds^2_{str} = -\frac{2}{h(r)}dx^+dx^- + \frac{f(r) + r^2(3\cos^2\theta - 1)}{h(r)}(dx^+)^2 + (k(r)l(r))^2(dr^2 + r^2 d\Omega_2^2), \quad (3.13)$$

$$e^{4\phi} = \frac{k(r)l(r)}{h^2(r)}, \quad (3.14)$$

where

$$f(r) = 1 + \frac{Q_1}{r}, \quad h(r) = 1 + \frac{Q_2}{r}, \quad k(r) = 1 + \frac{P_1}{r}, \quad l(r) = 1 + \frac{P_2}{r}. \quad (3.15)$$

The presence of the scalar ϕ means that this is not a vacuum solution, but it becomes a vacuum solution at large r, and it is easy to check that our boundary conditions are satisfied. The solution is independent of x^-, and it has $g^{(1)}_{+-}$ and $g^{(1)}_{IJ}$ going like $\mathcal{O}\left(r^{-1}\right)$, $g^{(1)}_{++}$ going like $\mathcal{O}\left(r\right)$, with the other components of $g^{(1)}_{\mu\nu}$ vanishing. We have written the string frame solution above but this statement will be true in either string or Einstein frame.

This was extended in [23] to construct a pure vacuum solution which is asymptotically plane wave, although it is not smooth in the interior:

$$ds^2 = \frac{1}{H(r)}\left[-2dx^+dx^- + f(r)(dx^+)^2 + \frac{H(r)^4}{r^4 H'(r)^2}(dr^2 + r^2 d\Omega_2^2)\right], \tag{3.16}$$

where

$$f(r) = 1 + \ln H(r) + \xi_2(x^+)\psi_2(r)(3\cos^2\theta - 1), \tag{3.17}$$

$$\psi_2(r) = (3r^2 + 2 + 3r^{-2})\left[\alpha_1 + \alpha_2 \ln\left(\frac{r-1}{r+1}\right)\right] + 6\alpha_2(r + r^{-1}), \tag{3.18}$$

$$H(r) = \left(\frac{r-1}{r+1}\right)^{\frac{2}{\sqrt{3}}}, \tag{3.19}$$

and α_1, α_2 are arbitrary constants and $\xi_2(x^+)$ is an arbitrary function of x^+. Again, it is easy to see that this satisfies our definition of asymptotically plane wave. The solution is independent of x^-, and it has $g^{(1)}_{+-}$ and $g^{(1)}_{IJ}$ going like $\mathcal{O}\left(r^{-1}\right)$, $g^{(1)}_{++}$ going like $\mathcal{O}\left(r\right)$, with the other components of $g^{(1)}_{\mu\nu}$ vanishing.

In [24], a solution was obtained by T-duality from a black hole in a Gödel universe. This solution reduces to a plane wave when the black hole mass parameter is set to zero, but it is not asymptotically plane wave as it has components $g^{(1)}_{IJ}$ going like $\mathcal{O}\left(r^0\right)$ at large r, so the sphere is deformed asymptotically. Thus, it does not satisfy our definition but this is unproblematic; we would not regard such a solution as a candidate for the appellation asymptotically plane wave.

Finally, another solution was obtained in [25] by a sequence of boosts and dualities known as the null Melvin twist. This is a solution in the common Neveu-Schwarz sector of the ten-dimensional superstring theories, and has

$$\begin{aligned} ds^2_{str} &= -\frac{f(r)(1+\beta^2 r^2)}{k(r)}dt^2 - \frac{2\beta^2 r^2 f(r)}{k(r)}dtdy + \left(1 - \frac{\beta^2 r^2}{k(r)}\right)dy^2 \\ &\quad + \frac{dr^2}{f(r)} + r^2 d\Omega_7^2 - \frac{\beta^2 r^4 (1 - f(r))}{4k(r)}\sigma^2, \end{aligned} \tag{3.20}$$

$$e^{\phi} = \frac{1}{\sqrt{k(r)}}, \tag{3.21}$$

and

$$B = \frac{\beta r^2}{2k(r)}(f(r)dt + dy) \wedge \sigma, \tag{3.22}$$

where

$$f(r) = 1 - \frac{M}{r^6}, \quad k(r) = 1 + \frac{\beta^2 M}{r^4}, \tag{3.23}$$

and the one-form σ is given in terms of Cartesian coordinates x^I by

$$\frac{r^2\sigma}{2} = x^1 dx^2 - x^2 dx^1 + x^3 dx^4 - x^4 dx^3 + x^5 dx^6 - x^6 dx^5 + x^7 dx^8 - x^8 dx^7. \tag{3.24}$$

This solution is not vacuum, even at large distances, but at large r it approaches a plane wave which [25] call $\mathcal{P}_{10}$, which is the two-form equivalent of an electromagnetic plane wave. We can then write the metric as $g = g^{(0)} + g^{(1)}$, where $g^{(0)}$ is the metric of the pure plane wave $\mathcal{P}_{10}$, which can be obtained by setting $M = 0$ in the above solution.

This solution lies outside of the scope of our analysis, since it is not a solution of the vacuum Einstein equations, even asymptotically. However, we can still observe that this solution does not satisfy our asymptotic fall-off conditions, as $g^{(1)}_{IJ} \sim \mathcal{O}\left(r^{-4}\right)$, so our input assumption that $g^{(1)}_{IJ} \sim \mathcal{O}\left(r^{2-d}\right)$ is not satisfied. That is, the spatial fall-off of the metric is not behaving as we would expect based on a localised source, which presumably means that there are source terms coming from the two-form field B which extend into the asymptotic region, additional to those associated with the plane wave $\mathcal{P}_{10}$. In addition, the relation between the different coefficients is not the same as we had; if we define $x^+ = t + y$, $x^- = t - y$, we will have $g^{(1)}_{+-} \sim \mathcal{O}\left(r^{-4}\right)$, but $g^{(1)}_{--} \sim \mathcal{O}\left(r^{-4}\right)$, and not $\mathcal{O}\left(r^{-6}\right)$ as we might have expected from the behaviour of $g^{(1)}_{IJ}$. It is not clear whether we should regard this solution as asymptotically plane wave or not; it asymptotically approaches the plane wave metric $\mathcal{P}_{10}$, but more slowly than we would expect. In particular, the slow fall-off of the spatial components $g^{(1)}_{IJ}$ is likely to make it difficult to define a finite action principle for such solutions. It would be very interesting to extend our analysis below to include form fields so that this case could be directly addressed.

We remark here that, although the particular vacuum solutions considered above satisfied our boundary conditions so they're not dynamically inconsistent, we will see in the next chapter that solutions describing generic sources on plane wave backgrounds are not asymptotically plane wave.

3.1.2 Conformal structure

We have given a definition of asymptotically plane wave spacetimes above, focusing on the behaviour of the solution at large r. Our decision to focus on the behaviour at large r is inspired in part by the previously known exact solutions which approach a plane wave only at large r, and by our interest in the construction of an appropriate action principle where it is the boundary at $r =$ constant which is expected to be problematic.

In special cases, however, we could take a different approach and define asymptotically plane wave spacetimes in terms of the existence of a suitable conformal completion. This would be closer in spirit to the usual treatments of asymptotic flatness. We will not develop this approach here; we simply want to make some remarks pointing out that it is really quite different to the approach we are taking.

In [10], a conformal completion was constructed for the maximally supersymmetric plane wave for which the metric is

$$ds^2 = -2dx^+dx^- - r^2(dx^+)^2 + dr^2 + r^2 d\Omega_7^2, \tag{3.25}$$

where $d\Omega_7^2$ denotes the unit metric on S^7. The conformal completion is obtained by making a coordinate transformation to rewrite this metric as a conformal factor times the metric on the Einstein static universe,

$$ds^2 = \frac{1}{|e^{i\psi} - \cos\alpha e^{i\beta}|^2}(-d\psi^2 + d\alpha^2 + \cos^2\alpha d\beta^2 + \sin^2\alpha d\Omega_7^2). \tag{3.26}$$

We thus see that the conformal boundary of this plane wave lies at $\alpha = 0$, $\psi = \beta$, and is a one-dimensional null line in the Einstein static universe. The explicit coordinate transformation is

$$r = \frac{\sin\alpha}{2|e^{i\psi} - \cos\alpha e^{i\beta}|}, \tag{3.27}$$

$$\tan x^+ = \frac{\sin\psi - \cos\alpha\sin\beta}{\cos\psi - \cos\alpha\cos\beta}, \tag{3.28}$$

$$x^- = \frac{1}{2}\left(\frac{\sin\psi + \cos\alpha\sin\beta}{\cos\psi - \cos\alpha\cos\beta} - r^2\tan x^+\right). \tag{3.29}$$

The point we want to stress is that when we approach the conformal boundary $\alpha = 0$, $\psi - \beta = 0$ along a generic direction, say $\alpha = \gamma(\psi - \beta)$ for some constant γ, r remains finite. In these generic directions, it is x^- which diverges. Thus, controlling the behaviour as $r \to \infty$ in a spacetime which asymptotically approaches this plane wave will give little information about whether there exists a conformal completion with (in some suitable sense) “the same structure” as for the pure plane wave. Rather, it is the behaviour at large x^- that one would have to study in detail to see if a suitable conformal completion exists.

Thus, the definition of asymptotically plane wave we have introduced is different in character from a definition based on conformal structure. If a definition based on conformal structure could be developed, it would presumably be suitable for addressing different questions from those which can be addressed with our definition. We would also remark that the above analysis suggests that the known exact solutions, which have a deformation away from the plane wave which is independent of x^-, are unlikely to qualify as asymptotically plane wave with respect to such a conformal definition of asymptotically plane wave.

3.2 Action for asymptotically plane wave spacetimes

We have put forward a definition of asymptotically plane wave spacetimes, using the linearised equations of motion to relate the fall-off of different components. In this section, we give the main result of this chapter, constructing an appropriate action principle for this class of spacetimes. We construct our action principle following Mann and Marolf [30] who recently introduced a new approach to specifying a well-defined action principle for vacuum gravity for asymptotically flat spacetimes.

For the asymptotically flat case, the action is [30]

$$S = -\frac{1}{16\pi G}\int_{\mathcal{M}} \sqrt{-g} R d^D x - \frac{1}{8\pi G}\int_{\partial\mathcal{M}} \sqrt{-h} K d^{D-1}x + \frac{1}{8\pi G}\int_{\partial\mathcal{M}} \sqrt{-h}\hat{K} d^{D-1}x, \tag{3.30}$$

where g is the determinant of the bulk metric, h is the determinant of the bulk metric pulled back to the boundary, R is the Ricci scalar, and $K = h^{\alpha\beta}K_{\alpha\beta}$ is the trace of the extrinsic curvature on the boundary. The final term is a new contribution introduced to cancel the divergences coming from the Gibbons-Hawking boundary term. The function $\hat{K}$ is defined implicitly by the solution of [6]

$$\mathcal{R}_{\alpha\beta} = \hat{K}_{\alpha\beta}\hat{K} - h^{\gamma\delta}\hat{K}_{\alpha\gamma}\hat{K}_{\delta\beta}, \tag{3.31}$$

where $\mathcal{R}_{\alpha\beta}$ is the Ricci tensor of the metric $h_{\alpha\beta}$ induced on $\partial\mathcal{M}$. Thus, this additional boundary term is determined locally by the induced metric on the boundary in the spirit of the boundary counterterm approach to constructing actions for asymptotically AdS spaces [33]. Alternative actions for asymptotically flat spacetimes with a similar philosophy appeared previously in [34, 35]. See also [59] for related work.

To apply this prescription to asymptotically plane wave spacetimes, we first need to introduce a cut-off to make the different terms in the action finite. We will cut off the spacetime by introducing a boundary at some large radial distance, $r =$ constant. Our main focus will be on boundary terms associated with this boundary; as in the asymptotically flat case, there is a divergence associated with the Gibbons-Hawking boundary term on this surface due to the extrinsic curvature of the sphere, and we need to introduce an appropriate local boundary term to cancel it.

Although our focus is mainly on the $r =$ constant boundary, to make the spacetime region we consider finite, we also need to introduce some cut-offs in the $x^{\pm}$ directions along the plane wave. The symmetry of the background under translations in x^- makes it natural to introduce cut-offs at two constant values of x^+, respecting this symmetry. In the simple case where μ_{IJ} are constants, which includes the cases of most interest for holography, there is an additional symmetry under translations in x^+, which suggests it is natural to take the other cut-off to be at constant values of x^-, respecting this translation invariance. We will also discuss the calculation of the action for the general case where $\mu_{IJ}(x^+)$ are not constants with this same cut-off. We will see that this choice of cut-off can give a satisfactory construction for an action even for general $\mu_{IJ}(x^+)$, although there are some additional subtleties associated with the surfaces at constant x^-. However, one should

[6]Refer to section 2.2 for a discussion of the origins of this equation.

bear in mind that there is no a priori justification for this choice of cut-off in the general case.

The action for the cut-off spacetime should contain a Gibbons-Hawking boundary term for each of these boundaries. In the case of the surfaces at $x^+ =$ constant, there is a subtlety as they are null surfaces, so the trace of the extrinsic curvature is not well-defined. However, this issue has been previously considered in [60] where it was shown that a suitable boundary term on a null boundary $x^+ =$ constant is

$$-\frac{1}{16\pi G}\int_{x^+=const} d^{d+1}x\sigma^\lambda\partial_\lambda x^+, \tag{3.32}$$

where $\sigma^\lambda = \frac{1}{\sqrt{-g}}\partial_\mu\left((-g)\,g^{\mu\lambda}\right)$, with g being the determinant of the metric on the full spacetime. We will adopt this prescription here. On the boundaries at $x^- =$ constant, we consider just the usual Gibbons-Hawking boundary term.

On the boundary at $r =$ constant, the Gibbons-Hawking boundary term gives a contribution which will diverge as we remove the cut-off. This divergence is associated with the intrinsic curvature of the boundary (the background plane wave spacetime has a flat spatial metric in the x^i directions, so the intrinsic and extrinsic curvatures of the $r =$ constant boundary are related), so we can try to cancel this divergence by adding a Mann-Marolf counterterm contribution to the action on this boundary.

Thus, the action we consider is

$$\begin{aligned} S \;=\; & -\frac{1}{16\pi G}\int_{\mathcal{M}} d^{d+2}x\sqrt{-g}R - \frac{1}{16\pi G}\int_{x^+=consts} d^{d+1}x\sigma^\lambda\partial_\lambda x^+ \\ & -\frac{1}{8\pi G}\int_{x^-=consts} d^{d+1}x\sqrt{|h|}K - \frac{1}{8\pi G}\int_{r=const} d^{d+1}x\sqrt{-h}\left(K-\hat{K}\right), \end{aligned} \tag{3.33}$$

where by the integral over $x^+ =$ constants we mean integrals over two surfaces at different values of x^+, with opposite orientations for the normal to the surface, and similarly for the integral over $x^- =$ constants.

Let us first of all consider the value of this action for the plane wave background (3.1). This is a vacuum solution, so $R = 0$. On the surface $x^+ =$ constant,

$$\sigma^\lambda\partial_\lambda x^+ = \sigma^+ = \partial_\mu g^{\mu+} = 0, \tag{3.34}$$

as $g^{(0)++} = 0$ and $g^{(0)+-} = -1$. So the boundary term at $x^+ =$ constant vanishes. On the surface $x^- =$ constant, if μ_{IJ} are constant, the only non-zero component of

$K_{\alpha\beta}$ is

$$K_{+I} = \frac{1}{2\sqrt{g^{(0)--}}}\partial_I g^{(0)}_{++}. \tag{3.35}$$

Since $h^{(0)+I} = 0$, this gives $K = 0$, and the boundary term at $x^- =$ constant vanishes as well.

In the more general case where $\mu_{IJ}(x^+)$ depend on x^+, we have

$$K = K_{++}h^{(0)++} = \frac{1}{2\sqrt{g^{(0)--}}}\partial_+ g^{(0)}_{++} h^{(0)++}, \tag{3.36}$$

and at $x^- =$ constant, $h^{(0)++} = 1/h^{(0)}_{++} = -1/(\mu_{IJ}(x^+)x^I x^J)$. Hence, this $K \sim \mathcal{O}(r^{-1})$, and the contribution to the action is

$$S_- = -\frac{1}{8\pi G}\int_{x^-=const} K\sqrt{|h|}dx^+ d^d x^I \sim \mathcal{O}(r^d), \tag{3.37}$$

so this boundary will make a divergent contribution to the action as we remove the cut-off at large r. However, in the full action, there are two boundaries at constant x^- (at say $x^- = \pm x_0^-$), and they contribute with opposite signs because of the opposite orientations of the outward normals, so this term will cancel between the two boundaries, making no contribution to the total action.

Finally, the boundary at $r =$ constant is what we want to focus on, so let us be more explicit and set up the notation we will use later. We define coordinates on the boundary $x^\alpha = \{x^-, x^+, \theta^i\}$, so the boundary metric is

$$h_{\alpha\beta} = \begin{pmatrix} 0 & -1 & \vec{0} \\ -1 & -\mu_{IJ}x^I x^J & \vec{0} \\ \vec{0} & \vec{0} & r^2\hat{h}_{ij} \end{pmatrix}, \tag{3.38}$$

with determinant $h = -r^{2d-2}\hat{h}$, where $\hat{h}$ is the determinant of the unit metric on S^{d-1}. The normal vector to the boundary is $n_\nu = \delta^r_\nu$. The non-zero components of the extrinsic curvature are

$$K_{ij} = r\hat{h}_{ij}, \quad K_{++} = -\frac{\mu_{IJ}x^I x^J}{r}, \tag{3.39}$$

so $K = \frac{d-1}{r}$. The Ricci tensor on the boundary is

$$\mathcal{R}_{\alpha\beta} = \begin{pmatrix} 0 & 0 & \vec{0} \\ 0 & \mathcal{R}_{++} & \vec{0} \\ \vec{0} & \vec{0} & (d-2)\,\hat{h}_{ij} \end{pmatrix}. \tag{3.40}$$

Solving (3.31) for $\hat{K}_{\alpha\beta}$, we find that the non-zero components are $\hat{K}_{ij} = r\hat{h}_{ij}$ and $\hat{K}_{++} = \frac{r\mathcal{R}_{++}}{d-1}$, and so $\hat{K} = \frac{d-1}{r}$. Thus $K - \hat{K} = 0$, hence there is no contribution to the action from the $r =$ constant surface.

Thus, we find that the on-shell action for the pure plane wave is zero. Note that the action vanishes for any plane wave solution, independent of the values of $\mu_{IJ}(x^+)$.

3.2.1 Finiteness of the action

Next, we consider an arbitrary asymptotically plane wave solution satisfying our asymptotic fall-off conditions, and show that the action of the solution will be finite. Since the metric g is still a solution of the vacuum equations, $R = 0$, the bulk term still makes no contribution to the action. For the boundaries at constant x^+, as in the pure plane wave,

$$S_+ = -\frac{1}{16\pi G}\int_{x^+=const} dx^- \left(dx^I\right)^d \partial_\mu g^{(1)\mu+}. \tag{3.41}$$

In the gauge we have chosen, $g^{++} = 0$, $g^{+-} = 1$, and $g^{+I} = 0$, so this term still vanishes.

For the boundaries at constant x^-, the contributions to the extrinsic curvature at linear order in the departure of the metric from the plane wave are

$$K = K^{(0)}_{++}h^{(1)++} + K^{(0)}_{+I}h^{(1)+I} + K^{(1)}_{++}h^{(0)++} + K^{(1)}_{IJ}h^{(0)IJ}. \tag{3.42}$$

On these boundaries, we have $h^{(1)++} \sim \mathcal{O}(r^{-d})$, $h^{(1)+I} \sim \mathcal{O}(r^{1-d})$, and

$$K^{(1)}_{++} = -\frac{1}{2}\frac{g^{(0)+-}}{\sqrt{g^{(0)--}}}\partial_+ g^{(1)}_{++} - \frac{1}{2}\frac{g^{(0)+-}g^{(1)--}}{(g^{(0)--})^{3/2}}\partial_+ g^{(0)}_{++} + \frac{1}{2}\sqrt{g^{(0)--}}\partial_- g^{(1)}_{++}, \tag{3.43}$$

$$K^{(1)}_{IJ} = -\frac{1}{2}\frac{g^{(0)+-}}{\sqrt{g^{(0)--}}}\left(\partial_J g^{(1)}_{I+} + \partial_I g^{(1)}_{J+} - \partial_+ g^{(1)}_{IJ}\right) + \frac{1}{2}\sqrt{g^{(0)--}}\partial_- g^{(1)}_{IJ}. \tag{3.44}$$

Thus, the terms which are independent of x^- will give a contribution to $K \sim \mathcal{O}(r^{1-d})$. This will make a divergent contribution to the integral over a single boundary, $S_- \sim \mathcal{O}(r^2)$. However, as in the action for the pure plane wave, this divergence cancels between the two boundaries so, for asymptotically plane wave solutions which are independent of x^-, the contribution to the action from these boundaries vanishes.

We require that any terms depending on x^- fall off at large x^-. This implies, in particular, that there cannot be any linear dependence on x^- near these boundaries, so the part of the components $g^{(1)}_{\mu\nu}$ involving x^- will fall off faster than the part that is independent of x^- by a factor of $1/r^4$. The contribution of the x^--dependent part of $g^{(1)}_{\mu\nu}$ to the terms in K that do not involve explicit derivatives ∂_- will then be $O(r^{-d-3})$. Thus, the contribution to the action from this part of K is finite and will go to zero as we take the cut-off in x^- to infinity. There are terms in $K^{(1)}_{++}$ and $K^{(1)}_{IJ}$ which involve explicit derivatives ∂_-; these make a contribution $K \sim \mathcal{O}(r^{-d-1})$, giving a contribution to the integral S_- which is logarithmically divergent at large r. However, this contribution comes with some negative power of x^-, so if we take the boundaries at constant x^- to infinity at the same time as we take the boundary at large r to infinity, this contribution will go to zero. This dependence on the order of limits is not entirely satisfactory but it allows us to define a finite action. It does not seem to conceal any particularly interesting deeper issues.

Finally, we consider the boundary at r = constant, for which the analysis will be similar to the asymptotically flat case considered in 2.2. We can write the linear order contribution to the boundary term in our gauge as

$$K^{(1)} - \hat{K}^{(1)} = K^{(1)}_{\alpha\beta} h^{(0)\alpha\beta} - \hat{K}^{(1)}_{\alpha\beta} h^{(0)\alpha\beta}. \tag{3.45}$$

As $\sqrt{-h} \sim \mathcal{O}(r^{d-1})$, we need $K^{(1)} - \hat{K}^{(1)} \sim \mathcal{O}(r^{1-d})$ to have a finite action. For the term involving the extrinsic curvature,

$$K^{(1)}_{\alpha\beta} = g^{(1)rr} K^{(0)}_{\alpha\beta} - \frac{1}{2}\left(g^{(1)}_{\beta r,\alpha} + g^{(1)}_{r\alpha,\beta} - g^{(1)}_{\alpha\beta,r}\right), \tag{3.46}$$

and substituting for $g^{(1)}_{\alpha\beta}$ it is easy to show that this term is $\mathcal{O}(r^{1-d})$.

As before, to evaluate $\hat{K}^{(1)}_{\alpha\beta}$, we linearise (3.31) to give

$$\mathcal{R}^{(1)}_{\alpha\beta} = \hat{K}^{(1)}_{\gamma\delta} L^{(0)\gamma\delta}_{\alpha\beta} + \left(\hat{K}^{(0)}_{\alpha\beta}\hat{K}^{(0)}_{\gamma\delta} - \hat{K}^{(0)}_{\alpha\gamma}\hat{K}^{(0)}_{\beta\delta}\right) h^{(1)\gamma\delta}, \tag{3.47}$$

where

$$L^{(0)\gamma\delta}_{\alpha\beta} = h^{\gamma\delta}\hat{K}_{\alpha\beta} + \frac{1}{2}\left(\delta^\gamma_\alpha\delta^\delta_\beta\hat{K} + \delta^\gamma_\beta\delta^\delta_\alpha\hat{K}\right) - \frac{1}{2}\left(\delta^\gamma_\alpha\hat{K}^\delta_\beta + \delta^\gamma_\beta\hat{K}^\delta_\alpha + \delta^\delta_\alpha\hat{K}^\gamma_\beta + \delta^\delta_\beta\hat{K}^\gamma_\alpha\right). \tag{3.48}$$

Inverting this will give us an expression for $\hat{K}^{(1)}_{\alpha\beta}$,

$$h^{(0)\alpha\beta}\hat{K}^{(1)}_{\alpha\beta} = M^{(0)\gamma\delta}\left(\mathcal{R}^{(1)}_{\gamma\delta} - \left(\hat{K}^{(0)}_{\alpha\beta}\hat{K}^{(0)}_{\gamma\delta} - \hat{K}^{(0)}_{\alpha\gamma}\hat{K}^{(0)}_{\beta\delta}\right)h^{(1)\alpha\beta}\right), \tag{3.49}$$

where $M^{\gamma\delta} = h^{\alpha\beta}\left(L^{-1}\right)^{\gamma\delta}_{\alpha\beta}$. Recall that the non-zero components in $\hat{K}^{(0)}_{\alpha\beta}$ are $\hat{K}^{(0)}_{++}$ and $\hat{K}^{(0)}_{ij}$, and note that in our gauge $h^{(1)++} = 0$ on the r = constant boundary. We thus have

$$h^{(0)\alpha\beta}\hat{K}^{(1)}_{\alpha\beta} = M^{(0)\alpha\beta}\mathcal{R}^{(1)}_{\alpha\beta} - M^{(0)ij}(\hat{K}^{(0)}_{ij}\hat{K}^{(0)}_{mn} - \hat{K}^{(0)}_{im}\hat{K}^{(0)}_{jn})h^{(1)mn}. \tag{3.50}$$

A lengthy explicit calculation gives that the only non-zero components of $M^{(0)\gamma\delta}$ are

$$M^{(0)+-} \sim \mathcal{O}(r), \quad M^{(0)--} \sim \mathcal{O}(r^2), \quad M^{(0)ij} = \frac{1}{2(d-2)r}\hat{h}^{ij} = \frac{r}{2(d-2)}h^{ij}. \tag{3.51}$$

For the second term in (3.50), we have $\hat{K}^{(0)}_{ij} \sim \mathcal{O}(r)$, and $h^{(1)mn} \sim \mathcal{O}(r^{-d})$, so this term is $\mathcal{O}(r^{1-d})$. For the first term, we express $\mathcal{R}^{(1)}_{\alpha\beta}$ by the analogue of (3.3),

$$\mathcal{R}^{(1)}_{\alpha\beta} = -\frac{1}{2}h^{(0)\gamma\delta}D^{(0)}_{\alpha}D^{(0)}_{\beta}h^{(1)}_{\gamma\delta} - \frac{1}{2}h^{(0)\gamma\delta}D^{(0)}_{\gamma}D^{(0)}_{\delta}h^{(1)}_{\alpha\beta} + h^{(0)\gamma\delta}D^{(0)}_{\gamma}D^{(0)}_{(\alpha}h^{(1)}_{\beta)\delta}, \tag{3.52}$$

where D_α is the covariant derivative compatible with $h_{\alpha\beta}$. Using this expression we can see that $\mathcal{R}^{(1)}_{+-} \sim \mathcal{O}(r^{-d})$, $\mathcal{R}^{(1)}_{--} \sim \mathcal{O}(r^{-d-2})$, and $\mathcal{R}^{(1)}_{ij} \sim \mathcal{O}(r^{2-d})$, so the first term also makes a finite contribution (in addition, many of these terms will actually be total derivatives, which make no contribution to the action).

Thus, we conclude that the on-shell action is finite for the asymptotically plane wave spacetimes.

3.2.2 Variations of the action

In addition to being finite on-shell, we would like to see that $\delta S = 0$ for arbitrary variations about a solution of the equations of motion. The variation of the usual Einstein-Hilbert plus Gibbons-Hawking action would give a boundary term

$$\delta S_{EH+GH} = -\frac{1}{16\pi G}\int d^{d+1}x\sqrt{-h}\pi^{\alpha\beta}\delta h_{\alpha\beta}, \tag{3.53}$$

where $\pi^{\alpha\beta} = K^{\alpha\beta} - h^{\alpha\beta}K$. On the boundaries at x^+ = constant and x^- = constant, we have just this term. Therefore if we require $\delta h_{\alpha\beta} = 0$ on these boundaries, they will make no contribution to the variation of the action. This is a reasonable boundary condition if we think of these as fixed cut-offs; that is, if we keep the coordinate position of the cut-off fixed as we vary the metric and do not intend to eventually send the cut-off to infinity. This is certainly an appropriate approach for the $x^+ =$ constant boundary. In some cases, however, it is more appropriate to eventually remove the cut-off on x^-. For this purpose, we could imagine relaxing this condition to require only that $\delta h_{\alpha\beta}$ decays as we go to large x^-. Since the background metric is independent of x^-, any $\delta h_{\alpha\beta}$ which goes to zero at large x^- will produce a contribution to δS which vanishes as we remove the cut-off on x^-. Thus, there is no problem with the variation of the action involving these boundaries.

We turn to the contribution to the variation of the action from the boundary at $r =$ constant, where we only want to require that the variation $\delta h_{\alpha\beta}$ falls off as quickly as $g^{(1)}_{\alpha\beta}$. On the $r =$ constant boundary, we have the above boundary contribution from the Einstein-Hilbert plus Gibbons-Hawking action and we have the contribution coming from the variation of the new boundary term,

$$\delta S_{MM} = \frac{1}{8\pi G} \int d^{d+1}x \sqrt{-h} \left(-\frac{1}{2} \hat{K} h^{\alpha\beta} \delta h_{\alpha\beta} + \hat{K}_{\alpha\beta} \delta h^{\alpha\beta} + h^{\alpha\beta} \delta \hat{K}_{\alpha\beta} \right). \quad (3.54)$$

To determine $h^{\alpha\beta}\delta\hat{K}_{\alpha\beta}$, we need to use the analogue of (3.47) for variations to write

$$h^{\alpha\beta}\delta\hat{K}_{\alpha\beta} = M^{\gamma\delta} \left(\delta\mathcal{R}_{\gamma\delta} - \left(\hat{K}_{\alpha\beta}\hat{K}_{\gamma\delta} - \hat{K}_{\alpha\gamma}\hat{K}_{\beta\delta} \right) \delta h^{\alpha\beta} \right), \quad (3.55)$$

where $\delta\mathcal{R}_{\gamma\delta}$ is given in terms of $\delta h_{\alpha\beta}$ by

$$\delta\mathcal{R}_{\alpha\beta} = -\frac{1}{2} h^{\gamma\delta} D_\alpha D_\beta \delta h_{\gamma\delta} - \frac{1}{2} h^{\gamma\delta} D_\gamma D_\delta \delta h_{\alpha\beta} + h^{\gamma\delta} D_\gamma D_{(\alpha} \delta h_{\beta)\delta}. \quad (3.56)$$

The variation can be taken to respect our choice of gauge, so $\delta h_{-\alpha} = 0$. Thus, we only need to consider the variations δh_{++}, δh_{+i} and δh_{ij}.

Let us first consider just δh_{++} non-zero. The term in δS_{EH+GH} involving δh_{++} is trivially zero, as $\pi^{++} = 0$ with our choice of gauge. For the new boundary term,

$$\delta S_{MM} = \frac{1}{8\pi G} \int d^{d+1}x \sqrt{-h} \left(\hat{K}^{++} \delta h_{++} + h^{\alpha\beta} \delta \hat{K}_{\alpha\beta} \right). \quad (3.57)$$

This expression involves the full metric of the asymptotically plane wave solution we are considering. For each term, we will explicitly calculate the result for the leading non-zero contribution (coming from either $g^{(0)}$ or $g^{(1)}$). Higher-order terms are suppressed, so if the first term gives zero contribution to the variation of the action, we do not need to consider higher orders. In the first term in (3.57), solving for $\hat{K}^{(1)++}$ using (3.47) gives $\hat{K}^{(1)++} \sim \mathcal{O}(r^{-d-1})$, and $\delta h_{++} \sim \mathcal{O}(r^{4-d})$, so $\hat{K}^{++}\delta h_{++} \sim \mathcal{O}(r^{3-2d})$, and the first term in the integral is $\mathcal{O}(r^{2-d})$, which vanishes for $d \geq 3$. For the second term, we use (3.55), where there will be a zeroth-order contribution to the first term and a first-order contribution to the second term. From (3.56), we find that δh_{++} gives only δR_{++}, δR_{+-} and δR_{+i} non-zero. Using our previous calculation of the components $M^{(0)\alpha\beta}$, we then have

$$h^{\alpha\beta}\delta\hat{K}_{\alpha\beta} = M^{(0)+-}\delta R^{(0)}_{+-} - M^{(0)ab}\hat{K}^{(0)}_{ab}\hat{K}^{(1)++}\delta h_{++}. \tag{3.58}$$

Now $\delta\mathcal{R}^{(0)}_{+-} = -\frac{1}{2}h^{(0)+-}\partial_-\partial_-\delta h_{++} \sim \mathcal{O}(r^{-d})$, so the first term is $\mathcal{O}(r^{1-d})$. Together with the factor of $\sqrt{-h}$ in the integral, this would give a finite contribution to the variation. However, this leading-order term is a total derivative because $h^{(0)}_{\alpha\beta}$ is independent of x^-, so it makes no contribution. Higher-order contributions from this term would not be a total derivative, but they are suppressed by further powers of r so their contribution to the action vanishes in the large r limit. The second term is of the same form as the contribution considered above, giving a contribution $h^{\alpha\beta}\hat{K}_{\alpha\beta} \sim \mathcal{O}(r^{3-2d})$. Thus all the terms coming from δh_{++} vanish in the large r limit.

We now evaluate terms involving δh_{i+}. We find

$$\delta S_{EH+GH} = -\frac{1}{16\pi G}\int d^{d+1}x\sqrt{-h}\pi^{i+}\delta h_{i+}. \tag{3.59}$$

At linear order, $\pi^{i+} \sim h^{ij}\partial_- h_{jr} \sim \mathcal{O}(r^{-d-1})$, and $\delta h_{i+} \sim \mathcal{O}(r^{4-d})$, so this term is vanishing for $d \geq 3$. For the new boundary term,

$$\delta S_{MM} = \frac{1}{8\pi G}\int d^{d+1}x\sqrt{-h}\left(\hat{K}^{i+}\delta h_{i+} + h^{\alpha\beta}\delta\hat{K}_{\alpha\beta}\right), \tag{3.60}$$

and (3.47) gives $\hat{K}^{(1)i+} \sim \mathcal{O}(r^{-d-1})$, so the first term also vanishes for $d \geq 3$. In the second term, having just δh_{i+} gives us all components of $\delta\mathcal{R}_{\alpha\beta}$ except $\delta\mathcal{R}_{--}$ non-zero. Using (3.55) and our previous calculation of the components $M^{(0)\alpha\beta}$, we then have

$$h^{\alpha\beta}\delta\hat{K}_{\alpha\beta} = M^{(0)+-}\delta\mathcal{R}^{(0)}_{+-} + M^{(0)ij}\delta\mathcal{R}^{(0)}_{ij} - M^{(0)ij}\hat{K}^{(0)}_{ij}\hat{K}^{(1)m+}\delta h_{m+}. \tag{3.61}$$

We have $\delta\mathcal{R}^{(0)}_{+-} = \frac{1}{2}h^{(0)mj}D^{(0)}_j\partial_-\delta h_{+m} \sim \mathcal{O}(r^{-d})$, and $\delta\mathcal{R}^{(0)}_{ij} = \frac{1}{2}h^{(0)+-}\partial_- D^{(0)}_j\delta h_{i+} \sim \mathcal{O}(r^{2-d})$. Thus, both of the first two terms in $h^{\alpha\beta}\delta\hat{K}_{\alpha\beta}$ would make finite contributions to the variation of the action. However, as they involve ∂_-, they are total derivatives so they actually make zero contribution. As in the previous case, when we analysed terms involving δh_{++}, higher-order contributions from this term would not be a total derivative, but they are suppressed by further powers of r so their contribution to the action vanishes in the large r limit. The final term in $h^{\alpha\beta}\delta\hat{K}_{\alpha\beta}$ is of the same form as the contribution to the variation coming from $\hat{K}^{i+}\delta h_{i+}$, so it goes like $\mathcal{O}(r^{3-2d})$, and all the terms in the variation of the action coming from δh_{i+} vanish in the large r limit.

Finally, we consider terms involving δh_{ij}. We find

$$\delta S_{EH+GH} = -\frac{1}{16\pi G}\int d^{d+1}x\sqrt{-h}\pi^{ij}\delta h_{ij}, \tag{3.62}$$

and since $\pi^{ij} \sim \mathcal{O}(r^{-3})$ and $\delta h_{ij} \sim \mathcal{O}(r^{4-d})$, this gives an r^0 term which does not vanish in the large r limit. This term needs to be cancelled by a corresponding term coming from δS_{MM}. The latter is

$$\begin{aligned}\delta S_{MM} &= \frac{1}{8\pi G}\int d^{d+1}x\sqrt{-h}\left(-\frac{1}{2}\hat{K}h^{\alpha\beta}\delta h_{\alpha\beta} + \hat{K}_{\alpha\beta}\delta h^{\alpha\beta} + h^{\alpha\beta}\delta\hat{K}_{\alpha\beta}\right)\\ &= \frac{1}{8\pi G}\int d^{d+1}x\sqrt{-h}\left(\frac{1}{2}\hat{\pi}^{ij}\delta h_{ij} + \frac{1}{2}\hat{K}^{ij}\delta h_{ij} + h^{\alpha\beta}\delta\hat{K}_{\alpha\beta}\right),\end{aligned} \tag{3.63}$$

where $\hat{\pi}^{ij} = \hat{K}^{ij} - h^{ij}\hat{K}$. To zeroth order, $\hat{\pi}^{(0)ij} = \pi^{(0)ij}$, so the first term in (3.63) cancels the non-zero contribution from (3.62). However, the second term in (3.63) also has a non-zero leading order part so we need to see that this can be cancelled by a contribution from the final term. Considering the variation δh_{ij},

$$\begin{aligned}h^{\alpha\beta}\delta\hat{K}_{\alpha\beta} &= M^{(0)+-}\delta\mathcal{R}^{(0)}_{+-} + M^{(0)--}\delta\mathcal{R}^{(0)}_{--} + M^{(0)ij}\delta\mathcal{R}^{(0)}_{ij}\\ &\quad -M^{(0)ij}\left(\hat{K}^{(0)}_{ij}\hat{K}^{(0)}_{mn} - \hat{K}^{(0)}_{im}\hat{K}^{(0)}_{jn}\right)\delta h^{mn}.\end{aligned} \tag{3.64}$$

The terms involving $\delta\mathcal{R}_{\alpha\beta}$ give finite contributions which are total derivatives, as before. For the first two terms,

$$\delta\mathcal{R}^{(0)}_{+-} = h^{(0)ij}D^{(0)}_+\partial_-\delta h_{ij} \sim \mathcal{O}(r^{-d}), \quad \delta\mathcal{R}^{(0)}_{--} = h^{(0)ij}\partial_-\partial_-\delta h_{ij} \sim \mathcal{O}(r^{2-d}), \tag{3.65}$$

and these are total derivatives because $h^{(0)}_{\alpha\beta}$ is independent of x^-. For the other term, $\delta\mathcal{R}^{(0)}_{ij} \sim \mathcal{O}(r^{2-d})$ involves covariant derivatives with respect to the unit metric on

S^{d-1}, $\hat{h}_{ij}$, and this term is a total derivative because the only θ_i dependence in the terms multiplying $\delta\mathcal{R}^{(0)}_{ij}$ is through the covariantly constant metric $\hat{h}_{ij}$. As in the previous two cases, higher-order contributions from these terms would not be total derivatives, but they are suppressed by further powers of r, so their contribution to the action vanishes in the large r limit. We are then left with evaluating the last term in (3.64). Using $\hat{K}^{(0)}_{ij} = r\hat{h}_{ij}$ and $M^{(0)ij} = \frac{1}{2(d-2)r}\hat{h}^{ij}$,

$$h^{\alpha\beta}\delta\hat{K}_{\alpha\beta} \to -M^{(0)ij}\left(\hat{K}^{(0)}_{ij}\hat{K}^{(0)}_{mn} - \hat{K}^{(0)}_{im}\hat{K}^{(0)}_{jn}\right)\delta h^{mn} = -\frac{1}{2}r\hat{h}^{ij}\delta h_{ij} = -\frac{1}{2}\hat{K}^{(0)ij}\delta h_{ij}. \tag{3.66}$$

This will cancel with the leading order part of the second term in (3.63), leaving us with no finite contributions to the variation of the action in the large r limit. Thus, this action gives a well-defined variational principle for our class of asymptotically plane wave spacetimes. Notice that this computation worked in a very similar way to the asymptotically flat case, reviewed in chapter 2.

In this chapter, we have given a definition of asymptotically plane wave spacetimes which is consistent with the known exact solutions. Using this definition, we then constructed a well-behaved action principle for asymptotically plane wave solutions. We discuss the interpretation of these results in chapter 5. In the following chapter we consider the construction of black holes and black strings in plane wave spacetimes.

Chapter 4

Black holes and black strings in plane waves

From the point of view of holography, it is clearly interesting to construct asymptotically plane wave black holes and black strings and look for interpretations of these spacetimes in field theory terms. Some exact solutions describing black strings in plane wave backgrounds have been obtained by applying solution-generating transformations [21–25]. A review of this work and the structure of horizons and plane waves can be found in [26]. However, such methods are available only in special cases and a solution describing the simplest situation, a regular black hole or black string in a vacuum plane wave background, has not been obtained by these methods. Constructing solutions by directly solving the equations of motion is challenging.

In this chapter, we adopt the method of matched asymptotic expansions to find approximate stationary solutions when the horizon size r_+ of the black hole or black string is small compared to the curvature scale μ^{-1} of the plane wave. This gives a separation of scales which can be exploited to solve the equations of motion in the linearised approximation in separate regions, matching the solutions in an overlap region. Such methods have been successfully applied to the construction of caged black holes in Kaluza-Klein theory [58] and to construct black ring solutions in more than five spacetime dimensions [56] and in anti-de Sitter space [61]. These ideas have been further developed in [51,52] as reviewed in chapter 2, where general extended black objects wrapping a submanifold in an arbitrary spacetime have been considered at leading order in the region far from the black object.

We proceed in a similar way to these previous examples, first finding the metric far from the source (for $r \gg r_+$) by studying the linearised approximation to gravity with an appropriate delta-function source. The wave equation in the plane wave background is rather complicated, so we focus on solving this problem in an intermediate region $r_+ \ll r \ll \mu^{-1}$ where the deviations from flat space due to both the source and the plane wave are small.

Solving the equation in this regime, we find that simple dimensional analysis indicates that the solutions will violate the asymptotic boundary conditions proposed in the previous chapter as a definition of asymptotically plane wave spacetimes. In fact, the perturbation due to the delta-function source becomes large relative to the background metric at large distances. An explicit analysis in four and five dimensions shows that the terms violating these boundary conditions are indeed non-zero. Thus, these solutions appear not to be asymptotically plane wave; we will refer to them as black holes or black strings in plane wave backgrounds. The fact that the linearised solutions for a delta-function source violate the asymptotic boundary conditions suggests that as in AdS_2 [62] and the Kerr/CFT correspondence [63–65], the space of asymptotically plane wave spacetimes may be highly restricted.

We then obtain the near horizon metric in the region $r \ll \mu^{-1}$ by solving the linearised Einstein equations on the background of the black object, treating the plane wave as a perturbation. For a black hole, we find that there is no linearised solution which is regular on the horizon. For the black string, we obtain a regular solution in the near region, and verify that it matches on to the solution in the intermediate region.

When solving the equations, we focus on vacuum plane waves in the lowest possible dimension, for simplicity, but the method of matched asymptotic expansion is more general and a similar analysis could be applied to construct black string solutions in any plane wave background of interest in arbitrary dimensions. We will remark on the extension to other waves and higher dimensions at appropriate points in the calculation. The calculation in the region $r \gg r_+$ is described in section 4.1, and the calculation in the region $r \ll \mu^{-1}$ is described in section 4.2.

4.1 Linearised solutions on a plane wave background

We want to construct solutions corresponding to a black hole or black string of radius r_+ in a general vacuum plane wave background in $D = d + 2$ dimensions

$$ds^2 = -dt^2 + dz^2 - \mu_{IJ}(t+z)x^I x^J (dt+dz)^2 + \delta_{IJ} dx^I dx^J, \tag{4.1}$$

where x^I, $I = 1, \ldots d$ are Cartesian coordinates on the transverse space. We will work in the parameter range $r_+ \ll \mu^{-1}$, where we take the matrix $\mu_{IJ}(t+z)$ characterising the wave to have a single characteristic scale μ for simplicity. The black object can then be treated as a small perturbation of the plane wave background for $r \gg r_+$. In this region of the spacetime, the problem of constructing a black hole or black string solution thus reduces to solving the linearised Einstein equations for a suitable source $T_{\mu\nu}$. In transverse gauge, the linearised equations are[1]

$$\Box \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}. \tag{4.2}$$

For a pointlike source, the relevant stress tensor is simply $T_{\mu\nu} = M V_\mu V_\nu \delta(x^\mu - x^\mu(\tau))$, where $x^\mu(\tau)$ is the particle's trajectory, $V^\mu = dx^\mu/d\tau$ is the tangent to this trajectory, and M is the proper mass. For a black string solution, the stress tensor can be determined by linearising the vacuum black string solution in $d+2$ dimensions,

$$ds^2 = -\left(1 - \frac{r_+^{d-2}}{r^{d-2}}\right) dt^2 + dz^2 + \left(1 - \frac{r_+^{d-2}}{r^{d-2}}\right)^{-1} dr^2 + r^2 d\Omega_{d-1}^2, \tag{4.3}$$

which gives the stress tensor in these coordinates as

$$T_{tt} = \frac{(d-1)r_+^{d-2}}{16\pi G}\delta^d(r), \quad T_{zz} = -\frac{r_+^{d-2}}{16\pi G}\delta^d(r). \tag{4.4}$$

The source is fixed to follow some appropriate trajectory in the plane wave background. For a pointlike source, the appropriate trajectory is a timelike geodesic of the background spacetime. To obtain a stationary black hole solution, we should require this geodesic to be the orbit of a timelike Killing vector in the spacetime.

[1] Note that, in our actual calculations we will not assume the transverse traceless gauge as it is more convenient to use the gauge freedom to fix particular components of the perturbation.

This forces us to restrict to plane waves with a constant matrix $\mu_{IJ}(t+z) = \mu_{IJ}$, so that the solution has a timelike Killing vector, and to consider the geodesic $z = 0$,[2] $x^I = 0$, which is the unique geodesic trajectory which is also an orbit of the Killing vector. The appropriate source is then $T_{tt} = M\delta(z)\delta^d(x^I)$, and the size of the black hole is $r_+^{d-1} \propto M$.

For the black string, as reviewed in chapter 2, the equation of motion for a probe string is [57]

$$K_{\mu\nu}{}^{\rho} T^{\mu\nu} = 0, \tag{4.5}$$

where $K_{\mu\nu}{}^{\rho}$ is defined in (2.81) and $T_{\mu\nu}$ is the stress tensor of the source. We will consider embedding the black string along the submanifold $x^I = 0$, which has $K_{\mu\nu}{}^{\rho} = 0$. As a result, there is no constraint on the form of the stress tensor. As for the black hole, we need to restrict to constant $\mu_{IJ}(t+z) = \mu_{IJ}$ so that this submanifold is an orbit of the spacetime isometries, so that we can expect to obtain a stationary uniform black string solution. We can then use boosts in the $t-z$ plane to choose the black string solution to be in its rest frame, setting $T_{tz} = 0$, without loss of generality. The appropriate source is thus (4.4). We want to find a uniform black string solution, so the components of the stress tensor are assumed to be constants along the worldvolume. The blackfold equations of [52] are hence trivially satisfied.

In each case, the problem thus reduces in principle to solving (4.2) on the plane wave background for an appropriate source. However, we do not have the Green's function for this differential equation in closed form, so we will content ourselves with studying this problem in the intermediate region $r_+ \ll r \ll \mu^{-1}$, where we can treat the plane wave itself as a small perturbation of flat space, and obtain the solution of (4.2) order by order in $\mu^2 r^2$.

4.1.1 Dimensional analysis

We first discuss the perturbation in general dimensions using a simple dimensional analysis argument. For the case of a point source, we find it convenient to rewrite

[2]We can make this choice without loss of generality by translation invariance in z.

the metric in spherical polar coordinates, introducing a radial coordinate

$$r^2 = z^2 + \delta_{IJ} x^I x^J, \tag{4.6}$$

and defining coordinates θ^i on the S^d at constant r. As in [66], we use a, b to denote coordinates on the two dimensional space spanned by r, t. By dimensional analysis, the form of the perturbation to first order in M and in μ^2 will be

$$\begin{aligned} h_{ab} &= \frac{M}{r^{D-3}} h^{(0)}_{ab} + \frac{M\mu^2}{r^{D-5}} h^{(1)}_{ab}(\theta^i), \\ h_{ai} &= \frac{M}{r^{D-4}} h^{(0)}_{ai} + \frac{M\mu^2}{r^{D-6}} h^{(1)}_{ai}(\theta^i), \\ h_{ij} &= \frac{M}{r^{D-5}} h^{(0)}_{ij} + \frac{M\mu^2}{r^{D-7}} h^{(1)}_{ij}(\theta^i), \end{aligned} \tag{4.7}$$

where $h^{(0)}_{\mu\nu}$ and $h^{(1)}_{\mu\nu}$ are dimensionless functions[3] depending only on the angles θ^i. In fact, since the spherical symmetry is only broken by the plane wave, $h^{(0)}_{ab}$ are constants, and the component on the sphere $h^{(0)}_{ij}$ will be proportional to the metric on the sphere γ_{ij}. We will always work in a gauge where $h^{(0)}_{ij}$ vanishes. Each addition of an i index raises the power of r by one because the coordinates on the sphere are written in terms of dimensionless angles.

This simple dimensional analysis already indicates a significant issue: this perturbation does not satisfy the boundary conditions introduced in chapter 3. There, it was assumed that components of the perturbation in the directions transverse to the wave would fall off at least as $1/r^{D-4}$ (corresponding to $h_{ij} \propto 1/r^{D-6}$, because of the extra factors of r from writing the perturbation in polar coordinates), characteristic of a localised source in a flat spacetime. However, we find that the term resulting from the interaction with the wave must grow more quickly than this on dimensional grounds. When we think of the plane wave as a perturbation around flat space, the plane wave background introduces corrections which grow more quickly with r than the original leading-order response.

Similarly, when we consider a black string source, it is convenient to write the metric in the directions transverse to the wave in polar coordinates, introducing a

[3]Note that, in chapter 3, $g^{(0)}$ denoted the zeroth order part of the metric, and $g^{(1)}$ denoted the perturbation. In this section, $h^{(0)}$ and $h^{(1)}$ denote the part of the perturbation of zeroth order and first order in μ^2 respectively.

radial coordinate

$$r^2 = \delta_{IJ} x^I x^J, \tag{4.8}$$

and introducing coordinates θ^i on the S^{d-1} at constant r, z. In the string source case, a, b will denote coordinates in the three dimensional space spanned by t, r, z. Then, to leading order in r_+ and μ^2, the perturbation sourced by a black string will have the form

$$\begin{aligned} h_{ab} &= \frac{r_+^{D-4}}{r^{D-4}} h_{ab}^{(0)} + \frac{r_+^{D-4}\mu^2}{r^{D-6}} h_{ab}^{(1)}(\theta^i), \\ h_{ai} &= \frac{r_+^{D-4}}{r^{D-5}} h_{ai}^{(0)} + \frac{r_+^{D-4}\mu^2}{r^{D-7}} h_{ai}^{(1)}(\theta^i), \\ h_{ij} &= \frac{r_+^{D-4}}{r^{D-6}} h_{ij}^{(0)} + \frac{r_+^{D-4}\mu^2}{r^{D-8}} h_{ij}^{(1)}(\theta^i), \end{aligned} \tag{4.9}$$

where $h_{ab}^{(0)}$ are constants and $h_{\mu\nu}^{(1)}$ are functions of the coordinates θ^i on the sphere only. Thus, as in the black hole case, the perturbation does not satisfy the boundary conditions introduced in chapter 3.

This is a significant issue because at least in low spacetime dimensions, the resulting perturbation actually grows more quickly with r than the background metric. In $D = 4$ for the black hole and $D = 5$ for the black string, the perturbation of the angular metric h_{ij} has a contribution that goes like $r_+\mu^2 r^3$, which is growing faster than the background metric on the sphere which goes like r^2. Furthermore, what we have discussed so far is just the leading order correction in μ^2. Higher order terms in μ^2 will come with additional powers of r. One might hope that when the problem is solved to all orders in μ^2, the resulting behaviour could be under better control, but it is hard to see how such a cancellation between different orders could be arranged. We will see later, in a particular example, that this does not occur.

Thus, we are faced with the odd situation that the linearised field of a point source may become more important than the background, signalling a breakdown of the linearised approximation far from the source itself. Thus, the solutions we construct should not be thought of as "asymptotically plane wave" black holes/strings, as the metric in the asymptotic regime is not close to the original plane wave metric. As a result, the analysis of chapter 3 will not apply to these spacetimes and, in particular, we do not expect that they will have finite action with respect to the action principle discussed there.

One might hope that the terms which violate those boundary conditions which are allowed by dimensional analysis may actually vanish. This hope would be encouraged by the fact that the specific examples of plane wave black strings constructed in [21–23] satisfied the asymptotic boundary conditions of chapter 3. However, the examples of [21–23] are special cases in that they are constructed by the Garfinkle-Vachaspati solution-generating transformation [67] and, by construction, can only differ from the seed solution in the metric components along the null direction. By contrast, the solution constructed in [25], which was obtained by a different method, has precisely the kinds of corrections that are predicted by this dimensional analysis argument.

In the next two subsections, we will consider the solution of the linearised equations of motion for the perturbation in detail for the lowest possible dimension for black hole and black string sources, and see in these particular examples that the terms which violate our asymptotic boundary conditions do indeed appear. Thus, the approximate solutions we obtain for black holes and black strings in plane wave backgrounds are not asymptotically plane wave in the sense defined in chapter 3. Given the above dimensional analysis arguments and the results below, it seems reasonable to expect that this is the generic case, so that the space of asymptotically plane wave solutions is very limited. We will comment on this in chapter 5.

4.1.2 Black hole

Let us consider the perturbation sourced by a point source in the lowest possible dimension, $D = 4$, in detail. By a choice of coordinates, the most general four dimensional plane wave can be written as

$$ds^2_{wave} = -dt^2 + dx^2 + dy^2 + dz^2 - \mu^2(x^2 - y^2)(dt + dz)^2. \tag{4.10}$$

We rewrite this in spherical polars by defining

$$z = r\cos\theta, \quad x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \tag{4.11}$$

so

$$\begin{aligned} ds^2_{wave} &= -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \\ &\quad -\mu^2 r^2 \sin^2\theta(\cos^2\phi - \sin^2\phi)(dt + \cos\theta dr - r\sin\theta d\theta)^2. \end{aligned} \tag{4.12}$$

As in the previous subsection, we can use dimensional analysis to fix the dependence of the perturbation on r. We can, in fact, determine the perturbation to zeroth order in μ^2 by simply linearising the Schwarzschild solution, which gives $h_{tt} = h_{rr} = \frac{2M}{r}$. This satisfies the linearised equations of motion for a delta-function point source, but not in the transverse traceless gauge which was assumed in writing (4.2). In what follows, we will not assume the transverse traceless gauge as it is more convenient to use the gauge freedom to fix some components of the perturbation.

For the terms of first order in μ^2, we can use the freedom to choose a gauge for the perturbation to set $h^{(1)}_{a\phi}$ and $h^{(1)}_{\theta\phi}$ to zero. Note that we have four gauge degrees of freedom but have only eliminated three components, hence we have one remaining degree of freedom which we will use later. We then make an ansatz for the ϕ dependence of the perturbation, and write our perturbation as

$$\begin{aligned} h_{ab} &= \frac{M}{r}h^{(0)}_{ab} + M\mu^2 r(\cos^2\phi - \sin^2\phi)h^{(1)}_{ab}(\theta), \\ h_{a\theta} &= M\mu^2 r^2(\cos^2\phi - \sin^2\phi)h^{(1)}_{a\theta}(\theta), \\ h_{ij} &= M\mu^2 r^3(\cos^2\phi - \sin^2\phi)h^{(1)}_{ij}(\theta), \end{aligned} \tag{4.13}$$

where the non-zero components of $h^{(0)}_{ab}$ are $h^{(0)}_{tt} = 2, h^{(0)}_{rr} = 2$, and the non-zero components of $h^{(1)}_{\mu\nu}(\theta)$ are $h^{(1)}_{tt}(\theta)$, $h^{(1)}_{tr}(\theta)$, $h^{(1)}_{t\theta}(\theta)$, $h^{(1)}_{rr}(\theta)$, $h^{(1)}_{r\theta}(\theta)$, $h^{(1)}_{\theta\theta}(\theta)$ and $h^{(1)}_{\phi\phi}(\theta)$.

We now want to substitute this ansatz into the linearised Einstein equations and solve for the undetermined functions $h^{(1)}_{\mu\nu}(\theta)$, requiring regularity on the sphere. In an arbitrary gauge, the linearised Einstein equations for $r \neq 0$ are

$$R^{(1)}_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}(\nabla_\rho\nabla_\mu h_{\nu\sigma} + \nabla_\rho\nabla_\nu h_{\mu\sigma} - \nabla_\mu\nabla_\nu h_{\rho\sigma} - \nabla_\rho\nabla_\sigma h_{\mu\nu}) = 0. \tag{4.14}$$

Substituting our ansatz, these equations become (where primes denote derivatives with respect to θ)

$$\begin{aligned} &-\sin^2\theta h^{(1)\prime\prime}_{tt}(\theta) - \sin\theta\cos\theta h^{(1)\prime}_{tt}(\theta) + 2(\cos^2\theta + 1)h^{(1)}_{tt}(\theta) \\ &\qquad -6\cos^6\theta - 2\cos^4\theta + 22\cos^2\theta - 14 = 0, \end{aligned} \tag{4.15}$$

$$-\sin^2\theta h_{tr}^{(1)\prime\prime}(\theta)-\sin\theta\cos\theta h_{tr}^{(1)\prime}(\theta)+2\sin^2\theta h_{t\theta}^{(1)\prime}(\theta)+4h_{t\theta}^{(1)}(\theta) \quad (4.16)$$
$$+2\sin\theta\cos\theta h_{t\theta}^{(1)}(\theta)-8\cos^5\theta+16\cos^3\theta-8\cos\theta=0,$$

$$-\sin^2\theta h_{rr}^{(1)\prime\prime}(\theta)-\sin\theta\cos\theta h_{rr}^{(1)\prime}(\theta)+4\sin^2\theta h_{r\theta}^{(1)\prime}(\theta)+2(3-\cos^2\theta)h_{rr}^{(1)}(\theta) \quad (4.17)$$
$$-2\sin^2\theta h_{\theta\theta}^{(1)}(\theta)-2\sin^2\theta h_{\phi\phi}^{(1)}(\theta)+10\cos^6\theta-26\cos^4\theta+22\cos^2\theta-6=0,$$

$$\sin^2\theta h_{rr}^{(1)\prime}(\theta)-\sin^2\theta h_{\phi\phi}^{(1)\prime}(\theta)-\sin\theta\cos\theta h_{\phi\phi}^{(1)}(\theta)+\sin\theta\cos\theta h_{\theta\theta}^{(1)}(\theta) \quad (4.18)$$
$$+2(\cos^2\theta+1)h_{r\theta}^{(1)}(\theta)-4\sin\theta(\cos^5\theta-2\cos^3\theta+\cos\theta)=0,$$

$$\sin^2\theta h_{tr}^{(1)\prime}(\theta)+2(\cos^2\theta+1)h_{t\theta}^{(1)}(\theta)+2\sin\theta(\cos^4\theta-4\cos^2\theta+3)=0, \quad (4.19)$$

$$h_{r\theta}^{(1)\prime}(\theta)-\cot\theta h_{r\theta}^{(1)}(\theta)-h_{\theta\theta}^{(1)}(\theta)+h_{rr}^{(1)}(\theta)-\cos^4\theta+2\cos^2\theta-1=0, \quad (4.20)$$

$$h_{t\theta}^{(1)\prime}(\theta)-\cot\theta h_{t\theta}^{(1)}(\theta)+h_{tr}^{(1)}(\theta)+2\cos\theta(1-\cos^2\theta)=0, \quad (4.21)$$

$$-\sin\theta(h_{rr}^{(1)\prime}(\theta)-h_{tt}^{(1)\prime}(\theta))+\cos\theta(h_{rr}^{(1)}(\theta)-h_{tt}^{(1)}(\theta))+2\sin\theta h_{r\theta}^{(1)}(\theta)+2\cos\theta(\cos^4\theta-1)=0, \quad (4.22)$$

$$-\sin^2\theta h_{\phi\phi}^{(1)\prime\prime}(\theta)+\sin^2\theta h_{tt}^{(1)\prime\prime}(\theta)+\sin\theta\cos\theta(h_{\theta\theta}^{(1)\prime}(\theta)-2h_{\phi\phi}^{(1)\prime}(\theta))$$
$$-\sin^2\theta h_{rr}^{(1)\prime\prime}(\theta)+6\sin^2\theta h_{r\theta}^{(1)\prime}(\theta)-5\sin^2\theta h_{\theta\theta}^{(1)}(\theta)+3\sin^2\theta h_{rr}^{(1)}(\theta) \quad (4.23)$$
$$-\sin^2\theta h_{\phi\phi}^{(1)}(\theta)+\sin^2\theta h_{tt}^{(1)}(\theta)+2\cos^2\theta(\cos^4\theta+3\cos^2\theta-5)+2=0,$$

$$\sin^2\theta h_{\phi\phi}^{(1)\prime\prime}(\theta)+\sin\theta\cos\theta(h_{rr}^{(1)\prime}(\theta)-h_{tt}^{(1)\prime}(\theta)+2h_{\phi\phi}^{(1)\prime}(\theta)-h_{\theta\theta}^{(1)\prime}(\theta))$$
$$+\cos^2\theta(3h_{rr}^{(1)}(\theta)-3h_{\theta\theta}^{(1)}(\theta)+h_{tt}^{(1)}(\theta))-2\sin^2\theta h_{r\theta}^{(1)\prime}(\theta)+3\sin^2\theta h_{\phi\phi}^{(1)}(\theta) \quad (4.24)$$
$$-7h_{rr}^{(1)}(\theta)+3h_{tt}^{(1)}(\theta)+2\cos^2\theta(3\cos^4\theta-7\cos^2\theta+9)-10=0.$$

We have a system of ten equations in seven unknown functions (in fact, there will be only six unknown functions once we have made use of the one remaining degree of gauge freedom) so it seems that our system is over-constrained. We find, however, that there are only six independent equations and, hence, that our system is in fact well-defined. It is convenient to subtract a multiple of (4.20) from (4.17) to simplify it to

$$-\sin^2\theta h_{rr}^{(1)\prime\prime}(\theta)-\sin\theta\cos\theta h_{rr}^{(1)\prime}(\theta)+2(1+\cos^2\theta)h_{rr}^{(1)}(\theta)+8\sin\theta\cos\theta h_{r\theta}^{(1)}(\theta) \quad (4.25)$$
$$+2\sin^2\theta h_{\theta\theta}^{(1)}(\theta)-2\sin^2\theta h_{\phi\phi}^{(1)}(\theta)+10\cos^2\theta-14\cos^4\theta+6\cos^6\theta-2=0.$$

By using combinations of (4.20), (4.22), (4.25) and their derivatives it is possible to reduce (4.15) to an algebraic equation

$$\begin{aligned} 2\sin\theta\cos\theta h^{(1)}_{r\theta}(\theta) + 2\sin^2\theta h^{(1)}_{\phi\phi}(\theta) + 3h^{(1)}_{tt}(\theta) - 5h^{(1)}_{rr}(\theta) \\ +2\cos^2\theta h^{(1)}_{rr}(\theta) + 2(-\cos^6\theta + 4\cos^4\theta + \cos^2\theta - 4) = 0. \end{aligned} \tag{4.26}$$

We find we can write (4.16), (4.18), (4.23) and (4.24) as linear combinations of (4.19), (4.20), (4.21), (4.22), (4.25) and (4.26) and hence that these equations are not independent. We now see that a convenient choice of gauge is one in which $h^{(1)}_{r\theta}(\theta) = 0$.

It is also useful to define new functions

$$c_{tt}(\theta) = h^{(1)}_{tt}(\theta) - h^{(1)}_{rr}(\theta) \tag{4.27}$$

$$c_{\phi\phi}(\theta) = h^{(1)}_{\phi\phi}(\theta) - h^{(1)}_{rr}(\theta) \tag{4.28}$$

$$c_{\theta\theta}(\theta) = h^{(1)}_{\theta\theta}(\theta) - h^{(1)}_{rr}(\theta). \tag{4.29}$$

We can now solve (4.22) for $c_{tt}(\theta)$, (4.26) for $c_{\phi\phi}(\theta)$, and (4.20) for $c_{\theta\theta}(\theta)$. We find the regular solutions are

$$c_{tt}(\theta) = 4\sin^2\theta - \frac{2}{3}\sin^4\theta, \tag{4.30}$$

$$c_{\phi\phi}(\theta) = -\sin^4\theta, \tag{4.31}$$

$$c_{\theta\theta}(\theta) = -\sin^4\theta. \tag{4.32}$$

Equations (4.21) and (4.19) are a set of coupled first order equations in two variables. We can therefore reduce this set to a single second order equation in one variable. We find the regular solutions of this system are

$$h^{(1)}_{tr}(\theta) = 2\sin^2\theta\cos\theta, \tag{4.33}$$

$$h^{(1)}_{t\theta}(\theta) = -2\sin^3\theta. \tag{4.34}$$

Finally we solve (4.25) for $h^{(1)}_{rr}(\theta)$. The regular solution is

$$h^{(1)}_{rr}(\theta) = \frac{1}{3}\sin^4\theta. \tag{4.35}$$

Using (4.27), (4.28) and (4.29) we now also have solutions for $h^{(1)}_{tt}(\theta), h^{(1)}_{\phi\phi}(\theta)$ and $h^{(1)}_{\theta\theta}(\theta)$.

Thus, in this gauge, the solution which is regular on the sphere is

$$\begin{aligned} h_{\mu\nu}dx^\mu dx^\nu &= \frac{2M}{r}dt^2 + \frac{2M}{r}dr^2 + M\mu^2 r\sin^2\theta(\cos^2\phi - \sin^2\phi)\times \\ & \left[(4-\frac{1}{3}\sin^2\theta)dt^2 + 4\cos\theta dtdr - 4r\sin\theta dtd\theta \right. \\ & \left. +\frac{1}{3}\sin^2\theta dr^2 - \frac{2}{3}r^2\sin^2\theta(d\theta^2+\sin^2\theta d\phi^2)\right]. \end{aligned} \tag{4.36}$$

We note that, as stated earlier, the regular solution for the terms of first order in μ^2 has non-zero components on the sphere which grow faster than the background metric on the sphere. These solutions are hence not asymptotically plane wave. While this leading order term would not grow faster than the background metric in higher dimensions, higher order terms in μ^2 will, in principle, do so.

4.1.3 Black string

We now consider the perturbation for a black string source in the lowest possible dimension, which is $D = 5$ for the black string. The most general plane wave solution in five dimensions is

$$ds^2_{wave} = -dt^2 + dx^2 + dy^2 + dz^2 + dw^2 - \mu^2(\alpha(x^2+y^2-2w^2) + \beta(x^2-y^2))(dt+dz)^2; \tag{4.37}$$

note that there is a two-parameter family of plane wave solutions here. We rewrite this in spherical polars in the directions transverse to the wave by writing

$$x = r\sin\theta\cos\phi, \quad y = r\sin\theta\sin\phi, \quad w = r\cos\theta, \tag{4.38}$$

so

$$\begin{aligned} ds^2_{wave} &= -dt^2 + dz^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \\ & -\mu^2(\alpha r^2(1-3\cos^2\theta) + \beta r^2\sin^2\theta(\cos^2\phi - \sin^2\phi))(dt+dz)^2. \end{aligned} \tag{4.39}$$

As in the previous subsection, we can determine the perturbation to zeroth order in μ^2 by simply linearising the Schwarzschild black string solution (4.3), which gives $h_{tt} = h_{rr} = \frac{2M}{r}$. We will again find it convenient to fix the gauge by choosing some components of the perturbation to vanish at each order in μ^2. We note that the background has an invariance under $t \to -t$, $z \to -z$ and a translational invariance in t and z which is not broken by the source, so the $h_{t\mu}$, $h_{z\mu}$ components for $\mu \neq t, z$ will automatically vanish.

At first order in μ^2, we can treat the two different components of the plane wave separately. We first consider the first-order terms in the perturbation associated to α. Let us therefore set $\alpha = 1$ and $\beta = 0$ in the plane wave background (4.37). There is then a translation invariance in ϕ and a symmetry under $\phi \to -\phi$, which imply that $h_{\phi\mu}$ vanish for $\mu \neq \phi$. We will make a choice of gauge to set $h^{(1)}_{rr}$ and $h^{(1)}_{r\theta}$ to zero. This gauge choice proves to be convenient for comparing to the solution in the near region to be obtained later. The form of the perturbation is then

$$\begin{aligned} h_{ab} &= \frac{M}{r} h^{(0)}_{ab} + M\mu^2 r h^{(1)}_{ab}(\theta), \\ h_{a\theta} &= M\mu^2 r^2 h^{(1)}_{a\theta}(\theta), \\ h_{ij} &= M\mu^2 r^3 h^{(1)}_{ij}(\theta), \end{aligned} \tag{4.40}$$

where the non-zero components of $h^{(0)}_{ab}$ are $h^{(0)}_{tt} = 2, h^{(0)}_{rr} = 2$, and the non-zero components of $h^{(1)}_{\mu\nu}(\theta)$ are $h^{(1)}_{tt}(\theta)$, $h^{(1)}_{tz}(\theta)$, $h^{(1)}_{zz}(\theta)$, $h^{(1)}_{\theta\theta}(\theta)$ and $h^{(1)}_{\phi\phi}(\theta)$.

We now want to substitute this ansatz into the linearised Einstein equations and solve for the undetermined functions $h^{(1)}_{\mu\nu}(\theta)$, requiring regularity on the sphere. In an arbitrary gauge, the linearised Einstein equations for $r \neq 0$ are

$$R^{(1)}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} (\nabla_\rho \nabla_\mu h_{\nu\sigma} + \nabla_\rho \nabla_\nu h_{\mu\sigma} - \nabla_\mu \nabla_\nu h_{\rho\sigma} - \nabla_\rho \nabla_\sigma h_{\mu\nu}) = 0. \tag{4.41}$$

Substituting our ansatz, these equations become

$$\partial^2_\theta h^{(1)}_{tt}(\theta) + \cot\theta \partial_\theta h^{(1)}_{tt}(\theta) + 2h^{(1)}_{tt}(\theta) + 16(1 - 3\cos^2\theta) = 0, \tag{4.42}$$

$$\partial^2_\theta h^{(1)}_{tz}(\theta) + \cot\theta \partial_\theta h^{(1)}_{tz}(\theta) + 2h^{(1)}_{tz}(\theta) + 12(1 - 3\cos^2\theta) = 0, \tag{4.43}$$

$$\partial^2_\theta h^{(1)}_{zz}(\theta) + \cot\theta \partial_\theta h^{(1)}_{zz}(\theta) + 2h^{(1)}_{zz}(\theta) + 8(1 - 3\cos^2\theta) = 0, \tag{4.44}$$

$$h^{(1)}_{\theta\theta}(\theta) + h^{(1)}_{\phi\phi}(\theta) + 2(1 - 3\cos^2\theta) = 0, \tag{4.45}$$

$$\tan\theta \partial_\theta h^{(1)}_{\phi\phi}(\theta) - h^{(1)}_{\theta\theta}(\theta) + h^{(1)}_{\phi\phi}(\theta) + 6\sin^2\theta = 0, \tag{4.46}$$

$$\begin{aligned} &\partial^2_\theta h^{(1)}_{tt}(\theta) - \partial^2_\theta h^{(1)}_{\phi\phi}(\theta) - \partial^2_\theta h^{(1)}_{zz}(\theta) + \cot\theta(\partial_\theta h^{(1)}_{\theta\theta}(\theta) - 2\partial_\theta h^{(1)}_{\phi\phi}(\theta))) \\ &+ h^{(1)}_{tt}(\theta) - h^{(1)}_{zz}(\theta) - 5h^{(1)}_{\theta\theta}(\theta) - h^{(1)}_{\phi\phi}(\theta) + 12\sin^2\theta - 2(1 - 3\cos^2\theta) = 0, \end{aligned} \tag{4.47}$$

$$\partial_\theta^2 h^{(1)}_{\phi\phi}(\theta) + \cot\theta(\partial_\theta h^{(1)}_{zz}(\theta) + \partial_\theta h^{(1)}_{\phi\phi}(\theta) - \partial_\theta h^{(1)}_{\theta\theta}(\theta) - \partial_\theta h^{(1)}_{tt}(\theta)) \\ +3h^{(1)}_{\theta\theta}(\theta) + 3h^{(1)}_{\phi\phi}(\theta) - h^{(1)}_{tt}(\theta) + h^{(1)}_{zz}(\theta) + 2(1-3\cos^2\theta) = 0. \tag{4.48}$$

We first solve equations (4.42),(4.43) and (4.44) for $h^{(1)}_{tt}(\theta), h^{(1)}_{zt}(\theta)$ and $h^{(1)}_{zz}(\theta)$ respectively. We then solve for $h^{(1)}_{\theta\theta}(\theta)$ and $h^{(1)}_{\phi\phi}(\theta)$ using equations (4.45) and (4.46). It is easy to verify that these solutions satisfy (4.47) and (4.48). Keeping only the regular part of the solution, we find

$$h^{(1)}_{tt}(\theta) = 4(1-3\cos^2\theta), \quad h^{(1)}_{tz}(\theta) = 3(1-3\cos^2\theta), \quad h^{(1)}_{zz}(\theta) = 2(1-3\cos^2\theta), \tag{4.49}$$

$$h^{(1)}_{\theta\theta}(\theta) = -(1-3\cos^2\theta), \quad h^{(1)}_{\phi\phi}(\theta) = -\sin^2\theta(1-3\cos^2\theta). \tag{4.50}$$

As in the black hole case, we see that terms that grow faster than the background metric at large r do indeed occur.

It turns out that, for this background, the linearised equations of motion can be solved exactly by including one further term at next order in μ^2. If we take

$$\begin{aligned} h_{ab} &= \frac{M}{r}h^{(0)}_{ab} + M\mu^2 r h^{(1)}_{ab}(\theta) + M\mu^4 r^3 h^{(2)}_{ab}(\theta), \\ h_{a\theta} &= M\mu^2 r^2 h^{(1)}_{a\theta}(\theta), \\ h_{ij} &= M\mu^2 r^3 h^{(1)}_{ij}(\theta), \end{aligned} \tag{4.51}$$

with $h^{(0)}_{\mu\nu}$ and $h^{(1)}_{\mu\nu}$ as given above, and

$$h^{(2)}_{tt} = h^{(2)}_{tz} = h^{(2)}_{zz} = \frac{1}{2}(3 - 30\cos^2\theta + 27\cos^4\theta), \tag{4.52}$$

this will solve the equations to linear order in M but to all orders in μ^2. This gives an approximation valid in the full far region $r \gg M$, demonstrating that the bad asymptotic behaviour of this solution is not resolved at higher order in μ^2.

We now consider briefly the similar analysis for the other independent component, setting $\alpha = 0$ and $\beta = 1$ in the plane wave background (4.37). The ϕ dependence in this background restricts our ability to simplify the form of the solution by general arguments, but the results from the previous case suggest we take an ansatz of the form

$$\begin{aligned} h_{ab} &= \frac{M}{r} h^{(0)}_{ab} + M\mu^2 r \sin^2\theta(\cos^2\phi - \sin^2\phi) h^{(1)}_{ab}, \\ h_{a\theta} &= M\mu^2 r^2 \sin^2\theta(\cos^2\phi - \sin^2\phi) h^{(1)}_{a\theta}, \\ h_{\theta\theta} &= M\mu^2 r^3 \sin^2\theta(\cos^2\phi - \sin^2\phi) h^{(1)}_{\theta\theta}, \\ h_{\phi\phi} &= M\mu^2 r^3 \sin^4\theta(\cos^2\phi - \sin^2\phi) h^{(1)}_{\phi\phi}, \end{aligned} \tag{4.53}$$

assuming the angular dependence at first order in μ^2 will reproduce the angular dependence of the background plane wave. The non-zero components of $h^{(0)}_{ab}$ are $h^{(0)}_{tt} = 2, h^{(0)}_{rr} = 2$, and we assume the $h^{(1)}_{\mu\nu}$ above are constants. We find that we can solve the linearised equations of motion to first order in μ^2 for this ansatz by setting $h^{(1)}_{tt} = 4, h^{(1)}_{tz} = 3, h^{(1)}_{zz} = 2, h^{(1)}_{\theta\theta} = -1, h^{(1)}_{\phi\phi} = -1$.

We can summarise these results in a more invariant fashion by saying that for a plane wave background of the form

$$ds^2_{wave} = -dt^2 + dz^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) - \mu^2 r^2 f(\theta,\phi)(dt+dz)^2, \tag{4.54}$$

a solution of the linearised equations of motion for a black string source, to linear order in μ^2, is

$$\begin{aligned} h_{tt} &= \frac{2M}{r} + 4M\mu^2 r f(\theta,\phi), \\ h_{tz} &= 3M\mu^2 r f(\theta,\phi), \\ h_{zz} &= 2M\mu^2 r f(\theta,\phi), \\ h_{rr} &= \frac{2M}{r}, \\ h_{\theta\theta} &= -M\mu^2 r^3 f(\theta,\phi), \\ h_{\phi\phi} &= -M\mu^2 r^3 \sin^2\theta f(\theta,\phi). \end{aligned} \tag{4.55}$$

We would expect that this generalises straightforwardly to higher dimensions. As in the black hole case, this demonstrates that these solutions are not asymptotically plane wave, as the perturbation is large compared to the background metric far from the source.

Let us now consider the action of this black string solution. Despite the fact that it does not satisfy our falloff conditions for asymptotically plane waves set out in chapter 3, we can still evaluate our action for this solution. To make use of the results of the previous chapter it is convenient to change to lightcone coordinates

$$x^+ = \frac{1}{\sqrt{2}}(t+z), \quad x^- = \frac{1}{\sqrt{2}}(t-z). \tag{4.56}$$

In these coordinates the plane wave background is

$$ds^2_{wave} = -2dx^+dx^- + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) - 2\mu^2 r^2 f(\theta,\phi)(dx^+)^2, \tag{4.57}$$

and the perturbations (4.55) become

$$\begin{aligned} h_{++} &= \frac{2M}{r} + 12\mu^2 r^2 f(\theta,\phi), & (4.58)\\ h_{+-} &= \frac{2M}{r} + 6\mu^2 r^2 f(\theta,\phi), & (4.59)\\ h_{--} &= \frac{2M}{r}, & (4.60) \end{aligned}$$

with $h_{rr}, h_{\theta\theta}$ and $h_{\phi\phi}$ unchanged. Let us consider the boundary at constant r. Since we have a vacuum solution, the action we need to evaluate consists of just the boundary terms

$$S = -\frac{1}{8\pi G}\int_{r=const} d^4x\sqrt{-h}\left(K - \hat{K}\right). \tag{4.61}$$

To leading order, we have $\sqrt{-h} \sim \mathcal{O}(r^2)$. In the previous chapter[4] we showed that $K^{(0)} - \hat{K}^{(0)} = 0$, so we focus on finding $K^{(1)} - \hat{K}^{(1)}$. We have[5]

$$K^{(1)} - \hat{K}^{(1)} = (K^{(0)}_{\alpha\beta} - \hat{K}^{(0)}_{\alpha\beta})h^{(1)\alpha\beta} - (K^{(1)}_{\alpha\beta} - \hat{K}^{(1)}_{\alpha\beta})h^{(0)\alpha\beta} \tag{4.62}$$

Using (3.39) and that the only non-zero components of $\hat{K}^{(0)}_{\alpha\beta}$ are $\hat{K}^{(0)}_{ij} = r\hat{h}_{ij}$ and $\hat{K}^{(0)}_{++} = \frac{r\mathcal{R}_{++}}{d-1}$, we find

$$(K^{(0)}_{\alpha\beta} - \hat{K}^{(0)}_{\alpha\beta})h^{(1)\alpha\beta} = (\mathcal{R}_{++} - 4\mu^2 f(\theta,\phi))M \sim \mathcal{O}(r^0) \tag{4.63}$$

To evaluate $K^{(1)}_{\alpha\beta}$, we use

$$K^{(1)}_{\alpha\beta} = g^{(1)rr}K^{(0)}_{\alpha\beta} + \frac{1}{2}\left(g^{(1)}_{\beta r,\alpha} + g^{(1)}_{r\alpha,\beta} - g^{(1)}_{\alpha\beta,r}\right). \tag{4.64}$$

[4]Please note that the $^{(0)}$ and $^{(1)}$ superscripts are being used here as in chapter 3, and not as elsewhere in this chapter.

[5]We have not fixed the gauge in the same way as in chapter 3; this will result in additional terms to evaluate.

We find

$$K^{(1)}_{\alpha\beta}h^{(0)\alpha\beta} = -2M\mu^2 f(\theta,\phi) \sim \mathcal{O}(r^0). \tag{4.65}$$

To evaluate $h^{(0)\alpha\beta}\hat{K}^{(1)}_{\alpha\beta}$, we use

$$h^{(0)\alpha\beta}\hat{K}^{(1)}_{\alpha\beta} = M^{(0)\alpha\beta}\mathcal{R}^{(1)}_{\alpha\beta} - M^{(0)ij}(\hat{K}^{(0)}_{ij}\hat{K}^{(0)}_{mn} - \hat{K}^{(0)}_{im}\hat{K}^{(0)}_{jn})h^{(1)mn}. \tag{4.66}$$

Note that, in the second term, we have made use of the fact that the only non-zero components of $M^{(0)\alpha\beta}$ are

$$M^{(0)+-} \sim \mathcal{O}(r), \quad M^{(0)--} \sim \mathcal{O}(r^2), \quad M^{(0)ij} = \frac{1}{2(d-2)r}\hat{h}^{ij}, \tag{4.67}$$

and that the only non-zero components of $\hat{K}^{(0)}_{\alpha\beta}$ are $\hat{K}^{(0)}_{ij}$ and $\hat{K}^{(0)}_{++}$. For the second term in (4.66), we have $\hat{K}^{(0)}_{ij} \sim \hat{K}^{(0)}_{++} \sim \mathcal{O}(r)$, and $h^{(1)mn} \sim h^{(1)++} \sim \mathcal{O}(r^{-1})$, so this term is $\mathcal{O}(r^0)$. As before, for the first term, we express $\mathcal{R}^{(1)}_{\alpha\beta}$ by,

$$\mathcal{R}^{(1)}_{\alpha\beta} = -\frac{1}{2}h^{(0)\gamma\delta}D^{(0)}_{\alpha}D^{(0)}_{\beta}h^{(1)}_{\gamma\delta} - \frac{1}{2}h^{(0)\gamma\delta}D^{(0)}_{\gamma}D^{(0)}_{\delta}h^{(1)}_{\alpha\beta} + h^{(0)\gamma\delta}D^{(0)}_{\gamma}D^{(0)}_{(\alpha}h^{(1)}_{\beta)\delta}, \tag{4.68}$$

where D_α is the covariant derivative compatible with $h_{\alpha\beta}$. Using this expression we can see that $\mathcal{R}^{(1)}_{+-} \sim \mathcal{O}(r^{-1})$, $\mathcal{R}^{(1)}_{--} \sim \mathcal{O}(r^{-3})$, and $\mathcal{R}^{(1)}_{ij} \sim \mathcal{O}(r^1)$, so the first term also makes a contribution $\mathcal{O}(r^0)$. Hence, we find $\hat{K}^{(1)} \sim K^{(1)} \sim \mathcal{O}(r^0)$. However, their coefficients will generically be different so there will be no cancellation between the $K^{(1)}$ and $\hat{K}^{(1)}$ terms and the action will diverge like r^2 in the large r limit. This result is not surprising, the solution is clearly not asymptotically plane wave so we would not expect it to have a finite action.

4.2 Near region analysis

Having explored the behaviour in the intermediate region, where we can use a linearised approximation about the plane wave background, we now turn to the analysis in the region $r \ll \mu^{-1}$ near the black hole or black string. In this region we can treat the plane wave as a small perturbation of the black object, and the problem reduces to linearised perturbations on the black hole or black string background, with boundary conditions at large distances determined from the previous solution in the intermediate region and a boundary condition at the horizon determined by requiring regularity of the perturbed solution there. We will find that there is no regular solution in the black hole case. For the black string, we find a regular solution which matches on to the solution we discussed above in the intermediate region. We will focus on the analysis for the black hole in four dimensions and the black string in five dimensions, as in the previous section, but the same techniques can easily be applied in higher dimensions. We will comment briefly on the extension of the analysis to higher dimensions for the black hole case.

4.2.1 Black hole

We first study the near horizon region of the black hole, treating the plane wave as a perturbation. We will do the analysis in the lowest possible dimension, $D = 4$, even though there is a simple symmetry argument that no regular solution exists in this case. The calculation is simplest in this dimension, and it serves to illustrate the method of calculation which will be very similar in higher dimensions.

Take the Schwarzschild black hole solution in four dimensions,

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{4.69}$$

with $f(r) = 1 - 2M/r$. We want to find a solution of the source-free linearised vacuum equations on this background which asymptotically approaches the four-dimensional plane wave (4.10). This implies that we want a perturbation $h_{\mu\nu}$ with asymptotic boundary conditions

$$\lim_{r\to\infty} h_{\mu\nu}dx^\mu dx^\nu = -\frac{\mu^2 r^2}{2}\sin^2\theta(e^{2i\phi} + e^{-2i\phi})(dt + \cos\theta dr - r\sin\theta d\theta)^2 + \dots, \tag{4.70}$$

where the $\dots$ denotes terms going like $\mu^2 M^n$ for $n > 0$. These terms are suppressed relative to the leading term because dimensional analysis tells us the mass will always appear in the combination M/r. At linear order in M, the sub-leading terms at large r should match onto the results of the analysis in the intermediate region obtained in the previous section.

At the horizon, the boundary condition is that the solution be regular there. Since the background metric is not regular at the horizon in the Schwarzschild coordinate system we are using, this condition is most easily applied by writing the perturbation in an orthonormal frame. A suitable frame is $e^{(0)} = \sqrt{f(r)}dt$, $e^{(1)} = f(r)^{-1/2}dr$, $e^{(2)} = rd\theta$, $e^{(3)} = r\sin\theta d\phi$. Requiring that the components of the perturbation in the orthonormal frame are regular at the horizon implies that we must require that as $r \to 2M$,

$$h_{tt} \sim (r-2M), \quad h_{t\mu} \sim (r-2M)^{1/2} \text{ for } \mu \neq t, r \tag{4.71}$$

$$h_{rr} \sim (r-2M)^{-1}, \quad h_{r\mu} \sim (r-2M)^{-1/2} \text{ for } \mu \neq r, t \tag{4.72}$$

$$h_{tr} \sim (r-2M)^{0}, \quad h_{ij} \sim (r-2M)^{0}. \tag{4.73}$$

These conditions can also be derived by requiring finiteness of $h_{\mu\nu}$ in a coordinate system which is well-behaved at $r = 2M$, such as Kruskal coordinates.

Matching the leading term written in (4.70) and imposing regularity at the horizon should determine the solution of the perturbation equations uniquely. In fact, as we mentioned above, we will find that there is no solution of the linearised perturbation equations that satisfies these two boundary conditions.

For the black hole case, the analysis of the components on the sphere is sufficiently complicated that it is useful to exploit the results of [66] on the spherical harmonic decomposition for perturbations of Schwarzschild and rewrite the linearised equations of motion in terms of gauge-invariant variables with respect to coordinate transformations on the sphere. We therefore want to convert (4.70) into boundary conditions for their gauge-invariant perturbations. Let $a, b = t, r$ and $i, j = \theta, \phi$. Then we have boundary conditions which are scalars h_{ab}, vectors h_{ai}, and a tensor h_{ij}, for which the boundary condition only has an $h_{\theta\theta}$ component. Following [66] we expand the perturbation in terms of harmonics on S^2: the scalar harmonics

$$\Box S = -l(l+1)S, \quad l = 0, 1, 2, \ldots, \tag{4.74}$$

the vector harmonics

$$\Box V_i = (-l(l+1)+1)V_i, \quad l = 1, 2, 3, \ldots, \tag{4.75}$$

with $D_i V^i = 0$, and the transverse traceless tensor harmonics

$$\Box T_{ij} = (-l(l+1)+2)T_{ij}, \quad l = 2, 3, 4, \ldots, \tag{4.76}$$

with $D_i T^i_j = 0, T^i_i = 0$. We use the notation $\Box = D_i D^i$ for the d'Alembertian operator on S^2, where D_i is the covariant derivative with respect to the metric γ_{ij} on the unit two-sphere.

In terms of these harmonics, the scalar components of the perturbation are

$$h_{ab} = \sum_{l,m} f_{ab} S^m_l. \tag{4.77}$$

Note that here and hereafter we will omit the l, m indices on the coefficients f_{ab} or equivalent in the general relations like this for brevity. The vector perturbations are decomposed into their scalar-derived and pure vector components $h_{ai} = h^S_{ai} + h^V_{ai}$, where

$$h^S_{ai} = r \sum_{l,m} f_a \left(-\frac{1}{k^2} D_i S^m_l\right), \tag{4.78}$$

where $k^2 = l(l+1)$, and

$$h^V_{ai} = r \sum_{l,m} f^V_a (V^m_l)_i. \tag{4.79}$$

Similarly, the tensor part of the perturbation is decomposed into scalar-derived, vector-derived and pure tensor components $h_{ij} = h^S_{ij} + h^V_{ij} + h^T_{ij}$, where

$$h^S_{ij} = 2r^2 \sum_{l,m} (H_L \gamma_{ij} S^m_l + H_T S_{ij}), \tag{4.80}$$

where $S_{ij} = \frac{1}{k^2} D_i D_j S^m_l + \frac{\gamma_{ij}}{2} S^m_l$,

$$h^V_{ij} = 2r^2 \sum_{l,m} H^V_T V_{ij}, \tag{4.81}$$

where $V_{ij} = -\frac{1}{2k_V^2}(D_i V_j + D_j V_i)$ with $k_V^2 = l(l+1) - 1$, and

$$h^T_{ij} = 2r^2 \sum_{l,m} H^T_T T_{ij}. \tag{4.82}$$

There are, however, no pure tensor harmonics T_{ij} on S^2.

Thus, to determine the boundary conditions for the gauge invariant variables, we must apply this expansion to (4.70) and find the asymptotic values for the unknown expansion coefficients. For scalar perturbations this is straightforward. Substituting (4.70) into (4.77) we are able to read off that

$$\lim_{r\to\infty} (f_{tt})^{\pm 2}_2 = -\frac{\mu^2 r^2}{2}, \quad \lim_{r\to\infty} (f_{rr})^{\pm 2}_2 = -\frac{\mu^2 r^2}{14}, \tag{4.83}$$

$$\lim_{r\to\infty}(f_{tr})_3^{\pm 2} = -\mu^2 r^2, \quad \lim_{r\to\infty}(f_{rr})_4^{\pm 2} = -\frac{\mu^2 r^2}{14}. \tag{4.84}$$

We now turn our attention to the vector perturbations. Since $D^i V_i = 0$ we have $D^i h^V_{ai} = 0$, so

$$D^i h_{ai} = D^i h^S_{ai} = r \sum_{l,m} f_a S^m_l, \tag{4.85}$$

where we have used $D^i D_i S = -k^2 S$. Explicit computation gives us the boundary conditions for the scalar-derived vector coefficients,

$$\lim_{r\to\infty}(f_r)_2^{\pm 2} = -\frac{\mu^2 r^2}{7}, \quad \lim_{r\to\infty}(f_t)_3^{\pm 2} = 2\mu^2 r^2, \quad \lim_{r\to\infty}(f_r)_4^{\pm 2} = \frac{5\mu^2 r^2}{14}. \tag{4.86}$$

To find the pure vector coefficients we write

$$h^V_{ai} = h_{ai} - h^S_{ai} = h_{ai} + r \sum_{l,m} f_a \frac{1}{k^2} D_i S^m_l = r \sum_{l,m} f^V_a (V^m_l)_i. \tag{4.87}$$

Again, by explicit computation we find,

$$\lim_{r\to\infty}(f^V_t)_2^{\pm 2} = \frac{\mu^2 r^2}{3}, \quad \lim_{r\to\infty}(f^V_r)_3^{\pm 2} = \frac{\mu^2 r^2}{6}. \tag{4.88}$$

Finally we consider the tensor perturbations. We can write

$$h^i{}_i = (h^S)^i{}_i = 4r^2 \sum_{l,m} H_L S^m_l, \tag{4.89}$$

where we have used $D^i V_i = 0, T^i{}_i = 0, S^i{}_i = 0$ and $\gamma^i{}_i = 2$. This allows us to easily show that

$$\lim_{r\to\infty}(H_L)_2^{\pm 2} = -\frac{3\mu^2 r^2}{28}, \quad \lim_{r\to\infty}(H_L)_4^{\pm 2} = \frac{\mu^2 r^2}{56}. \tag{4.90}$$

To find the scalar-derived transverse modes we will need the following results,

$$D^i D^j V_{ij} = 0, \tag{4.91}$$

$$D^i D^j S_{ij} = \frac{(k^2 - 2)}{2} S, \tag{4.92}$$

which are proved in appendix A. Using the above results along with $D^i T_{ij} = 0$, we find

$$\begin{aligned} D^i D^j h_{ij} &= D^i D^j h^S_{ij} \\ &= 2r^2 \sum_{l,m} (-k^2 H_L S^m_l + H_T D^i D^j S_{ij}) \\ &= 2r^2 \sum_{l,m} (-k^2 H_L + H_T \frac{(k^2 - 2)}{2}) S. \end{aligned} \tag{4.93}$$

We can now show that

$$\lim_{r\to\infty}(H_T)_2^{\pm 2} = -\frac{\mu^2 r^2}{7}, \quad \lim_{r\to\infty}(H_T)_4^{\pm 2} = -\frac{5\mu^2 r^2}{12\cdot 7}. \tag{4.94}$$

To find the vector-derived transverse modes we will use the identities

$$D^i S_{ij} = -\frac{1}{2k^2}(k^2-2)D_j S, \tag{4.95}$$

and

$$D^i V_{ij} = \frac{1}{2k_V^2}(k_V^2-1)V_j, \tag{4.96}$$

which we also prove in appendix A. Since $D^i T_{ij} = 0$, we have

$$D^i h_{ij} = D^i h_{ij}^S + D^i h_{ij}^V, \tag{4.97}$$

and using the results above we can write this as

$$D^i h_{ij} = 2r^2 \sum_{l,m}(H_L - \frac{1}{2k^2}(k^2-2)H_T)D_j S + 2r^2 \sum_{l,m} H_T^V \frac{1}{2k_V^2}(k_V^2-1)V_j. \tag{4.98}$$

We are now able to show that

$$\lim_{r\to\infty}(H_T^V)_3^{\pm 2} = -\frac{11\mu^2 r^2}{12}. \tag{4.99}$$

Using Maple we find that $h_{ij} = h_{ij}^S + h_{ij}^V$, so there are no pure tensor perturbations as expected.

We now want to translate this into boundary conditions for the gauge-invariant variables introduced in [66]. For vector perturbations the gauge-invariant variable is

$$F_a = f_a^V + \frac{r}{k_V^2}D_a H_T^V. \tag{4.100}$$

For $l=2$, $\lim_{r\to\infty}(f_t^V)_2^{\pm 2} = \frac{\mu^2 r^2}{3}$, so $\lim_{r\to\infty} F_t = \frac{\mu^2 r^2}{3}$. The vector master function Φ is defined by $F_a = r^{-1}\epsilon_{ab}D^b(r\Phi)$ [66], so the boundary condition for $\Phi_{l=2}^V$ is

$$\lim_{r\to\infty}\Phi_{l=2}^V = \frac{\mu^2 r^3}{12}. \tag{4.101}$$

For $l=3$, $\lim_{r\to\infty}(f_r^V)_3^{\pm 2} = \frac{\mu^2 r^2}{6}$ and $\lim_{r\to\infty}(H_T^V)_3^{\pm 2} = -\frac{11\mu^2 r^2}{12}$ so $F_a = 0$; this mode is pure gauge. This is as we might expect; the r^2 behaviour of the plane wave is typical of an $l=2$ spherical harmonic, so the higher l modes that seem to appear in our decomposition of the mode in terms of spherical harmonics ought to be pure gauge.

For scalar perturbations, the gauge-invariant variables are [66]

$$F = H_L + \frac{1}{2}H_T + \frac{1}{r}(D^a r)X_a \tag{4.102}$$

$$F_{ab} = f_{ab} + D_a X_b + D_b X_a \tag{4.103}$$

with

$$X_a = \frac{r}{k^2}(f_a + r D_a H_T). \tag{4.104}$$

The master variable Φ is

$$\Phi = \frac{2\tilde{Z} - r(X+Y)}{4}, \tag{4.105}$$

with

$$X = F_t^t - 2F \tag{4.106}$$

$$Y = F_r^r - 2F \tag{4.107}$$

$$\tilde{Z} = 0. \tag{4.108}$$

For $l = 2$ perturbations direct substitution gives us $\lim_{r\to\infty} X = \mu^2 r^2, Y = 0, \tilde{Z} = 0$, hence the boundary condition on Φ is

$$\lim_{r\to\infty} \Phi^S_{l=2} = -\frac{\mu^2 r^3}{4}. \tag{4.109}$$

For the $l = 3$ and $l = 4$ modes we find the gauge-invariant variables F and F_{ab} are zero, so these modes are pure gauge as expected. Thus, we are left with two non-trivial modes, the $l = 2$ scalar and the $l = 2$ vector modes.

Having established which modes are non-zero and their boundary conditions, we consider the bulk solution. For the vector mode the equation for the master field is [66]

$$\partial_r((1 - \frac{2M}{r})\partial_r \Phi) - \frac{1}{r^2}[l(l+1) - 3 \cdot \frac{2M}{r}]\Phi = 0. \tag{4.110}$$

The boundary condition is $\lim_{r\to\infty} \Phi^V_{l=2} = \frac{\mu^2 r^3}{12}$, therefore we set $\Phi = r^3\psi$. This allows us to reduce the master equation (4.110) to

$$\partial_r(r^6(1 - \frac{2M}{r})\partial_r \psi) = 0. \tag{4.111}$$

which has solution

$$\psi = a\left(\frac{1}{8Mr^4} + \frac{1}{12M^2r^3} + \frac{1}{16M^3r^2} + \frac{1}{16M^4r} + \frac{1}{32M^5}\ln(1-\frac{2M}{r})\right) + b. \quad (4.112)$$

Solutions with $a \neq 0$ are clearly not regular at $r = 2M$, therefore the solution for the vector master field is $\Phi^V = br^3$. The boundary condition at large r then requires $b = \frac{\mu^2}{12}$. However, the boundary condition at the horizon (4.71) requires that h_{tt} and h_{ti} vanish at the horizon. This implies that f_t^V and hence F_t^V also vanish at the horizon. Finally $F^t = r^{-1}D_r(r\Phi)$ implies that Φ too must vanish at the horizon, which would require $b = 0$. Hence, there is no solution which satisfies the boundary conditions at both the horizon and infinity.

Thus, there is no regular solution describing a four-dimensional black hole in the plane wave background (4.10). In fact, this is not a surprising result in four dimensions; the rigidity theorem [68] shows that regular black holes must be static or stationary axisymmetric, and the plane wave (4.10) is not static and does not preserve a $U(1)$ symmetry. Thus, the plane wave perturbation breaks too many of the symmetries of the black hole for a regular deformed black hole solution to be possible.

One might hope to avoid this problem by considering a non-vacuum plane wave solution. We can for example consider in four dimensions the electromagnetic plane wave

$$ds^2_{wave} = -dt^2 + dx^2 + dy^2 + dz^2 - \mu^2(x^2+y^2)(dt+dz)^2 \quad (4.113)$$

supported by the electric flux

$$F = 2\mu(dt + dz) \wedge dx. \quad (4.114)$$

This is also interesting as a simplified model of the maximally supersymmetric plane wave of [18]. Here, the metric perturbation preserves a $U(1)$ symmetry, but this is broken by the gauge field, and as a result, we again do not expect to find a regular black hole solution. In this case, the problem is that the equation of motion for the gauge field on the Schwarzschild black hole background has no solution which is regular on the horizon and satisfies the boundary condition at large r.

If we consider the situation in higher dimensions, the above rigidity argument does not apply, but there is still no regular solution. Take for example a six-dimensional Schwarzschild black hole and add as a perturbation the six-dimensional vacuum plane wave

$$ds^2_{wave} = -dt^2+dv^2+dw^2+dx^2+dy^2+dz^2-\mu^2(v^2+w^2-x^2-y^2)(dt+dz)^2. \quad (4.115)$$

This clearly preserves two $U(1)$ isometries, in the $x-y$ and $v-w$ planes. However, if we rewrite this in spherical polars, there is again an $l=2$ vector part to the perturbation in the decomposition into spherical harmonics. The analysis is very similar to the above four-dimensional case, and it is not possible to find a solution for the vector part of the perturbation that satisfies the plane wave boundary conditions at large distances and the regularity condition on the event horizon. In this case, the plane wave preserves two $U(1)$ isometries on the S^4 surrounding the black hole, so the above argument does not apply; a regular deformed black hole solution would not violate the conditions of [69]. This problem seems to be very general. In all cases we have explored in the vacuum Einstein equations, the plane wave has a vector part in the spherical harmonic decomposition, and it is not possible to find a regular perturbation of the black hole which satisfies the plane wave boundary condition. It would be interesting to understand the physical origins of this restriction further.

4.2.2 Black string

We next study the near horizon region of the black string, treating the plane wave as a perturbation. The background is the five-dimensional black string solution

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2+\sin^2\theta d\phi^2) + dz^2, \quad (4.116)$$

with $f(r) = 1-2M/r$. We want to find a solution of the source-free linearised vacuum equations on this background which asymptotically approaches the five-dimensional plane wave (4.37). This implies that we want a perturbation $h_{\mu\nu}$ with asymptotic boundary conditions

$$\lim_{r\to\infty} h_{\mu\nu}dx^\mu dx^\nu = -\mu^2 r^2[\alpha(1-3\cos^2\theta)+\beta\sin^2\theta(\cos^2\phi-\sin^2\phi)](dt+dz)^2+\dots, \quad (4.117)$$

where the $\dots$ denotes terms going like $\mu^2 M^n$ for $n\neq 0$. These terms are suppressed relative to the leading term because dimensional analysis tells us the mass will always appear in the combination M/r.

As in the analysis in the intermediate region, we will deal with the α and β components separately. It will turn out that the analysis is identical in these two cases. In terms of the spherical harmonic analysis on the two-sphere, these are scalar-type perturbations which excite the $l = 2, m = 0$ and $l = 2, m = 2$ harmonic modes respectively. In the linearised theory, we can assume that the perturbation has only these modes turned on. Since only scalar-type modes are excited, the analysis on the sphere is fairly simple, and we will follow the similar analysis by Emparan et al [56], reviewed in chapter 2.

The boundary conditions, and hence the perturbation, are invariant under simultaneously taking $t \to -t$, $z \to -z$ and under translations in t and z, so the only modes we need to consider are h_{tt}, h_{tz}, h_{zz}, h_{rr}, and the longitudinal and transverse scalar-derived perturbations on the sphere.

We first consider only the $l = 2, m = 0$ perturbation (we set $\beta = 0$). Assuming that only this spherical harmonic is excited, we can write the perturbation as

$$h_{tt} = \alpha(1 - 3\cos^2\theta)a(r), \quad h_{tz} = \alpha(1 - 3\cos^2\theta)b(r), \quad h_{zz} = \alpha(1 - 3\cos^2\theta)c(r), \tag{4.118}$$

$$h_{rr} = \alpha\frac{(1 - 3\cos^2\theta)}{(1 - 2M/r)}f(r), \tag{4.119}$$

$$h_{\theta\theta} = \alpha r^2[(1 - 3\cos^2\theta)g(r) - 3\sin^2\theta h(r)], \tag{4.120}$$

$$h_{\phi\phi} = \alpha r^2\sin^2\theta[(1 - 3\cos^2\theta)g(r) + 3\sin^2\theta h(r)]. \tag{4.121}$$

Note that $g(r)$ is the coefficient of the longitudinal mode on the sphere, and $h(r)$ is the coefficient of the transverse mode on the sphere. As in [56], there is a remaining coordinate freedom, under

$$r \to r + \gamma(r)(1 - 3\cos^2\theta), \quad \theta \to \theta + 6\beta(r)\cos\theta\sin\theta, \tag{4.122}$$

with

$$\beta'(r) = -\frac{\gamma(r)}{r(r - 2M)}, \quad \gamma(2M) = 0. \tag{4.123}$$

Similarly, for the $l = 2, m = 2$ perturbation (obtained by setting $\alpha = 0$), we define

$$h_{tt} = \beta\sin^2\theta(\cos^2\phi - \sin^2\phi)a(r), \quad h_{tz} = \beta\sin^2\theta(\cos^2\phi - \sin^2\phi)b(r), \tag{4.124}$$

$$h_{zz} = \beta \sin^2\theta(\cos^2\phi - \sin^2\phi)c(r), \tag{4.125}$$

$$h_{rr} = \beta \frac{\sin^2\theta(\cos^2\phi - \sin^2\phi)}{(1-2M/r)} f(r), \tag{4.126}$$

$$h_{\theta\phi} = \beta r^2 \sin\theta\cos\theta\sin\phi\cos\phi h(r), \tag{4.127}$$

$$h_{\theta\theta} = \beta r^2[\sin^2\theta(\cos^2\phi - \sin^2\phi)g(r) - (\cos^2\theta+1)(\cos^2\phi - \sin^2\phi)h(r)], \tag{4.128}$$

$$h_{\phi\phi} = \beta r^2 \sin^2\theta[\sin^2\theta(\cos^2\phi - \sin^2\phi)g(r) + (\cos^2\theta+1)(\cos^2\phi - \sin^2\phi)h(r)]. \tag{4.129}$$

Now we have remaining coordinate freedom under

$$r \to r + \gamma(r)\sin^2\theta(\cos^2\phi - \sin^2\phi), \tag{4.130}$$

$$\theta \to \theta + 2\beta(r)\sin\theta\cos\theta(\cos^2\phi - \sin^2\phi), \tag{4.131}$$

$$\phi \to \phi - 4\beta(r)\sin^2\theta\cos\phi\sin\phi, \tag{4.132}$$

with

$$\beta'(r) = -\frac{\gamma(r)}{r(r-2M)}, \quad \gamma(2M) = 0. \tag{4.133}$$

We find both coordinate transformations produce identical shifts

$$a(r) \to a(r) - \frac{2M}{r^2}\gamma(r), \quad f(r) \to f(r) + \left(2\gamma' - \frac{2M}{r}\frac{\gamma(r)}{r-2M}\right), \tag{4.134}$$

$$g(r) \to g(r) + \frac{2}{r}\gamma(r) - 6\beta(r), \quad h(r) \to h(r) + 2\beta(r), \tag{4.135}$$

while $b(r)$ and $c(r)$ are unchanged.

We want to consider combinations which are invariant under these coordinate transformations. $B = b(r)$ and $C = c(r)$ are already invariant. We define in addition

$$A = a(r) + \frac{M}{r}(g(r) + 3h(r)), \tag{4.136}$$

$$F = f(r) - \frac{d}{dr}\left(r(g(r) + 3h(r))\right) + \frac{M(g(r)+3h(r))}{(r-2M)}, \tag{4.137}$$

$$H' = \frac{dh}{dr} + \frac{g(r)+3h(r)}{(r-2M)}. \tag{4.138}$$

Note that in this section, primes denote derivatives with respect to r. As in [56], the constant part of $h(r)$ can be fixed using the constant part of $\beta(r)$. Using the gauge-invariant combinations basically amounts to setting $g(r) = -3h(r)$, which can be achieved for $r \neq 2M$ by an appropriate choice of gauge. Because of the boundary condition in (4.133), $g(2M)+3h(2M)$ is gauge-invariant. It will, however, not be determined by solving the equations of motion for the above gauge-invariant variables, and will have to be separately specified. It will turn out to be determined by requiring regularity of the solution at the horizon.

For either $\alpha = 0$ or $\beta = 0$, substituting into the linearised Einstein equations gives the same system of equations for the unknown functions A, B, C, F, H' (keeping terms up to $\mathcal{O}(\mu^2)$),

$$R^{(1)}_{tt} \propto r^2(r-2M)^2A'' + r(r-2M)(2r-5M)A' - M(r-2M)^2C' \tag{4.139}$$
$$-(6r(r-2M)-2M^2)A + M(r-2M)^2F' + 6M(r-2M)^2H',$$

$$R^{(1)}_{tz} \propto r(r-2M)B'' + 2(r-2M)B' - 6B, \tag{4.140}$$

$$R^{(1)}_{zz} \propto r(r-2M)C'' + 2(r-M)C' - 6C, \tag{4.141}$$

$$R^{(1)}_{rr} \propto r^2(r-2M)^2A'' - rM(r-2M)A' + 2M(2r-3M)A \tag{4.142}$$
$$-r(r-2M)^3C'' - M(r-2M)^2C' + (2r-3M)(r-2M)^2F'$$
$$+6(r-2M)^2F + 6r(r-2M)^3H'' + 6(2r-3M)(r-2M)^2H',$$

$$R^{(1)}_{r\theta} \propto -r^2(r-2M)A' + r(r-M)A + r(r-2M)^2C' - (r-2M)^2C \tag{4.143}$$
$$-(r-2M)(r-M)F - r(r-2M)^2H',$$

$$R^{(1)}_{\theta\theta} + \frac{1}{\sin^2\theta}R^{(1)}_{\phi\phi} \propto r(r-2M)A' - (3r+2M)A - (r-2M)^2C' + 3(r-2M)C$$
$$+(r-2M)^2F' + 5(r-2M)F + 3r(r-2M)^2H'' \tag{4.144}$$
$$+6(2r-3M)(r-2M)H',$$

$$R^{(1)}_{\theta\theta} - \frac{1}{\sin^2\theta}R^{(1)}_{\phi\phi} \propto -rA + (r-2M)C + (r-2M)F + r(r-2M)^2H'' + 2(r-M)(r-2M)H'. \tag{4.145}$$

In fact, it is easy to show that the linearised Einstein equations must be the same for both modes. The perturbation involves some $l = 2$ scalar harmonic, let's call it S, so

$$h_{ab} = f_{ab}(r)S, \quad h_{ai} = f_a(r)\nabla_i S, \quad h_{ij} = f(r)Sg_{ij} + f'(r)\nabla_i\nabla_j S, \tag{4.146}$$

where i, j are coordinates on the two-sphere and $a, b = t, r, z$. Then, the first order Ricci tensor constructed from the second covariant derivatives of $h_{\mu\nu}$ will also depend on angular coordinates only through S and its derivatives. Using $\nabla_i \nabla^i S = -6S$ and the fact that the sphere is an Einstein space, so $R_{ij} = g_{ij}$, one can eliminate extra derivatives of S to leave us with

$$R^{(1)}_{ab} = \epsilon_{ab}(r) S, \quad R^{(1)}_{ai} = \epsilon_a(r) \nabla_i S, \quad R^{(1)}_{ij} = \epsilon(r) S g_{ij} + \epsilon'(r) \nabla_i \nabla_j S. \tag{4.147}$$

Hence, the resulting equations $\epsilon_{ab}(r) = \epsilon_a(r) = \epsilon(r) = \epsilon'(r) = 0$ are independent of whether S is in the $m = 0$ or $m = 2$ mode. Thus, solving the equations (4.139-4.145) will give us the general solution for the perturbation in the near-horizon region for both modes.

The boundary conditions at large r imply that at order M^0, $a(r), b(r), c(r) \to -\mu^2 r^2$, and $f(r)$, $g(r), h(r)$ have no $\mu^2 M^0$ term. This implies that

$$A, B, C \to -\mu^2 r^2, \tag{4.148}$$

and F and H' have no $\mu^2 M^0$ term. Regularity at the horizon requires $a(r) \propto (r - 2M)$, $b(r) \propto \sqrt{r - 2M}$, and the other functions $c(r), f(r), g(r)$ and $h(r)$ are required to be finite there. In terms of the gauge-invariant combinations, these boundary conditions are best expressed in terms of the alternative combinations

$$\bar{A} = A - \frac{M}{r}(r - 2M) H', \quad \bar{F} = F - M H'. \tag{4.149}$$

The conditions for regularity at the horizon are then that $\bar{A} \to 0$, $\bar{F}$ is finite, and H' is allowed to diverge like $(r - 2M)^{-1}$.

We now want to solve this system of equations. We see that there are two decoupled equations, (4.140) and (4.141). The solutions of these satisfying our boundary conditions are

$$B(r) = -\mu^2 (r - M)(r - 2M) \tag{4.150}$$

and

$$C(r) = -\mu^2 (r^2 - 2Mr + \frac{2}{3} M^2). \tag{4.151}$$

It is also convenient to subtract a multiple of (4.141) from (4.142) to simplify it to

$$0 = r^2(r-2M)^2A'' - rM(r-2M)A' + 2M(2r-3M)A \tag{4.152}$$
$$+(2r-5M)(r-2M)^2C' - 6(r-2M)^2C + (2r-3M)(r-2M)^2F'$$
$$+6(r-2M)^2F + 6r(r-2M)^3H'' + 6(2r-3M)(r-2M)^2H'.$$

We first solve (4.145) for A,

$$A = \frac{(r-2M)}{r}\left[C + F + r(r-2M)H'' + 2(r-2M)H'\right], \tag{4.153}$$

and then solve $R^{(1)}_{tt} - (r-2M)^2R^{(1)}_{zz} - R^{(1)}_{rr}$ for F,

$$F = \frac{1}{6}\left[r(r-2M)^2H''' - 2(r-2M)(r+2M)H'' - 2(5r-7M)H' - MC'\right]. \tag{4.154}$$

The remaining equations then need to be solved for H'. By combining equations, we can obtain a second-order inhomogeneous equation for H',

$$-2r(r+M)(r-2M)^2H''' - 2(4r^2+3rM-4M^2)(r-2M)H'' \tag{4.155}$$
$$+2(4r^2-13rM+4M^2)H' = M[(r-2M)C' + 6C].$$

It's useful to note at this point that if $M = 0$, we have a solution with $F = H' = 0$ and $A = C = -\mu^2r^2$, which is precisely our original plane wave.

The general solution of (4.155) is

$$H' = \frac{\mu^2}{3}(r-M) + c_1\frac{r^2-2M^2}{r-2M} + c_2\frac{[-6rM(r+M) + 4M^3 + (6rM^2-3R^3)\ln(1-2M/r)]}{r(r-2M)}. \tag{4.156}$$

This then satisfies all of the equations. To get a solution which is both regular and has the correct asymptotics, i.e. has $A \to -\mu^2r^2$ at large r, we need to take $c_1 = -\frac{1}{3}\mu^2$ and $c_2 = 0$. We find

$$H' = -\frac{\mu^2M}{3}\frac{3r-4M}{r-2M}, \tag{4.157}$$

and

$$A = -\mu^2\left[r^2 - 4rM + \frac{16}{3}M^2 - 2\frac{M^3}{r}\right], \quad F = \frac{2\mu^2M}{3}\frac{3r^2-9rM+5M^2}{r-2M}. \tag{4.158}$$

In terms of the alternative combinations $\bar{A}$, $\bar{F}$,

$$\bar{A} = -\mu^2(r-2M)\left[r - 2M + \frac{M^2}{3r}\right], \quad \bar{F} = \mu^2 M(2r - M). \tag{4.159}$$

Thus, this solution satisfies the regularity conditions at the horizon. Regularity of the original functions $a(r), f(r), g(r), h(r)$ at $r = 2M$ further requires us to choose

$$g(2M) + 3h(2M) = -\frac{2\mu^2 M^2}{3}. \tag{4.160}$$

We now match the near horizon and intermediate region solutions in the intermediate region $\mu^{-1} \gg r \gg M$, where both approximations are valid. The contribution from the black string background is

$$ds^2_{NR,BG} \approx -(1 - \frac{2M}{r})dt^2 + (1 + \frac{2M}{r})dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) + dz^2. \tag{4.161}$$

We must now find the unknown functions $a(r), b(r), c(r), f(r), g(r), h(r)$ in this region to obtain the contribution from the perturbation. In addition to the solutions (4.150), (4.151) and (4.158) we must make a choice of gauge. We choose $g + 3h = -M\mu^2 r$ in order to make the rr-component of the perturbation vanish, matching our gauge choice in the intermediate region solution. We find, keeping just the terms up to $\mathcal{O}(M)$ and $\mathcal{O}(\mu^2)$,

$$a(r) \approx -\mu^2(r^2 - 4Mr), \quad b(r) \approx -\mu^2(r^2 - 3Mr), \quad c(r) \approx -\mu^2(r^2 - 2Mr), \tag{4.162}$$

$$f(r) \approx 0, \quad g(r) \approx -M\mu^2 r, \quad h(r) \approx 0. \tag{4.163}$$

Hence the near region perturbation is

$$\begin{aligned} ds^2_{NR,P} &\approx (\alpha(1 - 3\cos^2\theta) + \beta\sin^2\theta(\cos^2\phi - \sin^2\phi)) \times \\ &(-\mu^2 r^2(dt + dz)^2 + M\mu^2 r(4dt^2 + 6dtdz + 2dz^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2))). \end{aligned} \tag{4.164}$$

In the intermediate region the plane wave background is,

$$\begin{aligned} ds^2_{IR,BG} &= -dt^2 + dz^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \\ &-\mu^2 r^2(\alpha(1 - 3\cos^2\theta) + \beta\sin^2\theta(\cos^2\phi - \sin^2\phi))(dt + dz)^2. \end{aligned} \tag{4.165}$$

From section 4.1.3, the perturbation due to the black string is

$$\begin{aligned} ds^2_{IR,P} &= \frac{2M}{r}dt^2 + \frac{2M}{r}dr^2 + M\mu^2 r(\alpha(1 - 3\cos^2\theta) + \beta\sin^2\theta(\cos^2\phi - \sin^2\phi)) \times \\ &(4dt^2 + 6dtdz + 2dz^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2)). \end{aligned} \tag{4.166}$$

Thus the solution constructed in the near region $ds^2_{NR} = ds^2_{NR,BG} + ds^2_{NR,P}$ agrees with the solution constructed in the intermediate region $ds^2_{IR} = ds^2_{IR,BG} + ds^2_{IR,P}$ to the relevant order. This gives us an approximate solution describing a black string in a plane wave, valid when the size of the black string is small compared to the curvature scale of the wave, $r_+ \ll \mu^{-1}$.

As in [56], the perturbation does not affect the thermodynamic properties of the black hole at this order. The area of the horizon cannot be affected at this order because the perturbation is entirely in an $l = 2$ mode, which deforms the shape of the S^2 but does not change its area. The temperature cannot be affected because it is constant over the horizon. Since the perturbation is an $l = 2$ mode, it will vanish at some point on the horizon so the temperature at that point must be unaffected and, since it is constant, it must be unchanged over the whole horizon.

In this chapter, we have attempted to construct black hole and black string solutions in plane wave backgrounds using the method of matched asymptotic expansions. We have found that it is not possible to construct a regular black hole solution. The failure of regularity here is a counter-example to the conjecture in [52] that satisfying the blackfold equations implies horizon regularity.

We have successfully constructed an approximate solution describing a black string in a vacuum plane wave background in five dimensions. This solution exhibits an interesting property; the effect of a localised object in a plane wave background is not small, even far from the source. We discuss the interpretation of these results in the next chapter.

Chapter 5

Conclusions

In this thesis, we have proposed a definition of asymptotically plane wave spacetimes which is consistent with some known exact solutions, and constructed a well-behaved action principle for asymptotically plane wave solutions of the vacuum Einstein equations, following the work of [30]. Our definition of asymptotically plane wave solutions is valid for any solution which asymptotically approaches a vacuum plane wave. We have considered only the pure vacuum action and it would be interesting to extend this work to include appropriate matter fields. It is also interesting to ask if there are non-trivial physically relevant examples to which our ideas apply. For the asymptotically plane wave boundary conditions, (3.13) provides such an example, but this is not a pure vacuum solution so our discussion of the action does not apply to it. A more trivial example is provided by some pp-wave solutions. For example, consider the vacuum pp-wave metric

$$ds^2 = -2dx^+dx^- - F(x^+, x^I)\left(dx^+\right)^2 + \delta_{IJ}dx^I dx^J \tag{5.1}$$

with $\partial_I \partial^I F = 0$. If $F(x^+, x^I) \to \mu_{IJ}(x^+)x^I x^J + \mathcal{O}(r^{4-d})$ as $r \to \infty$, this solution is asymptotically plane wave according to our definition, and the action we have defined will be finite for it. However, this is a rather trivial example and it would be interesting to construct solutions really corresponding to localised sources in an asymptotically plane wave background. Unfortunately, the analysis of chapter 4 on the construction of black holes and black strings in plane waves suggests that the space of such solutions will be highly restricted. In this analysis we find an interesting general result; the effect of localised objects in a plane wave background is not small, even far from the source. The usual $1/r^{d-1}$ fall-off associated with a localised object in $d+1$ spatial dimensions is offset by the $\mu^2 r^2$ factors coming from the plane wave background. As a result, we find that the "perturbation" due to the source is larger than the background metric at sufficiently large r. This leads us to believe that these solutions should not be thought of as "asymptotically plane wave" spacetimes.

Our definition of "asymptotically plane wave" allows the construction of a well-behaved action principle. This still seems a useful definition. However, from the present results it seems that the phase space associated with those boundary conditions will not include solutions describing localised sources in a vacuum plane wave background, so it may not admit many physically interesting solutions. Understanding the space of asymptotically plane wave spacetimes is clearly important for attempts to construct a direct holographic duality directly for plane waves, so we would like to understand this issue better.

Similar problems have arisen in AdS_2 spacetimes [62], where there are no finite-energy asymptotically AdS_2 geometries, and in the study of near-horizon extremal Kerr solutions (NHEK) [63–65], where the space of metrics which are asymptotically NHEK consists only of the NHEK solution and solutions obtained from it by diffeomorphisms. It is interesting to note that plane waves, like AdS_2, have a one-dimensional boundary [10, 11]. Perhaps the problem is that there is in some sense "not enough space" near infinity to have interesting asymptotically plane wave solutions. It would be interesting to carry out a general analysis for asymptotically plane wave solutions along the lines of that in [64, 65].

We have only demonstrated that the action is well-behaved; an obvious extension of this work would be to go on to construct a boundary stress tensor $\langle T_{\alpha\beta}(x^+, x^-, \theta^i)\rangle$, as was done for the asymptotically flat case in [30] and for the linear dilaton case in [40]. This could then be used to calculate conserved quantities. The fact that different components of $g^{(1)}$ fall off at different rates at large r may lead to some interesting subtleties in extending the previous work to this case; perhaps, as in the asymptotically flat case, there will be more than one stress tensor associated with different orders in the asymptotic expansion.

A central motivation for work in this direction is to better understand holography for the plane wave. In [39], it was argued that a holographic dual of asymptotically flat space could be constructed on the hyperbola at spatial infinity, calculating two-point functions in the holographic dual from variations of the action. It is possible that similar ideas could be applied in this case, but there is no obvious connection between this notion of holography and the known example. String theory on the plane wave obtained from the Penrose limit of $\mathrm{AdS}_5 \times S^5$ is dual to a quantum mechanics so it has observables depending on a single coordinate, whereas if we were to construct a boundary stress tensor $\langle T_{\alpha\beta}(x^+, x^-, \theta^i)\rangle$ or two-point functions on the boundary at large r from our action, we would expect them to generically depend on all the boundary coordinates. Our remarks in section 3.1.2 on the relation between our notion of asymptotically plane wave and the conformal boundary of the maximally supersymmetric plane wave suggest that the boundary at large r we have focused on is not, at least, the whole story. To understand the relation to holography, we probably need to study the boundaries at constant x^- in more detail, and the information coming just from large r may be misleading.

This asymptotically plane wave example seems to have some interesting differences compared to previous attempts to study holography for more general spacetimes and we hope this work will shed some useful light on the relation between the bulk action and the holographic dual theory for other spacetimes which, in general, remains to be worked out.

We have also attempted to construct solutions describing black holes and black strings in plane wave backgrounds using the matched asymptotic expansion method. We have found that it is not possible to construct a regular black hole solution. In the approximation where the wave is thought of as a linearised perturbation on the black hole solution, we need a non-zero vector part in the spherical harmonic decomposition on the sphere, and it is not possible to make this vector part regular on the horizon. It would be interesting to have a deeper physical understanding of this failure of regularity. One might think that this is simply saying that the plane wave is exerting a force on the black hole so no stationary solution exists. However, we do not believe this is the correct interpretation of our result. The black hole was chosen to follow a geodesic in the plane wave background so there is no force on it at leading order. Finite size effects can be analysed in the asymptotic region using the classical effective field theory approach of [50, 70, 71]. In this approach, the work done by such finite size terms involves derivatives of the long wavelength background fields along the black hole world-line. Since our world-line is chosen to be an orbit of the isometries of the background, the work done will vanish. Thus, we would have expected the background to simply produce some deformation of the horizon.

The regularity problem seems to be simply an inconsistency between the symmetry structure of the black hole and the plane wave. In four dimensions, the problem is that the solution will not be axisymmetric, so there cannot be a regular black hole solution as all stationary four-dimensional black holes are required to be axisymmetric [68]. In higher dimensions, however, stationary axisymmetric solutions describing black holes in plane waves could, in principle, exist and the fact that our solutions are never regular is somewhat mysterious. Further exploration of this issue is an interesting project for the future.

The importance of this problem is reinforced by the fact that the failure of regularity here is a counter-example to the assumption in [52] that satisfying the blackfold equations implies horizon regularity. Understanding this issue in a more general context is clearly important for the blackfolds program [51, 52]. In considering the embedding of black branes in arbitrary backgrounds, we need to understand when the resulting deformation of the near-horizon region will preserve the regularity of the event horizon. Clearly we must require that the embedding of the blackfold in the background spacetime preserves enough symmetry to satisfy the rigidity theorems of [68, 69]. Our higher-dimensional examples indicate that this is a necessary but not a sufficient condition. Identifying sufficient conditions is an important general problem.

We have successfully constructed an approximate solution describing a black string in a vacuum plane wave background in five dimensions. It would clearly be interesting to extend this work to find black string solutions in the maximally supersymmetric plane wave background. It should be straightforward to extend our calculation to this case.

Appendix A

Harmonic identities on S^2

In this appendix we prove some harmonic identities needed for our analysis of black holes in the near horizon region. Definitions are given in section 4.2.1. We want to show that:

- $D^i D^j V_{ij} = 0$,

Proof:

$$\begin{aligned} D^i D^j V_{ij} &\propto D^i D^j D_i V_j + D^i D^j D_j V_i \\ &= [D^i, D^j] D_i V_j + 2 D^j D^i D_i V_j \\ &= -R^{k}{}_{i}{}^{ij} D_k V_j - R^{k}{}_{j}{}^{ij} D_i V_k - 2k_V^2 D^j V_j \\ &= R^{kj} D_k V_j - R^{ki} D_i V_k \\ &= 0. \end{aligned} \tag{A.0.1}$$

- $D^i D^j S_{ij} = \frac{(k^2-2)}{2} S$,

Proof:

$$\begin{aligned} D^i D^j S_{ij} &= \frac{1}{k^2} D^i D^j D_i D_j S + \frac{1}{2} D^j D_j S \\ &= \frac{1}{k^2} D^i [D^j, D_i] D_j S + \frac{1}{k^2} D^i D_i D^j D_j S + \frac{1}{2} D^j D_j S \\ &= -\frac{1}{k^2} D^i (R^{k}{}_{j}{}^{j}{}_{i} D_k S) + \frac{k^2}{2} S \\ &= \frac{1}{k^2} D^i (R^{k}{}_{i} D_k S) + \frac{k^2}{2} S \end{aligned} \tag{A.0.2}$$

for S^2, $R_{ij} = \gamma_{ij}$ so

$$\begin{aligned} D^iD^jS_{ij} &= \frac{1}{k^2}D^iD_iS + \frac{k^2}{2}S \\ &= \frac{(k^2-2)}{2}S. \end{aligned} \tag{A.0.3}$$

- $D^iS_{ij} = -\frac{1}{2k^2}(k^2-2)D_jS$,

Proof:

$$\begin{aligned} D^iS_{ij} &= \frac{1}{k^2}D^iD_iD_jS + \frac{1}{2}D_jS \\ &= \frac{1}{k^2}[D^i, D_j]D_iS + \frac{1}{k^2}D_jD^iD_iS + \frac{1}{2}D_jS \\ &= -\frac{1}{k^2}R^l{}_i{}^i{}_jD_lS - \frac{1}{2}D_jS \\ &= -\frac{1}{2k^2}(k^2-2)D_jS \end{aligned} \tag{A.0.4}$$

- $D^iV_{ij} = \frac{1}{2k_V^2}(k_V^2-1)V_j$,

Proof:

$$\begin{aligned} D^iV_{ij} &= -\frac{1}{2k_V^2}(D^iD_iV_j + D^iD_jV_i) \\ &= \frac{1}{2}V_j - \frac{1}{2k_V^2}[D^i, D_j]V_i - \frac{1}{2k_V^2}D_jD^iV_i \\ &= \frac{1}{2}V_j + \frac{1}{2k_V^2}R^k{}_i{}^i{}_jV_k \\ &= \frac{1}{2k_V^2}(k_V^2-1)V_j. \end{aligned} \tag{A.0.5}$$

Bibliography

[1] J. Le Witt and S. F. Ross, "Asymptotically Plane Wave Spacetimes and their Actions," *JHEP* **04** (2008) 084, arXiv:0801.4412 [hep-th].

[2] J. Le Witt and S. F. Ross, "Black holes and black strings in plane waves," arXiv:0910.4332 [hep-th].

[3] H. W. Brinkmann, "Einstein spaces which are mapped conformally on each other,"*Mathematische Annalen* **94** (Dec, 1925) 119–145.

[4] J. Bicak, "The role of exact solutions of Einstein's equations in the developments of general relativity and astrophysics selected themes," *Lect. Notes Phys.* **540** (2000) 1–126, arXiv:gr-qc/0004016.

[5] E. J. Jordan, P. and Kundt, "Strenge Losungen der Feldgleichungen der Allgemeinen Relativitatstheorie," *Math. Naturwiss. Kl.* (1960) .

[6] W. K. J. Ehlers, "Exact Solutions of the Gravitational Field Equations," *Gravitation: An Introduction to Current Research (ed. L. Witten),Wiley, New York* (1962) 49–101.

[7] M. Blau, "Plane Waves and Penrose Limits," (2006) .

[8] R. Penrose, "Any space-time has a plane wave as a limit," *Differential geometry and relativity* (1976) 271–275.

[9] R. Penrose, "A Remarkable property of plane waves in general relativity," *Rev. Mod. Phys.* **37** (1965) 215–220.

[10] D. Berenstein and H. Nastase, "On lightcone string field theory from super Yang-Mills and holography," `hep-th/0205048`.

[11] D. Marolf and S. F. Ross, "Plane waves: To infinity and beyond!," *Class. Quant. Grav.* **19** (2002) 6289–6302, `hep-th/0208197`.

[12] G. T. Horowitz and A. R. Steif, "Spacetime singularities in string theory,"*Phys. Rev. Lett.* **64** (Jan, 1990) 260–263.

[13] D. Sadri and M. M. Sheikh-Jabbari, "The plane-wave / super Yang-Mills duality," *Rev. Mod. Phys.* **76** (2004) 853, `hep-th/0310119`.

[14] G. 't Hooft, "Dimensional reduction in quantum gravity," `arXiv:gr-qc/9310026`.

[15] L. Susskind, "The World as a hologram," *J. Math. Phys.* **36** (1995) 6377–6396, `arXiv:hep-th/9409089`.

[16] J. D. Bekenstein, "Entropy bounds and black hole remnants,"*Phys. Rev. D* **49** (Feb, 1994) 1912–1921.

[17] J. M. Maldacena, "The large N limit of superconformal field theories and supergravity," *Adv. Theor. Math. Phys.* **2** (1998) 231–252, `arXiv:hep-th/9711200`.

[18] M. Blau, J. Figueroa-O'Farrill, C. Hull, and G. Papadopoulos, "Penrose limits and maximal supersymmetry," *Class. Quant. Grav.* **19** (2002) L87–L95, `hep-th/0201081`.

[19] M. Blau, J. M. Figueroa-O'Farrill, C. Hull, and G. Papadopoulos, "A new maximally supersymmetric background of IIB superstring theory," *JHEP* **01** (2002) 047, `arXiv:hep-th/0110242`.

[20] D. E. Berenstein, J. M. Maldacena, and H. S. Nastase, "Strings in flat space and pp waves from $N = 4$ super Yang Mills," *JHEP* **04** (2002) 013, `hep-th/0202021`.

[21] V. E. Hubeny and M. Rangamani, "No horizons in pp-waves," *JHEP* **11** (2002) 021, `hep-th/0210234`.

[22] N. Kaloper, R. C. Myers, and H. Roussel, "Wavy strings: Black or bright?," *Phys. Rev. D* **55** (1997) 7625–7644, `hep-th/9612248`.

[23] V. E. Hubeny and M. Rangamani, "Generating asymptotically plane wave spacetimes," *JHEP* **01** (2003) 031, `hep-th/0211206`.

[24] E. G. Gimon and A. Hashimoto, "Black holes in Gödel universes and pp-waves," *Phys. Rev. Lett.* **91** (2003) 021601, `hep-th/0304181`.

[25] E. G. Gimon, A. Hashimoto, V. E. Hubeny, O. Lunin, and M. Rangamani, "Black strings in asymptotically plane wave geometries," *JHEP* **08** (2003) 035, `hep-th/0306131`.

[26] V. E. Hubeny and M. Rangamani, "Horizons and plane waves: A review," *Mod. Phys. Lett.* **A18** (2003) 2699–2712, `hep-th/0311053`.

[27] D. Marolf and S. F. Ross, "A new recipe for causal completions," *Class. Quant. Grav.* **20** (2003) 4085–4118, `gr-qc/0303025`.

[28] D. Marolf and S. F. Ross, "Plane waves and spacelike infinity," *Class. Quant. Grav.* **20** (2003) 4119–4133, `hep-th/0303044`.

[29] G. W. Gibbons and S. W. Hawking, "Action integrals and partition functions in quantum gravity,"*Phys. Rev. D* **15** (May, 1977) 2752–2756.

[30] R. B. Mann and D. Marolf, "Holographic renormalization of asymptotically flat spacetimes," *Class. Quant. Grav.* **23** (2006) 2927–2950, `hep-th/0511096`.

[31] E. A. Martinez, "Quasilocal energy for a Kerr black hole," *Phys. Rev.* **D50** (1994) 4920–4928, `arXiv:gr-qc/9405033`.

[32] M. Henningson and K. Skenderis, "The holographic Weyl anomaly," *JHEP* **07** (1998) 023, `arXiv:hep-th/9806087`.

[33] V. Balasubramanian and P. Kraus, "A stress tensor for anti-de Sitter gravity," *Commun. Math. Phys.* **208** (1999) 413–428, `hep-th/9902121`.

[34] R. B. Mann, "Misner string entropy," *Phys. Rev. D* **60** (1999) 104047, `hep-th/9903229`.

[35] P. Kraus, F. Larsen, and R. Siebelink, "The gravitational action in asymptotically AdS and flat spacetimes," *Nucl. Phys.* **B563** (1999) 259–278, `hep-th/9906127`.

[36] R. B. Mann, D. Marolf, and A. Virmani, "Covariant counterterms and conserved charges in asymptotically flat spacetimes," *Class. Quant. Grav.* **23** (2006) 6357–6378, `gr-qc/0607041`.

[37] D. Astefanesei, R. B. Mann, and C. Stelea, "Note on counterterms in asymptotically flat spacetimes," *Phys. Rev.* **D75** (2007) 024007, `hep-th/0608037`.

[38] R. B. Mann, D. Marolf, R. McNees, and A. Virmani, "On the Stress Tensor for Asymptotically Flat Gravity," *Class. Quant. Grav.* **25** (2008) 225019, `arXiv:0804.2079 [hep-th]`.

[39] D. Marolf, "Asymptotic flatness, little string theory, and holography," *JHEP* **03** (2007) 122, `hep-th/0612012`.

[40] D. Marolf and A. Virmani, "Holographic renormalization of gravity in little string theory duals," *JHEP* **06** (2007) 042, `hep-th/0703251`.

[41] R. B. Mann and R. McNees, "Boundary Terms Unbound! Holographic Renormalization of Asymptotically Linear Dilaton Gravity," `arXiv:0905.3848 [hep-th]`.

[42] R. Emparan and H. S. Reall, "Black Holes in Higher Dimensions," *Living Rev. Rel.* **11** (2008) 6, `arXiv:0801.3471 [hep-th]`.

[43] R. C. Myers and M. J. Perry, "Black Holes in Higher Dimensional Space-Times," *Ann. Phys.* **172** (1986) 304.

[44] R. Emparan and H. S. Reall, "A rotating black ring in five dimensions," *Phys. Rev. Lett.* **88** (2002) 101101, arXiv:hep-th/0110260.

[45] R. Emparan and R. C. Myers, "Instability of ultra-spinning black holes," *JHEP* **09** (2003) 025, arXiv:hep-th/0308056.

[46] H. Elvang and P. Figueras, "Black Saturn," *JHEP* **05** (2007) 050, arXiv:hep-th/0701035.

[47] E. Witten, "Anti-de Sitter space, thermal phase transition, and confinement in gauge theories," *Adv. Theor. Math. Phys.* **2** (1998) 505–532, arXiv:hep-th/9803131.

[48] G. W. Gibbons, H. Lu, D. N. Page, and C. N. Pope, "Rotating black holes in higher dimensions with a cosmological constant," *Phys. Rev. Lett.* **93** (2004) 171102, arXiv:hep-th/0409155.

[49] T. Harmark, "Small black holes on cylinders," *Phys. Rev.* **D69** (2004) 104015, arXiv:hep-th/0310259.

[50] W. D. Goldberger and I. Z. Rothstein, "An effective field theory of gravity for extended objects," *Phys. Rev.* **D73** (2006) 104029, arXiv:hep-th/0409156.

[51] R. Emparan, T. Harmark, V. Niarchos, and N. A. Obers, "Blackfolds," *Phys. Rev. Lett.* **102** (2009) 191301, arXiv:0902.0427 [hep-th].

[52] R. Emparan, T. Harmark, V. Niarchos, and N. A. Obers, "Essentials of Blackfold Dynamics," arXiv:0910.1601 [Unknown].

[53] R. M. Wald, *General Relativity.* Chicago University Press, Chicago, USA, 1984.

[54] I. Papadimitriou and K. Skenderis, "AdS / CFT correspondence and geometry," arXiv:hep-th/0404176.

[55] I. Kanitscheider, K. Skenderis, and M. Taylor, "Precision holography for non-conformal branes," *JHEP* **09** (2008) 094, arXiv:0807.3324 [hep-th].

[56] R. Emparan, T. Harmark, V. Niarchos, N. A. Obers, and M. J. Rodriguez, "The Phase Structure of Higher-Dimensional Black Rings and Black Holes," *JHEP* **10** (2007) 110, `arXiv:0708.2181 [hep-th]`.

[57] B. Carter, "Essentials of classical brane dynamics," *Int. J. Theor. Phys.* **40** (2001) 2099–2130, `arXiv:gr-qc/0012036`.

[58] D. Gorbonos and B. Kol, "A dialogue of multipoles: Matched asymptotic expansion for caged black holes," *JHEP* **06** (2004) 053, `arXiv:hep-th/0406002`.

[59] R.-G. Cai and N. Ohta, "Surface counterterms and boundary stress-energy tensors for asymptotically non-anti-de Sitter spaces," *Phys. Rev. D* **62** (2000) 024006, `hep-th/9912013`.

[60] C. Barrabes and W. Israel, "Lagrangian brane dynamics in general relativity and Einstein-Gauss-Bonnet gravity," *Phys. Rev. D* **71** (2005) 064008, `gr-qc/0502108`.

[61] M. M. Caldarelli, R. Emparan, and M. J. Rodriguez, "Black Rings in (Anti)-deSitter space," *JHEP* **11** (2008) 011, `arXiv:0806.1954 [hep-th]`.

[62] J. M. Maldacena, J. Michelson, and A. Strominger, "Anti-de Sitter fragmentation," *JHEP* **02** (1999) 011, `arXiv:hep-th/9812073`.

[63] M. Guica, T. Hartman, W. Song, and A. Strominger, "The Kerr/CFT Correspondence," `arXiv:0809.4266 [hep-th]`.

[64] A. J. Amsel, G. T. Horowitz, D. Marolf, and M. M. Roberts, "No Dynamics in the Extremal Kerr Throat," *JHEP* **09** (2009) 044, `arXiv:0906.2376 [hep-th]`.

[65] O. J. C. Dias, H. S. Reall, and J. E. Santos, "Kerr-CFT and gravitational perturbations," *JHEP* **08** (2009) 101, `arXiv:0906.2380 [hep-th]`.

[66] H. Kodama and A. Ishibashi, "A master equation for gravitational perturbations of maximally symmetric black holes in higher dimensions," *Prog. Theor. Phys.* **110** (2003) 701–722, arXiv:hep-th/0305147.

[67] D. Garfinkle and T. Vachaspati, "Cosmic string traveling waves," *Phys. Rev.* **D42** (1990) 1960–1963.

[68] S. W. Hawking, "Black holes in general relativity," *Commun. Math. Phys.* **25** (1972) 152–166.

[69] S. Hollands, A. Ishibashi, and R. M. Wald, "A Higher Dimensional Stationary Rotating Black Hole Must be Axisymmetric," *Commun. Math. Phys.* **271** (2007) 699–722, arXiv:gr-qc/0605106.

[70] W. D. Goldberger and I. Z. Rothstein, "Dissipative effects in the worldline approach to black hole dynamics," *Phys. Rev.* **D73** (2006) 104030, arXiv:hep-th/0511133.

[71] B. Kol and M. Smolkin, "Classical Effective Field Theory and Caged Black Holes," *Phys. Rev.* **D77** (2008) 064033, arXiv:0712.2822 [hep-th].