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Fall 2008

# Uncertainty analysis and sensitivity analysis for multidisciplinary systems design

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UNCERTAINTY ANALYSIS AND SENSITIVITY ANALYSIS

FOR MULTIDISCIPLINARY SYSTEMS DESIGN


by


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A DISSERTATION

Presented to the Faculty of the Graduate School of the

MISSOURI UNIVERSITY OF SCIENCE & TECHNOLOGY

In Partial Fulfillment of the Requirements for the Degree


DOCTOR OF PHILOSOPHY

in

MECHANICAL ENGINEERING


2008

Approved by

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## PUBLICATION DISSERTATION OPTION

This dissertation consists of the following three articles that have been submitted for publication as follows:

Pages 7-54 have been published in AIAA JOURNAL, Vol.45, No.9, 2007.

Pages 55-93 have been submitted for publication to INTERNATIONAL JOURNAL OF NUMERICAL METHODS IN ENGINEERING. The revision is under review.

Pages 94-131 have been published in the proceedings of the 49th AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics, and Materials Conference, Schaumburg, IL, Apr.7-10, 2008, and submitted for publication to AIAA JOURNAL.

**ABSTRACT**

The objective of this research is to quantify the impact of both aleatory and epistemic uncertainties on performances of multidisciplinary systems. Aleatory uncertainty comes from the inherent uncertain nature and epistemic uncertainty comes from the lack of knowledge. Although intensive research has been conducted on aleatory uncertainty, few studies on epistemic uncertainty have been reported. In this work, the two types of uncertainty are analyzed. Aleatory uncertainty is modeled by probability distributions while epistemic uncertainty is modeled by intervals. Probabilistic analysis (PA) and interval analysis (IA) are integrated to capture the effect of the two types of uncertainty. The First Order Reliability Method is employed for PA while nonlinear optimization is used for IA. The unified uncertainty analysis, which consists of PA and IA, is employed to develop new sensitivity analysis methods for the mixture of the two types of uncertainty. The methods are able to quantify the contribution of each input variable with either epistemic uncertainty or aleatory uncertainty. The analysis results can then help better decision making on how to effectively mitigate the effect of uncertainty. The other major contribution of this research is the extension of the unified uncertainty analysis to the reliability analysis for multidisciplinary systems.

The major findings of this research are as follows. (1) Sensitivity analysis method is an effective tool for reducing the impact of epistemic uncertainty. (2) The proposed new reliability sensitivity indexes can easily measure the changes in output uncertainty with respect to those in input uncertainty. (3) The effect of aleatory uncertainty can be primarily measured by the distribution of a performance; and the effect of epistemic uncertainty can be measured by the bounds of the distribution. (4) The unified uncertainty analysis methods for single-disciplinary systems can be extended to the reliability analysis for multidisciplinary systems. (5) All the proposed methods can be ultimately integrated with multidisciplinary design optimization.

# ACKNOWLEDGMENTS

First of all, I would like to express my deepest gratitude to my academic advisor, Dr. Xiaoping Du, for his excellent guidance, patience, caring and providing me with an excellent atmosphere for my study and research through my years in Rolla. Without his help, I would never have been able to finish my dissertation.

I am also grateful to Dr. Frank Liou, Dr. K. Chandrashekhara, Dr. Xuerong Wen, and Dr. Serhat Hosder for serving on my advisory committee and teaching me many interesting knowledge and valuable skills during my study at the Missouri University of Science and Technology (formerly University of Missouri-Rolla).

I would like to thank the U.S. National Science Foundation, the Intelligent Systems Center at MST, and the University of Missouri Research Board for supporting this research.

Last but not least, I would like to thank all my friends and family members, especially, my parents, Zhonghong Guo and Caixia Cheng, for their endless encouragement and support during my graduate studies.

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# 1. INTRODUCTION

The rising demand for high reliability, robustness and safety of complex engineering systems, such as automobiles and aircrafts, requires engineers to understand and manage various uncertainties during their design process. Such uncertainties include manufacturing variation, imperfect approximation, imprecise estimates of loading, and limited samples. The ignorance of these uncertainties could lead to significant design bias, costly maintenance, even a catastrophic consequence, especially, for multidisciplinary systems. Therefore, it has become imperative to identify the sources of uncertainty and quantify the impact of multiple types of uncertainties in multidisciplinary systems design.

Uncertainty can be classified into two different types: aleatory uncertainty and epistemic uncertainty. Aleatory uncertainty is defined as the randomness or inherent variability of the nature, and it is objective and irreducible. Aleatory uncertainty is usually modeled by probability theory. Examples of this category include the dimensions of manufacturing parts and material properties. On the other hand, epistemic uncertainty is due to the lack of knowledge or the incompleteness of information. It is subjective and reducible. The assumptions made in building models are one example of epistemic uncertainty. Probability theory and non-probability theories such as evidence theory, possibility theory and fuzzy set can be used to model epistemic uncertainty.

In the past decades, much effort has been spent on exploring the effect of aleatory uncertainty on both single-disciplinary systems and multidisciplinary systems, while very few investigations have been reported in studying epistemic uncertainty and the mixture of aleatory and epistemic uncertainties. Aleatory and epistemic uncertainties exist simultaneously in real-world systems. Conventional uncertainty analysis methods are not

capable of handling such a situation. Hence, this research attempts to address this issue and answer the following question:

*Given the inputs to a multidisciplinary system with both aleatory and epistemic uncertainties, what will be the uncertain characteristics of outputs?*

The challenge to solve this question lies in several aspects:

1) A large number of uncertain variables are involved, including input variables with aleatory uncertainty, input variables with epistemic uncertainty, and coupling variables bridging different subsystems. In addition, each subsystem has its own local input variables and also shares system input variables with other subsystems. Solving such a high dimensional problem will need a huge computational cost.

2) All the subsystems are often highly coupled together. The output of one subsystem may be the input to other subsystems, and vice versa. This coupling makes the functional relationships between outputs and inputs complicated and highly nonlinear. Besides, uncertainty will be propagated from one subsystem to other subsystems through the interfaces over coupled subsystems. It becomes very difficult to capture the overall effect of accumulated uncertainties from all the subsystems.

3) A full range of uncertainty should be taken into account. New approaches are required to capture the effect of mixed aleatory and epistemic uncertainties on the performance of a multidisciplinary system. How to efficiently propagate the mixture of both uncertainties through all the subsystems is a major concern.

This research adopts the following approaches to address these challenges: 1) To model a full range of uncertainty, probability theory is used to model aleatory uncertainty, and evidence theory and intervals are chosen to represent epistemic uncertainty. 2) To

estimate the effect of the mixture of both uncertainties, probabilistic analysis and interval analysis are integrated with a unified uncertainty analysis framework. The First Order Reliability Method (FORM) is employed for probabilistic analysis because FORM has a good balance between accuracy and efficiency. Nonlinear optimization is used for interval analysis to ensure a higher accuracy. 3) To mitigate the effect of epistemic uncertainty, sensitivity analysis method is developed to find the most important input variables with epistemic uncertainty. Collecting more information on these variables instead of all the variables will reduce the effect of epistemic uncertainty in the most efficient way. 4) To efficiently propagate both uncertainties through various subsystems, sequential optimization and single loop strategies are used for integrating probabilistic analysis and interval analysis with multidisciplinary analysis. Expensive Monte Carlo Simulation can therefore be avoided.

The three articles included in this dissertation provide the details and major findings of this research on the above-mentioned issues. The framework of this dissertation is illustrated in Figure 1. The research consists of three research tasks. The first one is sensitivity analysis with the mixture of aleatory and epistemic uncertainty, the second one is reliability sensitivity analysis with random and interval variables, and the third one is reliability analysis for multidisciplinary systems with random and interval variables. The objective of the first research task is to determine the effect of a full range of uncertainty on a single-disciplinary system. This research task is based on the unified uncertainty analysis framework developed by Du [1]. The unified uncertainty analysis framework is shown as initial study in Fig.1. This initial study can calculate the belief and plausibility measures of output. With the help of the initial work, the dissertation

develops an effective sensitivity analysis method (the first paper) for epistemic uncertainty when both aleatory and epistemic uncertainties are involved. This method is able to estimate contributions of independent input variables with epistemic uncertainty to the model output and rank the importance of each variable. Guided by the results from sensitivity analysis, we can collect more information on the most significant variables and reduce the effect of epistemic uncertainty in the most efficient way.

| Single-disciplinary System | | Multidisciplinary System |
|---|---|---|
| **WHAT** will happen to the output with inputs of a full range of uncertainty? | **Initial Study**[1] Unified Uncertainty Analysis Framework | **HOW** to extend it to multidisciplinary systems? |
| **WHICH** are the most significant variables with epistemic uncertainty? | **Paper I** Sensitivity Analysis with the Mixture of Aleatory and Epistemic Uncertainty | **Paper III** Reliability Analysis for Multidisciplinary Systems with Random and Interval Variables |
| **HOW** will the output uncertainty change upon the change in input uncertainty? | **Paper II** Reliability Sensitivity Analysis with Random and Interval Variables | |

**Figure 1. Framework of this dissertation**

As shown above, collecting more information on variables with epistemic uncertainty will reduce the impact of input uncertainty. To do it effectively, we need to answer the following question:

*How will the characteristics of output uncertainty change when we change the characteristics of input uncertainty?*

Answering the above question is the objective of the second research task (the second article). In this task, we study reliability sensitivity analysis with random and interval variables. Here random variables are used for those with aleatory uncertainty, and their probability distributions are known. Interval variables are used for those with epistemic uncertainty when only the lower and upper bounds of those variables are known. Six new sensitivity indices are proposed to evaluate the sensitivities of the width and average of the probability of failure bounds with respect to the width and mean of each input interval variable, as well as the distribution parameters of each input random variable. These indices tell us what exact change will happen to the reliability bounds when we change the characteristics of uncertain variables.

Both of the above research tasks provide the effective tools to quantify the effect of a full range of uncertainty on outputs for single-disciplinary systems. With them as a basis, in research task 3 (the third article), we answer the question:

*How to extend methods for single-disciplinary systems to multidisciplinary systems*?

A unified reliability analysis is developed for multidisciplinary systems with random and interval variables (paper III). Three algorithms are proposed to get the better computational efficiency for different situations. Using these algorithms, we will be able to calculate the bounds of reliability or the probability of failure of each output from a multidisciplinary system.

To sum up, this research provides a group of effective and efficient analysis tools to deal with a full range of uncertainty in multidisciplinary systems design. With the sensitivity analysis methods, designers will be able to evaluate the effect of each input variable with epistemic uncertainty on the system outputs and determine the most significant input variables. Collecting more information on these most significant variables will efficiently reduce the effect of epistemic uncertainty. The smaller effect will help designers make more reliable decisions. The new reliability sensitivity analysis is a byproduct of reliability analysis where the calculation of sensitivity indexes does not require additional function evaluations. And the proposed sensitivity indexes will provide engineers with more exact understanding of how the uncertainty in the performance will change upon the changes in the input uncertainty. The unified reliability analysis for multidisciplinary systems accommodates a full range of uncertainty and facilitates the application of reliability analysis to a wider range of engineering fields with mixed aleatory and epistemic uncertainty. Designers of multidisciplinary systems are able to propagate a full range of uncertainty through all the coupled subsystems and quantify the effect of overall uncertainty on each output—the bounds of reliability.

**PAPER I**

# Sensitivity Analysis with the Mixture of Epistemic and Aleatory Uncertainties

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**Study on epistemic uncertainty due to the lack of knowledge has received increasing attention in risk assessment, reliability analysis, decision-making, and design optimization. Different theories have been applied to model and quantify epistemic uncertainty. Research on sensitivity analysis for epistemic uncertainty has also been initialized. Sensitivity analysis can identify the contributions of individual input variables with epistemic uncertainty to the model output. It then helps guide the collection of more information to reduce the effect of epistemic uncertainty. In this paper, an effective sensitivity analysis method for epistemic uncertainty is proposed when both epistemic and aleatory uncertainties exist in model inputs. This method employs the unified uncertainty analysis framework to calculate the plausibility measure and belief measure. The gap between belief and plausibility measures is used as an indicator of the effect of epistemic uncertainty on the model output. The Kolmogorov-Smirnov (KS) distance between the two measures is used to quantify the main effect and total effect of each independent variable with epistemic uncertainty. By the KS distance, the importance of each variable is ranked. The feasibility and effectiveness of the proposed method is demonstrated with two engineering examples.**

## Nomenclature

$Bel$ = belief
$C$ = subset of intervals
$d_{ks}$ = KS distance
$F$ = cumulative distribution function (CDF)

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$f$ = probability density function (PDF)
$G$ = output of a performance function
$g$ = performance function
$ME$ = main effect
$ME_{pf}$ = main effect on the probability of failure
$m_Y$ = basic probability assignment (BPA)
$P$ = probability
$Pl$ = plausibility
$p_f$ = probability of failure
$R$ = reliability
$TE$ = total effect
$TE_{pf}$ = total effect on the probability of failure
$\mathbf{U}$ = vector of standard normal variables
$U$ = standard normal variable
$u$ = realization of $U$
$\mathbf{u}^*$ = Most Probable Point in $u$-space (MPP)
$\mathbf{X}$ = vector of random variables
$X$ = random variable
$x$ = realization of $X$
$\mathbf{x}^*$ = Most Probable Point in $x$-space (MPP)
$\mathbf{Y}$ = vector of variables with epistemic uncertainty
$Y$ = variable with epistemic uncertainty
$y$ = realization of $Y$
$\beta$ = reliability index
$\Phi$ = cumulative distribution function of a standard normal distribution
$\Phi^{-1}$ = inverse function of $\Phi$
$\phi$ = probability density function of a standard normal distribution

# I. Introduction

UNCERTAINTY is ubiquitous in any engineering system, at any stage of product development, and throughout a product life cycle. Examples of uncertainty are manufacturing imprecision, usage variations, imperfect knowledge, and variability associated with loading, material properties, and geometric dimensions. Such uncertainties have a significant impact on product performance. A small variation in environment or design variables may lead to a significant quality loss. The ignorance of uncertainty may cause erroneous decision-making, low robustness and reliability, costly

warranty, low customer satisfaction, and even catastrophe.[1-5] With the intensive requirement of high product quality and reliability, understanding, identifying and managing various uncertainties have become imperative.

Uncertainty can be viewed as the difference between the present state of knowledge and the complete knowledge (Fig. 1). It is classified into aleatory and epistemic types.[6]
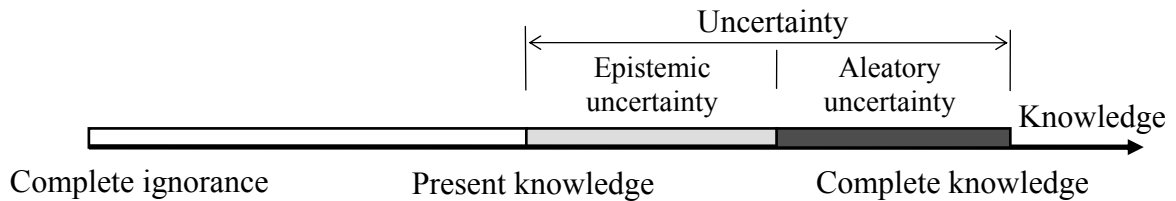


**Fig. 1 Uncertainty types.**

*Aleatory uncertainty*, also referred to as irreducible, objective or stochastic uncertainty, describes the inherent variability associated with a physical system or environment.[7-9] Aleatory uncertainty is modeled by random variables or stochastic processes by probability theory if information is sufficient to estimate probability distributions. For example, for a cantilever beam in Fig. 2, aleatory uncertainty exists in the dimensions $b$, $h$, and $l$ (due to manufacturing imprecision), external force $Q$ (due to variations in operation), and material properties (due to the stochastic physical nature). All the above quantities can be modeled as random variables if adequate statistical data are available. Aleatory uncertainty has been intensively researched and dealt with in a wide range of engineering fields.
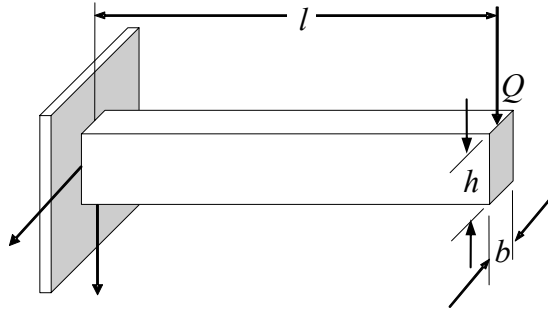
**Fig. 2 A cantilever beam.**

*Epistemic uncertainty*, on the other hand, is due to the lack of knowledge about a physical system or environment.[10, 11] In the above beam example, if we use different theories to calculate the stress and deflection, we may end up with different results. The reason is that each theory relies on various assumptions, which may not be completely valid. Epistemic uncertainty therefore exists in the model structure. Also, if the data of the external force $Q$ is scarce, the distribution of $Q$ may not be precisely known. This indicates that epistemic uncertainty may also exist in a parameter. Epistemic uncertainty is reducible because the collection of more information or an increase of knowledge would help decrease the level of uncertainty. In this work, we only focus on epistemic parameter uncertainty.

Different theories have been used to handle epistemic uncertainty. The theories include probability theory and non-probability theories such as evidence theory,[12] possibility theory,[13,14] and fuzzy set theory.[15] Evidence theory is widely used to deal with epistemic uncertainty. Intervals with evidence theory interpretation are especially of interest in engineering applications.[9] Although there has been a longtime debate on whether probability theory is universal for handling all types of uncertainty, intervals do

exist in many engineering applications, and its use is well justified in a vast amount of literature.[8] For example, for the above beam problem, the mean of the distribution of the external force $Q$ may be given by a confidence interval with limited samples. Engineers often specify their design variables in the form of nominal value ± tolerance. More interval examples are given by Du [16] and Du, et al.. [17]

Evidence theory is the generalization of probability theory and possibility theory. [18, 19] It can handle limited or even conflicting information. Most importantly, it is able to combine aleatory and epistemic uncertainties in a straightforward way.[19,20] Exploratory research on epistemic uncertainty by evidence theory has recently been conducted, including studies in risk assessment, decision-making, and design optimization.[8, 14, 17~25]

Most of the research focuses on uncertainty quantification and uncertainty analysis. A few investigations[18, 26, 27] have been conducted to explore sensitivity analysis with epistemic uncertainty. The purpose of such sensitivity analysis is to quantify the contribution of the input epistemic uncertainty to the model output. Bae, et al.[18, 26] develop a sensitivity analysis method for belief and plausibility measures. The method provides useful information to guide the future acquisition for more accurate reliability analysis and to reveal the most significant contributing factors in a sequential design phase. Helton, et al.[27] propose a three-step sampling-based sensitivity analysis for epistemic uncertainty. In their work, an initial exploratory analysis is employed to evaluate the model behavior, and then stepwise analyses are followed to show the incremental effects of uncertain variables on belief and plausibility measures.

The above sensitivity analysis methods deal with only epistemic uncertainty. In practical engineering applications, both aleatory and epistemic uncertainties often occur

simultaneously. Under this situation, a single probability measure (for instance, reliability) will not be available. Instead, its plausibility and belief measures must be used. Both of the measures will be discussed in the next section. The difference between the belief measure and plausibility measure indicates the effect of epistemic uncertainty. If the difference is too large, it will be difficult to make decisions. In this case, more information is needed in order to reduce the effect of epistemic uncertainty. Collecting more information on all the variables with epistemic uncertainty is costly. Collecting additional information on only the most important variables will be more efficient. Identifying variables with epistemic uncertainty that have the highest contribution to the uncertainty effect is the focus of sensitivity analysis in this paper. Since the proposed sensitivity analysis needs to quantify the uncertain characteristics of a model output given aleatory and epistemic uncertainties in model inputs, the unified uncertainty analysis framework[16] is used.

The paper is organized as follows. Brief introductions to sensitivity analysis, evidence theory, and unified uncertainty analysis, are provided in Section 2. The proposed sensitivity analysis method is discussed in Section 3. In Section 4, two examples are used for demonstration. Conclusions and future work are given in Section 5.

## II.    Sensitivity Analysis with Epistemic Uncertainty

### A. Sensitivity Analysis

Sensitivity analysis identifies the input uncertain variables that have the highest contribution to the uncertainty in output variables. So far most of research focuses on sensitivity analysis for aleatory uncertainty, which is mainly modeled by probability

theory. Such sensitivity analysis with a probabilistic representation is usually named *probabilistic sensitivity analysis*. Various probabilistic sensitivity analysis methods have been reported in a wide range of literature, including differential analysis,[28, 29] variance-based methods,[30] sampling-based methods,[30] and relative entropy based method.[31] Among them, the variance-based method is popular, which derives from the decomposition of the total variance of a model output into variances due to different input variables and their combinations. The Fourier Amplitude Sensitivity Test (FAST),[32, 33] correlation ratios,[34] importance measures,[35] and Sobol's indices[36] belong to this type of method.

Generally, these methods work well with the probabilistic representation. However, how to apply these methods to obtain the sensitivity information from epistemic uncertainty has not been well studied.

As mentioned in the introduction section, Bae, et al.,[18, 26] and Helton, et al.[27] have conducted exploratory research on sensitivity analysis with epistemic uncertainty. In this work, we are interested in the independent epistemic variables, and our goal is to develop a new sensitivity analysis method for identifying the most important variables with epistemic uncertainty when both aleatory and epistemic uncertainties are present. We employ the unified uncertainty analysis[16] to quantify both types of uncertainty. We then perform sensitivity analysis to identify the main effect and total effect of each variable with epistemic uncertainty by the once-at-a-time (OAT) strategy[37, 38] and the two-dimensional Kolmogorov-Smirnov (KS) distance.[39] Next, we provide a brief review of evidence theory and the unified uncertainty analysis.

## B. Evidence Theory

Intervals are widely used to characterize epistemic uncertainty. They can be naturally handled by evidence theory.[8] A good example of intervals is the periodic monitoring.[16] Suppose the status of a system is monitored at discrete time instants $t_0, t_1, t_2, \cdots$. If a failure is detected at $t_{i+1}$, then the failure could occur at any time in the interval between $t_i$ and $t_{i+1}$. In this case, we may not be able to determine the exact distribution of the failure time. But we can collect information to estimate the probability of the failure occurrence over each time interval. The probability assigned to an interval is defined as Basic Probability Assignment (BPA) in evidence theory. For example, for 20 systems, if 2 and 5 failures occurred over $[t_4, t_5]$ and $[t_9, t_{10}]$, respectively, the BPAs of intervals $[t_4, t_5]$ and $[t_9, t_{10}]$ would be $2/20 = 0.1$ and $5/20 = 0.4$, respectively.

In this paper, we use $Y$ to denote a variable with epistemic uncertainty. For brevity, we will call $Y$ an epistemic variable in the remainder of the paper. We also use this same symbol $Y$ to represent its frame of discernment, which is the sample space containing all the possible values of $Y$. We use $\mathcal{P}(Y)$ to denote the power set, the set that contains all the possible distinct subsets of $Y$. We also use $A$ to denote an element of the power set.

In evidence theory, a BPA is a mapping function, $\mathcal{P}(Y) \rightarrow [0,1]$, satisfying the following three axioms:

$$1)\ m_Y(A) \geq 0 \ \text{ for any } A \in \mathcal{P}(Y). \tag{1}$$

$$2)\ m_Y(\varnothing) = 0 \ . \tag{2}$$

$$3)\ \sum_{A \in \mathcal{P}(Y)} m_Y(A) = 1 \ . \tag{3}$$

For two epistemic variables $Y_1$ and $Y_2$ , if the change in $Y_1$ does not affect $Y_2$, and vice versa, $Y_1$ are $Y_2$ are said to be independent. Similar to the joint probability in probability theory, for two independent epistemic variables, $Y_1$ and $Y_2$, their joint BPA is also used. The joint BPA is defined by

$$m_Y(C) = \begin{cases} m_{Y_1}(A) \cdot m_{Y_2}(B) & \text{when } C = A \times B \\ 0 & \text{otherwise} \end{cases} \tag{4}$$

where $A \in \mathcal{P}(Y_1)$, $B \in \mathcal{P}(Y_2)$, $\mathbf{Y} = Y_1 \times Y_2$, and $C \in \mathcal{P}(Y)$ . $\mathbf{Y} = Y_1 \times Y_2$ denotes the joint space of $Y_1$ are $Y_2$.

Because of the interval nature, a single probability measure is not available. Instead, two measures, belief and plausibility measures, are used in evidence theory. In this paper, we consider that the BPAs of epistemic variables are from non-conflicting items of evidence and that only one BPA exists for one interval of an epistemic variable. Under these conditions, belief and plausibility measures can be considered as the lower and upper bounds of a probability measure[40]. Let a performance $G$ be expressed abstractly by a performance function $G = g(\mathbf{Y})$, where $\mathbf{Y} = (Y_1, Y_2, \cdots, Y_{n_Y})$ is the vector of epistemic variables. Let an event $E$ be defined by the performance less than a specific limit state $c$, namely, $E = \{\mathbf{Y} \mid g(\mathbf{Y}) < c\}$ .    Also let $m_\mathbf{Y}$ be the joint BPA over a frame $\mathbf{Y} = Y_1 \times Y_2 \times \cdots \times Y_{n_Y}$ .  The belief measure $Bel$ and the plausibility measure $Pl$ of the event $E \in \mathbf{Y}$ induced by $m_\mathbf{Y}$ are calculated by

$$Bel(E) = \sum_{A \in E} m_\mathbf{Y}(A), \tag{5}$$

and

$$Pl(E) = \sum_{A \cap E \neq \varnothing} m_\mathbf{Y}(A). \tag{6}$$

respectively.

$Bel(E)$ is interpreted as the degree of belief the event $E$ would occur. As shown in Eq. (5), it is calculated by adding the BPAs of the subsets entirely within the region $g(\mathbf{Y}) < c$. As indicated in Eq. (6), the degree of plausibility $Pl(E)$ is calculated by adding the BPAs of the subsets that are completely in the region $g(\mathbf{Y}) < c$ and the BPAs of the subsets that intersect with the region. The true probability $\Pr\{g(\mathbf{Y}) < c\}$ is bounded by *Bel* (*E*) and *Pl* (*E*) under the abovementioned condition.

Next, we give a short review of the unified uncertainty analysis,[16] which integrates probability and evidence theories to deal with the mixture of aleatory and epistemic uncertainties. The proposed sensitivity analysis relies on the unified uncertainty analysis.

## C. Unified Uncertainty Analysis

A framework of unified uncertainty analysis is given in Fig. 3.[16] The inputs to the framework are variables **X** with aleatory uncertainty defined by probability density functions (PDF) and epistemic variables **Y** represented by BPAs. Both types of uncertainty in the model inputs **X** and **Y** are propagated through the model $g(\mathbf{X}, \mathbf{Y})$ to the model output *G*. The outcomes of the uncertainty analysis are cumulative belief and plausibility functions (*CBF* and *CPF*).



**X**: joint PDF

$G = g(\mathbf{X}, \mathbf{Y})$

**Y**: joint BPA

*G*: Belief and plausibility functions
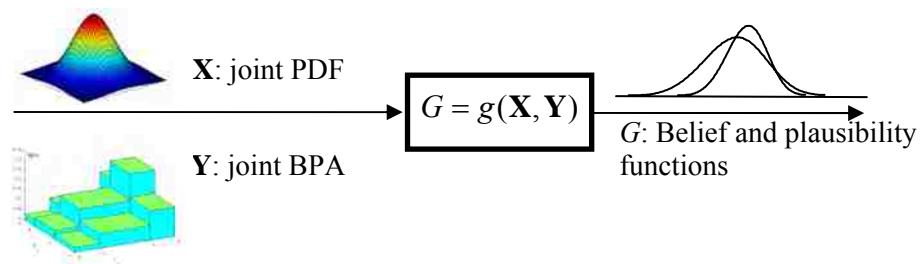
**Fig. 3 The unified uncertainty analysis framework.**

Let the subsets of $\mathbf{Y}$ be denoted by $\mathbf{C}_{\mathbf{Y}i}$ $(i = 1, 2, \cdots, n)$ with the corresponding joint BPA $m_{\mathbf{Y}}(\mathbf{C}_{\mathbf{Y}i})$. After appropriate information aggregation,[9, 12] $\mathbf{C}_{Yi}$ $(i = 1, 2, \cdots, n)$ can be disjoint. The entire input space therefore is partitioned into $n$ mutually exclusive subsets $\mathbf{C}_{\mathbf{XY}i} = (\mathbf{X}, \mathbf{C}_{\mathbf{Y}i})$ $(i = 1, 2, \cdots, n)$. In probability theory, the cumulative distribution function (CDF) of $G$ is defined by

$$F(c) = \Pr(E) = \Pr\{G = g(\mathbf{X}, \mathbf{Y}) < c\}, \tag{7}$$

where $F$ is the CDF of $G$ at $c$.

Let the product space of $\mathbf{X} = X_1 \times X_2 \times \cdots \times X_{n_X}$ be discretized into $k$ subsets (hypercubes) $\mathbf{C}_{\mathbf{X}j}$ $(j = 1, 2, \cdots, k)$ with $\Delta\mathbf{X} = \Delta X_1 \times \Delta X_2 \times \cdots \times \Delta X_{n_X}$, where $\Delta X_i$ $(i = 1, 2, \cdots, n_X)$ is the step size. Since the joint BPA of $\mathbf{C}_{\mathbf{X}j}$ is the probability of $\mathbf{X}$ in $\mathbf{C}_{\mathbf{X}j}$, the joint BPA of $\mathbf{X}$ is given by

$$m_{\mathbf{X}}(\mathbf{C}_{\mathbf{X}j}) = f_{\mathbf{X}}(\mathbf{x}|\mathbf{X} \in \mathbf{C}_{\mathbf{X}j})\Delta\mathbf{X}, \tag{8}$$

where $f_{\mathbf{X}}(\cdot)$ is the joint PDF of $\mathbf{X}$.

The joint BPA of $\mathbf{X}$ and $\mathbf{Y}$ is then derived as

$$\begin{aligned} m_{\mathbf{XY}}(\mathbf{C}_{\mathbf{Y}i}, \mathbf{C}_{\mathbf{X}j}) &= m_{\mathbf{Y}}(\mathbf{C}_{\mathbf{Y}i})\sum_{j=1}^{k} m_{\mathbf{X}}(\mathbf{C}_{\mathbf{X}j}) \\ &= m_{\mathbf{Y}}(\mathbf{C}_{\mathbf{Y}i})\sum_{j=1}^{k} f_{\mathbf{X}}(\mathbf{x}|\mathbf{X} \in \mathbf{C}_{\mathbf{X}j})\Delta\mathbf{X} \end{aligned}. \tag{9}$$

The belief measure of the failure event is then calculated by

$$\begin{aligned} Bel(c) &= \sum_{\substack{i=1 \\ (\mathbf{C}_{\mathbf{Y}i}, \mathbf{C}_{\mathbf{X}j}) \in E}}^{n} m_{\mathbf{XY}}(\mathbf{C}_{\mathbf{Y}i}, \mathbf{C}_{\mathbf{X}j}) = \sum_{\substack{i=1 \\ (\mathbf{C}_{\mathbf{Y}i}, \mathbf{C}_{\mathbf{X}j}) \in E}}^{n} \left[ m_{\mathbf{Y}}(\mathbf{C}_{\mathbf{Y}i})\sum_{j=1}^{k} m_{\mathbf{X}}(\mathbf{C}_{\mathbf{X}j}) \right] \\ &= \sum_{\substack{i=1 \\ (\mathbf{C}_{\mathbf{Y}i}, \mathbf{C}_{\mathbf{X}j}) \in E}}^{n} \left[ m_{\mathbf{Y}}(\mathbf{C}_{\mathbf{Y}i})\sum_{j=1}^{k} f_{\mathbf{X}}(\mathbf{x}|\mathbf{X} \in \mathbf{C}_{\mathbf{X}j})\Delta\mathbf{X} \right] \end{aligned} \tag{10}$$

When $k$ approaches infinity, the equation for the cumulative belief function (*CBF*), the degree of belief that the event $G < c$ would occur, becomes[16]

$$Bel(c) = F_G^{\min}(c) = \sum_{i=1}^{n} m_{\mathbf{Y}}(\mathbf{C}_{\mathbf{Y}i}) \Pr\left\{G_{\max} < c \,\middle|\, \mathbf{Y}_i \in \mathbf{C}_{\mathbf{Y}i}\right\}. \tag{11}$$

By analogy, the plausibility measure function (*CPF*), the degree of plausibility that the event $G < c$ would occur, can be computed by

$$Pl(c) = F_G^{\max}(c) = \sum_{i=1}^{n} m_{\mathbf{Y}}(\mathbf{C}_{\mathbf{Y}i}) \Pr\left\{G_{\min} < c \,\middle|\, \mathbf{Y}_i \in \mathbf{C}_{\mathbf{Y}i}\right\}, \tag{12}$$

respectively. $G_{\min}$ and $G_{\max}$ are respectively the global minimum and maximum values of $G$ in the subset $\mathbf{C}_{\mathbf{Y}i}$ given the values of $\mathbf{X}$.

Equations 11 and 12 are derived from evidence theory by dividing the random variables into infinite intervals. The same equation can also be derived from probability theory by using the total probability. See Ref. 16 for details. Equations 11 and 12 indicate that the evaluation of belief and plausibility measures with the mixture of probability distributions and BPAs is essentially the evaluation of the minimum and maximum probabilities of the performance function over the subsets of $\mathbf{Y}$. Therefore, traditional probabilistic analysis methods can be used for the unified uncertainty analysis. Hereby, we use the First Order Reliability Method (FORM) based uncertainty analysis method developed in Ref. 16.

### D. FORM-Based Unified Uncertainty Analysis

The First Order Reliability Method (FORM) is used to calculate a CDF or the probability of failure when only random variables $\mathbf{X}$ exist. If the joint probability density function (PDF) of $\mathbf{X}$ is $f_{\mathbf{X}}$, the probability of failure $p_f$ is calculated by

$$p_f = F(c) = \Pr\{G = g(\mathbf{X}) < c\} = \int\limits_{g(\mathbf{X})<c} f_{\mathbf{X}}(\mathbf{x})d\mathbf{x} \ . \tag{13}$$

FORM involves three steps to approximate the above integral: 1) transforming original random variables $\mathbf{X}$ to standard normal random variables $\mathbf{U}$, 2) searching the Most Probable Point (MPP), and 3) calculating $p_f$.

Step 1: Transformation, which is given by

$$u_i = \Phi^{-1}\left\{F_{X_i}(x_i)\right\}, \ \ i = 1, 2, \ldots, n_X \tag{14}$$

where $F_{X_i}$ is the CDF of $X_i$, and $\Phi^{-1}$ is the inverse CDF of a standard normal distribution.

Step 2: MPP search, where the MPP $\mathbf{u}^*$ is identified by

$$\min_{\mathbf{U}} \|\mathbf{U}\| \big| g(\mathbf{U}) = c \ , \tag{15}$$

where $\| \cdot \|$ stands for the norm (length) of a vector. $\beta = \|\mathbf{u}^*\|$ is termed as a reliability index.

Step 3: Estimation of $p_f$, which is given by

$$p_f = \Phi(-\beta) \ , \tag{16}$$

where $\Phi$ is the CDF of a standard normal distribution.

The key to FORM is the MPP search. The following recursive algorithm is used to search the MPP,

$$\begin{cases} \beta^{(k)} = \beta^{(k-1)} + \dfrac{\mathbf{u}^{(k-1)}}{\left\|\nabla g(\mathbf{u}^{(k-1)})\right\|} \\[4mm] \mathbf{u}^{(k)} = -\beta^{(k)} \dfrac{\nabla g(\mathbf{u}^{(k-1)})}{\left\|\nabla g(\mathbf{u}^{(k-1)})\right\|} \end{cases}, \tag{17}$$

where $\nabla g(\mathbf{u}^{(k-1)})$ is the gradient of $g$ at $\mathbf{u}^{(k-1)}$ and $\left\|\nabla g(\mathbf{u}^{(k-1)})\right\|$ is its magnitude, and $k$ is the iteration counter.

The above process is called *probabilistic analysis* (PA) because only random variables are involved. As shown in Eqs. (11) and (12), we need to find the maximum and minimum values of $G$ when interval variables $\mathbf{Y}$ exist. The process of finding the maximum and minimum $G$ is called *interval analysis* (IA). Solving Eqs. (11) and (12) directly requires a double-loop procedure where PA and IA are nested.[16] Given a set of interval variables $\mathbf{Y}$, the MPP is searched by the algorithm in Eq. (17). Then interval analysis is performed to find the maximum and minimum performance function values with the random variables fixed at the MPP. This process repeats till convergence is reached. This double-loop procedure is computationally inefficient. To improve computational efficiency, we need to embed IA into the MPP search algorithm. In this work, we focus on black-box performance functions where closed-form functions are not applicable. Since the traditional interval arithmetic is not applicable to a blackbox function, we employ nonlinear optimization to perform IA.

The flowchart for the minimum probability $\Pr\{G_{\max} < c | \mathbf{Y}_i \in \mathbf{C}_{\mathbf{Y}i}\}$ in the *CBF* equation is given in Fig. 4. The solution is the MPP $\mathbf{u}^*$ where $G$ is the maximum. The probability $\Pr\{G_{\max} < c | \mathbf{Y}_i \in \mathbf{C}_{\mathbf{Y}i}\}$ in Eq. (11) is then computed by

$$\Pr\{G_{\max} < c | \mathbf{Y}_i \in \mathbf{C}_{\mathbf{Y}i}\} = \Phi(-\beta) = \Phi(-\|\mathbf{u}^*\|). \tag{18}$$
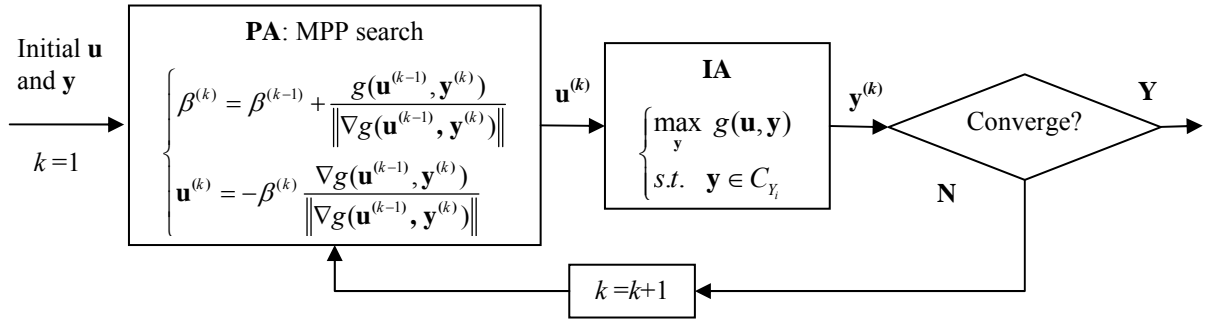
**Fig. 4 Flowchart of MPP search in *Bel* calculation.**

For the plausibility calculation, the model of the MPP search is the same as in Fig. 4, and IA becomes a minimization problem.

## III. Proposed Sensitivity Analysis Method

With only aleatory uncertainty, a single probability measure of a performance $G$ can be obtained. With both aleatory and epistemic uncertainties, the probability bounds, belief measure and plausibility measure can be obtained as shown in Fig. 5. The difference between belief and plausibility measures represents the effect of epistemic uncertainty. The wider the difference, the greater is the effect. If the difference is too wide, it will be difficult to make decisions.

For example, as shown in Fig. 5, the belief and plausibility are 0.016 and 0.64 at the limit state $G = 2$, respectively. If $G < 2$ is a failure event, then the minimum and maximum probabilities of failure $p_f$ are 0.016 and 0.64, respectively. The large gap between the two bounds makes the decision process too difficult. If one used the belief ($p_f = 0.016$), the design might be highly risky because the true $p_f$ may be much higher

than the minimum value. If one used the plausibility ($p_f = 0.64$), however, the design might be too conservative. In this case, more information about the epistemic variables is needed in order to reduce their effect. How to effectively collect more information is critical. In this work, we develop a sensitivity analysis method to identify the most important epistemic variables that have the highest impact on design performance. With this method, limited resources can be used to collect more information on the identified important epistemic variables.
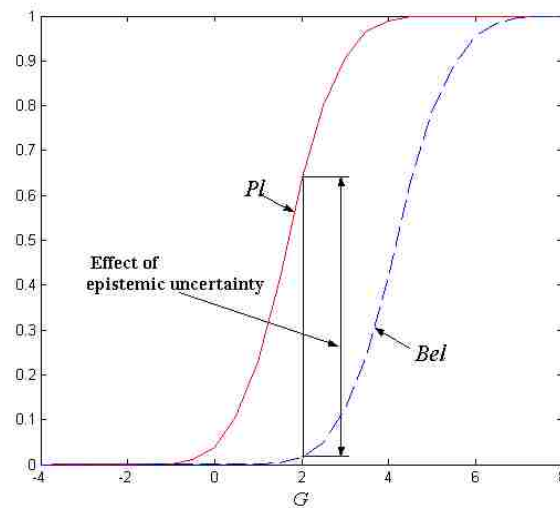


**Fig. 5 Effect of epistemic uncertainty and aleatory uncertainty.**

We adopt the OAT (one-at-a-time) strategy[38] to quantify the effect of each individual epistemic variable. The effect is measured by the difference between belief and plausibility measures. The difference is computed by Kolmogorov-Smirnov (KS) distance. [39]

The OAT strategy belongs to the simplest class of screening methods. The impact of uncertainty in each variable is evaluated one by one.[38] The sensitivity analysis is conducted by keeping one epistemic variable uncertain while the other epistemic

variables are fixed at their averages at one time. Then the impact of the varied variable on the performance can be isolated and evaluated. The average of an epistemic variable $Y_j$ $(j = 1, 2, \cdots, n_Y)$ is calculated by

$$\overline{Y}_j = \sum_{i=1}^{n} m(C_{Y_i}) \frac{Y_{ij}^u + Y_{ij}^l}{2}, \quad i = 1, 2, \ldots, n \tag{19}$$

where $m(C_{Y_i})$ is the BPA of the $i$-th subset $C_{Y_i}$, $Y_{ij}^u$ and $Y_{ij}^l$ are the upper and lower bounds of $Y_j$ on $C_{Y_i}$, respectively.

The Kolmogorov-Smirnov (KS) distance is a measure used in statistical test [39] and is defined as the maximum difference between the sample CDF and the hypothesized CDF. This distance measures how close the sample CDF to the hypothesized CDF. We adopt herein the same idea to measure the difference between *CPF* and *CBF*.

The proposed sensitivity analysis includes the following two steps:

*Step 1* - Uncertainty analysis: the unified uncertainty analysis is performed to calculate *CBF* and *CPF* when both aleatory variables **X** and epistemic variables **Y** exist.

*Step 2* - OAT analysis: the main effect and total effect of each epistemic variable are calculated. The main effect explores the impact on the performance from each single epistemic variable while the total effect measures the impact on the performance from the interactions of one epistemic variable with other epistemic variables.

To identify the main effect of the epistemic variable $Y_i$ $(i = 1, 2, \cdots, n_Y)$, we fix the rest of the epistemic variables $Y_j$ $(j = 1, 2, \cdots, n_Y, j \neq i)$ at their averages $\overline{Y}_j$ (see Eq. (19)). Only $Y_i$ is allowed to vary. To measure the total effect of the epistemic variable $Y_i$, we fix $Y_i$ at its average $\overline{Y}_i$, and keep the rest of epistemic variables $Y_j$ $(j = 1, 2, \cdots, n_Y, j \neq i)$.

After setting these different scenarios, we conduct the unified uncertainty analysis again to calculate *CBF* and *CPF* for each scenario. We then calculate the difference between *CBF* and *CPF* and rank the importance of epistemic variables by the difference.

In this work, we use sensitivity analysis for two applications, reliability analysis and uncertainty analysis for the entire range of a performance.

*Application 1 – Reliability Analysis*. Let a failure mode be defined by the event where the performance *G* is less than a threshold *c*, namely, $G < c$. The probability of failure $p_f$ can be calculated by Eq. (13) when only random variables **X** exist. When both aleatory and epistemic uncertainties are present, according to Eqs. (11) and (12), the minimum and maximum probabilities of failure are actually the *CBF* and *CPF* at *c*, namely,

$$p_f^{\min} = Bel(c) = F^{\min}(c), \tag{20}$$

and

$$p_f^{\max} = Pl(c) = F^{\max}(c). \tag{21}$$

The difference between $p_f^{\max}$ and $p_f^{\min}$ represents the effect of epistemic uncertainty on the probability of failure $p_f$. The difference is given by

$$d_{p_f} = p_f^{\max} - p_f^{\min} = Pl(c) - Bel(c). \tag{22}$$

The main effect of $Y_i$ $(i = 1, 2, \cdots, n_Y)$ on the probability of failure is given by

$$ME_{pf}^i = d_{p_f}^i, \tag{23}$$

where $d_{p_f}^i$ is the difference between $p_f^{max}$ and $p_f^{min}$ when $Y_i$ is kept as an epistemic variable and other variables $Y_j (j = 1, \cdots, n_Y, j \neq i)$ are fixed at their average $\overline{Y}_j$. $ME_{pf}^i$ is computed by

$$ME_{pf}^i = d_{p_f}^i = Pl(c \mid Y_j = \overline{Y}_j, j = 1, \cdots, n_Y, j \neq i) - Bel(c \mid Y_j = \overline{Y}_j, j = 1, \cdots, n_Y, j \neq i) \quad (24)$$

The smaller $d_{p_f}^i$ is, the weaker is the impact of $Y_i$ on $p_f$, and therefore $Y_i$ is less important.

The total effect of $Y_i$ on $d_{p_f}$ is given by

$$TE_{pf}^i = d_{p_f}^{\sim i} . \quad (25)$$

where $d_{p_f}^{\sim i}$ is the difference between $p_f^{max}$ and $p_f^{min}$ when $Y_i$ is fixed at its average $\overline{Y}_i$ and the other variables $Y_j (j = 1, 2, \cdots, n_Y, j \neq i)$ are kept as epistemic variables. $TE_{pf}^i$ is computed by

$$TE_{pf}^i = d_{p_f}^{\sim i} = Pl(c \mid Y_i = \overline{Y}_i) - Bel(c \mid Y_i = \overline{Y}_i) . \quad (26)$$

The smaller $d_{p_f}^{\sim i}$ means the larger influence of $Y_i$.

*Application 2 – Uncertainty analysis over the entire range of the performance G*. If we are interested in the effect of an epistemic variable on the entire range of the model output, we can calculate the KS distance between the *CBF* and *CPF* as follows,

$$d_{KS} = \max_c \left[ Pl(c) - Bel(c) \right]. \quad (27)$$

The equation implies that the KS distance is the maximum discrepancy between two curves of *CBF* and *CPF* as shown in the Fig. 6.
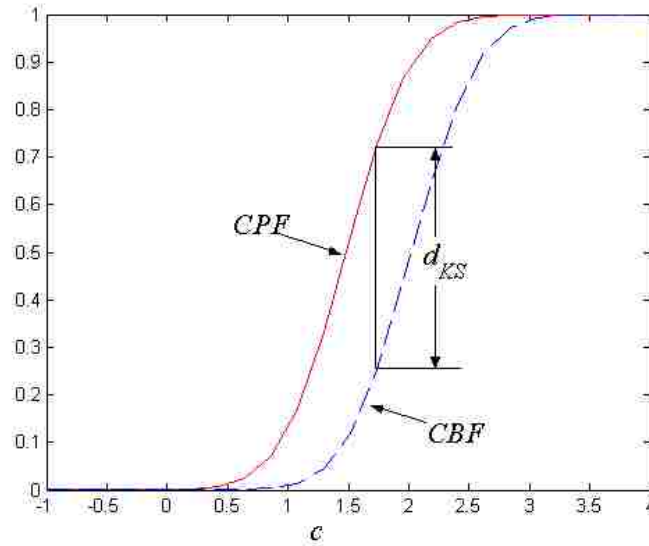
**Fig. 6 KS distance.**

The main effect of epistemic variable $Y_i$ on CDF is calculated as

$$
\begin{aligned}
ME^i &= d_{p_f}^i \\
&= \max_c \left[ Pl(c \mid Y_j = \bar{Y}_j, j = 1, 2, \cdots, n_Y, j \neq i) - Bel(c \mid Y_j = \bar{Y}_j, j = 1, 2, \cdots, n_Y, j \neq i) \right]
\end{aligned}
, \quad (28)
$$

where $d_{KS}^i$ is the KS distance between *CPF* and *CBF* when $Y_i$ is kept as an epistemic variable and other variables $Y_j (j = 1, 2, \cdots, n_Y, j \neq i)$ are fixed at their average $\bar{Y}_j$. The smaller $d_{KS}^i$ is, the closer are *CBF* and *CPF*; namely, the impact of $Y_i$ is weaker and $Y_i$ is less influential. Therefore, the smaller $ME^i$ is, the less significant is $Y_i$ to the uncertainty of the performance.

The total effect of epistemic variable $Y_i$ on CDF can be calculated as

$$
TE^i = d_{KS}^{\sim i} = \max_c \left[ Pl(c \mid Y_i = \bar{Y}_i) - Bel(c \mid Y_i = \bar{Y}_i) \right],
\quad (29)
$$

where $d_{KS}^{\sim i}$ is the KS distance when $Y_i$ is fixed at its average $\overline{Y}_i$ and other variables $Y_j (j = 1, 2, \cdots, n_Y, j \neq i)$ are kept as epistemic variables. In this case, a smaller discrepancy between *CPF* and *CBF* implies higher influence of $Y_i$ on *G*.

The flowchart of the proposed sensitivity analysis method is illustrated in Fig. 7.

From the above discussion, it is seen that one sensitivity analysis needs to call the unified analysis $2n_Y + 1$ times – one analysis is for the case with original uncertain variables, $n_Y$ analyses are for the main effects of the $n_Y$ epistemic variables, and the other $n_Y$ analyses are for the total effects of the $n_Y$ epistemic variables. The computation is intensive, and therefore efficiency is critical. To improve efficiency, we use the efficient MPP algorithm as shown in Eq. (17). In many engineering applications, a performance function is monotonic in terms of interval variables. In this case, it is not necessary to conduct nonlinear optimization for interval analysis. However, it is difficult to know whether the performance function is monotonic because of the black-box model. We therefore perform optimization for interval analysis in the first iteration. Thereafter, we check the Karush-Kuhn-Tucker (KKT) conditions[41] after the MPP is updated. If the KKT conditions are satisfied, there is no need to perform optimization again. We then proceed to the next iteration.

$$(\mathbf{X}, \mathbf{Y}) \qquad G = g(\mathbf{X}, \mathbf{Y})$$

**Initial Unified Uncertainty Analysis**

*CBF and CPF of G, $d_{p_f}$, $d_{KS}$*

**Sensitivity analysis**

**Main effect**

Keep one epistemic variable and fix all the others

**Unified Uncertainty Analysis**

*CBF and CPF of G*

$d_{p_f}^i$ $\qquad$ $d_{KS}^i$

**Application 1**

**Reliability**

$ME_{p_f}^i$

**Ranking by $ME_{p_f}^i$**

**Application 2**

**Whole range of G**

$ME^i$

**Ranking by $ME^i$**

**Total effect**

Fix one epistemic variable and keep all the others

**Unified Uncertainty Analysis**

*CBF and CPF of G*

$d_{p_f}^{\sim i}$ $\qquad$ $d_{KS}^{\sim i}$

**Application 1**

**Reliability**

$TE_{p_f}^i$

**Ranking by $TE_{p_f}^i$**

**Application 2**

**Whole range of G**

$TE^i$

**Ranking by $TE^i$**

**The importance ranking of the epistemic variables**

**Fig. 7 Flowchart of the proposed sensitivity analysis.**

# IV. Examples

## A. Example 1- Crank-Slider Mechanism

A crank-slider mechanism is used in a construction machine as shown in Fig. 8.[16] The length of the crank $AB$ $a$, the length of the coupler $BC$ $b$, the external force $Q$, the Young's modulus of the material of the coupler $E$, and the yield strength of the coupler $S$ are random variables. Their distributions are given in Table 1.
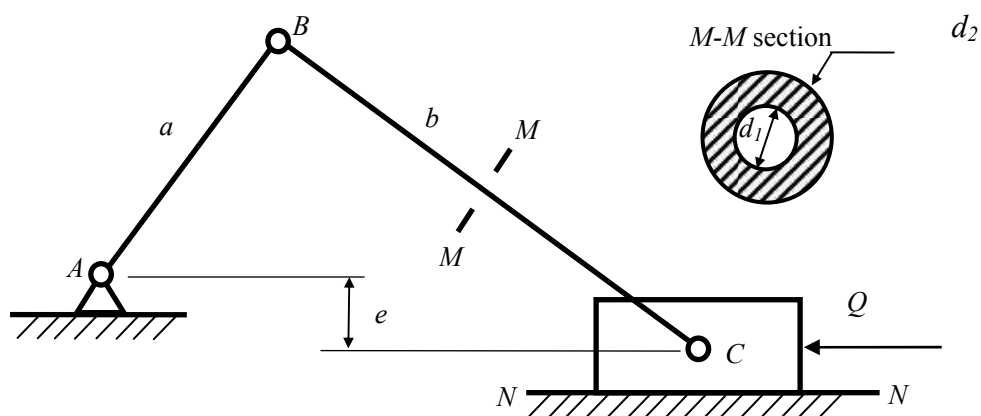


**Fig. 8 A crank-slider mechanism.**

**Table 1. Random variables X**

| Variables | Symbols in Fig.8 | Mean | Standard deviation | Distribution |
|---|---|---|---|---|
| $X_1$ | $a$ | 100 mm | 0.01 mm | Normal |
| $X_2$ | $b$ | 300 mm | 0.01 mm | Normal |
| $X_3$ | $Q$ | 250 kN | 25 kN | Normal |
| $X_4$ | $E$ | 200 GPa | 30 GPa | Normal |
| $X_5$ | $S$ | 390 MPa | 39 MPa | Normal |

Because of the harsh environment of the construction site, a precise distribution of the coefficient of friction $\mu$ between the ground $NN$ and the slider $C$ is not available, but its intervals and BPA are available based on the solicitation from experts. Because of the

different installation positions of the slider are required in various construction sites, the

intervals and BPA of the offset *e* are assigned based on limited historical data. Their

BPAs are provided in Table 2, and the joint BPA is also visualized in Fig. 9.

**Table 2. Uncertain variables with epistemic uncertainty**

| Variables | Symbols in Fig. 8 | Intervals | BPA |
|-----------|-------------------|-----------|-----|
| $Y_1$ | $e$ (mm) | [100, 120] | 0.2 |
| | | [120, 140] | 0.4 |
| | | [140, 150] | 0.4 |
| $Y_2$ | $\mu$ | [0.15, 0.18] | 0.3 |
| | | [0.18, 0.23] | 0.3 |
| | | [0.23, 0.25] | 0.4 |
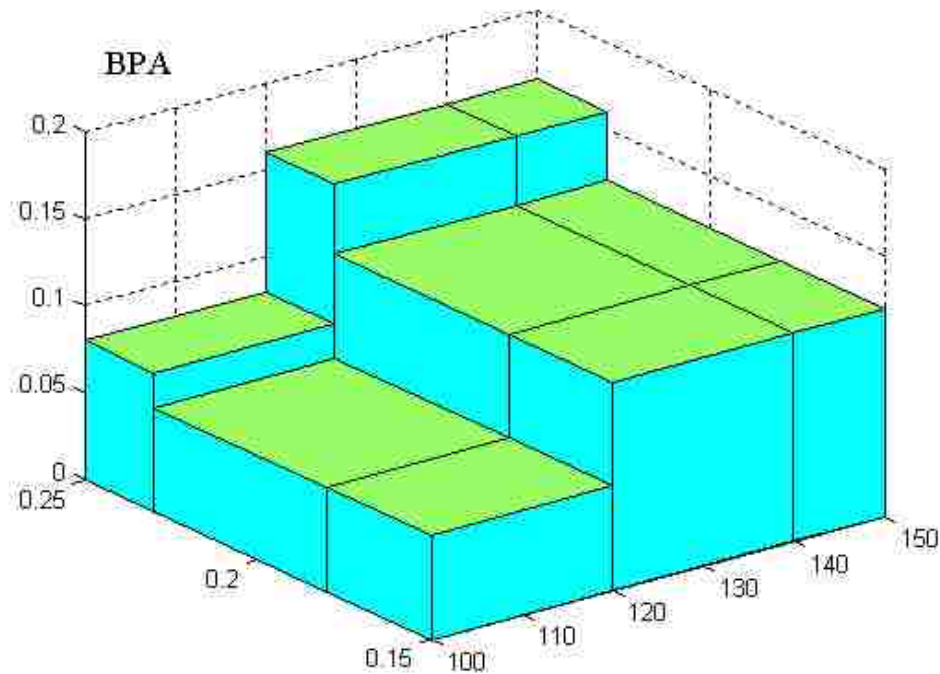


**Fig. 9 Joint BPA of Y with 3 Intervals.**

The two performance functions are the safety margins for strength and buckling

requirements of the coupler, which are defined by the difference between the material

strength and the maximum stress, and the difference between the critical load and the axial load, respectively. The equations are obtained at one of the positions when the crank *AB* and the coupler *BC* overlap. The functions are given by

$$G_1 = g_1(\mathbf{X}, \mathbf{Y}) = S - \frac{4P(b-a)}{\pi\left(\sqrt{(b-a)^2 - e^2} - \mu e\right)(d_2^2 - d_1^2)},$$

and

$$G_2 = g_2(\mathbf{X}, \mathbf{Y}) = \frac{\pi^3 E(d_2^4 - d_1^4)}{64b^2} - \frac{P(b-a)}{\sqrt{(b-a)^2 - e^2} - \mu e}.$$

The failure events are defined by $E_1 = \{\mathbf{X}, \mathbf{Y} | G_1 < 0\}$ and $E_2 = \{\mathbf{X}, \mathbf{Y} | G_2 < 0\}$. Our goal is to find out the most significant epistemic variable (offset $e$ or coefficient of friction $\mu$) which has the most dominant effect on the performance functions $G_1$ and $G_2$.

We first perform the unified uncertainty analysis for the two failure modes. The result is given in Table 3. The difference $d_{p_f}$ between the maximum and minimum probabilities of failure (or *Pl* and *Bel*) of $G_1$ is large, and the difference $d_{p_f}$ of $G_2$ is almost zero. Therefore, the effect of epistemic uncertainty on failure mode 1 ($G_1$) cannot be neglected, and the effect of epistemic uncertainty on failure mode 2 ($G_2$) is negligible. Sensitivity analysis on failure mode 1 is then necessary. Hence we only conduct sensitivity analysis on $G_1$.

In order to confirm the accuracy of the united uncertainty analysis, we solve the problem by Monte Carlo simulation (MCS). The result is also provided in Table 3, where $N$ is number of function evaluations. $N$ is used to measure computational efficiency. It is

seen that the unified uncertainty analysis employed in this paper is very accurate and efficient.

**Table 3 $d_{p_f}$ for $G_1$ and $G_2$**

| | United Uncertainty Analysis | | | | Monte Carlo Simulation | | | |
|---|---|---|---|---|---|---|---|---|
| | $p_f^{\min}$ (Bel) | $p_f^{\max}$ (Pl) | $d_{p_f}$ | $N$ | $p_f^{\min}$ (Bel) | $p_f^{\max}$ (Pl) | $d_{p_f}$ | $N$ |
| $G_1$ | 0.00735 | 0.0727 | 0.0654 | 468 | 0.0124 | 0.09424 | 0.0818 | $4 \times 10^6$ |
| $G_2$ | $\approx 0.0$ | $\approx 0.0$ | $\approx 0.0$ | 488 | $\approx 0.0$ | $\approx 0.0$ | $\approx 0.0$ | $4 \times 10^6$ |

We also perform the unified uncertainty analysis for the entire range of the two performance functions. The results of *CBF* and *CPF* for both $G_1$ and $G_2$ are shown in Table 4 and Figs. 10 and 11. It is also seen that the effect of epistemic uncertainty on $G_1$ is much larger than that on $G_2$ because the KS distance for $G_1$ is much larger than that for $G_2$. The numbers of function evaluations also indicate that the unified uncertainty analysis is much efficient than MCS.
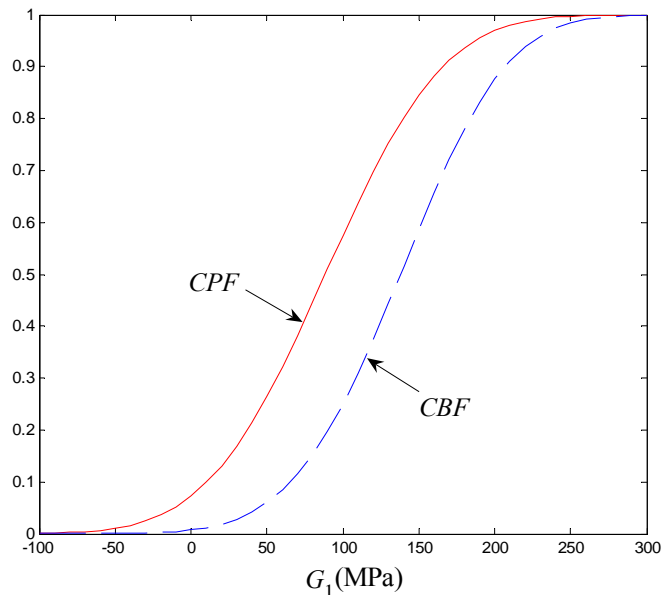


**Fig. 10 Initial unified uncertainty analysis for $G_1$.**
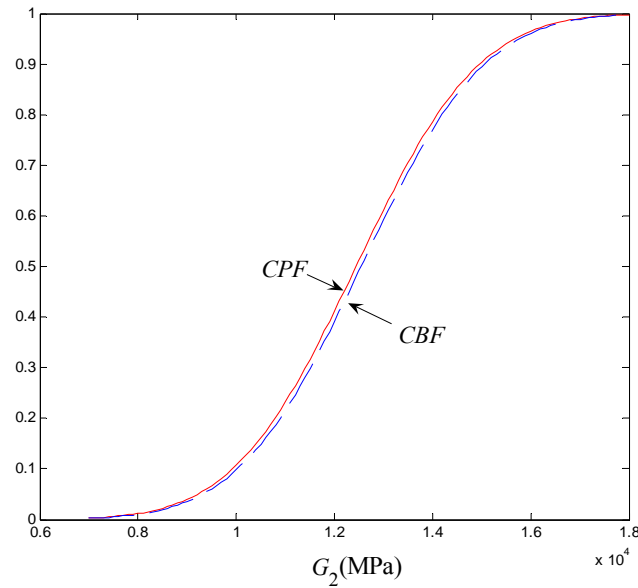
**Fig. 11 Initial unified uncertainty analysis for $G_2$.**

**Table 4 Results of unified uncertainty analysis for $G_1$**

| $G_1$ | United Uncertainty Analysis | | Monte Carlo Simulation | |
|---|---|---|---|---|
| | $CPF - CBF$ | $N$ | $CPF - CBF$ | $N$ |
| -100 | 0.0008 | 468 | 0.0013 | $36 \times 10^6$ |
| -50 | 0.0103 | 468 | 0.0156 | $36 \times 10^6$ |
| 0 | 0.0654 | 468 | 0.0818 | $36 \times 10^6$ |
| 50 | 0.2042 | 468 | 0.2050 | $36 \times 10^6$ |
| 100 | 0.3248 | 468 | 0.2804 | $36 \times 10^6$ |
| 150 | 0.2594 | 468 | 0.2322 | $36 \times 10^6$ |
| 200 | 0.0946 | 468 | 0.1017 | $36 \times 10^6$ |
| 250 | 0.0137 | 468 | 0.0181 | $36 \times 10^6$ |
| 300 | 0.0007 | 468 | 0.0011 | $36 \times 10^6$ |
| Total function calls | 4212 | | Total function calls | $36 \times 10^6$ |
| KS distance | 0.3248 | | KS distance | 0.2804 |

Next we perform sensitivity analysis on $G_1$ to find out the most influential epistemic variable. In this example, there are only two epistemic variables $Y_1$ and $Y_2$; no total effect is therefore needed. Thus we only analyze the main effect of each variable.

*Main Effect of $Y_1$*: Keep $Y_1$ as an epistemic variable and fix $Y_2$ at its average. The average of $Y_2$ is calculated by

$$\bar{Y}_2 = \frac{0.15+0.18}{2}\times 0.3 + \frac{0.18+0.23}{2}\times 0.3 + \frac{0.23+0.25}{2}\times 0.4 = 0.207 \,.$$

The *CBF* and *CPF* of $G_1$ are reevaluated by the unified uncertainty and are given in Fig. 12. The difference (main effect) between the maximum and minimum probabilities of failure $ME^1_{p_f} = d^1_{p_f}$, and the KS distance for the entire distribution $ME^1 = d^1_{KS}$ are given in Table 5.
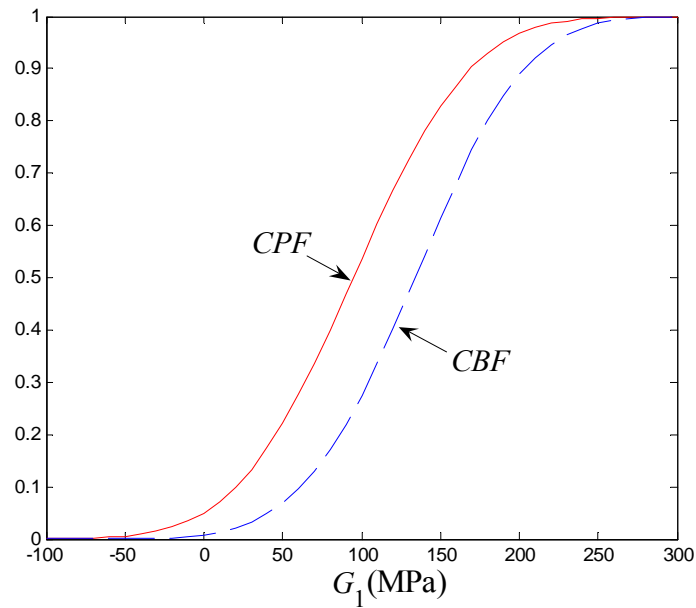


**Fig. 12 *CBF* and *CPF* from the main effect analysis for $Y_1$.**

**Table 5 Main effect of each epistemic variable**

| Main effect | $p_f^{min}$ (*Bel*) | $p_f^{max}$ (*Pl*) | $ME_{p_f}$ | *ME* |
|---|---|---|---|---|
| $Y_1$ | 0.008219 | 0.049754 | 0.041535 | 0.2979 |
| $Y_2$ | 0.004638 | 0.008164 | 0.003527 | 0.0761 |

*Main Effect of $Y_2$*: Keep $Y_2$ as an epistemic variable and fix $Y_1$ at its average. The average of $Y_1$ is calculated by

$$\overline{Y}_1 = \frac{100+120}{2} \times 0.2 + \frac{120+140}{2} \times 0.4 + \frac{140+150}{2} \times 0.4 = 132 .$$

The *CBF* and *CPF* of $G_1$ are illustrated in Fig. 13, and $ME_{p_f}^2 = d_{p_f}^2$ and $ME^2 = d_{KS}^2$ are also given in Table 5. The difference between the *CBF* and *CPF* is much narrower when $Y_1$ is fixed. The result indicates that the main effect of $Y_1$ is much greater than that of $Y_2$. Therefore $Y_1$ is the most influential contributor to the effect of epistemic uncertainty on the probability of failure $p_f$ of $G_1$, and it is also true for the entire range of $G_1$.
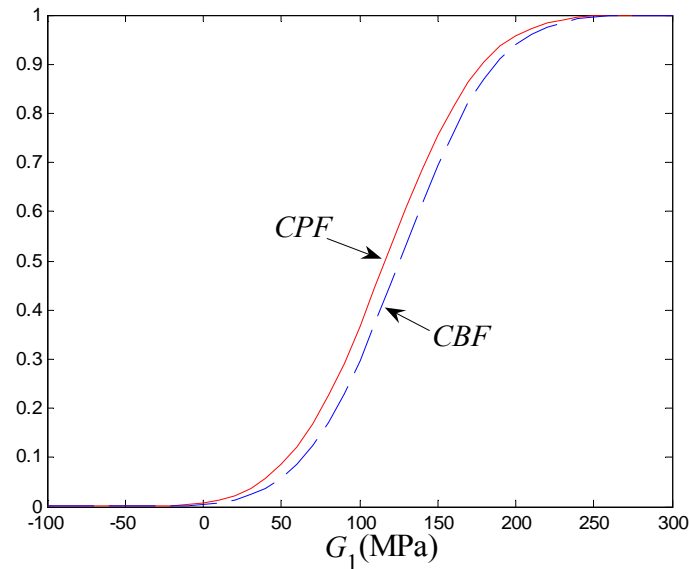


**Fig. 13 *CBF* and *CPF* from the main effect analysis for $Y_2$.**

If more information is needed to reduce the effect of epistemic uncertainty, we should collect more information on $Y_1$ instead of $Y_2$. After adequate information was collected on $Y_1$, $Y_1$ would become a random variable with only aleatory uncertainty. Suppose the available distribution of $Y_1$ is $N(125, 8.33)$ mm. Through the unified uncertainty analysis

again, the gap between *CBF* and *CPF* of $G_1$ becomes much narrower as shown in Fig.14 and Table 6.
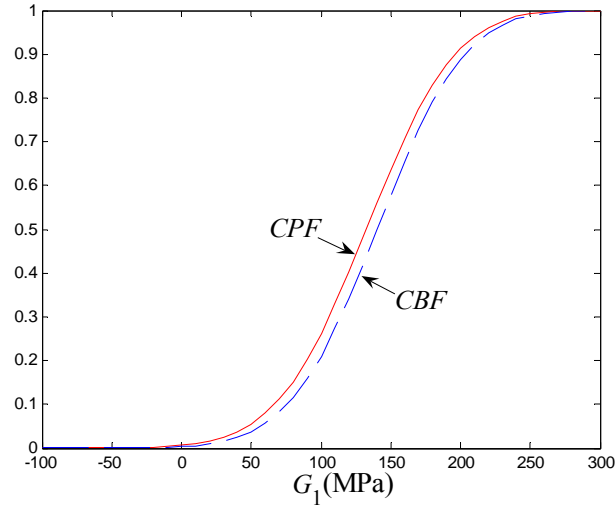


**Fig. 14** *CPF* and *CBF* of $G_1$ after more information on $Y_1$ is collected.

**Table 6** $ME^i_{p_f}$ and $ME^i$

| Scenarios | $p_f^{min}$ (*Bel*) | $p_f^{max}$ (*Pl*) | $ME^i_{p_f}$ | $ME^i$ |
|---|---|---|---|---|
| $Y_1$: 3 intervals, $Y_2$: 3 intervals | 0.007353 | 0.072707 | 0.065354 | 0.3288 |
| $Y_1$: aleatory, $Y_2$: 3 intervals | 0.003379 | 0.005881 | 0.002502 | 0.0622 |
| $Y_1$: 3 intervals, $Y_2$: aleatory | 0.007507 | 0.045416 | 0.037909 | 0.2625 |

If we did not conduct sensitivity analysis, we might arbitrarily choose to collect more information on $Y_2$. Suppose the distribution of $Y_2$ is $N(0.2, 0.017)$ after more information is collected. As shown in Fig. 15 and Table 6, the reduced effect of epistemic uncertainty (the gap between the *CPF* and *CBF*) is much less significant compared to the situation where the epistemic uncertainty of $Y_1$ is eliminated (see Fig. 14).
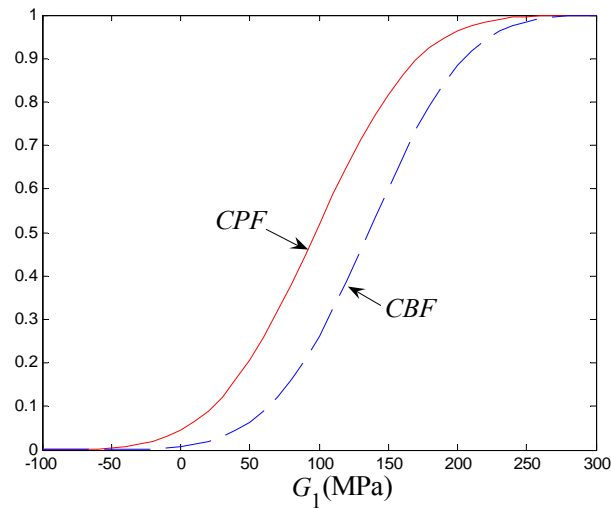
**Fig. 15** *CPF* and *CBF* of $G_1$ after collecting more information on $Y_2$ is collected.

For an easy comparison, all the information is provided in Table 6. The first row is the uncertainty analysis result with the original uncertain variables X and Y. The second row is the result when $Y_1$ becomes aleatory while the third row is the result when $Y_2$ becomes aleatory. The table verifies that $Y_1$ is more important than $Y_2$ in terms of the effect on $G_1$.

**B. Example 2 - Crowned Cam Roller-Follower's Contact**

A crowned cam roller-follower used in a transmission system is shown in Fig.16.[42] It has a gentle radius transverse to its rolling direction for eliminating the need for critical alignment of its axis with that of the cam. The roller radius is $R_1$ with a $R_1'$ crown radius at 90° to the roller radius. The cam's radius of curvature at the point of maximum load is $R_2$ and is flat axially so its crown radius $R_2'$ is infinite. The rotational axes of the cam and roller are parallel. The force is $Q$, normal to the contact plane. The materials of roller and follower are steel. Their Young's modulus $E$ is $30 \times 10^6$ psi, and the Poisson's ratio $v$ is 0.28. Due to the elastic deformation, the contact patch is an ellipse, and the pressure

distribution is a semi-ellipsoid, as illustrated in Fig. 17. $R_1$, $R_1'$, and $R_2$ are random

variables with the distributions listed in Table 7.
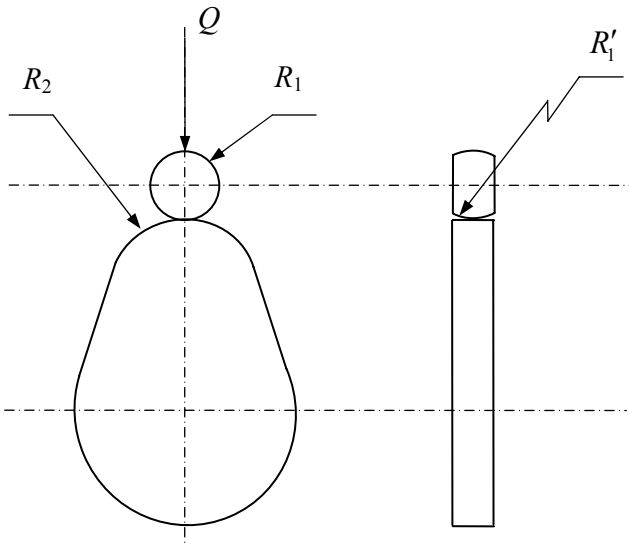


**Fig. 16 A crowned cam roller-follower under load $Q$.**
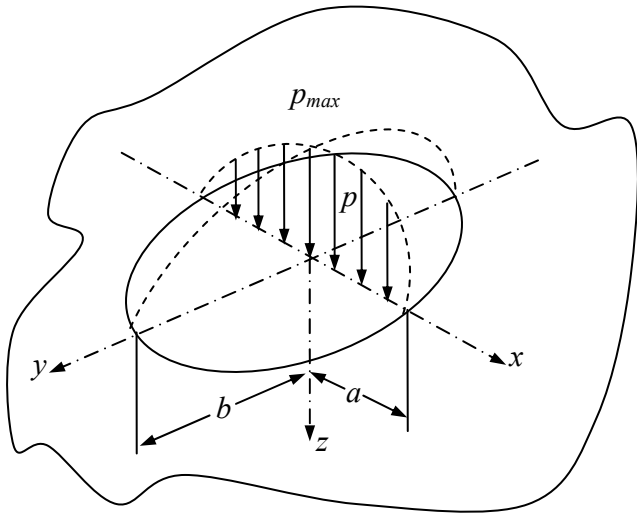


**Fig. 17 Contact patch ($p$ denotes the pressure distributed on the contact patch).**

**Table 7 Random variables X**

| Variable | Symbols in problem | Mean | Standard deviation | Distribution |
|---|---|---|---|---|
| $X_1$ | $R_1$ | 1 in | 0.01 in | Normal |
| $X_2$ | $R_1'$ | 20 in | 0.2 in | Normal |
| $X_3$ | $R_2$ | 3.46 in | 0.0346 in | Normal |
| $X_4$ | $S$ | 37.5 ksi | 0.375 ksi | Normal |

Because of limited information, an accurate measure or a distribution of $Q$ is not available. Its intervals and BPA are available based on the solicitation from experts, which is provided in Table 8.

**Table 8 Uncertain variables with epistemic uncertainty**

| Variable | Symbols in problem | Intervals | BPA |
|---|---|---|---|
| $Y_1$ | $k_a$ | [3.50, 3.60] | 1 |
| $Y_2$ | $k_b$ | [0.434, 0.440] | 1 |
| $Y_3$ | $Q$ (lb) | [246, 254] | 1 |

The half-width of the major axis $a$ and the half-width of the minor axis $b$ are determined by

$$a = k_a \sqrt[3]{\frac{3Q(m_1 + m_2)}{4A}},$$

$$b = k_b \sqrt[3]{\frac{3Q(m_1 + m_2)}{4A}},$$

where

$$m_1 = m_2 = \frac{1 - v^2}{E^2},$$

$$A = \frac{1}{2}(\frac{1}{R_1} + \frac{1}{R_1'} + \frac{1}{R_2} + \frac{1}{R_2'}).$$

The factors $k_a$ and $k_b$ are obtained from a table in Ref. 42 based on the value of $\phi$, which is calculated by

$$\phi = \cos^{-1}\left(\frac{A}{B}\right),$$

$$B = \frac{1}{2}\left[\left(\frac{1}{R_1} - \frac{1}{R_1'}\right)^2 + \left(\frac{1}{R_2} - \frac{1}{R_2'}\right)^2 + 2\left(\frac{1}{R_1} - \frac{1}{R_1'}\right)\left(\frac{1}{R_2} - \frac{1}{R_2'}\right)\right]^{\frac{1}{2}},$$

$\phi$ is a random variable due to the randomness in $R_1$, $R_1'$, and $R_2$, and $k_a$ and $k_b$ are tabulated in term of $\phi$. The values of $k_a$ and $k_b$ are not precisely known and are estimated within two intervals as shown in Table 8.

The performance function is the safety margin for shear yield strength of the roller and follower, defined by the difference between the shear yield strength and maximum shear stress, which is one-half of the tensile yield strength based on maximum shear-stress theory. The function is given by

$$G = g(\mathbf{X}, \mathbf{Y}) = \tau_{max} - S,$$

where $S$ is the shear yield strength, and $\tau_{max}$ is defined by

$$\tau_{max} = \max(\tau_a, \tau_u, \tau_{ma}, \tau_{mi}),$$

where $\tau_a$ is the maximum shear stress at the contact surface, $\tau_u$ is the largest shear stress under the contact surface, and $\tau_{ma}$ and $\tau_{mi}$ are the shear stresses at the ends of major and minor axis, respectively, on the contact surface. These stresses are calculated by

$$\tau_a = \left|\frac{\sigma_1 - \sigma_3}{2}\right|,$$

where $\sigma_1, \sigma_2$, and $\sigma_3$ are principal stresses at the contact surface, calculated by

$$\sigma_1 = -\left[2v + (1-2v)\frac{b}{a+b}\right]p_{max},$$

$$\sigma_2 = -\left[2v + (1-2v)\frac{a}{a+b}\right]p_{max},$$

$$\sigma_3 = -p_{max}, \text{ where } p_{max} = \frac{3Q}{2\pi ab},$$

and

$$\tau_u = 0.34 p_{max},$$

$$\tau_{ma} = (1-2v)\frac{k_3}{k_4^2}\left(\frac{1}{k_4}\tanh^{-1}k_4 - 1\right)p_{max}, \text{ where } k_3 = \frac{b}{a},$$

$$\tau_{mi} = (1-2v)\frac{k_3}{k_4^2}\left(1 - \frac{k_3}{k_4}\tanh^{-1}\left(\frac{k_4}{k_3}\right)\right)p_{max}, \text{ where } k_4 = \frac{1}{a}\sqrt{a^2 - b^2}.$$

The failure event is defined by $E = \{\mathbf{X}, \mathbf{Y} \mid G < 0\}$. The *CBF* and *CPF* are calculated by the unified uncertainty analysis and are shown in the Fig. 18. Also, a comparison with MCS is provided in Table 9. The result indicates that the impact of epistemic uncertainty is large and can not be ignored. In the table, $N$ is the number of function evaluations.

**Table 9 Results of unified uncertainty analysis**

| $G$ (ksi) | United Uncertainty Analysis | | Monte Carlo Simulation | |
|---|---|---|---|---|
| | $CPF - CBF$ | $N$ | $CPF - CBF$ | $N$ |
| -1 | 0 | 102 | 0 | $88 \times 10^6$ |
| -0.50 | 0.0018 | 88 | 0.0018 | $88 \times 10^6$ |
| 0 | 0.0412 | 88 | 0.0407 | $88 \times 10^6$ |
| 0.5 | 0.2890 | 83 | 0.2869 | $88 \times 10^6$ |
| 1 | 0.7331 | 83 | 0.7309 | $88 \times 10^6$ |
| 1.5 | 0.9627 | 84 | 0.9621 | $88 \times 10^6$ |
| 2.0 | 0.9675 | 84 | 0.9679 | $88 \times 10^6$ |
| 2.5 | 0.7575 | 83 | 0.7593 | $88 \times 10^6$ |
| 3.0 | 0.3205 | 83 | 0.3229 | $88 \times 10^6$ |
| 3.5 | 0.0517 | 88 | 0.0524 | $88 \times 10^6$ |
| 4.0 | 0.0027 | 88 | 0.0027 | $88 \times 10^6$ |
| | Total function calls | 954 | Total function calls | $88 \times 10^6$ |
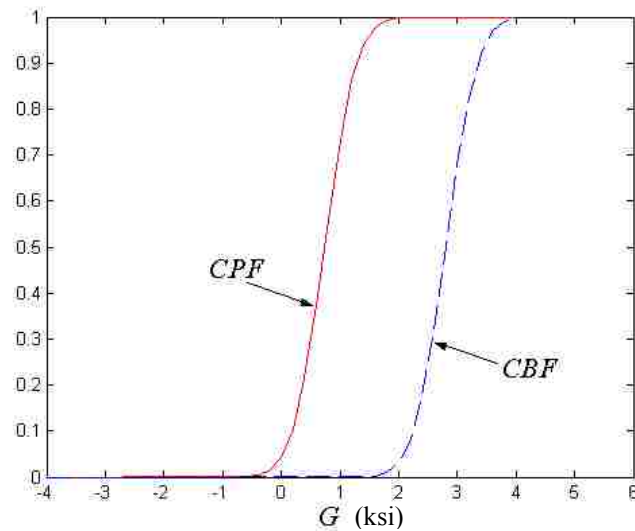
**Fig. 18 Initial unified uncertainty analysis of *G*.**

Sensitivity analysis is then conducted to identify the most influential variables. Total effect and main effect of each variable can be obtained respectively.

We conduct the main effect analysis as follows.

*Main effect of $Y_1$*: Keep $Y_1$ as an epistemic variable, and fix $Y_2$ and $Y_3$ at their averages.

*Main effect of $Y_2$*: Keep $Y_2$ as an epistemic variable, and fix $Y_1$ and $Y_3$ at their averages.

*Main effect of $Y_3$*: Keep $Y_3$ as an epistemic variable, and fix $Y_1$ and $Y_2$ at their averages.

The respective results for above three analyses are in Figs. 19, 20 and 21 and Table 10. As for the difference between the maximum and minimum probabilities of failure, since $ME_{p_f} = d_{p_f}^1$ is the largest, $Y_1$ therefore has the highest impact on $p_f$. As for the KS distance between the *CBF* and *CPF*, since $ME^1 = d_{KS}^1$ is the largest, $Y_1$ is also the most influential epistemic variable to the effect of epistemic uncertainty on *G*.
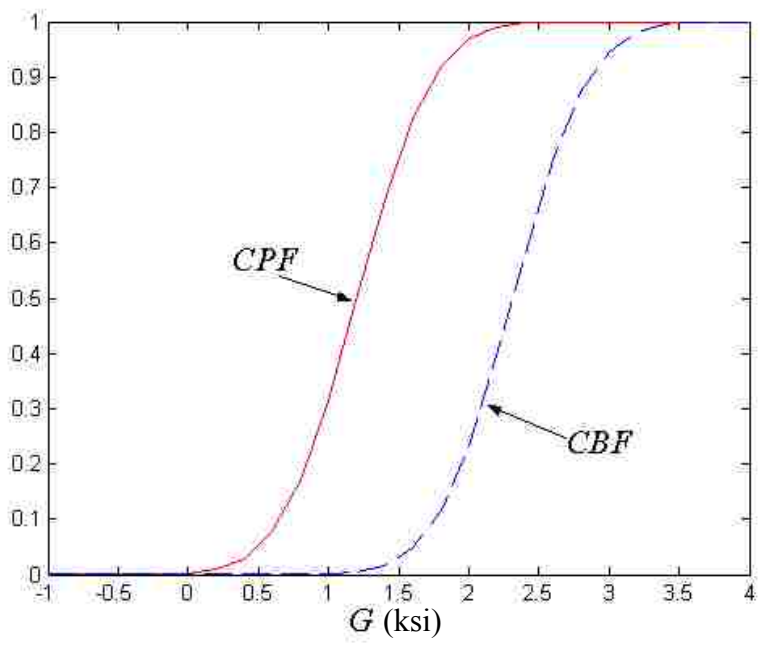
**Fig. 19** *CBF* **and** *CPF* **from the main effect analysis for** $Y_1$.
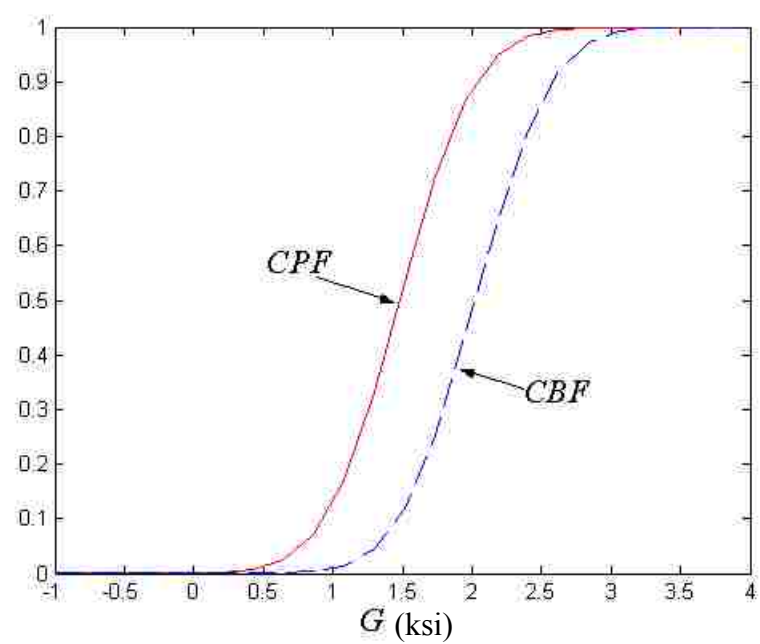


**Fig. 20** *CBF* **and** *CPF* **from the main effect analysis for** $Y_2$**.**
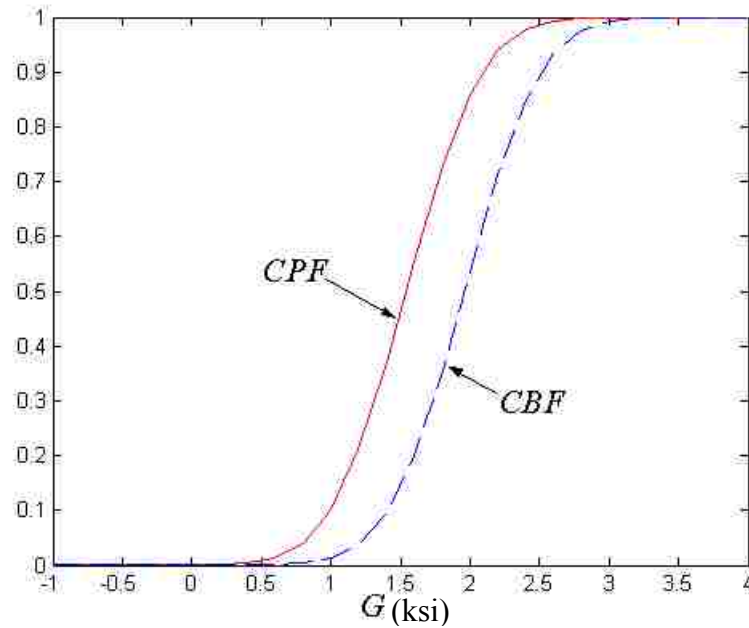
**Fig. 21 *CBF* and *CPF* from the main effect analysis for *Y₃*.**

**Table 10 Main effect of each epistemic variable**

| Main effect of | $p_f^{\min}$ (*Bel*) | $p_f^{\max}$ (*Pl*) | $ME_{p_f}$ | $ME^i$ |
|---|---|---|---|---|
| $Y_1$ | 2.91E-08 | 0.002245 | 0.002245 | 0.80278 |
| $Y_2$ | 1.01E-06 | 0.00024 | 0.000239 | 0.47207 |
| $Y_3$ | 2.08E-06 | 0.000144 | 0.000142 | 0.37393 |

We next perform the total effect analysis as follows.

*Total effect of $Y_1$*: Keep $Y_2$ and $Y_3$ as epistemic variables, and fix $Y_1$ at its average

*Total effect of $Y_2$*: Keep $Y_1$ and $Y_3$ as epistemic variables, and fix $Y_2$ at its average

*Total effect of $Y_3$*: Keep $Y_1$ and $Y_2$ as epistemic variables, and fix $Y_3$ at its average

The results are given in Figs. 22, 23 and 24 and Table 11. It can be seen that

$TE_{p_f}^1 = d_{p_f}^{-1}$ and $TE^1 = d_{KS}^{-1}$ are smallest, and therefore $Y_1$ is most influential, which is

consistent with the conclusion from the main effect analysis.
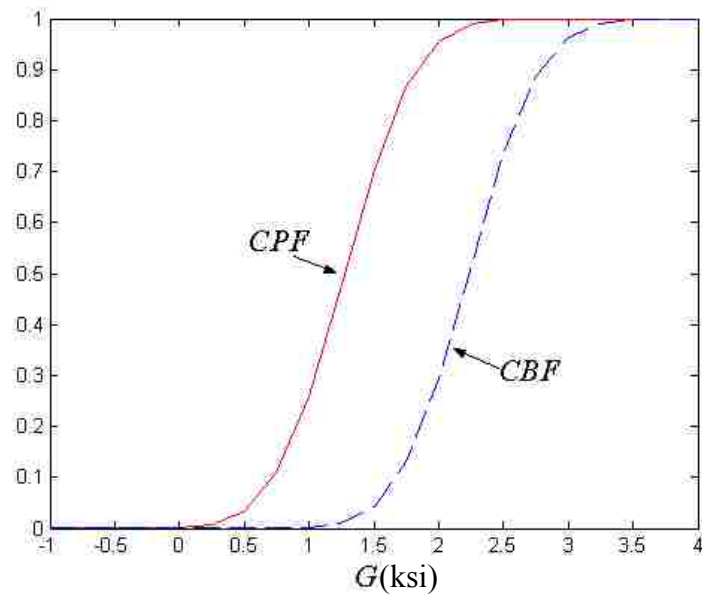
**Fig. 22** *CBF* **and** *CPF* **from the total effect analysis for** $Y_1$**.**
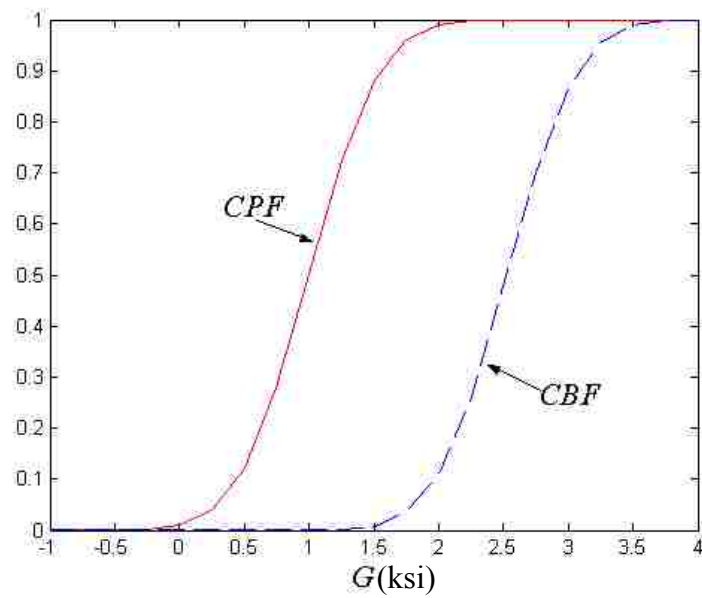


**Fig. 23** *CBF* **and** *CPF* **from the total effect analysis for** $Y_2$**.**
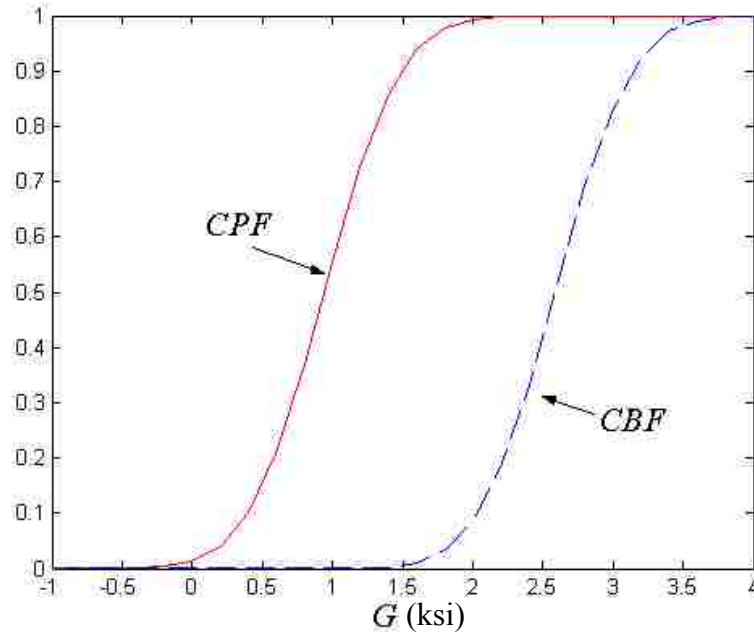
**Fig. 24 *CBF* and *CPF* from the total effect analysis for $Y_3$.**

**Table 11 Total effect of each epistemic variable**

| Total effect of | $p_f^{\min}$ (*Bel*) | $p_f^{\max}$ (*Pl*) | $TE_{p_f}$ | $TE^i$ |
|---|---|---|---|---|
| $Y_1$ | 8.08E-08 | 0.00134 | 0.00134 | 0.73815 |
| $Y_2$ | 1.66E-09 | 0.009275 | 0.009275 | 0.92589 |
| $Y_3$ | 6.65E-10 | 0.013141 | 0.013141 | 0.94513 |

To confirm the above sensitivity analyses results, we assume that more information could be collected on $Y_1$, $Y_2$ and $Y_3$. The distributions of $Y_1$, $Y_2$ and $Y_3$ after gathering more information are $N(3.55, 0.036)$, $N(0.437, 0.0044)$ and $N(250, 2.5)$, respectively. The unified uncertainty analysis is performed when one epistemic variable becomes aleatory. The *CBF* and *CPF* of $G$ are shown in Fig. 25, and $ME_{p_f}$ and *ME* in each case are provided in Table 12. It is seen that from Fig. 25 the gap between *CBF* and *CPF* becomes narrowest after the epistemic uncertainty in $Y_1$ is eliminated. The result confirms that collecting more information on the most influential variable $Y_1$ has the highest contribution for reducing the effect of epistemic uncertainty. The second highest

contribution is from gathering more information on $Y_2$. Collecting more information on $Y_3$ has the least contribution.

It is seen that after the epistemic uncertainty of the most important variable $Y_1$ has been eliminated, the gap between *CBF* and *CPF* is still large. The elimination of the epistemic uncertainty of one more variable may be needed. Since $Y_2$ is more important than $Y_3$, more information on $Y_2$ should be collected if further action needs to be taken.



Original *CBF* and *CPF*

*CBF* and *CPF* after $Y_1$ becomes aleatory

*CBF* and *CPF* after $Y_2$ becomes aleatory

*CBF* and *CPF* after $Y_3$ becomes aleatory

$G$ (ksi)
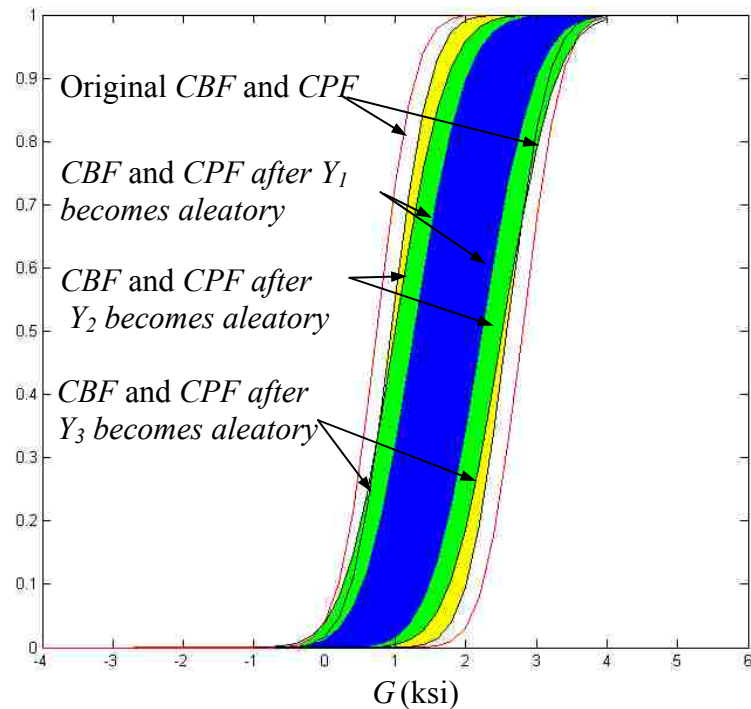
**Fig. 25 Comparison of uncertainty effect.**

**Table 12 Unified uncertainty analysis for confirmation**

| Scenarios | $p_f^{\min}$ (*Bel*) | $p_f^{\max}$ (*Pl*) | $ME_{p_f}^i$ | $ME^i$ |
|---|---|---|---|---|
| $Y_1$, $Y_2$ and $Y_3$ are aleatory | 2.74E-11 | 0.041172 | 0.041172 | 0.7836 |
| $Y_1$ is aleatory | 5.12E-05 | 0.012733 | 0.012682 | 0.5106 |
| $Y_2$ is aleatory | 5.89E-06 | 0.039711 | 0.039705 | 0.81014 |
| $Y_3$ is aleatory | 3.56E-09 | 0.0167 | 0.0167 | 0.93361 |

# V.  Conclusions

An effective sensitivity analysis method is developed to identify the most important input variables with epistemic uncertainty when aleatory uncertainty also exists. The importance of an epistemic variable is measured by its effect on the model output, including its main effect and total effect. These effects are indicated by the difference between belief and plausibility measures of an output variable. After the sensitivity analysis, all the epistemic variables are ranked by their importance. Then by collecting more information on the dominant epistemic variables, the effect of epistemic uncertainty can be reduced in the most efficient way as shown in the paper.

In the proposed sensitivity analysis procedure, an once-at-a-time strategy is used to set up different scenarios for the input epistemic variables in order to study their main effects and total effects. Then plausibility and belief measures of an output variable are calculated under each scenario by the unified uncertainty analysis framework. The Kolmogorov-Smirnov distance is used to quantify the discrepancy between the plausibility measure and belief measure, namely, the effect of epistemic uncertainty on the output. By comparing the main effects and total effects of the epistemic variables, their importance is ranked.

The proposed sensitivity analysis method is based on the First Order Reliability Method. The advantages of the proposed methods are as follows. (1) Engineers are familiar with First Order Reliability Method. (2) It is easy to quantify the contributions of the individual epistemic variables to the reliability or to the probability of failure. (3) Since optimization is used for interval analysis, the result in general is more accurate than that from interval arithmetic. (4) The process is efficient because the double-loop Monte

Carlo simulation is not involved. (5) The proposed method is applicable to black-box models.

When using the proposed method, one should also consider the other features of the method. (1) The method assumes the global optimal solution for the interval analysis. The method may not provide an accurate solution if a global optima is not reached. (2) The efficiency of the method depends on the number of subsets of the epistemic variables because First Order Reliability Method is performed for each subset. The efficiency also depends on the number of aleatory variables because the efficiency of First Order Reliability Method is directly proportional to the number of aleatory (random) variables.

Compared to the traditional probabilistic sensitivity analysis, sensitivity analysis with the mixture of epistemic and aleatory uncertainties is much more computationally expensive. Our future work will be the improvement of computational efficiency. We will also study the sensitivity of aleatory uncertainty and its interaction with epistemic uncertainty.

## Acknowledgements

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**PAPER II**

# Reliability sensitivity analysis with random and interval variables

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**SUMMARY**

In reliability analysis and reliability-based design, sensitivity analysis identifies the relationship between the change in reliability and the change in the characteristics of uncertain variables. Sensitivity analysis is also used to identify the most significant uncertain variables that have the highest contributions to reliability. Most of the current sensitivity analysis methods are applicable for only random variables. In many engineering applications, however, some of uncertain variables are intervals. In this work, a sensitivity analysis method is proposed for the mixture of random and interval variables. Six sensitivity indices are defined for the sensitivity of the average reliability and reliability bounds with respect to the averages and widths of intervals, as well as with respect to the distribution parameters of random variables. The equations of these sensitivity indices are derived based on the First Order Reliability Method (FORM). The proposed reliability sensitivity analysis is a byproduct of FORM without any extra function calls after reliability is found. Once FORM is performed, the sensitivity information is obtained automatically. Two examples are used for demonstration.

KEY WORDS: sensitivity analysis; random variable; interval variable; sensitivity index

## 1. INTRODUCTION

In reliability analysis [1~3] and reliability-based design [4~7], sensitivity analysis

provides information about the relationship between reliability and the distribution

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parameters of a random variable. Sensitivity analysis can therefore identify the most significant uncertain variables that have the highest contribution to reliability. When only random variables are involved, sensitivity analysis is usually performed for the probabilistic characteristics of a limit-state function, such as its moment, probability density function, and reliability. Such sensitivity analysis is usually named *probabilistic sensitivity analysis* (PSA). Various PSA approaches have been reported in a wide range of literature, including differential analysis [8, 9], variance-based methods [10], and sampling-based methods [10]. These types of probabilistic sensitivity analysis are briefly reviewed below.

(1) Differential analysis (probability sensitivity coefficient)

The probability-based sensitivity measure is defined as the rate of change in a probability ($P$) (reliability or the probability of failure) due to the change in a distribution parameter ($q_i$) of a random input, namely $\partial P / \partial q_i$. $\partial P / \partial q_i$ can be calculated by the finite difference method given by [2]:

$$S_{q_i} = \frac{P(q_i + \Delta q_i) - P(q_i)}{\Delta q_i} \tag{1}$$

where $q_i$ is a distribution parameter, such as the mean or the variance of a random variable; $\Delta q_i$ is a small step size of $q_i$.

Various probability sensitivity measures have been proposed in literature [11~14]. Wu [11] and Wu and Mohanty [12] propose a normalized *cumulative density function* (CDF)-based sensitivity coefficient for the probability of failure with respect to the distribution parameters of random variables. The sensitivity is defined by:

$$S_{q_i} = \frac{\partial p_f / p_f}{\partial q_i / q_i} = \int \cdots \int_\Omega \frac{q_i}{f_\mathbf{X}(\mathbf{X})} \frac{\partial f_\mathbf{X}(\mathbf{X})}{\partial q_i} \left[ \frac{f_\mathbf{X}(\mathbf{X})}{p_f} \right] d\mathbf{x} = E \left[ \frac{q_i \partial f_\mathbf{X}(\mathbf{X})}{f_\mathbf{X}(\mathbf{X}) \partial q_i} \right]_\Omega \qquad (2)$$

where $f_\mathbf{X}$ is the joint probability density function of all random variables, $p_f$ is the probability of failure, $\mathbf{X}$ is a vector of random variables, and $\Omega$ denotes the failure region. The calculation of this sensitivity measure involves evaluating a multidimensional integral. A sampling method is usually used to estimate this integral, which makes this method computationally expensive. Mavris et al. [13] extend Wu's method to evaluate the sensitivity of any probabilistic characteristics, such as the variance and mean of a limit-state function.

Another sensitivity measure related to reliability is the Most Probable Point (MPP) based sensitivity coefficients [14], defined as the gradient of a limit-state function at the MPP in the standard normal space, normalized by the reliability index. Let $G$ be a response calculated by a limit-state function $G = g(\mathbf{X})$, where $\mathbf{X}$ is the vector of random variables. After $\mathbf{X}$ is transformed into standard normal random variables $\mathbf{U}$, the MPP, $\mathbf{u}^* = \left( u_1^*, u_2^*, \cdots, u_{nx}^* \right)$, the shortest distance point from the limit state $g(\mathbf{U}) = c$, where $c$ is a limit state, to the origin O is identified. (The equation for the MPP search will be given in Eq. (4).) The sensitivity of reliability with respect to the $i$th random variables is then calculated by

$$S_i = \frac{\left( u_i^* \right)^2}{\beta^2} \qquad (3)$$

where $\beta$ is the magnitude of $\mathbf{u}^*$ or the reliability index. For the MPP-based reliability analysis, the probability sensitivity coefficient does not require any additional

computational efforts after the MPP is found. The sensitivity coefficient $S_i$ is just a byproduct of reliability analysis.

(2) Variance-based methods

Variance-based sensitivity analysis methods rely on the decomposition of the variance of a response into items contributed by various sources of input variations. These sources can be classified into two types: main effects and total effects. A main effect refers to the effect of only one random variable, while a total effect is used to include both the individual effect of a random variable and the interaction of the random variable with other random variables. Although the methods provide a global sensitivity measure, their major limitation is that a variance is assumed to be sufficient to describe the uncertainties encountered. This type of methods may lose accuracy when the variance is not a good measure of the distribution dispersion, such as in the case where a response distribution has high skewness and kurtosis [15].

(3) Sampling methods

Sampling approaches, such as Monte Carlo sampling for sensitivity analysis, usually involve three steps: (1) generating samples for uncertain input variables; (2) numerically evaluating a limit-state function and then obtaining samples of response variables; (3) statistically analyzing responses and quantifying their uncertainties, and then exploring the effects of the uncertainty of input variables on responses. Sampling methods are easy to use but computationally expensive when reliability is high. Because the probability of failure is low in this case, a large number of samples are required to capture a failure event.

The current PSA methods handle only random variables that are assumed to follow certain probability distributions. However, in many engineering applications, the information or knowledge might be too insufficient to build probability distributions. As discussed in [16, 17], uncertainty is sometimes represented by intervals due to the lack of knowledge. One example is that the true contact resistance in the vehicle crashworthiness design is hard to know; an interval is then used based on the engineers' best judgment [18]. Another example is in a new design. It is difficult to determine the precise distribution of design variables, such as dimensions. Engineers often define their design variables in the form of nominal values plus and minus certain tolerances, like 10±0.01mm. More examples of intervals can be found in [4, 16]. Sometimes even though a variable is random and follows a non-uniform distribution, only one interval estimate is available due to limited information or sparse samples. In this case, assigning an assumed distribution to the variable may lead to erroneous results [19]. When intervals are involved, the current PSA methods are no longer applicable.

Several methods of dealing with only interval variables have been reported for reliability analysis and reliability-based design [17, 20~34]. A few sensitivity analysis methods [35~38] for epistemic uncertainty (uncertainty due to the lack of knowledge) are potentially capable of dealing with interval variables. These methods use intervals to represent epistemic uncertainty. For example, a sensitivity analysis approach on the basis of belief and plausibility measures is proposed by Bae, et al [35, 36]. The results of this approach can help guide the data collection to improve the accuracy of reliability analysis and distinguish the dominant contributors of uncertainty. A sampling-based sensitivity analysis method is developed by Helton, et al [37]. It consists of three steps: an initial

analysis to explore the model behavior, a stepwise analysis to indicate the effects of uncertain variables on belief and plausibility functions, and a summary analysis to show a series of variance-based sensitivity analysis results. Considering the complexity of the mixture of aleatory and epistemic uncertainties, Guo and Du [38] propose an approach to conduct sensitivity analysis with this mixture. In their method, the most important epistemic variables are captured under the framework of the unified uncertainty analysis.

All of the above methods are capable of identifying the most significant interval variables, but they have some limitations. For example, it is difficult to use them to obtain information about how individual intervals impact reliability, especially how reliability bounds will change after narrowing interval bounds. In this work, we propose a sensitivity analysis method to handle the situation where both interval variables and random variables are involved. The intervals are treated as is without any distribution assumptions. With this method, we attempt to answer the following questions:

1)  How will the width of the reliability bounds change if the width of an interval is reduced or if the average of the interval is changed?

2)  How will the average reliability change if the width of an interval is reduced or if the average of the interval is changed?

3)  How will the width of the reliability bounds change if a distribution parameter of a random variable is changed?

4)  How will the average reliability change if a distribution parameter of a random variable is changed?

The answers to the above questions will provide useful information about improving reliability and reducing the impact of intervals and random variables on reliability.

Hence, six sensitivity indices are proposed for answering these questions. Equations for the sensitivity indices are then derived and corresponding computational procedures are developed. The calculation of sensitivity indices requires searching the minimum and maximum reliability, or the probabilities of failure, over the intervals. To alleviate the computational burden, we use an efficient FORM-based unified reliability analysis framework [39].

This paper is organized as follows: Sec. 2 provides a brief review of the unified reliability analysis. In Sec. 3, the six sensitivity indices are defined, and the equations for calculating these sensitivity indices are derived. In Sec. 4, two engineering examples are used to illustrate the proposed method. Conclusions and future work are summarized in Sec. 5.

## 2. UNIFIED RELIABILITY ANALYSIS

Reliability analysis is one of the main steps of reliability sensitivity analysis. The proposed sensitivity analysis is based on the First Order Reliability Method (FORM) [40, 41] which is applicable for random variables, and the unified reliability analysis (URA) [39], which is applicable for the mixture of random and interval variables. Both methods are briefly reviewed in this section.

### 2.1. Reliability analysis

In the reliability analysis where only random variables $\mathbf{X}$ are involved, reliability is defined by

$$R = \Pr\{G = g(\mathbf{X}) \geq c\} = 1 - \Pr\{G = g(\mathbf{X}) < c\} = 1 - p_f \qquad (4)$$

where $\Pr\{\cdot\}$ denotes a probability, $G$ is a response, $c$ is a specific limit state, $\mathbf{X} = (X_1, X_2, \cdots, X_i, \cdots, X_{nx})$ is a vector of random variables, $g$ is a performance function, also called a limit-state function [42], and $p_f$ is the probability of failure.

If the joint *probability density function* (PDF) of $\mathbf{X}$ is $f_{\mathbf{X}}$, the probability of failure $p_f$ is calculated by

$$p_f = \Pr\{G = g(\mathbf{X}) < c\} = \int\limits_{g(\mathbf{X})<c} f_{\mathbf{X}}(\mathbf{x})d\mathbf{x} \tag{5}$$

The limit-state function $g(\mathbf{X})$ is usually a nonlinear function of $\mathbf{X}$; therefore, the integration boundary is nonlinear. Since the number of random variables is usually high, multidimensional integration is involved. There is rarely a closed-form solution to Eq. (5). The First Order Reliability Method (FORM) is widely used to easily evaluate the integral in Eq. (5).

FORM involves three steps to approximate the probability integral: 1) transforming original random variables $\mathbf{X}$ into standard normal random variables $\mathbf{U}$, 2) searching for the Most Probable Point (MPP), and 3) calculating $p_f$.

Step 1: Transformation, which is given by

$$u_i = \Phi^{-1}\left\{F_{X_i}(x_i)\right\} \tag{6}$$

where $F_{X_i}$ is the CDF of $X_i$, and $\Phi^{-1}$ is the inverse CDF of a standard normal distribution.

Step 2: MPP search, where the MPP $\mathbf{u}^*$ is identified by

$$\begin{aligned} &\min_{\mathbf{U}} \|\mathbf{U}\| \\ &\text{s.t.} \ \ g(\mathbf{U}) = c \end{aligned} \tag{7}$$

in which $\|\cdot\|$ stands for the magnitude of a vector. Geometrically, the MPP is the shortest distance point from the limit state $g(\mathbf{U}) = c$ to the origin in U-space. The minimum distance $\beta = \|\mathbf{u}^*\|$ is called the *reliability index.*

Step 3: Estimation of $p_f$, which is given by

$$p_f = \Phi(-\beta) \tag{8}$$

where $\Phi$ is the CDF of a standard normal distribution.

The most computationally intensive work of FORM is the MPP search. The following recursive algorithm [43] is used for the MPP search,

$$\begin{cases} \beta^{(k)} = \beta^{(k-1)} + \dfrac{g(\mathbf{u}^{(k-1)})}{\left\|\nabla g(\mathbf{u}^{(k-1)})\right\|} \\[4mm] \mathbf{u}^{(k)} = -\beta^{(k)} \dfrac{\nabla g(\mathbf{u}^{(k-1)})}{\left\|\nabla g(\mathbf{u}^{(k-1)})\right\|} \end{cases} \tag{9}$$

where $\nabla g(\mathbf{u}^{(k)})$ is the gradient of $g$ at $\mathbf{u}^{(k)}$, $\left\|\nabla g(\mathbf{u}^{(k)})\right\|$ is its magnitude, and $k$ is the iteration counter.

## 2.2. Unified reliability analysis (URA)

When both random and interval variables are present, random variables $\mathbf{X}$ are characterized by probability distributions while interval variables $\mathbf{Y}$ reside on $[\mathbf{y}^l, \mathbf{y}^u]$. The unified uncertainty analysis framework and computational method proposed in [39] is applicable to handle this situation. As shown in [39], the cumulative distribution function (CDF) of the response $G = g(\mathbf{X}, \mathbf{Y})$ has its upper and lower bounds, and so does reliability $\Pr\{G \geq c\}$. The unified reliability analysis (URA) [39] is used to find the reliability bounds.

The URA framework is illustrated in Figure 1. The inputs to the framework are random variables **X** defined by a joint PDF and interval variables **Y**. The outputs are CDF bounds and reliability bounds.
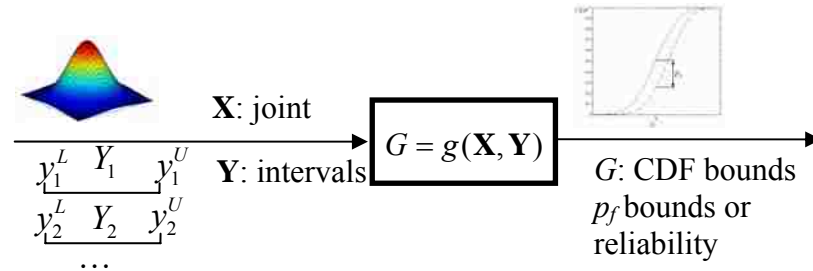


Figure 1. The unified reliability analysis framework.

The set of intervals **Y** is denoted by $\Delta_{\mathbf{Y}}$, and the event of failure is defined by $g(\mathbf{X},\mathbf{Y}) < c$. According to [39], the upper and lower bounds of the probability of failure, $p_f^U$ and $p_f^L$, are calculated by

$$p_f^L = \Pr\left\{G_{\max}(\mathbf{X},\mathbf{Y}) < c \,|\, \mathbf{Y} \in \Delta_{\mathbf{Y}}\right\} \tag{10}$$

and

$$p_f^U = \Pr\left\{G_{\min}(\mathbf{X},\mathbf{Y}) < c \,|\, \mathbf{Y} \in \Delta_{\mathbf{Y}}\right\} \tag{11}$$

respectively. $G_{\min}$ and $G_{\max}$ are the global minimum and maximum values, respectively, of $G$ over $\Delta_{\mathbf{Y}}$.

The evaluation of the upper and lower bounds of the probability of failure is essentially the evaluation of the minimum and maximum CDF of the limit-state function. Therefore, traditional probabilistic analysis methods can be used for the unified reliability analysis (URA). The First Order Reliability Method (FORM) is employed for the URA.

Figure 2 depicts the numerical procedure of the URA method. The procedure involves two types of analysis. The first one is *probabilistic analysis* (PA), which is responsible for the MPP search and the calculation of the probability of failure. The second one is *interval analysis* (IA), which is responsible for the search of the maximum and minimum values of $G$. The direct combination of PA and IA will involve a double loop procedure, where PA is an outer loop and IA is an inner loop. For example, to find the lower bound of $p_f$, at every iteration of the MPP search in the outer loop, interval analysis inner loops will be called to find the maximum $G$ in terms of **Y**. This method is inefficient due to the double-loop procedure. The efficient computational method is then developed in [44]. The method involves an efficient sequential single-loop procedure, where PA is decoupled from IA. The flowchart of this efficient procedure is shown in Figure 2 for the $p_f^L$ calculation. The solution is the MPP where $G$ is maximized. The MPP for $p_f^L$ is then named $\mathbf{u}^{*,L}$ in this paper. And the MPP for the maximum probability of failure $p_f^U$ is called $\mathbf{u}^{*,U}$.
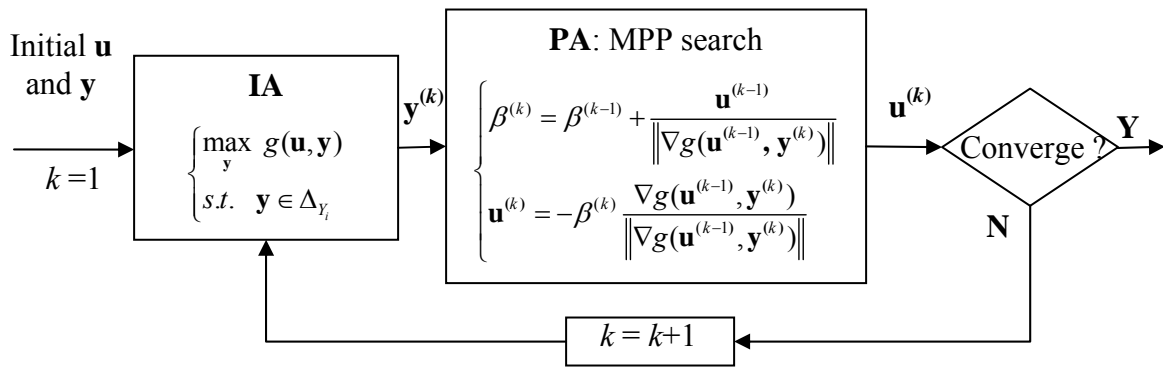


Figure 2. Flowchart of sequential single-loop procedure for $p_f^L$ calculation.

The probability $\Pr\{G_{\max}(\mathbf{X},\mathbf{Y}) < c \,|\, \mathbf{Y} \in \Delta_{\mathbf{Y}}\}$ in Eq. (10) is then computed by

$$\Pr\{G_{\max}(\mathbf{X},\mathbf{Y}) < c \,|\, \mathbf{Y} \in \Delta_{\mathbf{Y}}\} = \Phi(-\beta) = \Phi(-\|\mathbf{u}^*\|). \tag{12}$$

For the $p_f^U$ calculation, the model of the MPP search is the same as in Figure 2 except that IA becomes a minimization problem.

## 3. RELIABILITY SENSITIVITY ANALYSIS

When only random variables are involved, reliability sensitivity analysis is used to find the rate of change in the probability of failure (or reliability) due to the changes in distribution parameters (usually means and standard deviations). When both random variables and interval variables are involved, reliability analysis will generate two bounds of reliability or of the probability of failure $p_f$. The gap between the maximum probability of failure $p_f^U$ and the minimum probability of failure $p_f^L$ represents the effect of interval variables on the probability of failure. In addition to the traditional sensitivity analysis in terms of random variables, sensitivity analysis in terms of interval variables is also needed. In this work, six types of sensitivity are proposed with respect to both random variables and interval variables. The proposed sensitivity indexes are summarized in Table I.

Table I. Six sensitivity indices

| Sensitivity type | Description | Input |
|---|---|---|
| Type I $\partial \delta_p / \partial \delta_i$ | Sensitivity of the width of the $p_f$ bounds, $\delta_p$, with respect to the width of interval variable $Y_i$, $\delta_i$ | Interval |
| Type II $\partial \overline{p}_f / \partial \delta_i$ | Sensitivity of the average $p_f$, $\overline{p}_f$, with respect to the width of interval variable $Y_i$, $\delta_i$ | Interval |
| Type III $\partial \delta_p / \partial \overline{y}_i$ | Sensitivity of the width of the $p_f$ bounds, $\delta_p$, with respect to the average of interval variable $Y_i$, $\overline{y}_i$ | Interval |
| Type IV $\partial \overline{p}_f / \partial \overline{y}_i$ | Sensitivity of the average $p_f$, $\overline{p}_f$, with respect to the average of interval variable $Y_i$, $\overline{y}_i$ | Interval |
| Type V $\partial \delta_p / \partial q_i$ | Sensitivity of the width of the $p_f$ bounds, $\delta_p$, with respect to a distribution parameter, $q_i$, of random variable $X_i$ | Random |
| Type VI $\partial \overline{p}_f / \partial q_i$ | Sensitivity of the average $p_f$, $\overline{p}_f$, with respect to a distribution parameter, $q_i$, of random variable $X_i$ | Random |

### 3.1. Type I sensitivity $\partial \delta_p / \partial \delta_i$

$\partial \delta_p / \partial \delta_i$ is the sensitivity of the width of the $p_f$ bounds, $\delta_p$, with respect to the interval

width of the $i$th interval variable $Y_i$, $\delta_i$. $\delta_p$ is defined by

$$\delta_p = p_f^U - p_f^L \tag{13}$$

The width of $Y_i$ is calculated by

$$\delta_i = y_i^U - y_i^L \tag{14}$$

where $y_i^L$ and $y_i^U$ are the lower and upper bounds of $y_i$, respectively.

To obtain a unique sensitivity index, we define the change of $\delta_i$, $\Delta(\delta_i)$ in such a way

that $Y_i$ expands in both directions equally; namely, $y_i^L$ is decreased by $\dfrac{\Delta(\delta_i)}{2}$ and $y_i^U$ is

increased by $\dfrac{\Delta(\delta_i)}{2}$. There are infinite ways that $Y_i$ can change by $\Delta(\delta_i)$, for example,

$[y_i^L, y_i^U]$ can change to $\left[ y_i^L - \dfrac{3\Delta(\delta_i)}{4}, y_i^U + \dfrac{\Delta(\delta_i)}{4} \right]$ or $\left[ y_i^L - \dfrac{\Delta(\delta_i)}{4}, y_i^U + \dfrac{3\Delta(\delta_i)}{4} \right]$. Our

definition makes the change unique.

This type of sensitivity can identify interval variables that have the largest impact on the width of the $p_f$ bounds. If the gap of the $p_f$ bounds is too wide, decisions will be difficult to make. To narrow the width of $p_f$ bounds efficiently, more information about the important interval variables should be collected, and then their widths can be reduced. Sensitivity analysis will provide a useful guidance to the collection of more information.

To derive the equations for $\partial \delta_p / \partial \delta_i$, we consider all the situations where the maximum or minimum $p_f$ occurs on the lower bound, upper bound, or at an interior point of $Y_i$. Next we demonstrate how to derive $\partial \delta_p / \partial \delta_i$ when the maximum $p_f$ occurs on the upper bound of $Y_i$ and the minimum $p_f$ occurs on the lower bound of $Y_i$. The derivations of other cases are given in Appendix B, and the common equations used in derivations are given in Appendix A.

The problem can be stated as:

Given: $G = g(\mathbf{X}, \mathbf{Y})$, $\mathbf{Y}_{\sim i} = (Y_1, Y_2, \cdots, Y_{i-1}, Y_{i+1}, \cdots Y_{my})$, $\bar{y}_i = \dfrac{y_i^L + y_i^U}{2}$, $p_f^L$ occurs at $y_i^L$,

and $p_f^U$ occurs at $y_i^U$.

Find: $\partial \delta_p / \partial \delta_i$.

$$\frac{\partial \delta_p}{\partial \delta_i} = \frac{\partial \left( p_f^U - p_f^L \right)}{\partial \delta_i} = \frac{\partial \left[ p_f^U \left( \bar{y}_i + \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i} \right) - p_f^L \left( \bar{y}_i - \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i} \right) \right]}{\partial \delta_i}$$

$$= \frac{\partial \left[ p_f^U \left( \bar{y}_i + \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i} \right) \right]}{\partial \left( \bar{y}_i + \frac{1}{2}\delta_i \right)} \frac{\partial \left( \bar{y}_i + \frac{1}{2}\delta_i \right)}{\partial \delta_i} - \frac{\partial \left[ p_f^L \left( \bar{y}_i - \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i} \right) \right]}{\partial \left( \bar{y}_i - \frac{1}{2}\delta_i \right)} \frac{\partial \left( \bar{y}_i - \frac{1}{2}\delta_i \right)}{\partial \delta_i}$$

$$= \left( \frac{1}{2} \right) \frac{\partial \left[ p_f^U \left( \bar{y}_i + \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i} \right) \right]}{\partial \left( \bar{y}_i + \frac{1}{2}\delta_i \right)} - \left( -\frac{1}{2} \right) \frac{\partial \left[ p_f^L \left( \bar{y}_i - \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i} \right) \right]}{\partial \left( \bar{y}_i - \frac{1}{2}\delta_i \right)}$$

$$= \frac{1}{2} \left( \frac{\partial p_f^U}{\partial y_i^U} + \frac{\partial p_f^L}{\partial y_i^L} \right)$$

(15)

$\dfrac{\partial p_f^U}{\partial y_i^U}$ and $\dfrac{\partial p_f^L}{\partial y_i^L}$ then need to be calculated. In this case, the MPP's of $p_f^L$ or $p_f^U$ are on

one bound of $Y_i$. Let $h$ be the bound and $p_f$ be $p_f^U$ or $p_f^L$. Then,

$$\frac{\partial p_f}{\partial h} = \frac{\partial [\Phi(-\beta)]}{\partial h} = -\phi(-\beta) \frac{\partial \beta}{\partial h}$$

(16)

where $\phi(\cdot)$ is the PDF of a standard normal distribution. Next, we will show how to

calculate $\dfrac{\partial \beta}{\partial h}$.

Let the MPP be $\mathbf{u}^* = \left( u_1^*, u_2^*, \cdots, u_{nx}^* \right)$ and the corresponding intervals $\mathbf{Y}$ be $\mathbf{y}$. In the U-

space after $\mathbf{X}$ are transformed into $\mathbf{U}$, the limit-state function becomes $g(\mathbf{U}, \mathbf{Y})$, and at

the MPP the limit-state function is $g(\mathbf{u}^*, \mathbf{y})$, where $\mathbf{y}$ is the vector of $\mathbf{Y}$ at the MPP. Let

$\nabla g(\mathbf{u}^*)$ be the gradient of $g(\mathbf{U}, \mathbf{Y})$ in terms of $\mathbf{U}$ at the MPP; namely,

$\nabla g(\mathbf{u}^*) = \left( \left. \dfrac{\partial g}{\partial U_1} \right|_{\mathbf{u}^*, \mathbf{y}}, \left. \dfrac{\partial g}{\partial U_2} \right|_{\mathbf{u}^*, \mathbf{y}}, \cdots, \left. \dfrac{\partial g}{\partial U_n} \right|_{\mathbf{u}^*, \mathbf{y}} \right)$. For brevity, without losing generality, we

will drop **Y** or **y** in the limit-state function expression in the following derivations. At the MPP, the following equation holds [40, 41],

$$u_i^* = -\beta \frac{\nabla g(\mathbf{u}^*)}{\left\| \nabla g(\mathbf{u}^*) \right\|} \tag{17}$$

$\dfrac{\nabla g(\mathbf{u}^*)}{\left\| \nabla g(\mathbf{u}^*) \right\|}$ is the unit vector of the gradient, and the gradient is calculated at the MPP,

therefore a constant. Then,

$$\frac{\partial u_i}{\partial h} = -\frac{\partial \beta}{\partial h} \frac{\left. \dfrac{\partial g}{\partial U_i} \right|_{u_i^*}}{\left\| \nabla g(\mathbf{u}^*) \right\|} \tag{18}$$

Recall that $y_i$ is on one bound $h$ of the interval variable $Y_i$ at the MPP, where

$G = g(\mathbf{u}^*, h)$ reaches the limit state and hereby becomes a constant. Then

$$\frac{\partial G}{\partial h} = \sum_{i=1}^{n_x} \frac{\partial g}{\partial U_i} \frac{\partial U_i}{\partial h} + \frac{\partial g}{\partial h} = 0 \tag{19}$$

Therefore, Eq. (19) becomes

$$\sum_{i=1}^{n_x} \frac{\partial g}{\partial U_i} \frac{\partial U_i}{\partial h} + \frac{\partial g}{\partial h} = \sum_{i=1}^{n_x} \frac{\partial g}{\partial U_i} \left( -\frac{\partial \beta}{\partial h} \frac{\left. \dfrac{\partial g}{\partial U_i} \right|_{u_i^*}}{\left\| \nabla g(\mathbf{u}^*) \right\|} \right) + \frac{\partial g}{\partial h} = -\frac{\partial \beta}{\partial h} \frac{\sum_{i=1}^{n_x} \left( \left. \dfrac{\partial g}{\partial U_i} \right|_{u_i^*} \right)^2}{\left\| \nabla g(\mathbf{u}^*) \right\|} + \frac{\partial g}{\partial h}$$

$$= -\frac{\partial \beta}{\partial h} \left\| \nabla g(\mathbf{u}^*) \right\| + \frac{\partial g}{\partial h} = 0 \tag{20}$$

We then obtain

$$\frac{\partial \beta}{\partial h} = \frac{\dfrac{\partial g}{\partial h}}{\left\| \nabla g(\mathbf{u}^*) \right\|} \tag{21}$$

Substituting $\partial \beta / \partial h$ in Eq. (16) with Eq. (21) yields

$$\frac{\partial p_f}{\partial h} = -\phi(-\beta)\frac{\partial \beta}{\partial h} = -\phi(-\beta)\frac{\dfrac{\partial g}{\partial h}}{\left\|\nabla g(\mathbf{u}^*)\right\|} = \frac{-\phi(-\beta)}{\left\|\nabla g(\mathbf{u}^*)\right\|}\frac{\partial g}{\partial h} \tag{22}$$

Using the results from Eqs.(22) and (15), we get the equation of Type I sensitivity when $p_f^{\max}$ occurs on the upper bound of $Y_i$ and $p_f^{\min}$ occurs on the lower bound of $Y_i$ as follows:

$$\frac{\partial \delta_p}{\partial \delta_i} = \frac{1}{2}\left(\frac{\partial p_f^U}{\partial y_i^U} + \frac{\partial p_f^L}{\partial y_i^L}\right) = -\frac{1}{2}\left(\frac{\phi(-\beta^U)}{\left\|\nabla g(\mathbf{u}^{*,U})\right\|}\frac{\partial g}{\partial Y_i}\bigg|_{y_i^U} + \frac{\phi(-\beta^L)}{\left\|\nabla g(\mathbf{u}^{*,L})\right\|}\frac{\partial g}{\partial Y_i}\bigg|_{y_i^L}\right) \tag{23}$$

where $\beta^U$ is the reliability index at the maximum $p_f$, $\beta^L$ is the reliability index at the minimum $p_f$, $\mathbf{u}^{*,U}$ is the MPP for the maximum $p_f$, and $\mathbf{u}^{*,L}$ is the MPP for the minimum $p_f$. The equations of Type I sensitivity for other situations are given in Appendix B.

## 3.2. Type II sensitivity $\partial \overline{p}_f / \partial \delta_i$

$\partial \overline{p}_f / \partial \delta_i$ is the sensitivity of the average $p_f$, $\overline{p}_f$, with respect to $\delta_i$. $\overline{p}_f$ is defined by

$$\overline{p}_f = \frac{p_f^U + p_f^L}{2} \tag{24}$$

The relationship among $p_f^U$, $p_f^L$, $\delta_p$ and $\overline{p}_f$ is illustrated in Figure 3. This type of sensitivity quantifies the rate of change of the mean value of $p_f$ due to the change of the interval width of $Y_i$. The equations of this type of sensitivity are given in Appendix C.
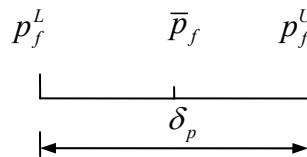


Figure 3. $p_f^U$, $p_f^L$, $\delta_i$, and $\overline{p}_f$.

*3.3. Type III sensitivity $\partial \delta_p / \partial \bar{y}_i$*

$\partial \delta_p / \partial \bar{y}_i$ is the sensitivity of the width of the probability of failure $\delta_p$ with respect to the average of the $i$th interval variable, $\bar{y}_i$. $\bar{y}_i$ is defined by

$$\bar{y}_i = \frac{y_i^U + y_i^L}{2} \tag{25}$$

The relationship among $y_i^L$, $y_i^U$, $\delta_i$ and $\bar{y}_i$ is illustrated in Figure 4. This type of sensitivity is useful when we can control the averages of the interval variables during reliability based-optimization. We can efficiently decrease the reliability gap by shifting averages of interval variables to which the probability of failure is highly sensitive. The equations of this type sensitivity are given in Appendix D.
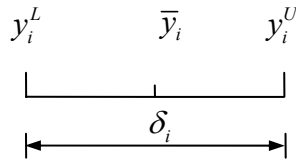
Figure 4. $y_i^L, y_i^U, \delta_i$, and $\bar{y}_i$.

*3.4. Type IV sensitivity $\partial \bar{p}_f / \partial \bar{y}_i$*

$\partial \bar{p}_f / \partial \bar{y}_i$ is the sensitivity of the average probability of failure $\bar{p}_f$ with respect to $\bar{y}_i$. It tells us how much the average probability of failure will change given the change in the midpoint of an interval variable. The equations of this type of sensitivity are given in Appendix E.

*3.5. Type V sensitivity $\partial\delta_p / \partial q_i$*

$\partial\delta_p / \partial q_i$ is the sensitivity of the width of the probability of failure $\delta_p$ with respect to a distribution parameter, $q_i$, of random variable $X_i$. For example, for a normal distribution, $q_i$ would be the mean $\mu_i$ or standard deviation $\sigma_i$ while for uniform distribution, $q_i$ could be one of the interval bounds. As shown previously, the $p_f$ gap $\delta_p$ is mainly caused by interval variables [38]. On the other hand, the value of $p_f$ primarily depends on random variables. The equations of this type of sensitivity are given in Appendix F.

*3.6. Type VI sensitivity $\partial\overline{p}_f / \partial q_i$*

$\partial\overline{p}_f / \partial q_i$ is the sensitivity of the average probability of failure $\overline{p}_f$ with respect to a distribution parameter, $q_i$, of random variable $X_i$. The equations of this type of sensitivity are given in Appendix G. The equations of Type V and VI sensitivities for a normal distribution are also given in Appendices F and G, respectively.

*3.7. Equations of all the sensitivity indices*

The equations for all the above sensitivity indices are summarized in Tables II, III and IV.

Table II. Type I and II sensitivities for intervals

| Case | Type I $\partial\delta_p/\partial\delta_i$ (Appendix B) | Type II $\partial\overline{p}_f/\partial\delta_i$ (Appendix C) |
|---|---|---|
| 1 — $p_f^U$ occurs at $y_i^U$; $p_f^L$ occurs at $y_i^L$ | $-\dfrac{1}{2}\left[\dfrac{\phi(-\beta^U)}{\|\nabla g(\mathbf{u}^{*,U})\|}\dfrac{\partial g}{\partial Y_i}\Big|_{y_i^U} + \dfrac{\phi(-\beta^L)}{\|\nabla g(\mathbf{u}^{*,L})\|}\dfrac{\partial g}{\partial Y_i}\Big|_{y_i^L}\right]$ | $-\dfrac{1}{4}\left[\dfrac{\phi(-\beta^U)}{\|\nabla g(\mathbf{u}^{*,U})\|}\dfrac{\partial g}{\partial Y_i}\Big|_{y_i^U} - \dfrac{\phi(-\beta^L)}{\|\nabla g(\mathbf{u}^{*,L})\|}\dfrac{\partial g}{\partial Y_i}\Big|_{y_i^L}\right]$ |
| 2 — $p_f^U$ occurs at $y_i^L$; $p_f^L$ occurs at $y_i^U$ | $\dfrac{1}{2}\left[\dfrac{\phi(-\beta^U)}{\|\nabla g(\mathbf{u}^{*,U})\|}\dfrac{\partial g}{\partial Y_i}\Big|_{y_i^L} + \dfrac{\phi(-\beta^L)}{\|\nabla g(\mathbf{u}^{*,L})\|}\dfrac{\partial g}{\partial Y_i}\Big|_{y_i^U}\right]$ | $\dfrac{1}{4}\left[\dfrac{\phi(-\beta^U)}{\|\nabla g(\mathbf{u}^{*,U})\|}\dfrac{\partial g}{\partial Y_i}\Big|_{y_i^L} - \dfrac{\phi(-\beta^L)}{\|\nabla g(\mathbf{u}^{*,L})\|}\dfrac{\partial g}{\partial Y_i}\Big|_{y_i^U}\right]$ |
| 3 — $p_f^U$ occurs at $y_i^U$; $p_f^L$ occurs at an interior point | $-\dfrac{1}{2}\left[\dfrac{\phi(-\beta^U)}{\|\nabla g(\mathbf{u}^{*,U})\|}\dfrac{\partial g}{\partial Y_i}\Big|_{y_i^U}\right]$ | $-\dfrac{1}{4}\left[\dfrac{\phi(-\beta^U)}{\|\nabla g(\mathbf{u}^{*,U})\|}\dfrac{\partial g}{\partial Y_i}\Big|_{y_i^U}\right]$ |
| 4 — $p_f^U$ occurs at $y_i^L$; $p_f^L$ occurs at an interior point | $-\dfrac{1}{2}\left[\dfrac{\phi(\beta^U)}{\|\nabla g(\mathbf{u}^{*,U})\|}\dfrac{\partial g}{\partial Y_i}\Big|_{y_i^L}\right]$ | $\dfrac{1}{4}\left[\dfrac{\phi(-\beta^U)}{\|\nabla g(\mathbf{u}^{*,U})\|}\dfrac{\partial g}{\partial Y_i}\Big|_{y_i^L}\right]$ |
| 5 — $p_f^L$ occurs at $y_i^U$; $p_f^U$ occurs at an interior point | $\dfrac{1}{2}\left[\dfrac{\phi(-\beta^L)}{\|\nabla g(\mathbf{u}^{*,L})\|}\dfrac{\partial g}{\partial Y_i}\Big|_{y_i^U}\right]$ | $-\dfrac{1}{4}\left[\dfrac{\phi(-\beta^L)}{\|\nabla g(\mathbf{u}^{*,L})\|}\dfrac{\partial g}{\partial Y_i}\Big|_{y_i^U}\right]$ |
| 6 — $p_f^L$ occurs at $y_i^L$; $p_f^U$ occurs at an interior point | $-\dfrac{1}{2}\left[\dfrac{\phi(-\beta^L)}{\|\nabla g(\mathbf{u}^{*,L})\|}\dfrac{\partial g}{\partial Y_i}\Big|_{y_i^L}\right]$ | $\dfrac{1}{4}\left[\dfrac{\phi(-\beta^L)}{\|\nabla g(\mathbf{u}^{*,L})\|}\dfrac{\partial g}{\partial Y_i}\Big|_{y_i^L}\right]$ |
| 7 — $p_f^U$ and $p_f^L$ both occurs at interior points | 0 | 0 |

Table III. Type III and IV sensitivities for intervals

| Case | Type III $\partial\delta_p/\partial\bar{y}_i$ (Appendix D) | Type IV $\partial\bar{p}_f/\partial\bar{y}_i$ (Appendix E) |
|---|---|---|
| 1 — $p_f^U$ occurs at $y_i^U$; $p_f^L$ occurs at $y_i^L$ | $\dfrac{-\phi(-\beta^U)}{\|\nabla g(\mathbf{u}^{*,U})\|}\dfrac{\partial g}{\partial Y_i}\Big|_{y_i^U} + \dfrac{\phi(-\beta^L)}{\|\nabla g(\mathbf{u}^{*,L})\|}\dfrac{\partial g}{\partial Y_i}\Big|_{y_i^L}$ | $-\dfrac{1}{2}\left[\dfrac{\phi(-\beta^U)}{\|\nabla g(\mathbf{u}^{*,U})\|}\dfrac{\partial g}{\partial Y_i}\Big|_{y_i^U} + \dfrac{\phi(-\beta^L)}{\|\nabla g(\mathbf{u}^{*,L})\|}\dfrac{\partial g}{\partial Y_i}\Big|_{y_i^L}\right]$ |
| 2 — $p_f^U$ occurs at $y_i^L$; $p_f^L$ occurs at $y_i^U$ | $\dfrac{-\phi(-\beta^U)}{\|\nabla g(\mathbf{u}^{*,U})\|}\dfrac{\partial g}{\partial Y_i}\Big|_{y_i^L} + \dfrac{\phi(-\beta^L)}{\|\nabla g(\mathbf{u}^{*,L})\|}\dfrac{\partial g}{\partial Y_i}\Big|_{y_i^U}$ | $-\dfrac{1}{2}\left[\dfrac{\phi(-\beta^U)}{\|\nabla g(\mathbf{u}^{*,U})\|}\dfrac{\partial g}{\partial Y_i}\Big|_{y_i^L} + \dfrac{\phi(-\beta^L)}{\|\nabla g(\mathbf{u}^{*,L})\|}\dfrac{\partial g}{\partial Y_i}\Big|_{y_i^U}\right]$ |
| 3 — $p_f^U$ occurs at $y_i^U$; $p_f^L$ occurs at an interior point | $\dfrac{-\phi(-\beta^U)}{\|\nabla g(\mathbf{u}^{*,U})\|}\dfrac{\partial g}{\partial Y_i}\Big|_{y_i^U}$ | $-\dfrac{1}{2}\left[\dfrac{\phi(-\beta^U)}{\|\nabla g(\mathbf{u}^{*,U})\|}\dfrac{\partial g}{\partial Y_i}\Big|_{y_i^U}\right]$ |
| 4 — $p_f^U$ occurs at $y_i^L$; $p_f^L$ occurs at an interior point | $\dfrac{-\phi(-\beta^U)}{\|\nabla g(\mathbf{u}^{*,U})\|}\dfrac{\partial g}{\partial Y_i}\Big|_{y_i^L}$ | $-\dfrac{1}{2}\left[\dfrac{\phi(-\beta^U)}{\|\nabla g(\mathbf{u}^{*,U})\|}\dfrac{\partial g}{\partial Y_i}\Big|_{y_i^L}\right]$ |
| 5 — $p_f^L$ occurs at $y_i^U$; $p_f^U$ occurs at an interior point | $\dfrac{\phi(-\beta^L)}{\|\nabla g(\mathbf{u}^{*,L})\|}\dfrac{\partial g}{\partial Y_i}\Big|_{y_i^U}$ | $-\dfrac{1}{2}\left[\dfrac{\phi(-\beta^L)}{\|\nabla g(\mathbf{u}^{*,L})\|}\dfrac{\partial g}{\partial Y_i}\Big|_{y_i^U}\right]$ |
| 6 — $p_f^L$ occurs at $y_i^L$; $p_f^U$ occurs at an interior point | $\dfrac{\phi(-\beta^L)}{\|\nabla g(\mathbf{u}^{*,L})\|}\dfrac{\partial g}{\partial Y_i}\Big|_{y_i^L}$ | $-\dfrac{1}{2}\left[\dfrac{\phi(-\beta^L)}{\|\nabla g(\mathbf{u}^{*,L})\|}\dfrac{\partial g}{\partial Y_i}\Big|_{y_i^L}\right]$ |
| 7 — $p_f^U$ and $p_f^L$ both occurs at interior points | 0 | 0 |

Table IV. Type V and VI sensitivities for random variables

| Case | Type V $\partial\delta_p / \partial q$ (Appendix F) | Type VI $\partial\overline{p}_f / \partial q$ (Appendix G) |
|---|---|---|
| General | $-\phi(-\beta^U)\dfrac{u_i^{*,U}}{\beta^U}\dfrac{\partial w}{\partial q_i}$ $+\phi(-\beta^L)\dfrac{u_i^{*,L}}{\beta^L}\dfrac{\partial w}{\partial q_i}$ | $-\dfrac{1}{2}\left[\phi(-\beta^U)\dfrac{u_i^{*,L}}{\beta^U}\dfrac{\partial w}{\partial q_i}\right.$ $\left.+\phi(-\beta^L)\dfrac{u_i^{*,L}}{\beta^L}\dfrac{\partial w}{\partial q_i}\right]$ |

In the above table, $w$ is given in Equation (A11) in Appendix A.

The procedure to calculate the sensitivity indices is illustrated in Figure 5. First, unified reliability analysis is conducted to obtain MPP's and interval variables at $p_f^U$ and $p_f^L$ . Then depending on the location of the interval variables, either interior or on a bound, at the MPP, the corresponding equations from Table II, III, and IV are used to calculate the sensitivity indices.
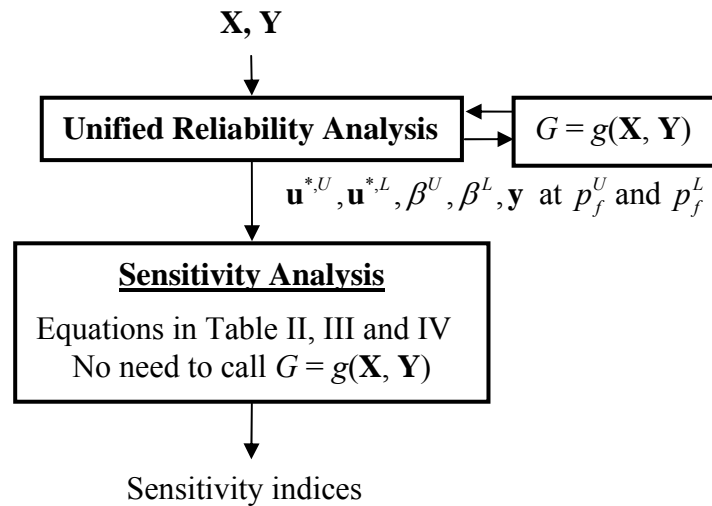


Figure 5. The procedure to calculate sensitivity indices.

## 4.  NUMERICAL EXAMPLES

Two examples are used to demonstrate our proposed sensitivity measures with random and interval variables. The first example deals with normally distributed variables while the second example handles random variables with non-normal distributions.

### *4.1. Example 1- adhesive bonding example*

A double-lap joint design of a rubber-modified epoxy based adhesive [45] is illustrated in Figure 6. The design consists of aluminum outer adherends and an inner steel adherend. The assembly is cured at 250 °F and is stress-free at temperature $T_1$. The completed bond is subjected to an axial load $P$ at a service temperature $T_2$. The coefficients of thermal expansion for the outer and inner adherend $\alpha_o$ and $\alpha_i$ are $6 \times 10^{-6}$ and $13 \times 10^{-6}$ in/(in $\cdot$ °F), respectively. The modulus $E_o$ and the thickness $t_o$, of the outer adherend, and the modulus $E_i$ and the thickness $t_i$, of the inner adherend, are random variables. The shear modulus $G$, width $b$, length $L$, of the adhesive, and the lap-shear strength of adhesive $S_a$ are also random variables. Their distributions are given in Table V.
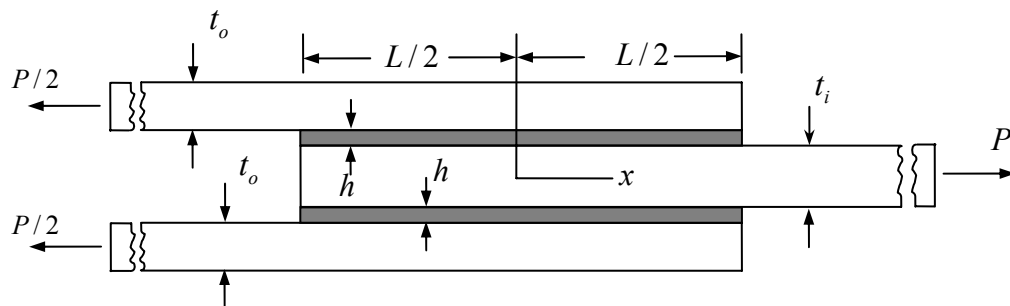


Figure 6. A double-lap joint design of adhesive

Table V. Random variables X

| Variable | Mean | Standard deviation | Distribution |
|----------|------|--------------------|--------------|
| $X_1 (E_o)$ | $10 \times 10^6$ psi | $0.1 \times 10^6$ psi | Normal |
| $X_2 (E_i)$ | $30 \times 10^6$ psi | $0.3 \times 10^6$ psi | Normal |
| $X_3 (t_o)$ | 0.15 in | 0.0015 in | Normal |
| $X_4 (t_i)$ | 0.10 in | 0.001 in | Normal |
| $X_5 (G)$ | $0.2 \times 10^6$ psi | $0.002 \times 10^6$ psi | Normal |
| $X_6 (b)$ | 1 in | 0.01 in | Normal |
| $X_7 (L)$ | 1.1 in | 0.011 in | Normal |
| $X_8 (P)$ | 2000 psi | 20 psi | Normal |
| $X_9 (S_a)$ | 4100 psi | 41 psi | Normal |

Because it is difficult to spread the adhesive uniformly over the surface, the thickness of the adhesive is estimated to be in an interval shown in Table VI. The temperature change, $\Delta T = T_2 - T_1$, is difficult to fit into some probability distribution since the temperature field is unknown. An interval is therefore assigned for $\Delta T$ as listed in Table VI.

Table VI. Interval variables

| Variable | Lower bound | Upper bound |
|----------|-------------|-------------|
| $Y_1 (h)$ | 0.0195 in | 0.0205 in |
| $Y_2 (\Delta T)$ | -131.0 °F | -129.0 °F |

The limit-state function is the safety margin for strength requirement of the joint, which is defined by the difference between the lap-shear strength of adhesive and the maximum shear stress $\tau_{max}$. The equation is obtained at $x = 0.5$ where the maximum shear stress occurs. The function is given by

$$G = g(\mathbf{X}, \mathbf{Y}) = S_a - \tau_{max}$$

where $\tau_{max} = \tau(0.5)$, and

$$\tau(x) = \frac{P\omega}{4b\sinh(\omega L/2)}\cosh(\omega x) + \left[\frac{P\omega}{4b\cosh(\omega L/2)}\left(\frac{2E_o t_o - E_i t_i}{2E_o t_o + E_i t_i}\right)\right.$$

$$\left. + \frac{(\alpha_i - \alpha_o)\Delta T\omega}{(1/E_o t_o + 2/E_i t_i)\cosh(\omega L/2)}\right]\sinh(\omega x)$$

and $\omega = \sqrt{\dfrac{G}{h}\left(\dfrac{1}{E_o t_o} + \dfrac{2}{E_i t_i}\right)}$.

The failure event is defined by $F = \{\mathbf{X}, \mathbf{Y}\,|\,g(\mathbf{X}, \mathbf{Y}) < 0\}$.

The analysis results are listed in Tables VII, VIII, IX and X. To verify the proposed method, additional reliability analyses are also conducted. The results are shown as "Numerical verification" in Table VIII (for interval variables) and Table X (for random variables). Each parameter (the average or width of an interval variable, or a distribution parameter of a random variable), with respect to which a sensitivity index would be calculated, is increased by a small step size. An additional reliability analysis for that parameter is then performed. The rate of change in the reliability analysis results with respect to the parameter was computed. The rate should be very close to the sensitivity index calculated from the proposed method. Both Tables VIII and X show good consistency and verify the accuracy of the proposed method.

The sign of a sensitivity index gives a possible direction for improvement. For example, in Table VIII $\partial\delta_p / \partial\delta_1$ and $\partial\delta_p / \partial\delta_2$ are both positive while $\partial\delta_p / \partial\bar{y}_1$ and $\partial\delta_p / \partial\bar{y}_2$ are both negative. Therefore, if we wish to reduce the bounds of $p_f$, we could narrow the intervals of thickness of adhesive ($\delta_1$) and the temperature change ($\delta_2$) or increase their averages of them ($\bar{y}_1$ and $\bar{y}_2$). A similar conclusion can be drawn for $\partial\delta_p / \partial\bar{y}_i$ and $\partial\bar{p}_f / \partial\bar{y}_i$.

To better interpret the sensitivity analysis results, the percentage change in Table IX is also included. $\Delta_{y_i}^{+1\%}$ indicates the change in $\delta_p$ or $\bar{p}_f$ corresponding to the 1% increase in $\delta_i$ or $\bar{y}_i$, respectively. For instance, if $\delta_1$ increased by 1%, or $\delta_1$ increased by $\left(y_1^U - y_1^L\right) \times 1\% = \left(0.0205 - 0.0195\right) \times 1\% = 1.0 \times 10^{-5}$ inch, the width of the probability of failure bounds $\delta_p$ would increase by $\left(5.009 \times 10^{-2}\right) \times \left(1.0 \times 10^{-5}\right) = 5.009 \times 10^{-7}$, where the multiplier is the change in $\delta_1$ while the multiplicand is the Type I sensitivity index. Similarly, the average probability of failure $\bar{p}_f$ would change by $\left(-2.494 \times 10^{-2}\right) \times \left(1.0 \times 10^{-5}\right) = -2.494 \times 10^{-7}$. Since the sign is negative, $\bar{p}_f$ would actually decrease. This example indicates how the change in input uncertainty impacts reliability or the probability of failure. A sensitivity index also tells us the relative importance of uncertain variables. For example, $Y_1$ has higher $\Delta^{+1\%}$ of Type I ~ IV sensitivity indices than those of $Y_2$; $Y_1$ is therefore more significant than $Y_2$ in terms of its impact on $\delta_p$ and $\bar{p}_f$.

Table VII. Bounds of the probability of failure

| Probability of Failure | $p_f^L$ | $p_f^U$ | $\bar{p}_f$ | $\delta_p$ |
|---|---|---|---|---|
| $p_f$ | $7.797 \times 10^{-5}$ | $1.067 \times 10^{-2}$ | $5.338 \times 10^{-3}$ | $1.059 \times 10^{-2}$ |

Table VIII. Sensitivity with respect to interval variables

| Type of sensitivity | Proposed method | | Numerical verification | |
|---|---|---|---|---|
| | $Y_1$ | $Y_2$ | $Y_1$ | $Y_2$ |
| Type I $\partial \delta_p / \partial \delta_i$ | $5.009 \times 10^{-2}$ | $9.309 \times 10^{-6}$ | $5.071 \times 10^{-2}$ | $9.379 \times 10^{-6}$ |
| Type II $\partial \overline{p}_f / \partial \delta_i$ | $-2.494 \times 10^{-2}$ | $-4.655 \times 10^{-6}$ | $-2.525 \times 10^{-2}$ | $-4.669 \times 10^{-6}$ |
| Type III $\partial \delta_p / \partial \overline{y}_i$ | $-9.978 \times 10^{-2}$ | $-1.862 \times 10^{-5}$ | $-1.001 \times 10^{-1}$ | $-1.868 \times 10^{-5}$ |
| Type IV $\partial \overline{p}_f / \partial \overline{y}_i$ | $5.009 \times 10^{-2}$ | $9.309 \times 10^{-6}$ | $5.071 \times 10^{-2}$ | $9.379 \times 10^{-6}$ |

Table IX. The change of $\delta_p$ and $\overline{p}_f$ with 1% increases in $\delta_i$ and $\overline{y}_i$

| Type of sensitivity | $\Delta_1^{+1\%}$ | $\Delta_2^{+1\%}$ |
|---|---|---|
| Type I $\partial \delta_p / \partial \delta_i$ | $5.009 \times 10^{-7}$ | $1.862 \times 10^{-7}$ |
| Type II $\partial \overline{p}_f / \partial \delta_i$ | $-2.494 \times 10^{-7}$ | $-9.310 \times 10^{-8}$ |
| Type III $\partial \delta_p / \partial \overline{y}_i$ | $-1.996 \times 10^{-3}$ | $2.241 \times 10^{-5}$ |
| Type IV $\partial \overline{p}_f / \partial \overline{y}_i$ | $1.002 \times 10^{-3}$ | $-1.210 \times 10^{-5}$ |

Table X shows sensitivities in terms of the mean and standard deviation of random variables. The positive signs of $\partial \delta_p / \partial q$ and $\partial \overline{p}_f / \partial q$ imply that the distribution parameters need to be lowered to reduce $\delta_p$ and $\overline{p}_f$. And the negative ones suggest that distribution parameters need to be increased to reduce $\delta_p$ and $\overline{p}_f$. From this table, it can be concluded that $X_7$ has the highest impact on $\delta_p$ and $\overline{p}_f$ because it has the highest sensitivity index values. Given the positive signs of Type V and VI sensitivity indices of $X_7$, reducing the mean and variance of $X_7$ would be more efficient than adjusting other random variables in order to lower $\delta_p$ and $\overline{p}_f$.

Table X. Sensitivity with respect to random variables

| | Proposed method | | Numerical validation | |
|---|---|---|---|---|
| | Type V $\partial\delta_p/\partial q$ | Type VI $\partial\overline{p}_f/\partial q$ | Type V $\partial\delta_p/\partial q$ | Type VI $\partial\overline{p}_f/\partial q$ |
| $X_1(\mu_1)$ | $1.091\times10^{-10}$ | $5.475\times10^{-11}$ | $1.097\times10^{-10}$ | $5.509\times10^{-11}$ |
| $X_1(\sigma_1)$ | $4.566\times10^{-11}$ | $2.295\times10^{-11}$ | $4.631\times10^{-11}$ | $2.328\times10^{-11}$ |
| $X_2(\mu_2)$ | $-2.097\times10^{-11}$ | $-1.054\times10^{-11}$ | $-2.083\times10^{-11}$ | $-1.046\times10^{-11}$ |
| $X_2(\sigma_2)$ | $5.066\times10^{-12}$ | $2.550\times10^{-12}$ | $4.997\times10^{-12}$ | $2.515\times10^{-12}$ |
| $X_3(\mu_3)$ | $7.270\times10^{-3}$ | $3.650\times10^{-3}$ | $7.315\times10^{-3}$ | $3.673\times10^{-3}$ |
| $X_3(\sigma_3)$ | $3.044\times10^{-3}$ | $1.530\times10^{-3}$ | $3.087\times10^{-3}$ | $1.552\times10^{-3}$ |
| $X_4(\mu_4)$ | $-6.292\times10^{-3}$ | $-3.161\times10^{-3}$ | $-6.250\times10^{-3}$ | $-3.139\times10^{-3}$ |
| $X_4(\sigma_4)$ | $1.520\times10^{-3}$ | $7.650\times10^{-4}$ | $1.499\times10^{-3}$ | $7.546\times10^{-4}$ |
| $X_5(\mu_5)$ | $9.818\times10^{-9}$ | $4.931\times10^{-9}$ | $9.780\times10^{-9}$ | $4.911\times10^{-9}$ |
| $X_5(\sigma_5)$ | $7.402\times10^{-9}$ | $3.723\times10^{-9}$ | $7.330\times10^{-9}$ | $3.687\times10^{-9}$ |
| $X_6(\mu_6)$ | $-1.913\times10^{-3}$ | $-9.608\times10^{-4}$ | $-1.912\times10^{-3}$ | $-9.601\times10^{-4}$ |
| $X_6(\sigma_6)$ | $1.405\times10^{-3}$ | $7.068\times10^{-4}$ | $1.404\times10^{-3}$ | $7.060\times10^{-4}$ |
| $X_7(\mu_7)$ | $-8.018\times10^{-3}$ | $-4.026\times10^{-3}$ | $-8.011\times10^{-3}$ | $-4.022\times10^{-3}$ |
| $X_7(\sigma_7)$ | $2.715\times10^{-2}$ | $1.365\times10^{-2}$ | $2.731\times10^{-2}$ | $1.373\times10^{-2}$ |
| $X_8(\mu_8)$ | $9.421\times10^{-7}$ | $4.731\times10^{-7}$ | $9.428\times10^{-7}$ | $4.735\times10^{-7}$ |
| $X_8(\sigma_8)$ | $6.815\times10^{-7}$ | $3.428\times10^{-7}$ | $6.817\times10^{-7}$ | $3.429\times10^{-7}$ |
| $X_9(\mu_9)$ | $-1.079\times10^{-6}$ | $-5.417\times10^{-7}$ | $-1.078\times10^{-6}$ | $-5.411\times10^{-7}$ |
| $X_9(\sigma_9)$ | $1.832\times10^{-6}$ | $9.213\times10^{-7}$ | $1.831\times10^{-6}$ | $9.210\times10^{-7}$ |

*4.2. Example 2- cantilever tube*

In Example 1, all random variables are normally distributed. In this example, some random variables follow non-normal distributions. The cantilever tube shown in Figure 7 is subject to external forces $F_1$, $F_2$, and $P$, and torsion $T$ [44]. The limit-state function is defined as the difference between the yield strength $S$ and the maximum stress $\sigma_{max}$, namely,

$$G = g(\mathbf{X,Y}) = S - \sigma_{max}$$

where $\sigma_{max}$ is the maximum von Mises stress on the top surface of the tube at the origin

and is given by

$$\sigma_{max} = \sqrt{\sigma_x^2 + 3\tau_{zx}^2} \, .$$
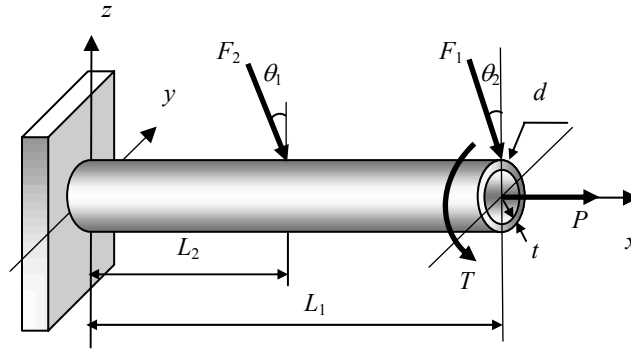


Figure 7. Cantilever tube.

The normal stress $\sigma_x$ is calculated by

$$\sigma_x = \frac{P + F_1 \sin\theta_1 + F_2\sin\theta_2}{A} + \frac{Mc}{I}$$

where the first term is the normal stress due to the axial forces, and the second term is the

normal stress due to the bending moment $M$, which is given by

$$M = F_1 L_1 \cos\theta_1 + F_2 L_2\cos\theta_2$$

and

$$A = \frac{\pi}{4}\left[d^2 - (d - 2t)^2\right]$$

$$c = d/2 \, ,$$

$$I = \frac{\pi}{64}\left[d^4 - (d - 2t)^4\right]$$

The torsional stress $\tau_{zx}$ at the same point is calculated by

$$\tau_{zx} = \frac{Td}{2J}$$

where $J = 2I$.

The random and interval variables are given in Tables XI and XII, respectively.

Table XI. Random variables

| Variable | Parameter 1 | Parameter 2 | Distribution |
|----------|-------------|-------------|--------------|
| $X_1$ ($t$) | 5 mm (mean) | 0.1 mm (std[*]) | Normal |
| $X_2$ ($d$) | 42 mm (mean) | 0.5 mm (std) | Normal |
| $X_3$ ($L_1$) | 119.75 mm (lb[**]) | 120.25 mm (ub[***]) | Uniform |
| $X_4$ ($L_2$) | 59.75 mm (lb) | 60.25 mm (ub) | Uniform |
| $X_5$ ($F_1$) | 3.0 kN (mean) | 0.3 kN (std) | Normal |
| $X_6$ ($F_2$) | 3.0 kN (mean) | 0.3 kN (std) | Normal |
| $X_7$ ($P$) | 12.0 kN (mean) | 1.2 kN (std) | Gumbel |
| $X_8$ ($T$) | 90.0 N·m (mean) | 9.0 N·m (std) | Normal |
| $X_9$ ($S_y$) | 220.0 MPa (mean) | 22.0 MPa (std) | Normal |

*: std – standard deviation

**: lb – the lower bound of a uniform distribution

***: ub – the upper bound of a uniform distribution

Table XII. Interval variables

| Variable | Lower bound | Upper bound |
|----------|-------------|-------------|
| $Y_1$ ($\theta_1$) | 0° | 10° |
| $Y_2$ ($\theta_2$) | 5° | 15° |

The results of reliability analysis and sensitivity are listed in Table XIII, XIV, and XV. It is noted that sensitivity indices of $\partial \delta_p / \partial \delta_i$, and $\partial \delta_p / \partial \overline{y}_i$ are all positive while sensitivity indices of $\partial \overline{p}_f / \partial \delta_i$ and $\partial \overline{p}_f / \partial \overline{y}_i$ are all negative. In this case, the direction of the change in $\delta_p$ will be opposite to the direction of change in $\overline{p}_f$ whenever we adjust $\delta_i$ and $\overline{y}_i$. For instance, decreasing $\delta_1$ will result in a lower $\delta_p$ and a higher $\overline{p}_f$.

Table XIII. Bounds of probability of failure

| Probability of Failure | $p_f^L$ | $p_f^U$ | $\bar{p}_f$ | $\delta_p$ |
|---|---|---|---|---|
| $p_f$ | $1.437\times10^{-4}$ | $1.631\times10^{-4}$ | $1.530\times10^{-4}$ | $1.940\times10^{-5}$ |

Table XIV. Sensitivity of interval variables

| Type of sensitivity | Proposed Method | | Numerical Validation | |
|---|---|---|---|---|
| | $Y_1$ | $Y_2$ | $Y_1$ | $Y_2$ |
| Type I $\partial\delta_p/\partial\delta_i$ | $1.038\times10^{-4}$ | $5.861\times10^{-5}$ | $1.034\times10^{-4}$ | $5.837\times10^{-5}$ |
| Type II $\partial\bar{p}_f/\partial\delta_i$ | $-5.192\times10^{-5}$ | $-2.930\times10^{-5}$ | $-5.170\times10^{-5}$ | $-2.919\times10^{-5}$ |
| Type III $\partial\delta_p/\partial\bar{y}_i$ | $2.077\times10^{-4}$ | $1.172\times10^{-4}$ | $2.068\times10^{-4}$ | $1.167\times10^{-4}$ |
| Type IV $\partial\bar{p}_f/\partial\bar{y}_i$ | $-1.038\times10^{-4}$ | $-5.861\times10^{-5}$ | $-1.034\times10^{-4}$ | $-5.837\times10^{-5}$ |

Table XV. The change of $\delta_p$ and $\bar{p}_f$ with 1% increases in $\delta_i$ and $\bar{y}_i$

| Type of sensitivity | $\Delta_{y_1}^{+1\%}$ | $\Delta_{y_2}^{+1\%}$ |
|---|---|---|
| Type I $\partial\delta_p/\partial\delta_i$ | $1.038\times10^{-5}$ | $5.861\times10^{-6}$ |
| Type II $\partial\bar{p}_f/\partial\delta_i$ | $-5.192\times10^{-6}$ | $-2.930\times10^{-6}$ |
| Type III $\partial\delta_p/\partial\bar{y}_i$ | $1.039\times10^{-5}$ | $5.860\times10^{-6}$ |
| Type IV $\partial\bar{p}_f/\partial\bar{y}_i$ | $-5.190\times10^{-6}$ | $-2.931\times10^{-6}$ |

In this example, uniform distributions and a Gumbel distribution are involved. In Table XVI, the sensitivities in terms of the parameters of these two distributions are also calculated. It is indicated that Type V and VI sensitivities of uniformly distributed variables, $X_3$ and $X_4$, are all positive. Hence, if we raise or lower the bounds of $X_3$ and $X_4$, the change of $\delta_p$ and $\bar{p}_f$ will follow the same direction.

Table XVI. Sensitivity of random variables

| | Proposed Method | | Numerical Validation | |
|---|---|---|---|---|
| | Type V $\partial \delta_p / \partial q$ | Type VI $\partial \overline{p}_f / \partial q$ | Type V $\partial \delta_p / \partial q$ | Type VI $\partial \overline{p}_f / \partial q$ |
| $X_1(\mu_1)$ | $-5.822\times10^{-2}$ | $-4.886\times10^{-1}$ | $-5.820\times10^{-2}$ | $-4.886\times10^{-1}$ |
| $X_1(\sigma_1)$ | $1.614\times10^{-2}$ | $1.457\times10^{-1}$ | $1.615\times10^{-2}$ | $1.458\times10^{-1}$ |
| $X_2(\mu_2)$ | $-2.413\times10^{-2}$ | $-1.888\times10^{-1}$ | $-2.413\times10^{-2}$ | $-1.888\times10^{-1}$ |
| $X_2(\sigma_2)$ | $1.393\times10^{-2}$ | $1.088\times10^{-1}$ | $1.394\times10^{-2}$ | $1.089\times10^{-1}$ |
| $X_3(a_3)$ | $1.093\times10^{-3}$ | $8.412\times10^{-3}$ | $1.093\times10^{-3}$ | $8.413\times10^{-3}$ |
| $X_3(b_3)$ | $1.130\times10^{-3}$ | $8.697\times10^{-3}$ | $1.137\times10^{-3}$ | $8.742\times10^{-3}$ |
| $X_4(a_4)$ | $1.123\times10^{-3}$ | $7.893\times10^{-3}$ | $1.124\times10^{-3}$ | $7.894\times10^{-3}$ |
| $X_4(b_4)$ | $1.162\times10^{-3}$ | $8.143\times10^{-3}$ | $1.167\times10^{-3}$ | $8.163\times10^{-3}$ |
| $X_5(\mu_5)$ | $7.630\times10^{-8}$ | $6.197\times10^{-7}$ | $7.631\times10^{-8}$ | $6.197\times10^{-7}$ |
| $X_5(\sigma_5)$ | $8.347\times10^{-8}$ | $7.033\times10^{-7}$ | $8.355\times10^{-8}$ | $7.040\times10^{-7}$ |
| $X_6(\mu_6)$ | $3.908\times10^{-8}$ | $3.117\times10^{-7}$ | $3.908\times10^{-8}$ | $3.117\times10^{-7}$ |
| $X_6(\sigma_6)$ | $2.192\times10^{-8}$ | $1.779\times10^{-7}$ | $2.193\times10^{-8}$ | $1.780\times10^{-7}$ |
| $X_7(\mu_7)$ | $5.002\times10^{-9}$ | $4.256\times10^{-8}$ | $5.002\times10^{-9}$ | $4.256\times10^{-8}$ |
| $X_7(\sigma_7)$ | $5.139\times10^{-10}$ | $5.670\times10^{-9}$ | $5.143\times10^{-10}$ | $5.674\times10^{-9}$ |
| $X_8(\mu_8)$ | $5.678\times10^{-8}$ | $5.050\times10^{-7}$ | $5.688\times10^{-8}$ | $5.049\times10^{-7}$ |
| $X_8(\sigma_8)$ | $1.363\times10^{-9}$ | $1.402\times10^{-8}$ | $1.363\times10^{-9}$ | $1.402\times10^{-8}$ |
| $X_9(\mu_9)$ | $-2.887\times10^{-12}$ | $-2.457\times10^{-11}$ | $-2.886\times10^{-12}$ | $-2.457\times10^{-11}$ |
| $X_9(\sigma_9)$ | $8.708\times10^{-12}$ | $8.108\times10^{-11}$ | $8.740\times10^{-12}$ | $8.146\times10^{-11}$ |

## 5. CONCLUSIONS

When information or knowledge is not adequate to build probability distributions, interval variables may be used. In this case, probabilistic sensitivity analysis approaches are no longer applicable. An effective sensitivity analysis method is proposed to handle the mixture of random variables and interval variables.

With the presence of both random and interval variables, reliability and the probability of failure resides between their lower and upper bounds. In this work, based on the

unified uncertainty analysis framework [39], we have explored various sensitivity indices with respect to both random and interval variables. Four new types of sensitivity for interval variables include the sensitivities of the width and average of the probability of failure bounds with respect to the interval width and with respect to the mean of each interval variable. Two new types of sensitivity for random variables include the sensitivities of the width and average of the probability of failure with respect to the distribution parameters of each random variable. Equations for the six sensitivity indices are derived. Through the unified reliability analysis and the First Order Reliability Method (FORM), the sensitivity indices are calculated after reliability analysis is completed without calling the limit-state function again. The sensitivity indices are therefore a byproduct of reliability analysis.

The advantages of the proposed methods are as follows: (1) The method is easy to use because it employs the First Order Reliability Method (FORM), which is widely used in industry. (2) Sensitivity information is just a byproduct of reliability analysis. (3) Both random and interval variables can be handled by reliability analysis at the same time. And (4) the computation is efficient without a double-loop procedure or Monte Carlo simulation involved.

The method has some limitations. Since it is based on only the First Order Reliability Method (FORM), the method cannot be directly extended to the Second Order Reliability Method (SORM). The method assumes the global optimal solution if optimization is used for interval analysis. The method may not provide an accurate solution if a global optima is not reached. It is well known that FORM may not be accurate when multiple

MPPs exist. The proposed method exhibits the same behavior for the multiple MPP's situation.

Future work would be the further improvement of efficiency and the inclusion of more sensitivity indices. For higher efficiency, the efficient interval arithmetic could be used for interval analysis. Other sensitivity methods, such as those suggested in [45], could also be incorporated.

## ACKNOWLEDGEMENTS

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**PAPER III**

# Reliability Analysis for Multidisciplinary Systems with Random and Interval variables

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**Tremendous efforts have been devoted to developing efficient approaches to reliability analysis for multidisciplinary systems. Most of the approaches, however, are only capable of dealing with random variables modeled by probability distributions. Both random and interval variables may exist in multidisciplinary systems. Their propagation through coupled subsystems makes reliability analysis computationally expensive. A unified reliability analysis framework with both random and interval variables is proposed for multidisciplinary systems in this work. The framework is an extension of the existing unified uncertainty analysis framework for single-disciplinary problems. The new framework involves probabilistic analysis (PA) and interval analysis (IA). Both PA and IA are decoupled from each other and are performed sequentially. Three supporting algorithms are developed. The effectiveness of these algorithms is demonstrated by a mathematical example and an engineering application.**

## Nomenclature

| | | |
|---|---|---|
| $c$ | = | limit state |
| $F_X$ | = | cumulative distribution function of $X$ |
| $f_X$ | = | joint probability function of $X$ |
| $G$ | = | response |
| $G_{\max}$ | = | maximum value of $G$ |
| $G_{\min}$ | = | minimum value of $G$ |
| $g$ | = | limit state function |
| $h$ | = | equality constraint |

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| Pr | = | probability |
|---|---|---|
| $p_f$ | = | probability of failure |
| $p_f^L$ | = | lower bound of probability of failure |
| $p_f^U$ | = | upper bound of probability of failure |
| $R$ | = | reliability |
| $\mathbf{U}$ | = | vector of standard normal random variables transformed from $\mathbf{X}$ |
| $\mathbf{u}^*$ | = | Most Probable Point |
| $\mathbf{W}_i$ | = | vector of interval input variables to $i$th discipline |
| $\mathbf{w}^L$ | = | vector of lower bounds of $\mathbf{W}$ |
| $\mathbf{w}^U$ | = | vector of upper bounds of $\mathbf{W}$ |
| $\mathbf{X}$ | = | vector of random variables |
| $\mathbf{X}_i$ | = | vector of random input variables to the $i$th discipline |
| $\mathbf{Y}_{ij}$ | = | vector of coupling variables from the $i$th discipline to the jth discipline |
| $\mathbf{Z}_i$ | = | vector of outputs from the $i$th discipline |
| $\beta$ | = | reliability index |
| $\Phi$ | = | cumulative distribution function of a standard normal variable |
| $\Phi^{-1}$ | = | inverse function of $\Phi$ |

# I. Introduction

COMPARED with single-disciplinary reliability analysis, multidisciplinary reliability analysis is much more complicated. A multidisciplinary system consists of a number of disciplines (subsystems), which are often highly coupled with each other. The output of one subsystem may be the input to other subsystems, and vice versa. Uncertainty in one discipline will be propagated to other disciplines through the interdisciplinary interfaces. The other complexity is that a large number of uncertain variables may be involved in a multidisciplinary system.

Due to these complexities, computationally efficient strategies and algorithms of reliability analysis become essential. Several multidisciplinary reliability analysis methods have been reported [1~11]. Sues et al. [1] use response surface models to replace the computationally expensive simulation models in reliability analysis for multidisciplinary optimization (MDO). A multi-stage, parallel implementation strategy is

developed to integrate reliability analysis and an MDO framework [2]. The reliability analysis methods proposed in [3, 4] use the concurrent subspace optimization framework. A similar approach, the collaborative reliability analysis [5], performs reliability analysis and multidisciplinary analysis (MDA) concurrently. Ahn et al. [6] employ a sequential approach to reliability analysis with MDA. They also develop a strategy to associate single-level reliability-based design with the bi-level integrated system synthesis; and in their method sequential single loops of reliability analysis and optimization are conducted based on the approximation of limit state functions [7]. To avoid the tremendous computational burden caused by the direct integration of reliability-based design (RBD) with MDO, a method of Sequential Optimization and Reliability Assessment (SORA) for MDO is developed in [8]. The strategy of SORA is to decouple reliability analysis from MDO.

Analytical Target Cascading (ATC) is reformulated for design optimization under uncertainty for hierarchically decomposed multilevel systems [9]. In this work, the advanced mean value (AMV)-based technique and a bottom-to-top coordination are used. ATC is also used in [10]. The method decomposes reliability-based MDO into several individual RBD problems at a sub-system level, and then SORA is used to solve the individual RBD problems. The study in [11] focuses on the tradeoff between the failure probabilities of subsystems and the tradeoff between system performance and subsystem failure probabilities. The study involves First Order Reliability Method (FORM) and multiobjective optimization with an all-in-one approach to the coupled analysis. A methodology for non-deterministic design optimization of hierarchically coupled

structural systems is proposed in [12], where parameter uncertainties are considered with modified deterministic multilevel decomposition formulations.

The aforementioned methods use probability distributions for uncertain variables, and those variables are hence treated as random variables. In many engineering applications, however, information or knowledge might not be sufficient to build probability distributions. Intervals are usually suitable to describe those uncertain variables, about which we may have too limited information to form distributions. Examples of using intervals in multidisciplinary systems are given in [13, 14], where epistemic uncertainty (due to lack of knowledge) is modeled by intervals and the evidence theory.

As indicated in [15], random variables and interval variables may present in a system simultaneously. A framework of Unified Reliability Analysis is developed to quantify the effect of random variables and intervals variables [15]. In this work, we extend the strategy in [15] to reliability analysis for multidisciplinary systems, where both random and interval variables are involved. In Section II, the Unified Reliability Analysis (URA) framework for a single disciplinary system and the First Order Reliability Method (FORM) are briefly reviewed. A multidisciplinary system model with random and interval input variables is also provided. In Section III, three algorithms, which support the extension of URA to multidisciplinary systems, are presented. These algorithms are demonstrated by a mathematical example and an aircraft wing design application in Section IV. Conclusions are given in Section V.

## II. Methodology and Modeling

### A. Reliability Analysis

In a single-disciplinary system, where only random variables $\mathbf{X}$ are involved, reliability is defined by

$$R = \Pr\{G = g(\mathbf{X}) \geq 0\} \tag{1}$$

where $\Pr\{\cdot\}$ denotes a probability, $G$ is a response, and $\mathbf{X} = \left(X_1, X_2, \cdots, X_i, \cdots, X_{n_X}\right)$ is a vector of random variables ($n_X$ is the number of random variables), $g$ is a limit-state function [16]. In this paper, we assume that $X_i$ ($i=1,2,...,n_X$) are independent.

If the joint *probability density function* (PDF) of $\mathbf{X}$ is $f_{\mathbf{X}}$, the probability of failure $p_f$ is calculated by

$$p_f = \Pr\{G = g(\mathbf{X}) < 0\} = \int_{g(\mathbf{X})<0} f_{\mathbf{X}}(\mathbf{x})d\mathbf{x} \tag{2}$$

It is obvious that $p_f = 1 - R$.

The limit state function $g(\mathbf{X})$ is usually a nonlinear function of $\mathbf{X}$; the integration boundary, $g(\mathbf{X}) = c$, therefore, is nonlinear. The probability integration in Eq. (2) is also multidimensional. There is rarely a close-form solution to Eq. (2) . Even a numerical integration method is computationally expensive or even impossible when the dimension is high. To this end, the efficient First Order Reliability Method (FORM) is widely used to obtain an approximate solution to Eq. (2).

FORM uses the following three steps.

Step 1: Transform random variables $\mathbf{X}$ into standard normal random variables $\mathbf{U}$. The $i$th random variable $X_i$ is transformed by

$$u_i = \Phi^{-1}\left[F_{X_i}(x_i)\right] \tag{3}$$

where $F_{X_i}$ is the cumulative distribution function (CDF) of $X_i$, and $\Phi^{-1}$ is the inverse CDF of a standard normal distribution.

Step 2: Search the Most Probable Point (MPP). The MPP $\mathbf{u}^*$ is found by

$$\begin{cases} \min_{\mathbf{u}} \|\mathbf{u}\| \\ s.t. \quad g(\mathbf{u}) = 0 \end{cases} \tag{4}$$

in which $\|\cdot\|$ stands for the norm (length) of a vector. Geometrically, the MPP is the shortest distance point from the limit state $g(\mathbf{U}) = 0$ to the origin in U-space. The minimum distance $\beta = \|\mathbf{u}^*\|$ is called a *reliability index*.

Step 3: Compute the probability of failure. $p_f$ is obtained by

$$p_f = \Phi(-\beta) \tag{5}$$

where $\Phi$ is the CDF of a standard normal distribution.

The most computation-intensive work of FORM is the MPP search. The following recursive algorithm [17] is commonly used for the MPP search,

$$\begin{cases} \beta^{(k)} = \beta^{(k-1)} + \dfrac{g(\mathbf{u}^{(k-1)})}{\left\|\nabla g(\mathbf{u}^{(k-1)})\right\|} \\ \mathbf{u}^{(k)} = -\beta^{(k)} \dfrac{\nabla g(\mathbf{u}^{(k-1)})}{\left\|\nabla g(\mathbf{u}^{(k-1)})\right\|} \end{cases} \tag{6}$$

where $\nabla g(\mathbf{u}^{(k)})$ is the gradient of $g$ at $\mathbf{u}^{(k)}$, $\left\|\nabla g(\mathbf{u}^{(k)})\right\|$ is its magnitude, and $k$ is the iteration counter.

## B. Unified Reliability Analysis Framework

The purpose of this work is to establish a Unified Reliability Analysis (URA) framework that can handle both random and interval variables in multidisciplinary systems. For this purpose, we employ the URA framework that has been developed for single-disciplinary systems [15]. The framework is illustrated in Fig.1. The input to the framework is random variables $\mathbf{X}$ characterized by probability distributions and interval variables $\mathbf{W}$ represented by their bounds $[\mathbf{w}^L, \mathbf{w}^U]$. It is obvious that the uncertain output (response) $G = g(\mathbf{X}, \mathbf{W})$ is also characterized by two bounds of its probability distributions [15]. Thus the reliability of the system will also be bounded within its maximum and minimum values.



$\mathbf{X}$: joint PDF

$w_1^L \quad w_1 \quad w_1^U$    $\mathbf{W}$: intervals

$w_2^L \quad w_2 \quad w_2^U$

$G = g(\mathbf{X}, \mathbf{W})$

$G$: CDF bounds, $p_f$ bounds or reliability bounds
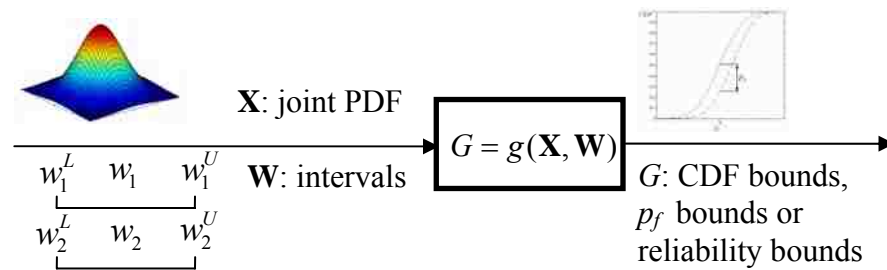
**Fig. 1  Unified reliability analysis framework.**

Reliability analysis calls the limit-state function $G = g(\mathbf{X}, \mathbf{W})$ a number of times. So does multidisciplinary analysis (MDA), which is responsible for solving the linking variables between subsystems. Various ways of integrating reliability analysis and MDA form various computational algorithms. In Section III, we present three computational algorithms that support the URA framework.

## C. FORM-Based URA

Let $\Delta_{\mathbf{w}}$ denote the set of intervals $\mathbf{W}$ and $g(\mathbf{X},\mathbf{W}) < 0$ denote a failure event. The lower and upper bounds of the probability of failure, $p_f^L$ and $p_f^U$, can then be calculated by

$$p_f^L = \Pr\left\{G_{\max} = \max g(\mathbf{X},\mathbf{W}) < 0 \,\middle|\, \mathbf{W} \in \Delta_{\mathbf{w}}\right\} \tag{7}$$

and

$$p_f^U = \Pr\left\{G_{\min} = \min g(\mathbf{X},\mathbf{W}) < 0 \,\middle|\, \mathbf{W} \in \Delta_{\mathbf{w}}\right\} \tag{8}$$

respectively [15]. $G_{\max}$ and $G_{\min}$ are respectively the global maximum and minimum values of $G$ over $\Delta_{\mathbf{w}}$.

According to Eqs. (7) and (8), the procedure to calculate $p_f^L$ and $p_f^U$ consists of two loops: one is interval analysis (IA) to search $G_{\min}$ and $G_{\max}$, and the other is probabilistic analysis (PA) to calculate the probabilities $\Pr\left\{G_{\min} < 0 \,\middle|\, \mathbf{W} \in \Delta_{\mathbf{w}}\right\}$ and $\Pr\left\{G_{\max} < 0 \,\middle|\, \mathbf{W} \in \Delta_{\mathbf{w}}\right\}$. If FORM is used for PA, the Most Probable Point (MPP) must be identified by solving the following model

$$\begin{cases} \min_{\mathbf{u}} \; \|\mathbf{u}\| \\ s.t. \quad g(\mathbf{u},\mathbf{w}) = 0 \end{cases} \tag{9}$$

where $\mathbf{w}$ is treated as a constant vector. For IA, an optimization problem can be formulated for $G_{\max}$:

$$\begin{cases} \max_{\mathbf{w}} \; g(\mathbf{u},\mathbf{w}) \\ s.t. \quad \mathbf{w} \in \Delta_{\mathbf{w}} \end{cases} \tag{10}$$

where $\mathbf{u}$ is treated as a constant vector.

For $G_{\min}$, Eq. (10) becomes a minimization problem.

To solve **u** in the PA problem in Eq. (9), **w** should be given; and to solve **w** in the IA problem in Eq. (10), **u** should be given. This indicates that both PA and IA are fully coupled. To reduce the computational cost, a FORM-based URA (FORM-URA) framework is proposed [15]. Under this framework, PA and IA are decoupled and are performed sequentially. This FORM-URA framework for the calculation of $p_f^L$ is illustrated in Fig.2. IA is performed after PA. After PA, the KKT conditions of IA are checked at the solution of PA. If the KKT conditions are satisfied, IA will be skipped. Skipping the IA loop will save the computational time dramatically.

The efficiency and robustness of an MPP search algorithm are very important for the FORM-URA method. The most commonly used MPP search algorithm HLRF [18, 19] may not converge for a nonlinear function. The improved version of HLRF algorithm [20], denoted by iHLRF, can be used. iHLRF is computationally efficient and guarantees to converge to a local MPP. We also use optimization for IA. Both FORM and optimization are capable of handling black-box performance functions, and therefore so is FORM-URA.
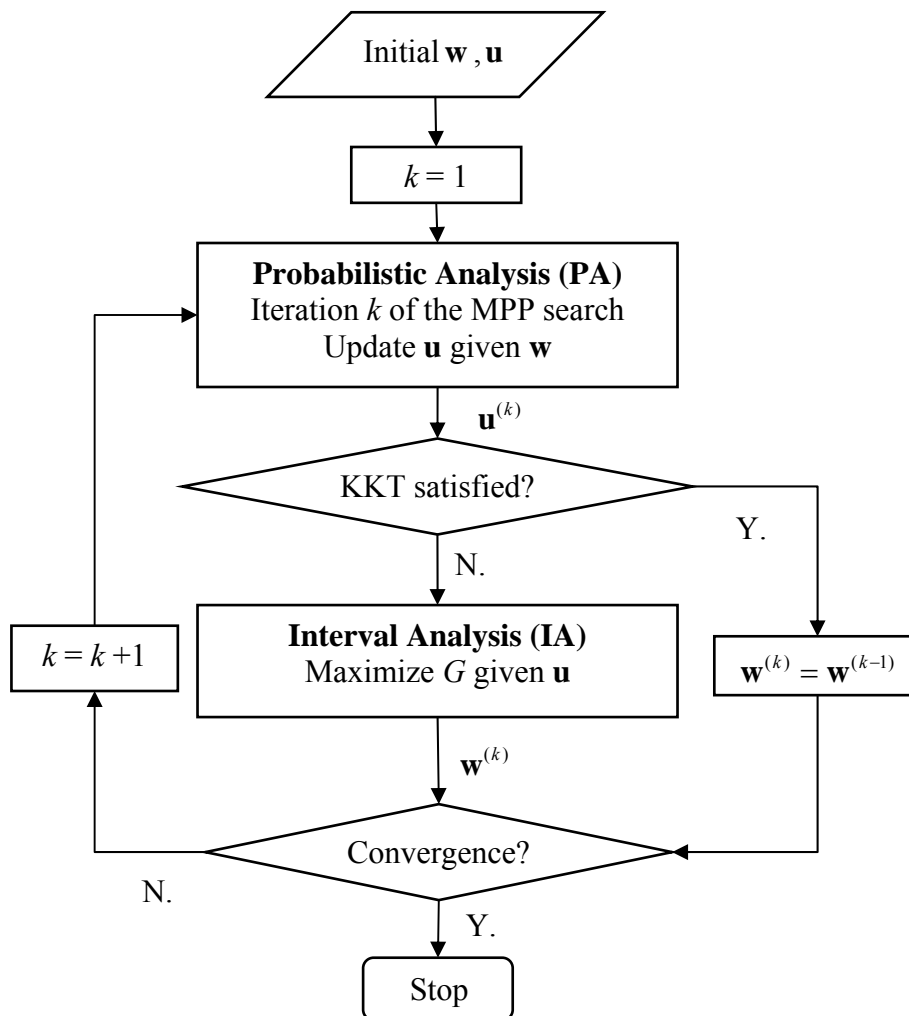
**Fig. 2 Flowchart of the FORM-URA method.**

## D. Multidisciplinary Analysis (MDA) with Random and Interval Variables

To integrate URA with MDA, we need to look at the relationship among random variables, interval variables, and coupling variables in a multidisciplinary system. A three-discipline system in Fig.3 illustrates such relationship. The notations are given below.

$\mathbf{X}_s$: sharing random input variables;

$\mathbf{X}_i$: local random input variables of discipline $i$;

$\mathbf{W}_s$: sharing interval input variables;

$\mathbf{W}_i$: local interval input variables of discipline $i$;

$\mathbf{Z}_i$: outputs of discipline $i$;

$\mathbf{Y}_{ij}$: coupling (linking) variables from discipline $i$ to discipline $j$.

$\mathbf{Z}_i$ and $\mathbf{Y}_{ij}$ are functions of random, interval and other coupling variables from each discipline. Multidisciplinary analysis (MDA) is responsible for solving output $\mathbf{Z}_i$ given all the input variables. Since $\mathbf{Z}_i$ depends on coupling variables, MDA must solve coupling variables $\mathbf{Y}_{ij}$ first. A complete set of coupling variables from the $i$th discipline is formulated as

$$\mathbf{Y}_{i\bullet} = (\mathbf{Y}_{ij}, j = 1, 2, \cdots, n; j \neq i) = \mathbf{Y}_{i\bullet}(\mathbf{X}_s, \mathbf{X}_i, \mathbf{W}_s, \mathbf{W}_i, \mathbf{Y}_{\bullet i}) \tag{11}$$

where $n$ is the number of disciplines, and $\mathbf{Y}_{i\bullet}$ represents dependent variables as on the left-hand side of Eq. (11) and also the functional relationships between dependent variables and independent variables. $\mathbf{Y}_{\bullet i}$ is the vector of coupling variables, which are the inputs to discipline $i$ and the outputs from other disciplines. $\mathbf{Y}_{\bullet i} = (\mathbf{Y}_{ji}, j = 1, 2, \cdots, n, j \neq i)$.
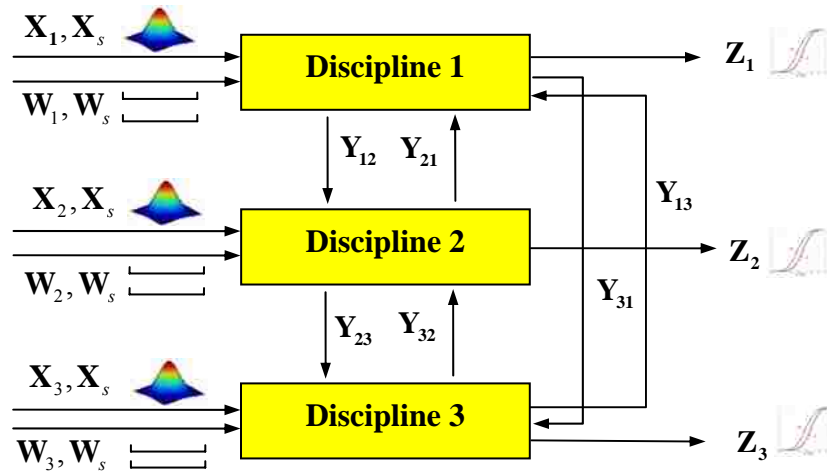
**Fig. 3 Multidisciplinary systems with random and interval variables.**

The system of simultaneous equations in Eq. (11) determines the system consistency over the interfaces among coupled disciplines. Expanding Eq. (11) over all disciplines, we obtain

$$
\begin{cases}
\mathbf{Y}_{12} = \mathbf{Y}_{12}(\mathbf{X}_1, \mathbf{X}_s, \mathbf{W}_1, \mathbf{W}_s, \mathbf{Y}_{\bullet 1}) \\
\mathbf{Y}_{13} = \mathbf{Y}_{13}(\mathbf{X}_1, \mathbf{X}_s, \mathbf{W}_1, \mathbf{W}_s, \mathbf{Y}_{\bullet 1}) \\
\mathbf{Y}_{21} = \mathbf{Y}_{21}(\mathbf{X}_2, \mathbf{X}_s, \mathbf{W}_2, \mathbf{W}_s, \mathbf{Y}_{\bullet 2}) \\
\mathbf{Y}_{23} = \mathbf{Y}_{23}(\mathbf{X}_2, \mathbf{X}_s, \mathbf{W}_2, \mathbf{W}_s, \mathbf{Y}_{\bullet 2}) \\
\mathbf{Y}_{31} = \mathbf{Y}_{31}(\mathbf{X}_3, \mathbf{X}_s, \mathbf{W}_3, \mathbf{W}_s, \mathbf{Y}_{\bullet 3}) \\
\mathbf{Y}_{32} = \mathbf{Y}_{32}(\mathbf{X}_3, \mathbf{X}_s, \mathbf{W}_3, \mathbf{W}_s, \mathbf{Y}_{\bullet 3})
\end{cases}
\tag{12}
$$

Solving the above system of equations is the task of multidisciplinary analysis (MDA).

Suppose $G_i$ is part of the outputs $\mathbf{Z}_i$ from discipline $i$ and the corresponding function is $g_i$. The function is given by

$$
G_i = g_i(\mathbf{X}_s, \mathbf{X}_i, \mathbf{W}_s, \mathbf{W}_i, \mathbf{Y}_{\bullet i})
\tag{13}
$$

If the failure event is defined as $G_i < 0$, then the task of reliability analysis is to find the probability $\Pr\{G_i < 0\}$. As describe in Sec.II.C, we need to quantify the lower and upper bounds: $\Pr\{G_i^{max} < 0\}$ and $\Pr\{G_i^{min} < 0\}$. It is apparent that the computational cost of reliability analysis (PA, IA, and MDO) for solving this problem will be very high. Hence, efficient computational tools are essential and desired. Next, we propose three algorithms based on different strategies.

## III. Algorithms

The key to this work is to extend the existing unified reliability analysis framework (URA) [15] to MDA. For this purpose, we propose three algorithms to integrate URA with MDA. In all the three algorithms, PA and IA are decoupled and are performed sequentially. PA is performed first while the interval variables are fixed, and then IA is performed while the random variables are fixed. The process of one PA and one IA is referred to as a *cycle*. After the first cycle, PA and IA are performed again in the second cycle. This process continues cycle by cycle till convergence.

The three algorithms differ from each other in the ways of how they call MDA. The first algorithm is the Sequential Double Loops (SDL) algorithm where PA and IA call MDA independently, and each of them therefore forms a double loop. The second one is the Sequential Single Loops (SSL) algorithm, which transforms the system consistency requirement in MDA into equality constraints in PA and IA and therefore eliminates the MDA loop. Each of PA and IA then forms a single loop. The last one is the Sequential Single-Single Loops (SSSL) algorithm. In this algorithm, IA is the same as in SSL and involves a single loop. In the PA loop, the MPP search and MDA are performed

sequentially. The three algorithms are outlined in Fig.4. The details are given in the following subsections.

**SDL – Sequential Double Loops algorithm**



**SSL – Sequential Single Loops algorithm**



**SSSL – Sequential Single-Single Loops algorithm**
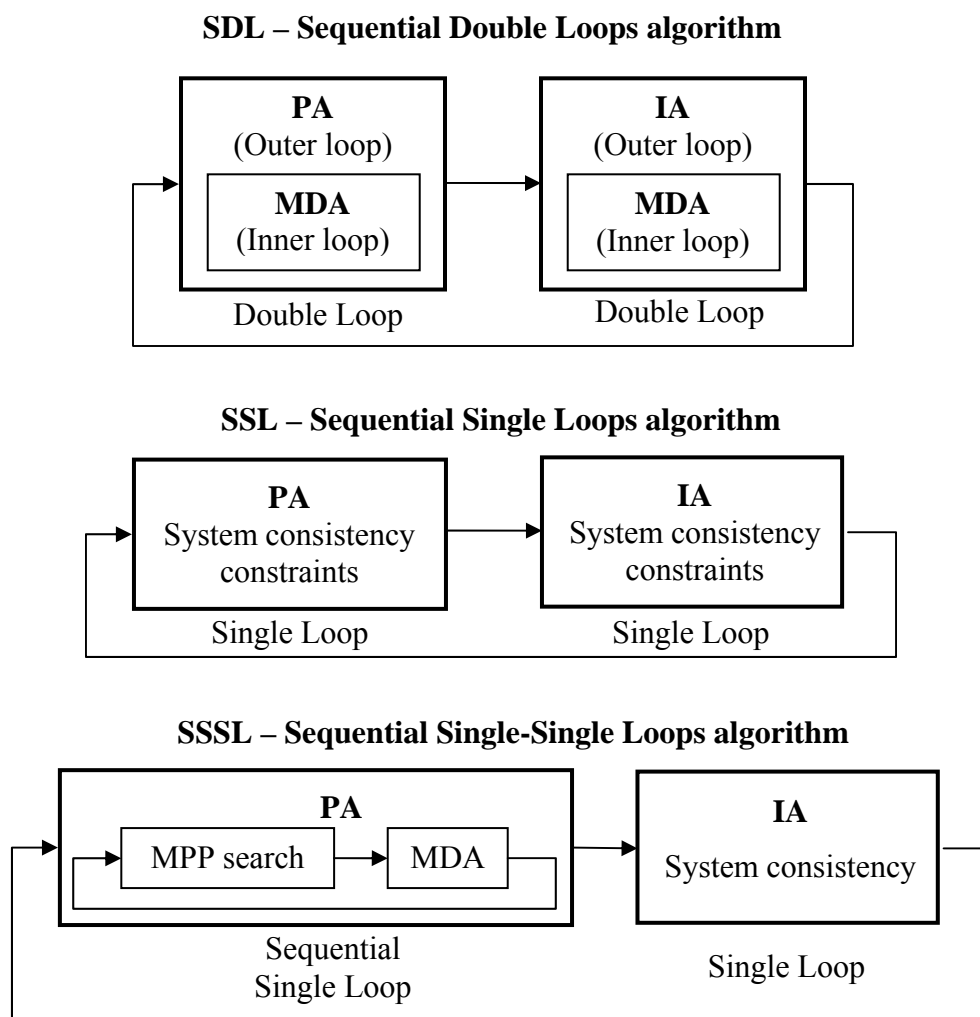


**Fig. 4  Outline of proposed algorithms.**

## A. Sequential Double Loops (SDL) Algorithm

In the SDL algorithm, the double-loop strategy is adopted to integrate the URA framework with MDA. SDL consists of the PA loop and IA loop, and each of them

involves a double-loop procedure with the MDA inner loop. At every iteration, both PA

and IA call MDA independently, and these two double loops are performed sequentially.

In both the PA and IA double loops, MDA is an inner loop for maintaining the system

consistency. In this work, FORM is used for PA. The flowchart of this algorithm for

searching the lower bound of probability of failure is given in Fig.5.
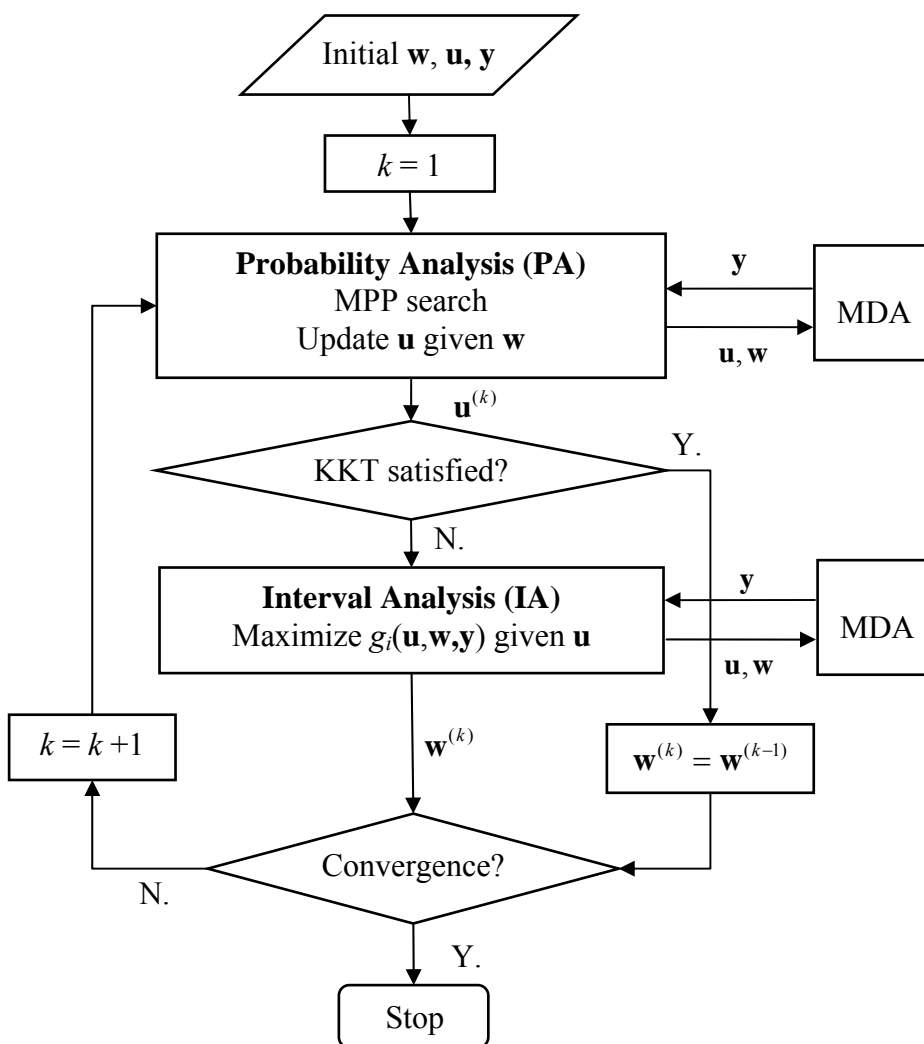


**Fig. 5 SDL algorithm for the lower bound of $p_f$**

Specifically, in PA the MPP search is the outer loop, which is modeled as an optimization problem and takes only random variables as design variables. The interval and coupling variables are treated as constant. Their values are from previous cycle. Suppose the current cycle of the overall reliability analysis is cycle $k$. The optimization problem is expressed by

$$\begin{cases} \min_{\mathbf{u}} \ \|\mathbf{u}\| \\ s.t. \ \ g_i(\mathbf{u}, \mathbf{w}^{(k-1)}, \mathbf{y}_{\bullet i}) = 0 \\ \quad \ \ \mathbf{y}_{\bullet i} \text{ is solved by MDA} \end{cases} \tag{14}$$

In the above model, $g_i$ is a limit-state function in the $i$th subsystem. Design variables $\mathbf{u}$ consist of not only the random input variables to the $i$th discipline, but also all the random input variables to other disciplines, namely, $\mathbf{u} = (\mathbf{u}_s, \mathbf{u}_1, \cdots, \mathbf{u}_n)$. All the interval variables $\mathbf{w}^{(k-1)} = \left( \mathbf{w}_s^{k-1}, \mathbf{w}_1^{k-1}, \mathbf{w}_2^{k-1}, \cdots, \mathbf{w}_n^{k-1} \right)$ are fixed, and they are from the IA in the last cycle. The MDA inner loop is used to solve coupling variables $\mathbf{y}_{\bullet i}$. In this work, we use FORM for PA, and the algorithm in Eq. (6) is used. With the interval and coupling variables, the algorithm becomes

$$\begin{cases} \beta^{(j)} = \beta^{(j-1)} + \dfrac{g_i(\mathbf{u}^{(j-1)}, \mathbf{w}^{(k-1)}, \mathbf{y}_{\bullet i})}{\left\| \nabla g_i(\mathbf{u}^{(j-1)}, \mathbf{w}^{(k-1)}, \mathbf{y}_{\bullet i}) \right\|} \\ \mathbf{u}^{(j)} = -\beta^{(j)} \dfrac{\nabla g_i(\mathbf{u}^{(j-1)}, \mathbf{w}^{(k-1)}, \mathbf{y}_{\bullet i})}{\left\| \nabla g_i(\mathbf{u}^{(j-1)}, \mathbf{w}^{(k-1)}, \mathbf{y}_{\bullet i}) \right\|} \end{cases} \tag{15}$$

where $j$ is the iteration counter of the PA loop, interval variables $\mathbf{w}^{(k-1)}$ are kept constant and are from the previous cycle of the overall reliability analysis. The coupling variables $\mathbf{y}_{\bullet i}$ are obtained from the following inner MDA loop.

$$\mathbf{y}_{q\bullet} = (\mathbf{y}_{qm}, q = 1, 2, \cdots, n; m = 1, 2, \cdots, n; m \neq q) = \mathbf{Y}_{q\bullet}(\mathbf{u}_s^{(j)}, \mathbf{u}_q^{(j)}, \mathbf{w}_s^{(k-1)}, \mathbf{w}_q^{(k-1)}, \mathbf{y}_{\bullet q}) \tag{16}$$

Then in IA, the minimization or maximization problem is an outer loop, which deals with only interval variables given random and coupling variables from PA. The MDA inner loop is also used to solve coupling variables $\mathbf{y}_{\cdot i}$. For the lower bound of $p_f$, IA is a maximization problem with the following formulation

$$\begin{cases} \max_{\mathbf{w}} \ g_i(\mathbf{u}^{(k)}, \mathbf{w}, \mathbf{y}_{\cdot i}) \\ s.t. \quad \mathbf{w} \in \Delta_{\mathbf{w}} \\ \qquad \mathbf{y}_{\cdot i} \text{ is solved by MDA} \end{cases} \tag{17}$$

where design variables are $\mathbf{w} = (\mathbf{w}_1, \mathbf{w}_2, \cdots, \mathbf{w}_n, \mathbf{w}_s)$. Random input variables $\mathbf{u}^{(k)}$ are obtained from the MPP search and are kept constant herein. Coupling variables $\mathbf{y}_{\cdot i}$ are solved by the following inner MDA loop.

$$\mathbf{y}_{q\cdot} = (\mathbf{y}_{qm}, q = 1, 2, \cdots, n; m = 1, 2, \cdots, n; m \neq q) = \mathbf{Y}_{q\cdot}(\mathbf{u}_s^{(k)}, \mathbf{u}_q^{(k)}, \mathbf{w}_s, \mathbf{w}_q, \mathbf{y}_{\cdot q}) \tag{18}$$

This algorithm integrates both PA and IA with MDA in a straightforward manner. It is therefore more robust than the other two algorithms that are presented next. Since the algorithm involves the direct combination of PA and MDA and the direct combination of IA and MDA, it requires calling MDA many times and may be computationally expensive. For instance, at the $j$th iteration of the MPP search in Eq. (14), MDA is performed whenever MPP is updated. When $\mathbf{u}^{(j)}$ is obtained, MDA is called to get $\mathbf{y}_{\cdot i}$ in order to calculate $g_i(\mathbf{u}^{(j-1)}, \mathbf{w}^{(k-1)}, \mathbf{y}_{\cdot i})$. ($\mathbf{w}^{(k-1)}$ is from IA of the previous cycle and is kept constant in the MPP search.) Besides, as shown in Eq. (15), MDA is also needed when the finite difference method is used to calculate the partial derivatives of $g_i$. The equation of the derivatives of $g_i$ with respect to a particular random variable $u_q$ ($q$-th element of $\mathbf{u}$) is given by

$$\frac{\partial g_i}{\partial u_q} = \frac{g_i(\mathbf{u}',\mathbf{w}^{(k-1)},\mathbf{y}'_{\cdot i}) - g_i(\mathbf{u},\mathbf{w}^{(k-1)},\mathbf{y}_{\cdot i})}{\Delta} \tag{19}$$

where $\mathbf{u}' = \left(u_1, u_2, \cdots, u_q + \Delta, \cdots, u_{nu}\right)$, $nu$ is length of $\mathbf{u}$, and $\Delta$ is a step size. $\mathbf{y}'_{\cdot i}$ is the new values of coupling variables associated with the new random variables $\mathbf{u}'$. MDA must be called again to obtain $\mathbf{y}'_{\cdot i}$.

The SDL algorithm suits the systems where the disciplinary analyses and MDA are computationally cheaper. In this work, PA is conducted by FORM. The general PA, however, is open to more methods, such as the Second Order Reliability Method (SORM) and the saddlepoint approximation method [21]. A nonlinear optimization can also be used for PA. For IA, a nonlinear optimization method or the interval arithmetic can be used. In this paper, we use the Sequential Quadratic Programming, which is one of the most popular nonlinear optimization methods.

**B. Sequential Single Loops (SSL) Algorithm**

As described above, when MDA is expensive, the SDL algorithm may not be efficient. To alleviate the computational demand from MDA, we use a single-loop strategy. The algorithm is then termed as the Sequential Single Loops (SSL) algorithm. As shown in Fig.5, the algorithm reformulates the optimization problems of both PA and IA by including the interdisciplinary equilibrium (consistency) as part of constraints. The constraints of the equilibrium are formulated by maintaining the simultaneous equations in Eq. (11) of coupling variables as the following equality constraints.

$$\mathbf{h}(\mathbf{u},\mathbf{w},\mathbf{y}) = \mathbf{Y}_{i\cdot} - \mathbf{Y}_{i\cdot}(\mathbf{u},\mathbf{w},\mathbf{y}_{\cdot i}) = \mathbf{0}, \; i = 1, 2, \cdots, n \tag{20}$$

where $\mathbf{y}$ contains all the coupling variables.

The model for the PA loop in the $k$th cycle of the overall reliability analysis is then reformulated as

$$\begin{cases} \min_{\mathbf{u},\mathbf{y}} \ \|\mathbf{u}\| \\ s.t. \quad g_i(\mathbf{u}, \mathbf{w}^{(k-1)}, \mathbf{y}_{\cdot i}) = 0 \\ \qquad \mathbf{h}(\mathbf{u}, \mathbf{w}^{(k-1)}, \mathbf{y}) = 0 \end{cases} \tag{21}$$

where $g_i$ is a limit-state function of subsystem $i$, interval variables $\mathbf{w}^{(k-1)}$ are given from the IA loop in the last cycle, and random variables $\mathbf{u}$ and coupling variables $\mathbf{y}$ are regarded as design variables and are solved by optimization. The solution is $\mathbf{u}^{(k)}$.

The optimization for the minimum probability of failure in the IA loop (see Eq.(7)) is modeled by

$$\begin{cases} \max_{\mathbf{w},\mathbf{y}} \ g_i(\mathbf{u}^{(k)}, \mathbf{w}, \mathbf{y}_{\cdot i}) \\ s.t. \quad \mathbf{w} \in \Delta_{\mathbf{w}} \\ \qquad \mathbf{h}(\mathbf{u}^{(k)}, \mathbf{w}, \mathbf{y}) = 0 \end{cases} \tag{22}$$

where random variables $\mathbf{u}^{(k)}$ is obtained from the PA loop and are constant. Interval variables $\mathbf{w}$ and coupling variables $\mathbf{y}$ are taken as design variables.

As shown in Fig. 6, there is no need to conduct separate MDA, but coupling variables become additional design variables that are solved in PA and IA. For example, in IA the interval and coupling variables become design variables and are solved given the random variables from the PA loop. The procedure is depicted in Fig.6. These two single loops are performed sequentially.
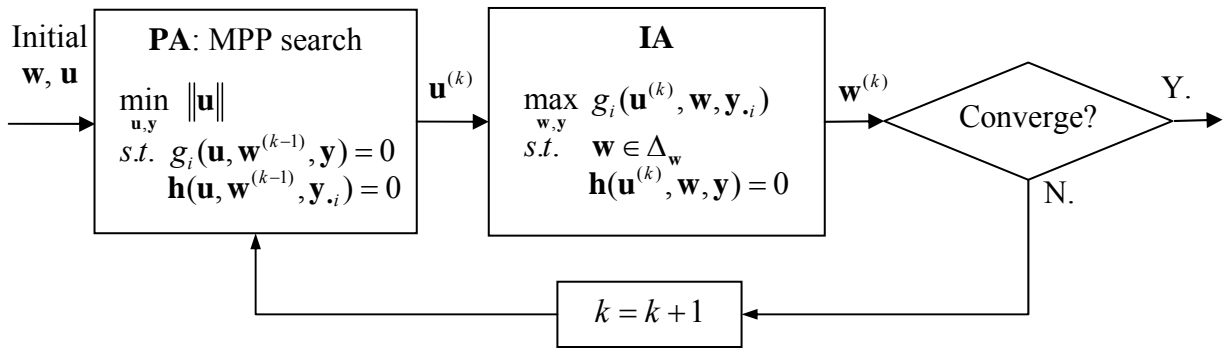
Fig. 6 SSL algorithm for the lower bound of $p_f$.

Compared to the first algorithm, the SSL algorithm does not call MDA directly. The task of MDA is implicitly embedded in the PA and IA loops. The algorithm is suitable for the situation when it is easy to perform disciplinary analyses concurrently. It is efficient for the systems that contain fewer coupling variables. However, when the number of coupling variables is large, this algorithm will contain a large number of design variables since the coupling variables are treated as design variables. This might diminish the efficiency of the SSL algorithm. The other disadvantage of the algorithm is the inclusion of equality constraints for the system consistency. Equality constraints make optimization hard to converge. Since additional constraints are added to the MPP search in PA, the existing MPP search algorithms are no longer applicable.

**C. Sequential Single-Single Loops (SSSL) Algorithm**

In the first algorithm, the SDL algorithm, an efficient MPP search method can be used for PA, while in the second algorithm, the SSL algorithm, only nonlinear optimization can be used for PA. Nonlinear optimization is usually not as efficient as specialized MPP search algorithms. To take advantage of the MPP search algorithms, we combine both of

the above two algorithms. The combination comes from the PA loop of the SDL algorithm and the IA loop of the SSL algorithm. An MPP search algorithm can then be used for PA. To save computational resources further, for PA, we change the double-loop procedure to a sequential single loop procedure where the MPP search and MDA are performed sequentially. The same double loop procedure for IA is used as in the SSL algorithm. The algorithm is illustrated in Fig. 7.
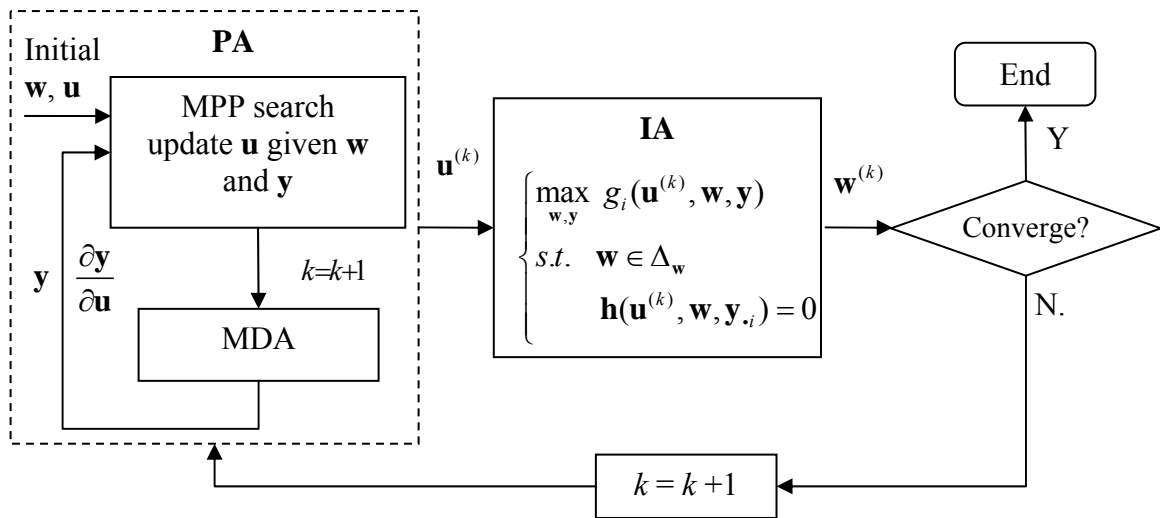


**Fig. 7    SSSL algorithm for the lower bound of $p_f$.**

In PA, only random variables are solved in the MPP search, and then the MDA loop is conducted to update the coupling variables. The MPP search and MDA are performed in a sequential manner till convergence is reached. In IA, the system consistency is part of constraints. Interval and coupling variables are solved simultaneously given the random variables from the PA loop. IA includes system consistency constraints and involves a

single-loop procedure. The overall reliability analysis is performed sequentially with the

sequential single-loop PA and the single-loop IA.

The MPP search algorithm in the SDL algorithm in Eq. (15) is modified as

$$
\begin{cases}
\beta^{(j)} = \beta^{(j-1)} + \dfrac{g_i(\mathbf{u}^{(j-1)}, \mathbf{w}^{(k-1)}, \mathbf{y}_{\cdot i}^{(q-1)})}{\left\| \nabla g_i(\mathbf{u}^{(j-1)}, \mathbf{w}^{(k-1)}, \mathbf{y}_{\cdot i}^{(q-1)}) \right\|} \\[4mm]
\mathbf{u}^{(j)} = -\beta^{(j)} \dfrac{\nabla g_i(\mathbf{u}^{(j-1)}, \mathbf{w}^{(k-1)}, \mathbf{y}_{\cdot i}^{(q-1)})}{\left\| \nabla g_i(\mathbf{u}^{(j-1)}, \mathbf{w}^{(k-1)}, \mathbf{y}_{\cdot i}^{(q-1)}) \right\|}
\end{cases}
\tag{23}
$$

The above equation is for the $j$th iteration of the MPP search in the $q$th iteration of the

PA loop and the $k$th cycle of the overall reliability analysis. The interval variables $\mathbf{w}^{(k-1)}$

are from the previous cycle (cycle $k$-1) of the overall reliability analysis and are kept

constant. The coupling variables $\mathbf{y}_{\cdot i}^{(q-1)}$ are from the last iteration (iteration $q$-1) of the PA

loop and are also kept constant. The solution is the MPP $\mathbf{u}^{(q)}$.

After the MPP loop is completed, MDA is performed. The coupling variables $\mathbf{y}_{\cdot i}^{(q)}$ are

obtained from the following model.

$$
\mathbf{y}_{p\cdot} = (\mathbf{y}_{pm}, p = 1,2,\cdots,n; m = 1,2,\cdots,n; m \neq p) = \mathbf{Y}_{p\cdot}(\mathbf{u}_s^{(q)}, \mathbf{u}_p^{(q)}, \mathbf{w}_s^{k-1}, \mathbf{w}_p^{k-1}, \mathbf{y}_{\cdot p}) \tag{24}
$$

If analytical derivatives are not available for the gradient $\nabla g_i(\mathbf{u}^{(j-1)}, \mathbf{w}^{(k-1)}, \mathbf{y}_{\cdot i}^{(q-1)})$ in

Eq. (23), the finite difference method in Eq. (19) can be used to estimate gradients

$\partial g_i / \partial u_p$, where $u_p$ is the $p$th element of $\mathbf{u}$. The equation is written below.

$$
\frac{\partial g_i}{\partial u_p} = \frac{g_i(\mathbf{u}^{(j-1)'}, \mathbf{w}^{(k-1)}, \mathbf{y}_{\cdot i}') - g_i(\mathbf{u}^{(j-1)}, \mathbf{w}^{(k-1)}, \mathbf{y}_{\cdot i})}{\Delta} \tag{25}
$$

where $\mathbf{y}_{\cdot i}'$ is the new values of coupling variables associated with the new random

variable $\mathbf{u}' = (u_1, u_2, \cdots, u_p + \Delta, \cdots, u_{nu})$.

It is noted that the coupling variables $\mathbf{y}_{\cdot i}$ are not constant. They are functions of $\mathbf{u}$. $\mathbf{y}_{\cdot i}'$ should therefore be re-calculated. However, $\mathbf{y}_{\cdot i}'$ cannot be obtained from the MPP search since it is decoupled from MDA. A first order Taylor's series expansion is used to estimate $\mathbf{y}_{\cdot i}'$, and the equation is given by

$$\mathbf{y}_{\cdot i}' = \mathbf{y}_{\cdot i} + \frac{\partial \mathbf{y}_{\cdot i}}{\partial u_p} \Delta \qquad (26)$$

where $\dfrac{\partial \mathbf{y}_{\cdot i}}{\partial u_p}$ is obtained from the MDA loop in the previous iteration (iteration $j$-1) of PA

and is kept constant in the MPP search.

This algorithm is suitable for problems where PA is relatively expensive and IA is relatively cheap. One may also choose this method when the number of random variables is large and the number of interval variables is small.

## IV. Examples

For demonstration, two examples are provided. The first one is a mathematical problem with two subsystems. Although the probabilistic constraints are simple and the number of variables is small, it is effective to show the formulations and procedures of the three algorithms. The second example is a wing design problem involving more complicated probabilistic constraints and more coupling variables and random variables. It indicates the potential use of the algorithms to real engineering applications.

### A.  Example 1 – A Mathematical Problem [8]

In this example the system consists of two subsystems. For demonstration, two local interval variables and one sharing interval variable are introduced to the original problem

in [8] where only random variables are involved. The new problem is demonstrated in Fig. 8 and is formulated as follows.

Subsystem 1:

$$G_1 = X_1 - (W_s + X_s + 2W_1 + 2Y_{21})$$

$$Y_{12} = W_s + X_s + W_1 + Y_{21}$$

Subsystem 2:

$$G_2 = (5W_s + 5X_s + 3W_2 - 4Y_{12}) - X_2$$

$$Y_{21} = W_s + X_s + W_2 - Y_{12}$$

where $W_s$ is a shared interval variable; $W_1$ and $W_2$ are local interval variables. $X_s$ is a sharing random variable; and $X_1$ and $X_2$ are local random variables. $X_s \sim N(0, 0.3)$, $X_1 \sim N(5, 0.5)$, and $X_2 \sim N(1, 0.1)$, where $N(\cdot, \cdot)$ stands for a normal distribution, and its first and second variables are mean and standard deviation, respectively. All the interval variables have the same bounds, and $W_s, W_1, W_2 \in [2.245, 2.255]$. The probabilities of failure are defined by $p_f = \Pr\{G_i < 0\}$ $(i = 1, 2)$.
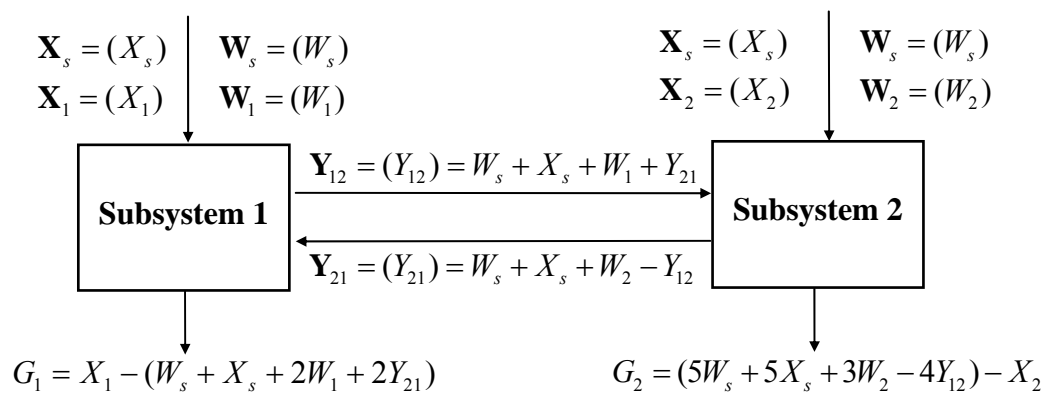


**Fig. 8 Mathematical example.**

To demonstrate the procedure of each algorithm, we provide the equations of the three algorithms for the lower bound of $p_f$ for $G_1$ at the $k$th cycle as follows. Recall that a cycle is a sequential process of PA and IA; in other word, it is one iteration of the overall reliability analysis.

*1. SDL algorithm*

1) PA loop

The MPP search is modeled by

$$\begin{cases} \min_{\mathbf{u}=(u_s,u_1,u_2)} \sqrt{u_s^2 + u_1^2 + u_2^2} \\ s.t. \quad G_1 = (\mu_1 + u_1\sigma_1) - \left[ w_s^{(k-1)} + (\mu_s + u_s\sigma_s) + 2w_1^{(k-1)} + 2y_{21} \right] = 0 \end{cases}$$

where the design variables are $\mathbf{u} = (u_s, u_1, u_2)$, $w_s^{(k-1)}$ and $w_1^{(k-1)}$ are the interval variables from the $(k$-1)th cycle, and $y_{21}$ is the coupling variable solved from MDA, which is solved by

$$\begin{cases} y_{12} = w_s^{(k-1)} + (\mu_s + u_s\sigma_s) + w_1^{(k-1)} + y_{21} \\ y_{21} = w_s^{(k-1)} + (\mu_s + u_s\sigma_s) + w_2^{(k-1)} - y_{12} \end{cases}$$

where interval variable $w_2^{(k-1)}$ is also from the $(k$-1)th cycle.

The above MPP search and MDA are nested and form a single-loop PA. The solution of the PA loop is the MPP $\mathbf{u}^{*,(k)} = \left( u_s^{*,(k)}, u_1^{*,(k)}, u_2^{*,(k)} \right)$. It is noted that in the above equations all the random variables are transformed into standard normal variables.

2) IA loop

The optimization model is given by

$$\begin{cases} \max_{\mathbf{w}=(w_s,w_1,w_2)} G_1 = (\mu_1 + u_1^{*,(k)}\sigma_1) - \left[ w_s^{(k-1)} + (\mu_s + u_s^{*,(k)}\sigma_s) + 2w_1^{(k-1)} + 2y_{21} \right] \\ s.t. \quad w_s, w_1, w_2 \in [2.245, 2.255] \end{cases}$$

where the design variables are $\mathbf{w} = (w_s, w_1, w_2)$, and $y_{21}$ is the coupling variable obtained by the following MDA.

$$\begin{cases} y_{12} = w_s + (\mu_s + u_s^{*,(k)}\sigma_s) + w_1 + y_{21} \\ y_{21} = w_s + (\mu_s + u_s^{*,(k)}\sigma_s) + w_2 - y_{12} \end{cases}$$

The above MDA and the optimization problem are nested; and they form a single-loop IA. The solution of the IA loop is the interval variables $\mathbf{w}^{(k)} = (w_s^{(k)}, w_1^{(k)}, w_2^{(k)})$.

*2. SSL algorithm*

1) PA loop

The MPP search and MDA are formulated together as a single-loop procedure. The formulation is given below.

$$\begin{cases} \min_{\mathbf{u},\mathbf{y}} \sqrt{u_s^2 + u_1^2 + u_2^2} \\ s.t. \quad G_1 = (\mu_1 + u_1\sigma_1) - \left[ w_s^{(k-1)} + (\mu_s + u_s\sigma_s) + 2w_1^{(k-1)} + 2y_{21} \right] = 0 \\ \quad h_1 = y_{12} - \left[ w_s^{(k-1)} + (\mu_s + u_s\sigma_s) + w_1^{(k-1)} + y_{21} \right] = 0 \\ \quad h_2 = y_{21} - \left[ w_s^{(k-1)} + (\mu_s + u_s\sigma_s) + w_2^{(k-1)} - y_{12} \right] = 0 \end{cases}$$

where the design variables are $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{y} = (y_{12}, y_{21})$.

2) IA loop

$$\begin{cases} \max_{\mathbf{w},\mathbf{y}} G_1 = (\mu_1 + u_1^{*,(k)}\sigma_1) - \left[ w_s^{(k-1)} + (\mu_s + u_s^{*,(k)}\sigma_s) + 2w_1^{(k-1)} + 2y_{21} \right] \\ s.t. \quad h_1 = y_{12} - \left[ w_s^{(k-1)} + (\mu_s + u_s\sigma_s) + w_1^{(k-1)} + y_{21} \right] = 0 \\ \quad h_2 = y_{21} - \left[ w_s^{(k-1)} + (\mu_s + u_s\sigma_s) + w_2^{(k-1)} - y_{12} \right] = 0 \\ \quad w_s, w_1, w_2 \in [2.245, 2.255] \end{cases}$$

where the design variables are $\mathbf{w} = (w_s, w_1, w_2)$ and $\mathbf{y} = (y_{12}, y_{21})$.

*3. SSSL algorithm*

1) PA loop

The MPP search and MDA are conducted sequentially. In the *j*th iteration of PA, the MPP search is formulated as

$$\min_{\mathbf{u}} \sqrt{u_s^2 + u_1^2 + u_2^2}$$

$$s.t. \quad G_1 = (\mu_1 + u_1\sigma_1) - \left[ w_s^{(k-1)} + (\mu_s + u_s\sigma_s) + 2w_1^{(k-1)} + 2y_{21}^{(j-1)} \right] = 0$$

where the design variables are $\mathbf{u} = (u_1, u_2, u_3)$. The interval variables $w_s^{(k-1)}$ and $w_1^{(k-1)}$ are from the (*k*-1)th cycle of the overall reliability analysis. (Recall the current cycle is the *k*th cycle.) The coupling variable $y_{21}^{(j-1)}$ is obtained from the previous MDA in the (*j*-1)th iteration. After the MPP search, MDA is performed to solve the coupling variable $y_{21}^{(j)}$ and is formulated by

$$\begin{cases} y_{12} - \left[ w_s^{(k-1)} + (\mu_s + u_s\sigma_s) + w_1^{(k-1)} + y_{21} \right] = 0 \\ y_{21} - \left[ w_s^{(k-1)} + (\mu_s + u_s\sigma_s) + w_2^{(k-1)} - y_{12} \right] = 0 \end{cases}$$

2) IA loop

The IA loop is the same as in the SSL algorithm.

The comparison of the reliability analysis results for probabilistic constraint $G_1$ and $G_2$ from the three algorithms are summarized in Table 1. The comparison is done with the same convergence criteria applied to each algorithm. Monte Carlo Simulation (MCS), as a sampling-based verification method, is also conducted. Latin Hypercube sampling is used to get the samples of the interval variables. The result from MCS is also listed in Table 1. The computational cost of all the methods is measured by the number of function evaluations (analyses at the subsystem level).

**Table 1 Bounds of $p_f$**

| Constraint | Bound | SDL | | SSL | | SSSL | | MCS |
|---|---|---|---|---|---|---|---|---|
| $G_1$ | $p_f^{max}$ | $1.463\times10^{-3}$ | | $1.463\times10^{-3}$ | | $1.463\times10^{-3}$ | | $1.412\times10^{-3}$ |
| | Funcall[a] | 1330 | 1210 | 370 | 370 | 406 | 358 | $10^9$ |
| | $p_f^{min}$ | $1.235\times10^{-3}$ | | $1.235\times10^{-3}$ | | $1.235\times10^{-3}$ | | $1.202\times10^{-3}$ |
| | Funcall | 1330 | 1210 | 370 | 370 | 406 | 358 | $10^9$ |
| $G_2$ | $p_f^{max}$ | $9.709\times10^{-4}$ | | $9.709\times10^{-4}$ | | $9.709\times10^{-4}$ | | $9.200\times10^{-4}$ |
| | Funcall | 2210 | 2430 | 310 | 310 | 438 | 502 | $10^8$ |
| | $p_f^{min}$ | $6.288\times10^{-4}$ | | $6.288\times10^{-4}$ | | $6.288\times10^{-4}$ | | $6.401\times10^{-4}$ |
| | Funcall | 2370 | 2606 | 310 | 310 | 438 | 502 | $10^8$ |

a. Funcall: the numbers of function evaluations, which are the numbers of analyses in subsystems 1 and 2, respectively.

It is noted that the results obtained from SDL, SSL and SSSL are identical. The results are also very close to the MCS solutions. All the algorithms therefore converge to an accurate solution. For this simple problem, SSL algorithm outperforms the rest in terms of efficiency since it calls subsystem analyses with the least number of times.

**B.  Example 2 - Aircraft Wing Design**

A wing design problem for a light aircraft [22] involves aerodynamic design and structural design. Aerodynamic design is responsible for selecting the external shape of the wing while structural design determines the structural size. The two disciplines are coupled with each other. A structural model is depicted in Fig. 9 [22], and the coupled subsystems are illustrated in Fig.10.
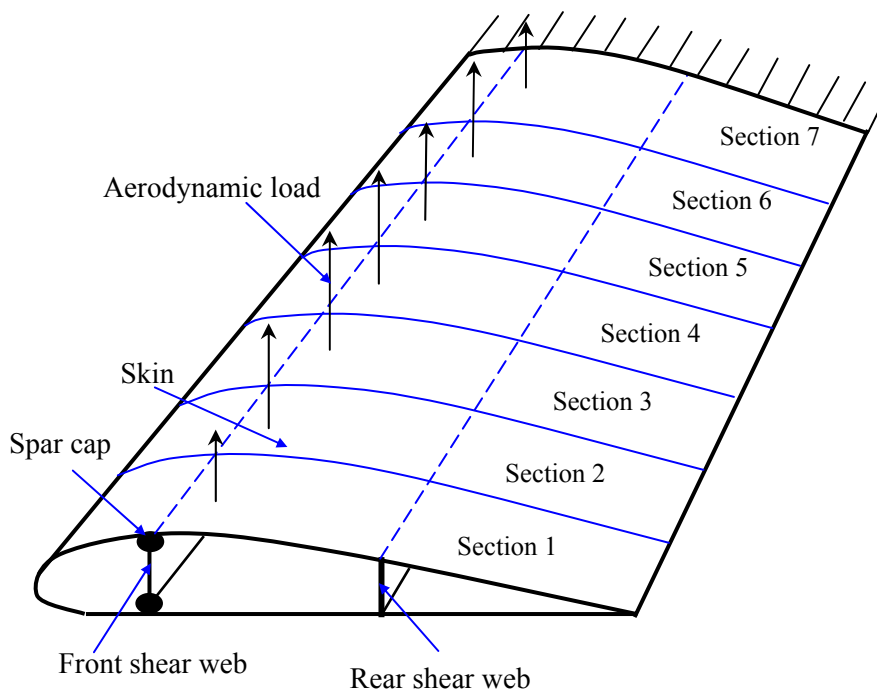
**Fig. 9 The wing structure model.**



$$\mathbf{W}_s = (AR) \qquad \mathbf{X}_s = (V, H)$$
$$\mathbf{W}_1 = (\theta, \alpha) \qquad \mathbf{X}_1 = (W_0)$$

$$\mathbf{W}_s = (AR)$$
$$\mathbf{W}_2 = (w_1, w_2, \cdots, w_{10}) \qquad \mathbf{X}_2 = (J, F, S_1, S_2, S_3)$$

**Subsystem 1 Aerodynamics**

$$\mathbf{Y}_{12} = (a_0, a_1, a_2, a_3)$$

**Subsystem 2 Structure**

$$\mathbf{Y}_{21} = (\delta)$$

$$\mathbf{Z}_1$$

$$\mathbf{Z}_2 = (G_2(1), G_2(2), \cdots, G_2(10))$$
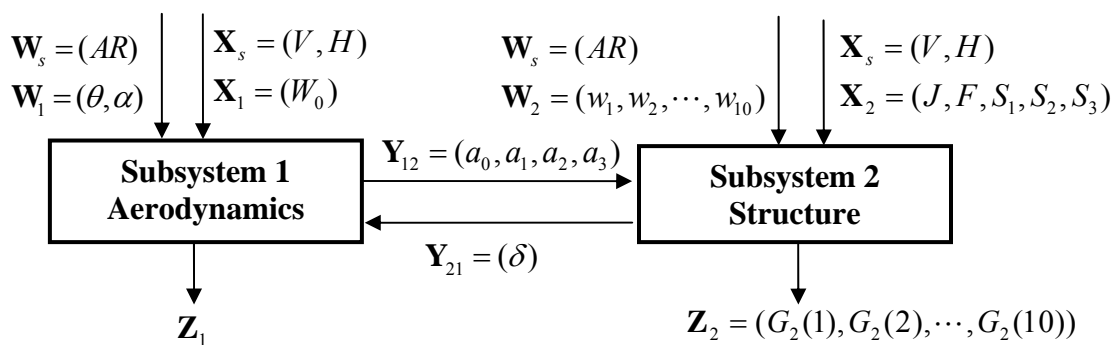
**Fig. 10 Coupled aerodynamic and structural subsystems**

The reliability associated with each of the following constraints in Subsystem 2 is to be evaluated. The probabilities of failure are given by

$$\Pr\{G_2(i) \le 0\} = \Pr\{\sigma_i - S_1 \le 0\} \quad (i = 1, 2, \ldots, 7)$$

$$\Pr\{G_2(8) \le 0\} = \Pr\{\tau_{skin} - S_2 \le 0\}$$

$$\Pr\{G_2(9) \le 0\} = \Pr\{\tau_{wf} - S_3 \le 0\}$$

$$\Pr\{G_2(10) \le 0\} = \Pr\{\tau_{wr} - S_3 \le 0\}$$

where $\sigma_i$ $(i = 1, 2, \ldots, 7)$ are the bending stresses in the spar cap for each section, $\tau_{skin}$ is

the maximum shear stress in the skin, $\tau_{wf}$ is the shear stress in the web, $\tau_{wr}$ is the shear

stress in the rear web, and $S_1$, $S_2$ and $S_3$ are the bending strength of the material of the

spar caps, the shear strength of the skin, and the shear strength of the spar web,

respectively. $S_1$, $S_2$ and $S_3$ are normally distributed, and their distribution parameters are

given in Table 2 along with other random variables.

**Table 2 Distributions of random variables**

| Variables | Mean | Standard deviation | Distribution |
|---|---|---|---|
| Flight altitude $H$ | 3000 m | 300 m | Normal |
| Flight speed $V$ | 200 km/h | 20 km/h | Normal |
| Wing area $S$ | 10 m$^2$ | 0.5 m$^2$ | Normal |
| Take-off weight $W_0$ | 700 kg | 70 kg | Normal |
| Shear modulus $J$ | $2.7 \times 10^{10}$ N/ mm$^2$ | $2.7 \times 10^9$ N/ mm$^2$ | Normal |
| Gust load factor $F$ | 4.0 | 0.4 | Normal |
| Bending strength $S_1$ | 450 N/mm$^2$ | 45 N/mm$^2$ | Normal |
| Shear strength of the skin $S_2$ | 200 N/mm$^2$ | 20N/mm$^2$ | Normal |
| Shear strength of the web $S_3$ | 250 N/mm$^2$ | 25 N/mm$^2$ | Normal |

The aspect ratio ($AR$), twist angle ($\theta$), areas ($w_1$-$w_{10}$) are considered as interval

variables. Their nominal values and widths are provided in Table 3.

**Table 3 Interval and deterministic variables**

| Design variables | Nominal values | Width | Disciplines |
|---|---|---|---|
| Aspect ratio | 5.7823 | 0.40 | Aerodynamics |
| Twist angle | 0.80406 (deg) | 0.2 (deg) | Aerodynamics |
| Angle of attack, $\alpha$ | 5.0877 (deg) | 0.05 (deg) | Aerodynamics |
| Area of spar cap in section 1, $w_1$ | 50.0 (mm$^2$) | 1.0 (mm$^2$) | Structure |
| Area of spar cap in section 2, $w_2$ | 54.81 (mm$^2$) | 1.0 (mm$^2$) | Structure |
| Area of spar cap in section 3, $w_3$ | 122.07 (mm$^2$) | 2.0 (mm$^2$) | Structure |
| Area of spar cap in section 4, $w_4$ | 215.23 (mm$^2$) | 2.0 (mm$^2$) | Structure |
| Area of spar cap in section 5, $w_5$ | 333.13 (mm$^2$) | 3.0 (mm$^2$) | Structure |
| Area of spar cap in section 6, $w_6$ | 472.83 (mm$^2$) | 5.0 (mm$^2$) | Structure |
| Area of spar cap in section 7, $w_7$ | 628.42 (mm$^2$) | 6.0 (mm$^2$) | Structure |
| Thickness of the skin, $w_8$ | 1.0 (mm) | 0.01 (mm) | Structure |
| Thickness of the front web, $w_9$ | 1.0 (mm) | 0.01 (mm) | Structure |
| Thickness of the rear web, $w_{10}$ | 1.0 (mm) | 0.01 (mm) | Structure |

A comparison of the analysis results from the three algorithms for limit-state functions $G_1 \sim G_{10}$ are summarized in Table 4. MCS is also conducted to confirm the accuracy of the results. Due to the high computational cost of evaluating the constraint functions in this example, MPP-based importance sampling [23, 24] is used in MCS. This method generates samples around the MPPs rather than over the whole random space, and therefore the number of simulations is much less than that of the general MCS. The results show that the three algorithms produce the same solutions, which are all close to the result from MCS. In this case, SSL algorithm requires the least disciplinary analyses.

**Table 4 Two bounds of $p_f$ obtained by different algorithms**

| Constraints | | SDL | | SSL | | SSSL | | Monte Carlo |
|---|---|---|---|---|---|---|---|---|
| $G_2(1)$ | $p_f^{max}$ | $\approx 0$ | | $\approx 0$ | | $\approx 0$ | | $\approx 0$ |
| | Funcall | 15759 | 16221 | 1966 | 1966 | 4838 | 5963 | $10^4$ |
| | $p_f^{min}$ | $\approx 0$ | | $\approx 0$ | | $\approx 0$ | | $\approx 0$ |
| | Funcall | 16048 | 16519 | 976 | 976 | 4816 | 5932 | $10^4$ |
| $G_2(2)$ | $p_f^{max}$ | $5.614\times10^{-3}$ | | $5.614\times10^{-3}$ | | $5.614\times10^{-3}$ | | $5.363\times10^{-3}$ |
| | Funcall | 12228 | 12586 | 1088 | 1088 | 3915 | 4509 | $10^4$ |
| | $p_f^{max}$ | $1.800\times10^{-3}$ | | $1.800\times10^{-3}$ | | $1.800\times10^{-3}$ | | $1.812\times10^{-3}$ |
| | Funcall | 10948 | 11269 | 1172 | 1172 | 3898 | 4528 | $10^4$ |
| $G_2(3)$ | $p_f^{max}$ | $5.622\times10^{-3}$ | | $5.622\times10^{-3}$ | | $5.622\times10^{-3}$ | | $5.549\times10^{-3}$ |
| | Funcall | 13447 | 13841 | 1088 | 1088 | 3915 | 4509 | $10^4$ |
| | $p_f^{min}$ | $1.802\times10^{-3}$ | | $1.802\times10^{-3}$ | | $1.802\times10^{-3}$ | | $1.808\times10^{-3}$ |
| | Funcall | 10957 | 11278 | 1172 | 1172 | 3898 | 4528 | $10^4$ |
| $G_2(4)$ | $p_f^{max}$ | $5.635\times10^{-3}$ | | $5.635\times10^{-3}$ | | $5.635\times10^{-3}$ | | $5.646\times10^{-3}$ |
| | Funcall | 11373 | 11706 | 1088 | 1088 | 3915 | 4518 | $10^4$ |
| | $p_f^{min}$ | $1.801\times10^{-3}$ | | $1.801\times10^{-3}$ | | $1.801\times10^{-3}$ | | $1.786\times10^{-3}$ |
| | Funcall | 12478 | 12844 | 1172 | 1172 | 3898 | 4528 | $10^4$ |
| $G_2(5)$ | $p_f^{max}$ | $5.658\times10^{-3}$ | | $5.658\times10^{-3}$ | | $5.658\times10^{-3}$ | | $5.498\times10^{-3}$ |
| | Funcall | 13515 | 13911 | 1088 | 1088 | 3915 | 4518 | $10^4$ |
| | $p_f^{min}$ | $1.800\times10^{-3}$ | | $1.800\times10^{-3}$ | | $1.800\times10^{-3}$ | | $1.741\times10^{-3}$ |
| | Funcall | 11356 | 11689 | 1172 | 1172 | 3898 | 4528 | $10^4$ |
| $G_2(6)$ | $p_f^{max}$ | $5.680\times10^{-3}$ | | $5.680\times10^{-3}$ | | $5.680\times10^{-3}$ | | $5.403\times10^{-3}$ |
| | Funcall | 11373 | 11706 | 1088 | 1088 | 3915 | 4527 | $10^4$ |
| | $p_f^{min}$ | $1.797\times10^{-3}$ | | $1.797\times10^{-3}$ | | $1.797\times10^{-3}$ | | $1.742\times10^{-3}$ |
| | Funcall | 12478 | 12844 | 1172 | 1172 | 3898 | 4537 | $10^4$ |
| $G_2(7)$ | $p_f^{max}$ | $5.451\times10^{-3}$ | | $5.451\times10^{-3}$ | | $5.451\times10^{-3}$ | | $5.703\times10^{-3}$ |
| | Funcall | 10761 | 11076 | 1074 | 1074 | 3915 | 4536 | $10^4$ |
| | $p_f^{min}$ | $1.794\times10^{-3}$ | | $1.794\times10^{-3}$ | | $1.794\times10^{-3}$ | | $1.777\times10^{-3}$ |
| | Funcall | 12070 | 12424 | 1158 | 1158 | 3898 | 4537 | $10^4$ |
| $G_2(8)$ | $p_f^{max}$ | $\approx 0$ | | $\approx 0$ | | $\approx 0$ | | $\approx 0$ |
| | Funcall | 14229 | 14646 | 1591 | 1591 | 4833 | 5787 | $10^4$ |
| | $p_f^{min}$ | $\approx 0$ | | $\approx 0$ | | $\approx 0$ | | $\approx 0$ |
| | Funcall | 13804 | 14209 | 920 | 920 | 4816 | 5734 | $10^4$ |

| Constraints | | SDL | | SSL | | SSSL | | Monte Carlo |
|---|---|---|---|---|---|---|---|---|
| $G_2(9)$ | $p_f^{max}$ | $\approx 0$ | | $\approx 0$ | | $\approx 0$ | | $\approx 0$ |
| | Funcall | 16303 | 16781 | 1256 | 1256 | 5445 | 6597 | $10^4$ |
| | $p_f^{min}$ | $\approx 0$ | | $\approx 0$ | | $\approx 0$ | | $\approx 0$ |
| | Funcall | 14739 | 15171 | 1273 | 1273 | 4833 | 5949 | $10^4$ |
| $G_2(10)$ | $p_f^{max}$ | $\approx 0$ | | $\approx 0$ | | $\approx 0$ | | $\approx 0$ |
| | Funcall | 12597 | 12966 | 906 | 906 | 4527 | 5346 | $10^4$ |
| | $p_f^{min}$ | $\approx 0$ | | $\approx 0$ | | $\approx 0$ | | $\approx 0$ |
| | Funcall | 12580 | 12949 | 1778 | 1778 | 3898 | 4645 | $10^4$ |

# V. Conclusion

A unified reliability analysis framework for multidisciplinary systems with both random and interval variables is developed. Given random and interval variables as inputs, the output of this framework is the bounds of reliability or the probability of failure. The framework consists of probabilistic analysis (PA) and interval analysis (IA). The framework requires integrating PA and IA with multidisciplinary analysis (MDA). Since the overall reliability analysis involves PA, IA and MDA, the computation is intensive. The direct integration of PA, IA, and MDA would require a triple loop procedure and would make the computational efficiency extremely low. To reduce the number of nested loops, PA and IA are performed sequentially. Three algorithms are designed based on how PA and IA loops call the MDA loop. In Table 5, the three algorithms are summarized in terms of their features, the possible algorithms that can be used for PA and IA, and when the three algorithms can be used.

**Table 5 Summary of the three algorithms**

| Algorithm | Features | PA and IA methods | When to use it |
|---|---|---|---|
| SDL: Sequential Double Loops | The MDA inner loop is nested with the PA and IA outer loops; PA and IA involve a double-loop procedure. | PA: any reliability analysis methods, including any MPP search algorithms. IA: nonlinear optimization, interval arithmetic, or other IA methods. | MDA is not computational expensive. |
| SSL: Sequential Single Loops | MDA is embedded as equality constraints within the PA and IA loop; All the coupling variables are treated as additional design variables in the PA or IA single loop. | PA: FORM with nonlinear optimization for the MPP search IA: nonlinear optimization | The number of coupling variables is small; concurrent subsystem analyses can be performed. |
| SSSL: Sequential Single-Single Loops | PA involves a sequence of MPP search and MDA and therefore forms a sequential single-loops procedure. IA requires a single-loop procedure as in SSL. | PA: any reliability analysis methods, including any MPP search algorithms. IA: nonlinear optimization | PA is relatively expensive and IA is relative cheap; concurrent subsystem analyses can be performed; the number of interval variables is small. |

As demonstrated in the two examples, the three algorithms are capable of producing identical solutions. But their efficiency differs from problem to problem. The efficiency depends on many factors, such as the number of disciplines, the number of random variables, the number of interval variables, the number of sharing variables, and the efficiency of analyses at the disciplinary level. When to use a specific algorithm is provided in Table 5.

Other algorithm variants can also be developed using the similar strategies of the proposed three algorithms. For example, the IA loop of the SSSL algorithm is a single-

loop procedure. It can be changed to a sequential single-loops procedure, where the search of the extreme values of the limit-state function and MDA are conducted sequentially. All the algorithms discussed in this paper are only for reliability analysis. Our future work will be their application in reliability based multidisciplinary design optimization.

## Acknowledgments

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## 2.CONCLUSIONS

This research attempts to explore the impact of aleatory and epistemic uncertainties on the performances of complex engineering systems. Both types of uncertainty occur simultaneously in many applications. However, most of current uncertainty analysis and sensitivity analysis methods are not applicable to analyze the effect of epistemic uncertainty or the joint effect of both types of uncertainty. This work investigates how to quantify the effect of both types of uncertainty. Probability theory is used to model aleatory uncertainty and evidence theory and intervals are employed to model epistemic uncertainty. Probabilistic analysis for aleatory uncertainty and interval analysis for epistemic uncertainty are integrated in a unified uncertainty analysis framework to propagate the mixed uncertainties and calculate the belief and plausibility measures of outputs. The First Order Reliability Method (FORM) is adopted for probabilistic analysis and nonlinear optimization is used for interval analysis.

A family of new uncertainty analysis and sensitivity analysis approaches are established in this dissertation:

An effective sensitivity analysis framework is established in the first paper to estimate the contributions of individual input variables with epistemic uncertainty to the model outputs and identify the most significant epistemic variables. The contribution of an epistemic variable is measured by its effect on the output, including main effect and total effect. And these effects are indicated by the discrepancy between belief and plausibility measures of the output (the lower and upper probability bounds).

Paper II aims to obtain more exact understanding of how the uncertain characteristics of outputs are related to both types of uncertainty in the inputs. Six new

types of sensitivity indexes are proposed and the equations associated with each sensitivity index for different scenarios are derived. Four of them are used to quantify the sensitivities of the width and average of the probability of failure bounds with respect to the characteristics of interval variables and the other two types are defined for random variables to evaluate the sensitivities of the width and average of the probability of failure bounds with respect to the parameters of probability distributions.

The third paper extends all the above methods from single-disciplinary systems to multidisciplinary systems. A unified reliability analysis framework for multidisciplinary systems with both random and interval variables is developed. The framework integrates probabilistic analysis and interval analysis with multidisciplinary analysis (MDA). In order to lower the computational burden of direct integration, three algorithms are designed by applying different strategies to call MDA in probabilistic analysis and interval analysis loops.

The major findings of this research include:

(1) Both uncertainties have a great impact on the system performance. The effect of aleatory uncertainty can be measured by the Cumulative Distribution Function (CDF) or Probability Distribution Function (PDF), while the effect of epistemic uncertainty can be measured by the gap between the lower and upper bounds of CDF or PDF.

(2) Both types of uncertainty can be quantified by the unified uncertainty analysis framework for both single-disciplinary and multidisciplinary systems.

(3) The proposed sensitivity analysis framework for epistemic uncertainty with the mixture of aleatory and epistemic uncertainties is an effective method for reducing the impact of epistemic uncertainty. This method identifies the most important input

variables with epistemic uncertainty. Collecting more data on those variables will mitigate the effect of epistemic uncertainty in the most efficient way.

(4) In different cases, the width and mean value of the probability bounds of an output have different relationships with the width and average of input interval variables and the distribution parameters of input random variables. And the six new sensitivity indexes are capable of quantifying these relationships.

(5) The calculation of these sensitivity indexes is just a byproduct of reliability analysis and does not require any additional evaluation of a performance function.

(6) Unified uncertainty analysis and sensitivity analysis methods can be integrated with Multidisciplinary Analysis (MDA) and then be extended to multidisciplinary systems design.

(7) All the methods can be used in design. The sensitivity analysis framework can be used to lower the overall uncertainty in the outputs and help designers to make more reliable judgments and decisions. The new sensitivity indexes will tell engineers what will happen to the output uncertainty if they change the input uncertainty. The unified reliability analysis for multidisciplinary systems provides an effective tool for industry to evaluate the reliability information of system performance in the development of complex products with a full range of uncertainty.

The effectiveness of the proposed methods has been demonstrated by mathematical examples and engineering applications. But some features of these methods could be improved in the future work:

(1) The computational cost of the sensitivity analysis methods will grow with the increasing number of aleatory variables and the subsets of epistemic variables. And

interval analysis uses nonlinear optimization, which is more accurate but less efficient than interval arithmetic. Efficient computational algorithms are desired to solve these issues.

(2) The computational efficiency of the algorithms in the unified reliability analysis for multidisciplinary systems varies from problem to problem. It also depends on the number of coupling variables, uncertainty variables and disciplines, as well as the efficiency of disciplinary analysis. The further investigation might discover more factors that have influence on the efficiency. Based on such discovery, more strategies could be developed to improve the performance of the algorithms and design more algorithm variants.

(3) Probabilistic analysis in this research is conducted by FORM. More accurate methods like the Second Order Reliability Method (SORM) and Saddlepoint Approximation method [2] might be considered for the purpose of higher accuracy.

(4) The ultimate goal of this research is to provide analysis tools for reliability-based multidisciplinary systems design. The major future research work is the integration of the proposed unified reliability analysis with multidisciplinary design optimization.

(5) This research only considers both aleatory and epistemic uncertainties in model input parameters. Model structure uncertainty is not included in this work. Model structure uncertainty is a special type of epistemic uncertainty. To take a full advantage of reliability-based multidisciplinary design optimization, model structure uncertainty should be incorporated.

APPENDIX A.

COMMON EQUATIONS

1. Derivative of $p_f$ with respect to one bound of an interval variable $Y_i$

$\dfrac{\partial p_f}{\partial h}$ is given in Eq. (22) and is rewritten below.

$$\frac{\partial p_f}{\partial h} = \frac{-\phi(-\beta\ )}{\left\| \nabla g(\mathbf{u}^*) \right\|}\frac{\partial g}{\partial h} \tag{A1}$$

where $p_f$ could be $p_f^L$ or $p_f^U$, and $\beta$ could be $\beta^L$ and $\beta^U$.

2. Derivative of $p_f$ with respect to the width of an interval, $\delta_i$

If $p_f$ occurs at $y_i^U$,

$$\frac{\partial p_f}{\partial \delta_i} = \frac{\partial p_f\left(y_i^U, \mathbf{Y}_{\sim i}\right)}{\partial \delta_i} = \frac{\partial p_f\left(\overline{y}_i + \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i}\right)}{\partial \delta_i} = \frac{\partial p_f\left(\overline{y}_i + \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i}\right)}{\partial(\overline{y}_i + \frac{1}{2}\delta_i)}\frac{\partial(\overline{y}_i + \frac{1}{2}\delta_i)}{\partial \delta_i} \tag{A2}$$

Eq. (A2) can then be simplified to

$$\frac{\partial p_f}{\partial \delta_i} = \frac{1}{2}\frac{\partial p_f\left(\overline{y}_i + \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i}\right)}{\partial(\overline{y}_i + \frac{1}{2}\delta_i)} = \frac{1}{2}\frac{\partial p_f}{\partial y_i^U} \tag{A3}$$

Similarly, if $p_f$ occurs at $y_i^L$, the equation becomes

$$\frac{\partial p_f}{\partial \delta_i} = -\frac{1}{2}\frac{\partial p_f\left(\overline{y}_i - \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i}\right)}{\partial(\overline{y}_i - \frac{1}{2}\delta_i)} = -\frac{1}{2}\frac{\partial p_f}{\partial y_i^L} \tag{A4}$$

If $p_f$ occurs at an interior point $\widehat{y}_i$, which is not a function of $\delta_i$, it can then be sh

own that

$$\frac{\partial p_f}{\partial \delta_i} = \frac{\partial p_f\left(\widehat{y}_i, \mathbf{Y}_{\sim i}\right)}{\partial \delta_i} = \frac{\partial p_f\left(\widehat{y}_i, \mathbf{Y}_{\sim i}\right)}{\partial \widehat{y}_i}\frac{\partial \widehat{y}_i}{\partial \delta_i} = \frac{\partial p_f\left(\widehat{y}_i, \mathbf{Y}_{\sim i}\right)}{\partial \widehat{y}_i}\cdot 0 = 0 \tag{A5}$$

3. Derivative of $p_f$ with respect to the average of an interval, $\bar{y}_i$

If $p_f$ occurs at $y_i^U$, one can obtain

$$\frac{\partial p_f}{\partial \bar{y}_i} = \frac{\partial p_f\left(y_i^U, \mathbf{Y}_{\sim i}\right)}{\partial \bar{y}_i} = \frac{\partial p_f\left(\bar{y}_i + \dfrac{1}{2}\delta_i, \mathbf{Y}_{\sim i}\right)}{\partial \bar{y}_i} = \frac{\partial p_f\left(\bar{y}_i + \dfrac{1}{2}\delta_i, \mathbf{Y}_{\sim i}\right)}{\partial\left(\bar{y}_i + \dfrac{1}{2}\delta_i\right)} \frac{\partial\left(\bar{y}_i + \dfrac{1}{2}\delta_i\right)}{\partial \bar{y}_i} \qquad (A6)$$

and therefore

$$\frac{\partial p_f}{\partial \bar{y}_i} = \frac{\partial p_f\left(\bar{y}_i + \dfrac{1}{2}\delta_i, \mathbf{Y}_{\sim i}\right)}{\partial\left(\bar{y}_i + \dfrac{1}{2}\delta_i\right)} = \frac{\partial p_f}{\partial y_i^U} \qquad (A7)$$

Similarly, if $p_f$ occurs at $y_i^L$,

$$\frac{\partial p_f}{\partial \bar{y}_i} = \frac{\partial p_f\left(\bar{y}_i - \dfrac{1}{2}\delta_i, \mathbf{Y}_{\sim i}\right)}{\partial\left(\bar{y}_i - \dfrac{1}{2}\delta_i\right)} = \frac{\partial p_f}{\partial y_i^L} \qquad (A8)$$

If $p_f$ occurs at an interior point $\hat{y}_i$,

$$\frac{\partial p_f}{\partial \bar{y}_i} = \frac{\partial p_f\left(\hat{y}_i, \mathbf{Y}_{\sim i}\right)}{\partial \bar{y}_i} = \frac{\partial p_f\left(\hat{y}_i, \mathbf{Y}_{\sim i}\right)}{\partial \hat{y}_i} \frac{\partial \hat{y}_i}{\partial \bar{y}_i} = \frac{\partial p_f\left(\hat{y}_i, \mathbf{Y}_{\sim i}\right)}{\partial \hat{y}_i} \cdot 0 = 0 \qquad (A9)$$

4. Derivative of $p_f$ bound with respect to a distribution parameter $q_i$

$$\frac{\partial p_f}{\partial q_i} = \frac{\partial \Phi(-\beta)}{\partial q_i} = -\phi(-\beta)\frac{\partial \beta}{\partial u_i^*}\frac{\partial u_i^*}{\partial q_i} \qquad (A10)$$

If the CDF of $X_i$ is $F_{X_i}\left(x_i\right)$, then

$$u_i^* = \Phi^{-1}\left[F_{X_i}\left(x_i^*\right)\right] = w(q_1, q_2, \cdots, q_i, \cdots, q_n) \qquad (A11)$$

where $n$ is the number of distribution parameters.

Then from $\beta = \left\| \mathbf{u}_i^* \right\|$, one obtains

$$\frac{\partial p_f}{\partial q_i} = -\phi(-\beta)\frac{u_i^*}{\beta}\frac{\partial w}{\partial q_i} \tag{A12}$$

APPENDIX B.


EQUATIONS FOR TYPE I SENSITIVITY $\partial \delta_p / \partial \delta_i$

**Case 1**: $p_f^L$ occurs at $y_i^L$ and $p_f^U$ occurs at $y_i^U$ (see Paper II Section 3).

**Case 2**: $p_f^L$ occurs at $y_i^U$ and $p_f^U$ occurs at $y_i^L$.

$$\frac{\partial \delta_p}{\partial \delta_i} = \frac{\partial \left[ p_f^U \left( \bar{y}_i - \frac{1}{2}\delta_i, y_{\sim i} \right) - p_f^L \left( \bar{y}_i + \frac{1}{2}\delta_i, y_{\sim i} \right) \right]}{\partial \delta_i} \tag{B1}$$

Using Eqs. (A3) and (A4) gives

$$\frac{\partial \delta_p}{\partial \delta_i} = -\frac{1}{2} \left( \frac{\partial p_f^U}{\partial y_i^L} + \frac{\partial p_f^L}{\partial y_i^U} \right) \tag{B2}$$

Then from Eq. (A1),

$$\frac{\partial \delta_p}{\partial \delta_i} = -\frac{1}{2} \left( \frac{\partial p_f^U}{\partial y_i^L} + \frac{\partial p_f^L}{\partial y_i^U} \right) = -\frac{1}{2} \left[ \frac{-\phi(-\beta^U)}{\left\| \nabla g(\mathbf{u}^{*,U}) \right\|} \frac{\partial g}{\partial Y_i} \Big|_{y_i^L} + \frac{-\phi(-\beta^L)}{\left\| \nabla g(\mathbf{u}^{*,L}) \right\|} \frac{\partial g}{\partial Y_i} \Big|_{y_i^U} \right] \tag{B3}$$

**Case 3**: $p_f^L$ occurs at an interior point $\hat{y}_i$ and $p_f^U$ occurs at $y_i^U$.

$$\frac{\partial \delta_p}{\partial \delta_i} = \frac{\partial \left[ p_f^U \left( \bar{y}_i + \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i} \right) - p_f^L \left( \hat{y}_i, \mathbf{Y}_{\sim i} \right) \right]}{\partial \delta_i} \tag{B4}$$

Using Eqs. (A3) and (A5), one obtains

$$\frac{\partial \delta_p}{\partial \delta_i} = \frac{1}{2} \frac{\partial p_f^U}{\partial y_i^U}. \tag{B5}$$

Applying the results from Eq. (A1) yields

$$\frac{\partial \delta_p}{\partial \delta_i} = \frac{1}{2} \frac{\partial p_f^U}{\partial y_i^U} = \frac{1}{2} \left[ \frac{-\phi(-\beta^U)}{\left\| \nabla g(\mathbf{u}^{*,U}) \right\|} \frac{\partial g}{\partial Y_i} \Big|_{y_i^U} \right] \tag{B6}$$

**Case 4**: $p_f^L$ occurs at an interior point $\hat{y}_i$ and $p_f^U$ occurs at $y_i^L$.

$$\frac{\partial \delta_p}{\partial \delta_i} = \frac{\partial \left[ p_f^U \left( \overline{y}_i - \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i} \right) - p_f^L \left( \widehat{y}_i, \mathbf{Y}_{\sim i} \right) \right]}{\partial \delta_i} \tag{B7}$$

Using Eqs. (A4) and (A5) yields

$$\frac{\partial \delta_p}{\partial \delta_i} = -\frac{1}{2}\frac{\partial p_f^U}{\partial y_i^L} \tag{B8}$$

Applying Eq. (A1) yields

$$\frac{\partial \delta_p}{\partial \delta_i} = -\frac{1}{2}\left[ \frac{-\phi(-\beta^U)}{\left\| \nabla g(\mathbf{u}^{*,U}) \right\|}\frac{\partial g}{\partial Y_i}\Big|_{y_i^L} \right] \tag{B9}$$

**Case 5**: $p_f^L$ occurs at $y_i^U$ and $p_f^U$ occurs at an interior point $\widehat{y}_i$.

$$\frac{\partial \delta_p}{\partial \delta_i} = \frac{\partial \left[ p_f^U \left( \widehat{y}_i, \mathbf{Y}_{\sim i} \right) - p_f^L \left( \overline{y}_i + \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i} \right) \right]}{\partial \delta_i} \tag{B10}$$

Using Eqs. (A3) and (A5), one obtains

$$\frac{\partial \delta_p}{\partial \delta_i} = -\frac{1}{2}\frac{\partial p_f^L}{\partial y_i^U} \tag{B11}$$

Using Eq. (A1) yields

$$\frac{\partial \delta_p}{\partial \delta_i} = -\frac{1}{2}\left[ \frac{-\phi(-\beta^L)}{\left\| \nabla g(\mathbf{u}^{*,L}) \right\|}\frac{\partial g}{\partial Y_i}\Big|_{y_i^U} \right] \tag{B12}$$

**Case 6**: $p_f^L$ occurs at $y_i^L$ and $p_f^U$ occurs at an interior point $\widehat{y}_i$.

$$\frac{\partial \delta_p}{\partial \delta_i} = \frac{\partial \left[ p_f^U \left( \widehat{y}_i, \mathbf{Y}_{\sim i} \right) - p_f^L \left( \overline{y}_i - \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i} \right) \right]}{\partial \delta_i} \tag{B13}$$

Using Eqs. (A4) and (A5) gives

$$\frac{\partial \delta_p}{\partial \delta_i} = \frac{1}{2} \frac{\partial p_f^L}{\partial y_i^L} = \frac{1}{2} \left[ \frac{-\phi(-\beta^L)}{\left\| \nabla g(\mathbf{u}^{*,L}) \right\|} \frac{\partial g}{\partial Y_i} \bigg|_{y_i^L} \right] \tag{B14}$$

Using Eq. (A1) yields

$$\frac{\partial \delta_p}{\partial \delta_i} = \frac{1}{2} \left[ \frac{-\phi(-\beta^L)}{\left\| \nabla g(\mathbf{u}^{*,L}) \right\|} \frac{\partial g}{\partial Y_i} \bigg|_{y_i^L} \right] \tag{B15}$$

**Case 7**: $p_f^L$ and $p_f^U$ occur at two interior points $\hat{y}_{i1}$ and $\hat{y}_{i2}$, respectively.

$$\frac{\partial \delta_p}{\partial \delta_i} = \frac{\partial \left[ p_f^U(\hat{y}_{i1}, \mathbf{Y}_{\sim i}) - p_f^L(\hat{y}_{i2}, \mathbf{Y}_{\sim i}) \right]}{\partial \delta_i} \tag{B16}$$

Using Eq. (A5) yields

$$\frac{\partial \delta_p}{\partial \delta_i} = 0 . \tag{B17}$$

APPENDIX C.


EQUATIONS FOR TYPE II SENSITIVITY $\partial \overline{p}_f / \partial \delta_i$

**Case 1**: $p_f^L$ occurs at $y_i^L$ and $p_f^U$ occurs at $y_i^U$.

$$\frac{\partial \overline{p}_f}{\partial \delta_i} = \frac{\partial \left\{ \frac{1}{2}\left[ p_f^U\left( \overline{y}_i + \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i}\right) + p_f^L\left( \overline{y}_i - \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i}\right)\right]\right\}}{\partial \delta_i} \tag{C1}$$

Using Eqs. (A3) and (A4) yields

$$\frac{\partial \overline{p}_f}{\partial \delta_i} = \frac{1}{4}\left( \frac{\partial p_f^U}{\partial y_i^U} - \frac{\partial p_f^L}{\partial y_i^L}\right) \tag{C2}$$

From Eq. (A1)

$$\frac{\partial \overline{p}_f}{\partial \delta_i} = \frac{1}{4}\left( \frac{-\phi(-\beta^U)}{\left\|\nabla g(\mathbf{u}^{*,U})\right\|}\frac{\partial g}{\partial Y_i}\bigg|_{y_i^U} - \frac{-\phi(-\beta^L)}{\left\|\nabla g(\mathbf{u}^{*,L})\right\|}\frac{\partial g}{\partial Y_i}\bigg|_{y_i^L}\right) \tag{C3}$$

**Case 2**: $p_f^L$ occurs at $y_i^U$ and $p_f^U$ occurs at $y_i^L$.

$$\frac{\partial \overline{p}_f}{\partial \delta_i} = \frac{\partial \left\{ \frac{1}{2}\left[ p_f^U\left( \overline{y}_i - \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i}\right) + p_f^L\left( \overline{y}_i + \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i}\right)\right]\right\}}{\partial \delta_i} \tag{C4}$$

Using Eqs. (A3) and (A4) yields

$$\frac{\partial \overline{p}_f}{\partial \delta_i} = \frac{1}{4}\left( -\frac{\partial p_f^U}{\partial y_i^L} + \frac{\partial p_f^L}{\partial y_i^U}\right) \tag{C5}$$

From Eq. (A1)

$$\frac{\partial \overline{p}_f}{\partial \delta_i} = \frac{1}{4}\left[ -\frac{-\phi(-\beta^U)}{\left\|\nabla g(\mathbf{u}^{*,U})\right\|}\frac{\partial g}{\partial Y_i}\bigg|_{y_i^L} + \frac{-\phi(-\beta^L)}{\left\|\nabla g(\mathbf{u}^{*,L})\right\|}\frac{\partial g}{\partial Y_i}\bigg|_{y_i^U}\right] \tag{C6}$$

**Case 3**: $p_f^L$ occurs at an interior point $\widehat{y}_i$ and $p_f^U$ occurs at $y_i^U$.

$$\frac{\partial \overline{p}_f}{\partial \delta_i} = \frac{\partial \left\{ \frac{1}{2}\left[ p_f^U\left( \overline{y}_i + \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i}\right) + p_f^L(\widehat{y}_i, \mathbf{Y}_{\sim i})\right]\right\}}{\partial \delta_i} \tag{C7}$$

Using Eqs. (A3) and (A5) yields

$$\frac{\partial \overline{p}_f}{\partial \delta_i} = \frac{1}{4}\frac{\partial p_f^U}{\partial y_i^U} \qquad \text{(C8)}$$

From Eq. (A1)

$$\frac{\partial \overline{p}_f}{\partial \delta_i} = \frac{1}{4}\left[\frac{-\phi(-\beta^U)}{\left\|\nabla g(\mathbf{u}^{*,U})\right\|}\frac{\partial g}{\partial Y_i}\Big|_{y_i^U}\right] \qquad \text{(C9)}$$

**Case 4**: $p_f^L$ occurs at an interior point $\widehat{y}_i$ and $p_f^U$ occurs at $y_i^L$.

$$\frac{\partial \overline{p}_f}{\partial \delta_i} = \frac{\partial\left\{\frac{1}{2}\left[p_f^U\left(\overline{y}_i - \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i}\right) + p_f^L\left(\widehat{y}_i, \mathbf{Y}_{\sim i}\right)\right]\right\}}{\partial \delta_i} \qquad \text{(C10)}$$

Using Eqs.(A4) and (A5) yields

$$\frac{\partial \overline{p}_f}{\partial \delta_i} = -\frac{1}{4}\frac{\partial p_f^U}{\partial y_i^L} \qquad \text{(C11)}$$

Applying Eq. (A1), one obtains

$$\frac{\partial \overline{p}_f}{\partial \delta_i} = -\frac{1}{4}\left[\frac{-\phi(-\beta^U)}{\left\|\nabla g(\mathbf{u}^{*,U})\right\|}\frac{\partial g}{\partial Y_i}\Big|_{y_i^L}\right] \qquad \text{(C12)}$$

**Case 5**: $p_f^L$ occurs at $y_i^U$ and $p_f^U$ occurs at an interior point $\widehat{y}_i$

$$\frac{\partial \overline{p}_f}{\partial \delta_i} = \frac{\partial\left\{\frac{1}{2}\left[p_f^U(\widehat{y}_i, \mathbf{Y}_{\sim i}) + p_f^L(\overline{y}_i + \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i})\right]\right\}}{\partial(\delta_i)} \qquad \text{(C13)}$$

Using Eqs. (A3) and (A5) gives

$$\frac{\partial \overline{p}_f}{\partial \delta_i} = \frac{1}{4}\frac{\partial p_f^L}{\partial y_i^U} \qquad \text{(C14)}$$

Applying Eq. (A1) yields

$$\frac{\partial \overline{p}_f}{\partial \delta_i} = \frac{1}{4}\left[\frac{-\phi(-\beta^L)}{\left\|\nabla g(\mathbf{u}^{*,L})\right\|}\frac{\partial g}{\partial Y_i}\bigg|_{y_i^U}\right]$$

(C15)

**Case 6**: $p_f^L$ occurs at $y_i^L$ and $p_f^U$ occurs at an interior point $\widehat{y}_i$.

$$\frac{\partial \overline{p}_f}{\partial \delta_i} = \frac{\partial\left\{\frac{1}{2}\left[p_f^U(\widehat{y}_i, \mathbf{Y}_{\sim i}) + p_f^L(\overline{y}_i - \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i})\right]\right\}}{\partial \delta_i}$$

(C16)

Using Eqs. (A4) and (A5) yields

$$\frac{\partial \overline{p}_f}{\partial \delta_i} = -\frac{1}{4}\frac{\partial p_f^L}{\partial y_i^L}$$

(C17)

From Eq. (A1)

$$\frac{\partial \overline{p}_f}{\partial \delta_i} = -\frac{1}{4}\left[\frac{-\phi(-\beta^L)}{\left\|\nabla g(\mathbf{u}^{*,L})\right\|}\frac{\partial g}{\partial Y_i}\bigg|_{y_i^L}\right]$$

(C18)

**Case 7**: $p_f^L$ and $p_f^U$ occur at two interior points $\widehat{y}_{i1}$ and $\widehat{y}_{i2}$, respectively.

$$\frac{\partial \delta_p}{\partial \delta_i} = \frac{\partial\left\{\frac{1}{2}\left[p_f^U(\widehat{y}_{i1}, \mathbf{Y}_{\sim i}) + p_f^L\left(\widehat{y}_{i2}, \mathbf{Y}_{\sim i}\right)\right]\right\}}{\partial \delta_i}$$

(C19)

Using Eq. (A5) yields

$$\frac{\partial \delta_p}{\partial \delta_i} = 0$$

(C20)

APPENDIX D.

EQUATIONS FOR TYPE III SENSITIVITY $\partial \delta_p / \partial \overline{y}_i$

**Case 1**: $p_f^L$ occurs at $y_i^L$ and $p_f^U$ occurs at $y_i^U$.

$$\frac{\partial \delta_p}{\partial \overline{y}_i} = \frac{\partial\left(p_f^U - p_f^L\right)}{\partial \overline{y}_i} = \frac{\partial\left[ p_f^U\left(\overline{y}_i + \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i}\right) - p_f^L\left(\overline{y}_i - \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i}\right)\right]}{\partial \overline{y}_i} \tag{D1}$$

Using Eqs. (A7) and (A8) yields

$$\frac{\partial \delta_p}{\partial \overline{y}_i} = \frac{\partial p_f^U}{\partial \overline{y}_i} - \frac{\partial p_f^L}{\partial \overline{y}_i} = \frac{\partial p_f^U}{\partial y_i^U} - \frac{\partial p_f^L}{\partial y_i^L} \tag{D2}$$

From Eq. (A1)

$$\frac{\partial \delta_p}{\partial \overline{y}_i} = \frac{-\phi(-\beta^U)}{\left\|\nabla g(\mathbf{u}^{*,U})\right\|}\left.\frac{\partial g}{\partial Y_i}\right|_{y_i^U} - \frac{-\phi(-\beta^L)}{\left\|\nabla g(\mathbf{u}^{*,L})\right\|}\left.\frac{\partial g}{\partial Y_i}\right|_{y_i^L} \tag{D3}$$

**Case 2**: $p_f^L$ occurs at $y_i^U$ and $p_f^U$ occurs at $y_i^L$.

$$\frac{\partial \delta_p}{\partial \overline{y}_i} = \frac{\partial\left(p_f^U - p_f^L\right)}{\partial \overline{y}_i} = \frac{\partial\left[ p_f^U\left(\overline{y}_i - \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i}\right) - p_f^L\left(\overline{y}_i + \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i}\right)\right]}{\partial \overline{y}_i} \tag{D4}$$

Using Eqs. (A7) and (A8) gives

$$\frac{\partial \delta_p}{\partial \overline{y}_i} = \frac{\partial p_f^U}{\partial \overline{y}_i} - \frac{\partial p_f^L}{\partial \overline{y}_i} = \frac{\partial p_f^U}{\partial y_i^L} - \frac{\partial p_f^L}{\partial y_i^U} \tag{D5}$$

Applying the results of Eq. (A1) yields

$$\frac{\partial \delta_p}{\partial \overline{y}_i} = \frac{-\phi(-\beta^U)}{\left\|\nabla g(\mathbf{u}^{*,U})\right\|}\left.\frac{\partial g}{\partial Y_i}\right|_{y_i^L} - \frac{-\phi(-\beta^L)}{\left\|\nabla g(\mathbf{u}^{*,L})\right\|}\left.\frac{\partial g}{\partial Y_i}\right|_{y_i^U} \tag{D6}$$

**Case 3**: $p_f^L$ occurs at an interior point $\hat{y}_i$ and $p_f^U$ occurs at $y_i^U$.

$$\frac{\partial \delta_p}{\partial \overline{y}_i} = \frac{\partial(p_f^U - p_f^L)}{\partial \overline{y}_i} = \frac{\partial\left[ p_f^U\left(\overline{y}_i + \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i}\right) - p_f^L\left(\hat{y}_i, \mathbf{Y}_{\sim i}\right)\right]}{\partial \overline{y}_i} \tag{D7}$$

Using Eqs. (A7) and (A9), one obtains

$$\frac{\partial \delta_p}{\partial \overline{y}_i} == \frac{\partial p_f^U}{\partial \overline{y}_i} = \frac{\partial p_f^U}{\partial y_i^U} \tag{D8}$$

By Eq. (A1)

$$\frac{\partial \delta_p}{\partial \overline{y}_i} = \frac{-\phi(-\beta^U)}{\left\|\nabla g(\mathbf{u}^{*,U})\right\|} \left.\frac{\partial g}{\partial Y_i}\right|_{y_i^U} \tag{D9}$$

**Case 4**: $p_f^L$ occurs at an interior point $\widehat{y}_i$ and $p_f^U$ occurs at $y_i^L$.

$$\frac{\partial \delta_p}{\partial \overline{y}_i} = \frac{\partial \left( p_f^U - p_f^L \right)}{\partial \overline{y}_i} = \frac{\partial \left[ p_f^U \left( \overline{y}_i - \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i} \right) - p_f^L \left( \widehat{y}_i, \mathbf{Y}_{\sim i} \right) \right]}{\partial \overline{y}_i} \tag{D10}$$

Using Eqs. (A8) and (A9) gives

$$\frac{\partial \delta_p}{\partial \overline{y}_i} = \frac{\partial p_f^U}{\partial \overline{y}_i} = \frac{\partial p_f^U}{\partial y_i^L} \tag{D11}$$

By Eq. (A1)

$$\frac{\partial \delta_p}{\partial \overline{y}_i} = \frac{-\phi(-\beta^U)}{\left\|\nabla g(\mathbf{u}^{*,U})\right\|} \left.\frac{\partial g}{\partial Y_i}\right|_{y_i^L} \tag{D12}$$

**Case 5**: $p_f^L$ occurs at $y_i^U$ and $p_f^U$ occurs at an interior point $\widehat{y}_i$.

$$\frac{\partial \delta_p}{\partial \overline{y}_i} = \frac{\partial \left( p_f^U - p_f^L \right)}{\partial \overline{y}_i} = \frac{\partial \left[ p_f^U \left( \widehat{y}_i, \mathbf{Y}_{\sim i} \right) - p_f^L \left( \overline{y}_i + \frac{1}{2}\delta_i, \mathbf{Y}_{\sim i} \right) \right]}{\partial \overline{y}_i} \tag{D13}$$

Using Eqs. (A7) and (A9) gives

$$\frac{\partial \delta_p}{\partial \overline{y}_i} = -\frac{\partial p_f^L}{\partial \overline{y}_i} = -\frac{\partial p_f^L}{\partial y_i^U} \tag{D14}$$

Using Eq. (A1) yields

$$\frac{\partial \delta_p}{\partial \overline{y}_i} == -\frac{-\phi(-\beta^L)}{\left\|\nabla g(\mathbf{u}^{*,L})\right\|} \left.\frac{\partial g}{\partial Y_i}\right|_{y_i^U} \tag{D15}$$

**Case 6**: $p_f^L$ occurs at $y_i^L$ and $p_f^U$ occurs at an interior point $\hat{y}_i$.

$$\frac{\partial \delta_p}{\partial \overline{y}_i} = \frac{\partial \left( p_f^U - p_f^L \right)}{\partial \overline{y}_i} = \frac{\partial \left[ p_f^U \left( \hat{y}_i, \mathbf{Y}_{\sim i} \right) - p_f^L \left( \overline{y}_i - \frac{1}{2} \delta_i, \mathbf{Y}_{\sim i} \right) \right]}{\partial \overline{y}_i} \tag{D16}$$

Using Eqs. (A8) and (A9) gives

$$\frac{\partial \delta_p}{\partial \overline{y}_i} = -\frac{\partial p_f^L}{\partial \overline{y}_i} = -\frac{\partial p_f^L}{\partial y_i^L} \tag{D17}$$

Using Eq. (A1) yields

$$\frac{\partial \delta_p}{\partial \overline{y}_i} = -\frac{-\phi(-\beta^L)}{\left\| \nabla g(\mathbf{u}^{*,L}) \right\|} \frac{\partial g}{\partial Y_i}\bigg|_{y_i^L} \tag{D18}$$

**Case 7**: $p_f^L$ and $p_f^U$ occur at two interior points $\hat{y}_{i1}$ and $\hat{y}_{i2}$, respectively.

$$\frac{\partial \delta_p}{\partial \overline{y}_i} = \frac{\partial \left[ p_f^U \left( \hat{y}_{i1}, \mathbf{Y}_{\sim i} \right) - p_f^L \left( \hat{y}_{i2}, \mathbf{Y}_{\sim i} \right) \right]}{\partial \overline{y}_i} \tag{D19}$$

Using Eq. (A9) yields

$$\frac{\partial \delta_p}{\partial \overline{y}_i} = 0 \tag{D20}$$

APPENDIX E.


EQUATIONS FOR TYPE IV SENSITIVITY $\partial \overline{p}_f / \partial \overline{y}_i$

**Case 1**: $p_f^L$ occurs at $y_i^L$ and $p_f^U$ occurs at $y_i^U$.

$$\frac{\partial \overline{p}_f}{\partial \overline{y}_i} = \frac{\partial \left( \dfrac{p_f^U + p_f^L}{2} \right)}{\partial \overline{y}_i} = \frac{\partial \left\{ \dfrac{1}{2} \left[ p_f^U \left( \overline{y}_i + \dfrac{1}{2}\delta_i, \mathbf{Y}_{\sim i} \right) + p_f^L \left( \overline{y}_i - \dfrac{1}{2}\delta_i, \mathbf{Y}_{\sim i} \right) \right] \right\}}{\partial \overline{y}_i} \tag{E1}$$

Using Eqs. (A7) and (A9) gives

$$\frac{\partial \overline{p}_f}{\partial \overline{y}_i} = \frac{1}{2}\left( \frac{\partial p_f^U}{\partial \overline{y}_i} + \frac{\partial p_f^L}{\partial \overline{y}_i} \right) = \frac{1}{2}\left( \frac{\partial p_f^U}{\partial y_i^U} + \frac{\partial p_f^L}{\partial y_i^L} \right) \tag{E2}$$

Using Eq. (A1) yields

$$\frac{\partial \overline{p}_f}{\partial \overline{y}_i} = \frac{1}{2}\left[ \frac{-\phi(-\beta^U)}{\left\| \nabla g(\mathbf{u}^{*,U}) \right\|} \frac{\partial g}{\partial Y_i}\bigg|_{y_i^U} + \frac{-\phi(-\beta^L)}{\left\| \nabla g(\mathbf{u}^{*,L}) \right\|} \frac{\partial g}{\partial Y_i}\bigg|_{y_i^L} \right]. \tag{E3}$$

**Case 2**: $p_f^L$ occurs at $y_i^U$ and $p_f^U$ occurs at $y_i^L$.

$$\frac{\partial \overline{p}_f}{\partial \overline{y}_i} = \frac{\partial \left( \dfrac{p_f^U + p_f^L}{2} \right)}{\partial \overline{y}_i} = \frac{\partial \left\{ \dfrac{1}{2} \left[ p_f^U \left( \overline{y}_i - \dfrac{1}{2}\delta_i, \mathbf{Y}_{\sim i} \right) + p_f^L \left( \overline{y}_i + \dfrac{1}{2}\delta_i, \mathbf{Y}_{\sim i} \right) \right] \right\}}{\partial \overline{y}_i} \tag{E4}$$

Using Eqs. (A7) and (A8) yields

$$\frac{\partial \overline{p}_f}{\partial \overline{y}_i} = \frac{1}{2}\left( \frac{\partial p_f^U}{\partial \overline{y}_i} + \frac{\partial p_f^L}{\partial \overline{y}_i} \right) = \frac{1}{2}\left( \frac{\partial p_f^U}{\partial y_i^L} + \frac{\partial p_f^L}{\partial y_i^U} \right) \tag{E5}$$

Using Eq. (A1) yields

$$\frac{\partial \overline{p}_f}{\partial \overline{y}_i} = \frac{1}{2}\left[ \frac{-\phi(-\beta^U)}{\left\| \nabla g(\mathbf{u}^{*,U}) \right\|} \frac{\partial g}{\partial Y_i}\bigg|_{y_i^L} + \frac{-\phi(-\beta^L)}{\left\| \nabla g(\mathbf{u}^{*,L}) \right\|} \frac{\partial g}{\partial Y_i}\bigg|_{y_i^U} \right] \tag{E6}$$

**Case 3**: $p_f^L$ occurs at an interior point $\widehat{y}_i$ and $p_f^U$ occurs at $y_i^U$.

$$\frac{\partial \overline{p}_f}{\partial \overline{y}_i} = \frac{\partial \left( \dfrac{p_f^U + p_f^L}{2} \right)}{\partial \overline{y}_i} = \frac{\partial \left\{ \dfrac{1}{2} \left[ p_f^U \left( \overline{y}_i + \dfrac{1}{2}\delta_i, \mathbf{Y}_{\sim i} \right) + p_f^L \left( \widehat{y}_i, \mathbf{Y}_{\sim i} \right) \right] \right\}}{\partial \overline{y}_i} \tag{E7}$$

Using Eqs. (A7) and (A9) gives

$$\frac{\partial \overline{p}_f}{\partial \overline{y}_i} = \frac{1}{2}\frac{\partial p_f^U}{\partial \overline{y}_i} = \frac{1}{2}\frac{\partial p_f^U}{\partial y_i^U} \tag{E8}$$

Using Eq. (A1) yields

$$\frac{\partial \overline{p}_f}{\partial \overline{y}_i} = \frac{1}{2}\frac{-\phi(-\beta^U)}{\left\|\nabla g(\mathbf{u}^{*,U})\right\|}\frac{\partial g}{\partial Y_i}\bigg|_{y_i^U} \tag{E9}$$

**Case 4**: $p_f^L$ occurs at an interior point $\widehat{y}_i$ and $p_f^U$ occurs at $y_i^L$.

$$\frac{\partial \overline{p}_f}{\partial \overline{y}_i} = \frac{\partial\left(\dfrac{p_f^U + p_f^L}{2}\right)}{\partial \overline{y}_i} = \frac{\partial\left\{\dfrac{1}{2}\left[p_f^U\left(\overline{y}_i - \dfrac{1}{2}\delta_i, \mathbf{Y}_{\sim i}\right) + p_f^L\left(\widehat{y}_i, \mathbf{Y}_{\sim i}\right)\right]\right\}}{\partial \overline{y}_i} \tag{E10}$$

Using Eqs. (A8) and (A9) yields

$$\frac{\partial \overline{p}_f}{\partial \overline{y}_i} = \frac{1}{2}\frac{\partial p_f^U}{\partial \overline{y}_i} = \frac{1}{2}\frac{\partial p_f^U}{\partial y_i^L} \tag{E11}$$

Using Eq. (A1) yields

$$\frac{\partial \overline{p}_f}{\partial \overline{y}_i} = \frac{1}{2}\frac{-\phi(-\beta^U)}{\left\|\nabla g(\mathbf{u}^{*,U})\right\|}\frac{\partial g}{\partial Y_i}\bigg|_{y_i^L} \tag{E12}$$

**Case 5**: $p_f^L$ occurs at $y_i^U$ and $p_f^U$ occurs at an interior point $\widehat{y}_i$.

$$\frac{\partial \overline{p}_f}{\partial \overline{y}_i} = \frac{\partial\left(\dfrac{p_f^U + p_f^L}{2}\right)}{\partial \overline{y}_i} = \frac{\partial\left\{\dfrac{1}{2}\left[p_f^U\left(\widehat{y}_i, \mathbf{Y}_{\sim i}\right) + p_f^L\left(\overline{y}_i + \dfrac{1}{2}\delta_i, \mathbf{Y}_{\sim i}\right)\right]\right\}}{\partial \overline{y}_i} \tag{E13}$$

Using Eqs. (A7) and (A9) gives

$$\frac{\partial \overline{p}_f}{\partial \overline{y}_i} = \frac{1}{2}\frac{\partial p_f^L}{\partial \overline{y}_i} = \frac{1}{2}\frac{\partial p_f^L}{\partial y_i^U} \tag{E14}$$

Using Eq. (A1) yields

$$\frac{\partial \overline{p}_f}{\partial \overline{y}_i} = \frac{1}{2} \frac{-\phi(-\beta^L)}{\left\| \nabla g(\mathbf{u}^{*,L}) \right\|} \frac{\partial g}{\partial Y_i} \bigg|_{y_i^U}$$

(E15)

**Case 6**: $p_f^L$ occurs at $y_i^L$ and $p_f^U$ occurs at an interior point $\widehat{y}_i$.

$$\frac{\partial \overline{p}_f}{\partial \overline{y}_i} = \frac{\partial \left( \dfrac{p_f^U + p_f^L}{2} \right)}{\partial \overline{y}_i} = \frac{\partial \left\{ \dfrac{1}{2} \left[ p_f^U \left( \widehat{y}_i, \mathbf{Y}_{\sim i} \right) + p_f^L \left( \overline{y}_i - \dfrac{1}{2}\delta_i, \mathbf{Y}_{\sim i} \right) \right] \right\}}{\partial \overline{y}_i}$$

(E16)

Using Eq. (A8) and (A9), one obtains

$$\frac{\partial \overline{p}_f}{\partial \overline{y}_i} = \frac{1}{2} \frac{\partial p_f^L}{\partial \overline{y}_i} = \frac{1}{2} \frac{\partial p_f^L}{\partial y_i^L}$$

(E17)

Applying Eq. (A1) yields

$$\frac{\partial \overline{p}_f}{\partial \overline{y}_i} = \frac{1}{2} \frac{-\phi(-\beta^L)}{\left\| \nabla g(\mathbf{u}^{*,L}) \right\|} \frac{\partial g}{\partial Y_i} \bigg|_{y_i^L}$$

(E18)

**Case 7**: $p_f^L$ and $p_f^U$ occur at two interior points $\widehat{y}_{i1}$ and $\widehat{y}_{i2}$, respectively.

$$\frac{\partial \overline{p}_f}{\partial \overline{y}_i} = \frac{\partial \left\{ \dfrac{1}{2} \left[ p_f^U (\widehat{y}_{i1}, \mathbf{Y}_{\sim i}) + p_f^L \left( \widehat{y}_{i2}, \mathbf{Y}_{\sim i} \right) \right] \right\}}{\partial \overline{y}_i}$$

(E19)

Using Eq. (A9) gives

$$\frac{\partial \overline{p}_f}{\partial \overline{y}_i} = 0$$

(E20)

APPENDIX F.

EQUATIONS FOR TYPE V SENSITIVITY $\partial \delta_p / \partial q_i$

$$\frac{\partial \delta_p}{\partial q_i} = \frac{\partial(p_f^U - p_f^L)}{\partial q_i} = \frac{\partial p_f^U}{\partial q_i} - \frac{\partial p_f^L}{\partial q_i} \tag{F1}$$

Using Eq. (A12) gives

$$\frac{\partial \delta_p}{\partial q_i} = -\phi(-\beta^U)\frac{u_i^{*,U}}{\beta^U}\frac{\partial w}{\partial q_i} + \phi(-\beta^L)\frac{u_i^{*,L}}{\beta^L}\frac{\partial w}{\partial q_i} \tag{F2}$$

where $u_i^{*,U}$ is the MPP at $p_f^U$ and $u_i^{*,L}$ is the MPP at $p_f^L$.

Specifically, for a normal distributed random variable $X_i \sim N(\mu_i, \sigma_i)$,

$$w(\mu_i, \sigma_i) = \Phi^{-1}\left[F_{X_i}\left(x_i^*\right)\right] = \Phi^{-1}\left[\Phi\left(\frac{x_i^* - \mu_i}{\sigma_i}\right)\right] = \frac{x_i^* - \mu_i}{\sigma_i}, \tag{F3}$$

so it can be obtained that

$$\frac{\partial w}{\partial \mu_i} = -\frac{1}{\sigma_i}, \frac{\partial w}{\partial \sigma_i} = -\frac{x_i^* - \mu_i}{\sigma_i^2} = -\frac{u_i^*}{\sigma_i}. \tag{F4}$$

Therefore, from Eq. (F2), we can obtain the following sensitivities.

1) $q_i = \mu_i$

$$\frac{\partial \delta_p}{\partial \mu_i} = -\phi(-\beta^U)\frac{u_i^{*,U}}{\beta^U}\frac{\partial w}{\partial \mu_i} + \phi(-\beta^L)\frac{u_i^{*,L}}{\beta^L}\frac{\partial w}{\partial \mu_i} = \phi(-\beta^U)\frac{u_i^{*,U}}{\beta^U\sigma_i} - \phi(-\beta^L)\frac{u_i^{*,L}}{\beta^L\sigma_i} \tag{F5}$$

2) $q_i = \sigma_i$

$$\frac{\partial \delta_p}{\partial \sigma_i} = -\phi(-\beta^U)\frac{u_i^{*,U}}{\beta^U}\frac{\partial w}{\partial \sigma_i} + \phi(-\beta^L)\frac{u_i^{*,L}}{\beta^L}\frac{\partial w}{\partial \sigma_i}$$
$$= \phi(-\beta^U)\frac{u_i^{*,U}}{\beta^U}\frac{u_i^{*,U}}{\sigma_i} - \phi(-\beta^L)\frac{u_i^{*,L}}{\beta^L}\frac{u_i^{*,L}}{\sigma_i} = \phi(-\beta^L)\frac{(u_i^{*,U})^2}{\beta^L\sigma_i} - \phi(-\beta^L)\frac{(u_i^{*,L})^2}{\beta^L\sigma_i} \tag{F6}$$

APPENDIX G.

EQUATIONS FOR TYPE VI SENSITIVITY $\partial \overline{p}_f / \partial q_i$

$$\frac{\partial(\bar{p}_f)}{\partial q_i} = \frac{\partial\left(\dfrac{p_f^U + p_f^L}{2}\right)}{\partial q_i} = \frac{1}{2}\left(\frac{\partial p_f^U}{\partial q_i} + \frac{\partial p_f^L}{\partial q_i}\right) \tag{G1}$$

Using Eq. (A12), it can be easily shown that

$$\frac{\partial \bar{p}_f}{\partial q_i} = -\frac{1}{2}\left[\phi(-\beta^U)\frac{u_i^{*,L}}{\beta^U}\frac{\partial w}{\partial q_i} + \phi(-\beta^L)\frac{u_i^{*,L}}{\beta^L}\frac{\partial w}{\partial q_i}\right] \tag{G2}$$

Applying the results from Eq. (F4) for a normal distributed random variable $X_i \sim N(\mu_i, \sigma_i)$, the following sensitivities are obtained.

1) $q_i = \mu_i$

$$\begin{aligned}
\frac{\partial \bar{p}_f}{\partial \mu_i} &= -\frac{1}{2}\left[\phi(-\beta^U)\frac{u_i^{*,U}}{\beta^U}\frac{\partial w}{\partial \mu_i} + \phi(-\beta^L)\frac{u_i^{*,L}}{\beta^L}\frac{\partial w}{\partial \mu_i}\right] \\
&= \frac{1}{2}\left[\phi(-\beta^U)\frac{u_i^{*,U}}{\beta^U \sigma_i} + \phi(-\beta^U)\frac{u_i^{*,L}}{\beta^U \sigma_i}\right]
\end{aligned} \tag{G3}$$

2) $q_i = \sigma_i$

$$\begin{aligned}
\frac{\partial \bar{p}_f}{\partial \sigma_i} &= -\frac{1}{2}\left[\phi(-\beta^U)\frac{u_i^{*,U}}{\beta^U}\frac{\partial w}{\partial \sigma_i} + \phi(-\beta^L)\frac{u_i^{*,L}}{\beta^L}\frac{\partial w}{\partial \sigma_i}\right] \\
&= \frac{1}{2}\left[\phi(-\beta^U)\frac{u_i^{*,U}}{\beta^U}\frac{u_i^{*,U}}{\sigma_i} + \phi(-\beta^L)\frac{u_i^{*,L}}{\beta^L}\frac{u_i^{*,L}}{\sigma_i}\right] \\
&= \frac{1}{2}\left[\phi(-\beta^U)\frac{(u_i^{*,U})^2}{\beta^U \sigma_i} + \phi(-\beta^L)\frac{(u_i^{*,L})^2}{\beta^L \sigma_i}\right]
\end{aligned} \tag{G4}$$

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**VITA**

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