Curves in the Minkowski plane and Lorentzian surfaces

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Dedicated to

Anyone might benefit from this thesis.

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Abstract

We investigate in this thesis the generic properties of curves in the Minkowski plane $\mathbb{R}^2_1$ and of smooth Lorentzian surfaces. The generic properties of curves in $\mathbb{R}^2_1$ are obtained by studying the contacts of curves in $\mathbb{R}^2_1$ with lines and pseudo-circles. These contacts are captured by the singularities of the families of height and distance-squared functions on the curves. On the other hand, the generic properties of smooth Lorentzian surfaces are obtained by studying certain Binary Differential Equations defined on the surfaces.

Declaration

The work in this thesis is based on research carried out at the Pure Mathematics Group, the Department of Mathematical Sciences, University of Durham, England. No part of this thesis has been submitted elsewhere for any other degree or qualification and it all my own work unless referenced to the contrary in the text.

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Contents

Chapter 1

Introduction

We investigate in this thesis the generic properties of curves in the Minkowski plane $\mathbb{R}^2_1$ and of smooth Lorentzian surfaces. The generic properties of curves in $\mathbb{R}^2_1$ are obtained by studying the contacts of curves in $\mathbb{R}^2_1$ with lines and pseudo-circles. These contacts are captured by the singularities of the families of height and distance-squared functions on the curves. On the other hand, the generic properties of smooth Lorentzian surfaces are obtained by studying certain Binary Differential Equations defined on the surfaces.

Chapter 2 contains some preliminary results about the Minkowski plane $\mathbb{R}^2_1$ and curves in $\mathbb{R}^2_1$ which we use in Chapters 3, 4 and 5. We also recall some basic concepts about transversality to clarify the meaning of the word "generic" which is used extensively in the thesis.

In Chapter 3 we study the caustics of curves in $\mathbb{R}^2_1$. The caustic is the bifurcation set of the family of distance-squared functions on the curve. It coincides with the evolute of the curve away from the curve's lightlike points. We observe that the caustic is well defined at all points of the curve while the evolute is not well defined at lightlike points. We show that the caustic of the oval lies in the complement of the interior of the oval. We also prove that any simple and closed curve in $\mathbb{R}^2_1$ has at least four lightlike points.

In Chapter 4 we consider the parallels of curves in $\mathbb{R}^2_1$. We prove that the parallels at an ordinary vertex of a curve in $\mathbb{R}^2_1$ have a distinct configuration to that of the parallels at an ordinary vertex of a curve in the Euclidean plane. The parallels at an ordinary vertex of a curve in the Euclidean plane are as Figure 4.3(left) while they are as Figure 4.3(right) at an ordinary vertex of a curve in $\mathbb{R}^2_1$.

In Chapter 5 we define the Minkowski symmetry set (MSS) in an analogous way to the symmetry set (SS) of a curve in the Euclidean plane [25, 43]. We consider the geometry of MSS and deal in some details with the MSS of an ellipse in $\mathbb{R}^2_1$. The remaining chapters are about certain Binary Differential Equations (BDEs) on smooth Lorentzian surfaces. In Chapter 6 we gather all the results about the local singularities of codimension ≤ 1 of BDEs and the way they bifurcate in generic 1-parameter families of BDEs. We present them in a way that makes their identification more apparent.

In Chapter 7 we recall some basic notions of differential geometry of surfaces in the Euclidean and Lorentzian 3-space which can be found in books on elementary differential geometry, see for example [21]. These notions are generalised to Lorentzian surfaces [31] as shown in section 7.2.

In Chapter 8 we study natural 1-parameter families of BDEs associated to a self-adjoint operator $\mathbb{A}$ on a Lorentzian surface M. The $\mathbb{A}$-principal, $\mathbb{A}$-asymptotic and $\mathbb{A}$-characteristic curves on M, which are defined by BDEs, are associated to $\mathbb{A}$. These three pairs of foliations determine the natural 1-parameter families of $\mathbb{A}$-conjugate curve congruences denoted by $\mathcal{LC}^i_\alpha, i = 1, 2$ and reflected $\mathbb{A}$-conjugate curve congruences denoted by $\mathcal{LR}^i_\alpha, i = 1, 2$ [35]. The curves $\mathcal{LC}^i_\alpha$ and $\mathcal{LR}^i_\alpha$ are the solution curves of their BDEs and the parameter α is the oriented hyperbolic angle between certain directions. We start the chapter by recalling some basic properties of oriented hyperbolic angles and then proceeding to study the families $\mathcal{LC}^i_\alpha$ and $\mathcal{LR}^i_\alpha, i = 1, 2$.

In Chapter 9 we determine the local topological configurations of $\mathcal{LC}^i_\alpha$ and $\mathcal{LR}^i_\alpha, i = 1, 2$, for α fixed, and the way they bifurcate locally as α varies. The BDEs $\mathcal{LC}^i_\alpha$ and $\mathcal{LR}^i_\alpha, i = 1, 2$ determine either pairs of transverse foliations or none away from their discriminant curves. We study the local configurations of $\mathcal{LC}^i_\alpha$ and $\mathcal{LR}^i_\alpha, i = 1, 2$ at points on their discriminant curves. We examine the related conditions given in Chapter 6 for each singularity to occur.

In the Appendix we give examples of calculations carried out to prove some of the results in Chapters 8 and 9.

Chapter 2

Curves in the Minkowski plane and transversality

In this chapter we give some preliminary results about the Minkowski plane $\mathbb{R}^2_1$ and curves γ in $\mathbb{R}^2_1$. Away from lightlike points, we can define the curvature of γ in a similar way to the case of curves in the Euclidean plane. Despite the concept of curvature cannot be defined at lightlike points, the four-vertex theorem is proved for smooth closed curves in $\mathbb{R}^2_1$ [46]. In section 2.4 we recall the basic concepts about transversality and explain what is meant by the word "generic" which is used extensively in the thesis.

2.1 The Minkowski plane

The *Minkowski plane* $\mathbb{R}^2_1$ is the vector space $\mathbb{R}^2$ endowed with the pseudo-scalar product $\langle u, v\rangle = -u_0v_0 + u_1v_1$, for any $u = (u_0, u_1)$ and $v = (v_0, v_1)$ in $\mathbb{R}^2$. We say that a non-zero vector $u \in \mathbb{R}^2_1$ is *spacelike* if $\langle u, u\rangle > 0$, *lightlike* if $\langle u, u\rangle = 0$ and *timelike* if $\langle u, u\rangle < 0$. The norm of a vector $u \in \mathbb{R}^2_1$ is defined by $\|u\| = \sqrt{|\langle u, u\rangle|}$. We have the following pseudo-circles in $\mathbb{R}^2_1$ with centre $p \in \mathbb{R}^2_1$ and radius $r > 0$,

$$\begin{aligned} H^1(p, -r) &= \{u \in \mathbb{R}^2_1 \mid \langle u - p, u - p\rangle = -r^2\}, \\ S^1_1(p, r) &= \{u \in \mathbb{R}^2_1 \mid \langle u - p, u - p\rangle = r^2\}, \\ LC^*(p) &= \{u \in \mathbb{R}^2_1 \mid \langle u - p, u - p\rangle = 0\}. \end{aligned}$$

We denote by $H^1(-r)$, $S^1_1(r)$ and LC^* the pseudo-circles centred at the origin in $\mathbb{R}^2_1$.

Definition 2.1.1 Two vectors u, v in $\mathbb{R}^2_1$ are orthogonal if and only if $\langle u, v\rangle = 0$.

Theorem 2.1.1 [41] Let u and v be non-zero orthogonal vectors in $\mathbb{R}^2_1$. Then, either

(i) u is timelike and v is spacelike or vice-versa; or

(ii) u is lightlike and $v = \lambda u$, $\lambda \in \mathbb{R} - \{0\}$.

We denote by $u^\perp$ the vector given by $u^\perp = (u_1, u_0)$. Thus, $u^\perp$ is orthogonal to u. We have $u^\perp = \pm u$ if and only if u is lightlike.

2.2 Properties of parametrised curves in $\mathbb{R}^2_1$

In this section we provide the preliminary material to study curves in $\mathbb{R}^2_1$. Some of the material is similar to the case of curves in the Euclidean plane.

Definition 2.2.1 A curve in the Minkowski plane is a C^∞-map $\gamma : I \to \mathbb{R}^2_1$ of an open interval I of the real line $\mathbb{R}$ into $\mathbb{R}^2_1$. The trace of γ is the set of points $\gamma(I)$, and we still refer to it as the curve γ.

The variable t is called the *parameter* of the curve γ. If we write $\gamma(t) = (X(t), Y(t))$, then the vector $(X'(t), Y'(t)) = \gamma'(t) \in \mathbb{R}^2_1$ is called the *tangent vector* to the curve γ at t.

Definition 2.2.2 A curve $\gamma : I \to \mathbb{R}^2_1$ is said to be *regular* if $\gamma'(t) \neq 0$ for all $t \in I$.

The point $\gamma(t)$ is called *spacelike*, *timelike* or *lightlike* if the tangent vector $\gamma'(t)$ is spacelike, timelike or lightlike respectively. At lightlike points of γ, the lightlike tangent vector $\gamma'(t)$ is orthogonal to itself.

Let γ be a spacelike or a timelike curve (i.e., $\gamma'(t)$ is spacelike or timelike for all $t \in I$). The *arc length* from a point t_0 is, by the definition,

$$s = l(t) = \int_{t_0}^{t} \|\gamma'(t)\| dt.$$

Since $\gamma'(t) \neq 0$, the arc length $s = l(t)$ is a differentiable function of t and $\frac{dl}{dt} = \|\gamma'(t)\|$. The function $s = l(t)$ has an inverse $t = l^{-1}(s), s \in l(I) = J \subset \mathbb{R}$.

Away from lightlike points, the curve γ can always be parametrised locally by arc length s by taking $\gamma(l^{-1}(s))$ as a new parametrisation of the curve. If γ is parametrised by arc length, then $\|\gamma'(s)\| = 1$.

Let $\gamma(s) = (X(s), Y(s))$ be parametrised by arc length. Then the unit tangent vector is $\gamma'(s) = \mathbf{t}(s) = (X'(s), Y'(s))$, and we have $\langle \mathbf{t}(s), \mathbf{t}(s)\rangle = (-1)^\beta$, where $\beta = 1$ if $\gamma(s)$ is timelike and $\beta = 2$ if $\gamma(s)$ is spacelike.

We define the unit normal vector $\mathbf{n}(s)$ as the vector obtained from the Euclidean reflection of $\mathbf{t}(s)$ anticlockwise with respect to one of the lightlike lines of LC^* as follows. If $\mathbf{t}(s)$ is spacelike, then the timelike vector $\mathbf{n}(s) = (Y'(s), X'(s))$ is the Euclidean reflection of $\mathbf{t}(s)$ with respect to $u_1 = u_2$. If $\mathbf{t}(s)$ is timelike, then the spacelike vector $\mathbf{n}(s) = (-Y'(s), -X'(s))$ is the Euclidean reflection with respect to $u_1 = -u_2$. In short,

$$\mathbf{n}(s) = (-1)^\beta (Y'(s), X'(s)),$$

with β is as above. The unit normal vector $\mathbf{n}(s)$ satisfies $\langle \mathbf{n}(s), \mathbf{n}(s)\rangle = (-1)^{\beta+1}$ and $\langle \mathbf{t}(s), \mathbf{n}(s)\rangle = 0$. Thus we can associate an orthonormal frame $\{\mathbf{t}(s), \mathbf{n}(s)\}$ of $\mathbb{R}^2_1$ along the spacelike or timelike curve γ.

Away from lightlike points, the orthonormal basis $\{\mathbf{t}(s), \mathbf{n}(s)\}$ associated to each point of the curve $\gamma(s)$ varies when we move along the curve. This change can be described in terms of the derivatives $\mathbf{t}'(s)$ and $\mathbf{n}'(s)$. Differentiating the following equations

$$\langle \mathbf{t}(s), \mathbf{t}(s)\rangle = (-1)^\beta, \ \langle \mathbf{n}(s), \mathbf{n}(s)\rangle = (-1)^{\beta+1}, \text{ and } \ \langle \mathbf{t}(s), \mathbf{n}(s)\rangle = 0, \tag{2.1}$$

with respect to s gives

$$\langle \mathbf{t}'(s), \mathbf{t}(s)\rangle = \langle \mathbf{n}'(s), \mathbf{n}(s)\rangle = 0 \text{ and } \langle \mathbf{t}'(s), \mathbf{n}(s)\rangle + \langle \mathbf{t}(s), \mathbf{n}'(s)\rangle = 0. \tag{2.2}$$

The first two equations in (2.2) gives that $\mathbf{t}'(s)$ is parallel to $\mathbf{n}(s)$ and $\mathbf{n}'(s)$ is parallel to $\mathbf{t}(s)$, so there exist real numbers $\kappa(s)$ and $\omega(s)$ such that

$$\mathbf{t}'(s) = \kappa(s)\mathbf{n}(s) \text{ and } \mathbf{n}'(s) = \omega(s)\mathbf{t}(s). \tag{2.3}$$

Substituting equations (2.3) in the last equation in (2.2) gives

$$(-1)^{\beta+1}\kappa(s) + (-1)^{\beta}\omega(s) = 0,$$

which implies that $\omega(s) = \kappa(s)$. Therefore,

$$\mathbf{t}'(s) = \kappa(s)\mathbf{n}(s), \tag{2.4}$$

and

$$\mathbf{n}'(s) = \kappa(s)\mathbf{t}(s), \tag{2.5}$$

which are the *Frenet equations* of the spacelike or timelike curve γ in $\mathbb{R}^2_1$. We also have

$$\kappa(s) = (-1)^{\beta+1}\langle \mathbf{t}'(s), \mathbf{n}(s)\rangle. \tag{2.6}$$

The function $\kappa : J \to \mathbb{R}$ is differentiable. The real number $\kappa(s)$ is called the *curvature of the curve* γ at s. We have $\kappa(s) = \pm\|\mathbf{t}'(s)\|$, and from equation (2.6) we get

$$\kappa(s) = X'(s)Y''(s) - X''(s)Y'(s). \tag{2.7}$$

Let t be an arbitrary parameter of a spacelike or timelike regular curve $\gamma(t) = (X(t), Y(t))$. Then, we have

$$\gamma'(t) = \frac{d\gamma}{dt} = \frac{d\gamma}{ds}\frac{ds}{dt} = \mathbf{t}(s)\|\frac{d\gamma}{dt}\| = \mathbf{t}(s)\|\gamma'(t)\|,$$

which implies

$$\mathbf{t}(s) = \frac{\gamma'(t)}{\|\gamma'(t)\|}.$$

Then $\mathbf{t}(t) = \frac{\gamma'(t)}{\|\gamma'(t)\|}$ is the unit tangent vector to γ at t. Since $\|\gamma'(t)\| > 0$, $\mathbf{t}(t)$ has the same direction of $\mathbf{t}(s)$. The unit normal vector of γ at t is given by

$$\mathbf{n}(t) = (-1)^{\beta}\frac{(Y'(t), X'(t))}{\|\gamma'(t)\|}.$$

We also have

$$\gamma''(t) = \frac{d^2\gamma}{dt^2} = \frac{d^2\gamma}{ds^2}\frac{ds^2}{dt^2} = \gamma''(s)\|\gamma'(t)\|^2. \tag{2.8}$$

Since $\gamma(s)$ is parametrised by arc length, so $\mathbf{t}'(s) = \gamma''(s)$ is the curvature times the unit normal. Taking the pseudo-scalar product for the both sides of equation (2.8) with $\mathbf{n}(t)$ gives

$$\langle \gamma''(t), \mathbf{n}(t)\rangle = (-1)^{\beta+1}\kappa(t)\|\gamma'(t)\|^2.$$

Therefore,

$$\kappa = \frac{X'Y'' - X''Y'}{(|X'^2 - Y'^2|)^{\frac{3}{2}}}.$$

Definition 2.2.3 A non-lightlike point $\gamma(t_0)$ is a *vertex* of γ if $\kappa(t_0) \neq 0$ and $\kappa'(t_0) = 0$. It is an *ordinary vertex* if $\kappa'(t_0) = 0$ and $\kappa''(t_0) \neq 0$. It is called an *inflection* point if $\kappa(t_0) = 0$.

Remark 2.2.1 The curvature of a curve γ in $\mathbb{R}^2_1$ is undefined at the lightlike points of γ. For instance, if $\gamma(t_0)$ is an isolated lightlike point and $\gamma''(t_0)$ is not parallel to $\gamma'(t_0)$, then $\langle \gamma''(t_0), \gamma'(t_0)^{\perp} \rangle \neq 0$ and the curvature at points on the spacelike and timelike components of γ tends to infinity as t tends to t_0.

Definition 2.2.4 A curve $\gamma : [a, b] \to \mathbb{R}^2_1$ is *closed* if $\gamma(a) = \gamma(b)$ and all its derivatives agree at a and b. The curve γ is *simple* if it has no self-intersection; that is, if $t_1, t_2 \in (a, b), t_1 \neq t_2$, then $\gamma(t_1) \neq \gamma(t_2)$.

If γ is a simple closed curve in $\mathbb{R}^2_1$, then by Jordan curve theorem, $\mathbb{R}^2_1 - \gamma$ has exactly two connected components with common boundary γ. One of them is bounded by γ and called the *interior of* γ*,* and the other is non-bounded and called the *exterior of* γ*.* Away from lightlike points, we call $\mathbf{n}(s)$ the *inward-pointing* unit normal at $\gamma(s)$ if it points to the interior of γ, and *outward- pointing* if it points to the exterior of γ. The curve γ is *positively oriented* if $\mathbf{n}(s)$ is an inward-pointing for all $s \in [a, b]$, and *negatively oriented* if $\mathbf{n}(s)$ is an outward-pointing for all $s \in [a, b]$.

2.3 The four-vertex theorem in $\mathbb{R}^2_1$

Tabachinkov in [46] generalized the classical four-vertex theorem which states that a simple, closed, and convex plane curve has at least four vertices. Tabachinkov considered a smooth closed strictly convex parametrised curve $\gamma(t)$ in the oriented affine plane. The acceleration vectors $\gamma''(t)$ generate a smooth line field $l(t)$ along the curve. Assume that these lines rotate in the same sense along γ; which means that γ is strictly convex. We have

Theorem 2.3.1 [46] For a generic curve $\gamma(t)$, the envelope of the 1-parameter family of the lines $l(t)$ has at least four cusp singularities.

Away from lightlike points, if $\gamma(t)$ is a strictly convex curve in the arc length parametrisation, then the lines $l(t)$ are perpendicular to γ and their envelope is the caustic of the curve. The singularities of the caustic correspond to the vertices of the curve. Tabachinkov showed that the statement of Theorem 2.3.1 holds without considering the convexity assumption.

2.4 Transversality

For references for this section see [11, 27, 32].

Definition 2.4.1 Let X and Y be smooth manifolds and $f : X \to Y$ be a smooth (C^∞) mapping. Let W be a submanifold of Y and x a point in X. Then f intersects W transversally at x (denoted by $f \pitchfork W$ at x) if either

(i) $f(x) \notin W$, or

(ii) $f(x) \in W$ and $T_{f(x)}Y = T_{f(x)}W + (df)_x(T_xX)$.

In addition, f intersects W transversally if $f \pitchfork W$ for every $x \in X$.

Proposition 2.4.1 Let X and Y be smooth manifolds, $W \subset Y$ a submanifold. Suppose dim $X <$ codim W. Let $f : X \to Y$ be smooth and suppose that $f \pitchfork W$. Then $f(X) \cap W = \phi$.

Theorem 2.4.1 Let X and Y be smooth manifolds, $W \subset Y$ a submanifold. Let $f : X \to Y$ be smooth and suppose that $f \pitchfork W$. Then $f^{-1}(W)$ is a submanifold of X, and codim $f^{-1}(W) =$ codim W. In particular, if dim $X =$ codim W, then $f^{-1}(W)$ consists only of isolated points.

Denote by $J^k(m,n)$ ($k \geq 1$) the vector space of polynomial mappings of degree $\leq k$ from $\mathbb{R}^m$ to $\mathbb{R}^n$, without a constant term. Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a smooth mapping. For every integer $k \geq 1$, there is a smooth mapping

$$j^k f : \mathbb{R}^m \to J^k(m,n),$$

defined by $j^k f(x)$= the degree k Taylor polynomial of $f - f(x)$.

This polynomial is called the *k-jet* of f at x, and $J^k(m,n)$ is called the *space of k-jets from* $\mathbb{R}^m$ *to* $\mathbb{R}^n$.

Definition 2.4.2 Let F be a topological space. Then

(i) A subset G of F is *residual* if it is the countable intersection of open and dense subsets of F.

(ii) F is a *Baire space* if every residual subset is dense.

Theorem 2.4.2 (Thom's Transversality Theorem) Let X and Y be smooth manifolds and W be a submanifold of $J^k(X,Y)$. Let

$$T_W = \{f \in C^\infty(X,Y) : j^k f \pitchfork W\}.$$

Then T_W is a residual subset of $C^\infty(X,Y)$ endowed with the Whitney C^∞-topology.

Since $C^\infty(X,Y)$ is a Baire space, then T_W is a dense subset of $C^\infty(X,Y)$. If W is closed, then T_W is an open and dense subset of $C^\infty(X,Y)$, see [27].

Definition 2.4.3 The property P in $C^\infty(X,Y)$ is said to be *generic* if it holds in a residual subset of $C^\infty(X,Y)$.

The property $j^k f \pitchfork W$ (i.e., $f \in T_W$) is generic for $C^\infty(X,Y)$ for any submanifold W of $J^k(X,Y)$.

Theorem 2.4.3 Let $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^n$ be smooth manifolds and U an open set in $\mathbb{R}^t$ with $G : X \times U \to \mathbb{R}^n$ a smooth map transverse to Y. Then for almost all $a \in U$ the maps $G_a : X \to \mathbb{R}^n$ given by $G_a(x) = G(x,a)$ are transverse to Y.

Chapter 3

Caustics and evolutes of curves in the Minkowski plane

We study in this chapter properties of curves γ in the Minkowski plane $\mathbb{R}^2_1$ captured by contact with lines and pseudo-circles. These contacts are measured by the singularity of the families of height and distance-squared functions. The case of curves in $\mathbb{R}^2_1$ is slightly different from the case of curves in the Euclidean plane: firstly, there are three types of pseudo-circles in $\mathbb{R}^2_1$ and secondly there are the lightlike points on γ where, for instance, the curvature is not defined. The results of this chapter are part of the paper [42].

3.1 Lightlike points and contact with lines

We consider embeddings $\gamma : S^1 \to \mathbb{R}^2_1$. The set $Emb(S^1, \mathbb{R}^2_1)$ of such embeddings is endowed with the Whitney C^∞-topology. We say that a property is *generic* if it is satisfied by curves in an open and dense subset of $Emb(S^1, \mathbb{R}^2_1)$. If we consider $\gamma : I \to \mathbb{R}^2_1$, where I is an open interval in $\mathbb{R}$, then a property is said to be *generic* if it is satisfied by curves in a residual subset of $Emb(I, \mathbb{R}^2_1)$. A curve that satisfies a generic property is called a generic curve.

Let $\gamma \in Emb(I, \mathbb{R}^2_1)$. We recall that γ is spacelike (resp. timelike) if $\gamma'(t)$ is a spacelike (resp. timelike) vector for all $t \in I$. A point $\gamma(t)$ is called a lightlike if $\gamma'(t)$ is a lightlike vector.

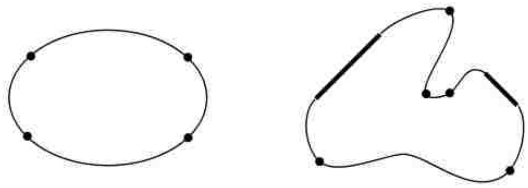


Figure 3.1: Examples of closed curves with lightlike points (dots and thick lines). The ellipse on the *left* has exactly four lightlike points and the curve on the *right* has two line segments of lightlike points and other isolated lightlike points.

Proposition 3.1.1 The set of lightlike points of a curve $\gamma \in Emb(S^1, \mathbb{R}^2_1)$ is the union of at least four disjoint non-empty and closed subsets of γ (see Figure 3.1).

Proof: The lightlike points are those where the tangent line to γ is parallel to $(\pm 1, 1)$. We change the metric in $\mathbb{R}^2$ and consider γ as a curve $\tilde{\gamma}$ in the Euclidean plane $\mathbb{R}^2$. Since $\tilde{\gamma}$ is closed, the image of its Gauss map $N : \tilde{\gamma} \to S^1$ is the whole unit circle S^1. The pre-images of the points $\frac{\sqrt{2}}{2}(\pm 1, \pm 1)$ by N have tangent lines parallel to $(\pm 1, 1)$, i.e., they are lightlike points on γ. It follows by the fact that N is a continuous map that the set of lightlike points of γ is the union of at least four disjoint non-empty and closed subsets of γ. $\Box$

Remark 3.1.1 If $\gamma : I \to \mathbb{R}^2_1$, then γ either have lightlike points or not. For example, the pseudo-circles $H^1(p, -r)$ and $S^1_1(p, r)$ where $r > 0$ are open curves with no lightlike points. On the other hand, the parabola (t, t^2), $t \in \mathbb{R}^2$ has two lightlike points.

We apply tools from singularity theory to obtain geometric information about curves in $\mathbb{R}^2_1$. Given a smooth (i.e., C^∞) function $f : J \to \mathbb{R}$ ($J = I$ or S^1), we say that f is singular at $t_0 \in J$ if $f'(t_0) = 0$. We consider the $\mathcal{R}$-singularities of f at $t_0 \in J$, where $\mathcal{R}$ is the group of local changes of parameters in the source that fix t_0. The models for the local $\mathcal{R}$-singularities of functions are $\pm(t - t_0)^{k+1}$, $k \geq 1$, and these are labelled A_k-singularities. The necessary and sufficient conditions for a function f to have an A_k-singularity at t_0 are

$$f'(t_0) = f''(t_0) = \ldots = f^{(k)}(t_0) = 0,\ f^{(k+1)}(t_0) \neq 0.$$

The only stable singularity (ignoring the constant terms) is $\pm(t-t_0)^2$, i.e., the A_1-singularity. (See [11] for more on singularities of functions and their applications to the geometry of curves in the Euclidean plane.)

The contact of a curve $\gamma \in Emb(J, \mathbb{R}^2_1)$ ($J = I$ or S^1) with lines is captured by the singularities of the family of height functions on γ. Let v be a non-zero vector in $\mathbb{R}^2_1$ and consider the parallel lines

$$L^v_c = \{p \in \mathbb{R}^2_1 \,|\, \langle p, v\rangle = c\},$$

with $c \in \mathbb{R}$, which are (pseudo)-orthogonal to v. The contact of γ with the lines L^v_c is measured by the singularities of the height function $h_v : J \to \mathbb{R}$, given by

$$h_v(t) = \langle \gamma(t), v\rangle.$$

An important observation is that the function h_v is defined for all non-zero vectors v including lightlike vectors, and at all points on γ including its lightlike points.

We say that the curve γ has an A_k-contact (resp. $A_{\geq k}$-contact) with L^v_c at $\gamma(t_0) \in L^v_c$ if h_v has an A_k (resp. A_l, $l \geq k$)-singularity at t_0. Thus, the contact of the curve γ with L^v_c at $\gamma(t_0) \in L^v_c$ is of type

A_1 if and only if $v = \lambda\gamma'(t_0)^\perp$ and $\langle \gamma''(t_0), \gamma'(t_0)^\perp\rangle \neq 0$;

A_2 if and only if $v = \lambda\gamma'(t_0)^\perp$, $\langle \gamma''(t_0), \gamma'(t_0)^\perp\rangle = 0$ and $\langle \gamma'''(t_0), \gamma'(t_0)^\perp\rangle \neq 0$;

$A_{\geq 2}$ if $v = \lambda\gamma'(t_0)^\perp$ and $\langle \gamma''(t_0), \gamma'(t_0)^\perp\rangle = 0$.

It follows that γ has an $A_{\geq 1}$-contact with L^v_c at $\gamma(t_0) \in L^v_c$ if and only if L^v_c is the tangent line to γ at $\gamma(t_0)$.

We call a point $\gamma(t_0)$ where γ has an A_2-contact with its tangent line an (ordinary) *inflection* point if $\gamma(t_0)$ is not a lightlike point and a *lightlike inflection point* if $\gamma(t_0)$ is a lightlike point. At such points the curve γ lies on both sides of its tangent line.

3.2 The evolutes of spacelike and timelike curves

Let $\gamma : I \to \mathbb{R}^2_1$ be a spacelike or a timelike curve and suppose that it is parametrised by arc length (i.e., $\|\gamma'(s)\| = 1$ for all $s \in I$). Recall that $\mathbf{t}(s)$ is the unit tangent vector to γ and $\mathbf{n}(s)$ is the unit normal vector to γ such that $\{\mathbf{t}(s), \mathbf{n}(s)\}$ is oriented anticlockwise. The vector $\mathbf{n}(s)$ is timelike (resp. spacelike) if γ is spacelike (resp. timelike).

The *evolute* of γ, with its inflection points removed, is defined as the curve in $\mathbb{R}^2_1$ given by

$$e(t) = \gamma(t) - \frac{1}{\kappa(t)}\mathbf{n}(t).$$

We have the following elementary result.

Proposition 3.2.1 (i) The evolute of a spacelike (resp. timelike) curve is a timelike (resp. spacelike) curve.

(ii) The evolute of γ is singular at precisely the vertices of γ.

Proof: We suppose that γ is parametrised by arc length. Then,

$$e'(s) = \frac{\kappa'(s)}{\kappa^2(s)}\mathbf{n}(s)$$

and the proof follows from the fact that the vectors $\mathbf{t}(s)$ and $\mathbf{n}(s)$ are of different types (one is spacelike while the other is timelike or vice-versa). □

Proposition 3.2.2 Let $\gamma : I \to \mathbb{R}^2_1$ be a connected spacelike or timelike curve. Then γ does not intersect its evolute e.

Proof: Suppose that γ intersects its evolute e. Then, there exist $t_1, t_2 \in I$, with $t_1 \neq t_2$ (and assume for simplicity that $t_1 < t_2$), such that

$$\gamma(t_1) - \frac{1}{\kappa(t_1)}\mathbf{n}(t_1) = \gamma(t_2).$$

It follows that

$$\gamma(t_1) - \gamma(t_2) = \frac{1}{\kappa(t_1)}\mathbf{n}(t_1).$$

But there exists $t_3 \in (t_1, t_2)$ such that $\gamma(t_1) - \gamma(t_2)$ is parallel to $\mathbf{t}(t_3)$. This is a contradiction as $\mathbf{t}(t_3)$ and $\mathbf{n}(t_1)$ are of different types. Therefore, γ cannot intersect its evolute. □

3.3 Caustics of curves in $\mathbb{R}^2_1$

We consider a curve $\gamma \in Emb(S^1, \mathbb{R}^2_1)$. To study the local properties of γ at $\gamma(t_0)$, we consider the germ $\gamma : \mathbb{R}, t_0 \to \mathbb{R}^2_1$ of γ at t_0.

The family of distance-squared functions $f : S^1 \times \mathbb{R}^2_1 \to \mathbb{R}$ on γ is given by

$$f(t, v) = \langle \gamma(t) - v, \gamma(t) - v \rangle.$$

We denote by $f_v : S^1 \to \mathbb{R}^2_1$ the function given by $f_v(t) = f(t, v)$. The $\mathcal{R}$-singularity type of f_v at t_0 measures the contact of γ at $\gamma(t_0)$ with the pseudo-circle of centre v and radius $||\gamma(t_0) - v||$. The type of the pseudo-circle is determined by the sign of $\langle \gamma(t_0) - v, \gamma(t_0) - v \rangle$.

The catastrophe set of f is defined by

$$\Sigma(f) = \{(t, v) \in S^1 \times \mathbb{R}^2_1 \,|\, f_v'(t) = 0\}.$$

We also define

$$Bif(f) = \{v \in \mathbb{R}^2_1 \,|\, \exists (t, v) \in \Sigma(f) \text{ such that } f_v''(t) = 0\}.$$

The set $Bif(f)$ is the local stratum of the *bifurcation set* of the family f, i.e., it is the set of points $v \in \mathbb{R}^2_1$ for which there exists $t \in S^1$ such that f_v has a degenerate (non-stable) singularity at t, i.e., a singularity of type $A_{\geq 2}$.

The function $g(t, v) = f_v'(t) = 2\langle \gamma(t) - v, \gamma'(t) \rangle$ is not singular at any point in $\Sigma(f)$. Indeed, if we write $\gamma(t) = (x(t), y(t))$, then the gradient of g is a multiple of

$$(\langle \gamma(t) - v, \gamma''(t) \rangle + \langle \gamma'(t), \gamma'(t) \rangle, x'(t), -y'(t))$$

and is never a zero vector as γ is a regular curve. Therefore, $\Sigma(f)$ is a smooth and regular 2-dimensional submanifold of $S^1 \times \mathbb{R}^2_1$ and the family f is a generating family (see [4] for terminology). We write $v = (v_0, v_1)$ and denote by $T^*\mathbb{R}^2_1$ the cotangent bundle of $\mathbb{R}^2_1$ endowed with the canonical symplectic structure (which is metric independent). We denote by $\pi : T^*\mathbb{R}^2_1 \to \mathbb{R}^2_1$ the canonical projection. Then, the map $L(f) : \Sigma(f) \to T^*\mathbb{R}^2_1$, given by

$$L(f)(t, v) = (v, (\frac{\partial f}{\partial v_0}(t, v), \frac{\partial f}{\partial v_1}(t, v))),$$

is a Lagrangian immersion, so the map $\pi \circ L(f) : \Sigma(f) \to \mathbb{R}^2_1$ given by $(t, v) \to v$ is a Lagrangian map.

The *caustic* $C(\gamma)$ of γ is the set of critical values of the Lagrangian map $\pi \circ L(f)$, and is precisely $Bif(f)$ (see [4] for details). It follows that for a generic curve γ, the caustic $C(\gamma)$ is locally either a regular curve or has a cusp singularity. The local models of the caustic at v corresponding to $t \in S^1$ depend on the $\mathcal{R}$-singularity type of f_v at t. For a generic γ, f_v has local singularities of type A_1, A_2 or A_3. The caustic is the empty set at an A_1-singularity of f_v. It is a regular curve at an A_2-singularity of f_v and has a cusp singularity at an A_3-singularity of f_v.

We can obtain a parametrisation of the caustic as follows. We have $f_v(t) = \langle \gamma(t) - v, \gamma(t) - v \rangle$, so

$$\frac{1}{2} f_v'(t) = \langle \gamma(t) - v, \gamma'(t) \rangle.$$

It follows that f_v is singular at t if and only if $\langle \gamma(t) - v, \gamma'(t) \rangle = 0$, equivalently, if and only if $\gamma(t) - v = \mu \gamma'(t)^{\perp}$ for some scalar μ. (This condition includes the lightlike points of γ where $\gamma'(t)^{\perp}$ is parallel to $\gamma'(t)$.)

Differentiating again we get

$$\begin{aligned} \tfrac{1}{2} f_v''(t) &= \langle \gamma(t) - v, \gamma''(t) \rangle + \langle \gamma'(t), \gamma'(t) \rangle \\ &= \mu \langle \gamma'(t)^{\perp}, \gamma''(t) \rangle + \langle \gamma'(t), \gamma'(t) \rangle. \end{aligned}$$

The singularity of f_v at $\gamma(t)$ is degenerate if and only if $f_v'(t) = f_v''(t) = 0$, equivalently, if and only if $\gamma(t) - v = \mu \gamma'(t)$ and

$$\mu \langle \gamma'(t)^{\perp}, \gamma''(t) \rangle + \langle \gamma'(t), \gamma'(t) \rangle = 0. \tag{3.1}$$

It follows that the caustic of γ is given by

$$C(\gamma) = \{\gamma(t) - \mu \gamma'(t)^{\perp} \,|\, t \in S^1 \text{ and } \mu \text{ is a solution of equation (3.1)}\}.$$

Away from the lightlike points of γ, we can write $\gamma(t) - v = \lambda \mathbf{n}(t)$, where $\lambda = \mu ||\gamma'(t)||$ and $\mathbf{n}(t) = (-1)^{\beta} \gamma'(t)^{\perp} / ||\gamma'(t)||$ is the unit normal vector ($\beta = 1$ if $\gamma(t)$ is timelike and $\beta = 2$ if it is spacelike). Then a singularity of f_v is degenerate if and only if

$$v = \gamma(t) - \frac{1}{\kappa(t)} \mathbf{n}(t). \tag{3.2}$$

This is precisely the evolute of the spacelike and timelike components of γ. As in the case of curves in the Euclidean plane, the evolute of γ (minus its lightlike points) is the locus of its centres of curvature. It is a subset of the caustic, which is the locus of centres of "*osculating*" pseudo-circles (i.e., pseudo-circles that have an $A_{\geq 2}$-contact with γ).

We define the subset Ω of $Emb(S^1, \mathbb{R}^2_1)$ such that a curve γ is in Ω if and only if $\langle \gamma''(t), \gamma'(t) \rangle \neq 0$ whenever $\langle \gamma'(t), \gamma'(t) \rangle = 0$ (i.e., the lightlike points of $\gamma \in \Omega$ are not lightlike inflection points). One can show, using Thom's transversality results (see for example Chapter 9 in [11] for an analogous proof), that Ω is an open dense subset of $Emb(S^1, \mathbb{R}^2_1)$.

Proposition 3.3.1 Let $\gamma \in \Omega$. Then,

(i) the lightlike points of γ are isolated points;

(ii) the caustic of γ is a regular curve at a lightlike point of γ and has ordinary tangency with γ at such point. Furthermore, γ and its caustic lie locally on opposite sides of their common tangent line at the lightlike point.

Proof: (i) Since the curve γ is in Ω, we have $g'(t) = 2\langle \gamma''(t), \gamma'(t) \rangle \neq 0$ whenever $g(t) = \langle \gamma'(t), \gamma'(t) \rangle = 0$. This implies that the lightlike points, given by $g(t) = 0$, are isolated points.

(ii) For $\gamma \in \Omega$, we can solve equation (3.1) at a lightlike point $\gamma(t_0)$ to get

$$\mu(t) = -\frac{\langle \gamma'(t), \gamma'(t) \rangle}{\langle \gamma'(t)^\perp, \gamma''(t) \rangle}$$

for t near t_0. Then, $\mu(t_0) = 0$ and the caustic $C(\gamma)$ is parametrised locally at t_0 by

$$c(t) = \gamma(t) - \mu(t)\gamma'(t)^\perp.$$

We have

$$\mu'(t) = -\frac{2\langle \gamma'(t), \gamma''(t) \rangle}{\langle \gamma'(t)^\perp, \gamma''(t) \rangle} - \left(\frac{1}{\langle \gamma'(t)^\perp, \gamma''(t) \rangle}\right)' \langle \gamma'(t), \gamma'(t) \rangle$$

and

$$\begin{aligned} \mu''(t) \quad = \quad & -\frac{2\langle \gamma'(t), \gamma'''(t) \rangle}{\langle \gamma'(t)^\perp, \gamma''(t) \rangle} - \frac{2\langle \gamma''(t), \gamma''(t) \rangle}{\langle \gamma'(t)^\perp, \gamma''(t) \rangle} - 4\left(\frac{1}{\langle \gamma'(t)^\perp, \gamma''(t) \rangle}\right)' \langle \gamma'(t), \gamma''(t) \rangle \\ & -\left(\frac{1}{\langle \gamma'(t)^\perp, \gamma''(t) \rangle}\right)'' \langle \gamma'(t), \gamma'(t) \rangle. \end{aligned}$$

At the lightlike point $\gamma(t_0)$ we have $\gamma'(t_0)^\perp = (-1)^\epsilon \gamma'(t_0)$, where $\epsilon = 2$ if $\gamma'(t_0) = (\lambda, \lambda)$ and $\epsilon = 1$ if $\gamma'(t_0) = (-\lambda, \lambda)$. Thus,

$$\mu'(t_0) = -\frac{2(-1)^\epsilon \langle \gamma'(t_0), \gamma''(t_0)\rangle}{\langle \gamma'(t_0), \gamma''(t_0)\rangle} = -2(-1)^\epsilon$$

and

$$\begin{aligned}
\mu''(t_0) &= -\frac{2(-1)^\epsilon \langle \gamma'(t_0), \gamma'''(t_0)\rangle}{\langle \gamma'(t_0), \gamma''(t_0)\rangle} - \frac{2(-1)^\epsilon \langle \gamma''(t_0), \gamma''(t_0)\rangle}{\langle \gamma'(t_0), \gamma''(t_0)\rangle} \\
&\quad -4\left(\frac{1}{\langle \gamma'(t)^\perp, \gamma''(t)\rangle}\right)'|_{t=t_0} \langle \gamma'(t_0), \gamma''(t_0)\rangle \\
&= \frac{2(-1)^\epsilon \langle \gamma'(t_0), \gamma'''(t_0)\rangle}{\langle \gamma'(t_0), \gamma''(t_0)\rangle} - \frac{2(-1)^\epsilon \langle \gamma''(t_0), \gamma''(t_0)\rangle}{\langle \gamma'(t_0), \gamma''(t_0)\rangle}.
\end{aligned}$$

It follows now that

$$\begin{aligned}
c'(t_0) &= \gamma'(t_0) - \mu'(t_0)\gamma'(t_0)^\perp - \mu(t_0)\gamma''(t_0)^\perp \\
&= \gamma'(t_0) + 2(-1)^\epsilon(-1)^\epsilon \gamma'(t_0) \\
&= 3\gamma'(t_0).
\end{aligned}$$

Therefore, γ and $C(\gamma)$ are tangential at $\gamma(t_0)$. Differentiating again, we get

$$\begin{aligned}
c''(t_0) &= \gamma''(t_0) - 2\mu'(t_0)\gamma''(t_0)^\perp - \mu''(t_0)\gamma'(t_0)^\perp - \mu(t_0)\gamma'''(t_0)^\perp \\
&= \gamma''(t_0) + 4(-1)^\epsilon \gamma''(t_0)^\perp - 2\left(\frac{\langle \gamma'(t_0), \gamma'''(t_0)\rangle}{\langle \gamma'(t_0), \gamma''(t_0)\rangle} - \frac{\langle \gamma''(t_0), \gamma''(t_0)\rangle}{\langle \gamma'(t_0), \gamma''(t_0)\rangle}\right)\gamma'(t_0).
\end{aligned}$$

We can take $\{\gamma'(t_0), \gamma''(t_0)\}$ as a system of coordinate of $\mathbb{R}^2_1$ at $\gamma(t_0)$. Then, we can write $c''(t_0) = \alpha\gamma'(t_0) + w\gamma''(t_0)$ with

$$\begin{aligned}
w &= \frac{\langle c''(t_0), \gamma'(t_0)\rangle}{\langle \gamma'(t_0), \gamma''(t_0)\rangle} \\
&= \frac{\langle \gamma''(t_0) + 4(-1)^\epsilon \gamma''(t_0)^\perp, \gamma'(t_0)\rangle}{\langle \gamma'(t_0), \gamma''(t_0)\rangle} \\
&= 1 - 4(-1)^\epsilon \frac{\langle \gamma''(t_0), \gamma'(t_0)^\perp\rangle}{\langle \gamma'(t_0), \gamma''(t_0)\rangle} \\
&= 1 - 4(-1)^\epsilon(-1)^\epsilon \frac{\langle \gamma''(t_0), \gamma'(t_0)\rangle}{\langle \gamma'(t_0), \gamma''(t_0)\rangle} \\
&= -3.
\end{aligned}$$

We have then

$$\gamma(t) - \gamma(t_0) = ((t-t_0) + h.o.t)\gamma'(t_0) + (\frac{1}{2}(t-t_0)^2 + h.o.t)\gamma''(t_0)$$

and

$$c(t) - c(t_0) = c(t) - \gamma(t_0) = (3(t-t_0) + h.o.t)\gamma'(t_0) + (-\frac{3}{2}(t-t_0)^2 + h.o.t)\gamma''(t_0).$$

This shows that γ and its caustic have an ordinary tangency at $\gamma(t_0)$ and that the two curves lie on opposite sides of their common tangent line at $\gamma(t_0)$. $\square$

Remark 3.3.1 At a lightlike inflection point $\gamma(t_0)$ of a curve $\gamma \notin \Omega$, the tangent line to γ is always a component of the caustic $C(\gamma)$ (any $\mu \in \mathbb{R}$ is a solution of equation (3.1) at a lightlike inflection point). The caustic has another component if and only if $\mathrm{ord}(\langle \gamma'(t), \gamma'(t)\rangle) \geq \mathrm{ord}(\langle \gamma'(t)^\perp, \gamma''(t)\rangle)$ at $t = t_0$. Then equation (3.1) can be solved for μ and we obtain a parametrisation of this other component of $C(\gamma)$. This component, which could be singular, passes through $\gamma(t_0)$ if and only if $\mu(t_0) = 0$.

We consider now some special curves in $\mathbb{R}^2_1$. An oval in the Euclidean plane is defined as a closed and simple curve with everywhere non-vanishing curvature. The curvature of a curve in $\mathbb{R}^2_1$ is not defined at the lightlike points of the curve. However, we can still define the concept of an oval in $\mathbb{R}^2_1$ using the contact of the curve with lines. We say that a closed and simple curve in $\mathbb{R}^2_1$ is an *oval* if it has an A_1-contact with all its tangent lines. (This definition includes the lightlike points. An example of an oval is the "circle" $S^1 = \{(u_0, u_1) \in \mathbb{R}^2_1 \,|\, u_0^2 + u_1^2 = 1\}$.)

Because the oval is a simple closed curve, by Jordan curve theorem, its complement consists of two open and connected subsets of $\mathbb{R}^2$. These subsets are the interior and the exterior of the oval, see Chapter 2.

Theorem 3.3.1 Let γ be an oval in the Minkowski plane. Then,

(i) γ has exactly four lightlike points;

(ii) the caustic of γ is a closed curve which lies in the complement of the interior of γ;

(iii) the evolute of each spacelike and timelike component of γ has at least one singular point.

Proof: (i) We use the arguments in the proof of Proposition 3.1.1. The curve $\tilde{\gamma}$ has nowhere vanishing (Euclidean) curvature as its contact with its tangent lines is of type A_1 (the contact of γ with lines is an affine property and is independent of the metric in $\mathbb{R}^2$). Therefore, the Gauss map N is a diffeomorphism and the result follows.

(ii) The curve γ is an oval, so it has neither inflection points nor lightlike inflection points. Therefore, its caustic is defined everywhere and is a closed curve. It follows

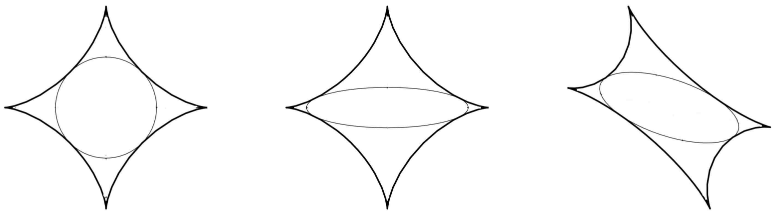


Figure 3.2: The caustic (thick curve) of a circle (left) and of an ellipse (centre and right) drawn using Maple.

from Proposition 3.3.1(ii) and from the fact that γ is an oval that the caustic of γ, minus the lightlike points, lies in the exterior of γ near the lightlike points. By Proposition 3.2.2, the evolute of a spacelike or a timelike component of γ does not intersect that component. Thus, the evolute of γ remains in the exterior of γ.

(iii) Let $I = (a, b)$ be an interval parametrising a spacelike or timelike component of γ, with $\gamma(a)$ and $\gamma(b)$ lightlike points. As $\langle \gamma''(t), \gamma'(t)^{\perp} \rangle \neq 0$, the curvature goes to infinity as t tends a or b. The curve γ is an oval, so its curvature has constant sign in I. Therefore, $\lim_{t\to a} \kappa(t) = \lim_{t\to b} \kappa(t) = \pm\infty$ (with $t \in I$). It follows that there exists $t \in I$ such that $\kappa'(t) = 0$, so γ has a vertex at t, and this corresponds to a singular point on the evolute (Proposition 3.2.1(ii)). $\square$

Example 3.3.1 An ellipse $\gamma(t) = (a\cos(t), b\sin(t))$, $t \in \mathbb{R}$ is an oval in $\mathbb{R}^2_1$. Figure 3.2 shows Maple plots of the caustics of some ellipses. The caustic of a "circle" ($a = b = 1$) is shown in Figure 3.2(left). (Recall that the caustic/evolute of a circle in the Euclidean plane is the centre of the circle.) We take $a = 2$ and $b = 1$ in Figure 3.2(centre). In Figure 3.2(right), we apply an Euclidean rotation to the ellipse and draw its caustic.

Chapter 4

Parallels of curves in the Minkowski plane

The family of distance-squared functions gives information about the parallels of γ. They are defined away from lightlike points of γ. We prove in this chapter that the parallels undergo swallowtail transitions at a vertex of γ, which when considered together in $\mathbb{R}^2_1$, give a distinct configuration to that of the parallels of a curve in the Euclidean plane. We also prove that the parallels of curves in the Euclidean plane at an ordinary vertex of γ are as Figure 4.3(left) while they are as Figure 4.3(right) at an ordinary vertex of γ in the Minkowski plane. (To our knowledge it is not proved before that the configuration of parallels at an ordinary vertex of a curve in the Euclidean plane are always as Figure 4.3(left).) The results of this chapter are part of the paper [42].

4.1 Parallels as wave fronts

Definition 4.1.1 A parallel of a curve γ in the Minkowski plane, with its lightlike points removed, is the curve obtained by moving each point on γ by a fixed distance r along the unit normal $\mathbf{n}$ to γ. Thus, a parametrisation of a parallel is given by

$$\eta_r(t) = \gamma(t) + r\mathbf{n}(t).$$

It is worth observing that the parallels are not defined at lightlike points (as we require a unit normal vector). Parallels are wave fronts and can be studied following the same approach for curves in the Euclidean plane using the family of distance-squared functions (see for example [8]).

Consider the map $F : (S^1 \setminus L) \times \mathbb{R}^2_1 \to \mathbb{R} \times \mathbb{R}^2_1$, given by $F(t,v) = (f(t,v), v)$, where L denotes the set of the lightlike points of γ and f is the family of distance-squared functions.

The set of critical points $\Sigma(F)$ of F coincides with $\Sigma(f) \cap ((S^1 \setminus L) \times \mathbb{R}^2_1)$, and thus is a smooth surface (section 3.3). The wave fronts (parallels) associated to γ are the sets

$$\eta_r = F(\Sigma(F)) \cap (\{r\} \times \mathbb{R}^2_1).$$

Wave fronts have generic Legendrian singularities [2, 8] apart from a discrete set of distances r. There are three possible transitions at these values of r [2]. However, it is shown in [8] that only the A_3-transition occurs (i.e., the swallowtail transitions in Figure 4.1(right), and this happens at an ordinary vertex of γ.)

4.2 Parallels at vertices of curves

The A_3-transition in wave fronts is studied by considering (locally) the big front $F(\Sigma(F))$. This big front is a swallowtail surface, that is, $F(\Sigma(F))$ is diffeomorphic to the discriminant of the polynomial $t^4 + \lambda_1 t^2 + \lambda_2 t + \lambda_3$ which is the surface (Figure 4.1(left))

$$S = \{(\lambda_1, -4t^3 - 2\lambda_1 t, 3t^4 + \lambda_1 t^2),\ t, \lambda_1 \in \mathbb{R}, 0\}.$$

To recover the individual wave fronts, one has to consider generic sections of the surface S. This is done by Arnold [2], where he considered functions $f : \mathbb{R}^3, 0 \to \mathbb{R}, 0$ and allowed changes of coordinates in $\mathbb{R}^3, 0$ that preserve S. Then a generic function is equivalent, under these changes of coordinates, to $f(\lambda_1, \lambda_2, \lambda_3) = \lambda_1$. Therefore, the individual wave fronts undergo the transitions in Figure 4.1(right).

Our concern here is how the individual fronts are stacked together in $\mathbb{R}^2_1$. For this, one needs to project the sections of S by f to a plane. Then, the problem becomes that of considering the divergent diagram (f, g)

$$\mathbb{R}^2, 0 \xleftarrow{g} \mathbb{R}^3, 0 \xrightarrow{f} \mathbb{R}, 0.$$

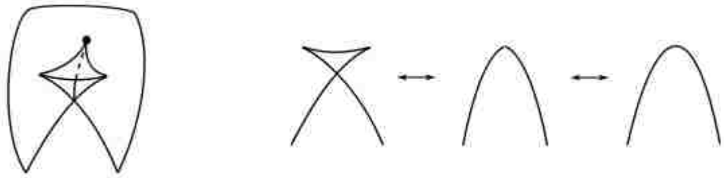


Figure 4.1: The swallowtail surface (left) and its generic sections (right).

Bruce proved in [9] that there are no stable pairs (f, g). (As a consequence, he showed that there are no discrete smooth models for an implicit differential equation (IDE) of cusp type. Davydov [20] showed that there is in fact a functional modulus for an IDE of cusp type even for the topological equivalence. Dara [18] pointed out that there are two possible configurations of the solutions of the IDE of cusp type and these are as in Figure 4.3.)

Theorem 4.2.1 [42] There are two generic configurations for the family of curves $g(f^{-1}(c) \cap S)$, $c \in \mathbb{R}, 0$. The two configurations are distinguished by $g(f^{-1}(0) \cap S)$ and the image of the singular set of S by g, which is a cusp. These are as in Figure 4.3(left) if the cusp and $g(f^{-1}(0) \cap S)$ are in the same semi-plane delimited by the limiting tangent line to $g(f^{-1}(0) \cap S)$ and as in Figure 4.3(right) if they are in different semi-planes.

Proof: By Arnold's result [2], we can take $f(\lambda_1, \lambda_2, \lambda_3) = \lambda_1$. Then the zero section of f in S is a curve with a singularity of type (t^3, t^4).

We assume that the kernel of dg_0 is transverse to the plane $\lambda_1 = 0$. This ensures that the restriction of g to the planes $f^{-1}(c)$ is a local diffeomorphism, so it preserves the structure of the curves $f^{-1}(c) \cap S$. We also assume that the kernel of dg_0 is not parallel to the direction $(1, 0, 0)$. This ensures that the image by g of the singular set of S is cusp curve. A map g that satisfy both of the above conditions is a generic map.

Suppose that $f^{-1}(c) \cap S$ has a self-intersection and denote by Δ_c the triangular region whose vertices are the origin and the two cusps of $g((f^{-1}(c) \cap S))$, and whose edges are formed by the image of the singular set of S by g and the segment of $g((f^{-1}(c) \cap S))$ delimited by its singular points (shaded regions in Figure 4.2).

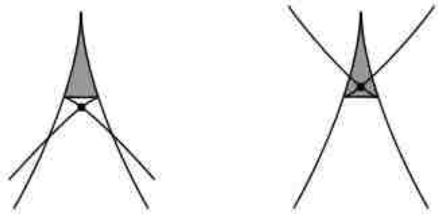


Figure 4.2: The two generic positions of the intersection point of $g(f^{-1}(c) \cap S)$, outside the shaded region (left) and inside it (right).

Then the two configurations of $g(f^{-1}(c) \cap S)$, $c \in \mathbb{R}, 0$, are distinguished by the fact that the self-intersection point of $g(f^{-1}(c) \cap S)$ is inside or outside the triangle Δ_c (Figures 4.3 and 4.2). This property depends only on dg_0. To show this, write $g = dg(0) + h$ where h is a smooth map with no linear terms. Let $g_s = dg(0) + sh$, $s \in [0, 1]$. Then $dg_s(0) = dg(0)$ for all $s \in [0, 1]$, so the map $g_s|f^{-1}(c)$ is a local diffeomorphism and maps the singular set of S to a cusp curve. For g_0 and g_1 to give two different configurations, there must exist $s \in [0, 1]$ such that $g_s|f^{-1}(c)$ is not a diffeomorphism (as g_s maps the curve $f^{-1}(c) \cap S$ to one which is not diffeomorphic to it), which is not the case.

We can therefore assume that g is a linear projection along a direction $u = (u_1, u_2, u_3) \in \mathbb{R}^3$, with $u_1^2 + u_2^2 + u_3^2 = 1$, to a transverse plane. As we assume that the kernel of g is not parallel to $(1, 0, 0)$ we can take, for simplicity, $u_3 \neq 0$ and project to the (u_1, u_2)-plane.

The projection of $f^{-1}(0) \cap S$ is the curve

$$l_0(t) = (4u_1u_2t^3 - 3u_1u_3t^4, 4(u_2^2 - 1)t^3 - 3u_2u_3t^4)$$

and the projection of the singular set of S is the cusp curve

$$c_0(t) = (6(u_1^2 - 1)t^2 - 8u_1u_2t^3 + 3u_1u_3t^4, 6u_1u_2t^2 - 8(u_2^2 - 1)t^3 + 3u_2u_3t^4).$$

The limiting tangent directions of the two curves are transverse as $u_1^2+u_2^2-1 \neq 0$. Then the position of the two curves with respect to the limiting tangent line L_0 to l_0 at $t = 0$ is determined by the sign of u_1u_3 (positive for the two curves to be in the same semi-plane determined by L_0 and negative if they lie in different semi-planes).

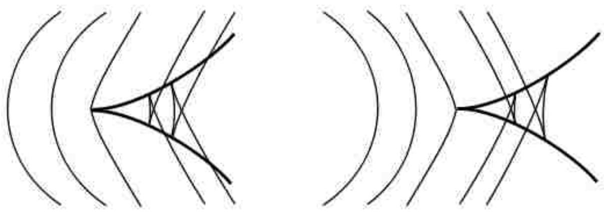


Figure 4.3: Parallels of a plane curve at a swallowtail transition: (left) in the Euclidean plane and (right) in the Minkowski plane.

The fibre $f^{-1}(c) \cap S$ is singular if $c < 0$. The singular points are given by $6t^2 + c = 0$ and the self-intersection point is given by $2t^2 + c = 0$. We project these points along u to the (u_1, u_2)-plane. It is not difficult to show that the projection of the self-intersection point is inside the triangle Δ_c if and only if $u_1u_3 < 0$ and outside if and only if $u_1u_3 > 0$. Thus, the configuration of the curves $g(f^{-1}(c) \cap S)$ is determined by the positions of the curves $l_0(t)$ and $c_0(t)$ with respect to the limiting tangent line L_0 to l_0 at $t = 0$. $\square$

Theorem 4.2.2 [42] (a) The parallels of a curve γ in the Euclidean plane are as in Figure 4.3(left) at an ordinary vertex of γ.

(b) The parallels of a curve γ in the Minkowski plane are as in Figure 4.3(right) at an ordinary vertex of γ.

Proof: We suppose that γ is parametrised by arc length and apply Theorem 4.2.1. The projection of the singular set of the big front is the evolute of γ.

(a) The evolute of a curve γ in the Euclidean plane is given by $e(t) = \gamma(t) + 1/\kappa(t)\mathbf{n}(t)$. Suppose that $t = 0$ is an ordinary vertex of γ, that is $\kappa'(0) = 0$ and $\kappa''(0) \neq 0$. Then $e'(0) = 0$ and $e''(0) = -\kappa''(0)/\kappa^2(0)\mathbf{n}(0)$.

We take $\{\mathbf{t}(0), \mathbf{n}(0)\}$ as a coordinate system at $e(0)$. Then the evolute is above the axis parallel to $\mathbf{t}(0)$ if $\kappa''(0) < 0$ and below it if $\kappa''(0) > 0$.

The parallel of interest is $\eta_{r_0}(t) = \gamma(t) + r_0\mathbf{n}(t)$, with $r_0 = 1/\kappa(0)$. We have $\eta'_{r_0}(0) = \eta''_{r_0}(0) = 0$ and

$$\begin{aligned} \eta'''_{r_0}(0) &= -\tfrac{\kappa''(0)}{\kappa(0)}\mathbf{t}(0), \\ \eta^{(4)}_{r_0}(0) &= -\tfrac{\kappa'''(0)}{\kappa(0)}\mathbf{t}(0) - 3\kappa''(0)\mathbf{n}(0). \end{aligned}$$

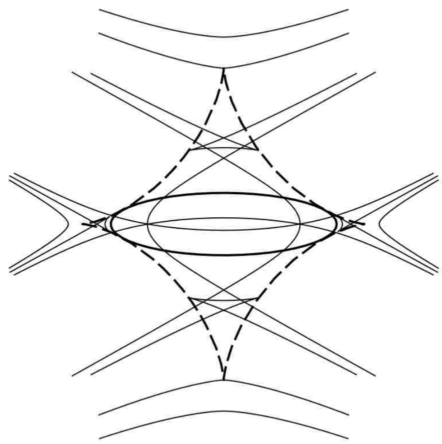


Figure 4.4: The parallels to an ellipse with its lightlike points removed. (The dashed curve is the caustic of the ellipse.)

Then

$$\eta_{r_0}(t) = (-\frac{\kappa''(0)}{3!\kappa(0)}t^3 + h.o.t)\mathbf{t}(0) + (-\frac{3}{4!}\kappa''(0)t^4 + h.o.t)\mathbf{n}(0),$$

so the parallel η_{r_0} is above the axis parallel to $\mathbf{t}(0)$ if $\kappa''(0) < 0$ and below it if $\kappa''(0) > 0$. That is, the parallel η_{r_0} and the evolute are always on the same side of the limiting tangent direction to the parallel. It follows by Theorem 4.2.1 that the parallels of γ have the configuration in Figure 4.3(left).

(b) The evolute of a curve γ in the Minkowski plane is given by $e(t) = \gamma(t) - \frac{1}{\kappa(t)}\mathbf{n}(t)$. At an ordinary vertex $t = 0$, we have $e'(0) = 0$ and $e''(0) = \kappa''(0)/\kappa^2(0)\mathbf{n}(0)$.

The parallel of interest is $\eta_{r_0}(t) = \gamma(t) + r_0\mathbf{n}(t)$, with $r_0 = -1/\kappa(0)$. Here we have $\eta'_{r_0}(0) = \eta''_{r_0}(0) = 0$ and

$$\begin{aligned} \eta'''_{r_0}(0) &= -\tfrac{\kappa''(0)}{\kappa(0)}\mathbf{t}(0), \\ \eta^{(4)}_{r_0}(0) &= -\tfrac{\kappa'''(0)}{\kappa(0)}\mathbf{t}(0) - 3\kappa''(0)\mathbf{n}(0). \end{aligned}$$

Following the same argument above, we conclude that the parallel η_{r_0} and the evolute are always on opposite sides of the limiting tangent direction to the parallel. It follows by Theorem 4.2.1 that the parallels of γ have the configuration in Figure

4.3(right). □

Figure 4.4 shows a Maple plot of the parallels of an ellipse with its lightlike points removed. Observe that the tangent lines to the ellipse at the lightlike points are asymptotes of its parallels (this is also the case at an isolated lightlike point of any curve in the Minkowski plane). Indeed, away from the lightlike point $\gamma(t_0)$, a parallel is given by $\gamma(t) + r\frac{\gamma'(t)^\perp}{\|\gamma'(t)\|}$. As $t \to t_0$ we have $\frac{r}{\|\gamma'(t)\|} \to \pm\infty$.

Chapter 5

Symmetry set of curves in the Minkowski plane

The symmetry set (SS) of a curve in the Euclidean plane is defined as the closure of the locus of centres of bi-tangent circles to the curve [25, 43]. In this chapter we define its analogue for a curve in the Minkowski plane and call it the Minkowski symmetry set (MSS). We consider the geometry of MSS and deal in some details with the MSS of an ellipse. The results of this chapter are part of the paper [42].

5.1 Contact of curves with pseudo-circles

Definition 5.1.1 The Minkowski symmetry set (MSS) of a curve γ in the Minkowski plane is the closure of the locus of centres of bi-tangent pseudo-circles to γ.

The pseudo-circles $H^1(p, -r)$ and $S^1_1(p, r)$ have two connected components and γ can be tangent to either a single component at two distinct points or to each component of these pseudo-circles. If γ is bi-tangent to $LC^*(p)$, then generically it is tangent to each line of $LC^*(p)$ at a single point. (One can show using Thom's transversality theorem that bi-tangency with a single line of $LC^*(p)$ is not a generic property of curves in $\mathbb{R}^2_1$.)

The contact of γ with pseudo-circles is measured by the family of distance-squared functions f (section 3.3). The multi-local stratum of the bifurcation set of f is the set of points v such that f_v has singularities at two distinct points t_1 and t_2

with $f_v(t_1) = f_v(t_2)$.

The MSS has the following properties, some of which are similar to those of the SS.

Theorem 5.1.1 (i) The MSS of γ is the closure of the multi-local stratum of the bifurcation set of the family of distance-squared functions on γ.

(ii) If γ is spacelike or timelike and is parametrised by arc length, then there is a bi-tangent pseudo-circle to γ at $\gamma(t_1)$ and $\gamma(t_2)$ if and only if

$$\langle \gamma(t_1) - \gamma(t_2), \mathbf{t}(t_1) \pm \mathbf{t}(t_2) \rangle = 0,$$

where $+$ or $-$ is determined by the orientation of γ at $\gamma(t_1)$ and $\gamma(t_2)$.

(iii) The MSS is a regular curve at p if and only if the bi-tangent pseudo-circle to γ at $\gamma(t_1)$ and $\gamma(t_2)$ is not osculating at $\gamma(t_1)$ or at $\gamma(t_2)$. If this is the case, the tangent line to the MSS at p is the perpendicular bisector to the chord joining $\gamma(t_1)$ and $\gamma(t_2)$.

(iv) The MSS is a spacelike curve at a point p in the following cases: (1) the curve γ is tangent to each component of a pseudo-circle $H^1(p, -r)$; (2) the curve γ is bi-tangent to a single component of a pseudo-circle $S^1_1(p, r)$; (3) the curve γ is tangent to one line of $LC^*(p)$ at $\gamma(t_1)$ and to the other line at $\gamma(t_2)$, and $\gamma(t_2)-\gamma(t_1)$ is timelike.

(v) The MSS is a timelike curve at a point p in the following cases: (1) the curve γ is tangent to each component of a pseudo-circle $S^1_1(p, r)$; (2) the curve γ is bi-tangent to a single component of a pseudo-circle $H^1(p, -r)$; (3) the curve γ is tangent to one line of $LC^*(p)$ at $\gamma(t_1)$ and to the other line at $\gamma(t_2)$, and $\gamma(t_2)-\gamma(t_1)$ is spacelike.

(vi) The MSS has generically no lightlike points.

Proof: The proof of (i) follows from the definition of the MSS and the proof of (ii) is identical to that for the symmetry set of a curve in the Euclidean plane (see [25]).

For (iii), we consider the case where the bi-tangent pseudo-circles are of type $H^1(p, -r)$ and suppose that γ is tangent to both components of these pseudo-circles (the other cases follow similarly). We give the pieces of γ at $\gamma(t_1)$ (resp. $\gamma(t_2)$) the orientation of $p + (r\cosh(t), r\sinh(t))$ (resp. $p + (-r\cosh(t), r\sinh(t))$). To simplify

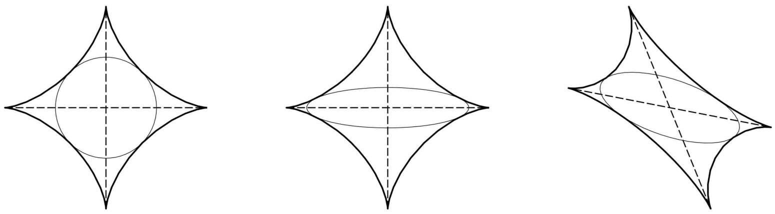


Figure 5.1: The caustic (thick curve) of a circle (left) and of an ellipse (centre and right) drawn using Maple. The dashed curves are the Minkowski symmetry sets.

notation, we write γ_1 for $\gamma(t_1)$ and γ_2 for $\gamma(t_2)$ and similarly for all information at $\gamma(t_1)$ and $\gamma(t_2)$. The condition for bi-tangency is then given by $g(t_1, t_2) = \langle \gamma_1 - \gamma_2, \mathbf{t}_1 + \mathbf{t}_2 \rangle = 0$. As $\langle \mathbf{t}_2 + \mathbf{t}_1, \mathbf{n}_2 + \mathbf{n}_1 \rangle = 0$, $g(t_1, t_2) = 0$ if and only if $\gamma_1 - \gamma_2 = r(\mathbf{n}_1 + \mathbf{n}_2)$. The radius r of the bi-tangent pseudo-circle can then be given explicitly in the form

$$r(t_1, t_2) = \frac{\langle \gamma_1 - \gamma_2, \mathbf{n}_1 + \mathbf{n}_2 \rangle}{2(\langle \mathbf{n}_1, \mathbf{n}_2 \rangle - 1)}.$$

(We observe that $\langle \mathbf{n}_1, \mathbf{n}_2 \rangle - 1 \neq 0$.) We have

$$\begin{aligned} g_{t_1}(t_1, t_2) &= -(\langle \mathbf{n}_1, \mathbf{n}_2 \rangle - 1)(1 - r\kappa_1), \\ g_{t_2}(t_1, t_2) &= (\langle \mathbf{n}_1, \mathbf{n}_2 \rangle - 1)(1 + r\kappa_2). \end{aligned}$$

So the MSS is a regular curve at p if and only if $1 - r\kappa_1 \neq 0$ or $1 + r\kappa_2 \neq 0$, equivalently, if and only if $H^1(p, -r)$ is not osculating at both $\gamma(t_1)$ and $\gamma(t_2)$.

Suppose that $H^1(p, -r)$ is not osculating at $\gamma(t_1)$. Then we can parametrise locally $g^{-1}(0)$ by $(t_1, t_2(t_1))$ for some smooth function $t_2(t_1)$. The MSS is then parametrised by

$$c(t_1) = \gamma(t_1) - r(t_1, t_2(t_1))\mathbf{n}(t_1).$$

We have

$$c' = (1 - r\kappa_1)\mathbf{t}_1 - (r_{t_1} + t_2' r_{t_2})\mathbf{n}_1$$

and

$$\begin{aligned} r_{t_1} &= \frac{(1 - r\kappa_1)\mathbf{t}_1\mathbf{n}_2}{2(\langle \mathbf{n}_1, \mathbf{n}_2 \rangle - 1)}, \\ r_{t_2} &= -\frac{(1 + r\kappa_2)\mathbf{t}_2\mathbf{n}_1}{2(\langle \mathbf{n}_1, \mathbf{n}_2 \rangle - 1)}, \\ t_2' &= \frac{1 - r\kappa_1}{1 + r\kappa_2}. \end{aligned}$$

Therefore $\langle c', \mathbf{n}_1 + \mathbf{n}_2 \rangle = 0$, that is $c'(t_1)$ is orthogonal to $\gamma(t_1) - \gamma(t_2)$. To show that the tangent line to the MSS is the perpendicular bisector to the chord joining $\gamma(t_1)$ and $\gamma(t_2)$, it is enough to consider these points on the pseudo-circle $H^1(p, -r)$ and observe the said perpendicular bisector passes through p.

For (iv) and (v), the results are immediate using (iii) for bi-tangency with $LC^*(p)$. For the other cases, also using (iii), it is enough to choose any two distinct points q_1 and q_2 on $H^1(p, -r)$ (resp. $S^1_1(p, r)$) and consider the vector $\overrightarrow{q_1 q_2}$. We can take, without loss of generality, $r = 1$ and p to be the origin. Then, a parametrisation of the components of $H^1(-1)$ are given by $(\cosh(s), \sinh(s))$ and $(-\cosh(s), -\sinh(s))$, and those of $S^1_1(1)$ by $(\sinh(s), \cosh(s))$ and $(-\sinh(s), -\cosh(s))$. The result now follows by straightforward calculations.

For (vi), the vector $\overrightarrow{q_1 q_2}$ (with q_1 and q_2 as above) is never a lightlike vector, so the only possible case for a point $p \in MSS$ to be lightlike is when one of the lines of LC^* is bi-tangent to the curve γ. However, this does not occur for generic curves in the Minkowski plane. $\square$

We consider now the example of an ellipse in the Minkowski plane.

Proposition 5.1.1 The MSS of an ellipse consists of the two segments of lines joining opposite cusps of the caustic of the ellipse (the dashed lines in Figure 5.1). These segments contain the diagonals of the parallelogram formed by the four tangent lines to the ellipse at its lightlike points (Figure 5.2(left)).

Proof: We make an affine transformation A and work with a circle (C) tangent to the four sides of a parallelogram (P) (Figure 5.2(right)). The images by A of the families of hyperbolas $H^1(p, -r)$ and $S^1_1(p, r)$ are families of hyperbolas $l_1 l_2 = c$ ($c \in \mathbb{R}$) with asymptotes $l_1 = 0$ and $l_2 = 0$ parallel to the sides of the parallelogram (P). The MSS of the ellipse is the pre-image by A of the locus of bi-tangency of the circle (C) with the hyperbolas $l_1 l_2 = c$.

Given a bi-tangent hyperbola $l_1 l_2 = c$ to the circle (C), the centre of (C) belongs to the Euclidean symmetry set of $l_1 l_2 = c$. Now, the symmetry set of a hyperbola $l_1 l_2 = c$ consists of the pair of lines which bisect the lines $l_1 = 0$ and $l_2 = 0$. It follows that the point of intersection of $l_1 = 0$ and $l_2 = 0$ is on a diagonal of the

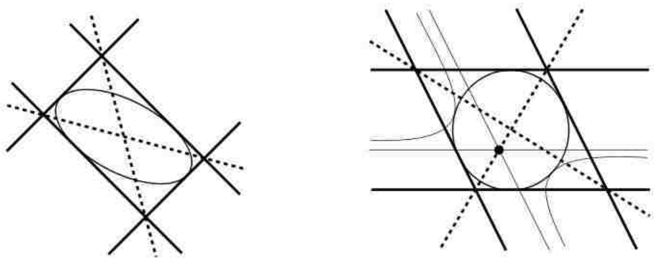


Figure 5.2: Constructing the MSS of an ellipse (dashed line).

parallelogram (P) (Figure 5.2(right)). As the diagonals of the parallelogram are preserved under affine transformations, it follows that the MSS of the ellipse is a subset of the lines containing the diagonals of the parallelogram formed by the four tangent lines to the ellipse at its lightlike points. The result follows now using the fact that the MSS has endpoints at the cusps of the evolute of the ellipse. (See Figure 5.1 for the MSS of various ellipses in the Minkowski plane. Observe that, in general, the MSS of an ellipse is not along the axes of the ellipse; Figure 5.1(right).) $\square$

Remark 5.1.1 (1) The concepts of evolute, caustic and MSS can be associated to a curve in any Lorentzian plane $(\mathbb{R}^2, g)$. We can find a g-orthonormal basis $\{u_1, u_2\}$ of $\mathbb{R}^2$ so that the expression for g is given, with respect to this basis, by $g(u, v) = -u_0v_0 + u_1v_1$, for any $u = (u_0, u_1)$ and $=(v_0, v_1)$ in $\mathbb{R}^2$ (so we are back to the Minkowski plane).

(2) If we write the g-lightlike lines as $l_i = a_ix + b_iy = 0$, $i = 1, 2$, where (x, y) are the coordinates with respect to the standard basis in $\mathbb{R}^2$, then the g-pseudo-circles centred at the origin are the family of hyperbolas (including their asymptotes) $l_1l_2 = c$, $c \in \mathbb{R}$. Therefore, the results can be interpreted in the affine setting. They provide information about the contact of a curve in the affine plane $\mathbb{R}^2$ with a given family of hyperbolas $l_1l_2 = c$, translated by any vector in $\mathbb{R}^2$.

Chapter 6

Binary differential equations

In this chapter we gather all the results about singularities of codimension ≤ 1 of Binary Differential Equations (BDEs) and the way they bifurcate in generic 1-parameter families of BDEs. We present them in a way that makes their identifications more apparent. Binary and implicit differential equations are studied by many authors. We refer to the survey article [55] for references and also to the books [3, 19].

A *binary differential equation* is an equation in the form

$$a(x,y)dy^2 + 2b(x,y)dydx + c(x,y)dx^2 = 0, \tag{6.1}$$

where a,b,c are called the *coefficients* of the BDE (6.1). These coefficients are smooth functions defined in an open set U in $\mathbb{R}^2$.

Since our study is local, we shall consider the origin, without loss of generality, as the point of interest in the entire study. The *discriminant* of the BDE (6.1) is the set $\Delta = \{(x,y) \in U : \delta(x,y) = (b^2 - ac)(x,y) = 0\}$. A BDE determines two directions at each point (x,y) in U where $\delta(x,y) > 0$, a unique direction at points on the discriminant and no directions at points where $\delta(x,y) < 0$. Consequently, a BDE determines a pair of transverse foliations (solution curves) or no foliations away from its discriminant. Therefore, the important local geometric features of the foliations of a BDE arise on its discriminant. The *configuration* of a BDE is its discriminant together with the pair of foliations it determines.

We denote by ω the quadratic form associated to the BDE (6.1) and also use ω

to refer to the equation $\omega = 0$. Since our study is local, the coefficients a, b, c are taken as germs of functions $\mathbb{R}^2, 0 \to \mathbb{R}$.

Definition 6.0.2 Two germs of BDEs ω_1 and ω_2 are said to be *smoothly equivalent* if there exists a germ of a diffeomorphism $H : \mathbb{R}^2, 0 \to \mathbb{R}^2, 0$ and a non-zero function $r : \mathbb{R}^2, 0 \to \mathbb{R}, 0$ such that

$$\omega_2 = r(H^*\omega_1).$$

Two germs of BDEs are said to be *topologically equivalent* if there exists a germ of a homeomorphism that takes the configuration of one to the configuration of the other.

We associate to a germ of a BDE $\omega(x, y) = (a, b, c)$ the jet-extension map

$$j^k\omega : \mathbb{R}^2, 0 \to J^k(2, 3)$$
$$(x, y) \mapsto j^k\omega,$$

where $J^k(2, 3)$ denotes the vector space of polynomials maps of degree $\leq k$ from $\mathbb{R}^2$ to $\mathbb{R}^3$ and $j^k\omega = (j^k a, j^k b, j^k c)$ is the k-jet of (a, b, c) at (x, y).

Although the solution curves are singular at all points of the discriminant, we define a singularity of a BDE to be a point of the discriminant at which the unique solution is tangent to the discriminant or a point at which the discriminant itself is singular.

Definition 6.0.3 A singularity of a BDE is said to be of codimension m if the conditions that define it determine a submanifold W in $J^k(2, 3)$ of codimension $m + 2$, for any $k \geq k_0$ for a given fixed k_0.

BDEs are divided to two types. The first type is when the coefficients of the BDE do not all vanish at the origin; the BDE in this case is labelled *Type 1*. The second one is when all the coefficients of the BDE vanish at the origin; the BDE in this case is labelled *Type 2*.

6.1 BDEs of Type 1

The study of the BDEs of Type 1 follows from the general study of Implicit Differential Equations (IDEs). Suppose, without loss of generality, that $a(0,0) \neq 0$ so $dx = 0$ is not a solution of the BDE (6.1) at the origin. Then, we can divide equation (6.1) by dx^2 and consider the BDE (6.1) as an IDE

$$a(x,y)p^2 + 2b(x,y)p + c(x,y) = 0,\ p = \frac{dy}{dx}. \tag{6.2}$$

In general an IDE is an equation in the form

$$F(x,y,p) = 0,\ p = \frac{dy}{dx}, \tag{6.3}$$

where F is a differentiable function of $(x,y,p) \in \mathbb{R}^3$. The IDE (6.3) defines a surface

$$M = \{(x,y,p) \in \mathbb{R}^3 : F(x,y,p) = 0\}.$$

in the 3-dimensional space of 1-jets of functions endowed with the contact structure $\alpha = dy - pdx$. Consider the projection $\pi : M \to \mathbb{R}^2$, $\pi(x,y,p) = (x,y)$. In general, M is smooth surface and $\pi|_M$ is either a local diffeomorphism, a fold or a cusp map. The projection π is a double cover of the set $\{(x,y) : \delta(x,y) > 0\}$. The set of critical points of the projection given by $F = F_p = 0$ is called the *criminant* of the IDE and the set of critical values of the projection is called the *discriminant* of the IDE which is obtained by eliminating p from the equations $F = F_p = 0$.

The bi-valued direction field defined by the BDE in the plane lifts to a single-valued direction field ξ on M. The direction field ξ is given by

$$\xi = F_p\frac{\partial}{\partial x} + pF_p\frac{\partial}{\partial y} - (F_x + pF_y)\frac{\partial}{\partial p}. \tag{6.4}$$

The direction field ξ is tangent to M at (x,y,p) and projects to a line through (x,y) with slope p. If $\pi|_M$ is a local diffeomorphism at (x,y,p), then the integral curves of ξ around (x,y,p) projects to a family of smooth curves around (x,y).

Suppose that $\pi|_M$ has a fold singularity at (x,y,p), i.e., we can choose local coordinates in M and $\mathbb{R}^2$ for which π has the form (u,v^2). This means $F = F_p = 0$ but $F_{pp} \neq 0$ at (x,y,p). Then, by using the Division Theorem, the IDE is locally a

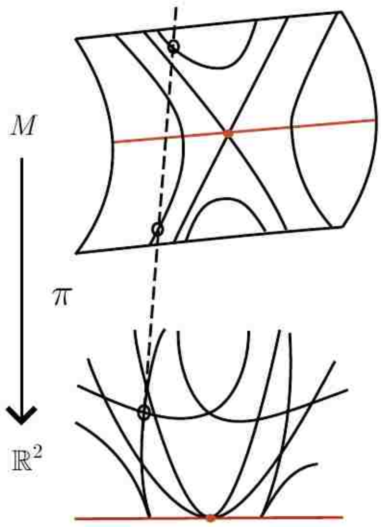


Figure 6.1: The lifted field ξ and the involution σ on M.

BDE at (x, y, p) and the discriminant is a smooth curve. (The discriminant of the IDE is precisely that of the BDE.)

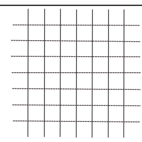
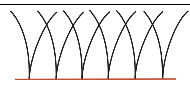
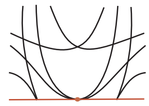
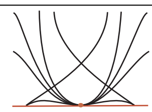
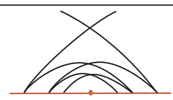
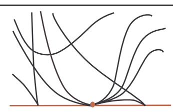
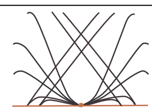
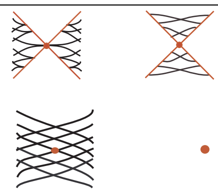
Every point in the plane near (x, y) which is not on the discriminant has two pre-images on M under π or none. This defines an involution σ on M near (x, y, p), which interchanges pairs of points with the same image under σ (Figure 6.1). The criminant is the set of fixed points of σ. Thus, locally at (x, y, p), we have a pair (ξ, σ) of a vector field and an involution on M. The classification (smooth or topological) of IDEs is the same as the classification (smooth or topological) of the pairs (ξ, σ). The singularity of ξ occurs on the criminant. Observe that for BDEs of Type 1, $F_{pp} \neq 0$, so $\pi|_M$ cannot have a cusp singularity.

We summarise below the classification of BDEs of Type 1.

Theorem 6.1.1 A germ of BDE of Type 1 with a local singularity of codimension ≤ 1 is topologically equivalent to $dy^2 + f_0(x, y)dx^2 = 0$ with $f_0(x, y)$ as in Table 6.1, second column. The configurations of the solution curves of the BDE are as in Table 6.1, fourth column.

Remark 6.1.1 The models $dy^2 \pm dx^2 = 0$ and $dy^2 - xdx^2 = 0$ of the first two cases in Table 6.1 are up to smooth equivalence [18]. The folded singularities of

Table 6.1: The stable and codimension ≤ 1 configurations of BDEs of Type 1.

Name	f_0 in topological normal forms	Codim	Figures
Transverse foliations or empty set	± 1	0	 ϕ
Family of cusps	$-x$	0	
Folded saddle	$-y + \lambda x^2, \lambda < 0$	0	
Folded node	$-y + \lambda x^2, 0 < \lambda < 1/16$	0	
Folded focus	$-y + \lambda x^2, 1/16 < \lambda$	0	
Folded saddle-node	$-y + x^3$	1	
Folded node-focus	$-y + \frac{1}{16}x^2$	1	
Morse Type 1	$\pm x^2 \pm y^2$	1	 A_1^- (top) and A_1^+ (bottom).

types (saddle/node/focus) are part of a family $dy^2 + (-y + \lambda x^2)dx^2 = 0$ where $\lambda \neq 0, 1/16$, and this model is up to smooth equivalence provides that the vector field ξ is linearisable at its singular point [19, 20]. For the topological models, we take any fixed λ in the given intervals.

6.1.1 Recognition of codimension ≤ 1 singularities

We apply the classification in Theorem 6.1.1 to 1-parameter families of BDEs on a smooth surface. Our task is to determine the type of the singularities and to check if the family is generic. In order to do so and to carry out the calculation in Chapter 9 we need to identify algebraically the singularities of codimension ≤ 1. The BDE (6.2) is of Type 1, we can suppose, without loss of generality, that $a(0,0) \neq 0$ and divide equation (6.2) by $a(x,y)$. Then the BDE (6.2) could be written in the form

$$p^2 + r_1(x,y)p + r_2(x,y) = 0. \tag{6.5}$$

We write the 3-jets of $r_i(x,y), i = 1,2$ as follows

$$j^3 r_1(x,y) = b_0 + b_{10}x + b_{11}y + \sum_{i=0}^{2} b_{2i}x^{2-i}y^i + \sum_{i=0}^{3} b_{3i}x^{3-i}y^i, \tag{6.6}$$

$$j^3 r_2(x,y) = c_0 + c_{10}x + c_{11}y + \sum_{i=0}^{2} c_{2i}x^{2-i}y^i + \sum_{i=0}^{3} c_{3i}x^{3-i}y^i. \tag{6.7}$$

The 2-jet of the discriminant ($\delta(x,y) = r_1^2(x,y) - 4r_2(x,y)$) is given by

$$\begin{aligned} j^2\delta(x,y) = & b_0^2 - 4c_0 + 2(b_{10}b_0 - 2c_{10})x + 2(b_{11}b_0 - 2c_{11})y + (b_{10}^2 + 2b_0b_{20} - 4c_{20})x^2 \\ & + 2(b_0b_{21} - 2c_{21} + b_{11}b_{10})xy + (b_{11}^2 + 2b_0b_{22} - 4c_{22})y^2. \end{aligned}$$

We summarise the geometric characterisations of the singularities of codimension $\leq$ 1 of BDEs of Type 1 shown in Table 6.1. We give also the algebraic conditions for them to occur.

Theorem 6.1.2 The geometric characterisations and the algebraic conditions of the local codimension ≤ 1 singularities of BDEs of Type 1 to occur are as shown in Table 6.2, where the expressions for $C_i s, i = 1, ..., 10$ are:

Table 6.2: The geometric characterisations and the algebraic conditions of the local codimension ≤ 1 singularities of BDEs of Type 1.

Name	On the surface M	In $\mathbb{R}^2$	The algebraic conditions
Transverse foliations or empty set	M is smooth, $\pi\|_M$ is a local diffeomorphism, ξ is regular.	The discriminant Δ is empty.	$C_1 \neq 0$.
Family of cusps	M is smooth, $\pi\|_M$ has a fold singularity, ξ is regular.	Δ is a smooth curve, the unique direction determined by the BDE is transverse to Δ, ξ is regular.	$C_1 = 0$, $C_j \neq 0, j = 2, 5$.
Folded (saddle/node/focus)	M is smooth, $\pi\|_M$ has a fold singularity, ξ has an elementary singularity of type saddle, node or focus.	Δ is smooth, the unique direction is tangent to Δ, ξ has an elementary singularity of type folded saddle, node or focus.	$C_1 = 0$, $C_2 = 0$, $C_j \neq 0, j = 3, 4, 5$.
Folded saddle-node	M is smooth, $\pi\|_M$ has a fold singularity, and ξ has a saddle-node singularity.	Δ is smooth, the unique direction is tangent to Δ, ξ has a folded saddle-node singularity.	$C_1 = 0$, $C_2 = 0$, $C_3 = 0$, $C_j \neq 0, j = 5, 6$.
Folded node-focus	M is smooth, $\pi\|_M$ has a fold singularity, ξ has a node-focus singularity.	Δ is smooth, the unique direction is tangent to Δ, ξ has a folded node-focus singularity.	$C_1 = 0$, $C_2 = 0$, $C_4 = 0$, $C_j \neq 0, j = 3, 5$.
Morse Type 1	M has an $A_1^{\pm}$-singularity.	Δ has a Morse singularity, the unique direction is transverse to the branches of Δ in case it has an A_1^--singularity.	$C_1 = 0$, $C_7 = 0$, $C_8 = 0$, $C_j \neq 0, j = 9, 10$.

$$
\begin{aligned}
C_1 &= b_0^2 - 4c_0,\\
C_2 &= 4c_{10} - 2b_0c_{11} - 2b_0b_{10} + b_0^2b_{11},\\
C_3 &= 8c_{20} - 4b_0c_{21} + 2b_0^2c_{22} + (b_{11}b_0 - 2b_{10})c_{11} - 4b_0b_{20} + 2b_0^2b_{21} - b_0^3b_{22} - 2b_{10}^2 + 3b_{11}b_{10}b_0\\
&\quad - b_{11}^2b_0^2,\\
C_4 &= 64c_{20} - 32b_0c_{21} + 16b_0^2c_{22} - 4c_{11}^2 + (12b_{11}b_0 - 16b_{10})c_{11} - 32b_0b_{20} + 16b_0^2b_{21} - 8b_0^3b_{22}\\
&\quad - 9b_0^2b_{11}^2 + 24b_0b_{10}b_{11} - 16b_{10}^2,\\
C_5 &= (2c_{11} - b_{11}b_0)^2 + (2c_{10} - b_{10}b_0)^2,\\
C_6 &= 48c_{30} - 24b_0c_{31} + 12b_0^2c_{32} - 6b_0^3c_{33} + 6(b_{11}b_0 - 2b_{10})c_{21} - 6b_0(b_{11}b_0 - 2b_{10})c_{22}\\
&\quad + (4b_0b_{21} + 2b_{11}b_{10} - 8b_{20} - b_{11}^2b_0 - 2b_0^2b_{22})c_{11} - 3b_0(8b_{30} - 4b_0b_{31} + 2b_0^2b_{32}\\
&\quad - b_0^3b_{33}) + 8(2b_{11}b_0 - 3b_{10})b_{20} + (18b_{10}b_0 - 11b_{11}b_0^2)b_{21} + (7b_{11}b_0^3 - 12b_{10}b_0^2)b_{22}\\
&\quad + (6b_{10}^2 - 7b_{10}b_{11}b_0 + 2b_0^2b_{11}^2)b_{11},\\
C_7 &= 2c_{11} - b_{11}b_0,\\
C_8 &= 2c_{10} - b_{10}b_0,\\
C_9 &= (b_0b_{21} - 2c_{21} + b_{11}b_{10})^2 - (b_{10}^2 + 2b_0b_{20} - 4c_{20})(b_{11}^2 + 2b_0b_{22} - 4c_{22}),\\
C_{10} &= 16c_{20} - 4b_{10}^2 + b_0(4b_0c_{22} - 8c_{21} - 8b_{20} + 4b_0b_{21} + 4b_{11}b_{10} - b_{11}^2b_0 - 2b_0^2b_{22}),
\end{aligned}
$$

and

$$\lambda = \frac{C_3}{2(b_{11}b_0 - 2c_{11})^2}.$$

6.1.2 Generic families of BDEs of Type 1

Suppose that we have a 1-parameter family of BDEs of Type 1. We can write the equation of the family, without loss of generality, in the form

$$F(x,y,p,t) = p^2 + r_1(x,y,t)p + r_2(x,y,t) = 0, \tag{6.8}$$

and suppose that $j^3(r_1^t(x,y), r_2^t(x,y))$ are as in the expressions (6.6) and (6.7) respectively, with the coefficients b_{ij} and c_{ij}, $i,j = 1,2,3$ are functions of t. We denote by F_0 the BDE $F(x,y,p,0) = 0$.

We associate to the family of BDEs in $F(x,y,p,t)=0$ the jet-extension map

$$\begin{aligned}\varphi: \ & \mathbb{R}^2\times\mathbb{R},0 \to J^k(2,2)\\ & (x,y,t)\mapsto j^k(r_1(x,y),r_2(x,y))|_t.\end{aligned}$$

Let W be a submanifold of codimension $m+2$ in $J^k(2,2)$ defining the singularity of F_0. The family F is said to be a *generic* family if the map φ is transverse to W in $J^k(2,2)$ at j^kF_0, see [50].

Theorem 6.1.3 [50] A 1-parameter family F of BDEs of Type 1 is a generic family of the codimension 1 singularity of F_0 of the given type if and only if the following condition is satisfied:

(1) F_0 has a folded saddle-node singularity:

$$(1+b_{10})\frac{\partial r_1}{\partial t}+(2c_{21}-(b_{10}+1)b_{11})\frac{\partial r_2}{\partial t}-2\frac{\partial^2 r_2}{\partial t\partial x}\neq 0. \tag{6.9}$$

(2) F_0 has a folded node-focus singularity:

$$\lambda_1\frac{\partial r_1}{\partial t}+\lambda_2\frac{\partial^2 r_1}{\partial t\partial x}+\lambda_3\frac{\partial r_2}{\partial t}+\frac{\lambda_4}{2}\frac{\partial^2 r_2}{\partial t\partial y}-\frac{1}{8}\frac{\partial^3 r_2}{\partial t\partial x^2}\neq 0, \tag{6.10}$$

where the partial derivatives are evaluated at the origin and

$$\begin{aligned}
\lambda_1 &= -\frac{3}{2}(b_{10}+1)C_6+\frac{1}{16}(b_{10}+2)c_{21}+\frac{1}{16}b_{20}-\frac{1}{128}b_{11}(4b_{10}^2+10b_{10}+3),\\
\lambda_2 &= \frac{1}{32}(1+2b_{10}),\\
\lambda_3 &= 3(\frac{1}{2}b_{11}(b_{10}+1)-c_{21})c_{30}+\frac{1}{8}c_{31}+\frac{1}{8}(1+6b_{10})c_{21}^2-\frac{1}{8}(3b_{10}+1)(2b_{11}b_{10}+b_{11}-4b_{20})c_{21}\\
&\quad -\frac{1}{32}(1+2b_{10})c_{22}-\frac{1}{16}b_{11}(6b_{10}+5)(1+2b_{10})b_{20}-\frac{1}{32}(1+2b_{10})b_{21}+\frac{3}{128}b_{11}^2(1+2b_{10})^3,\\
\lambda_4 &= 3C_6+\frac{1}{16}b_{11}b_{10}-\frac{1}{8}c_{21}.
\end{aligned}$$

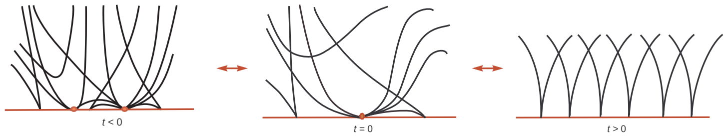
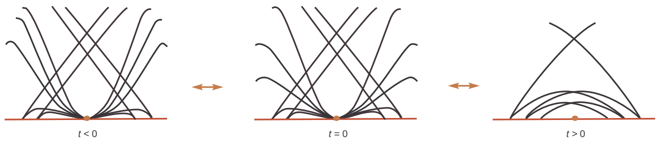
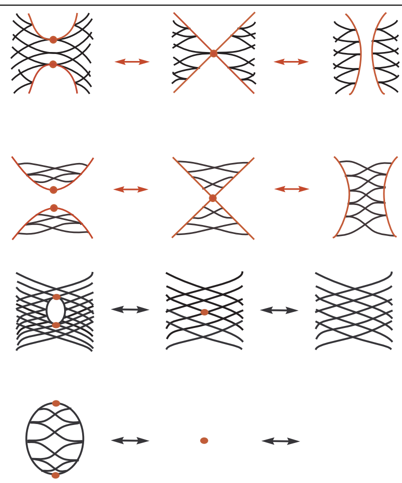
(3) F_0 has a Morse Type 1 singularity:

$$b_0\frac{\partial r_1}{\partial t}-2\frac{\partial r_2}{\partial t}\neq 0. \tag{6.11}$$

Bifurcations of codimension 1 singularities in generic 1-parameter families of BDEs are studied in [52, 49, 50].

Theorem 6.1.4 A generic family of codimension 1 singularity of BDEs of Type 1 is fibre topologically equivalent to one of the normal forms in Table 6.3, second column. See also Table 6.3, third column, for the bifurcations in the families.

Table 6.3: Generic bifurcations of codimension 1 singularities of BDEs of Type 1.

Name	Generic family	Figures
Folded saddle-node	$-y + x^3 + tx$	
Folded node-focus	$-y + (\frac{1}{16} + t)x^2$	
Morse Type 1	$\pm x^2 \pm y^2 + t$	 A_1^- (top) and A_1^+ (bottom).

6.2 BDEs of Type 2

The lifted field method is one of the methods for seeking topological models for BDEs of Type 2, see [10, 12]. When all of the coefficients of the BDE (6.1) vanish at an isolated point (origin), all the directions at the origin are solutions of the BDE. The associated surface to the BDE in this case is

$$N = \{(x, y, [\alpha : \beta]) \in \mathbb{R}^2, 0 \times \mathbb{R}P^1 : a\beta^2 + 2b\alpha\beta + c\alpha^2 = 0\}.$$

When Δ has a Morse singularity the surface N is smooth [12] and the BDE has a singularity of type *Morse Type 2* at the origin. The set $\pi^{-1}(0) = \{0\} \times \mathbb{R}P^1 \subset \pi^{-1}(\Delta)$ is called the *exceptional fibre*. The criminant is the closure of the set $\pi^{-1}(\Delta) - (\{0\} \times \mathbb{R}P^1)$. Consider the affine chart $p = \beta/\alpha$ (we also consider the chart $q = \alpha/\beta$) and set

$$F(x, y, p) = a(x, y)p^2 + 2b(x, y)p + c(x, y).$$

The bi-valued field in the plane determined by the BDE lifts to a single-valued direction field ξ (6.4) on the surface N. The vector field ξ extends smoothly to $\pi^{-1}(0)$ which is an integral curve of ξ.

The singularities of ξ on $\pi^{-1}(0)$ $(F(0, 0, p) = F_p(0, 0, p) = 0)$ are given by the roots of the cubic

$$\phi(p) = (F_x + pF_y)(0, 0, p) = a_2p^3 + (2b_2 + a_1)p^2 + (2b_1 + c_2)p + c_1,$$

where $j^1(a, b, c) = (a_1x + a_2y, b_1x + b_2y, c_1x + c_2y)$. The eigenvalues of the linear part of ξ at a singularity are $-\phi'(p)$ and $\alpha_1(p)$, where

$$\alpha_1(p) = 2(a_2p^2 + (b_2 + a_1)p + b_1).$$

The cubic $\phi(p)$ and $\alpha_1(p)$ determine the number and the type of the singularities of ξ, see [10, 12].

Remark 6.2.1 It is shown in [12] that if $\phi(p)$ and $\alpha_1(p)$ have no common roots and $\phi(p)$ has no repeated roots, then one can reduce the 1-jet of a BDE of Type 2 to the following

$$j^1(a, b, c) = (y, b_1x + b_2y, \epsilon y), \epsilon = \pm 1.$$

The codimension 1 singularities of BDEs of Type 2 are classified by the number and the type of the singularities of ξ when (b_1, b_2) is away from some special curves in the (b_1, b_2)-plane where the singularities of codimension ≥ 2 occur. These curves are:

1. The non-Morse curve: $b_1 = 0$.
2. The double root curve of ϕ: $2b_1 + \epsilon$=0 or $b_1 = \frac{1}{2}(b_2^2 - \epsilon)$.
3. The α_1 and ϕ have common root curve: $b_1 = \pm b_2 - 1$ for the case $\epsilon = 1$.

The three curves partition the (b_1, b_2)-plane into regions where the topological type of the BDE is constant and is one of the cases in Theorem 6.2.1.

Theorem 6.2.1 [12] Suppose that the discriminant has a Morse singularity, the lifted field ξ has no repeated root and ϕ and α_1 have no common root. Then, there is a germ of homeomorphism $h : \mathbb{R}^2, 0 \to \mathbb{R}^2, 0$ taking the integral curves of the BDE (6.1) to the integral curves of one of the normal forms:

(I) The discriminant has an A_1^+-singularity (Figure 6.2): (Theorem 0.1, [10])

- (a) 1 saddle: $ydy^2 + 2xdxdy - ydx^2 = 0$,
- (b) 3 saddles: $ydy^2 - 2xdxdy - ydx^2 = 0$,
- (c) 2 saddles + 1 node: $ydy^2 + \frac{1}{2}xdxdy - ydx^2 = 0$.

(II) The discriminant has an A_1^--singularity (Figure 6.3):

- (a) 1 saddle: $ydy^2 + 2xdxdy + ydx^2 = 0$,
- (b) 1 node: $ydy^2 - \frac{1}{2}xdxdy + ydx^2 = 0$,
- (c) 3 saddles: $ydy^2 - 4xdxdy + ydx^2 = 0$,
- (d) 2 saddles + 1 node: $ydy^2 + 2(y - x)xdxdy + ydx^2 = 0$,
- (e) 1 saddle + 2 nodes: $ydy^2 - \frac{4}{3}xdxdy + ydx^2 = 0$.

The type is determined by the 1-jet of the functions a, b, c.

It is shown in [13] that a 1-parameter family of BDEs, depending on the parameter t, of Morse Type 2 at $t = 0$ is generic if and only if

$$\begin{vmatrix} a_x & a_y & a_t \\ b_x & b_y & b_t \\ c_x & c_y & c_t \end{vmatrix} \neq 0, \tag{6.12}$$

where the partial derivatives are evaluated at the origin.

Theorem 6.2.2 [13] Suppose that the family of BDEs is of Morse Type 2 at $t = 0$, the lifted field ξ has no repeated roots, ϕ and α_1 have no common root and the family satisfies the versality condition (6.12). Then, the family is fibre topologically equivalent to one of the following normal forms.

(I) The discriminant has an A_1^+-singularity (Figure 6.4):

(a) 1 saddle: $(y + t)dy^2 + 2xdxdy - ydx^2 = 0$,

(b) 3 saddles: $(y + t)dy^2 - 2xdxdy - ydx^2 = 0$,

(c) 2 saddles + 1 node: $(y + t)dy^2 + \frac{1}{2}xdxdy - ydx^2 = 0$.

(II) The discriminant has an A_1^--singularity (Figure 6.5):

(a) 1 saddle: $(y + t)dy^2 + 2xdxdy + ydx^2 = 0$,

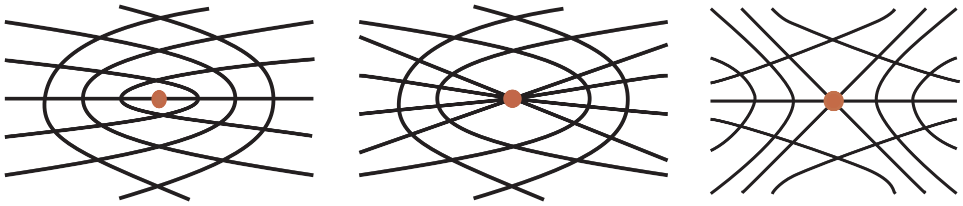
(b) 1 node: $(y + t)dy^2 - \frac{1}{2}xdxdy + ydx^2 = 0$,

(c) 3 saddles: $(y + t)dy^2 - 4xdxdy + ydx^2 = 0$,

(d) 2 saddles + 1 node: $(y + t)dy^2 + 2(y - x)xdxdy + ydx^2 = 0$,

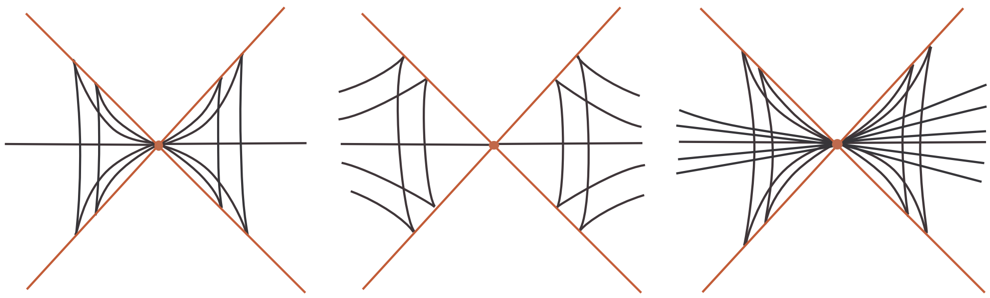
(e) 1 saddle + 2 nodes: $(y + t)dy^2 - \frac{4}{3}xdxdy + ydx^2 = 0$.

The figures on both sides of the bifurcations are equivalent, so only one side of the bifurcations is drawn in Figures 6.4 and 6.5.

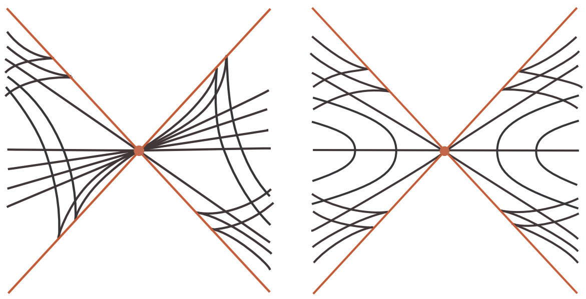


(a) 1S (b) 2S+1N (c) 3S

Figure 6.2: Morse Type 2 singularities of type A_1^+.

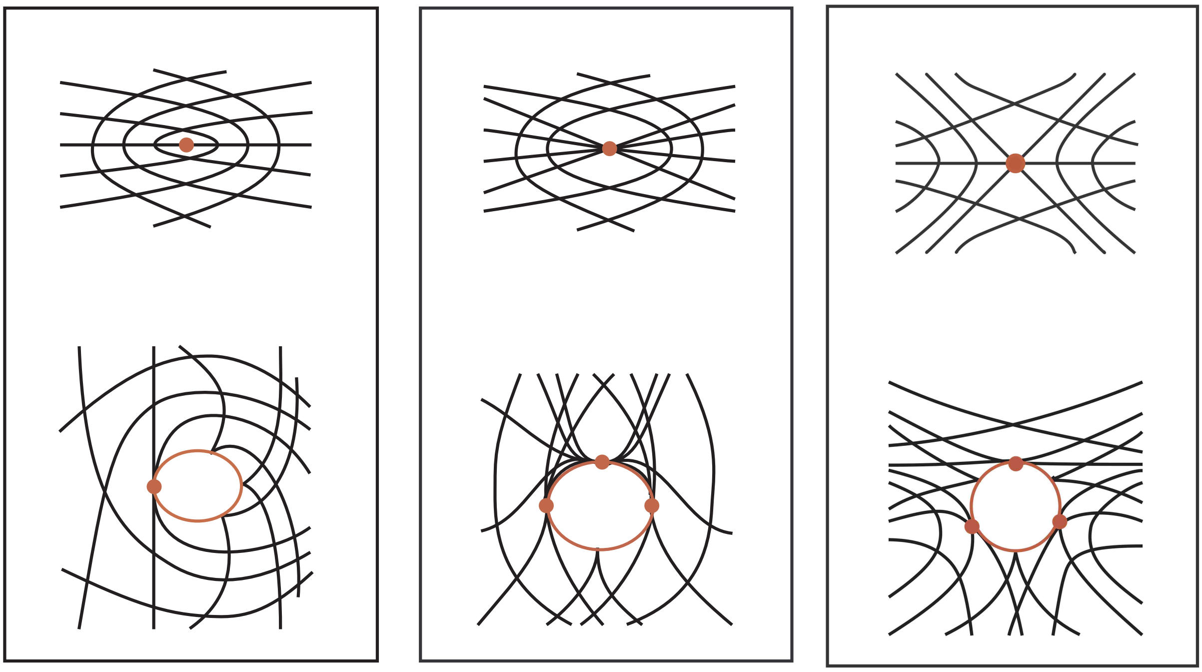


(a) 1N (b) 1S (c) 1S+2N



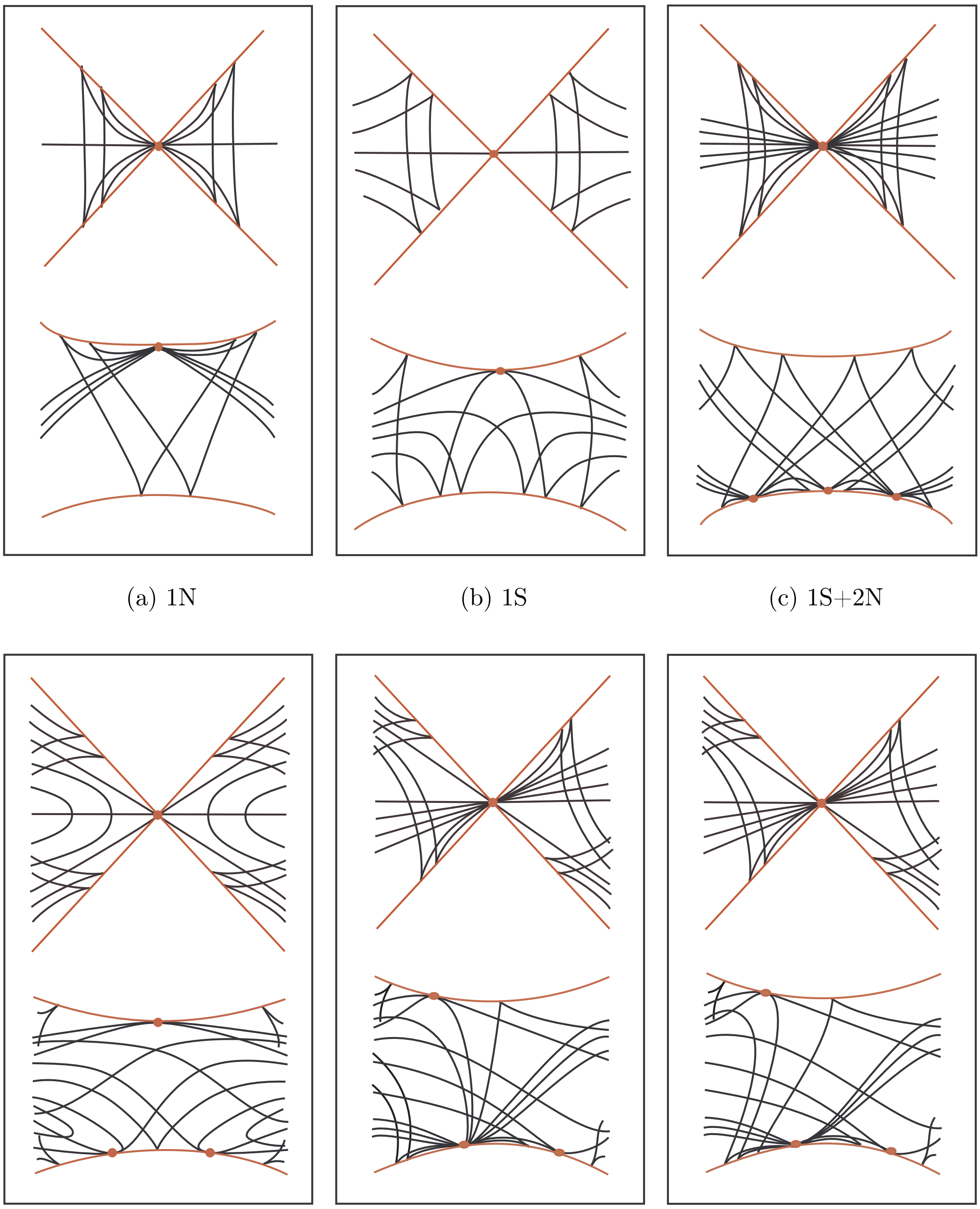
(d) 2S+1N (e) 3S

Figure 6.3: Morse Type 2 singularities of type A_1^-.



(a) 1S (b) 2S+1N (c) 3S

Figure 6.4: Bifurcations at Morse Type 2 singularities of type A_1^+.



(a) 1N (b) 1S (c) 1S+2N

(d) 3S (e) 2S+1N (case 1) (f) 2S+1N (case 2)

Figure 6.5: Bifurcations at Morse Type 2 singularities of type A_1^-.

Chapter 7

Pairs of foliations associated to self-adjoint operators

Lines of principal curvature and asymptotic curves are classical pairs of foliations on a smooth surface M in $\mathbb{R}^3$. More recently another pair of foliations called the characteristic curves, first introduced in [22], is studied in [16, 17, 24, 39]. The above three pairs of foliations are defined by the shape-operator $-dN(p)$ on M which is a self-adjoint operator on T_pM for $p \in M$. It is shown in [53] that one can associate the concept of those three pairs of foliations to any self-adjoint operator on a Riemannian surface and their behaviours are the same as the behaviours of those three pairs of foliations on M. The work in [53] is generalised in [31] to the case of self-adjoint operators on Lorentzian surfaces. The concepts of the three pairs of foliations are defined, however, the behaviours of these pairs are quite different than those of pairs on Riemannian surfaces. We recall some basic notions of differential geometry in section 7.1 which can be found in books on elementary differential geometry, see for example [21]. These notions are generalised to Lorentzian surfaces [31] as shown in section 7.2.

7.1 Pairs of foliations on surfaces in $\mathbb{R}^3$

Let M be a smooth and orientable surface in $\mathbb{R}^3$ and let S^2 be the unit sphere. The Gauss map

$$N : M \to S^2$$

associates a unit normal vector $N(p)$ to each $p \in M$. The Weingarton map (or shape-operator) is given by

$$W_p = -dN(p) : T_pM \to T_pM,$$

where T_pM in the target is equivalent to $T_{N(p)}S^2$. The Weingarten map W_p provides information about the local shape of M in $\mathbb{R}^3$. It is a self-adjoint operator that is a linear operator satisfies that $W_p(u) \cdot v = u \cdot W_p(v)$, for any $u, v \in T_pM$, where "$\cdot$" denotes the scalar product in $\mathbb{R}^3$.

The first and second fundamental forms of M are the quadratic forms on T_pM given by $I_p(u, u) = u \cdot u$ and $II_p(u, u) = W_p(u) \cdot u$ respectively. If M is parametrised by $\mathbf{r}(x, y)$ with $(x, y) \in U$ and U is an open set in $\mathbb{R}^2$, then for $u = a\mathbf{r}_x + b\mathbf{r}_y \in T_pM$, we have

$$I_p(u) = Ea^2 + 2Fab + Gb^2,$$

where

$$E = \mathbf{r}_x \cdot \mathbf{r}_x, \quad F = \mathbf{r}_x \cdot \mathbf{r}_y, \quad G = \mathbf{r}_y \cdot \mathbf{r}_y$$

are called the coefficients of I_p with respect to the parametrisation $\mathbf{r}$, and

$$II_p(u) = la^2 + 2mab + nb^2,$$

where

$$l = W_p(\mathbf{r}_x) \cdot \mathbf{r}_x = N \cdot \mathbf{r}_{xx},\ m = W_p(\mathbf{r}_x) \cdot \mathbf{r}_y = N \cdot \mathbf{r}_{xy},\ n = W_p(\mathbf{r}_y) \cdot \mathbf{r}_y = N \cdot \mathbf{r}_{yy}$$

are the coefficients of $\mathrm{II_p}$. The Gaussian curvature is given by

$$K = det(W_p) = \frac{ln - m^2}{EG - F^2},$$

and the mean curvature is given by

$$H = \frac{1}{2}tr(W_p) = \frac{lG - 2mF + nE}{2(EG - F^2)}.$$

There are three pairs of foliations given by BDEs on the surface M determined by the shape-operator W_p.

We start with a pair of foliations called the *lines of principal curvatures.* The shape-operator W_p has always real eigenvalues $k_i(p), i = 1, 2$ called the *principal curvatures* $(K(p) = (k_1k_2)(p))$*.* When $k_1(p) \neq k_2(p)$, we have two orthogonal eigenvectors called the *principal* directions. The integral curves of the principal directions on M are called the lines of principal curvature. Points where $k_1(p) = k_2(p)$ are called *umbilic* points. At such points every direction is a principal direction. They are isolated points on generic surfaces in $\mathbb{R}^3$.

The lines of principal curvature are given by the following BDE

$$(Fn - Gm)dy^2 + (En - Gl)dydx + (Em - Fl)dx^2 = 0. \tag{7.1}$$

The discriminant of the principal BDE (7.1) given by $(En-Gl)^2-4(Fn-Gm)(Em-Fl)$, which is equivalent to $H^2 - K = 0$, consists of the umbilic points. The configurations of the lines of principal curvature at umbilic points were first drawn by Darboux and rigorous studies were carried out in [10, 44]. There are three distinct generic topological configurations of the lines of principal curvature at umbilic points and these are illustrated in Figure 6.2.

Two directions $u, v \in T_pM$ are *conjugate* if $W_p(u) \cdot v = 0$. A direction in T_pM is an asymptotic direction if it is a self-conjugate direction, i.e., $W_p(u) \cdot u = 0$. There are two asymptotic directions (resp. none) at hyperbolic (resp. elliptic) points. The integral curves of the asymptotic directions are called the *asymptotic* curves. The equation of the asymptotic curves is given by the BDE

$$ndy^2 + 2mdydx + ldx^2 = 0. \tag{7.2}$$

The discriminant of the asymptotic BDE (7.2) is give by $K = 0$, equivalently, by $ln - m^2 = 0$ which is precisely the parabolic set of M. For a generic surface, at points on the parabolic set, the asymptotic direction is repeated and the asymptotic curves form a family of cusps with cusps tracing the parabolic set except at some isolated points. These isolated points are the cusps of Gauss where the repeated asymptotic direction is tangent to the parabolic set. The stable topological configurations of the integral curves at points on the parabolic set are shown in Table 6.1.

At a non-umbilic elliptic point, there is a unique pair of conjugate directions which the included angle is extremal. These directions are called the *characteristic* directions. The integral curves of the characteristic directions are the *characteristic* curves. Characteristic directions on surfaces in $\mathbb{R}^3$ are defined in [22] and studied more recently in [16, 17, 24, 39]. The equation of the characteristic curves is given by the BDE

$$\begin{aligned}&(2m(Gm - Fn) - n(Gl - En))dy^2 + 2(l(Gm - Fn) - n(Fl - Em))dydx \\ &+ (l(Gl - En) - 2m(Fl - Em))dx^2 = 0.\end{aligned} \tag{7.3}$$

The discriminant of the characteristic BDE (7.3) is the parabolic set union the umbilic points [16]. At a parabolic point, the characteristic direction is repeated. At umbilic points, every direction is a characteristic direction.

There are several properties of the characteristic curves analogous to those of the asymptotic curves. The characteristic curves foliate the elliptic region of the surface M while the asymptotic curves foliate the hyperbolic region. At parabolic points, the repeated asymptotic and the repeated characteristic directions coincide and are principal directions. At cusps of Gauss, the values of λ (Table 6.1) for the two BDEs have opposite signs but equal absolute values. This means if the asymptotic BDE has a folded saddle, the characteristic BDE has a folded node or focus and vice-versa [16].

It is shown in [16] that the BDEs of the principal, asymptotic and characteristic curves are related. A BDE (6.1) can be viewed as a quadratic form and represented at each point in U by the point $(a(x, y) : 2b(x, y) : c(x, y))$ in the projective plane. Let Γ denote the conic of degenerate quadratic forms. To each point in the projective plane is associated a unique polar line with respect to Γ and vice-versa. A triple of points is called self-polar triangle if the polar line of any point of the triple contains the remaining two points. If (x, y) is neither parabolic nor umbilic point, the triple principal, asymptotic and characteristic curves BDEs form a self-polar triangle. That means any two of them determine the third one, see [16] for more details.

Remark 7.1.1 It is observed in [53] that the concepts of principal, asymptotic and characteristic curves depend on the self-adjoint operator and the induced metric on a smooth surface M. This leads to the definitions of such concepts for any given self-adjoint operator on a smooth surface endowed with a Riemannian metric. This is generalised in [31] to the case of Lorentzian surfaces as we can see in the below section.

7.2 Pairs of foliations on Lorentzian surfaces

A *Lorentzian* surface M is a smooth and orientable surface endowed with metric g of signature 1. A non zero tangent direction $u \in T_pM$ is *spacelike* if $g(u,u) > 0$, *timelike* if $g(u,u) < 0$ and *lightlike* if $g(u,u) = 0$. The norm of the direction $u \in T_pM$ is defined by $\|u\| = \sqrt{|g(u,u)|}$. Two directions u and v in T_pM are orthogonal if $g(u,v) = 0$.

Let $\mathbf{r} : U \to M$ be a local parametrisation of M, where U is an open subset of $\mathbb{R}^2$. The first fundamental form of M at p is the quadratic form $I_p : T_pM \to \mathbb{R}$ given by $I_p(v) = g(v,v)$ for any $v \in T_pM$. If $v = a\mathbf{r}_x + b\mathbf{r}_y$, then $I_p(v) = Ea^2 + 2Fab + Gb^2$ where

$$E = g(\mathbf{r}_x, \mathbf{r}_x), \quad F = g(\mathbf{r}_x, \mathbf{r}_y), \quad G = g(\mathbf{r}_y, \mathbf{r}_y)$$

are the coefficients of I_p with respect to the parametrisation $\mathbf{r}$.

Since g is of signature 1, $EG - F^2 < 0$ on M. This means the *lightlike* BDE

$$Gdy^2 + 2Fdydx + Edx^2 = 0 \tag{7.4}$$

determines two distinct lightlike directions in T_pM at each point $p \in \mathbf{r}(U)$. Their integral curves are called the *lightlike* curves.

We can take a local chart at any point on M in such a way that the coordinate curves are the lightlike curves. In fact, this local parametrisation facilitates our study remarkably.

Theorem 7.2.1 [31] Let M be a Lorentzian surface. For every $p \in M$, there is a local parametrisation of a neighbourhood V of p, such that for any $p' \in V$, the coordinate curves through p' are the tangents to the lightlike directions. Equivalently, there exists a local parametrisation $\mathbf{r} : U \to V \subset M$ with $E = G = 0$ in U.

Let $\mathbb{A} : TM \longrightarrow TM$ be a self-adjoint operator, i.e., $\mathbb{A}$ is a linear map satisfies that $g(\mathbb{A}(u), v) = g(u, \mathbb{A}(v))$ for any $u, v \in T_pM$; and $\mathbb{A}_p$ denotes the restriction of $\mathbb{A}$ to T_pM. If $u = a\mathbf{r}_x + b\mathbf{r}_y$, the $\mathbb{A}$-second fundamental form is defined as $II_p = g(\mathbb{A}_p(u), u) = la^2 + 2mab + nb^2$ where

$$l = g(\mathbb{A}_p(\mathbf{r}_x), \mathbf{r}_x),\ m = g(\mathbb{A}_p(\mathbf{r}_x), \mathbf{r}_y),\ n = g(\mathbb{A}_p(\mathbf{r}_y), \mathbf{r}_y)$$

are referred to as the coefficients of II_p. In the basis $\{\mathbf{r}_x, \mathbf{r}_y\}$, $\mathbb{A}_p$ is given by the following matrix

$$\mathbb{A}_p = \frac{1}{EG - F^2} \begin{pmatrix} G & -F \\ -F & E \end{pmatrix} \begin{pmatrix} l & m \\ m & n \end{pmatrix}.$$

The $\mathbb{A}$-*Gaussian curvature* of M at p is defined as $\det(\mathbb{A}_p)$ and it is given by

$$K(p) = \frac{ln - m^2}{EG - F^2}.$$

The $\mathbb{A}$-*mean curvature* of M at p is defined as $1/2tr(\mathbb{A}_p)$ and is given by

$$H(p) = \frac{lG - 2mF + nE}{2(EG - F^2)}.$$

There are three pairs of foliations on M corresponding to $\mathbb{A}$ represented by BDEs [31].

7.2.1 The lines of $\mathbb{A}$-principal curvature

As the metric g is of signature 1, the self-adjoint operator $\mathbb{A}_p$ does not always have real eigenvalues. When it does they are denoted by $k_i(p), i = 1, 2$ and called the $\mathbb{A}$-*principal curvatures*; and $K(p) = k_1(p)k_2(p)$. The eigenvectors of $\mathbb{A}_p$ are called the $\mathbb{A}$-*principal directions* and the integral curves of their associated line fields are called the *lines of* $\mathbb{A}$-*principal curvature*. The BDE of the lines of $\mathbb{A}$-principal curvature is analogous to equation (7.1) and is given by

$$(Fn - Gm)dy^2 + (En - Gl)dydx + (Em - Fl)dx^2 = 0. \tag{7.5}$$

If we consider the local parametrisation of the surface in Theorem 7.2.1, the BDE (7.5) becomes

$$ndy^2 - ldx^2 = 0. \tag{7.6}$$

The discriminant of the BDE (7.5) is given by $H^2 - K = 0$ ($ln = 0$ for equation (7.6)). This is, when not empty, a curve labelled $\mathbb{A}$-*Lightlike Principal Locus* (LPL). It is a locus of points where the two $\mathbb{A}$-principal directions coincide and become lightlike. The $\mathbb{A}$-principal directions, when distinct, are orthogonal and one of them is spacelike and the other is timelike [31].

Proposition 7.2.1 [31]

1. The LPL divides the surface into two regions. In one region, there is no $\mathbb{A}$-principal direction and in the other region there are two distinct $\mathbb{A}$-principal directions at each point.

2. For generic $\mathbb{A}$, the LPL is a smooth curve except possibly at some isolated points where the LPL has Morse singularities of type A_1^-. Those points are where the matrix of $\mathbb{A}_p$ is a multiple of the identity and are labelled timelike umbilic points.

The local topological configurations of the lines of the $\mathbb{A}$-principal curvature are given in [31]. The $\mathbb{A}$-principal BDE (7.5) determines a pair of transverse foliations or none away from the LPL. On the LPL, the lines of $\mathbb{A}$-principal curvature are a family of cusps with cusps tracing the LPL except at some isolated points where the $\mathbb{A}$-principal BDE has, for generic $\mathbb{A}$, folded singularities of type saddle, node or focus (Table 6.1). At a timelike umbilic point, the $\mathbb{A}$-principal BDE has generically a Morse Type 2 singularity of type A_1^-. On the contrary, the principal BDE has a Morse Type 2 singularity of type A_1^+ at the umbilic point on Riemannian surfaces.

7.2.2 The $\mathbb{A}$-asymptotic curves

Two directions $u, v \in T_pM$ are $\mathbb{A}$-*conjugate* if $g(\mathbb{A}_p(u), v) = 0$ for $u, v \in T_pM$. A direction u is called $\mathbb{A}$-*asymptotic* if it is $\mathbb{A}$-conjugate to itself, i.e., $g(\mathbb{A}_p(u), u) = 0$. The $\mathbb{A}$-asymptotic curves are the integral curves of the $\mathbb{A}$-asymptotic directions and are determined by the BDE

$$ndy^2 + 2mdydx + ldx^2 = 0. \tag{7.7}$$

The discriminant of the $\mathbb{A}$-asymptotic BDE (7.7) is given by $K = 0$. For generic $\mathbb{A}$, it is a regular curve when not empty. The set of points where $K = 0$ is called the $\mathbb{A}$-parabolic set. Since $EG - F^2 < 0$, there are two distinct $\mathbb{A}$-asymptotic directions in the region where $K > 0$ and none in the region where $K < 0$. At a point on the $\mathbb{A}$-parabolic set, the $\mathbb{A}$-asymptotic direction is repeated.

Proposition 7.2.2 [31]

1. An $\mathbb{A}$-asymptotic direction at $p \in M$ is also an $\mathbb{A}$-principal direction if and only if p is a point on the $\mathbb{A}$-parabolic set or on the LPL. On the LPL, the $\mathbb{A}$-asymptotic direction is lightlike.

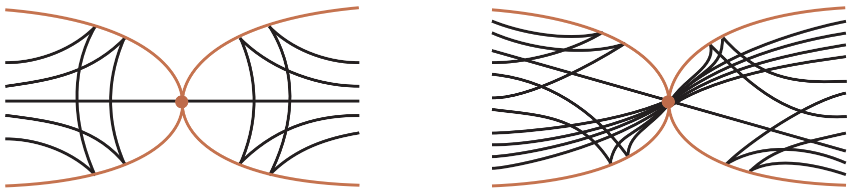
2. The $\mathbb{A}$-parabolic set and the LPL are tangential at their intersection points. On the $\mathbb{A}$ parabolic set, the unique $\mathbb{A}$-asymptotic direction is spacelike on one side of the point of tangency with the LPL and is timelike in the other side of that point.

Away from the $\mathbb{A}$-parabolic set, the $\mathbb{A}$-asymptotic curves form a pair of transverse foliations or none. On the $\mathbb{A}$-parabolic set, the $\mathbb{A}$-asymptotic curves form a family of cusps with cusps tracing the $\mathbb{A}$-parabolic set except possibly at some isolated points where the $\mathbb{A}$-asymptotic BDE has folded singularities of type saddle, node or focus (Table 6.1). For generic $\mathbb{A}$, the integral curves of the $\mathbb{A}$-principal and the $\mathbb{A}$-asymptotic BDEs form a family of cusps in the neighbourhood of the point of tangency of the LPL and the $\mathbb{A}$-parabolic set (Table 6.1).

7.2.3 The $\mathbb{A}$-characteristic curves

For given self-adjoint operator $\mathbb{A}$, the BDE of the $\mathbb{A}$-*characteristics* curves is the BDE which forms a self-polar triangle with the $\mathbb{A}$-principal and the $\mathbb{A}$-asymptotic BDEs [31]. It is given as the Jacobian of the $\mathbb{A}$-principal and $\mathbb{A}$-asymptotic BDEs. The $\mathbb{A}$-characteristics BDE is given by

$$\begin{aligned}(2m(Gm - Fn) - n(Gl - En))dy^2 + 2(l(Gm - Fn) - n(Fl - Em))dydx \\ +(l(Gl - En) - 2m(Fl - Em))dx^2 = 0. \qquad (7.8)\end{aligned}$$



(a) (b)

Figure 7.1: The local topological configurations of a BDE when its discriminant curve has an A_3^--singularity [31].

When considering the local parametrisation where $E = G = 0$, equation (7.8) becomes

$$mndy^2 + 2lndydx + mldx^2 = 0. \tag{7.9}$$

The discriminant of the $\mathbb{A}$-characteristic BDE (7.8) is given by the equation $(H^2 - K)K = 0$ ($ln(ln - m^2) = 0$ for equation (7.9)). It is the union of the $\mathbb{A}$-parabolic set and of the LPL. The $\mathbb{A}$-characteristic curves form a pair of transverse foliations in the region where $K > 0$ and $H^2 - K < 0$ as well as the region where $K < 0$ and $H^2 - K > 0$, and none elsewhere. Observe that the $\mathbb{A}$-characteristic and the $\mathbb{A}$-asymptotic curves foliate a common region where $K > 0$ and $H^2 - K < 0$.

The local configurations of the $\mathbb{A}$-characteristic BDE at points on the discriminant ($\{K = 0\} \cup \{H^2 - K = 0\}$) are obtained in [31]. On the $\mathbb{A}$-parabolic set, away from the point of tangency with the LPL, the $\mathbb{A}$-characteristic curves form a family of cusps with cusps tracing the $\mathbb{A}$- parabolic set except at some isolated points. At these isolated points, the $\mathbb{A}$-characteristics BDE has, for generic $\mathbb{A}$, a folded singularity of type saddle, node or focus (Table 6.1). The folded singularities of the $\mathbb{A}$-characteristic and $\mathbb{A}$-asymptotic BDEs are common and have opposite indices.

At timelike umbilic points, the $\mathbb{A}$-characteristic BDE has generically a Morse Type 2 singularity of type A_1^-. All the five generic topological models of such singularities may occur (Figure 6.3). Moreover, it is shown in [31] that at the point of tangency of the LPL and the $\mathbb{A}$-parabolic set the $\mathbb{A}$-characteristic BDE has an A_3^--singularity for generic $\mathbb{A}$. At that point, the $\mathbb{A}$-characteristic BDE is topologically equivalent to

$$ydy^2 + xdydx + y^3dx^2 = 0, \text{ Figure 7.1}(a) \text{ or to}$$
$$ydy^2 - xdydx + y^3dx^2 = 0. \text{ Figure 7.1}(b).$$

The results above can be applied to timelike surfaces in the de Sitter space $S^3_1 \subset \mathbb{R}^4_1$ and to timelike surfaces in the Minkowski space $\mathbb{R}^3_1$. The pairs of foliations on M in $S^3_1 \subset \mathbb{R}^4_1$ are determined by the shape-operator $\mathbb{A}_p(u) = -d\mathbb{E}_p(u)$ where $\mathbb{E} : U \subset \mathbb{R}^2 \to S^3_1$ is the de Sitter Gauss map of M [30, 31]. On the other hand, the pairs of foliations on timelike surfaces in $\mathbb{R}^3_1$ are determined by the shape-operator $\mathbb{A}_p(u) = -dN_p(u)$ where $N : U \subset \mathbb{R}^2 \to \mathbb{H}^2$ is the hyperbolic Gauss map of the timelike surface M [31].

Chapter 8

Families of curve congruences on Lorentzian surfaces

We recalled in Chapter 7 the three pairs of foliations on smooth surfaces in $\mathbb{R}^3$ namely the principal, asymptotic and characteristic curves. These curves were treated independently until Fletcher [23] discovered a natural 1-parameter family of BDEs linking the principal BDE to the asymptotic BDE. Fletcher called the family *conjugate curve congruences* and denoted it by $\mathcal{C}_\alpha$. Basically, for each point $\alpha \in [0, \pm\frac{\pi}{2}]$ the BDE $\mathcal{C}_\alpha$ determines two directions that have an oriented angle α between the directions and their conjugate directions. Then $\mathcal{C}_0$ is the asymptotic BDE (7.2) and $\mathcal{C}_{\pm\frac{\pi}{2}}$ is the principal BDE (7.1).

The result of Fletcher is interpreted in [16] using pencils of quadratic forms. As a result, another natural 1-parameter family was introduced. This family is called *the reflected conjugate curve congruences* and denoted by $\mathcal{R}_\alpha$. This family links the principal BDE (7.1) to the characteristic BDE (7.3).

In this chapter we study the natural 1-parameter family of $\mathbb{A}$-conjugate curve congruences denoted by $\mathcal{LC}^i_\alpha, i = 1, 2$ (resp. reflected $\mathbb{A}$-conjugate curve congruences denoted by $\mathcal{LR}^i_\alpha, i = 1, 2$) which is defined in [35]. In this case, α is the oriented hyperbolic angle. We start with recalling some basic propositions of oriented hyperbolic angles and then proceeding to study the families $\mathcal{LC}^i_\alpha$ and $\mathcal{LR}^i_\alpha, i = 1, 2$.

8.1 Oriented hyperbolic angles

Let $g(u, v) = u_1v_1 - u_2v_2$ denote the pseudo-scalar product for $u = (u_1, u_2)$ and $v = (v_1, v_2)$ in $\mathbb{R}^2_1$. A timelike vector u is said to be future-pointing if $u_2 > 0$, and past-pointing otherwise. The norm or length of u is $\|u\| = \sqrt{|g(u, u)|}$.

Let G be the proper Lorentz group of $\mathbb{R}^2_1$, i.e., the group consisting of all orientation-preserving linear transformation of $\mathbb{R}^2$ which preserve the pseudo-scalar product and the time-orientation. It consists all matrices of the form

$$R(\alpha) = \begin{pmatrix} \cosh(\alpha) & \sinh(\alpha) \\ \sinh(\alpha) & \cosh(\alpha) \end{pmatrix}$$

where $\alpha \in \mathbb{R}$. The oriented hyperbolic angle $\alpha = \angle(u, v)$ between two non-lightlike vectors u, v in $\mathbb{R}^2_1$ is defined in [6, 7, 36, 37] as follows, and has similar properties to the oriented Euclidean angle.

If u, v are both future or past pointing unit timelike vectors, then α is defined by the relation $R(\alpha)u = v$ and satisfies

$$\cosh(\alpha) = -g(u, v), \quad \sinh(\alpha) = -g(u, Sv) \tag{8.1}$$

where Sv is the spacelike vector obtained from v by the Euclidean reflection

$$S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with respect to the lightlike line $u_1 = u_2$. If u, v are both unit timelike vectors but one is a future-pointing and the other is a past-pointing, then $\angle(u, v) := \angle(u, -v)$, where $-v$ is the unit timelike vector with time-orientation similar to u relevant to the vector v. Therefore, $\alpha = \angle(u, v)$ is defined by the relation $R(\alpha)u = -v$ and satisfies

$$\cosh(\alpha) = g(u, v), \quad \sinh(\alpha) = g(u, Sv).$$

If u, v are unit spacelike vectors with u_1 and v_1 are of the same sign, then Su and Sv are unit timelike vectors with the same time-orientation relevant to u and v. By substituting Su and Sv in (8.1), we get $\angle(u, v) := -\angle(Su, Sv)$. On the other hand, if u_1 and v_1 are of different sign, then the vectors Su and Sv are unit timelike vectors of different time-orientations relevant to u and v, and then $\angle(u, v) := \angle(Su, Sv)$.

If u, v are unit vectors of different types, and supposing that u is timelike, then $\angle(u, v) := -\angle(u, Sv)$.

For non-lightlike vectors u, v of arbitrary lengths, the above formulas are divided by the product of the lengths of u and v. For example, if u, v are future-pointing timelike vectors, then

$$\cosh(\alpha) = \frac{-g(u,v)}{\|u\|\|v\|}, \quad \sinh(\alpha) = \frac{-g(u,Sv)}{\|u\|\|v\|}.$$

The notation of oriented hyperbolic angles can be defined in any Lorentzian plane with a metric g of signature 1. Let l_1 and l_2 be two lightlike independent vectors which are basis of such a Lorentzian plane. Then, $e_1 = \frac{2(l_1+l_2)}{g(l_1+l_2,l_1+l_2)}$ and $e_2 = \frac{2(l_1-l_2)}{g(l_1-l_2,l_1-l_2)}$ form an orthonormal basis of this plane. Suppose, without loss of generality, that e_1 is spacelike, then e_2 is timelike. Then, the plane with the basis $\{e_1, e_2\}$ can be identified as the Minkowski plane $\mathbb{R}^2_1$.

Remarks 8.1.1 [35]

1. The oriented hyperbolic angle between a lightlike vector and any other vector is not well defined. If u is fixed as a non-lightlike vector and v tends to a lightlike vector, then $\angle(u, v)$ tends to $\pm\infty$.

2. For a Lorentzian surface M, the oriented hyperbolic angle between two tangent directions at $p \in M$ is the oriented hyperbolic angle between any of their respective directional vectors. The oriented hyperbolic angle does not depend on the choice of the vectors as $\angle(u, v) = \angle(-u, v) = \angle(u, -v) = \angle(-u, -v)$.

8.2 The Lorentzian $\mathbb{A}$-conjugate curve congruences

In [35], a natural 1-parameter family of BDEs linking the $\mathbb{A}$-asymptotic curves to the $\mathbb{A}$-principal curves BDEs on a Lorentzian surface M is constructed. The approach of the construction follows that in [23] for Riemannian surfaces with considering oriented hyperbolic angles instead. The directions $u \in T_pM$ that make a fixed oriented hyperbolic angle with their $\mathbb{A}$-conjugate directions $\bar{u}$ are considered. There are two cases depending on whether u and $\bar{u}$ are of the same type (both spacelike or timelike) or are of different types (one is spacelike and the other is timelike). Each case produces a distinct family of curve congruences on M.

Definition 8.2.1 [35] Let M be a smooth oriented Lorentzian surface. Given a self-adjoint operator $\mathbb{A}$ on M, define $\Theta_i : TM \longrightarrow \mathbb{R}$, $i = 1, 2$, by $\Theta_i(p, u) = \alpha = \angle(u, \bar{u})$, with $i = 1$ when u and its $\mathbb{A}$-conjugate direction $\bar{u}$ are of the same type, and $i = 2$ when they are of different types. The Lorentzian $\mathbb{A}$-conjugate curve congruences, for a fixed α, is defined to be $\Theta_i^{-1}(\alpha)$, $i = 1, 2$, and is denoted by $\mathcal{LC}^i_\alpha$.

Remark 8.2.1 In Definition 8.2.1, note that Θ_1 is not well defined at points corresponding to $\mathbb{A}$-asymptotic directions on the $\mathbb{A}$-parabolic set. Moreover, Θ_2 is not well defined at points on the LPL where the unique $\mathbb{A}$-principal is a self $\mathbb{A}$-conjugate lightlike direction. In addition, Θ_2 is not well defined at timelike umbilic points on the LPL where any direction is an $\mathbb{A}$-principal direction.

It is shown in [35] that $\mathcal{LC}^i_\alpha, i = 1, 2$ are given by BDEs. In the following theorem, (P) and (A) are respectively the BDEs (7.5) and (7.7) given in Chapter 7.

Theorem 8.2.1 [35] The Lorentzian $\mathbb{A}$-conjugate curve congruences are given by BDEs and are as follows.

If u and $\bar{u}$ are both spacelike or both timelike:

$$(\mathcal{LC}^1_\alpha) : \sinh(\alpha)\, P + \sqrt{F^2 - EG}\, \cosh(\alpha)\, A = 0. \tag{8.2}$$

If u is spacelike and $\bar{u}$ is timelike or vice-versa:

$$(\mathcal{LC}^2_\alpha) : \cosh(\alpha)\, P + \sqrt{F^2 - EG}\, \sinh(\alpha)\, A = 0. \tag{8.3}$$

Remark 8.2.2 [35]

1. According to Theorem 8.2.1, there are at most two directions in T_pM which make a fixed oriented hyperbolic angle α with their $\mathbb{A}$-conjugate directions.

2. For shape-operators on smooth surfaces in $\mathbb{R}^3$ (and for self-adjoint operators on Riemannian surfaces in general) the BDE of the family of conjugate curve congruences is given by

$$(\mathcal{C}_\alpha) :\ \sin(\alpha)\, P - \sqrt{EG - F^2}\, \cos(\alpha)\, A = 0,$$

where $\alpha \in [0, \pm\pi/2]$, P is the principal BDE (7.1) and A is the asymptotic BDE (7.2), see [23]. The family $(\mathcal{C}_\alpha)$ contains the asymptotic BDE $(\mathcal{C}_0)$ and the principal BDE $(\mathcal{C}_{\pm\pi/2})$. However, from Theorem 8.2.1, the family $(\mathcal{LC}^1_\alpha)$ contains the $\mathbb{A}$-asymptotic BDE $(\mathcal{LC}^1_0)$, but not the $\mathbb{A}$-principal BDE. On the other hand, the family $(\mathcal{LC}^2_\alpha)$ contains the $\mathbb{A}$-principal BDE $(\mathcal{LC}^2_0)$, but not the $\mathbb{A}$-asymptotic BDE. In fact, the families $(\mathcal{LC}^1_\alpha)$ and $(\mathcal{LC}^2_\alpha)$ are best understood by considering pencils of forms in the projective plane. At each point on M, for $\alpha \in \mathbb{R}$, the families $(\mathcal{LC}^1_\alpha)$ and $(\mathcal{LC}^2_\alpha)$ are two disjoint open intervals of the pencil joining the $\mathbb{A}$-asymptotic BDE and the lines of $\mathbb{A}$-principal curvature BDE. The union of the closure of these intervals is the full pencil. The boundary BDEs $\mathcal{LC}^1_{-\infty} = \mathcal{LC}^2_{-\infty} = \mathcal{LC}_{-\infty}$ and $\mathcal{LC}^1_{+\infty} = \mathcal{LC}^2_{+\infty} = \mathcal{LC}_{+\infty}$ have the property that one of the solution curves of one BDE is a lightlike foliation and one of the solution curves of the other is the other lightlike foliations of M. The $\mathbb{A}$-asymptotic BDE is a member of $(\mathcal{LC}^1_\alpha)$ and the $\mathbb{A}$-principal BDE is a member of $(\mathcal{LC}^2_\alpha)$. Therefore, the BDEs $\mathcal{LC}_{\pm\infty}$ form an obstruction for linking the $\mathbb{A}$-asymptotic and the $\mathbb{A}$-principal curves BDEs via the families of $\mathbb{A}$-conjugate curve congruences. This phenomenon is explained in more details in [35].

We shall find the equations of the discriminants of $\mathcal{LC}^i_\alpha, i = 1, 2$ and denote them by $\Delta^\alpha_{\mathcal{LC}^i}, i = 1, 2$.

Proposition 8.2.1 The equation of the discriminant of $\mathcal{LC}^1_\alpha$ is given by

$$\Delta^\alpha_{\mathcal{LC}^1} : \; H^2 \sinh^2(\alpha) + K = 0, \tag{8.4}$$

while the equation of the discriminant of $\mathcal{LC}^2_\alpha$ is given by

$$\Delta^\alpha_{\mathcal{LC}^2} : \; H^2 \cosh^2(\alpha) - K = 0, \tag{8.5}$$

where K and H are the $\mathbb{A}$-Gaussian curvature and the $\mathbb{A}$-mean curvature respectively.

Proof: From equation (8.2), the coefficients of $\mathcal{LC}^1_\alpha$ are given by

$$a = (Fn - Gm)\sinh\alpha + n\sqrt{F^2 - EG}\,\cosh\alpha,$$

$$b = \frac{1}{2}((En - Gl)\sinh\alpha + 2m\sqrt{F^2 - EG}\ \cosh\alpha),$$
$$c = (Em - Fl)\sinh\alpha + l\sqrt{F^2 - EG}\ \cosh\alpha.$$

By substituting a, b and c in the equation of the discriminant $b^2 - ac = 0$ and expanding, we get

$$(\frac{1}{4}E^2n^2 - \frac{1}{2}nlEG + \frac{1}{4}G^2l^2 + m^2EG - GmFl - FnEm + nlF^2)\sinh^2(\alpha)$$
$$+ (nlEG + m^2F^2 - m^2EG - nlF^2)\cosh^2(\alpha) = 0.$$

Replacing $\cosh^2(\alpha)$ by $1 + \sinh^2(\alpha)$ in the above equation and dividing by $(EG - F^2)^2$ yields

$$(\frac{lG - 2mF + nE}{2(EG - F^2)})^2\sinh^2(\alpha) + \frac{ln - m^2}{EG - F^2} = 0,$$

which is equation (8.4) in the proposition. Similar steps lead to the discriminant equation (8.5) of $\mathcal{LC}^2_\alpha$. $\square$

The local geometric properties of the BDEs are clearly brought up when we consider the special local parametrisation in Theorem 7.2.1 that is when the coordinate curves are lightlike ($E = G = 0$). For this purpose, we express the equations of the BDEs $\mathcal{LC}^i_\alpha, i = 1, 2$ and their discriminants in that local coordinate and suppose, without loss of generality, that $F > 0$. In this case, the set where the $\mathbb{A}$-mean curvature vanishes is given by $m(x, y) = 0$.

Remark 8.2.3 Consider the local parametrisation where $E = G = 0$. Then equations (8.2) and (8.3) of $\mathcal{LC}^i_\alpha, i = 1, 2$ and equations (8.4) and (8.5) of their discriminants are given by

$$\mathcal{LC}^1_\alpha : n(1 + \tanh(\alpha))dy^2 + 2mdxdy + l(1 - \tanh(\alpha))dx^2 = 0, \tag{8.6}$$

$$\Delta^\alpha_{\mathcal{LC}^1} : m^2\cosh^2(\alpha) - ln = 0, \tag{8.7}$$

and

$$\mathcal{LC}^2_\alpha : n(1 + \tanh(\alpha))dy^2 + 2m\tanh(\alpha)dxdy + l(\tanh(\alpha) - 1)dx^2 = 0, \tag{8.8}$$

$$\Delta^\alpha_{\mathcal{LC}^2} : m^2\sinh^2(\alpha) + ln = 0 \tag{8.9}$$

Proposition 8.2.2 Away from the discriminants $\Delta^{\alpha}_{\mathcal{LC}^i}, i = 1, 2$, we have the following.

1. In the region where $H^2 - K > 0$, the directions determined by $\mathcal{LC}^1_\alpha$ (resp. $\mathcal{LC}^2_\alpha$) are of the same type spacelike or timelike (resp. of different types). In the region where $H^2 - K < 0$, the directions determined by $\mathcal{LC}^1_\alpha$ (resp. $\mathcal{LC}^2_\alpha$ for $\alpha \neq 0$) are of different types (resp. of the same type).

2. The BDEs $\mathcal{LC}^i_\alpha$ and $\mathcal{LC}^i_{-\alpha}$ are $\mathbb{A}$-conjugate BDEs for fixed i.

Proof: We consider the local parametrisation where $E = G = 0$. The directions determined by $\mathcal{LC}^1_\alpha$ are along

$$u_j = n(1 + \tanh(\alpha))\, r_x - m(-1)^j \sqrt{m^2 - ln(1 - \tanh^2(\alpha))}\, r_y, \tag{8.10}$$

in T_pM where $j = 1, 2$. Since we have

$$g(u_1, u_1)g(u_2, u_2) = 4F^2 n^2 (ln)(1 + \tanh(\alpha))^2 (1 - \tanh^2(\alpha)),$$

we conclude that u_1, u_2 are of the same type when $ln > 0$ and of different types when $ln < 0$. (Observe that with the local parametrisation given in Theorem 7.2.1 ln has the same sign as $H^2 - K$.) Similar calculations for the case of $\mathcal{LC}^2_\alpha$ lead to the result in the first assertion.

Suppose that α is positive in the expression of u_j (8.10), then $u_j, j = 1, 2$ are the directions determined by the BDE $\mathcal{LC}^1_\alpha$. Without loss of generality, the direction u_2 and its $\mathbb{A}$-conjugate $\bar{u_2}$ can be written as $u_2 = r_x + \xi r_y$, where

$$\xi = \frac{-m + \sqrt{m^2 - ln(1 - \tanh^2(\alpha))}}{n(1 + \tanh(\alpha))},$$

and $\bar{u_2} = r_x + \eta r_y$. Since the directions u_2 and $\bar{u_2}$ satisfy $g(\mathbb{A}_p(u_2), \bar{u_2}) = 0$, we get $\eta = -(\xi m + l)/\xi n + m$, see [35]. That implies

$$\eta = \frac{-m + \sqrt{m^2 - ln(1 - \tanh^2(\alpha))}}{n(1 - \tanh(\alpha))}.$$

Therefore, the direction $\bar{u_2}$ is one of the directions determine by $\mathcal{LC}^1_{-\alpha}$. Similar calculations show that the $\mathbb{A}$-conjugate of u_1 is the other direction determined by $\mathcal{LC}^1_{-\alpha}$. The proof for $\mathcal{LC}^2_{\pm\alpha}$ follows similarly. $\square$

Remark 8.2.4 Fix $\alpha \neq 0$, despite $\mathcal{LC}^i_{+\alpha} \neq \mathcal{LC}^i_{-\alpha}$, we have $\Delta^{+\alpha}_{\mathcal{LC}^i} = \Delta^{-\alpha}_{\mathcal{LC}^i}, i = 1, 2$.

8.3 The Lorentzian reflected $\mathbb{A}$-conjugate curve congruences

The family of curve congruences on M given by replacing the involution $C(u)$ with $(R \circ C)(u)$ is also investigated in [35]. The aim is to construct a family of BDEs which links the $\mathbb{A}$-characteristic BDE to the $\mathbb{A}$-principal BDE. It is proved in [35] that there are at most two directions in T_pM that make an oriented hyperbolic angle α with the reflection of their $\mathbb{A}$-conjugate directions with respect to either of the $\mathbb{A}$-principal directions when the $\mathbb{A}$-principal directions exist at p. This gives two families of BDEs called the *Lorentzian reflected $\mathbb{A}$-conjugate curve congruences* denoted by $\mathcal{LR}_{\alpha}^{i}, i = 1, 2$.

Definition 8.3.1 [35] Let M be a smooth oriented Lorentzian surface. Given a self-adjoint operator $\mathbb{A}$ on M, define $\Phi_i : TM \longrightarrow \mathbb{R}, i = 1, 2$ by $\Phi_i(p, u) = \alpha = \angle(u, R(\bar{u}))$, with $i = 1$ when u and $R(\bar{u})$ are of the same type, and $i = 2$ when u and $R(\bar{u})$ are of different types. The Lorentzian reflected $\mathbb{A}$-conjugate curve congruences, for a fixed α, is defined as $\Phi_i^{-1}(\alpha), i = 1, 2$, and is denoted by $\mathcal{LR}_{\alpha}^{i}, i = 1, 2$.

Remark 8.3.1 In Definition 8.3.1, note that Φ_i are not well defined at points (p, u) with $\delta(p) \leq 0$ where $\delta(p) = 0$ is the LPL. In addition, Φ_1 is not well defined at points corresponding to the unique $\mathbb{A}$-characteristic direction on the $\mathbb{A}$-parabolic set.

It is noted in [35] that if the oriented hyperbolic angle α between u and $R(\bar{u})$ is well defined, it can be considered as the sum of the oriented hyperbolic angles between u and an $\mathbb{A}$-principal direction e and $\bar{u}$ and e. This does not depend on the choice of the $\mathbb{A}$-principal direction. The following theorem expresses the BDEs $\mathcal{LR}_{\alpha}^{i}, i = 1, 2$, where (P) and (C) are respectively the BDEs (7.5) and (7.8) in Chapter 7.

Theorem 8.3.1 [35] The Lorentzian reflected $\mathbb{A}$-conjugate curve congruences are given by BDEs and are as follows.

If u and $\bar{u}$ are both spacelike or both timelike:

$$(\mathcal{LR}_{\alpha}^{1}) : \cosh(\alpha)\, C + \frac{2Fm - Gl - En}{\sqrt{F^2 - EG}} \sinh(\alpha)\, P = 0. \tag{8.11}$$

If u is spacelike and $\bar{u}$ is timelike or vice-versa:

$$(\mathcal{LR}^2_\alpha) : \sinh(\alpha)\, C + \frac{2Fm - Gl - En}{\sqrt{F^2 - EG}} \cosh(\alpha)\, P = 0. \tag{8.12}$$

Remarks 8.3.1 1. When there is no $\mathbb{A}$-principal directions or when there is a unique $\mathbb{A}$-principal direction at $p \in M$, the directions defined by the BDEs $\mathcal{LR}^i_\alpha, i = 1, 2$ given in Theorem 8.3.1 at p do not have the geometric characterisation in terms of the oriented hyperbolic angle provided in Definition 8.3.1. The directions, however, have a characterisation in terms of pencils of quadratic forms, see [35] for details.

2. For the shape-operators on smooth surfaces in $\mathbb{R}^3$ (and for self-adjoint operators on Riemannian surfaces in general), the reflected curve congruences $\mathcal{R}_\alpha$ is given by
$$\cos(\alpha)C + \frac{2Fm - Gl - En}{\sqrt{EG - F^2}} \sin(\alpha)P = 0.$$
where $\alpha \in [0, \pm\pi/2]$, see [16]. The family $\mathcal{R}_\alpha$ contains the characteristic curves BDE ($\mathcal{R}_0$) and the principal curves BDE ($\mathcal{R}_{\pm\pi/2}$). However, according to Theorem 8.3.1, the family $\mathcal{LR}^1_\alpha$ contains the $\mathbb{A}$-characteristic curves BDE ($\mathcal{LR}^1_0$) but not the $\mathbb{A}$-principal curve BDE. On the other hand, the family $\mathcal{LR}^2_\alpha$ contains the $\mathbb{A}$-principal curve BDE ($\mathcal{LR}^2_0$) but not the $\mathbb{A}$-characteristic curves BDE, the phenomenon is justified in similar way to that of $\mathcal{LC}^i_\alpha$ but with considering $\mathcal{LR}_{\pm\infty}$ as the obstructions, see [35].

3. The BDE $\mathcal{LR}^2_0$ gives $HP = 0$, where H is the $\mathbb{A}$-mean curvature and P is the $\mathbb{A}$-principal BDE. Observe that multiplying the $\mathbb{A}$-principal BDE with the function H results nothing but changing the direction of the vector field of the $\mathbb{A}$-principal BDE according to the sign of H. Moreover, the set of points on M satisfy that $H = 0$ constitutes a curve of singular points of the vector field. The existence of such a singular curve does not influence the configurations of the integral curves determined by the $\mathbb{A}$-principal BDE.

We shall find the equations of the discriminants of $\mathcal{LR}^i_\alpha, i = 1, 2$.

Proposition 8.3.1 The equation of the discriminant of the BDE $\mathcal{LR}^1_\alpha$ is given by

$$\Delta^\alpha_{\mathcal{LR}^1} : \ (H^2 \sinh^2(\alpha) - K \cosh^2(\alpha))(H^2 - K) = 0, \tag{8.13}$$

while the equation of the discriminant of $\mathcal{LR}^2_\alpha$ is given by

$$\Delta^{\alpha}_{\mathcal{LR}^2}: \ (H^2\cosh^2(\alpha) - K\sinh^2(\alpha))(H^2 - K) = 0. \tag{8.14}$$

where K and H are the $\mathbb{A}$-Gaussian curvature and the $\mathbb{A}$-mean curvature respectively.

Proof: To simplify the calculation, we rewrite the BDE $\mathcal{LR}^1_\alpha$ (8.11) in the form

$$\frac{\cosh(\alpha)}{2\sqrt{F^2-EG}}\,C + H\sinh(\alpha)P = 0, \tag{8.15}$$

The coefficients of the BDE (8.15) are

$$\begin{aligned}
a &= \frac{\cosh(\alpha)}{2\sqrt{F^2-EG}}(2m^2G - 2mnF - nlG + n^2E) + H\sinh(\alpha)(Fn - Gm),\\
b &= \frac{1}{2}(\frac{\cosh(\alpha)}{\sqrt{F^2-EG}}(mlG + mnE - 2nlF) + H\sinh(\alpha)(En - Gl)),\\
c &= \frac{\cosh(\alpha)}{2\sqrt{F^2-EG}}(l^2G - nlE - 2mlF + 2m^2E) + H\sinh(\alpha)(Em - Fl),
\end{aligned}$$

By expanding the equation of the discriminants $b^2 - ac = 0$ with the values above, and rearranging terms, the equation of $\Delta^{\alpha}_{\mathcal{LR}^1}$ is as given in the proposition. Similarly, if we rewrite the BDE $\mathcal{LR}^2_\alpha$ (8.12) in the form

$$\frac{\sinh(\alpha)}{2\sqrt{F^2-EG}}\,C + H\cosh(\alpha)P = 0, \tag{8.16}$$

then the coefficients of $\mathcal{LR}^2_\alpha$ (8.16) are given by

$$\begin{aligned}
a &= \frac{\sinh(\alpha)}{2\sqrt{F^2-EG}}(2m^2G - 2mnF - nlG + n^2E) + H\cosh(\alpha)(Fn - Gm),\\
b &= \frac{1}{2}(\frac{\sinh(\alpha)}{\sqrt{F^2-EG}}(mlG + mnE - 2nlF) + H\cosh(\alpha)(En - Gl)),\\
c &= \frac{\sinh(\alpha)}{2\sqrt{F^2-EG}}(l^2G - nlE - 2mlF + 2m^2E) + H\cosh(\alpha)(Em - Fl).
\end{aligned}$$

Similar steps to those of finding the discriminant of $\mathcal{LR}^1_\alpha$ lead to the equation of the discriminant of $\mathcal{LR}^2_\alpha$ as shown in the proposition. $\square$

We denote by D^1_α (resp. D^2_α) the sets given by $H^2\cosh^2(\alpha) - K\sinh^2(\alpha) = 0$ (resp. $H^2\cosh^2(\alpha) - K\sinh^2(\alpha) = 0$). We derive from equations (8.13) and (8.14) that $\Delta^{\alpha}_{\mathcal{LR}^i} = LPL \cup D^i_\alpha$, $i = 1, 2$. The discriminant $\Delta^0_{\mathcal{LR}^1}$ is the union of the $\mathbb{A}$-parabolic set and the LPL while $\Delta^0_{\mathcal{LR}^2}$ is the union of the singular curve $H = 0$ and the LPL.

Remark 8.3.2 The equation of the Lorentzian reflected $\mathbb{A}$-conjugate curve congruences and their discriminants are as the following in the local parametrisation where the coordinate curves are lightlike:

$$(\mathcal{LR}^1_{\alpha}) : \; mn(\tanh(\alpha) - 1)dy^2 - 2lndxdy + ml(-\tanh(\alpha) - 1)dx^2 = 0, \quad (8.17)$$

$$(\Delta^{\alpha}_{\mathcal{LR}^1}) : (ln \cosh^2(\alpha) - m^2) \; ln = 0, \quad (8.18)$$

and

$$(\mathcal{LR}^2_{\alpha}) : mn(1-\tanh(\alpha))dy^2 - 2ln \tanh(\alpha)dxdy + ml(-\tanh(\alpha) - 1)dx^2 = 0, \quad (8.19)$$

$$(\Delta^{\alpha}_{\mathcal{LR}^2}) : (ln \sinh^2(\alpha) + m^2) \; ln = 0. \quad (8.20)$$

Proposition 8.3.2 Away from the discriminants $\Delta^{\alpha}_{\mathcal{LR}^i}, i = 1, 2$, the directions determined by the BDEs $\mathcal{LR}^i_{\alpha}, i = 1, 2$ are $\mathbb{A}$-conjugate to each other. In the region where $H^2 - K > 0$, the type (spacelike/timelike) of the $\mathbb{A}$-conjugate directions determined by $\mathcal{LR}^i_{\alpha}, i = 1, 2$ follows Theorem 8.3.1. Nevertheless, in the region where $H^2 - K < 0$, the $\mathbb{A}$-conjugate directions defined by $\mathcal{LR}^1_{\alpha}$ are of different types and those determined by $\mathcal{LR}^2_{\alpha}$ are of the same type.

Proof: The technique of the proof is similar to that of Proposition 8.2.2. □

Remark 8.3.3

1. Let u_1, u_2 be the $\mathbb{A}$-conjugate directions determined by $\mathcal{LR}^i_{\alpha}, i = 1, 2$, for fixed i. Then, the oriented hyperbolic angle $\alpha = \angle(u_1, R(u_2)) = \angle(u_1, e) + \angle(e, u_2)$ in the region where $H^2 - K > 0$.

2. On the discriminants of $\mathcal{LR}^i_{\alpha}, i = 1, 2$, the unique direction u determined by $\mathcal{LR}^i_{\alpha}, i = 1, 2$, for fixed i, is a self $\mathbb{A}$-conjugate direction. On D^1_{α}, for $\alpha \neq 0$, the oriented hyperbolic angle α satisfies $\angle(u, R(\bar{u})) = 2\angle(u, e)$.

3. Fix $\alpha \neq 0$, despite $\mathcal{LR}^i_{+\alpha} \neq \mathcal{LR}^i_{-\alpha}$, we have $\Delta^{+\alpha}_{\mathcal{LR}^i} = \Delta^{-\alpha}_{\mathcal{LR}^i}, i = 1, 2$.

8.4 Properties of $\Delta^{\alpha}_{\mathcal{LC}^i}$ and $\Delta^{\alpha}_{\mathcal{LR}^i}, i = 1, 2$

In this section we analyse the properties of the discriminants for both families $\mathcal{LC}^i_{\alpha}$ and $\mathcal{LR}^i_{\alpha}, i = 1, 2$. We use Thom's transversality Theorem 2.4.2 and Theorem 2.4.3 to study the generic properties of the discriminants.

Proposition 8.4.1 Let $\mathbb{A}$ be a generic self-adjoint operator on a Lorentzian surface M. There is a discrete set $I_i \subset \mathbb{R} \setminus \{0\}$ such that if $\alpha \notin I_i$ the discriminants $\Delta^{\alpha}_{\mathcal{LC}^i}, i = 1, 2$ are regular curves. For $\alpha \in I_i$, the discriminants $\Delta^{\alpha}_{\mathcal{LC}^i}, i = 1, 2$ have a Morse singularity of type $A_1^{\pm}$. Furthermore, $\Delta^{0}_{\mathcal{LC}^2}$ has only a Morse singularity of type A_1^{-}.

Proof: We consider the submesion jet-extension map

$$\begin{aligned} j^1\phi: \ & \mathbb{R}^2, 0 \times \mathbb{R} \times C^{\infty}(M, \mathbb{R}^3) \to J^1(2,3) \\ & (q, \alpha), \mathbb{A} \mapsto j^1(a_{\alpha}(q), b_{\alpha}(q), c_{\alpha}(q)), \end{aligned}$$

where $a_{\alpha}(q), b_{\alpha}(q), c_{\alpha}(q)$ are the coefficients of $\mathcal{LC}^1_{\alpha}$ depend on $l(q), m(q), n(q)$ and α. Let $\delta(q, \alpha) = 0$ be the equation of the family of discriminants of the BDEs $\mathcal{LC}^1_{\alpha}$ given by (8.7). For each fixed $\alpha (\alpha \notin I_1)$, the equation $\delta_{\alpha}(q) = (m^2 \cosh^2(\alpha) - ln)(q) = 0$ defines a submanifold W of codimension 1 in $j^1(2, 3)$. Since $j^1\phi \pitchfork W$, by Theorem 2.4.3, we have $j^1\phi_{\mathbb{A}} \pitchfork W$ and that implies $(j^1\phi_{\mathbb{A}})^{-1}(W)$ is a germ of a curve in $\mathbb{R}^2$ for generic $\mathbb{A}$ and a dense set J of values α. The transversality is an open property, so the set J is also open, which implies that I_1 is a discrete set.

Assume that q is the origin and write $j^1 l = l_0 + l_{10}x + l_{11}y$, $j^1 m = m_0 + m_{10}x + m_{11}y$, and $j^1 n = n_0 + n_{10}x + n_{11}y$. Let $\alpha \in I_1$, we consider the map

$$\begin{aligned} & j^1\phi : \mathbb{R}^2, 0 \times I_1 \times C^{\infty}(M, \mathbb{R}^3) \to J^1(2,3) \\ & (q, \alpha) \mapsto j^1(a_{\alpha}(q), b_{\alpha}(q), c_{\alpha}(q)) \end{aligned}$$

Equation $\delta_{\alpha}(0) = 0$ gives $\cosh^2(\alpha_0) = \frac{m_0^2}{l_0 n_0}(0)$. Substituting that condition in $\frac{\partial \delta_{\alpha}}{\partial x}(0) = 0$ and $\frac{\partial \delta_{\alpha}}{\partial y}(0) = 0$ gives respectively

$$\begin{aligned} -2l_0 n_0 m_{10} + n_0 m_0 l_{10} + l_0 m_0 n_{10} = 0, \\ -2l_0 n_0 m_{11} + n_0 m_0 l_{11} + l_0 m_0 n_{11} = 0. \end{aligned}$$

The above equations define a submanifold W of codimension 3 in $J^1(2, 3)$. By Theorem 2.4.1, $j^1\phi_{\mathbb{A}} \pitchfork W$ means that $(j^1\phi_{\mathbb{A}})^{-1}(W)$ is an isolated point in the family $\mathcal{LC}^1_{\alpha}$. Therefore, for $\alpha \in I_1$, the discriminants $\Delta^{\alpha}_{\mathcal{LC}^1}$ could have singularities, which are generically of type Morse $A_1^{\pm}$, for some isolated values of α. The proof for the discriminants $\Delta^{\alpha}_{\mathcal{LC}^2}$, for $\alpha \neq 0$, follows similarly.

If $\alpha = 0$, then $\mathcal{LC}^1_0$ is the $\mathbb{A}$-asymptotic BDE, and $\delta(q) = 0$ is the $\mathbb{A}$-parabolic set. The $\mathbb{A}$-parabolic set has a singularity of type Morse at the origin if and only if the following conditions satisfied; $(m_0^2 - l_0 n_0)(q) = 0$ and the other two conditions $\frac{\partial}{\partial x}\delta(q) = 0$ and $\frac{\partial}{\partial y}\delta(q) = 0$ mean $n_0^2 l_{11} + m_0^2 n_{11} - 2m_0 n_0 m_{11} = 0$ and $n_0^2 l_{10} + m_0^2 n_{10} - 2m_0 n_0 m_{10} = 0$. These conditions define a submanifold W of codimension 3 in $J^1(2,3)$. In this case, $\phi_0 \pitchfork W$ means that ϕ_0 avoids W. Therefore, the $\mathbb{A}$-parabolic set is a regular curve.

The BDE $\mathcal{LC}^2_0$ is the $\mathbb{A}$-principal BDE, and $\delta(q) = (ln)(q) = 0$ is the LPL. It is proved in [31] that for generic $\mathbb{A}$, the LPL is a regular curve except at timelike umbilic points (where $l(q) = n(q) = 0$). Generically, the singularity in this case is Morse of type A_1^-. □

The discriminants $\Delta^{\alpha}_{\mathcal{LR}^i}, i = 1, 2$ are curves. These curves can have singularities at some isolated points.

Proposition 8.4.2 For a generic $\mathbb{A}$, the discriminants $\Delta^{\alpha}_{\mathcal{LR}^i}, i = 1, 2$ are curves which possibly are singular at isolated points. To be more precise, we have the following.

1. There is a discrete set $I_i \subset \mathbb{R} \setminus \{0\}$ such that if $\alpha \notin I_i$ the discriminants $\Delta^{\alpha}_{\mathcal{LR}^i}, i = 1, 2$ are regular curves. For $\alpha \in I_i$, the discriminants $\Delta^{\alpha}_{\mathcal{LR}^i}, i = 1, 2$ have Morse singularity of type $A_1^{\pm}$ at some isolated points on D^i_{α}. On the LPL, the discriminants $\Delta^{\alpha}_{\mathcal{LR}^i}, i = 1, 2$ can have only a Morse singularity of type A_1^-.

2. At the intersection point of the LPL and D^i_{α}, the discriminant curves $\Delta^{\alpha}_{\mathcal{LR}^i}, i = 1, 2$ have a singularity of type A_3^- for any α if $i = 1$, and for $\alpha \neq 0$ if $i = 2$.

Proof: The proof of the first assertion is similar to that of Proposition 8.4.1 and is omitted. To prove the second assertion, assume that the origin is the intersection point of the LPL and D^i_{α}. At that point we have two conditions $m_0 = l_0 = 0$ (if the LPL is given by $l(q) = 0$) which means that this occurs at isolated points for generic $\mathbb{A}$. Except $\Delta^0_{\mathcal{LR}^2}$, by change of coordinates (detailed calculation is given in the Appendix, section A.1), the discriminant functions of $\Delta^{\alpha}_{\mathcal{LR}^i}, i = 1, 2$ are $\mathcal{R}$-

equivalent to $\pm(x^2-k_iy^4), i=1,2$ with

$$k_1=\frac{((2l_{11}^2n_0l_{20}-2l_{10}l_{11}n_0l_{21}+2l_{10}^2n_0l_{22})\cosh(\alpha)^2-l_{10}^2m_{11}^2-l_{11}^2m_{10}^2+2l_{10}l_{11}m_{11}m_{10})^2}{4l_{10}^4\cosh(\alpha)^2},$$

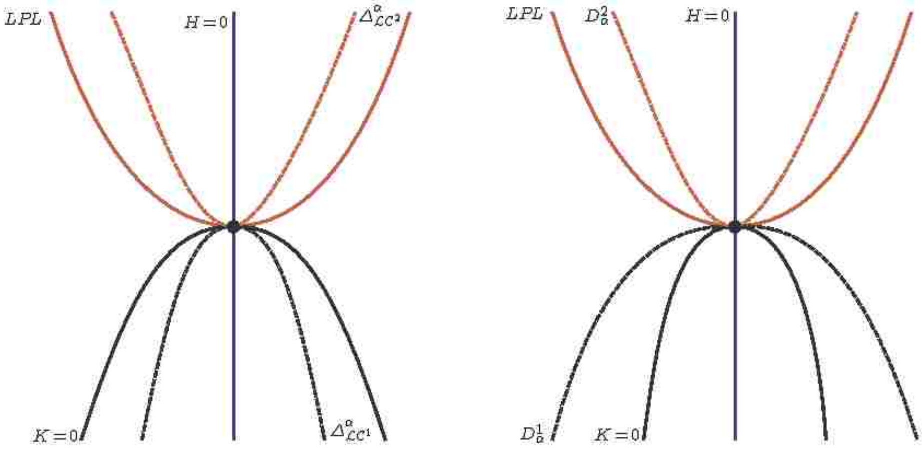
and

$$k_2=\frac{((-2l_{10}l_{11}n_0l_{21}+2l_{10}^2n_0l_{22}+2l_{11}^2n_0l_{20})\sinh(\alpha)^2+l_{10}^2m_{11}^2+l_{11}^2m_{10}^2-2l_{10}l_{11}m_{11}m_{10})^2}{4l_{10}^4\sinh(\alpha)^2}.$$

Therefore, the discriminants curves $\Delta^\alpha_{\mathcal{LR}^i}, i=1,2$ have an A_3^--singularity at the origin. For the case of $\mathcal{LR}_0^2=mP$ the discriminant curve $\Delta^0_{\mathcal{LR}^2}$ is regular and the behaviour of the BDE (mP) is as same as that of the BDE (P). □

Proposition 8.4.3 Let $\mathbb{A}$ be a generic self-adjoint operator on a Lorentzian surface M. The discriminant curves $\Delta^\alpha_{\mathcal{LC}^i}$ and $\Delta^\alpha_{\mathcal{LR}^i}, i=1,2$ intersect tangentially at their point of intersection. They intersect the curve $H=0$ transversally at such a point (Figure 8.1).

Proof: We consider the local parametrisation where $E=G=0$, and assume that the point of intersection is the origin. The discriminant curves $\Delta^\alpha_{\mathcal{LC}^i}$, $\Delta^\alpha_{\mathcal{LR}^i}$, and the curve $H=0$ given by $m=0$ intersect at the origin if and only if $m_0=l_0=0$ and $n_0\neq 0$. In fact, the gradient of $\Delta^0_{\mathcal{LC}^1}$ is $(-nl_{10},-nl_{11})$ is parallel to the gradient of the $\Delta^0_{\mathcal{LC}^2}$, and therefore, the two curves intersect tangentially. The proof follows similarly for the discriminant curves $\Delta^\alpha_{\mathcal{LC}^i}, i=1,2$ for $\alpha\neq 0$. The gradients of $\Delta^\alpha_{\mathcal{LC}^i}$ is $(-1)^i(nl_{10},nl_{11})$. It is parallel to the gradients of $\Delta^0_{\mathcal{LC}^i}, i=1,2$ at the origin. On the other hand, we know that the discriminant curves $\Delta^\alpha_{\mathcal{LR}^i}$ are $LPL\cup D^i_\alpha$. From Proposition 8.4.2(2), except $\Delta^0{}_{mathcalLR^2}$, the discriminant curves $\Delta^\alpha_{\mathcal{LR}^i}$ have an A_3^--singularity at the origin. Therefore, the LPL and D^i_α intersect tangentially at that point. The discriminant curves $\Delta^\alpha_{\mathcal{LC}^i}, i=1,2$ and the sets D^i_α intersect the LPL tangentially at the origin, which implies that, they are tangential to each other. By contrast, suppose that the curve $H=0$ and the LPL intersect tangentially at the origin. That implies $m_0=0$, $l_0=0$ and $m_{10}l_{11}-m_{11}l_{10}=0$. These conditions define a submanifold W of codimension 3 in $J^1(2,3)$. Since codim $W>2$, the genericity of $\mathbb{A}$ implies that $j^1\mathbb{A}(0,0)$ misses W. Hence these curves are generically transversal, that is $H=0$ and the discriminant curves $\Delta^\alpha_{\mathcal{LC}^i}$ and $\Delta^\alpha_{\mathcal{LR}^i}, i=1,2$ are generically transversal at the origin. □



(a) (b)

Figure 8.1: The distribution of the discriminant curves $\Delta^{\alpha}_{\mathcal{LC}^i}$ (a) and $\Delta^{\alpha}_{\mathcal{LR}^i}$ (b).

Remarks 8.4.1

1. The discriminant curves $\Delta^{\alpha}_{\mathcal{LC}^1}$ foliate the region where $K \leq 0$, with $\Delta^{0}_{\mathcal{LC}^1}$ is the $\mathbb{A}$-parabolic set. The discriminant curves $\Delta^{\alpha}_{\mathcal{LC}^2}$ foliate the region where $H^2 - K \leq 0$ where $\Delta^{0}_{\mathcal{LC}^2}$ is the LPL. Neither of the discriminant curves $\Delta^{\alpha}_{\mathcal{LC}^1}$ nor $\Delta^{\alpha}_{\mathcal{LC}^2}$ occur in the region where $K > 0$ and $H^2 - K > 0$ (Figure 8.1(a)).

2. The sets D^1_α occur in the region where $K \geq 0$ and $H^2 - K > 0$, with $\Delta^{0}_{\mathcal{LR}^1}$ is the $\mathbb{A}$-parabolic set union the LPL. The sets D^2_α occur in the region where $H^2 - K < 0$ with $\Delta^{0}_{\mathcal{LR}^2}$ is the singular curve $H = 0$ union the LPL. Neither of the discriminant curves $\Delta^{\alpha}_{\mathcal{LR}^1}$ nor $\Delta^{\alpha}_{\mathcal{LR}^2}$ occur in the region where $K < 0$ (Figure 8.1(b)).

3. The discriminant curves $\Delta^{\alpha}_{\mathcal{LR}^2}$ and $\Delta^{\alpha}_{\mathcal{LC}^2}$ foliate the same region where $H^2 - K < 0$. Then there are two discriminant curves for these families at each point in this region, one for each family.

Proposition 8.4.4 On the discriminant curves $\Delta^{\pm\alpha}_{\mathcal{LC}^i}, i = 1, 2$, for fixed $\alpha \neq 0$, the unique directions determined by $\mathcal{LC}^i_\alpha$ and by $\mathcal{LC}^i_{-\alpha}$, for fixed i, are the $\mathbb{A}$-characteristic directions. The $\mathbb{A}$-characteristic directions on $\Delta^{\pm\alpha}_{\mathcal{LC}^1}$ are of the same type (spacelike/timelike) while they are of different types on $\Delta^{\pm\alpha}_{\mathcal{LC}^2}$.

Proof: Consider the local parametrisation where $E = G = 0$. For fixed $\alpha \neq 0$, the unique directions determined by $\mathcal{LC}^i_\alpha$ and by $\mathcal{LC}^i_{-\alpha}$ on the discriminant curve $\Delta^{\pm\alpha}_{\mathcal{LC}^i}$, for fixed i, are $(mn, -ln \pm \sqrt{(ln)^2 - m^2 ln})$ (+ for α and $-$ for $-\alpha$) on the plane. These directions are the $\mathbb{A}$-characteristic directions. From Proposition 8.3.2, the $\mathbb{A}$-characteristic directions are $\mathbb{A}$-conjugate. They are of the same type in the region foliated by the discriminant curves $\Delta^{\alpha}_{\mathcal{LC}^1}$ $(ln > 0)$ and of different types in the region foliated by the discriminant curves $\Delta^{\alpha}_{\mathcal{LC}^2}$ $(ln < 0)$. $\square$

Remark 8.4.1 On the discriminant curve $\Delta^{0}_{\mathcal{LC}^1}$ the unique $\mathbb{A}$-characteristic direction determined by $\mathcal{LC}^1_0$ is a non-lightlike coincides with the unique $\mathbb{A}$-asymptotic direction. On the discriminant curve $\Delta^{0}_{\mathcal{LC}^2}$, which is the LPL, away from timelike umbilic points, the unique $\mathbb{A}$-characteristic direction determined by $\mathcal{LC}^2_0$ is lightlike coincides with the unique lightlike $\mathbb{A}$-principal direction (which is also $\mathbb{A}$-asymptotic). At timelike umbilic points, every direction is an $\mathbb{A}$-characteristic direction.

We have the following for $\mathcal{LR}^i_\alpha, i = 1, 2$.

Proposition 8.4.5 On the discriminant curves $\Delta^{\pm\alpha}_{\mathcal{LR}^i}, i = 1, 2$, for fixed $\alpha \neq 0$, the unique directions determined by $\mathcal{LR}^i_\alpha$ and by $\mathcal{LR}^i_{-\alpha}$, for fixed i, are the $\mathbb{A}$-asymptotic directions. On $D^1_{\pm\alpha}$ (resp. $D^2_{\pm\alpha}$), the $\mathbb{A}$-asymptotic directions are of the same type (spacelike/timelike) (resp. different types). On the LPL, away from timelike umbilic points, one of the $\mathbb{A}$-asymptotic directions is lightlike.

Proof: Consider the local parametrisation where $E = G = 0$. For fixed $\alpha \neq 0$, the unique directions determined by $\mathcal{LR}^i_\alpha$ and by $\mathcal{LR}^i_{-\alpha}$, for fixed i, are $(n, -m \pm \sqrt{m^2 - ln})$ (+ for α and $-$ for $-\alpha$) on the plane. These directions are the $\mathbb{A}$-asymptotic directions. Since the $\mathbb{A}$-asymptotic directions are determined by $\mathcal{LC}^1_0$, their types on $D^i_{\pm\alpha}$ follows from Proposition 8.3.2. On the LPL, away from timelike umbilic points, one of the $\mathbb{A}$-asymptotic directions coincides with the unique lightlike $\mathbb{A}$-principal direction. At timelike umbilic points, both of the $\mathbb{A}$-asymptotic directions are lightlike. $\square$

8.5 The limits of $\mathcal{LC}^i_\alpha$ and $\mathcal{LR}^i_\alpha, i = 1,2$ as $\alpha \to \pm\infty$

The analysis of the limit of $\mathcal{LC}^i_\alpha$ and $\mathcal{LR}^i_\alpha, i = 1,2$ when $\alpha \to \pm\infty$ is started in [35]. Equations (8.2) and (8.3) of $\mathcal{LC}^i_\alpha$ and (8.11) and (8.12) of $\mathcal{LR}^i_\alpha$ are divided by $\cosh(\alpha)$ to get

$$\lim_{\alpha\to\pm\infty} \mathcal{LC}^1_\alpha = \lim_{\alpha\to\pm\infty} \mathcal{LC}^2_\alpha = \mathcal{LC}_{\pm\infty},$$

and

$$\lim_{\alpha\to\pm\infty} \mathcal{LR}^1_\alpha = \lim_{\alpha\to\pm\infty} \mathcal{LR}^2_\alpha = \mathcal{LR}_{\pm\infty},$$

where

$$(\mathcal{LC}_{\pm\infty}) : \pm P + \sqrt{F^2 - EG}\, A = 0, \tag{8.21}$$

$$(\mathcal{LR}_{\pm\infty}) : C \pm \frac{2Fm - Gl - En}{\sqrt{F^2 - EG}} P = 0. \tag{8.22}$$

In order to find the discriminants $\Delta^{\pm\infty}_{\mathcal{LC}}$ and $\Delta^{\pm\infty}_{\mathcal{LR}}$, we divide equations (8.4) and (8.5) of the discriminant curves $\Delta^\alpha_{\mathcal{LC}^i}$ and equations (8.13) and (8.14) of the discriminant curves $\Delta^\alpha_{\mathcal{LR}^i}$ by $\cosh^2(\alpha)$. As $\alpha \to \pm\infty$ we have $\tanh^2(\alpha) \to 1$ and that implies

$$\lim_{\alpha\to\pm\infty} \Delta^\alpha_{\mathcal{LC}^1} = \lim_{\alpha\to\pm\infty} \Delta^\alpha_{\mathcal{LC}^2} = \Delta^{\pm\infty}_{\mathcal{LC}},$$

which is $H^2 = 0$. Furthermore, we have

$$\lim_{\alpha\to\pm\infty} \Delta^\alpha_{\mathcal{LR}^1} = \lim_{\alpha\to\pm\infty} \Delta^\alpha_{\mathcal{LR}^2} = \Delta^{\pm\infty}_{\mathcal{LR}},$$

which is $(H^2 - K)^2 = 0$.

Remarks 8.5.1 1. Consider the local parametrisation where $E = G = 0$. Then equation (8.21) becomes

$$(\mathcal{LC}_{+\infty}) : dy\,(mdx + ndy) = 0, \tag{8.23}$$

$$(\mathcal{LC}_{-\infty}) : dx\,(ldx + mdy) = 0, \tag{8.24}$$

and $\Delta^{\pm\infty}_{\mathcal{LC}}$ is the curve $m^2 = 0$.

2. The pairs of 1-forms $\mathcal{LC}_{\pm\infty}$ given by equations (8.23) and (8.24) determine two distinct directions at each point $p \in M$ except at points on $\Delta_{\mathcal{LC}}^{\pm\infty}$. One of the directions is lightlike and the $\mathbb{A}$-conjugate of the other direction is the other lightlike direction at p. Therefore, in this case, the hyperbolic angle between the directions and their $\mathbb{A}$-conjugates is not well defined.

3. On $\Delta_{\mathcal{LC}}^{\pm\infty}$, the unique directions defined by $\mathcal{LC}_{+\infty}$ and by $\mathcal{LC}_{-\infty}$ are lightlike $\mathbb{A}$-conjugate to each other. They are the lightlike $\mathbb{A}$-characteristic directions.

4. Consider the local parametrisation where $E = G = 0$, equation (8.22) becomes

$$(\mathcal{LR}_{+\infty}) : ldx\,(ndy + mdx) = 0, \tag{8.25}$$

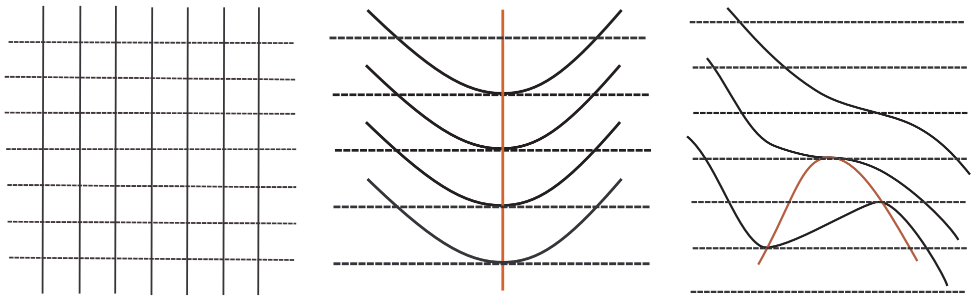
$$(\mathcal{LR}_{-\infty}) : ndy\,(ldx + mdy) = 0. \tag{8.26}$$

   The local configurations of $\mathcal{LR}_{+\infty}$ is that of the pair of 1-forms $(dx, ndy + mdx)$ and that of $\mathcal{LR}_{-\infty}$ is that of the pair of 1-forms $(dy, ldx + mdy)$. The discriminant curve $\Delta_{\mathcal{LR}}^{+\infty}$ (resp. $\Delta_{\mathcal{LR}}^{-\infty}$) is the LPL given by $n = 0$ (resp. $l = 0$) union the singular curve $l = 0$ (resp. $n = 0$).

5. The pairs of 1-forms $\mathcal{LR}_{\pm\infty}$ determine two directions at each point $p \in M$ except at points on the discriminant curves $\Delta_{\mathcal{LR}}^{\pm\infty}$. One of the directions is lightlike and the other direction is its $\mathbb{A}$-conjugate. Therefore, in this case, the hyperbolic angle between the directions and their $\mathbb{A}$-conjugates is not well defined.

6. On $\Delta_{\mathcal{LR}}^{+\infty}$ and $\Delta_{\mathcal{LR}}^{-\infty}$, the unique directions defined by $\mathcal{LR}_{+\infty}$ and by $\mathcal{LR}_{-\infty}$ coincide with the lightlike $\mathbb{A}$-asymptotic direction.

We are interested in studying the local topological configurations of $\mathcal{LC}_{\pm\infty}$ and $\mathcal{LR}_{\pm\infty}$. We study the case of $\mathcal{LC}_{+\infty}$ in details; the study of the cases $\mathcal{LC}_{-\infty}$ and $\mathcal{LR}_{\pm\infty}$ follows in the same way. We consider equation (8.23), and set $\beta_1 = dy$ and $\beta_2 = m(x, y)dx + n(x, y)dy$. We assume that the point of interest to be the origin. The 1-form β_1 is regular, and β_2 is regular if and only if either $m(0, 0) \neq 0$ or $n(0, 0) \neq 0$; and is singular if $m(0, 0) = n(0, 0) = 0$. This is the case when the origin is the intersection point of the LPL and $H = 0$.



(a) (dx, dy) (b) $(dy, d(y - x^2))$ (c) $(dy, d(y + xy + x^3))$

Figure 8.2: Regular pairs of 1-forms. The discriminant curve is in red.

The discriminant of (β_1,β_2) consists of points (x, y) where β_1 is a multiple of β_2. This is the case if and only if $m(x, y) = 0$. The local topological configurations of the pairs of regular 1-forms in the plane are given in [38].

Theorem 8.5.1 [38] Let β_1 and β_2 be two germs of regular 1-forms. Then the stable pairs (β_1,β_2) are topologically equivalent to one of the following:

1. (dx, dy): if the two 1-forms are transverse. Figure 8.2(a).

2. $(dy, d(y - x^2))$: if the two 1-forms have 2-point contact at the origin, equivalently, if the discriminant and the two 1-forms have 1-point contact at the origin. Figure 8.2(b).

3. $(dy, d(y + xy + x^3))$: if the two 1-forms have 3-point contact at the origin, equivalently, if the discriminant and the two 1-forms have 2-point contact at the origin. Figure 8.2(c).

Proposition 8.5.1 The generic local configuration of the foliations of $\mathcal{LC}_{\pm\infty}$ and $\mathcal{LR}_{\pm\infty}$ away from their discriminant curves is as in Figure 8.2(a). On their discriminant curves, away from the intersection point of the LPL and $H = 0$, the generic local configurations of $\mathcal{LC}_{\pm\infty}$ and $\mathcal{LR}_{\pm\infty}$ are as in Figure 8.2(b) and (c).

Proof: We consider $\mathcal{LC}_{+\infty}$ at the origin, and set $\beta_1 = dy$ and $\beta_2 = mdx + ndy$ with $j^2m = m_0 + m_{10}x + m_{11}y + m_{20}x^2 + m_{21}xy + m_{22}y^2$ and $j^1n = n_0 + n_{10}x + n_{11}y$. The discriminant curve $\Delta_{\mathcal{LC}}^{+\infty}$, given by $m = 0$, is regular at the origin unless $m_0 =$

$m_{10} = m_{11} = 0$. These 3 conditions define a submanifold W of codimension 3 in $J^1(2,3)$. Since $\mathbb{A}$ is generic, the genericity implies that $j^1(\mathbb{A}_p)$ misses W, i.e., $m = 0$ is a regular curve. The 1-form β_2 is regular if and only if either $n_0 \neq 0$ or $m_0 \neq 0$. Away from the discriminant curve $m = 0$, $\mathcal{LC}_{+\infty}$ determines two directions which are $(1,0)$ and $(n_0, -m_0)$. Therefore, the pair of foliations determined by $\mathcal{LC}_{+\infty}$ are transverse (Figure 8.2(a)) as $m_0 \neq 0$. Suppose that the origin is on the discriminant curve $m = 0$ and is not the point of intersection with the LPL ($n_0 \neq 0$). The discriminant curve $m = 0$ is tangent to the line $m_{10}x + m_{11}y = 0$ at the origin. The leaf of β_1 and the discriminant curve have 1-point contact at the origin if and only if $m_{10} \neq 0$ (Figure 8.2(b)). If $m_{10} = 0$ and $m_{20} \neq 0$, then the leaf of β_1 and the discriminant curve have 2-point contact at the origin (Figure 8.2(c)). $\square$

We turn to the case when the 1-form $\beta_2 = mdx + ndy$ is singular and the singularity is of type saddle, node, or focus. This case occurs precisely when the origin is the intersection point of $H = 0$ and the LPL. The topological normal forms in this case follow from Theorem 8.5.2.

Theorem 8.5.2 [38] Let β_1 be a germ at the origin of a regular 1-form and β_2 a germ of a 1-form with a saddle/node/focus singularity at the origin. The discriminant is then a smooth curve. If the leaf of β_1 at the origin is transverse to the separatrices of β_2, then there exists a homeomorphism taking the foliations of (β_1, β_2) to those of

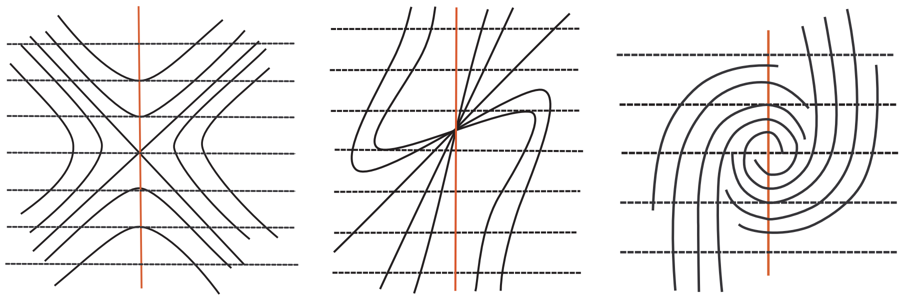
1. $(dy, (x - y)dy + xdx)$: if β_2 is a saddle. Figure 8.3(a).

2. $(dy, (x + \frac{1}{8}y)dy + xdx)$: if β_2 is a node. Figure 8.3(b).

3. $(dy, (x + y)dy + xdx)$: if β_2 is a focus. Figure 8.3(c).

We observe that $\mathcal{LC}_{+\infty}$ and $\mathcal{LR}_{+\infty}$ (resp. $\mathcal{LC}_{-\infty}$ and $\mathcal{LR}_{-\infty}$) have a common direction $mdx + ndy$ (resp. $ldx + mdy$).

Proposition 8.5.2 For generic $\mathbb{A}$, at the point of intersection of the LPL and the curve $H = 0$, the configuration of the foliations of $\mathcal{LC}_{+\infty}$ and $\mathcal{LR}_{+\infty}$ (resp. $\mathcal{LC}_{-\infty}$ and $\mathcal{LR}_{-\infty}$) are locally topologically equivalent to those in Theorem 8.5.2.



(a) (R,S) (b) (R,N) (c) (R,F)

Figure 8.3: (regular, singular) pairs of 1-forms. The discriminant curve is in red. The abbreviations R/S/N/F are for regular/saddle/node/focus.

Proof: Consider the local parametrisation where $E = G = 0$ and the origin to be the point of intersection of the regular curves $m(x, y) = 0$ and the LPL $(n = 0)$. We investigate the case of $\mathcal{LC}_{+\infty}$. At the origin we have $m_0 = n_0 = 0$, and therefore, the 1-form $\beta_2 = mdx + ndy$ is singular. Given that $j^1m = m_{10}x + m_{11}y$ and $j^1n = n_{10}x + n_{11}y$, we consider the matrix

$$A = \begin{pmatrix} n_{10} & n_{11} \\ -m_{10} & -m_{11} \end{pmatrix}.$$

From Proposition 8.4.3, for generic $\mathbb{A}$, the discriminant curve $m(x, y) = 0$ and the LPL are regular curves which intersect transversally at the origin. Therefore, the matrix A is non-singular, i.e., A has non-zero eigenvalues. The eigenvalues of A are the values of λ for which

$$\lambda^2 - tr(A)\lambda + det(A) = 0.$$

Here tr(A)=$n_{10} - m_{11}$ and det(A)=$m_{10}\, n_{11} - m_{11}\, n_{10}$. Hence, the eigenvalues of A are $\lambda_1 = \frac{1}{2}(tr(A) + \sqrt{\Delta})$ and $\lambda_2 = \frac{1}{2}(tr(A) - \sqrt{\Delta})$, with

$$\Delta = (tr(A))^2 - 4det(A).$$

If $\Delta > 0$, then λ_1 and λ_2 are real distinct eigenvalues. Besides if det$(A)$$> 0$, then λ_1 and λ_2 have the same sign, and the singularity of β_2 is node. If det$(A)$$< 0$, λ_1 and λ_2 have opposite signs, and the singularity of β_2 is saddle. If $\Delta < 0$, then

λ_1 and λ_2 are complex eigenvalues and the singularity of β_2 is focus. The proof is analogous for $\mathcal{LC}_{-\infty}$ and $\mathcal{LR}_{\pm\infty}$. $\square$

Chapter 9

The local configurations of the Lorentzian curve congruences

We determine in this chapter the local topological configurations of $\mathcal{LC}_\alpha^i$ and $\mathcal{LR}_\alpha^i, i = 1, 2$, for α fixed, and the way they bifurcate locally as α varies in a neighbourhood of a given fixed α. We recall that away from the discriminant curves, the BDEs $\mathcal{LC}_\alpha^i$ and $\mathcal{LR}_\alpha^i, i = 1, 2$ determine either a pair of transverse foliations or none. We study the local configurations of $\mathcal{LC}_\alpha^i$ and $\mathcal{LR}_\alpha^i, i = 1, 2$ at points on their discriminant curves. We examine the related conditions given in Chapter 6 for each singularity to occur for generic self-adjoint operators $\mathbb{A}$.

9.1 The local configurations of $\mathcal{LC}_\alpha^i, i = 1, 2$ at their codimension ≤ 1 singularities

Since our study is local, we assume that the point of interest is the origin. We deal first with the case when the coefficients of $\mathcal{LC}_\alpha^i, i = 1, 2$ do not vanish simultaneously at the origin. The BDEs in this case are of Type 1. We can assume, without loss of generality, that $n(0, 0) \neq 0$, set $p = dy/dx$ and write the BDEs $\mathcal{LC}_\alpha^i, i = 1, 2$ given by equations (8.6) and (8.8) in the forms

$$(\mathcal{LC}_\alpha^1) : p^2 + \frac{2m(x, y)}{n(x, y)(\tanh(\alpha) + 1)}p + \frac{l(x, y)(1 - \tanh(\alpha))}{n(x, y)(\tanh(\alpha) + 1)} = 0, \qquad (9.1)$$

$$(\mathcal{LC}^2_\alpha) : p^2 + \frac{2m(x,y)\tanh(\alpha)}{n(x,y)(\tanh(\alpha)+1)}p + \frac{l(x,y)(\tanh(\alpha)-1)}{n(x,y)(\tanh(\alpha)+1)} = 0, \qquad (9.2)$$

where $r^\alpha_1(x,y)$ are the coefficients of p and $r^\alpha_2(x,y)$ are the coefficients of p^0 in equations (9.1) and (9.2). Now, the equations of $\mathcal{LC}^i_\alpha$ are adapted to compute the conditions for codimension ≤ 1 singularities to occur at the origin using the calculations in Chapter 6. (When the coefficients of $\mathcal{LC}^i_\alpha$ all vanish at the origin, then the BDEs are of Type 2. In this case, equations (8.6) and (8.8) are considered. See for example the proof of Theorem 9.1.2(2).)

The coefficients of the 3-jets of $r^\alpha_1(x,y)$ and $r^\alpha_2(x,y)$ shown in Chapter 6 are differentiable functions of α. In fact, solving the condition $C_1 = 0$ for α specifies the discriminant curve $\Delta^{\alpha_0}_{LC^i}$ that passes through the origin. Substituting the value of α_0 in the remaining conditions provides equations that depend purely on $j^3(l,m,n)$ at the origin.

Theorem 9.1.1 Let $\mathbb{A}$ be a generic self-adjoint operator on a Lorentzian surface M. Away from timelike umbilic points, the BDEs $\mathcal{LC}^i_\alpha, i = 1,2$ have the codimension 0 singularities shown in Table 6.1 and their conditions to occur are as in Table 6.2.

Proof: We consider the local parametrisation where $E = G = 0$. A direct calculation for the discriminant equation of $\mathcal{LC}^1_\alpha$ (9.1) gives

$$(m^2 - ln(1-\tanh^2(\alpha))(q) = 0. \qquad (9.3)$$

So for each fixed $\alpha \in \mathbb{R}$, we have a family of discriminant curves $\Delta^\alpha_{\mathcal{LC}^1}$ given by equation (9.3), see Proposition 8.4.1. In fact, equation (9.3) presents the condition $C_1 = 0$. Supposing that q is the origin and solving equation (9.3) for α give $\tanh(\pm\alpha_0) = \pm\sqrt{\frac{l_0 n_0 - m_0^2}{l_0 n_0}}$ (+ for α_0 and $-$ for $-\alpha_0$). We substitute the value of $\tanh(\alpha_0)$ in the conditions $C_j, j = 2,3,4,5$. If $C_j \neq 0, j = 2,5$, then the local configuration of $\mathcal{LC}^1_{\alpha_0}$ is a family of cusps with cusps tracing the discriminant curve $\Delta^{\alpha_0}_{\mathcal{LC}^1}$ in a neighbourhood of the origin. If the origin is the point of intersection of the LPL and $K = 0$ (where $m_0 = l_0 = 0$), the integral curves of the BDE $\mathcal{LC}^1_{\alpha_0}$ is generically a family of cusps.

If $C_i = 0, i = 1,2$ and $C_j \neq 0, j = 3,4,5$, then $\mathcal{LC}^1_{\alpha_0}$ has a folded singularity of type saddle, node or focus at the origin. In fact, the condition $C_2 = 0$ depends purely on $j^1(l, m, n)$. For each fixed α_0, the conditions $C_i = 0, i = 1, 2$ define a submanifold W of codimension 2 in $J^1(2, 2)$. For generic $\mathbb{A}$ and each fixed α_0, we have $(j^1(\phi_{\mathbb{A}}))^{-1}(W)$ (where $j^1\phi_{\mathbb{A}}$ is the jet-extension map in the proof of Proposition 8.4.1) is an isolated folded singularitiy on the discriminant curve $\Delta^{\alpha_0}_{\mathcal{LC}^1}$; the technique used here is similar to that of the proof of Proposition 8.4.1. Any type of the folded singularity can occur, refer to the Appendix (section A.2) for detailed calculations. The study of $\mathcal{LC}^1_{-\alpha_0}$ and $\mathcal{LC}^2_{\pm\alpha_0}$ follows similarly. □

We turn to the case when $\mathcal{LC}^i_\alpha, i = 1, 2$ can have codimension 1 singularities.

Theorem 9.1.2 Let $\mathbb{A}$ be a generic self-adjoint operator on a Lorentzian surface M. There is a discrete set $I_i \subset \mathbb{R} \setminus \{0\}, i = 1, 2$ such that:

1. For $\alpha_0 \in I_i$, the BDEs $\mathcal{LC}^i_{\alpha_0}, i = 1, 2$ have codimension 1 singularity of type folded saddle-node, folded node-focus or Morse Type 1 at some isolated points on $\Delta^{\alpha_0}_{\mathcal{LC}^i}$ (Table 6.1 and 6.2). Furthermore, the families $\mathcal{LC}^i_\alpha, i = 1, 2$ are generic for α near α_0 in sense of Theorem 6.1.3. The bifurcations in the families $\mathcal{LC}^i_\alpha, i = 1, 2$ of each type of codimension 1 singularity are shown in Table 6.3.

2. At timelike umbilic points, the BDE $\mathcal{LC}^2_0$ can have only a Morse Type 2 singularity of type A_1^-. All the five generic cases of such a singularity can occur (Figure 6.3). Moreover, the family $\mathcal{LC}^2_\alpha$ is generic for α near $\alpha_0 = 0$ in the sense of Theorem 6.2.2. The bifurcations in the family $\mathcal{LC}^2_\alpha$ of each case of Morse Type 2 of type A_1^- singularity are as shown in Figure 6.5.

Proof: Let $\alpha_0 \in I_1$. The BDE $\mathcal{LC}^1_{\alpha_0}$ has a codimension 1 singularity of type folded saddle-node at the origin if and only if $C_i = 0, i = 1, 2, 3$ and $C_j \neq 0, j = 5, 6$. These conditions define a submanifold W of codimension 3 in $J^2(2, 2)$. For generic $\mathbb{A}$ and fixed α_0, we have $(j^1(\phi_{\mathbb{A}}))^{-1}(W)$ is an isolated point on the discriminant curve $\Delta^{\alpha_0}_{\mathcal{LC}^1}$ in the family $\mathcal{LC}^1_\alpha$; the technique is similar to that of the proof of Proposition 8.4.1. Similarly, the BDE $\mathcal{LC}^1_{\alpha_0}$ has a codimension 1 singularity of type folded node-focus at the origin in the family $\mathcal{LC}^1_\alpha$ if and only if $C_i = 0, i = 1, 2, 4$ and $C_j \neq 0, j = 3, 5$. Furthermore, the BDE $\mathcal{LC}^1_{\alpha_0}$ has an isolated singularity of type Morse Type 1 of type $A_1^\pm$ if and only if $C_i = 0, i = 1, 7, 8$ and $C_j \neq 0, j = 9, 10$.

A 1-parameter family of BDEs $\mathcal{LC}^1_\alpha$ with $\mathcal{LC}^1_{\alpha_0}$ having a folded saddle-node singularity at the origin is generic if and only if the genericity condition (6.9) is satisfied. In order to simplify the calculation, we make a linear change of coordinates and compute the right hand side of the genericity condition (6.9). We find that the genericity condition (6.9) is satisfied, refer to the Appendix (section A.2) for calculations. The studies of the genericity of the 1-parameter family of the BDEs $\mathcal{LC}^1_\alpha$ with $\mathcal{LC}^1_{\alpha_0}$ having the other types of codimension 1 singularities at the origin follow similarly using the genericity conditions in Theorem 6.1.3. The results for $\mathcal{LC}^1_{-\alpha_0}$ and $\mathcal{LC}^2_{\pm\alpha_0}$ are proved in similar manners.

The LPL $(\Delta^0_{\mathcal{LC}^2})$ has generically a Morse singularity of type A_1^- at the origin if and only if $n_0 = l_0 = 0$. These two conditions define a submanifold W of codimension 2 in $J^0(2, 3)$. By Thom's transversality theorem 2.4.2, for generic $\mathbb{A}$, such a singularity can occur at isolated points on the LPL. As a result, the $\mathbb{A}$-principal BDE $(\mathcal{LC}^0_2)$ has a Morse Type 2 singularity of type A_1^- and the five generic cases can occur [31]. We examine the genericity condition (6.12) in Chapter 6 and that gives $m_0(n_{10}l_{11} - n_{11}l_{10}) \neq 0$. Therefore, the family $\mathcal{LC}^2_\alpha$ is generic in this case. □

Proposition 9.1.1 Away from the intersection point of the LPL and the $\mathbb{A}$-parabolic set, we have the following for fixed $\alpha_0 \neq 0$ and $q \in \Delta^{\alpha_0}_{\mathcal{LC}^i}, i = 1, 2$ for fixed i.

1. If $\mathcal{LC}^i_{\alpha_0}$ has a folded singularity (saddle/node/focus), a folded saddle-node or a folded node-focus at q, then $\mathcal{LC}^i_{-\alpha_0}$ form a family of cusps in a neighbourhood of q and vice-versa.

2. The local configurations of $\mathcal{LC}^i_{\alpha_0}$ and $\mathcal{LC}^i_{-\alpha_0}$ are equivalent in case that one of them has a Morse Type 1 singularity at q.

Proof: Suppose that q is the origin. From Proposition 8.4.4, the unique directions determined by $\mathcal{LC}^i_{\alpha_0}$ and $\mathcal{LC}^i_{-\alpha_0}$ at $q \in \Delta^{\pm\alpha_0}_{\mathcal{LC}^i}$ are the $\mathbb{A}$-characteristic directions. Therefore, if the $\mathbb{A}$-characteristic direction determined by $\mathcal{LC}^i_{\alpha_0}$, for fixed i, is tangent to the discriminant curve $\Delta^{\alpha_0}_{\mathcal{LC}^i}$ at the origin, then the other one determined by $\mathcal{LC}^i_{-\alpha_0}$ is transverse.

Calculations, done by using Maple, show that the conditions $C_i = 0, i = 1, 7, 8$ and $C_j \neq 0, j = 9, 10$ for $\mathcal{LC}^i{}_{\alpha_0}$ and $\mathcal{LC}^i{}_{-\alpha_0}$, for fixed i, to have a Morse Type 1 singularity at the origin are equivalent. Furthermore, their Morse Type 1 singularity has the same bifurcation, and the two possible bifurcations can occur (two saddles or two foci). $\square$

9.2 The local configurations of $\mathcal{LR}^i_\alpha, i = 1, 2$ at their codimension ≤ 1 singularities

The study of the local configurations of $\mathcal{LR}^i_\alpha, i = 1, 2$ at their codimension $\leq$1 singularities follows similarly to that of $\mathcal{LC}^i_\alpha, i = 1, 2$. Note that the discriminant curves $\Delta^\alpha_{\mathcal{LR}^i}, i = 1, 2$ are the union of the LPL and the sets D^i_α. When $q \in D^i_\alpha$, the results are similar to those in Theorem 9.1.1 and Theorem 9.1.2. There is extra information when $q \in LPL$.

We deal firstly with the case when the coefficients of the BDEs $\mathcal{LR}^i_\alpha, i = 1, 2$ do not vanish simultaneously at the origin. The BDEs in this case are of Type 1. We can assume, without loss of generality, that $mn(0, 0) \neq 0$, set $p = dy/dx$ and write the BDEs $\mathcal{LR}^i_\alpha, i = 1, 2$ given by equations (8.17) and (8.19) in the forms

$$(\mathcal{LR}^1_\alpha) : p^2 - \frac{2l(x, y)}{m(x, y)(\tanh(\alpha) - 1)}p - \frac{l(x, y)(\tanh(\alpha) + 1)}{n(x, y)(\tanh(\alpha) - 1)} = 0, \tag{9.4}$$

and

$$(\mathcal{LR}^2_\alpha) : p^2 - \frac{2l(x, y)\tanh(\alpha)}{m(x, y)(1 - \tanh(\alpha))}p - \frac{l(x, y)(\tanh(\alpha) + 1)}{n(x, y)(1 - \tanh(\alpha))} = 0, \tag{9.5}$$

where $r^\alpha_1(x, y)$ are the coefficients of p and $r^\alpha_2(x, y)$ are the coefficients of p^0 in equations (9.1) and (9.2). Now, the equations of $\mathcal{LR}^i_\alpha$ are adapted to compute the conditions for codimension ≤ 1 singularities to occur at the origin using the calculations in Chapter 6. (When all the coefficients of $\mathcal{LR}^i_\alpha$ vanish at the origin, then the BDEs are of Type 2. In this case, equations (8.17) and (8.19) are considered. See for example the proof of Theorem 9.2.3.)

Theorem 9.2.1 Let $\mathbb{A}$ be a generic self-adjoint operator on a Lorentzian surface M. Away from timelike umbilic points, and from the intersection points of the LPL and the $\mathbb{A}$-parabolic set, we have the following.

1. For $q \in D^i_\alpha$, the BDEs $\mathcal{LR}^i_\alpha, i = 1, 2$ have codimension 0 singularities (Table 6.1 and 6.2).

2. For $q \in LPL$, the BDEs $\mathcal{LR}^i_\alpha, i = 1, 2$ have codimension 0 singularity (Table 6.1 and 6.2). Moreover, for each fixed α_0, the folded singularities (saddle/node/focus) of $\mathcal{LR}^1_{\alpha_0}$ and $\mathcal{LR}^2_{\alpha_0}$ occur at the same points and have opposite indices.

Proof: We consider the local parametrisation where $E = G = 0$, and suppose that q is the origin. If the origin belongs to D^i_α, the proof is similar to that of Theorem 9.1.1 and is omitted. If the origin belongs to the LPL, for each fixed α_0, the unique directions determined by $\mathcal{LR}^1_{\alpha_0}$ and $\mathcal{LR}^2_{\alpha_0}$ coincide with the unique lightlike $\mathbb{A}$-principal direction. The integral curves of the BDEs $\mathcal{LR}^i_{\alpha_0}, i = 1, 2$ form families of cusps with cusps tracing the LPL given by $l(x, y) = 0$ if and only if $l_0 = 0$ and $l_{10} \neq 0$. They have folded singularity of type saddle, node or focus at the origin if and only if $l_0 = l_{10} = 0$ and $l_{11} \neq 0$ in addition to

$$\lambda_1 = \frac{-2n_0 l_{20}(-1 + \tanh(\alpha_0))}{(\tanh(\alpha_0) + 1)l_{11}^2} \neq 0, \frac{1}{16}, \tag{9.6}$$

in the case of $\mathcal{LR}^1_{\alpha_0}$, and in the case of $\mathcal{LR}^2_{\alpha_0}$

$$\lambda_2 = \frac{2n_0 l_{20}(-1 + \tanh(\alpha_0))}{(\tanh(\alpha_0) + 1)l_{11}^2} \neq 0, \frac{1}{16}. \tag{9.7}$$

It follows from the expressions of λ_1 and λ_2 that the three types of the folded singularity can occur. For each fixed α_0, we have $\lambda_2 = -\lambda_1$, therefore, the folded singularities of $\mathcal{LR}^1_{\alpha_0}$ and $\mathcal{LR}^2_{\alpha_0}$ have opposite indices. $\square$

Corollary 9.2.1 For a generic self-adjoint operator $\mathbb{A}$, the folded singularities of the $\mathbb{A}$-characteristic and the $\mathbb{A}$-principal curves BDEs coincide and have opposite indices.

Proof: The result follows from the fact that the $\mathbb{A}$-characteristic and the $\mathbb{A}$-principal curves BDEs are the BDEs $\mathcal{LR}^1_0$ and $\mathcal{LR}^2_0$ respectively. $\square$

We turn to the case when $\mathcal{LR}^i_\alpha, i = 1, 2$ have codimension 1 singularities.

Theorem 9.2.2 Let $\mathbb{A}$ be a generic self-adjoint operator on a Lorentzian surface M. Away from timelike umbilic points and from the intersection point of the LPL and the $\mathbb{A}$-parabolic set, we have the following.

1. There is a discrete set $I_i \subset \mathbb{R} \setminus \{0\}, i = 1, 2$ such that for $\alpha_0 \in I_i$ and $q \in D^i_{\alpha_0}$, the BDEs $\mathcal{LR}^i_{\alpha_0}, i = 1, 2$ have a codimension 1 singularity of type folded saddle-node, folded node-focus or Morse type 1 (Table 6.1 and 6.2). Moreover, the families $\mathcal{LR}^i_\alpha, i = 1, 2$ are generic for α near α_0 in sense of Theorem 6.1.3. The bifurcations in the families $\mathcal{LR}^i_\alpha, i = 1, 2$ as α varies near α_0 for each type of codimension 1 singularity are shown in Table 6.3.

2. For $q \in LPL$, the BDEs $\mathcal{LR}^i_{\alpha_0}, i = 1, 2$ can have only a codimension 1 singularity of type folded node-focus for some values of α_0.

Proof: We consider the local parametrisation where $E = G = 0$ and assume that q is the origin. When the origin belongs to D^i_α, the technique of the proof is similar to that of Theorem 9.1.2(1) and is omitted.

When the origin belongs to the LPL, for any fixed α_0, the BDEs $\mathcal{LR}^i_{\alpha_0}, i = 1, 2$ have a folded saddle-node (resp. a Morse Type 1) singularity if and only if $l_0 = l_{10} = l_{20} = 0$ (resp. $l_0 = l_{10} = l_{11} = 0$) at the origin. These conditions define a submanifold W of codimension 3 in $J^2(2, 3)$. Since $\mathbb{A}$ is generic, then $j^2(\mathbb{A}_p) \pitchfork W$ implies that $j^2(\mathbb{A}_p)$ misses W, and therefore, the BDEs $\mathcal{LR}^i_{\alpha_0}, i = 1, 2$ do not have such a singularity on the LPL.

The BDE $\mathcal{LR}^1_{\alpha_0}$ (resp. $\mathcal{LR}^2_{\alpha_0}$) have an isolated singularity of type folded node-focus at the origin on the LPL if and only if $l_0 = l_{10} = 0$ and $\tanh(\alpha_0) = \frac{-l_{11}^2+16n_0l_{20}}{16n_0l_{20}+l_{11}^2}$ (resp. $\tanh(\alpha_0) = \frac{16l_{20}n_0+l_{11}^2}{16l_{20}n_0-l_{11}^2}$). The technique of the proof follows that of Theorem 9.1.2(1) and is omitted. $\square$

Proposition 9.2.1 Let $\mathbb{A}$ be a generic self-adjoint operator on a Lorentzian surface M. Away from timelike umbilic points and from the intersection point of the LPL and the $\mathbb{A}$-parabolic set, we have the following for fixed $\alpha_0 \neq 0$.

1. For $q \in D^i_{\alpha_0}, i = 1, 2$, if $\mathcal{LR}^i_{\alpha_0}$, for fixed i, has a folded singularity, a folded saddle-node or a folded node-focus at q, then $\mathcal{LR}^i_{-\alpha_0}$ form a family of cusps in

a neighbourhood of q and vice-versa. Nevertheless, the local configurations of $\mathcal{LR}^i_{\alpha_0}$ is equivalent to $\mathcal{LR}^i_{-\alpha_0}$, for fixed i, in case that one of them has a Morse Type 1 singularity at q.

2. For $q \in LPL$, if $\mathcal{LR}^i_{\alpha_0}$, for fixed i, has a folded singularity or a folded node-focus at q, then $\mathcal{LR}^i_{-\alpha_0}$ form a family of cusps in a neighbourhood of q and vice-versa.

Proof: Suppose that q is the origin. From Proposition 8.4.5, the unique directions determined by $\mathcal{LR}^i_{\alpha_0}$ and $\mathcal{LR}^i_{-\alpha_0}$, for fixed i, at the origin are the $\mathbb{A}$-asymptotic directions. If the origin is on $D^i_{\alpha_0}$, then the proof of this case is similar to that of Proposition 9.1.1(1) and (2). If the origin is on the LPL, one of the $\mathbb{A}$-asymptotic directions is lightlike. Therefore, if the BDE which determines the lightlike $\mathbb{A}$-asymptotic direction has a folded singularity or folded node-focus, then the other BDE forms a family of cusps in a neighbourhood of the origin. □

Theorem 9.2.3 Let $\mathbb{A}$ be a generic self-adjoint operator on a Lorentzian surface M. At timelike umbilic points, the BDEs $\mathcal{LR}^i_\alpha, i=1,2$ have a Morse Type 2 singularity of type A_1^- for any $\alpha \in \mathbb{R}$. For each fixed α, the five generic cases can occur (Figure 6.3). Moreover, the families $\mathcal{LR}^i_\alpha, i=1,2$ are not generic of a Morse Type 2 singularity of $\mathcal{LR}^i_{\alpha_0}, i=1,2$ in sense of Theorem 6.2.2.

Proof: We know that the LPL is part of the discriminant curves $\Delta^\alpha_{\mathcal{LR}^i}$. Suppose that the origin is a timelike umbilic point (i.e., $l_0=n_0=0$). Then all the coefficients of $\mathcal{LR}^i_\alpha, i=1,2$ vanish at this point. From Proposition 8.4.2(1), the LPL has a Morse singularity of type A_1^- at that point. Subsequently, the BDEs $\mathcal{LR}^i_\alpha, i=1,2$ have generically an isolated singularity of type Morse Type 2 of type A_1^- for any $\alpha \in \mathbb{R}$. The 1-jet of $\mathcal{LR}^1_\alpha$ is

$$(\tanh(\alpha)-1)j^1n(x,y)dy^2-(\tanh(\alpha)+1)j^1l(x,y)dx^2. \tag{9.8}$$

As highlighted in Chapter 6, the local configuration of $\mathcal{LR}^1_\alpha$ are determined by the number and type of the singularities of the vector field ξ. The singularities of ξ are the roots of the cubic

$$\phi(p)=(\tanh(\alpha)-1)n_{11}p^3+(\tanh(\alpha)-1)n_{10}p^2-(\tanh(\alpha)+1)l_{11}p-(\tanh(\alpha)+1)l_{10}.$$

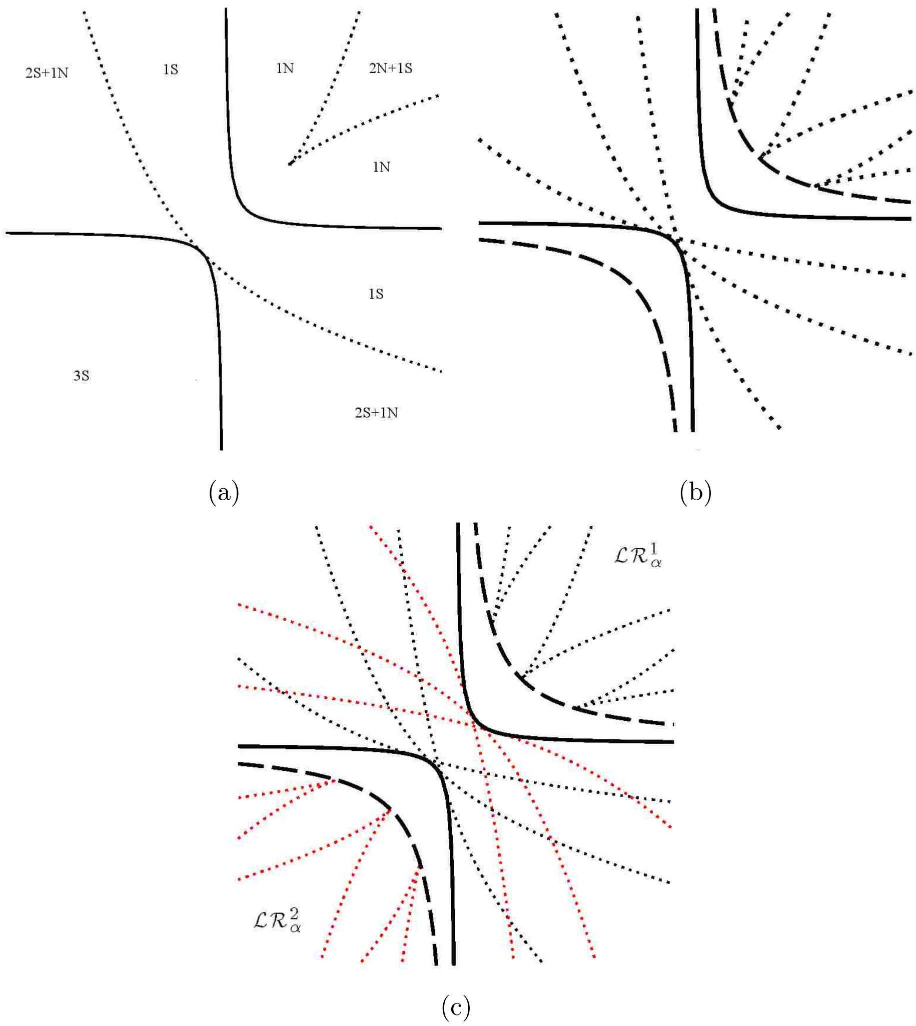


Figure 9.1: (a) Maple plots of the partition of the (n_{10}, l_{11})-plane for $\mathcal{LR}^1_0$. The dotted curve is the double roots of ϕ. (b) The variation of the double root curve of the cubic ϕ in the case of $\mathcal{LR}^1_\alpha$ as α varies. (c) The double root curves of ϕ in the cases of $\mathcal{LR}^i_\alpha, i = 1, 2$. The branches of the dashed hyperbola $n_{10}l_{11} = 9$ are the locus of the cusps of the double root curves of ϕ of $\mathcal{LR}^i_\alpha, i = 1, 2$. The double root curves of ϕ of $\mathcal{LR}^2_\alpha$ are in red.

The eigenvalues of the linear part of ξ at a singularity are $-\phi'(p)$ and

$$\alpha_1(p) = 2p(\tanh(\alpha) - 1)(n_{11}p + n_{10}).$$

The cubic $\phi(p)$ has repeated roots if and only if

$$(-18n_{11}l_{10}n_{10}l_{11} - 4n_{10}^3 l_{10} - n_{10}^2 l_{11}^2 + 27n_{11}^2 l_{10}^2 - 4l_{11}^3 n_{11})\tanh(\alpha)^2$$
$$+ (8n_{10}^3 l_{10} - 8l_{11}^3 n_{11})\tanh(\alpha) + 18n_{11}l_{10}n_{10}l_{11} + n_{10}^2 l_{11}^2 - 4n_{10}^3 l_{10} - 27n_{11}^2 l_{10}^2 - 4l_{11}^3 n_{11} = 0.$$

Moreover, $\alpha_1(p)$ and $\phi(p)$ have a common root if and only if

$$n_{11}l_{10}(n_{11}l_{10} - l_{11}n_{10}) = 0.$$

The singularity of the LPL at the origin is worse than Morse if and only if

$$n_{11}l_{10} - l_{11}n_{10} = 0.$$

The above conditions can be avoided at the origin if we take the genericity of $\mathbb{A}$ in consideration. (This means generically $\phi(p)$ has simple roots, $\phi(p)$ and $\alpha_1(p)$ has no common roots and the LPL has no singularity worse than Morse at the origin.) We need to simplify the above conditions in order to visualise the exceptional curves in $\mathbb{R}^2$ following the concept in Chapter 6. By considering that $n_{11}l_{10} \neq 0$ and by making a linear change of coordinates in the source, we obtain an equivalent BDE to that of $\mathcal{LR}^1_\alpha$ with 1-jet given by

$$(a_1x + y)dy^2 + (x + c_2y)dx^2 \tag{9.9}$$

with

$$a_1 = \frac{-n_{10}(\tanh(\alpha)+1)^{2/3}(\tanh(\alpha)-1)^{4/3}}{(l_{10})^{1/3}(n_{11})^{2/3}(\tanh(\alpha)^2-1)}$$

and

$$c_2 = \frac{-l_{11}(\tanh(\alpha)+1)^{1/3}}{(l_{10})^{2/3}(n_{11})^{1/3}(\tanh(\alpha)-1)^{1/3}},$$

where $n_{10}, l_{11} \in \mathbb{R}$ and $n_{11}, l_{10} \in \mathbb{R} - \{0\}$. Then the cubic $\phi(p)$ is given by

$$\phi(p) = p^3 + a_1p^2 + c_2p + 1,$$

and

$$\alpha_1(p) = 2p(p + a_1).$$

Therefore, $\mathcal{LR}^1_\alpha$ has a codimension 1 singularity of type Morse Type 2 of type A_1^- away from the following curves in the (n_{10}, l_{11})-plane:

1. The non-Morse curve: $1 - l_{11}n_{10} = 0$.

2. The double root curve of ϕ: $(18l_{11}n_{10} + 4l_{11}^3 + l_{11}^2 n_{10}^2 + 4n_{10}^3 - 27)\tanh(\alpha)^2 + 8(l_{11}^3 - n_{10}^3)\tanh(\alpha) - 18l_{11}n_{10} + 4n_{10}^3 - l_{11}^2 n_{10}^2 + 4l_{11}^3 + 27 = 0$.

Note that both of the non-Morse and the α_1,ϕ common root curves are given by the same equation. These curves divide the (n_{10}, l_{11})-plane into five regions. The calculation to determine the number and type of the singularity of ξ in each region is given in the Appendix (section A.2).

We conclude that for each fixed α, the five generic cases shown in Figure 6.3 can occur and their distribution on the (n_{10}, l_{11})-plane is as in Figure 9.1(a).

For each fixed α, the double root curve of ϕ has a cusp at the point

$$\left(\frac{3(1+\tanh(\alpha))}{-(-1+\tanh(\alpha))^{1/3}(1+\tanh(\alpha))^{2/3}}, \frac{-3(-1+\tanh(\alpha))^{1/3}(1+\tanh(\alpha))^{2/3}}{1+\tanh(\alpha)}\right).$$

As α varies, the locus of these cusps trace a branch of the hyperbola $n_{10}l_{11} = 9$. (The other branch of the hyperbola is traced by the cusps of the double root curves of ϕ of $\mathcal{LR}^2_\alpha$.)

For each fixed $\alpha \in \mathbb{R}$, the double root curve of $\phi(p)$ and the non-Morse curve are tangential at the point

$$\left(\frac{1+\tanh(\alpha)}{(-1+\tanh(\alpha))^{1/3}(1+\tanh(\alpha))^{2/3}}, \frac{(-1+\tanh(\alpha))^{1/3}(1+\tanh(\alpha))^{2/3}}{1+\tanh(\alpha)}\right).$$

The double root curve of $\phi(p)$ varies with α (Figure 9.1(b)).

Finally, the genericity condition (6.12) for the family $\mathcal{LR}^1_\alpha$ with $\mathcal{LR}^1_{\alpha_0}$ has a Morse Type 2 of type A_1^- is violated in this case since the right hand side of the condition vanish. The case of $\mathcal{LR}^2_\alpha$ follows in similar way. □

Remark 9.2.1

1. As α varies, the case 3S remains the same, the case $2S + 1N$ changes to $1S$ and vice-versa while the case $2N + 1S$ changes to $1N$ and vice-versa (Figure 9.1(b)).

2. For any fixed α, the change of coordinate $(n_{10}, l_{11}, \alpha) \to (-l_{11}, -n_{10}, -\alpha)$ applied to the equation of the double root of $\phi(p)$ of $\mathcal{LR}^1_\alpha$ gives the equation of the double roots of ϕ of $\mathcal{LR}^2_\alpha$ and vice-versa (Figure 9.1(c)).

We aim to figure out the local configurations of $\mathcal{LR}^i_\alpha, i = 1, 2$ at the point of tangency of the LPL and the sets D^i_α. It is the point where the discriminant curves of $\mathcal{LR}^i_\alpha, i = 1, 2$ (except the discriminant curve of $\mathcal{LR}^2_0$) have an A_3^--singularity, see Proposition 8.4.2.

Proposition 9.2.2 [14] Suppose a BDE has 1-jet $(a_1x + y, \epsilon x, 0), \epsilon = \pm 1$. Then for almost all values of a_1 the BDE can be reduced by a formal diffeomorphism to the form

$$(a_1x + y)dy^2 + \epsilon xdydx + c(y)dx^2 = 0, \tag{9.10}$$

where $c(y)$ is a formal power series in y with zero 1-jet.

Remark 9.2.2 The result in Proposition 9.2.2 was proved in [14] for a BDE with 1-jet $(y, 2(\epsilon x + b_2y), 0)$. One can show that the 1-jets $(y, 2(\epsilon x + b_2y), 0)$ and $(a_1x + y, \epsilon x, 0)$ are equivalent, so the results in [14] also hold for BDEs with 1-jet $(a_1x + y, \epsilon x, 0)$ as stated in Proposition 9.2.2.

It is shown in [31] that the BDE with 1-jet $(a_1x + y, \epsilon x, c(y))$ is topologically determined by $j^3(a_1x + y, \epsilon x, c(y))$ where $j^3c(y) \neq 0$.

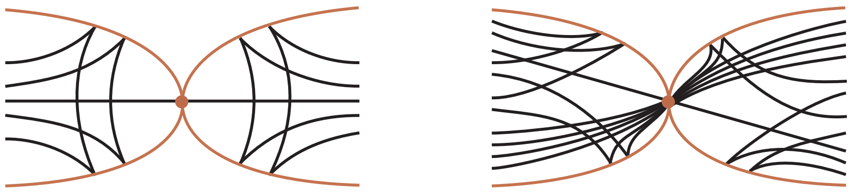
Theorem 9.2.4 [31] A BDE with 1-jet equivalent to $(a_1x + y, \epsilon x, 0)$, where $\epsilon = \pm 1$, and with a discriminant has an $A_3^\pm$-singularity is generically topologically equivalent to one of the following cases.

(i) Discriminant has an A_3^--singularity:

$$(A) : ydy^2 + xdydx + y^3dx^2 = 0, \text{ or to}$$
$$(B) : ydy^2 - xdydx + y^3dx^2 = 0.$$

(ii) Discriminant has an A_3^+-singularity:

$$(C) : ydy^2 + xdydx - y^3dx^2 = 0, \text{ or to}$$
$$(D) : ydy^2 - xdydx - y^3dx^2 = 0.$$



(a) (b)

Figure 9.2: Topological models of the BDE when its discriminant curve has an A_3^--singularity [31].

Theorem 9.2.5 At the point of tangency of the sets $D^i_\alpha, i = 1, 2$ and the LPL, the BDEs $\mathcal{LR}^i_\alpha, i = 1, 2$, except the BDE $\mathcal{LR}^2_0$, are topologically equivalent to

$$(A) : ydy^2 + xdydx + y^3dx^2 = 0, \text{ Figure 9.2}(a) \text{ or to}$$
$$(B) : ydy^2 - xdydx + y^3dx^2 = 0. \text{ Figure 9.2}(b).$$

Since the 1-jets of $\mathcal{LR}^i_\alpha, i = 1, 2$ are equivalent to $(a_1x + y,\epsilon x,0)$, the local configurations of $\mathcal{LR}^i_\alpha, i = 1, 2$ are as Figure 9.2(b) when $\epsilon = -1$. When $\epsilon = 1$, their local configurations can be either as Figure 9.2(a) or (b).

Proof: We consider equations (8.17) and (8.19) for the BDEs $\mathcal{LR}^1_\alpha$ and $\mathcal{LR}^2_\alpha$ and equations (8.18) and (8.20) for their discriminants respectively, and take the origin to be the point of concern. The origin is on the LPL and on D^i_α if and only if $m_0 = l_0 = 0$ or $m_0 = n_0 = 0$. We assume that $m_0 = l_0 = 0$ and $n_0 \neq 0$ (otherwise the origin is a timelike umbilic point and this case does not occur for generic $\mathbb{A}$). We proved in Proposition 8.4.2 that the discriminants of $\mathcal{LR}^i_\alpha, i = 1, 2$ have generically an A_3^--singularity at the origin. We study the case of $\mathcal{LR}^1_\alpha$ and similar procedure will be carried out for $\mathcal{LR}^2_\alpha$. From equation (8.17), the 1-jet of $\mathcal{LR}^1_\alpha$ is

$$j^1(a, b, c) = (m_{10}n_0(-1+\tanh(\alpha))x+m_{11}n_0(-1+\tanh(\alpha))y, -2l_{10}n_0x-2l_{11}n_0y, 0).$$

(Since we consider $m_0 = l_0 = 0$ and $n_0 \neq 0$, the 1-jet of $mathcalLR^1_\alpha$ depends on the 1-jet of l and m, however, if we consider $m_0 = n_0 = 0$ and $l_0 \neq 0$, then the 1-jet of $\mathcal{LR}^1_\alpha$ will depend on the 1-jet of n and m.) The 1-jet of $\mathcal{LR}^1_\alpha$ is in the form

$(a_1x + a_2y, b_1x + b_2y, 0)$. Since the LPL is regular at the origin for generic $\mathbb{A}$, that implies $l_{10} \neq 0$. Then by a linear change of coordinates in the form

$$(x, y) \mapsto (x + \frac{m_{10}(-1 + \tanh(\alpha))}{2l_{10}} y, y),$$

and change of scale, we can reduce the 1-jet of $\mathcal{LR}^1_\alpha$ to $(y, \epsilon x + b_2y, 0)$ where

$$b_2 = \frac{(-m_{10}n_0(-1 + \tanh(\alpha)) - 2l_{11}n_0)\sqrt{-2l_{10}^4 n_0^2(l_{10}\tanh(\alpha)m_{11} - m_{11}l_{10} - \tanh(\alpha)m_{10}l_{11} + m_{10}l_{11})}}{2(l_{10}^2 n_0^2(-1 + \tanh(\alpha))(m_{11}l_{10} - m_{10}l_{11})}. \tag{9.11}$$

Further linear change of coordinate in the form $(x, y) \mapsto (x + qy, y)$ where

$$q = \frac{(-m_{10} + m_{10}\tanh(\alpha) + 2l_{11})\sqrt{-2l_{10}^4 n_0^2(l_{10}\tanh(\alpha)m_{11} - m_{11}l_{10} - \tanh(\alpha)m_{10}l_{11} + m_{10}l_{11})}}{2l_{10}^2 n_0(l_{10}\tanh(\alpha)m_{11} - m_{11}l_{10} - \tanh(\alpha)m_{10}l_{11} + m_{10}l_{11})}, \tag{9.12}$$

transforms the 1-jet of $\mathcal{LR}^1_\alpha$ in the form $(y, \epsilon x + b_2y, 0)$ to the form $(a_1x + y, \epsilon x, 0)$ where

$$a_1 = \frac{(-m_{10} + m_{10}\tanh(\alpha) + 2l_{11})\sqrt{2}\sqrt{l_{10}^4 n_0^2(m_{11}l_{10} - m_{10}l_{11})(-1 + \tanh(\alpha))}}{2(-1 + \tanh(\alpha))(m_{11}l_{10} - m_{10}l_{11})l_{10}^2 n_0}. \tag{9.13}$$

We aim to distinguish the cases that the BDEs $\mathcal{LR}^1_\alpha$ have the topological configurations (A) or (B) in Theorem 9.2.4 shown in Figure 9.2(a) or (b). In order do so, we need to reduce the 3-jet of $\mathcal{LR}^1_\alpha$ with 1-jet $(a_1x+y, \epsilon x, 0)$ to the normal forms in Theorem 9.2.4. Proposition 9.2.2 assures that the 3-jet of the BDEs $\mathcal{LR}^1_\alpha$ can be reduced to the normal form $(a_1x + y, \epsilon x, \lambda y^3)$ ($c(y)$ in this case is with zero 2-jet). We consider, firstly, a change of coordinate of the form $(x, y) \longmapsto (x+P_2(x, y), y+Q_2(x, y))$ and multiply the new BDE by a non zero function of the form $1 + r_1(x, y)$, the subscripts 2 and 1 refer to homogeneous polynomials in (x, y) of degrees 2 and 1. We aim to eliminate the homogeneous parts of degree 2 in the 2-jet of the coefficients of the new BDE $(a_1x + y + A_2(x, y), \epsilon x + B_2(x, y), C_2(x, y))$. Solving a linear system of 2 equations which are the coefficients of $C_2(x, y)$ eliminates this homogeneous part. After that, we solve a linear system of 6 equations which are the coefficients of $A_2(x, y)$ and $B_2(x, y)$ and that eliminates them. Likewise, a change of coordinates in the form $(x, y) \longmapsto (x + P_3(x, y), y + Q_3(x, y))$, multiplication of the new BDE by a non zero function of the form $1 + r_2(x, y)$, and then solving linear systems of the coefficients of $C_3(x, y)$, $A_3(x, y)$ and $B_3(x, y)$ give the normal form $(a_1x + y, \epsilon x, \lambda y^3)$ with

$$\lambda = \frac{-(m_{11}l_{10} - m_{10}l_{11})^3(-1 + \tanh(\alpha))(\tanh(\alpha) + 1)^2}{4n_0^2 l_{10}^4}.$$

Since $m_{11}l_{10} - m_{10}l_{11} \neq 0$ at the origin, then $\lambda \neq 0$.

The required conditions to determine the topological models of the BDEs $\mathcal{LR}^1_\alpha$ given in Theorem 9.2.4 are obtained by using the blowing-up technique of the proof of Theorem 9.2.4 in [31] (see the Appendix, section A.3). The BDE $\mathcal{LR}^1_\alpha$ is topologically equivalent to (A) (resp. (B)) if $a_1^2 - 8\epsilon > 0$ (resp. $a_1^2 - 8\epsilon < 0$). When $\epsilon = 1$, it is clear from a_1 (9.13) that we can choose $j^1(l, m)$ in such a way that we can have either $a_1^2 - 8 > 0$ or $a_1^2 - 8 < 0$, i.e., both cases can occur. On the other hand, when $\epsilon = -1$, then we have $a_1^2 + 8 > 0$, i.e., we have always case (B). Similar study for the case of $\mathcal{LR}^2_\alpha$, where $\alpha \neq 0$, gives similar results. $\square$

Appendix A

Calculations of some results in Chapters 8 and 9

A.1 The A_3^--singularity of $\Delta^{\alpha}_{\mathcal{LR}^i}$

We present below the calculation carried out to show that the discriminant curves $\Delta^{\alpha}_{\mathcal{LR}^1}$ have an A_3^--singularity at the intersection point of the LPL and D^1_{α} (Proposition 8.4.2). The calculation for the case of $\Delta^{\alpha}_{\mathcal{LR}^2}$, where $\alpha \neq 0$, follows similarly.

> $restart:$
> $with(linalg):$
> $with(Student[Precalculus]):$

> $m(x,y) := m[0]+m[10]\cdot x+m[11]\cdot y+m[20]\cdot x^2+m[21]\cdot x\cdot y+m[22]\cdot y^2 + m[30]\cdot x^3+m[31]\cdot x^2\cdot y+m[32]\cdot x\cdot y^2+m[33]\cdot y^3:$
> $l(x,y) := l[0]+l[10]\cdot x+l[11]\cdot y+l[20]\cdot x^2+l[21]\cdot x\cdot y+l[22]\cdot y^2+l[30]\cdot x^3+l[31]\cdot x^2\cdot y+l[32]\cdot x\cdot y^2+l[33]\cdot y^3:$
> $n(x,y) := n[0]+n[10]\cdot x+n[11]\cdot y+n[20]\cdot x^2+n[21]\cdot x\cdot y+n[22]\cdot y^2 + n[30]\cdot x^3+n[31]\cdot x^2\cdot y+n[32]\cdot x\cdot y^2+n[33]\cdot y^3:$

> #The equation of the discriminant curve of LR^1_\alpha when considering the local parametrisation where E=G=0 at the origin when it is the intersection point of the LPL and D^1_\alpha
> $dis := \left(l(x,y)\cdot n(x,y)\cdot\cosh^2(\alpha) - m^2(x,y)\right)\cdot l(x,y)\cdot n(x,y):$

> $dis := mtaylor\left(subs\left(m_0=0,\, l_0=0,\, dis\right),\, \{x,y\},\, 5\right):$

> $mtaylor\left(subs\left(m_0=0,\, l_0=0,\, dis\right),\, \{x,y\},\, 3\right)$

$$l_{10}^2\, n_0^2 \cosh(\alpha)^2 x^2 + 2\, l_{11}\, y\, l_{10}\, n_0^2 \cosh(\alpha)^2 x + l_{11}^2\, y^2\, n_0^2 \cosh(\alpha)^2 \quad \textbf{(1)}$$

> $\left(2\, l_{11}\; l_{10}\, n_0^2\cosh(\alpha)^2\right)^2 - 4\cdot l_{10}^2\, n_0^2\cosh(\alpha)^2\cdot l_{11}^2\, n_0^2\cosh(\alpha)^2$

$$0 \quad \textbf{(2)}$$

> #We have the j^1 dis(0)=0 and the origin is degenerate point for the discriminants. By a change of coordinates we can write the j^2 dis(0) in the form (a_0 x^2)

> $CompleteSquare\left(mtaylor\left(subs\left(m_0=0,\, l_0=0,\, dis\right),\, \{x,y\},\, 3\right),\, x\right)$

$$l_{10}^2\, n_0^2\cosh(\alpha)^2\left(x+\frac{l_{11}\, y}{l_{10}}\right)^2 \quad \textbf{(3)}$$

> $dis := mtaylor\left(subs\left(x=\frac{X}{l_{10}\, n_0\cosh(\alpha)} - \frac{l_{11}\, y}{l_{10}},\, y=Y,\, dis\right),\, \{X,Y\},\, 4\right):$

> $simplify(coeff(coeff(dis, X, 3), Y, 0)):$
> $simplify(coeff(coeff(dis, X, 2), Y, 1)):$
> $simplify(coeff(coeff(dis, X, 1), Y, 2)):$
> $simplify(coeff(coeff(dis, X, 0), Y, 3)):$

```
> #Since the coefficiant (simplify(coeff(coeff(dis, X, 0), Y, 3))
  )) of y^3 vanishes, by completeing squares, we can kill off the
  terms beyond x^2 which involves x and arrive at a 4-jet as
  follows
```

> $dis2 := mtaylor\Big(subs\Big(X=x-\frac{1}{2}\cdot\big(simplify(coeff(coeff(dis, X, 3), Y, 0))\cdot x^2 + simplify(coeff(coeff(dis, X, 2), Y, 1))\cdot x\cdot y + simplify(coeff(coeff(dis, X, 1), Y, 2))\cdot y^2\big), Y=y, dis\Big), \{x, y\}, 5\Big):$

> $simplify(coeff(coeff(dis2, x, 3), y, 0))$

$$0 \tag{4}$$

> $simplify(coeff(coeff(dis2, x, 2), y, 1))$

$$0 \tag{5}$$

> $simplify(coeff(coeff(dis2, x, 1), y, 2))$

$$0 \tag{6}$$

> $simplify(coeff(coeff(dis2, x, 4), y, 0))$

$$-\frac{5}{4}\frac{1}{l_{10}^4 n_0^4 \cosh(\alpha)^6}\Big(4\cosh(\alpha)^4 l_{10}^2 n_{10}^2 + 8\cosh(\alpha)^4 l_{10} n_{10} l_{20} n_0 + 4\cosh(\alpha)^4 l_{20}^2 n_0^2 - 4 m_{10}^2 \cosh(\alpha)^2 l_{10} n_{10} - 4 m_{10}^2 \cosh(\alpha)^2 l_{20} n_0 + m_{10}^4\Big) \tag{7}$$

> $simplify(coeff(coeff(dis2, x, 3), y, 1)):$

> $simplify(coeff(coeff(dis2, x, 2), y, 2)):$

> $simplify(coeff(coeff(dis2, x, 1), y, 3)):$

> $factor(simplify(coeff(coeff(dis2, x, 0), y, 4)))$

$$-\frac{1}{4}\frac{1}{l_{10}^4 \cosh(\alpha)^2}\Big(2 l_{11}^2 n_0 \cosh(\alpha)^2 l_{20} - l_{11}^2 m_{10}^2 - 2 l_{10} l_{11} n_0 \cosh(\alpha)^2 l_{21} + 2 l_{10}^2 n_0 \cosh(\alpha)^2 l_{22} - l_{10}^2 m_{11}^2 + 2 l_{10} l_{11} m_{11} m_{10}\Big)^2 \tag{8}$$

> $dis3 := mtaylor\Big(subs\Big(x=X-\frac{1}{2}\cdot\big(simplify(coeff(coeff(dis2, x, 4), y, 0))\cdot X^3 + simplify(coeff(coeff(dis2, x, 3), y, 1))\cdot X^2\cdot Y + simplify(coeff(coeff(dis2, x, 2), y, 2))\cdot X\cdot Y^2 + simplify(coeff(coeff(dis2, x, 1), y, 3))\cdot Y^3\big), y=Y, dis2\Big), \{X, Y\}, 5\Big):$

> $simplify(coeff(coeff(dis3, X, 4), Y, 0))$

$$0 \tag{9}$$

> $simplify(coeff(coeff(dis3, X, 3), Y, 1))$

$$0 \tag{10}$$

> $factor(simplify(coeff(coeff(dis3, X, 0), Y, 4)))$

$$-\frac{1}{4}\frac{1}{l_{10}^4\cosh(\alpha)^2}\left(2\,l_{11}^2\,n_0\cosh(\alpha)^2\,l_{20}-l_{11}^2\,m_{10}^2-2\,l_{10}\,l_{11}\,n_0\cosh(\alpha)^2\,l_{21}+2\,l_{10}^2\,n_0\cosh(\alpha)^2\,l_{22}-l_{10}^2\,m_{11}^2+2\,l_{10}\,l_{11}\,m_{11}\,m_{10}\right)^2 \quad \textbf{(11)}$$

> $subs(\,l_{11}=0,\,m_{11}=1,\,m_{10}=0,\,l_{22}=0,\,\%)$

$$-\frac{1}{4\cosh(\alpha)^2} \quad \textbf{(12)}$$

> **#The step above shows that the coefficiant of y^4 (-k_1) is not equal to zero at the origin (m_10 l_11- m_11 l_10 is not equal to zero).**

> $collect\left(2\,l_{11}^2\,n_0\cosh(\alpha)^2\,l_{20}-l_{11}^2\,m_{10}^2-2\,l_{10}\,l_{11}\,n_0\cosh(\alpha)^2\,l_{21}+2\,l_{10}^2\,n_0\cosh(\alpha)^2\,l_{22}-l_{10}^2\,m_{11}^2+2\,l_{10}\,l_{11}\,m_{11}\,m_{10},\,\cosh^2(\alpha)\right)$

$$\left(-2\,l_{10}\,l_{11}\,n_0\,l_{21}+2\,l_{10}^2\,n_0\,l_{22}+2\,l_{11}^2\,n_0\,l_{20}\right)\cosh(\alpha)^2-l_{10}^2\,m_{11}^2-l_{11}^2\,m_{10}^2+2\,l_{10}\,l_{11}\,m_{11}\,m_{10} \quad \textbf{(13)}$$

A.2 The local configurations of $\mathcal{LC}_\alpha^i$ and $\mathcal{LR}_\alpha^i, i = 1, 2$

We show below an example of the calculations carried out to determine the topological configurations of $\mathcal{LC}_\alpha^i$ and $\mathcal{LR}_\alpha^i, i = 1, 2$ for fixed α and the way they bifurcate locally as α varies in a neighbourhood of a given fixed α.

```
> restart :
> with(linalg) :

> m(x, y) := m[0] + m[10]·x + m[11]·y + m[20]·x^2 + m[21]·x·y + m[22]·y^2
      + m[30]·x^3 + m[31]·x^2·y + m[32]·x·y^2 + m[33]·y^3 :
> l(x, y) := l[0] + l[10]·x + l[11]·y + l[20]·x^2 + l[21]·x·y + l[22]·y^2 + l[30]
      ·x^3 + l[31]·x^2·y + l[32]·x·y^2 + l[33]·y^3 :
> n(x, y) := n[0] + n[10]·x + n[11]·y + n[20]·x^2 + n[21]·x·y + n[22]·y^2
      + n[30]·x^3 + n[31]·x^2·y + n[32]·x·y^2 + n[33]·y^3 :

> #LC^1_alpha:=p^2+r_1^alpha(x,y) p+r_2^alpha=0, p=dy/dx,
  r_1^alpha(x,y)=B, r_2^alpha(x,y)=C.
```

$$B := mtaylor\left(simplify\left(subs\left(f = \frac{2}{(1+\tanh(\alpha))}, mtaylor\left(f\cdot\frac{m(x,y)}{n(x,y)}, [x,y], 4\right)\right)\right), [x,y], 4\right):$$

```
> b0 := coeff(coeff(mtaylor(simplify(B), [x, y], 4), x, 0), y, 0) :
> b10 := coeff(coeff(mtaylor(simplify(B), [x, y], 4), x, 1), y, 0) :
> b11 := coeff(coeff(mtaylor(simplify(B), [x, y], 4), x, 0), y, 1) :
> b20 := coeff(coeff(mtaylor(simplify(B), [x, y], 4), x, 2), y, 0) :
> b21 := coeff(coeff(mtaylor(simplify(B), [x, y], 4), x, 1), y, 1) :
> b22 := coeff(coeff(mtaylor(simplify(B), [x, y], 4), x, 0), y, 2) :
> b30 := coeff(coeff(mtaylor(simplify(B), [x, y], 4), x, 3), y, 0) :
> b31 := coeff(coeff(mtaylor(simplify(B), [x, y], 4), x, 2), y, 1) :
> b32 := coeff(coeff(mtaylor(simplify(B), [x, y], 4), x, 1), y, 2) :
> b33 := coeff(coeff(mtaylor(simplify(B), [x, y], 4), x, 0), y, 3) :
```

$$C := subs\left(g = \frac{(-\tanh(\alpha)+1)}{1+\tanh(\alpha)}, mtaylor\left(g\cdot\frac{l(x,y)}{n(x,y)}, [x,y], 4\right)\right):$$

```
> c0 := coeff(coeff(mtaylor(simplify(C), [x, y], 4), x, 0), y, 0) :
> c10 := coeff(coeff(mtaylor(simplify(C), [x, y], 4), x, 1), y, 0) :
> c11 := coeff(coeff(mtaylor(simplify(C), [x, y], 4), x, 0), y, 1) :
> c20 := coeff(coeff(mtaylor(simplify(C), [x, y], 4), x, 2), y, 0) :
> c21 := coeff(coeff(mtaylor(simplify(C), [x, y], 4), x, 1), y, 1) :
> c22 := coeff(coeff(mtaylor(simplify(C), [x, y], 4), x, 0), y, 2) :
> c30 := coeff(coeff(mtaylor(simplify(C), [x, y], 4), x, 3), y, 0) :
> c31 := coeff(coeff(mtaylor(simplify(C), [x, y], 4), x, 2), y, 1) :
> c32 := coeff(coeff(mtaylor(simplify(C), [x, y], 4), x, 1), y, 2) :
```

> $c33 := coeff(coeff(mtaylor(simplify(C), [x, y], 4), x, 0), y, 3)$:

> #Codimension 0 singularity of LC^1_alpha

> $C1 := 4 \cdot c0 - b0^2$:

> $factor(numer(\%))$:

> #The condition C_1=0

> $S := \{solve(4 \cdot c0 - b0^2, \tanh(\alpha))\}$;

$$S := \left\{ \frac{\sqrt{l_0 n_0 (l_0 n_0 - m_0^2)}}{l_0 n_0}, -\frac{\sqrt{l_0 n_0 (l_0 n_0 - m_0^2)}}{l_0 n_0} \right\} \quad (1)$$

> #tanh(+\alpha_0)=

> $s1 := op(1, S)$

$$s1 := \frac{\sqrt{l_0 n_0 (l_0 n_0 - m_0^2)}}{l_0 n_0} \quad (2)$$

> $expand(simplify(subs(m_0 = 2, n_0 = 3, l_0 = 3, \%)))$

$$\frac{1}{3}\sqrt{5} \quad (3)$$

> #tanh(-\alpha_0)=

> $s2 := op(2, S)$

$$s2 := -\frac{\sqrt{l_0 n_0 (l_0 n_0 - m_0^2)}}{l_0 n_0} \quad (4)$$

> $simplify(expand(simplify(subs(m_0 = 2, n_0 = 3, l_0 = 3, \%))))$

$$-\frac{1}{3}\sqrt{5} \quad (5)$$

> #The conditions C_1=0, C_2=0 to have a folded singularity at the origin in the case of tanh(+\alpha_0)

> $C2 := 4 \cdot c10 - 2 \cdot b0 \cdot c11 - 2 \cdot b0 \cdot b10 + b0^2 \cdot b11$:

> $newC2 := factor(subs(\tanh(\alpha) = s1, C2))$:

> $denom(newC2)$:

> $simplify(numer(newC2))$:

> $numC2 := factor(\%)$:

> #The conditions C_1=0, C_2=0, C_3≠0, C_4≠0 (lambda ≠0,1/16)

> $N11 := solve(numC2, n[11])$:

> $denom(N11)$:

> $C3 := 8 \cdot c20 - 4 \cdot b0 \cdot c21 + 2 \cdot b0^2 \cdot c22 + (b11 \cdot b0 - 2 \cdot b10) \cdot c11 - 4 \cdot b0 \cdot b20 + 2 \cdot b0^2 \cdot b21 - b0^3 \cdot b22 - 2 \cdot b10^2 + 3 \cdot b11 \cdot b10 \cdot b0 - b11^2 \cdot b0^2$:

> $\lambda := factor\left(simplify\left(subs\left(\tanh(\alpha) = s1, n[11] = N11, \frac{C3}{2(b11 \cdot b0 - 2 \cdot c11)}\right)\right)\right)$:

> $expand(simplify(subs(m_0 = 2, n_0 = 3, l_0 = 3, m_{10} = 1, n_{10} = 2, l_{10} = -3, m_{11} = 1, n_{11} = 1, l_{11} = -1, m_{20} = 1, n_{20} = 0, l_{20} = -1, m_{21} = 1, n_{21} = 0, l_{21} = -1, m_{22} = 1, n_{22} = 0, l_{22} = -1, \lambda)))$#

#Less than zero (saddle) = -1.83502721

$$-\frac{827}{6(3+\sqrt{5})^3} - \frac{337}{6}\frac{\sqrt{5}}{(3+\sqrt{5})^3} \tag{6}$$

> $expand(simplify(subs(m_0 = 2, n_0 = 3, l_0 = 3, m_{10} = 1, n_{10} = 2, l_{10} = -3, m_{11} = 1, n_{11} = 1, l_{11} = -1, m_{20} = -2, n_{20} = 0, l_{20} = 1, m_{21} = 1, n_{21} = 0, l_{21} = -1, m_{22} = 1, n_{22} = 0, l_{22} = -1, \lambda)))$

#greater than (1/16) (focus) = .26578585

$$\frac{97}{6(3+\sqrt{5})^3} + \frac{59}{6}\frac{\sqrt{5}}{(3+\sqrt{5})^3} \tag{7}$$

> $expand(simplify(subs(m_0 = 2, n_0 = 3, l_0 = 3, m_{10} = -1, n_{10} = .003, l_{10} = -.004, m_{11} = .07, n_{11} = -1, l_{11} = .002, m_{20} = -.05, n_{20} = -.0000000001, l_{20} = .1, m_{21} = 1, n_{21} = .000003, l_{21} = 0, m_{22} = .00001, n_{22} = .1, l_{22} = -.0009, \lambda)))$

#Between 0 and 1/16 (node)

$$0.03638120325 \tag{8}$$

> #The conditions C1 =0, C2=0, C3 = 0 (folded saddle-node) singularity

> $factor(subs(\tanh(\alpha) = s1, n[11] = N11, C3))$:

> $numC3 := simplify(numer(\%))$:

> $L22 := solve(numC3, l[22])$:

> #The conditions C1=0, C2=0, C4=0 (folded node-focus) singularity

> $C4 := 64 \cdot c20 - 32 \cdot b0 \cdot c21 + 16 \cdot b0^2 \cdot c22 - 4 \cdot c11^2 + (12 \cdot b11 \cdot b0 - 16 \cdot b10) \cdot c11 - 32 \cdot b0 \cdot b20 + 16 \cdot b0^2 \cdot b21 - 8 \cdot b0^3 \cdot b22 - 9\, b0^2 \cdot b11^2 + 24 \cdot b0 \cdot b10 \cdot b11 - 16 \cdot b10^2$:

> $denom(C4)$:

> $numC4 := simplify(factor(subs(\tanh(\alpha) = s1, n[11] = N11, C4)))$:

> $L20 := simplify(solve(numer(numC4), l[20]))$:

```
> denom(%) :

> #The conditions C1=0, C5=0, C6=0 (Morse Type 1) singularity

> C5 := 2·c11 − b11·b0 :
> denom(C5) :

> numC5 := factor(subs(tanh(α) = s1, numer(C5))) :

> L11MT1 := solve(numC5, l[11]) :

> C6 := 2·c10 − b10·b0 :

> denom(C6) :
> numC6 := subs(tanh(α) = s1, numer(C6)) :

> L10MT1 := solve(numC6, l[10]) :

> #The test of the genericity condition of the family LC^1_alpha
  with F_0=LC^1_alpha_0 having a folded saddle-node singularity
  at the origin. To simplify the calculations, we make a linear
  change of coordiantes to get b_0=c_0=c_10=0 and c_11=1.

> r1 := B :
> r2 := C :
> r1 := subs({n[11] = N11, l[22] = L22}, r1) :
> r2 := subs({n[11] = N11, l[22] = L22}, r2) :

> F := dy^2 + r1·dy·dx + r2·dx^2 :
> #Firstly, we want to get b_0=0.
> p := X :
> q := Y + q1·X :

> F1 := mtaylor(mtaylor(subs( x = p, dx = diff(p, X) * dX + diff(p, Y) * dY, y = q, dy
      = diff(q, X) * dX + diff(q, Y) * dY, F), {X, Y}, 4), {dX, dY}) :

> mtaylor(coeff(coeff(F1, dX), dY), {X, Y}, 4) :
> coeff(coeff(mtaylor(coeff(coeff(F1, dX), dY), {X, Y}, 4), X, 0), Y, 0) :
```

```
> q10 := subs(tanh(α) = s1, solve(coeff(coeff(mtaylor(coeff(coeff(F1, dX), dY), {X, Y},
    4), X, 0), Y, 0), q1)) :
> p := X:
> q := Y + q10·X:

> F1 := subs( mtaylor(mtaylor(subs(x = p, dx = diff(p, X) * dX + diff(p, Y) * dY, y = q, dy
    = diff(q, X) * dX + diff(q, Y) * dY, F), {X, Y}, 4), {dX, dY})) :

> AF1 := mtaylor(coeff(F1, dY, 2), {X, Y}, 4) :
> BF1 := mtaylor(coeff(coeff(F1, dX, 1), dY, 1), {X, Y}, 4) :
> CF1 := mtaylor(coeff(F1, dX, 2), {X, Y}, 4) :

> b0 := subs(tanh(α) = s1, simplify(coeff(coeff(mtaylor(coeff(coeff(F1, dX), dY) , {X, Y},
    1), X, 0), Y, 0))) :
> NAF1 := mtaylor(AF1/AF1, {X, Y}, 4) :
> NBF1 := mtaylor(BF1/AF1, {X, Y}, 4) :
> NCF1 := mtaylor(CF1/AF1, {X, Y}, 4) :
> F2 := NAF1·dY^2 + NBF1·dY·dX + NCF1·dX^2 :

> #To get c_11=1.

> F3 := mtaylor(mtaylor(subs(X = x, dX = dx, Y = q11· y, dY = q11·dy, F2), {x, y}, 4), {dx,
    dy}) :
> mtaylor(coeff(F3, dy, 2), {x, y}, 4) :

> mtaylor(coeff(F3, dx, 2), {x, y}, 2) :
> c1 := coeff(coeff(%, y, 1), x, 0) :
> S := {solve(c1 = q11^2, q11)} :
> s := simplify(subs(tanh(α) = s1, op(2, S))) :
> F3 := mtaylor(mtaylor(subs(X = x, dX = dx, Y = s ·y, dY = s ·dy, F2), {x, y}, 4), {dx,
    dy}) :

> NAF3 := mtaylor(coeff(F3, dy, 2), {x, y}, 4) :
> simplify(subs(tanh(α) = s1, %)) :

> NBF3 := mtaylor(coeff(coeff(F3, dx), dy), {x, y}, 4) :
```

> $NCF3 := mtaylor(coeff(F3, dx, 2), \{x, y\}, 4):$

> $Nc11 := simplify(coeff(coeff(coeff(F3, dx, 2), y, 1), x, 0)):$

> $simplify\left(subs\left(\tanh(\alpha) = s1, simplify\left(\frac{Nc11}{NAF3}\right)\right)\right):$

> $NB := mtaylor\left(\frac{NBF3}{NAF3}, \{x, y\}, 4\right):$

> $NC := mtaylor\left(\frac{NCF3}{NAF3}, \{x, y\}, 4\right):$

> $PNB := simplify\left(subs\left(\{\tanh(\alpha) = s1, x = 0, y = 0\}, \frac{\partial}{\partial \alpha} NB\right)\right):$

> $PNC := simplify\left(subs\left(\{\tanh(\alpha) = s1, x = 0, y = 0\}, \frac{\partial}{\partial \alpha} NC\right)\right):$

> $PPNCX := factor\left(simplify\left(subs\left(\{\tanh(\alpha) = s1, x = 0, y = 0\}, \frac{\partial}{\partial x}\left(\frac{\partial}{\partial \alpha} NC\right)\right)\right)\right):$

> $B10 := coeff(coeff(NB, x, 1), y, 0):$

> $B11 := coeff(coeff(NB, x, 0), y, 1):$

> $C21 := coeff(coeff(NC, x, 1), y, 1):$

> **#The genericity condition:**

> $(1 + B10) \cdot PNB + (2 \cdot C21 - (B10 + 1) \cdot B11) \cdot PNC - 2 \cdot PPNCX:$

> $condition := factor(simplify(subs(\tanh(\alpha) = s1, \%))):$

> $simplify(subs(m_0 = 2, n_0 = 3, l_0 = 3, m_{10} = 1, n_{10} = 2, l_{10} = -3, m_{11} = 1, n_{11} = 1, l_{11} = -1, m_{20} = 1, n_{20} = 0, l_{20} = -1, m_{21} = 1, n_{21} = 0, l_{21} = -1, m_{22} = 1, n_{22} = 0, l_{22} = -1, \%))$

$$-\frac{4\left(123281 + 55133\sqrt{5}\right)\left(7 + 3\sqrt{5}\right)}{\left(3 + \sqrt{5}\right)^{9}} \qquad \textbf{(9)}$$

The calculations below are carried out to determine the local configuration of $\mathcal{LR}^1_0$ at its Morse Type 2 singularity of type A_1^- on the LPL. It is determined by the number and type of the singularity of ξ in each region of the (n_{10}, l_{11})-plane (Theorem 9.2.3).

```
> restart :
> with(plots) : with(plottools) :
> with(plots, implicitplot) :
> with(linalg) :
> with(Student[Precalculus]) :
> with(Maplets[Elements]) :
> m(x, y) := m[0] + m[10]·x + m[11]·y + m[20]·x^2 + m[21]·x·y + m[22]·y^2
      + m[30]·x^3 + m[31]·x^2·y + m[32]·x·y^2 + m[33]·y^3 :
> l(x, y) := l[10]·x + l[11]·y + l[20]·x^2 + l[21]·x·y + l[22]·y^2 + l[30]·x^3
      + l[31]·x^2·y + l[32]·x·y^2 + l[33]·y^3 :
> n(x, y) := n[10]·x + n[11]·y + n[20]·x^2 + n[21]·x·y + n[22]·y^2 + n[30]
      ·x^3 + n[31]·x^2·y + n[32]·x·y^2 + n[33]·y^3 :

> #The 1-jet of the A-characteristic BDE.

> A := mtaylor(simplify(mtaylor(n(x, y)·m(x, y), [x, y], 2)), [x, y], 2)
```

$$A := n_{10}\, m_0\, x + n_{11}\, y\, m_0 \tag{1}$$

```
> B := mtaylor(simplify(mtaylor(2· n(x, y)·l(x, y), [x, y], 2)), [x, y], 2)
```

$$B := 0 \tag{2}$$

```
> C := mtaylor(m(x, y)·l(x, y), [x, y], 2)
```

$$C := m_0\, l_{10}\, x + m_0\, l_{11}\, y \tag{3}$$

```
> char := subs(m[0] = 1, A·dy^2 + B·dy·dx + C·dx^2)
```

$$char := (n_{10}\, x + n_{11}\, y)\, dy^2 + (l_{10}\, x + l_{11}\, y)\, dx^2 \tag{4}$$

```
> #The 2-jet of the discriminant.
> mt := expand(factor(simplify(subs(m[0] = 1, expand(B^2 - 4 A·C)))))
```

$$mt := -4\, n_{10}\, x^2\, l_{10} - 4\, n_{10}\, x\, l_{11}\, y - 4\, n_{11}\, y\, l_{10}\, x - 4\, n_{11}\, y^2\, l_{11} \tag{5}$$

```
> coeff(mt, x, 2)
```

$$-4\, n_{10}\, l_{10} \tag{6}$$

```
> coeff(mt, y, 2)
```

$$-4\, n_{11}\, l_{11} \tag{7}$$

```
> coeff(coeff(mt, x, 1), y, 1)
```

$$-4\, n_{10}\, l_{11} - 4\, n_{11}\, l_{10} \tag{8}$$

```
> #The discriminant has zero 1-jet and it has a singularity worse
  than Morse if the following expresion equal to zero.
> mt1 := factor((coeff(coeff(mt, x, 1), y, 1))^2 - 4·coeff(mt, x, 2)·coeff(mt, y, 2))
```

$$mt1 := 16\, (-n_{10}\, l_{11} + n_{11}\, l_{10})^2 \tag{9}$$

> $\{solve(mt1, l[11])\}$

$$\left\{\frac{n_{11}\, l_{10}}{n_{10}}\right\} \qquad (10)$$

> #The singularity of \xi is given by the roots of the cubic \phi (p) which is given by

> $\mathrm{Cub} := n_{11}\cdot p^3 + n_{10}\cdot p^2 + l_{11}\cdot \mathrm{p} + l_{10}:$

> # The cubic \phi(p) has repeated roots if and only if its discriminant is equal to zero.

> $disp := factor(\ simplify(\ discrim(Cub, p)\ )\ )$

$$disp := -27\, n_{11}^2\, l_{10}^2 + 18\, n_{10}\, l_{10}\, n_{11}\, l_{11} + n_{10}^2\, l_{11}^2 - 4\, n_{10}^3\, l_{10} - 4\, n_{11}\, l_{11}^3 \qquad (11)$$

> # The eigenvalues of \xi are -\phi(p)' and \alpha_1(p) where \alpha_1(p) is given by

> $\alpha := 2\cdot p\cdot \left(n[11]\cdot p + n_{10}\right):$

> # The cubic \phi(p) and \alpha_1(p) have common roots if and only if the following is equal to zero.

> $simplify\left(resultant(\mathrm{Cub}, \alpha, \mathrm{p})\right)$

$$8\, l_{10}\, n_{11}^2\, \left(-n_{10}\, l_{11} + n_{11}\, l_{10}\right) \qquad (12)$$

> #Form the above, for generic A, n[11]*l[10] is not equal to zero. To visualise the special curves and to show the distribution of the five generic cases in R^2, we can make the following change of coordinate

> $p1 := a1\cdot X:$

> $q := a2\cdot Y:$

> $char1 := mtaylor\left(mtaylor(subs(\ x = p1, dx = a1\cdot dX, y = q, dy = a2\cdot dY, char), \{X, Y\}, 2),\ \{dX, dY\}\right)$

$$char1 := \left(l_{10}\, a1^3\, X + l_{11}\, a2\, Y\, a1^2\right) dX^2 + \left(n_{10}\, a1\, a2^2\, X + n_{11}\, a2^3\, Y\right) dY^2 \qquad (13)$$

> $\{solve(coeff(coeff(char1, dX, 2), X, 1) = 1, a1)\}:$

> $a33 := op(1, \%):$

> $char11 := subs(a1 = a33, char1)$

$$char11 := \left(X + \frac{l_{11}\, a2\, Y}{l_{10}^{(2/3)}}\right) dX^2 + \left(\frac{n_{10}\, a2^2\, X}{l_{10}^{(1/3)}} + n_{11}\, a2^3\, Y\right) dY^2 \qquad (14)$$

> $\{solve(coeff(coeff(char11, dY, 2), Y, 1) = 1, a2)\}:$

> $a11 := op(1, \%):$

> # The 1-jet of the equivalent BDE to the A-characteristic one.

> $carac := subs(a2 = a11, char11)$

$$carac := \left(X + \frac{l_{11}\,Y}{n_{11}^{(1/3)}\,l_{10}^{(2/3)}}\right) dX^2 + \left(\frac{n_{10}\,X}{l_{10}^{(1/3)}\,n_{11}^{(2/3)}} + Y\right) dY^2 \quad \textbf{(15)}$$

> $c := coeff(coeff(carac, dX, 2), Y, 1)$

$$c := \frac{l_{11}}{n_{11}^{(1/3)}\,l_{10}^{(2/3)}} \quad \textbf{(16)}$$

> $a := coeff(coeff(carac, dY, 2), X, 1)$

$$a := \frac{n_{10}}{l_{10}^{(1/3)}\,n_{11}^{(2/3)}} \quad \textbf{(17)}$$

> **# The cubic associated to the 1-jet of the equivalent BDE.**

> $\mathrm{Cub2} := p^3 + \left(\frac{n_{10}}{l_{10}^{(1/3)}\,n_{11}^{(2/3)}}\right)\cdot p^2 + \left(\frac{l_{11}}{n_{11}^{(1/3)}\,l_{10}^{(2/3)}}\right)\cdot \mathrm{p} + 1 :$

> $disp := factor(\ simplify(\ discrim(Cub2, p)))$

$$disp := -\frac{l_{10}^{(4/3)}\,n_{11}^{(4/3)}\left(27\,n_{11}^2\,l_{10}^2 - 18\,n_{10}\,l_{10}\,n_{11}\,l_{11} - n_{10}^2\,l_{11}^2 + 4\,n_{10}^3\,l_{10} + 4\,n_{11}\,l_{11}^3\right)}{\left(l_{10}^3\right)^{2/3}\left(n_{11}^3\right)^{2/3}\left(l_{10}^2\right)^{2/3}\left(n_{11}^2\right)^{2/3}} \quad \textbf{(18)}$$

> **# The repeated roots of the cubic curve is the locus of the zeros of the discriminant of the cubic.**

> $dispp := subs(n[11] = 1, l[10] = 1, \%)$

$$dispp := -27 + 18\,n_{10}\,l_{11} + n_{10}^2\,l_{11}^2 - 4\,n_{10}^3 - 4\,l_{11}^3 \quad \textbf{(19)}$$

> $\alpha := 2\cdot p\cdot\left(p + \frac{n_{10}}{l_{10}^{(1/3)}\,n_{11}^{(2/3)}}\right) :$

> $simplify(resultant(\mathrm{Cub2}\ , \alpha, \mathrm{p}))$

$$\frac{8\,(-n_{10}\,l_{11} + n_{11}\,l_{10})}{l_{10}\,n_{11}} \quad \textbf{(20)}$$

> **# The common root of the cubic and \alpha_1 curve is the zeros of the following function. (It is also the non-Morse curve.)**

> $res := subs(n[11] = 1, l[10] = 1, \%)$

$$res := -8\,n_{10}\,l_{11} + 8 \quad \textbf{(21)}$$

> $\phi := subs(n[11] = 1, l[10] = 1, -diff(Cub2, p))$

$$\phi := -3\,p^2 - 2\,n_{10}\,p - l_{11} \tag{22}$$

> $expand(simplify(subs(n[11] = 1, l[10] = 1, \phi\cdot\alpha)))$

$$-6\,p^4 - 10\,n_{10}\,p^3 - 4\,n_{10}^2\,p^2 - 2\,l_{11}\,p^2 - 2\,p\,l_{11}\,n_{10} \tag{23}$$

```
> # The calculation to determine the number and type of the
singularity of \xi in the (n_10,l_11)-plane.
```

> $n1 := -5.36$

$$n1 := -5.36 \tag{24}$$

> $l1 := -3.95$

$$l1 := -3.95 \tag{25}$$

> $subs(n[10] = \mathrm{n1}, l[11] = l1, \mathrm{res})$

$$-161.3760 \tag{26}$$

> $subs(n[10] = \mathrm{n1}, l[11] = l1, dispp)$

$$1664.831708 \tag{27}$$

> $p0 := \{solve(y2, p)\}:$

> $p1 := simplify(subs(n[10] = \mathrm{n1}, l[11] = l1, n[11] = 1, l[10] = 1, op(1, p0)))$

$$p1 := 5.991418825 - 3.000000000\ 10^{-10}\,\mathrm{I} \tag{28}$$

> $p2 := simplify(subs(n[10] = \mathrm{n1}, l[11] = l1, n[11] = 1, l[10] = 1, op(2, p0)))$

$$p2 := 0.2006023235 + 1.000000000\ 10^{-10}\,\mathrm{I} \tag{29}$$

> $p3 := simplify(subs(n[10] = \mathrm{n1}, l[11] = l1, n[11] = 1, l[10] = 1, op(3, p0)))$

$$p3 := -0.8320211455 + 1.000000000\ 10^{-10}\,\mathrm{I} \tag{30}$$

> $\delta := simplify(subs(n[10] = \mathrm{n1}, l[11] = l1, n[11] = 1, l[10] = 1, p = \mathrm{p1}, (\alpha + \phi)^2 - 4\cdot\phi\cdot\alpha))$

$$\delta := 2216.477255 - 0.000001086806821\,\mathrm{I} \tag{31}$$

> $dete := simplify(subs(n[10] = \mathrm{n1}, l[11] = l1, n[11] = 1, l[10] = 1, p = p1, \phi\cdot\alpha))$

$$dete := -298.9650218 + 2.142791692\ 10^{-7}\,\mathrm{I} \tag{32}$$

> $\delta := simplify(subs(n[10] = \mathrm{n1}, l[11] = l1, n[11] = 1, l[10] = 1, p = \mathrm{p2}, (\alpha + \phi)^2 - 4\cdot\phi\cdot\alpha))$

$$\delta := 64.79778850 + 3.128756514\ 10^{-8}\,\mathrm{I} \tag{33}$$

> $dete := simplify(subs(n[10] = \mathrm{n1}, l[11] = l1, n[11] = 1, l[10] = 1, p = p2, \phi\cdot\alpha))$

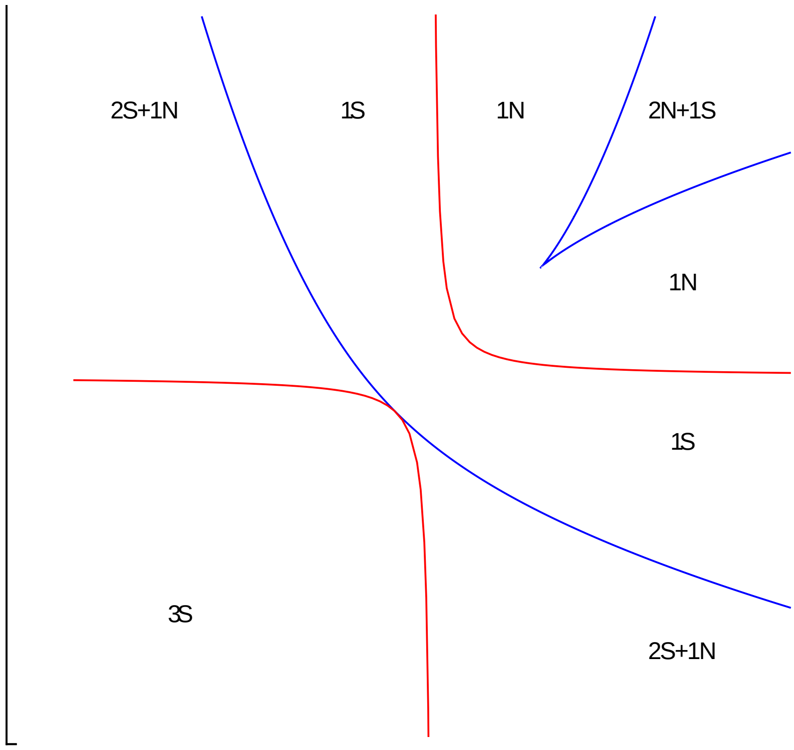
$$dete := -12.37789384 - 7.900321952\ 10^{-9}\,\mathrm{I} \tag{34}$$

> $\delta := simplify(subs(n[10] = \mathrm{n1}, l[11] = l1, n[11] = 1, l[10] = 1, p = p3, (\alpha + \phi)^2 - 4\cdot\phi\cdot\alpha))$

$$\delta := 301.0165765 - 1.032669177\ 10^{-7}\,\mathrm{I} \tag{35}$$

```
> dete := simplify(subs(n[10]=n1, l[11]=l1, n[11]=1, l[10]=1, p=p3, ϕ·α))
                    dete := -72.60092532 + 2.608778034 10^-8 I                    (36)

> ###########################################################################\
    ######
> disc := implicitplot([subs(l[11]=y, n[10]=x, -27 + 18 l_11 n_10 + l_11^2 n_10^2 - 4 l_11^3 - 4 n_10^3
    =0)], x=-10 .. 10, y=-10 .. 10, color=[blue, red], gridrefine=5) :
> resu := plot([1/x], x=-10 .. 10, y=-10 .. 10, color=[red], discont=true) :
> #coor:=implicitplot([y=0, x=0], x=-10 .. 10, y=-10 .. 10, color=[black, black],
    gridrefine=6) :
> t1 := textplot([7, 7, typeset("2N+1S")], align=above) :
> t2 := textplot([2.21, 7, typeset("1N")], align=above) :
> t3 := textplot([7, 2.21, typeset("1N")], align=above) :
> t4 := textplot([-2.21, 7, typeset("1S")], align=above) :
> t5 := textplot([7,-2.21, typeset("1S")], align=above) :
> t6 := textplot([-8, 7, typeset("2S+1N")], align=above) :
> t7 := textplot([7,-8, typeset("2S+1N")], align=above) :
> t8 := textplot([-7,-7, typeset("3S")], align=above) :
> display( {disc, resu, t1, t2, t3, t4, t5, t6, t7, t8}, axes=none )
```


2S+1N
1S
1N
2N+1S
1N
1S
3S
2S+1N

A.3 The proof of Theorem 9.2.4

The calculations below are from the first version of [31]. We reproduce them here to make the conditions required for determining the local configuration of $\mathcal{LR}^i_\alpha, i = 1, 2$, except $\mathcal{LR}^2_0$, at the intersection point of the LPL and D^i_α apparent (Theorem 9.2.5).

Theorem A.3.1 A BDE with 1-jet equivalent to $(\alpha u + v, \pm u, 0)$ and with a discriminant with an $A_3^{\pm}$-singularity is topologically equivalent to one of the following cases.

(i) Discriminant has an A_3^--singularity:

$$vdv^2 + udvdu + v^3du^2 = 0 \quad \textit{Figure A.1 A, or to}$$
$$vdv^2 - udvdu + v^3du^2 = 0 \quad \textit{Figure A.1 B}$$

(ii) Discriminant has an A_3^+-singularity

$$vdv^2 + udvdu - v^3du^2 = 0 \quad \textit{Figure A.1 C, or to}$$
$$vdv^2 - udvdu - v^3du^2 = 0 \quad \textit{Figure A.1 D}$$

Proof: We take the coefficients of the BDE $\omega = (a, b, c)$ in the form

$$a = \alpha u + v + M_1(u, v)$$
$$b = \epsilon u + M_2(u, v)$$
$$a = \lambda v^3 + M_3(u, v)$$

where M_i are germs, at the origin, of smooth functions with $j^1M_1 = j^1M_2 = 0$, $j^3M_3 = 0$, $\epsilon = \pm 1$ and $\lambda = \pm 1$.

We consider a blowing up of the BDE following the method introduced in [44, 28] for BDEs whose discriminants are isolated points, and extended in [49, 51] for general BDEs.

Following the notation in [28], let $f_i(w)$, $i = 1, 2$ denote the foliation associated to the BDE $\omega = (a, b, c)$, which is tangent to the vector field $a\frac{\partial}{\partial u} + (-b + (-1)^i\sqrt{b^2 - ac}\,)\frac{\partial}{\partial v}$. If ψ is a diffeomorphism and $\mu(u, v)$ is a non-vanishing real valued function, then ([28]) for $k = 1, 2$

1. $\psi(f_k(w)) = f_k(\psi^*(\omega))$, if ψ is orientation preserving;

2. $\psi(f_k(w)) = f_{3-k}(\psi^*(\omega))$, if ψ is orientation reversing;
3. $f_k(\lambda w) = f_k(\omega)$, if $\mu(u,v)$ is positive;
4. $f_k(\lambda w) = f_{3-k}(\omega)$, if $\mu(u,v)$ is negative.

We consider the directional blowing-up

$$(1) : \begin{cases} u = y^2 \\ v = xy \end{cases} \quad (2) : \begin{cases} u = -y^2 \\ v = xy \end{cases} \quad (3) : \begin{cases} u = xy^2 \\ v = y \end{cases}$$

The blowing up (1) (resp. (2)) is a diffeomorphism from the region $y > 0$ (resp. $y < 0$) in the (x,y)-plane to the region $u > 0$ (resp. $u < 0$) in the (u,v)-plane and is orientation reversing (resp. preserving) in this region. Applying the blowing up (2) gives similar results to applying (1). So we deal in more details with blowing up (1). We also consider the blowing up (3) but this does not reveal any extra information.

Consider the blowing-up (1). Then the new BDE $\omega_0 = (x,y)^*\omega$ has coefficients

$$\begin{aligned} \bar{a} = &\quad y(x(x^2 + axy + 4\epsilon y^2) + N_1(x,y)), \\ \bar{b} = &\quad y^2((x^2 + axy + 2\epsilon y^2) + N_2(x,y)), \\ \bar{c} = &\quad y^3(x + av + N_3(x,y)), \end{aligned}$$

where N_i, $i = 1,2,3$, are smooth functions with $j^3 N_1 = 0$, $j^2 N_2 = 0$ and $j^1 N_3 = 0$. We can write $(\bar{a}, \bar{b}, \bar{c}) = y(A_1, yB_1, y^2 C_1)$ with

$$\begin{aligned} A_1 = &\quad x(x^2 + axy + 4\epsilon y^2) + N_1(x,y), \\ B_1 = &\quad x^2 + axy + 2\epsilon y^2 + N_2(x,y), \\ C_1 = &\quad x + ay + N_3(x,y). \end{aligned}$$

and consider the quadratic form $\omega_1 = (A_1, yB_1, y^2 C_1)$.

The discriminant of ω_1, which is the blowing up of the discriminant of ω, is the exceptional fibre $v = 0$ when $\lambda = -1$ and the union of the exceptional fibre with two smooth curves C_i, $i = 1,2$, meeting transversaly the exceptional fibre at $(x_i, 0)$, where x_i are solutions of $(b_{22}^2 - 1)x^4 + 2\epsilon b_{22} x^2 + 1 = 0$.

We decompose ω_1 into two 1-forms, and to these 1-forms are associated the vector fields

$$X_i = A_1 \frac{\partial}{\partial x} + (-yB_1 + (-1)^i \sqrt{y^2(B_1^2 - A_1 C_1)}\,)\frac{\partial}{\partial y}, \quad i = 1,2.$$

These vector fields are tangent to the foliations defined by ω_1 and have the exceptional fibre $v=0$ (or part of it in the case $\lambda=+1$) as an integral curve. We deal now with the cases A_3^+ and A_3^- separately.

The case A_3^- $(\lambda=+1)$

We have $A_1(x,0)=x^3$, so $A(x_i,0)\neq 0$, $i=1,2$ with x_i as above. This means that near $(x_i,0)$, $i=1,2$, the integral curves of X_i form segments of smooth curves ending transeversaly on C_i, $i=1,2$. It also means that both vector fields X_1 and X_2 are smooth away from $(0,0)$ and the points $(x_i,0)$, $i=1,2$. So we need to analyse the configurations of these vector fields at $(0,0)$, with $y\geq 0$ (we are dealing with the blowing up (1) which is a diffeomorphism from the region $y>0$ delimited by the exceptional fibre and the curves C_1 and C_2 to the region in the (u,v)-plane delimited by the discriminant and with $u>0$). We have, at the origin,

$$X_1=(x(x^2+axy+4\epsilon y^2)+h.o.t.)\frac{\partial}{\partial x}+(-xy(x+ay)+h.o.t.)\frac{\partial}{\partial y}.$$

The first component A_1 of X_1 has a D_4-singularity provided $a^2-16\epsilon\neq 0$ (i.e., equivalent to $x(x^2\pm y^2)$). So $A_1=(x-g_1(x,y))(x^2+axy+4\epsilon y^2+g_2(x,y))$ for some germ of smooth functions g_1 and g_2 with $j^1g_1=j^2g_2=0$. We observe that when $x-g_1(x,y)=0$ the second component of X_1 also vanishes. Therefore $X_1=(x-g_1(x,y))\bar{X}_1$ with

$$\bar{X}_1=(x^2+axy+4\epsilon y^2+h.o.t.)\frac{\partial}{\partial x}+(-y(x+ay)+h.o.t.)\frac{\partial}{\partial y}.$$

The 2-jet of the vector field $\bar{X}_1$ satisfies the general position condition in Proposition 3.5 in [47]. Therefore, by Takens' result, $\bar{X}_1$ is topologically equivalent to $(x^2+axy+4\epsilon y^2)\frac{\partial}{\partial x}-y(x+ay)\frac{\partial}{\partial y}$ near the origin. Using a polar blowing up, we find that the configuration of $\bar{X}_1$ at the origin and hence of X_1 is as in Figure A.1. (The singularities of the blown up field are either saddles or nodes and are as shown in Figure A.1.)

We now consider the vector field X_2. We have, at the origin,

$$X_2=(x(x^2+axy+4\epsilon y^2)+h.o.t.)\frac{\partial}{\partial x}+(-y(x^2+axy+4\epsilon y^2)+h.o.t.)\frac{\partial}{\partial y}.$$

If $a^2-16\epsilon>0$, we can proceed as for X_1 above. We can factor out a term $x^2+axy+4\epsilon y^2+h.o.t$ from both component of X_2 and consider the vector field $\bar{X}_2=$

$(x + h.o.t.)\frac{\partial}{\partial x} + (-y + h.o.t.)\frac{\partial}{\partial y}$ which has a saddle-singularity at the origin. So X_2 has a saddle singularity at the origin.

If $a^2 - 16\epsilon < 0$, we consider a blowing up of the singularity of X_2 and find again that it has a saddle singularity at the origin.

Applying the blowing up (2) gives similar result, with the position of the nodes and saddles of X_1 inverted when $a^2 - 8\epsilon > 0$. So we have the configuration of the solution curves of ω in region (u, v)-plane delimited by the discriminant and with $u < 0$ as shown in Figure A.1. We also need to invert the colours of the foliations as we factored out a term y in ω (see comments at the beginning of the proof).

Therefore the configurations of the solutions of the BDE ω are as in shown in Figure A.1.

The case A_3^+ $(\lambda = -1)$

The situation here is similar to the case A_3^- (the 3-jets of X_1 and X_2 do not depend on λ). The difference is that the whole exceptional fibre is an integral curve of both X_1 and X_2. The configurations are as in Figure A.1. $\square$

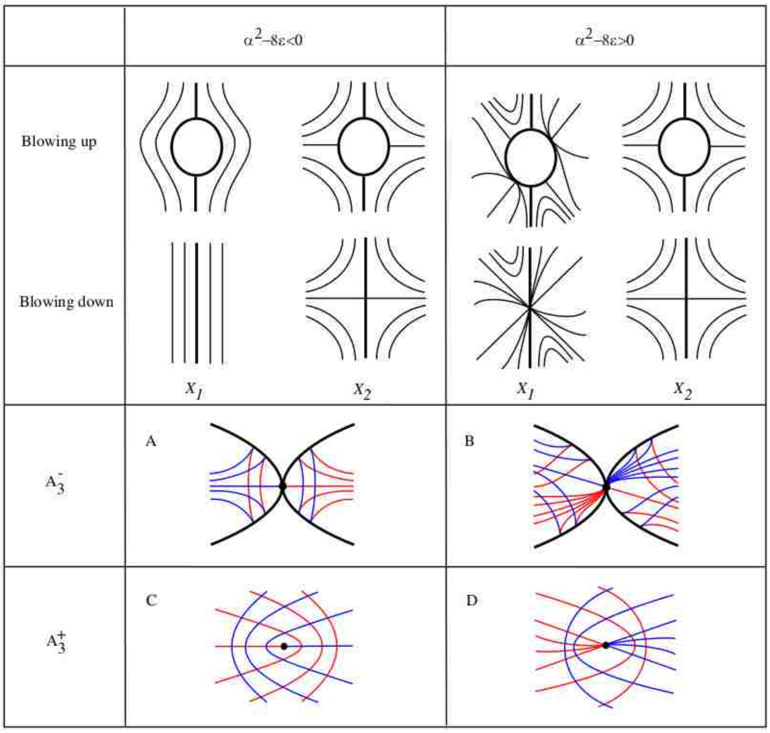


Figure A.1: Topological models of BDEs with 1-jet equivalent to $(\alpha u + v, \epsilon u, 0)$, $\epsilon = \pm 1$, and with discriminant with an A_3-singularity (bottom two rows). The figures in the top row give the configurations of the vector fields X_1 and X_2 at the origin and those of their blowing up.

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