Virtual knot homology and concordance

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A Thesis presented for the degree of
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for Peter Rushworth

and

for Margaret Young

30·11·05·

Mr Cook,

Will has disrupted my class is an a C3 and has completed ~~No~~ work as yet.

pg 97 – substitution.

A1)

a. $2h^2 = 2 \times 16 = 32$

b) $100 - h^2 = 100 - 16 = 84$

c) $\frac{100}{5h} = 100 \div 20 = 5$

d) $\frac{64}{h^2} = 64 \div 16 = 4$

e) $\frac{1-h^2}{5} = 3$

A2

a)

A3

a) ~~D = 6~~

A5

a)

a) $2 - 4x^2 = 98$

$10(x+3) = 130$

b) $2(10-x) = 310$

$x^2 - 2 = 12.5$

.. $25 \div x - 4 = 25$

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Abstract: We construct and investigate the properties of a new extension of Khovanov homology to virtual links, known as *doubled Khovanov homology*. We describe a perturbation of doubled Khovanov homology, analogous to Lee homology, and produce a *doubled Rasmussen invariant*; we use it obtain a number of results regarding virtual knot and link concordance. For instance, we demonstrate that the doubled Rasmussen invariant can obstruct the existance of a concordance between a virtual knot and a classical knot (i.e. a knot in S^3).

Kawamura and independently Lobb defined easily-computable bounds on the Rasmussen invariant of classical knots; we generalise these bounds to both the doubled Rasmussen invariant and to a distinct concordance invariant known as the *virtual Rasmussen invariant*, due to Dye, Kaestner, and Kauffman. We use the new bounds to compute or estimate the slice genus of all virtual knots of 4 classical crossings or less.

Finally, we use doubled Khovanov homology as a framework to construct a homology theory of links in thickened surfaces (objects closely related to, but distinct from, virtual links). This homology theory of links in thickened surfaces feeds back to the study of virtual knot concordance, as we are able to use it to investigate a refinement of the notion of sliceness of virtual knots.

Declaration

The work in this thesis is based on research carried out in the Topology Group, Department of Mathematical Sciences, Durham University, United Kingdom. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all my own work unless referenced to the contrary in the text.

The work of Chapter 4 has been accepted for publication in the *Canadian Journal of Mathematics* [Rus17b]. The work of Chapter 5 is contained in the preprint [Rus17a]. The work of Sections 6.1 and 6.2 makes up my contribution to a joint work with Vassily Manturov [MR17]. The work of Section 6.3 is contained in the preprint [Rus18]. All preprints are under review.

Acknowledgements

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I thank Ellen, who helps me much more than I let on. It makes me very happy to think about what we'll do next.

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Contents

Chapter 1

Introduction

This thesis is concerned with the extension of Khovanov homology to virtual links and applications thereof. The important terms in the previous sentence are covered fully in Chapters 2 and 3; here we briefly introduce them.

Since its birth in 1999 knot homology has grown to become a central topic within low dimensional topology, encompassing analytically and algebraically defined invariants of many different flavours. Homology theories of links in S^3 have been used to obtain a number of topological results, in addition to becoming objects of intense study themselves. The first such theory, Khovanov homology, exemplifies this: it lifts the celebrated Jones polynomial to a group-valued invariant, can detect the unknot, yields a lower bound on the slice genus, and has been generalised in many different directions.

The fecundity of Khovanov homology is further evidenced by its connections to physics, which are myriad. The act of associating an algebraic object (the homology group) to a topological one (the link) is, in a physical sense, quantization. This is more than just cosmetic, as Khovanov homology fits into an existing mathematical axiomitisation of physical quantization: it is a topological quantum field theory. A deep consequence of this is that Khovanov homology is a functor from the category of links and link cobordisms to the category of modules and module-maps (physically, this functoriality allows the theory to describe time evolution). Exploiting the functoriality of Khovanov homology allowed Rasmussen to unlock topological information contained in the theory, and define his powerful concordance invariant (which yields the aforementioned slice genus bound). A part of this thesis is concerned with generalising Rasmussen's techniques.

We are interested in the extension of Khovanov homology to virtual links, generalised knotted objects introduced by Kauffman in the late 90's. Virtual knot theory contains the theory of knots and links in S^3 (henceforth refered to as classical links) as a proper subset; virtual links can be studied in a diagrammatic manner just like classical links by adding a third type of crossing, distinct to the over/undercrossing decoration. A virtual link is then an appropriate equivalence class of virtual link diagrams.

Kauffman's original motivation to study virtual knot theory came from the problem of Gauss code realisation. As is well known, a classical knot defines a Gauss diagram, unique up to certain equivalence. The converse is not true, however: there are many Gauss diagrams that cannot be realised by a classical knot. Kauffman sought a generalised knot theory which realised the set of all Gauss codes, and arrived at virtual knot theory.

While much of the initial work in virtual knot theory followed this combinatorial precedent, virtual links have another interpretation as topological objects: they are embeddings of disjoint unions of S^1 into $\Sigma \times I$ (where Σ denotes a closed orientable surface), considered up to self-diffeomorphism and certain handle stabilisations of Σ. This viewpoint is predominant in this thesis, and exhibits virtual links as equivalence clases of embeddings into *equivalence classes of* 3*-manifolds*, distinguishing the theory from the study of links in a fixed 3-manifold.

The world of virtual links has many similarities to that of classical links, but it also exhibits new counter-intuitive phenomena. The non-triviality of the fundamental group of the target 3-manifold is responsible, in one way or another, for many of them; much of the new topological information contained in virtual links is therefore intrinsically 3-dimensional. These and other phenomena produce obstacles to extending invariants of classical links to virtual links, which require novel methods to overcome.

1.1 Organisation and original results

Conventions: All manifolds and embeddings are smooth, Σ_g denotes the closed orientable surface of genus g, and the labelling of virtual knots is that of Green's table [Gre]. To nip any confusion in the bud we point out here that the *virtual Rasmussen invariant* and the *doubled Rasmussen invariant* are two distinct concordance invariants of virtual knots; the former due to Dye, Kaester, and Kauffman, and the latter due to the author.

1.1.1 Chapters 2 and 3

Chapter 2 contains a review of the definition of classical Khovanov homology, along with the relevant definitions of cobordism and concordance of classical links, and a description of the definition of the classical Rasmussen invariant.

Chapter 3 contains an introduction to virtual knot theory and the associated theory of cobordism and concordance. It also contains a description of a virtual extension of Khovanov homology originally due to Manturov, and reformulated by Dye, Kaestner, and Kauffman (therefore known as MDKK homology), and the extraction of a virtual Rasmussen invariant from this homology theory.

1.1.2 Chapter 4

Chapter 4 contains the first original work of the thesis: the definition and investigation of an extension of Khovanov homology to virtual links known as *doubled Khovanov homology*. For instance, we show that doubled Khovanov homology is distinct to MDKK homology - in particular, it can sometimes be used to show that a given virtual link is not a classical link, something which MDKK homology is unable to do.

Further, we define a perturbation of doubled Khovanov homology analogous to that of Lee in the classical case and use it to define a *doubled Rasmussen invariant*. We then determine various properties of the invariant, and use it to obtain results regarding concordance of virtual links.

The chapter concludes by addressing the issue of the ill-defined nature of connect sum of virtual knots (as covered in Chapter 3); it is possible for a virtual knot to be realised as the connect sum of two unknot diagrams. We show that doubled Khovanov homology yields a condition met by such knots.

1.1.3 Chapter 5

We investigate the doubled and virtual Rasmussen invariants further, and convert bounds on the classical Rasmussen invariant due to Lobb and independently Kawamura to bounds on the new invariants (the bounds on the virtual and doubled Rasmussen invariants differ in structure, owing to the differences in the invariants themselves).

The slice genus of a virtual knot is defined in much the same way as that of a classical knot (as detailed in Chapter 3). We use the new bounds to compute or estimate the slice genus of every virtual knot of 4 classical crossings or less. These computations also demonstrate that there are a number of virtual knots detected by the doubled Rasmussen invariant which are not detected by the virtual Rasmussen invariant, including 6.8909, 6.9825, 6.28566, 6.37329, and 6.58375.

As an aside, we prove that the virtual Rasmussen invariant is additive with respect to connect sum.

1.1.4 Chapter 6

We use doubled Khovanov homology as a framework to define an invariant of links in thickened surfaces (objects closely related to, but distinct from, virtual links). While doubled Khovanov homology is bigraded, the new theory is trigraded; we use the cohomology of surfaces to define the new grading. We investigate the interaction of this new theory with concordance.

To conclude this chapter and the thesis, we show that the invariant of links in thickened surfaces feeds back to the study of virtual knot concordance: we use it to investigate a refinement of the notion of sliceness of virtual knots.

Chapter 2

Classical Khovanov homology

We review the definition of classical Khovanov homology, as given by Bar-Natan [BN05], before progressing to Rasmussen's construction of his concordance invariant, following [Ras10].

Khovanov homology associates to an oriented classical link a bigraded finitely generated Abelian group, and to a cobordism between classical links a map between the groups assigned to them. That is, Khovanov homology is a functor from an appropriate category of links and cobordisms to that of modules and module-maps. This functoriality is an important feature of the Khovanov package exploited in this thesis. Cobordisms appear also in the construction of the invariant itself: an abstract chain complex is produced in a category of 1-manifolds and cobordisms between them, before being turned into an algebra. For these reasons we begin this chapter with concrete definitions of cobordism and concordance.

2.1 Classical cobordism

Classical knot theory considers copies of S^1 embedded in S^3; classical knot concordance considers an enrichment of this setup by associating to such embeddings a surface embedded in B^4. We arrive at a quintessential problem of low dimensional topology: a 1-manifold inside a 3-manifold, appearing as the boundary of a 2-manifold inside a 4-manifold. In this section we give concrete definitions of this setup, and describe how we treat cobordisms throughout this thesis.

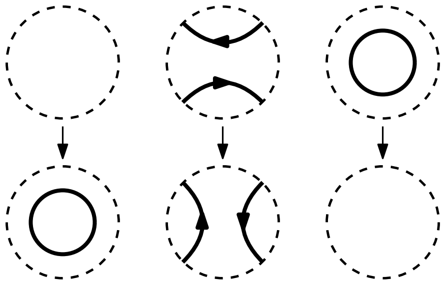


Figure 2.1: Oriented handle additions on classical link diagrams. From left to right: a 0-handle (a *birth of a circle*), a 1-handle addition (an *oriented saddle*), and a 2-handle (a *death of a circle*).

Definition 2.1.1. A *cobordism between classical links L_1 and L_2* is a compact oriented surface S properly embedded into $S^3 \times I$, such that $\partial S = D_1 \sqcup D_2$, where D_i is a diagram of L_i[1]. If a cobordism exists between L_1 and L_2 we say that they are *cobordant.* ◊

To get our hands on a cobordism we present it as a sequence of diagrams.

Definition 2.1.2. A *movie* is a one-parameter family D_t, $t \in [0, 1]$ such that D_t is a classical link diagram except for a finite number of values of t, the set of which is denoted $P = \{p_1, p_2, \dots, p_n \mid p_i < p_{i+1}\}$. The behaviour around the exceptional values is as follows. For $t, t' \notin P$ such that

$$p_i < t < p_{i+1} < t' < p_{i+2},$$

D_t is related to $D_{t'}$ by a classical Reidemeister move or an oriented handle addition, as depicted in Figure 2.1. For

$$p_i < t, t' < p_{i+1}$$

the diagrams D_t and $D_{t'}$ are related by planar isotopy. ◊

We use simple movies as building blocks to produce general cobordisms.

Definition 2.1.3. A movie is *elementary* if the set of exceptional t values has exactly zero or one element. That is, if it contains exactly zero or one classical Reidemeister move or handle addition. The realisations of elementary movies as cobordisms are known as *elementary cobordisms*, and those associated to handle additions are depicted in Figure 2.2

[1] the orientation of the cobordism induces an orientation on each of the links appearing on its boundary

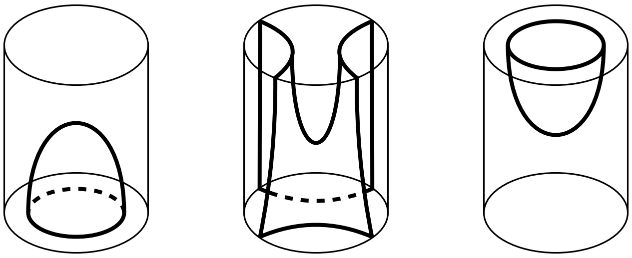


Figure 2.2: The cobordisms induced by oriented handle additions on classical link diagrams (embedded in $S^3 \times I$). From left to right: a 0-handle, a 1-handle, and a 2-handle. Taking the boundary, one arrives back at Figure 2.1.

(those associated to Reidemeister moves are obtained by taking the trace of the move). ◊

It is clear that given two movies M_1 and M_2, we may compose them to produce a third movie $M_2 \circ M_1$, provided the terminal diagram of M_1 is equal to the initial diagram of M_2. It follows that a movie may be written as the composition of a finite number of elementary movies. We use this as a recipe to recover a cobordism from a movie.

Definition 2.1.4. Let M be a movie with initial diagram D_0 and terminal diagram D_1, and $M_n \circ M_{n-1} \circ \cdots \circ M_1$ be a decomposition of it into elementary movies. Properly embed the cobordism associated to M_1 into $S^3 \times I$, and glue the cobordism associated to M_2 to it along their common boundary component (the initial diagram of M_2 and the terminal diagram of M_1). Repeating this process for the remaining M_i's, we obtain a cobordism between the links represented by D_0 and D_1[2]. If a cobordism S can be obtained in this manner from M, we say that *M is a movie presentation of S*. ◊

For our purposes the difference between a cobordism and a movie presentation of it may be ignored. For the rest of this thesis we shall freely interchange between them, using the term cobordism to refer to both the embedded surface and a sequence of diagrams presenting it.

Definition 2.1.1 allows for arbitrary genus surfaces, and under this definition it follows that any two classical links are cobordant; this is equivalent to showing that all links are cobordant to the unknot, which is immediate from the fact that a crossing change can be realised as a genus 1 cobordism. To see this, consider the cobordism with movie

[2] it is therefore clear that a cobordism may be written as a finite composition of elementary cobordisms

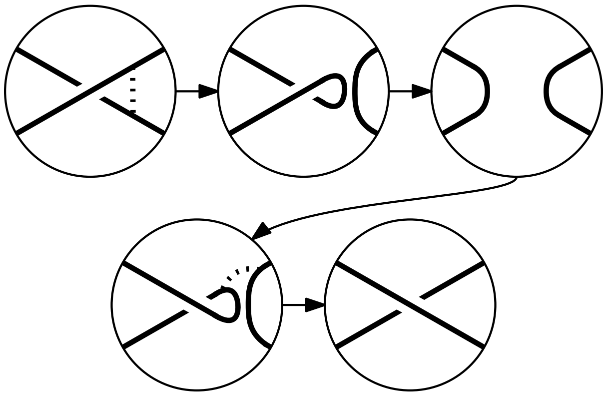


Figure 2.3: Realising a crossing change as a genus 1 cobordism. The order of slides is denoted by the arrows, and a dotted line indicates the location of a 1-handle addition.

presentation given in Figure 2.3: in the order dictated by the arrows it is a 1-handle addition, followed by two Reidemeister 1 moves, and a final 1-handle addition.

The following is a natural restriction of the definition of cobordism.

Definition 2.1.5. Let L_1 and L_2 be classical links such that $|L_1| = |L_2|$, where $|L_i|$ denotes the number of components of the link L_i. A *concordance between* L_1 *and* L_2 is a cobordism S, such that S is a disjoint union of annuli, with each annulus having one boundary component in L_1 and another in L_2. If a concordance exists between L_1 and L_2 we say that they are *concordant*. ◊

A concordance between classical knots is simply a single annulus.

Using the genus of cobordisms to the unknot we give a quantative analysis of the 4-dimensional complexity of knots.

Definition 2.1.6. Let K be a classical knot. Define the *slice genus of* K, denoted $g^*(K)$, to be

$$g^*(K) := \min\left(\{g(S) \mid S \text{ is a cobordism from } K \text{ to the unknot}\right).$$

◊

Of course, we may cap off the unknot in ∂S (without altering the genus) to obtain a surface whose boundary is exactly K. The question "what is the slice genus of K?" is then "what is the least genus of surfaces in B^4 which bound K?". We say that

$$\begin{aligned} K \text{ is } \textit{slice} &\Leftrightarrow g^*(K) = 0 \\ &\Leftrightarrow K \text{ is concordant to the unknot} \\ &\Leftrightarrow K \text{ bounds a disc in } B^4. \end{aligned}$$

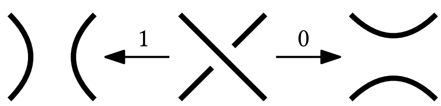
The computation of the slice genus of classical knots is a difficult problem with a long history. The definition of sliceness can traced back to a 1925 paper of Artin [Art25], and appears also in 1957 and 1966 papers of Fox and Milnor [FM57; FM66]; this latter paper also demonstrates that the Alexander polynomial yields a necessary condition for sliceness. Half a century later, there is now a menagerie of other invariants which yield obstructions to and conditions on sliceness, in addition to lower bounds on the slice genus. In Section 2.3 we shall describe the construction of a concordance invariant which is extracted from Khovanov homology, the *Rasmussen invariant*; much of this thesis is concerned with its generalisation.

2.2 Construction

We shall now briefly describe the construction of Khovanov homology for classical links. We follow Bar-Natan [BN05]; for further details, consult [BN02; Kho99].

The construction is formed of two main parts. First, an abstract chain complex of diagrams is produced. This is then converted into algebra using a topological quantum field theory (TQFT). To begin we describe the chain complex of diagrams.

Definition 2.2.1. A *resolution* of a crossing within a classical link diagram is the diagram formed by replacing a local neighbourhood with one of the following configurations:



The two resolutions are known as the 0-resolution and the 1-resolution, as labelled above. A *smoothing* of a classical link diagram is the diagram formed by resolving all of its crossings, yielding a collection of disjoint circles embedded in the plane. ◊

Definition 2.2.2 (Cube of smoothings)**.** Let D be an oriented classical link diagram. Arbitrarily order the crossings of D. Under this ordering, an element of $\{0,1\}^n$ bijectively defines a smoothing of D: the string $e_1 e_2 \cdots e_n \in \{0,1\}^n$, is associated to the smoothing obtained by resolving the i-th crossing into its e_i-resolution.

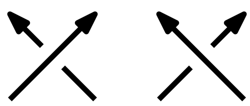
In other words, we may decorate the vertices of the n-dimensional cube $\{0,1\}^n$ with the smoothings of D; having done so, we shall no longer distinguish between vertices and smoothings. The edges of the cube are decorated with cobordisms between smoothings as follows. Denote by $e_1 e_2 \cdots e_{i-1} * e_{i+1} \cdots e_n$ the directed edge which starts at the smoothing obtained by setting $* = 0$ and ends at that obtained by setting $* = 1$. It is clear that the initial and terminal smoothing are related by a 1-handle addition, similar to that depicted in Figure 2.1 (in this case, there are no fixed orientations, however). The cobordism assigned to the edge $e_1 e_2 \cdots e_{i-1} * e_{i+1} \cdots e_n$ is therefore obtained by taking the product cobordism on the initial smoothing, and replacing a neighbourhood of the i-th crossing with the saddle cobordism, as depicted in Figure 2.2. It is easy to see that this cobordism will be a disjoint union of annuli and a pair of pants, so that the number of circles in the initial smoothing and the terminal smoothing differ by exactly 1.

The fully decorated cube produced from D is denoted $[\![D]\!]$, and is known as the *cube of smoothings associated to D.* ◊

Remark. The cube of resolutions is a concrete chain complex in (the category of chain complexes over the additive closure of) the category whose objects are smoothings and morphisms are cobordisms.

An example of a cube of smoothings of a classical link diagram is given in the upper half of Figure 2.4. The cube of smoothings is converted into algebra by assigning modules to the circles of a smoothing, sending disjoint unions to tensor products.

Definition 2.2.3 (Algebraic chain complex)**.** Let D be an oriented classical link diagram. The crossings of D are seperated into two types, *positive* and *negative*, as follows:



$+$ $-$

Let n_+ (n_-) denote the number of positive (negative) crossings, so that $n = n_+ + n_-$ is the total number of crossings. Form the cube of resolutions of D as in Definition 2.2.2. Given a smoothing $e_1e_2\cdots e_n$, define the *height* to be

$$|e_1e_2\cdots e_n| := \left(\sum_{i=1}^{n} e_i\right) - n_- \tag{2.2.1}$$

i.e. it is the number of 1-resolutions appearing in the smoothing minus the number of negative crossings of the diagram.

Let the smoothing $e_1e_2\cdots e_n$ be a disjoint union of m circles, and assign to it the module $\bigotimes^m \mathcal{A}$, where $\mathcal{A} = \mathcal{R}[X]/X^2 = \langle v_-, v_+\rangle$ (under the identification $X = v_-$, $1 = v_+$) and $\mathcal{R}$ is a commutative unital ring[3]. Denote by $CKh_i(D)$ the direct sum of the modules assigned to the smoothings of height i.

The modules $CKh_i(D)$ are the chain spaces of a chain complex, denoted $CKh(D)$; the differential is built as follows. As observed in Definition 2.2.2, the number of circles in a smoothing changes by exactly ± 1 along an edge of the cube, depending on whether two circles merge into one, or one circle splits into two along the edge. In the case of a merge, assign to the edge the map m, and in the case of a split, the map Δ, defined as follows:

$$\begin{aligned} &m(v_+\otimes v_+) = v_+ && \Delta(v_+) = v_+\otimes v_- + v_-\otimes v_+ \\ &m(v_+\otimes v_-) = m(v_-\otimes v_+) = v_- && \Delta(v_-) = v_-\otimes v_- \\ &m(v_-\otimes v_-) = 0 && \end{aligned} \tag{2.2.2}$$

Further, signs are added to the m and Δ maps to ensure that the faces of the cube are anti-commutative. Let $e_1e_2\cdots e_{i-1} * e_{i+1}\cdots e_n$ be an edge; the map assigned to it inherits the sign

$$\mathrm{sgn} = (-1)^{\sum_{k=1}^{i-1} e_k}. \tag{2.2.3}$$

We then define the differential, $d : CKh_i(D) \to CKh_{i+1}(D)$, to be a matrix of the appropriate $\pm m$ and $\pm\Delta$ maps. The chain complex $CKh(D)$ is known as *the Khovanov complex of D*. ◊

The lower half of Figure 2.4 gives an example of the Khovanov complex of a link diagram.

[3] in this thesis it shall be $\mathbb{Q}$ or $\mathbb{Z}$.

Remark. The set $(\mathcal{A}, m, \Delta, \iota, \epsilon)$ - where ι and ϵ are additional maps associated to 0- and 2-handle additions - defines the algebraic structure known as a *Frobenius algebra*. The particular association of modules to smoothings and maps to cobordisms described above satisfies the axioms of a TQFT, owning to the correspondence between Frobenius algebras and TQFTs. While we shall not purse it much further in this thesis, that Khovanov homology is a TQFT is a deep and still mysterious aspect connecting it to many other areas of mathematics and physics, and may offer an explanation of its functoriality.

There is a convenient basis of $CKh(D)$, produced as follows. Given a smoothing of D we may decorate each of its circles with a v_+ or v_-. A *state* is the algebraic element $x = v_\pm \otimes v_\pm \otimes \cdots \otimes v_\pm$, where $v_\pm$ is as dictated by the decoration. Using this basis we define two integer gradings on $CKh(D)$, the *homological grading* i, and the *quantum grading* j.

Given a state x, $i(x)$ is defined to be the height of the decorated smoothing defining x (as given in Equation (2.2.1)). The quantum grading is defined as

$$j(x) := \#(v_+) - \#(v_-) + i(x) + wr(D), \tag{2.2.4}$$

where $wr(D) = n_+ - n_-$ denotes the writhe of D. Shifting the quantum grading by the homological grading and the writhe ensures that the resulting bigraded homology groups are invariant under the classical Reidemeister moves. Also, shifting by the homological grading causes the components of the differential, as given in Equation (2.2.2), to preserve the quantum grading.

Theorem 2.2.4 ([Kho99]). *The chain homotopy type of the graded complex $CKh(D)$ is an invariant of the link, L, represented by D. The homology is therefore also an invariant, known as* the Khovanov homology of *L, and denoted $Kh(L)$.*

The Khovanov homology of the link depicted in Figure 2.4 is given in Figure 2.5.

2.3 The Rasmussen invariant

As demonstrated in the previous section, the definition of Khovanov homology is purely combinatorial. One might suspect, as a consequence, that it does not contain geometric

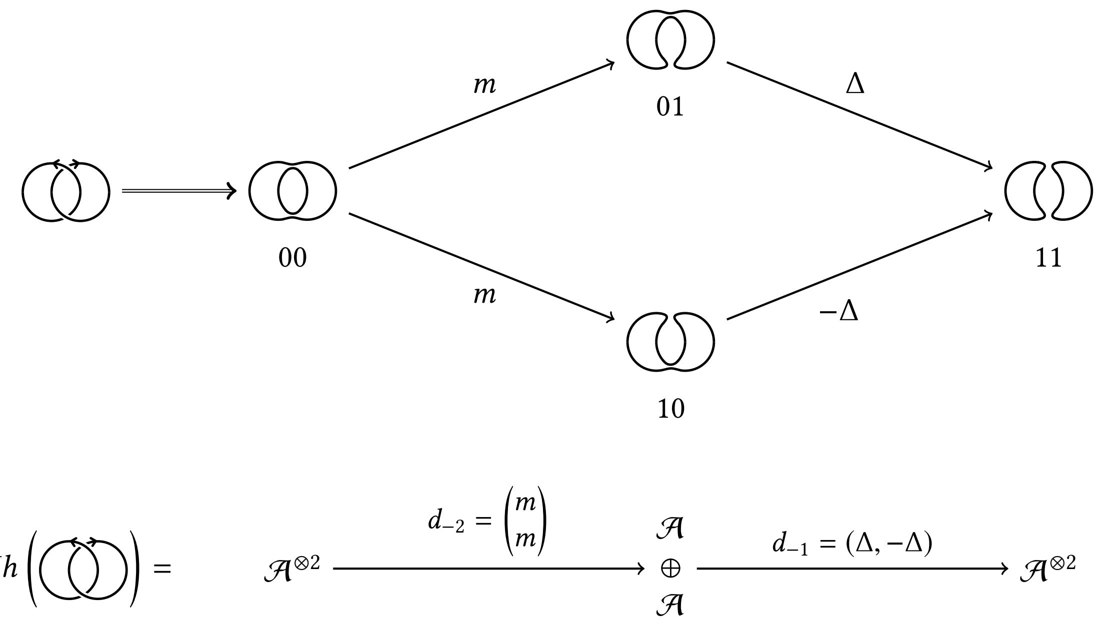


Figure 2.4: The cube of smoothings and the Khovanov complex of the given oriented diagram of the Hopf link.

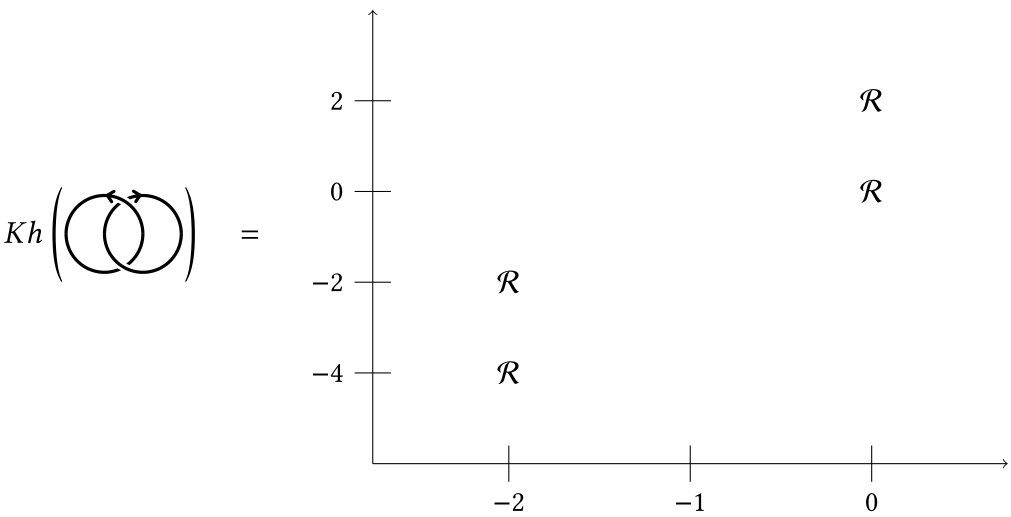


Figure 2.5: The Khovanov homology of the oriented Hopf link of Figure 2.4, split by homological grading (horizontal axis), and quantum grading (vertical axis).

or topological information regarding the argument link. Rasmussen demonstrated that Khovanov homology does, in fact, yield such information, using it to obtain a concordance invariant which produces a bound on the slice genus of a classical knot [Ras10]. In this section we outline the methods he used to do so, which inspire a part of this thesis.

2.3.1 Lee's perturbation

We start by describing a perturbation of Khovanov homology due to Lee [Lee05]. It is a perturbation in the sense that the new theory has the same chain spaces as Khovanov homology but an altered differential, formed by adding terms to the maps given in Equation (2.2.2); as such, the resulting Lee homology can be expressed as the E_∞ page of a spectral sequence whose E_2 is page Khovanov homology. The perturbation is drastic: in the case of classical knots it produces a theory which is almost trivial[4]. Remarkably, the remaining information is enough to produce a powerful concordance invariant.

For further details consult [BNM06].

Definition 2.3.1 (Lee homology)**.** Let D be a diagram of an oriented classical link L. Set $\mathcal{A} = \mathbb{Q}[X]/X^2 - 1 = \langle v_-, v_+ \rangle$ and denote by $CKh'(D)$ the chain complex whose chain spaces are produced identically to those of $CKh(D)$, but with altered differential. The components of the new differential are defined as follows:

$$\begin{aligned} m'(v_+ \otimes v_+) &= v_+ & \Delta'(v_+) &= v_+ \otimes v_- + v_- \otimes v_+ \\ m'(v_+ \otimes v_-) &= m'(v_- \otimes v_+) = v_- & \Delta'(v_-) &= v_- \otimes v_- + v_+ \otimes v_+ \\ m'(v_- \otimes v_-) &= v_+. \end{aligned} \tag{2.3.1}$$

Notice that the m' and Δ' maps split into a part which preserves the quantum grading (the part shared with m and Δ), and a part which raises it by 4. Thus the complex $CKh'(D)$ is no longer graded, but *filtered*. The chain homotopy type of $CKh'(D)$ is an invariant of the link represented by D, and we define $Kh'(L)$ to be its homology, referred to as the *Lee homology of* L. ◊

Unlike that of Khovanov homology, the rank of Lee homology depends only on the number of components of the argument link, and its homological support can be determined

[4] in fact, if one ignores the quantum grading the theory is trivial

easily. This is due to the behaviour of Lee homology with respect to the following type of smoothing.

Definition 2.3.2. A smoothing of a classical link diagram is *alternately coloured* if its circles are coloured exactly one of two colours in such a way that in a neighbourhood of each crossing the two incident arcs are different colours. A smoothing which can be coloured in such a way is known as *alternately colourable.* $\lozenge$

The smoothings labelled 00 and 11 in Figure 2.4 are alternately colourable, while those labelled 01 and 10 are not.

Theorem 2.3.3 ([Lee05; BNM06])**.** *Let D be a diagram of an oriented classical link L. Then*

$$\begin{aligned} \textit{rank}\,(Kh'(L)) &= \#(\textit{alternately coloured smoothings of } D) \\ &= \#(\textit{orientations of } L) \\ &= 2^{|L|}. \end{aligned} \tag{2.3.2}$$

Further, $Kh'(L)$ is supported in exactly those homological degrees which are equal to the height of an alternately colourable smoothing of D.

2.3.2 The invariant

It is clear from Theorem 2.3.3 that the rank of the Lee homology of a classical knot is equal to 2, as a classical knot diagram has only 2 alternately coloured smoothings. It is easy to see that these smoothings are, in fact, the two possible colourings of the *oriented smoothing*: the smoothing obtained by resolving every crossing in agreement with the orientation of the diagram (the 0-(1-)resolution for a positive (negative) crossing)[5]. By construction, this smoothing is always at height 0, so that the Lee homology of a classical knot is supported in homological degree 0.

As explored later, virtual links exhibit behaviour different to that of classical links with respect to their alternately coloured smoothings, and the consequences of this for doubled Khovanov homology are the focus of Section 4.4.

[5] this is verified by considering chequerboard colourings of the plane [BNM06].

Remaining with the classical case, Theorem 2.3.3 makes clear that all of the non-trivial information contained in the Lee homology of a knot is encoded in the quantum grading, and that the rank of the homology is 2. Rasmussen showed that, in fact, the quantum grading of one generator determines that of the other, and used this observation to define a concordance invariant.

Theorem 2.3.4 ([Ras10]). *Let K be a classical knot. The quantum grading information contained in $Kh'(K)$ is equivalent to an even integer. This integer is known as the* Rasmussen invariant *of K, and is denoted $s(K)$. The Rasmussen invariant is a homomorphism from the smooth knot concordance group to the even integers. That is, it is invariant under concordance, additive with respect to connect sum, and $s(K) = -s(\overline{K})$ (where $\overline{K}$ denotes the mirror image of K).*

Further, the Rasmussen invariant yields a lower bound on the slice genus. Specifically,

$$\frac{|s(K)|}{2} \leq g^*(K). \tag{2.3.3}$$

We shall conclude this chapter with a description of the techniques used to obtain Equation (2.3.3), as they inform much of the work of Chapter 4.

The main ingredient is the fact that Lee homology, like Khovanov homology, is functorial with respect to cobordism. Concretely, given classical links L_1 and L_2, a cobordism between them, S, defines a filtered map $\phi_S : Kh'(L_1) \longrightarrow Kh'(L_2)$. This map is built up by decomposing S into elementary pieces, as described in Section 2.1.

It is crucial to verify that ϕ_S is non-zero. This is done by demonstrating that it behaves well with respect to the following basis of $Kh'(L_1)$. Let $\partial S = D_1 \sqcup D_2$, where D_i is a diagram of L_i; given an alternately coloured smoothing of D_1 with circles coloured either red or green, an *alternately coloured generator* of $Kh'(L_1)$ is the state formed by decorating the red circles of the smoothing with $r = \frac{1}{2}(v_+ + v_-)$, and the green circles with $g = \frac{1}{2}(v_+ - v_-)$.

By an induction on the elementary cobordisms making up S, Rasmussen showed that an alternately coloured generator of $Kh'(L_1)$ is sent to a linear combination of those of $Kh'(L_2)$, so that ϕ_S is non-zero. In addition, the homological degree of ϕ_S is 0 by construction, and it can be determined that it is of quantum (filtration) degree $-2g(S)$ (for $g(S)$ the genus of S).

If S is a cobordism between classical knots K_1 and K_2, the above results are enough to determine the relationship between $s(K_1)$ and $s(K_2)$; specifically, that $s(K_2) \geq s(K_1) - 2g(S)$. Combining this with the behaviour of the Rasmussen invariant with respect to mirror image, one arrives at Equation (2.3.3). We shall follow this blueprint later on in order to produce obstructions to the sliceness of virtual knots (among other things), but must take into account a number of new phenomena one encounters in virtual knot theory and virtual concordance, many of which are the subject of the next chapter.

Chapter 3

Virtual knot theory

This chapter contains an introduction to virtual knot theory, and the associated theory of cobordism and concordance. To help to put the work of this thesis into context (and because we need it later), we also describe an extension of Khovanov homology to virtual links due to Manturov [Man07]. We focus on the reformulation of his theory due to Dye, Kaestner, and Kauffman [DKK17].

3.1 Virtual knot theory

First we outline the combinatorial description of virtual links, following Kauffman [Kau99], before presenting the complementary topological viewpoint (this material is informed by Turaev [Tur07] and Carter-Kamada-Saito [CKS02]).

3.1.1 Virtual link diagrams

The study of virtual knot theory was initiated by Kauffman in the late 1990's [Kau99], his original motivation being combinatorial. A Gauss diagram is an oriented signed chord diagram; that is, it is a circle together with a set of chords whose endpoints lie on the circle, each chord possessing both an orientation and a sign. Every classical knot defines a Gauss diagram (up to appropriate equivalence), but not every Gauss diagram defines a classical knot (an example is given on the left of Figure 3.1); it is therefore natural to ask if there exists a generalised knot theory which corresponds to the set of all Gauss codes

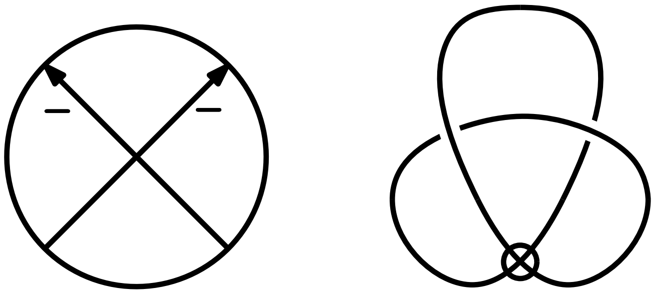


Figure 3.1: On the left, a Gauss code which does not correspond to a classical knot (this can be shown by verifying that it is a non-trivial code, and recalling that there exist no two-crossing classical knots). On the right, a diagram of virtual knot 2.1 realising the code on the left.

(up to appropriate equivalence). Virtual knot theory provides an affirmative answer to this question.

As in the classical case, we get at virtual links via their diagrams.

Definition 3.1.1. A *virtual link diagram* is a 4-valent planar graph, the vertices of which are decorated with either the classical overcrossing and undercrossing decorations, or a new decoration, $\otimes$, known as a *virtual crossing*. ◊

An example of a virtual link diagram is given on the right of Figure 3.1. A virtual link diagram represents a *virtual link* in essentially the same manner as a classical diagram does a classical link.

Definition 3.1.2. A *virtual link* is an equivalence class of virtual link diagrams, up to the *virtual Reidemeister moves*. These moves consists of those of classical knot theory, together with four new moves involving virtual crossings, depicted in Figure 3.2. ◊

Goussarov, Polyak, and Viro demonstrated that virtual knots (one component virtual links) are in bijection with the set of all Gauss codes (up to appropriate equivalence) [GPV98]. Further, they verified that classical knot theory is a well-defined subset of virtual knot theory.

Theorem 3.1.3 ([GPV98]). *Two classical link diagrams are related by the virtual Reidemeister moves if and only if they are related by the classical Reidemeister moves. That is, the inclusion of classical knot theory into virtual knot theory is injective.*

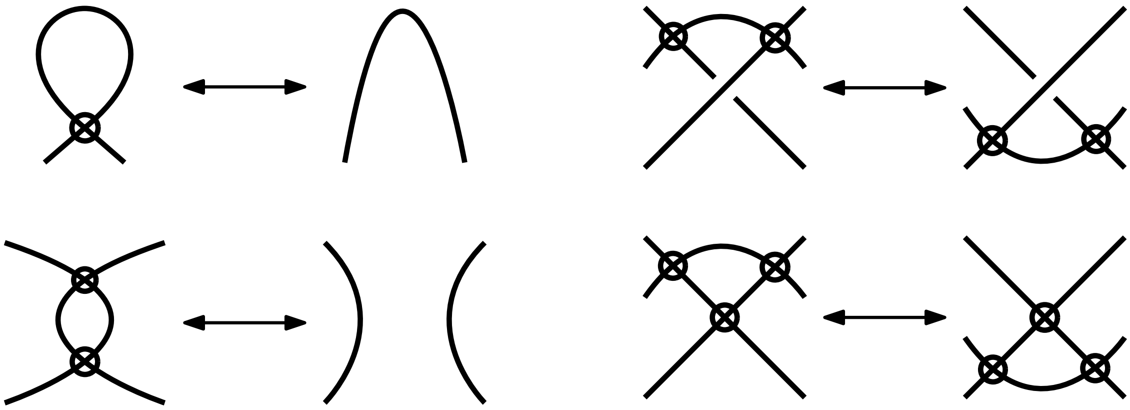


Figure 3.2: The non-classical moves making up the virtual Reidemeister moves. Anticlockwise from the top left: virtual Reidemeister 1, virtual Reidemeister 2, virtual Reidemeister 3, and the mixed move.

3.1.2 A topological viewpoint

Much of the early work in virtual knot theory utilises the combinatorial viewpoint described above, and diagrams remain the best way of working with virtual links. There are a number of other ways in which to interpret virtual links, however. Predominant in this thesis is an interpretation which places virtual links firmly in the topological world. This interpretation is also originally due to Kauffman, and was legitimised by Kuperberg [Kup02]. It recasts virtual crossings as artifacts of the knotting of a virtual link about the topology of another manifold; specifically, it describes virtual links as embeddings into non-simply connected 3-manifolds.

Definition 3.1.4. A virtual link is an equivalence class of embeddings $\bigsqcup S^1 \hookrightarrow \Sigma_g \times I$, up to self-diffeomorphism of Σ_g, and handle stabilisations of Σ_g such that the annulus formed by the product of the attaching sphere with I is disjoint from the image of the embedding.

A *representative D of a virtual link* is a particular embedding $D : \bigsqcup S^1 \hookrightarrow \Sigma_g \times I$; we abbreviate notation to write $D \hookrightarrow \Sigma_g \times I$. $\Diamond$

Given a representative $D \hookrightarrow \Sigma_g \times I$ of a virtual link L, we make contact with the interpretation described in Section 3.1.1 as follows. Under a generic projection to Σ_g, D is sent to a 4-valent graph on Σ_g; we keep track of the overcrossing and undercrossing information with the classical crossing decorations on vertices. A second generic projection, this time to $\mathbb{R}^2$, again produces double points, but of a different nature: they are

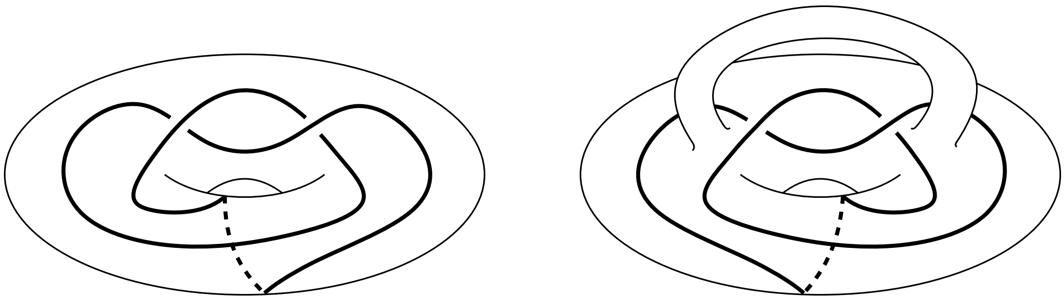


Figure 3.3: Two representatives of the virtual knot 2.1.

a consequence of the (possible) failure of the 4-valent graph to be planar, not of how the link is knotted about itself. As such, we distinguish these points by decorating them with the virtual crossing. The result is a virtual link diagram representing L.

It is clear that this process can be reversed, by lifting a virtual link diagram first to a graph on a surface (one piece of genus for each virtual crossing), and then to an embedding into a thickened surface. Examples of the relationship between representatives in thickened surfaces and diagrams can be seen by comparing Figures 3.1 and 3.3.

The consequences of Definition 3.1.4 may not be apparent at first glance, particularly those due to handle stabilisation; for instance, the two embeddings depicted in Figure 3.3 are representatives of the same virtual knot. The incorporation of handle stabilisation means that the objects of study in virtual knot theory are equivalence classes of embeddings into *equivalence classes of* 3*-manifolds* (this separates it from more traditional theories of links in fixed 3-manifolds other than S^3). As a result, we may ask questions about the (set of) 3-manifolds appearing as targets for representatives of a given virtual link. An example is as follows.

Definition 3.1.5. Let L be a virtual link. The *minimal supporting genus* of L, denoted $m(L)$, is the minimal genus of all surfaces Σ_g such that L has a representative in $\Sigma_g \times I$. ◊

Kuperberg showed that the minimal supporting genus is well-defined, and identified a useful property of genus-minimal representatives of virtual links.

Theorem 3.1.6 ([Kup02])**.** *Let L be a virtual link. There exists a g such that L has a representative in Σ_g but not $\Sigma_{g'}$ for all $g' < g$. Further, if $D, D' \hookrightarrow \Sigma_g \times I$ are two genus-minimal representatives of L, then they are related by self-diffeomorphism of Σ_g (no handle stabilisations are required).*

The minimal supporting genus is an example of a quantity which can only be defined in light of the topological interpretation of virtual links. Despite this, diagrams contain information regarding it. For example, recalling the process of lifting a diagram to a representative in a thickened surface described above: one notices that the number of virtual crossings in a diagram of a virtual link is an upper bound on the minimal supporting genus.

This is a simple example of the interplay between the two interpretations: the topological viewpoint informs the direction of study and motivates new questions, while the combinatorial interpretation allows one to work hands-on to answer them.

3.1.3 New phenomena

Before progressing to the theory of virtual cobordism, it is important to point out a number of new phenomena one encounters when transitioning from classical to virtual knot theory; they may be counter-intuitive to the reader comfortable with classical knot theory. It is reasonable to suspect that the following new phenomena, and many others, are ultimately inherited from the non-triviality of $\pi_1(\Sigma_g)$.

Infinite unknotting number

As in the classical case we define the *classical unknotting number* of a virtual knot as the minimum number of crossing changes needed to convert a diagram of the knot to a trivial diagram (a crossing change being the interchange of the underpass and overpass at a classical crossing). Unlike the classical case, however, there exist virtual knots with infinite classical unknotting number. The virtual knot known as Kishino's knot, depicted in Figure 3.4, is an example [KS04]. It is clear that if a virtual knot has infinite classical unknotting number, then $[D] \neq 1 \in \pi_1(\Sigma_g \times I)$ for all representatives $D \hookrightarrow \Sigma_g \times I$[1].

Connect sum

For concreteness we begin with the definition of the connect sum of virtual knot diagrams.

[1]the converse is not true, however, owing to the ability to (de)stabilise.

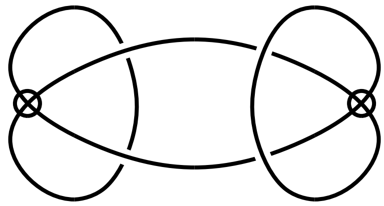


Figure 3.4: Kishino's knot.

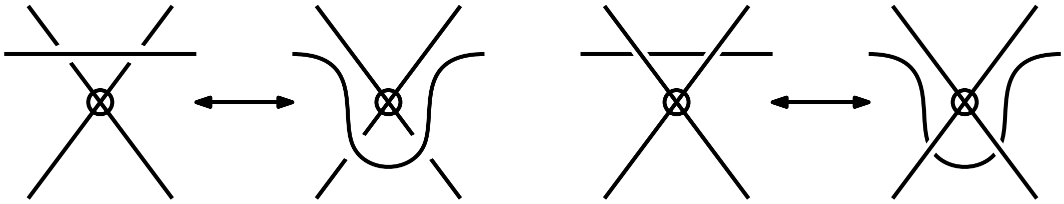


Figure 3.5: The forbidden moves.

Definition 3.1.7. Let D_1 and D_2 be oriented virtual knot diagrams. If $D_1 \sqcup D_2 \looparrowright \mathbb{R}^2$ is such that there exists a disc $B \hookrightarrow \mathbb{R}^2$ with $B \cap D_1 = I$ and $B \cap D_2 = \bar{I}$ (where $\bar{I}$ denotes an interval with reverse orientation) then we denote by $D_1 \#_B D_2$ the diagram produced by 1-handle addition with attaching sphere $I \sqcup \bar{I}$. ◊

The connect sum of classical knots is well-defined with respect to both the diagrams used and the site at which the connect sum is conducted. This fails completely in the virtual case: the virtual knot represented by the diagram $D_1 \#_B D_2$ depends on the choices of D_1 and D_2, and on the disc B.

We can understand the ill-defined nature of virtual connect sum in both the combinatorial and topological interpretations outlined in this chapter. Diagrammatically, no longer can one area of a diagram be freely moved over all others, due to presence of the forbidden moves. These are moves on diagrams, depicted in Figure 3.5, which do not follow from the virtual Reiedemeister moves (in fact, they can be used to unknot any virtual knot [Nel01]). Classically, Reidemeister moves commute, in a certain sense, with handle addition: for example, let D_1 and D_2 be classical unknot diagrams. Then D_2 can be treated as a small neighbourhood of $D_1 \# D_2$ and slid under (or over) the rest of the diagram. Thus the sequence of Reidemeister moves which takes D_1 to the crossingless unknot diagram can be replicated on $D_1 \# D_2$, taking it to D_2, which is itself an unknot diagram. Nontrivial diagrams are treated similarly. Virtually, however, this cannot be replicated, as areas of a diagram cannot always be moved across others.

We are able to obtain a deeper explanation using the topological viewpoint; specifically, it can be viewed as a consequence of the higher-dimensional topological information contained in a virtual knot. From the topological viewpoint, we see that the connect sum operation is not only a 2-dimensional 1-handle addition between the copies of S^1, but that it also induces a 3-dimensional 1-handle addition on the thickened surfaces involved[2]. This contrasts with the classical case in which both copies of S^1 can be contained in a single S^3 and only a 2-dimensional 1-handle need be added. Different choices of the disc B (as in Definition 3.1.7) correspond to different choices of 3-dimensional handles.

A novel manifestation of this ill-definedness is that there exist non-trivial virtual knots which are connect sums of a pair of trivial virtual knots; again Figure 3.4 provides an example. (In Section 4.6 a condition met by such virtual knots is derived using doubled Khovanov homology.) The decomposition of a nontrivial object into the sum of two trivial objects is a bizarre phenomenon wherever it is found in mathematics; much of the work of this thesis relates to concordance, however, where this phenomenon evaporates [3].

Alternately coloured smoothing behaviour

As mentioned in Section 2.3.2 alternately coloured smoothings of virtual link diagrams behave differently to those of classical diagrams. In particular, a smoothing of an oriented classical diagram is an oriented smoothing (formed by resolving all of its crossings in agreement with the orientation) if and only if it is an alternately colourable smoothing.

In the case of virtual diagrams, we form smoothings by resolving classical crossings and leaving virtual crossings untouched. The behaviour described above for classical diagrams is not replicated in the virtual case: arbitrarily orienting the diagram in either Figure 3.1 or Figure 3.4, one sees that the *unoriented smoothing* (formed by swapping all the resolutions of the oriented smoothing) is, in fact, alternately colourable (so that the oriented smoothing is not). There exist virtual knot diagrams for which the alternately colourable smoothing is neither the oriented nor the unoriented smoothing (take the

[2] it is possible for the connect sum operation not to induce a 3-dimensional handle addition but a slightly more complicated operation. We refer the reader to [MI13, page 41, Fig. 2.7].

[3] a virtual knot which is the connect sum of two unknots is clearly slice.

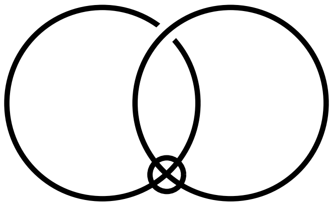


Figure 3.6: The so-called virtual Hopf link, which has no alternately coloured smoothings.

connect sum of a classical diagram with either of those in Figure 3.1 or Figure 3.4, for example). This phenomenon also occurs in virtual link diagrams.

In a further departure from the classical case, there exist virtual links which possess no alternately coloured smoothings, so that the right hand side of Equation (2.3.2) fails for generic virtual links. That is, there exists a diagram D of a virtual link L such that

$$\#(\text{alternately coloured smoothings of } D) \neq \#(\text{orientations of } L)\ (= 2^{|L|}).$$

An example of such a diagram is given in Figure 3.6. In Section 4.4 it is determined that a virtual link has either 0 or $2^{|L|}$ alternately coloured smoothings, and that which case holds can be determined easily from a diagram of L.

In Chapter 4 the above phenomena are characterised completely. The number of alternately coloured smoothings of a generic virtual link is determined, which boils down to a simple check on Gauss diagrams. In addition, a necessary and sufficient condition for a smoothing to be resolved into its unoriented resolution in the alternately colourable smoothing of a virtual link diagram is determined. This condition is given in terms of *crossing parity*, a notion due to Manturov [Man10b].

3.2 Virtual knot concordance

This section contains defintions analogous to those of Section 2.1, translated to the virtual setting.

Definition 3.2.1. Let $L \hookrightarrow \Sigma_g \times I$ and $L' \hookrightarrow \Sigma_{g'} \times I$ be virtual links. We say that L and L' are *cobordant* if there exists a compact oriented surface S and an oriented 3-manifold M, such that $\partial S = L \sqcup L'$, $\partial M = \Sigma_g \sqcup \Sigma_{g'}$, and $S \hookrightarrow M \times I$. We refer to S as a *cobordism between L and L'*. ◊

Definition 3.2.2. Let $|L|$ denote the number of components of a virtual link L. We say that say that L and L' are *concordant* there exists a cobordism S between them, which is a disjoint union of $|L|$ annuli, such that each annulus has a boundary component in L and another in L'. We refer to such an S as a *concordance between L and L'*. $\lozenge$

One notices that a cobordism between virtual links is a pair consisting of a surface and a 3-manifold; as such, we shall often denote a cobordism as a pair (S, M) (where S and M are as in Definition 3.2.1). The 3-manifold M may be given a Morse decomposition and described in terms of level surfaces and critical points. Let $f : M \to I$ be a Morse function: starting from Σ_g, level surfaces of f are Σ_g until we pass a critical point, after which they are $\Sigma_{g\pm1}$. Critical points correspond to handle stabilisations. A finite number of handle stabilisations are made to reach $\Sigma_{g'}$. In other words, M may be any compact connected oriented 3-manifold with boundary $\Sigma_g \sqcup \Sigma_{g'}$.

For completeness we include the definition of the movie description of a virtual cobordism, in direct analogy to Definition 2.1.2.

Definition 3.2.3. A *virtual movie* is a one-parameter family D_t, $t \in [0, 1]$ such that D_t is a virtual link diagram except for a finite number of values of t, the set of which is denoted $P = \{p_1, p_2, \ldots, p_n \mid p_i < p_{i+1}\}$. The behaviour around the exceptional values is as follows. For $t, t' \notin P$ such that

$$p_i < t < p_{i+1} < t' < p_{i+2},$$

D_t is related to $D_{t'}$ by a virtual Reidemeister move or an oriented handle addition, as depicted in Figure 2.1. For

$$p_i < t, t' < p_{i+1}$$

the diagrams D_t and $D_{t'}$ are related by planar isotopy. $\lozenge$

In a classical movie exceptional values correspond to Reidemeister moves and handle additions to the cobordism surface. In a virtual movie, we have the additional possibility that exceptional values may correspond to 2-dimensional handle additions to the level surfaces making up M (as described above); such handle additions are represented in the movie by the purely virtual Reidemeister moves (given in Figure 3.2). This correspond-

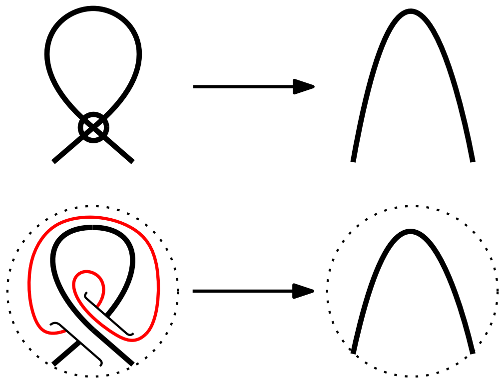


Figure 3.7: Realising virtual Reidemeister move 1 as a handle addition. The top row depicts the move on virtual link diagrams, while the bottom depicts the associated handle addition: a neighbourhood of the surface is depicted, with the link depicted in black and the attaching sphere in red. Attaching a handle and pushing the link over it, one obtains the situation in the bottom right of the figure.

ence in the case of virtual Reidemeister move 1 is demonstrated in Figure 3.7 (the other moves are realised in a similiar fashion).

Given a movie description of a cobordism (S, M) between virtual links, we say that a virtual link L *appears in* (S, M) if a diagram of it is a member of the family of diagrams D_t. It is instructive to consider the following topological (movie-free) version of this definition.

Definition 3.2.4. Let (S, M) be a cobordism. Fix a Morse function $f : M \longrightarrow I$ such that the restriction of f to S is a Morse function also. We say that a virtual link $J \hookrightarrow \Sigma_l \times I$ *appears in* S if $S \cap (f^{-1}(t) \times I) = J$, for some $t \in I$ with $f^{-1}(t) = \Sigma_l$. The situation is depicted in Figure 3.8. ◊

Definition 3.2.5. Let K be a virtual knot. Define the *slice genus of* K, denoted $g^*(K)$, to be

$$g^*(K) := \min\left(\{g(S) \mid S \text{ a cobordism between } K \text{ and the unknot}\}\right).$$

If $g^*(K) = 0$ (so that K is concordant to the unknot) we say that K is *slice*. ◊

Given a cobordism from a virtual knot to the unknot we can simply cap the unknot with a disc to yield a surface whose boundary is exactly the knot, as in the classical case. Therefore, for a virtual knot $K \hookrightarrow \Sigma_g \times I$, the question "what is the slice genus of K?" reads: "what is the least genus of oriented surfaces $S \hookrightarrow M \times I$ with $\partial S = K$, where M is

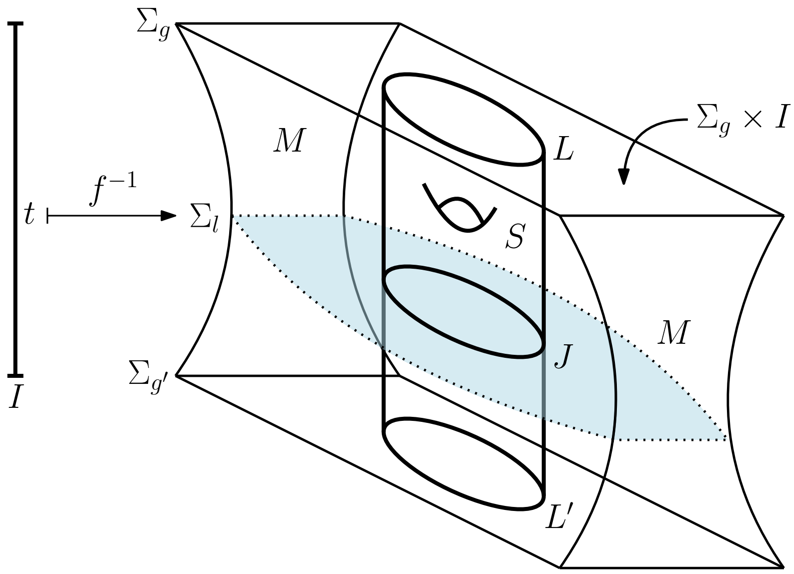


Figure 3.8: A cobordism between virtual links L and L'. The manifold $M \times I$ is depicted dimensionally reduced, and the blue plane depicts the submanifold $\Sigma_l \times I \subset M \times I$. The virtual link J is the intersection of this submanifold with the surface S.

an oriented 3-manifold with $\partial M = \Sigma_g$?". Chapter 5 is concerned with the computation of estimation of the slice genus of virtual knots.

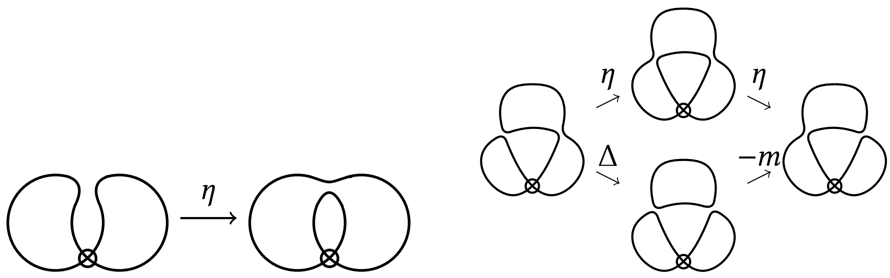
Repeating the theme outlined in Section 3.1.2, we may ask new questions of the 3-manifolds appearing in cobordisms between virtual links, rather than of the surfaces: Section 6.3 is concerned with a question of exactly this nature.

3.3 MDKK Homology and the virtual Rasmussen invariant

To conclude this chapter we describe the construction of an extension of Khovanov homology to virtual links due to Manturov [Man07], as reforumlated by Dye, Kaestner, and Kauffman [DKK17][4]. As such, we refer to it as *MDKK homology*. We also cover the *virtual Rasmussen invariant* due to Dye, Kaestner, and Kauffman, which extends the concordance invariant outlined in Section 2.3.

We begin with a discussion of the problems encountered when attempting to extend

[4]Tubbenhauer has also developed a virtual Khovanov homology using non-orientable cobordisms [Tub14], but there are compatibility issues with MDKK homologyy



(A) The single-cycle smoothing. (B) The problem face.

Figure 3.9

Khovanov homology to virtual links.

3.3.1 Extending Khovanov homology

Any extension of Khovanov homology to virtual links must deal with the fundamental problem presented by the *single-cycle smoothing*, also known as the *one-to-one bifurcation*. This is depicted in Figure 3.9(A): altering the resolution of a classical crossing no longer necessarily splits one circle or merges two circles, but can in fact take one circle to one circle. The realisation of this as a cobordism between smoothings is a once-punctured Möbius band. How does one associate an algebraic map, η, to this? Looking at the quantum grading we notice that

$$\begin{array}{ccc} 0 & & v_+ \\ v_+ & \xrightarrow{\eta} & 0 \\ 0 & & v_- \\ v_- & & 0 \end{array}$$

from which we observe that the map $\eta : \mathcal{A} \to \mathcal{A}$ must be the zero map if it is to be grading-preserving (we have arranged the generators vertically by quantum grading). This is the approach taken by Manturov and subsequently Dye et al.

However, setting η to be the zero map causes collateral damage to the rest of the chain complex. Consider the cube of smoothings depicted in Figure 3.9(B) (it is the cube associated to the diagram Figure 3.11(A)): along the upper two edges we have $\eta \circ \eta = 0$, but along the lower two edges we have $-m \circ \Delta$. The latter is non-zero (as $m \circ \Delta = 2v_-$),

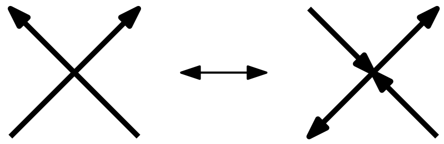


Figure 3.10: The source-sink decoration.

so that the maps making up this face fail to anticommute. In turn, attempting to follow Definition 2.2.3 fails to produce a chain complex (except in the case $\mathcal{R} = \mathbb{Z}_2$).

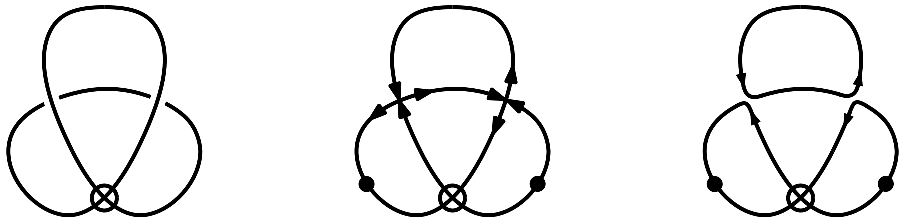
Therefore, if we wish to overcome the single-cycle smoothing by setting $\eta = 0$, we must fix the problem face by recovering anticommutativity. In the case of MDKK homology, this is done using two pieces of new diagrammatic technology (as described in Section 3.3.2). A central part of this thesis is the construction of doubled Khovanov homology, which makes the necessary modifications in the realm of alegebra, rather than diagrammatics; it is detailed in Chapter 4. To conclude this chapter we shall describe a method of overcoming the problems described above, originally due to Manturov.

3.3.2 MDKK homology

We follow Dye, Kaestner, and Kauffman in their reformulation of the theory due to Manturov [DKK17]. Their strategy for overcoming the obstacles outlined in the previous section is as follows. New diagrammatic technology is added to the cube of smoothings of a virtual link diagram, in order to detect the problem faces within it. At a problem face, the composition $m \circ \Delta$ is set to zero (of course, this could not be done globally without destroying the link invariance of the theory). The way in which the composition is set to zero is intricate, and to speed up the verification that it really does recover anticommutativity a second piece of diagrammatic technology is added[5]. The result is a theory which is a virtual link invariant in arbitrary coefficients, and which recovers the Khovanov homology of classical links. However, it is much more labour intensive to compute than classical Khovanov homology, owing to the heavily decorated cube of smoothings and involved construction of the differential.

Let $\mathscr{A}$ and $\mathcal{R}$ be as defined in Chapter 2. The first piece of technology allows for the

[5]this extra piece of technology does not alter the isomorphism class of the resulting homology, as it is simply fixing the way signs are added to the differentials



(A) A virtual knot diagram.

(B) The source-sink diagram of the virtual knot diagram on the left, marked with cut loci.

(C) A smoothing of the virtual knot diagram on the far left, marked with cut loci and source-sink orientations.

Figure 3.11

exploitation of a symmetry present in $\mathcal{A}$ (which corresponds to the two possible orientations of S^1), using the following automorphism.

Definition 3.3.1. The *barring operator* is the map

$$\overline{} : \mathcal{A} \to \mathcal{A}, \; X \mapsto -X.$$

Applying the barring operator is referred to as *conjugation*. ◊

How the barring operator is applied within the Khovanov complex is determined using an extra decoration on link diagrams, the *source-sink decoration*.

Definition 3.3.2. Let D be an oriented virtual link diagram. Denote by $S(D)$ the diagram formed by replacing the classical crossings of D with the source-sink decoration, depicted in Figure 3.10. It is clear that the source-sink decoration at each classical crossing induces an orientation of the arcs of $S(D)$. An arc on which the orientations induced by distinct crossings disagree is marked with a *cut locus*. We refer to $S(D)$ as the *source-sink diagram of D*. ◊

An example of a source-sink diagram is given in Figure 3.11(B). For the remainder of this section we shall assume that all smoothings of D are also marked with cut loci, in positions as dictated by those of $S(D)$ (we can do this as cut loci are placed away from classical crossings), as exemplified in Figure 3.11(C).

Further, given a smoothing of a virtual link diagram, the source-sink decoration at each classical crossing induces an orientation on the (at most two) circles of the smoothing which are incident to it; this is depicted in Figure 3.12. This orientation is known as the

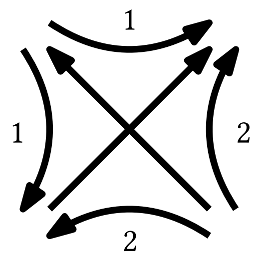


Figure 3.12: The recipe for the source-sink orientation (the curved directed arrows), and the local order (the labels 1 and 2) at a classical crossing.

source-sink orientation. Note that the orientations induced by distinct classical crossings may disagree; this is captured by the cut loci, and it is this disagreement which allows for the detection of the problem face. An example of the source-sink orientation of a smoothing is given in Figure 3.11(C).

The second piece of new technology is the *order* construction. It is used to add signs to the edges of the cube of smoothings, and to greatly speed up the verification that MDKK homology is an invariant of virtual links.

Definition 3.3.3. Given a smoothing of a virtual link diagram, the *global order* is an arbitrary labelling $1, \ldots, r$ of the r circles making up the smoothing. The *local order* is the labelling of the circles produced by the recipe given in Figure 3.12. ◊

With both the source-sink decorations and orders in place we are able to define MDKK homology.

Definition 3.3.4 (MDKK homology [Man07; DKK17])**.** Let D be a diagram of an oriented virtual link L. Form the source-sink diagram $S(D)$, and decorate the smoothings of D with the cut loci, source-sink orientations, and local and global orders. In addition to these decorations, arbitrarily mark a point on each circle of the smoothing away from crossing neighbourhoods and cut loci (this is depicted by a cross). Denote by $\widetilde{[\![D]\!]}$ the cube of resolutions of D, formed exactly as in Definition 2.2.2, but using the fully decorated smoothings; an example is given in Figure 3.13.

Form the chain groups $vCKh_i(D)$ as in the classical case, by assigning to a smoothing of r circles the module $\bigotimes^r \mathcal{A}$, and taking the direct sum of the modules assigned to the smoothings of height i. The differentials are matrices of maps, the entries of which are determined by the procedure given in Table 3.1. To read the table, recall that the entries of the matrix correspond to edges of the cube of smoothings, and that along an edge one classical crossing changes from its 0-resolution to its 1-resolution; we refer to the area of the smoothing surrounding this crossing as the *crossing neighbourhood.* The map assigned to an edge is built by composing the results of Stages (1) through to (5), so that the final result is given by

$$d(*) = \text{sign}\,(\rho_5)\overline{f\left(\overline{\text{sign}\,(\rho_1)*}^{\,i}\right)}^{\,j}$$

where ρ_1, ρ_5 are the permutations of Stages (1) and (5), respectively, f is either m, Δ, or η, as specified in Stage (3), and $\overline{}^{\,i}$ denotes the barring operator applied to the i-th tensor factor.

This definition yields a chain complex, denoted $vCKh(D)$, the chain homotopy equivalence class of which is an invariant of L. The homology of $vCKh(D)$ is an invariant of L, denoted $vKh(L)$ and refered to as the *MDKK homology of L*. It is bigraded, with homological and quantum gradings defined identically to those of classical Khovanov homology. ◊

Remark. The marked points on each cycle must be placed in order to break the symmetry of S^1 and hence $\mathcal{A}$: their placing corresponds to fixing the orientation of the circle to be that of the arc on which the marked point sits. Upon passing a cut locus one moves into a segment of the circle with reverse orientation. This is replicated in $\mathcal{A}$ by applying the barring operator. The marked points can be placed arbitrarily without changing the anticommutativity of a face: moving the point along an arc without passing a cut locus results in no change in any differentials, while moving the point past a cut locus results in a change to the number of times the barring operator is applied on both the incoming and outgoing differentials. These changes cancel pairwise so that anticommutativity is left unchanged.

It is apparent from Definition 3.3.4 that the construction of MDKK homology is substantially more cumbersome than that of classical Khovanov homology. The source-sink decorations used to detect and fix the problem face themselves make link invariance hard to verify, which must be rectified with the addition of the order construction. As both pieces of technology are diagrammatic, MDKK homology can be unwieldy. Doubled

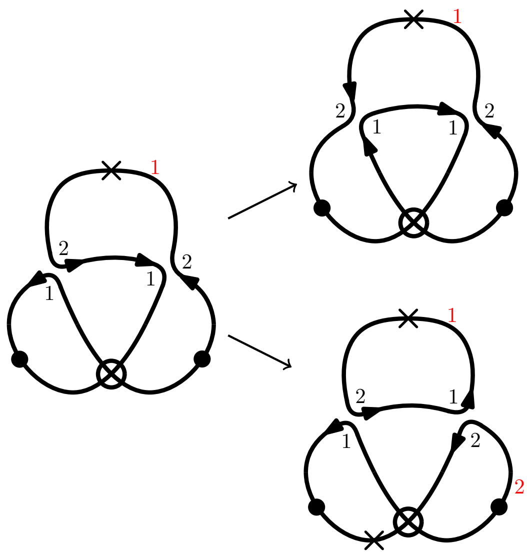


Figure 3.13: The cube of resolutions of the diagram given in Figure 3.11(A), with fully decorated smoothings. The local order is denoted by the black labels, and the global order the red.

Khovanov homology, defined in Chapter 4, solves the problems of Section 3.3.1 algebraically, resulting in a more streamlined and computable theory.

3.3.3 The virtual Rasmussen invariant

In Section 2.3.1 we outlined the degeneration of classical Khovanov homology due to Lee. There is an analogous degeneration of MDKK homology also, which yields a *virtual Rasmussen invariant*. To conclude this chapter we shall outline its definition, due to Dye, Kaestner, and Kauffman.

The perturbation of MDKK homology is defined identically to Lee homology: the chain spaces are left unchanged (with $\mathcal{R} = \mathbb{Q}$), and a new term is added to the differential (the η map remains zero, and the others are as in Equation (2.3.1)). We denote the perturbed homology by $vKh'(L)$ and refer to it as MDKK′ homology.

We make extensive use of an interpretation of virtual links due to Carter, Kamada, and Saito, known as *abstract links* [CKS02; KK00] (abstract links are the focus of much of Chapter 5 also). Given a virtual link diagram, we form an abstract link diagram as follows

(i) About the classical crossings place a disc as shown in Figure 3.14.

(ii) About the virtual crossings place two discs as shown in Figure 3.15.

Stage	Property of source or target smoothing	component of the differential
(1) Source	At the crossing neighbourhood compare the local order to the global order in one of two ways. If the labels 1 and 2 given by the recipe in Figure 3.12 appear on the same circle, permute the global order so that the circle on which they appear is in the first position and the other circles are in the same relative position they were before the permutation. If the labels 1 and 2 appear on separate circles permute the global order so that these circles are in the first and second positions, respectively, and the other circles are in the same relative position they were in before the permutation.	Multiplication by the sign of the permutation.
(2) Source	On each circle incident to the crossing neighbourhood, follow the source-sink orientation from the crossing neighbourhood to the marked point on that circle. Let the number of cut loci passed be n_i, for $1 \leq i \leq 2$ (there may be two circles involved in the smoothing site).	Application of the barring operator on the tensor factor corresponding to the i-the circle n_i mod 2 times.
(3)	As in the classical case, check if altering the resolution of the classical crossing splits one circle into two, merges two circles into one, or sends one circle to one circle.	A split yields the Δ map, a merge the m map, and a single-cycle smoothing the η map (where $\eta = 0$).
(4) Target	As in Stage (2) follow the source-sink orientation along each circle incident to the crossing neighbourhood to the marked point on that circle. As before, let the number of cut loci passed be n'_j, for $1 \leq j \leq 2$.	Application of the barring operator on the tensor factor corresponding to the j-the circle n'_i mod 2 times.
(5) Target	As in Stage (1), compare the global order to the local order at the crossing neighbourhood.	Multiplication by the sign of the permutation.

Table 3.1: The procedure for determining the components of the differential in MDKK homology.

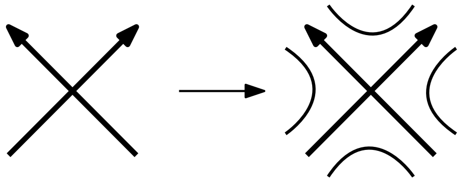


Figure 3.14: Component of the surface of an abstract link diagram about a classical crossing.

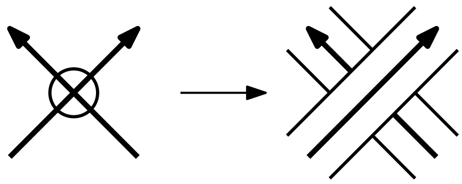


Figure 3.15: Component of the surface of an abstract link diagram about a virtual crossing.

(iii) Join up these discs with collars about the arcs of the diagram.

The result is a knot diagram on a surface which deformation retracts onto the underlying curve of the diagram; an example is given in Figure 3.19. We shall denote abstract link diagrams by (F, D) for D a knot diagram and F a compact, oriented surface (which deformation retracts on to the underlying curve of D). We treat such diagrams up to stable equivalence, defined below.

Definition 3.3.5 (Definition 3.2 of [CKS02])**.** Let (F_1, D_1) and (F_2, D_2) be abstract link diagrams. We say that (F_1, D_1) and (F_2, D_2) are equivalent, denoted $(F_1, D_1) \leftrightsquigarrow (F_2, D_2)$, if there exists a closed, connected, oriented surface F_3 and orientation-preserving embeddings $f_1 : F_1 \to F_3$, $f_2 : F_2 \to F_3$ such that $f_1(D_1)$ and $f_2(D_2)$ are related by Reidemeister moves on F_3. We say that two abstract link diagrams (F, D) and (F', D') are *stably equivalent* if there is a chain of equivalences

$$(F, D) = (F_0, D_0) \leftrightsquigarrow (F_1, D_1) \leftrightsquigarrow \cdots \leftrightsquigarrow (F_n, D_n) = (F', D')$$

for some $n \in \mathbb{N}$. $\lozenge$

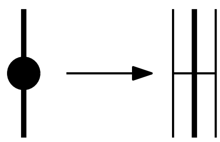


Figure 3.16: Cross cuts on an abstract link diagram inherited from cut loci.

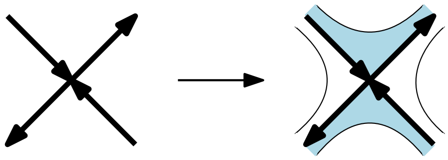


Figure 3.17: Checkerboard colouring at a crossing.

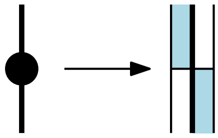


Figure 3.18: Checkerboard colouring at a cut locus.

Stable equivalence classes of abstract link diagrams are in bijective correspondence to equivalence classes of virtual link diagrams [KK00]. Smoothings of abstract link diagrams are defined exactly as those of virtual link diagrams; an example is given in Figure 3.19.

We need to keep track of the source-sink structure of a virtual link diagram on its associated abstract link diagram. We do so using *cross cuts*, which are added in the following way: before beginning the procedure described above mark the virtual link diagram with cut loci as inherited from the source-sink orientation and preserve them on the abstract link diagram. Replace each cut locus with a cross cut which bisects the surface as shown in Figure 3.16. Henceforth by *abstract link diagram* we mean an *abstract link diagram with cross cuts*.

Using the source-sink decoration we add yet more information to abstract link diagrams in the form of a *chequerboard colouring*. This extra information will allow us to define generators of a relevant homology theory in a canonical way, which will allow us to define bounds on a virtual generalisation of the Rasmussen invariant.

Definition 3.3.6. From an abstract link diagram (F, D) form its associated *chequerboard coloured* abstract link diagram from the surface and curve pair $(F, S(D))$ (where $S(D)$ denotes the source-sink diagram formed by replacing each crossing by the source-sink decoration) by colouring the surface F using the recipe given in Figure 3.17 and Figure 3.18.

Notice that Figure 3.17 allows us to induce a chequerboard colouring of smoothings of abstract link diagrams by simply joining the shaded or unshaded areas produced by smoothing the crossing. ◊

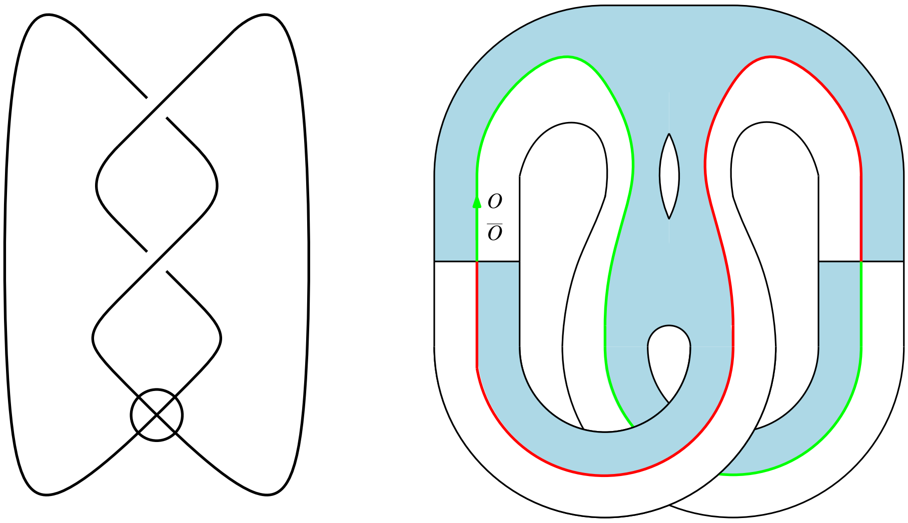


Figure 3.19: On the left, a virtual knot diagram, and on the right an alternately coloured smoothing of its associated abstract link diagram.

An example of a chequerboard coloured smoothing of an abstract link diagram is given in Figure 3.19.Henceforth by *abstract link diagram* we mean a *chequerboard coloured abstract link diagram with cross cuts* (likewise smoothings of abstract link diagrams).

As mentioned in Section 2.3, virtual links behave differently to classical links with respect to their alternately coloured smoothings; understanding this new behaviour is crucial to the definition and exploitation of doubled Khovanov homology. Dye, Kaestner, and Kauffman take a different approach, however: adding cross cuts and chequerboard colourings, one can force abstract links (and therefore the associated virtual links) to behave identically to classical links with respect to alternately coloured smoothings, and define a concordance invariant almost exactly as is done by Rasmussen. As in the classical case we employ the following basis of $\mathcal{A}$.

Definition 3.3.7. Let $\{r, g\}$ be the basis for $\mathcal{A}$ where

$$\begin{aligned} \text{“red"} = r &= \frac{1+X}{2} \\ \text{“green"} = g &= \frac{1-X}{2}. \end{aligned}$$

On the level of diagrams, arcs of a smoothing are coloured red or green to denote which generator they are labelled with. ◊

The properties of r and g are listed in Lemma 4.1 of [DKK17]. The most important for our purposes is that r and g are conjugates with respect to the barring operator. That is

$$\overline{r} = g \text{ and } \overline{g} = r.$$

We augment slightly the definition of alternately coloured smoothings to incorporate the cross cut decorations.

Definition 3.3.8 (Analogue of Definition 1.1 of [BNM06]). An *alternately coloured smoothing of an abstract link diagram* is a smoothing for which the arcs have been coloured either red or green such that the arcs passing through each crossing neighbourhood are coloured different colours. At a cross cut the colouring of an arc switches. ◊

Notice that this definition allows for a circle of a smoothing to possess two colours, as in Figure 3.19. Recall from Section 2.3 the alternately coloured generators of classical Lee homology: they are algebraic elements read off from alternately coloured smoothings of the argument diagram. For alternately coloured smoothings of an abstract link diagram containing a circle which possesses two colours (i.e. a circle with at least two cut loci on it), there is no clear way to repeat this process and push them to algebraic elements. In Chapter 5 we describe a method of doing just that, but Dye, Kaestner, and Kauffman are able to use the alternately coloured smoothings of abstract link diagrams themselves to prove structural properties of MDKK′ homology, and define the virtual Rasmussen invariant.

Theorem 3.3.9 (Theorem 4.2 of [DKK17]). *Let L be a virtual link. Then rank $vKh'(L) =$ #(alternately coloured smoothings of an abstract link diagram representing L).*

Theorem 3.3.10 (Theorem 4.3 of [DKK17]). *A virtual link L with $|L|$ components has exactly $2^{|L|}$ alternately coloured smoothings of an abstract link diagram. These smoothings are in bijective correspondence with the $2^{|L|}$ orientations of L.*

These results show that the rank of MDKK′ homology behaves exactly as that of classical Lee homology; in addition, we recover the triviality of the homological grading of the MDKK′ homology of a virtual knot. In Chapter 5 we describe the bijective correspondence of Theorem 3.3.10, and go into further detail regarding smoothings of abstract link diagrams, but we conclude this section by stating the definition of the virtual Rasmussen invariant and some of its properties.

Definition 3.3.11. Let K be a virtual knot. Then $vKh'(K)$ is of rank two, and is supported in homological degree 0. The information contained in the quantum grading is equivalent to an even integer, and we may define the *virtual Rasmussen invariant of* K, denoted $s(K) \in 2\mathbb{Z}$, as in the classical case. ◊

Still working with diagrammatic objects, Dye, Kaestner, and Kauffman determined the following properties of the virtual Rasmussen invariant.

Proposition 3.3.12 (Parts of Proposition 6.5 and Theorem 5.6 of [DKK17])**.** *The virtual Rasmussen invariant satisfies the following*

(i) $s(\overline{K}) = -s(K)$, *where* $\overline{K}$ *denotes the virtual knot represented by a diagram obtained by applying a crossing change to all classical crossings of a diagram of* K.

(ii) $|s(K)| \leq 2g^*(K)$.

(iii) If K *is a classical knot, then* $s(K)$ *is equal to the classical Rasmussen invariant.*

Notice that the virtual Rasmussen invariant lacks the out-of-the-box additivity of its classical counterpart (a consequence of the ill-defined nature of the connect sum operation on virtual knots). In Chapter 5 we show that the invariant is indeed additive.

In summary, by using alternately coloured smoothings of abstract link diagrams (rather than simply those of virtual link diagrams) together with cross cuts inherited from the source-sink decoration, Dye, Kaestner, and Kauffman arrive at a theory which behaves exactly as classical Lee homology, complete with a concordance invariant. In the next chapter we describe a homology theory which naturally incorporates the alternately coloured smoothing behaviour of virtual link diagrams (as described in Section 3.1.3), and, among other things, yields an extension of the Rasmussen invariant distinct to that which has been described in this section.

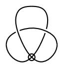
Chapter 4

Doubled Khovanov homology

In his chapter we define and investigate the properties of a homology theory of virtual links, the titular *doubled Khovanov homology*. Before diving into the construction of the invariant we give an overview of the chapter and the results obtained.

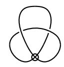
4.1 Overview

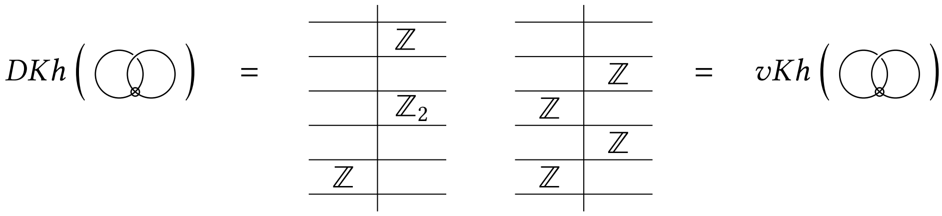
For a virtual link L we denote by $DKh(L)$ its doubled Khovanov homology (which is a bigraded finitely generated Abelian group). Below are two examples of the doubled Khovanov homologies of links, with the first (homological) grading represented on the horizontal axis, and the second (quantum) grading on the vertical. The position of = indicates 0 in the quantum grading, and the right-most column of the first pair of grids is at homological degree 0:

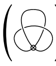
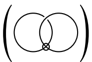
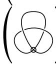
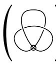
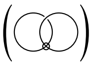
$DKh\Big($  $\Big)$ =

		$\mathbb{Z}$
		$\mathbb{Z}_2$
$\mathbb{Z}$		
$\mathbb{Z}$	$\mathbb{Z}$	
$\mathbb{Z}$		
$\mathbb{Z}$		

		$\mathbb{Z}$
	$\mathbb{Z}$	
		$\mathbb{Z}$
	$\mathbb{Z}_2$	
$\mathbb{Z}$		

= $vKh\Big($  $\Big)$



Also depicted is the aforemention MDKK homology, denoted by vKh. One observes that, while there are bigradings in which the groups assigned to the links by each theory are identical, they differ substantially overall. Specifically, we see that vKh () and vKh () both contain a $\mathbb{Z}^{\oplus 2}$ term for each component of the argument, and that vKh () also contains the knight's move familiar from classical Khovanov homology [BN02]. In contrast DKh () contains a knight's move and a $\mathbb{Z}^{\oplus 4}$ term, whereas DKh () contains only a single knight's move.

Unlike MDKK homology, doubled Khovanov homology can sometimes detect non-classicality of a virtual link.

Theorem (Corollary 4.3.6 of Section 4.3.2)**.** *Let L be a virtual link. If*

$$DKh(L) \neq G \oplus G\{-1\}$$

for G a non-trivial bigraded Abelian group, then L is non-classical.

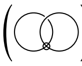
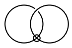
The connect sum operation on virtual knots exhibits more complicated behaviour than that of the classical case: the result of a connect sum between two virtual knots depends on both the diagrams used and the site at which the connect sum is conducted (as explained in Chapter 3). Indeed, there are multiple inequivalent virtual knots which can be obtained as connect sums of a fixed pair of virtual knots. A surprising consequence of this that there are non-trivial virtual knots which can be obtained as a connect sum of a pair of unknots. Doubled Khovanov homology yields a condition met by such knots.

Theorem (Theorem 4.6.11 of Section 4.6.2)**.** *Let K be a virtual knot which is a connect sum of two trivial knots. Then $DKh(K) = DKh\,(\bigcirc)$.*

Further, there is a perturbation of doubled Khovanov homology akin to Lee's perturbation of Khovanov homology; we denote it by $DKh'(L)$ and refer to it as *doubled Lee* *homology*. Unlike the classical case, however, doubled Lee homology vanishes for certain links. We show this in two steps. Firstly, we prove that the rank of doubled Lee homology behaves analogously to that of classical Lee homology.

Theorem (Theorem 4.4.4 of Section 4.4.1)**.** *Given a virtual link L*

$$rank\,(DKh'(L)) = 2\,|\{alternately\ coloured\ smoothings\ of\ L\}|\ .$$

Secondly, in Theorem 4.4.11 of Section 4.4.1, we determine the number of alternately coloured smoothings of a virtual link. In abbreviated form, Theorem 4.4.11 states that a virtual link L has either $2^{|L|}$ or 0 alternately coloured smoothings, and that one can determine which case holds via a simple check on a (Gauss diagram of a) diagram of L. This explains why DKh () is a single knight's move: a knight's move cancels when we pass to doubled Lee homology and  has no alternately coloured smoothings.

Kauffman related alternately coloured smoothings of virtual link diagrams to perfect matchings of 3-valent graphs [Kau04], and using that correspondence we can show that doubled Lee homology yields an equivalent to the Four Colour Theorem. Specifically, Kauffman showed first that the following statement is equivalent to the Four Colour Theorem:

> Let G be a planar, bridgeless, 3-valent graph. Then G has an even perfect matching.

Let $\mathcal{E}$ be a perfect matching of G. Associated to the pair $(G, \mathcal{E})$ is a family of virtual link diagrams. Denote a member of this family by $D(G, \mathcal{E})$. Kauffman next showed that the following is also equivalent to the Four Colour Theorem:

> Let G be a planar, bridgeless, 3-valent graph. Then every $D(G, \mathcal{E})$ has an alternately coloured smoothing.

Combining this with Theorem 4.4.4, we obtain the following equivalent to the Four Colour Theorem.

Theorem. *Let G be a planar, bridgeless, 3-valent graph and $\mathcal{E}$ a perfect matching of G. Then there exists a perfect matching $\mathcal{E}$ such that*

$$DKh'(D(G, \mathcal{E})) \neq 0$$

for all $D(G, \mathcal{E})$.

Doubled Lee homology cannot vanish for virtual knots, however; we show that a virtual knot has exactly 2 alternately coloured smoothings, so that its homology is of rank 4. In Section 4.5 we show that the information contained in $DKh'(K)$ is equivalent to a pair of integers, denoted $\mathbb{s}(K) = (s_1(K), s_2(K))$, and referred to as the *doubled Rasmussen invariant*[1]; $s_1(K)$ contains information regarding the quantum grading, $s_2(K)$ the homological grading. Using $\mathbb{s}(K)$ we are able to give the following obstructions to the existence of various kinds of cobordisms.

Theorem (Theorem 4.6.3 of Section 4.6.1)**.** *Let K_1 and K_2 be a pair of virtual knots with $s_2(K_1) = s_2(K_2)$, and S be a certain type of cobordism between them such that every link appearing in S has a generator in homological degree $s_2(K)$. Then*

$$\frac{|s_1(K_1) - s_1(K_2)|}{2} \leq g(S).$$

Theorem (Theorem 4.6.6 of Section 4.6.1)**.** *Let L be a virtual link of $|L|$ components. Further, let S be a connected genus 0 cobordism between L and a virtual knot K such that $DKh'_{s_2(K)}(L) \neq 0$. Let $M(L)$ be the maximum non-trivial quantum degree of elements $x \in DKh'(L)$ such that $\phi_S(x) \neq 0$. Then*

$$M(L) \leq s_1(K) + |L|.$$

Both components of the doubled Rasmussen invariant are concordance invariants and obstructions to sliceness; in Sections 4.4.2 and 4.6.1 we use the functorial nature of doubled Lee homology to show this. In addition, the homological degree information contained in the invariant is equivalent to the odd writhe - an easy-to-compute combinatorial invariant of virtual knots - so that we are able to show that this well known invariant is also an obstruction to sliceness.

Theorem (Proposition 4.5.11 of Section 4.5.3)**.** *Let K be a virtual knot. Then $s_2(K) = J(K)$, where $J(K)$ is the odd writhe of K.*

[1] as is demonstrated below this invariant is distinct from the aforementioned virtual Rasmussen invariant

Theorem (Theorem 4.6.8 of Section 4.6.1)**.** *Let K be a virtual knot and $J(K)$ its odd writhe. If $J(K) \neq 0$ then K is not slice.*

Theorem (Theorem 4.6.4 of Section 4.6.1)**.** *Let K and K' be virtual knots such that $s_2(K) = s_2(K')$. If $s_1(K) \neq s_1(K')$ then K and K' are not concordant.*

Finally, using the above results, we show that there exist virtual knots which are not concordant to any classical knots.

Theorem (Corollary 4.6.10 of Section 4.6.1)**.** *Let K be a virtual knot. If $J(K) \neq 0$ then K is not concordant to a classical knot.*

The virtual Rasmussen invariant (mentioned above and described in detail Chapter 3) is unable to obstruct the existence of a concordance between a virtual knot and classical knot. Further, there exist virtual knots whose nonsliceness is undetected by the virtual Rasmussen invariant and the odd writhe, but are detected by the doubled Rasmussen invariant, including 6.8909, 6.9825, 6.28566, 6.37329, and 6.58375 (the computations of the doubled and virtual Rasmussen invariants are given in Chapter 5).

4.1.1 Organisation

The chapter is organised as follows. In Section 4.2 we outline an alternative method of extending Khovanov homology to virtual links to that described in Section 3.3. In Section 4.3 we define the doubled Khovanov homology theory and describe some of its properties: we find the doubled Khovanov homology of classical links, and illustrate a method to produce an infinite number of non-trivial virtual knots with doubled Khovanov homology of the unknot. In Section 4.4 we define a perturbation analogous to Lee homology of classical links and show that, as in the classical case, the rank of this perturbed theory can be computed in terms of alternately coloured smoothings. As such, we also get to the bottom of the strange behaviour of alternately coloured smoothings of virtual links, outlined in Section 3.1.3. We then investigate the functorial nature of the perturbed theory. Section 4.5 contains the definition of the doubled Rasmussen invariant and a description of its properties. Finally, in Section 4.6 the invariant is put to use, yielding topological applications.

4.2 An alternative method of extension

As discussed in Section 3.3.1, dealing with the single-cycle smoothing (as in Figure 3.9(A)) must be the first accomplishment of any attempt to extend Khovanov homology to virtual links. Recall that if one associates the module $\mathcal{A}$ to a circle, the algebraic map η, assigned to a single-cycle smoothing, must be the zero map if it is to be quantum degree preserving.

Another way to overcome the problem is to "double up" the complex associated to a link diagram in order to plug the gaps in the quantum grading, so that the η map may be non-zero. The notion of "doubling up" will be made precise in Section 4.3, but for now let us return to the single-cycle smoothing: if we take the direct sum of the standard Khovanov chain complex with itself, but shifted in quantum grading by -1 (so that a circle is associated the module $\mathcal{A} \oplus \mathcal{A}\{-1\}$), we obtain $\eta : \mathcal{A} \oplus \mathcal{A}\{-1\} \to \mathcal{A} \oplus \mathcal{A}\{-1\}$, that is

$$\begin{array}{ccc} 0 & & v_+^{\mathrm{u}} \\ v_+^{\mathrm{u}} & & v_+^{\mathrm{l}} \\ v_+^{\mathrm{l}} & \xrightarrow{\eta} & v_-^{\mathrm{u}} \\ v_-^{\mathrm{u}} & & v_-^{\mathrm{l}} \\ v_-^{\mathrm{l}} & & 0 \end{array}$$

where $\mathcal{A} = \langle v_+^{\mathrm{u}}, v_-^{\mathrm{u}} \rangle$ and $\mathcal{A}\{-1\} = \langle v_+^{\mathrm{l}}, v_-^{\mathrm{l}} \rangle$ (u for "upper" and l for "lower") are graded modules and for W a graded module $W_{l-k} = W\{k\}_l$. Thus η may now be non-zero while still degree-preserving.

This approach is a fruitful one. Recall that setting $\eta = 0$ has the collateral effect of destroying anticommutativity, and much effort is given to its recovery in the construction of MDKK homology: the approach outlined above yields anticommutative faces (of the cube of smoothings) out of the box (whose anticommutativity is quickly verified), so that the issue of the problem face does not occur. That no diagrammatic technology need to be added to construction produces a theory which is easier to work with; in addition, doubled Khovanov homology is sometimes able to show that a given virtual link is not a classical link, which MDKK homology is unable to do.

4.3 Doubled Khovanov homology

4.3.1 Definition

In the tradition of classical Khovanov homology and its descendants *doubled Khovanov homology* assigns to an oriented virtual link diagram a bigraded Abelian group which is the homology of a chain complex; the result is an invariant of the link represented. In contrast to MDKK homology (as described in Chapter 3) the work of dealing with the single-cycle smoothing is done in the realm of algebra so that certain verifications require no new technology to complete (c.f. with the *order* construction of Section 3.3).

Definition 4.3.1 (Doubled Khovanov complex)**.** Let L be an oriented virtual link diagram with n_+ positive classical crossings and n_- negative classical crossings. Form the cube of smoothings associated to L in the standard manner by resolving classical crossings and leaving virtual crossings unchanged – see the example given in Figure 4.1.

Let $\mathcal{A} = \mathcal{R}[X]/X^2 = \langle v_-, v_+ \rangle$ (under the identification $X = v_-$, $1 = v_+$) where $\mathcal{R}$ is either $\mathbb{Q}$ or $\mathbb{Z}$. Form a chain complex by associating to a smoothing consisting of m cycles (that is, m copies of S^1 immersed in the plane) a vector space in the following way

$$\bigsqcup_{1 \leq i \leq m} S^1_i \longmapsto \left(\mathcal{A}^{\otimes m} \right) \oplus \left(\mathcal{A}^{\otimes m} \right) \{-1\}. \tag{4.3.1}$$

We refer to the unshifted (shifted) summand as the *upper (lower)* summand and denote elements in the upper summand by a superscript u and those in the lower summand by a superscript l. We also suppress tensor products, concatenating them into one subscript e.g.

$$v^u_{--+-} := (v_- \otimes v_- \otimes v_+ \otimes v_-)^u \in \mathcal{A}^{\otimes 4}$$

or

$$v^l_{++} := (v_+ \otimes v_+)^l \in \left(\mathcal{A}^{\otimes 2} \right) \{-1\}.$$

The components of the differential are built in the standard way as matrices with entries the maps Δ, m, and η, whose positions are read off from the cube of smoothings. The Δ

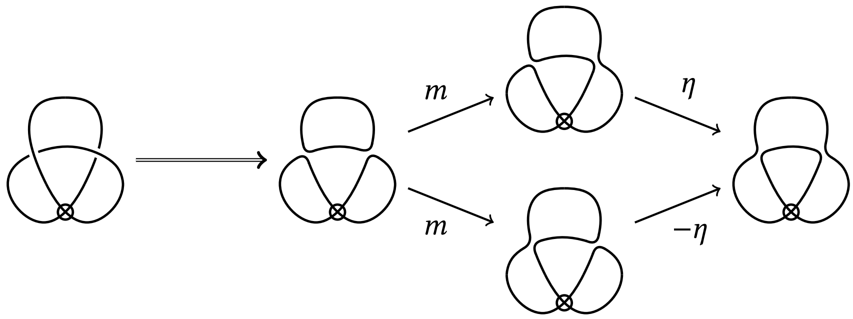
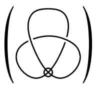


Figure 4.1: The cube of smoothings associated to the virtual knot diagram depicted on the left of the figure.

$CDKh\Bigg($  $\Bigg)$

$$
= \begin{array}{c} \mathcal{A}^{\otimes 2} \\ \oplus \\ \mathcal{A}^{\otimes 2}\{-1\} \end{array} \xrightarrow{d_{-2} = \begin{pmatrix} m \\ m \end{pmatrix}} \begin{array}{c} \mathcal{A} \\ \oplus \\ \mathcal{A}\{-1\} \\ \oplus \\ \mathcal{A} \\ \oplus \\ \mathcal{A}\{-1\} \end{array} \xrightarrow{d_{-1} = (\eta, -\eta)} \begin{array}{c} \mathcal{A} \\ \oplus \\ \mathcal{A}\{-1\} \end{array}
$$

$$
\begin{array}{ccc} -2 & \qquad\qquad -1 \qquad\qquad & 0 \end{array}
$$

Figure 4.2: The chain complex associated to the cube of smoothings depicted in Figure 4.1 (homological degree is denoted beneath the chain groups).

and m maps are given by

$$
\begin{aligned}
& m(v_+^{u/l} \otimes v_+^{u/l}) = v_+^{u/l} && \Delta(v_+^{u/l}) = v_+^{u/l} \otimes v_-^{u/l} + v_-^{u/l} \otimes v_+^{u/l} \\
& m(v_+^{u/l} \otimes v_-^{u/l}) = m(v_-^{u/l} \otimes v_+^{u/l}) = v_-^{u/l} && \Delta(v_-^{u/l}) = v_-^{u/l} \otimes v_-^{u/l} \\
& m(v_-^{u/l} \otimes v_-^{u/l}) = 0
\end{aligned} \tag{4.3.2}
$$

(so that they do not map between the upper and lower summands). The map associated to the single-cycle smoothing as in Figure 3.9(A) is given by

$$
\begin{aligned}
\eta(v_+^u) &= v_+^l & \qquad \eta(v_+^l) &= 2v_-^u \\
\eta(v_-^u) &= v_-^l & \qquad \eta(v_-^l) &= 0.
\end{aligned} \tag{4.3.3}
$$

The coefficients in Equation (4.3.3) are dictated by the requirement that faces of the cube of smoothings anticommute; in fact, this is the only choice which works.

The effect of the η map on tensor products is (possibly) to alter the superscript of entire string and the subscript of the tensorand in question. For example, if the cycle undergoing the single-cycle smoothing is corresponds to the second tensor factor

$$\begin{aligned}\eta(v^u_{-+-}) &= v^l_{-+-}\\ \eta(v^l_{++-}) &= 2v^u_{+--}.\end{aligned}$$

Any assignment of signs to the maps within the cube of smoothings which yields anti-commutative faces produces isomorphic chain complexes.

Let C_i denote the direct sum of the vector spaces assigned to the smoothings with exactly i 1-resolutions. Define the chain spaces of the *doubled Khovanov complex* to be

$$CDKh_i(L) = C_i[-n_-]\{n_+ - 2n_-\} \tag{4.3.4}$$

(where $[-n_-]$ denotes a shift in homological degree and $\{n_+ - 2n_-\}$ a shift in quantum degree). An example of such a chain complex is given in Figure 4.2. ◊

Remark. The map given in Equation (4.3.3) is not an $\mathcal{R}$-module map, so that $(\mathcal{A}, m, \Delta, \eta)$ is not an extended Frobenius algebra in the sense of [TT06], and doubled Khovanov homology seemingly cannot be interpreted as an (unoriented) TQFT. Also, doubled Lee homology, as defined in Section 4.4, does not satisfy the multiplicativity axiom of a TQFT. Perhaps a deeper reason for doubled Khovanov homologys failure to be a TQFT is that the domain category (the cobordism category) is incorrect. That is, in the case of virtual cobordism we are using *manifolds with corners*; there is no reason to suspect that the category of such objects will behave in a similar fashion to the category use in the classical construction.

Proposition 4.3.2. *Equipped with the differential given by matrices of maps as described in Definition 4.3.1 $CDKh(L)$ is a chain complex.*

Proof. It is enough to verify the commutativity of the faces

$$\begin{tikzcd} & {} \arrow[dr, "\eta"] & \\ {} \arrow[ur, "\eta"] \arrow[dr, "\Delta"'] & & {} \\ & {} \arrow[ur, "m"'] & \end{tikzcd} \qquad \begin{tikzcd} & {} \arrow[dr, "\Delta"] & \\ {} \arrow[ur, "\eta"] \arrow[dr, "\Delta"'] & & {} \\ & {} \arrow[ur, "\eta"'] & \end{tikzcd} \qquad \begin{tikzcd} & {} \arrow[dr, "m"] & \\ {} \arrow[ur, "\eta"] \arrow[dr, "m"'] & & {} \\ & {} \arrow[ur, "\eta"'] & \end{tikzcd}$$

as the face

$$
\begin{tikzcd}
 & {} \arrow[dr, "\eta"] & \\
{} \arrow[ur, "\eta"] \arrow[dr, "m"'] & & {} \\
 & {} \arrow[ur, "\Delta"'] &
\end{tikzcd}
$$

cannot occur (set up a 2-crossing smoothing so that is has an outgoing η map and an outgoing Δ map, and it becomes clear that it cannot feature in a face which contains an m map). We leave the algebra to the reader and note that, as in the classical case, sprinkling signs appropriately yields a chain complex. □

Theorem 4.3.3. *Given an oriented virtual link diagram D the chain homotopy equivalence class of $CDKh(D)$ is an invariant of the oriented link represented by D. The homology of $CDKh(D)$, denoted $DKh(D)$, is therefore also an invariant of the link represented by D.*

Proof. We are required to construct chain homotopy equivalences for each of the virtual Reidemeister moves. It is readily observed that if two diagrams D_1 and D_2 are related by a finite sequence of the purely virtual moves and mixed move (depicted in Figure 3.2) then $CDKh(D_1) = CDKh(D_2)$ as these moves do not alter the number of cycles in a smoothing nor the incoming and outcoming maps.

Concerning the classical moves, we follow Bar-Natan [BN02], using [BN02, Lemma 3.7] and Gauss elimination (specifically, [BNBS14, Lemma 3.2]). We leave the details to the reader. □

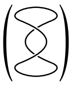
The homology of the complex given in Figure 4.2 is depicted in Figure 4.4.

Although the module assigned to a smoothing in the construction of doubled Khovanov homology is not equal to that of MDKK homology, the Euler characteristics of the two theories contain equivalent information.

Proposition 4.3.4. *Let L be a virtual link. Denote by $\chi_q(DKh(L))$ the graded Euler characteristic of $DKh(L)$ with respect to the quantum grading. Then $\chi_q(DKh(L)) = (1+q^{-1})V_L(q)$, for $V_L(q)$ the Jones polynomial of L.*

Proof. The Jones polynomial of L is the graded Euler characteristic of $vKh(L)$ [Man07]. That is

$$\chi_q(vKh(L)) = V_L(q).$$

$CDKh\Bigg($  $\Bigg) =$

$$\begin{array}{ccccc} & & \mathcal{A}^{\otimes 3} & & \\ & & \oplus & & \\ \mathcal{A}^{\otimes 2} & & \mathcal{A}^{\otimes 3}\{-1\} & & \mathcal{A}^{\otimes 2} \\ \oplus & \xrightarrow{d_{-2} = \begin{pmatrix}\Delta \\ m\end{pmatrix}} & \oplus & \xrightarrow{d_{-1} = (m, -\Delta)} & \oplus \\ \mathcal{A}^{\otimes 2}\{-1\} & & \mathcal{A} & & \mathcal{A}^{\otimes 2}\{-1\} \\ & & \oplus & & \\ & & \mathcal{A}\{-1\} & & \\ -1 & & -0 & & 1 \end{array}$$

Figure 4.3: The doubled Khovanov complex of a classical diagram.

As we are dealing with bounded, finitely generated chain complexes, the graded Euler characteristics of $DKh(L)$ and $vKh(L)$ depend only on the chain complex used to define them. Therefore, to prove the claim we need only consider $\chi_q(CDKh(L))$. Noticing that $CDKh(L)$ is simply a direct sum of two copies of the chain complex whose homology is $vKh(L)$, with one copy shifted in the quantum degree by -1, we obtain

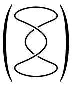
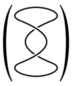
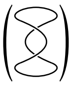
$$\begin{aligned} \chi_q(CDKh(L)) &= \chi_q(vKh(L)) + q^{-1}\chi_q(vKh(L)) \\ &= (1 + q^{-1})V_L(q). \end{aligned}$$

□

4.3.2 Detection of non-classicality

We say that a virtual link is *non-classical* if all diagrams representing it have a virtual crossing. Conversely, we say that a virtual link is *classical* if it has a diagram with no virtual crossings. Doubled Khovanov homology can sometimes be used to detect non-classicality.

Consider the complex associated to the classical diagram of the unknot given in Figure 4.3: the reader notices immediately that not only do the chain spaces decompose as direct sums, the entire complex does also (as there are no η maps). That is

$CDKh\Bigg($  $\Bigg) = CKh\Bigg($  $\Bigg) \oplus CKh\Bigg($  $\Bigg)\{-1\}$ (4.3.5)

where $CKh(D)$ denotes the classical Khovanov complex of a diagram D. This motivates the following proposition.

Proposition 4.3.5. *Let L be a virtual link. If L is classical then there exists a diagram of*

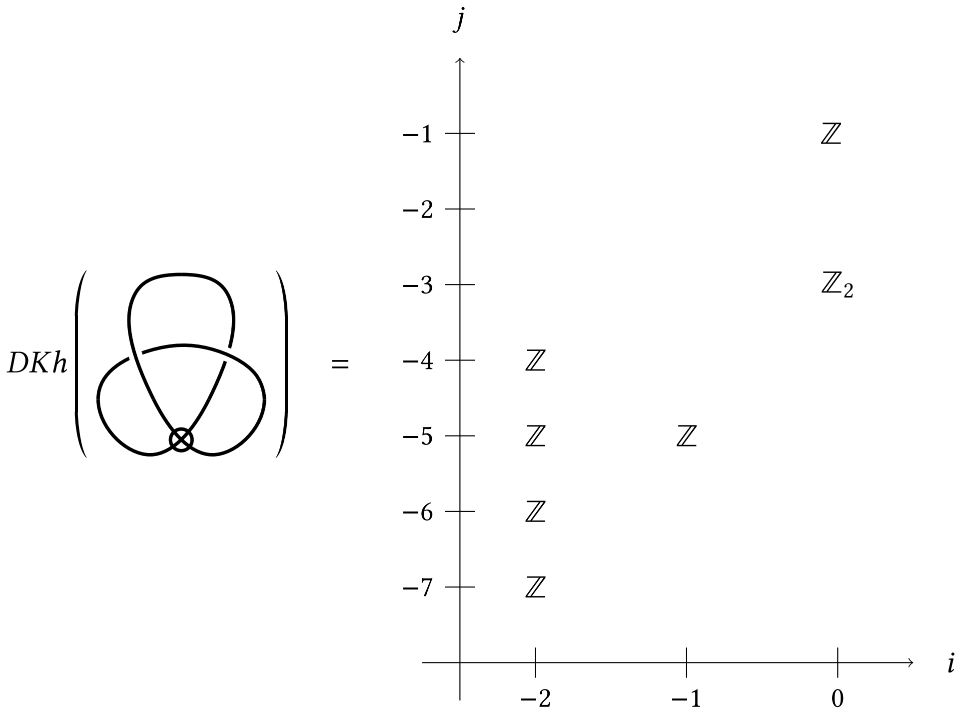


Figure 4.4: The doubled Khovanov homology of the virtual knot 2.1.

L, denoted D, which has only classical crossings. Then

$$DKh(L) = Kh(D) \oplus Kh(D)\{-1\}$$

where $Kh(D)$ *denotes the standard Khovanov homology of a classical link.*

Proof. This is an obvious consequence of Equation (4.3.5), which holds for all classical diagrams. □

The contrapositive statement to that of Proposition 4.3.5 is

Corollary 4.3.6. *Let L be a virtual link. If*

$$DKh(L) \neq G \oplus G\{-1\} \tag{4.3.6}$$

for G a non-trivial bigraded Abelian group, then L is non-classical.

As an example consider the virtual knot 2.1, depicted in Figure 4.4, along with its doubled Khovanov homology, split by homological grading (horizontal axis) and quantum grading (vertical axis).

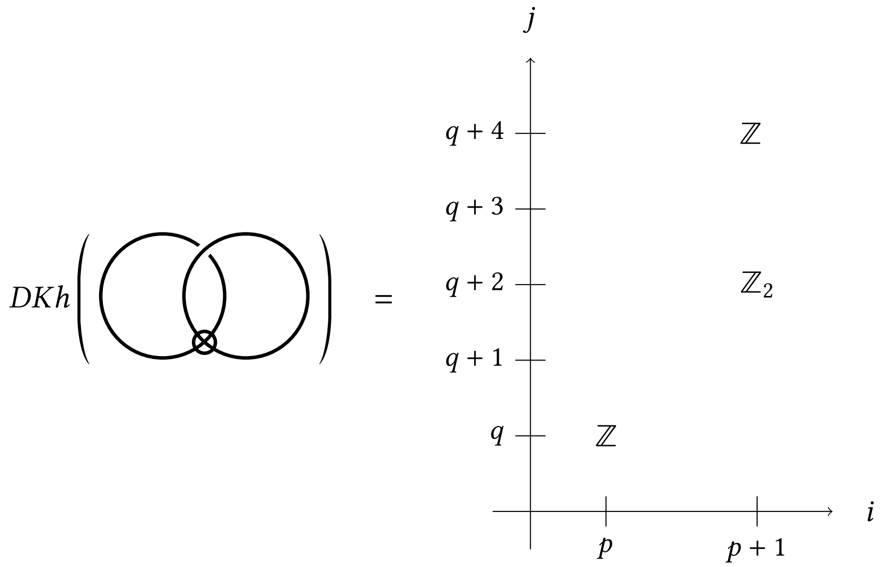
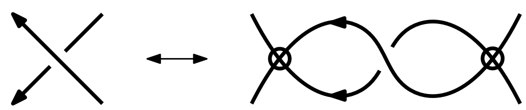


Figure 4.5: The doubled Khovanov homology of the virtual Hopf link (p and q depend on the orientation of the components).

Another interesting example is given by the so-called virtual Hopf link, given in Figure 4.5; we shall look into it further in Section 4.4.

The statement within Corollary 4.3.6 cannot be upgraded to an equivalence, however. A counterexample is given by the virtual knot 3.7, depicted on the right of Figure 4.6 (the non-classicality of 3.7 is demonstrated by its generalised Alexander polynomial [KR03]). The cube of smoothings associated to 3.7 does not contain any η maps, and therefore $DKh(3.7) = G \oplus G\{-1\}$ for some non-trivial Abelian group G. In fact, $DKh(3.7) = Kh(\bigcirc) \oplus Kh(\bigcirc)\{-1\} = DKh(\bigcirc)$. This follows from the fact that 3.7 can be obtained from a diagram of the unknot by applying the following move on diagrams

Definition 4.3.7. Within an oriented virtual link diagram one may place a virtual crossing on either side of a classical crossing in the following manner



This move is known as *flanking*. ◊

Flanking is also known *virtualization*, but as there is some confusion in the literature regarding that term we avoid it.

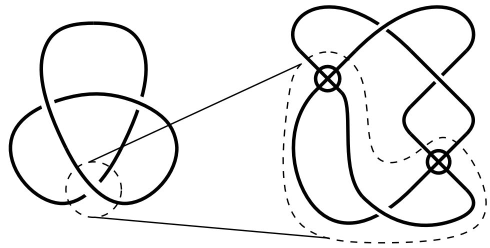


Figure 4.6: Obtaining virtual knot 3.7 from the unknot via flanking.

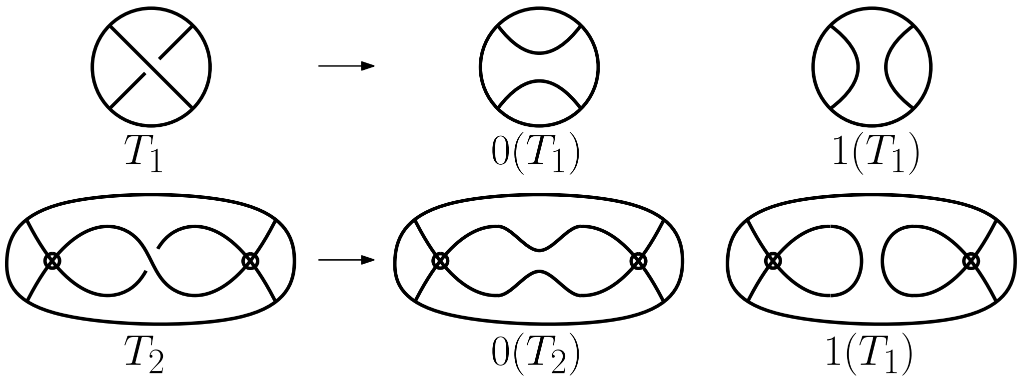


Figure 4.7: Smoothings of the tangle diagrams related to the flanking move.

Proposition 4.3.8. *If a virtual link diagram D_2 can be obtained from another, D_1, by a flanking move then $CDKh(D_1) = CDKh(D_2)$.*

Proof. Let D_1 and D_2 be as in the proposition. Consider the tangle diagrams produced by isolating a neighbourhood of the classical crossing undergoing the flanking move in D_1 and a neighbourhood of the result of the flanking move in D_2. We construct an identification of the smoothings of D_1 with those of D_2 using the smoothings of the tangle diagrams depicted in Figure 4.7: a smoothing of D_1 must contain either $0(T_1)$ or $1(T_1)$, and we associate to it the smoothing of D_2 formed by replacing $0(T_1)$ with $0(T_2)$, or $1(T_1)$ with $1(T_2)$. One readily sees that this identification is a bijection which does not change the number of cycles in a smoothing nor how those cycles are linked. Thus the chain spaces of $CDKh(D_1)$ and $CDKh(D_2)$ are equal, and so are the components of the differential. □

Corollary 4.3.9. *There is an infinite number of non-trivial virtual knots with doubled Khovanov homology equal to that of the unknot.*

Proof. There is an infinite number of non-trivial virtual knot diagrams with unit Jones polynomial, produced via flanking [Dye05; Kau99; SW04]. Each of these knots must also have the doubled Khovanov homology of the unknot. □

4.4 Doubled Lee homology

In Section 4.4.1 we define doubled Lee homology and determine some of its properties, and in Section 4.4.2 we investigate aspects of the functorial nature of the theory.

4.4.1 Definition

The reader may have noticed that there are generators of the homologies depicted in Figure 4.4 and Figure 4.5 which are 4 apart in quantum degree. Quantum degree separations of length 4 are important in classical Khovanov homology; Lee's perturbation of Khovanov homology is defined by adding to the differential a component of degree 4 (as described in Chapter 2). Such a perturbation of doubled Khovanov homology exists also.

Definition 4.4.1 (Doubled Lee homology)**.** Let D be an oriented virtual link diagram and $CDKh'(D)$ denote the chain complex with the chain spaces of $CDKh(D)$ but with altered differential, and $\mathcal{R} = \mathbb{Q}$. The components of this differential are as follows

$$\begin{aligned}
&m'(v_+^{u/l} \otimes v_+^{u/l}) = v_+^{u/l} &&\Delta'(v_+^{u/l}) = v_+^{u/l} \otimes v_-^{u/l} + v_-^{u/l} \otimes v_+^{u/l}\\
&m'(v_+^{u/l} \otimes v_-^{u/l}) = m'(v_-^{u/l} \otimes v_+^{u/l}) = v_-^{u/l} &&\Delta'(v_-^{u/l}) = v_-^{u/l} \otimes v_-^{u/l} + v_+^{u/l} \otimes v_+^{u/l}\\
&m'(v_-^{u/l} \otimes v_-^{u/l}) = v_+^{u/l}
\end{aligned}$$

and

$$\begin{aligned}
\eta'(v_+^u) &= v_-^u & \eta'(v_+^l) &= 2v_-^u\\
\eta'(v_-^u) &= v_-^l & \eta'(v_-^l) &= 2v_+^u.
\end{aligned}$$

The above maps are no longer graded, but filtered (as in the classical case). That $CDKh'(D)$ is a chain complex is verified as in Proposition 4.3.2. Setting $DKh'(D)$ to be the homology of $CDKh'(D)$, define the doubled Lee homology of L

$$DKh'(L) \coloneqq DKh'(D)$$

where L is the link represented by D. ◊

The proof of invariance of doubled Lee homology follows in essentially the same manner as that of doubled Khovanov homology, and we obtain the following.

Theorem 4.4.2. *For a virtual link diagram D, $DKh'(D)$ is an invariant of the link represented by D.*

As in the classical case, doubled Khovanov homology and doubled Lee homology are related in the following manner.

Theorem 4.4.3. *For any virtual link L there is a spectral sequence with E_2 page $DKh(L)$ converging to $DKh'(L)$.*

As described in Section 2.3.1, the rank of the classical Lee homology of a link depends only on the number of its components. Precisely, for a classical link L_c

$$\operatorname{rank}(Kh'(L_c)) = 2^{|L_c|} \tag{4.4.1}$$

where $|L_c|$ denotes the number of components of L_c and $Kh'(L_c)$ its classical Lee homology. In fact, Equation (4.4.1) follows from the following two statements [BNM06]:

$$\operatorname{rank}(Kh'(L_c)) = |\{\text{alternately coloured smoothings of } L_c\}| \tag{4.4.2}$$

and

$$\{\text{alternately coloured smoothings of } L_c\} = \{\text{orientations of } L_c\}\,. \tag{4.4.3}$$

(Any potential issue raised by the fact that the definition of alternately coloured smoothings regards diagrams while Equations (4.4.2) and (4.4.3) are statements about links is resolved by the fact that the number of alternately coloured smoothings is a link invariant.)

In the virtual case we recover Equation (4.4.2) (up to a scalar) but not Equation (4.4.3).

Theorem 4.4.4. *Given a virtual link L*

$$\mathit{rank}(DKh'(L)) = 2\,|\{\textit{alternately coloured smoothings of } L\}|\,.$$

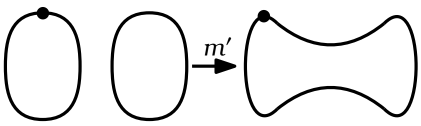
We postpone stating the virtual generalisation of Equation (4.4.3) until we have proved Theorem 4.4.4, for which we require the following analogue of a classical result.

Lemma 4.4.5. *Let D be a diagram of a virtual link L. There is an action of $\mathcal{A}$ on $CDKh'(D)$ which descends to an action on $DKh'(L)$.*

Proof. Given a virtual link diagram D define an action of $\mathcal{A}$ on $CDKh'(D)$ in the following manner: mark a point on D and maintain it across the smoothings of D. The action $\mathcal{A} \times CDKh'(D) \to CDKh'(D)$ is given by

$$s \cdot \Big((x_1 \otimes x_2 \otimes \ldots \otimes x_n)^{\mathrm{u}} + (x_1 \otimes x_2 \otimes \ldots \otimes x_n)^{\mathrm{l}} \Big) = (x_1 \otimes x_2 \otimes \ldots \otimes s{\cdot}x_i \otimes \ldots \otimes x_n)^{\mathrm{u}} + (x_1 \otimes x_2 \otimes \ldots \otimes s{\cdot}x_i \otimes \ldots \otimes x_n)^{\mathrm{l}}$$

where the i-th cycle is marked (component-wise multiplication $\cdot : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$ is given by m'). Clearly this action endows $CDKh'(D)$ with the structure of an $\mathcal{A}$-module. To show that $DKh'(D)$ is also an $\mathcal{A}$-module it suffices to show that the action defined above commutes with the differential. We verify this in the case of m' and multiplication by v_-, with the marked point on the cycle corresponding to the first tensor factor:



$$\begin{aligned}
v_- \cdot m'((v_+ \otimes v_+)^{\mathrm{u/l}}) &= v_- \cdot v_+^{\mathrm{u/l}} = m'((v_- \otimes v_+)^{\mathrm{u/l}}) = m'(((v_- \cdot v_+) \otimes v_+)^{\mathrm{u/l}}) \\
v_- \cdot m'((v_+ \otimes v_-)^{\mathrm{u/l}}) &= v_- \cdot v_-^{\mathrm{u/l}} = m'((v_- \otimes v_-)^{\mathrm{u/l}}) = m'(((v_- \cdot v_+) \otimes v_-)^{\mathrm{u/l}}) \\
v_- \cdot m'((v_- \otimes v_+)^{\mathrm{u/l}}) &= v_- \cdot v_-^{\mathrm{u/l}} = m'((v_- \otimes v_-)^{\mathrm{u/l}}) = m'(((v_- \cdot v_-) \otimes v_+)^{\mathrm{u/l}}) \\
v_- \cdot m'((v_- \otimes v_-)^{\mathrm{u/l}}) &= v_- \cdot v_+^{\mathrm{u/l}} = m'((v_- \otimes v_p)^{\mathrm{u/l}}) = m'(((v_- \cdot v_-) \otimes v_-)^{\mathrm{u/l}})
\end{aligned}$$

as required. The other cases are left to the reader. □

Recall the familiar "red" and "green" basis first given by Bar-Natan and Morrison.

Definition 4.4.6. Let $\{r, g\}$ be the basis for $\mathcal{A}$ where

$$\begin{aligned}
\text{“red”} = r &= \frac{v_+ + v_-}{2} \\
\text{“green”} = g &= \frac{v_+ - v_-}{2}.
\end{aligned}$$

We denote the corresponding generators of $\mathcal{A} \oplus \mathcal{A}\{-1\}$ as r^{u}, r^{l}, g^{u}, and g^{l}. ◊

We denote which generator a cycle of a smoothing is labelled with by colouring that cycle either red or green. Thus alternately coloured smoothings are such that given any two cycles which share a crossing (i.e. they pass through the same crossing neighbourhood) one is coloured red and the other green.

We shall use the following definition in the remainder of this work.

Definition 4.4.7. Let D be an oriented virtual link diagram with n_- negative classical crossings, and $\mathscr{S}$ an alternately coloured smoothing in which m classical crossings (positive or negative) are resolved into their 1-resolution. Define the *height* of $\mathscr{S}$ to be $|\mathscr{S}| := m - n_-$. ◊

Proof of Theorem 4.4.4. Let D be a diagram of L and $\mathscr{S}$ be an alternately coloured smoothing of D, with cycles coloured either red or green, and $\mathfrak{s}^{\mathrm{u}}$ be the algebraic element given by

$$\mathfrak{s}^{\mathrm{u}} = \bigotimes_{\text{cycles of } S} \square_i^{\mathrm{u}} \tag{4.4.4}$$

where

$$\square_i^{\mathrm{u}} = \begin{cases} r^u, & \text{if the } i\text{-th cycle is coloured red} \\ g^u, & \text{if the } i\text{-th cycle is coloured green} \end{cases}$$

and likewise define $\mathfrak{s}^{\mathrm{l}}$, so that to each alternately coloured smoothing we associate two algebraic objects. We refer to such $\mathfrak{s}^{\mathrm{u/l}}$'s as *alternately coloured generators*[2], a term we justify in two steps: we shall show that such elements are homologically non-trivial and distinct, and that they do indeed generate $DKh'(L)$.

First notice that alternately coloured smoothings have restricted incoming and outgoing differentials: if a smoothing has an η' map either incoming or outgoing then it must have a crossing neighbourhood with only one cycle passing through it. Such a crossing neighbourhood cannot satisfy the alternately coloured condition. Likewise, if a smoothing has an incoming m' map or an outgoing Δ' map it must have a crossing neighbourhood with only one cycle passing through. Thus an alternately coloured smoothing has only incoming Δ' maps and outgoing m' maps and homological non-triviality of the associated $\mathfrak{s}^{\mathrm{u/l}}$ is equivalent to $\mathfrak{s}^{\mathrm{u/l}} \in \ker(m')$ and $\mathfrak{s}^{\mathrm{u/l}} \notin \operatorname{im}(\Delta')$. With respect to the $\{r, g\}$ basis we

[2] Of course, if $\mathfrak{s}^{\mathrm{u/l}}$ are the alternately coloured generators assigned to $\mathscr{S}$ then $|\mathscr{S}| = i(\mathfrak{s}^{\mathrm{u/l}})$.

have

$$\begin{aligned} m'(r \otimes r) &= r & \Delta'(r) &= 2r \otimes r \\ m'(g \otimes g) &= g & \Delta'(g) &= -2g \otimes g \\ m'(r \otimes g) &= m'(g \otimes r) = 0 \end{aligned} \tag{4.4.5}$$

so that clearly $[\mathfrak{s}^{\mathrm{u/l}}] \neq 0 \in DKh'(L)$.

Let $\mathscr{S}_1$ and $\mathscr{S}_2$ be two alternately coloured smoothings of L and $\mathfrak{s}_1^{\mathrm{u/l}}$, $\mathfrak{s}_2^{\mathrm{u/l}}$ their associated alternately coloured generators. Notice that it is possible that $\mathscr{S}_1$ and $\mathscr{S}_2$ are alternately coloured smoothings associated to the same uncoloured smoothing of L. We shall consider the two cases: (i) $\mathscr{S}_1$ and $\mathscr{S}_2$ are not alternately coloured smoothings associated to the same uncoloured smoothing of D and (ii) they are.

(i): It is possible that $\mathscr{S}_1$ and $\mathscr{S}_2$ are at different height (that is, they have a different number of 1-resolutions). Then $[\mathfrak{s}_1^{\mathrm{u/l}}] \neq [\mathfrak{s}_2^{\mathrm{u/l}}]$ as they are of differing homological grading. If $\mathscr{S}_1$ and $\mathscr{S}_2$ are at the same height, i, say, we recall that $CDKh'_i(L)$ is a direct sum of the modules associated to all smoothings of height i so that $\mathfrak{s}_1^{\mathrm{u/l}} - \mathfrak{s}_2^{\mathrm{u/l}}$ can be written

$$\mathfrak{s}_1^{\mathrm{u/l}} - \mathfrak{s}_2^{\mathrm{u/l}} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \mathfrak{s}_1^{\mathrm{u/l}} \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ \vdots \\ \vdots \\ 0 \\ \mathfrak{s}_2^{\mathrm{u/l}} \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \mathfrak{s}_1^{\mathrm{u/l}} \\ 0 \\ \vdots \\ 0 \\ -\mathfrak{s}_2^{\mathrm{u/l}} \\ 0 \\ \vdots \end{pmatrix}$$

so that $\mathfrak{s}_1^{\mathrm{u/l}} - \mathfrak{s}_2^{\mathrm{u/l}} \notin \mathrm{im}(\Delta')$.

(ii): Mark a point on L such that the cycles of $\mathscr{S}_1$ and $\mathscr{S}_2$ on which the point lies are opposite colours (such a point always exists as $\mathscr{S}_1 \neq \mathscr{S}_2$), and define the action of $\mathcal{A}$ as in Lemma 4.4.5. Notice that $v_- \cdot r = r$ and $v_- \cdot g = -g$ so that

$$\begin{aligned} v_- \cdot \mathfrak{s}_1^{\mathrm{u/l}} &= \pm \mathfrak{s}_1^{\mathrm{u/l}} \\ v_- \cdot \mathfrak{s}_2^{\mathrm{u/l}} &= \mp \mathfrak{s}_2^{\mathrm{u/l}} \end{aligned}$$

as if the marked cycle is red in $\mathscr{S}_1$ then it is green in $\mathscr{S}_2$ and vice versa. As the action descends to an action on $DKh'(L)$ we see that $[\mathfrak{s}_1^{\mathrm{u/l}}]$ is an eigenvector of the action of v_- of eigenvalue ± 1 and $[\mathfrak{s}_2^{\mathrm{u/l}}]$ is an eigenvector of eigenvalue ∓ 1, so that $[\mathfrak{s}_1^{\mathrm{u/l}}] \neq [\mathfrak{s}_2^{\mathrm{u/l}}]$.

At this point we have

$$\operatorname{rank}(DKh'(L)) \geq 2 \left| \{\text{alternately coloured smoothings of } L\} \right| .$$

In order to tighten this to an equality we shall again employ Gauss elimination along with the observation that the differential restricted to elements corresponding to non-alternately coloured smoothings is an isomorphism. In the case of the Δ' and m' maps this is evident from Equation (4.4.5). Regarding the η' map, we have

$$\begin{aligned} \eta'(r^{\mathrm{u}}) &= r^{\mathrm{l}} & \qquad \eta'(r^{\mathrm{l}}) &= 2r^{\mathrm{u}} \\ \eta'(g^{\mathrm{u}}) &= g^{\mathrm{l}} & \qquad \eta'(g^{\mathrm{l}}) &= -2g^{\mathrm{u}} \end{aligned} \tag{4.4.6}$$

so that η' is an isomorphism (we are working over $\mathbb{Q}$). Thus we Gauss eliminate elements associated to non-alternately coloured smoothings of L and arrive at the desired equality.

□

We now return to Equation (4.4.3), in order to generalise it to the virtual case. It is clear that we have some work to do, as the virtual Hopf link (as depicted in Figure 4.5), for example, has no alternately coloured smoothings (one readily sees that the generators on the right of Figure 4.5 will cancel in doubled Lee homology). Before describing the virtual situation we make some preliminary definitions.

Definition 4.4.8. Let D be a virtual link diagram. Denote by $S(D)$ the diagram obtained from D be removing the decoration at classical crossings; we refer to $S(D)$ as the *shadow* of D. Let a *component* of $S(D)$ be an S^1 embedded in such a way that at a classical or virtual crossing we have exactly one of the following:

- All the incident arcs are contained in the component.
- The arcs contained in the component are not adjacent.
- None of the arcs are contained in the component.

Thus components of $S(D)$ are in bijection with those of D. ◊

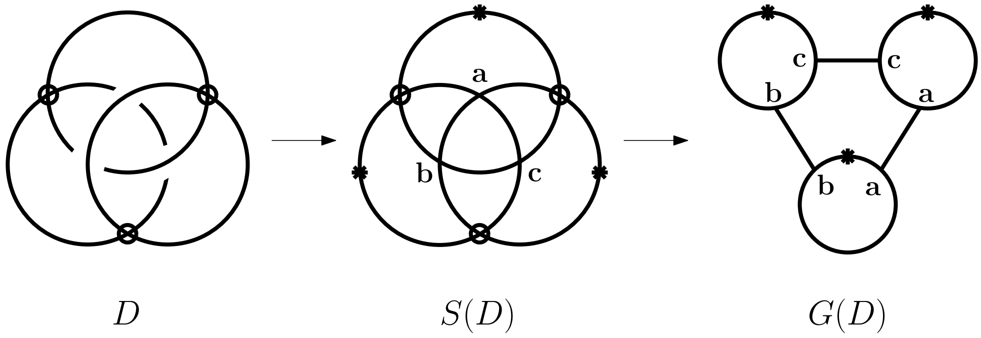


Figure 4.8: The shadow and Gauss diagram of a virtual link diagram.

Definition 4.4.9. Let D be an n-component virtual link diagram and $Sh(D)$ its shadow. Denote by $G(D)$ the *Gauss diagram of* D, formed in the following manner:

(i) Place n copies of S^1 disjoint in the plane. A copy of S^1 is known as a *circle* of $G(D)$.

(ii) Fix a bijection between the components of $S(D)$ and the circles of $G(D)$.

(iii) Arbitrarily pick a basepoint on each component of $Sh(D)$ and on the corresponding circle of $G(D)$.

(iv) Pick a component of $Sh(D)$ and progress from the basepoint around that component (in either direction). When meeting a classical crossing label it and mark that label on the corresponding circle of $G(D)$ (virtual crossings are ignored). Continue until the basepoint is returned to.

(v) Repeat for all components of $Sh(D)$; if a crossing is met which already has a label, use it.

(vi) Add a chord linking the two incidences of each label. These chords may intersect and have their endpoints on different circles of $G(D)$. ◊

Gauss diagrams are more commonly defined for diagrams, rather than shadows, of links but this definition contains all the information we require. An example of a shadow and of a Gauss diagram can be found in Figure 4.8.

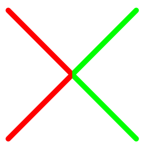
Definition 4.4.10. A circle within a Gauss diagram is known as *degenerate* if it contains an odd number of chord endpoints. ◊

Theorem 4.4.11. *Given a diagram D of a virtual link L*

$$|\{\textit{alternately coloured smoothings of } L\}| = |\{\textit{alternately coloured smoothings of } D\}|$$
$$= \begin{cases} 2^{|L|}, & \textit{if } G(D) \textit{ contains no degenerate circles} \\ 0, & \textit{otherwise.} \end{cases}$$

Proof. As demonstrated in Theorems 4.4.3 and 4.4.4, the number of alternately coloured smoothings is a link invariant, so that we are free to use the Gauss diagram associated to any diagram of L.

As observed by Kauffman [Kau04] alternately coloured smoothings of a link diagram are in bijection with particular colourings of the shadow of the diagram: colouring the arcs of the shadow either red or green such that at every classical crossing we have the following (up to rotation):



Such a colouring is known as a *proper colouring*. Given a virtual link diagram D and a proper colouring of $Sh(D)$, one produces an alternately coloured smoothing of D by resolving each classical crossing in the manner dictated by the proper colouring i.e. joining red to red and green to green. Two examples are given in Figure 4.9. It is easy to see that this association defines a bijection between the set of proper colourings and the set of alternately coloured smoothings.

Next, notice that a proper colouring of $Sh(D)$ induces a colouring of the circles of $G(D)$: colour the connected components of the complement of the chord endpoints in the manner dictated by the colouring of the shadow (so that when an endpoint is passed the colour changes). A Gauss diagram coloured in such a way is known as *alternately coloured*. Examples are given in Figure 4.9. It is again easy to see that alternately coloured Gauss diagrams are in bijection with proper colourings, so that we have a bijection between alternately coloured smoothings of D and alternate colourings of $G(D)$.

In light of the above we see that we are required to verify that a Gauss diagram of n circles has 2^n alternate colourings if and only it has no degenerate circles, and none otherwise.

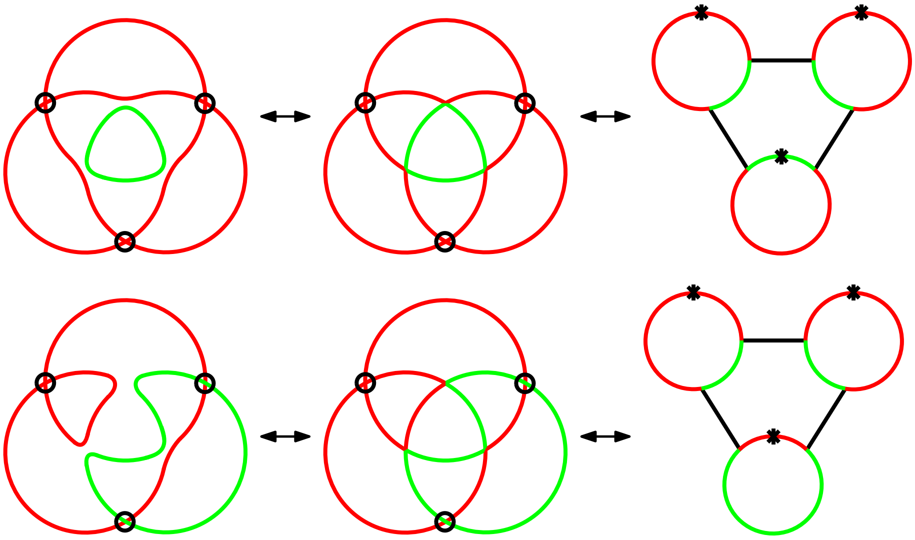


Figure 4.9: Alternately coloured smoothings (left) and Gauss diagrams (right) associated to proper colourings of a shadow (centre).

Let $G(D)$ contain a degenerate circle. On this circle the number of connected components of the complement of the end points is odd, from which we deduce that it cannot be alternately coloured (as the colour must change when passing an endpoint). That there are 2^n alternate colourings if there is no degenerate circle follows from the observation that there are two possible configurations for each circle, and that given an alternate colouring flipping the configuration on one circle yields a new alternate colouring. □

Corollary 4.4.12. *Let K be a virtual knot. Then rank$(DKh'(K)) = 4$ and $DKh'(K)$ is supported in homological degree equal to the height of the alternately colourable smoothing.*

Proof. Let D be a virtual knot diagram. Then $G(D)$ satisfies the condition of Theorem 4.4.11 as it contains only one circle, on which all chord endpoints must lie. Of course, every chord has two endpoints so that this circle must contain an even number of them. The statement then follows from Theorem 4.4.4. □

Classically, the alternately colourable smoothing of an oriented knot diagram is its oriented smoothing. Classical Khovanov homology is rigged so that this smoothing is at height 0, and subsequently classical Lee homology of a knot is supported in homological degree 0. This is no longer the case with doubled Lee homology: virtual knot 2.1 (given in Figure 4.4) provides an example of a knot for which the alternately colourable smoothing is, in fact, the unoriented smoothing. Taking the connect sum of 2.1 with any classical knot yields a virtual knot for which the alternately colourable smoothing is neither the oriented nor the unoriented smoothing. The height of the alternately colourable smoothing of a knot shall be used in the definition of the doubled Rasmussen invariant in Section 4.5, and is shown to be equal to the odd writhe of the knot in Section 4.5.3.

Corollary 4.4.13. *Let L be a virtual link such that any two distinct components are split. Then rank* $(DKh'(L)) = 2^{|L|+1}$.

4.4.2 Interaction with cobordisms

A cobordism between classical links defines a map on classical Lee homology; this behaviour is replicated by doubled Lee homology. Unlike the classical case, however, many connected cobordisms must be assigned the zero map, a consequence, for example, of the vanishing of $DKh'(L)$ for certain links or of the possibility of doubled Lee homology of knots being supported in non-zero homological degrees. Nevertheless, there are classes of cobordisms for which the associated maps are non-zero (some of which we use in Section 4.6). We wish to associate maps to cobordisms such that, where they are non-zero, the maps respect the filtration and send alternately coloured generators (of the homology of the intitial link) to linear combinations of alternately coloured generators (of the homology of the final link).

Recall the simple building blocks of general cobordisms.

Definition 4.4.14. Let S be a cobordism between two virtual links L_1 and L_2 which is presented by a movie consisting of exactly one virtual Reidemeister move or one oriented 0-,1-, or 2-handle addition. Such a cobordism is known as *elementary.* ◊

Of course, any cobordism can be built by gluing elementary cobordisms end to end, so we shall first investigate these simple cobordisms. In all there are ten of them: four given by the purely virtual Reidemeister moves and the mixed move, three given by the classical Reidemeister moves, and three given by the 0-, 1−, and 2-handle additions. We separate the work into elementary cobordisms which contain virtual Reidemeister moves and those which contain handle additions.

(*Virtual Reidemeister moves*): Let D_1 and D_2 be diagrams of virtual links L_1 and L_2, and S an elementary cobordism between them which contains a purely virtual Reidemeister move or mixed move. Then $CDKh'(D_1) = CDKh(D_2)$, as such moves preserve the number of cycles in a smoothing and the incoming and outgoing differentials. Thus we associate to S the map $\phi_S = \mathrm{id} : DKh'(L_1) \to DKh'(L_2)$. It is also clear that such a cobordism sends alternately coloured smoothings of D_1 to those of D_2, so that alternately coloured generators of $DKh'(L_1)$ are sent to those of $DKh'(L_1)$.

If S contains a classical Reidemeister move then ϕ_S is one of the maps defined in [Ras10, Section 6], with the addition of the appropriate ${}^{u/l}$ superscripts. We satisfy ourselves with a quick demonstration that classical Reidemeister moves send alternately coloured smoothings to alternately coloured smoothings, via proper colourings of shadows. As mentioned above, given a virtual link diagram D, the set of its alternately coloured smoothings is in bijection with the set of proper colourings of its shadow. Let D and D' be related by a classical Reidemeister move. Then D and D' are identical except within a neighbourhood of the move. Given a proper colouring of $Sh(D)$ define a proper colouring of $Sh(D')$ which is identical to that of $Sh(D)$ outside the prescribed neighbourhood; the colouring within is dictated by that of arcs incident to the neighbourhood. Some examples are given in Figure 4.10. It is clear that this defines a bijection between the proper colourings of $Sh(D)$ and those of $Sh(D')$, and it follows that the maps associated to the classical Reidemeister moves are isomorphisms on doubled Lee homology.

(*Handle additions*): Let D_1 and D_2 be diagrams of virtual links L_1 and L_2, and S an elementary cobordism between them which contains a handle addition. Then S defines a map of cubes between the cube of smoothings of D_1 and that of D_2: removing a neighbourhood of the classical crossings of D_1 and D_2, both diagrams look identical except in the region in which the handle is attached. Moreover, as handle additions do not change the number of crossings of a diagram, the smoothings of D_1 and D_2 are in bijection (a string of 0's and 1's defines uniquely a smoothing of D_1 and of D_2). Let the map of cubes defined by S be the map which sends a smoothing of D_1 to the associated smoothing of D_2. As the diagrams are identical exept in a small region this map acts simply on smoothings, and depends on the handle addition contained in S:

- 0-handle: a cycle is added which shares no crossings with any other cycle or itself.

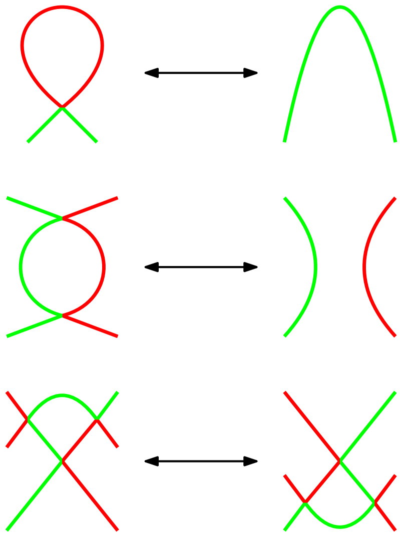


Figure 4.10: Examples of the effects of classical Reidemeister moves on proper colourings of shadows. Notice the endpoints of the arcs are coloured the same colour on the left- and right-hand sides.

- 1-handle: two cycles are merged into one cycle, one cycle is split into two, or one cycle is sent to one cycle (while the 1-handle is necessarily oriented, it is nonetheless possible for it to induce such a term as a map of cubes.)
- 2-handle: a cycle which shares no crossings with any other cycle or itself is removed.

Thus we define a map $\psi : CDKh'(D_1) \to CDKh'(D_2)$, whose affect on the specific cycle or cycles involved is as follows (and acts as the identity on the uninvolved cycles)

- 0-handle: $\iota' : \mathbb{Q} \to \mathscr{A}$ where $\iota'(1) = v_+^{\mathrm{u/l}}$, so that $\iota(1) \otimes v_+^{\mathrm{u}} = (v_{++})^{\mathrm{u}}$, for example.
- 1-handle: either m', Δ', or η' as dictated by the corresponding entry in map of cubes.
- 2-handle: $\epsilon' : \mathscr{A} \to \mathbb{Q}$ where $\epsilon'(v_+^{\mathrm{u/l}}) = 0$, $\epsilon'(v_-^{\mathrm{u/l}}) = 1$.

We define $\phi_S : DKh'(L_1) \to DKh'(L_2)$ to be the map induced by ψ. Notice that ϕ_S is filtered of degree 1 for 0- and 2-handle additions and filtered of degree -1 for 1-handle additions, and that it preserves homological degree.

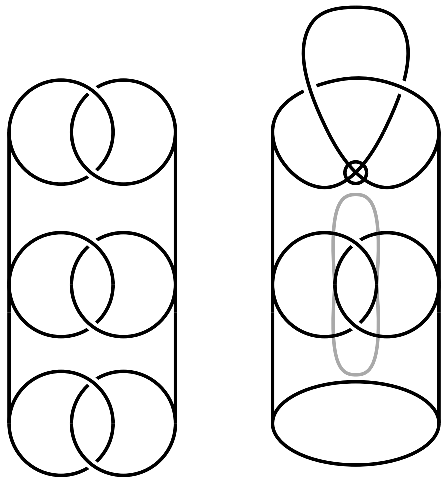


Figure 4.11: A cobordism with shared degree −2 which is assigned the zero map. The grey lines denote that the right-hand component is a twice-punctured torus: it is formed by a 1-handle between a single component followed by a 1-handle between two components. In other words, we have glued two pairs of pants along their ankles.

Definition 4.4.15. Let D_1 and D_2 be diagrams of virtual links L_1 and L_2, and S a cobordism between them. Then S can be decomposed as a finite union of elementary cobordisms, so that

$$S = S_1 \cup S_2 \cup \cdots \cup S_n$$

where S_i is an elementary cobordism. Define $\phi_S = \phi_{S_n} \circ \phi_{S_{n-1}} \circ \cdots \circ \phi_{S_1}$. ◊

It is possible that a map associated to a cobordism is necessarily zero, owing to the doubled Lee homology of a link (or links) appearing in it being trivial in particular degrees (or possibly every degree). Homological degrees which survive throughout a cobordism are important, therefore.

Definition 4.4.16. Let D_1 and D_2 be diagrams of virtual links L_1 and L_2, and S a cobordism between them such that the doubled Lee homology of every link appearing in it is non-trivial in homological degree k. Such a homological degree is known as a *shared* degree (of S). ◊

The existence of shared homological degrees is not enough to guarantee that a cobordism is assigned a non-zero map, however. Consider the cobordism depicted in Figure 4.11: the left-hand component is the identity cobordism on the classical Hopf link, while the right-hand component is a genus 1 cobordism from virtual knot 2.1 to the unknot. It can be quickly verified that -2 is a shared degree of this cobordism, but that the map assigned to it is zero.

The remainder of this section is concerned with the task of verifying that concordances and some arbitrary genus cobordisms are assigned non-zero maps, as advertised above. In what follows, a cobordism is said to be *weakly connected* if every connected component has a boundary component in the initial link.

Theorem 4.4.17. *Let S be a genus 0 cobordism between a virtual knot K and a virtual link L. Suppose that S contains no closed components and that $DKh'(L) \neq 0$. Then ϕ_S is non-zero.*

Theorem 4.4.18. *Let S_1 and S_2 be weakly connected genus 0 cobordisms which contain only virtual Reidemeister moves and 1-handles between single link components. Further, let S_i be between a virtual knot K_i and a virtual link L. Denote by $\overline{S_2}$ the reverse cobordism to S_2 i.e. it is a genus 0 cobordism from L to K_2. We may glue S_1 to $\overline{S_2}$ along L to produce a cobordism (now with non-zero genus) between K_1 and K_2; denote this cobordism by S (so that $S = S_1 \underset{L}{\cup} \overline{S_2}$). Then ϕ_S is non-zero if and only if $DKh'(L) \supseteq \mathrm{img}(\phi_{S_1}) \cap \mathrm{img}(\phi_{S_2}) \neq \emptyset$.*

A cobordism satisfying the criteria of Theorem 4.4.18 is known as a *targeted* cobordism.

We begin our path to the proofs of Theorems 4.4.17 and 4.4.18 by investigating elementary cobordisms; many maps assigned to them are non-zero automatically.

Proposition 4.4.19. *Let D_1 and D_2 be diagrams of virtual links L_1 and L_2, and S an elementary cobordism between them which is a 0- or 2-handle addition, or a 1-handle addition between two distinct link components. If L_1 has a non-zero number of alternately coloured smoothings, then S has shared degrees and ϕ_S is non-zero in them.*

Proof. We are required to verify two criteria (i): that D_2 has at least one alternately coloured smoothing at the same height as one of the alternately coloured smoothings of D_1, and (ii): that ϕ_S sends at least one alternately coloured generator of $DKh'(L_1)$ to a linear combination of those of $DKh'(L_2)$. For 0- and 2-handles (i) follows from the fact that the cycle being added or removed does not take part in any of the crossings in D_1 or D_2, and thus places no restrictions on a smoothing being alternately coloured. As a handle addition does not change the number of classical crossings it is clear that an alternately coloured smoothing of D_1 is sent to an alternately coloured smoothing of D_2 of the same height. Further, noticing that

$$\begin{aligned} \iota'(1) &= (r+g)^{\mathrm{u/l}} \\ \epsilon'(r^{\mathrm{u/l}}) &= -\frac{1}{2} \\ \epsilon'(g^{\mathrm{u/l}}) &= \frac{1}{2} \end{aligned} \tag{4.4.7}$$

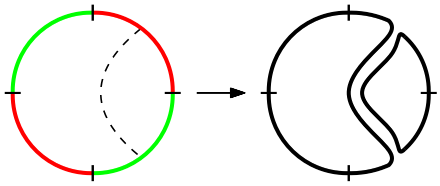
we see that (ii) is satisfied. (Note that 0-handles double the number of alternately coloured smoothings, while 2-handles halve it.)

For 1-handle additions between two distinct link components we verify (i) in the following manner: consider the Gauss diagrams $G(D_1)$ and $G(D_2)$. By assumption $G(D_1)$ contains no degenerate circles. As the 1-handle consituting S is between two distinct link components, $G(S(D_2))$ can be obtained from $G(D_1)$ by combining two circles (those corresponding to the components between which the handle is added) and adding all chord endpoints which lie on them to the new circle, leaving the other circles unchanged. Thus the number of chord endpoints lying on the new circle must be a multiple of 4 and it is not degenerate. As the other circles are unchanged it is clear that $G(D_2)$ has no degenerate circles and D_2 has alternately coloured smoothings - note that it has half the number that D_1 has, however. That there are heights at which both D_1 and D_2 have alternately coloured smoothings again follows from the fact that handle additions do not change the number of classical crossings. The statement (ii) follows from Equations (4.4.5) and (4.4.6): it is clear from the form of the components of the map that at least one alternately coloured generator of the intitial link is sent to a linear combination of those of the final link. □

In the case of 1-handles involving a single link component we are able to determine whether they preserve the existence of alternately coloured smoothings by looking at their effect on Gauss diagrams. Using this we can specify the handle additions which are associated non-zero maps.

Lemma 4.4.20. *Let D_1 and D_2 be diagrams of virtual links L_1 and L_2, and S an elementary cobordism between them which is a 1-handle addition involving a single link component. Further, assume $DKh'(L_1)$ is non-trivial. Then $DKh'(L_2)$ is trivial if and only if there is a* *proper colouring of $S(D_1)$ such that the handle addition is between two strands of opposite colour.*

Proof. It follows from Theorem 4.4.4 that $DKh'(L_2)$ is trivial if and only if D_2 has no alternately coloured smoothings. Consider the Gauss diagram of the shadow of D_1: as the handle addition is between a single link component it can be represented in the following manner:



On the left the circle of $G(D_1)$ corresponding to the component of D_1 undergoing the handle addition is depicted; the dotted line shows the location of the handle addition. Clearly, if the handle is added between two regions of opposite colour the dotted line must enclose an odd number of chord endpoints, so that the newly created circles are degenerate (as depicted on the right). Conversely, it is easy to see that if the handle is between two regions of the same colour then the newly created circles are non-degenerate. To conclude, note the regions are either both coloured the same colour in all proper colourings of $S(D_1)$ or are coloured opposite colours in all proper colourings, as all proper colourings are related by flipping the colours on a finite number of circles. □

Corollary 4.4.21. *Let D_1 and D_2 be diagrams of virtual links L_1 and L_2, and S an elementary cobordism between them which contains a 1-handle addition between a single link component. Further, assume that $DKh'(L_1)$ is non-trivial and that the 1-handle addition is between strands of the same colour in $S(D_1)$. Then S has shared degrees and ϕ_S is non-zero in them.*

We omit the proof of Corollary 4.4.21 as it uses very similar ideas to that of Proposition 4.4.19 along with Equations (4.4.5) and (4.4.6).

Using the map of cubes defined by a handle addition (see page 65) we continue to investigate the maps associated to 1-handle additions further. In what follows we shall suppress the upper/lower subscripts of the generators $\mathfrak{s}$, as it easy to see that $\mathfrak{s}^{\mathrm{u}} \in \mathrm{im}(\phi_S)$

if and only if $\mathfrak{s}^1 \in \operatorname{im}(\phi_S)$. Also, whenever we state equalities such as $\phi_S(\mathfrak{s}) = \mathfrak{s}'$, for example, we shall always mean equality up to a (non-zero) scalar.

Proposition 4.4.22. *Let D_1 and D_2 be diagrams of virtual links L_1 and L_2, and S an elementary cobordism between them which is a 1-handle addition. Further let $DKh'(L_1)$ and $DKh'(L_2)$ be non-trivial. (Recall that the smoothings of D_1 and D_2 are in bijection.) There are two cases:*

(i) if the 1-handle is between two distinct components of L_1, then every alternately coloured smoothing of D_2 is associated to an alternately coloured smoothing of D_1.

(ii) if the 1-handle involves a single component of L_1, then every alternately coloured smoothing of D_1 is associated to an alternately coloured smoothing of D_2.

(A smoothing of D_1 is associated to a smoothing of D_2 if and only if it is sent to it under the map of cubes defined by S.)

Proof. As observed in Section 4.4 the alternately coloured smoothings of a diagram are in bijection with the proper colourings of the shadow of the diagram. In case (ii) one readily observes that a proper colouring of $S(D_1)$ defines a proper colouring of $S(D_2)$ (as the handle must join two strands of the same colour, a consequence of Lemma 4.4.20). Moreover this proper colouring of $S(D_2)$ induces the same crossing resolutions as those of the proper colouring of $S(D_1)$, so that corresponding alternately coloured smoothings are associated. In case (i), notice that the reverse cobordism (from L_2 to L_1) satisfies (ii). □

Corollary 4.4.23. *Let D_1 and D_2 be diagrams of virtual links L_1 and L_2, and S an elementary cobordism between them which is a 1-handle addition with shared degrees. Then, for k a shared degree*

(i) If the handle addition is between two distinct components of L_1 then ϕ_S surjects onto $\bigoplus_i DKh'_k(L_2)$.

(ii) If the handle addition is between a single component of L_1 then for all $\mathfrak{s} \in \bigoplus_i DKh'_k(L_1)$ $\phi_S(\mathfrak{s}) \neq 0$.

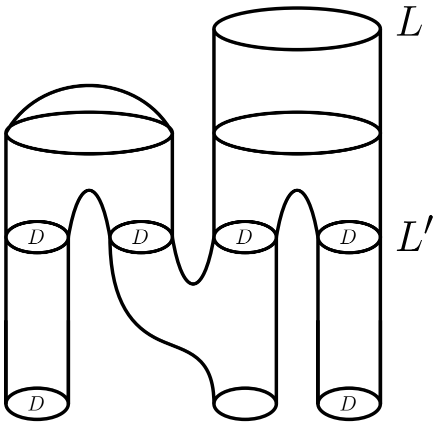


Figure 4.12: Cancelling degenerate components. The label D denotes a degenerate component.

Proof. (i): Let $\mathfrak{s}_2 \in DKh'_k(L_2)$ be defined by an alternately coloured smoothing $\mathscr{S}_2$ of D_2. Then by Proposition 4.4.22 $\mathscr{S}_2$ is associated to $\mathscr{S}_1$, an alternately coloured smoothing of D_1 (and is mapped to it under the map of cubes defined by S). Let $\mathfrak{s}_1$ denote the alternately coloured generator of $DKh'_k(L_1)$ defined by $\mathscr{S}_1$. If ϕ_S acts by either Δ' or η' on $\mathfrak{s}_1$ then $\phi_S(\mathfrak{s}_1) = \mathfrak{s}_2$ automatically (by Equations (4.4.5) and (4.4.6)). If it acts by m', then it is possible that $\phi_S(\mathfrak{s}_1) = 0$, if the cycles undergoing the merge map are coloured opposite colours. Notice that if $\mathscr{S}_2$ is obtained from $\mathscr{S}_1$ by merging two cycles, then $\mathscr{S}_1$ is obtained from $\mathscr{S}_2$ by splitting two cycles. As observed in the proof of Proposition 4.4.22, by looking at proper colourings $S(D_2)$ and $S(D_1)$ associated to $\mathscr{S}_2$ and $\mathscr{S}_1$, respectively, we see that the relevant cycles cannot be coloured opposite colours in $\mathscr{S}_1$; thus $\phi_S(\mathfrak{s}_1) = \mathfrak{s}_2$ (again by Equation (4.4.5)).

(ii): Let $\mathfrak{s} \in \bigoplus_i DKh'_k(L_1)$ be defined by the alternately coloured smoothing $\mathscr{S}$ of D_1. By Lemma 4.4.20 the handle addition must be between cycles of the same colour in $\mathscr{S}$ so that $\phi_S(\mathfrak{s}) \neq 0$ by Equations (4.4.5) and (4.4.6). □

Proof of Theorem 4.4.17. First we shall prove a fact about links appearing in genus 0 cobordisms, before using this fact and an induction argument to prove the theorem in this restricted case.

Let S be a genus 0 cobordism between a virtual knot K and a virtual link L such that $DKh'(L) \neq 0$. Assume towards a contradiction that a link, $\widetilde{L}$, appearing in S is such that $DKh'(\widetilde{L}) = 0$. By Theorems 4.4.4 and 4.4.11, $G(D)$ must contain a degenerate circle, for D any diagram of $\widetilde{L}$. Further, by Lemma 4.4.20, we see that degenerate circles are always created in pairs in a cobordism, and that degenerate circles can be cancelled against one another to produce non-degenerate circles (see Figure 4.12). This cancelling process is as follows: add a 1-handle between the components of $\widetilde{L}$ which correspond to the degenerate circles, producing a new circle. Let the two initial degenerate circles be C_1 and C_2, and N_i denote the number of chord endpoints lying on C_i. It is easy to see that the number of chord endpoints lying on the newly created circle is $N = N_1 + N_2$, and that N must be even as N_1 and N_2 are odd. Thus the newly created circle is non-degenerate.

In what follows we shall call a component of a link diagram *degenerate* if the circle corresponding to it in the associated Gauss diagram is degenerate. We may also speak of *degenerate* components of links, as virtual Reidemeister moves cannot change the mod 2 number of chord endpoints lying on a circle.

As K has non-trivial doubled Lee homology (it is a knot), no diagram of it contains a degenerate component. Therefore at least one 1-handle involving a single link component must occur in S to produce $\widetilde{L}$ (recall again Lemma 4.4.20). As L also has non-trivial doubled Lee homology, we see that we must remove all degenerate link components (by the process outlined above) in order to reach L from $\widetilde{L}$. But degenerate circles are always formed in pairs, and we see that an attempt to cancel them all against one another without introducing genus (which we are prohibited from doing as S is a of genus 0) leads to a non-compact situation; consider Figure 4.12. As we are considering only compact cobordisms we arrive at the desired contradiction.

We now present the aforementioned induction argument: we shall build up genus 0 cobordisms with elementary cobordisms. Let S' be a genus 0 cobordism between a virtual knot J and virtual link L_1 (distinct from K, L, and $\widetilde{L}$ above) such that S' contains no closed components, $DKh'(L_1) \neq 0$ and ϕ_S is non-zero. We claim that if S_e is an elementary cobordism between L_1 and L_2 such that $g(S' \cup S_e) = 0$ then $\phi_{S' \cup S_e}$ is non-zero also. Note that the argument above implies that we may restrict to the case in which $DKh'(L_2) \neq 0$: if this did not hold then $S' \cup S_e$ could not form part of a concordance between links which both have non-trivial doubled Lee homology.

If S_e is a virtual Reidemeister move or a 0-handle addition then $\phi_{S' \cup S_e}$ is non-zero as ϕ_{S_e} has trivial kernel. If S_e is a 2-handle addition then $\ker(\phi_{S_e})$ is spanned by the image of the map associated to a 0-handle addition. But if a 0-handle addition preceeds S_e then $S' \cup S_e$

would contain a closed component, which it does not by assumption, so that $\phi_{S' \cup S_e}$ is non-zero.

If S_e is a 1-handle involving a single link component we see that ϕ_{S_e} has trivial kernel by Corollary 4.4.23, as we are working in the case in which $DKh'(L_2) \neq 0$.

We are left with the case in which S_e is a 1-handle between distinct link components. If $S' \cup S_e$ is to have genus 0 the link components of L_1 involved in S_e must belong to different connected components of S'. As S' begins with J, a virtual knot, at least one of the components of S' involved in S_e must have no boundary component in J i.e. its first appearance in S' is a 0-handle. (Cutting the cobordism depicted in Figure 4.12 at the link labelled L' yields an example.)

Let $x \in \text{im}(\phi_{S'})$. We can write $x = \sum_i \mathfrak{s}_i$, where $\mathfrak{s}_i$ is an alternately coloured generator of L_1. Let $\mathscr{S}_i$ denote the alternately coloured smoothing of L_1 which defines $\mathfrak{s}_i$, and $\mathscr{C}_i$ the associated proper colouring of the shadow (of the appropriate diagram) of L_1. Then $\phi_{S' \cup S_e}(x) = 0$ if and only if the link components of L_1 involved in S_e are coloured opposite colours in every $\mathscr{C}_i$ (recall the bijection between components of a link diagram and components of its shadow given in Definition 4.4.8). This can be seen from Equation (4.4.5).

As observed above, at least one of the connected components of S' involved in S_e begins with a 0-handle, and Equation (4.4.7) shows that the image of the map assigned to a 0-handle is a linear combination of both red and green. Therefore, given an arc of $S(L_1)$ lying on a component which begins with a 0-handle, if $\mathscr{C}_i$ has the arc labelled a particular colour, there must exist a $\mathscr{C}_j$ in which the arc is coloured the opposite colour, and $\phi_{S' \cup S_e}$ is non-zero.

The base cases of the induction are the elementary cobordisms: they are all clearly of genus 0 and satisfy the induction hypothesis, under our assumption that both the initial and terminal links have non-trivial doubled Lee homology. Thus, given a genus 0 cobordism between a virtual knot and a virtual link with non-trivial doubled Lee homology, the assigned map is non-zero. □

Proof of Theorem 4.4.18. ($\Rightarrow$): Both ϕ_{S_1} and ϕ_{S_2} satisfy Theorem 4.4.17, so $\text{img}(\phi_1), \text{img}(\phi_2) \neq \emptyset$. Let $\text{img}(\phi_{S_1}) \cap \text{img}(\phi_{S_2}) \neq \emptyset$. We shall show that there is at least one element of $DKh'(K_1)$ whose image is non-zero under ϕ_S.

Pick $y \in \operatorname{img}(\phi_{S_1}) \cap \operatorname{img}(\phi_{S_2})$ with $\phi_{S_1}(x) = y$, for some $x \in DKh'(K_1)$. Further, let $x' \in DKh'(K_2)$ be such that $\phi_{S_2}(x') = y$. We claim that $\phi_S(x) = x'$ up to a non-zero scalar (in what follows we shall always mean equality up to a non-zero scalar).

As $\phi_S = \phi_{\overline{S_2}} \circ \phi_{S_1}$, we have that $\phi_S(x) = \phi_{\overline{S_2}}(y)$, and we are required to show that $\phi_{\overline{S_2}}(y) = x'$. We do this by verifying that $\phi_{\overline{S_2}} \circ \phi_{S_2}$.

Consider the following decomposition of S_2 into elementary cobordisms, $S_2 = S^e_n \cup S^e_{n-1} \cup S^e_{n-2} \cup \cdots \cup S^e_1$, for S^e_i an elementary cobordism (recall that S_2 contains only virtual Reidemeister moves and 1-handles between single link components). This induces the decomposition of $\overline{S_2}$ as $\overline{S_2} = \overline{S^e_1} \cup \overline{S^e_2} \cup S^e_3 \cup \cdots \cup \overline{S^e_n}$ (where $\overline{S^e_i}$ denotes the reverse S^e_i). From these decompositions we obtain

$$\phi_{\overline{S_2}} \circ \phi_{S_2} = \phi_{\overline{S^e_1}} \circ \cdots \circ \phi_{\overline{S^e_n}} \circ \phi_{S^e_n} \circ \cdots \circ \phi_{S^e_1}$$

which can be reduced to the identity by considering Equation (4.4.5).

($\Leftarrow$): We prove the contrapositive. Let S_1 and S_2 be as in the theorem statement. We shall show that if $\phi_S = \phi_{\overline{S_2}} \circ \phi_{S_1}$ is the zero map then $\operatorname{img}(\phi_{S_1}) \cap \operatorname{img}(\phi_{S_2}) = \emptyset$.

Let $\phi_S = \phi_{\overline{S_2}} \circ \phi_{S_1}$ be the zero map and assume towards a contradiction that there exists a $y \in \operatorname{img}(\phi_{S_1}) \cap \operatorname{img}(\phi_{S_2})$. By the argument outlined above this implies that there exist $x \in DKh'(K_1)$ and $x' \in DKh'(K_2)$ with $\phi_{S_1}(x) = y$ and $\phi_{S_2}(x') = y$, such that $\phi_S(x) = x'$. But ϕ_S is zero, and we arrive at the desired contradiction. □

Remark. In proving Theorems 4.4.17 and 4.4.18 we could not follow Rasmussen's approach of propagating orientations through the cobordism, as we no longer necessarily have the relationship between orientations of a link and its alternately coloured smoothings. Also, while all the maps associated to elementary cobordisms are non-zero (as long as the homologies do not vanish), the full map associated to S may fail to be non-zero without requiring a non-empty set of shared degrees (in the classical case every cobordism has shared degree 0). Moreover, the proof in the classical case is concerned only with this degree, while we must investigate the map associated to cobordisms in every homological degree.

4.5 A doubled Rasmussen invariant

As demonstrated in the preceding section, for an oriented virtual knot, K, $DKh'(K)$ is a rank 4 bigraded group, supported in a single homological degree which can be determined easily from any diagram of K. In Section 4.5.1 we show that the data provided by the quantum gradings in which $DKh'(K)$ is supported are equivalent to a single integer (in the classical case this integer is necessarily even), so that the information contained in $DKh'(K)$ is equivalent to a pair of integers. In Section 4.5.2 we give some properties of this pair of integers. and in Section 4.5.3 we show that one of the members of the pair is equal to the odd writhe of the given knot. Finally, in Section 4.5.4 we describe a class of knots for which the invariant can be quickly calculated.

4.5.1 Definition

We referred to a filtration of $CDKh'(K)$ in Definition 4.4.1 - let us concretise it (following Rasmussen [Ras10]). Let D be an oriented virtual knot diagram of K with n_+ positive classical crossings and n_- classical crossings. The homological grading on $CDKh'(K)$, denoted i, is as defined in Equation (4.3.4). The quantum grading is the standard one: define $p(v_+^{\mathrm{u}}) = 1, p(v_-^{\mathrm{u}}) = -1, p(v_+^{\mathrm{l}}) = 0, p(v_-^{\mathrm{l}}) = -2, p(\bigotimes x) = \sum p(x)$, then the quantum grading is $j(x) = p(x) + i(x) + n_+ - n_-$. Let $\mathcal{F}_k = \{x \in CDKh'(K) \mid j(x) \geq k\}$, so that we have the filtration

$$0 = \mathcal{F}_n \subset \mathcal{F}_{n-1} \subset \cdots \subset \mathcal{F}_m = CDKh'(K)$$

for some $n, m \in \mathbb{Z}$; let s denote the associated grading i.e. $s(x) = k$ if $x \in \mathcal{F}_k$ and $x \notin \mathcal{F}_{k+1}$.

Definition 4.5.1. For a virtual knot K let

$$\begin{aligned} s^{\mathrm{u}}_{\max}(K) &= \max\{s(x) \mid x \in DKh'(K),\ x \neq 0,\ x \in \mathcal{A}^{\otimes n}\} \\ s^{\mathrm{l}}_{\max}(K) &= \max\{s(x) \mid x \in DKh'(K),\ x \neq 0,\ x \in \mathcal{A}^{\otimes n}\{-1\}\} \end{aligned} \tag{4.5.1}$$

and similarly define $s^{\mathrm{u/l}}_{\min}(K)$. ◊

That $s^{\mathrm{u/l}}_{\max}(K)$ can be determined from $s^{\mathrm{u/l}}_{\min}(K)$ (and vice versa) follows in large part from the following augmented version of [Ras10, Lemma 3.5].

Lemma 4.5.2. *For a virtual knot K*

$$DKh'(K) = DKh'_1(K) \oplus DKh'_2(K) \oplus DKh'_3(K) \oplus DKh'_0(K)$$

where $DKh'_i(K)$ is generated by elements of quantum grading congruent to $i \mod 4$. Further

(i) Either

$$\mathfrak{s}^u \pm \bar{\mathfrak{s}}^u \in DKh'_1(K)$$
$$\mathfrak{s}^u \mp \bar{\mathfrak{s}}^u \in DKh'_3(K)$$

or

$$\mathfrak{s}^u \pm \bar{\mathfrak{s}}^u \in DKh'_0(K)$$
$$\mathfrak{s}^u \mp \bar{\mathfrak{s}}^u \in DKh'_2(K).$$

(ii) Either

$$\mathfrak{s}^u \pm \bar{\mathfrak{s}}^u \in DKh'_{1/3}(K)$$
$$\mathfrak{s}^l \pm \bar{\mathfrak{s}}^l \in DKh'_{0/2}(K)$$

or

$$\mathfrak{s}^u \pm \bar{\mathfrak{s}}^u \in DKh'_{0/2}(K)$$
$$\mathfrak{s}^l \pm \bar{\mathfrak{s}}^l \in DKh'_{3/1}(K).$$

Here $\mathfrak{s}^{u/l}$ denotes an alternately coloured generator as defined in Equation (4.4.4), and $\bar{\mathfrak{s}}^{u/l}$ denotes the generator formed by replacing r with g and g with r.

Proof. That $DKh'(K)$ decomposes into the given direct sum follows from the form of the differential: a part graded of degree 0 and other graded of degree 4 (recall Definition 4.4.1). The statements within (ii) are obvious consequence of the construction of $\mathfrak{s}^{\mathrm{u/l}}$; in particular, the fact that $j(\mathfrak{s}^{\mathrm{u}}) = j(\mathfrak{s}^{\mathrm{l}}) + 1$.

We are left with (i): the mod 4 behaviour of the quantum grading is complicated by the fact that doubled Khovanov homology is supported in both odd and even quantum gradings, a departure from the classical case. We shall prove the case when $s(\mathfrak{s}^{\mathrm{l}}) \in 2\mathbb{Z}$; this corresponds to the first statement in (i), the second follows identically modulo a grading shift.

Following Rasmussen, define $\iota : DKh'(K) \to DKh'(K)$ so that ι acts by the identity on $DKh'_0(K) \oplus DKh'_1(K)$ and by multiplication by -1 on $DKh'_2(K) \oplus DKh'_3(K)$. Next, define $\mathfrak{i} : \mathcal{A} \to \mathcal{A}$ by $\mathfrak{i}(v_+) = v_+$ and $\mathfrak{i}(v_-) = -v_-$. Then $\mathfrak{i}(r) = g$ and $\mathfrak{i}(g) = r$, and $\mathfrak{i}^{\otimes n} : \mathcal{A}\{-1\} \to \mathcal{A}\{-1\}$ acts as the identity on $DKh'_0(K)$ and by multiplication by -1 on $DKh'_2(K)$. Thus we have

$$\iota(\mathfrak{s}^{\mathrm{l}}) = \mathfrak{i}(\mathfrak{s}^{\mathrm{l}}) = \overline{\mathfrak{s}}^{\mathrm{l}}$$

which yields

$$\iota(\mathfrak{s}^{\mathrm{l}} + \overline{\mathfrak{s}}^{\mathrm{l}}) = \mathfrak{i}(\mathfrak{s}^{\mathrm{l}} + \overline{\mathfrak{s}}^{\mathrm{l}}) = \mathfrak{s}^{\mathrm{l}} + \overline{\mathfrak{s}}^{\mathrm{l}}$$
$$\iota(\mathfrak{s}^{\mathrm{l}} - \overline{\mathfrak{s}}^{\mathrm{l}}) = \mathfrak{i}(\mathfrak{s}^{\mathrm{l}} - \overline{\mathfrak{s}}^{\mathrm{l}}) = -\left(\mathfrak{s}^{\mathrm{l}} - \overline{\mathfrak{s}}^{\mathrm{l}}\right)$$

from which we deduce that $\mathfrak{s}^{\mathrm{l}} + \overline{\mathfrak{s}}^{\mathrm{l}} \in DKh'_0(K)$ and $\mathfrak{s}^{\mathrm{l}} - \overline{\mathfrak{s}}^{\mathrm{l}} \in DKh'_2(K)$. We conclude by invoking (ii). □

Corollary 4.5.3. *Let K be a virtual knot. Then*

$$s^{u/l}_{max}(K) > s^{u/l}_{min}(K).$$

Proposition 4.5.4. *Let K be a virtual knot. Then*

$$s^{u/l}_{max}(K) = s^{u/l}_{min}(K) + 2.$$

Proof. Consider the map

$$\partial : DKh'(K \sqcup \bigcirc) \to DKh'(K)$$

induced by the connect sum $K \# \bigcirc = K$ (this is well-defined as it is between K and a crossingless unknot diagram). This is well-defined, preserves homological degree, and with respect to the quantum degree is graded of degree -1 (as it is simply $\mathrm{id} \otimes m'$). Again we follow Rasmussen and denote the alternately coloured generators of $DKh'(K)$ by their decoration at the connect sum site i.e. $\mathfrak{s}^{\mathrm{u/l}}_r$ and $\mathfrak{s}^{\mathrm{u/l}}_g$. The alternately coloured generators of $DKh'(K \sqcup \bigcirc)$ are then $\mathfrak{s}^{\mathrm{u/l}}_r \otimes r^{\mathrm{u/l}}$, $\mathfrak{s}^{\mathrm{u/l}}_r \otimes g^{\mathrm{u/l}}$, $\mathfrak{s}^{\mathrm{u/l}}_g \otimes r^{\mathrm{u/l}}$, and $\mathfrak{s}^{\mathrm{u/l}}_g \otimes g^{\mathrm{u/l}}$. Under ∂ we have

$$\partial(\mathfrak{s}^{\mathrm{u/l}}_r \otimes g^{\mathrm{u/l}}) = \partial(\mathfrak{s}^{\mathrm{u/l}}_g \otimes r^{\mathrm{u/l}}) = 0$$
$$\partial(\mathfrak{s}^{\mathrm{u/l}}_r \otimes r^{\mathrm{u/l}}) = \mathfrak{s}^{\mathrm{u/l}}_r$$
$$\partial(\mathfrak{s}^{\mathrm{u/l}}_g \otimes g^{\mathrm{u/l}}) = \mathfrak{s}^{\mathrm{u/l}}_g.$$

Noticing that $s^{\mathrm{u/l}}_{\mathrm{max}}(K) = s(\mathfrak{s}^{\mathrm{u/l}}_r \pm \mathfrak{s}^{\mathrm{u/l}}_g)$ and

$$\partial((\mathfrak{s}^{\mathrm{u/l}}_r \pm \mathfrak{s}^{\mathrm{u/l}}_g) \otimes r^{\mathrm{u/l}}) = \mathfrak{s}^{\mathrm{u/l}}_r$$

we obtain

$$s((\mathfrak{s}_r^{\mathrm{u/l}} \pm \mathfrak{s}_g^{\mathrm{u/l}}) \otimes r^{\mathrm{u/l}}) \leq s(\mathfrak{s}_r^{\mathrm{u/l}}) + 1$$

$$s_{\max}^{\mathrm{u/l}}(K) - 1 \leq s_{\min}^{\mathrm{u/l}}(K) + 1$$

as ∂ is graded of degree -1 (that $s_{\min}^{\mathrm{u/l}}(K) = s(\mathfrak{s}_r^{\mathrm{u/l}})$ follows from Lemma 4.5.2). □

Thus any of the four quantities defined in Definition 4.5.1 determines all of the others and we able to make the following definition.

Definition 4.5.5. For a virtual knot K let $\mathbb{s}(K) = (s_1(K), s_2(K)) \in \mathbb{Z} \times \mathbb{Z}$ where

$$s_1(K) = s_{\max}^{\mathrm{l}}(K)$$

$$s_2(K) = i(\mathfrak{s}^{\mathrm{u/l}}) = |\mathscr{S}|$$

where i denotes homological grading and $\mathfrak{s}^{\mathrm{u/l}}$ an alternately coloured generator of K associated to the alternately coloured smoothing $\mathscr{S}$. We refer to $\mathbb{s}(K)$ as the *doubled Rasmussen invariant* of K. ◊

4.5.2 Properties

Proposition 4.5.6. *For a classical knot K $\mathbb{s}(K) = (s(K), 0)$, where $s(K)$ denotes the classical Rasmussen invariant.*

Proof. For K a classical knot $DKh'(K)$ decomposes as

$$DKh'(K) = Kh'(K) \oplus Kh'(K)\{-1\}$$

so that clearly $s_{\max}^{\mathrm{u}} = s_{\max}(K)$, where $s_{\max}(K)$ denotes the classical quantity. Then

$$\begin{aligned} s(K) &= s_{\max}(K) - 1 \\ &= s_{\max}^{\mathrm{u}}(K) - 1 \\ &= s_{\max}^{\mathrm{l}}(K). \end{aligned}$$

That $s_2(K) = 0$ is observed on page 64. □

The doubled Rasmussen invariant exhibits the same behaviour with respect to mirror image and connect sum as its classical counterpart.

Proposition 4.5.7. *Let K be a virtual knot $\overline{K}$ denote its mirror image. Then $\mathbb{s}(K) = -\mathbb{s}(\overline{K})$.*

Proof. The statement $s_1(K) = -s_1(\overline{K})$ follows, as in the classical case, from the existence of the isomorphism of dual complexes

$$r : (\mathcal{A} \oplus \mathcal{A}\{-1\}, m', \Delta', \eta') \to ((\mathcal{A} \oplus \mathcal{A}\{-1\})^*, \Delta'^*, m'^*, \eta'^*).$$

That $s_2(K) = -s_2(\overline{K})$ is seen as follows: let D be a diagram of K with n_+ positive classical crossings and n_- negative classical crossings. Let $\mathscr{S}$ be the alternately colourable smoothing of D, so that $s_2(K) = |\mathscr{S}|$, the height of $\mathscr{S}$. Further, notice that

$$\begin{aligned} |\mathscr{S}| &= n_p^u + n_n^o - n_- \\ &= n_p^u + n_n^o - (n_n^u + n_n^o) \\ &= n_p^u - n_n^u \end{aligned}$$

where

n_p^u = the number of positive crossings resolved into their unoriented smoothing

n_p^o = the number of positive crossings resolved into their oriented smoothing

and likewise n_n^u and n_n^o (for a classical knot $n_p^u = n_n^u = 0$, of course). It is quickly observed that

$$\begin{aligned} \overline{n}_n^u &= n_p^u \\ \overline{n}_p^o &= n_n^o \end{aligned}$$

where $\overline{n}_*^*$ denote the corresponding quantities for $\overline{D}$. Then

$$\begin{aligned} |\overline{\mathscr{S}}| &= \overline{n}_p^u - \overline{n}_n^u \\ &= \overline{n}_+ - \overline{n}_p^o - n_p^u \\ &= n_- - n_n^o - n_p^u \\ &= n_n^u - n_p^u \\ &= -|\mathscr{S}|. \end{aligned}$$

□

Proposition 4.5.8. *Let K_1 and K_2 be virtual knots and denote by $K_1 \# K_2$ any of their connect sums. Then*

$$\mathbb{s}(K_1 \# K_2) = \mathbb{s}(K_1) + \mathbb{s}(K_2).$$

Proof. It is readily apparent that $\mathscr{S} = \mathscr{S}_1 \sqcup \mathscr{S}_2$, where $\mathscr{S}$ / $\mathscr{S}_1$ / $\mathscr{S}_2$ is the alternately colourable smoothing of $K_1 \# K_2$ / K_1 / K_2. Then $|\mathscr{S}| = |\mathscr{S}_1| + |\mathscr{S}_2|$, which proves the claim regarding $s_2(K_1 \# K_2)$.

Let $\varrho : DKh'(K_1 \# K_2) \longrightarrow DKh'(K_1 \sqcup K_2)$ be the map realised by applying Δ' to the appropriate tensorand dictated by the connect sum. Regarding $s_1(K_1 \# K_2)$, the proof follows in identical fashion to the classical proof when one notices that we only require the existence of ϱ (as opposed to the short exact sequence used in [Ras10]). □

4.5.3 Relationship with the odd writhe

Kauffman defined the odd writhe of a virtual knot in terms of Gauss diagrams [Kau04]. In this section we show that the doubled Rasmussen invariant contains the odd writhe.

Definition 4.5.9. Let D be a diagram of a virtual knot and $G(D)$ its Gauss diagram. A classical crossing of D, associated to the chord labelled c in $G(D)$, is known as *odd* if the number of chord endpoints appearing between the two endpoints of c is odd. Otherwise it is known as *even*. The *odd writhe* of D is defined

$$J(D) = \sum_{\text{odd crossings of } D} \text{sign of the crossing}.$$

◊

Theorem 4.5.10. *Let D be a virtual knot diagram of K. The odd writhe is an invariant of K and we define*

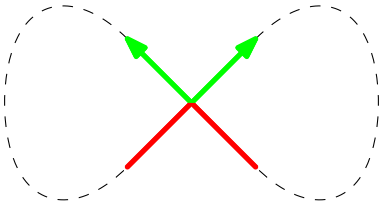
$$J(K) \coloneqq J(D).$$

The odd writhe of a virtual knot K provides a quick way to calculate $s_2(K)$.

Proposition 4.5.11. *Let be D a diagram of a virtual knot K. Then $s_2(K) = J(K)$.*

Proof. We claim that a classical crossing in D is odd if and only if it is in its unoriented resolution in the alternately colourable smoothing of D.

($\Rightarrow$): Let c denote an odd classical crossing of D. Leaving the crossing from either of the outgoing arcs we must return to a specified incoming arc. Between leaving and returning we have passed through an odd number of classical crossings (which are not c). Thus the incoming arc must be coloured the opposite colour to the outgoing, and c is resolved into its unoriented resolution in the both of the alternately coloured smoothings of D, as depicted here:



($\Leftarrow$): Let c denote a classical crossing of D which is resolved into its unoriented smoothing in the alternately colourable smoothing of D. The colouring at c must be as depicted above. Again, leaving c from either outgoing arc and returning at the specified incoming arc, we see that, as the colours of the arcs are opposite, an odd number of classical crossings must have been passed.

The contributions of odd and even crossings to $J(K)$ and $s_2(K)$ are summarised in the following table, from which the result follows. The contributions to $s_2(K)$ are clear when one recalls that the height of a smoothing contains the shift $-n_-$, the total number of negative classical crossings of D.

sign	parity	reso.	$J(K)$	$s_2(K)$
$+$	odd	1	$+1$	$+1$
$+$	even	0	0	0
$-$	odd	0	-1	-1
$-$	even	1	0	0

□

Corollary 4.5.12. *Let K_1 and K_2 be virtual knots and $K_1 \# K_2$ denote any of their connect sums. Then*

$$J(K_1 \# K_2) = J(K_1) + J(K_2).$$

4.5.4 Leftmost knots and quick calculations

To conclude this section we identity a class of knots for which the calculation of the doubled Rasmussen invariant is trivial, a generalisation of the case of computation of the classical Rasmussen invariant of positive classical knots. The key here, as in the classical case, is that the alternately coloured smoothings of the class of knots in question have no incoming differentials.

Definition 4.5.13. Let D be a virtual knot diagram. We say that D is *leftmost* if it contains only positive even and negative odd classical crossings. A virtual knot is leftmost if it has a leftmost diagram. ◊

Proposition 4.5.14. *Let D be a leftmost diagram of a virtual knot K with n_- negative classical crossings. Then $s_2(K) = -n_-$, the minimal non-trivial homological grading of $DKh'(K)$.*

Proof. Let D be a leftmost diagram of a virtual knot. By Proposition 4.5.11 we have $s_2(K) = J(K) = -n_-$, as a crossing in D is odd if and only if it is negative. □

Proposition 4.5.15. *Let D be a leftmost diagram of a virtual knot K. Then $s_1(K) = max\{s(\mathfrak{s} + \bar{\mathfrak{s}}), s(\mathfrak{s} - \bar{\mathfrak{s}})\}$, where $\mathfrak{s}$ is an alternately coloured generator associated to the alternately colourable smoothing of D.*

Proof. By Proposition 4.5.14 the alternately colourable smoothing of D is at the minimal non-trivial height of the cube of resolutions. By construction there is only one smoothing at this height. Further, this smoothing has no incoming differentials. Recalling Definition 4.5.5, we obtain the result. □

4.6 Applications

We shall now describe some applications of the invariants $DKh(L)$ and $\mathbb{s}(K)$. All of the given applications are related to virtual link concordance, to a greater or lesser extent.

4.6.1 Cobordism obstructions

As mentioned in Section 4.4.2, we can use the information contained in the quantum degree of $DKh'(L)$ to obtain obstructions to the existence of cobordisms between L and other links. First we repeat the procedure used to show that the classical Rasmussen invariant yields a bound on the slice genus to obtain a bound on the genus of a certain class of cobordisms from a knot to the unknot, and between two given knots. We then obtain an obstruction to the existence of a genus 0 cobordism between a link and a given knot. Finally, we use doubled Lee homology to show that virtual knots with non-zero odd writhe are not slice.

Genus bounds

In this section we use the fact that concordances and targeted cobordisms are assigned non-zero maps to obtain obstructions to the existence of cobordisms of certain genera between pairs of virtual knots. First we obtain a lower bound on the genus of targeted cobordisms between pairs of knots whose s_2 invariants agree (the definition of a targeted coboridism is given in Theorem 4.4.18).

Theorem 4.6.1. *Let K be a virtual knot with $s_2(K) = 0$ and S a targeted cobordism from K to the unknot such that 0 is a shared degree of S. Then*

$$\frac{|s_1(K)|}{2} \leq g(S). \tag{4.6.1}$$

Proof. Let K and S be as in the theorem statement. Then, by Theorem 4.4.18, ϕ_S is a non-zero map. As in the classical case, it is easy to see that ϕ_S is filtered of degree $-2g(S)$. Let $x \in DKh'(K)$ realise $s^{\mathrm{u}}_{\max}(K)$ so that

$$1 \geq s(\phi_S(x)) \geq s^{\mathrm{u}}_{\max}(K) - 2g(S)$$

as $s^{\mathrm{u}}_{\max}(\bigcirc) = 1$. This yields

$$2g(S) + 1 \geq s^{\mathrm{u}}_{\max}(K)$$

$$2g(S) \geq s_1(K).$$

Repeating the argument for $\overline{K}$, and using Proposition 4.5.7, we obtain

$$-2g(S) \leq s_1(K)$$

which yields the desired result. □

Corollary 4.6.2. *Let K be a virtual knot with $s_2(K) = 0$ and S a targeted cobordism from K to the unknot such that $2g(S) \leq |s_1(K)|$. Then there exists a link L which appears in S with $DKh'_0(L) = 0$.*

In a very similar manner we able to show the following.

Theorem 4.6.3. *Let K_1 and K_2 be a pair of virtual knots with $s_2(K_1) = s_2(K_2)$, and S be a targeted cobordism between them such that $s_2(K)$ is a shared homological degree of S. Then*

$$\frac{|s_1(K_1) - s_1(K_2)|}{2} \leq g(S).$$

Further, concordances between virtual knots are obstructed by the quantum degree component of the doubled Rasmussen invariant, s_1 (in Section 4.6.1 we show that the homological component is such an obstruction, also).

Theorem 4.6.4. *Let K and K' be virtual knots such that $s_2(K) = s_2(K')$. If $s_1(K) \neq s_1(K')$ then K and K' are not concordant.*

The proof of Theorem 4.6.4 follows almost exactly along the lines of that of Theorem 4.6.1, which itself is very similar to the classical case; all we require is that the map assigned to a concordance is non-zero, which is verified in Theorem 4.4.17.

Corollary 4.6.5. *Let K be a virtual knot with $s_2(K) = 0$. If $s_1(K) \neq 0$ then K is not slice.*

Obstructions to genus 0 cobordisms between knots and links

We can extend Theorem 4.6.4 to the case in which one end of the genus 0 cobordism is a link, provided the homologies of the knot and link in question are compatible, and the genus 0 cobordism is connected.

Theorem 4.6.6. *Let L be a virtual link of $|L|$ components. Further, let S be a connected genus 0 cobordism between L and a virtual knot K such that $DKh'_{s_2(K)}(L) \neq 0$. Let $M(L)$ be the maximum non-trivial quantum degree of elements $x \in DKh'(L)$ such that $\phi_S(x) \neq 0$. Then*

$$M(L) \leq s_1(K) + |L|.$$

Proof. Let L, K, and S be as in the theorem statement. Then ϕ_S is non-zero by Theorem 4.4.17. It is clear that ϕ_S is filtered of degree $-(|L| - 1)$: a minimum of $|L| - 1$ 1-handles are needed to take a $|L|$-component link to a knot, and any surplus 1-handles must be paired with 2-handles. It is also clear that if $x \in DKh'(L)$ is such that $\phi_S(x) \neq 0$ then $x \in DKh'_{s_2(K)}(L)$. For such an x we have that $s(x) \geq M(L)$ and

$$M(L) - (|L| - 1) \leq s(x) - (|L| - 1) \leq s(\phi_S(x)) \leq s^{\mathrm{u}}_{\max}(K)$$

so that

$$M(L) - |L| + 1 \leq s_1(K) + 1$$

as required. □

Corollary 4.6.7. *Let L be a virtual link of $|L|$ components such that $DKh'(L) \neq 0$. Further, let K a virtual knot such that $DKh'(L)$ is trivial in homological degree $s_2(K)$ or*

$$M(L) \geq s_1(K) + |L|.$$

Then any genus 0 cobordism from L to K is disconnected.

A particular consequence of Corollary 4.6.7 is that, given a virtual link L for which $DKh'(L) \neq 0$ and $DKh'_0(L) = 0$, all genus 0 cobordisms from L to classical knots must be disconnected: no classical knots can be obtained from L by simply merging its components.

The odd writhe is an obstruction to sliceness

The odd writhe of a knot is very easy to calculate. Despite this it can detect non-classicality (and hence non-triviality) and chirality of many virtual knots [Kau04]. Here we show that it also contains information regarding the concordance class of a virtual knot.

Theorem 4.6.8. *Let K be a virtual knot. If $J(K) \neq 0$ then K is not slice.*

Proof. We prove the contrapositive. Assume towards a contradiction that K is a slice virtual knot such that $J(K) \neq 0$. Then $s_2(K) \neq 0$ by Proposition 4.5.11. Let S realise a slice disc so that ϕ_S is non-zero by Theorem 4.4.17. Recall that ϕ_S preserves homological degree by construction. There must exist $x \in DKh'_{s_2(K)}(K)$ such that $\phi_S(x) \neq 0$. But then

$$\phi_S(x) \neq 0 \in DKh'_{s_2(K)}(\bigcirc) = 0$$

as $s_2(K) \neq 0$, a contradiction. □

The proof of Theorem 4.6.8 can be used *mutatis mutandis* to show that the set of concordance classes of virtual knots is partitioned by the odd writhe.

Theorem 4.6.9. *Let K_1 and K_2 be virtual knots. If $J(K_1) \neq J(K_2)$ then K_1 and K_2 are not concordant.*

Corollary 4.6.10. *Let K be a virtual knot. If $J(K) \neq 0$ then K is not concordant to a classical knot.*

Examples

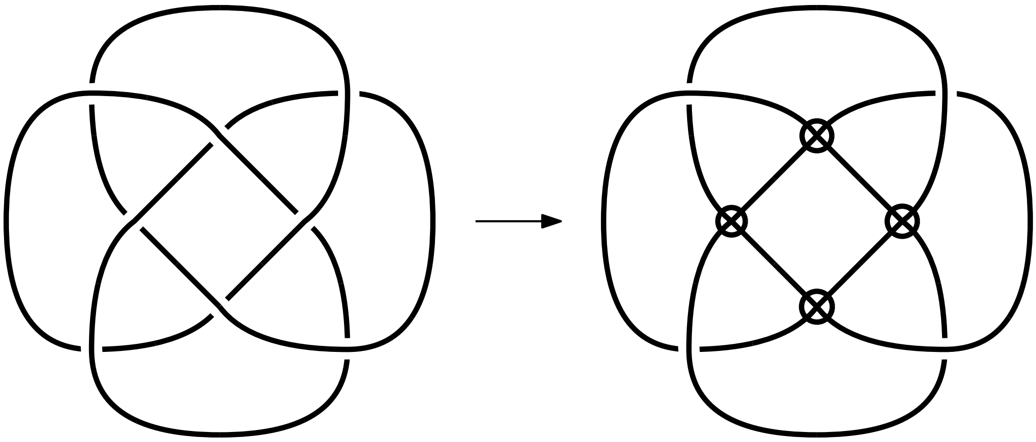
Consider the classical knot $T(4,3)$, as given in Figure 4.13. By converting a particular subset of its crossings to virtual crossings we are able to produce a virtual knot, K, whose alternately colourable smoothing is its oriented smoothing (K is also positive, as $T(4,3)$ is). Thus $s_2(K) = 0$ and the odd writhe provides no obstruction to sliceness. However, K is a leftmost knot (as defined in Section 4.5.4), so that $s_1(K) = \max\{s(\mathfrak{s}^1 + \overline{\mathfrak{s}}^1), s(\mathfrak{s}^1 - \overline{\mathfrak{s}}^1)\}$ by Proposition 4.5.15. It can be quickly verified that $\max\left(s(\mathfrak{s}^1 + \overline{\mathfrak{s}}^1), s(\mathfrak{s}^1 - \overline{\mathfrak{s}}^1)\right) = 1$ so that K is not slice by Corollary 4.6.5.

Further, consider the classical two-component link 9^2_{61}, as depicted in Figure 4.14, and let L denote the virtual link on the right of the figure. By an argument identical to that used in the case of leftmost knots we can show that the maxiumum quantum degree of all elements in $DKh'_0(L)$ is 5. In the context of Theorem 4.6.6, considering connected concordances from L to the unknot, $M(L) = 5$ and as $|L| = 2$, $s_1(\bigcirc) = 0$, it follows that there does not exist a connected genus 0 cobordism from L to the unknot.

The method used in both the above examples can be applied to many positive oriented classical link diagrams in order to produce virtual link diagrams for which the quantum degree information (at particular homological degrees) is easy to compute.

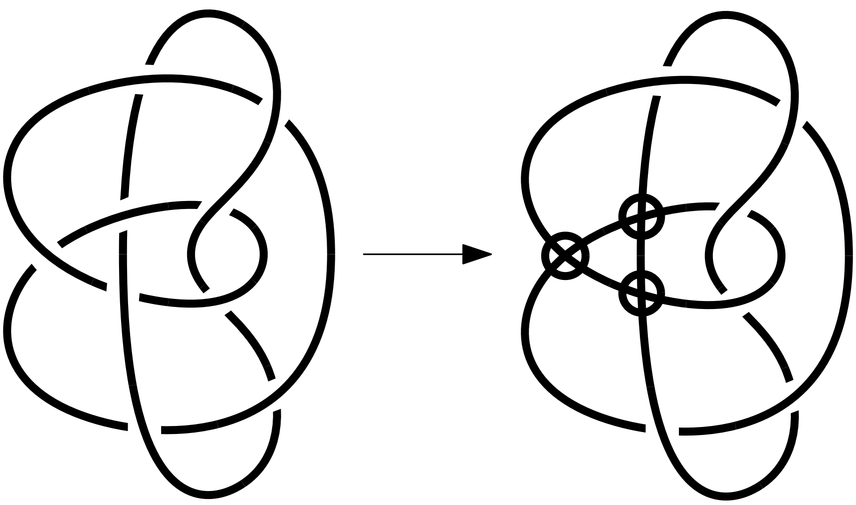
4.6.2 Connect sums of trivial diagrams

As discussed in Section 3.1.3, the connect sum of virtual knot diagrams is not well-defined: the result of a connect sum depends on both diagrams used and on the site



$T(4,3)$ K

Figure 4.13: The classical torus knot $T(4,3)$, on the left, and a virtual knot K formed by converting a subset of its crossings to virtual crossings, on the right.



9^2_{61} L

Figure 4.14: The classical link 9^2_{61}, on the left, and a virtual link L formed by converting a subset of its crossings to virtual crossings, on the right.

at which it is conducted. As a consequence, there exist non-trivial virtual knots which can be represented by a connect sum of trivial diagrams. Doubled Khovanov homology yields a condition met by such virtual knots.

Theorem 4.6.11. *Let K be a virtual knot which is a connect sum of two trivial knots. Then $DKh(K) = DKh(\bigcirc)$.*

In order prove to Theorem 4.6.11 we shall define a reduction of doubled Khovanov homology, in direct analogy to the classical case [Kho99; Shu11b].

Definition 4.6.12 (Reduced doubled Khovanov homology)**.** Let L be an oriented virtual link diagram with a marked point on one component (away from the crossings of L). Distribute the marked point across the cube of smoothings so that each smoothing of L contains one marked point. Define $C(L)$ to be the chain subcomplex of $CDKh(L)$ spanned by those states in which all the marked cycles are decorated with either v^{u}_{-} or v^{l}_{-}. That C is a subcomplex is evident from Equations (4.3.2) and (4.3.3) (it is also graded).

Let $\mathcal{H}(L)$ denote the homology of $C(L)$. We refer to $\mathcal{H}(L)$ as the *reduced doubled Khovanov homology* of L. ◊

The proof of invariance of $\mathcal{H}(L)$ under virtual Reidemeister moves follows as in the classical case. There is a dependence of $\mathcal{H}(L)$ on the choice of marked point, but as we need only consider knots in what follows, we need not take this into account.

Lemma 4.6.13. *Let L be a virtual link diagram. Then $CDKh(L)/C(L) \cong C(L)\{2\}$.*

Proof. We prove the statement for a virtual knot diagram K (link diagrams follow essentially identically). Let $CDKh(K)/C(K) = C'(K)$. The isomorphism $g : C'(K) \longrightarrow C(K)$ is straightforward to define. Given a representative, x, of an element of $C'(K)$ the marked cycle must be decorated with either v^{u}_{+} or v^{l}_{+} i.e. we must have $x = x_1 \otimes x_2 \otimes \ldots \otimes v^{\mathrm{u/l}}_{+} \otimes \ldots x_n$. Define

$$g(x_1 \otimes x_2 \otimes \ldots \otimes v^{\mathrm{u/l}}_{+} \otimes \ldots x_n) = x_1 \otimes x_2 \otimes \ldots \otimes v^{\mathrm{u/l}}_{-} \otimes \ldots x_n$$
$$g^{-1}(x_1 \otimes x_2 \otimes \ldots \otimes v^{\mathrm{u/l}}_{-} \otimes \ldots x_n) =_1 \otimes x_2 \otimes \ldots \otimes v^{\mathrm{u/l}}_{+} \otimes \ldots x_n.$$

That g is well defined is clear and that it is a chain map is apparent when one considers the schematic given in Figure 4.15: the only issue that could arise is due to the factor of

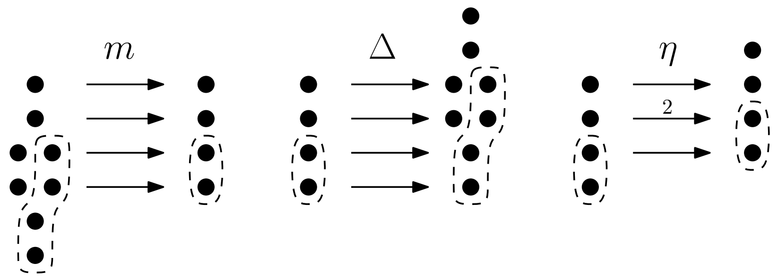


Figure 4.15: A schematic for the interaction between the map g and the differential. The enclosed dots depict generators of $C(K)$; g sends a dot outside an enclosure to the corresponding dot inside.

2 in the η map, the position of which ensures that it does not cause any trouble. That the degree of g is -2 is obvious. □

Proof of Theorem 4.6.11 . Let K be as in the proposition. By an abuse of notation let $K = D_1 \# D_2$ be the diagram which is the result of a connect sum between D_1 and D_2, both of which are unknot diagrams. We are free to pick marked points on the diagrams K and $D_1 \sqcup D_2$ so that the situation is as in Figure 4.16, from which we observe that there is a chain complex isomorphism from $f : C(K) \to C(D_1 \sqcup D_2)$. The isomorphism is defined as follows

$$f(x_1 \otimes x_2 \otimes \ldots \otimes v_-^{\mathrm{u/l}} \otimes \ldots \otimes x_n) = x_1 \otimes x_2 \otimes \ldots \otimes v_-^{\mathrm{u/l}} \otimes v_-^{\mathrm{u/l}} \otimes \ldots \otimes x_n$$
$$f^{-1}(x_1 \otimes x_2 \otimes \ldots \otimes v_-^{\mathrm{u/l}} \otimes v_-^{\mathrm{u/l}} \otimes \ldots \otimes x_n) = x_1 \otimes x_2 \otimes \ldots \otimes v_-^{\mathrm{u/l}} \otimes \ldots \otimes x_n$$

where $v_-^{\mathrm{u/l}}$ and $v_-^{\mathrm{u/l}} \otimes v_-^{\mathrm{u/l}}$ decorate the marked cycles. That f is a chain map follows from the observation that if $\mathfrak{s}^{\mathrm{u/l}}$ is a state of $C(K)$ then $f(\mathfrak{s}^{\mathrm{u/l}})$ has the same incoming and outgoing differentials. It is clear that f is graded of degree -1.

We have established the isomorphism $C(K) \cong C(D_1 \sqcup D_2)\{1\}$; further, there is a chain homotopy equivalence between $C(D_1 \sqcup D_2)$ and $C\left(\dot{\bigcirc}\dot{\bigcirc}\right)$ as D_1 and D_2 are unknot diagrams. It is easy to see that $C\left(\dot{\bigcirc}\dot{\bigcirc}\right) = C\left(\dot{\bigcirc}\right)\{-1\}$ so that

$$C(K) \cong C(D_1 \sqcup D_2)\{1\} \simeq C\left(\dot{\bigcirc}\dot{\bigcirc}\right)\{1\} \simeq \left(C\left(\dot{\bigcirc}\right)\{-1\}\right)\{1\} = C\left(\dot{\bigcirc}\right)$$

and

$$\mathcal{H}(K) = \mathcal{H}\left(\dot{\bigcirc}\right). \tag{4.6.2}$$

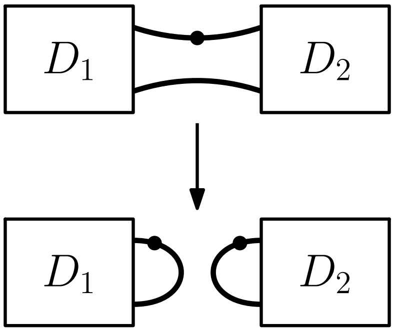


Figure 4.16: Marked diagrams of K (above) and $D_1 \sqcup D_2$ (below).

In addition, there is an exact triangle

$$\begin{tikzcd} \mathcal{H}(K) \arrow[rr] & & DKh(K) \arrow[dl] \\ & \mathcal{H}(K)\{2\} \arrow[ul] & \end{tikzcd} \tag{4.6.3}$$

which is arrived at via the short exact sequence

$$0 \longrightarrow C(K) \longrightarrow CDKh(K) \longrightarrow CDKh(K)/C(K) \longrightarrow 0,$$

Lemma 4.6.13 and the observation that Equation (4.6.2) implies that $\mathcal{H}(K)$ is supported in homological degree 0. Also by Equation (4.6.2) we obtain $\operatorname{rank}(\mathcal{H}(K)) = 2$ so that the triangle splits and

$$DKh(K) = \mathcal{H}(K) \oplus \mathcal{H}(K)\{2\} = \mathcal{H}(\bigcirc) \oplus \mathcal{H}(\bigcirc)\{2\} = DKh(\bigcirc).$$

□

Proposition 4.6.14. *Let K and K' be virtual knots which are connect sums of the same pair of initial virtual knots J and J': that is, there exist diagrams D_1 and D_2 of J and D_3 and D_4 of J' such that $K = D_1 \# D_3$ and $K' = D_2 \# D_4$. Then $C(K) \simeq C(K')$.*

Proof. We have $C(K) \cong C(D_1 \sqcup D_3) \simeq C(D_2 \sqcup D_4) \cong C(K')$, as $D_1 \sqcup D_3$ and $D_2 \sqcup D_4$ are both diagrams of $J \sqcup J'$ and the isomorphisms are essentially identical to that given in the proof of Theorem 4.6.11. □

Remark. Of course, there is still a pair of short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{C}(K) \arrow[r] \arrow[d, "\wr"] & CDKh(K) \arrow[r] & \mathcal{C}(K)\{2\} \arrow[r] \arrow[d, "\wr"] & 0, \\ 0 \arrow[r] & \mathcal{C}(K') \arrow[r] & CDKh(K') \arrow[r] & \mathcal{C}(K')\{2\} \arrow[r] & 0, \end{tikzcd}$$

but the associated long exact sequences no longer split. Indeed, it is not true in general that

$$DKh(K) = \mathcal{H}(K) \oplus \mathcal{H}(K)\{2\},$$

the aforementioned virtual knot 2.1 provides a counterexample.

Chapter 5

Computation and estimation of the slice genus of virtual knots

In this chapter we construct bounds on the virtual and doubled Rasmussen invariants, and identify classes of virtual knots for which these bounds are minimised. In contrast to the invariants these bounds are easily computable from diagrams. We use the bounds to compute or estimate the slice genus of every virtual knot of four classical crossings or less. In addition, we also compute or estimate the slice genus of 45 virtual knots of 5 or 6 classical crossings whose slice status is undetermined by Boden, Chrisman, and Gaudrea [BCG17b].

The chapter is organised as follows. In Section 5.0.1 we outline the *strong slice-Bennequin* bounds, which we shall generalise to the virtual and doubled Rasmussen invariants. We also identify, in Section 5.0.2, a class of virtual knots for which the two extensions of the Rasmussen invariant are equal.

Next, in Section 5.1, we produce canonical chain-level generators of MDKK′ homology (as defined in Chapter 3). This is done by simplifying the decorated diagrammatic generators defined in Section 3.3, so that elements of the algebraic chain complex may be read off from them. These canonical generators are required in Section 5.2, in which we construct the strong slice-Bennequin bounds on both the virtual and the doubled Rasmussen invariants. In this we follow much the same path as Lobb [Lob11]; in fact, in the case of the virtual Rasmussen invariant, we recover formulae identical to his. In the case of the doubled Rasmussen invariant, however, the formulae arrived at are substantially different, a consequence of the structural differences between doubled Khovanov homology and its classical predecessor.

Finally, in Section 5.3, we use the tools we have developed to compute or estimate the slice genus of the virtual knots, as described above. The computations and estimations are made as follows. Let D be the diagram of a virtual knot K given, then:

(i) Compute the generalised strong slice-Bennequin bounds using D.

(ii) Take the greatest of the lower bounds on $g^*(K)$ provided by the resulting estimations or computations of the virtual and doubled Rasmussen invariants.

(iii) Attempt to find a cobordism from D to the unknot of genus equal to the greatest lower bound on $g^*(K)$, thus computing $g^*(K)$.

(iv) Failing that, find a cobordism of higher genus so that a region in which $g^*(K)$ lies is identified.

5.0.1 The slice-Bennequin bounds

The Rasmussen invariant of a classical knot extracts geometric information from Khovanov homology, yielding a lower bound on the slice genus. Given a classical knot K it is, in principle, difficult to compute its classical Rasmussen invariant, as it is equivalent to the maximal filtration grading of all elements homologous to a certain generator of the Lee homology of K.

Kawamura [Kaw15] and Lobb [Lob11] independently defined diagram-dependent upper bounds on $s(K)$, denoted $U(D)$ (for D a diagram of K), which are easily computable by hand, along with an error term, $\Delta(D)$, the vanishing of which implies that $s(K) = U(D)$. Precisely,

$$U(D) - 2\Delta(D) \leq s(K) \leq U(D).$$

The bounds $U(D)$ are henceforth referred to as the *strong slice-Bennequin bounds*; in Section 5.2 we construct analogous bounds on the virtual and doubled Rasmussen invariants.

5.0.2 Even knots

As we are interested in producing bounds which allow for easier computation or estimation of the virtual and doubled Rasmussen invariants, it is useful to identify a class of virtual knots on which the two invariants agree.

Recall the definitions of even and odd classical crossings given in Section 4.5.3 (on page 81); a virtual diagram is *even* if it possesses only even crossings (*odd* diagrams are defined similarly), and a virtual knot is *even* if it possesses an even diagram. Here we prove a fact about the cube of resolutions associated to even virtual knot diagrams.

Proposition 5.0.1. *Let D be an even virtual knot diagram. Then $vCKh(D)$ and $CDKh(D)$ contain no η maps.*

Proof. As D is even it possesses a global source-sink orientation i.e. applying the source-sink decoration does not yield any cut loci (as defined in Definition 3.3.2). (In fact, possessing a global source-sink structure is equivalent to being even, but here we only need one direction.) To see this, orient D with either of its orientations (the usual notion of orientation, not source-sink), and consider leaving a classical crossing of D and returning to the arc proscribed by the usual orientation. One sees in Figure 3.10 (on page 30) that passing through a classical crossing reverses the source-sink orientation. As all classical crossings of D are even, one passes through an even number of crossings between leaving and returning at the proscribed arc. Thus the source-sink orientation has been reversed an even number of times, yielding no overall change. This argument can be applied at every crossing to show that D has a global source-sink orientation.

Next, notice that every smoothing of D inherits an orientation from the global source-sink orientation of D: looking again at Figure 3.10 one sees that both resolutions of the classical crossing inherit an orientation from the source-sink decoration. That the orientation inherited is consistent between distinct classical crossings of D follows from that fact that D has no cut loci.

Finally, we notice that if every smoothing of D inherits a coherent orientation from the global source-sink orientation of D then every circle within a smoothing must look as in the left or center of Figure 5.1, as the configuration on the right is prohibited for reasons of (source-sink) orientation. But we see that the configurations on the left and center

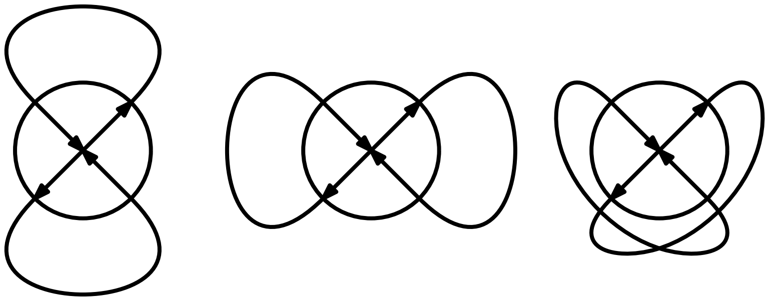


Figure 5.1: Configurations of circles within a smoothing of a diagram possessing a global source-sink orientation. Two possible configurations are at the left and center, while an impossible configuration is at the right.

correspond to either a merge or a split, while the configuration on the right corresponds to the single-cycle smoothing. Thus no single-cycle smoothings can occur in the cube of resolutions of D and we arrive at the desired result. □

Corollary 5.0.2. *Let K be an even virtual knot. Then $DKh(K) = vKh(K) \oplus vKh(K)\{-1\}$ so that $s(K) = s_1(K)$.*

Proof. Let D be an even diagram of K. Then both $vCKh(D)$ and $CDKh(D)$ contain no η maps by Proposition 5.0.1. As m and Δ do not map between the shifted and unshifted summands of $CDKh(D)$, the complex splits as the direct sum $CDKh(D) = vCKh(D) \oplus vCKh(D)\{-1\}$. □

5.1 Chain-level generators of vKh'

As outlined in Section 3.3, Dye, Kaestner, and Kauffman use diagram-level generators of vKh' to define and investigate the virtual Rasmussen invariant i.e. such generators are alternately coloured smoothings of chequerboard-coloured abstract link diagrams with cross cuts. These generators are sufficient to define the virtual Rasmussen invariant, but there is not clear way to push them to algebra as they may contain circles which possess more than one colour.

Below, we give a method to produce the corresponding chain-level generators of vKh', which allows us to generalise the strong slice-Bennequin bounds. Before doing so, however, it is instructive to recall the bijection of Theorem 3.3.10 between orientations of a

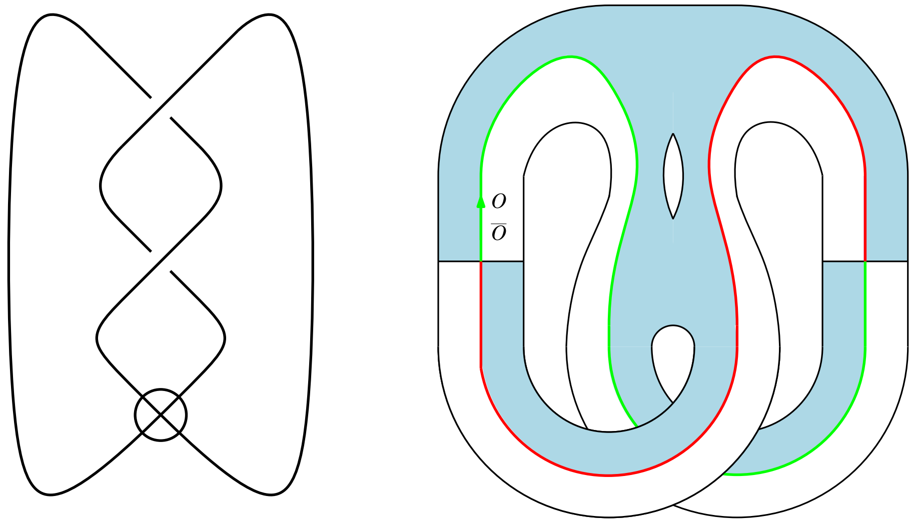
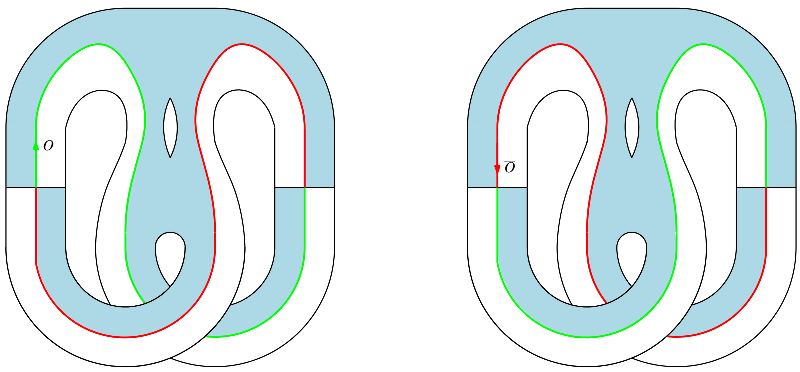


Figure 5.2: On the left, a virtual knot diagram, and on the right an alternately coloured smoothing of its associated abstract link diagram.



(A) The alternately coloured smoothing associated to orientation o.

(B) The alternately coloured smoothing associated to orientation $\overline{o}$.

Figure 5.3: The alternately coloured smoothings on abstract link diagrams corresponding to the two possible orientations of the virtual knot diagram.

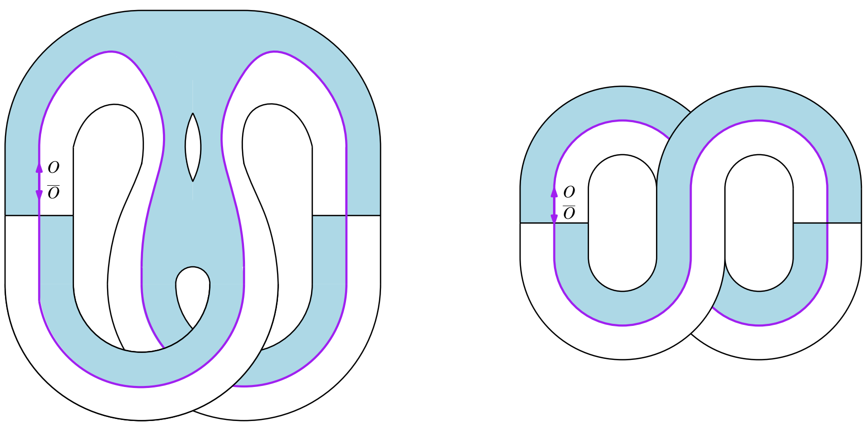
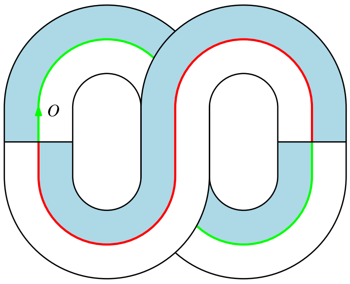
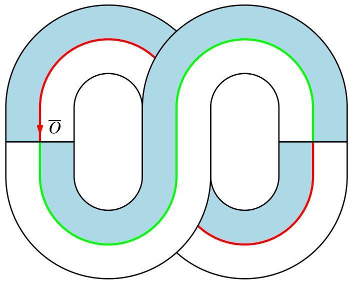


Figure 5.4: Two representatives of the stable equivalence class of smoothings of the checkboard coloured abstract link diagram depicted in Figure 5.2, with orientations o and $\overline{o}$.



(A) A smoothing stably equivalent to that of Figure 5.3(A).



(B) A smoothing stably equivalent to that of Figure 5.3(B).

Figure 5.5: Alternately coloured smoothings stably equivalent to those of Figure 5.3.

virtual link and alternately coloured smoothings of the associated abstract link diagram as given in [DKK17]. We use the diagram on the left of Figure 5.2 as an example.

(i) Given a virtual link diagram D construct the chequerboard coloured abstract link diagram as described in Chapter 3. Note that for a virtual knot the chequerboard colouring is independent of the orientation, a consequence of the invariance of the source-sink decoration under 180° rotations. See Figure 5.2.

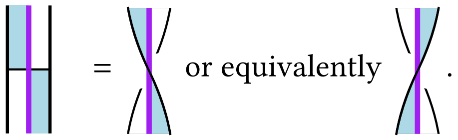
(ii) For a given orientation o of D form the corresponding oriented smoothing on the chequerboard coloured abstract link diagram. See Figure 5.3.

(iii) Place a clockwise orientation on the shaded regions of the oriented smoothing, which in turn induces a new orientation on the arcs of the smoothing. On each arc compare this orientation to that induced by o. If these two orientations agree colour the arc red, if they disagree colour the arc green (as in Definition 3.3.7). See Figure 5.4.

At this stage we have produced alternately coloured smoothings on chequerboard-coloured abstract link diagrams with cross cuts. We need a way of reading off from these diagrams elements of $vCKh'_0(K)$ (as the oriented resolution is always at height 0), which will be the chain-level canonical generators of $vKh'(K)$. We are unable to do so at this point as the cycles of the alternately coloured smoothings possess more than one colour. We now describe a process by which single coloured smoothings can be produced, and hence chain-level generators of $vKh'(K)$.

Firstly, we utilise the stable equivalence relation given in Definition 3.3.5 (on page 36) to work with alternately coloured smoothings of abstract link diagrams for which the surface deformation retracts onto the curve of the smoothing, for example the abstract link diagrams given in Figure 5.5. We can always do this as the curve of the smoothing is simply a disjoint union of copies of S^1. Note that the resulting smoothing of a chequerboard coloured abstract link diagram may not be connected.

Next, we interpret the cross cuts as half-twists with the parity of the twist ignored. That is



The author learnt of this interpretation in the talks of Dye and of Kaestner during Special Session 35, "Low Dimensional Topology and Its Relationships with Physics", of the 2015 AMS/EMS/SPM Joint Meeting.

Replacing cross cuts with appropriate half-twists we are able to view the surface of the smoothing (of a chequerboard coloured abstract link diagram) as a two-sided surface such that the curve of the smoothing appears on both sides. That cross cuts always come in pairs ensures that the surface has two sides. Importantly, on each side of the surface the curve of the smoothing is coloured exactly one colour. This is because passing a cross cut causes the arc to change to change colour (c.f. Definition 3.3.8), and to pass a cut locus is to pass on to the other side of the surface. (From this one can see that passing a cut locus, or equivalently moving on to the other side of the surface, is replicated in $\mathscr{A}$ by applying the barring operator.)

In summary, we view alternately coloured smoothings of chequerboard coloured abstract link diagrams, such as those in Figure 5.5, as two sided surfaces such that the curve of the smoothing is coloured exactly one colour on each side. At this point it is clear that in order to read off generators of $vCKh'_0(K)$ from such objects we must make a choice of side of the surface to read (or sides, if the surface is disconnected). Further, we must also ensure that this choice is the same for both the alternately coloured smoothing of an abstract link diagram associated to o and that associated to $\overline{o}$. We must have this as they are both coloured versions of the same smoothing of an abstract link diagram (the oriented smoothing) c.f. the left hand smoothing of Figure 5.4 with Figure 5.3. In effect we are making the choice on this uncoloured smoothing, which the alternately coloured smoothings then inherit.

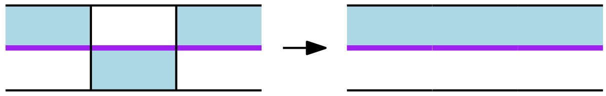


Figure 5.6: Removing a strand by cancelling cross-cuts.

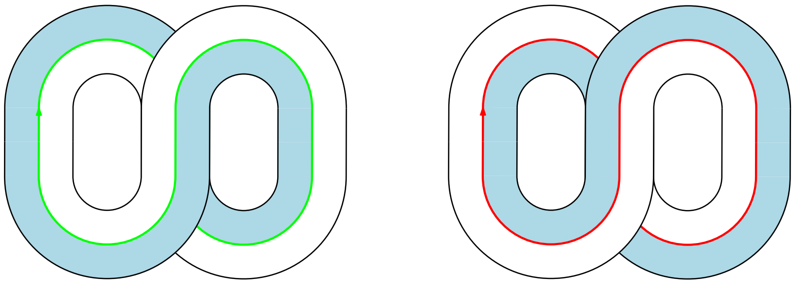


Figure 5.7: The possible ways to cancel the alternately coloured smoothing corresponding to orientation o of K.

With all this in mind, let us make a choice: given a virtual knot diagram K with orientations o and $\bar{o}$, let $\mathscr{S}$ denote the oriented smoothing of the chequerboard coloured abstract link diagram associated to K. On $\mathscr{S}$ cancel an arbitrary pair of adjacent cross cuts against one another so that the strand they bound is removed. An example is given in Figure 5.6. This cancellation of cross cuts is simply 'flipping' the segment of the surface they bound so that the other side of the surface is shown. Continue cancelling available arbitrary pairs of cross cuts until all have been removed. In our interpretation, that the smoothing has no cross cuts means that we are looking at exactly one side of surface. Now return to part (iii) of the process given on page 98, and colour the cycles of the oriented smoothings associated to o and $\bar{o}$ as dictated there. Denote by $\mathscr{S}_o$ and $\mathscr{S}_{\bar{o}}$ the resulting alternately coloured smoothings of abstract link diagrams associated to o and $\bar{o}$, respectively. That the cycles of $\mathscr{S}_o$ and $\mathscr{S}_{\bar{o}}$ are coloured with opposite colours follows from the fact that their orientations are opposite but the chequerboard colouring of $\mathscr{S}_o$ and $\mathscr{S}_{\bar{o}}$ is the same.

Examples of such single coloured smoothings are given in Figure 5.7 and Figure 5.8. In this case a choice of top and bottom is equivalent to picking either the two smoothings on the left of the Figures, or the two on the right.

After all that we are left with smoothings of abstract link diagrams the cycles of which are coloured with exactly one colour, either red or green. We form the canonical generators of $vKh'(K)$, denoted $\mathfrak{s}_o$ for o an orientation of K, by taking the appropriate tensor product of r and g as dictated by the colours of the cycles. In this way we obtain two

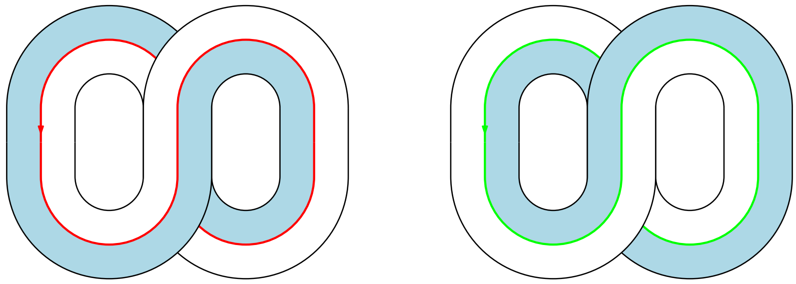


Figure 5.8: The possible ways to cancel the alternately coloured smoothing corresponding to orientation $\bar{o}$ of K.

distinct algebraic generators.

We conclude by remarking that the virtual Rasmussen invariant is independent of this choice of which side of the surface to read. Making another choice results in an application of the barring operator to one or more tensor factors of $\mathfrak{s}_o$ and $\mathfrak{s}_{\bar{o}}$, because if a cycle is coloured green on one side of the surface it must be coloured red on the other. But conjugation does not interact with the filtration; that is

$$j(\bar{r}) = j(g) \text{ and } j(\bar{g}) = j(r).$$

To conclude this section we prove a Lemma analogous to Lemma 3.5 of Rasmussen [Ras10] which we will use in both the following sections.

Lemma 5.1.1. *Let n be the number of components of K. There is a direct sum decomposition $vKh'(K) \cong vKh'_o(K) \oplus vKh'_e(K)$, where $vKh'_o(K)$ is generated by all states with q-grading conguent to $2+n \mod 4$, and $vKh'_e(K)$ is generated by all states with q-grading congruent to $n \mod 4$. If o is an orientation on K, then $\mathfrak{s}_o+\mathfrak{s}_{\bar{o}}$ is contained in one of the two summands, and $\mathfrak{s}_o - \mathfrak{s}_{\bar{o}}$ is contained in the other.*

Proof. The first statement follows exactly as in the classical case. Regarding the second statement, following [Ras10] let $\iota : vCKh'(K) \longrightarrow vCKh'(K)$ be the map which acts by the identity on $vCKh'_e(K)$ and multiplication by -1 on $vCKh'_o(K)$. We claim that $\iota(\mathfrak{s}_o) = \pm\mathfrak{s}_{\bar{o}}$. To show this we define a new grading on $\mathcal{A}$ with respect to which X has grading 2 and 1 has grading 4. We have that $\overline{X} = -X$ and $\overline{1} = 1$ so that $\bar{r} = g$ and $\bar{g} = r$, and the map

$$\overline{}^{\otimes n} : \mathcal{A}^{\otimes n} \longrightarrow \mathcal{A}^{\otimes n}$$

(which applies the barring operator to all tensor factors) acts as the identity on elements with new grading congruent to $0 \mod 4$ and multiplication by -1 on elements with new grading congruent to $2 \mod 4$. The new grading differs from the q-grading by an overall shift so that

$$\iota(\mathfrak{s}_o) = \pm\overline{\mathfrak{s}_o}^{\otimes n} = \pm\mathfrak{s}_{\overline{o}}$$

as in the classical case. □

A direct corollary of Lemma 5.1.1 is that $\mathfrak{s}_o$ is not of top filtered degree, that is:

$$s(\mathfrak{s}_o) = s(\mathfrak{s}_{\overline{o}}) = s_{min}(K). \tag{5.1.1}$$

5.1.1 Additivity of the virtual Rasmussen invariant

We can use the chain-level generators of $vKh'(K)$ to show that the virtual Rasmussen invariant is additive with respect to connect sum, confirming that the virtual invariant behaves in the same way as its classical counterpart in this respect.

As discussed in Chapter 3, the connect sum operation on virtual knots is ill-defined. By an abuse of notation we shall denote by $K_1 \# K_2$ any of the knots obtained as a connect sum of virtual knots K_1 and K_2.

Theorem 5.1.2. *For virtual knots K_1 and K_2*

$$s(K_1 \# K_2) = s(K_1) + s(K_2). \tag{5.1.2}$$

Proof. With the chain-level generators in place, along with Lemma 5.1.1, the proof follows much the same path as that in [Ras10]. For all connect sums $K_1 \# K_2$ there exists the map

$$vKh'(K_1 \# K_2) \xrightarrow{\Delta'} vKh'(K_1 \sqcup K_2) \cong vKh'(K_1) \otimes vKh'(K_2).$$

It sends a canonical generator $\mathfrak{s}_o$ of $vKh'(K_1 \# K_2)$ to a canonical generator of $vKh'(K_1) \otimes vKh'(K_2)$ of the form $\mathfrak{s}^1 \otimes \mathfrak{s}^2$ where $\mathfrak{s}^i$ is a generator of $vKh'(K_i)$ for $i = 1, 2$. As in the classical case, the map is of filtered degree -1 and we obtain

$$\begin{aligned} s(\mathfrak{s}_o) - 1 &\leq s(\mathfrak{s}^1 \otimes \mathfrak{s}^2) = s(\mathfrak{s}^1) + s(\mathfrak{s}^2) \\ s_{min}(K_1 \# K_2) &\leq s_{min}(K_1) + s_{min}(K_2), \text{ by Equation (5.1.1).} \end{aligned} \tag{5.1.3}$$

From this point the proof proceeds as in that of the analogous statement in [Ras10]: utilising the fact that $s_{min}(K) = -s_{max}(\overline{K})$ we are able to obtain from Equation (5.1.3) that

$$s_{min}(K_1 \# K_2) = s_{min}(K_1) + s_{min}(K_2) + 1$$
$$s_{max}(K_1 \# K_2) = s_{max}(K_1) + s_{max}(K_2) - 1$$

as required. □

In light of Theorem 5.1.2 we see that the Rasmussen invariant cannot distinguish between connect sums of a fixed pair of virtual knots. In general it is not known, for K_1 and K_2 both (possibly inequivalent) connect sums of a fixed pair of virtual knots, if K_1 is concordant to K_2. It is known, however, that neither the Jones polynomial [MI13] nor either of the virtual or doubled Rasmussen invariants can distinguish them. This leads one to posit whether Khovanov homology can; in the case of connect sums of trivial diagrams it was shown in Chapter 4 that doubled Khovanov homology cannot.

5.2 Computable bounds

In this section we extend the strong slice-Bennequin bounds to the virtual and doubled Rasmussen invariants. The bounds are constructed, and cases in which they vanish partly or wholly are described.

5.2.1 The virtual Rasmussen invariant

Formulation

Definition 5.2.1. Given a virtual link diagram D denote by $O(D)$ the oriented smoothing of D. Denote by $T_O(D)$ the signed graph with a vertex for each cycle of $O(D)$ and an edge for each classical crossing of D, decorated with the sign of the crossing. The edge associated to a crossing is between the vertex or vertices associated to the cycles involved in the smoothing site of that crossing. The subgraph of $T_O(D)$ formed by removing all the edges labelled with $+$ (respectively $-$) is denoted $T_O^-(D)$ (respectively $T_O^+(D)$). ◊

The graph $T_O(D)$ is often called the *Seifert graph* of D, but in order to avoid confusion with a graph defined in Section 5.2.2 we shall not use that term.

Definition 5.2.2. Given a virtual knot diagram D the quantities $U_v(D),\ \Delta_v(D) \in \mathbb{Z}$ are given by

$$U_v(D) = \#\text{ vertices }(T_O(D)) - 2\#\text{ components }(T_O^-(D)) + wr(D) + 1$$
$$\Delta_v(D) = \#\text{ vertices }(T_O(D)) - \#\text{ components }(T_O^+(D)) - \#\text{ components }(T_O^-(D)) + 1.$$

The quantities $U_v(D)$ and $\Delta_v(D)$ are dependent on the diagram D and are not invariants of the virtual knot. ◊

Theorem 5.2.3 (Analogue of Theorem 1.2 of Lobb [Lob11]). *For D a diagram of a virtual knot K*

$$s(K) \leq U_v(D).$$

Notice that the left hand side is a knot invariant whereas the right is diagram-dependent.

To prove this we require Lemma 5.1.1, as we have canonical generators in terms of r and g instead of $a = 2r$ and $b = -2g$ and the proof given in [Ras10] relies on the sign of a and b.

Proof. (of Theorem 5.2.3) The proof is practically identical to that of the classical case given in [Lob11]. Form the diagram K^- from K by smoothing all the positive classical crossings of K to their oriented resolution, and suppose that K^- is the disjoint union of l virtual link diagrams. Label these diagrams $K_1^-, K_2^-, \dots, K_l^-$. Then the canonical generator $\mathfrak{s}_o$ splits as a tensor product of canonical generators of $vKh'(K_r^-)$ as

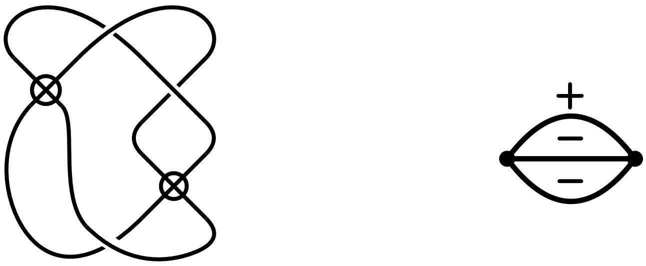
$$\mathfrak{s}_o = \mathfrak{s}_1 \otimes \mathfrak{s}_2 \otimes \cdots \otimes \mathfrak{s}_l.$$

Classically, $\mathfrak{s}_r$ can either be $\mathfrak{s}_{o'}$ or $\mathfrak{s}_{\overline{o'}}$ where o' denotes the induced orientation on K_r^-, as we are possibly altering the number of cycles separating others from infinity. In the virtual case, however, $\mathfrak{s}_r = \mathfrak{s}_{o'}$ by construction as we use abstract link diagrams to produce the canonical generators rather than the method due to Lee.

Where the proof given in [Lob11] invokes Theorem 3.5 of [Ras10] we invoke Lemma 5.1.1 as given above. □

Theorem 5.2.4 (Analogue of Theorem 1.10 of Lobb [Lob11]). *If $\Delta_v(D) = 0$ then $s(K) = U_v(D)$, where K is the virtual knot represented by D. In fact*

$$U_v(D) - 2\Delta_v(D) \leq s(K) \leq U_v(D).$$



(A) A diagram of virtual knot 3.7 which is alternating but not homogeneous.

(B) The graph $T_O(D)$ of virtual knot 3.7.

Figure 5.9

The proof of Theorem 5.2.4 is identical to that of the classical case, owing to the identical behaviour of the virtual and classical Rasmussen invariants with respect to the mirror image.

The case $\Delta_v(D) = 0$

Cromwell defined *homogeneous knots* [Cro89]. Here we recap his definition, which works equally well for virtual knots.

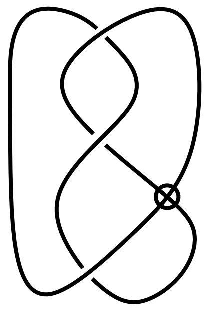
Definition 5.2.5. A *cut vertex* of a graph G is a vertex such that the graph obtained by removing the vertex along with its boundary edges has more connected components than G. ◊

Definition 5.2.6. A *block* of a graph G is a maximal connected subgraph of G containing no cut vertices. ◊

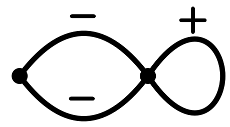
Definition 5.2.7. A signed graph G is *homogeneous* if every block B of G is such that all edges contained in B are decorated with the same sign. ◊

Definition 5.2.8. A virtual link diagram K is *homogeneous* if $T_O(K)$ is homogeneous. A virtual link is homogeneous if there exists a diagram of it which is homogeneous. ◊

Positive and negative virtual knots are homogeneous trivially (as $T_O(D)$ possesses only one kind of decoration). In the classical case alternating knots are also homogeneous [Kau83]. In the virtual case, however, this no longer holds. For example, the virtual knot diagram given in Figure 5.9(A) is alternating but not homogeneous.



(A) A diagram of virtual knot 3.2.



(B) The graph $T_O(D)$ of virtual knot 3.2.

Figure 5.10

Abe showed that for a classical knot diagram D $\Delta_v(D) = 0$ if and only if D is homogeneous [Abe11]. However, Abe's proof relies on $T_O(D)$ containing no loops (an edge which begins and ends at the same vertex); classically, this is always the case as the oriented resolution is the alternately coloured resolution, so that $T_O(D)$ is bipartite. Virtually, however, this is not the case, as discussed previously. An example is given in Figure 5.10. For now, it suffices to notice that the quantity Δ_v can be expressed as the first Betti number of the graph, G_O, defined as follows.

Definition 5.2.9. Let $T_O(D)$ be associated to the virtual knot diagram D. Form the graph G_O in the following way:

(i) For each connected component of $T_O^+(D)$ place a vertex, and a vertex for each connected component of $T_O^-(D)$.

(ii) Each vertex of $T_O(D)$ lies in exactly one connected component of $T_O^+(D)$, and exactly one connected component of $T_O^-(D)$. For each vertex of $T_O(D)$ place an edge linking the vertices of G_Δ corresponding to the connected components in which it lies. ◊

Proposition 5.2.10. *Let $T_O(D)$ be associated to the virtual knot diagram D, and $\widetilde{T}_O(D)$ be a graph obtained from $T_O(D)$ by adding or removing a loop (of arbitrary sign). Further, let $\widetilde{G}_O$ be the graph formed from $\widetilde{T}_O(D)$ following the method given in Definition 5.2.9, where $\widetilde{T}_O^+(D)$ and $\widetilde{T}_O^-(D)$ are formed in the obvious way. Then $G_O = \widetilde{G}_O$.*

Proof. It is clear that

$$\#\text{components}(T_O^{+/-}(D)) = \#\text{components}(\widetilde{T}_O^{+/-}(D))$$

(we have only added or removed a loop) so that

$$\#\text{vertices}(G_O) = \#\text{vertices}(\widetilde{G}_O).$$

Further, as loops do not connect distinct vertices, two vertices are linked in G_O if and only if they are linked in $\widetilde{G}_O$. □

In light of Proposition 5.2.10 it is clear that we need only consider homogeneity of $T_O(D)$ up to the addition and removal of loops.

Definition 5.2.11. Let G be a signed graph and let $\overline{G}$ be the graph formed by removing all loops of G. Then G is *l-homogenous* if $\overline{G}$ is homogenous. A virtual knot diagram is l-homogenous if $T_O(D)$ is, and a virtual knot is l-homogenous if it has an l-homogenous diagram. ◊

Theorem 5.2.12 (Analogue of Theorem 3.4 of Abe [Abe11])**.** *A virtual knot diagram D is l-homogeneous if and only if $\Delta_v(D) = 0$. Hence, for an l-homogeneous diagram D of a virtual knot K*

$$U(D) = s(K).$$

Proof. Abe's original proof yields the following statement: if D is such that $T_O(D)$ is loopless, then D is homogenous if and only if $\Delta_v(D) = 0$. By Proposition 5.2.10 we may remove any loops from $T_O(D)$, leaving the associated G_O unchanged. Noticing that $\Delta_v(D) = b_1(G_O)$, the first Betti number of G_O, we obtain the desired result. □

5.2.2 The doubled Rasmussen invariant

Formulation

In formulating the bounds on the doubled Rasmussen invariant we follow much the same path as in Section 5.2.1. The formulae arrived at in this section exhibit important differences between those of Section 5.2.1, however, owing to the structural differences between MDKK homology and doubled Lee homology.

In the construction of MDKK homology source-sink decorations are used to ensure that the oriented resolution of a virtual knot diagram is, in fact, alternately colourable;

doubled Khovanov homology does not do so. In the definition below, therefore, we consider the graph associated to the alternately coloured smoothing of a virtual knot diagram.

Definition 5.2.13. Given a virtual link diagram D denote by $\mathscr{S}(D)$ the alternately coloured smoothing of D. Denote by $T_{\mathscr{S}}(D)$ the graph with a vertex for each cycle of $\mathscr{S}(D)$ and an edge for each classical crossing of D, decorated with the sign and parity of the crossing: every edge is decorated with an element of $\{(e,+),(e,-),(o,+),(o,-)\}$, where $(e,+)$ denotes an even positive crossing, $(o,+)$ an odd positive crossing, and so on. The edge associated to a crossing is between the vertex or vertices associated to the cycles involved in the smoothing site of that crossing. The subgraph of $T_{\mathscr{S}}(D)$ formed by removing all the edges labelled with either $(e,+)$ or $(o,-)$ is denoted $T_{\mathscr{S}}^{\text{◖}}(D)$. The subgraph of $T_{\mathscr{S}}(D)$ formed by removing all the edges labelled with either $(e,-)$ or $(o,+)$ is denoted $T_{\mathscr{S}}^{\text{◗}}(D)$. ◊

Definition 5.2.14. Let D be a virtual knot diagram with n_+^o (n_-^o) odd positive (negative) classical crossings. Define the quantities

$$
\begin{aligned}
U_d(D) &= \#\text{vertices}(T_{\mathscr{S}}(D)) - 2\#\text{comp}(T_{\mathscr{S}}^{\text{◖}}(D)) + wr(D) + J(D) + n_+^o + 1 \\
\Delta_d(D) &= 2(\#\text{vertices}(T_{\mathscr{S}}(D)) - \#\text{comp}(T_{\mathscr{S}}^{\text{◖}}(D)) - \#\text{comp}(T_{\mathscr{S}}^{\text{◗}}(D)) + 1) \\
&\quad + n_+^o + n_-^o
\end{aligned} \tag{5.2.1}
$$

where #comp denotes the number of components of a graph. ◊

In direct analogy to Theorem 5.2.3 we have the following.

Theorem 5.2.15. *Let D be a diagram of a virtual knot K. Then*

$$
U_d(D) - \Delta_d(D) \leq s_1(K) \leq U_d(D). \tag{5.2.2}
$$

Proof. We shall go through the proof of Theorem 5.2.15 in more detail than that of it's counterpart Theorem 5.2.3, owing to the aforementioned differences between the theories vKh' and DKh'. The gist of the proof is unchanged, however: as computation of $s_1(K)$ only requires knowledge of the partial chain complex

$$
CDKh_{s_2(K)-1}(D)' \xrightarrow{d_{s_2(K)-1}} CDKh_{s_2(K)}(D)'
$$

we ignore (by resolving them) classical crossings whose alternately coloured resolution is the 0-resolution; such crossings are associated to outgoing maps from the alternately coloured resolution of D and do not contribute to $d_{s_2(K)-1}$. This comes at the price, of course: we lose a large amount of the information contained in $CDKh'_{s_2(K)}(D)$. Nevertheless, the trade is a worthwhile one, as we are able to use what's left to obtain bounds on $s_1(K)$.

Let D be a diagram of a virtual knot K, with n_+ (n_-) positive (negative) classical crossings. Further, let $n_+ = n^e_+ + n^o_+$ and $n_- = n^e_- + n^o_-$, where a superscript e (o) denotes the number of even (odd) crossings. Form a virtual link diagram, $\widetilde{D}$, by resolving all even positive crossings and all odd negative crossings of D into their alternately coloured resolutions. (One readily observes that such crossings are those with alternately coloured resolution the 0-resolution, as mentioned above.) We can write

$$\widetilde{D} = \widetilde{D}_1 \sqcup \widetilde{D}_2 \sqcup \ldots \sqcup \widetilde{D}_r$$

where $\widetilde{D}_i$ is a virtual link diagram with n^i_+ positive and n^i_- negative classical crossings (the parity of positive (negative) crossings is necessarily odd (even), of course). Further, for $\mathscr{S}$ the alternately colourable smoothing of D, we have

$$\mathscr{S} = \mathscr{S}_1 \sqcup \mathscr{S}_2 \sqcup \ldots \sqcup \mathscr{S}_r$$

where $\mathscr{S}_i$ is the unique alternately colourable smoothing of $\widetilde{D}_i$ formed by resolving all crossings into the resolution they are resolved into in $\mathscr{S}$.

Notice that while $CDKh'(D)$ does not split as a tensor product of the $CDKh'(\widetilde{D}_i)$'s, the alternately coloured generators of $DKh'(K)$ do. That is, if $\mathfrak{s}^u$ is associated to $\mathscr{S}$, then

$$\mathfrak{s}^{\mathrm{u}} = \mathfrak{s}^{\mathrm{u}}_1 \otimes \mathfrak{s}^{\mathrm{u}}_2 \otimes \cdots \otimes \mathfrak{s}^{\mathrm{u}}_r \tag{5.2.3}$$

where $\mathfrak{s}^{\mathrm{u}}_i$ is the alternately coloured generator defined by $\mathscr{S}_i$.

We have $J(\widetilde{D}_i) = n^i_+$ (as all negative crossings of $\widetilde{D}_i$ are even), so that the highest non-trivial quantum grading of $CDKh'_{n^i_+}(\widetilde{D}_i)$ containing $\left[\mathfrak{s}^{\mathrm{u}}_i\right]$ is $e_i + n^i_+ + n^i_+ - n^i_-$, where e_i denotes the number of cycles of $\mathscr{S}_i$. Further, as a corollary to Lemma 4.5.2 (on page 77), we determine that $\left[\mathfrak{s}^{\mathrm{u}}_i\right]$ is not of top degree, and that $e_i + n^i_+ + n^i_+ - n^i_- - 2$ is the highest non-trivial degree of $CDKh'_{n^i_+}(\widetilde{D}_i)$ containing it. By Equation (5.2.3) and an argument directly analogous to Lobb's we obtain

$$\begin{aligned} s^{\mathrm{u}}_{\min}(K) &\leq n^e_+ - n^o_- + \sum_{i=1}^{r} \left(e_i + n^i_+ + n^i_+ - n^i_- - 2 \right) \\ &= wr(D) + J(D) + n^o_+ + \#\mathrm{vertices}(T_{\mathscr{S}}(D)) - 2\#\mathrm{comp}(T^{\leftmoon}_{\mathscr{S}}(D)). \end{aligned}$$

Recalling that $s^{\mathrm{u}}_{\min}(K) = s_1(K) + 1$, we arrive at

$$s_1(K) \leq U_d(D).$$

To see that

$$U_d(D) - \Delta_d(D) \leq s_1(K)$$

repeat the proof of Theorem 5.2.4, which we are free to do as the doubled Rasmussen invariant replicates the behaviour of its classical counterpart with respect to the mirror image. □

Simplifying $\Delta_d(D)$

Much of the analysis used in the Section 5.2.1 may be repeated in order to characterise a case in which the Δ_d formula simplifies. However, we do not recover the vanishing result as in the case of Δ_v.

Definition 5.2.16. Let D be a virtual knot diagram and $T_{\mathscr{S}}(D)$ the graph associated to it. Recall that each edge of $T_{\mathscr{S}}(D)$ is decorated with exactly one element of

$$\{(e,+),(e,-),(o,+),(o,-)\}.$$

Let $\rightmoon = \{(e,-),(o,+)\}$ and $\leftmoon = \{(e,+),(o,-)\}$. The graph $T_{\mathscr{S}}(D)$ is *d-homogenous* if every block is decorated with elements of either $\rightmoon$ or $\leftmoon$, but not both.

The diagram D is d-homogenous if $T_{\mathscr{S}}(D)$ is d-homogenous. A virtual knot is d-homogenous if it has a d-homogenous diagram. ◊

Proposition 5.2.17. *Let D be a virtual link diagram and $T_{\mathscr{S}}(D)$ the graph associated to it. Then D is d-homogenous if and only if*

$$\#vertices(T_{\mathscr{S}}(D)) - \#comp(T^{\leftmoon}_{\mathscr{S}}(D)) - \#comp(T^{\rightmoon}_{\mathscr{S}}(D)) + 1 = 0.$$

Proof. Let $G_{\mathscr{S}}$ denote the graph formed from $T_{\mathscr{S}}(D)$ in direct analogy to G_O, as given in Definition 5.2.9, with $T^{\text{◖}}_{\mathscr{S}}(D)$ and $T^{\text{◗}}_{\mathscr{S}}(D)$ taking the place of $T^{+}_{O}(D)$ and $T^{-}_{O}(D)$. The graph $T_{\mathscr{S}}(D)$ is bipartite as $\mathscr{S}(D)$ is alternately coloured. Thus it is loopless and Abe's proof may be employed to show that $T_{\mathscr{S}}(D)$ is homogenous if and only if $b_1(G_{\mathscr{S}}) = 0$. We conclude by noticing that

$$b_1(G_{\mathscr{S}}) = \#\text{vertices}(T_{\mathscr{S}}(D)) - \#\text{comp}(T^{\text{◖}}_{\mathscr{S}}(D)) - \#\text{comp}(T^{\text{◗}}_{\mathscr{S}}(D)) + 1,$$

which follows exactly as in the case of Δ_v and G_O. □

Corollary 5.2.18. *Let D be diagram of a virtual knot K. If D is d-homogenous then*

$$U_d(D) - n^o_+ - n^o_- \leq s_1(K) \leq U_d(D)$$

where n^o_+ (n^o_-) denotes the number of odd positive (negative) classical crossings of D.

5.3 Computation and estimation of the slice genus

In this section we use the bounds U_v and U_d to compute or estimate the slice genus of a number of virtual knots. The computations are made by finding a surface of appropriate genus between the given knot and the unknot.

The following table contains the results of the analysis for the virtual knots of 4 crossing or less in Green's table. A blank entry denotes an unknown, and most computations of s, s_1, and s_2 (or the region in which they lie) are made by computing $U_{v/d}$, $\Delta_{v/d}$, and J for the diagram given in the table. The exceptions to this are $s_1(3.3)$, which the author computed by hand from $DKh'(3.3)$, and leftmost knots, for which the definition and the method of computation of s_1 are given in Section 4.5.4. Further, many computations of s, s_2, and s_2 are made by spotting that the knot in question is a connect sum of two other knots, and employing the additivity of the invariants along with their invariance under flanking Definition 4.3.7. (As observed in Section 5.0.2, s and s_1 coincide for even knots, so that the invariants are buy one get one free in this case.)

Exact values of g^* are obtained by constructing a cobordism which attains a lower bound given by s, s_1, or s_2. Upper bounds on g^* are obtained by constructing a cobordism of the given genus, and employing the fact that half the crossing number bounds the slice genus of a knot from above (as in the classical case) [BCG17a]. Shortly after posting a paper which contains much of the material of this chapter to the arXiv, the author learned of the work of Boden, Chrisman, and Gaudreau in which they compute or estimate the slice genus of a very large number of the 92800 virtual knots of 6 crossings or less [BCG17a; BCG17b]. In the table below we do not include the values of g^* they arrive at in order to demonstrate the infomation that can be obtained using the bounds U_v, U_d, and the properties of the virtual and doubled Rasmussen invariants.

Knot	l-hom.	d-hom.	s	s_1	s_2	g^*
0.1	Y	Y	0	0	0	0
2.1	Y	Y	-2	-5	-2	1
3.1			$[-2, 0]$	$[-3, 1]$	0	$[0, 2]$
3.2	Y		0	-4	-2	1
3.3	Y		-2	-6	-2	1
3.4			$[-2, 0]$	-4	-2	1
3.5			-2	-2	0	1
3.6	Y	Y	-2	-2	0	1
3.7			0	0	0	$[0, 2]$
4.1	Y		-4	-10	-4	2
4.2	Y		0	0	0	$[0, 2]$
4.3	Y		-4	-10	-4	2
4.4	Y		-2	-5	-2	1
4.5	Y		-2	-5	-2	1
4.6	Y	Y	0	0	0	$[0, 2]$
4.7	Y		-4	-10	-4	2
4.8	Y	Y	0	0	0	0
4.9	Y		-4	$[-9, -5]$	-2	2
4.10	Y		-2	$[-4, 0]$	0	$[1, 2]$
4.11	Y		-2	$[-7, -2]$	-2	$[1, 2]$
4.12	Y		0	$[-2, 2]$	0	$[0, 2]$
4.13	Y		0	$[-2, 2]$	0	$[0, 2]$

Knot	l-hom.	d-hom.	s	s_1	s_2	g^*
4.14	Y		0	-3	-2	$[1, 2]$
4.15	Y		-4	$[-9, -5]$	-2	2
4.16	Y		-2	$[-4, 0]$	0	1
4.17	Y		0	$[-3, 1]$	0	$[0, 2]$
4.18	Y		-2	-5	-2	1
4.19						$[0, 2]$
4.20	Y	Y	0	-3	-2	1
4.21	Y		0	$[0, 2]$	0	$[0, 2]$
4.22	Y	Y	0	$[-5, -2]$	-2	$[1, 2]$
4.23	Y		-2	$[-4, 0]$	0	$[1, 2]$
4.24	Y		2	$[0, 4]$	0	$[1, 2]$
4.25	Y	Y	-2	-9	-4	1
4.26			$[-2, 0]$	$[-5, 2]$	0	$[0, 2]$
4.27	Y		0	$[-6, -2]$	-2	1
4.28			$[-2, 0]$	$[-5, 2]$	0	$[0, 2]$
4.29	Y		-4	$[-11, -5]$	-2	2
4.30	Y		-2	$[-9, -2]$	-2	1
4.31	Y		-2	$[-6, 0]$	0	1
4.32	Y		-2	$[-6, 0]$	-2	$[1, 2]$
4.33	Y		-2	$[-9, -2]$	-2	1
4.34	Y	Y	0	-3	-2	$[1, 2]$
4.35	Y		0	$[-4, 2]$	0	$[0, 1]$
4.36	Y		0	$[1, 7]$	2	$[1, 2]$
4.37	Y		-4	$[-11, -5]$	-2	2
4.38	Y		-2	$[-9, -2]$	-2	1
4.39	Y		-2	$[-8, -2]$	-2	$[1, 2]$
4.40	Y	Y	0	-3	-2	1
4.41	Y		-2	$[-6, 0]$	0	1
4.42	Y		0	$[-2, 2]$	0	$[0, 2]$
4.43	Y	Y	-2	-9	-4	1

Knot	l-hom.	d-hom.	s	s_1	s_2	g^*
4.44			$[-2,0]$	$[-10,-2]$	-2	1
4.45			$[-2,0]$	$[-10,-2]$	-2	1
4.46			$[-2,0]$	$[-4,4]$	0	$[0,2]$
4.47			$[0,2]$	$[-5,2]$	0	$[0,2]$
4.48	Y		-4	$[-13,-5]$	-2	2
4.49	Y		-2	$[-9,-2]$	-2	$[0,2]$
4.50	Y		-2	$[-6,0]$	0	$[1,2]$
4.51	Y		-2	$[-6,0]$	0	$[1,2]$
4.52	Y	Y	0	$[-5,-2]$	-2	1
4.53	Y	Y	-4	-10	-4	2
4.54	Y		-2	-10	-2	$[1,2]$
4.55	Y		0	0	0	$[0,2]$
4.56	Y		0	0	0	$[0,1]$
4.57	Y		-2	$[-6,0]$	0	$[1,2]$
4.58	Y		0	$[-4,2]$	0	$[0,1]$
4.59	Y		0	$[-4,2]$	0	$[0,1]$
4.60	Y	Y	0	-3	-2	1
4.61	Y		-4	$[-10,-6]$	-2	2
4.62	Y		-2	$[-7,-2]$	-2	$[1,2]$
4.63	Y		-4	$[-7,-2]$	-2	2
4.64	Y	Y	-4	-3	-2	2
4.65	Y		-2	$[-4,0]$	0	$[1,2]$
4.66	Y		0	$[-2,2]$	0	$[1,2]$
4.67	Y		0	$[-2,2]$	0	$[0,1]$
4.68	Y		2	$[-4,0]$	0	1
4.69	Y		-4	$[-9,-5]$	-2	2
4.70	Y		-2	$[-4,0]$	0	1
4.71	Y		0	0	0	0
4.72	Y		0	0	0	0
4.73			-4	-10	-4	2

Knot	l-hom.	d-hom.	s	s_1	s_2	g^*
4.74			-2	-5	-2	1
4.75			0	0	0	0
4.76			0	0	0	0
4.77			0	0	0	0
4.78	Y		-4	$[-9, -5]$	-2	2
4.79	Y		-2	$[-4, 0]$	0	$[1, 2]$
4.80	Y	Y	-2	-9	-4	1
4.81	Y		0	$[-8, -2]$	-2	$[1, 2]$
4.82	Y	Y	-2	-8	-2	1
4.83			$[-4, 0]$	$[-8, -4]$	-2	$[1, 2]$
4.84		Y	$[0, 2]$	$[-2, 0]$	2	$[1, 2]$
4.85			$[-2, 2]$	$[-2, 2]$	0	$[0, 1]$
4.86	Y	Y	0	0	0	$[0, 1]$
4.87	Y	Y	-2	$[-8, -6]$	-2	$[1, 2]$
4.88		Y	$[0, 2]$	$[4, 6]$	2	1
4.89	Y	Y	-2	-2	0	1
4.90	Y	Y	0	0	0	0
4.91	Y	Y	-4	-11	-4	2
4.92	Y	Y	-2	$[-8, -6]$	-2	1
4.93			$[-2, 0]$	$[-3, 1]$	0	$[0, 1]$
4.94	Y	Y	-2	$[-8, -6]$	-2	1
4.95			$[-2, 0]$	$[-8, -4]$	-2	1
4.96			$[-2, 0]$	$[-3, 1]$	0	$[0, 1]$
4.97			$[-2, 2]$	$[-3, 2]$	0	$[0, 1]$
4.98			$[-2, 2]$	$[-2, 2]$	0	$[0, 1]$
4.99	Y	Y	0	0	0	$[0, 1]$
4.100	Y		0	$[-3, 2]$	0	$[0, 1]$
4.101	Y		-2	$[-10, -4]$	-2	$[1, 2]$
4.102	Y		0	$[-3, 2]$	0	$[0, 2]$
4.103			$[-2, 0]$	$[-3, 1]$	0	$[0, 2]$

Knot	l-hom.	d-hom.	s	s_1	s_2	g^*
4.104		Y	$[0, 2]$	$[4, 6]$	2	1
4.105			-2	-2	0	1
4.106			$[-2, 2]$	$[-2, 2]$	0	$[0, 1]$
4.107			$[-2, 2]$	$[-2, 2]$	0	$[0, 1]$
4.108			0	0	0	1

Table 1: computations of U_v, U_d, and s_2, and computations or estimations of s, s_1, and g^* for all virtual knots of four classical crossings or less.

From the table we are able to make some observations regarding the two extensions of the Rasmussen invariant. We see that only s_1 is able to distinguish between 2.1 and 3.3. Further, there are a number of knots for which the easy to compute s_2 obstructs sliceness while the harder to compute s does not. The virtual and doubled Rasmussen invariants are also able to distinguish many pairs of knots which have the same positive slice genus, showing that they are not concordant to one another.

From the table above we make the following conjecture.

Conjecture 5.3.1. *Let K be a virtual knot. If $s_2(K) \neq 0$ then $s_1(K) \neq 0$.*

A resolution of this conjecture in the affirmative would make the computation of $s_1(K)$ redundant for the purposes of slice genus computation in the case of a virtual knot for which $s_2(K) \neq 0$.

We also give presentations of the surfaces of genus 0, 1, and 2 used to determine the slice genus of the knots 4.8, 3.5, and 4.15 respectively; they are contained in Figures 5.11 to 5.13. Unlabeled arrows denote virtual Reidemeister moves, while those which denote 1-handle additions are so labelled. Red arcs between strands denote the locations of such handle additions within individual diagrams.

To conclude we list the results of similar analysis as that used to produce the previous table, this time on the virtual knots for which Boden, Chrisman, and Gaudreau's methods are unable to obstruct sliceness but the virtual or doubled Rasmussen invariants can. The upper bounds on g^* are those given by Boden, Chrisman, and Gaudreau [BCG17b]. As in

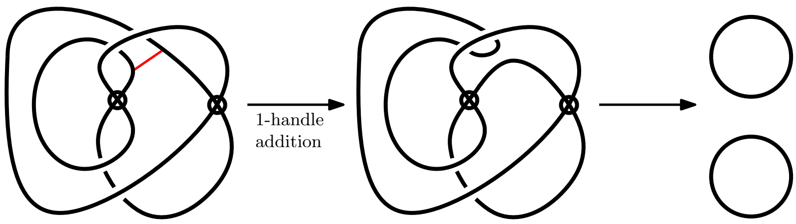


Figure 5.11: A slice disc for virtual knot 4.8.

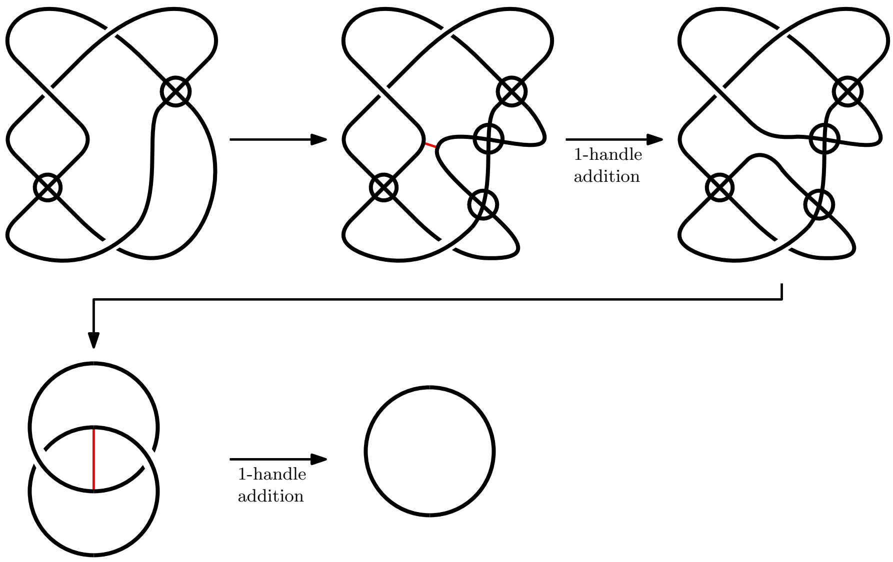


Figure 5.12: A genus 1 cobordism to the unknot from virtual knot 3.5.

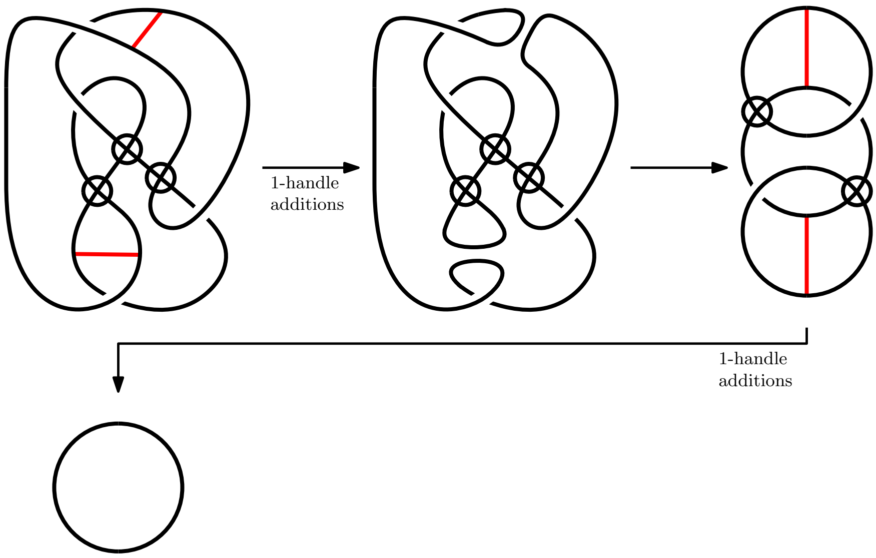


Figure 5.13: A genus 2 cobordism to the unknot from virtual knot 4.15.

the case of knots of 4 or less crossings many of the computations are made by spotting connect sums.

Knot	l-hom.	d-hom.	s	s_1	s_2	g^*
5.114			-2	-1	0	$[1,2]$
5.344			2	9	0	$[1,2]$
5.2351	Y	Y	-2	$[-3,1]$	0	$[1,2]$
6.1617			-2	-1	0	$[1,2]$
6.2414			-2	$[-2,3]$	0	$[1,2]$
6.3036			0	$[1,3]$	0	1
6.3452			2	$[0,4]$	0	1
6.3536			-2	$[-6,0]$	0	1
6.3537			-2	$[-6,0]$	0	1
6.3780			-2	$[-6,0]$	0	1
6.3781			-2	$[-6,0]$	0	1
6.3972			-2	$[-6,0]$	0	1
6.3973			-2	$[-6,0]$	0	1
6.5252			-2	$[-6,0]$	0	1
6.5253			-2	$[-6,0]$	0	1
6.5738			-2	$[-6,0]$	0	1
6.5740			-2	$[-6,0]$	0	1
6.6176	Y		-2	$[-4,0]$	0	$[1,2]$
6.6508			-2	$[-6,0]$	0	1
6.6509			-2	$[-6,0]$	0	1
6.7805			-2	$[-6,0]$	0	1
6.7807			-2	$[-6,0]$	0	1
6.8909			0	-1	0	$[1,2]$
6.9825			0	-1	0	$[1,2]$
6.12069			2	$[-1,3]$	0	$[1,2]$
6.13061			2	$[-1,3]$	0	$[1,2]$
6.14012	Y		2	$[-3,3]$	0	1
6.28566			0	-1	0	$[1,2]$

Knot	l-hom.	d-hom.	s	s_1	s_2	g^*
6.35229	Y	Y	2	$[0, 4]$	0	$[1, 2]$
6.37329			0	3	0	$[1, 2]$
6.37570	Y		2	$[-1, 5]$	0	1
6.38605	Y		2	$[-2, 4]$	0	1
6.42015	Y		-2	$[-4, 0]$	0	1
6.46580	Y		2	$[-4, 4]$	0	1
6.46684	Y		2	$[-4, 4]$	0	1
6.49730	Y		-2	$[-4, 0]$	0	1
6.58375			0	-3	0	1
6.58930	Y		2	$[0, 4]$	0	1
6.70672	Y	Y	-2	-2	0	$[1, 2]$
6.75192	Y		2	$[0, 4]$	0	1
6.78145	Y		-2	$[-4, 0]$	0	1
6.85784	Y		-2	-2	0	$[1, 2]$
6.90115	Y	Y	-2	-2	0	$[1, 2]$
6.90150	Y	Y	-2	-2	0	$[1, 2]$
6.90209	Y	Y	-2	-2	0	$[1, 2]$

Table 2: Computations or estimations of g^* for 45 virtual knots, whose slice status is undetermined in the work of Boden, Chrisman, and Gaudrea.

Chapter 6

Augmenting doubled Khovanov homology

We begin this chapter by shifting our focus from virtual knot theory to the intimately related theory of *links in thickened surfaces* - outlined in Section 6.1 - in order to produce an augmented version of doubled Khovanov homology for such objects. The new theory exhibits a third grading, which is constructed from the cohomology of surfaces. We investigate the properties of this trigraded homology theory, especially the interaction between various filtered versions derived from it, and produce cobordism obstructions.

In Section 6.3 these cobordism obstructions for links in thickened surfaces feed back to the study of virtual knot concordance. We show that they yield a condition which implies *ascent sliceness*, a refinement of the notion of sliceness for virtual knots, which analyses the complexity of the 3-manifolds appearing in a cobordism between a virtual knot and the unknot, rather than that of the surfaces.

6.1 Links in thickened surfaces

Here we give the essential details of links in thickened surfaces and their relationship to virtual links (for full details see, for example, [CW14; Tur07]). To differentiate between the two types of object, we denote links in thickened surfaces with `\mathfrak` letters, while virtual links remain uppercase Roman letters.

A *link in a thickened surface* is an isotopy class of embeddings $\bigsqcup S^1 \hookrightarrow \Sigma_g \times I$. Let $\mathfrak{L}$ be a link in a thickened surface, and consider the 4-valent graph (on Σ_g) obtained from a generic projection to Σ_g of a particular embedding $\bigsqcup S^1 \hookrightarrow \Sigma_g \times I$ representing $\mathfrak{L}$. A *diagram of a link in a thickened surface*, denoted $\mathfrak{D}$, is obtained by decorating the vertices of this graph with the appropriate overcrossing and undercrossing information. Two diagrams represent the same link in a thickened surface if they are related by a finite sequence of Reidemeister moves (those familiar from classical knot theory), applied on Σ_g. An example of a diagram of a link in a thickened surface is given in Figure 6.2.

By the Isotopy Extension Theorem an isotopy of an embedding $\bigsqcup S^1 \hookrightarrow \Sigma_g \times I$ extends to a self-diffeomorphism of $\Sigma_g \times I$. Of course, there are many self-diffeomorphisms which cannot be realised as extensions of isotopies of embeddings; in this case, Dehn twists on Σ_g. Therefore, one can obtain a virtual link from a link in a thickened surface by simply considering the latter up to Dehn twists and the permitted handle stabilisations (described in Chapter 3). If we can obtain a virtual link L from a link in a thickened surface $\mathfrak{L}$ in this manner we say that $\mathfrak{L}$ *projects to* L.

A representative of a virtual link $D \hookrightarrow \Sigma_g \times I$ defines a link in a thickened surface simply by considering D up to isotopy. It is important to note that two representatives of the same virtual link may define non-equivalent links in thickened surfaces, even if the representatives are in the same thickened surface. For example, given D and D', two representatives of a virtual link L, we may need to apply handle stabilisations and Dehn twists to obtain D from D'. Recall the minimal supporting genus, $m(L)$, as defined in Section 3.1.2 (on page 20); Theorem 3.1.6 ensures that if D and D' are representatives in $\Sigma_{m(L)} \times I$, then we need only apply self-diffeomorphisms (only isotopies and Dehn twists), so that no stabilisations are required.

The relationship between virtual links and links in thickened surfaces is depicted in Figure 6.1.

The notions of cobordism and concordance of links in thickened surfaces are defined identically to those of virtual links (see Section 3.2 and [Tur07]): they are pairs consisting of a surface S and an oriented 3-manifold M with appropriate boundary, such that $S \hookrightarrow M \times I$. In particular, a cobordism between two links in thickened surfaces $\mathfrak{L}_1$ and $\mathfrak{L}_2$ is also a cobordism between the virtual links projected to by $\mathfrak{L}_1$ and $\mathfrak{L}_2$.

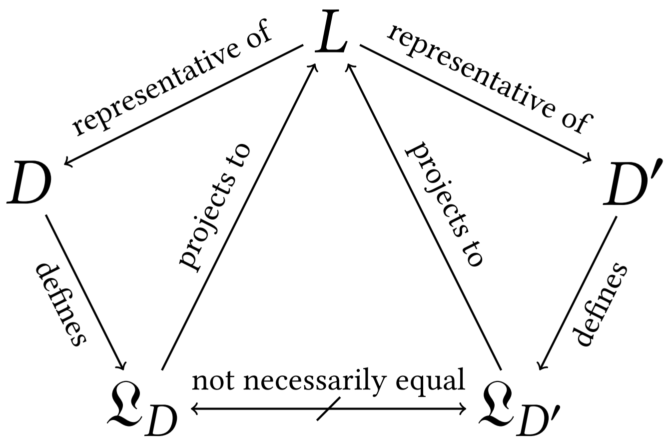


Figure 6.1: A schematic picture of the relationship between virtual links and links in thickened surfaces; L is a virtual link, D and D' two representatives of L which define the links in thickened surfaces $\mathfrak{L}_D$, $\mathfrak{L}_{D'}$. In general $\mathfrak{L}_D$ and $\mathfrak{L}_{D'}$ are inequivalent.

6.2 A grading from surface cohomology

In this section we describe the construction of an additional grading on the doubled Khovanov homology of a link in a thickened surface, using the first cohomology of the surface to produce the grading. This is a companion to a similar construction in the case of MDKK homology, due to Manturov [Man08a].

In Section 6.2.4 we describe the assignment of maps on homology to cobordisms between links in thickened surfaces, and in Section 6.2.5 use them to obtain an obstruction to knots in $\Sigma_g \times I$ bounding a disc in $\Sigma_g \times I \times I$.

6.2.1 Extra decoration on the cube of resolutions

In light of the relationship between links in thickened surfaces and virtual links described above, it is clear that the methods used to construct doubled Khovanov homology may be applied without modification to links in thickened surfaces. That is, a diagram of a link in a thickened surface is a decorated 4-valent graph on Σ_g, and we may form smoothings of such objects exactly as we form those of virtual link diagrams (an example of the resulting cube is given in Figure 6.3). Of course, another consequence of the above relationship is that the resulting theory cannot distinguish between $\mathfrak{L}_D$ and $\mathfrak{L}_{D'}$, in the notation of Figure 6.1: both project to the same virtual link, of which doubled Khovanov homology is an invariant.

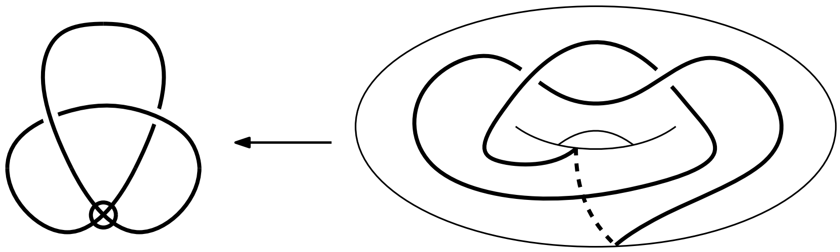


Figure 6.2: A knot in $\Sigma_1 \times I$ which projects to the virtual knot 2.1.

Motivation to augment the construction of doubled Khovanov homology is as follows. Conceptually, we wish to make use of the way in which an embedding $\bigsqcup S^1 \hookrightarrow \Sigma_g \times I$ is knotted around Σ_g in order to enchance the standard doubled Khovanov complex. Doubled Khovanov homology is quite insensitive to this information, as evidenced by its invariance under the purely virtual Reidemeister moves and flanking (given in Definition 4.3.7). This increase in sensitivity comes at the cost of moving from virtual links to links in thickened surfaces.

In any case, we begin by decorating the cube of resolutions in a manner which captures some of this information.

Definition 6.2.1 (The dotted cube of resolutions)**.** Let $\mathfrak{D}$ be a diagram of an oriented link in a thickened surface $\mathfrak{L} \hookrightarrow \Sigma_g \times I$. Form the cube of resolutions of $\mathfrak{D}$ in the same way as for a virtual link diagram: resolutions are embeddings of disjoint unions of S^1 into Σ_g. An example is given in Figure 6.3.

Pick an element $c \in H^1(\Sigma_g; \mathbb{Z}_2)$. Decorate the cube of resolutions as follows: a circle within a resolution is decorated with a *dot* if it has non-zero image under c. The assignment of dots to all circles of all resolutions withing the cube is referred to as the *dotting associated to* c.

Two examples of dottings are given in Figure 6.3; green dots represent the dotting associated to the element of $H^1(\Sigma_g; \mathbb{Z}_2)$ coloured green, and the element coloured red does not produce any dots. The fully decorated cube is refered to as the *dotted cube of resolutions of* $\mathfrak{D}$ *with respect to* c, denoted $[\![\mathfrak{D}, h]\!]$. $\Diamond$

Of course, the dotted cube of resolutions defined above depends on the choice of $c \in H^1(\Sigma_g; \mathbb{Z}_2)$. We shall show that the resulting homology is an invariant of the pair $(\mathfrak{L}, c)$.

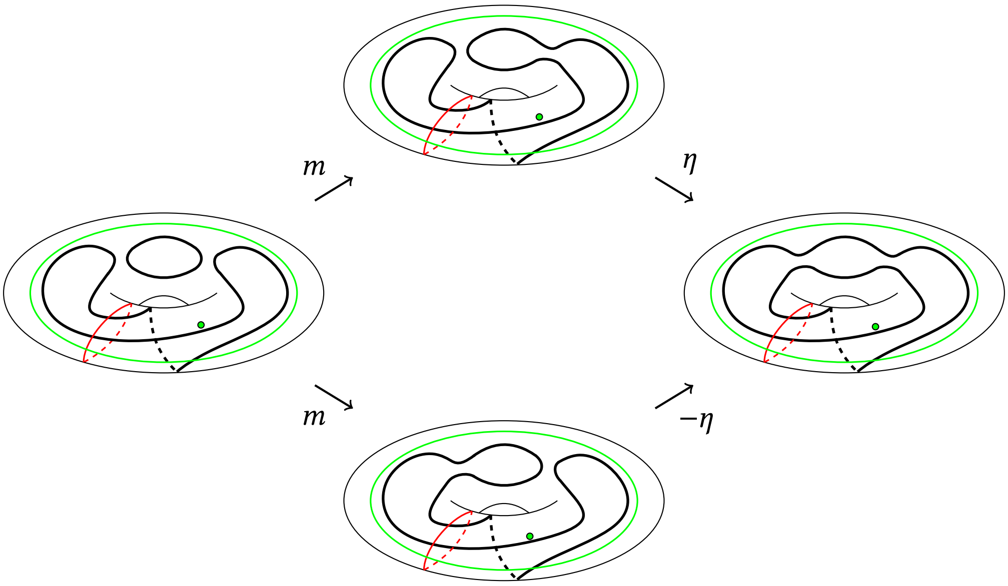


Figure 6.3: The dotted cube of resolutions of the diagram depicted in Figure 6.2.

6.2.2 Doubled Khovanov homology with dots

We now incorporate the higher dimensional information, in the form of dots, into the algebraic complex. We keep track of the dots of the resolutions by placing a dot above the module associated to a dotted circle.

Definition 6.2.2 (The dotted doubled Khovanov complex)**.** Let $\mathfrak{D}$ be a diagram of an oriented link in a thickened surface $\mathfrak{L} \hookrightarrow \Sigma_g \times I$. Pick an element $c \in H^1(\Sigma_g; \mathbb{Z}_2)$ and form the dotted cube $[\![\mathfrak{D}, c]\!]$ as in Definition 6.2.1. We form the *doubled Khovanov complex of* $\mathfrak{D}$ *with respect to* c in the manner of Chapter 4, but augmented by adding dots above modules assigned to circles which are dotted. These dots persist to elements of the dotted module; that is, we denote the elements of $\overset{\bullet}{\mathcal{A}}$ as $v_{\overset{\bullet}{+}}$ and $v_{\overset{\bullet}{-}}$.

As in unaugmented doubled Khovanov homology, the components of the differential are matrices of the appropriate maps, which are assigned signs in the standard way. The resulting chain complex is denoted $CDKh(\mathfrak{D}, c)$, and an example of such a complex is given in Figure 6.4. ◊

Definition 6.2.3. Let $\mathfrak{D}$ be a diagram of an oriented link in a thickened surface $\mathfrak{L} \hookrightarrow \Sigma_g \times I$ and $CDKh(\mathfrak{D}, c)$ its dotted doubled Khovanov complex with respect to $c \in H^1(\Sigma_g; \mathbb{Z}_2)$. By an abuse of notation we denote by c both the cohomology class and a $\mathbb{Z}[\frac{1}{2}]$-grading on $CDKh(\mathfrak{D}, c)$ defined in the following manner. Given $x \in \overset{(\bullet)}{\mathcal{A}} \otimes \overset{(\bullet)}{\mathcal{A}} \otimes \cdots \otimes \overset{(\bullet)}{\mathcal{A}}$ (where

$$
\begin{array}{ccccc}
\begin{array}{c} \mathcal{A}\otimes\dot{\mathcal{A}} \\ \mathcal{A}\otimes\dot{\mathcal{A}}\{-1\} \end{array}
& \xrightarrow{d_{-2}=\begin{pmatrix} m \\ m \end{pmatrix}} &
\begin{array}{c} \dot{\mathcal{A}} \\ \oplus \\ \dot{\mathcal{A}}\{-1\} \\ \oplus \\ \dot{\mathcal{A}} \\ \oplus \\ \dot{\mathcal{A}}\{-1\} \end{array}
& \xrightarrow{d_{-1}=(\eta,-\eta)} &
\begin{array}{c} \dot{\mathcal{A}} \\ \oplus \\ \dot{\mathcal{A}}\{-1\} \end{array} \\
-2 & & -1 & & 0
\end{array}
$$

Figure 6.4: The dotted complex associated to the cube of resolutions depicted in Figure 6.3, with respect to the green cohomology class.

the copies of $\mathcal{A}$ may or may not be dotted) define

$$c(x) := \#\left(v_{\underset{-}{\bullet}}\right) - \#\left(v_{\underset{+}{\bullet}}\right) + \frac{1}{2}j(x) \tag{6.2.1}$$

where $\#\left(v_{\underset{+}{\bullet}}\right)$ denotes the number of $v_{\underset{+}{\bullet}}$ in x (likewise $\#\left(v_{\underset{-}{\bullet}}\right)$ the number of $v_{\underset{-}{\bullet}}$), and j the standard quantum degree. We refer to this grading as the c-*grading*. $\lozenge$

Of course, the c-grading contains no new information if c is a trivial cohomology class or if no circles within the cube of resolutions are assigned dots (see Proposition 6.2.5).

The components of the differential of $CDKh(\mathfrak{D}, c)$ split with respect to the c-grading as

$$d = d^0 + d^{+2}$$

where d^i is c-graded of degree i. For the m, Δ, and η maps, the particular splitting depends on the configuration of the dots assigned to the circles involved. We shall denote by $\_ \to \bullet\otimes\bullet$ a Δ map taking an undotted circle to two dotted circles, and so forth. We have suppressed the u/l superscripts for the m and Δ maps (as they do not interact with them). (Terms in parentheses denote components of the differential of doubled Lee homology, and are required later.)

$$m : \bullet\otimes\bullet \longrightarrow \_ \begin{cases} v_{\underset{++}{\bullet\bullet}} \xrightarrow{m^0} 0 & v_{\underset{++}{\bullet\bullet}} \xrightarrow{m^{+2}} v_+ \\ v_{\underset{+-}{\bullet\bullet}}, v_{\underset{-+}{\bullet\bullet}} \xrightarrow{m^0} v_- & v_{\underset{+-}{\bullet\bullet}}, v_{\underset{-+}{\bullet\bullet}} \xrightarrow{m^{+2}} 0 \\ v_{\underset{--}{\bullet\bullet}} \xrightarrow{m^0} 0\ (v_+) & v_{\underset{--}{\bullet\bullet}} \xrightarrow{m^{+2}} 0 \end{cases} \tag{6.2.2}$$

$$m : \bullet \otimes \_ \to \bullet \begin{cases} v_{\overset{\bullet}{+}+} \xrightarrow{m^0} v_{\overset{\bullet}{+}} & \qquad v_{\overset{\bullet}{+}+} \xrightarrow{m^{+2}} 0 \\ v_{\overset{\bullet}{+}-} \xrightarrow{m^0} 0 & \qquad v_{\overset{\bullet}{+}-} \xrightarrow{m^{+2}} v_{\overset{\bullet}{-}} \\ v_{\overset{\bullet}{-}+} \xrightarrow{m^0} v_{\overset{\bullet}{-}} & \qquad v_{\overset{\bullet}{-}+} \xrightarrow{m^{+2}} 0 \\ v_{\overset{\bullet}{-}-} \xrightarrow{m^0} 0\ (v_{\overset{\bullet}{+}}) & \qquad v_{\overset{\bullet}{-}\overset{\bullet}{-}} \xrightarrow{m^{+2}} 0 \end{cases} \tag{6.2.3}$$

In the case $m : \_ \otimes \_ \to \_$ we have $m^{+2} = 0$ so that $m^0 = m$.

$$\Delta : \bullet \to \bullet \otimes \_ \begin{cases} v_{\overset{\bullet}{+}} \xrightarrow{\Delta^0} v_{\overset{\bullet}{+}-} & \qquad v_{\overset{\bullet}{+}} \xrightarrow{\Delta^{+2}} v_{\overset{\bullet}{-}+} \\ v_{\overset{\bullet}{-}} \xrightarrow{\Delta^0} v_{\overset{\bullet}{-}-}\ (+v_{\overset{\bullet}{+}+}) & \qquad v_{\overset{\bullet}{-}} \xrightarrow{\Delta^{+2}} 0 \end{cases} \tag{6.2.4}$$

$$\Delta : \_ \to \bullet \otimes \bullet \begin{cases} v_+ \xrightarrow{\Delta^0} v_{\overset{\bullet}{+}\overset{\bullet}{-}} + v_{\overset{\bullet}{-}\overset{\bullet}{+}} & \qquad v_+ \xrightarrow{\Delta^{+2}} 0 \\ v_- \xrightarrow{\Delta^0} 0\ (v_{\overset{\bullet}{+}\overset{\bullet}{+}}) & \qquad v_- \xrightarrow{\Delta^{+2}} v_{\overset{\bullet}{-}\overset{\bullet}{-}} \end{cases} \tag{6.2.5}$$

Again, for $\Delta : \_ \to \_ \otimes \_$ we have $\Delta^{+2} = 0$.

$$\eta : \bullet \to \bullet \begin{cases} v^{\mathrm{u}}_{\overset{\bullet}{+}} \xrightarrow{\eta^0} v^{\mathrm{l}}_{\overset{\bullet}{+}} & \qquad v^{\mathrm{u}}_{\overset{\bullet}{+}} \xrightarrow{\eta^{+2}} 0 \\ v^{\mathrm{l}}_{\overset{\bullet}{+}} \xrightarrow{\eta^0} 0 & \qquad v^{\mathrm{l}}_{\overset{\bullet}{+}} \xrightarrow{\eta^{+2}} 2v^{\mathrm{u}}_{\overset{\bullet}{-}} \\ v^{\mathrm{u}}_{\overset{\bullet}{-}} \xrightarrow{\eta^0} v^{\mathrm{l}}_{\overset{\bullet}{-}} & \qquad v^{\mathrm{u}}_{\overset{\bullet}{-}} \xrightarrow{\eta^{+2}} 0 \\ v^{\mathrm{l}}_{\overset{\bullet}{-}} \xrightarrow{\eta^0} 0\ (2v^{\mathrm{u}}_{\overset{\bullet}{+}}) & \qquad v_{\overset{\bullet}{-}\overset{\bullet}{-}} \xrightarrow{\eta^{+2}} 0 \end{cases} \tag{6.2.6}$$

Finally, for $\eta : \_ \to \_$ we have $\eta^{+2} = 0$, as usual.

Theorem 6.2.4. *Let $\mathfrak{D}$ be a diagram of an oriented link in a thickened surface $\mathfrak{L} \hookrightarrow \Sigma_g \times I$ and $CDKh(\mathfrak{D}, c)$ its dotted doubled Khovanov complex with respect to $c \in H^1(\Sigma_g; \mathbb{Z}_2)$. The homology of $CDKh(\mathfrak{D}, c)$ with respect to d^0, the c-grading preserving component of the differential, is well-defined and is an invariant of $\mathfrak{L}$. It is denoted by $DKh(\mathfrak{L}, c)$, and refered to as the* doubled Khovanov homology of $\mathfrak{L}$ with respect to *c.*

Proof. We need only show that $DKh(\mathfrak{L}, c)$ is invariant under the Reidemeister moves. Consider the two cubes of resolutions associated to the two tangle diagrams involved in a Reidemeister move; in order for $DKh(\mathfrak{L}, c)$ to be an invariant we must have that any circles appearing in the cubes are not assigned dots. This holds as they are necessarily homologically trivial. We can then apply the now standard Bar-Natan proof [BN02;

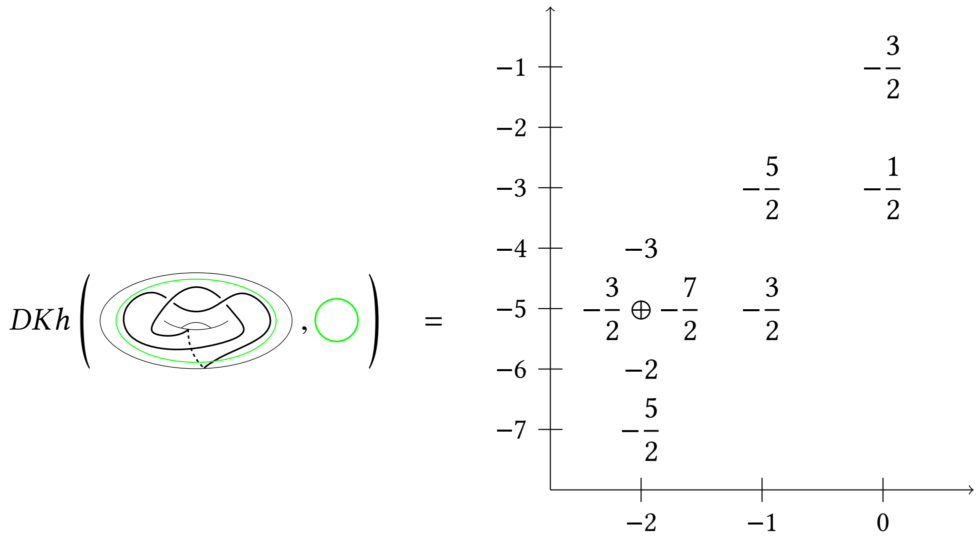


Figure 6.5: The homology of the complex given in Figure 6.4. The horizontal (vertical) axis denotes the homological (quantum) grading, and the terms on the grid points denote copies of $\mathbb{Z}$ which generate the homology, along with their c-grading.

MI13]; this hinges on the fact that m is surjective and Δ injective. For full details we refer the reader to [Man08a, Section 4]. □

An example of the resulting homology is given in Figure 6.5.

Of course, if the dotting associated to a cohomology class is trivial the resulting homology contains no new information.

Proposition 6.2.5. *Let $\mathfrak{D}$ be a diagram of an oriented link in a thickened surface $\mathfrak{L} \hookrightarrow \Sigma_g \times I$. Suppose that $c \in H^1(\Sigma_g; \mathbb{Z}_2)$ is homologically trivial, or is such that no circle in any resolution in $[\![\mathfrak{D}, c]\!]$ is assigned a dot. Then $DKh(\mathfrak{L}, c) = DKh(L)$, where $\mathfrak{L}$ projects to the virtual link L. This holds also for doubled Lee homology.*

6.2.3 Spectral sequences

As mentioned previously, Lee defined a spectral sequence on Khovanov homology whose E_∞ page is known as Lee homology. Rasmussen used Lee homology to define the s-invariant of classical knots. There is a similar spectral sequence on doubled Khovanov homology also, and the E_∞ page is doubled Lee homology, as discussed in Chapter 4. These spectral sequences are defined by adding extra terms to the differential. We can use this technique with respect to the c-grading defined in Section 6.2.2 also.

First, consider the homology with respect to the differential which includes the terms in parentheses in Equations (6.2.2) to (6.2.6). These terms raise the quantum grading by 4.

Theorem 6.2.6. *Denote by $\widetilde{d^0}$ the differential obtained by adding the terms in parentheses (in Equations (6.2.2) to (6.2.6)) to d^0. Let $CDKh'(\mathfrak{D}, c)$ be the chain complex with chain spaces equal to $CDKh(\mathfrak{D}, c)$ but with differential given by $\widetilde{d^0}$. The homology of $CDKh'(\mathfrak{D}, c)$ with respect to $\widetilde{d^0}$ is an invariant of the link represented by $\mathfrak{D}$, and is denoted $DKh'(\mathfrak{L}, c)$. We refer to $DKh'(\mathfrak{L}, c)$ as the* doubled Lee homology of $\mathfrak{L}$ with respect to c.

This homology is filtered with respect to the quantum grading, but graded with respect to the c-grading. Next, we introduce a filtration of the c-grading also.

Definition 6.2.7. Let $\mathfrak{D}$ be a diagram of a link in a thickened surface $\mathfrak{L} \hookrightarrow \Sigma_g \times I$, and pick $c \in H^1(\Sigma_g; \mathbb{Z}_2)$. Let $CDKh''(\mathfrak{D}, c)$ be the chain complex with chain spaces equal to $CDKh'(\mathfrak{D}, c)$, and differential obtained from $\tilde{d}^0$ by adding d^{+2} (defined in Equations (6.2.2) to (6.2.6)). Denote this differential by d'', and define $DKh''(\mathfrak{L}, c)$ to be the homology of $CDKh''(\mathfrak{D}, c)$ with respect to it. ◊

Theorem 6.2.8. *The homology $DKh''(\mathfrak{L}, c)$ is an invariant of $\mathfrak{L}$, and is refered to as the* totally reduced homology of $\mathfrak{L}$ with respect to c.

Proof. A diagram of a link in a thickened surface projects to a diagram of virtual link. It is easy to see that the chain complex $CDKh''(\mathfrak{D}, c)$ is equal to the standard doubled Lee complex associated to the virtual link diagram to which $\mathfrak{D}$ projects to; we have added the components which raise the c-grading, recovering the full differential. That doubled Lee homology is invariant under the virtual Reidemeister moves shows that $DKh''(\mathfrak{L}, c)$ is invariant also. □

Corollary 6.2.9. *Let $\mathfrak{L} \hookrightarrow \Sigma_g \times I$ be a link in a thickened surface. Forgetting the c-grading $DKh''(\mathfrak{L}, c) \cong DKh'(L)$, where L denotes the virtual link projected to by $\mathfrak{L}$.*

As a result of Corollary 6.2.9 we see that, ignoring the c-grading, DKh'' behaves identically to the doubled Lee homology of virtual links. As explored in Chapter 4, an important trait of doubled Lee homology is that its rank is determined by the number of *alternately coloured resolutions* of the argument link; thus it is possible for DKh'' to vanish. When using DKh'' to define invariants of surfaces we must take care of this phenomenon.

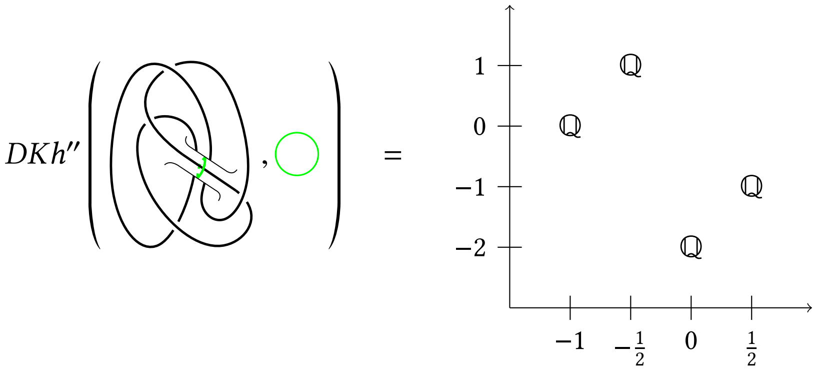


Figure 6.6: The totally reduced homology of the knot depicted with respect to the generator depicted in green; the knot is a lift of the virtual knot 4.12. All of the generators are at homological grading 0, and the horizontal (vertical) axis denotes the c-grading (quantum grading).

Of course, we could have added the terms in Equations (6.2.2) to (6.2.6) which raise the c-grading first then added the terms in parentheses second, to arrive at the totally reduced homology. This naturally leads to the conjecture that there exists the commutative square of spectral sequences:

$$\begin{tikzcd}[column sep=small] & DKh(\mathfrak{L}, c) \arrow[dl] \arrow[dr] & \\ \mathcal{H}(\mathfrak{L}, c) \arrow[dr] & & DKh'(\mathfrak{L}, c) \arrow[dl] \\ & DKh''(\mathfrak{L}, c) & \end{tikzcd}$$

where $\mathcal{H}(\mathfrak{L}, c)$ is the homology obtained from $DKh(\mathfrak{L}, c)$ by adding the terms labelled d^{+2} in Equations (6.2.2) to (6.2.6). We conclude by remarking that the groups $\mathcal{H}(\mathfrak{L}, c)$ and $DKh'(\mathfrak{L}, c)$ are mysterious; understanding their structure is an interesting direction of further research.

6.2.4 Interaction with cobordisms

In this section we describe the process of associating maps between homology groups to cobordisms between links in thickened surfaces, and a prove result analogous to one of Chapter 4.

In the theory of links in thickened surfaces, cobordisms are pairs consisting of a surface S and an oriented 3-manifold M such that $S \hookrightarrow M \times I$ (identically to the case of virtual concordance, described in Chapter 3). In this section, however, we shall be restricting ourselves to cobordisms of the form $S \hookrightarrow \Sigma_g \times I \times I$ i.e. where $M = \Sigma_g \times I$. This is be done to ensure that we have a meaningful way of comparing the homologies of the inital and terminal link: the homology theory takes as argument a link in a thickened surface $\mathfrak{L} \hookrightarrow \Sigma_g \times I$ and an element of $H^1(\Sigma_g; \mathbb{Z}_2)$, and if allowing for more general 3-manifolds there is no obvious way to coherently select a cocycle throughout a cobordism.

Once this restriction has been made the assignment of maps to cobordisms is straightforward, and is done in the manner described in Section 4.4.2: maps associated to elementary cobordisms are defined, and the map assigned to a generic cobordism is the appropriate composition of elementary maps.

Given $\mathfrak{L} \hookrightarrow \Sigma_g \times I$, $c \in H^1(\Sigma_g; \mathbb{Z}_2)$, the three theories $DKh(\mathfrak{L}, c)$, $DKh'(\mathfrak{L}, c)$, and $DKh''(\mathfrak{L}, c)$ are nested i.e. there is the relationship:

$$DKh(\mathfrak{L}, c) \xrightarrow[\text{add Lee components}]{\text{filtration in } j\text{-grading}} DKh'(\mathfrak{L}, c) \xrightarrow[\text{add } d^{+2} \text{ components}]{\text{filtration in } c\text{-grading}} DKh''(\mathfrak{L}, c)$$

Thus it is sufficient to describe the process of assigning maps on DKh to cobordisms, as the process is identical for DKh' and DKh'' modulo taking the appropriate filtration and adding the appropriate components to the maps assigned to 1-handles (the maps assigned to 0- and 2-handles remain the same).

The process is identical to that of Section 4.4.2; we comment only on the interaction of the elementary handle cobordisms with the c-grading.

Definition 6.2.10. Let $S \hookrightarrow \Sigma_g \times I \times I$ be an elementary handle addition (i.e. a cobordism containing exactly one Morse critical points, see Definition 4.4.14). Denote by ϕ_S the associated map on DKh, as given below.

(0-*handles*): If S is a 0-handle then $\phi_S = \iota$, where $\iota : \mathbb{Q} \rightarrow \mathcal{A}$, $\iota(1) = v_+^{\mathrm{u/l}}$, so that $\iota(1) \otimes v_+^{\mathrm{u}} = v_{++}^{\mathrm{u}}$, for example. The newly created circle cannot possess a dot, as it is contractible. Thus ι is c-filtered of degree $+\frac{1}{2}$.

(1-*handles*): If S is a 1-handle then ϕ_S acts as either m^0, Δ^0, or η^0 (as defined in Equations (6.2.2) to (6.2.6)) - which map is determed by the effect of S on individual resolutions. It is clear that ϕ_S is c-filtered of degree $-\frac{1}{2}$.

(2-*handles*): If S is a 2-handle then $\phi_S = \epsilon$, where $\epsilon : \mathcal{A} \to \mathbb{Q}$, $\epsilon(v_+^{\mathrm{u/l}}) = 0$, $\epsilon(v_-^{\mathrm{u/l}}) = 1$. As the circle being killed is contractible, ϵ is also c-filtered of degree $+\frac{1}{2}$. ◊

We repeat this method to assign maps $\phi'_S : DKh'(L,c) \to DKh'(L',c)$ and $\phi''_S : DKh''(L,c) \to DKh''(L',c)$ by taking filtrations and adding terms to the differential. They are defined as follows.

Definition 6.2.11. (0-*handles*): ϕ_S, ϕ'_S, and ϕ''_S are all of the same form, as ι is unchanged.

(1-*handles*): ϕ'_S is obtained from ϕ_S by adding the terms in parentheses in Equations (6.2.2) to (6.2.6), and ϕ''_S from ϕ'_S by adding the d^{+2} terms.

(2-*handles*): ϕ_S, ϕ'_S, and ϕ''_S are all of the same form, as ϵ is unchanged. ◊

The maps ϕ'_S and ϕ''_S are of the same c-degree as ϕ_S. In the case of 0- and 2-handles this is obvious. In the case of 1-handles, one can see this by noting that, although the components of d^{+2} raise the c-grading by $\frac{3}{2}$ (as cobordism maps), we have taken an (upward) filtration, so the filtration degree of ϕ_S, ϕ'_S, and ϕ''_S depends only on terms whose c-grading is lowered.

As in the case of cobordism maps on classical or doubled Khovanov homology, ϕ_S is homologically graded of degree 0, and quantum filtered of degree $0, -1$, or $+1$, depending on its type.

In summary, we have the following maps assigned to elementary cobordisms.

Definition 6.2.12. Assigned to an elementary cobordism S between $\mathfrak{L}$ and $\mathfrak{L}'$, we have the three maps $\phi_S : DKh(\mathfrak{L},c) \to DKh(\mathfrak{L}',c)$, $\phi'_S : DKh'(\mathfrak{L},c) \to DKh'(\mathfrak{L}',c)$ and $\phi''_S : DKh''(\mathfrak{L},c) \to DKh''(\mathfrak{L}',c)$: they are all trigraded of degree $(0, x, \frac{1}{2}x)$, $x \in \{0, \pm 1\}$, where (i, j, c) denotes the trigrading given by the homological, quantum, and c-gradings (in that order). ◊

Using these maps we can define the map assigned to a generic cobordism, exactly as in Section 4.4.2.

Definition 6.2.13. Let $S \hookrightarrow \Sigma_g \times I \times I$ be a cobordism between links $\mathfrak{L}, \mathfrak{L}' \hookrightarrow \Sigma_g \times I$, such that

$$S = S_1 \cup S_2 \cup \cdots \cup S_n$$

where S_i is an elementary cobordism. Define $\phi_S : DKh(\mathfrak{L}, c) \to DKh(\mathfrak{L}', c)$ as

$$\phi_S = \phi_{S_n} \circ \phi_{S_{n-1}} \circ \cdots \circ \phi_{S_1}$$

and likewise $\phi'_S : DKh'(\mathfrak{L}, c) \to DKh'(\mathfrak{L}', c)$ and $\phi''_S : DKh''(\mathfrak{L}, c) \to DKh''(\mathfrak{L}', c)$. ◊

Proposition 6.2.14. *Let $S \hookrightarrow \Sigma \times I \times I$ be a genus 0 cobordism between a knot $\mathfrak{K} \hookrightarrow \Sigma_g \times I$ and a link $\mathfrak{L} \hookrightarrow \Sigma_g \times I$, such that $DKh''(\mathfrak{L}, c)$ is non-trivial and S contains no closed components. Then $\phi''_S : DKh''(\mathfrak{K}, c) \to DKh''(\mathfrak{L}, c)$ is c-filtered of degree 0, and has trivial kernel. If S is between two knots then ϕ_S is an isomorphism.*

Proof. It is shown in Theorem 4.4.17 that the map on doubled Khovanov homology assigned to S is non-zero. Combining this with Corollary 6.2.9, in particular the fact that the terms of the differential of DKh are equal to those of DKh'', we see that ϕ''_S is non-trivial. It is not stated explicitly in the proof of Theorem 4.4.17, but this implies that ϕ_S has trivial kernel: carefully applying the proof applied to cobordisms which begin with a knot (as S does here) while keeping the relationship between the elements of the canonical basis in mind makes this clear. In the case in which S is between two knots, we see that ϕ_S is an injective linear map with domain and codomain a vector space of rank 4; by the Rank-Nullity Theorem it is surjective.

To see that ϕ''_S is c-filtered of degree 0, we recall that in any decomposition of S into elementary cobordisms the number of 0- and 2-handles must equal the number of 1-handles. To conclude, we notice that the degree of the map assigned to a 0- or a 2-handle cancels exactly with that of the map assigned to 1-handles (analogously to the quantum degree situation). □

6.2.5 Obstructions to the existence of embedded discs from DKh''

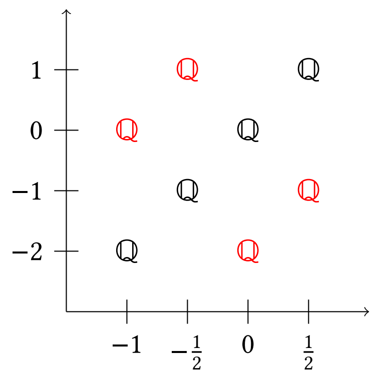
Let $\mathfrak{K} \hookrightarrow \Sigma_g \times I$ be a knot in a thickened surface. All three gradings of $DKh''(\mathfrak{K}, c)$ obstruct the existence of a disc bounding $\mathfrak{K}$ in $\Sigma_g \times I \times I$; that the homological and quantum gradings do follows from the properties of the doubled Rasmussen invariant. In this section we show that the c-grading does also.

Theorem 6.2.15. *Let $\mathfrak{K} \hookrightarrow \Sigma_g \times I$ be a knot in a thickened surface. Pick $c \in H^1(\Sigma_g; \mathbb{Z}_2)$ and compute $DKh''(\mathfrak{K}, c)$. If $DKh''(\mathfrak{K}, c)$ is non-trivial then $\mathfrak{K}$ does not bound a disc in $\Sigma_g \times I \times I$.*

Proof. As mentioned above, a knot $\mathfrak{K}$ for which $DKh''(\mathfrak{K}, c)$ has non-trivial homological or quantum degree is not slice as a virtual knot, so that in particular it cannot bound a disc in $\Sigma_g \times I \times I$. As such, we shall focus on the case in which $DKh''(\mathfrak{K}, c)$ has trivial homological and quantum gradings, but non-trivial c-grading.

First we shall show that if there exists a quantum degree, $\mathfrak{j}$, such that for all $x \in DKh''(\mathfrak{K}, c)$ with $\mathfrak{j} \leq j(x)$ then $c(x) < j(x) - \frac{1}{2}$, then $\mathfrak{K}$ does not bound a disc in $\Sigma_g \times I \times I$.

Let such a $\mathfrak{j}$ exist and assume towards a contradiction that there exists a disc embedded in $\Sigma_g \times I \times I$ which bounds $\mathfrak{K}$. Then there exists a concordance, S, from a contractible loop in Σ_g to $\mathfrak{K}$, formed by cutting the disc open. Denote this loop by U, so that $\phi_S : DKh''(U, c) \longrightarrow DKh''(\mathfrak{K}, c)$. The totally reduced homology of U is denoted by the black generators below:



All of the generators are at homological degree 0, and the horizontal (vertical) axis denotes the c-grading (quantum grading). By Proposition 6.2.14 ϕ_S is j- and c-filtered of degree 0. Thus generators of $DKh''(U, c)$ cannot decrease in either j- or c-grading, and one may think of them as being permitted to move only up and to right under the action of ϕ_S (when depicted on grids such as the one above). However, as $\mathfrak{j}$ exists, one sees that there must be generators of $DKh''(U, c)$ such that there are no available generators of $DKh''(\mathfrak{K}, c)$ above and to the right of them. Also by Proposition 6.2.14 we have that ϕ_S is an isomorphism so that such generators cannot be sent to zero, yielding the desired contradiction.

To conclude we claim that if $\mathfrak{K}$ is such that $DKh''(\mathfrak{K}, c)$ is not that of the unknot, then it must be equal to the homology depicted by the red generators on the grid above. This can be shown using essentially identical arguments to those used to determine analogous properties of the j-grading; see Lemma 4.5.2. Clearly $\mathfrak{j} = 0$ for the homology depicted by the red generators, so that if $\mathfrak{K}$ does not have the totally reduced homology of an unknot then it does not bound a disc in $\Sigma_g \times I \times I$. □

6.3 Ascent sliceness

In this section we introduce the notion of *ascent sliceness* of virtual knots. It represents an attempt to conduct a finer investigation of sliceness by analysing the 3-manifolds which appear in concordances from a given slice virtual knot to the unknot.

As mentioned in Chapter 3, passing to virtual knot theory allows one to ask new questions of a 3-dimensional flavour. One example is the minimal supporting genus (Definition 3.1.5 on page 21), a measure of how a virtual link $L \hookrightarrow \Sigma_g \times I$ is knotted about the topology of Σ_g. It is natural to ask how much of this knotting may be removed through a concordance.

Definition 6.3.1. Let $K \hookrightarrow \Sigma_g \times I$ be a virtual knot. Define the *minimal concordance genus of* K, denoted $m^*(K)$, as

$$m^*(K) = \min\left(\{m(K') \mid K \text{ is concordant to } K'\}\right) \tag{6.3.1}$$

where $m(K')$ denotes the minimal supporting genus of K'.

The minimal concordance genus is an interesting property, which current invariants of virtual knots contain very little information on [1]. Moreover, a slice virtual knot has minimal concordance genus 0, clearly. The introduction of ascent sliceness is an attempt at producing a non-trivial refinement of sliceness for virtual knots, by considering the complexity of the 3-manifolds appearing in concordances to the unknot, as opposed to that of the surfaces.

It is not known whether the set of ascent slice virtual knots is nonempty; it it were, it would be a manifestation of the ubiqituous principle of "increase before decrease", as seen in the hard unknot diagrams of Kauffman and Lambropoulou [KL12], and handlebody decompositions of manifolds, among many other contexts. Whilst we do not

[1] any slice obstruction also obtructs a virtual knot from having minimal concordance genus 0, of course, but that's about as much as we can say currently.

present an ascent slice virtual knot, in Section 6.3.2 we use the totally reduced homology defined in Section 6.2 to define a property which implies ascent sliceness for slice virtual knots of minimal supporting genus 1.

6.3.1 Definition

We give the formal definition of ascent concordance, which specialises to the case of ascent sliceness.

Definition 6.3.2. Let (S, M) be a cobordism. Fix a Morse function $f : M \longrightarrow I$ such that the restriction of f to S is a Morse function also. We say that a virtual link $J \hookrightarrow \Sigma_l \times I$ *appears in* S if $S \cap (f^{-1}(t) \times I) = J$, for some $t \in I$ with $f^{-1}(t) = \Sigma_l$. ◊

Definition 6.3.3. Let L and L' be concordant virtual links, and S a concordance between them. We say that S is *ascent* if a (representative of a) virtual link, $J \hookrightarrow \Sigma_g \times I$, appears in S such that $g > m(L), m(L')$. If every concordance between L and L' is ascent we say that L and L' are *ascent concordant*. ◊

That is, a concordance $S \hookrightarrow M \times I$ is ascent if the genus of surfaces appearing as level surfaces of Morse functions on M is at some point greater than the minimal supporting genera of both L and L'.

Definition 6.3.4. Let K be a slice virtual knot. If every concordance from K to the unknot is ascent, then K is *ascent slice*. ◊

Of course, the unknot has minimal supporting genus 0, so that a slice virtual knot K is ascent slice if, in every concordance from K to the unknot, a virtual link appears whose minimal supporting genus is greater than that of K.

Boden and Nagel showed that if a classical knot is slice when treated as a virtual knot, then it was already slice as a classical knot [BN16]. Equivalently, there are no ascent slice classical knots.

6.3.2 A source of potential examples

In this section we define a property which implies ascent sliceness for slice virtual knot of minimal supporting genus 1; the property is defined using the totally reduced ho-

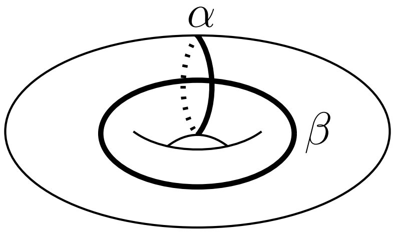


Figure 6.7: The generators α and β.

mology of Section 6.2; in particular it employs the obstructions to embedded discs of Section 6.2.5.

For ease, we shall fix a basis of $H^1(\Sigma_1; \mathbb{Z}_2)$. Let α be the class represented by the curve of that label in Figure 6.7, and likewise β, so that $\{\alpha, \beta\}$ forms a basis. We shall abuse notation and denote by α both the curve and the cohomology class represented by it (likewise β). Let $\bigcirc$ denote the unique knot in $\Sigma_1 \times I$ which bounds a disc, whose totally reduced homology $DKh''(\bigcirc, \gamma)$ is as follows: it is of rank 4, supported in homological grading 0, quantum gradings $\{1, 0, -1, -2\}$, and γ-gradings $\{\frac{1}{2}, 0, -\frac{1}{2}, -1\}$, for all $\gamma \in \{\alpha, \beta\}$.

For the remainder of this section all virtual knots and links have minimal supporting genus equal to 1 unless otherwise stated.

Definition 6.3.5. Let $\mathfrak{K}$ be a knot in $\Sigma_1 \times I$. We say that $\mathfrak{K}$ is *totally non-trivial* if $DKh''(\mathfrak{K}, \gamma) \neq DKh''(\bigcirc, \gamma)$, for all $\gamma \in \{\alpha, \beta, \alpha + \beta\}$. ◊

A non-trivial example of the totally reduced homology of a knot in $\Sigma_1 \times I$ is given in Figure 6.6 (this knot projects to the virtual knot 4.12 in Green's table [Gre]). The totally reduced homology with respect to the red curve is equal to that of $\bigcirc$, however, so the knot depicted is not totally non-trivial.

Proposition 6.3.6. *Let $D \hookrightarrow \Sigma_1 \times I$ and $D' \hookrightarrow \Sigma_1 \times I$ be representatives of a virtual knot K. As described in Section 6.1 D and D' define knots in $\Sigma_1 \times I$, denoted $\mathfrak{K}_D$ and $\mathfrak{K}_{D'}$, respectively. If $\mathfrak{K}_D$ is totally non-trivial, then $\mathfrak{K}_{D'}$ is also.*

Proof. As D and D' are genus-minimal representatives of the same virtual knot, Kuperberg's Theorem implies that $\mathfrak{K}_D$ and $\mathfrak{K}_{D'}$ are related by finite sequence of isotopies and Dehn twists i.e. there is the sequence of diagrams

$$\mathfrak{D}_1 \longrightarrow \mathfrak{D}_2 \longrightarrow \cdots \longrightarrow \mathfrak{D}_n$$

where $\mathfrak{D}_i$ is a diagram of $\mathfrak{K}_i$, $\mathfrak{K}_1 = \mathfrak{K}_D$, $\mathfrak{K}_n = \mathfrak{K}_{D'}$, and $\mathfrak{D}_{i+1}$ is obtained from $\mathfrak{D}_i$ by a Reidemeister move or Dehn twist. We may also assume that if $\mathfrak{D}_i \longrightarrow \mathfrak{D}_{i+1}$ is a Dehn twist, it is a twist about α or β (all twists may be written as a composition of these elementary twists).

If $\mathfrak{D}_i \longrightarrow \mathfrak{D}_{i+1}$ is a Reidemeister move, then the proposition holds as the totally reduced homology is an invariant of links in thickened surfaces.

Let $\mathfrak{D}_i \longrightarrow \mathfrak{D}_{i+1}$ be a Dehn twist. This twist sends $\mathfrak{D}_i$ to $\mathfrak{D}_{i+1}$, and as it is a self-diffeomorphism of Σ_1 it must permute the elements of the set $\{\alpha, \beta, \alpha + \beta\}$, from which the proposition follows. □

We may then define a virtual knot to be *totally non-trivial* if it has a totally non-trivial representative, and state the following theorem.

Theorem 6.3.7. *Let K be a slice virtual knot. If K is totally non-trivial then K is ascent slice.*

A slice virtual knot K is, of course, concordant to a classical knot. We prove Theorem 6.3.7 by focussing on cobordisms from K embedded into $\Sigma_1 \times I \times I$, in order to demonstrate that if a link appears in such a cobordism, then it cannot be destabilised (so that a stabilisation to higher genus surface must be made in any concordance between K and the unknot). The crux of the proof is that a link can only be destabilised along simple closed curves to which it is disjoint, and the c-grading of the totally reduced homology is sensitive to the intersection between links and a simple closed curve representing c.

Proof of Theorem 6.3.7. As K is a slice virtual knot, Corollary 6.2.9 shows that the homological and quantum gradings of $DKh''(\mathfrak{K}_D, c)$ are those of the unknot (for any representative D of K)[2]. Therefore, the totally non-trivial condition is equivalent to the homology groups $DKh''(\mathfrak{K}_D, \alpha)$, $DKh''(\mathfrak{K}_D, \beta)$, and $DKh''(\mathfrak{K}_D, \alpha + \beta)$ having non-trivial α-, β- and $(\alpha + \beta)$-gradings, respectively.

By Theorem 6.2.15 K does not possess a representative D such that $\mathfrak{K}_D$ bounds a disc in $\Sigma_1 \times I \times I$. To show that K is ascent slice, therefore, we must show that it does not exist a

[2]both the homological and quantum gradings of doubled Lee homology are slice obstructions, so that the homology of a slice virtual knot must be trivial in both.

genus 0 cobordism in $\Sigma_1 \times I \times I$ which cobounds (a representative) of K and a link which can be destabilised. That is, we must show that if there exists a link, $\mathfrak{L}$, in $\Sigma_1 \times I$ and a genus 0 cobordism, $S \hookrightarrow \Sigma_1 \times I \times I$, with $\partial S = \mathfrak{K}_D \sqcup \mathfrak{L}$, then for γ a simple closed curve on Σ_1 we have $\gamma \cap \mathfrak{L} \neq \emptyset$.

Assume towards a contradiction that there exists such a link and genus 0 cobordism pair, $\mathfrak{L}$ and S, and a simple closed curve γ on Σ_1 such that $\gamma \cap \mathfrak{L} = \emptyset$. Then, by the contrapositive to Proposition 6.2.5, we have that $DKh''(\mathfrak{L}, [\gamma])$ possesses no non-trivial $[\gamma]$-gradings. Further, $[\gamma] \in \{\alpha, \beta, \alpha + \beta\}$, and by assumption $DKh''(\mathfrak{K}_D, [\gamma])$ has non-trivial $[\gamma]$-gradings. But by Proposition 6.2.14 the map ϕ_S is an isomorphism onto its image in $DKh''(\mathfrak{L}, [\gamma])$. As $DKh''((K)_D, [\gamma])$ has non-trivial $[\gamma]$-gradings, while $DKh''(\mathfrak{L}, [\gamma])$ does not, the existence of this graded isomorphism yields the desired contradiction.

Therefore, there does not exist a representative of K which is concordant (in $\Sigma_1 \times I \times I$) to a link which can be destabilised, and any concordance (of virtual knots) from K to the unknot must exhibit a handle stabilisation to at least $\Sigma_2 \times I$. $\square$

We conclude by remarking that totally non-trivial property can be defined for virtual knots of higher minimal supporting genus than 1, and that it can be demonstrated that it implies ascent sliceness for such virtual knots in essentially identical fashion to the genus 1 case. However, for a virtual knot of minimal supporting genus g, determining if it is totally non-trivial requires the computation of $\sum_{i=0}^{g}(g-i)$ homology groups, rendering the technique impractical.

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