Evaluation of Correlation Functions in Integrable Quantum Field Theories

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A Thesis presented for the degree of
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Dedicated to

My parents, brothers and Sylvie.

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Abstract

The aim of this thesis is to explore correlation functions in two dimensional quantum field theories in two distinct ways. In part I a new method for calculating the differential equations parametrising the correlation functions of twist fields associated with the $U(1)$ symmetry of the Dirac model is presented. While developing this method a new family of descendent twist fields are identified and their form factors calculated. This provides a novel way of calculating the vacuum expectation values of the primary twist fields and is shown to be entirely consistent with known results. The method of calculating the correlation functions of twist fields provides a parametrisation of several other correlation functions for various quantum states. Since this method relies on the Ward identities found in a double copy model it is hoped to have wider applications in other free fermion models. Part II concerns the truncated conformal space approach which has been developed to approximate perturbed conformal field theories. In this part the theory underpinning the approach is discussed and a working algorithm is developed for both bulk and boundary perturbed minimal models. The energy levels, mass gaps and one point functions of various models are computed using the truncated conformal space approach and are shown to be in good agreement with previous calculations. A possible method for using this approach to approximate two point functions in perturbed conformal field theories is discussed.

Declaration

The work in this thesis is based on research carried out at the Center for Particle Theory, the Department of Mathematical Sciences, England. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all my own work unless referenced to the contrary in the text. The material presented in part I was published as [28], in collaboration with Dr. B. Doyon, and [72].

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Chapter 1

General Introduction

Quantum field theory is one of the most widely used and studied tools in modern theoretical physics. This popularity is, in part, due to the number of applications it has found, being used to describe fundamental particles, electronic excitations in materials and thermal fluctuations in statistical systems to name three. Despite all this interest it is often very difficult to obtain accurate numerical results from quantum field theories, with many relativistic quantum field theories not being well-defined.

There are, however, classes of quantum field theories that are well-defined and provide accurate numerical results. Two such classes are free relativistic quantum field theories and two dimensional quantum field theories with conformal invariance. Indeed it is often the case that properties of more complicated models are obtained by treating them as perturbations of free or conformal quantum field theories. The general structure of quantum field theory is inferred from the properties of free and conformal quantum field theories.

Alongside free and conformal models there is a special class of models, two dimensional integrable quantum field theories, in which it is possible to evaluate certain quantities exactly. These models all contain an infinite number of symmetries and it is this property that allows objects, like the scattering matrix or correlation functions of local fields, to be evaluated exactly. While some integrable models have direct applications they are worthy of study in their own right as they can be used to verify expected qualities of quantum field theory and find new properties and

techniques for dealing with interacting theories.

The correlation functions of local fields of quantum field theories contain all the physical information of the model as all other quantities can, in principle, be obtained from the correlation functions. It is these useful objects that are the focus of this thesis. A new method for the calculation of correlation functions of twist fields in the massive Dirac model is described in part I, while the important principles of integrable models and twist fields are introduced in section 1.1 below. Part II of this thesis describes an attempt to use the truncated conformal space approach (TCSA) to approximate correlation functions in perturbed conformal field theories, this is introduced in section 1.2. This thesis concludes in chapter 8 with a general discussion of what has been achieved and some open problems. Note that this thesis will only deal with $(1+1)$ dimensional quantum field theories.

1.1 Twist Fields

1.1.1 Integrable Models

It is well know that integrable models of quantum field theory often allow their spectrum of particles and scattering matrices to be calculated exactly [19, 62]. In general it is a non-trivial task to obtain the correlation functions of local fields from this on-shell description but there are methods, applicable to integrable quantum field theories, that allow the calculation of correlation functions. One such method is based on the asymptotic states and the form factors of local fields [44, 75].

The asymptotic states form a basis of the Hilbert space of our theory. These states describe the particles of the model in the infinite past or infinite future, where they are assumed to be infinitely separated. As all the interactions in our models are local these asymptotic states propagate freely and so represent a set of non-interacting particles with well-defined momenta. As we wish to examine the interactions of the particles in our model we define an *in*-state as an asymptotic state in the infinite past, with the particles ordered from left to right with decreasing rapidities. Similarly an *out*-state has particles its particles ordered from left to right with increasing rapidities. Thus the *in* (*out*) asymptotic states represent particle

configurations in which no more interactions occur in the past (future).

Asymptotic states are constructed from creation and annihilation operators in the infinite past or future acting on a vacuum state. These operators create and destroy particles of fixed rapidity and are eigenoperators of the Hamiltonian and momentum operator, so the states they create are eigenstates of these operators. Thus the Hilbert space of our model is just the Fock space over the algebra of all the *in* creation and annihilation operators, which is isomorphic to the Fock space over all *out* operators.

Alongside energy and momentum, integrable models possess an infinite number of local conserved quantities, all in involution. When applied to asymptotic states these conservation laws relate to the free propagation of the particles and so are referred to as deformations of the free laws [51]. The free conservation laws that are deformed are those whose action on a state is given by an integral power of the momentum multiplied by an integral power of the energy for each particle, note the powers of the energy and momentum may be different but must be the same for each particle. The infinite set of integer spins, i.e. the power of the momentum mentioned above, for which there is a conserved charge is a characterising feature of an integrable model. All these conserved charges impose the stringent condition that the number of particles and the final set of rapidities in the *out*-state is the same as that of the corresponding *in*-state. This tells us that the scattering in these models is purely elastic [51, 65].

The arguments of [71] tell us that the conserved charges of higher spins generate changes in the impact parameters of a scattering process. As the action of these charges is independent of time, final states with different impact parameters are proportional. In this way we may shift the impact parameters so as to separate a scattering process into several well defined sub processes involving only two particles. This renders the scattering process factorisable [71, 81, 86] so that the multi-particle scattering matrix, or S-matrix, can be written in terms of two particle S-matrix elements:

$$|A_{a_1}(\theta_1)A_{a_2}(\theta_2)\rangle = S_{a_1a_2}^{b_1b_2}(\theta_1-\theta_2)|A_{b_1}(\theta_2)A_{b_2}(\theta_1)\rangle \tag{1.1.1}$$

where $S_{a_1a_2}^{b_1b_2}(\theta_1-\theta_2)$ is the two particle S-matrix and $A_i(\theta)$ represents a particle of

type i with rapidity θ.

In one space dimension the particles propagate on a line so all the necessary scattering required to convert an *in*-state to an *out*-state must occur. As the conserved charges of an integrable model allow us to alter the impact parameters it is possible to move from an *in*-state to an *out*-state in more than one way, i.e. by shifting the particles so that different combinations collide first. This constrains the two particle S-matrix to satisfy the Yang-Baxter equations:

$$S_{a_1a_2}^{b_1b_2}(\theta_1-\theta_2)S_{b_1a_3}^{c_1b_3}(\theta_1-\theta_3)S_{b_2b_3}^{c_2c_3}(\theta_2-\theta_3)=S_{a_2a_3}^{b_2b_3}(\theta_2-\theta_3)S_{a_1b_3}^{b_1c_3}(\theta_1-\theta_3)S_{b_1b_2}^{c_1c_2}(\theta_1-\theta_2) \tag{1.1.2}$$

where repeated indices are summed over.

This factorisation of the S-matrix can be implemented by identification of the *in* and *out* bases of the Hilbert space with two bases of Zamolodchikov's algebra [86] whose generating elements have the exchange relations

$$A_{a_1}(\theta_1)A_{a_2}(\theta_2)=S_{a_1a_2}^{b_1b_2}A_{b_2}(\theta_2)A_{b_1}(\theta_1). \tag{1.1.3}$$

The *in* basis is identified with products of As in descending order from left to right and the *out* basis increasing from left to right and the associativity of this algebra gives the Yang-Baxter equations.

When considering models invariant under charge conjugation there are additional constraints on the two particle S-matrix [29, 86]:

Real analyticity: $\left(S_{a_1a_2}^{b_1b_2}(\theta)\right)^*=S_{a_1a_2}^{b_1b_2}(-\theta)$

Unitarity: $S_{a_1a_2}^{b_1b_2}(\theta)S_{b_1b_2}^{c_1c_2}(-\theta)=\delta_{a_1}^{c_1}\delta_{a_2}^{c_2}$

Crossing symmetry: $S_{a_1a_2}^{b_1b_2}(i\pi-\theta)=C_{a_1c}S_{da_2}^{cb_2}(\theta)C^{db_1}$

Analytic structure: $S_{a_1a_2}^{b_1b_2}(\theta)$ is analytic in the strip $0<\Im(\theta)<\pi$ except for poles at θ and $i\pi-\theta$, where $\theta=i\arccos\left(\frac{M^2-M_1^2-M_2^2}{2M_1M_2}\right)$, corresponding to particles of mass M formed by bound states of particles of masses M_1 and M_2 in the s- and t-channel respectively and poles corresponding to Feynman diagrams with all the internal momenta on-shell.

The presence of bound states already in the spectrum of the model leads to a set of constraints called the bootstrap equations. These constraints affect the S-matrix, the masses of particles in the model, the set of spins of the conserved charges and the eigenvalues of the conserved charges. These constraints will not be discussed further here but are treated in the review [62].

The S-matrices and particles involved in several integrable models have been found using internal symmetries and bootstrap equations, see e.g. [14, 82]. We also note that properties of integrable models can sometimes be found using semi-classical techniques which are exact in this case. The spectrum of particles may be found in this way [18, 58]. The spectrum and S-matrix may also be computed via Bethe ansatz [2, 49] or the quantum inverse scattering method, for details see the book [50].

When dealing with integrable models, one way of connecting the on-shell description with knowledge of local fields is to consider matrix elements of local operators in the asymptotic basis. Matrix elements of an operator $\mathcal{O}$ of the form

$$F_{a_1\cdots a_n}(\theta_1,\cdots,\theta_n) = \langle 0|\mathcal{O}(x,y)|A_{a_1}(\theta_1)\cdots A_{a_n}(\theta_n)\rangle_{in} \tag{1.1.4}$$

are known as the form factors of $\mathcal{O}$. Crossing symmetry and knowledge of the scattering matrix allows all matrix elements to be written in terms of these form factors and so they give a complete description of an operator once they are known.

The form factors of local fields in integrable models can be constructed from their expected properties, which are a consequence of the factorised scattering. In [44] the consequences of factorised scattering were considered alongside the general principles of quantum field theory in order to develop a set of axioms for form factors. These axioms were then used to calculate form factors in the sine-Gordon model, but this method only gave access to form factors with three or fewer particles. It was not until the quantum version of the Gelfand-Levitan-Marchenko (GLM) equations were found in [17] and used to calculate both soliton and breather form factors in the sine-Gordon model [73, 74] that further progress was made. After an analysis of this method and its applications it was realised that the GLM equations are not essential to prove that an operator is local [46, 47], all that is required is that the

form factors satisfy a set of axioms. The form factors of local fields are constructed from these axioms [75]:

1. The function $F_{a_1\cdots a_n}(\theta_1,\cdots,\theta_n)$ is analytic in $\theta_i-\theta_j$ inside the strip $0<\Im(\theta_i-\theta_j)<2\pi$ except for simple poles.

2. Relativistic invariance implies that the F satisfies

$$F_{a_1\cdots a_n}(\theta_1+\theta,\cdots,\theta_n+\theta)=e^{s\theta}F_{a_1\cdots a_n}(\theta_1,\cdots,\theta_n) \tag{1.1.5}$$

   where s is the spin of the operator $\mathcal{O}$.

3. F has the symmetry property

$$F_{a_1\cdots a_n}(\theta_1,\cdots,\theta_n)= S^{b_kb_{k+1}}_{a_ka_{k+1}}(\theta_k-\theta_{k+1})F_{a_1\cdots b_{k+1}b_k\cdots a_n}(\theta_1,\cdots,\theta_{k+1},\theta_k,\cdots,\theta_n) \tag{1.1.6}$$

   which is a generalisation of Watson's theorem.

4. F has the locality property

$$F_{a_1\cdots a_{n-1}a_n}(\theta_1,\cdots,\theta_{n-1},\theta_n)=e^{2\pi i\omega(\mathcal{O},\Psi)}F_{a_na_1\cdots a_{n-1}}(\theta_n,\theta_1,\cdots,\theta_{n-1}) \tag{1.1.7}$$

   where $\omega(\mathcal{O},\Psi)$ is the mutual locality index between the operator $\mathcal{O}$ and the elementary field Ψ which is associated with the particle a_n. This index is the phase factor acquired by correlation functions when Ψ is brought, anti-clockwise, once around $\mathcal{O}$.

5. As a function of θ_n, F has simple poles at $\theta_n=\theta_j+i\pi$ for $j=1,\cdots,n-1$ and the residue of these poles is given by

$$\begin{aligned} &iF_{a_1\cdots a_n}(\theta_1,\cdots,\theta_n)\sim\\ &\big(\,\delta^{b_1}_{a_1}\cdots\delta^{b_{j-1}}_{a_{j-1}}S^{b_{j+1}c_j}_{a_{j+1}a_j}(\theta_{j+1}-\theta_j)S^{b_{j+2}c_{j+1}}_{a_{j+2}c_j}(\theta_{j+2}-\theta_j)\cdots S^{b_{n-1}b_j}_{a_{n-1}c_{n-3}}(\theta_{n-1}-\theta_j)-\\ &e^{2\pi i\omega(\mathcal{O},\Psi)}\delta^{b_{n-1}}_{a_{n-1}}\cdots\delta^{b_{j+1}}_{a_{j+1}}S^{c_{j-1}b_{j-1}}_{a_ja_{j-1}}(\theta_j-\theta_{j-1})S^{c_{j-1}b_{j-2}}_{c_ja_{j-2}}(\theta_j-\theta_{j-2})\cdots S^{b_jb_1}_{c_3a_1}(\theta_j-\theta_1)\,\big)\times\\ &C_{a_nb_j}\frac{F_{b_1\cdots\hat{b}_j\cdots b_n}(\theta_1,\cdots,\hat{\theta}_j,\cdots,\theta_n)}{\theta_n-\theta_j-i\pi} \end{aligned} \tag{1.1.8}$$

   where hats indicate that entry has been removed. These poles are known as kinematic poles and the form factors also contain poles at $\theta_n-\theta_j=i\arccos\frac{M^2-M_n^2-M_j^2}{2M_nM_j}\in i[0,\pi]$ corresponding to bound states of mass M.

It is axiom 5 which allows all form factors to be constructed using a bootstrap approach and the subsequent equations relate the residues at bound state poles with other form factors. It is thought that the space of solutions to the form factor axioms and bootstrap equations may be identified with the space of local fields in a integrable model [15, 56].

Knowledge of the form factors of a model allows long-distance expansions of correlation functions to be written down. This is accomplished by inserting a resolution of the identity, in the *in*- or *out*-basis:

$$\langle \mathcal{O}(x,t)\mathcal{O}(0,0)\rangle = \sum_{n=0}^{\infty}\sum_{a_1\cdots a_n}\int \frac{\mathrm{d}\theta_1\cdots\mathrm{d}\theta_n}{n!(2\pi)^n} e^{-r\sum_j M_j\cosh(\theta_j)} \times \langle 0|\mathcal{O}(0,0)|A_{a_1}(\theta_1)\cdots A_{a_n}(\theta_n)\rangle\langle A_{a_1}(\theta_1)\cdots A_{a_n}(\theta_n)|\mathcal{O}(0,0)|0\rangle \quad (1.1.9)$$

where $r = \sqrt{x^2 - t^2}$ and the factors of 2π are related to the normalisation of the basis.

The form factor axioms have been solved for various models [62]. In the chapter 2 these axioms are used to construct form factors for a new family of twist fields which are then used to verify relations obtained from conformal field theory in chapter 3.

1.1.2 Twist Fields in Free Fermion Models

Twist fields are present in any free fermion model which possesses an internal global symmetry and there is one twist field for every element of the symmetry group. These fields, sometimes referred to as disorder fields [43], are interacting fields which are not local with respect to the fundamental free fermion fields, that is to say that when a fermion field is brought smoothly once round a twist field the product acquires an extra phase factor. This type of twist field first appeared as the $\mathbb{Z}_2$ monodromy field of the Majorana fermion corresponding to the spin field of the Ising model [69].

The twist field σ_g, associated with symmetry group element g, is essentially defined by the property that when any other local field is brought round it once, anti-clockwise, inside a correlation function, it transforms according to the element g of the symmetry group. As this transformation is necessarily a symmetry of the

model the twist field is local with respect to the Hamiltonian and so is a field of the theory.

In the path integral formulation twist fields have a simple definition. Any correlation functions containing the twist field $\sigma_g(x)$, at the point x, are defined as the path integrals over field configurations with appropriate monodromy conditions. That is to say that the path integral is over all field configurations with the fundamental fields transformed according to g when they are taken once round x:

$$\langle \sigma_g(x) \cdots \rangle = \int_{\Psi \to \hat{g}\Psi \text{around} x} [\mathcal{D}\Psi]\, e^{-S[\Psi]\cdots} \tag{1.1.10}$$

which can easily be extended to correlation functions containing multiple twist fields. A cut in the plane over which the integration is taken is required in order for the fermion fields to be well defined there. This cut must start at x, and in massive models may be taken to end at infinity otherwise must end at the position of a conjugate twist field $\sigma_{g^{-1}}$, and across this cut fundamental fields are discontinuous.

The path integral definition above is independent of the shape of the cut and altering this only requires fields which the cut crosses to be transformed by g or g^{-1} depending on which way they cross the cut. To see this let us consider two cuts $\mathcal{C}_1$ and $\mathcal{C}_2$ such that $\mathcal{C} = \mathcal{C}_1 \cup \mathcal{C}_2$ is a closed path. Now when moving the cut in the path integral from $\mathcal{C}_1$ to $\mathcal{C}_2$ we simply transform all the fields inside $\mathcal{C}$, that is all fields inside the region bounded by $\mathcal{C}$ going from $\mathcal{C}_2$ to $\mathcal{C}_1$ anti-clockwise, according to the symmetry group element g. The only effect of this procedure is to add a contribution along $\mathcal{C}$ to the action:

$$\langle \sigma_g(x) \cdots \rangle_{\mathcal{C}_1} = \langle e^{\int_{\mathcal{C}} \mathrm{d}s^\mu j^\nu \epsilon_{\mu\nu}} \sigma_g(x) \cdots \rangle_{\mathcal{C}_2} \tag{1.1.11}$$

where $\mathrm{d}s^\mu$ is the line element along $\mathcal{C}$ and j^ν is the current associated with the charge generated by g. This extra contribution to the action is equivalent to applying the transformation g to all the fields inside $\mathcal{C}$ as promised.

Two twist fields are local with respect to each other if and only if their corresponding symmetry group elements commute. The action of the transformation g_1 on σ_{g_2} is to produce the twist field $\sigma_{g_1 g_2 g_1^{-1}}$. So from (1.1.11) the two fields σ_{g_1} and σ_{g_2} are local with respect to each other when $g_1 g_2 g_1^{-1} = g_2$, so we see that twist fields are self local.

For twist fields related to a $U(1)$ symmetry in free fermion models it is sometimes possible to calculate the one point functions of the associated twist fields [24, 57] and also to re-sum the form factor expansion of correlation functions in terms of Fredholm determinants [3, 7, 67, 68]. It is also known that in certain models it is possible to express these non-trivial correlation functions as solutions of integrable partial differential equations. This remarkable property was first observed in [79] for the two point spin-spin correlation function in the thermally perturbed lattice Ising model. It was later found that this result could also be obtained via alternate methods and in different models; using Dirac operators for the Dirac theory in flat and curved space-time with a magnetic field [55, 63], by considering a double model of the Ising spin chain and Ising quantum field theory at zero and finite temperature and on curved space times [27, 33, 64]. The existence of these differential equations allows the corresponding correlation functions to be evaluated to a very high accuracy once initial conditions have been fixed via conformal perturbation theory or form factor analysis [25, 27, 55].

Part I of this thesis is devoted to examining twist fields corresponding to the $U(1)$ symmetry of the massive Dirac theory. The aim of this part is to reproduce the differential equations satisfied by their two point functions, calculated previously using Fredholm determinants [7, 8], using a more direct method inspired by [33]. In [33] differential equations for the Ising spin-spin correlation functions are obtained by considering the conserved charges in a model containing two non-interacting copies of the Ising field theory. By using the Ward identities associated to conserved charges in this double model, alongside properties of the Ising model, differential equations for the spin-spin correlation functions are found.

Before deriving the differential equations we first develop the theory of Dirac $U(1)$ twist fields. The spinless, chargeless twist fields σ_α are constructed for all non-integer $\alpha \in \mathbb{R}$. These fields are characterised by their $U(1)$ monodromy $e^{2\pi i\alpha}$ and dimension α^2, as well as their fermionic descendents defined via the operator product expansion (OPE), $\sigma_{\alpha+1,\alpha} \sim \Psi_R^\dagger \sigma_\alpha$ and $\sigma_{\alpha-1,\alpha} \sim \Psi_R \sigma_\alpha$ (where Ψ_R is the right-moving component of the Dirac spinor). The construction is done both using explicit form factors, and using a CFT description that takes into account the constraints provided

by the massive perturbation. We evaluate the action of the double-model conserved charges using these two pictures.

After calculating the form factors of the descendent twist fields and examining the OPEs, a recursion relation for the VEVs $\langle \text{vac}|\sigma_\alpha|\text{vac}\rangle = m^{\alpha^2} c_\alpha$ is found:

$$c_\alpha = \frac{\Gamma(\alpha+1)}{\Gamma(-\alpha)} c_{\alpha+1}. \tag{1.1.12}$$

This relation is consistent with known results [4, 57] but can also be viewed as a method for evaluating c_α, given the normalisation $c_0 = 1$ and the symmetry $c_{-\alpha} = c_\alpha$

The main results of this part are as follows. Consider correlation functions in a quantum state whose density matrix is an exponential of a bilinear form in fermions, and which is translation and parity invariant. Writing the following correlation function as

$$\langle \sigma_\alpha(x,y)\sigma_\alpha(0,0)\rangle = c_\alpha^2 m^{2\alpha^2} e^{\Sigma_\alpha(x,y)} \tag{1.1.13}$$

we show that it is the solution of the differential equations:

$$\partial\bar{\partial}\psi = \frac{m^2}{2}\sinh(2\psi) \tag{1.1.14}$$

$$\partial\bar{\partial}\Sigma_\alpha = \frac{m^2}{2}(1-\cosh(2\psi)). \tag{1.1.15}$$

This result was first found implicitly in [68], where solutions to the Dirac equation satisfying the monodromy conditions of twist fields were studied, and in [7] the same result was obtained explicitly using the method of Fredholm determinants. Both methods, however, restricted the state where this result held and the fact that these equations apply more generally is a new result. We further find that the correlation function involving the descendent field $\sigma_{\alpha+1}$ along with σ_α is obtained from

$$\langle \text{vac}|\sigma_\alpha(x,y)\sigma_{\alpha+1}(0,0)|\text{vac}\rangle = c_\alpha c_{\alpha+1} m^{2\alpha^2+2\alpha+1} e^{\Sigma'_\alpha(x,y)} \tag{1.1.16}$$

via

$$\partial\bar{\partial}\Sigma'_\alpha = \frac{2\tanh^2(\psi)}{\cosh(2\psi)-1}\partial\psi\bar{\partial}\psi. \tag{1.1.17}$$

Note that this is different from the equation found for the correlation functions $\langle \text{vac}|\sigma_\alpha(x,y)\sigma_\beta(0,0)|\text{vac}\rangle$ in [7], but there is no contradiction, as there there was the

restriction $0 < \alpha, \beta < 1$. Finally for this quantum state, we find that the correlation function of fermionic descendents,

$$\langle \sigma_{\alpha+1,\alpha}(x,y)\sigma_{\alpha,\alpha+1}(0,0)\rangle = e^{-i\pi\alpha} c_\alpha c_{\alpha+1} m^{2\alpha^2+2\alpha+1}\Omega(x,y), \tag{1.1.18}$$

is algebraically expressed in terms of the previous correlation functions:

$$\Omega = \sqrt{e^{2\Sigma_\alpha} - e^{2\Sigma'_\alpha}}. \tag{1.1.19}$$

It is also possible to reproduce the full results of [7], that is after setting

$$\langle \sigma_\alpha(x,y)\sigma_\beta(0,0)\rangle = c_\alpha c_\beta m^{\alpha^2+\beta^2} e^{\Sigma(x,y)} \tag{1.1.20}$$

the function Σ is a solution of the equations

$$\left(\partial_r^2 + \frac{1}{r}\partial_r\right)\Sigma = \frac{m^2}{2}(1-\cosh(2\psi)) \tag{1.1.21}$$

$$\left(\partial_r^2 + \frac{1}{r}\partial_r\right)\psi = \frac{m^2}{2}\sinh(2\psi) + \frac{(\alpha-\beta)^2}{r^2}\tanh\psi(1-\tanh^2\psi). \tag{1.1.22}$$

This result, however, requires the quantum state under consideration to have rotational symmetry, as without this extra symmetry the Ward identities we derive do not provide enough constraints to solve the system. There is a discrepancy between our result and the results of [7], namely a factor of 4 in the $(\alpha-\beta)^2$ term in (1.1.22). This discrepancy is discussed in section 3.2.6 where strong evidence is presented that equation (1.1.22) is indeed satisfied by the two-point correlation functions.

With rotational symmetry imposed we also find differential equations for correlation functions of more general descendent twist fields:

$$\begin{aligned}\langle \sigma_{\alpha,\alpha+1}(x,y)\sigma_{\beta+1,\beta}(0,0)\rangle =& p e^{i\theta(\alpha-\beta)} c_\alpha c_\beta m^{\alpha^2+\alpha+\beta^2+\beta+1} e^{\Sigma'(r)} \\ \langle \sigma_{\alpha+1,\alpha}(x,y)\sigma_{\beta,\beta+1}(0,0)\rangle =& q e^{i\theta(\beta-\alpha)} c_\alpha c_\beta m^{\alpha^2+\alpha+\beta^2+\beta+1} e^{\Sigma'(r)}\end{aligned} \tag{1.1.23}$$

involving the same auxiliary function ψ as (1.1.22). Both correlators are shown to have the same r dependence and the constants p and q are discussed in more detail in section 3.2.2, but essentially depend on the positions of the fields through their braiding relations. It is shown the function Σ' satisfies

$$\left(\partial_r^2 + \frac{1}{r}\partial_r\right)\Sigma' = \frac{(\partial_r\psi)^2}{\sinh^2\psi} - m^2 - \frac{(\alpha-\beta)^2}{r^2\cosh^2\psi} \tag{1.1.24}$$

which is a new result that appears naturally from the Ward identities.

Part I is set out as follows. In chapter 2 we set the conventions of this part and the Dirac model is introduced and known results regarding primary twist fields are reviewed in section 2.1.2. In section 2.2 the massless limit of the Dirac theory is examined and through bosonisation techniques descendent twist fields are identified. These descendent twist fields are the subject of section 2.3 where they are defined and their form factors are calculated in the massive theory. It is these form factors which lead to a recursion relation for the VEV constant c_α. In chapter 3 we present a derivation of the differential equations mentioned above. Firstly the double model is introduced in section 3.1 and its conserved charges and their action are calculated and verified using form factors. In section 3.2 the quantum states of interest are first discussed and the consequences of space-time symmetries on the correlation functions are examined. It is in section 3.2.3 that Ward identities relating to the conserved charges in the double model are written down and in the following two sections the differential equations described above are found. This chapter ends in section 3.2.6 with a verification of the results presented here as there is a discrepancy between these and previous results. Finally chapter 4 contains a discussion of the outlook for this part.

1.2 Perturbed Conformal Field Theory

We now summarise some key aspects of perturbed conformal field theories and the truncated conformal space approach as these are key topics in part II of this thesis.

Quantum field theories in two dimensions which possess conformal symmetry have been a topic interest to mathematicians and physicists for some time. The attraction of these theories is due to the infinite dimension of the conformal group in two dimensions. This infinite symmetry group allows conformal field theories to be solved exactly, unlike most other quantum field theories. This means that, besides their direct applications to condensed matter and string theory, a study of conformal field theories may lead to a better understanding of more general quantum field theory.

Since the birth of modern conformal field theory in the paper of Belavin, Polyakov and Zamolodchikov [6] a lot of effort has gone into the study of conformal field theories as the reviews [34, 36, 38] show. As we know so much about conformal field theories in two dimensions it is natural to ask what happens when we break conformal symmetry and move away from an exactly solvable model. This raises several issues, the first of which is how conformal symmetry is broken, as there are many quantum field theories without conformal symmetry. It is natural that we want to break conformal symmetry in such a way that we may bring some of the vast arsenal of conformal field theory techniques to bear on our new theory, so in some sense we need our model to be 'close' to a conformal field theory. With this in mind the perturbed conformal field theories considered in this thesis have actions of the form

$$\mathcal{A} = \mathcal{A}_{CFT} + \lambda \int \mathrm{d}^2 x \varphi(x) \tag{1.2.1}$$

where $\mathcal{A}_{CFT}$ is the action of the unperturbed conformal field theory, φ is a field of the conformal field theory and λ is a constant with dimensions related to those of φ. Perturbations of this kind do not introduce any new field into the model and preserve the operator algebra and representation theory of the fields involved. In general it is not known how to calculate correlation functions in models of this kind and so developing a method to approximate these was what instigated the research into the truncated conformal space approach (TCSA) presented in part II of this thesis.

1.2.1 Truncated Conformal Space Approach

The truncated conformal space approach was first developed by Yurov and Zamolodchikov to explore the scaling Lee-Yang model [80] and results compared well with those obtained through the thermodynamic Bethe Ansatz [83]; the TCSA was shown to reproduce the expected energy level distribution and behaviour. Since this first investigation TCSA has provided good approximations of quantities such as one-point functions, in other minimal models [41, 45, 53]. With the accuracy of the method verified it has also been used to analyse the spread of energy levels in minimal models [9]. Alongside bulk models, TCSA has been applied to conformal field theories

in the presence of boundaries and boundary perturbations [20, 21, 40], modelling one point functions and renormalisation group flows between different conformal boundary conditions.

After proving successful in the above areas attention has more recently focused on improving the accuracy of TCSA approximations. In the papers [39, 78] the renormalisation group for TCSA with and without boundaries is developed and demonstrated to improve TCSA predictions but this is outside the scope of this thesis.

As has been discussed correlation functions are important quantities in quantum field theories but it is not known how to evaluate two point correlation functions in many perturbed conformal field theories, in particular those with boundaries. It was the author's intention to develop a method for approximating these two point functions using TCSA; part II summarises the author's progress towards this goal and is set out as follows.

Chapter 5 gives a brief review of the relevant areas of conformal field theory and minimal models are introduced in section 5.1.2. In section 5.2 conformal field theory in the presence of a boundary is discussed and some useful results are presented. Chapter 6 introduces the truncated conformal space approach and it is then applied to the scaling Lee-Yang model and the tri-critical Ising models in bulk and boundary cases in sections 6.1 and 6.2. Section 6.2.3 describes a possible approach to use TCSA to approximate two point functions and finally we conclude with a discussion of what has been achieved and future outlook in chapter 7.

Part I

Twist Fields

Chapter 2

Twist Fields in the Dirac Model

We begin this part by introducing the Dirac model and its associated twist fields. In this part, $x \in \mathbb{R}$ is a space coordinate and $y \in \mathbb{R}$ is a Euclidean time coordinate ($y = it$, where t is real time). Space and Euclidean time will be put into complex coordinates $z := -\frac{i}{2}(x + iy)$ and $\bar{z} := \frac{i}{2}(x - iy)$, with derivatives $\partial := \partial_z = i\partial_x + \partial_y$ and $\bar{\partial} := \partial_{\bar{z}} = -i\partial_x + \partial_y$. Also, when only one coordinate is specified for a quantum field, it is understood as the space coordinate, the Euclidean time coordinate being set to 0.

2.1 The Dirac Model

2.1.1 Dirac Fermions

In two dimensions and in the quantisation on the line, the free Dirac Fermi field of mass m is an operator solution $\Psi_R(x, y),\ \Psi_L(x, y)$ to the equations of motion

$$\begin{aligned} \bar{\partial}\Psi_R &= -im\Psi_L, \quad \bar{\partial}\Psi_R^\dagger = -im\Psi_L^\dagger \\ \partial\Psi_L &= im\Psi_R, \qquad \partial\Psi_L^\dagger = im\Psi_R^\dagger \end{aligned} \tag{2.1.1}$$

and to the equal-time anti-commutation relations

$$\{\Psi_R(x_1), \Psi_R^\dagger(x_2)\} = 4\pi\delta(x_1 - x_2), \quad \{\Psi_L(x_1), \Psi_L^\dagger(x_2)\} = 4\pi\delta(x_1 - x_2) \tag{2.1.2}$$

(other combinations of fields anti-commutating), subject to the condition that there exists a state $|\text{vac}\rangle$, the vacuum state, such that

$$\lim_{y\to-\infty}\Psi_{R,L}(x,y)|\text{vac}\rangle=\lim_{y\to-\infty}\Psi^\dagger_{R,L}(x,y)|\text{vac}\rangle=0. \tag{2.1.3}$$

The normalisation in (2.1.2) is chosen in such a way that the fields have the so-called CFT normalisation, in terms of the z variables, as $|z_1-z_2|\to 0$,

$$\begin{aligned}\langle\text{vac}|\mathcal{T}\left[\Psi^\dagger_R(x_1,y_1)\Psi_R(x_2,y_2)\right]|\text{vac}\rangle &\sim \frac{1}{z_1-z_2},\\ \langle\text{vac}|\mathcal{T}\left[\Psi^\dagger_L(x_1,y_1)\Psi_L(x_2,y_2)\right]|\text{vac}\rangle &\sim \frac{1}{\bar z_1-\bar z_2}.\end{aligned}$$

Here, the symbol $\mathcal{T}$ is the time-ordering symbol: in its bracket, operators at later Euclidean times are placed to the left of operators at earlier Euclidean times (getting a sign, in general, if the operators exchanged have fermionic statistics).

As usual, the solution can be given in terms of creation and annihilation operators (Fourier modes of the fields), whose time evolution is simple:

$$\begin{aligned}\Psi_R(x,y)=&\sqrt{m}\int \mathrm{d}\theta\, e^{\theta/2}\left(D^\dagger_+(\theta)e^{yE_\theta-ixp_\theta}-iD_-(\theta)e^{-yE_\theta+ixp_\theta}\right)\\ \Psi_L(x,y)=&\sqrt{m}\int \mathrm{d}\theta\, e^{-\theta/2}\left(iD^\dagger_+(\theta)e^{yE_\theta-ixp_\theta}-D_-(\theta)e^{-yE_\theta+ixp_\theta}\right)\end{aligned} \tag{2.1.4}$$

where

$$E_\theta=m\cosh\theta,\quad p_\theta=m\sinh\theta. \tag{2.1.5}$$

It is worth noting that since y is a Euclidean time the conjugate fields, $\Psi^\dagger_{R,L}(x,y)$, are not just the Hermitian conjugates of $\Psi_{R,L}(x,y)$, but in fact obtained by Hermitian conjugation followed by a change in sign of y:

$$\begin{aligned}\Psi^\dagger_R(x,y) &= \sqrt{m}\int \mathrm{d}\theta\, e^{\theta/2}\left(iD^\dagger_-(\theta)e^{yE_\theta-ixp_\theta}+D_+(\theta)e^{-yE_\theta+ixp_\theta}\right)\\ \Psi^\dagger_L(x,y) &= \sqrt{m}\int \mathrm{d}\theta\, e^{-\theta/2}\left(-D^\dagger_-(\theta)e^{yE_\theta-ixp_\theta}-iD_+(\theta)e^{-yE_\theta+ixp_\theta}\right).\end{aligned} \tag{2.1.6}$$

From (2.1.2), the creation and annihilation operators satisfy

$$\{D^\dagger_+(\theta_1),D_+(\theta_2)\}=\delta(\theta_1-\theta_2),\quad \{D^\dagger_-(\theta_1),D_-(\theta_2)\}=\delta(\theta_1-\theta_2) \tag{2.1.7}$$

(with all other combinations anti-commuting), and from (2.1.3), the vacuum condition

$$D_\pm(\theta)|\text{vac}\rangle=0.$$

The Hilbert space is simply the Fock space over (2.1.7), with a basis of multi-particle states denoted by

$$|\theta_1 \cdots \theta_n\rangle_{\epsilon_1 \cdots \epsilon_n} := D^\dagger_{\epsilon_1}(\theta_1) \cdots D^\dagger_{\epsilon_n}(\theta_n)|\text{vac}\rangle \quad \text{for} \quad \theta_1 \geqslant \cdots \geqslant \theta_n, \ \epsilon_i \in \{+,-\}. \tag{2.1.8}$$

where $\theta_i > \theta_{i+1}$ when $\epsilon_i = \epsilon_{i+1}$ and $\theta_i \geqslant \theta_{i+1}$ when $\epsilon_i \neq \epsilon_{i+1}$. These states have total energies $\sum_j E_{\theta_j}$ and total momenta $\sum_j p_{\theta_j}$. The dual vectors will be denoted by ${}^{\epsilon_1 \cdots \epsilon_n}\langle\theta_1 \cdots \theta_n| := |\theta_1 \cdots \theta_n\rangle^\dagger_{\epsilon_1 \cdots \epsilon_n}$, and clearly we have

$${}^{\epsilon'_1 \cdots \epsilon'_n}\langle\theta'_1 \cdots \theta'_n|\theta_1 \cdots \theta_n\rangle_{\epsilon_1 \cdots \epsilon_n} = \prod_{j=1}^{n} \delta^{\epsilon'_j}_{\epsilon_j} \delta(\theta_j - \theta'_j). \tag{2.1.9}$$

Writing as in (2.1.8) vectors with different orderings of the creation operators, the identity operator can be decomposed into

$$1 = \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{\epsilon_1 \cdots \epsilon_N} \int_{-\infty}^{\infty} d\theta_1 \cdots \int_{-\infty}^{\infty} d\theta_N \, |\theta_1 \cdots \theta_N\rangle_{\epsilon_1 \cdots \epsilon_N} {}^{\epsilon_N \cdots \epsilon_1}\langle\theta_N \cdots \theta_1|. \tag{2.1.10}$$

The solution (2.1.4), (2.1.6), defining the mode operators from the fermion operators, is chosen in such a way that it satisfies the equations of motion, and that the resulting mode operators obey the canonical anti-commutation relations (2.1.7). However, the choice of such mode operators is not unique: we may re-define the mode operators via $D_\pm(\theta) \mapsto u_\pm D_\pm(\theta)$ for some pure phases $u_\pm$, $|u_\pm| = 1$, and still have a solution to the equations of motion and canonical anti-commutation relations. This amounts to a phase-redefinition of the states in the Hilbert space. The mode operators are partially fixed by requiring further the crossing symmetry condition

$$\langle\text{vac}|\Psi_{R,L}(0)|\theta + i\pi\rangle_\epsilon = {}^{-\epsilon}\langle\theta|\Psi_{R,L}(0)|\text{vac}\rangle \tag{2.1.11}$$

where the form factor on the left hand side is obtained by analytic continuation in θ. This condition leaves one phase ambiguity: there is still an invariance under the change $D_\pm(\theta) \mapsto u^{\pm 1} D_\pm(\theta)$ for $|u| = 1$. This phase ambiguity will not play any role in the following, hence we fix it arbitrarily by choosing the solution (2.1.4), (2.1.6).

2.1.2 Primary Twist Fields

The Dirac theory enjoys a $U(1)$ internal symmetry: $\Psi_{R,L} \mapsto e^{2\pi i\alpha}\Psi_{R,L}$, $\alpha \in [0,1)$. The associated twist fields, $\sigma_\alpha(x,y)$, are local, spin-less, $U(1)$ neutral quantum fields

with bosonic statistics, and are known to have scaling dimensions α^2 [68]. Their twist property, characterised by a locality index $e^{2\pi i\alpha}$, can be expressed via a monodromy property of correlation functions of time-ordered operators,

$$C(z) := \langle \mathrm{vac}|\mathcal{T}\left[\cdots \Psi_{R,L}(x,y)\sigma_\alpha(0)\cdots\right]|\mathrm{vac}\rangle.$$

Seeing $C(z)$ as a function of the complex variable z smoothly continued through the $y = 0$ line, its continuation along a loop γ surrounding once counter-clockwise the origin is given by

$$C(e^{2\pi i}z) = e^{-2\pi i\alpha}C(z). \tag{2.1.12}$$

This is true for any loop γ that can be contracted to the origin without intersecting the position of other twist fields possibly present in the correlation function. A similar relation holds for the Hermitian conjugates $\Psi^\dagger_{R,L}$, but with the factor $e^{2\pi i\alpha}$ on the right-hand side, instead of $e^{-2\pi i\alpha}$. This twist property can also be expressed operatorially via equal-time exchange relations:

$$\Psi_{R,L}(x)\sigma_\alpha(0) = \begin{cases} \sigma_\alpha(0)\Psi_{R,L}(x) & (x<0) \\ e^{2\pi i\alpha}\sigma_\alpha(0)\Psi_{R,L}(x) & (x>0) \end{cases} \tag{2.1.13a}$$

$$\Psi^\dagger_{R,L}(x)\sigma_\alpha(0) = \begin{cases} \sigma_\alpha(0)\Psi^\dagger_{R,L}(x) & (x<0) \\ e^{-2\pi i\alpha}\sigma_\alpha(0)\Psi^\dagger_{R,L}(x) & (x>0). \end{cases} \tag{2.1.13b}$$

These naturally lead us to define primary twist fields with negative index via Hermitian conjugates:

$$\sigma_{-\alpha} := \sigma^\dagger_\alpha, \quad \alpha \in [0,1). \tag{2.1.14}$$

These exchange relations along with the property of being primary immediately gives the form factors of the $U(1)$ twist fields [30, 44, 59]. The simplest derivation of the form factors can be found in [7], where the braiding relations (2.1.13) are Fourier transformed to show that the twist fields induce a Bogoliubov transform on the creation and annihilation operators. The properties of Bogoliubov transformations are then used to extract the two particle form factors of the twist fields, with higher particle form factors being calculated via Wick's theorem.

As twist fields are $U(1)$ neutral they only have non-vanishing form factors with $U(1)$ neutral states and from the method described above we have, for $-1 < \alpha < 1$,

$$\langle \mathrm{vac}|\sigma_\alpha(0)|\theta_1\theta_2\cdots\theta_{2n}\rangle_{+\cdots+-\cdots-} = c_\alpha m^{\alpha^2}(-1)^{n(n-1)/2}\left(\frac{\sin(\pi\alpha)}{\pi i}\right)^n \times \left(\prod_{i=1}^n (u_i)^{\frac{1}{2}+\alpha}(u_{i+n})^{\frac{1}{2}-\alpha}\right)\frac{\prod_{i<j\leq n}(u_i-u_j)\prod_{n+1\leq i<j}(u_i-u_j)}{\prod_{r=1}^n\prod_{s=n+1}^{2n}(u_r+u_s)} \quad (2.1.15)$$

where there are n $-$'s and n $+$'s in the state, and $u_i := \exp(\theta_i)$. In particular, the two-particle form factor is

$$\langle \mathrm{vac}|\sigma_\alpha(0)|\theta_1\theta_2\rangle_{+-} = c_\alpha m^{\alpha^2}\frac{\sin(\pi\alpha)}{2\pi i}\frac{e^{\alpha(\theta_1-\theta_2)}}{\cosh\frac{\theta_1-\theta_2}{2}}. \quad (2.1.16)$$

Note that the form factors (2.1.15) may differ from those written in other publications in the choice of the sign of α and in the choice of the sign of the two-particle form factor (the latter amounting to an overall sign $(-1)^n$). These choices relate to our choice of exchange relations (2.1.13) (defining the operator σ_α), and to our choice of crossing symmetry relation (2.1.11) (defining the asymptotic states). Calculations below, using the two-particle form factor, will show that the two-particle form factor (2.1.16) leads to a behaviour of $\Psi_R^\dagger(x)\sigma_\alpha(0)$ and $\Psi_R(x)\sigma_\alpha(0)$ as $x \to 0$ exhibiting a branch point in accordance to the exchange relations (2.1.13). Further calculations will show that the disconnected, delta-function term in a crossing symmetry relation similar to (2.1.11) is correct, this being true only with the sign chosen for the two-particle form factor.

The constant c_α is chosen, up to a phase, in order to guarantee the CFT normalisation of the fields:

$$\langle \mathrm{vac}|\sigma_\alpha(x_1,y_1)\sigma_{-\alpha}(x_2,y_2)|\mathrm{vac}\rangle \sim \frac{1}{|z_1-z_2|^{2\alpha^2}} \quad \text{as} \quad |z_1-z_2|\to 0. \quad (2.1.17)$$

The phase of c_α is fixed by requiring that it be real and positive. The evaluation of the constant c_α was first explained in [57] (this constant was first obtained in an unpublished note by Al.B. Zamolodchikov). This constant, or rather its generalisation to the Dirac Fermi field on the Poinaré disk, was identified with Barnes' G-function in [25]:

$$c_\alpha = \frac{1}{G(1-\alpha)G(1+\alpha)}. \quad (2.1.18)$$

Recall that the main property of Barnes' G-function is $G(1+z) = \Gamma(z)G(z)$, with the normalisation $G(1) = 1$. In the next section we provide an alternative way of evaluating c_α.

Other matrix elements can be obtained by analytic continuation in the rapidities, as is standard in the context of 1+1-dimensional QFT [75]. Note that the form factors (2.1.15) and the Hermiticity relation (2.1.14) are in agreement with the analytic-continuation formula

$$\langle \mathrm{vac}|\sigma_\alpha(0)|(\theta_1 + i\pi)\cdots(\theta_{2n} + i\pi)\rangle_{+\cdots+-\cdots-} = {}^{+\cdots+-\cdots-}\langle \theta_{2n}\cdots\theta_1|\sigma_\alpha(0)|\mathrm{vac}\rangle, \quad (2.1.19)$$

where the analytic continuation is simultaneous on all rapidities.

2.2 The Massless Limit

If the mass of the model is set to zero in equations (2.1.1) then the fields $\Psi_R^{(\dagger)}$ become holomorphic, $\Psi_L^{(\dagger)}$ become anti-holomorphic and the model is reduced to a conformal field theory. In this limit the non-vanishing two point functions of the Dirac fields become

$$\begin{aligned} \langle \mathrm{vac}|\mathcal{T}\left[\Psi_R^\dagger(x_1,y_1)\Psi_R(x_2,y_2)\right]|\mathrm{vac}\rangle_{m=0} &= \frac{1}{z_1 - z_2} \\ \langle \mathrm{vac}|\mathcal{T}\left[\Psi_L^\dagger(x_1,y_1)\Psi_L(x_2,y_2)\right]|\mathrm{vac}\rangle_{m=0} &= \frac{1}{\bar{z}_1 - \bar{z}_2} \end{aligned} \quad (2.2.1)$$

with multi-point functions evaluated via Wick's theorem. It is known that reproducing these correlation functions using bosonisation can prove insightful and this is indeed the case here.

2.2.1 Bosonisation

We introduce holomorphic and anti-holomorphic operators $e_\alpha(x,y)$ and $\bar{e}_\alpha(x,y)$, $\alpha \in \mathbb{R}$. They have bosonic statistics and the following correlation functions:

$$\begin{aligned} \langle \mathrm{vac}|e_{\alpha_1}(x_1,y_1)\cdots e_{\alpha_n}(x_n,y_n)|\mathrm{vac}\rangle_{m=0} &= \delta_{0,\sum_j \alpha_j} \prod_{j<k}(-i(z_j - z_k))^{\alpha_j\alpha_k}, \\ \langle \mathrm{vac}|\bar{e}_{\alpha_1}(x_1,y_1)\cdots \bar{e}_{\alpha_n}(x_n,y_n)|\mathrm{vac}\rangle_{m=0} &= \delta_{0,\sum_j \alpha_j} \prod_{j<k}(i(\bar{z}_j - \bar{z}_k))^{\alpha_j\alpha_k} \end{aligned} \quad (2.2.2)$$

for $y_1 > y_2 > \cdots > y_n$, where the power functions are on their principal branches. These correlation functions lead directly to the following operator product expansion [34]:

$$\begin{aligned}\mathcal{T}\left[e_\alpha(x,y)e_{\alpha'}(x',y')\right] \sim & (-i(z-z'))^{\alpha\alpha'}\left(1+\frac{\alpha}{\alpha+\alpha'}(z-z')\partial'\right)e_{\alpha+\alpha'}(x',y')\\ \mathcal{T}\left[\bar{e}_\alpha(x,y)\bar{e}_{\alpha'}(x',y')\right] \sim & (i(\bar{z}-\bar{z}'))^{\alpha\alpha'}\left(1+\frac{\alpha}{\alpha+\alpha'}(\bar{z}-\bar{z}')\bar{\partial}'\right)\bar{e}_{\alpha+\alpha'}(x',y'),\end{aligned} \quad (2.2.3)$$

as well as the following equal-time exchange relations for the operators $e_\alpha(x)$ and $\bar{e}_\alpha(x)$:

$$\begin{aligned}e_{\alpha_1}(x_1)e_{\alpha_2}(x_2) &= e^{i\pi\alpha_1\alpha_2\mathrm{sgn}(x_2-x_1)}e_{\alpha_2}(x_2)e_{\alpha_1}(x_1),\\ \bar{e}_{\alpha_1}(x_1)\bar{e}_{\alpha_2}(x_2) &= e^{i\pi\alpha_1\alpha_2\mathrm{sgn}(x_1-x_2)}\bar{e}_{\alpha_2}(x_2)\bar{e}_{\alpha_1}(x_1)\end{aligned} \quad (2.2.4)$$

in both cases for $x_1 \neq x_2$.

The principle of bosonisation rests on there being appropriate choices of the αs which provide an equivalence between these correlation functions and those of the Dirac fields. If we identify

$$\begin{aligned}\Psi_R = e^{-i\pi/4}\omega_R^{-1}e_{-1}, \qquad \Psi_R^\dagger = e^{-i\pi/4}\omega_R e_1\\ \Psi_L = e^{i\pi/4}\omega_L\bar{e}_1 \qquad \Psi_L^\dagger = e^{i\pi/4}\omega_L^{-1}\bar{e}_{-1}\end{aligned} \quad (2.2.5)$$

where $\omega_{R,L}$ are pure phases the correlation functions (2.2.1) are recovered. This is not the whole story, however, since these operators do not have the correct exchange relations between left and right movers. To remedy this situation we introduce Klein factors to guarantee fermionic exchange relations. In the present case the Klein factors will take the form of quaternionic elements $\hat{\imath}$, $\hat{\jmath}$ and $\hat{k}$, with both Ψ_R and $\Psi_R^\dagger$ proportional to $\hat{\jmath}$, and both Ψ_L and $\Psi_L^\dagger$ proportional to $\hat{\imath}$. In each sector these elements can be seen as part of the pure phases and hence do not affect these correlation functions and in mixed sectors the anti-commutation relation $\hat{\imath}\,\hat{\jmath} = -\hat{\jmath}\,\hat{\imath}$ guarantees the correct statistics for the Dirac fields. As the only non-zero correlation functions contain even numbers of both left and right moving fields the quaternionic element $\hat{k}$ is never present. The precise identification can then be written as follows:

$$\begin{aligned}\Psi_R = -e^{-i\pi/4}\omega^{-1}\,\hat{\jmath}\,e_{-1}, \qquad \Psi_R^\dagger = e^{-i\pi/4}\omega\,\hat{\jmath}\,e_1,\\ \Psi_L = e^{i\pi/4}\omega\,\hat{\imath}\,\bar{e}_1, \qquad \Psi_L^\dagger = -e^{i\pi/4}\omega^{-1}\,\hat{\imath}\,\bar{e}_{-1}.\end{aligned} \quad (2.2.6)$$

Here, we have used the $U(1)$ invariance of the Dirac theory in order to remain with only one arbitrary phase ω.

Using this bosonisation procedure we can put the primary twist fields discussed in section 2.1.2 into the same framework. The basis for this identification is the comparison between the exchange relations (2.1.13) and (2.2.4). Letting Q be the $U(1)$ charge of the Dirac theory we see that $[Q, \Psi^\dagger_{R,L}] = -\Psi^\dagger_{R,L}$ and $[Q, \Psi_{R,L}] = \Psi_{R,L}$ and we can simply make the identification:

$$\sigma_\alpha = e_\alpha \bar{e}_\alpha e^{-i\pi\alpha Q}. \tag{2.2.7}$$

The action of Q on the operators e_α and $\bar{e}_\alpha$ is given by

$$e^{icQ} e_\alpha e^{-icQ} = e^{-ic\alpha} e_\alpha, \quad e^{icQ} \bar{e}_\alpha e^{-icQ} = e^{ic\alpha} \bar{e}_\alpha. \tag{2.2.8}$$

Note that relation (2.2.7) along with the correlation functions (2.2.2) are in agreement with the normalisation (2.1.17).

Correlation functions (2.2.2) can be realised using the Heisenberg algebra: one usually writes $e_\alpha(x,y) = e^{i\alpha\varphi_R(x,y)}$ and $\bar{e}_\alpha(x,y) = e^{-i\alpha\varphi_L(x,y)}$ where $\varphi_{R,L}$ are (appropriately normalised) holomorphic and anti-holomorphic free massless bosonic field, respectively. The $U(1)$ symmetry in terms of such fields corresponds to a shift of both φ_R and φ_L by the same constant, and the arbitrariness of the phase ω in (2.2.6) can be thought of as an arbitrariness under shifts of φ_R and φ_L by opposite constants. Also, in terms of the bosonic fields, the $U(1)$-twist field is simply $\sigma_\alpha = e^{i\alpha\varphi} e^{-i\pi\alpha Q}$, with $\varphi := \varphi_R - \varphi_L$.

2.2.2 Operator Product Expansions

From the relation (2.2.7) and the correlation functions (2.2.2) the leading coefficients of the operator product expansions (OPEs) of Dirac fields with twist fields can be computed. Accounting for the exchange relations (2.1.13) and the correlation functions (2.2.2) we observe that

$$\begin{aligned} \mathcal{T}\left[\Psi^\dagger_R(x,y)\sigma_\alpha(0)\right] &\sim (-iz)^\alpha \sigma_{\alpha+1,\alpha}(0), \\ \mathcal{T}\left[\Psi_R(x,y)\sigma_\alpha(0)\right] &\sim (-iz)^{-\alpha} \sigma_{\alpha-1,\alpha}(0), \end{aligned} \tag{2.2.9}$$

where we have defined,

$$\begin{aligned}\sigma_{\alpha+1,\alpha} &:= e^{-i\pi/4}\omega\,\hat{\jmath}\,e_{\alpha+1}\bar{e}_{\alpha}e^{-i\pi\alpha Q},\\ \sigma_{\alpha-1,\alpha} &:= -e^{-i\pi/4}\omega^{-1}\,\hat{\jmath}\,e_{\alpha-1}\bar{e}_{\alpha}e^{-i\pi\alpha Q}.\end{aligned} \tag{2.2.10}$$

for $\alpha \in \mathbb{R}$. The power functions on the right-hand sides in (2.2.9) are on their principal branches which is in agreement with the phases occurring in the exchange relations (2.1.13) and the phase differences occurring through the cuts of the principal branches. Further, we also find

$$\begin{aligned}\mathcal{T}\left[\Psi_L^{\dagger}(x,y)\sigma_{\alpha}(0)\right] &\sim -i\omega^{-2}\,\hat{\mathrm{k}}\,e^{-i\pi Q}(i\bar{z})^{-\alpha}\sigma_{\alpha,\alpha-1}(0)\\ \mathcal{T}\left[\Psi_L(x,y)\sigma_{\alpha}(0)\right] &\sim -i\omega^{2}\,\hat{\mathrm{k}}\,e^{i\pi Q}(i\bar{z})^{\alpha}\sigma_{\alpha,\alpha+1}(0).\end{aligned} \tag{2.2.11}$$

In order to simplify the OPEs (2.2.11), we need to specify both $\hat{\mathrm{k}}$ and ω.

As the CFT has both a $U(1)$ symmetry and a symmetry under constant shifts in φ there is no way to determine the phase ω in the CFT context. But there are non-zero correlation functions containing odd numbers of left and right movers if twist fields are inserted and such functions will involve factors of $\hat{\mathrm{k}}$ and ω. While fixing these correlation functions would remove these ambiguities there is no clear principle to fix them by.

In the massive theory $\hat{\mathrm{k}}$ can be written as an (almost) unambiguous operator. We already know that $\hat{\mathrm{k}}$ anti-commutes with all Dirac fields and squares to -1, so we may write it as

$$\hat{\mathrm{k}} = \epsilon\, i\, e^{i\pi Q} \tag{2.2.12}$$

for some sign $\epsilon = \pm$. In particular we have $\langle \mathrm{vac}|\,\hat{\mathrm{k}}\,|\mathrm{vac}\rangle = \epsilon i$. Fixing $\hat{\mathrm{k}}$ is important since there are unambiguous non-zero correlation functions containing left and right movers. In the bosonised language these functions involve a remaining operator $\hat{\mathrm{k}}$ which needs to be well defined. For our purposes the sign ϵ can be left undetermined.

As well as fixing $\hat{\mathrm{k}}$ the massive theory also breaks the symmetry under shifts in φ and this should allow us to evaluate ω. The mass term in the Hamiltonian is

$$\frac{im}{4\pi}\int dx\,\left(\Psi_R^{\dagger}(x)\Psi_L(x) - \Psi_L^{\dagger}(x)\Psi_R(x)\right) = -\frac{im\,\hat{\mathrm{k}}}{2\pi}\int dx\,\cos(\varphi(x)+2\nu) \tag{2.2.13}$$

with $\omega =: e^{i\nu}$ and the symmetry breaking is clear to see. ω is fixed by defining φ such that the twist field σ_{α} has a real and positive vacuum expectation value. The field

$e^{i\varphi}$ has a real and positive vacuum expectation value if and only if the integrand in the mass term has a minimum at $\varphi = 0 \bmod 2\pi$. This ensures the main contributions in the path integral formulation are at values of φ located symmetrically about 0, and these values add to a positive real number. Hence the integrand in the mass term must be $\cos\varphi$ and the coefficient in front of the integral must be negative in bosonic sectors. So finally we see that

$$\omega^2 = -\epsilon. \tag{2.2.14}$$

Combining (2.2.12), (2.2.14) with (2.2.11), we find, independently of ϵ, the OPEs

$$\begin{aligned} \mathcal{T}\left[\Psi_L^\dagger(x,y)\sigma_\alpha(0)\right] &\sim -(i\bar{z})^{-\alpha}\sigma_{\alpha,\alpha-1}(0) \\ \mathcal{T}\left[\Psi_L(x,y)\sigma_\alpha(0)\right] &\sim -(i\bar{z})^{\alpha}\sigma_{\alpha,\alpha+1}(0). \end{aligned} \tag{2.2.15}$$

In writing the twist fields in this form it is understood that

$$\sigma_{\alpha,\alpha} := \sigma_\alpha. \tag{2.2.16}$$

It is now simple to use these definitions to define other non-derivative descendent twist fields of the Dirac theory and then determine the leading and next to leading terms in their OPEs with fermion fields. We define recursively $\sigma_{\alpha\pm n,\alpha}$ for integer $n \geq 0$ via

$$\begin{aligned} \mathcal{T}\left[\Psi_R^\dagger(x,y)\sigma_{\alpha+n,\alpha}(0)\right] &\sim (-iz)^{\alpha+n}\sigma_{\alpha+n+1,\alpha}(0), \\ \mathcal{T}\left[\Psi_R(x,y)\sigma_{\alpha-n,\alpha}(0)\right] &\sim (-iz)^{n-\alpha}\sigma_{\alpha-n-1,\alpha}(0). \end{aligned} \tag{2.2.17}$$

This gives the expressions, for $\alpha \in \mathbb{R}$ and $n \geq 0$ integer,

$$\begin{aligned} \sigma_{\alpha+n,\alpha} &:= (e^{-i\pi/4}\omega\,\hat{\jmath}\,)^n e_{\alpha+n}\bar{e}_\alpha e^{-i\pi\alpha Q}, \\ \sigma_{\alpha-n,\alpha} &:= (-e^{-i\pi/4}\omega^{-1}\,\hat{\jmath}\,)^n e_{\alpha-n}\bar{e}_\alpha e^{-i\pi\alpha Q} \end{aligned} \tag{2.2.18}$$

(these are in agreement with (2.2.7) and (2.2.10) in the cases $n = 0$ and $n = 1$ respectively). From these expressions along with the rules (2.2.6), (2.2.12), (2.2.14)

and (2.2.8), and with the OPEs (2.2.3), we find the OPEs (for $\alpha-\beta\in\mathbb{Z}$):

$$\begin{aligned}
\mathcal{T}\left[\Psi_R^\dagger(x,y)\sigma_{\alpha,\beta}(0)\right] \sim&(-i)^{\Theta(\beta-\alpha-1/2)}(-iz)^{\alpha}\left(1+\frac{1}{1+\alpha}z\partial\right)\sigma_{\alpha+1,\beta}(0)\\
\mathcal{T}\left[\Psi_R(x,y)\sigma_{\alpha,\beta}(0)\right] \sim&(-i)^{\Theta(\alpha-\beta-1/2)}(-iz)^{-\alpha}\left(1+\frac{1}{1-\alpha}z\partial\right)\sigma_{\alpha-1,\beta}(0)\\
\mathcal{T}\left[\Psi_L^\dagger(x,y)\sigma_{\alpha,\beta}(0)\right] \sim&(-i)^{\Theta(\beta-\alpha-1/2)}(-1)^{\alpha-\beta+1}(i\bar z)^{-\beta}\left(1+\frac{1}{1-\beta}\bar z\bar\partial\right)\sigma_{\alpha,\beta-1}(0)\\
\mathcal{T}\left[\Psi_L(x,y)\sigma_{\alpha,\beta}(0)\right] \sim&(-i)^{\Theta(\alpha-\beta-1/2)}(-1)^{\alpha-\beta+1}(i\bar z)^{\beta}\left(1+\frac{1}{1+\beta}\bar z\bar\partial\right)\sigma_{\alpha,\beta+1}(0)
\end{aligned}\tag{2.2.19}$$

where Θ is the Heaviside step-function, $\Theta(\gamma)=1$ $(\gamma>0)$, 0 $(\gamma<0)$, and the derivatives on the right-hand side are with respect to the arguments of the twist fields, which are afterwards put to 0. In these OPEs, the next correction term is equivalent to a term of order z^2 (for the first two) or $\bar z^2$ (for the last two) inside the large parenthesis (but generally involve non-derivative descendents of the twist fields written).

We may write these OPEs more generally by putting the twist fields at position x',y' and by replacing z by $z-z'$ and $\bar z$ by $\bar z-\bar z'$ in the power functions on the right-hand side. From this, taking derivatives with respect to z' and $\bar z'$, we obtain OPEs with first-derivatives of twist fields:

$$\begin{aligned}
\mathcal{T}\left[\Psi_R^\dagger(x,y)\partial\sigma_{\alpha,\beta}(0)\right] =&(-i)^{\Theta(\beta-\alpha-1/2)}(i\alpha)(-iz)^{\alpha-1}\sigma_{\alpha+1,\beta}(0)+O(z^{\alpha+2})\\
\mathcal{T}\left[\Psi_R(x,y)\partial\sigma_{\alpha,\beta}(0)\right] =&(-i)^{\Theta(\alpha-\beta-1/2)}(-i\alpha)(-iz)^{-\alpha-1}\sigma_{\alpha-1,\beta}(0)+O(z^{-\alpha+2})\\
\mathcal{T}\left[\Psi_L^\dagger(x,y)\bar\partial\sigma_{\alpha,\beta}(0)\right] =&(-i)^{\Theta(\beta-\alpha-1/2)}(-1)^{\alpha-\beta+1}(i\beta)(i\bar z)^{-\beta-1}\sigma_{\alpha,\beta-1}(0)+O(\bar z^{-\beta+2})\\
\mathcal{T}\left[\Psi_L(x,y)\bar\partial\sigma_{\alpha,\beta}(0)\right] =&(-i)^{\Theta(\alpha-\beta-1/2)}(-1)^{\alpha-\beta+1}(-i\beta)(i\bar z)^{\beta-1}\sigma_{\alpha,\beta+1}(0)+O(\bar z^{\beta+2}).
\end{aligned}\tag{2.2.20}$$

Note that the first non-trivial corrections to the leading OPEs, corresponding to pure derivatives of twist fields, are exactly zero.

2.3 Descendent Twist Fields

When deriving differential equations for the two point correlation functions of primary twist fields the intermediate steps involve descendent twist fields. So it is nec-

essary to provide unambiguous definitions for these fields and evaluate their form factors.

2.3.1 Definitions

Firstly we notice that (2.1.15) is an analytic function of α on $\mathbb{C} \setminus \mathbb{Z}^*$, for fixed rapidities; the points $\mathbb{Z}^* := \mathbb{Z} \setminus \{0\}$ corresponding in general to poles. Hence, it is natural to use the formula (2.1.15) to define the fields σ_α for all $\alpha \in \mathbb{R} \setminus \mathbb{Z}^*$, and by analytic continuation in the rapidities, all other matrix elements. Here we are implicitly assuming that c_α is also analytic in this region. Now the two-point function of such twist fields is, for any fixed distance between the fields, an analytic function of α on a neighbourhood of $\mathbb{R} \setminus \mathbb{Z}^*$. In fact, we expect that the coefficients of operator product expansions with the stress-energy tensor are "uniformly" analytic in α at all distances. This means that the new field defined by analytic continuation has dimension α^2; we also expect the correct CFT normalisation (2.1.17).

The descendent twist fields with fermionic statistics, and in particular those whose form factors with even numbers of particles vanish, are obtained through regularised limits where the position of a Dirac field approaches that of a primary twist field, or equivalently as coefficients of the operator product expansion. For α in particular subsets of $\mathbb{R} \setminus \mathbb{Z}^*$, we define two one-parameter families of Fermionic twist fields via the coefficients occurring in the leading short-distance asymptotics associated to the Dirac fields Ψ_R and $\Psi_R^\dagger$:

$$\begin{aligned} \mathcal{T}[\Psi_R^\dagger(x,y)\sigma_\alpha(0)] &\sim (-iz)^\alpha \sigma_{\alpha+1,\alpha}(0) && \text{for } \alpha < \frac{1}{2} \\ \mathcal{T}[\Psi_R(x,y)\sigma_\alpha(0)] &\sim (-iz)^{-\alpha} \sigma_{\alpha-1,\alpha}(0) && \text{for } \alpha > -\frac{1}{2}. \end{aligned} \tag{2.3.1}$$

As before, $\mathcal{T}$ is the time-ordering operation. The power functions on the right-hand sides are on their principal branches; note that there is agreement between the phases occurring from exchange relations (2.1.13) and the phase differences occurring through the cuts of the principal branches. The requirement of phase agreement is not quite enough to fully determine the particular power functions occurring: these are consequences of CFT considerations of section 2.2, or of form-factor calculations (performed below). From (2.3.1), we see that the new fields $\sigma_{\alpha\pm1,\alpha}$ have dimensions

$\alpha^2 \pm \alpha + 1/2$, spins $\pm\alpha + \frac{1}{2}$ and charges ∓ 1. They also satisfy the hermiticity relation

$$\sigma^\dagger_{\alpha\pm1,\alpha} = \sigma_{-\alpha\mp1,-\alpha}. \tag{2.3.2}$$

The range of α shown is a consequence of the equations of motion, as we explain shortly. We define fields for all $\alpha \in \mathbb{R} \setminus \mathbb{Z}^*$ again by analytic continuation beyond the range of α shown.

The CFT considerations of section 2.2 also suggest that the same fermionic twist field families can be defined using Ψ_L and $\Psi_L^\dagger$:

$$\begin{aligned} \mathcal{T}\left[\Psi_L^\dagger(x,y)\sigma_\alpha(0)\right] &\sim -(i\bar{z})^{-\alpha}\sigma_{\alpha,\alpha-1}(0) && \text{for } \alpha > -\frac{1}{2} \\ \mathcal{T}\left[\Psi_L(x,y)\sigma_\alpha(0)\right] &\sim -(i\bar{z})^{\alpha}\sigma_{\alpha,\alpha+1}(0) && \text{for } \alpha < \frac{1}{2}. \end{aligned} \tag{2.3.3}$$

Together with the equations of motion (2.1.1), this suggests the form of the leading OPE for all $\alpha \in \mathbb{R} \setminus \mathbb{Z}^*$:

$$\begin{aligned} \mathcal{T}\left[\Psi_R^\dagger(x,y)\sigma_\alpha(0)\right] &\sim (-iz)^{\alpha}\sigma_{\alpha+1,\alpha}(0) + \frac{m}{1-\alpha}(i\bar{z})^{1-\alpha}\sigma_{\alpha,\alpha-1}(0) \\ \mathcal{T}\left[\Psi_R(x,y)\sigma_\alpha(0)\right] &\sim (-iz)^{-\alpha}\sigma_{\alpha-1,\alpha}(0) + \frac{m}{1+\alpha}(i\bar{z})^{1+\alpha}\sigma_{\alpha,\alpha+1}(0) \end{aligned} \tag{2.3.4}$$

and

$$\begin{aligned} \mathcal{T}\left[\Psi_L^\dagger(x,y)\sigma_\alpha(0)\right] &\sim -(i\bar{z})^{-\alpha}\sigma_{\alpha,\alpha-1}(0) - \frac{m}{1+\alpha}(-iz)^{1+\alpha}\sigma_{\alpha+1,\alpha}(0) \\ \mathcal{T}\left[\Psi_L(x,y)\sigma_\alpha(0)\right] &\sim -(i\bar{z})^{\alpha}\sigma_{\alpha,\alpha+1}(0) - \frac{m}{1-\alpha}(-iz)^{1-\alpha}\sigma_{\alpha-1,\alpha}(0), \end{aligned} \tag{2.3.5}$$

where the meaning of $\sim$ is that for any value of α, the term on the right-hand side that is dominant gives the leading OPE. Note that according to these equations, the leading term of the OPE occurring in the CFT is modified in certain ranges of α by lower-dimensional "ghost fields", like $m\sigma_{\alpha,\alpha-1}$, which make no contribution in the massless limit. This expansion can be verified by using the techniques of the next section. We also note that sub-leading terms in these OPEs will have poles at other integer values of α. It is, in principle, possible to calculate these OPEs from perturbation theory [84] but this has not been attempted here. There may also be more singular terms in the OPEs above, however, we assume no such terms are present.

Moreover, we notice that further applications of Dirac fields may lead back to the original family of twist fields, σ_α. For instance, for $\alpha < 0$ we have

$$\mathcal{T}\left[\Psi_R^\dagger(x,y)\sigma_{\alpha,\alpha+1}(0)\right] \sim -i(-iz)^\alpha \sigma_{\alpha+1}(0) \tag{2.3.6}$$

which is a consequence of CFT considerations. Note in particular that the double application of Dirac fields preserved the CFT normalisation, because the Dirac fields themselves are CFT normalised.

The next subsection is devoted to computing the form factors of the Fermionic twist fields defined in (2.3.1), and to showing the correctness of (2.3.4) and (2.3.5). In combination with these form factors, the consistency of (2.3.1) and (2.3.3) leads to the non-trivial recursion relation (2.3.22) for the normalisation constant c_α. Equivalently, relation (2.3.6) can be used (since (2.3.6) is a simpler consequence of CFT, this is a slightly more direct way of obtaining the recursion relation (2.3.22)). We will show that it is indeed satisfied by (2.1.18). Interestingly, such a recursion relation can be seen as a novel way of evaluating this constant.

2.3.2 Form Factors

We start this subsection by giving details of the computation of $\langle \text{vac}|\sigma_{\alpha+1,\alpha}(0)|\theta\rangle_+$. The form factor is evaluated by expanding $\langle \text{vac}|\Psi_R^\dagger(x)\sigma_\alpha(0)|\theta\rangle_+$ and finding the term proportional to z^α, using (2.3.1). The result for the form factor $\langle \text{vac}|\sigma_{\alpha-1,\alpha}(0)|\theta\rangle_-$ is obtained similarly, by expanding $\langle \text{vac}|\Psi_R(x)\sigma_\alpha(0)|\theta\rangle_-$ and so only one calculation is presented here.

In order to obtain the leading short-distance behaviour of

$$\langle \text{vac}|\Psi_R^\dagger(x,y)\sigma_\alpha(0)|\theta\rangle_+ \tag{2.3.7}$$

we insert (2.1.10) between the two fields. For simplicity we consider $x < 0,\ y = 0$, as this is sufficient to extract the one-particle form factors. The full z dependence of the leading behaviour can be restored using rotation (or relativistic boost) covariance. There is only one non-zero form factor of $\Psi_R^\dagger$ given by

$$\langle \text{vac}|\Psi_R^\dagger(x)|\theta\rangle_+ = \sqrt{m}e^{\theta/2}e^{ixp_\theta} \tag{2.3.8}$$

This means that there is only one term involving σ_α to be evaluated, which can be accomplished using the form factor rules of [75] and the two-particle form factor (2.1.16):

$$\begin{aligned}{}^+\langle\phi|\sigma_\alpha(0)|\theta\rangle_+ &= \langle\mathrm{vac}|\sigma_\alpha(0)|\theta,\phi+i\pi-i0^+\rangle_{+-}+\langle\mathrm{vac}|\sigma_\alpha(0)|\mathrm{vac}\rangle\delta(\theta-\phi)\\ &= c_\alpha m^{\alpha^2}\frac{\sin(\pi\alpha)}{2\pi}e^{-i\pi\alpha}\frac{e^{\alpha(\theta-\phi)}}{\sinh\left(\frac{\theta-\phi+i0^+}{2}\right)}+c_\alpha m^{\alpha^2}\delta(\theta-\phi).\end{aligned}\tag{2.3.9}$$

The $+i0^+$ prescription is to be understood in terms of distributions: it is a prescription as to how an integral should avoid the pole of the function. Putting these together we see that

$$\begin{aligned}\langle\mathrm{vac}|\Psi_R^\dagger(x)\sigma_\alpha(0)|\theta\rangle_+ &= c_\alpha m^{\alpha^2+1/2}\frac{\sin(\pi\alpha)}{2\pi}e^{-i\pi\alpha}\int\mathrm{d}\phi\, e^{\phi/2}\frac{e^{\alpha(\theta-\phi)}}{\sinh\left(\frac{\theta-\phi+i0^+}{2}\right)}e^{ixp_\phi}\\ &\quad+c_\alpha m^{\alpha^2+1/2}e^{\theta/2}e^{ixp_\theta}.\end{aligned}\tag{2.3.10}$$

The integral is conditionally convergent. In order to make it a convergent integral, we make a change of contour, shifting $\phi\mapsto\phi-i\pi/2$. This shift of contour does not give any pole contribution thanks to the $i0^+$ prescription, and the result is indeed a convergent integral for $x<0$:

$$\begin{aligned}\langle\mathrm{vac}|\Psi_R^\dagger(x,0)\sigma_\alpha(0)|\theta\rangle_+ &= c_\alpha m^{\alpha^2+1/2}\tfrac{\sin(\pi\alpha)}{2\pi}e^{-i\pi\alpha/2}\int\mathrm{d}\phi\, e^{-i\pi/4}e^{\phi/2}\tfrac{e^{\alpha(\theta-\phi)}}{\sinh(\frac{\theta-\phi}{2}+\frac{i\pi}{4})}e^{xE_\phi}\\ &\quad+c_\alpha m^{\alpha^2+1/2}e^{\theta/2}e^{ixp_\theta}\\ &= c_\alpha m^{\alpha^2+1/2}\tfrac{\sin(\pi\alpha)}{\pi}e^{-i\pi\alpha/2}\int\mathrm{d}\phi\,\tfrac{e^{\alpha\theta-(\alpha-1/2)\phi+xE_\phi}}{i\,e^{\frac{\theta-\phi}{2}}-e^{\frac{\phi-\theta}{2}}}\\ &\quad+c_\alpha m^{\alpha^2+1/2}e^{\theta/2}e^{ixp_\theta}.\end{aligned}\tag{2.3.11}$$

In general, the leading behaviour at small x will be obtained from the large-ϕ behaviour of the integrand. Let us analyse what happens at $x=0$ in different regions of α. In the regions $\alpha<0$ and $\alpha>1$, the resulting integral is divergent. In these cases, the leading small-x behaviour is growing, and can be obtained from the leading large-$|\phi|$ behaviour of the integrand only. On the other hand, in the region $0<\alpha<1$, the integral converges, and the resulting constant is cancelled by the term arising from the delta function as is discussed further in subsection 2.3.3. Note that this cancellation requires in particular the correct sign of the two-particle form factor (2.1.16). Hence, in this case, the leading small-x behaviour is

decaying, and can be extracted again from the leading large-$|\phi|$ behaviour by first rendering the integral divergent by taking its first x-derivative. As explained before, we expect the field $\sigma_{\alpha+1,\alpha}$ to occur in the leading OPE for $\alpha < 1/2$, and to have an appropriate ghost field in the range $\alpha > 1/2$, as per (2.3.4). We will see below that the one-particle form factors agree with this. In order to evaluate the form factor, it is sufficient to consider the region $\alpha < 0$.

In order to extract the small-x behaviour in the region $\alpha < 0$, we split the integral into two regions, $\phi \in (-\infty, 0)$ and $\phi \in (0, \infty)$. In each region we pull a factor out of the fraction to leave a denominator of the form $1 + \eta$ where $|\eta| \to 0$ as $|\phi| \to \infty$, and we then Taylor expand each fraction. This gives

$$\begin{aligned}\langle \mathrm{vac}|\Psi_R^\dagger(x,0)\sigma_\alpha(0)|\theta\rangle_+ \sim c_\alpha m^{\alpha^2+1/2}\frac{\sin(\pi\alpha)}{\pi}e^{-i\pi\alpha/2}\times \\ \left[\int_0^\infty \mathrm{d}\phi\left(-e^{(\alpha+1/2)\theta-\alpha\phi+xE_\phi} - ie^{(\alpha+3/2)\theta-(\alpha+1)\phi+xE_\phi}+\ldots\right)+\right. \\ \left.\int_{-\infty}^0 \mathrm{d}\phi\left(-ie^{(\alpha-1/2)\theta-(\alpha-1)\phi+xE_\phi} - e^{(\alpha-3/2)\theta-(\alpha-2)\phi+xE_\phi}+\ldots\right)\right]. \quad (2.3.12)\end{aligned}$$

The leading divergent term for $\alpha < 0$ occurs in the region $(0, \infty)$. This term can be rewritten using a modified Bessel function:

$$\begin{aligned}\int_0^\infty \mathrm{d}\phi\,(e^{-\alpha\phi+xE_\phi}) &= \int_{-\infty}^\infty \mathrm{d}\phi\,(e^{-\alpha\phi+mx\cosh\phi}) - \int_{-\infty}^0 \mathrm{d}\phi\,(e^{-\alpha\phi+mx\cosh\phi}) \\ &= 2K_{|\alpha|}(-mx) + O(1) \quad (2.3.13)\end{aligned}$$

as $x \to 0^-$. Hence, the integral diverges like

$$\Gamma(-\alpha)\left(\frac{2}{-mx}\right)^{-\alpha}. \quad (2.3.14)$$

The first equation in (2.3.1) allows us to identify the coefficient of the divergence with the one-particle form factor we are seeking. We obtain:

$$\langle \mathrm{vac}|\sigma_{\alpha+1,\alpha}(0)|\theta\rangle_+ = c_\alpha\frac{e^{-i\pi\alpha/2}}{\Gamma(1+\alpha)}m^{\alpha^2+\alpha+1/2}e^{(\alpha+1/2)\theta}. \quad (2.3.15)$$

By analytic continuation, this provides the form factors for all $\alpha \in \mathbb{R} \setminus \mathbb{Z}^*$.

In order to verify the full z-dependence of the leading behaviour (2.3.1), we may use the spins of the operators involved and examine the effects of a relativistic boost

of rapidity β about the origin. Acting with the boost operator R_β, defined essentially by $R_\beta^\dagger|\theta\rangle = |\theta+\beta\rangle$, we see that

$$R_\beta \Psi_R^\dagger(x,y) R_\beta^\dagger = e^{\beta/2}\Psi_R^\dagger(x_\beta, y_\beta) \quad \text{and} \quad R_\beta \sigma_\alpha(0) R_\beta^\dagger = \sigma_\alpha(0) \tag{2.3.16}$$

as the fields have spins $1/2$ and 0 respectively. Here, x_β and y_β are obtained using $z_\beta = e^\beta z$ and $\bar{z}_\beta = e^{-\beta} z$. Moreover, from the form factor (2.3.15), we find that

$$R_\beta \sigma_{\alpha+1,\alpha}(0) R_\beta^\dagger = e^{(\alpha+1/2)\beta}\sigma_\alpha(0). \tag{2.3.17}$$

Hence, the field $\sigma_{\alpha+1,\alpha}$ has spin $\alpha+1/2$ (as mentioned above). When acting with the rotation operators on $\Psi_R^\dagger(x,y)\sigma_{\alpha,\alpha}(0)$, we can either perform the boost first and then use the OPE:

$$\begin{aligned} R_\beta \Psi_R^\dagger(x,y)\sigma_\alpha(0) R_\beta^\dagger &= e^{\beta/2}\Psi_R^\dagger(x_\beta,y_\beta)\sigma_\alpha(0) \\ &\sim e^{\beta/2}(-ize^\beta)^\alpha \sigma_{\alpha+1,\alpha}(0) \end{aligned} \tag{2.3.18}$$

or vice versa:

$$\begin{aligned} R_\beta \Psi_R^\dagger(x,y)\sigma_\alpha(0) R_\beta^\dagger &\sim (-iz)^\alpha R_\beta \sigma_{\alpha+1,\alpha}(0) R_\beta^\dagger \\ &= (-iz)^\alpha e^{(\alpha+1/2)\beta}\sigma_{\alpha+1,\alpha}(0). \end{aligned} \tag{2.3.19}$$

Agreement here is in line with the first equation of (2.3.1) and so gives us confidence in our method.

The one particle form factor of the field $\sigma_{\alpha-1,\alpha}(0)$ can be calculated in a similar way, using the second relation of (2.3.1). The calculation, in the range $\alpha > 0$, gives

$$\langle \text{vac}|\sigma_{\alpha-1,\alpha}(0)|\theta\rangle_- = -ic_\alpha \frac{e^{i\pi\alpha/2}}{\Gamma(1-\alpha)} m^{\alpha^2-\alpha+1/2} e^{(-\alpha+1/2)\theta}, \tag{2.3.20}$$

which, by analytic continuation, holds for all $\alpha \in \mathbb{R}\setminus\mathbb{Z}^*$. Note that

$$\langle \text{vac}|\sigma_{\alpha-1,\alpha}(0)|\theta\rangle_- = -i\langle \text{vac}|\sigma_{-\alpha+1,-\alpha}(0)|\theta\rangle_+.$$

With the crossing symmetry relation $\langle \text{vac}|\sigma_{\alpha-1,\alpha}(0)|\theta+i\pi\rangle_- = {}^+\langle\theta|\sigma_{\alpha-1,\alpha}(0)|\text{vac}\rangle$ along with the hermiticity relation (2.3.2), this implies

$$-i\langle \text{vac}|\sigma_{\alpha-1,\alpha}(0)|\theta+i\pi\rangle_+ = (\langle \text{vac}|\sigma_{\alpha-1,\alpha}(0)|\theta\rangle_+)^*$$

which is indeed satisfied by (2.3.15).

All higher particle form factors are obtained using Wick's theorem. The form factor

$$\langle \mathrm{vac}|\sigma_{\alpha\pm1,\alpha}(0)|\theta_1,\dots,\theta_{2n+1}\rangle_{\epsilon_1,\dots,\epsilon_{2n+1}}$$

is the sum over every term obtained by a product of n (normalised) two-particle form factors of σ_α associated to n disjoint pairs of particles (contractions), multiplied by the one-particle form factor of $\sigma_{\alpha\pm1,\alpha}$ associated to the remaining unpaired particle, with an appropriate sign depending on the number of crossings of the contractions:

$$\langle \mathrm{vac}|\sigma_{\alpha\pm1,\alpha}(0)|\theta_1,\dots,\theta_{2n+1}\rangle_{\epsilon_1,\dots,\epsilon_{2n+1}} = \sum_{\substack{\text{partition } \{q\,;(s_j,t_j):j=1,\dots,n\} \\ \text{of } \{1,\dots,2n-1\}}} (-1)^{\text{crossings}} \left(\prod_{j=1}^{n} \frac{\langle \mathrm{vac}|\sigma_\alpha(0)|\theta_{s_j},\theta_{t_j}\rangle_{\epsilon_{s_j},\epsilon_{t_j}}}{c_\alpha m^{\alpha^2}}\right) \langle \mathrm{vac}|\sigma_{\alpha\pm1,\alpha}(0)|\theta_q\rangle_{\epsilon_q}.$$

We now have a working definition of both primary and descendent twist fields and have evaluated their form factors. These will be used in chapter 3 to verify the action of conserved charges in a double model Dirac theory. This is a key step towards our goal of deriving a set of differential equations for the two point functions of twist fields. First however we use the methods applied in this section to further verify the short-distance behaviour of twist fields and find ways of evaluating the constant c_α.

2.3.3 A Recursion Relation For c_α

We now proceed to verify the relations (2.3.4) and (2.3.5), as well as (2.3.6) and other similar relations. As we will show, these in fact provide a non-trivial recursion relation for the constant c_α. This recursion relation can in fact be used to *evaluate* the constant c_α, since, for instance, relations (2.3.4) were deduced solely from CFT arguments. We note, in particular, that this way of evaluating c_α implements the CFT normalisation of the twist fields somewhat more clearly than in previous methods [57].

First, we may verify the first relation of (2.3.4) in the range $\alpha > 1$ by evaluating the leading diverging term of $\langle \mathrm{vac}|\Psi_R^\dagger(x)\sigma_\alpha(0)|\theta\rangle_+$ for α in that range. Starting with (2.3.12), we need only to consider the region of integration $(-\infty, 0)$. We can

re-write the integral of the first term displayed using a modified Bessel function, in a similar fashion to what was done in the previous subsection. Comparing with the mass term in the first relation of (2.3.4), we find the equation

$$\langle \mathrm{vac}|\sigma_{\alpha,\alpha-1}(0)|\theta\rangle_+ = ic_\alpha \frac{e^{-i\pi\alpha/2}}{\Gamma(1-\alpha)} m^{\alpha^2-\alpha+1/2} e^{\theta(\alpha-1/2)} \tag{2.3.21}$$

After a shift $\alpha \mapsto \alpha+1$, comparison with the form factor (2.3.15) gives the relation

$$\frac{c_\alpha}{c_{\alpha+1}} = \frac{\Gamma(\alpha+1)}{\Gamma(-\alpha)}. \tag{2.3.22}$$

It is a simple matter to verify that this relation is indeed satisfied by (2.1.18) using the properties of the Gamma function and of Barnes' G-function. We remark that this provides a very non-trivial check of the validity of the definition of the fields $\sigma_{\alpha\pm1,\alpha}$ by analytic continuation.

The recursion relation (2.3.22) along with the symmetry property $c_\alpha = c_{-\alpha}$ gives a way of evaluating c_α. Indeed, although these relations have many solutions, the solution is made unique by imposing additionally the normalisation condition $c_0 = 1$ and the conditions of monotonicity and convexity in the range $\alpha \in [0,1)$. These extra conditions are indeed satisfied by (2.1.18). The normalisation condition $c_0 = 1$ is a simple consequence of the fact that σ_α is the identity operator at $\alpha = 0$. However, we do not have clear arguments for monotonicity and convexity from quantum field theory. We note that all calculations in later sections are unaffected by the explicit choice of the solution c_α, and only follow from the recursion relation (2.3.22) (along with, implicitly, the symmetry property and the normalisation condition).

Second, we may complete the verification of the first relation of (2.3.4) by analysing the region $0 < \alpha < 1$. We first note that the $x = 0$ value of the integral in the first line of (2.3.11) is

$$\int \mathrm{d}\phi\, e^{-i\pi/4} e^{\phi/2} \frac{e^{\alpha(\theta-\phi)}}{\sinh(\frac{\theta-\phi}{2} + \frac{i\pi}{4})} = -\frac{2\pi e^{i\pi\alpha/2}}{\sin(\pi\alpha)} e^{\theta/2}. \tag{2.3.23}$$

This can be obtained by first shifting $\phi \mapsto \phi+\theta$ in order to extract the θ-dependent factor, then by shifting the contour $\phi \mapsto \phi - 2i\pi$, getting the pole at $\phi = -3i\pi/2$, and noticing that the integral with a shifted contour is just the initial integral up to a phase (solving the resulting equation gives the answer). Hence, we indeed

find that the quantity (2.3.11) decays as $x \to 0$. The power law of the decay can be obtained simply by taking a derivative with respect to x: it is then a divergent behaviour, hence can be obtained by the techniques illustrated above, using (2.3.12). We then integrate over x in order to recover the leading OPE. The leading term is different in the regions $\alpha < 1/2$ and $\alpha > 1/2$. For $\alpha < 1/2$, we recover the first equation of (2.3.1), with (2.3.15). For $\alpha > 1/2$, taking into account the conditions on the spin, the power law is $(i\bar{z})^{1-\alpha}$, and the coefficient is identified with a multiple of the one-particle form factor of the field $m\sigma_{\alpha,\alpha-1}$. The proportionality constant can be evaluated using (2.1.18), and this again only involves verifying the recursion relation (2.3.22). The resulting leading OPE is in agreement with the first equation of (2.3.4) in the range $1/2 < \alpha < 1$. The case $\alpha = 1/2$ simply involves the two terms calculated, again in agreement with (2.3.4).

Third, in a similar way, we may verify agreement of the form factor (2.3.20) with the second relation of (2.3.4) by considering the ranges $\alpha < -1$ and $-1 < \alpha < 0$, and by using similar arguments as those above. For the range $\alpha < -1$, the mass term of the second relation of (2.3.4) leads to the equation

$$\langle \mathrm{vac}|\sigma_{\alpha,\alpha+1}(0)|\theta\rangle_- = c_\alpha \frac{e^{i\pi\alpha/2}}{\Gamma(1+\alpha)} m^{\alpha^2+\alpha+1/2} e^{\theta(-\alpha-1/2)}. \tag{2.3.24}$$

Again, comparing with (2.3.20), this leads to the recursion relation (2.3.22), hence is satisfied by (2.1.18). Likewise, the relations (2.3.5) can be verified along entirely similar lines.

An alternative way of deriving a recursion relation for the constant c_α is to notice that multiple applications of the Fermi fields can lead back to the family of primary twist fields σ_α. By rearranging (2.3.6) we have that

$$\sigma_{\alpha+1}(0) \sim i(-iz)^{-\alpha} \mathcal{T}[\Psi_R^\dagger(x,y)\sigma_{\alpha,\alpha+1}(0)] \tag{2.3.25}$$

as $z \to 0$ (for $\alpha < 0$). The vacuum expectation value of $\sigma_{\alpha+1}(0)$ can be calculated from this relation in the same way as the one particle form factors were calculated

in subsection 2.3.2. Firstly (2.1.10) is inserted between the fields to give

$$\begin{aligned}\langle \mathrm{vac}|\Psi_R^\dagger(x,y)\sigma_{\alpha,\alpha+1}(0)|\mathrm{vac}\rangle =& \int \mathrm{d}\theta\, \langle \mathrm{vac}|\Psi_R^\dagger(x,y)|\theta\rangle_+{}^+\langle\theta|\sigma_{\alpha,\alpha+1}(0)|\mathrm{vac}\rangle \\ =&\sqrt{m}\int \mathrm{d}\theta\, e^{\theta/2}e^{-yE_\theta+ixp_\theta}\langle \mathrm{vac}|\sigma_{\alpha,\alpha+1}(0)|\theta+i\pi\rangle_- \\ =&-ic_\alpha m^{\alpha^2+\alpha+1}\frac{e^{-i\pi\alpha/2}}{\Gamma(1+\alpha)}\int \mathrm{d}\theta\, e^{-\alpha\theta-yE_\theta+ixp_\theta}. \quad (2.3.26)\end{aligned}$$

Doing as before, we set $y=0$ and shift the integration contour, $\theta\mapsto\theta-i\pi/2$. We then take the small x expansion to obtain:

$$\begin{aligned}\langle \mathrm{vac}|\Psi_R^\dagger(x,0)\sigma_{\alpha,\alpha+1}(0)|\mathrm{vac}\rangle =& -ic_\alpha m^{\alpha^2+\alpha+1}\frac{1}{\Gamma(1+\alpha)}\int \mathrm{d}\theta\, e^{-\alpha\theta+xE_\theta} \\ \sim& -ic_\alpha m^{\alpha^2+\alpha+1}\frac{\Gamma(-\alpha)}{\Gamma(1+\alpha)}\left(\frac{2}{-mx}\right)^{-\alpha}, \quad (2.3.27)\end{aligned}$$

where on the second line we assumed $\alpha<0$. Hence (2.3.25) gives

$$\langle \mathrm{vac}|\sigma_{\alpha+1}(0)|\mathrm{vac}\rangle = c_{\alpha+1}m^{(\alpha+1)^2} = c_\alpha m^{\alpha^2+2\alpha+1}\frac{\Gamma(-\alpha)}{\Gamma(\alpha+1)} \quad (2.3.28)$$

which gives again the recursion relation (2.3.22).

Finally, it is worth mentioning that the matrix elements $\langle \mathrm{vac}|\sigma_\alpha(0)\Psi_{R,L}^\dagger(x)|\theta\rangle_\epsilon$ and $\langle \mathrm{vac}|\sigma_\alpha(0)\Psi_{R,L}(x)|\theta\rangle_\epsilon$ can be evaluated by similar methods, but without the need for using crossing symmetry. The result is in agreement with the exchange relations: for instance, for $x<0$, the same function of x and θ is obtained as for the matrix elements $\langle \mathrm{vac}|\Psi_{R,L}^\dagger(x)\sigma_\alpha(0)|\theta\rangle_\epsilon$ and $\langle \mathrm{vac}|\Psi_{R,L}(x)\sigma_\alpha(0)|\theta\rangle_\epsilon$ respectively.

Chapter 3

Correlation Functions of Twist Fields

Our approach to deriving the differential equations satisfied by the twist field two point function takes inspiration from [33] where two non-interacting copies of Ising field theory are considered. Since the two copies are identical there is invariance under rotations between them. This extra continuous symmetry leads to Ward identities which allow differential equations for the two-point spin-spin function, and other objects, to be written down. Here we apply the same approach to twist fields in the Dirac theory.

3.1 The Double Model

Our double model consists of two copies of the Dirac theory defined in section 2.1 and we denote the Fermi fields of each model as Ψ and Φ. The creation and annihilation operators are $D_{\pm}^{(\dagger)}(\theta)$ and $E_{\pm}^{(\dagger)}(\theta)$ respectively and necessarily satisfy all the relations of section 2.1 independently of each other. As the two copies do not interact they satisfy the anti-commutation relations

$$\{\Psi_{R,L}^{(\dagger)}(z), \Phi_{R,L}^{(\dagger)}(w)\} = 0 \quad \text{and} \quad \{D_{\pm}^{(\dagger)}(\theta_1), E_{\pm}^{(\dagger)}(\theta_2)\} = 0. \tag{3.1.1}$$

There are also twist fields associated with the independent $U(1)$ symmetries of the two copies. Where relevant we use a superscript to denote the copy each field

belongs to: σ_α^Ψ and σ_α^Φ for example. The same notation will be used more generally to denote which copy any other operator belongs to.

3.1.1 Conserved Charges

Both energy and momentum are conserved charges of the single-copy model. They are associated to the dynamical invariance under space and time translation: the equations of motion (2.1.1) possesses this invariance. In the double-copy model, the energy and momentum conserved charges are the sums of the corresponding charges of each copy, however, there are other conserved charges in the double-copy model that can be constructed. As the two copies are non-interacting, the energy-momentum operators for each are still independently conserved quantities, and can be combined in various ways to give new conserved quantities. The two specific conserved charges that are of interest to us are the differences of the momenta of the single copies. We define P and $\bar{P}$ via the following action (these conserved charges are chosen to be anti-Hermitian):

$$\begin{aligned}[P, \mathcal{O}^\Psi \mathcal{O}^\Phi] =& i\partial\mathcal{O}^\Psi\mathcal{O}^\Phi - i\mathcal{O}^\Psi\partial\mathcal{O}^\Phi \\ [\bar{P}, \mathcal{O}^\Psi \mathcal{O}^\Phi] =& -i\bar{\partial}\mathcal{O}^\Psi\mathcal{O}^\Phi + i\mathcal{O}^\Psi\bar{\partial}\mathcal{O}^\Phi.\end{aligned} \tag{3.1.2}$$

This holds for any local fields $\mathcal{O}^\Psi$ and $\mathcal{O}^\Phi$ interacting non-trivially with fields in copy Ψ and Φ respectively. In particular, the actions on the creation and annihilation operators are

$$\begin{aligned}[P, D_\pm(\theta)] &= -ime^\theta D_\pm(\theta), \quad [P, E_\pm(\theta)] = ime^\theta E_\pm(\theta), \\ [\bar{P}, D_\pm(\theta)] &= ime^{-\theta} D_\pm(\theta), \quad [\bar{P}, E_\pm(\theta)] = -ime^{-\theta} E_\pm(\theta)\end{aligned}$$

from which it is simple to derive an explicit expression for P and $\bar{P}$ through bilinears in the creation and annihilation operators.

In the double-copy model, there is yet another conserved charge, Z, related to the $O(2)$ rotation symmetry amongst the copies. Written in terms of the Fermi fields Ψ and Φ, the charge Z is (again, chosen to be anti-Hermitian):

$$Z = \frac{1}{4\pi}\int \mathrm{d}x(\Psi_R\Phi_R^\dagger + \Psi_L\Phi_L^\dagger + \Psi_R^\dagger\Phi_R + \Psi_L^\dagger\Phi_L). \tag{3.1.3}$$

The action of this charge on the creation and annihilation operators is

$$[Z, D_{\pm}^{\dagger}(\theta)] = -E_{\pm}^{\dagger}(\theta) \quad [Z, E_{\pm}^{\dagger}(\theta)] = D_{\pm}^{\dagger}(\theta) \tag{3.1.4}$$

and similarly for $D_{\pm}(\theta)$ and $E_{\pm}(\theta)$.

It is of course possible to obtain higher order conserved charges by commutations amongst P, $\bar{P}$ and Z. In the approach of [33], where the Ising model was studied, it is assumed that P, $\bar{P}$ and Z all preserve the quantum state where correlation functions are evaluated, and the Ward identities corresponding to certain higher order conserved charges are used to derive differential equations for spin-spin correlation functions (it was also observed there that the higher order conserved charges form a $\widehat{sl(2)}$ algebra). However, it turns out that this is not necessary, as was noticed in [26]. Indeed, the Ward identities coming from the invariance under the Z-action alone are sufficient to obtain non-trivial equations for correlation functions – we do not need to assume that the dynamical space-time translation symmetry is a symmetry of the quantum state under study. When the quantum state does possess this symmetry, the space-time dependence of correlation functions simplifies, and the resulting equations are the integrable differential equations describing the Ising correlation functions. As we will see, the same structure occurs in the present case of the Dirac theory, with additional conditions. It turns out that we need parity invariance in order to further simplify the space-time dependence of correlation functions and in the most general case considered rotation symmetry is also required. This is an important remark, as it shows that the method may have applicability beyond the cases where the quantum states are invariant under these space-time symmetries.

In the next section, we will derive the integrable differential equations from the Ward identities associated to Z, and use the simpler space-time dependence of correlation functions coming from translation, parity and rotation symmetries. In appendix A the special case of $\langle \sigma_\alpha(x_1, y_1)\sigma_\alpha(x_2, y_2)\rangle$ is considered as it is possible to write down the differential equations in the absence of these symmetries, however we have not been successful in identifying a structure to these equations.

As was observed in [26] (in the generalised situation of a theory on the Poincaré

disk), it is the relation

$$[P,[\bar{P},Z]] = [\bar{P},[P,Z]] = 4m^2 Z \tag{3.1.5}$$

which provides the dependence on the mass in the differential equations. Relation (3.1.5) corresponds essentially to the equations of motion of the theory, as expressed using the charge Z. It can be derived from (3.1.2) and (3.1.3) using the equations of motion (2.1.1). Relation (3.1.5) is the only one where the massive theory is used: besides it, we only need the action of Z on products of twist fields and their first derivatives in order to find the differential equations, and this action, as is verified below, is solely a consequence of CFT calculations (i.e. calculations using the massless limit of the theory).

3.1.2 The Action of Z

In order to determine the action of Z on products of twist fields we look to the massless CFT description our model given in section 2.2. The OPEs (2.2.19) and (2.2.20) combined with the integral expression for Z, (3.1.3), allow us to accomplish this task.

Firstly we note that in the CFT Z decomposes into a sum of left and right moving conserved charges, $Z = Z_R + Z_L$. We can then evaluate the commutators of the local charges Z_R and Z_L with local fields as contour integrals:

$$\begin{aligned}
[Z_R, \mathcal{O}(0)] &\mapsto \frac{1}{2\pi}\oint ds \left(\Psi_R(x,y)\Phi_R^\dagger(x,y) + \Psi_R^\dagger(x,y)\Phi_R(x,y)\right)\mathcal{O}(0) \\
[Z_L, \mathcal{O}(0)] &\mapsto -\frac{1}{2\pi}\oint ds \left(\Psi_L(x,y)\Phi_L^\dagger(x,y) + \Psi_L^\dagger(x,y)\Phi_L(x,y)\right)\mathcal{O}(0)
\end{aligned}$$

where we have set $-iz \mapsto s$ and $i\bar{z} \mapsto s$ respectively. By replacing $\mathcal{O}$ with a product of twist fields of the form $\sigma_\alpha^\Psi \sigma_\alpha^\Phi$ and using the OPEs (2.2.19) and (2.2.20), as applicable, we can then evaluate these integrals using the residue theorem.

This method will only provide useful results when the pair of twist fields are at the same space-time point, otherwise the result of the action of Z will be non-local and the Ward identities arising from this will not be useful. As a result in the rest of this section all the fields are evaluated at the same point, $(0,0)$ for instance, and we neglect to write this location with the fields to make our equations easier to read.

Putting all this together we see that the action of Z on pairs of twist fields involving no derivatives is given by:

$$
\begin{aligned}
[Z, \sigma^{\Psi}_{\alpha}\sigma^{\Phi}_{\alpha}] &= 0 && \text{(3.1.6a)}\\
[Z, \sigma^{\Psi}_{\alpha}\sigma^{\Phi}_{\alpha-1}] &= i(\sigma^{\Psi}_{\alpha-1,\alpha}\sigma^{\Phi}_{\alpha,\alpha-1} - \sigma^{\Psi}_{\alpha,\alpha-1}\sigma^{\Phi}_{\alpha-1,\alpha}) && \text{(3.1.6b)}\\
[Z, \sigma^{\Psi}_{\alpha}\sigma^{\Phi}_{\alpha+1}] &= i(\sigma^{\Psi}_{\alpha+1,\alpha}\sigma^{\Phi}_{\alpha,\alpha+1} - \sigma^{\Psi}_{\alpha,\alpha+1}\sigma^{\Phi}_{\alpha+1,\alpha}) && \text{(3.1.6c)}\\
[Z, \sigma^{\Psi}_{\alpha+1,\alpha}\sigma^{\Phi}_{\alpha,\alpha+1}] &= i(\sigma^{\Psi}_{\alpha}\sigma^{\Phi}_{\alpha+1} - \sigma^{\Psi}_{\alpha+1}\sigma^{\Phi}_{\alpha}) && \text{(3.1.6d)}\\
[Z, \sigma^{\Psi}_{\alpha+1}\sigma^{\Phi}_{\alpha-1}] &= \frac{1}{\alpha}\big((\partial\sigma^{\Psi}_{\alpha,\alpha+1})\sigma^{\Phi}_{\alpha,\alpha-1} - \sigma^{\Psi}_{\alpha,\alpha+1}(\partial\sigma^{\Phi}_{\alpha,\alpha-1}) + \\
&\qquad (\bar{\partial}\sigma^{\Psi}_{\alpha+1,\alpha})\sigma^{\Phi}_{\alpha-1,\alpha} - \sigma^{\Psi}_{\alpha+1,\alpha}(\bar{\partial}\sigma^{\Phi}_{\alpha-1,\alpha})\big) && \text{(3.1.6e)}\\
[Z, \sigma^{\Psi}_{\alpha-1}\sigma^{\Phi}_{\alpha+1}] &= \frac{1}{\alpha}\big(\sigma^{\Psi}_{\alpha,\alpha-1}(\partial\sigma^{\Phi}_{\alpha,\alpha+1}) - (\partial\sigma^{\Psi}_{\alpha,\alpha-1})\sigma^{\Phi}_{\alpha,\alpha+1} + \\
&\qquad \sigma^{\Psi}_{\alpha-1,\alpha}(\bar{\partial}\sigma^{\Phi}_{\alpha+1,\alpha}) - (\bar{\partial}\sigma^{\Psi}_{\alpha-1,\alpha})\sigma^{\Phi}_{\alpha+1,\alpha}\big). && \text{(3.1.6f)}
\end{aligned}
$$

It should be noted that such relations can only be obtained when the twist fields involved have the same monodromy factor, modulo 2π, otherwise the residue theorem cannot be applied.

Similarly we find the action of Z on products of twist fields involving one derivative to be:

$$
\begin{aligned}
[Z, (\partial\sigma^{\Psi}_{\alpha})\sigma^{\Phi}_{\alpha}] &= \alpha(\sigma^{\Psi}_{\alpha-1,\alpha}\sigma^{\Phi}_{\alpha+1,\alpha} - \sigma^{\Psi}_{\alpha+1,\alpha}\sigma^{\Phi}_{\alpha-1,\alpha}) && \text{(3.1.7a)}\\
[Z, (\bar{\partial}\sigma^{\Psi}_{\alpha})\sigma^{\Phi}_{\alpha}] &= \alpha(\sigma^{\Psi}_{\alpha,\alpha-1}\sigma^{\Phi}_{\alpha,\alpha+1} - \sigma^{\Psi}_{\alpha,\alpha+1}\sigma^{\Phi}_{\alpha,\alpha-1}) && \text{(3.1.7b)}\\
[Z, (\partial\sigma^{\Psi}_{\alpha})\sigma^{\Phi}_{\alpha+1}] &= i((\partial\sigma^{\Psi}_{\alpha,\alpha+1})\sigma^{\Phi}_{\alpha+1,\alpha} - \sigma^{\Psi}_{\alpha+1,\alpha}(\partial\sigma^{\Phi}_{\alpha,\alpha+1})) && \text{(3.1.7c)}\\
[Z, (\bar{\partial}\sigma^{\Psi}_{\alpha})\sigma^{\Phi}_{\alpha+1}] &= i(\sigma^{\Psi}_{\alpha,\alpha+1}(\bar{\partial}\sigma^{\Phi}_{\alpha+1,\alpha}) - (\bar{\partial}\sigma^{\Psi}_{\alpha+1,\alpha})\sigma^{\Phi}_{\alpha,\alpha+1}) && \text{(3.1.7d)}\\
[Z, (\partial\sigma^{\Psi}_{\alpha})\sigma^{\Phi}_{\alpha-1}] &= i((\partial\sigma^{\Psi}_{\alpha,\alpha-1})\sigma^{\Phi}_{\alpha-1,\alpha} - \sigma^{\Psi}_{\alpha-1,\alpha}(\partial\sigma^{\Phi}_{\alpha,\alpha-1})) && \text{(3.1.7e)}\\
[Z, (\bar{\partial}\sigma^{\Psi}_{\alpha})\sigma^{\Phi}_{\alpha-1}] &= i(\sigma^{\Psi}_{\alpha,\alpha-1}(\bar{\partial}\sigma^{\Phi}_{\alpha-1,\alpha}) - (\bar{\partial}\sigma^{\Psi}_{\alpha-1,\alpha})\sigma^{\Phi}_{\alpha,\alpha-1}) && \text{(3.1.7f)}\\
[Z, (\partial\sigma^{\Psi}_{\alpha,\alpha+1})\sigma^{\Phi}_{\alpha+1,\alpha}] &= i(\sigma^{\Psi}_{\alpha+1}(\partial\sigma^{\Phi}_{\alpha}) - (\partial\sigma^{\Psi}_{\alpha})\sigma^{\Phi}_{\alpha+1}) && \text{(3.1.7g)}\\
[Z, (\partial\sigma^{\Psi}_{\alpha+1,\alpha})\sigma^{\Phi}_{\alpha,\alpha+1}] &= i(\sigma^{\Psi}_{\alpha}(\partial\sigma^{\Phi}_{\alpha+1}) - (\partial\sigma^{\Psi}_{\alpha+1})\sigma^{\Phi}_{\alpha}) && \text{(3.1.7h)}\\
[Z, (\bar{\partial}\sigma^{\Psi}_{\alpha,\alpha+1})\sigma^{\Phi}_{\alpha+1,\alpha}] &= i((\bar{\partial}\sigma^{\Psi}_{\alpha+1})\sigma^{\Phi}_{\alpha} - \sigma^{\Psi}_{\alpha}(\bar{\partial}\sigma^{\Phi}_{\alpha+1})) && \text{(3.1.7i)}\\
[Z, (\bar{\partial}\sigma^{\Psi}_{\alpha+1,\alpha})\sigma^{\Phi}_{\alpha,\alpha+1}] &= i((\bar{\partial}\sigma^{\Psi}_{\alpha})\sigma^{\Phi}_{\alpha+1} - \sigma^{\Psi}_{\alpha+1}(\bar{\partial}\sigma^{\Phi}_{\alpha})). && \text{(3.1.7j)}
\end{aligned}
$$

The relations (3.1.6) and (3.1.7) were calculated in the CFT and we wish to examine correlation functions in the massive Dirac theory, so we must be sure these

relations also hold in this instance. To verify that these relations can be carried over to the massive case we use the form factors presented in chapter 2 together with the knowledge that $Z|\text{vac}\rangle = 0$. Armed with this information we can place the vacuum to the left and a multi-particle state to the right and evaluate both sides of each relation. For example when verifying (3.1.6c) with two particles we may examine

$$\langle \text{vac}|[Z, \sigma^{\Psi}_{\alpha}\sigma^{\Phi}_{\alpha+1}]|\theta^{\Psi}_1\theta^{\Phi}_2\rangle_{+-} = \langle \text{vac}|i(\sigma^{\Psi}_{\alpha+1,\alpha}\sigma^{\Phi}_{\alpha,\alpha+1} - \sigma^{\Psi}_{\alpha,\alpha+1}\sigma^{\Phi}_{\alpha+1,\alpha})|\theta^{\Psi}_1\theta^{\Phi}_2\rangle_{+-}. \quad (3.1.8)$$

The left hand side can be evaluated using the action of Z on the creation operators together with the form factors (2.1.16), while the right hand side is evaluated using the form factors of section 2.3. These expressions are then found to be equal using the recursion relation for c_α (2.3.22).

This operation has been performed on all of the relations (3.1.6) and (3.1.7) with up to four particles in the state and agreement is found for each. As this agreement is very non-trivial it provides strong evidence that these relations do indeed hold for all matrix elements in the massive theory.

The origins of equations (3.1.6) are not obvious without reference to the CFT. Equation (3.1.6a), however, is easy to verify, and using this (3.1.6b) and (3.1.6c) can be 'derived' from the relations of the type

$$\Psi^{\dagger}_R\Psi_L\sigma_\alpha \mapsto \sigma_{\alpha+1} \quad \text{and} \quad \Psi_R\Psi^{\dagger}_L\sigma_\alpha \mapsto \sigma_{\alpha-1}$$

(see (2.3.3) and (2.3.6) for the exact relations). The other equations do not appear to have such simple origins but have all been verified by the method described above. We also note that the left-hand sides of (3.1.6b) and (3.1.6c) are the same up to a shift in indices and a relabelling of the fields and that the right-hand sides are also consistent with these operations, indicating that our relations are consistent.

Finally, the action of Z on fields involving higher numbers of derivatives are not in general expected, in the massive theory, to be in agreement with calculations from CFT. However, the only relations required at present can be obtained by using those above along with the equation of motion of the charge Z (3.1.5). Indeed, we only need the action of Z on double-derivative fields of the form $[P, [\bar{P}, \mathcal{O}^{\Psi}\mathcal{O}^{\Phi}]]$, which

we can evaluate using

$$\begin{aligned}
&[Z,[P,[\bar{P},\mathcal{O}^{\Psi}\mathcal{O}^{\Phi}]]] \\
&= [[P,[\bar{P},Z]],\mathcal{O}^{\Psi}\mathcal{O}^{\Phi}]+[\bar{P},[Z,[P,\mathcal{O}^{\Psi}\mathcal{O}^{\Phi}]]]+[P,[Z,[\bar{P},\mathcal{O}^{\Psi}\mathcal{O}^{\Phi}]]]-[P,[\bar{P},[Z,\mathcal{O}^{\Psi}\mathcal{O}^{\Phi}]]] \\
&= 4m^2[Z,\mathcal{O}^{\Psi}\mathcal{O}^{\Phi}]+[\bar{P},[Z,[P,\mathcal{O}^{\Psi}\mathcal{O}^{\Phi}]]]+[P,[Z,[\bar{P},\mathcal{O}^{\Psi}\mathcal{O}^{\Phi}]]]-[P,[\bar{P},[Z,\mathcal{O}^{\Psi}\mathcal{O}^{\Phi}]]]
\end{aligned} \tag{3.1.9}$$

This is how the mass dependence will appear in the equations for correlation functions.

3.2 The Correlation Functions

The aim of this section is to derive differential equations for the two point correlation function of twist fields. This is accomplished using the double model described in section 3.1 via the action of the conserved charges P, $\bar{P}$ and Z. We also endeavour to keep the state where the correlation function is evaluated as general as possible.

3.2.1 Correlation Functions of Interest

Firstly, the states we consider must be invariant under the action of Z. The first general form of a state we consider here is the mixed state where the average of an operator A is given by

$$\frac{\mathrm{Tr}\left(e^{V}A\right)}{\mathrm{Tr}\left(e^{V}\right)}, \tag{3.2.1}$$

where V is a bi-linear in the creation and annihilation operators, or equivalently the Dirac operators. In the double copy model $V^{\Psi}+V^{\Phi}$ is invariant under $O(2)$ rotations between the copies and hence

$$\frac{\mathrm{Tr}\left(e^{V^{\Psi}+V^{\Phi}}A\right)}{\mathrm{Tr}\left(e^{V^{\Psi}+V^{\Phi}}\right)} \tag{3.2.2}$$

vanishes when A is of the form $[Z,\cdots]$, which is the form of the Ward identities associated to Z. This type of state includes the finite temperature state $V=-H/T$, at temperature T as well as the zero temperature limit, that is to say the vacuum state.

Averages of quantum states can also be given by

$$\langle B'|A|B\rangle \tag{3.2.3}$$

where $|B\rangle = e^V|\text{vac}\rangle$ and $|B'\rangle = e^{V'}|\text{vac}\rangle$ are boundary states with V and V' again being bi-linear in the creation and annihilation operators. As before

$$\langle (B')^\Psi (B')^\Phi |A| B^\Psi B^\Phi\rangle \tag{3.2.4}$$

vanishes when $A = [Z, \cdots]$. The integrable boundary states [37] of the Dirac theory have this form.

As the action of Z on local fields is also local the exact space where the quantisation holds is irrelevant and we may choose an alternate Hilbert space quantised on a finite line segment, the circle of length L for example, while still considering the states mentioned above.

In the rest of this section the notation $\langle A\rangle$ is used to represent (3.2.1), (3.2.2), (3.2.3) or (3.2.4), it being clear when we are referring to the single or double models.

3.2.2 Symmetries

Before discussing the symmetries of correlation functions we first discuss some observations arising from the definitions of the twist fields.

Firstly we observe that correlators of the form

$$\langle \sigma_{\alpha,\alpha\pm1}(x,y)\sigma_{\beta,\beta}(0)\rangle, \quad \langle \sigma_{\alpha,\alpha+1}(x,y)\sigma_{\alpha,\alpha+1}(0)\rangle \quad \text{and} \quad \langle \sigma_{\alpha,\alpha-1}(x,y)\sigma_{\alpha,\alpha-1}(0)\rangle$$

vanish as a consequence of $U(1)$ charge conservation.

Secondly, since primary twist fields commute we see that

$$\langle \sigma_\alpha(x_1,y_1)\sigma_\beta(x_2,y_2)\rangle = \langle \sigma_\beta(x_2,y_2)\sigma_\alpha(x_1,y_1)\rangle. \tag{3.2.5}$$

To obtain a similar relation for descendent twist fields we must look to the braiding relations (2.1.13), and recall that by definition

$$\begin{aligned} \sigma_{\alpha+1,\alpha}(x,y) &= \lim_{z\to w}(-i(z-w))^{-\alpha}\Psi_R^\dagger(x',y')\sigma_\alpha(x,y) \\ \sigma_{\alpha,\alpha+1}(x,y) &= \lim_{z\to w}(i(\bar z-\bar w))^{-\alpha}\Psi_L(x',y')\sigma_\alpha(x,y) \end{aligned} \tag{3.2.6}$$

where $z = -\frac{i}{2}(x' + iy')$ and $w = -\frac{i}{2}(x + iy)$ are appropriately time ordered ($y' > y$). The factors $(-i(z-w))^{-\alpha}$ and $(i(\bar{z}-\bar{w}))^{-\alpha}$ are taken on the principal branch, and so are continuous exactly where $\Psi_R^\dagger(x',y')\sigma_\alpha(x,y)$ and $\Psi_L(x',y')\sigma_\alpha(x,y)$ are continuous. Using these definitions and (2.1.13) we obtain

$$\langle \sigma_{\beta,\beta+1}(x_2,y_2)\sigma_{\alpha+1,\alpha}(x_1,y_1)\rangle = \left\{ \begin{array}{ll} -e^{2\pi i\beta} & \text{if } x_1 > x_2 \\ -e^{2\pi i\alpha} & \text{if } x_1 < x_2 \end{array} \right\} \langle \sigma_{\alpha+1,\alpha}(x_1,y_1)\sigma_{\beta,\beta+1}(x_2,y_2)\rangle. \quad (3.2.7)$$

We now discuss how space-time symmetries affect the form of our correlation functions. Space and time translation invariance reduces the space-time dependence of the correlators to their relative positions, effectively allowing us to place one of the twist fields at the origin.

Space and time parity symmetry allow further simplification of certain correlation functions. In two dimensions these symmetries can be combined to give invariance under a rotation of π. Since this symmetry is Euclidean it cannot be expressed solely through the action of a unitary operator but is instead a relation connecting vacuum expectation values. This symmetry says that there is a unitary operator R which performs a parity transformation on local fields such that

$$\langle \text{vac}|\mathcal{O}_1(x_1,y_1)\cdots\mathcal{O}_n(x_n,y_n)|\text{vac}\rangle = \langle \text{vac}|R\mathcal{O}_n(x_n,y_n)\cdots\mathcal{O}_1(x_1,y_1)R^\dagger|\text{vac}\rangle \quad (3.2.8)$$

whenever the local fields $\mathcal{O}_j(x_j,y_j)$ are time-ordered: $y_1 > \ldots > y_n$. R does not act trivially on the vacuum, rather it transforms an in-vacuum to an out-vacuum and vice-versa but it does act locally on local fields, multiplying time and space coordinates by -1 so the correlation function on the right hand side is still time ordered. Relation (3.2.8) originates from the Euclidean path integral picture which is only connected to the operational picture when the operators are time ordered. This relation represents the invariance of Euclidean correlation functions under a rotation of the Euclidean plane by π.

The operator R can of course be seen as a shift of rapidities by $i\pi$ but in the present circumstances there is a better way of representing it. We may view R as a combination of two symmetries, space parity S and time parity T:

$$R = TS. \quad (3.2.9)$$

The space parity operator S is linear and satisfies

$$S\Psi_{R,L}(x,y)S = \pm i\Psi_{L,R}(-x,y), \quad S^2 = 1, \tag{3.2.10}$$

while the time parity operator T is also linear and satisfies

$$T\Psi_{R,L}(x,y)T = \Psi_{L,R}(x,-y), \quad T^2 = 1. \tag{3.2.11}$$

Note that T is not an anti-linear operator because it inverts the Euclidean time instead of the real time and so is akin to a parity operator. These symmetries clearly preserve the equations of motion (2.1.1) and so are dynamical symmetries of the model. From the relations above we see that the actions of S and T on $D_\pm(\theta)$ are

$$SD_\pm(\theta)S = \mp D_\pm(-\theta) \qquad \text{and} \qquad TD_\pm(\theta)T = -iD^\dagger_\mp(-\theta) \tag{3.2.12}$$

and the combined parity transformation has the action

$$TS\Psi_{R,L}(x,y)ST = \pm i\Psi_{R,L}(-x,-y), \quad TSD_\pm(\theta)ST = \pm iD^\dagger_\mp(\theta), \tag{3.2.13}$$

with the following simple operator representation:

$$TS = \exp\left[\frac{i\pi}{2}\int \mathrm{d}\theta\,(D^\dagger_-(\theta)D^\dagger_+(\theta) + D_+(\theta)D_-(\theta))\right]. \tag{3.2.14}$$

From (3.2.13) we see that creation operators are transformed into annihilation operators of opposite charge and vice-versa. When combined with (3.2.8) the invariance property can be extended to correlation functions involving arbitrary states. Again with time ordering $y_1 > \ldots > y_n$ we have

$$\langle B'|\mathcal{O}_1(x_1,y_1)\cdots\mathcal{O}_n(x_n,y_n)|B\rangle = \langle\bar{B}|R\mathcal{O}_n(x_n,y_n)\cdots\mathcal{O}_1(x_1,y_1)R^\dagger|\bar{B}'\rangle \tag{3.2.15}$$

where $|\bar{B}\rangle$ and $|\bar{B}'\rangle$ are the states obtained from $|B\rangle$ and $|B'\rangle$, respectively, by inverting the charges of all particles, with extra factors $-i$ and $+i$ for each particle if the new charge is $-$ or $+$ respectively. Thus a rotation by π in the Euclidean picture takes particles coming from the past and rotates them to come from the future, which can then be interpreted as going towards the future if there charge is conjugated (a particle moving forward in time is the same as its anti-particle moving

backward in time). Parity invariance can be generalised to mixed states using the trace definition and (3.2.15) on the summand in the trace.

Next we want to examine how twist fields transform under the parity symmetry R. Since σ_α is spinless, and since $\sigma_{\alpha+1,\alpha}$ and $\sigma_{\alpha,\alpha+1}$ have opposite spins, it is clear that $\sigma_\alpha(x,y)\sigma_{\alpha+1}(0)$ and $\sigma_{\alpha+1,\alpha}(x,y)\sigma_{\alpha,\alpha+1}(0)$ transform under rotation by rotating the coordinates, without phase factors associated to the spin. There is an extra subtlety, however, relating to the fact that we only ask for invariance of the state under rotation by the unique angle π. As a consequence, the correlation functions may depend on the direction of the cut emanating from the twist field, which we have chosen to be towards the right according to (2.1.13). After a π rotation, the cut goes in the opposite direction. This can be represented by the presence of an extra $U(1)$ transformation operator, so that we have

$$R\sigma_\alpha(x,y)R^\dagger = \sigma_\alpha(-x,-y)e^{2\pi i\alpha Q} \tag{3.2.16}$$

where Q is the Hermitian $U(1)$ charge. The operator $e^{2\pi i\alpha Q}$ is effectively a branch cut along the whole x-axis, cancelling out the twist fields cut to the right and reinstating the cut to the left. From (3.2.6), and remembering that relation (3.2.8) inverts the positions of operators, this leads to

$$R\sigma_{\alpha+1,\alpha}(x,y)R^\dagger = e^{-i\pi(\alpha+1/2)}\sigma_{\alpha+1,\alpha}(-x,-y)e^{2\pi i\alpha Q} \tag{3.2.17}$$

$$R\sigma_{\alpha,\alpha+1}(x,y)R^\dagger = e^{i\pi(\alpha+1/2)}\sigma_{\alpha,\alpha+1}(-x,-y)e^{2\pi i\alpha Q}. \tag{3.2.18}$$

Hence, a parity transformation gives us, using translation invariance of the state,

$$\begin{aligned}\langle\sigma_{\alpha+1,\alpha}(x,y)\sigma_{\alpha,\alpha+1}(0)\rangle &= -\langle\sigma_{\alpha,\alpha+1}(0)e^{2\pi i\alpha Q}\sigma_{\alpha+1,\alpha}(-x,-y)e^{2\pi i\alpha Q}\rangle^c \\ &= -e^{-2i\pi\alpha}\langle\sigma_{\alpha,\alpha+1}(x,y)\sigma_{\alpha+1,\alpha}(0,0)e^{4\pi i\alpha Q}\rangle^c\end{aligned}$$

where the superscript c means that the state is as in the right-hand side of (3.2.15) (or its generalisation to mixed states). Likewise,

$$\begin{aligned}G(x,y) &= \langle\sigma_\alpha(x,y)\sigma_{\alpha+1}(0)\rangle \\ &= \langle\sigma_{\alpha+1}(0)\sigma_\alpha(-x,-y)e^{4\pi i\alpha Q}\rangle^c \\ &= \langle\sigma_{\alpha+1}(x,y)\sigma_\alpha(0)e^{4\pi i\alpha Q}\rangle^c.\end{aligned} \tag{3.2.19}$$

Hence, parity invariance for these twist field correlation functions requires the following condition on the state:

$$\langle \cdots e^{4\pi i\alpha Q}\rangle^c = \langle \cdots \rangle. \tag{3.2.20}$$

This condition is assumed to hold for all the states below.

This parity symmetry is enough to simplify correlation functions of certain primary twist fields and those involving certain descendents, as discussed below, but in order to derive equations for more general correlation functions we will also require our states to be rotation invariant. This will allow the simplification of general correlators of descendent twist fields.

From the definitions given in section 2.3 the spin properties of the descendent twist fields are evident. Imposing rotation invariance on the states considered allows us to use this knowledge to write

$$\langle \sigma_{\alpha+1,\alpha}(w_1)\sigma_{\beta,\beta+1}(w_2)\rangle = (i(w_1-w_2))^{\beta-\alpha} f^{\beta,\beta+1}_{\alpha+1,\alpha}(|w_1-w_2|). \tag{3.2.21}$$

The choice of factor $(i(w_1-w_2))^{\beta-\alpha}$ is in some sense arbitrary, but defined on the principal branch, it is convenient as, from (3.2.7), we can also write

$$\langle \sigma_{\beta,\beta+1}(w_2)\sigma_{\alpha+1,\alpha}(w_1)\rangle = -e^{2\pi i\zeta}(i(w_1-w_2))^{\beta-\alpha} f^{\beta,\beta+1}_{\alpha+1,\alpha}(|w_1-w_2|) \tag{3.2.22}$$

where the same function $f^{\beta,\beta+1}_{\alpha+1,\alpha}(|w_1-w_2|)$ is involved and $\zeta=\alpha$ if $x(w_1)<x(w_2)$ or $\zeta=\beta$ if $x(w_1)>x(w_2)$. This choice will not influence any future calculations as we will always consider products of the form

$$\langle \sigma_{\alpha,\alpha+1}(x,y)\sigma_{\beta+1,\beta}(0,0)\rangle\langle \sigma_{\alpha+1,\alpha}(x,y)\sigma_{\beta,\beta+1}(0,0)\rangle, \tag{3.2.23}$$

so the same constants will occur everywhere.

Now setting $z=re^{i\theta}$ and $\bar{z}=re^{-i\theta}$ it is possible to factorise the r and θ dependence of these correlation functions:

$$\begin{aligned}\langle \sigma_{\alpha,\alpha+1}(x,y)\sigma_{\beta+1,\beta}(0,0)\rangle &= pe^{i\theta(\alpha-\beta)} f_1(r)\\ \langle \sigma_{\alpha+1,\alpha}(x,y)\sigma_{\beta,\beta+1}(0,0)\rangle &= qe^{i\theta(\beta-\alpha)} f_2(r)\end{aligned} \tag{3.2.24}$$

where p and q are phases, as in (3.2.22), coming from (2.1.13), and the functions f_1 and f_2 will be discussed in section 3.2.5.

Following the same arguments we also see that when rotational invariance is imposed there is no θ dependence in correlation functions of primary twist fields, as they are spin-less, so in this case

$$\langle \sigma_\alpha(x,y)\sigma_\beta(0,0)\rangle = F(r). \tag{3.2.25}$$

3.2.3 Ward Identities

We are now ready to write down the Ward identities coming from the actions of P, $\bar{P}$ and Z and simplify the resulting equations. The Ward identities of interest here are:

$$\langle [Z, \sigma^{\Psi}_{\alpha,\alpha}(z)\sigma^{\Phi}_{\alpha+1,\alpha+1}(z)\sigma^{\Psi}_{\beta+1,\beta}(0)\sigma^{\Phi}_{\beta,\beta+1}(0)]\rangle = 0 \tag{3.2.26a}$$

$$\langle [Z, [P, \sigma^{\Psi}_{\alpha,\alpha}(z)\sigma^{\Phi}_{\alpha+1,\alpha+1}(z)]\sigma^{\Psi}_{\beta+1,\beta}(0)\sigma^{\Phi}_{\beta,\beta+1}(0)]\rangle = 0 \tag{3.2.26b}$$

$$\langle [Z, [\bar{P}, \sigma^{\Psi}_{\alpha,\alpha}(z)\sigma^{\Phi}_{\alpha+1,\alpha+1}(z)]\sigma^{\Psi}_{\beta+1,\beta}(0)\sigma^{\Phi}_{\beta,\beta+1}(0)]\rangle = 0 \tag{3.2.26c}$$

$$\langle [Z, [P, [\bar{P}, \sigma^{\Psi}_{\alpha,\alpha}(z)\sigma^{\Phi}_{\alpha+1,\alpha+1}(z)]]\sigma^{\Psi}_{\beta+1,\beta}(0)\sigma^{\Phi}_{\beta,\beta+1}(0)]\rangle = 0 \tag{3.2.26d}$$

$$\langle [Z, [P, \sigma^{\Psi}_{\alpha,\alpha}(z)\sigma^{\Phi}_{\alpha+1,\alpha+1}(z)][\bar{P}, \sigma^{\Psi}_{\beta+1,\beta}(0)\sigma^{\Phi}_{\beta,\beta+1}(0)]]\rangle = 0. \tag{3.2.26e}$$

At this point it is useful to introduce the notation

$$F^{\gamma,\delta}_{\alpha,\beta}(x,y) = \langle \sigma_{\alpha,\beta}(x,y)\sigma_{\gamma,\delta}(0,0)\rangle \tag{3.2.27}$$

to make the functional equations resulting from the Ward identities (3.2.26) easier to write. Applying the actions of P, $\bar{P}$ and Z we see that these equations become, respectively:

$$F^{\beta+1,\beta}_{\alpha,\alpha+1}(z)F^{\beta,\beta+1}_{\alpha+1,\alpha}(z) - F^{\beta+1,\beta+1}_{\alpha,\alpha}(z)F^{\beta,\beta}_{\alpha+1,\alpha+1}(z) + F^{\beta,\beta}_{\alpha,\alpha}(z)F^{\beta+1,\beta+1}_{\alpha+1,\alpha+1}(z) = 0 \tag{3.2.28a}$$

$$\begin{aligned}\partial F^{\beta+1,\beta}_{\alpha,\alpha+1}(z)F^{\beta,\beta+1}_{\alpha+1,\alpha}(z) - F^{\beta+1,\beta}_{\alpha,\alpha+1}(z)\partial F^{\beta,\beta+1}_{\alpha+1,\alpha}(z) + \\ \partial F^{\beta+1,\beta+1}_{\alpha,\alpha}(z)F^{\beta,\beta}_{\alpha+1,\alpha+1}(z) - F^{\beta+1,\beta+1}_{\alpha,\alpha}(z)\partial F^{\beta,\beta}_{\alpha+1,\alpha+1}(z) = 0\end{aligned} \tag{3.2.28b}$$

$$\begin{aligned}\bar{\partial} F^{\beta+1,\beta}_{\alpha,\alpha+1}(z)F^{\beta,\beta+1}_{\alpha+1,\alpha}(z) - F^{\beta+1,\beta}_{\alpha,\alpha+1}(z)\bar{\partial} F^{\beta,\beta+1}_{\alpha+1,\alpha}(z) \\ - \bar{\partial} F^{\beta+1,\beta+1}_{\alpha,\alpha}(z)F^{\beta,\beta}_{\alpha+1,\alpha+1}(z) + F^{\beta+1,\beta+1}_{\alpha,\alpha}(z)\bar{\partial} F^{\beta,\beta}_{\alpha+1,\alpha+1}(z) = 0\end{aligned} \tag{3.2.28c}$$

$$\begin{aligned}
&\partial\bar{\partial}F^{\beta+1,\beta}_{\alpha,\alpha+1}(z)F^{\beta,\beta+1}_{\alpha+1,\alpha}(z) - \partial F^{\beta+1,\beta}_{\alpha,\alpha+1}(z)\bar{\partial}F^{\beta,\beta+1}_{\alpha+1,\alpha}(z) - \bar{\partial}F^{\beta+1,\beta}_{\alpha,\alpha+1}(z)\partial F^{\beta,\beta+1}_{\alpha+1,\alpha}(z)\\
&\quad + F^{\beta+1,\beta}_{\alpha,\alpha+1}(z)\partial\bar{\partial}F^{\beta,\beta+1}_{\alpha+1,\alpha}(z) - 4m^2F^{\beta+1,\beta}_{\alpha,\alpha+1}(z)F^{\beta,\beta+1}_{\alpha+1,\alpha}(z) + \partial\bar{\partial}F^{\beta+1,\beta+1}_{\alpha,\alpha}(z)F^{\beta,\beta}_{\alpha+1,\alpha+1}(z)\\
&-\partial F^{\beta+1,\beta+1}_{\alpha,\alpha}(z)\bar{\partial}F^{\beta,\beta}_{\alpha+1,\alpha+1}(z) - \bar{\partial}F^{\beta+1,\beta+1}_{\alpha,\alpha}(z)\partial F^{\beta,\beta}_{\alpha+1,\alpha+1}(z) + F^{\beta+1,\beta+1}_{\alpha,\alpha}(z)\partial\bar{\partial}F^{\beta,\beta}_{\alpha+1,\alpha+1}(z)\\
&\quad - \partial\bar{\partial}F^{\beta,\beta}_{\alpha,\alpha}(z)F^{\beta+1,\beta+1}_{\alpha+1,\alpha+1}(z) + \partial F^{\beta,\beta}_{\alpha,\alpha}(z)\bar{\partial}F^{\beta+1,\beta+1}_{\alpha+1,\alpha+1}(z) + \bar{\partial}F^{\beta,\beta}_{\alpha,\alpha}(z)\partial F^{\beta+1,\beta+1}_{\alpha+1,\alpha+1}(z)\\
&\qquad - F^{\beta,\beta}_{\alpha,\alpha}(z)\partial\bar{\partial}F^{\beta+1,\beta+1}_{\alpha+1,\alpha+1}(z) = 0 \quad (3.2.28\text{d})
\end{aligned}$$

$$\begin{aligned}
&\partial\bar{\partial}F^{\beta+1,\beta}_{\alpha,\alpha+1}(z)F^{\beta,\beta+1}_{\alpha+1,\alpha}(z) - \partial F^{\beta+1,\beta}_{\alpha,\alpha+1}(z)\bar{\partial}F^{\beta,\beta+1}_{\alpha+1,\alpha}(z) - \bar{\partial}F^{\beta+1,\beta}_{\alpha,\alpha+1}(z)\partial F^{\beta,\beta+1}_{\alpha+1,\alpha}(z)\\
&\quad + F^{\beta+1,\beta}_{\alpha,\alpha+1}(z)\partial\bar{\partial}F^{\beta,\beta+1}_{\alpha+1,\alpha}(z) + \partial\bar{\partial}F^{\beta,\beta}_{\alpha,\alpha}(z)F^{\beta+1,\beta+1}_{\alpha+1,\alpha+1}(z) - \partial F^{\beta,\beta}_{\alpha,\alpha}(z)\bar{\partial}F^{\beta+1,\beta+1}_{\alpha+1,\alpha+1}(z)\\
&- \bar{\partial}F^{\beta,\beta}_{\alpha,\alpha}(z)\partial F^{\beta+1,\beta+1}_{\alpha+1,\alpha+1}(z) + F^{\beta,\beta}_{\alpha,\alpha}(z)\partial\bar{\partial}F^{\beta+1,\beta+1}_{\alpha+1,\alpha+1}(z) + \partial\bar{\partial}F^{\beta+1,\beta+1}_{\alpha,\alpha}(z)F^{\beta,\beta}_{\alpha+1,\alpha+1}(z)\\
&\quad - \partial F^{\beta+1,\beta+1}_{\alpha,\alpha}(z)\bar{\partial}F^{\beta,\beta}_{\alpha+1,\alpha+1}(z) - \bar{\partial}F^{\beta+1,\beta+1}_{\alpha,\alpha}(z)\partial F^{\beta,\beta}_{\alpha+1,\alpha+1}(z)\\
&\qquad + F^{\beta+1,\beta+1}_{\alpha,\alpha}(z)\partial\bar{\partial}F^{\beta,\beta}_{\alpha+1,\alpha+1}(z) = 0. \quad (3.2.28\text{e})
\end{aligned}$$

We now have a choice to make. By setting $\alpha = \beta$ we can proceed to calculate differential equations for specific correlation functions of primary and descendent twist fields requiring only translation and parity symmetric states. Alternatively we can consider states with rotation symmetry and find equations for more general correlation functions of primary twist fields. The former case will be considered first.

3.2.4 Correlation Functions Without Rotational Symmetry

To make progress in the absence of rotational symmetry we must set $\alpha = \beta$. This reduces the correlation functions we must consider to:

$$\begin{aligned}
F(x,y) &= \langle\sigma_\alpha(x,y)\sigma_\alpha(0)\rangle,\\
G(x,y) &= \langle\sigma_\alpha(x,y)\sigma_{\alpha+1}(0)\rangle,\\
H(x,y) &= \langle\sigma_{\alpha+1,\alpha}(x,y)\sigma_{\alpha,\alpha+1}(0)\rangle. \quad (3.2.29)
\end{aligned}$$

Where the dependence on α is implicit, in order not to clutter the equations. In the same spirit, we will also use the notation

$$\tilde{F}(x,y) = \langle\sigma_{\alpha+1}(x,y)\sigma_{\alpha+1}(0))\rangle. \quad (3.2.30)$$

Parity symmetry, as discussed in section 3.2.2, tells us that

$$\begin{aligned} F(-x,-y) &= F(x,y), \quad \tilde{F}(-x,-y) = \tilde{F}(x,y), \\ G(-x,-y) &= G(x,y), \quad H(-x,-y) = H(x,y). \end{aligned} \tag{3.2.31}$$

The commutation properties of the twist fields also simplify the Ward identities, they tell us that

$$\langle \sigma_{\alpha+1}(x,y)\sigma_{\alpha}(0)\rangle = G(-x,-y), \quad \langle \sigma_{\alpha,\alpha+1}(x,y)\sigma_{\alpha+1,\alpha}(0)\rangle = -e^{2\pi i\alpha}H(-x,-y). \tag{3.2.32}$$

In order to further ease our notation it is useful to introduce the operator

$$\mathcal{D}(f,g) := \frac{1}{2}\left(f\partial\bar{\partial}g + g\partial\bar{\partial}f - \partial f\bar{\partial}g - \partial g\bar{\partial}f\right) \tag{3.2.33}$$

and also write $\mathcal{D}(f) := \mathcal{D}(f,f)$, noting that

$$\mathcal{D}(e^f) = e^{2f}\partial\bar{\partial}f. \tag{3.2.34}$$

We can rewrite equations (3.2.28) in terms of our new functions (3.2.29):

$$e^{2\pi i\alpha}H^2 + G^2 - F\tilde{F} = 0 \tag{3.2.35a}$$

$$F\partial\tilde{F} - (\partial F)\tilde{F} = 0 \tag{3.2.35b}$$

$$(\bar{\partial}F)\tilde{F} - F\bar{\partial}\tilde{F} = 0 \tag{3.2.35c}$$

$$e^{2\pi i\alpha}\left(\mathcal{D}(H) - 2m^2H^2\right) - \mathcal{D}(G) + \mathcal{D}(F,\tilde{F}) = 0 \tag{3.2.35d}$$

$$e^{2\pi i\alpha}\mathcal{D}(H) - \mathcal{D}(G) - \mathcal{D}(F,\tilde{F}) = 0. \tag{3.2.35e}$$

The above equations are similar in structure to those of [33], for the Ising spin-spin correlator but here we have four functions to solve for. We begin this process by noting that (3.2.35b) and (3.2.35c) only involve F and $\tilde{F}$ and so these relations can be solved, giving

$$\tilde{F} = k^2F \tag{3.2.36}$$

where k is the constant of integration. This relation is somewhat surprising as it tells us the two correlation functions behave in the same way, at both short and long distances, despite the fields having different dimensions. Using the twist field's form factors k can be calculated exactly. Inserting (2.1.10) between the fields in

the correlation functions allows us to compare terms involving equal numbers of particles. Clearly the zero particle terms differ by the constant k^2 where

$$k = \frac{c_{\alpha+1}}{c_\alpha} m^{2\alpha+1} \tag{3.2.37}$$

and it is a simple exercise to show that the same is true for all subsequent terms. We do not have yet a convincing intuitive explanation of the relation (3.2.36); it simply follows from our analytic-continuation definition of descendent twist fields.

Our next step is to change variables, defining the functions χ and φ by

$$\begin{aligned} kF + G &= e^{\chi} \cosh(\varphi) \\ kF - G &= e^{\chi} \sinh(\varphi) \end{aligned} \tag{3.2.38}$$

noting that (3.2.35a) becomes

$$e^{2\pi i\alpha} H^2 = \frac{1}{2} e^{2\chi} \sinh(2\varphi). \tag{3.2.39}$$

Using this relation to eliminate H we find that (3.2.35d) and (3.2.35e) become

$$\begin{aligned} &\sinh(2\varphi)\left(\partial\bar{\partial}\chi + \partial\varphi\bar{\partial}\varphi\right) \\ &\qquad + \cosh(2\varphi)\left(\partial\bar{\partial}\varphi - \coth(2\varphi)\partial\varphi\bar{\partial}\varphi\right) - m^2 \sinh(2\varphi) = 0 \end{aligned} \tag{3.2.40a}$$

$$\begin{aligned} &\sinh(2\varphi)\left(\partial\bar{\partial}\chi + 2\partial\varphi\bar{\partial}\varphi - \partial\bar{\partial}\varphi\right) \\ &\qquad + \cosh(2\varphi)\left(\partial\bar{\partial}\varphi - 2\coth(2\varphi)\partial\varphi\bar{\partial}\varphi - \partial\bar{\partial}\chi\right) = 0. \end{aligned} \tag{3.2.40b}$$

We next use (3.2.40a) to eliminate χ from (3.2.40b), leaving

$$\partial\bar{\partial}\varphi - (1 + \coth(2\varphi))\partial\varphi\bar{\partial}\varphi + m^2 \sinh^2(2\varphi)(1 - \coth(2\varphi)) = 0. \tag{3.2.41}$$

In order to retrieve (1.1.22) from (3.2.41) the $\partial\varphi\bar{\partial}\varphi$ term must be eliminated. This is done using by noting that

$$\frac{f''(\varphi)}{f'(\varphi)} = 1 + \coth(2\varphi) \tag{3.2.42}$$

admits the solution

$$f(\varphi) = i\pi + \log\left(\frac{\sqrt{1 - e^{4\varphi}} - 1}{\sqrt{1 - e^{4\varphi}} + 1}\right) \equiv 2\psi. \tag{3.2.43}$$

It is then straightforward to show that ψ satisfies

$$\partial\bar{\partial}\psi = \frac{m^2}{2}\sinh(2\psi), \tag{3.2.44}$$

giving (1.1.22) when $\alpha = \beta$. To obtain (1.1.21) we set

$$e^{\Sigma_\alpha} = 2kF(x,y) = e^{\chi}(\cosh(\varphi) + \sinh(\varphi)) \tag{3.2.45}$$

so that

$$\partial\bar{\partial}\Sigma_\alpha = \partial\bar{\partial}\chi + \partial\bar{\partial}\varphi. \tag{3.2.46}$$

Using (3.2.40a) and (3.2.41) it is possible to show that

$$\partial\bar{\partial}\Sigma_\alpha = \frac{m^2}{2}(1 - \cosh(2\psi)) \tag{3.2.47}$$

and so we have derived the known formula for the correlation functions (up to a constant factor which can be absorbed into the definitions of either the twist fields or the co-ordinates z and $\bar{z}$).

Following this method we notice that $G(x,y)$ the solution of a differential equation involving the same function ψ. Setting

$$e^{\Sigma'_\alpha} = 2G(x,y) = e^{\chi}(\cosh(\varphi) - \sinh(\varphi)) \tag{3.2.48}$$

gives

$$\begin{aligned}\partial\bar{\partial}\Sigma'_\alpha =& -2\partial\varphi\bar{\partial}\varphi(1 + \coth(2\varphi)) \\ =& \frac{2\tanh^2(\psi)}{\cosh(2\psi) - 1}\partial\psi\bar{\partial}\psi.\end{aligned} \tag{3.2.49}$$

It is also worth noting that $H(x,y)$ can be written, algebraically, in terms of Σ_α and Σ'_α, via (3.2.35a), and so it is also given by ψ.

3.2.5 Correlation Functions With Rotational Symmetry

By insisting that the states we consider have rotational invariance we may keep α and β arbitrary in (3.2.28) and use (3.2.24) to simplify these equations. As we are imposing rotational invariance it is sensible to change coordinates, using r and θ defined by

$$z = re^{i\theta} \qquad \text{and} \qquad \bar{z} = re^{-i\theta}. \tag{3.2.50}$$

After this change of coordinates we can use (3.2.28b) and (3.2.28c) together with (3.2.25) to find $f_1 = f_2$, up to an irrelevant constant, in (3.2.24) thus we let

$$f(r) := f_1(r) = f_2(r). \tag{3.2.51}$$

As we have this equality but no relation between $F^{\beta+1,\beta+1}_{\alpha,\alpha}$ and $F^{\beta,\beta}_{\alpha+1,\alpha+1}$ we must employ a different approach to that of subsection 3.2.4, eliminating $F^{\beta+1,\beta+1}_{\alpha,\alpha}$ and $F^{\beta,\beta}_{\alpha+1,\alpha+1}$ instead of $F^{\beta+1,\beta}_{\alpha,\alpha+1}$ and $F^{\beta,\beta+1}_{\alpha+1,\alpha}$. In doing this it will be useful to reduce our notation by defining the differential operator

$$\mathcal{D}_r = \partial_r^2 + \frac{1}{r}\partial_r. \tag{3.2.52}$$

and the function

$$F(r) := F^{\beta,\beta}_{\alpha,\alpha}(r). \tag{3.2.53}$$

Note that the α and β dependence of F is assumed, so $F \neq F^{\beta+1,\beta+1}_{\alpha,\alpha}$ and $F \neq F^{\beta,\beta}_{\alpha+1,\alpha+1}$.

Now using (3.2.28a,3.2.28b,3.2.28c) to eliminate $F^{\beta+1,\beta+1}_{\alpha,\alpha}$ and $F^{\beta,\beta}_{\alpha+1,\alpha+1}$ from (3.2.28d) and (3.2.28e) leaves

$$\begin{aligned} -pqf\mathcal{D}_r f + pq\frac{(\alpha-\beta)^2}{r^2}f^2 + pqm^2f^2 + (\partial_r F)^2 \\ - \frac{1}{F^2 - pqf^2}\big(F^2(\partial_r F)^2 - 2pqFf\partial_r F\partial_r f \\ + (pq)^2 f^2(\partial_r f)^2 - (pq)^2(\alpha-\beta)^2 f^4/r^2 \big) = 0. \end{aligned} \tag{3.2.54a}$$

and similarly (3.2.28e) becomes

$$\begin{aligned} -pqf\mathcal{D}_r f + pq\frac{(\alpha-\beta)^2}{r^2}f^2 + F\mathcal{D}_r F \\ - \frac{1}{F^2 - pqf^2}\big(F^2(\partial_r F)^2 - 2pqFf\partial_r F\partial_r f \\ + (pq)^2 f^2(\partial_r f)^2 - (pq)^2(\alpha-\beta)^2 f^4/r^2 \big) = 0. \end{aligned} \tag{3.2.54b}$$

As in the case for $\alpha = \beta$ further progress is made by setting

$$F + \sqrt{pq}f = e^{\chi}\cosh(\varphi) \qquad F - \sqrt{pq}f = e^{\chi}\sinh(\varphi) \tag{3.2.55}$$

which reduces (3.2.54a) and (3.2.54b), respectively, to:

$$\begin{aligned}(\mathcal{D}_r\chi - \mathcal{D}_r\varphi)(\sinh(2\varphi) - \cosh(2\varphi)) \\ + (\partial_r\varphi)^2(2\sinh(2\varphi) - \frac{(\cosh^2\varphi + \sinh^2\varphi)^2}{\cosh\varphi\sinh\varphi}) \\ + m^2(\cosh(2\varphi) - \sinh(2\varphi)) \\ + \frac{(\alpha-\beta)^2}{r^2}(\cosh(2\varphi) - \sinh(2\varphi))(1 + \frac{1}{4}(\coth\varphi - 1)(1 - \tanh\varphi)) = 0 \end{aligned} \quad (3.2.56a)$$

$$\begin{aligned}\sinh(2\varphi)\mathcal{D}_r\chi + \cosh(2\varphi)\mathcal{D}_r\varphi \\ + (\partial_r\varphi)^2(\sinh(2\varphi) - \cosh(2\varphi)\coth(2\varphi)) \\ + \frac{(\alpha-\beta)^2}{4r^2}(\cosh(2\varphi) - \sinh(2\varphi))(\coth(2\varphi) + 1) = 0. \end{aligned} \quad (3.2.56b)$$

Again we eliminate χ from these equations, leaving

$$\begin{aligned}\mathcal{D}_r\varphi - (\partial_r\varphi)^2(1 + \coth(2\varphi)) - m^2\sinh(2\varphi)(\sinh(2\varphi) - \cosh(2\varphi)) \\ + \frac{(\alpha-\beta)^2}{4r^2}(1 + \coth(2\varphi)) = 0 \end{aligned} \quad (3.2.57)$$

which is to become (1.1.22). In analogy with the previous case this is accomplished by noticing that

$$\frac{h''}{h'} = -1 - \coth(2\varphi) \quad (3.2.58)$$

is solved by

$$2\psi = h(\varphi) = \ln\left(\frac{\sqrt{1-e^{4\varphi}} - 1}{\sqrt{1-e^{4\varphi}} + 1}\right) \quad (3.2.59)$$

and it is then easy to show that

$$\mathcal{D}_r\psi = \frac{m^2}{2}\sinh(2\psi) + \frac{(\alpha-\beta)^2}{r^2}\tanh(\psi)\left(1 - \tanh^2(\psi)\right). \quad (3.2.60)$$

It is also straight forward to show that by setting

$$e^{\Sigma} = 2F(r) = e^{\chi+\varphi} \quad (3.2.61)$$

we find

$$\begin{aligned}\mathcal{D}_r\Sigma &= \mathcal{D}_r\chi + \mathcal{D}_r\varphi \\ &= \frac{m^2}{2}(1 - \cosh(2\psi)) \end{aligned} \quad (3.2.62)$$

completing our derivation of known results. As with the previous derivation we can also write a new result. By setting

$$e^{\Sigma'} = 2\sqrt{pq}f(r) = e^{\chi-\varphi} \tag{3.2.63}$$

we find that f is also given by the same function, ψ, via the equation

$$\mathcal{D}_r\Sigma' = \frac{(\partial_r\psi)^2}{\sinh^2\psi} - m^2 - \frac{(\alpha-\beta)^2}{r^2\cosh^2\psi}. \tag{3.2.64}$$

In the present situation, however, we cannot write down any equations for the functions $F^{\beta+1,\beta+1}_{\alpha,\alpha}$ and $F^{\beta,\beta}_{\alpha+1,\alpha+1}$ as these functions always occur as a product and we have no equation relating the two.

3.2.6 Verification of Results

It should be noted that the differential equations for the correlation functions differ from those presented in [7] where, using a method of Fredholm determinants, it is shown that

$$\begin{aligned} \mathcal{D}_r\Sigma &= \frac{m^2}{2}(1-\cosh(2\psi)) \\ \mathcal{D}_r\psi &= \frac{m^2}{2}\sinh(2\psi) + \frac{4(\alpha-\beta)^2}{r^2}\tanh\psi(1-\tanh^2\psi) \end{aligned} \tag{3.2.65}$$

with Σ as in (3.2.61). In order to be sure that our equations are correct we undertake numerical and analytical approximations of the correlation function

$$\langle\sigma_\alpha(x,y)\sigma_\beta(0,0)\rangle \tag{3.2.66}$$

by examining its form factor expansion at large r and investigating how well the differential equations (3.2.60) and (3.2.65) are satisfied. Strong evidence will be found using up to only two particles in this expansion.

To begin we insert the resolution of the identity (2.1.10) in between the fields of

(3.2.66) and then expand the two particle term:

$$\begin{aligned}
\langle\sigma_\alpha(x,y)\sigma_\beta(0,0)\rangle &= \langle \mathrm{vac}|\sigma_\alpha(x,y)|\mathrm{vac}\rangle\langle \mathrm{vac}|\sigma_\beta(0,0)|\mathrm{vac}\rangle \\
&+ \int \mathrm{d}\theta_1\mathrm{d}\theta_2 \langle \mathrm{vac}|\sigma_\alpha(x,y)|\theta_1,\theta_2\rangle_{+-}{}^{-+}\langle\theta_2,\theta_1|\sigma_\beta(0,0)|\mathrm{vac}\rangle + \cdots \\
&= \langle \mathrm{vac}|\sigma_\alpha(x,y)|\mathrm{vac}\rangle\langle \mathrm{vac}|\sigma_\beta(0)|\mathrm{vac}\rangle \\
&+ \int \mathrm{d}\theta_1\mathrm{d}\theta_2 e^{y(E_{\theta_1}+E_{\theta_2})-ix(p_{\theta_1}+p_{\theta_2})} \\
&\times \left(\langle \mathrm{vac}|\sigma_\alpha(0,0)|\theta_1,\theta_2\rangle_{+-}\langle \mathrm{vac}|\sigma_\beta(0,0)|\theta_2+i\pi,\theta_1+i\pi\rangle_{+-} \right) + \cdots \\
&= c_\alpha c_\beta m^{\alpha^2+\beta^2} \\
&- \frac{c_\alpha c_\beta m^{\alpha^2+\beta^2}\sin(\pi\alpha)\sin(\pi\beta)}{4\pi^2} \\
&\times \left(\int \mathrm{d}\theta_1\mathrm{d}\theta_2 e^{my(E_{\theta_1}+E_{\theta_2})-ix(p_{\theta_1}+p_{\theta_2})} \frac{e^{(\theta_1-\theta_2)(\alpha-\beta)}}{\cosh^2(\frac{\theta_1-\theta_2}{2})} \right) + \cdots
\end{aligned} \tag{3.2.67}$$

Since the first term is constant it is the second term which will eventually provide the leading large distance behaviour of ψ. It is useful to use boost invariance to replace the (x,y) dependence in this term with r dependence, so that the second term in (3.2.67) becomes

$$-\frac{c_\alpha c_\beta m^{\alpha^2+\beta^2}\sin(\pi\alpha)\sin(\pi\beta)}{4\pi^2}\int \mathrm{d}\theta_1\mathrm{d}\theta_2 e^{-r(E_{\theta_1}+E_{\theta_2})}\frac{e^{(\theta_1-\theta_2)(\alpha-\beta)}}{\cosh^2(\frac{\theta_1-\theta_2}{2})}. \tag{3.2.68}$$

This expression can be further simplified using a change of variables inside the integral. Setting

$$\phi_1 = \frac{\theta_1+\theta_2}{2} \quad \text{and} \quad \phi_2 = \frac{\theta_1-\theta_2}{2} \tag{3.2.69}$$

we see that the integral in (3.2.68) becomes

$$\begin{aligned}
&\frac{1}{2}\int \mathrm{d}\phi_1\mathrm{d}\phi_2 e^{-rm(\cosh(\phi_1+\phi_2)+\cosh(\phi_1-\phi_2))}\frac{e^{2\phi_2(\alpha-\beta)}}{\cosh^2\phi_2} \\
&= \frac{1}{2}\int \mathrm{d}\phi_1\mathrm{d}\phi_2 e^{-2rm\cosh(\phi_1)\cosh(\phi_2)}\frac{e^{2\phi_2(\alpha-\beta)}}{\cosh^2\phi_2} \\
&= \int \mathrm{d}\phi_2 \frac{e^{2\phi_2(\alpha-\beta)}}{\cosh^2\phi_2}K_0(2rm\cosh\phi_2)
\end{aligned} \tag{3.2.70}$$

where $K_n(x)$ is a modified Bessel function.

We wish to use (3.2.70) to examine Σ at large values of r. From the definition (3.2.61) we can write

$$\begin{aligned}
\Sigma &= \ln(1-\frac{\sin(\pi\alpha)\sin(\pi\beta)}{4\pi^2}\int \mathrm{d}\theta\ \frac{e^{2\theta(\alpha-\beta)}}{\cosh^2\theta}K_0(2mr\cosh\theta)) + \cdots \\
&= \ln(1-g(r)) + \cdots
\end{aligned} \tag{3.2.71}$$

where $g(r)$ is defined implicitly. To ease our notation we introduce the constant

$$\xi = \frac{\sin(\pi\alpha)\sin(\pi\beta)}{4\pi^2}. \tag{3.2.72}$$

In order to approximate ψ, as this is where our equations differ from (3.2.65), we need to calculate

$$\left(\partial_r^2 + \frac{1}{r}\partial_r\right)\Sigma \tag{3.2.73}$$

which quickly becomes very cumbersome when using (3.2.71). However when r is large $g(r)$ becomes small so we expect the Taylor series expansion of the log to still provide a good approximation. Thus we define our approximate solution for Σ as

$$\tilde{\Sigma} = -g(r) \tag{3.2.74}$$

and thus

$$\begin{aligned}\left(\partial_r^2 + \frac{1}{r}\partial_r\right)\tilde{\Sigma} &= -(g''(r) + \frac{1}{r}g'(r)) \\ &= -4m^2\xi\int \mathrm{d}\theta\ e^{2\theta(\alpha-\beta)}K_0(2mr\cosh\theta).\end{aligned} \tag{3.2.75}$$

This is a useful approximation to the left hand side of (3.2.62) so we must now examine the right hand side to obtain a usable expression for ψ. As $r \to \infty$ (3.2.66) is expected to tend to $c_\alpha c_\beta m^{\alpha^2+\beta^2}$, a constant, and so in this limit it must be that $\psi \to 0$. With this in mind, at sufficiently large r we may Taylor expand the right hand side of (3.2.62) and only keep the leading term:

$$\frac{m^2}{2}\left(1 - \cosh(2\psi)\right) = -m^2\psi^2 + \cdots \tag{3.2.76}$$

Letting our approximate solution be $\tilde{\psi}$ we see that

$$\tilde{\psi}^2 = 4\xi\int \mathrm{d}\theta\ e^{2\theta(\alpha-\beta)}K_0(2mr\cosh\theta) = J(r) \tag{3.2.77}$$

where the function $J(r)$ is defined for later convenience. Our goal now is to see how well this approximate solution solves the equations (3.2.60) and the corresponding equation from (3.2.65).

As a first indicator as to which equation is correct we examine how well our approximate solution (3.2.77) satisfies both (3.2.60) and (3.2.65) numerically. As

the left hand sides of both equations are the same it is sufficient to examine the three functions

$$(\partial_r^2 + \frac{1}{r}\partial_r)\tilde{\psi} \tag{3.2.78}$$

$$\frac{m^2}{2}\sinh(2\tilde{\psi}) + \frac{4(\alpha-\beta)^2}{r^2}\tanh\tilde{\psi}(1-\tanh^2\tilde{\psi}) \tag{3.2.79}$$

$$\frac{m^2}{2}\sinh(2\tilde{\psi}) + \frac{(\alpha-\beta)^2}{r^2}\tanh\tilde{\psi}(1-\tanh^2\tilde{\psi}). \tag{3.2.80}$$

at large enough values of r.

These three functions can be calculated numerically for a fixed value or $(\alpha-\beta)$ and plotted for a range of r. We note that we need $(\alpha-\beta) \neq 0$ otherwise the two equations coincide and our calculations tell us nothing. In figure 3.1 these curves are plotted with $(\alpha-\beta) = 0.3$ and $r \in (7, 7.25)$. From this plot it is clear that for modest values of r the expression presented in section 3.2.3, (3.2.80), is a better approximation of (3.2.78) than (3.2.79). The same behaviour is observed for different values of $(\alpha-\beta)$.

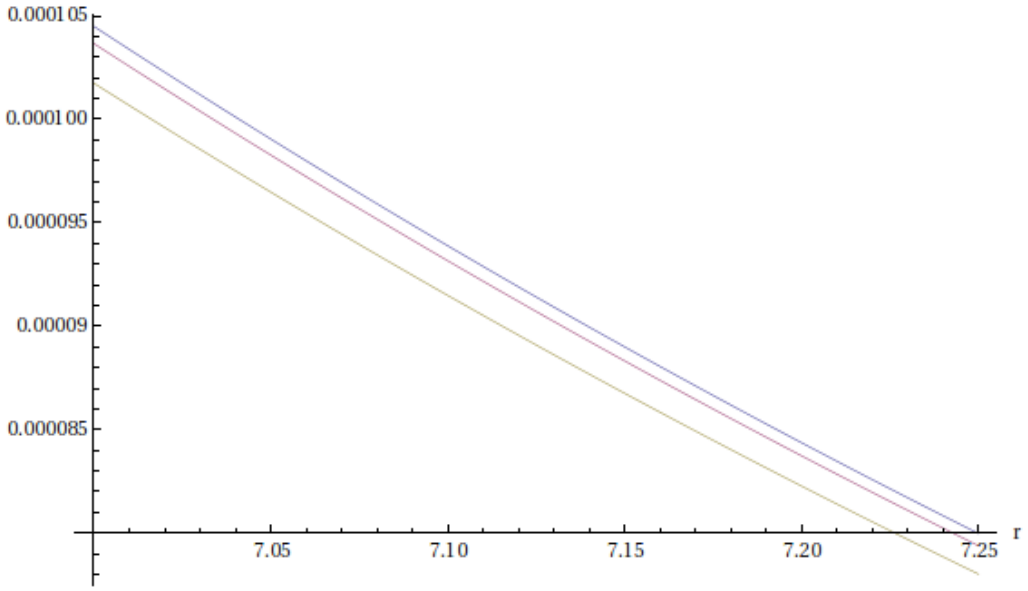

Figure 3.1: Plot of (3.2.78) (blue), (3.2.79) (yellow) and (3.2.80) (red).

From plots such as figure 3.1 we see that (3.2.80) provides a better approximate solution for all values of r. At $r = 10$, for example (3.2.78) is 4.36163×10^{-6}, (3.2.80) is 4.34464×10^{-6} while (3.2.79) is 4.30542×10^{-6}. So in this case and using the equation presented in this thesis the approximate solution $\tilde{\psi}$ is out by 0.39%.

When the equation (3.2.79) is used this difference jumps to 1.39%. Again the same pattern appears when different values of $(\alpha - \beta)$ and r are used.

With the strong numerical evidence pointing to (3.2.60) being the correct equation for ψ we will now look for some analytic evidence to support this. The objective is to expand both sides of (3.2.60) and test whether or not the leading r terms cancel when the approximation $\psi = \tilde{\psi}$ is applied.

To begin with, as we know that ψ is small at large values of r we may expand the right hand side of (3.2.60) and only keep the linear term:

$$\frac{m^2}{2}\sinh(2\psi) + \frac{(\alpha-\beta)^2}{r^2}\tanh\psi(1-\tanh^2\psi) = \left(m^2 + \frac{(\alpha-\beta)^2}{r^2}\right)\psi + \cdots \quad (3.2.81)$$

We now want to insert our approximation of ψ and examine the equation

$$(\partial_r + \frac{1}{r}\partial_r)\tilde{\psi} = \left(m^2 + \frac{(\alpha-\beta)^2}{r^2}\right)\tilde{\psi} + \cdots \quad (3.2.82)$$

It is convenient at this point to use the function $J(r)$ defined in (3.2.77) so that this equation becomes, neglecting higher order terms,

$$\frac{J''J - \frac{1}{2}(J')^2 + \frac{1}{r}J'J}{2J^{3/2}} \simeq \left(m^2 + \frac{(\alpha-\beta)^2}{r^2}\right)\sqrt{J} \quad (3.2.83)$$

and to further ease our calculations this expression can be rearranged to give

$$J''J - \frac{1}{2}(J')^2 + \frac{1}{r}J'J \simeq 2\left(m^2 + \frac{(\alpha-\beta)^2}{r^2}\right)J^2. \quad (3.2.84)$$

In order to progress further we need:

$$\begin{aligned} J(r) &= 4\xi\int \mathrm{d}\theta\ e^{2\theta(\alpha-\beta)}K_0(2mr\cosh\theta) && (3.2.85)\\ J'(r) &= -8m\xi\int \mathrm{d}\theta\ e^{2(\alpha-\beta)\theta}\cosh\theta K_1(2mr\cosh\theta) && (3.2.86)\\ J''(r) &= 16m^2\xi\int \mathrm{d}\theta\ e^{2(\alpha-\beta)\theta}\cosh^2\theta K_0(2mr\cosh\theta) \\ &\quad +8m\xi\int \mathrm{d}\theta\ e^{2(\alpha-\beta)\theta}\frac{\cosh\theta}{r}K_1(2mr\cosh\theta). && (3.2.87)\end{aligned}$$

From this we see that the second term in the $J''J$ term of (3.2.84) cancels with the $J'J/r$ term so there are only two terms left to deal with.

Next we note that there are no $(\alpha-\beta)^2$ terms in these derivatives so we need to eliminate these from (3.2.84). This is done by integrating $J(r)$ by parts, integrating

the exponential term to introduce factors of $1/(\alpha-\beta)$:

$$\begin{aligned}\int \mathrm{d}\theta\ e^{2(\alpha-\beta)\theta}K_0(2mr\cosh\theta) =& -\frac{mr}{\alpha-\beta}\int \mathrm{d}\theta\ e^{2(\alpha-\beta)\theta}K_1(2mr\cosh\theta)\sinh\theta\\ =&\frac{mr}{2(\alpha-\beta)^2}\int \mathrm{d}\theta\ e^{2(\alpha-\beta)\theta}\Big(2mrK_0(2mr\cosh\theta)\sinh^2\theta\\ &+\frac{\sinh^2\theta}{\cosh\theta}K_1(2mr\cosh\theta)-K_1(2mr\cosh\theta)\cosh\theta\Big)\\ =&\frac{m^2r^2}{(\alpha-\beta)^2}\int \mathrm{d}\theta\ e^{2(\alpha-\beta)\theta}K_0(2mr\cosh\theta)(\cosh^2\theta-1)\\ &-\frac{mr}{2(\alpha-\beta)^2}\int \mathrm{d}\theta\ e^{2(\alpha-\beta)\theta}\frac{1}{\cosh\theta}K_1(2mr\cosh\theta).\end{aligned} \tag{3.2.88}$$

Now writing the right hand side of (3.2.84) as

$$m^2J^2+\frac{(\alpha-\beta)^2}{r^2}J^2 \tag{3.2.89}$$

and using the above integration by parts to rewrite one of the J's multiplying the $(\alpha-\beta)^2$ term we see that the m^2 term is cancelled by part of the first integral in (3.2.88) and the other part of this integral cancels some of the integral from the $J''J$ term on the left hand side. So bringing all the remaining terms of (3.2.84) together we have

$$\begin{aligned}&32m^2\xi^2\left(\int \mathrm{d}\theta\ e^{2(\alpha-\beta)\theta}K_0(2mr\cosh\theta)\right)\\ &\quad\times\left(\int \mathrm{d}\theta\ e^{2(\alpha-\beta)\theta}\Big(\frac{1}{2mr\cosh\theta}K_1(2mr\cosh\theta)+\cosh^2\theta K_0(2mr\cosh\theta)\Big)\right)\\ &\qquad-32m^2\xi^2\left(\int \mathrm{d}\theta\ e^{2(\alpha-\beta)\theta}\cosh\theta K_1(2mr\cosh\theta)\right)^2\simeq 0\end{aligned} \tag{3.2.90}$$

from which we want to extract the leading large r behaviour. Expanding the Bessel functions for large r gives leading terms proportional to $1/\sqrt{r}$:

$$\begin{aligned}&32m^2\xi^2\left(\int \mathrm{d}\theta\ e^{2(\alpha-\beta)\theta}\sqrt{\frac{\pi}{4mr\cosh\theta}}e^{-2mr\cosh\theta}\right)\\ &\qquad\times\left(\int \mathrm{d}\theta\ e^{2(\alpha-\beta)\theta}\cosh^2\theta\sqrt{\frac{\pi}{4mr\cosh\theta}}e^{-2mr\cosh\theta}\right)\\ &\qquad-32m^2\xi^2\left(\int \mathrm{d}\theta\ e^{2(\alpha-\beta)\theta}\cosh\theta\sqrt{\frac{\pi}{4mr\cosh\theta}}e^{-2mr\cosh\theta}\right)^2.\end{aligned} \tag{3.2.91}$$

While these terms do not cancel directly we observe that the main contribution from each integrand is from the region around $\theta = 0$ and so the first order saddle point approximation of these integrals does indeed vanish, as we had hoped.

It is worth noting that if we had started from (3.2.65) there would be an extra term proportional to $m^2 J^2$ in (3.2.90) and this would lead to extra terms in (3.2.91) which would have nothing to cancel against, either directly or in the saddle point approximation.

Chapter 4

Discussion

As stated in the introduction, differential equations satisfied by the correlation functions

$$\langle \sigma_\alpha(x,y)\sigma_\beta(0,0)\rangle \tag{4.0.1}$$

in the Dirac theory were derived. The method employed led us to consider a new family of descendent twist fields, whose properties and form factors were discussed in section 2.3 and first written down in [28]. By considering these fields a novel way of evaluating the vacuum expectation values of primary twist fields was discovered, namely through the recursion relation (2.3.22). While extra assumptions, for which we have no clear arguments, are needed to fix c_α it transpired that this relation was sufficient for our purposes.

The techniques used to calculate the form factors of descendent twist fields may be applicable to other free field models and so could give a method for calculating the vacuum expectation values of general twist fields. As well as applying to twist fields with a more general symmetry group this method could be used in conjunction with finite temperature form factors to give vacuum expectation values of twist fields at finite temperatures.

Taking inspiration from [33] meant that a double copy model of the Dirac theory was considered and the action of conserved charges in this theory had to be found. In order to do this the CFT limit of the Dirac theory, where the action of these conserved charges could be calculated exactly, was examined. The relations calculated in this limit were then carried over to the massive case and verified using form

factors with up to four particles.

The success of this approach was unexpected because it was thought that the special form of the OPEs in the Ising model permitted the reduction of the Ward identities to integrable equations. Despite having more complicated OPEs the differential equations for the correlators of twist fields in the Dirac model were calculated and, as only standard techniques were used to obtain the result, that is knowledge of symmetries of the states in question and the action of the conserved charges, there is hope that this approach could have more general applications. It has already been shown that in the Dirac model the quantum state under consideration only requires the density matrix to be an exponential of a quadratic form to produce useful Ward identities. However, only by imposing certain symmetries can the Ward identities be reduced to an integrable form in the most general case presented in this thesis.

As the method presented above is very direct and very general it may prove fruitful when applied to other free field models possessing non-trivial twist fields. Applying this method to models with non-abelian symmetries would be an interesting avenue of research.

Part II

Truncated Conformal Space Approach

Chapter 5

Perturbed Conformal Field Theory

In this chapter we discuss the ingredients necessary to implement the Truncated Conformal Space Approach. The key elements are all based in Conformal Field Theory (CFT) and so we begin with a short review of the aspects of CFT relevant to our investigations. This is by no means a full review of CFT as this is beyond the scope of this thesis and many good reviews already exist: [34, 36, 38].

5.1 Conformal Field Theory

The power of Conformal Field Theory in two dimensions is due to its infinite dimensional symmetry group. The two dimensional spaces where a CFT is defined may be equipped with a Minkowski or Euclidean metric. In what follows we will only consider flat Euclidean spaces unless otherwise stated.

Conformal Field Theory is easiest to construct after identifying our two dimensional Euclidean space with the complex plane:

$$z = x + iy \qquad \bar{z} = x - iy. \tag{5.1.1}$$

In all that follows we will consider z and $\bar{z}$ to be independent variables and so we often consider only the z dependence of functions, keeping $\bar{z}$ fixed and setting $\bar{z} = z^*$, the complex conjugate, only at the end of calculations.

The group of global conformal transformations on the Riemann sphere is just the Möbius transformations

$$z \mapsto f(z) = \frac{az+b}{cz+d} \tag{5.1.2}$$

where $a, b, c, d \in \mathbb{C}$ and $ad - bc = 1$. These transformations form a finite group but we also require our Conformal Field Theories to be invariant under infinitesimal conformal transformations, which have an infinite algebra in two-dimensions.

A primary field of our theory, ϕ, with scaling dimension Δ, spin s has holomorphic dimension h and anti-holomorphic dimension $\bar{h}$ given by

$$h = \frac{1}{2}(\Delta + s) \qquad \bar{h} = \frac{1}{2}(\Delta - s) \tag{5.1.3}$$

is defined to be a field which under any arbitrary conformal mapping $z \mapsto w(z)$, $\bar{z} \mapsto \bar{w}(\bar{z})$ transforms as

$$\phi'(w, \bar{w}) = \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^{-h} \left(\frac{\mathrm{d}\bar{w}}{\mathrm{d}\bar{z}}\right)^{-\bar{h}} \phi(z, \bar{z}). \tag{5.1.4}$$

A quasi-primary field is one which transforms as (5.1.4) only for maps which are Möbius transformations, (5.1.2). This transformation property fixes the two and three point functions up to some overall structure constants. Two point functions are given by

$$\langle \phi_1(z_1, \bar{z}_1)\phi_2(z_2, \bar{z}_2)\rangle = \frac{C_{12}}{(z_1 - z_2)^{2h}(\bar{z}_1 - \bar{z}_2)^{2\bar{h}}} \qquad \text{if} \begin{cases} h_1 = h_2 = h \\ \bar{h}_1 = \bar{h}_2 = \bar{h} \end{cases} \tag{5.1.5}$$

and vanish otherwise, while three point functions become

$$\langle \phi_1(z_1, \bar{z}_1)\phi_2(z_2, \bar{z}_2)\phi_3(z_3, \bar{z}_3)\rangle = C_{123} \frac{1}{z_{12}^{h_1+h_2-h_3} z_{23}^{h_2+h_3-h_1} z_{13}^{h_3+h_1-h_2}} \times \frac{1}{\bar{z}_{12}^{\bar{h}_1+\bar{h}_2-\bar{h}_3} \bar{z}_{23}^{\bar{h}_2+\bar{h}_3-\bar{h}_1} \bar{z}_{13}^{\bar{h}_3+\bar{h}_1-\bar{h}_2}} \tag{5.1.6}$$

where $z_{ij} = z_i - z_j$. In the cases which concern us, i.e. that of minimal models, the constants C_{ij} will be fixed through a choice of basis for the fields of the model and the structure constants C_{ijk} also occur in the OPEs of the relevant fields. This will be discussed in more detail in section 5.1.2.

5.1.1 The Operator Formalism

In the present work the operator formalism is the most convenient way to describe the relevant conformal field theories so we give a brief review of the key points.

To construct our Hilbert space we use radial quantisation. After picking a point z_0 the vectors in the Hilbert space are states on concentric circles around z_0 which is usually set to the origin. We must assume the existence of a vacuum state $|0\rangle$ upon which the Hilbert space is constructed by the application of operators. For free theories this means that the vacuum is the state annihilated by the positive frequency part of the field and we assume that interacting theories are free in the infinite past and future. This allows us to define 'in' states as

$$|\phi_{\text{in}}\rangle = \lim_{z,\bar{z}\to 0} \phi(z,\bar{z})|0\rangle = |h,\bar{h}\rangle \tag{5.1.7}$$

and then use Hermitian conjugation to define 'out' states as

$$\langle\phi_{\text{out}}| = |\phi_{\text{in}}\rangle^{\dagger} \tag{5.1.8}$$

where h and $\bar{h}$ are the holomorphic and anti-holomorphic dimensions, respectively, of ϕ. In the following chapters we will normalise these states such that

$$\langle h_i,\bar{h}_i|h_j,\bar{h}_j\rangle = \delta_{i,j}. \tag{5.1.9}$$

To build a full basis of states we need some knowledge of the conformal generators which will act non-trivially on these highest weight states.

The conformal generators may be found by examining the components of the energy-momentum tensor. In what follows we only consider the holomorphic part, T, but all the following arguments also apply to the anti-holomorphic part, $\bar{T}$. Translation, rotation and scale invariance restrict the form of operator product expansion (OPE) of the energy momentum tensor with itself. The most general form of this OPE is

$$T(z)T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} \tag{5.1.10}$$

where c is the central charge and is a fundamental constant which helps define a model of conformal field theory. As the energy momentum tensor has dimension 2

we may expand it in terms of its modes L_n:

$$T(z) = \sum_{n\in\mathbb{Z}} z^{-n-2} L_n \tag{5.1.11}$$

which may also be defined through the integral

$$L_n = \frac{1}{2\pi i} \oint \mathrm{d}z \; z^{n+1} T(z) \tag{5.1.12}$$

where the integration is taken around a circle centred on the origin. The modes L_n are the generators of local conformal transformations on the Hilbert space and their commutation relations can be evaluated using the OPE (5.1.10):

$$\begin{aligned}
[L_n, L_m] =& \frac{1}{(2\pi i)^2} \oint_0 \mathrm{d}w \; w^{m+1} \oint_w \mathrm{d}z \; z^{n+1} T(z)T(w) \\
=& \frac{1}{(2\pi i)^2} \oint_0 \mathrm{d}w \; w^{m+1} \oint_w \mathrm{d}z \; z^{n+1} \left(\frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} + \cdots \right) \\
=& \frac{1}{2\pi i} \oint_0 \mathrm{d}w \; w^{m+1} \left(\frac{cn(n^2-1)}{12} w^{n-2} + 2(n+1) w^n T(w) + w^{n+1} \partial T(w) \right) \\
=& \frac{cn(n^2-1)}{12} \delta_{n+m,0} + 2(n+1) L_{m+n} - \oint_0 \mathrm{d}w \; (n+m+2) w^{n+m+1} T(w) \\
=& \frac{cn(n^2-1)}{12} \delta_{n+m,0} + (n-m) L_{n+m}.
\end{aligned} \tag{5.1.13}$$

When combined with their anti-holomorphic counterparts, $\bar{L}_n$, we see that these generators satisfy the Virasoro algebra:

$$\begin{aligned}
[L_n, L_m] =& (n-m) L_{n+m} + \frac{c}{12} n(n^2-1) \delta_{n+m,0} \\
[L_n, \bar{L}_m] =& 0 \\
[\bar{L}_n, \bar{L}_m] =& (n-m) \bar{L}_{n+m} + \frac{c}{12} n(n^2-1) \delta_{n+m,0}.
\end{aligned} \tag{5.1.14}$$

Hermitian conjugation of these operators is given by

$$(L_n)^\dagger = L_{-n} \tag{5.1.15}$$

and similarly for the anti-holomorphic operators. The operator $L_0 + \bar{L}_0$ is the generator of dilations.

The generators L_n and $\bar{L}_n$ are used to construct a basis of the Hilbert space. The vacuum state, $|0\rangle$, must be invariant under the global conformal transformations and so must be annihilated by the operators L_{-1}, L_0 and L_1 and the anti-holomorphic

counterparts. The well defined action of the stress energy momentum tensor on the vacuum thus implies that

$$\begin{aligned} L_n|0\rangle &= 0 \\ \bar{L}_n|0\rangle &= 0 \end{aligned} \qquad \text{for} \quad n \geq -1. \tag{5.1.16}$$

It can also be shown, using a method similar to that of (5.1.13), that in this operator language a primary field ϕ of dimensions $(h, \bar{h})$ satisfies the following relations:

$$\begin{aligned} [L_n, \phi(w,\bar{w})] =& h(n+1)w^n\phi(w,\bar{w}) + w^{n+1}\partial\phi(w,\bar{w}) \\ [\bar{L}_n, \phi(w,\bar{w})] =& \bar{h}(n+1)\bar{w}^n\phi(w,\bar{w}) + \bar{w}^{n+1}\bar{\partial}\phi(w,\bar{w}). \end{aligned} \tag{5.1.17}$$

Applying these to the asymptotic state (5.1.7) we see that

$$L_0|h,\bar{h}\rangle = h|h,\bar{h}\rangle \qquad \bar{L}_0|h,\bar{h}\rangle = \bar{h}|h,\bar{h}\rangle \tag{5.1.18}$$

and thus $|h,\bar{h}\rangle$ is an eigenstate of the Hamiltonian and is annihilated by L_n and $\bar{L}_n$ for $n > 0$. Excited states are obtained by acting on $|h,\bar{h}\rangle$ with so called ladder operators L_{-m} and $\bar{L}_{-m}$ with $m > 0$, each operator increasing the holomorphic or anti-holomorphic dimensions, respectively, of the state by m. Considering only the holomorphic part for simplicity, excited states of the form

$$L_{-k_1}L_{-k_2}\cdots L_{-k_n}|h\rangle \tag{5.1.19}$$

are referred to as descendents of $|h\rangle$. This state has dimension

$$h' = h + k_1 + k_2 + \cdots + k_n \equiv h + N \tag{5.1.20}$$

and the integer N is the level of the state. The full set of descendent states of $|h\rangle$ form a subset of the full Hilbert space which is closed under the action of the Virasoro generators and so forms a module of the Virasoro algebra called a Verma module.

5.1.2 Minimal Models

Minimal models are some of the simplest conformal field theories as their Hilbert space consists of a finite number of Verma modules. These models are considered

in the original work of Belavin, Polyakov and Zamolodchikov [6] as well as the reviews already mentioned where more details can be found. We simply present some information which will be useful in later calculations.

Minimal models possess a finite number of primary fields and the operator algebra is closed. This closing of the operator algebra only occurs for certain values of the central charge, namely

$$c = 1 - 6\frac{(p-q)^2}{pq} \tag{5.1.21}$$

where $p > q > 1$. The conformal weights of primary fields in this model are

$$h_{r,s} = \frac{(pr-qs)^2 - (p-q)^2}{4pq} \tag{5.1.22}$$

where $1 \leq r < q$ and $1 \leq s < p$. This clearly implies the symmetry $h_{r,s} = h_{q-r,p-s}$ and so we may identify the corresponding fields:

$$\phi_{(r,s)} = \phi_{(q-r,p-s)} \tag{5.1.23}$$

and we see that there are $(p-1)(q-1)/2$ distinct fields in the diagonal modular invariant theory. Each of these fields corresponds to an irreducible representation of the Virasoro algebra which is degenerate, that is to say it contains null vectors. These null vectors are members of the Verma module which, other than the highest weight vector, are annihilated by L_m for $m > 0$. The first null vector in the module corresponding to $\phi_{(r,s)}$ is at level rs and the number of linearly independent vectors at each level, $\dim(h+n)$, is given by coefficients of the generating function $\chi_{(r,s)}(x)$. This generating function is also referred to as the character of the module and is given by [34]:

$$\begin{aligned}\chi_{(r,s)}(x) =& \sum_{n=0}^{\infty} \dim(h+n) x^{n+h_{r,s}-c/24} \\ =& K^{p,q}_{r,s}(x) - K^{p,q}_{r,-s}(x)\end{aligned} \tag{5.1.24}$$

where

$$\begin{aligned}K^{p,q}_{r,s}(x) =& \frac{x^{-1/24}}{\varphi(x)} \sum_{n\in\mathbb{Z}} x^{(2pqn+pr-qs)^2/4pq} \qquad \text{and} \\ \frac{1}{\varphi(x)} =& \prod_{n=1}^{\infty} \frac{1}{1-x^n}.\end{aligned} \tag{5.1.25}$$

The operator product expansions for primary fields are given by

$$\phi_i(z,\bar{z})\phi_j(w,\bar{w}) = \sum_k C_{ij}^k (z-w)^{h_k-h_i-h_j}(\bar{z}-\bar{w})^{\bar{h}_k-\bar{h}_i-\bar{h}_j}\phi_k(w,\bar{w}) + \cdots \tag{5.1.26}$$

where $\cdots$ represents the descendent fields. The sum is over all the fields of the theory, of which there are infinitely many, and we note that if a primary field is not present in the OPE then none of its descendents will appear either. Therefore in an abuse of notation we may write express the OPE between primary fields through the fusion rule

$$\phi_i \times \phi_j = \sum_k \mathcal{N}_{ij}^k \phi_k \tag{5.1.27}$$

where ϕ_i represents the family of vectors in the highest weight representation descended from ϕ_i and $\mathcal{N}_{ij}^k$ vanishes if ϕ_k does not occur in the OPE and 1 otherwise. In the case of minimal models these values can be calculated using the Verlinde formula [77]:

$$\mathcal{N}_{ij;kl}^{mn} = \sum_{(rs)} \frac{S_{ij;rs}S_{kl;rs}S_{rs;mn}}{S_{11;rs}} \tag{5.1.28}$$

where the sum is over all fields in the model and

$$S_{ij;kl} = 2\sqrt{\frac{2}{pq}}(-1)^{1+jk+il}\sin(\pi\frac{p}{q}ik)\sin(\pi\frac{q}{p}jl). \tag{5.1.29}$$

The structure constants C_{ij}^k in (5.1.26) are not fixed by the representation theory of the Virasoro algebra and so must be computed another way. They were first calculated in [23] but we will use the expression given in [66],

$$C_{ij}^k = \left(\mathsf{F}_{k1}\begin{bmatrix} i & i \\ j & j \end{bmatrix}\right)^{-1} \tag{5.1.30}$$

where i, j and k represent fields in the model given by their indices (r,s) and the power -1 is interpreted as one over the matrix element, not the inverse matrix. If we set $j=(r,s)$ then define $j\pm(2,1) = (r\pm 1, s)$ and $j\pm(1,2) = (r, s\pm 1)$ we then express the F-matrix entries involving the field $(2,1)$ as

$$\begin{aligned}\mathsf{F}\begin{bmatrix} j & (2,1) \\ i & l \end{bmatrix} &= \begin{pmatrix} \mathsf{F}_{l-(2,1),j-(2,1)} & \mathsf{F}_{l-(2,1),j+(2,1)} \\ \mathsf{F}_{l+(2,1),j-(2,1)} & \mathsf{F}_{l+(2,1),j+(2,1)} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\Gamma(d_j)\Gamma(1-d_l)}{\Gamma(\frac{1}{2}(1-d_i+d_j-d_l))\Gamma(\frac{1}{2}(1+d_i+d_j-d_l))} & \frac{\Gamma(-d_j)\Gamma(1-d_l)}{\Gamma(\frac{1}{2}(1-d_i-d_j-d_l))\Gamma(\frac{1}{2}(1+d_i-d_j-d_l))} \\ \frac{\Gamma(d_j)\Gamma(1+d_l)}{\Gamma(\frac{1}{2}(1-d_i+d_j+d_l))\Gamma(\frac{1}{2}(1+d_i+d_j+d_l))} & \frac{\Gamma(-d_j)\Gamma(1+d_l)}{\Gamma(\frac{1}{2}(1-d_i-d_j+d_l))\Gamma(\frac{1}{2}(1+d_i-d_j+d_l))} \end{pmatrix}\end{aligned} \tag{5.1.31}$$

where $d_j = rt - s$ and $t = p/q$. Similarly the F-matrix entries involving $(1,2)$ are

$$\mathsf{F}\begin{bmatrix} j & (1,2) \\ i & l \end{bmatrix} = \begin{pmatrix} \mathsf{F}_{l-(1,2),j-(1,2)} & \mathsf{F}_{l-(1,2),j+(1,2)} \\ \mathsf{F}_{l+(1,2),j-(1,2)} & \mathsf{F}_{l+(1,2),j+(1,2)} \end{pmatrix}$$
$$= \begin{pmatrix} \frac{\Gamma(-\frac{1}{t}d_j)\Gamma(1+\frac{1}{t}d_l)}{\Gamma(\frac{1}{2t}(t+d_i-d_j+d_l))\Gamma(\frac{1}{2t}(t-d_i-d_j+d_l))} & \frac{\Gamma(\frac{1}{t}d_j)\Gamma(1+\frac{1}{t}d_l)}{\Gamma(\frac{1}{2t}(t+d_i+d_j+d_l))\Gamma(\frac{1}{2t}(1-d_i+d_j+d_l))} \\ \frac{\Gamma(-\frac{1}{t}d_j)\Gamma(1-\frac{1}{t}d_l)}{\Gamma(\frac{1}{2t}(t+d_i-d_j-d_l))\Gamma(\frac{1}{2t}(t-d_i-d_j-d_l))} & \frac{\Gamma(\frac{1}{t}d_j)\Gamma(1-\frac{1}{t}d_l)}{\Gamma(\frac{1}{2t}(t+d_i+d_j-d_l))\Gamma(\frac{1}{2t}(t-d_i+d_j-d_l))} \end{pmatrix}. \quad (5.1.32)$$

All other F-matrices can be calculated using the recursion relation

$$\mathsf{F}_{a,b}\begin{bmatrix} j & k+\Delta \\ i & l \end{bmatrix} = \sum_{r,s} \mathsf{F}_{k+\Delta,r}\begin{bmatrix} l & \Delta \\ a & k \end{bmatrix} \mathsf{F}_{a,s}\begin{bmatrix} j & k \\ i & r \end{bmatrix} \mathsf{F}_{r,b}\begin{bmatrix} s & \Delta \\ i & l \end{bmatrix} \mathsf{F}_{s,k+\Delta}\begin{bmatrix} \Delta & k \\ b & j \end{bmatrix} \quad (5.1.33)$$

where Δ is $(2,1)$ or $(1,2)$, together with the symmetry relations

$$\mathsf{F}_{a,b}\begin{bmatrix} j & k \\ i & l \end{bmatrix} = \mathsf{F}_{a,b}\begin{bmatrix} i & l \\ j & k \end{bmatrix} = \mathsf{F}_{a,b}\begin{bmatrix} l & i \\ k & j \end{bmatrix} \quad (5.1.34)$$

which can be found in [61] along with many other F-matrix relations.

5.2 Boundary Conformal Field Theory

Adding a boundary to a conformal field theory brings boundary conditions into play. Boundary conformal field theory on the upper half plane was first studied by Cardy and Lewellen in a series of articles, [10, 11, 13]. As in the previous section we only discuss ideas and results relevant to TCSA applications and leave a full discussion to the literature, [12, 34] and [5] for example.

5.2.1 The Upper Half Plane

Following the literature chronologically we begin with a discussion of conformal field theories defined on the upper half plane. The boundary of these models runs along the real axis and to ensure these models are physical energy must be conserved and therefore an appropriate boundary condition must be applied. This condition is most simply expressed in terms of the stress tensor:

$$T(z)|_{z=\bar{z}} = \bar{T}(\bar{z})|_{z=\bar{z}}. \quad (5.2.1)$$

While this equation does not fix the boundary condition uniquely it does reduce the symmetries allowed in the bulk theory. The upper half plane model can be mapped to the unit disc and, recalling the definitions (5.1.12), in the radial quantisation picture the symmetry (5.2.1) implies that $L_n = \bar{L}_{-n}$. This means that the boundary state living on the unit circle must satisfy the condition

$$L_n|B\rangle = \bar{L}_{-n}|B\rangle. \tag{5.2.2}$$

This equation implies that the holomorphic representation of the Virasoro algebra with highest weight i is equivalent to the anti-holomorphic representation with the same highest weight. There is an independent boundary state, $|j\rangle\!\rangle$, corresponding to each Virasoro representation, with highest weight j, in the model. These states are known as the Ishibashi states [42].

There are subtleties associated to calculating inner products of the Ishibashi states [11, 42] but these subtleties will not come into play in this thesis. It is enough to know that the inner product can be defined and that any general boundary state can be written as

$$|a\rangle = \sum_j \frac{\psi_a^j}{\sqrt{S_{1j}}}|j\rangle\!\rangle \tag{5.2.3}$$

where the sum runs over all the highest weights present in the model. These states, however, are not yet the correct boundary states. While they satisfy the gluing condition (5.2.2) there are constraints on the reflection factors ψ_a^j which can be deduced from the nature of the Hilbert space.

5.2.2 The Cylinder

In order to relate the Hilbert space of a model with its general boundary conditions it is easiest to consider the theory on a finite cylinder. There are two equivalent quantisation schemes on the cylinder. The first has time flowing around the cylinder so that the Hamiltonian H_{ab} depends on the boundary conditions at each edge. The second has time flowing along the cylinder so that the boundary conditions are represented as initial and final states, $|a\rangle$ and $|b\rangle$, and the Hamiltonian is obtained from the whole complex plane.

For the rest of this section we will use a different normalisation of the fields in order to be consistent with the literature. Thus we set

$$\langle 0|0\rangle = S_{11} \qquad \text{and} \qquad \langle i|i\rangle = C^1_{ii}\langle 0|0\rangle \tag{5.2.4}$$

giving $C^1_{ii} = S_{1i}/S_{11}$. In order to return to the normalisation of the rest of this part we perform the transformation

$$\phi_i \mapsto \alpha_i \phi_i \qquad \text{and} \qquad \langle 0|0\rangle \mapsto \mu^2 \langle 0|0\rangle \tag{5.2.5}$$

where $\alpha_i = \sqrt{S_{11}/S_{1i}}$ and $\mu = 1/\sqrt{S_{11}}$ before using these results in any TCSA programme.

In the picture where time is periodic the partition function on the finite cylinder may be written as

$$Z_{ab}(x) = \sum_i n^i_{ab} \chi_i(x) \tag{5.2.6}$$

since the spectrum falls into irreducible representations of the Virasoro algebra. The integers n^i_{ab} count the number of copies of the representation with highest weight i in the model.

Using a modular transformation it is possible to map this picture to the alternate picture where space is periodic. As we are only interested in minimal models we may use their transformation properties together with our knowledge of the structure of boundary conformal field theories to obtain the relation

$$\sum_i S_{ij} n^i_{ab} = \langle a|j\rangle\langle j|b\rangle \tag{5.2.7}$$

or alternatively the Cardy relation

$$n^i_{ab} = \sum_j \frac{S_{ij}}{S_{1j}} (\psi^j_a)^* \psi^j_b. \tag{5.2.8}$$

Bearing in mind that there is a boundary state corresponding to each irreducible representation present in the bulk minimal model we may define a set of allowed boundary states as

$$|\tilde{i}\rangle = \sum_j \frac{S_{ij}}{S_{1j}} |j\rangle\rangle \tag{5.2.9}$$

where i and j can be any highest weight in the minimal model. It is then easy to show that $n^i_{\tilde{k}\tilde{l}}$ satisfies the Verlinde formula:

$$\sum_i S_{ij} n^i_{\tilde{k}\tilde{l}} = \frac{S_{kj} S_{lj}}{S_{1j}}. \tag{5.2.10}$$

So we conclude that

$$n^i_{\tilde{k}\tilde{l}} = \mathcal{N}^i_{kl} \tag{5.2.11}$$

and thus the number of times representation i occurs in the Hilbert space $\mathcal{H}_{\tilde{k}\tilde{l}}$ is simply given by fusion coefficient $\mathcal{N}^i_{kl}$, an integer as required.

5.2.3 Boundary Fields

As we are now able to write down the Hilbert space for a conformal field theory with boundary conditions $\tilde{k}$ and $\tilde{l}$, we may ask if it is possible for these boundary conditions to change. The most natural setting for this in on a strip of finite width with time running along the infinite length of the strip. In this picture the Hilbert space contains $\mathcal{N}^i_{kl}$ copies of the representation i. If the boundary condition $\tilde{l}$ is to change it must change to another allowed boundary condition, $\tilde{m}$ say, instantaneously, at $t = t_0$. For $t > t_0$ the Hilbert space contains $\mathcal{N}^i_{\tilde{k}\tilde{m}}$ copies of the representation i which is generally different from $\mathcal{N}^i_{\tilde{k}\tilde{l}}$. This discontinuity is achieved by the insertion of a local operator $\phi^j_{\tilde{l}\tilde{m}}$ on the boundary at $t = t_0$. These fields transform in the representation j of the Virasoro algebra where the fusion coefficient $\mathcal{N}^j_{lm}$ is non-zero.

Fields living on a boundary can be treated in the same way as boundary changing fields. While it may be more natural to consider the field $\phi^j_{\tilde{l}\tilde{l}}(t)$ as a degree of freedom on the boundary it is equally valid to consider it a field changing the $\tilde{l}$ boundary condition into the $\tilde{l}$ boundary condition.

In the upper half plane, when a bulk field $\phi(x+iy)$, with dimensions $(h, \bar{h})$, gets close to the boundary at $y = 0$, with boundary condition $\tilde{k}$, it may be expressed as a sum of boundary fields $\phi^j_{\tilde{k}\tilde{k}}$:

$$\phi(x+iy) = \sum_j \; {}^{\tilde{k}}B^j_\phi (2y)^{h_k - h - \bar{h}} \phi^j_{\tilde{k}\tilde{k}}(x) + \cdots \tag{5.2.12}$$

where $\cdots$ represents descendent fields. Two boundary fields on the real axis $\phi^i_{\tilde{l}\tilde{m}}(x)$

and $\phi^j_{\tilde{m}\tilde{n}}(y)$ with $x > y$ have an operator product expansion given by

$$\phi^i_{l\tilde{m}}(x)\phi^j_{\tilde{m}\tilde{n}}(y) = \sum_k C_{ij}^{(lmn)k}(x-y)^{h_k-h_i-h_j}\phi^k_{l\tilde{n}}(y) + \cdots \tag{5.2.13}$$

These two expansions define the structure constants ${}^{\tilde{k}}B^j_\phi$ and $C_{ij}^{(lmn)k}$. They have been calculated for the minimal models considered here in [66]:

$${}^{\tilde{k}}B^j_i = e^{i\frac{\pi}{2}h_j}\left(\mathsf{F}_{k1}\begin{bmatrix} k & k \\ j & j \end{bmatrix}\right)^{-1}\frac{S^i_k(j)}{S^1_k} \tag{5.2.14}$$

$$C_{ij}^{(lmn)k} = \mathsf{F}_{mk}\begin{bmatrix} l & n \\ i & j \end{bmatrix} \tag{5.2.15}$$

where $S^i_k(j)$ are the S-matrix elements on the torus with one operator insertion. These structure constants are a necessary ingredient of the TCSA as on the cylinder we have

$$\langle i|\phi_k(0)|j\rangle = C^i_{kj} \tag{5.2.16}$$

in the bulk and

$$\langle i|\phi^k_{ll}(1)|j\rangle = C_{kj}^{(lll)i} \tag{5.2.17}$$

on the boundary for appropriately normalised states.

Chapter 6

Truncated Conformal Space Approach

Before exploring the Truncated Conformal Space Approach we need a theory to apply it to. This will be a minimal model defined on a cylinder or strip which is perturbed by a diagonal, relevant field in the bulk or, in the case of the strip, on the boundary.

6.1 Bulk Perturbations

We start by following the original paper of Yurov and Zamolodchikov [80] where the authors considered the perturbed scaling Lee-Yang model on a cylinder. On a cylinder with circumference R in the x direction the perturbed Hamiltonian is

$$H = H_{\text{CFT}} + \lambda H_{\text{Bulk}}. \tag{6.1.1}$$

The CFT Hamiltonian on the cylinder is expressed in terms of the operators on the plane, as in (5.1.11), as

$$H_{\text{CFT}} = \frac{2\pi}{R}(L_0 + \bar{L}_0 - \frac{c}{12}) \tag{6.1.2}$$

and the bulk perturbation is

$$H_{\text{Bulk}} = \int_0^R \phi_p(x, 0)\mathrm{d}x. \tag{6.1.3}$$

The translation symmetry of the cylinder allows us to arbitrarily fix the y coordinate of the perturbing field, so 0 is chosen for convenience. Also, only perturbations by diagonal fields, that is fields ϕ_p which satisfy $h_p = \bar{h}_p$, are considered.

Choosing a basis of the Hilbert space given by eigenstates of H_{CFT} we now need a method for calculating matrix elements of H_{CFT} and H_{Bulk}. Since $L_0 + \bar{L}_0$ simply count the holomorphic and anti-holomorphic dimensions of states the eigenstates of H_{CFT} are simply

$$|\psi_i\rangle = L_{-n_1} L_{-n_2} \cdots L_{-n_k} \bar{L}_{-m_1} \cdots \bar{L}_{-m_l} |\phi_i\rangle \tag{6.1.4}$$

and we define $x_i = n_1 + \cdots + n_k$ and $\bar{x} = m_1 + \cdots + m_l$ with the spin of the state given by $s = x - \bar{x}$. The matrix element of H_{Bulk} with two states $|\psi_i\rangle$ and $|\psi_j\rangle$ is given by

$$\langle\psi_i|H_{\text{Bulk}}|\psi_j\rangle = 2\pi \left(\frac{R}{2\pi}\right)^{1-2h_p} \delta_{s_i,s_j} \langle\psi_i|\phi_p(0,0)|\psi_j\rangle. \tag{6.1.5}$$

The calculation of these matrix elements is left to appendix B.2 but is essentially an application of the commutation rules (5.1.17).

The truncated conformal space approach can now be used to examine the theory at finite R. This is done by truncating the space of conformal states (6.1.4) to only include those states with $x, \bar{x} \leq N$. With this restriction we may find the eigenvalues of the Hamiltonian, $H^{(N)}$, restricted to these states, numerically. Finding the basis of states to use is a non-trivial task which is covered in appendix B.1. Diagonalising this matrix is simplified since states with different spins give a zero matrix element and so different spin sectors decouple.

We expect the eigenvalues of H, E_i for $i = 1, 2, 3 \cdots$, to have different scaling behaviour for different values of R. The ultraviolet spectrum is dominated by the unperturbed CFT Hamiltonian and so, letting ξ denote the scaling length of the model,

$$E_i \simeq \frac{2\pi}{R}\left(x_i - \frac{c}{12}\right) \qquad \text{for } R \ll \xi \tag{6.1.6}$$

where x_i is the eigenvalue of $L_0 + \bar{L}_0$. For larger values of R it should be that

$$E_i \simeq \frac{\epsilon_0}{\xi^2} R + m_i \qquad \text{for } R \gg \xi \tag{6.1.7}$$

where m_i is the mass gap for the ith level. The constant ϵ_0 can be interpreted as the universal contribution to the vacuum energy density. This dimensionless constant can be calculated exactly for integrable massive systems. The interpretation of m_i as the mass of a particle of the model certainly holds for the lowest energy levels, which represent one particle states with zero momentum. For higher energy eigenvalues the same scaling behaviour will be observed but m_i has a different meaning. Because the theory is defined on a cylinder the allowed momenta are quantised and so for higher energy levels the allowed values of m_i will still be discrete and should be interpreted as the total energy above the ground state.

By truncating the Hilbert space we are inserting a new length scale ρ into the problem, as the truncation imposes an upper bound on the eigenvalues of H_0. For large values of R the matrix elements of $H^{(N)}$ will therefore be dominated by H_{Bulk} and so we expect the eigenvalues of $H^{(N)}$, $E_i^{(N)}$, to scale like

$$E_i^{(N)} \simeq \lambda R^{1-2*h_\phi} \qquad \text{for } R \gg \rho. \tag{6.1.8}$$

We now see that in order to obtain useful approximations from the TCSA calculation we must include enough states so that $\rho \gg \xi$. When this is the case the eigenstates of $H^{(N)}$ will display three distinct regions characterised by their dependence on R, namely (6.1.6), (6.1.7) and (6.1.8). These three regions will be discussed in more detail in the case of the tri-critical Ising model in section 6.1.2.

6.1.1 The Lee-Yang Model

We begin with a check on our algorithms by following the original work of [80] and applying the TCSA to the scaling Lee-Yang model. This model has $p = 5$, $q = 2$ and central charge $c = -22/5$. The negative central charge means that this model is non-unitary but it is the simplest non-trivial minimal model as it contains only two irreducible representations of the Virasoro algebra, corresponding to the identity, $\mathbb{1}$ with weight $h = \bar{h} = 0$, and only one relevant field ϕ with weight $h = \bar{h} = -1/5$.

The Hamiltonian on a cylinder of circumference R is

$$H = \frac{2\pi}{R}(L_0 + \bar{L}_0 - \frac{c}{12}) + i\lambda \int_0^R \phi(x, 0)\mathrm{d}x \tag{6.1.9}$$

where the i is included to keep the eigenvalues of H real for real values of λ. The spectrum of this Hamiltonian contains a mass gap which is the mass, M, of the single massive particle in the model. This mass is linked to the perturbing parameter λ through the equation [83, 85]

$$\lambda = (0.097048456\cdots)M^{12/5} \tag{6.1.10}$$

so we may set $M = 1$ by setting $\lambda = 0.09704845636\cdots$

As the ground state and first excited state are found in the spin 0 sector we will only consider spin 0 states here. Truncating the space of states at level 5 gives 17 states and the 17 eigenvalues of the Hamiltonian (6.1.9) are plotted in figure 6.1. This figure shows several expected phenomena.

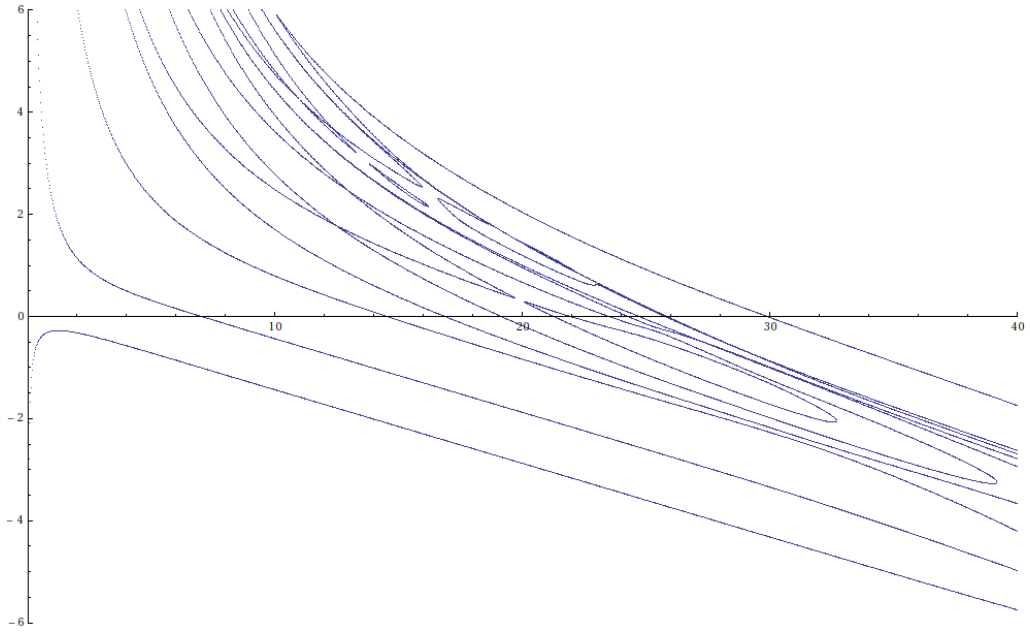

Figure 6.1: The spin 0 energy levels plotted against R from states truncated at level 5.

Firstly the energy levels in the spin 0 sector diverge as $R \to 0$ as expected from (6.1.6). As R increases we see that the lowest energy levels, which are expected to be the best approximated, become parallel straight lines as expected from (6.1.7). The slope of these lines has been calculated from the Thermodynamic Bethe Ansatz to be $-\sqrt{3}/12$ [83] and this is the slope of the lowest energy levels shown in figure 6.1. The mass gap between the two lowest energy levels in this region gives $m_1 = 1$

as expected from (6.1.10).

The final characteristic of figure 6.1 we mention is the behaviour of crossing energy levels. It is the integrability of the full Hamiltonian which allows energy levels to cross but as this is lost with the truncation of the Hilbert space the energy levels of the truncated, non-unitary, Hamiltonian form complex conjugate pairs at the crossing points. The range of values of R for which the energy levels are complex at each crossing point decreases as the truncation level is increased [80], indicating that raising the truncation level is bringing us closer to the full theory.

6.1.2 The Tri-Critical Ising Model

We next turn our attention to the more complicated tri-critical Ising model which we shall investigate in greater detail. This minimal model has $p = 5$ and $q = 4$, central charge $c = 7/10$ and contains 6 representations of the Virasoro algebra, given together with their highest weights in table 6.1. As this model is unitary the Hamiltonian, given by

$$H = \frac{2\pi}{R}(L_0 + \bar{L}_0 - \frac{c}{12}) + \lambda \int_0^R \phi_{i,j}(x,0)\mathrm{d}x, \tag{6.1.11}$$

has real eigenvalues. In order to be sure our algorithm is working correctly with this model we will first reproduce some of the results of [53] and [41]. We also note that as with the Lee-Yang model sectors of the Hilbert space with different spins decouple and the states that are of interest to us at the moment are all in the spin 0 sector. It is sufficient for us to truncate the Hilbert space at level 6, giving 465 states.

Following [53] we start with the leading magnetic perturbation by setting $\lambda = 1$ and $\phi_{i,j} = \phi_{2,2}$. The first 10 eigenvalues of $H^{(6)}$ are plotted in figure (6.2) where the region given by (6.1.7) is clearly visible. The behaviour of the energy levels as $R \to 0$ is highlighted in figure 6.3 which clearly shows that as $R \to 0$, $RE_i \to$ constant as expected from (6.1.6). The three scaling regions are easiest to see in figure 6.4 where the effective scaling exponent of the ground state

$$a = \frac{\mathrm{d}\log_{10} E_0}{\mathrm{d}\log_{10} R} \tag{6.1.12}$$

Field	Conformal Weight
$\phi_{1,1}$	0
$\phi_{2,1}$	$\frac{7}{16}$
$\phi_{2,2}$	$\frac{3}{80}$
$\phi_{3,1}$	$\frac{3}{2}$
$\phi_{3,2}$	$\frac{3}{5}$
$\phi_{3,3}$	$\frac{1}{10}$

Table 6.1: The primary fields of the tri-critical Ising model and their conformal weights.

is plotted against $\log_{10} R$. From this plot it is easy to distinguish the UV region where $a = -1$, the IR regime where $a = 1$ and the truncation dominated region where $a = 0.925$.

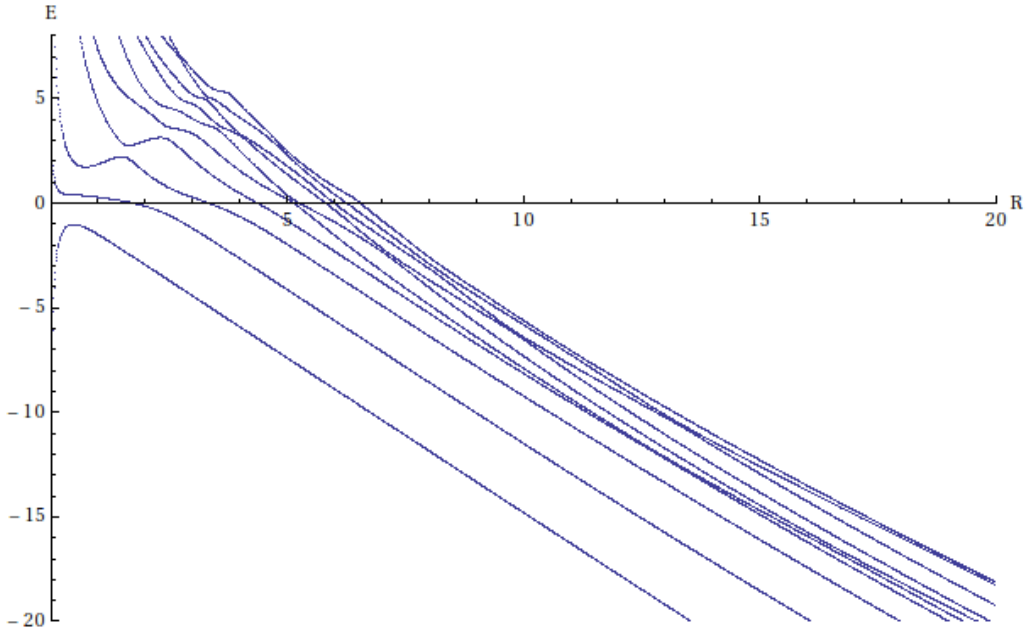

Figure 6.2: The lowest 10 spin 0 energy levels of the leading magnetic perturbation truncated at level 6.

Now that we have clearly identified the IR scaling region we may extract the masses of stable particles in the model. Particles are stable if their mass is less than twice the mass of the lightest particle, so it is impossible for them to decay. At

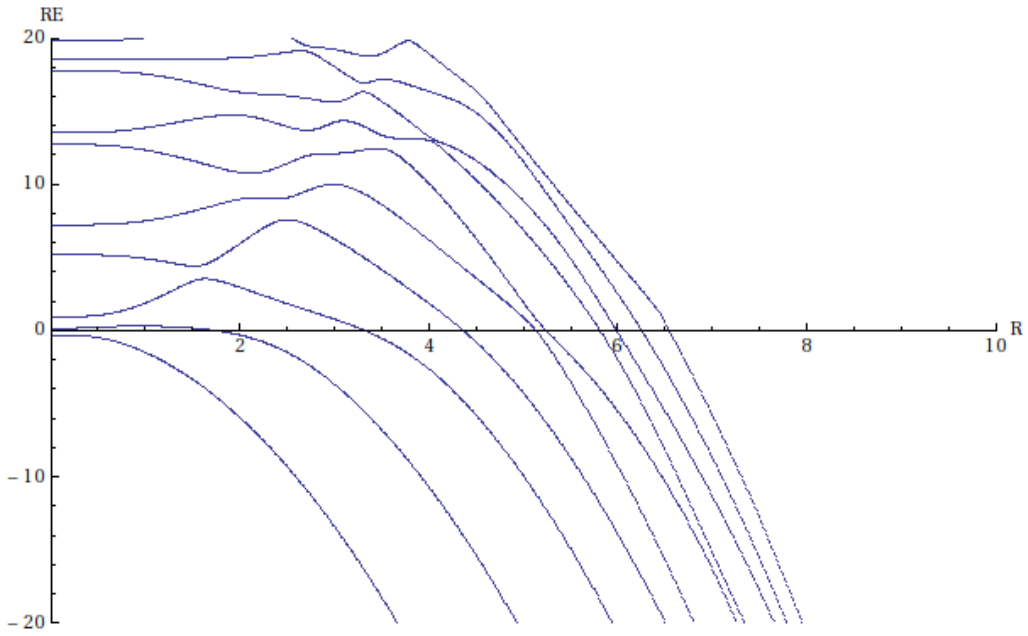

Figure 6.3: RE_i against R for the lowest 10 spin 0 energy levels of the leading magnetic perturbation.

$R = 2.2$, which is within the IR scaling region as shown in figure 6.4, we find

$$m_1 = 1 \qquad m_2 = 1.6316 \qquad m_3 = 1.9683 \tag{6.1.13}$$

where we have normalised the masses so that $m_1 = 1$. These masses are very close to those calculated in [53]. The same quantities can be calculated when the perturbing field $\phi_{i,j}$ is any of the other relevant primary fields and in all these cases good agreement is found between our calculations and those of [53].

As a final check of our algorithms we calculate vacuum expectation values in the perturbed models as was done in [41]. It is known, [60], that the one point function of a field ϕ_Δ in the perturbed theory defined by the Hamiltonian (6.1.11) is given by

$$\langle \phi_\Delta \rangle = A(\Delta, (i,j))|\lambda|^{\frac{h_\Delta}{1-h_{i,j}}} \tag{6.1.14}$$

when $h_\Delta + h_{i,j} < 1$, that is the field ϕ has no multiplicative renormalisation. It is the constant A that we will calculate and in certain cases compare with TBA data. This is possible as we may use the TCSA Hamiltonian $H^{(N)}$ to find an approximation of the ground state of the model at finite a R inside the IR scaling region, we denote

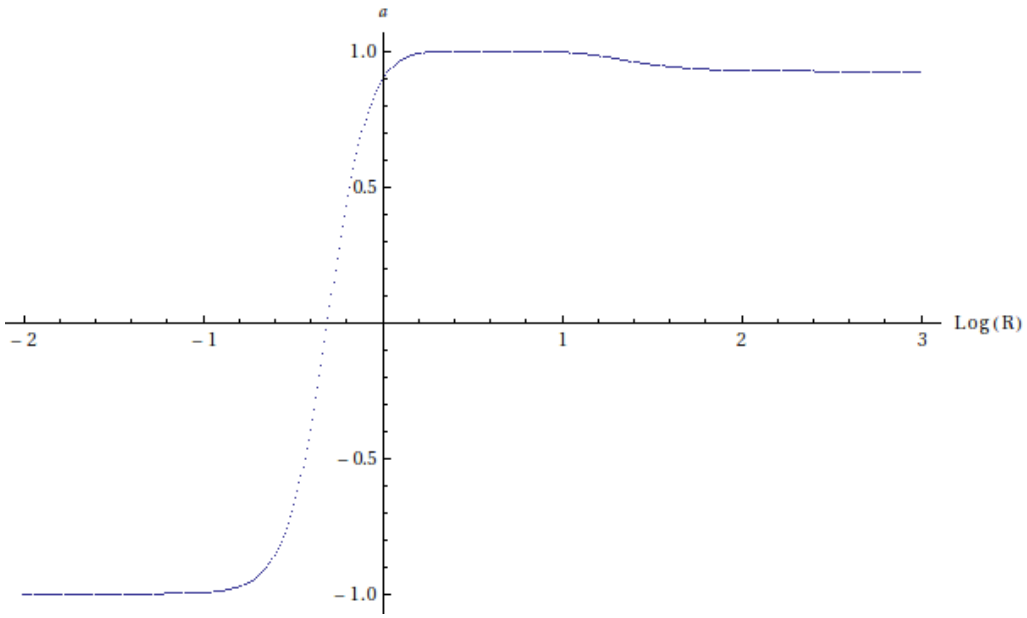

Figure 6.4: The effective scaling exponent of the ground state energy of the leading magnetic perturbation.

this state by

$$|0\rangle_N = \sum_i \psi_i^{(N)} |i\rangle \tag{6.1.15}$$

where $|i\rangle$ are the conformal states at or below the truncation level.

We next note that the finite constant A is given by

$$A(\Delta,(i,j)) = \frac{\langle\phi_\Delta\rangle}{|\lambda|^{\frac{h_\Delta}{1-h_{i,j}}}} = (\frac{2\pi}{R})^{2h_\Delta} \lim_{R,N\to\infty} \frac{{}_N\langle 0|\phi_\Delta|0\rangle_N}{|\lambda|^{\frac{h_\Delta}{1-h_{i,j}}}} \tag{6.1.16}$$

where the matrix elements in the final sum are evaluated on the cylinder of radius R. These are calculated using (6.1.15)

$${}_N\langle 0|\phi_\Delta|0\rangle_N = \frac{\psi_i^{(N)}\psi_j^{(N)}\langle i|\phi_\Delta|j\rangle}{\psi_i^{(N)}\psi_j^{(N)}\langle i|j\rangle} \tag{6.1.17}$$

where the sums over i and j are implicit. As well as taking a value of R inside the scaling region we also check the consistency of our results by verifying that ${}_N\langle 0|\phi_\Delta|0\rangle_N \sim R^{2h_\Delta}$ as must be the case when A is finite.

With the perturbing field $\phi_{2,2}$ and truncation level of 6, as with all previous calculations, we plot

$$b = \frac{1}{2h_{2,2}} \frac{\mathrm{d}\log_{10}{}_N\langle 0|\phi_{2,2}|0\rangle_N}{\mathrm{d}\log_{10} R} \tag{6.1.18}$$

against $\log R$ in figure 6.5. This plot clearly show the three scaling regions of the truncated conformal space approach and it is easy to identify the region where the scaling of the one point function is $2h_{2,2} = 6/80$, that is where $b = 1$, and verify that this coincides with the region where the effective scaling exponent of the ground state is 1. A similar plot can be produced for all the fields whose vacuum expectation value we wish to examine so we can be sure that our estimates of A are as accurate as possible at the given truncation level. For the model perturbed by $\phi_{2,2}$, with $\lambda = 1$, the values of A are

$$A((2,1),(2,2)) = -1.55001 \tag{6.1.19}$$

$$A((2,2),(2,2)) = -1.53968 \tag{6.1.20}$$

$$A((3,2),(2,2)) = 1.90413 \tag{6.1.21}$$

$$A((3,3),(2,2)) = 1.33585 \tag{6.1.22}$$

which are in good agreement with the same TCSA calculations performed in [41].

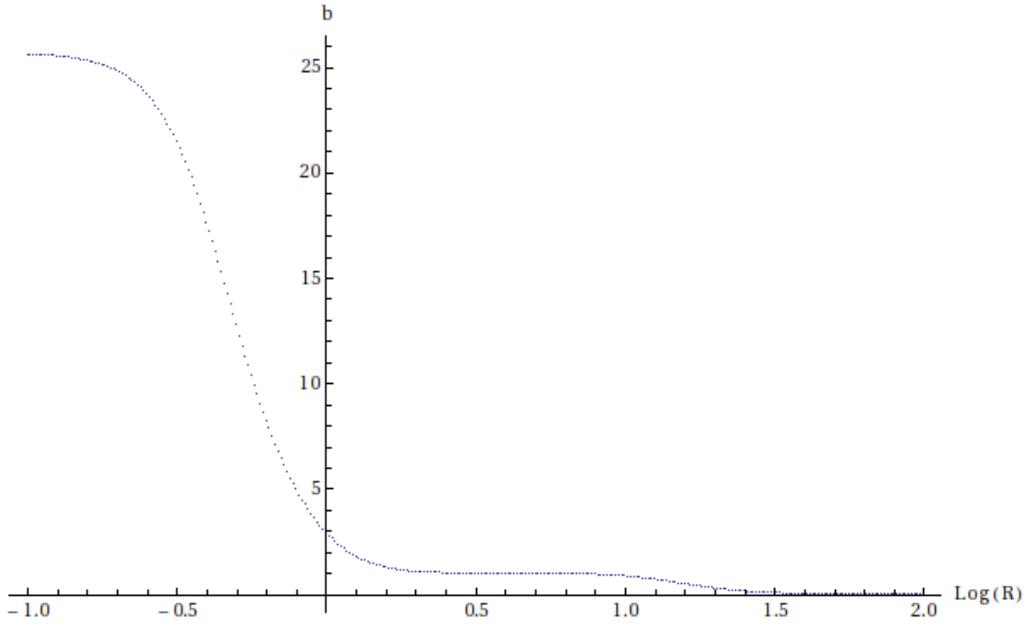

Figure 6.5: The effective scaling of the one point function of $\phi_{2,2}$ on the cylinder.

The vacuum expectation values of certain fields in the tri-critical Ising model with a specific perturbation have been calculated using the thermodynamic Bethe Ansatz and so we may use these results as an independent check on the validity of

the TCSA method. For example in [31] it is shown that

$$A((3,3),(3,3)) = -1.4683954240276\cdots \tag{6.1.23}$$

for $\lambda > 0$ and using the TCSA approximation described above we find that

$$A((3,3),(3,3)) = -\,1.46524 \qquad \text{with } \lambda = 1, \tag{6.1.24}$$

in good agreement with (6.1.23).

We are now confident that the TCSA method outlined in this chapter provides a good approximation of perturbed minimal models in the bulk as our algorithms give the expected results. However, we have not attempted to analyse the renormalisation group flows initiated by these perturbations which smoothly transform the spectrum of the initial minimal model into that of another with fewer relevant fields. This is beyond the scope of this thesis but is discussed in [39].

6.2 Boundary Perturbations

We next consider the same minimal models defined on a strip but the edges of the strip are no longer identified. As there are now boundaries in the system we must impose conformal boundary conditions as discussed in section 5.2, which restricts the representations of the Virasoro algebra occurring in the model. In this section we will examine perturbations of the boundary conditions which was first explored via TCSA in [21] where the Lee-Yang model is considered.

The Hamiltonian of a minimal model with boundary conditions a and b on the left and right boundaries, perturbed by fields ϕ_L and ϕ_R respectively, is most simply expressed when mapped to the unit semi-circle in the upper half plane:

$$H_{(a,b)} = \frac{\pi}{R}\left(L_0 - \frac{c}{24} + \lambda_L\left(\frac{R}{\pi}\right)^{1-h_L}\phi_L(-1) + \lambda_R\left(\frac{R}{\pi}\right)^{1-h_R}\phi_R(1)\right). \tag{6.2.1}$$

As all the perturbations considered here are on the boundaries the unperturbed bulk Hilbert space, which is massless and obtained using the methods described in section 5.2, must remain massless. Thus perturbing by a relevant boundary field must produce a renormalisation group flow from one conformal boundary condition

to another with fewer relevant operators. This effect means that massless boundary flows can be observed using TCSA, with a modest truncation level, despite the truncation errors. Bulk massless flows are less well modelled by bare TCSA, as was observed in [53], and only provide accurate results at modest truncation levels when the divergent ground state energy is subtracted [48] and the couplings are renormalised [39].

6.2.1 The Lee-Yang Model

As in the previous section we start by examining the Lee-Yang model on a strip of width R. As discussed in section 5.2 the choice of boundary conditions dictates the representations of the Virasoro algebra occurring in the model as well as the perturbing fields allowed.

The Lee-Yang model admits only two conformal boundary conditions which we label $\mathbb{1}$ and Φ, following the notation of [21]. The $\mathbb{1}$ boundary has only one primary field, which is the identity, while the Φ boundary has two, the identity and a field ϕ with dimensions $h_\phi = -1/5$. As there are only two possible boundary conditions there are only three independent combinations. Firstly, with boundary conditions $(\mathbb{1}, \mathbb{1})$, the only irreducible representation in the model has highest weight 0 and there are no relevant fields living on the boundaries, so there is nothing to examine here.

The next case has boundary conditions $(\mathbb{1}, \Phi)$ and so the only irreducible representation present has highest weight $-1/5$. In this situation our model is described by (6.2.1) with $(a, b) = (\mathbb{1}, \Phi)$, $\lambda_L = 0$ and $\phi_R = \phi$. Here the Φ boundary is perturbed by the field ϕ and we observed at the beginning of this section this perturbation must produce a massless boundary flow. As there is only one other allowed conformal boundary condition this boundary must flow to the $\mathbb{1}$ boundary and this is what we observe in figure 6.6 for $\lambda_R > 0$ where the spectrum of the model flows smoothly from the character of the irreducible representation with highest weight $-1/5$ at $\lambda_R = 0$ to that of the irreducible representation with highest weight 0 at $\lambda_R = 10$.

For $\lambda_R < 0$ we see that some of the energy levels become complex, an effect also

observed with a bulk perturbation of the Lee-Yang model [80] where some energy levels also form complex conjugate pairs. The spectrum in this region may still be identified with the character χ_0 [21], but not in the same way as for $\lambda_R > 0$. In the present case we can identify this character twice, once in the real energy levels, shown in blue in figure 6.6, and again in the real part of the complex conjugate pairs, as shown in red.

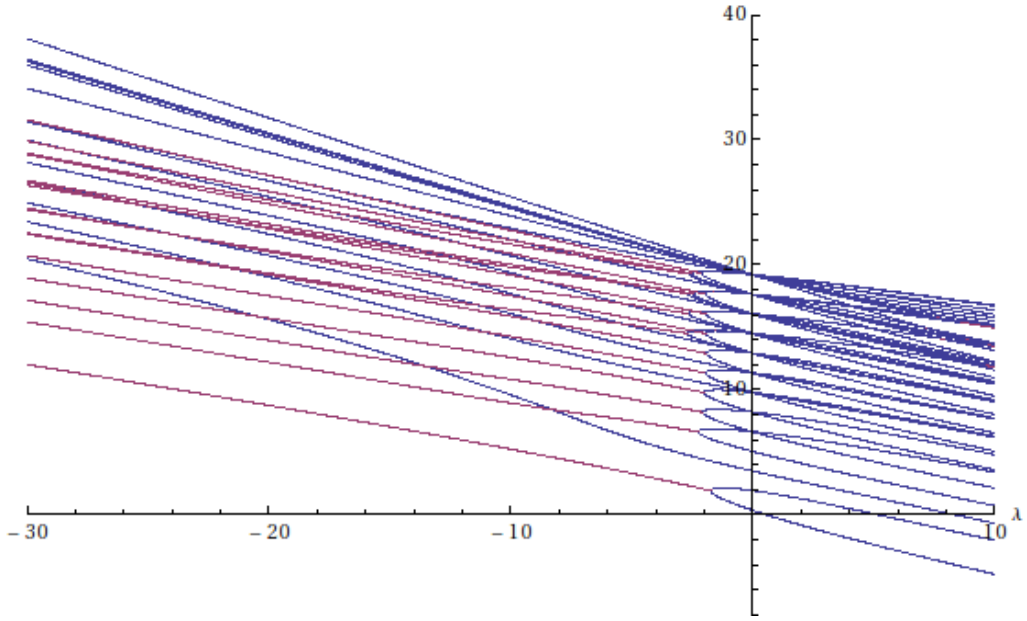

Figure 6.6: The spectrum of the Lee-Yang model with $(\mathbb{1}, \Phi)$ boundary conditions for different values of the perturbation parameter λ_R. Where two energy levels combine and form a complex conjugate pair the real part is the plotted in red.

6.2.2 The Tri-Critical Ising Model

As in the case of the Lee-Yang model massless boundary flows can be observed in the tri-critical Ising model using TCSA. The tri-critical Ising model contains more representations of the Virasoro algebra and so there are more conformal boundary conditions [11, 16], which are listed along with the fields living on each boundary in table 6.2. The boundary flows between these conformal boundary conditions were first given in [1] and explored in the context of renormalisation group flows in [78].

Figure 6.7 shows the re-scaled spectrum of the tri-critical Ising model with

Virasoro Label	Conformal Weight	Representations of Boundary Fields
$(1,1)$	0	$(1,1)$
$(2,1)$	$\frac{7}{16}$	$(1,1),(3,1)$
$(2,2)$	$\frac{3}{80}$	$(1,1),(3,2),(3,3),(3,1)$
$(3,1)$	$\frac{3}{2}$	$(1,1)$
$(3,2)$	$\frac{3}{5}$	$(1,1),(3,2)$
$(3,3)$	$\frac{1}{10}$	$(1,1),(3,2)$

Table 6.2: The representations of conformal boundary conditions in the tri-critical Ising model and the fields living on those boundaries.

boundary conditions $(1,1)$ on the left and $(3,2)$ on the right. The model is perturbed by the field $\phi_{3,2}$ on the right boundary and we plot the scaled spectrum

$$F_i = \frac{E_i - E_0}{E_1 - E_0} \tag{6.2.2}$$

for $\lambda_R \in (-1,1)$. This plot clearly shows the spectrum of the model given by the character $\chi_{(3,2)}$ at $\lambda_R = 0$ and flowing to the characters $\chi_{(3,1)}$ for $\lambda_R > 0$ and $\chi_{(3,3)}$ for $\lambda_R < 0$. We note here that as observed in [32] and [76] the TCSA spectrum extends beyond the IR fixed point in a sequence of flows that was first found in [54] and also observed in [35] and [22].

Similar behaviour is observed when perturbing any of the boundary conditions by any of the boundary fields listed in table 6.2.

6.2.3 Two Point Functions in Perturbed Minimal Models

Now that we are confident in our TCSA algorithms we will outline a method for attempting to use TCSA to approximate two point functions in perturbed minimal models with boundaries.

As we are able to accurately approximate the eigenvalues of the Hamiltonians of these perturbed minimal models it may be possible to use the corresponding eigenvectors to approximate the identity operator:

$$\mathbb{1} = \sum_n |E_n\rangle\langle E_n| \tag{6.2.3}$$

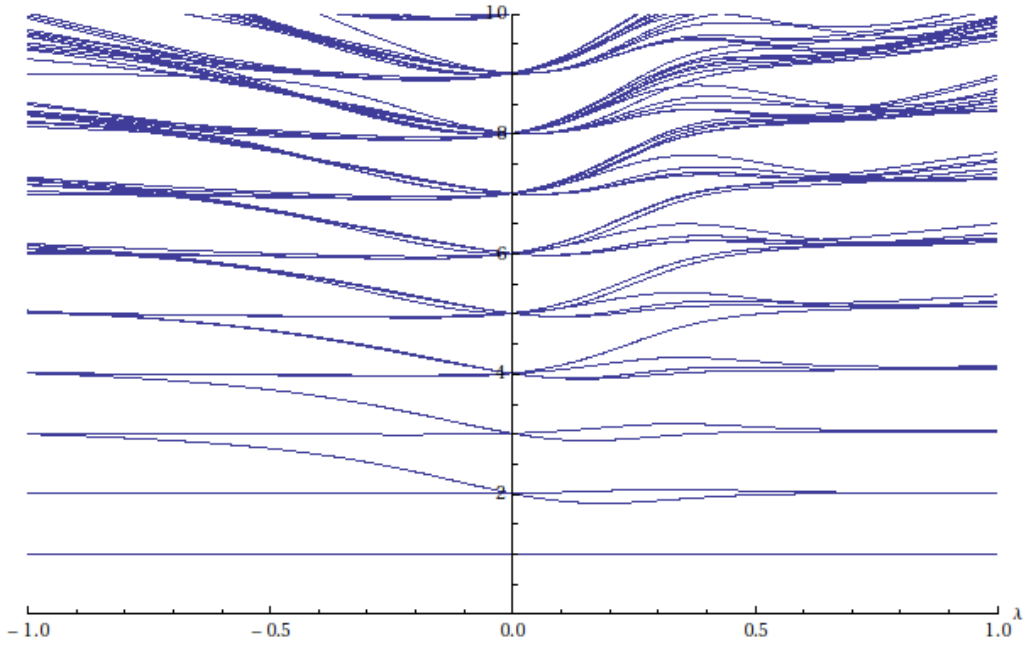

Figure 6.7: The re-scaled spectrum of the tri-critical Ising model with $((1,1),(3,2))$ boundary conditions perturbed by $(3,2)$ for different values of the perturbation parameter λ_R.

where $|E_n\rangle$ are the eigenstates of H, with eigenvalues E_n respectively, and are normalised such that $\langle E_i|E_j\rangle = \delta_{ij}$. As we have already shown that in certain circumstances TCSA does a good job of approximating the ground state, so it is not unreasonable to suppose that the TCSA will provide a good approximation of several other low energy eigenstates of the model. With these approximate eigenstates we have an approximation of the identity,

$$\mathbb{1}^{(N)} = \sum_{n=0}^{k} |E_n\rangle_{(N)\ (N)}\langle E_n| \tag{6.2.4}$$

where $|E_n\rangle_{(N)}$ is the TCSA approximation, with truncation level N, of the eigenvector $|E_n\rangle$. Only the first k eigenvectors should be included, where k is less than the total number of states used since it is not reasonable to expect the TCSA approximation of the highest energy eigenstates approximated to be accurate. This approximation of the identity can be inserted between the fields in a two point

correlation function to give

$$\begin{aligned}\langle\phi_i(x_1,y_1)\phi_j(x_2,y_2)\rangle =& \langle 0|\phi_i(x_1,y_1)\mathbb{1}\phi_j(x_2,y_2)|0\rangle \\ \simeq_{(N)}& \langle 0|\phi_i(x_1,y_1)\mathbb{1}^{(N)}\phi_j(x_2,y_2)|0\rangle_{(N)} \\ =& \sum_{n=0}^{k} {}_{(N)}\langle 0|\phi_i(x_1,y_1)|E_n\rangle_{(N)(N)}\langle E_n|\phi_j(x_2,y_2)|0\rangle_{(N)}\end{aligned} \tag{6.2.5}$$

Our task is now reduced to evaluating the matrix elements

$${}_N\langle 0|\phi_i(x,y)|E_n\rangle \tag{6.2.6}$$

which can be reduced to calculating matrix elements of the form

$$\langle 0|\phi_i(x,y)L_{-n_1}L_{-n_2}\cdots L_{-n_m}|\phi_j\rangle \tag{6.2.7}$$

which can in turn be evaluated using the commutation relations (5.1.14) and (5.1.17) as shown in appendix B.2. Using this approach we may be able to approximate two point functions in these perturbed conformal field theories.

No results are presented here as the author has not yet implemented this method or compiled any results to compare the TCSA calculations to. There are also several possible areas of complication and ways the TCSA method can be improved which should be examined before any results are presented. This is discussed in the next section.

Chapter 7

Discussion

So far in this part we have developed a TCSA algorithm, applied it to the scaling Lee-Yang and the tri-critical Ising models and achieved results consistent with those obtained by other works on TCSA and with results from different methods. This includes calculating one point functions in bulk models and observing renormalisation group flows in perturbed boundary models.

The author has, however, not yet attempted to calculate two point correlation functions using the TCSA method as outlined in section 6.2.3. The results of such calculations could then be compared to the results of [70] where two point spin-spin functions of the Ising model with a boundary were calculated using form factors. If this consistency check were passed we could hope to use the TCSA method to obtain information about spin-spin functions close to the boundary and also approximate two point functions in other perturbed conformal field theories.

As mentioned in the previous section there are some recent advances in TCSA and some possible areas of complication which need to be explored in greater detail before we attempt to approximate two point functions in general perturbed minimal models.

The first of these issues, as noted in [53], is that the Hilbert space may divide into several sectors and the eigenvalues in each sector may differ by unphysical amounts. This problem was addressed for bulk TCSA in [39] where formula to calculate the rescaling of energy levels are derived and found to be different in these different sectors. This rescaling must be taken into account when constructing the resolution

of the identity (6.2.4) as it will contain states from many sectors of the theory.

It should also be noted that when the weight of the perturbing field is greater than 1/2 the TCSA eigenvalues become divergent [48] and so TCSA results from these perturbations are unreliable. An expression for this divergence was found in [39] and would need to be accounted for in order to make our methods as widely applicable as possible.

Improvements to the TCSA approximations for the energy levels can also be obtained from an analysis of the renormalisation group flows in both the bulk [39] and boundary [78] cases. This would allow us to calculate more accurate estimates of quantities without increasing the truncation level, something which rapidly increases the time needed to complete the calculations.

Chapter 8

General Conclusion

This thesis has explored correlation functions in two-dimensional conformal field theories in two distinct ways. In part I two point functions of twist fields in the Dirac theory were calculated directly from the Ward identities of a double copy model. These Ward identities were first found in the conformal limit of the Dirac theory and then carried over to the massive case where, in order to verify their applicability we had to consider a new family of descendent twist fields. When considering this family of fermionic twist fields a new method for evaluating the vacuum expectation values of the primary twist fields was developed. It would be interesting to investigate whether the techniques developed to calculate the form factors of the descendent twist fields and vacuum expectation values of primary twist fields have applications to twist fields with more general symmetry groups. The generality of the quantum state considered also suggests that the method used to calculated differential equations for the correlation functions of twist fields could be applied at finite temperature. It may also be possible to apply the methods used to calculate the constant c_α in conjunction with finite temperature form factors to calculate the vacuum expectation values of twist fields at finite temperature. All of these open problems would provide interesting avenues of research.

In part II the truncated conformal space approach was introduced as a method for approximating quantities in perturbed conformal field theories. After introducing the relevant concepts from conformal field theory a truncated conformal space approach algorithm was developed and used to reproduce known results in both

bulk and boundary conformal field theories. These results could be improved by applying recent results regarding the renormalisation group flows of the coupling constant and it would be best to impliment these changes before attempting any further calculations. The aim of this part was to use the truncated conformal space approach to approximate two point functions in perturbed conformal field theories by constructing an approximation of the identity operator which could then be inserted between the fields in the two point function. By evaluating all the terms in this sum we would obtain an approximation of the two point function which could be compared with existing results. It would be interesting to see if this approach does indeed provide a good approximation of two point functions in these perturbed conformal field theories.

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Appendix A

Twist Fields in the Absence of Space Time Symmetries

As mentioned in section 3.1 it is possible write down differential equations relating correlation functions of certain twist fields in the absence of translation and parity symmetry. These equations come from the Ward identities (3.2.26) but the lack of symmetries requires different notation as we cannot apply the simplifications of section 3.2.2. For this reason we introduce the notation

$$F^{\gamma,\delta}_{\alpha,\beta}(x_1,y_1,x_2,y_2) := \langle \sigma_{\alpha,\beta}(x_1,y_1)\sigma_{\gamma,\delta}(x_2,y_2)\rangle \tag{A.0.1}$$

and first note that the lack of translation symmetry prevents us from always placing the second field at $(0,0)$, so the correlation functions must remain functions of four variables. It should also be noted that

$$\begin{aligned} F^{\beta,\beta}_{\alpha,\alpha}(x_1,y_1,x_2,y_2) = \langle \sigma_{\alpha,\alpha}(x_1,y_1)\sigma_{\beta,\beta}(x_2,y_2)\rangle \quad &= \langle \sigma_{\beta,\beta}(x_2,y_2)\sigma_{\alpha,\alpha}(x_1,y_1)\rangle \\ &\neq F^{\alpha,\alpha}_{\beta,\beta}(x_1,y_1,x_2,y_2) \end{aligned} \tag{A.0.2}$$

and so the Ward identities do not simplify as much as they did in chapter 3.

To ease our notation the dependence on (x_1,y_1,x_2,y_2) will be implicit in all $F^{\gamma,\delta}_{\alpha,\beta}$, as operators at (x_1,y_1) will always be placed to the left of those at (x_2,y_2). We also introduce the symmetric, bilinear differential operator

$$\mathcal{D}_{i,j}(u,v) = u\partial_i\bar{\partial}_j v + v\partial_i\bar{\partial}_j u - \partial_i u\bar{\partial}_j v - \bar{\partial}_j u\partial_i v \tag{A.0.3}$$

where $z_j = \frac{-i}{2}(x_j + iy_j)$ and

$$\partial_i = \frac{\partial}{\partial z_i} \quad \text{and} \quad \bar{\partial}_i = \frac{\partial}{\partial \bar{z}_i}. \tag{A.0.4}$$

A.1 Equations for Correlation Functions Without Symmetries

Now starting from the Ward identities

$$\langle [Z, \sigma^{\Psi}_{\alpha,\alpha}(x_1,y_1)\sigma^{\Phi}_{\alpha+1,\alpha+1}(x_1,y_1)\sigma^{\Psi}_{\alpha+1,\alpha}(x_2,y_2)\sigma^{\Phi}_{\alpha,\alpha+1}(x_2,y_2)]\rangle = 0 \tag{A.1.5a}$$

$$\langle [Z, [P, \sigma^{\Psi}_{\alpha,\alpha}(x_1,y_1)\sigma^{\Phi}_{\alpha+1,\alpha+1}(x_1,y_1)]\sigma^{\Psi}_{\alpha+1,\alpha}(x_2,y_2)\sigma^{\Phi}_{\alpha,\alpha+1}(x_2,y_2)]\rangle = 0 \tag{A.1.5b}$$

$$\langle [Z, [\bar{P}, \sigma^{\Psi}_{\alpha,\alpha}(x_1,y_1)\sigma^{\Phi}_{\alpha+1,\alpha+1}(x_1,y_1)]\sigma^{\Psi}_{\alpha+1,\alpha}(x_2,y_2)\sigma^{\Phi}_{\alpha,\alpha+1}(x_2,y_2)]\rangle = 0 \tag{A.1.5c}$$

$$\langle [Z, [P, [\bar{P}, \sigma^{\Psi}_{\alpha,\alpha}(x_1,y_1)\sigma^{\Phi}_{\alpha+1,\alpha+1}(x_1,y_1)]]\sigma^{\Psi}_{\alpha,\alpha+1}(x_2,y_2)\sigma^{\Phi}_{\alpha+1,\alpha}(x_2,y_2)]\rangle = 0 \tag{A.1.5d}$$

$$\langle [Z, [P, \sigma^{\Psi}_{\alpha,\alpha}(x_1,y_1)\sigma^{\Phi}_{\alpha+1,\alpha+1}(x_1,y_1)][\bar{P}, \sigma^{\Psi}_{\alpha,\alpha+1}(x_2,y_2)\sigma^{\Phi}_{\alpha+1,\alpha}(x_2,y_2)]]\rangle = 0 \tag{A.1.5e}$$

we find that

$$F^{\alpha+1,\alpha}_{\alpha,\alpha+1}F^{\alpha,\alpha+1}_{\alpha+1,\alpha} - F^{\alpha+1,\alpha+1}_{\alpha,\alpha}F^{\alpha,\alpha}_{\alpha+1,\alpha+1} + F^{\alpha,\alpha}_{\alpha,\alpha}F^{\alpha+1,\alpha+1}_{\alpha+1,\alpha+1} = 0 \tag{A.1.6a}$$

$$\begin{aligned}\partial_{z_1}F^{\alpha+1,\alpha}_{\alpha,\alpha+1}F^{\alpha,\alpha+1}_{\alpha+1,\alpha} - F^{\alpha+1,\alpha}_{\alpha,\alpha+1}\partial_{z_1}F^{\alpha,\alpha+1}_{\alpha+1,\alpha} + \partial_{z_1}F^{\alpha+1,\alpha+1}_{\alpha,\alpha}F^{\alpha,\alpha}_{\alpha+1,\alpha+1} - F^{\alpha+1,\alpha+1}_{\alpha,\alpha}\partial_{z_1}F^{\alpha,\alpha}_{\alpha+1,\alpha+1}\\ - \partial_{z_1}F^{\alpha,\alpha}_{\alpha,\alpha}F^{\alpha+1,\alpha+1}_{\alpha+1,\alpha+1} + F^{\alpha,\alpha}_{\alpha,\alpha}\partial_{z_1}F^{\alpha+1,\alpha+1}_{\alpha+1,\alpha+1} = 0\end{aligned} \tag{A.1.6b}$$

$$\begin{aligned}\bar{\partial}_{z_1}F^{\alpha+1,\alpha}_{\alpha,\alpha+1}F^{\alpha,\alpha+1}_{\alpha+1,\alpha} - F^{\alpha+1,\alpha}_{\alpha,\alpha+1}\bar{\partial}_{z_1}F^{\alpha,\alpha+1}_{\alpha+1,\alpha} + \bar{\partial}_{z_1}F^{\alpha,\alpha}_{\alpha+1,\alpha+1}F^{\alpha+1,\alpha+1}_{\alpha,\alpha} - F^{\alpha,\alpha}_{\alpha+1,\alpha+1}\bar{\partial}_{z_1}F^{\alpha+1,\alpha+1}_{\alpha,\alpha}\\ - \bar{\partial}_{z_1}F^{\alpha+1,\alpha+1}_{\alpha+1,\alpha+1}F^{\alpha,\alpha}_{\alpha,\alpha} + F^{\alpha+1,\alpha+1}_{\alpha+1,\alpha+1}\bar{\partial}_{z_1}F^{\alpha,\alpha}_{\alpha,\alpha} = 0\end{aligned} \tag{A.1.6c}$$

$$\begin{aligned}\mathcal{D}_{1,1}(F^{\alpha,\alpha+1}_{\alpha+1,\alpha}, F^{\alpha+1,\alpha}_{\alpha,\alpha+1}) - m^2F^{\alpha,\alpha+1}_{\alpha+1,\alpha}F^{\alpha+1,\alpha}_{\alpha,\alpha+1}\\ - \mathcal{D}_{1,1}(F^{\alpha,\alpha}_{\alpha,\alpha}, F^{\alpha+1,\alpha+1}_{\alpha+1,\alpha+1}) + \mathcal{D}_{1,1}(F^{\alpha+1,\alpha+1}_{\alpha,\alpha}, F^{\alpha,\alpha}_{\alpha+1,\alpha+1}) = 0\end{aligned} \tag{A.1.6d}$$

$$\mathcal{D}_{1,2}(F^{\alpha+1,\alpha}_{\alpha,\alpha+1}, F^{\alpha,\alpha+1}_{\alpha+1,\alpha}) - \mathcal{D}_{1,2}(F^{\alpha,\alpha}_{\alpha,\alpha}, F^{\alpha+1,\alpha+1}_{\alpha+1,\alpha+1}) - \mathcal{D}_{1,2}(F^{\alpha+1,\alpha+1}_{\alpha,\alpha}, F^{\alpha,\alpha}_{\alpha+1,\alpha+1}) = 0. \tag{A.1.6e}$$

It should be noted that this system is under defined as there are 6 independent functions, each of two variables, and only 5 equations. It is possible to find more Ward identities leading to non-trivial relations between the correlation functions

but as no integrable structure has been identified in these equations we only include the above equations for completeness. These equations can also be generalised to relate correlation functions of twist operators with different twist indices but this only introduces more functions and so this system is even more under determined.

Appendix B

TCSA Algorithm

All the TCSA calculations were performed with Mathematica and the algorithms follow the general structure of [80] and [52]. The basic structure of all the programmes is outlined in section 6.1 and can be roughly broken down into the following steps:

- The model is entered and all structure constants are calculated and relevant fields found.
- A basis for the conformal holomorphic states is found.
- In the case of periodic boundary conditions a basis of full, holomorphic and anti-holomorphic states is found.
- The Hamiltonian is calculated and any further calculations performed.

These steps hide some technical computations, the more involved of which are the subject of this appendix.

After a model is chosen the necessary structure constants are calculated using the formulae set out in sections 5.1.2 and 5.2.3. If boundary conditions are involved the irreducible representations present in the model are found using the Verlinde formula set out in section 5.2.2. Finding a basis of states for the model is the next challenge and is discussed in section B.1. Calculating the matrix elements as in (6.1.5),

$$M_{ij} = \langle \psi_i | \phi_p(0,0) | \psi_j \rangle, \tag{B.0.1}$$

and their boundary equivalent, is the subject of section B.2. The final generic step involves constructing the Hamiltonian and this is discussed in section B.3. The output of this operation may then be manipulated and used to calculate various quantities as seen throughout chapter 6.

B.1 Basis of States

Once the minimal model under consideration had been defined the irreducible representations present can be easily identified using the methods of sections 5.1.2 and 5.2.2. Our task is to develop an algorithm to find a basis of states for each irreducible representation up to the cutoff level.

For a generic module with highest weight $h_{r,s}$ we begin by calculating the generating function $\chi_{(r,s)}(x)$, (5.1.24), as the coefficients of this polynomial give the number of linearly independent states present at each level.

When calculating a basis of states at level N we first generate a list of all the possible states at this level, which is simply all the positive integer partitions of N. We first take the state

$$L_{-N}|\phi_{(r,s)}\rangle \tag{B.1.2}$$

and calculate its norm. If this is non-zero we have the first state of our basis.

We next select another state from our list, $L_{-m_1}L_{-m_2}|\phi_{(r,s)}\rangle$ say, and calculate the matrix of inner products of this state with the first, (B.1.2). If the states are not linearly independent then this matrix will have determinant 0 and we discard the state $L_{-m_1}L_{-m_2}|\phi_{(r,s)}\rangle$, otherwise, we add this state to our basis.

This process of selecting a state, calculating the matrix of inner products of everything already in our basis plus the new state, calculating its determinant and checking whether it vanishes is repeated until we have found the required number of states.

The procedure outlined here is repeated for every level up to the cutoff N, for every irreducible representation in the model to create a basis for all the holomorphic states below level N. In the case of boundary theories these are all the states we need to perform further calculations but in the periodic case we must combine

the holomorphic and anti-holomorphic states to give full states with the required spin. This is easily accomplished as the anti-holomorphic states are just copies of the holomorphic ones so constructing full states is just a matter of ensuring that holomorphic and anti-holomorphic components have appropriate levels.

B.2 Matrix Element Calculation

The subject of this section is the calculation of the matrix elements M_{ij}. As in the previous section only the holomorphic case will be discussed as the anti-holomorphic case is identical. It is also worth noting that it does not matter whether the field ϕ_p is a bulk field in the periodic model, as is the case in (B.0.1), or a boundary field, as this will only alter the structure constant used in the final stage.

The evaluation of these matrix elements is simply a repeated application of the commutation relation

$$[L_n - L_0, \phi_p(0,0)] = nh_p\phi(0,0) \tag{B.2.3}$$

which follows simply from (5.1.14) and (5.1.17). Focusing on the holomorphic components only we let

$$|\psi_i\rangle = L_{-n_1}L_{-n_2}\cdots L_{-n_k}|\phi_i\rangle \quad \text{and} \quad |\psi_j\rangle = L_{-m_1}L_{-m_2}\cdots L_{-m_l}|\phi_j\rangle. \tag{B.2.4}$$

We now see it is possible to use the commutation rule (B.2.3) to write:

$$\begin{aligned}\langle\phi_i|L_{n_k}\cdots L_{n_2}L_{n_1}\phi_p(0,0)|\psi_j\rangle = \langle\phi_i|L_{n_k}\cdots L_{n_2}\,(\, nh_p\phi_p(0,0) + \phi_p(0,0)L_{n_1} \\ + L_0\phi_p(0,0) - \phi_p(0,0)L_0\,)\,|\psi_j\rangle.\end{aligned} \tag{B.2.5}$$

The first term is already better as contains has one less Virasoro generator. The final two terms can be similarly reduced as $|\psi_j\rangle$ and $\langle\phi_i|L_{n_k}\cdots L_{n_2}$ are eigenvectors of L_0 and it is very easy to calculate their eigenvalues. This just leaves the term

$$\langle\phi_i|L_{n_k}\cdots L_{n_2}\phi_p(0,0)L_{n_1}|\psi_j\rangle. \tag{B.2.6}$$

The level of the state on the right hand side is lower than that of $|\psi_j\rangle$ and we may commute L_{n_1} through all the Virasoro generators in $|\psi_j\rangle$ to leave states at a lower

level. In this way all the Virasoro generators on the left hand side of (B.2.5) can be commuted to the right, reducing the matrix element (B.0.1) to a sum over terms of the form

$$\langle\phi_i|\phi_p(0,0)|\psi_h\rangle = \langle\phi_i|\phi_p(0,0)L_{-m'_1}L_{-m'_2}\cdots L_{-m'_{l'}}|\phi_k\rangle. \tag{B.2.7}$$

We can now use the same commutation relation (B.2.3) to further reduce this expression:

$$\begin{aligned}\langle\phi_i|\phi_p(0,0)L_{-m'_1}L_{-m'_2}\cdots L_{-m'_{l'}}|\phi_k\rangle = \langle\phi_i|\,(\,&L_{-m'_1}\phi_p(0,0) - L_0\phi_p(0,0)\\ &+ \phi_p(0,0)L_0 - nh_p\phi_p(0,0)\,)\,L_{-m'_2}\cdots L_{-m'_{l'}}|\phi_k\rangle.\end{aligned} \tag{B.2.8}$$

Here the first term vanishes as the Virasoro generator annihilates the out state. As before the terms involving L_0 are simply reduced as $\langle\phi_i|$ and $L_{-m'_2}\cdots L_{-m'_{l'}}|\phi_k\rangle$ are again eigenvectors of this operator and the final term already contains one less Virasoro generator. Thus with repeated application we may reduce (B.2.8) to a sum of terms which only involve the matrix element

$$\langle\phi_i|\phi_p(0,0)|\phi_j\rangle \tag{B.2.9}$$

and this constant can be evaluated using the formulae discussed in section 5.1.2 for the periodic case, or 5.2.2 for the boundary case. Using this method we can construct the full matrix M_{ij}.

We also note that the calculation of the CFT matrix elements is very straight forward as all the states of our basis are eigenvectors of H_{CFT} and so the calculation of the matrix of the unperturbed part of the Hamiltonian, H_{ij}^{CFT}, is simple.

B.3 Constructing the Hamiltonian

In this section we make some comments regarding the final construction and diagonalisation of the Hamiltonian. The Hamiltonian is constructed by adding together the matrices H_{ij}^{CFT} and M_{ij}, both multiplied by the appropriate factors of R and π depending on the model and perturbation under consideration, to give H_{ij}. We now note that the Hamiltonian we wish to examine has one covariant and one contravariant index, while H_{ij} has two covariant indices. Thus we must multiply this

matrix by the inverse metric g^{ij}, where g_{ij} is the matrix of inner products of the basis vectors, to obtain the final TCSA approximation for the Hamiltonian:

$$H^i{}_j = g^{ik} H_{kj}. \tag{B.3.10}$$

This operation is necessary as the basis vectors chosen in section B.1 are not orthonormal.

It is the eigenvalues of $H^i{}_j$ that give the TCSA approximation for the spectrum of the model and these are then used in any subsequent calculations.