Near-symplectic 2n-manifolds

VERA-SANCHEZ, RAMON

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Ramón Vera

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Abstract

We give a generalization of the concept of near-symplectic structures to $2n$ dimensions. According to our definition, a closed 2-form on a $2n-$manifold M is near-symplectic, if it is symplectic outside a submanifold Z of codimension 3, where ω^{n-1} vanishes. We depict how this notion relates to near-symplectic 4-manifolds and broken Lefschetz fibrations via some examples. We define a generalized broken Lefschetz fibration, or BLF, as a singular map with indefinite folds and Lefschetz-type singularities. We show that given such a map on a $2n$-manifold over a symplectic base of codimension 2, then the total space carries such a near-symplectic structure, whose singular locus corresponds precisely to the singularity set of the fibration. A second part studies the geometry around the codimension–3 singular locus Z . We describe a splitting property of the normal bundle N_Z that is also present in dimension four. A tubular neighbourhood for Z is provided, which has as a corollary a Darboux-type theorem for near- symplectic forms.

Declaration

The work in this thesis is based on research carried out in the Pure Mathematics Group at the Department of Mathematical Sciences, Durham University. No part of this thesis has been submitted elsewhere for any other degree or qualification and it is all my own work unless referenced to the contrary in the text.

Para mi familia

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Contents

Chapter 1

Introduction

> To see a world in a grain of sand
> And a heaven in a wild flower,
> Hold infinity in the palm of your hand
> And eternity in an hour
>
> William Blake, *Auguries of Innocence*

The motivation for near-symplectic manifolds arose from a programme initiated by Taubes to study 4-manifolds equipped with symplectic forms that vanish on circles with the goal of obtaining smooth invariants of non-symplectic 4-manifolds [Tau98]. A 4-manifold is called near-symplectic if it is equipped with a closed 2-form that is non-degenerate outside a disjoint union of circles, where it vanishes. These structures where studied in detail in the work of Auroux, Donaldson and Katzarkov [ADK05] using broken Lefschetz fibrations (BLFs). It was shown that there is a direct correspondence between BLFs and near-symplectic 4-manifolds. These results extended the theorems of Donaldson [Don99] and Gompf [GS99] on Lefschetz fibrations and symplectic manifolds, which in turn expanded Thurston's theorem on symplectic fibrations [Thu76]. Broken Lefschetz fibrations have found fruitful application in low-dimensional topology, for example in holomorphic quilts [WW] and Lagrangian matching invariants [Per07, Per08]. A significant existence result states that every smooth closed oriented 4-manifold admits a BLF [GK07, Bay09, Lek09, AK08]. The geometric structure induced by a near-symplectic 4-manifold on the boundary of the tubular neighbourhood of its singular locus is an overtwisted structure as studied by Honda, Gay and Kirby [Hon04b, GK04]. This shows that these manifolds are not fillable as that would require removing remove all singular circles, which Perutz proved not to be possible [Per06].

This work aims to find a good notion to generalize near-symplectic structures on higher dimensions. We propose a definition on manifolds of dimension $2n$ and use singular maps in higher dimensions that resemble broken Lefschetz fibrations. we also study the underlying geometric structure, induced by the near-symplectic form, on the boundaries of tubular neighbourhoods, which are codimension 1 submanifolds in this setting.

In section 3.1, we suggest a definition of a near-symplectic structure in dimension $2n$. The goal is to relax the non-degeneracy condition of the symplectic form in a controlled way so that it degenerates exclusively on a certain submanifold. The idea starts by considering a closed 2-form ω on a smooth, orientable, $2n$-manifold M, such that $\omega^n \geq 0$. At the points where the degeneracy occurs, that is where $\omega^n = 0$, we impose a transversality condition on the gradient or differential map of ω. This transversality condition tells us that the singular locus Z_ω is a submanifold of codimension 3, where $\omega_p^{n-1} = 0$ for all $p \in Z_\omega$. We call these 2-forms near-symplectic. Examples of near-symplectic $2n$-manifolds are given in sections 3.2 and 4.4.

Next, we study the question of the existence of these structures using singular fibrations, analogous to BLFs. We define a generalized *BLF* as a submersion $f \colon M^{2n} \to X^{2n-2}$ with two types of sets of singularities, both of which lie in M. First, we have codimension 4 submanifolds of extended Lefschetz type singularities locally modelled by complex coordinate charts $\mathbb{C}^n \to \mathbb{C}^{n-1}$ such that $(z_1, \ldots, z_n) \mapsto (z_1, \ldots, z_{n-2}, z_{n-1}^2 + z_n^2)$. The second singularities are codimension 3-submanifolds Σ of indefinite folds modelled by real coordinate charts $\mathbb{R}^{2n} \to \mathbb{R}^{2n-2}, (t_1, \ldots, t_{2n-3}, x_1, x_2, x_3) \mapsto (t_1, \ldots, t_{2n-3}, -x_1^2 + x_2^2 + x_3^2)$. For more details see definition 4.1.1. We state our first result.

Theorem 1. *Let $f \colon M \to X$ be a generalized BLF from a smooth closed oriented $2n$-manifold M to a compact symplectic $(2n-2)-$manifold (X, ω_X). Denote by Σ the set of fold singularities of f. Assume that there is a class $\alpha \in H^2(M)$, such that it pairs positively with every component of every fibre, and that $\alpha|_\Sigma = [\omega_X|_\Sigma]$. Then, there is a near-symplectic form ω on M, with singular locus Z_ω equal to Σ, and symplectic fibres outside Σ.*

The proof appearing in section 4.2 starts by constructing an explicit closed 2-form on the fibres that vanishes at the set of singularities of the mapping. Then it pulls back the symplectic form of the base, Both 2-forms are combined and glued together into a global 2-form representing the class α. This statement follows a similar line of reasoning as Auroux-Donaldson Katrzarkov [ADK05] construction of near-symplectic forms using BLFs in dimension 4.

The last two chapters concern the geometric structure on the boundary of the neighbourhood of the singular locus. We study two geometric structures that appear on a codimension 1 submanifold of M. Firstly, we look at Hamiltonian structures. A Hamiltonian structure on an $(2n-1)$-dimensional manifold N is a closed 2-form Ω such that $\Omega^{n-1} \neq 0$ everywhere. In the presence of a Hamiltonian structure, there is a 1-dimensional distribution associated to Ω through its kernel $\ker(\Omega)$. A 1-form λ on N is called a stabilizing 1-form, if $\lambda \wedge \Omega^{n-1} > 0$ and $\ker(\Omega) \subset \ker(d\lambda)$. The pair (λ, Ω) is known as a stable Hamiltonian structure. A near-symplectic form naturally equips the singular locus Z with a Hamiltonian structure. Moreover, if Z carries a stable Hamiltonian structure so does the boundary of a small tubular neighbourhood in case that the normal bundle is trivial.

We conclude by examining the properties of the normal bundle of Z that are defined by the near-symplectic form. As in dimension 4, there is a decomposition of the normal bundle N_Z in 2 eigenssubbundles, a rank 1 bundle L^- and a rank 2 bundle L^+. In section 5.3, we give a neigbourhood theorem for near-symplectic forms around their singular locus.

Theorem 2. *Let $(M_0, \omega_0), (M_1, \omega_1)$ be two near-symplectic manifolds with diffeomorphic singular locus $Z_0 \cong Z_1$ and equal symplectic forms on them, $\omega_0|_{Z_0} = \omega_1|_{Z_1}$. Assume that there is an isomorphism on the normal bundles $N_{Z_0} \simeq N_{Z_1}$, such that it restricts to an isomorphism on the positive subbundles $L_0^+ \simeq L_1^+$. Denote by $\mathcal{U}_0 \subset M_0$ and $\mathcal{U}_1 \subset M_1$ the corresponding tubular neighbourhoods of Z_0 and Z_1. Then, there is a homeomorphism $\varphi\colon \mathcal{U}_0 \to \mathcal{U}_1$ that is a diffeomorphism away from Z, such that $\varphi^*\omega_1 = \omega_0$.*

As a corollary, we obtain a local Darboux-type theorem which describes a near-symplectic form around a point of Z.

Corollary 1. *Let (M, ω) be a near-symplectic manifold and p a point of the singular locus $Z \subset M$. There is a coordinate neighbourhood $U \subset M$ around p, such that on*

U

$$\omega = \omega_Z - 2x_1(dz_0 \wedge dx_1 + dx_2 \wedge dx_3) + x_2(dz_0 \wedge dx_2 - dx_1 \wedge dx_3) + x_3(dz_0 \wedge dx_3 + dx_1 \wedge dx_2)$$

where $\omega_Z := i^*\omega$ is a 2-form of maximal rank on Z.

Chapter 2

Preliminaries

Memory is the space in which a thing happens for the second time.

Paul Auster, *The Invention of Solitude*

Symplectic geometry arose in the Hamiltonian formulation of classical mechanics. A prototypical example of a symplectic manifold is the cotangent bundle of any smooth oriented manifold. Under the physical interpretation, the cotangent bundle plays the role of the phase space of its base manifold, where it represents the possible positions and momenta of a particle in a system. To give a more accurate definition, an even-dimensional manifold M^{2n} is called *symplectic*, if it is endowed with a closed, non-degenerate 2-form $\omega \in \Omega^2(M)$. A prototypical example of a symplectic manifold is the cotangent bundle T^*L of any n-manifold L. The symplectic form of T^*L is naturally defined as the exterior derivative of the canonical 1-form $\lambda \in \Omega^1(T^*L)$, that is, $\omega = d\lambda$. In coordinates, the symplectic form of the cotangent bundle is represented $\omega = \sum_{i=1}^n dp_i \wedge dq_i$, where q_i are the coordinates of the base L and p_i the coordinates of the fibre.

Contact topology can be seen as the odd-dimensional sister of symplectic topology. A contact structure ξ on an odd-dimensional manifold N^{2n-1} is a maximally non-integrable hyperplane field. This plane field is defined by the kernel of a global 1-form α, which satisfies $\alpha \wedge d\alpha^{n-1} > 0$. The origins of contact geometry can be traced back to the work of Huygens on optics, although the term seems to be coined by Sophus Lie in the late 19^{th} century. A rapid development of the field appeared in the 1990's and 2000's, in particular in dimension 3. To mention a few of the relevant results from the last two decades, there is the classification of 3-dimensional contact manifolds into tight and overtwisted, the theory of convex surfaces, and Giroux's 1-1 correspondence between open book decompositions and contact 3-manifolds.

In this chapter, we give a short synopsis of concepts in symplectic and contact topology that will serve as a background of the next chapters. We start with some concepts from differential geometry and topology.

2.1 Differential geometric and topological concepts

2.1.1 Hodge $*$-operator

The following information is based on [[Bal06] Ch. 1, [MT97] Ch.9, Ch. 15, Ch. 16 ,[Mor01] Ch. 2, Ch. 4, Ch. 5]. Let V be a vector space. Using a positive-definite inner product given on V, we may induce an isomorphism $V \cong V^*$, and hence an inner product in the dual vector space V^*. Let k be an integer bigger than 1, and let $\alpha_i, \beta_j \in V^*$. For any two elements of the form $\alpha_1 \wedge \cdots \wedge \alpha_k$ and $\beta_1 \wedge \ldots \beta_k$, we define the value of their inner product to be $\langle \alpha_1 \wedge \cdots \wedge \alpha_k, \beta_1 \wedge \ldots \beta_k \rangle = \det((\alpha_i, \beta_j))$. We now extend the inner product to the whole space $\Lambda^k V^*$ by linearity. If $e_1, \ldots, e_n$ is an orthonormal basis of V and $\theta_1, \ldots, \theta_n$ is the dual basis in V^*, then the elements of the shape

$$\theta_{i_1} \wedge \ldots, \wedge \theta_{i_k}, \qquad 1 \leq i_1 < \cdots < i_k \leq n$$

form an orthonormal basis of $\Lambda^k V^*$.

In this way for any two differential k-forms ω and η defined on a smooth n-dimensional Riemannian manifold M, we have an inner product $\langle \omega_p, \eta_p \rangle$ for each $p \in M$, and thus a function $\langle \omega, \eta \rangle$ on M. In the case of $k = 0$, we define the inner product between functions f and g at each point p as the product of the values of f and g at p. We also define the inner product between two differential forms of different degrees to be 0.

For any integer k, $\Lambda^k T_p^* M$ and $\Lambda^{n-k} T_p^* M$ have the same dimension as vector spaces, and they are isomorphic. If M has a Riemannian metric g and is also oriented, then there is a natural isomorphism associated to g

$$\Lambda^k T_p^* M \simeq \Lambda^{n-k} T_p^* M$$

for each point $p \in M$. By varying $p \in M$, we get a linear isomorphism

$$\Omega^k(M) \to \Omega^{n-k}(M)$$

where $\Omega^k(M)$ and $\Omega^{n-k}(M)$ denote the vector spaces of all k and $(n-k)$ forms, respectively. At a point $p \in M$, we follow the same definition as explained at the beginning of this section in the case of a vector space V, by substituting V for T_pM. Let $\{\theta_1, \ldots, \theta_n\}$ be an arbitrary positively oriented orthonormal basis of T_p^*M. Then we can get a linear map

$$*\colon \Lambda^k T_p^*M \to \Lambda^{n-k} T_p^*M \tag{2.1}$$

by setting

$$*(\theta_1 \wedge \cdots \wedge \theta_k) = \theta_{k+1} \wedge \cdots \wedge \theta_n$$

In particular, we have

$$*1 = \theta_1 \wedge \cdots \wedge \theta_n$$
$$*(\theta_1 \wedge \cdots \wedge \theta_n) = 1$$

The *Hodge $*$-operator* $*\colon \Omega^k(M) \to \Omega^{n-k}(M)$ can be defined globally. This operator is well-defined for a given metric. If $\omega \in \Omega^k(M)$, then $*\omega$ is an element in $\Omega^{n-k}(M)$, whose value at p is equal to $*\omega_p$ for every p.

Definition 2.1.1. On an oriented Riemannian manifold M the *Hodge $*$-operator* $*\colon \Omega^k(M) \to \Omega^{n-k}(M)$ interchanges forms of complementary degrees. The Hodge $*$-operator is defined by comparing the natural metric on the forms with the wedge product:

$$\omega \wedge *\beta = \langle \omega, \beta \rangle vol_M$$

where vol_M is the Riemannian volume element. This definition corresponds to the previous explanation using the orthonormal basis.

Proposition 2.1.2. *Properties of the $*$-operator. For any $f, g \in C^\infty(M)$ and for any $\omega, \eta \in \Omega^k(M)$, we have*

1. $**\omega = (-1)^{k(n-k)}\omega$
2. $\omega \wedge *\eta = \eta \wedge *\omega = \langle \omega, \eta \rangle vol_M$

3. $*\omega \wedge *\eta = *(\eta \wedge *\omega) = \langle \omega, \eta \rangle$
4. $*(f\omega + g\eta) = f(*\omega) + g(*\eta)$
5. $\langle *\omega, *\eta \rangle = \langle \omega, \eta \rangle$

2.1.2 Harmonic Forms

Let (M, g) be a closed Riemannian manifold. In the previous section we defined an inner product for two k-forms ω, η for each point in M. We introduce an inner product in $\Omega^k(M)$ by integrating the function $\langle \omega_p, \eta_p \rangle$ over M, that is

$$(\omega, \eta) = \int_M \langle \omega, \eta \rangle vol_M$$

According to the property 2 of proposition 2.1.2, this inner product can also be expressed as

$$(\omega, \eta) = \int_M \omega \wedge *\eta = \int_M \eta \wedge *\omega$$

By convention, we define the inner product between differential forms of different degrees to be zero, so that the whole space $\Omega^*(M)$ is endowed with an inner product.

Now we consider a relation between the Hodge operator and the exterior differentiation $d\colon \Omega^k(M) \to \Omega^{k+1}(M)$. Define the linear operator

$$\delta = (-1)^k *^{-1} d* = (-1)^{n(k+1)+1} * d*$$

by requiring that the following diagram be commutative

$$\begin{tikzcd}
\Omega^k(M) \arrow[r, "*"] \arrow[d, "\delta"'] & \Omega^{n-k}(M) \arrow[d, "d"] \\
\Omega^{k-1}(M) \arrow[r, "(-1)^k *"'] & \Omega^{n-k+1}(M)
\end{tikzcd}$$

It follows from the definition that

$$*\delta = (-1)^k d* \,, \quad \delta * = (-1)^{k+1} * d \,, \quad \delta \circ \delta = 0$$

Definition 2.1.3. Relative to the inner product $(,)$ in $\Omega^*(M)$, δ is the *adjoint operator* of exterior differentiation d. That is, we have

$$(d\omega, \eta) = (\omega, \delta\eta) \tag{2.2}$$

Conversely, d is the adjoint operator of δ. The adjoint operator can be computed using the $*$-operator using the equation

$$\delta = (-1)^{n(k+1)+1} * d*$$

Definition 2.1.4. For a Riemannian manifold M, the operator defined by

$$\Delta = d\delta + \delta d \colon \Omega^k(M) \to \Omega^k(M)$$

is called the *Laplace operator*. A form $\omega \in \Omega^*(M)$ with the property that

$$\Delta\omega = 0$$

is called a *harmonic form*.

A necessary and sufficient condition for a k-form ω to satisfy $\Delta\omega = 0$ is that $d\omega = 0$ and $\delta\omega = 0$ [[Mor01] Ch 4. Prop 4.12]. We explain why this is the case. If $d\omega = \delta\omega = 0$, then clearly $\Delta = 0$. To show the converse, notice that

$$(\Delta\eta, \eta) = ((d\delta + \delta d)\eta, \eta) = (\delta\eta, \delta\eta) + (d\eta, d\eta) = 0$$

This shows that $\Delta\eta = 0$, which implies $d\eta = \delta\eta = 0$.

Proposition 2.1.5. *The Laplacian Δ has the following properties:*

1. $*\Delta = \Delta *$.

   If ω is a harmonic form, so is $*\omega$.

2. *Δ is self-adjoint, that is*

$$(\Delta\omega, \eta) = (\omega, \Delta\eta) \qquad \forall \omega, \eta \in \Omega^*(M)$$

3. *A necessary and sufficient condition for $\Delta\omega = 0$ is that $d\omega = 0$ and $\delta\omega = 0$.*

If M is closed, then $*$ maps harmonic forms to harmonic forms. Denote by $\mathcal{H}^k(M, R)$ be the space of harmonic forms of degree k. The previous proposition tells us that every harmonic form is closed. Then, we get a linear map $\mathcal{H}^k(M, R) \to H^k_{DR}(M)$ by taking the de Rham cohomology. This map is in fact an injection. The Hodge theorem, stated below, asserts that actually $\mathcal{H}^k(M, R)$ is isomorphic to $H^k_{DR}(M)$. For a closed M, $*\colon \mathcal{H}^k(M, R) \to \mathcal{H}^{n-k}(M, R)$ is an isomorphsim. This is the Poincaré duality on the level of harmonic forms.

Theorem 2.1.6 (Hodge Theorem, [[Mor01] Ch. 4, pg 159). *An arbitrary de Rham cohomology class of an oriented compact Riemannian manifold can be represented by a unique harmonic form. In other words, the natural map $\mathcal{H}^k(M) \to H^k_{DR}(M)$ is an isomorphism.*

2.1.3 Self-dual and Anti-self-dual decomposition

The following note is based on the books [[Don97] (Ch.1, Ch. 2), [GS99](Ch. 2.4, Ch. 10), [Tau11] (Ch. 19)]. Suppose (X, g) is a Riemannian 4-manifold. The Hodge $*$-operator takes 2-forms to 2-forms and we have

$$** = id_{\Lambda^2 T^* X}$$

$\Lambda^2 T^* X$ decomposes as the orthogonal direct sum $\Lambda^2_+ T^* X \oplus \Lambda^2_- T^* X$. These are the bundles associated to the ± 1 eigenvalues of $*$. Each of these summands is a real vector bundle over X with fibre of dimension 3. The bundle $\Lambda^2_+ T^* X$ consists of the 2-forms ω with $*\omega = \omega$, and $\Lambda^2_- T^* X$ consists of the 2-forms ω with $*\omega = -\omega$. The 2-forms in $\Lambda^2_+ T^* X$ are called *self-dual*, and the 2-forms in $\Lambda^2_- T^* X$ are called *anti-self dual*.

On any compact Riemannian manifold (X, g), Hodge Theory gives preferred representatives for cohomology classes by harmonic differential forms. Recall the adjoint operator

$$\delta \colon \Omega^p(X) \to \Omega^{p+1}(X)$$

associated with the exterior derivative, so that $(d\beta, \alpha) = (\beta, \delta\alpha)$. In the oriented case $\delta = \pm * d *$. The Hodge theorem asserts that a real cohomology class has a unique representative β with

$$d\beta = \delta\beta = 0$$

For a compact, oriented 4-manifold there is an interaction between the splitting of $\Lambda^2 T^* X$ and Hodge theory. First, the harmonic 2-forms are preserved by the $*$-operator (Proposition 2.1.5 (1)), which interchanges $\ker d$ and $\ker \delta$.

Since the $*$-operator maps the space $\mathcal{H}^2(X)$ of harmonic 2-forms to itself, this vector space also has a pointwise orthogonal direct sum splitting as $\mathcal{H}^2(X) = \mathcal{H}^2_+(X) \oplus \mathcal{H}^2_-(X)$, where $\mathcal{H}^2_+(X)$ consists of harmonic 2-forms with $*\omega = \omega$. On the other hand, $\mathcal{H}^2_-(X)$ is composed of harmonic, anti-self-dual 2-forms.

The dimension of $\mathcal{H}^2_+(X)$ and $\mathcal{H}^2_-(X)$ depend only on the smooth structure of X. The De Rham cohomology $H^2_{DR}(X)$ has the symmetric pairing that associates classes $a, \tilde{a}$ the number

$$Q(a, \tilde{a}) = \int_M \omega \wedge \tilde{\omega}$$

as computed using any pair of representative closed 2-forms ω and $\tilde{\omega}$ for a and $\tilde{a}$. The dimension of $\mathcal{H}_+(X)$ is equal to the dimension of the maximal vector subspace in $H^2_{DR}(X)$ on which Q is positive definite. The dimension of $\mathcal{H}_-(X)$ corresponds to that of the maximal subspace on which Q is negative definite. These dimensions depend only on the underlying topological structure of X [[Don97](Ch.1), [Tau98](Ch. 19.5, pg 286)]. The difference of the dimension n_+ of $\mathcal{H}_+(X)$ and n_- of $\mathcal{H}_-(X)$ is called the *signature* of X.

2.1.4 Morse 1-forms

We will introduce the concept of closed 1-forms with Morse singularities, also known as Morse 1-forms. This object will be useful in generating examples of near-symplectic manifolds. We use the reference of Farber [[Far04] Ch. 9, pg 125-149].

Let M be a closed manifold. A 1-form on M is a smooth section of the cotangent bundle T^*M. In local coordinates $(x_1, x_2, \dots, x_n)$ defined in an open subset $U \subset M$ any 1-form μ is given by $\mu = a_1(x)dx_1 + a_2(x)dx_2 + \cdots + a_n(x)dx_n$, where $a_1(x), \dots a_n(x)$ are smooth real-valued functions defined in U. In local coordinates, if $\mu = \sum_{i=1}^n a_i dx_i$, then

$$\begin{aligned} d\mu &= \sum_{i=1}^n da_i \wedge dx_i = \sum_{i,j}^n \frac{\partial a_i}{\partial x_j} dx_j \wedge dx_i \\ &= \sum_{i<j} \left(\frac{\partial a_j}{\partial x_i} - \frac{\partial a_i}{\partial x_j} \right) \cdot dx_i \wedge dx_j \end{aligned}$$

If μ is closed, i.e. $d\mu = 0$. Thus, the closedness condition is equivalent to the equations

$$\frac{\partial a_j}{\partial x_i} = \frac{\partial a_i}{\partial x_j} \qquad \forall i, j$$

If μ is a closed 1-form, then by Poincaré Lemma and by the De Rham theorem, for any simply connected domain $U \subset M$, the restriction $\mu|_U$ is an exact 1-form. That is, we can respresent it as the differential df_U, where $f_U : U \to \mathbb{R}$ is a smooth function. If U is connected, then the function f_U is determined by $\mu|_U$ up to a constant. Thus, locally a closed 1-form is the same as a smooth function determined up to a constant.

Properties of smooth functions have immediate meaning for closed 1-forms. For instance, the notion of a critical point translates to the notion of a *zero* of a closed 1-form.

Definition 2.1.7. A *zero* of a closed 1-form is a point $p \in M$ such that $\mu_p = 0$. Zeroes of μ lying in a simply connected open set U are exactly the critical points of any function $f_U : U \to \mathbb{R}$ with $df_U = \mu|_U$.

A *non-degenerate zero* $p \in M$ is one such that $\mu_p = 0$, and the 1-form viewed as a smooth section $M \to T^*M$ is transversal to the zero section $M \subset T^*M$ at the point p. This condition is equivalent to the requirement that $p \in U$ is a non-degenerate critical point of any function $f_U : U \to \mathbb{R}$ with $df_U = \mu|_U$.

A closed 1-form is *Morse* if all its zeros are non-degenerate. We denote the set of zeros of a Morse 1-form by Crit (μ). Any zero p of a Morse closed 1-form has well-defined *Morse index* lying in $\{0, 1, \ldots, n\}$, which is defined as the index of p viewed as a critical point of f_U. We say that a Morse closed 1-form (and also a Morse function) is of *indefinite type* if it has no zero or critical point of index 0, that is a minimum, nor a zero or critical point of index n, that is a maximum.

2.1.5 Gradient and Taylor expansion

Let M be a smooth manifold, U a neighbourhood of a point z in M and $\omega \in \Omega^2(M)$ a differential 2-form. Locally, consider ω as a section $\omega \colon U \to \Lambda^2 T^*U$. As any smooth map between manifolds, we can consider the differential on tangent spaces $\nabla\omega_z \colon T_zM \to \Lambda^2 T^*_z M$. In the context of near-symplectic 4-manifolds (defined in the next section) this local definition is known in the literature as intrinsic gradient [ADK05, Per06, GK04, GS09]. Gay-Symington[GS09] provide a further explanation. In this situation, let M be a 4-manifold and z be a point lying in the 1-submanifold $Z \subset M$. Identify $\omega \colon U \to \Lambda^2 T^*U$ as a smooth map $\omega \colon \mathbb{R}^4 \to \mathbb{R}^4$, and the linearization on tangent spaces as $D\omega_z \colon \mathbb{R}^4 \to \mathbb{R}^6$. If $\phi \colon \mathbb{R}^4 \to \mathbb{R}^4$ is a change of coordinates in U and $\psi \colon \mathbb{R}^6 \to \mathbb{R}^6$ is a change of coordinates on $\Lambda^2 T^*U$, then restricted to Z, we have $D_z\omega \circ \phi = \psi \circ D_z\omega$. Consequently on Z, we have that $D_z\omega$ represents what they call an intrinsically defined derivative, the derivative denoted by $\nabla\omega_z \colon T_zM \to \Lambda^2 T_z^* M$ in the work of [ADK05, Per06].

Throughout this work we will also use a Taylor expansion for 2-forms. Lang provides the following general setting for the Taylor expansion [[Lan95] Ch.1 §1.3-1.4, pg 6-11]. Let E, F be vector spaces. Let U be open in E. Let x, y be two points in U such that the segment $x + t \cdot y$ lies in U for $0 \leq t \leq 1$. Let $f \colon U \to F$ be a smooth map , and denote by $y^{(p)}$ the vector $(y, \dots, y)$ p–times. Then the function $D^p f(x + t \cdot y) \cdot y(p)$ is continuous in t, and we have

$$\begin{aligned} f(x+y) =& f(x) + Df(x)y + \cdots + \frac{D^{p-1}f(x)y^{(p-1)}}{(p-1)!} \\ &+ \int_0^1 \frac{(1-t)^{p-1}}{(p-1)!} D^p f(x+ty) y^{(p)} dt \end{aligned}$$

The derivatives have the following meaning. The first derivative $Df \colon U \to L(E, F)$ is a mapping from U to the space of linear maps from E to F. $L(E, F)$ is itself a vector space, so it can take derivatives as well. The second derivative $Df^2 \colon U \to L^2(E, F) \simeq L(E, L(E, F))$ takes values in the space of symmetric bilinear maps from E to F, and similarly for higher order derivatives $Df^p \colon U \to L^p(E, F)$.

In the context of differential 2-forms we have the following setting. Let $\omega \colon U \to \Lambda^2\mathbb{R}^{2n}$ be a 2-form, where $U \subset \mathbb{R}^{2n}$. Consider v to be a vector in U, $s \in [0, 1]$ such that $s \cdot v$ lies in U. We express the Taylor approximation of ω around $0 \in U$ as

$$\omega(s \cdot v) = \omega(0) + s \cdot D\omega(0)v + \cdots + s^{p-1}\frac{D^{p-1}\omega(0)v^{(p-1)}}{(p-1)!} + \dots$$

The element $D\omega(0)v$ lies in $\Lambda^2\mathbb{R}^{2n}$, as the first derivative is a map $D\omega \colon U \to L(\mathbb{R}^{2n}, \Lambda^2\mathbb{R}^{2n})$. This gives us an object $D\omega(0) \colon \mathbb{R}^{2n} \to \Lambda^2\mathbb{R}^{2n}$. Thus, the derivative $D\omega(0)$ acts as a differential 2-form as well. In this work, we will only focus on the first derivative. The higher order derivatives will not play a role as they will be negligible, which will become clear from the context. The following local definition regarding the differential of ω takes its name from the literature of near-symplectic geometry.

Definition 2.1.8. Let M be a smooth manifold, z a point in M and $\omega \in \Omega^2(M)$. We denote the differential of ω on tangent spaces as $\nabla\omega(z) \colon T_zM \to \Lambda^2 T_z^* M$ and following the convention of [ADK05, Per06, GS09] we call it *intrinsic gradient*.

Performing a Taylor expansion around a point, $\nabla_v \omega(z)$ corresponds to $D\omega(0)v$, that is the first derivative of ω at a point z in the direction of a vector v.

2.2 Near-symplectic 4-manifolds

Let X be a smooth, oriented 4-manifold, and let ω be a closed 2-form on X such that $\omega^2 \geq 0$ on all X. Generically at every point $p \in X$, a 2-form ω can have rank 0, 2, or 4. If ω only has rank 4 for every point in M, then it is symplectic. Relax the non-degeneracy condition of the symplectic form but going beyond the generic behaviour. Thus, we consider closed 2-forms with $\omega^2 \geq 0$ and with ω having rank 0 or 4, but not rank 2. Then, due to the non-negativity of ω^2, we ask for a natural transversality condition on the gradient $\nabla\omega_p$ that guarantees that the set where $\omega = 0$ is a smooth 1-submanifold of X [Per06]. We say that ω is *near-symplectic*, if for every $p \in X$ it is either non-degenerate, or it vanishes transversally along circles.

Definition 2.2.1 ([ADK05])**.** Let X be a smooth oriented 4-manifold. Consider a closed 2-form $\omega \in \Omega^2(X)$ such that $\omega^2 \geq 0$ and such that ω_p only has rank 4 or rank 0 at any point $p \in X$, but never rank 2. The form ω is called *near-symplectic*, if for every $p \in X$, either

1. $\omega_p^2 > 0$, or
2. $\omega_p = 0$, and Rank$(\nabla\omega_p) = 3$, where $\nabla\omega_p \colon T_pX \to \Lambda^2 T_p^*X$ denotes the intrinsic gradient of ω.

Lemma 2.2.2. *The zero set $Z_\omega = \{x \in X \mid \omega_x = 0\}$ of a near-symplectic form $\omega \in \Omega^2(X)$ is a smooth 1-dimensional submanifold.*

Proof. We work over a small ball B containing z. Recall that there is a decomposition of $\Lambda^2 T^*B$ into 2 subbundles; $\Lambda^2_+ T^*B$ a rank 3 bundle of self dual 2- forms, and $\Lambda^2_- T^*B$ the rank 3 bundle of anti self dual forms.
We want to show that ω intersects the negative bundle transversely. The point is that the bundle $\Lambda^2_- T^*B$ is 9-dimensional, X is 4-dimensional, and $\Lambda^2 T^*B$ is 12-dimensional. So they intersect in a 1-dimensional manifold [[Bre97], Ch II.15, pg 114-115]. Moreover, ω is symplectic away from the zero set, and it does not

intersect the negative bundle over there. The zero set intersects it, so this is the 1-dimensional manifold. The reason why they intersect transversely lies on the rank condition of the derivative having 3 dimensional image. These three dimensions cannot point into the negative bundle, giving the remaining 3 dimensions.
Without the rank condition, the intrinsic gradient $\nabla\omega_p$ defined on a neighbourhood U of point in Z could have rank greater or equal to 3. Looking at tangent spaces, we can see that if Rank$(\nabla\omega_p) = 3$ then the tangent spaces Im$(\nabla\omega_p)$ and $T_{\nabla\omega_p}\Lambda^2_- T^*B$ span the whole space $T_{\nabla\omega_p}\Lambda^2 T^*B$.
To show that this 1-submanifold is Z_ω, recall our original condition (3.1): $\omega^2 \geq 0$. Let $v \in T_pU$ be a non-zero in the tangent space of a neighbourhood $p \in U$. Assume that $\nabla\omega_p(v) \in \Lambda^2_- T^*B$ for a point $\omega(p) \in$ Im$(\omega) \cap \Lambda^2_- T^*B$. Then, it would follow that $\omega_p^2 \leq 0$. However, this is not possible by the non-negative condition (3.1), so we conclude that $\omega_p^2 = 0$. Hence, the 1-submanifold coming from the transverse intersection corresponds to those points where ω vanishes, that is, Im$(\omega) \cap \Lambda^2_- T^*B = \{p \in X^4 \mid \omega(p) = 0\} = Z_\omega$. $\square$

Example of a near-symplectic 4-manifold

Example 2.2.3. A prototypical example of a near-symplectic 4-manifold is given by $X = S^1 \times Y^3$, where Y is a closed Riemannian 3-manifold. Consider a closed Morse 1-form $\beta \in \Omega^1(Y)$ with indefinite zeroes, and let t be the parameter of S^1. To guarantee that ω is closed, we need that β is intrinsically harmonic. The 2-form

$$\omega = dt \wedge \beta + *(dt \wedge \beta)$$

is near-symplectic, where $*$ is the Hodge operator defined with respect to the product metric on S^1 and Y. The singular locus $Z_\omega = \{p \in X \mid \omega_p = 0\}$ is in this case $S^1 \times$Crit(β). The property of ω being closed will be discussed in the remaining part of this section.

We proceed now to explain harmonicity and transitiveness of 1-forms with Morse critical points, which is related to the closedness property of the near-symplectic form in the previous example.

Definition 2.2.4. A differential 1-form α is harmonic if $\Delta\alpha = 0$, where $\Delta = d\delta + \delta d$ denotes the Laplace operator as previously defined more generally in 2.1.4. A closed 1-form α on a smooth manifold M is called *intrinsically harmonic* if it is harmonic with respect to some Riemannian metric on M.

Indefinite indices in a closed 1-form imply harmonicity and transitivity.

Definition 2.2.5 ([Far04] Def. 9.10, pg 132)**.** A closed 1-form α is called *transitive* if for any two points $p, q \in M$ which are not zeroes of α, there exists a path $\gamma\colon [0,1] \to M$ with $\gamma(0) = p$ and $\gamma(1) = q$, such that $\alpha(\dot\gamma(t)) > 0$ for any $t \in [0,1]$. Geometrically, this means that the velocity vector $\dot\gamma(t)$ of the path $\gamma(t)$ always points in the direction in which the 1-form increases, where the function f increases, seen locally as $df = \alpha$.

Theorem 2.2.6 ([Cal69])**.** *A closed 1-form α on a smooth manifold M is intrinsically harmonic if and only if (i) α does not have zeroes of index 0 or n, and (ii) it is transitive.*

Theorem 2.2.7 ([Hon04b])**.** *Let α be a closed 1-form on a closed smooth manifold M having Morse zeroes with indices different from 0 or $n = \dim(M)$. Then there exists an intrinsically harmonic Morse closed 1-form $\tilde\alpha$ on M such that $[\tilde\alpha] = [\alpha] \in H^1(M, \mathbb{R})$, and same critical points* $\mathrm{Crit}_i(\tilde\alpha) = \mathrm{Crit}_i(\alpha)$ *for all $i = 0, 1, \ldots, n$*

As it is shown in the following lemma, given a regular Morse function $f\colon Y \to S^1$, that is one without critical points, then df is transitive. On the other hand, any closed 1-form with no local maxima nor minima can be replaced by an intrinsically harmonic 1-form having equal Morse numbers and lying in the same cohomology class.

Lemma 2.2.8. *Let $f : Y \to S^1$ be a fibration on a smooth, closed, connected, manifold Y. Then the closed 1-form df is transitive.*

Proof. **Step 1**
Assume that f_* is the zero map on fundamental groups. If that was the case, then we could get a lift $\tilde f$ of f to the reals with $p \circ \tilde f = f$, where p is the projection $p\colon \mathbb{R} \to S^1$. The mapping $\tilde f$ has the same critical points as f, since $df_q = dp \circ d\tilde f_q = 0 \Leftrightarrow df_q = 0$. Thus, if f is regular $(df_q \neq 0)$, then $\tilde f$ is regular. However, Y is closed. Thus, $\tilde f$ should have a maximum and a minimum, i.e. critical points of index n or 0. This means that f_* cannot be trivial on fundamental group, and the assumption

$f_* = 0$ is incorrect. That is, $\ker(f_*)$ cannot be the whole $\pi_1(Y)$. Since the kernel of f is a normal subgroup of $\pi_1(Y)$ there is a regular connected covering space Y_K with $\pi_1 = \ker f$ by Hatcher [Ch.1.3, Thm 1.38].

Step 2

Let $\hat{Y}_K := \hat{f}^{-1}([a,b])$ and consider $\hat{f} : \hat{Y}_K \to [a,b] \subset \mathbb{R}$. We choose a, b so that $p(a) = p(b)$. Since our assumption is that f is a fibration, that is regular over $[a,b]$, we have that $\hat{f}$ is a Morse function with no critical points. It follows then from the regular interval theorem that $\hat{Y}_K$ is diffeomorphic to $\hat{f}^{-1}(a) \times I$ with $f^{-1}(a)$ being connected.

$$\begin{tikzcd} Y_K \arrow[r, "\hat{f}"] \arrow[d, "g"] & \mathbb{R} \arrow[d, "p"] \\ Y \arrow[r, "f"'] & S^1 \end{tikzcd} \tag{2.3}$$

Step 3

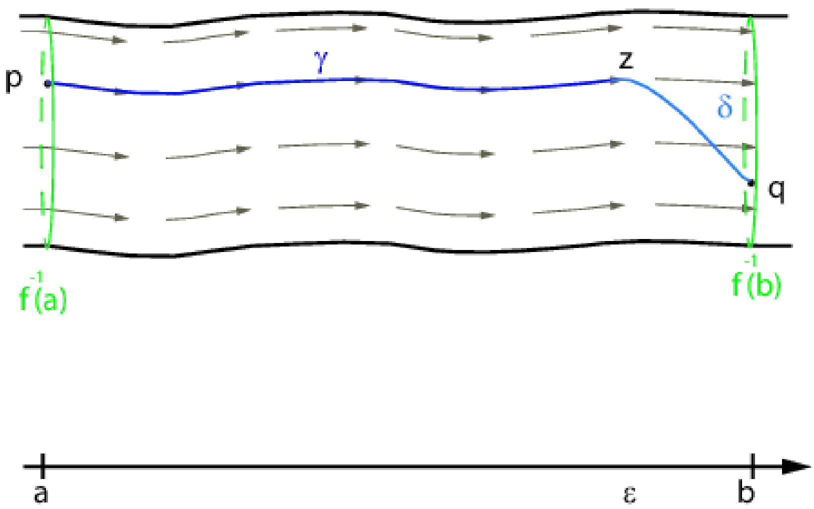

Figure 2.1:

Now we define a path joining 2 points. Let $p, q \in Y_K$ be two different points lying at different level sets. To join these points, we can take the flow line γ generated by the gradient vector field of $\hat{f}$. Since $\hat{Y}_K \approx \hat{f}^{-1}(a) \times I$, then all flow lines starting at $\hat{f}^{-1}(0)$ reach $\hat{f}^{-1}(1)$ in unit time. Along this path it follows that for the closed Morse 1-form df, we have $df(\gamma(t)) = 1 > 0$. If p, q lie at different level sets $\hat{f}^{-1}(a)$ and $\hat{f}^{-1}(b)$, but also at different heights, then we first take the flow line γ, and then join them with a path δ in $\hat{f}^{-1}([\epsilon, b])$ from z to q so that $df(\dot{\delta}(t)) > 0$. By defining a path $\eta = \gamma \circ \delta$ via the flow line γ and the path δ, we can build a path $\eta : [0,1] \to Y_K$ from $\eta(0) = p$ to $\eta(1) = q$ with $d\hat{f}(\dot{\eta}(t)) > 0 \forall t \in [0,1]$, i.e. df is transitive. □

2.3 Broken Lefschetz fibrations

A (topological) *Lefschetz fibration* on a simply connected 4-manifold X is a smooth map $f\colon X \to S^2$ whose generic fiber is a surface. The map f is allowed to have isolated critical pints, known as Lefschetz singularities, which are modeled in local complex coordinates by $f\colon (z_1, z_2) \to z_1^2 + z_2^2$. Regular fibers are smooth and convex, but singular fibres present an isolated nodal singularity.

A Lefschetz pencil on a 4-manifold X is a map $f\colon X \setminus B \to S^2$, which is not defined at a finite number of base points $b_1, \ldots, b_m \in B$. Around each base point, f is modeled in local complex coordinates by $f\colon (z_1, z_2) \mapsto z_1/z_2$. Alternatively, thinking of S^2 as CP^1, then $f(z_1, z_2) \mapsto [z_1 : z_2]$. The fibers of f are punctured surfaces, to which one adds the base points to obtain closed surfaces, called the fibers of the pencil. Near a base point, the mapping looks like the slicing of $\mathbb{C}^2$ into complex planes passing throught the origin. If one blows up a Lefschetz pencil at all its base points, then one obtains a Lefschetz fibration [[Sco05] pg 416-418]

The work from Donaldson and Gompf [Don99, GS99] show that Lefschetz fibrations are in one to one correspondence to symplectic 4-manifolds. The natural object associated to a near-symplectic 4-manifold is a *broken Lefschetz fibration* or BLF. By a BLF, we understand a submersion $f\colon X \to S^2$ with two type of singularities: isolated points (Lefschetz singularities) and circles (indefinite folds). In [ADK05] these mappings were studied under the name of singular Lefschetz pencils. It was shown that there is a relation between BLFs and near-symplectic manifolds. Up to blow ups, a near-symplectic 4-manifold X can be decomposed into a BLF. In the other direction, given a BLF on smooth oriented 4-manifold, we can obtain a near-symplectic structure on X such that the singular locus of ω_{ns} is exactly the same as the fold singular set of f.

Definition 2.3.1. On a smooth, closed 4-manifold X^4, a *broken Lefschetz fibration* or *BLF* is a smooth map to the 2-sphere $f\colon X^4 \to S^2$ from a closed 4-manifold X^4

to S^2 with two types of singularities:

1. isolated *Lefschetz-type* singularities, contained in the finite subset of points $B \subset X^4$, which are locally modeled by complex charts

$$\mathbb{C}^2 \longrightarrow \mathbb{C} \quad , \quad (z_1, z_2) \longmapsto z_1^2 + z_2^2$$

2. *indefinite fold* singularities, also called *broken*, contained in the smooth embedded 1-dimensional submanifold $\Gamma \subset X^4 \setminus B$, which are locally modelled by the real charts

$$\mathbb{R}^4 \longrightarrow \mathbb{R}^2 \quad , \quad (t, x_1, x_2, x_3) \longmapsto (t, x_1^2 + x_2^2 - x_3^2)$$

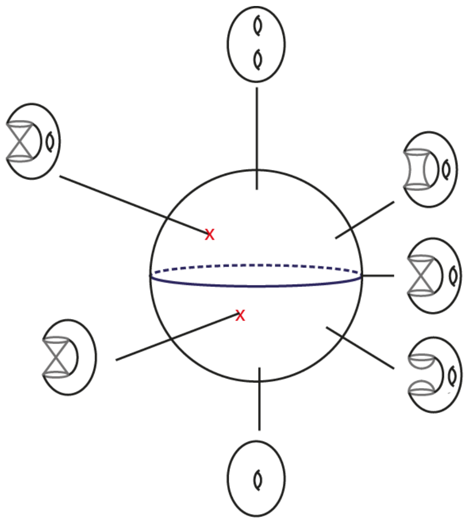

Figure 2.2: Example of a BLF with 1 circle of folds and 2 Lefschetz points

BLFs are naturally related to near-symplectic manifolds. Up to blow ups, a near-symplectic 4-manifold X can be decomposed into a BLF. The other direction is given by the following theorem.

Theorem 2.3.2 ([ADK05] Theorem 3). *Given a BLF with singularity set $\Gamma \sqcup B$ on a closed oriented 4-manifold X, with the property that there is a class $\alpha \in H^2(X)$, such that it pairs positively with every component of every fiber, then X carries a near-symplectic structure with zero-locus being equal to the set of broken singularities of f.*

Our theorem 1 will show an analogous statement in higher dimensions.

2.4 Overtwisted contact structures

Let M be a manifold of dimension $2n-1$. A contact structure on M is a hyperplane field $\xi = \ker(\alpha) \subset TM$ with the defining differential 1-form $\alpha \in \Omega^1(M)$ satisfying $\alpha \wedge d\alpha^n \neq 0$. In $\dim(M) = 3$, contact structures are classified as *tight* and *overtwisted*. A contact structure ξ on a 3-manifold is called *overtwisted* if it contains a disk D^2 with a specific singular foliation arising from the intersection of the tangent planes of the disk and the contact planes from ξ. Denote by $\Delta = T_pD \cap \xi_p$ the characteristic foliation.

Definition 2.4.1. A contact structure is called *overtwisted* if (M^3, ξ) contains an embedding of a disk $D^2 \hookrightarrow M^3$ such that for its characteristic foliation: (i) the boundary ∂D is a closed leaf, and (ii) there is a unique elliptic singular point in the interior. If we cannot find such an overtwisted disk, then the contact structure is called *tight*.

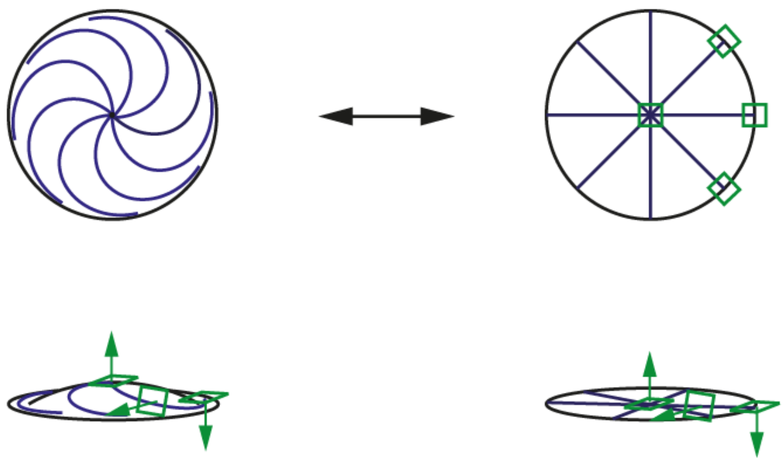

Figure 2.3: Two equivalent representation of the overtwisted disk.
Left: the disk has a bump. Right: the disk is flat.

Definition 2.4.2. A *half-torsion domain* is a thickened torus $[0,1] \times T^2 \ni (r, (x, y))$ with contact structure $\xi = \ker\{\sin(\pi r)dx + \cos(\pi r)dy\}$.

The half-torsion domain can be represented as a cube. Since (x, y) are the torus coordinates, we identify the top and bottom faces, as well as the front and back ones. The contact structure is horizontal on the left and right faces, and the

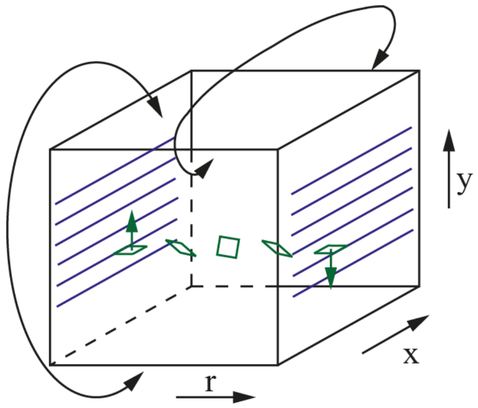

Figure 2.4: Half-Torsion Domain

r-direction is always tangent to ξ. Taking any line-r, we obtain a half twist of the contact plane. This happens everywhere, whatever line we choose.

Definition 2.4.3. (M^3, ξ) has *positive Giroux torsion*, if it contains two half-torsion domains glued along one face. This means that along a line-r, inside the two blocks, the contact plane makes a full turn from side to side. If (M^3, ξ) is overtwisted, then it has positive torsion.

Now we consider the situation in higher dimensions. In higher dimensions, Niederkrüger introduced the concept of *plastikstufe*, which generalizes the idea of an overtwisted disk [Nie06].

Definition 2.4.4. A contact manifold (V^{2n+1}, ξ) is called *PS-overwtisted* if it contains a submanifold $N \subset B^{2n+1} \subset V$ with the properties:

1. $\dim(N) = n + 1$, where N is compact with boundary
2. $\xi \cap TN$ induces a singular foliation that looks like an open book
3. ∂N lies in the singular set of the foliation

The binding B of the open book is a codimension 2 submanifold in the interior of N with trivial normal bundle. In addition, $\theta\colon N \backslash B \to S^1$ is a fibration whose fibers are transverse to the boundary of N, and which coincides in a neighbourhood $B \times D^2$ of $B = B \times \{0\}$ with the normal angular coordinate.

Recently, Massot, Niederkrüger, and Wendl gave a generalization of the plastikstufe with an object called *bordered Legendrian open books* or *bLob* [MNW12].

For completion we only state that the higher dimensional analogue of the Giroux torsion is called Giroux domain. A more precise definition of these objects appears in [MNW12].

Going back to the context of near-symplectic 4-manifolds, we notice that there is a connection with overtwisted contact structures.

Theorem 2.4.5. *[Hon04b, GK04] Let (X, ω) be a near-symplectic 4-manifold. The normal sphere bundle of the singular locus Z, diffeomorphic to $S^1 \times S^2$, has an overtwisted contact structure.*

Honda provides a direct argument by first showing a local model of a near-symplectic form on the tubular neighbourhood of the singular locus Z_ω. Then a 1-form is obtained, which is contact on the boundary of the normal disk bundle, and finally he shows that this is of the overtwisted type. Gay and Kirby's method is more elaborate involving open book decompositions and 4-dimensional techniques from Kirby calculus. In higher dimensions, it is not known if a near-symplectic manifold, as it will be defined in chapter 3, induces a PS-overtwisted contact structure on the normal sphere bundle of Z.

2.5 Symplectic Fillings

In this section we give some notions regarding the interaction of symplectic manifolds with contact structures at their boundary. The definitions and theorems follow [[Gei08] Ch 5.1, 269-273]. However, before we introduce the concept of the Reeb vector field, a unique vector field associated with a contact form.

Let M is a closed manifold of dimension $2n-1$ with cooriented contact structure $\xi = \ker(\alpha)$. Due to the contact condition, $d\alpha|_{T_pM}$ is a skew-symmetric form of maximal rank $2n-2$ when restricted to the hyperplane distribution defined by the kernel of α. Hence, $d\alpha|_{T_pM}$ has 1-dimensional kernel. The condition $\iota_R d\alpha = 0$ defines a vector field R uniquely up to scaling, that is, it is a unique line field $\langle R \rangle \subset TM$.

Definition 2.5.1. Associated with a contact form α, there is a unique vector field R called the *Reeb vector field* defined by

1. $\iota_R d\alpha = d\alpha(R, \cdot) = 0$

2. $\alpha(R) = 1$

The first condition tells us the direction where R points at. Since $\alpha \wedge (d\alpha)^{n-1} > 0$, the second condition specifies a non-vanishing section and normalizes R. The following definition gives associations between symplectic and contact structures.

Definition 2.5.2.

1. A compact symplectic manifold (W^{2n}, ω) is a *weak symplectic filling* of (M, ξ) if
   - $\partial M = W$ as oriented manifolds and
   - $\omega^{n-1}|_{\xi} > 0$
2. A compact symplectic manifold (W, ω) *dominates* (M, ξ) if
   - $\partial M = W$ as oriented manifolds and
   - $\omega|_{\xi} = f \cdot d\alpha|_{\xi}$, where $f \colon M \to \mathbb{R}_+$ is the conformal factor they differ.
3. A compact symplectic manifold (W^{2n}, ω) is a *strong symplectic filling* of (M, ξ) if
   - $\partial M = W$ as oriented manifolds and
   - there is a Liouville vector field Y defined near ∂W, pointing outwards along ∂W. By Liouville vector field, we mean a vector field Y such that $\mathcal{L}_Y \omega = \omega$, where $\mathcal{L}$ denotes the Lie derivative of the symplectic form in the direction of Y. In particular, this means that Y defines a contact form by $\alpha = \iota_Y \omega$, thus a contact structure $\xi = \ker(\alpha|_{TM})$. In this case the contact manifold (M, ξ) is called the *convex boundary* of (W, ω).

Remark 2.5.3. In dimension 3, the boundaries of the singular loci of a near-symplectic 4-manifold are overtwisted, which implies that they are not fillable. This, in turn implies that the singular points cannot be easily removed. It would be interesting to see if something similar might be true in higher dimensions.

Chapter 3

Near-symplectic 2n-manifolds

"How can the mind be so imperfect?" she says with a smile.[...]
"It may well be imperfect," I say, "but it leaves traces. And we can follow those traces, like footsteps in the snow."
"Where do they lead?"
"To oneself," I answer. "That's where the mind is. Without the mind, nothing leads anywhere."

Haruki Murakami, *Hard-Boiled Wonderland and the End of the World*

3.1 Definition

Let M be an oriented $2n$-manifold, and $\omega \in \Omega^2(M)$ a closed 2-form such that

$$\omega^n \geq 0 \tag{3.1}$$

everywhere. Suppose that at some point p, the kernel K of ω seen as a subspace of the tangent space has dimension 4.

$$K = \{v \in T_pM \mid \omega_p(v, \cdot) = 0\}$$

We have an intrinsic gradient $\nabla\omega\colon K \to \Lambda^2 T_p^*M$. We can restrict this map to bivectors in K and consider the composition $K \to \Lambda^2 T_p^*M \to \Lambda^2 K^*$, where the map $\Lambda^2 T_p^*M \to \Lambda^2 K^*$ corresponds to the dual of the inclusion $K \hookrightarrow T_pM$ in the corresponding exterior algebra. We denote this composition as

$$D_K\colon K \to \Lambda^2 K^* \tag{3.2}$$

Then, the wedge square gives us a non-degenerate quadratic form $q\colon \Lambda^2 K^* \otimes \Lambda^2 K^* \to \Lambda^4 K^*$.

Proposition 3.1.1. *The image* $\mathrm{Im}(D_K)$ *is of dimension 3.*

Proof. Take an arbitrary tangent vector $v \in T_pM$ and choose coordinates such that $p = 0$ is the point at the origin. By our assumption on ω, we have $\omega^n(t \cdot v) \geq 0$ for all scalars t, where $t \cdot v$ points into the manifold. Yet, if we use a Taylor expansion to write $\omega(t \cdot v) = \omega(0) + t \cdot \nabla_v \omega(0) + O(t^2)$ and take $v \in K$, we have

$$\omega^n(t \cdot v) = \underbrace{\omega^n(0)}_{=0} + t\binom{n}{1}\underbrace{\omega^{n-1}(0)}_{=0} \wedge \nabla_v \omega(0) + t^2\binom{n}{2}\omega(0)^{n-2} \wedge (\nabla\omega_v(0))^2 + O(t^3)$$

The forms $\omega^n(0)$ and $\omega^{n-1}(0)$ vanish since they necessarily take vectors $\partial_{k_1}, \ldots, \partial_{k_4}$ from K, whereas in the linear combination of ω^{n-2} there will be vectors outside from K where the form remains non-zero. This gives us

$$\omega^n(t \cdot v) = \binom{n}{2} \cdot t^2 \cdot \omega(0)^{n-2} \wedge (\nabla\omega_v(0))^2 + O(t^3)$$

We work in local coordinates using the tangent space at $p = 0$ for the local coordinate system. The space T_pM/K has a symplectic structure and we can combine an orientation on it with an orientation of K to get an orientation of T_pM, which has a natural orientation. With respect to this chosen orientation we want to show that $D_K(v) \wedge D_K(v) \geq 0$ for a $v \in K$. Let $e_i = \left(\frac{\partial}{\partial x_i}\right)_{1 \leq i \leq 2n}$ be an oriented basis. Since $\omega^n(tv) \geq 0$ from our original consideration (3.1), then we have that $\omega^n(tv)(e_1, \ldots, e_{2n}) \geq 0$, thus

$$\omega^n(t \cdot v) \approx C \cdot \omega(0)^{n-2} \wedge (\nabla\omega_v(0))^2\,(e_1, \ldots, e_{2n}) \geq 0$$

with the constant $C = \binom{n}{2} \cdot t^2$. The form $\omega(0)^{n-2}$ has a sign on the complementary subspace to K in T_pM, since we have chosen an orientation. However, from (3.2) by restricting to vectors in K, then

$$\omega^n(t \cdot v) \approx C \cdot t^2 \cdot \omega(0)^{n-2}(e_1, \ldots, e_{2n-4}) \wedge \underbrace{(\nabla\omega_v(0))^2}_{D_K(v) \wedge D_K(v)}(\partial_{k_1}, \ldots, \partial_{k_4}) \geq 0$$

We can see now that the image of D_K is a positive semi-definite subspace of $\Lambda^2 K^*$. Hence, $\mathrm{Im}(D_K)$ has dimension at most 3. In particular, $D_K(v) \wedge D_K(v)$ is a non-negative 4-form with respect to K. $\square$

As seen chapter 2, $\Lambda^2\mathbb{R}^4$ splits into a positive and negative subbundles. The positive subbundle consists of 2-forms ω such that $*\omega = \omega$ and the negative subbundle of those 2-forms such that $*\omega = -\omega$. Each of these subbundles is of dimension 3. Then, the image of D_K has dimension at most 3. For example, with respect to the standard basis of $\mathbb{R}^4$ with coordinates (x_1, x_2, x_3, x_4) such bundles are given by

$$\begin{array}{ll}
\Lambda_+ : & \Lambda_- : \\
\beta_1 = dx_1 \wedge dx_2 + dx_3 \wedge dx_4 & \beta_4 = dx_1 \wedge dx_2 - dx_3 \wedge dx_4 \\
\beta_2 = dx_1 \wedge dx_3 - dx_2 \wedge dx_4 & \beta_5 = dx_1 \wedge dx_3 + dx_2 \wedge dx_4 \\
\beta_3 = dx_1 \wedge dx_4 + dx_2 \wedge dx_3 & \beta_6 = dx_1 \wedge dx_4 - dx_2 \wedge dx_3
\end{array}$$

With this in mind we come to our defintion.

Definition 3.1.2. The 2-form $\omega \in \Omega^2(M^{2n})$ is *near-symplectic*, if it is closed, $\omega^n \geq 0$, and at a point p where $\omega^n = 0$, one has that the kernel K is 4-dimensional and that the Im(D_K) has dimension 3.

Remark 3.1.3. Informally, the definition implies that a closed 2-form $\omega \in \Omega^2(M)$ is *near-symplectic*, if for every $p \in M$, either

(i) $\omega_p^n > 0$, or

(ii) $\omega_p^{n-1} = 0$, but $\omega_p^{n-2} \neq 0$ at a codimension 3 submanifold of M.

In the remaining part of this section we will explain why the degeneracy locus is a codimension 3 submanifold.

The image of the map $D_K \colon K^4 \to \Lambda^2 K^*$ is of dimension 3, thus it has 1-dimensional kernel. If we look at ω^{n-1} then it vanishes at p, since it takes at least 2 vectors from K. Moreover, $G = \nabla\omega^{n-1}(p)$ is defined. Choose coordinates (x_k) so that K is defined by the vanishing of all but the last four dx_k. Take the derivative of ω^{n-1} and apply the chain rule to obtain

$$G = (n-1)\omega(p)^{n-2}\nabla\omega_p$$

where the gradient on the right is defined using the coordinates. The 2-form ω is symplectic on the submanifold Z where the last 4 coordinates are zero (for instance, using the previously chosen basis (e_i) we get that $\omega^{n-2}(p)(e_1, \ldots, e_{2n-4}) \neq 0$). We can adjust the coordinates to Darboux form, so that ω is constant on Z,

$\omega|_p = dx_1 \wedge dx_2 + \cdots + dx_{2n-5} \wedge dx_{2n-4}$ for $p \in Z$. Hence $\nabla\omega_p(\partial x_i) = 0$ for $i = 1, \ldots, 2n-4$. However,

$$\ker G = \ker(\nabla\omega_p)$$

and now one sees that this is a codimension 3 subspace containing the line $\ker(D_K)$. Hence the degeneracy locus Z of the near-symplectic form is a codimension 3-submanifold of M^{2n}.

Lemma 3.1.4. *The singular locus $Z_\omega = \{p \in M \mid \omega_p^{n-1} = 0\}$ is a codimension 3 submanifold of M.*

Proof. We want to show that at a singular point $x \in U \subset M$, the section $\bar{\omega}\colon U \to \Lambda^2_+ K^*$ is a submersion, since the map on tangent spaces is of maximal rank due to our definition. Since 0 is a regular value of $\bar{\omega}$, the preimage of $0 \in \Lambda^2_+ K^*$ is a submanifold of dimension equal to $(\dim(M) - \dim(\Lambda^2_+ K^*))$ in M. The idea of the proof is to show that a zero of ω_x^{n-1} corresponds to a zero of $\bar{\omega}_x$. Since the kernel of $G = \omega_p^{n-2}\nabla\omega_p$ is actually the kernel of $\nabla\omega_p$, then we can work with $\bar{\omega}$ and conclude our statement using the properties of a submersion.
Let $x \in M$ be a point where $\omega_x^n = 0$, that is we have 4-dimensional K and $\dim(\mathrm{Im}(D_K)) = 3$. In particular, the 4-dimensional kernel implies that $\omega_x^{n-1} = 0$ and $\omega_x^{n-2} \neq 0$. Furthermore, since $\ker G = \ker(\nabla\omega_p)$ we can look at ω instead of ω^{n-1}. Consider $\mathbb{R}^{2n} = K \oplus N$, where N is the complement of K. Locally, we have the following diagram

$$\begin{tikzcd}
\mathbb{R}^{2n} \arrow[r, "\omega"] \arrow[ddr, "f"'] & \Lambda^2\mathbb{R}^{2n} \arrow[d, "p"] \\
& \Lambda^2 K^* \arrow[d, "\pi"] \\
& \Lambda^2_+ K^* \simeq \mathbb{R}^3
\end{tikzcd}$$

Define $g = \pi \circ p\colon \Lambda^2\mathbb{R}^{2n} \to \mathbb{R}^3$ and $f = g \circ \omega$. The gradient $\nabla f_x = \iota \circ \nabla\omega_x$ is surjective. The map f is similar to $D_K\colon K \to \Lambda^2 K^*$ with the only difference that it is not restricted to K, but it is still surjective. In fact, we have $\ker(D_K) \subset \ker(f_*)$, as f is a more general map. Now, we want to deduce that $f^{-1}(0)$ is a submanifold of codimension 3 by showing

$$\omega_x^{n-1} = 0 \Longleftrightarrow f_x := g \circ \omega_x = 0$$

"$\Rightarrow$": Since $\omega_x = \omega_{\mathbb{R}^{2n-4}}$.
"$\Leftarrow$": Assume to the contrary that $\omega_x^{n-1} \neq 0$ (given that $g \circ \omega_x = 0$). Then by the assumptions of the definition, it follows that $\omega_x^n \neq 0$. Thus, this will also be a non-zero element in $\Lambda^2\mathbb{R}^{2n}$, and consequently also non-zero in $\Lambda^2 K^*$ and $\Lambda^2_+ K^* \simeq \mathbb{R}^3$. That is, if $\omega_x^{n-1} \neq 0$, then $\omega_x^n \neq 0$, and hence $g \circ \omega_x \neq 0$, thus contradicting the assumption. Thus, by contradiction a zero of $g \circ \omega_x$ needs to be a zero of ω_x^{n-1}.

Since $\mathrm{Im}(D_K)$ is 3-dimensional, the linearized map map on tangent spaces $f \colon \mathbb{R}^{2n} \to \mathbb{R}^3$ is surjective. Hence, $\bar{\omega} \colon \mathbb{R}^{2n} \to \Lambda^2_+ K^*$ is a submersion. By the submersion theorem [[Bre97] Theorem 7.3-7.4, pg 83-84], then $f^{-1}(0)$ is a submanifold of dimension $2n-3$.

□

Remark 3.1.5. The property of $\omega|^n_{M\setminus Z} > 0$ guarantees that the whole M^{2n} is orientable. This is due to the fact that Z is a submanifold of codimension 3. In fact, it follows from a standard algebraic topological argument that this orientability property is true on any dimension if the codimension of the submanifold is greater or equal to two. That is to say, if ω is a 2-form on a $2n$-manifold M, K is a k-dimensional submanifold of M, and $\omega^n > 0$ on $M \setminus K$, then M is oriented if $\mathrm{codim}(K) \geq 2$.

Remark 3.1.6. In dimension 4, near-symplectic structures are related to self-dual harmonic forms. An obvious obstacle in dimensions 6 and above is that there is no analogue of self-duality for 2-forms. Some local models of near-symplectic forms on 6-manifolds M^6 might indicate that near-symplectic forms could be equivalent to $\omega = *\omega^2$ for some generic metric, outside the singular locus Z.

3.2 Examples

Example 3.2.1. On $\mathbb{R}^{2n}$ with coordinates $(q_1, p_1, \ldots, q_{n-2}, p_{n-2}, x_0, x_1, x_2, x_3)$, the following 2-form is near-symplectic

$$\begin{aligned}\omega = \sum_{i=1}^{n-2} dq_i \wedge dp_i - 2x_1(dx_0 \wedge dx_1 + dx_2 \wedge dx_3) \\ + x_2(dx_0 \wedge dx_2 - dx_1 \wedge dx_3) + x_3(dx_0 \wedge dx_3 + dx_1 \wedge dx_2)\end{aligned}$$

The singular locus where $\omega^{n-1} = 0$ is given by $Z_\omega = \{p \in \mathbb{R}^{2n} \mid x_1 = x_2 = x_3 = 0\}$. We also have that $\omega^n > 0$ away from Z_ω.

Our next example is a compact manifold. For this we make use of Tischler's theorem stating that if a closed manifold M admits a non-vanishing closed 1-form, then M is a fiber bundle over S^1 [Tis70]. Let β be a closed non-vanishing 1-form on an odd-dimensional manifold N, which can be thought as a map from the fundamental group to the reals

$$\begin{aligned} \pi_1(N) &\to \mathbb{R} \\ [\gamma] &\mapsto \int_\gamma \beta \end{aligned}$$

We add very small closed 1-forms so that the induced homomorphism maps into the rationals. These closed 1-forms are so small that the new closed 1-form is still non-singular. Then we multiply everything with a rational so that the homomorphism maps into the integers. The proper reference for this argument appears in [Tis70]. In his work, Tischler shows that we can modify β to obtain a map from $\pi_1(N) \to \mathbb{Z}$, thus we are basically considering a map to the circle $N \to S^1$. This is a fibre bundle because of the Ehresmann Fibration Theorem [Ehr51].

Honda's theorem 2.2.7 tells us that closed 1-forms with indefinite Morse singularities are harmonic. This statement will be useful in the following examples.

For the next example, let $(Q, \bar{\omega})$ be a symplectic manifold and $\phi\colon Q \to Q$ a symplectomorphism. Form the mapping torus $N = Q(\phi) = Q \times [0,1]/((x,0) \simeq (\phi(x),1))$. The mapping torus is in particular a fibre bundle over S^1 and it carries a non-vanishing closed 1-form $\beta = dt$.

We can extend $\bar{\omega}$ from Q to N. There is a 2-form defined on $Q \times \mathbb{R}$. The $\mathbb{Z}$-action on this manifold given by $(x,t) \mapsto (\phi(x), t+1)$ leaves the 2-form invariant, hence it descends to the quotient. Thus, $\bar{\omega}$ is a well-defined 2-form on N that is symplectic on Q.

Example 3.2.2. Consider the $2n$-manifold $M = N \times Y$ obtained by crossing N with a closed, connected, orientable, smooth 3-manifold Y^3. Let $\alpha \in \Omega^1(Y)$ be a closed 1-form with indefinite Morse singular points, that is no maximum, nor minimum. By Calabi's and Ko Honda's theorems this form can be replaced by an intrinsically harmonic 1-form lying in the same cohomology class and having the same Morse numbers [Hon04b], [Hon99] . Thus, we may assume that $\Delta\alpha = 0$ for some Riemannian metric on Y. Equip the 2n-manifold with the following 2-form:

$$\omega = \beta \wedge \alpha + \bar{\omega} + (*_Y \alpha) \tag{3.3}$$

We can see that this 2-form is closed since each of its elements is closed. The kernel $K = \varepsilon_p \oplus T_pY$ is of dimension 4, where ε denotes the line $\ker(\bar{\omega}) \subset TN$ transverse to the fibre. The image of D_K is maximal where ω degenerates, and it iof dimension 3. For a $p \in N \times$ Crit(α), we have

$$\omega_p^n = \beta_p \wedge \underbrace{\alpha_p}_{=0} \wedge \bar{\omega}_p^{n-2} \wedge (\underbrace{*_Y \alpha_p}_{=0}) = 0$$

$$\omega_p^{n-1} = \beta_p \wedge \underbrace{\alpha_p}_{=0} \wedge \bar{\omega}_p^{n-2} + \bar{\omega}_p^{n-2} \wedge (\underbrace{*_Y \alpha_p}_{=0}) + \beta_p \wedge \underbrace{\alpha_p}_{=0} \wedge *_Y \alpha_p \wedge \bar{\omega_p}^{n-3} = 0$$

but $\omega_p^{n-2} = \bar{\omega}_p^{n-2} \neq 0$. Thus, the singular locus, the submanifold where the 2-form degenerates, is given by $Z_\omega = N \times$ Crit(α). Now we show the condition on $\nabla\omega$. For this we look on a neighbourhood U around a $p \in Z_\omega$. Locally, the Morse 1-form α can be represented as the differential df of a Morse function $f\colon U \to \mathbb{R}$. Since α is a 1-form on Y, the representing Morse function acts only on the 3 dimensions corresponding to Y with coordinates (x_1, x_2, x_3) complementary to N in M. Let $f\colon (z, x) \mapsto (0, -C_1 \cdot x_1^2 + C_2 \cdot x_2^2 + C_3 \cdot x_3^2)$ be the parametrization of the Morse function. In local coordinates, the kernel K is spanned by $\langle t, x_1, x_2, x_3 \rangle$, where t corresponds to the coordinate of the closed 1-form β. With respect to this basis, we can write a basis for $\Lambda_+^2 K^* \simeq \Lambda_+^2 \mathbb{R}^4$ with generators: $\beta_1 = dt \wedge dx_1 \wedge dx_2 + dx_2 \wedge dx_3$, $\beta_2 = dt \wedge dx_1 - dx_2 \wedge dx_3$, $\beta_3 = dt \wedge dx_3 + dx_1 \wedge dx_2$. Then, we have

$$\begin{aligned} \omega &= dt \wedge (-2C_1 \cdot x_1 dx_1 + 2C_2 \cdot x_2 dx_2 + 2C_3 \cdot x_3 dx_3) + \bar{\omega} \\ &\quad + *_Y(-2C_1 \cdot x_1 dx_1 + 2C_2 \cdot x_2 dx_2 + 2C_3 \cdot x_3 dx_3) \end{aligned}$$

That is,

$$\omega = \bar{\omega} - 2C_1(\beta_1) + 2C_2(\beta_2) + 2C_3(\beta_3)$$

If we look at the matrix of partial derivatives, denoted by $D\omega$, coming from the linearization of $\nabla\omega$ we can see that it has rank 3 at the singular points. The rows correspond to the derivatives $\partial x_1, \partial x_2, \partial x_3, \partial t, \ldots$.

$$D\omega = \begin{array}{c} \begin{array}{ccccc} \beta_1 & \beta_2 & \beta_3 & & \mathrm{Symp}_{\bar{\omega}} \end{array} \\ \left(\begin{array}{cccc|c} -2C_1 & 0 & 0 & 0 & \\ 0 & 2C_2 & 0 & 0 & \mathbf{0} \\ 0 & 0 & 2C_3 & 0 & \\ 0 & 0 & 0 & 0 & \\ \hline \vdots & \vdots & \vdots & \vdots & \mathbf{0} \\ 0 & 0 & 0 & 0 & \end{array}\right) \end{array}$$

We give one more example in dimension 6. The following example is similar to the previous one, but there is no assumption of a mapping torus of a symplectomorphism coming from a symplectic manifold. This is reflected in the near-symplectic form which is not defined with an induced symplectic form (such as $\bar{\omega}$ from the previous case).

Example 3.2.3. Let $M^6 = N^3 \times Y^3$, where N and Y are closed, connected, orientable, smooth 3-manifolds. If N fibers over S^1, we can get a nowhere vanishing closed 1-form $\beta \in \Omega^1(N)$, and it is transitive [[Far04] § 9.2]. This implies that it is intrinsically harmonic. Let $\alpha \in \Omega^1(Y)$ be a closed 1-form with indefinite Morse singular points. Equip the 6-manifold with the following 2-form:

$$\omega = \beta \wedge \alpha + (*_N \beta) + (*_Y \alpha)$$

This 2-form is closed, $\omega^3 \geq 0$ on M, and the singular locus where $\omega^2 = 0$ is at $N^3 \times \text{Crit}(\alpha)$, thus near-symplectic. Using local coordinate charts, the transversality condition on the intrinsic gradient can be seen on the (6×15)-matrix coming from the linearization $D\omega^2$ of $\nabla \omega^2$, which has rank 3 at the singular points.

Chapter 4

Generalized Broken Lefschetz Fibrations

> Es aventurado pensar que una coordinación de palabras (otra cosa no son las filosfías) pueda parecerse mucho al universo. También es aventurado pensar que de esas coordinaciones ilustres, alguna no se parezca un poco más que otras.
>
> Jorge Luis Borges, *Avatares de la tortuga*

In this chapter we will define a map analogous to a BLF in higher dimensions. We will use this map to prove theorem 1 and give examples of near-symplectic manifolds in different dimensions.

4.1 Definition

The map under consideration is a submersion with folds and Lefschetz-type singularities. A submersion is a smooth map $f\colon M \to X$ with $\dim(M) \geq \dim(X)$, such that its differential on tangent spaces is of maximal rank. For a submersion, a set of fold singularities is a set $\Sigma_f = \{p \in M \mid \text{Rank}_f = \dim(X) - 1\}$, such that $T_p\Sigma_f + \ker(f) = T_pM$. A mapping $f\colon M \to X$ is a *submersion with folds*, if it is a submersion outside the set of singularities, which are folds. A submersion with folds restricted to its fold locus is an immersion [[GG73] p87, Lemma 4.3]. Furthermore, around a point $p \in \Sigma_f$, there exists a nice coordinate neighbourhood with a parametrization described as in the next definition 4.1.1.

Submersion with folds are related to stable maps. By stable we mean the following. If f and $\tilde{f}$ are elements of $C^\infty(M, X)$, then f is equivalent to $\tilde{f}$ if there exists diffeomorphisms $g\colon M \to M$ and $h\colon X \to X$ such that the following diagram commutes

$$\begin{tikzcd} M \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "h"] \\ M \arrow[r, "\tilde{f}"] & X \end{tikzcd}$$

Then f is *stable* if there is a neighbourhood U_f of f in $C^\infty(M, X)$, such that each $\tilde{f}$ in U_f is equivalent to f [[GG73], pg 72]. Submersions with folds are stable, if f restricted to its fold set is an immersion with normal crossings [[GG73] p87, Theorem 4.4 and Ch3 §3]. An immersion with normal crossings is a mapping $g\colon W \to Y$ such that g intersects transversally Y. Moreover, the set of mappings of W to Y with normal crossings is dense in $C^\infty(W, Y)$ [[GG73], pg 82, Proposition 3.2]. Motivated by these concepts from singularity theory and by broken Lefschetz fibrations on 4-manifolds, we proceed to the following definition.

Definition 4.1.1. Let M be a smooth, $2n$-manifold M and X a smooth $(2n-2)$-manifold. By a *broken Lefschetz fibration* we mean a submersion $f\colon M \to X$ with two type of singularities:

(1) "extended" *Lefschetz-type* singularities, locally modelled by

$$\mathbb{C}^n \to \mathbb{C}^{n-1}$$
$$(z_1, \ldots, z_n) \to (z_1, \ldots z_{n-2}, z_{n-1}^2 + z_n^2)$$

These singularities are contained in codimension 4 submanifolds cross a Lefschetz singular point. Each singular fibre presents at most one singularity on each fibre. On a piece of the fibre, this can be depicted as a local cone that collapses at the origin where $z_{n-1}^2 + z_n^2 = 0$. Nearby fibres are smooth. In the local description on a piece of a fibre, the cone opens up again and it is convex.

(2) *indefinite fold* singularities, locally modeled by

$$\mathbb{R}^{2n} \to \mathbb{R}^{2n-2}$$
$$(t_1, \ldots, t_{2n-3}, x_1, x_2, x_3) \mapsto (t_1, \ldots, t_{2n-3}, -x_1^2 + x_2^2 + x_3^2)$$

The fold locus is an embedded codimension 3 submanifold, and we denote it by Σ. Singular fibres have again at most one singularity on each fibre, but this time crossing Σ changes the genus of the regular fibre by one. Throughout this work we assume that the singular fibres do not intersect each other.

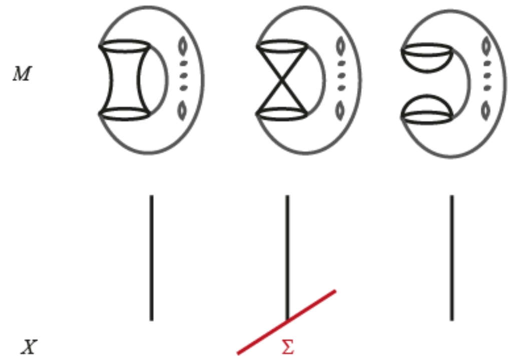

Figure 4.1: Fibres around an indefinite fold.
Left and right are regular. Middle fibre is singular.

If we consider the total space to be near-symplectic, then we will refer to the previous map as a *near-symplectic fibration*.

In the context of near-symplectic geometry, broken Lefschetz fibrations will be referred as near-symplectic fibration under the following setting.

Definition 4.1.2. Let $f\colon M \to X$ be a BLF as described above with (X, ω) being a symplectic manifold and (M, ω_{ns}) near-symplectic. If the singularity set of folds Σ of the map f corresponds to the singlar locus Z_ω of the near-symplectic form, and the symplectic form on the base pullsback to the total space under f, then we will call f a *near-symplectic fibration*.

4.2 Proof of theorem 1

Outline

The idea of the proof consists of constructing first a local near-symplectic form around the singularity set of f, and then gradually extend it to larger neighbourhoods, until it is defined on the whole total space M. In step 1, we construct a local closed 2-form τ on a piece of a singular fibre F_q around its singular point.

This form is positive on the regular part of F_q and vanishes at the singularity set Σ. Then, the 2-form is extended to the fibre. By summing up 2-forms τ_{p_k} over a finite cover of Σ and adding the pullback $f^*\omega_X$, we obtain a near-symplectic form ω_A on the tubular neighbourhood of Σ. In the second step, the 2-form ω_A is extended to the neighbourhood of the fibres. Using the cohomological assumption $\alpha|_\Sigma = \omega_X|_\Sigma \in H^2(\Sigma)$, we obtain a 2-form η on M that agrees with ω_A on the tubular neighbourhood U of the singularities of f. Then, we proceed to equip the fibres with a closed 2-form σ_F that is symplectic outside the singular point, and it is equal to η on the intersection of the tubular neighbourhood U with the fibre. Using the 2-form σ_F of the fibre, we extend the construction to a 2-form β the neighbourhood V_q of F_q. For this we use a map $\pi\colon V_q \to V_q$. With π we pullback the 2-form η, from the intersection $V_q \cap U$, and the 2-form σ_F from the fibre. One of the properties of the 2-form β is that on V_q, we have $\beta_q = \eta + d\mu_q$ for a nice 1-form μ_q, whose properties will be depicted in the proof.

In step 3, we follow the idea of Thurston to build a global near-symplectic form on M. We sum up η and $d\mu_q$ using a partition of unity over all M to build a global closed 2-form Ω. This 2-form Ω induces a near-symplectic structure on each fibre, it is positive outside Σ, and degenerates on Σ. In the fourth step, we conclude constructing the near-symplectic form $\omega_K = \Omega + K \cdot f^*\omega_X$ by adding the pullback and multiplying it by a positive constant K to preserve positivity on the horizontal spaces.

Step 1: Constructing the local 2-form

First we want to define the local near-symplectic form near the singular sets $\Sigma \sqcup C$, where Σ denotes the singularity set of folds and C the set of extended Lefschetz-type singularities. We begin by defining a singular symplectic form vanishing at Σ, and then we pull back the symplectic form of the base. Consider the local model $\tilde{f}\colon (z_1, \dots, z_{2n-3}, x) \mapsto (z_1, \dots, z_{2n-3}, -x_1^2 + \frac{1}{2}(x_2^2 + x_3^2))$ arounda fold point $p \in \Sigma$ of index 1. Since the fibres are 2-dimensional, we can take a similar local model as the near-symplectic forms on 4-manifolds. Define the following 2-form on a piece of the tubular neighbourhood of Σ containing p:

$$\tau_p = d\left(\chi(z)x_1(x_2dx_3 - x_3dx_2)\right) \tag{4.1}$$

This 2-form is closed, vanishes at the singularity set, is non-degenerate outside Σ, and evaluates positive on the fibres. The positivity of τ_p on the fibres of f can be checked by integrating τ_p over a piece of the fibre, where τ_p is defined (see Appendix A.1.1). The map $\chi(z)$ is a smooth cut-off function depending on coordinates $(z_1, \ldots, z_{2n-3})$ on Z. This cut-off function that will help us in the gluing process when summing up the 2-forms τ_{p_i} to build a local 2-form on the whole tubular neighbourhood of Σ. Sum up the forms τ_{p_k} over a finte cover of Σ, and pullback the symplectic form from the base. We obtain

$$\omega_A = \sum_{p_k} \tau_{p_k} + f^*\omega_X \tag{4.2}$$

This closed 2-form is defined on the tubular neighbourhood of Σ. It is non-degenerate outside Σ and posiitve on the fibres. At the degenaracy points, $K_p = N_p\Sigma \oplus \varepsilon_p$ is of dimension four, where $\varepsilon = \ker(f^*\omega_X) \subset T\Sigma$.

Around the elements of C, we are in the situation of a Lefschetz fibration, and we proceed as in [[GS99] Thm 10.2.18 pg 401-403]. For any $v_1, v_2 \in T_pF$, we get $\omega_A|_{B_k}(v_1, v_2) > 0$ away from the singularity (see Appendix A.1.2). The symplectic form $\omega_A|_{B_k}$ can be extended to the fibre F_q as a symplectic form for all $q \in f(B_k) \subset X$.

Step 2: Extension over the neighbourhoods of the fibres

In this step we want to construct local 2-forms on the neighbourhood of the fibres. We extend the 2-form to a local model over the neighbourhood of the fibres, such that it agrees with ω_A near $\Sigma \sqcup C$. Let U be the tubular neighbourhood of $\Sigma \sqcup C$. Choose a closed 2-form $\zeta \in \Omega^2(M)$ with a class being represented by α. Since $\alpha|_\Sigma = \omega_X|_\Sigma \in H^2(\Sigma)$, over U there exists a 1-form $\bar{\mu} \in \Omega^1(U)$, such that $\omega_A - \zeta = d\bar{\mu}$. We extend now $\bar{\mu}$ to an arbitrary 1-form on the manifold, $\mu \in \Omega^1(M)$, supported in a neighbourhood W of U. By substituting $\eta = \zeta + d\mu$ on U, we can regard η to be ω_A when restricted to U.

By assumption, we have a positive pairing, $\langle \alpha, F \rangle > 0$ over each component of the fibre, $[\eta] = \alpha$, and the fibres have a symplectic form σ_F. We equip the fibres with a closed singular 2-form σ_q such that

(a) $\sigma_q|_{F_q \cap U_1} = \eta$, that is, restricted to U, σ is near-symplectic, since $\eta|_U = \omega_A$. The form σ_q is defined on the fibre, so $\sigma_q|_{F_q \cap U_1}$ is near-symplectic.

(b) $\sigma_q|_{F_q}$ is positive over the smooth part of F_q, since ω_A takes care of piece close to the singularities and the regular part by the area form of the 2-dimensional fibres.

(c) $\int_F \sigma_q = \langle \alpha, F \rangle > 0$, since $[\sigma_q - \eta|_{F_q}] = 0$ in $H^2(F_q, F_q \cap U_1) \overset{PD}{\simeq} H_0(F_q, F_q \cap U_1) \simeq 0$, assuming F_q connected. Then $(\sigma_q - \eta|_{F_q})$ is exact in $F_q \cap U_1$, that is $[\sigma_q] = [\eta] = \alpha$.

Now, we describe some properties of the neighbourhood of the fibres in order to extend the 2-form. For any $q \in X$ we can find a tubular neighbourhood V_q of the fibre F_q and neighbourhoods $U_2 \subset U_1 \subset U$ of the fold singularity set Σ. A $q \in X$ can be engulfed by an m-disk D^m. Around a fibre F_q, take $f^{-1}(D^m) = V_q$. As in [ADK05], after removing a small neigbourhood of the critical set, we have that $V_q \setminus (V_q \cap U_2)$ is diffeomorphic to $D^{m-1} \times (F_q \setminus (F_q \cap U_2))$. This follows from Ehresmann theorem, since we have a nice smooth map locally without critical points. To extend the 2-form on the neighbourhood the fibre, we build a smooth map

$$\pi \colon V_q \to V_q$$

by interpolating between two maps:

(i) Close to the singular point of the fibre inside the neighbourhood $V_q \cap U_1$, we use the identity map, so that π is $\mathrm{id}_{V_q \cap U_1}$. Since V_q is a neighbourhood of a fibre F_q and $V_q \cap U_1$ retracts to $F_q \cap \Sigma$, we want that π maps down to the piece of the fibre close to the singularity together with the intersection of the neighbourhoods V_q and U_1. That is,

$$\mathtt{Im}(\pi) \subset F_q \cup (V_q \cap U_1)$$

(ii) Farther away from the singular region, that is on the smooth part $F_q \setminus (F_q \cap U_2)$, we use the projection map $\mathtt{pr} \colon V_q \setminus (V_q \cap U_2) \to F_q \setminus F_q \cap U_2$ that comes from the product structure.

We use the map π to construct a near-symplectic form β on V_q. With π, we pull back the 2-form η on $V_q \cap U_1$ and the 2-form σ_q on F_q

$$\beta = \pi^* \sigma_q \, + \, \pi^* \eta$$

This 2-form has the following features:

1. $d\beta = 0$ and $[\beta] = \alpha|_{V_q}$
2. $\beta|_{V_q \cap U_2} = \eta$
3. there exists a 1-form μ_q on V_q, such that $\beta - \eta = d\mu_q$, since $[\beta - \eta] = 0$ in $H^2(V_q, V_q \cap U_2) \simeq H^2(F_q, F_q \cap U_2)$. Thus, on V_q

$$\beta_q = \eta + d\mu_q$$

4. $\beta_q|_{F_q} > 0$ restricts positively to the fibre for every regular point $q \in V_q$.

Step 3: Patching into a global form

We expand the near-symplectic form over the whole manifold M. Since our base is compact, we can find a finite subset $Q \subset X$ and choose a finite cover $\mathcal{D}$ with open subsets $(D_q)_{q \in Q}$, such that $f^{-1}(D_q) \subset V_q$ for each $q \in X$. Consider a smooth partition of unity $\rho\colon X \to [0,1]$, $\sum_{q\in Q} \rho_q = 1$, subordinate to the cover $\mathcal{D}$ with $\mathtt{supp}(\rho_q) \subset D_q$. We build a global 2-form Ω on M by patching the local 1-forms μ_q previously defined on V_q. Thus, we define the following closed 2-form

$$\Omega = \eta + d\left(\sum_{q\in Q} (\rho_q \circ f)\mu_q\right) \tag{4.3}$$

Since f is constant on the fibres, the 1-form $d((\rho_q \circ f)\mu_q) = 0$ when evaluated on the vectors tangent to the fibre. From the second step, η agrees with ω_A when restricted to U. Thus, the first summand takes care of the part near the critical set. Let $\bar{U}_2$ be the intersection of all neighbourhoods U_2 for all $q \in Q$, that is, $\bar{U}_2 = U_2 \bigcap_{q\in Q} f^{-1}(D_q)$. The global form Ω agrees with η when restricted to $\bar{U}_2$, so it agrees with the local model of ω_A at U_2. Thus, Ω is globally well-defined over M.

The 2-form Ω restricts to a fibre F_q in the following way

$$\Omega|_{F_q} = \eta|_{F_q} + \textstyle\sum_{q\in Q} \rho \circ f(p) d\mu_q|_{F_q} = \sum_{q\in Q} \rho \circ f(p)(\eta + d\mu_q)|_{F_q}$$
$$= \textstyle\sum_{q\in Q} (\rho \circ f(p))\, \beta_q|_{F_q}$$

This is a convex combination of near-symplectic 2-forms. On each fibre, Ω is closed, positive outside the singular locus, and degenerates at Σ, inducing a symplectic structure on each fibre outside the singularities.

Step 4: Positivity on vertical and horizontal tangent subspaces

To conclude the global construction, we can apply a similar argument as in the symplectic case [Thu76]. The 2-form Ω is positive on the vertical tangent subspaces to the fibre $\text{Ver}_p = \ker df(p) = T_pF \subset T_pM$, outside the singularity set. To guarantee positivity on the horizontal spaces, we multiply the pullback from the symplectic form of the base by a sufficiently large real number $K > 0$ to obtain the 2-form

$$\omega_K = \Omega + K \cdot f^*\omega_X \tag{4.4}$$

If we restrict ω_K to the vertical tangent subspaces to the fibre, it agrees with Ω. The 2-form ω_K defines a near-symplectic structure on M. $\square$

4.3 Local Compatibility of Fold and Darboux Charts

This section describes a local representation of the the standard symplectic form using standard parametrization of folds. We point out that this section is independent of the thereom's proof. For purposes of local models, we notice that charts of folds and Darboux charts present some compatibility. In other words, locally we can pullback the symplectic form of the base, with the standard representation coming from a Darboux chart, via the standard parametrization of the indefinite fold singularities. Inspired by Arnol'd, the following lemma guarantees that the normal forms of the fold singularity can be pullbacked symplectically [[Arn89] Theorem B, pg 230].

Lemma 4.3.1. *Let $p \in \Sigma$ be a point of a submanifold σ of codimension 1 embedded in $(\mathbb{R}^{2n}, \omega_{\text{st}})$. There is a symplectomorphism $\phi\colon \Sigma \subset \mathbb{R}^{2n} \to \tilde{\Sigma} \subset \mathbb{R}^{2n}$ mapping Σ to $\tilde{\sigma} \subset \mathbb{R}^{2n-1} \times 0 \subseteq \mathbb{R}^{2n}$.*

Proof. This proof follows the lines of [[Arn89] Appendix B, pg 230-232]. Let Σ be represented as a graph in $(\mathbb{R}^{2n}, (y_1, \dots y_{2n}))$, with the standard symplectic form ω_{st}

$$\Sigma = \{(y_1, \dots, y_{2n-1}, h(y_1, \dots, y_{2n-1})\}$$

We proceed now to the construction of a symplectomorphism $\mathbb{R}^{2n} \to \mathbb{R}^{2n}$ sending the graph Σ to $\mathbb{R}^{2n-1} \times 0$. Define the first coordinate of the smooth map to be the projection to the $(2n-1)$-th coordinate,

$$p_1\colon (y_1, \dots, y_{2n}) \mapsto y_{2n-1}$$

Its Hamiltonian vector field $V_{p_1} = -\frac{\partial}{\partial y_{2n}} = (0, \ldots, 0, -1)$ can be easily modified to be transverse to Σ. Take a point $a = (y_1, \ldots, y_{2n-1}, h(y)) \in \Sigma$. Consider the Hamiltonian flow $V_{p_1}^t$. Move the point a along this flow until it reaches the point $V_{p_1}^t(a) = (y_1, \ldots, h(y) - t) := z$ in time t. Define the second coordinate q_1 as a function of the point $z = V_{p_1}^t(a)$ under the action of the Hamiltonian flow $V_{p_1}^t$, which measures the time taken from a to z. More precisely, we are defining q_1 as

$$q_1 \colon (y_1, \ldots, y_{2n}) \mapsto y_{2n} - h(y) = t$$

On the hypersurface we have $y_{2n} = h(y)$, thus $q_1|_\Sigma = 0$. The derivative of q_1 in the direction of the vector field V_{p_1} is equal to 1. Thus, the Poisson bracket of the constructed coordinates p_1 and q_1 is one:

$$\{q_1, p_1\} = 1$$

The Hamiltonian vector field of q_1 is:

$$V_{q_1} = \left(-\frac{\partial h}{\partial y_2}, \frac{\partial h}{\partial y_1}, -\frac{\partial h}{\partial y_4}, \frac{\partial h}{\partial y_3}, \ldots, 1, \frac{\partial h}{\partial y_{2n-1}} \right)$$

It lies in $T_a\Sigma$. The remaining coordinates are defined as: $p_i \colon (y_1, \ldots, y_{2n}) \mapsto y_{2i-3}$ and $q_i \colon (y_1, \ldots, y_{2n-1}) \mapsto y_{2i-2}$, with the corresponding Hamiltonian vector fields being $V_{p_i} = -\frac{\partial}{\partial y_{2i-2}}$ and $V_{q_i} = \frac{\partial}{\partial y_{2i-3}}$. Since the derivative of p_i in the direction of the Hamiltonian vector field V_{q_i} is 1, then the Poisson bracket is

$$\{p_i, q_i\} = 1$$

Furthermore, V_{p_i} and V_{q_i} do not act in the direction of any of the remaining $2n - 2$ coordinates. Thus, neither the coordinates (p_i, p_j), (q_i, q_j), nor (p_i, q_j) commute with each other. We have

$$\{p_i, p_j\} = 0 \quad , \quad \{q_i, q_j\} = 0 \quad , \quad , \{p_i, q_j\} = 0 \qquad (i \neq j)$$

Hence, $\{p_1, q_1, \ldots, p_n, q_n\}$ form a symplectic basis in which $\tilde{\omega} = \sum dp_i \wedge dq_i$. With these coordinates we obtain a symplectomorphism $\phi \colon (\mathbb{R}^{2n}, \omega_{st}) \longrightarrow (\mathbb{R}^{2n}, \tilde{\omega})$ sending $\Sigma = (y_1, \ldots, y_{2n-1}, h(y))$ to $\Sigma_0 = (\bar{y}_1, \ldots, \bar{y}_{2n-1}, 0) \simeq U \subset \mathbb{R}^{2n-1} \times 0$ $\square$

Remark 4.3.2. In the remaining part of this section, we show how the standard symplectic form can be pullbacked via a standard fold map.

Let $\tilde{f}\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n-2}$ be the standard fold map. Choose contractible open subsets U_k in a finite cover $\{U_k\}_k$ of the image of Σ in X, and let $\varphi_k\colon U_k \to \mathbb{R}^{2n-2}$ be Darboux charts so that $\left(\varphi_k^{-1}\right)^* \omega_X = \omega_{\mathbb{R}^{2n-2}}$. The maps φ_k could modify the local parametrization of the folds coming from $\tilde{f}$ and the position of the critical value set. Nevertheless, this can be fixed. We say now a word of how this can be done.

Start with $f(p) \in X$ for $p \in \Sigma$. Let $m = 2n-2$. Choose standard coordinates $(y_1, \dots, y_m)$ in $\mathbb{R}^m$, near $\tilde{f}(p)$ such that $\omega_{st} = dy_1 \wedge dy_2 + \cdots + dy_{m-1} \wedge dy_m$. Around a point $p \in \Sigma$, we have $\mathtt{Rank}_f(p) \geq m-1$. By the rank theorem we can find coordinates such that $\tilde{f}\colon (t,x) \mapsto (t_1, \dots, t_{m-1}, h(t,x))$. A more detailed explanation of this fact appears in [[Tu11] Appendix B, pg 343-344]. To obtain the nice representation of the folds singularities, we need that the image of Σ is described by $h(t,x) = 0$. At the moment, the image of Σ sits as a graph inside $\mathbb{R}^m$, but not necessarily looking as $\mathbb{R}^{m-1} \times 0$. Denote by S the image of Σ in $\mathbb{R}^m$ moved by φ. By the previous lemma, we can modify S symplectomorphically so that $\Sigma \simeq \mathbb{R}^{m-1} \times 0 \subset \mathbb{R}^m$.

$$\begin{tikzcd}
M^{2n} \supset V \arrow[r, "f"] \arrow[d, "\psi"'] & \tilde{V} \subset X^m \arrow[d, "\nu"'] & U_k \supset f(\Sigma) \arrow[l, hook', "i"'] \arrow[dl, "\varphi_k"] \\
\mathbb{R}^{2n} \arrow[r, "\tilde{f}"] \arrow[d, "\zeta"'] & \mathbb{R}^{2n-2} \arrow[loop below, "\phi"] & \\
\mathbb{R}^{2n} \arrow[ur, "\pi"'] & &
\end{tikzcd}$$

The line of reasoning is now a fold version of the Morse Lemma [[GG73] Proof of Theorem 4.5, pg 89-90]. The restriction $f|_\Sigma\colon \Sigma \to (t_1, \dots, t_{m-1}, 0)$ is a local diffeomorphism near p. Thus, we can choose coordinates in the domain so that Σ is defined near p by $x_1 = x_2 = x_3 = 0$. By the properties of the singular set, Σ is also described by $\frac{\partial h}{\partial x_i}(0) = 0$, and $\frac{\partial^2 h}{\partial x_i^2}(0) \neq 0$. After applying the symplectomorphism ϕ on $\mathbb{R}^m$, S is described by $y_m = 0$. Thus, we get $h(t,x) = 0$, when $x_1 = x_2 = x_3 = 0$, and we can express h as

$$h(t,x) = \sum_{1 \leq i,j \leq 3} b_{ij}(x) x_i x_j$$

where $b_{ij}(x)$ are smooth functions. Moreover, the (3×3)-matrix $b_{ij}(0)$ is nonsingular. This means that at f_m, on the m-th component of the map f being equal to $h(t,x)$, we are in a situation of a function with a non-degenerate critical point. We only need to perform ú coordinate changes in the domain $\zeta\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n}$ to obtain the normal representation of the folds [[GG73] Theorems 4.4,4.5]. For the sake of clarity, we can assume that this change of coordinates has already occurred and $\tilde{f}$ has the nice parametrization. All of this happens while preserving the shape of the symplectic form. The pull back $\psi^*\left(\tilde{f}^*\left((\varphi_k^{-1})^*\omega_X\right)\right) = f^*\omega_X$ looks locally like $\tilde{f}^*\omega_{\mathbb{R}^m}$, and drops to rank $(2n-4)$ at the points of Σ. By summing up the forms τ_{p_k} over a finite cover of Σ together with the pullback, $\omega_A = \sum_{p_k}\tau_{p_k} + \psi^*\left(\tilde{f}^*\left((\varphi_k^{-1})^*\omega_X\right)\right)$, we obtain the local model

$$\omega_A = \tau + f^*\omega_X \tag{4.5}$$

This 2-form defines the near-symplectic form on the tubular neighbourhood of Σ. It is closed, non-degenerate outside Σ, and positive on the fibres. Moreover, $\omega_A^n|_p = \omega_A^{n-1}|_p = 0$ for all $p \in \Sigma$, that it degenerates to rank $2n-4$. At the degeneracy points we have a four dimensional kernel $K_p = N_p\Sigma \oplus \varepsilon_p$, where $\varepsilon = \ker(f^*\omega_X) \subset T\Sigma$.

Remark 4.3.3. Even though, lemma 4.3.1 together with step 1 tell us that the fold map pulls back symplectomorphically, other types of singularities might need a different treatment. For instance, if we would like to consider deformations of near-symplectic fibrations, in a similar fashion as Lekili [Lek09], then it would be necessary to consider all stable singularities of maps from $\mathbb{R}^{2n}$ to $\mathbb{R}^{2n-2}$. For maps going from a 6 dimensional source to a 4 dimensional target, there are 4 stable singularities: folds, cusps, swallowtails, and butterflies [[AGZV12] (pg 46-49), [Mar82] (pg 156)]. For higher dimensions the list becomes longer and more complicated.

4.4 Examples

4.4.1 Pullback bundle

We can obtain examples of near-symplectic manifolds and near-symplectic fibrations via a pullback bundle construction. Let M and X be oriented, closed manifolds of dimension $(2n-2)$, and B an oriented, closed, connected manifold of dimension $(2n-4)$. Furthermore, let f and g be smooth mappings, and consider the pullback $W = \{(x,m) \in X \times M \mid f(x) = g(m)\}$.

$$\begin{array}{ccc} W & \xrightarrow{\tilde{f}} & M \\ {\scriptstyle \tilde{g}}\downarrow & & \downarrow{\scriptstyle g} \\ X & \xrightarrow[f]{} & B \end{array}$$

Before going to the near-symplectic case, we briefly comment on the symplectic one. The work of Thurston provides a starting point. By a compact symplectic fibration it is understood a fibration whose fibres are compact symplectic manifolds and the transition functions are symplectomorphisms [[MS98] Ch 6.1]. A theorem from Thurston tells us, that if g is a compact symplectic fibration over a closed connected symplectic manifold B, and there is a class $\alpha \in H^2(M)$ such that $\iota^*\alpha = [\sigma_b] \forall b \in B$, where $\sigma_b \in \Omega^2(F_b)$ is the canonical form of the fibre, then M is symplectic [[Thu76], [MS98] Theorem 6.3, pg 199]. Assuming that we have such a class, then we can pullback this information to W via $\tilde{f}$. We obtain a class $\tilde{\alpha} = \tilde{f}^*\alpha \in H^2(W)$ with the same property. Thus, we only need X to be symplectic in order that W is a symplectic manifold via the induced map $\tilde{g}$. Now we discuss the near-symplectic scenario.

Throughout these examples we assume that the critical set of g form regular points for f, so that f behaves like a bundle near the critical sets by Ehresmann theorem (whenever there is a critcal set for g). The first example follows from theorem 1. If g is a BLF (thus $\tilde{g}$ a generalized BLF), and X is symplectic, then W is near-symplectic via $\tilde{g}$ assumming that the cohomological condition of theorem 1 is satisfied. A second case appears when the base X is near-symplectic. Keeping a "vertical" view of the diagram, now we do not consider g and $\tilde{g}$ to be BLFs. The following proposition explains this situation.

Proposition 4.4.1. *Let $g\colon M \to B$ be a compact symplectic fibration with symplectic total space M and (X, ω_X) a closed, near-symplectic manifold over a closed connected symplectic base B of codimension 2. Let W be the pullback bundle as defined in the previous parragraph. Then, W carries a near-symplectic structure induced by $\tilde{g}\colon W \to X$.*

Proof. Let Γ be the singular locus of ω_X, that is a codimension 3 submanifold in X. Its preimage under $\tilde{g}$ is a surface bundle over Γ, and we will denote by Z its total space. This bundle will become the singular locus of the near-symplectic form of W. Let $\mathcal{U}$ be the tubular neighbourhood of Γ and let $E = \tilde{g}^{-1}(\mathcal{U})$. E is a surface bundle. We will also consider a small tubular neighbourhood $\bar{E}$ inside E.

Now we construct a closed 2-form $\tilde{\eta} \in \Omega^2(W)$ such that it is positive on the fibres of $\tilde{g}$ in W and whose wedge power $\tilde{\eta}^{n-1}$ is zero on E. Since g is a symplectic fibration, we have a cohomology class $\alpha \in H^2(M)$ that pairs pairs positively with the fibre class. We choose $\tilde{\eta}$ such that $[\tilde{\eta}] = \tilde{f}^*\alpha \in H^2(W)$ with $\iota^*\tilde{\alpha} = \tilde{f}^*[\sigma]$. Secondly, as $\bar{E}$ and E are cohomologically $(2n-3)$-dimensional, we can select $\tilde{\eta}$ with the property that $\tilde{\eta}^{n-1}|_{\bar{E}} = 0$.

Let U_k be contractible open subsets of a cover of B with trivializations ϕ_k, such that $\phi_k \circ \phi_j^{-1}$ are symplectomorphisms over $U_k \cap U_j$. We bring these neighbourhoods to W as $(\tilde{g} \circ f)^{-1}(U_k) = \tilde{U}_k$. Define $\psi_k := (\text{proj} \circ \tilde{\phi}_k \circ \tilde{f})\colon \tilde{U}_k \to F$. Over $\tilde{U}_k$ there is a 1-form μ_k such that $d\mu_k = \psi^*\tilde{\sigma}_k - \tilde{\eta}_k$, since $[\tilde{\eta}] = \tilde{f}^*|_F(\alpha) = [\psi^*\tilde{\sigma}]$.

The rest of the proof follows similarly as in step 3 and 4 of theorem's 1 proof. Choose a partition of unity $\rho\colon W \to [0,1]$ in such a way that its open subsets do not touch $\bar{E}$, and with it define a closed 2-form $\beta = \tilde{\eta} + \sum_k \rho_k d\mu_k$ on W. This form has the properties that: $\beta|_{\bar{E}} = \tilde{\eta}|_{\bar{E}}$ and $\beta|_F = \sigma_b$, where σ_b is the form of the fibre F_b. Finally, we build up our global form by adding $\tilde{g}^*\omega_X$. If K is a sufficiently large positive real number, then we have a closed 2-form ω_K, which is non-degenerate away from Z

$$\omega_K = \beta + K \cdot \tilde{g}^*\omega_X$$

□

4.4.2 Near-symplectic manifolds coming from BLFs

Broken Lefschetz fibrations provide also ways to obtain near-symplectic fibrations on $2n$-manifolds over near-symplectic $(2n-2)$-manifolds. Let $g\colon M \to B$ be a BLF as defined previously with singular fold set $\Sigma_{\tilde{g}}$, where M is near-symplectic of $\dim(M) \geq 4$ and B is a closed, connected, symplectic manifold of $\dim(B) \geq 2$. Furthermore, consider (X, ω_X) to be a symplectic manifold of $\dim(X) \geq 4$. Assume that there is a class $\alpha \in H^2(M)$ such that $\langle \alpha, F \rangle > 0$ and $\tilde{\alpha}|_{\Sigma_{\tilde{g}}} = \omega_X|_{\Sigma_{\tilde{g}}}$. Then, W is near-symplectic via a generalized BLF $\tilde{g}$.

If both $f\colon X \to B$ and $g\colon M \to B$ are two BLFs, then we require the intersection of their critical images to be transversal in B, but not necessarily disjoint. In that case, it follows from standard differential topology that W is a $2n$-dimensional manifold (see the remark at the end of this section). The maps $\tilde{f}$ and

$\tilde{g}$ become near-symplectic fibrations, carrying the same type and number of fold and Lefschetz-type singularities as f and g respectively. Around a critical point in f^*M, the maps $\tilde{f}$ and $\tilde{g}$ are locally modelled by coordinate charts φ and π defined as

$$\begin{aligned}
\varphi\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n-2} \qquad &, \quad (r_1, \dots, r_{2n}) \mapsto (r_1^2 + r_2^2 - r_3^2, r_4, \dots, r_{2n}) \\
\pi\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n-2} \qquad &, \quad (r_1, \dots, r_{2n}) \mapsto (r_1, \dots, r_{2n-3}, -r_{2n-2}^2 + r_{2n-1}^2 + r_{2n}^2)
\end{aligned}$$

Assume the cohomological condition on the class $\tilde{\alpha} \in H^2(W)$ as in theorem 1. Denote by Γ the singular locus of ω_X in X, and by Σ, the set of fold singularities of $\tilde{g}$. The mapping $\tilde{g}$ becomes a a near-symplectic fibration over a near-symplectic base (X, ω_X), if $\tilde{g}^{-1}(\Gamma) \not\subset \Sigma$ in W. This construction gives 2 generalized BLFs, one for each pullback mapping.

Remark 4.4.2. To see that W is a 2n-dimensional manifold, we write an explicit chart for W using $2n$ coordinates and show that $\tilde{g}$ is a bundle projection map. Following [[Bre97] § 14, pg 111-114], let $\phi\colon U \times F \to g^{-1}(U)$ be a chart over the open set $U \subset X$. To see that $\tilde{g}$ is a projection we want to show that $\tilde{g}(\phi(b, y)) = b$ for all $b \in U$ and $y \in F$. Define

$$\begin{aligned}
\psi\colon f^{-1}(U) \times F &\longrightarrow (f \circ \tilde{g})^{-1}(U) \\
(v, y) &\longmapsto (v, \phi(f(v), y))
\end{aligned} \tag{4.6}$$

Since $\tilde{g}\colon W \to X$ is given by $\tilde{g}(a, x) = a$ and $\tilde{f}\colon W \to M$ by $\tilde{f}(a, x) = x$, then we obtain $g\phi(f(v), y) = f(v) \in U$ and $f\tilde{g}(v, \varphi(f(v), y)) = f(v)$. The inverse of the chart ψ is defined as

$$\begin{aligned}
\lambda := \psi^{-1}\colon (f \circ \tilde{g})^{-1}(U) &\longrightarrow f^{-1}(U) \times F \\
(a, x) &\longmapsto (a, p_F\phi^{-1}(x))
\end{aligned}$$

for $g(x) = f(a) \in U$, and where $p_F\colon U \times F \to F$ is the projection onto F. To check this, we calculate

$$\lambda \circ \psi(v, y) = \lambda(v, \phi(f(v), y)) = (v, p_F\phi^{-1}[\phi(f(v), y)]) = (v, p_F(f(v), y)) = (v, y)$$

and noticing that $f(a) = p(x)$, then we check the other direction

$$\begin{aligned}
\psi \circ \lambda(a, x) &= \psi(a, p_F\phi^{-1}(x)) = (a, \phi[f(a), p_F\phi^{-1}(x)]) = (a, \phi[p(x), p_F\phi^{-1}(x)]) \\
&= (a, \phi[p_U\phi^{-1}(x), p_F\phi^{-1}(x)]) = (a, \phi\phi^{-1}(x)) = (a, x)
\end{aligned}$$

Now we can conclude that $\tilde{g}$ is a bundle map with fibre F. Coming back to our particular example of a pullback back with broken Lefschetz fibrations, we give the explicit charts of the around a fold singularity. The maps $\tilde{f}$ and $\tilde{g}$ become near-symplectic fibrations, carrying the same type and number of fold and Lefschetz-type singularities as f and g respectively. Around a critical point in f^*M, the maps $\tilde{f}$ and $\tilde{g}$ are locally modelled by coordinate charts φ and π respectively defined as

$$\begin{aligned}
\varphi\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n-2} \qquad &, \quad (r_1, \dots, r_{2n}) \mapsto \underbrace{(-r_1^2 + r_2^2 + r_3^2, r_4, \dots, r_{2n})} \\
&\qquad\qquad\qquad = (u_{2n-5}, u_1, \dots, u_{2n-6}, y_1, y_2, y_3) \\
\pi\colon \mathbb{R}^{2n} \to \mathbb{R}^{2n-2} \qquad &, \quad (r_1, \dots, r_{2n}) \mapsto \underbrace{(-r_1, \dots, r_{2n-3}, +r_{2n-2}^2 + r_{2n-1}^2 + r_{2n}^2)} \\
&\qquad\qquad\qquad = (x_1, x_2, x_3, t_2, \dots, t_m, t_1)
\end{aligned}$$

The coordinate charts of f are given by

$$\mathbb{R}^{2n-2} \to \mathbb{R}^{2n-4}$$

$$(r_1, \dots, r_{2n-3}, -r_{2n-2}^2 + r_{2n-1}^2 + -r_{2n}^2) \mapsto (-r_1^2 + r_2^2 + r_3^2, r_4, \dots, r_{2n-3}, -r_{2n-2}^2 + r_{2n-1}^2 + -r_{2n}^2)$$

corresponding to

$$(x_1, x_2, x_3, t_2, \dots, t_m) \mapsto (t_1, \dots, t_m, -x_1^2 + x_2^2 + x_3^2$$

and the coordinates of g by

$$\mathbb{R}^{2n-2} \to \mathbb{R}^{2n-4}$$

$$(-r_1^2 + r_2^2 + r_3^2, r_4, \dots, r_{2n}) \mapsto (r_1^2 + r_2^2 - r_3^2, r_4, \dots, r_{2n-3}, -r_{2n-2}^2 + r_{2n-1}^2 + -r_{2n}^2)$$

corresponding to

$$(u_{2n-5}, u_1, \dots, u_{2n-6}, y_1, y_2, y_3) \mapsto (-y_1^2 + y_2^2 + y_3^2, u_1, \dots, u_{2n-5})$$

The corresponding chart ψ as in 4.7 can be defined as

$$\begin{aligned}
\psi\colon f^{-1}(U) \times F &\longrightarrow (f \circ \tilde{g})^{-1}(U) \qquad (4.7) \\
((x_1, x_2, x_3, t_2, \dots, t_m), (y_1, y_2, y_3)) &\longmapsto ((x_1, x_2, x_3, t_2, \dots, t_m), \phi(f(x,t), y))
\end{aligned}$$

We now do a small computation to check that the $2n$-coordinates $(x,t), y)$ hit all $(4n-4)$-coordinates $(x, m) \in X \times M$

$$\begin{aligned}
\psi((x,t), y) &= ((x,t), \phi[f(x,t), y]) \\
&= ((x_1, x_2, x_3, t_2, \dots, t_m), \phi[f_1(x,t), \dots, f_{2n-4}(x,t), y_1, , y_2, y_3]) \\
&= ((x_1, x_2, x_3, t_2, \dots, t_m), \phi[t_1, u_1, \dots, u_{2n-5}, y_1, , y_2, y_3])
\end{aligned}$$

Here, we are using $f_1(x,t) = -y_1^2+y_2^2+y_2^2 = t_1$, $f_2(x,t) = t_2 = u_1, \ldots, f_{2n-5}(x,t) = t_m = u_{2n-6}$, $f_{2n-4}(x,t) = -x_1^2+x_2^2+x_3^2 = u_{2n-5}$

4.5 Lemma for folds a la Morse

In this section we want to show an adaptation of Nicolaescu's proof of the Morse lemma for the situation of fold singularities [[Nic11] Ch1.1 pg 12-17]. This lemma is of a pure differential topological nature and does not concern the proof of theorem 1. It could be a useful tool when working with broken Lefschetz fibrations.

Lemma 4.5.1. *Let $\varphi\colon Z\times\mathbb{R}^3 \to Z\times\mathbb{R}$ be a map with fold singularity, and $p=(z,0)$ a non-degenerate critical point of $\varphi_z\colon \mathbb{R}^3\to\mathbb{R}$ of index λ for a fixed z. Then there is a neighbourhood U covering the whole $Z\times\mathbb{R}^3$ with coordinates $(t_1,t_2,t_3,x_1,x_2,x_3)$ such that $x_1(0)=x_2(0)=x_3(0)=0$ and $\varphi(t,x)=\varphi(p)+H_z^{\varphi}(x)$, where $H_z^{\varphi}(x)$ is a quadratic polynomial varying in z.*

Proof. Let $\varphi\colon Z\times\mathbb{R}^3\to Z\times\mathbb{R}$ be a local representation of the previous map, and φ_s a 1-parameter family with $s\in[0,s]$ defined for a fixed $z\in Z$ by

$$\varphi_s(z,x)|_z = \varphi_{s,z}(x) = (1-s)\varphi_z(x) + sQ_z(x)$$

where $Q_Z = \frac{1}{2}\sum\frac{\partial\varphi_z}{\partial x_i\partial x_j}(z,0)x_ix_j$ over a point $z\in Z$. Let $\psi_s\colon Z\times U\to Z\times\mathbb{R}^3$ be a s-parameter path of embeddings, where U is a neighbourhood of a point $(z,0)$ such that on U

$$\psi_s(z,0)|_z = (z,0) \quad , \quad \varphi_{s,z}\circ\psi_{s,z}(x) = \varphi|_z(x) \tag{4.8}$$

This path of embeddings is determined by a s-dependent vector field $V_s(z,x) = \frac{d}{ds}\psi_s(z,x)$. Working over a point $z\in Z$ and differentiating with respect to s, we obtain

$$\dot{\varphi}_{s,z}\circ\psi_{s,z}(x) + (V_{s,z}(x)\,\varphi_{s,z})\circ\psi_{s,z} = 0 \iff Q_z-\varphi_z = V_{s,z}\varphi_z \tag{4.9}$$

on $\psi_s(U), \forall s\in[0,1]$. For every $z\in Z$, we want to have a ψ_s varying smoothly along Z. Thus, we look for a vector field to obtain ψ_s satisfying:

1. $V_s(z,0)=(z,0) \qquad \forall s\in[0,1]$ and for a fixed z

2. $Q_z - \varphi_z = V_{s,z}\varphi_z$ on a neighbourhood A of $(z,0)$

and parametrized for each $z \in Z$ such that it varies smoothly for every $z \in Z$. Since Z is compact, we can integrate V_s to an isotopy $\psi_{s,z}\colon \mathbb{R}^3 \to \mathbb{R}^3$ without touching z. Then $K = \bigcap_{s\in[0,1]} \psi_{s,z}^{-1}(A)$ is a neighbourhood of $(z,0)$ and $\psi_{s,z}$ satisfies (4.8) on K.

To show the existence of the vector field, we use some singularity theoretic notation. For a fixed $z \in Z$, denote by $[\varphi_z]$ a germ of $(z,0)$, and by $\mathcal{E}_z$ the collection of germs at the same point. We want to give a bundle-theoretic argument to obtain the representation of folds. Thus, we consider the bundle ϵ over Z with fibre $\mathcal{E}_z$. Each section $\sigma\colon Z \to Z \times C^\infty(U)\,,\ z \mapsto (z,\varphi_z)$ can be regarded locally as the germ $[\varphi_z]$ for a fixed z. For any point z, consider $\Phi\colon Z \times C^\infty(U) \to Z \times \mathbb{R}\,,\ (z,\varphi_z) \mapsto (z,\varphi_z(0))$ as the evaulation map. This map induces a surjective morphism $Z \times \mathcal{E} \to Z \times \mathbb{R}$, and at a point z, $\ker(\Phi_z) = \mathfrak{m}_z$ is a maximal ideal in $\mathcal{E}_z$. There is a natural subbundle $m \hookrightarrow \epsilon$ over Z with fibre $\mathfrak{m}_z$.

We follow not the arguments of Nicolaescu [[Nic11] pg 12-16]. With this setting, we can apply a modified version of Hadamard's lemma for this bundle situation [1]. It follows from it, that for a fixed $z \in Z$, a section is generated by coordinate functions x_i, i.e. $\sigma_z = [\varphi_z] = \sum_i [x_i][u_i]$. Moreover, the germ $[\varphi_z]$ at $(z,0)$ of the partial deriviatives $\frac{\partial\varphi_z}{\partial x_1}, \frac{\partial\varphi_z}{\partial x_2}, \frac{\partial\varphi_z}{\partial x_3}$ generates the Jacobian ideal J_{φ_z} in $\mathcal{E}_x$. And, since $(z,0)$ is a non-degenerate critical point then $J_{\varphi_z} = \mathfrak{m}_z$, thus also a subbundle of ϵ.

To conclude, we consider the following initial value problem. Set $\delta_z = \varphi_z - Q_z$ such that $\varphi_{s,z} = \varphi_z - s\delta_z$. Equation (4.9) can be expressed as $V_s \cdot (\varphi_z - s\delta_z) = -\delta_z$. For a every section g_z of the bundle ϵ, consider $\forall s \in [0,1]$

$$V_s(z,0) = (z,0) \tag{4.10}$$

$$V_s \cdot (\varphi_z - s\delta_z) = g_z \tag{4.11}$$

Applying a similar argument as in [[Nic11] Ch.1.1, pg 12-17] modified for our bundle situation, we can conclude that for each section g_z of the subbundle m, there is a vector field V_s satisfying (4.10) and (4.11) for each s. We can form a

[1] Hadamard's Lemma [[Nic11] Lemma 1.13, pg 14]: The ideal $\mathfrak{m}$ is generated by the germs of the coordinate functions x_i.

convex combination of vector fields, and then glue them together with a partition of unity to obtain a global vector field for the whole Z. $\square$

Chapter 5

Geometry of the Singular Locus

> Boundaries have to be continuously sealed off, but it is a hopeless job, for everything touches everything else in this world. A beginning never disappears, not even with the ending.
>
> ---
>
> Harry Mulisch, *The Assault*

In this chapter we study the geometry around the singular locus induced by the near-symplectic form. First, we observe that the singular locus carries a natural Hamiltonian structure. Then we show that if Z admits a stable Hamiltonian structure, so does its normal sphere bundle $Z \times S^2$, in the case where the normal bundle is trivial. In the second section, we describe the splitting property of the the normal bundle following from a near-symplectic structure, which happens also in dimension 4. Then, we give a neighbourhood theorem that can serve as a tool for further normal forms. This instrument could be useful in the future to study the contact geometry on the tubular neighbourhood of the singular locus. As a corollary of this statement we find a local Darboux-type theorem. This extends the results of Honda [Hon04a] and Perutz [Per06] to higher dimensions.

5.1 Stable Hamiltonian Structures

The following definitions are based on the work from Cieliebak and Volkov [CV10].

Definition 5.1.1. A *Hamiltonian structure* (HS) on an oriented $(2n-1)$-dimensional manifold M is a closed 2-form ω such that $\omega^{n-1} \neq 0$ everywhere. Associated to

ω is its 1-dimensional *kernel distribution* $\ker(\omega) := \{v \in TM \mid \iota_v\omega = 0\}$. We orient $\ker(\omega)$ using the orientation on M together with the orientation on the local transversal to $\ker(\omega)$ given by ω^{n-1}. A *stabilizing 1-form* for ω is a 1-form λ on M such that

1. $\lambda \wedge \omega^{n-1} > 0$, and

2. $\ker(\omega) \subset \ker(d\lambda)$

A Hamiltonian structure ω is called *stabilizable* if it admits a stabilizing 1-form λ. A *stable Hamiltonian structure (SHS)* is the pair (ω, λ).

A SHS (ω, λ) induces a canonical *Reeb vector field* R generating $\ker(\omega)$ and normalized by $\lambda(R) = 1$. Not that if (ω, λ) is a SHS, then $(\omega, -\lambda)$ is a SHS inducing the opposite orientation.

Example 5.1.2.

1. *Contact manifolds*: (M, λ) is a contact manifold, R is the Reeb vector field, and $\omega = \pm d\lambda$.

2. *Mapping tori*: $M := W_\phi = \mathbb{R} \times W/(t,x) \sim (t+1, \phi(x))$ is the mapping torus of a symplectomorphism ϕ of a symplectic manifold $(W, \bar{\omega}), R = \frac{\partial}{\partial t}, \lambda = dt$, and ω is the form on M induced by $\bar{\omega}$. Note that $d\lambda = 0$, so $\ker(\lambda)$ defines a foliation. Notice that $W_\phi = [0,1] \times W/(0,x) \simeq (1, \phi(x))$

3. *Circle bundles*: $\pi \colon M \to W$ is a principal circle bundle over a symplectic manifold $(W, \bar{\omega})$, R is the vector field generating the circle action, λ is the connection form, and $\omega = \pi^*\bar{\omega}$.

Proposition 5.1.3. *A near-symplectic structure induces a Hamiltonian structure on its singular locus Z.*

Proof. This follows directly from the definiton of a near-symplectic form, since ω_Z is a 2-form of maximal rank on Z. □

Proposition 5.1.4. *Let $(Z \times \mathbb{R}^3, \omega_{\text{ns}})$ be a near-symplectic manifold, where Z is an oriented $(2n-1)$–manifold. If ε is a stabilizing 1-form for ω_Z on Z, then the normal sphere bundle $Z \times S^2$ has a stable Hamiltonian structure.*

Proof. By assumption, we have that $\varepsilon \wedge \omega_Z^{n-2} > 0$ on Z and $\ker(\omega_Z) \subset \ker(d\varepsilon)$. Let σ_{S^2} be the symplectic form of S^2. The boundary of a piece of the tubular neighbourhood $\partial(Z \times B^3) = Z \times S^2$ can be equipped with a Hamiltonian structure by

$$\bar{\omega} = \omega_Z + \sigma_{S^2} \tag{5.1}$$

This is a closed 2-form of maximal rank on $Z \times S^2$, since $\bar{\omega}^{n-1} = \omega_Z^{n-2} \wedge \sigma_{S^2} > 0$. The stabilizing 1-form on $Z \times S^2$ is defined by $\lambda = \varepsilon$. We have

$$\lambda \wedge \bar{\omega}^{n-1} = \varepsilon \wedge (\bar{\omega}^{n-2} \wedge \sigma_{S^2}) > 0$$

This shows the first condition of a SHS. Now, for the second property observe that

$$\ker(\bar{\omega}) = \{v \in TM \mid \iota_v \bar{\omega} = \iota_v(\omega_Z + \sigma_{S^2}) = 0\} \simeq \mathcal{E} = \ker(\omega_Z)$$

In this case $\ker(\bar{\omega}) \subset \ker(d\lambda)$. The pair $(\bar{\omega}, \lambda)$ is a stable Hamiltonian structure for $Z \times S^2 \subset (M, \omega_{\text{ns}})$. □

Stable Hamiltonian in BLF case

Proposition 5.1.5. *Let $(Z, \xi_Z = \ker(\alpha_Z))$ be a contact manifold of dimension $2n-1$, and $(Z \times \mathbb{R}, \omega_B = d(e^t \alpha_Z))$ its symplectization. Let $f\colon Z \times \mathbb{R}^3 \to Z \times \mathbb{R}$ be a (generalized) broken Lefschetz fibration. The total space $Z \times \mathbb{R}^3$ is near-symplectic inducing a stable Hamiltonian structure on $Z \times S^2$.*

Proof. We now equip $M = Z \times \mathbb{R}^3$ with a near-symplectic form along the lines of [ADK05] and theorem 1. Over the regular neighbourhood of Z, using the coordinates (x_i) of the fibre, define the 2-form

$$\tau = d(x_1(x_2 dx_3 - x_3 dx_2)) \tag{5.2}$$

We obtain a closed 2-form, positive on the fibres and non-degenerate outside Z. Define the 2-form $\omega \in \Omega^2(Z \times \mathbb{R}^3)$ as

$$\omega = \tau + f^* \omega_B$$

At the points where $\omega^n = 0$ we have a 4-dimensional kernel $K = \{v \in T_p M \mid \omega_p(v, \cdot) = 0\} \simeq \varepsilon \oplus TY^3$, where $\varepsilon = \ker(f^* \omega_B)$. The image of the intrinsic gradient

$\nabla\omega_p|_K$ is of dimension 3. The 2-form ω defines a near-symplectic structure on $Z \times \mathbb{R}^3$.

Let $\mathcal{U}$ be the tubular neighbourhood of Z in M. Define on the boundary of $\mathcal{U}$ the 2-form

$$\bar{\omega} = d\alpha_Z + \sigma_{S^2} \tag{5.3}$$

The contact form α_Z will work as the stabilizing 1-form $\lambda = \alpha_Z$. A simple computation shows that

$$\lambda \wedge \bar{\omega}^{n-1} = \alpha_Z \wedge d\alpha_Z^{n-2} \wedge \sigma_{S^2} > 0$$

Moreover, since $\ker(\bar{\omega}) \simeq \varepsilon \simeq \ker(d\alpha_Z)$, the second property is also satisfied. Hence, the pair $(\bar{\omega}, \alpha_Z)$ defines a stable Hamiltonian structure on the boundary of the singular locus $Z \times S^2$.

□

5.2 Normal bundle of Z

In this section, we will first show that the definition of near-symplectic form reflects properties on the normal bundle of the singular locus similar to dimension 4. In particular, we obtain a splitting of the normal bundle N_Z into two subbundles L_+, L_-, where L_+ is a rank 2-bundle corresponding to the positive eigenspace of an endomorphism $N_Z \to N_Z$, and L_- is a rank 1 bundle corresponding to the negative one.

Let $K := \epsilon \oplus N_Z$ be defined by the normal bundle N_Z and $\epsilon = \ker(\omega_Z)$. Fix a metric g on K such that $\omega|_K$ is self-dual. Identify the intrinsic normal bundle N_Z with the complement $(TZ)^\perp$ using the metric g. From the transversality of ω, the image of the intrisic gradient $D_K := \nabla\omega|_K$ is 3-dimensional. In fact, we have that Im$(D_k) = \Lambda^2_+ K^*$. Thus, we have a natural identification with the bundle of self-dual 2-forms. This follows from the non-negativity property of the near-symplectic form as shown in chapter 3. This implies that D_K defines an isomorphism

$$N_Z \to \Lambda^2_+ K^*$$

Let $X = \frac{\partial}{\partial z_0}$ be unit vector field defined on the line $\ker(\omega|_Z) \subset TZ$. The interior derivative defines a bundle isomorphism

$$\Lambda^2_+ K^* \to N_Z^*$$
$$\beta \mapsto \iota_X \beta$$

Its inverse $N_Z^* \to \Lambda_+$ is given by $\nu \mapsto \zeta \wedge \nu + *(\zeta \wedge \nu)$, where ζ is a 1-form that is non-vanishing on ϵ. Using the metric we can define an isomorphism $N_Z^* \to N_Z$. The endomorphism

$$F \colon N_Z \to N_Z$$

defined by the composition

$$N_Z \xrightarrow{D_K} \Lambda_+ \xrightarrow{\iota_X} N_Z^* \xrightarrow{g} N_Z$$

is a self-adjoint, trace-free automorphism as in dimension 4 [Per06, Hon04a]. The matrix A representing this map is symmetric and trace-free. Consequently, at each point $p \in Z$, A has three real eigenvalues, two of the same sign and one of the opposite sign. Following the convention used in low dimensions [Per06, Hon04a, Tau98], we choose the signs of the eigenvalues to be two positive and one negative. We obtain a splitting of the normal bundle in 2 eigensubbundles defined by the negative and positive eigenspaces

$$N_Z \simeq L_- \oplus L_+ \tag{5.4}$$

L_- is a rank 1 bundle, locally trivial, and L_+ is a rank 2 bundle orthogonal complementary to L_-. After a choice of basis the linear map F can be represented by a trace-free symmetric matrix $A = A_+ \oplus A_-$, where A_+ is a 2×2 positive-definite matrix, and $A_- < 0$.

Remark 5.2.1. The fact that the endomorphism $F \colon N_Z \to N_Z$ is represented by a symmetric and trace-free matrix A follows from the property of ω being closed. The intrinsic gradient is a bundle map from K to $\Lambda^2_+ K^*$. The derivative $\nabla\omega|_K$ is a section of $N_Z^* \otimes \Lambda^2_+ K^*$, where $N_Z^* \subset T^*M|_Z$ is the conormal bundle of Z and $\Lambda^2_+ K^* \subset \Lambda^2 K^*$ is the bundle of self-dual 2-forms.

We have that

$$\nabla\omega|_K = \sum A_{ij} v \otimes \beta$$

where $v \in N_Z^*$, and $\beta \in \Lambda^2_+ K^*$.

Let e^1, e^2, e^3 be an orthonormal basis for N^* and let e^4 be another 1-form such that $e^1, \dots, e^4$ are an orthonormal basis for $\varepsilon^* \oplus N_Z^* \subset T^*M|_Z$. Let

$$\theta^1 = e^2 \wedge e^3 \, , \, \theta^2 = e^1 \wedge e^3 \, , \, \theta^3 = e^1 \wedge e^2$$

With respect to the previous chosen basis this can be rewritten as

$$\nabla\omega|_K = \sum_{ij} A_{ij} \left(e^i \otimes (e^j \wedge e^4) + e^i \otimes \theta^j \right)$$

Both, terms can be multiplied by the same matrix A_{ij} since $\omega|_K$ is self-dual. In order to show more clearly where the symmetric and trace-less property come from, rewrite the previous equation as

$$\nabla\omega|_K = \sum_{ij} \bar{A}_{ij}(e^i \otimes (e^j \wedge e^4)) + \bar{\bar{A}}_{ij}(e^i \otimes \theta^j)$$

Notice that $\bar{A}_{ij} = \bar{\bar{A}}_{ij}$. We are just expressing the matrix in this way to see the properties more clearly. To see the symmetry, observe that for ω to be closed, then the terms outside the diagonal $\bar{a}_{kl}(k \neq l)$ have to be the same as the terms $\bar{a}_{lk}$ as they are multiplying terms with the same basis elements. The same is true for the terms multiplied by $\bar{\bar{A}}_{ij}$, although in the latter case they even vanish trivially. For the trace-less property let us focus on $\bar{\bar{A}}_{ij}$. The terms that do not vanish trivially lie on the diagonal and they are composed by the same basis elements. Since ω is closed then the sum $\bar{\bar{A}}_{11} + \bar{\bar{A}}_{22} + \bar{\bar{A}}_{33} = 0$.

Remark 5.2.2. The endomorphism $N_Z \to N_Z$ induces an orientation on the line $\varepsilon := \ker(\omega_Z)$, and its complementary $P \subset TZ$ is oriented by ω_Z, thus $TZ = \varepsilon \oplus P$ is oriented. Since Z and M^{2n} are oriented, then the normal bundle $N_Z = M \setminus Z$ is oriented.

5.3 Neighbourhood Theorem

Theorem 5.3.1. *Let $(M_0, \omega_0), (M_1, \omega_1)$ be two near-symplectic manifolds with diffeomorphic singular locus $Z_0 \cong Z_1$ and equal symplectic forms on them, $\omega_0|_{Z_0} = \omega_1|_{Z_1}$. Assume that there is an isomorphism on the normal bundles $N_{Z_0} \simeq N_{Z_1}$, such that it restricts to an isomorphism on the positive subbundles $L_0^+ \simeq L_1^+$. Denote by $\mathcal{U}_0 \subset M_0$ and $\mathcal{U}_1 \subset M_1$ the corresponding tubular neighbourhoods of Z_0 and Z_1. Then, there* *is a homeomorphism $\varphi\colon \mathcal{U}_0 \to \mathcal{U}_1$ which is a diffeomorphism away from Z, such that $\varphi^*\omega_1 = \omega_0$.*

Proof. **Step 1: Family of near-symplectic forms**

Define $\omega_t = (1-t)\omega_0 + t\cdot\omega_1$. We want to show that this is a family of near-symplectic forms. The closedness property follows from the fact that this family is a linear combination of closed 2-forms. The symplectic subspaces defined by ω_{Z_0} and ω_{Z_1} are the same on $TZ_0 \simeq \mathrm{Symp}_0 \oplus \varepsilon_0$ and $TZ_1 \simeq \mathrm{Symp}_1 \oplus \varepsilon_1$. This defines the same complementary line bundle $\varepsilon = \ker(\omega_{Z_0}) = \ker(\omega_{Z_1})$.

The kernels $K_0 \simeq \varepsilon \oplus N_{Z_0}$ and $K_1 \simeq \varepsilon \oplus N_{Z_0}$ are 4-dimensional. Interpolating between ω_0 and ω_1 leaves $\dim(K_t) = 4 \quad \forall t$. Thus, up to scaling the intrinsic gradients $D_{K_0} := \nabla\omega|_{K_0}$ and $D_{K_1} := \nabla\omega|_{K_1}$ agree and so their images. Hence, it follows that at a point $p = 0$ in Z we have that $\omega_t^n = 0$. Notice that this property can also be computed directly by looking at the expansion

$$\omega_t^n(0) = c_n(t)\omega_0^n(0) + c_{n-1}(t)\binom{n}{1}\cdot\omega_0^{n-1}\wedge\omega_1(0) + c_{n-2}(t)\binom{n}{2}\omega_0^{n-2}\wedge\omega_1^2(0) + \dots$$
$$+ c_0(t)\omega_1^n(0)$$

where $c_k(t) = (1-t)^k \cdot t^{n-k}$, with $k \in \mathbb{Z}, k \in [0,n]$. In the previous expression all terms vanish, since each of them necessarily takes four vectors from K_t.

Now we show that ω_t^n is non-negative. Let v a vector in N_Z and $s \in \mathbb{R}$. Consider the Taylor expansion around $p \in Z$.

$$\begin{aligned}
&\omega_t^n(s\cdot v)\\
&= \underbrace{\omega_0^n(0)}_{=0} \quad + s\cdot\underbrace{\omega_0^{n-1}}_{=0}\wedge\nabla_v\omega_0 + s^2\cdot\omega_0^{n-2}\wedge(\nabla_v\omega_0)^2 + \dots + \underbrace{\omega_0^k\wedge\omega_1^{n-k}(0)}_{=0}\\
&+ s\cdot\underbrace{\omega_0^{k-1}\wedge\omega_1^{n-k}}_{=0}\wedge\nabla\omega_0 + s\cdot\underbrace{\omega_0^k\wedge\omega_1^{n-k-1}}_{=0}\wedge\nabla\omega_1\\
&+ s^2\cdot(\omega_0^{k-2}\wedge\omega_1^{n-k}\wedge(\nabla_v\omega_0)^2 + \omega_0^{k-1}\wedge\omega_1^{n-k-1}\wedge(\nabla_v\omega_0\wedge\nabla_v\omega_1)\\
&+ \omega_0^k\wedge\omega_1^{n-k-2}\wedge(\nabla_v\omega_1)^2) + \dots\\
&= \underbrace{\omega_1^n(0)}_{=0} \quad + s\cdot\underbrace{\omega_1^{n-1}}_{=0}\wedge\nabla_v\omega_1 + s^2\cdot\omega_1^{n-2}\wedge(\nabla_v\omega_1)^2 + \dots
\end{aligned}$$

The terms of the form $\omega_0^k\wedge\omega_1^{n-k}$ for $k \in \{0,\dots,n\}$ vanish identically as explained in the previous paragraph. The linear terms of the form $\omega_0^{k-1}\wedge\omega_1^{n-k+1}\wedge\nabla_v\omega_i$ for $i = 0,1$ are also zero, since from the $2n-2$ vectors v_i which are allocated in $\omega_0^{k-1} \wedge \omega_1^{n-k+1}(v_1, \ldots, v_{2n-2})$ at least 2 of those vectors should come from K_t. This leaves us with the following expression with leading terms of the order s^2

$$\begin{aligned}
&\omega_t^n(s \cdot v)\\
&= s^2 \cdot (\omega_0^{n-2}(0) \wedge (\nabla_v \omega_0)^2 + \ldots\\
&+ \omega_0^{n-2} \wedge (\nabla_v \omega_0 \wedge \nabla_v \omega_1) + \cdots + \omega_0^{n-3} \wedge \omega_1 (\nabla_v \omega_0)^2 + \ldots\\
&+ \omega_0^{n-2} \wedge (\nabla_v \omega_1)^2 + \cdots + \omega_0^{n-3} \wedge \omega_1 (\nabla_v \omega_0 \wedge \nabla_v \omega_1) + \cdots + \omega_0^{n-4} \wedge \omega_1^2 \wedge (\nabla_v \omega_0)^2 + \ldots\\
&+ \omega_0^{n-3} \wedge \omega_1 \wedge (\nabla_v \omega_0)^2 + \cdots + \omega_0^{n-4} \wedge \omega_1^2 (\nabla_v \omega_0 \wedge \nabla_v \omega_1) + \cdots + \omega_0^{n-5} \wedge \omega_1^3 \wedge (\nabla_v \omega_0)^2 + \ldots\\
&\vdots\\
&+ \omega_1^{n-2} \wedge (\nabla_v \omega_0 \wedge \nabla_v \omega_1) + \cdots + \omega_0 \wedge \omega_1^{n-3} (\nabla_v \omega_0)^2 + \ldots\\
&+ \omega_1^{n-2}(0) \wedge (\nabla_v \omega_1)^2 + \ldots) \qquad (5.5)
\end{aligned}$$

Factorizing the $(n-2)$-forms which are symplectic on Z, we can rewrite the previous expression as

$$\begin{aligned}
\omega_t^n(s \cdot v) = s^2 \cdot (&\omega_0^{n-2}(0) \wedge \big((\nabla_v \omega_0)^2 + \nabla_v \omega_0 \wedge \nabla_v \omega_1 + (\nabla_v \omega_1)^2\big) + \ldots\\
&+ \omega_0^{n-k} \wedge \omega_1^k \wedge \big((\nabla_v \omega_0)^2 + \nabla_v \omega_0 \wedge \nabla_v \omega_1 + (\nabla_v \omega_1)^2\big) + \ldots\\
&+ \omega_1^{n-2}(0) \wedge \big((\nabla_v \omega_0)^2 + \nabla_v \omega_0 \wedge \nabla_v \omega_1 + (\nabla_v \omega_1)^2\big)\big) \qquad (5.6)
\end{aligned}$$

As shown in chapter 3, by restricting the terms $(\nabla_v \omega_0)^2$ and $(\nabla_v \omega_1)^2$ to vectors ∂_{k_i} in K_t we have $(\nabla_v \omega_0)^2(\partial_{k_1}, \ldots, \partial_{k_4}) = D_{K_0}^2 \geq 0$ and $(\nabla_v \omega_1)^2(\partial_{k_1}, \ldots, \partial_{k_4}) = D_{K_1}^2 \geq 0$. Thus, in the equation 5.6, the square binomial terms are non-negative

$$((\nabla_v \omega_0)|_K^2 + \nabla_v \omega_0 \wedge \nabla_v \omega_1|_K + (\nabla_v \omega_1)|_K^2) = (\nabla_v \omega_0|_K + \nabla_v \omega_1|_K)^2 := \nabla_v \omega_t^2|_K \geq 0$$

and the forms $\omega_0^{n-k} \wedge \omega_1^k$, for $k \in \{0, 1, \ldots, n\}$ are positive on the symplectic subspace in Z, from which we conclude that $\omega_t^n \geq 0$ on the tubular neighbourhood of the singular locus.

Step 2: Poincaré Lemma

These next two steps follow the lines of Perutz [Per06]. Recall the following general version of the Poincaré Lemma. Let $\pi\colon N_Z \to Z$ be the bundle projection and $i\colon Z \to N_Z$ the zero-section. Let $h_t\colon N_Z \to N_Z$, $x \to t \cdot x$ be the fibrewise dilation and R the Euler vector field, that is the vertical vector field defined on any fibre N_x by $R(y) = \sum y_j \frac{\partial}{\partial y_j}$, where y_j are orthonormal Euclidean coordinates on N_x. Such a vector field always exist on any vector bundle. The De Rham homotopy operator

$$Q\colon \Omega^k \to \Omega^{k-1}$$
$$Q\Omega = \int_0^1 h_t^*(\iota_R \Omega) dt \tag{5.7}$$

satisfies

$$\mathrm{Id}(\Omega) - (\iota \circ \pi)^*(\Omega) = dQ(\Omega) + Qd(\Omega)$$

Applying this lemma to a neighbourhood of the zero section $\mathcal{U}_0 \subset N_Z$ we find a 1-form $\lambda_t := Q(\omega_t)$ satisfying $d\lambda_t = \omega_t$ on $\mathcal{U}_0 \setminus Z$, and vanishing to second order along Z. To see this, notice that ω_t vanishes up to degree 1 on K_t. Inserting the Euler vector field R into ω_t adds one degree more and produces a 1-form $\iota_R \omega_t$ that vanishes on Z up to degree 2.

Step 3: Moser equation

On $\mathcal{U}_0 \setminus Z$, where ω_t is near-symplectic, introduce vector fields X_t defined by

$$\iota_{X_t} \omega_t + \lambda_t = 0 \tag{5.8}$$

We want to show that on the tubular neighbourhood X_t shrinks as it approaches Z. On the other four complementary directions defined by K_t, we have that $\nabla \lambda_t(u) = 0$ for all non-zero vectors $u \in N_{Z_0}$, since λ_t vanishes to the second order along Z. Furthermore, ω_t degenerates on K_t, and a Taylor expansion shows that $\nabla \omega_t \neq 0$ on K, so that $|X^t_{\mathrm{K}}(x)| \leq C|x|$ for a constant C [Per06].

On the symplectic subspace in Z we have, $\lambda_t|_{\mathrm{Symp}_Z} = 0$ but $\omega_t|_{\mathrm{Symp}_Z}$ is non-degenerate. Thus, in order to satisfy equation 5.8, the vector field X_t needs to vanish on Symp_Z. In particular, the components of the vector field along the symplectic subspace $|X^t_{\mathrm{Symp}}(x)| \leq c|x|$ for a constant c.

The family $\{X_t\}_{t\in[0,1]}$ generates a flow $\{\psi_t\}_{t\in[0,1]}$ on $\mathcal{U}_0$ outside Z. A trajectory x_s defined on some interval $[0, \tilde{s}]$ satisfies $\frac{d}{ds}(\log |x_s|) \geq -C$ Integrating over $[0, \tilde{s}]$ we obtain $|x_{\tilde{s}}| \geq e^{C\tilde{s}}|x_0|$. This shows that the trajectory stays inside $\mathcal{U}_{\prime} \setminus Z_0$, hence the flow ψ_s is well defined.

Step 4

Define on $\mathcal{U}_0 \setminus Z_0$

$$\tilde{\omega_t} := \psi_t^* \omega_t$$

and for $p \in Z$

$$\tilde{\omega_t} := \omega_{Z_0}$$

Moser's argument shows that $\tilde{\omega}_t = \omega_t$ in some neighbourhood of Z. The diffeomorophism ψ_1 is not defined on Z. Extend it to Z by the identity. At the level of the singular locus, we can take the diffeomorphism to be the one from the theorem's assumption $Z_0 \approx Z_1$. This leads to a homeomorphism, which is a diffeomorphism away from Z, since we do not know if both diffeomorphism extend smoothly to each other. Finally, set $\varphi = \psi_1$, and $\psi_1(\mathcal{U}_0) = \mathcal{U}_1$. Then we have that $\varphi^*\omega_1 = \omega_0$ away from Z, but by assumption ω_1 and ω_0 agree on Z. □

5.4 Local Darboux-type theorem

Corollary 5.4.1. *Let (M, ω) be a near-symplectic manifold and p a point of the singular locus $Z \subset M$. There is a coordinate neighbourhood $U \subset M$ around p, such that on U*

$$\omega = \omega_Z - 2x_1(dz_0 \wedge dx_1 + dx_2 \wedge dx_3) + x_2(dz_0 \wedge dx_2 - dx_1 \wedge dx_3) + x_3(dz_0 \wedge dx_3 + dx_1 \wedge dx_2) \tag{5.9}$$

The following proof follows from the previous theorem and an adaptation of an' argument from [Per06].

Proof. Let γ be a closed interval inside the line $\varepsilon = \ker(\omega_Z)$. Consider $\kappa := \gamma \times B^3 \subset K := \varepsilon \oplus N_Z$. Identify an open subset of Z with $V \times \{0\} \subset U \simeq V \times B_0^3(R)$ inside M, such that $\kappa \subset U$. Denote by z_0 the coordinate on γ and by ∂_{z_0} a positively oriented vector field on γ for the orientation determined by ω.

Take a metric g for which $\omega|_\kappa$ is self-dual. We can find an orthonormal frame (e_1, e_2, e_3) for N_Z such that $L_- = \text{span}(e_1), L_+ = \text{span}(e_2, e_3)$. The metric and the choice of e_i provide normal coordinates $(\bar{z}, z_0, x_1, x_2, x_3)$ on a small neighbourhood of p inside U, where $\bar{z}$ correspond to the $(2n-4)$-complementary coordinates to z_0 on Z. Using these coordinates we can write three basis elements β_i of $\Lambda_+^2 K^*$.

$$\beta_1 = dz_0 \wedge dx_1 + dx_2 \wedge dx_3$$
$$\beta_2 = dz_0 \wedge dx_2 - dx_1 \wedge dx_3$$
$$\beta_3 = dz_0 \wedge dx_3 + dx_1 \wedge dx_2$$

Let $\tilde{F} = \tilde{F}_- \oplus \tilde{F}_+$ be a matrix representing the linear map $F \in \text{End}(N_Z)$ with respect to (e_1, e_2, e_3), and let $x = (x_1, x_2, x_3)$ and $\beta = (\beta_1, \beta_2, \beta_3)^T$. Expand ω near

Z to obtain

$$\begin{aligned}\omega(z,x) &= \omega|_Z + \Big(x \cdot \tilde{F} \cdot \beta^T + O(x^2)\Big)\\ &= \omega|_Z + \left(x_1 \tilde{F}_- \beta_1 + (x_2, x_3)\tilde{F}_+ \begin{pmatrix}\beta_2\\ \beta_3\end{pmatrix} + O(x^2)\right)\end{aligned}$$

Define on a small neighbourhood of Z a family of near-symplectic forms with common singular locus Z by

$$\omega_t = (1-t)\omega + t \cdot (\omega|_Z - 2x_1\beta_1 + x_2\beta_2 + x_3\beta_3)$$

Following the same reasoning as in the proof of the previous theorem in a local setting, we can show that this is a family of near-symplectic forms with common degeneracy locus Z. The next steps follow as in the previous proof. □

Chapter 6

Conclusion

> Everything, everything seemed once-upon-a-time.
>
> Haruki Murakami, *Hard-Boiled Wonderland and the End of the World*

In this work we have proposed a definition of near-symplectic manifolds for all even dimensions, used broken Lefschetz fibrations as a fibration of near-symplectic manifolds, and studied some geometrical properties of the singular locus. We conclude this thesis by stating some questions that have arisen from studying near-symplectic manifolds that could lead to some potential future directions.

The first question relates to contact topology in higher dimensions. Does a near-symplectic $2n$-manifold (M, ω) induce an overtwisted contact structure on the normal sphere bundle of the singular locus Z? In dimension 4 this is known to be true. Honda [Hon04b], Gay and Kirby [GK04] showed that a near-symplectic form on a 4-manifold induces an overtwisted contact structure on the boundary of the tubular neighbourhood of Z, which is diffeomorophic to $S^1 \times S^2$. It is not known whether in higher dimensions this relation is also true, and under what conditions this occurs. Some difficulties arise from dimensionality reasons. For dimensions $2n > 4$, the normal bundle of the codimension 3 singular locus Z is not trivial in general, and Z is not a particular manifold as in dimension four, when it is S^1. Moreover, what line of overtwistedness could be applied for such a construction? In recent times, different tools and techniques have appeared to construct overtwisted contact structures in higher dimensions, such as Lutz twist [EP11], blow down operations [Mor09], and Giroux domains [MNW12] among others. Massot, Niederkrüger and Wendl [MNW12] have introduced the concept of a Giroux domain, a generalization of Giroux torsion. They have shown that a closed contact manifold containing a compact subdomain N, obtained by gluing and blowing down Giroux domains, is PS-overtwisted. This might be a helpful object.

In dimension 4, there is a direct correspondence between near-symplectic structures and broken Lefschetz fibrations. An analogous relation could be established in dimensions 4 and higher by showing the other direction of theorem 1. That is, whether given a near-symplectic $2n$-manifold $(M, \omega_{\mathrm{ns}})$, a blow up of M admits a generalized BLF $f\colon (M, \omega_{\mathrm{ns}}) \to (X, \omega)$ with fold singularities exactly on the degenerate locus Z. The work of Auroux could provide a starting point. In [Aur00] and [AK00], the authors show branched coverings of symplectic 4-manifolds over $\mathbb{CP}^2$ using approximately holomorphic techniques.

Appendix A

Appendix

A.1 Proof of Theorem 1

Remark A.1.1. The 2-form τ_p is defined on a piece of the tubular neighborhood of Σ that intersects the fibre. To show that τ_p evaluates positively with the fibres, we need just need to check that τ_p integrates positively over the piece of a fibre, where τ_p is defined. Fix a $c < 0$ defined by $c = -x_1^2 + \frac{1}{2}(x_2^2 + x_3^2))$ so that $F_c = f^{-1}(c)$. We proceed to integrate over a piece of F_c, which we call F_c^s, where $s \in \mathbb{R}$ given by $s := x_1^2$. Locally, a small piece of a fibre looks like a compact convex cone with ∂F_c^s being two circles. By Stokes theorem, with $\tau_p = d\alpha$, we obtain

$$\begin{aligned}
\int_{F_c^s} \tau_p &= \int_{\partial F_c^s} \alpha \\
&= \int_{\partial F_c^s} \underbrace{\chi(t)}_{=K=\text{constant}} x_1(x_2dx_3 - x_3dx_2) \\
&= \int_{S^1} x_1(x_2dx_3 - x_3dx_2) - \int_{-S^1} x_1(x_2dx_3 - x_3dx_2) \\
&= \int_{S^1} x_1(x_2dx_3 - x_3dx_2) - \int_{-S^1} x_1(x_2dx_3 - x_3dx_2) \\
&= \int_{S^1} (+\sqrt{s})(x_2dx_3 - x_3dx_2) - \int_{-S^1} (-\sqrt{s})(x_2dx_3 - x_3dx_2) \qquad \text{(A.1)}
\end{aligned}$$

We continue as in [[MT97] pg 22, Example 3.13]. Let $\eta := x_1dx_2 - x_2dx_1$ and $t \in [0,1]$. Set $\phi\colon [0,1] \to (\cos(2\pi t), \sin(2\pi t))$ and focus on one of integrals $\int_{S^1}(x_2dx_3 -$

$x_3dx_2) = \int_{S^1}\eta$. Then,

$$\begin{aligned}\int_{S^1}\eta &= \int_{[0,1]}\phi*\eta\\ &= \int_{[0,1]}[(\cos(2\pi t)(\cos(2\pi t)\cdot 2\pi - (\sin(2\pi t))(-\sin(2\pi t))\cdot 2\pi)]dt\\ &= 2\pi\int_0^1[\cos^2(2\pi t) - \sin^2(2\pi t)]dt\\ &= 2\pi\end{aligned}$$

Summing both integrands of A.1, we obtain

$$\begin{aligned}\int_{F_c^s}\tau_p &= K\cdot\left((+\sqrt{s})\int_{S^1}\eta - (-\sqrt{s})\int_{S^1}\eta\right)\\ &= K\cdot\left((+\sqrt{s})\int_{S^1}\eta - (-\sqrt{s})\int_{S^1}\eta\right)\\ &= K\cdot\big((+\sqrt{s})[2\pi] + \sqrt{s}[2\pi]\big)\\ &= 2K\sqrt{s}(2\pi)\end{aligned}$$

Remark A.1.2. In this remark we explain why near a Lefschetz-type singularity, the 2-form ω_0 is positive. Take ω_0 to be the standard Kähler form of $\omega_{\mathbb{R}^{2n}} = dx_1\wedge dx_2 + \cdots + dx_{2n-1}\wedge dx_{2n}$ $\mathbb{R}^{2n}$, in some local coordinates, in which f is given by the standard models $(z_1, z_2, \ldots, z_n)\mapsto(z_1^2 + z_2^2, z_3, \ldots, z_n)$. We want to show that $\iota^*\omega_X \neq 0$, where $\iota : F\hookrightarrow X$. In other words, $\omega(X_1, X_2)\neq 0$, for tangent vectors $X_1, X_2 \in T_pF = \ker(df)$.

$$df_1 = 2z_1dz_1 + 2z_2dz_2, df_2 = 1, \ldots, df_{n-1} = 1$$

With kernels

$$\ker(df_1) = \langle z_1\frac{\partial}{\partial z_2}, -z_2\frac{\partial}{\partial z_1}\rangle, \ker(df_2) = 0, \ldots, \ker(df_{n-1}) = 0$$

Consider the vector fields $X_1 = z_1\frac{\partial}{\partial z_2}, X_2 = -z_2\frac{\partial}{\partial z_1}$ in T_pF. Take $z_1 = x_1 + ix_2, z_2 = x_3 + ix_4$ and consider the realification:

$$\begin{aligned}X_1 &= x_1\frac{\partial}{\partial x_3} - x_2\frac{\partial}{\partial x_4} - x_3\frac{\partial}{\partial x_1} + x_4\frac{\partial}{\partial x_2}\\ X_2 &= x_1\frac{\partial}{\partial x_4} + x_2\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_2} - x_4\frac{\partial}{\partial x_1}\end{aligned}$$

Then

$$\omega_{\mathbb{R}^{2n}}(X_1, X_2) = (-x_4dx_1 - x_3dx_2 + x_2dx_3 + x_1dx_4)(X_2) = x_4^2 + x_3^2 + x_2^2 + x_1^2 > 0$$

Remark A.1.3. The role of the smooth cut-off function is to Let U_1 and U_2 be two neighbourhoods with non-empty intersection around points p_1 and p_2 respectively, where the forms τ_{p_1}, τ_{p_2} are defined. Consider a diffeomorphism

$$\begin{aligned} \phi\colon U_1 &\to U_2 \\ (z_1, \dots, z_{2n-3}, x_1, x_2, x_3) &\mapsto (\phi_1(z,x), \dots, \phi_{2n-3}(z,x), \phi_{2n-2}(z,x), \phi_{2n-1}(z,x), \phi_{2n}(z,x)) \end{aligned} \tag{A.2}$$

We are interested in looking at the 2-forms τ_{p_1} and $\phi^*\tau_{p_2}$ on the intersection $U_1 \cap U_2$. The key role of $\chi(z)$ appears when checking the non-degeneracy of $\tau_{p_1} \wedge \phi^*\tau_{p_2}$. To see this, we are going to expand $\tau_{p_1} \wedge \phi^*\tau_{p_2}$. To simplify the formulation, we use the notation

$$dz = \sum_{1}^{2n-3} dz_i \quad , \quad d\phi_k = \sum_{i=1}^{2n} \frac{\partial \phi_k}{\partial u_i} du_i$$

$$(\phi_z, \phi_{x_1}, \phi_{x_2}, \phi_{x_3})) = (\phi_1(z,x), \dots, \phi_{2n-3}(z,x), \phi_{2n-2}(z,x), \phi_{2n-1}(z,x), \phi_{2n}(z,x))$$

with $\phi_z = (\phi_1(z,x), \dots, \phi_{2n-3}(z,x)), \phi_{x_1} = \phi_{2n-2}(z,x), \phi_{x_2} = \phi_{2n-1}(z,x)$, and $\phi_{x_3} = \phi_{2n}(z,x)$
Then with

$$\phi^*\tau_{p_2} = d\left(\chi(\phi_z)\phi_{x_1}(\phi_{x_2} d\phi_{x_3} - \phi_{x_3} d\phi_{x_2})\right)$$

we can expand

$$\begin{aligned} \tau_{p_1} \wedge \phi^*\tau_{p_2} = {} & \dot\chi_1(z)\chi_2(\phi_z)\underbrace{x_0 x_1 \phi_{x_2}}_{P_1^3}(dz \wedge dx_2 \wedge d\phi_{x_1} \wedge d\phi_{x_1}) \\ & - \chi_1(z)\dot\chi_2(\phi_z)\underbrace{x_1 \phi_{x_1} \phi_{x_2}}_{P_2^3}(dx_1 \wedge dx_3 \wedge d\phi_z \wedge d\phi_{x_1}) \\ & + 2\chi_1(z)\dot\chi_2(\phi_z)\underbrace{x_1 \phi_{x_1} \phi_{x_3}}_{P_3^3}(dx_2 \wedge dx_3 \wedge d\phi_z \wedge d\phi_{x_3}) \\ & - \dot\chi_1(z)\chi_2(\phi_z)\underbrace{x_0 x_2 \phi_{x_1}}_{P_4^3}(dz \wedge dx_1 \wedge d\phi_{x_1} \wedge d\phi_{x_3}) \\ & - \chi_1(z)\dot\chi_2(\phi_z)\underbrace{x_2 \phi_{x_1} \phi_{x_2}}_{P_5^3}(dx_1 \wedge dx_2 \wedge d\phi_z \wedge d\phi_{x_2}) \end{aligned}$$

Each of these summands has a polynomial of degree 3, expressed as P_i^3. This is then multiplied by a polynomial of degree 1 coming from the Taylor expansion of $d\phi_i$. Thus, every summand is being multiplied by polynomials of degree at least

4. On the other hand, the terms coming from the wedge square $\tau_{p_1} \wedge f^*\omega_X$ and $\tau_{p_2} \wedge f^*\omega_X$ have a nice leading polynomial of degree 2 in the form of $(x_1^2 + x_2^2 + x_3^2)$. We want that the terms of $\tau_{p_i} \wedge f^*\omega_X$ dominate over $\tau_{p_1} \wedge \phi^*\tau_{p_2}$. Thus, we select the cut-off function $\chi(z)$, so that$\chi = 0$ after the intersection value of the the degree 2 polynomial and the degree 4 polynomial.

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