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EXACT LOCALIZED SOLUTIONS OF THE NONLINEAR DIRAC EQUATION

Yaser Hasan Mohammad Sabbah

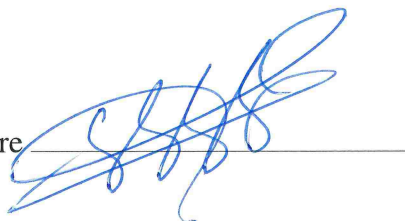This thesis is submitted in partial fulfillment of the requirements for the degree of Master of Science in Physics

Under the Supervision of Professor Usama Al Khawaja

November 2017

Declaration of Original Work

I, Yaser Hasan Mohammad Sabbah, the undersigned, a graduate student at the United Arab Emirates University (UAEU), and the author of this thesis entitled *"Exact Localized Solutions of the Nonlinear Dirac Equation"*, hereby, solemnly declare that this thesis is my own original research work that has been done and prepared by me under the supervision of Professor Usama Al Khawaja, in the College of Science at UAEU. This work has not previously been presented or published, or formed the basis for the award of any academic degree, diploma or a similar title at this or any other university. Any materials borrowed from other sources (whether published or unpublished) and relied upon or included in my thesis have been properly cited and acknowledged in accordance with appropriate academic conventions. I further declare that there is no potential conflict of interest with respect to the research, data collection, authorship, presentation and/or publication of this thesis.

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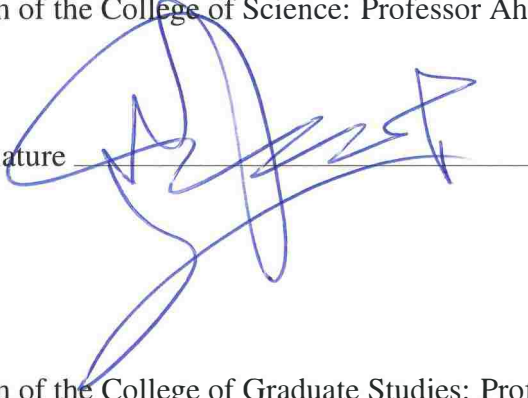Institution: Abu Dhabi university (visiting faculty)

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Declaration of Original Work

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Abstract

The nonlinear Dirac equation is a relativistic classical field theory that describes the behavior of a system of self-interacting spinor fields. According to this theory, the interactions among spinor fields are represented by additional Kerr-nonlinearity added to the Dirac equation, which justifies and models the noticed solitonic behavior of the systems.

There are various models of the nonlinear Dirac equation which differ from each other in the factors taken into account in the modelling, especially the mode of the coupling among the spinor fields as well as the nature of the system represented by the model.

In this present thesis, a special form of the nonlinear Dirac equation (NLDE) is considered, namely, the massive Thirring model (MTM) in (1+1)-dimensions, which models the vector-vector coupling mode of interactions among spinor fields in condensed matter. Here, exact localized stationary solutions are obtained using analytical methods. The physical properties of MTM and the corresponding conserved physical quantities are discussed through developing the continuity equation of the current density together with evaluating explicitly the elements of the energy-momentum tensor, which are then used to calculate some properties like charge and energy of the fields. Also, the same analytical methods are used to find stationary exact solutions of another model of NLDE, the Gross-Neveu model, which is of interest in high-energy physics.

Keywords: Nonlinear Dirac equation, Massive Thirring model, Gross-Neveu model, Self-interacting spinor fields, Energy-momentum tensor, Solitonic behaviour, Exact localized solutions.

Title and Abstract (in Arabic)

الحلول المحددة التامة لمعادلة ديراك غير الخطية

الملخص

تعتبر معادلة ديراك غير الخطية نظرية نسبية خاصة كلاسيكية في المجال حيث أنها تصف سلوك أنظمة مكونة من مجالات العزم المغزلي التي تكون متفاعلة ذاتيا فيما بينها. حسب هذه النظرية، فإن التفاعلات الذاتية الحاصلة بين مجالات العزم المغزلي يتم التعبير عنها بحدود كيرر غير الخطية التي يتم إضافتها إلى معادلة ديراك الخطية، و التي تفسر و تمثل السلوك السوليتوني لمثل تلك الأنظمة

هنالك العديد من النماذج لمعادلة ديراك غير الخطية و التي تختلف فيما بينها في العوامل التي تؤخذ بعين الإعتبار عند وضع النموذج، و خاصة الطريقة التي يتم فيها الاقتران بين مجالات العزم المغزلي بالإضافة إلى طبيعة النظام الذي يمثله النموذج

تتناول هذه الأطروحة واحدا من نماذج معادلة ديراك غير الخطية، و هو نموذج ثيرينغ الكتلي في بعد مكاني واحد بالإضافة إلى بعد الزمن، و الذي تصف التفاعل بين مجالات العزم المغزلي الاتجاهية في مجال فيزياء الحالة المكثفة. يتم هنا ايجاد حلول موضعية محددة و تامة باستخدام طرق تحليلية. كذلك تتم مناقشة كل من الخصائص الفيزيائية لمعادلة ثيرينغ و الكميات المحفوظة الناتجة عنها بالإضافة إلى ايجاد جميع عناصر مصفوفة الطاقة و الزخم، و التي تستخدم بعد ذلك لحساب بعض الخصائص مثل الشحنة و الطاقة للمجالات. أيضا نفس الطرق التحليلية يتم استخدامها لايجاد حلول نموذج آخر لمعادلة ديراك غير الخطية، و هو نموذج غروس و نيفيو و الذي له تطبيقات في فيزياء الطاقة العالية

مفاهيم البحث الرئيسية : معادلة ديراك غير الخطية، نموذج ثيرينغ الكتلي، نموذج غروس و نيفيو الكتلي، مجالات العزم المغزلي المتفاعلة، سلوك سوليتوني، مصفوفة الطاقة و الزخم

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My special thanks go to my loyal wife for providing me with the continuous support, to my two wonderful children, Maryam and Tamim, for inspiring me, and to my brothers and sisters for encouraging me during the years of study.

Last but not the least important, I am grateful to my parents. This accomplishment would not have been possible without them. Thank you.

Dedication

To my beloved parents and teachers

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List of Abbreviations and Notations

- NLDE stands for *nonlinear Dirac equation*, MTM for *massive Thirring model* And MGNM for *massive Gross-Neveu model*.

- The letter i is the imaginary unit $i = \sqrt{-1}$.

- Partial derivatives $\partial_x = \frac{\partial}{\partial x}$, $\partial_{xx} = \frac{\partial^2}{\partial x^2}$.

- All bold face symbols define vectors or higher-rank tensors.

- $g^{\mu\nu}$ is the metric tensor.

- Latin superscripts define contravariant spatial vectors and run from 1 to 3 (depending on the dimensionality of the case in hand), whereas Greek superscripts define contravariant vectors in the spacetime and rum from 0 to 3. The subscripts define the corresponding covariant vectors.

- Einstein summation convention is implied, i.e. $A^\mu B_\mu$ implies summation on μ.

- γ^μ are the Dirac matrices while σ^i are Pauli matrices.

- a^* is the complex conjugate of a, A^T and A^{-1} are the transpose and the inverse of A, respectively.

- $\bar{\Psi} = (\Psi^*)^T \gamma^0$.

Chapter 1: Foundations of the Nonlinear Dirac Equation (NLDE)

1.1 Introduction

A century is about to finish since the trials of describing notions of quantization in a general settings are crowned to success by the formulation of the general quantum theory. Although it has been developed in two different formalisms parallel to each other: The old quantum theory, represented mainly by Schrödinger equation [1]; and the relativistic one, mainly represented by Dirac equation [2]. The old theory gained more reputation and acceptance in the early days, especially in the experimental fields of physics, chemistry and materials science. One reason was the fact that Schrödinger equation, together with introduction of spin; were enough to predict much of the natural phenomena and justify the outcomes of many experiments at that time. Another reason was the need of relativistic quantum theory to more interpretation and even to modify the axioms so that the way they describe phenomena could fit with relativity. Besides the fact that relativistic quantum mechanics was compatible with the quantum field theory, which was developed later; rather than with a system composed of a fixed number of particles. Even after developing a consistent relativistic theory, the old quantum mechanics kept its prominence due to the successful predictions in many fields of research.

Apart from all the achievements mentioned above of Schrödinger equation, some other phenomena and experimental outcomes were still left at that time without being described by a satisfactory model; particularly those related to nuclear and ele-

mentary particle physics. In fact, there has been an evidence that such fields of research have been in need of a relativistic formalism. Keep in mind also the advantage of Dirac equation of predicting the spin as a property resulting naturally of the particles. Moreover, if quantization is to be imposed on electrodynamics; then the relativistic quantization will be more suitable. So, it was reasonable to proceed with relativistic theory in parallel with the old one. Even physicists realized from the beginning that both of the formulations were promising and even inspiring for more development, while the Dirac equation became the dominant theory in the fields of modern physics.

Theoretical representation of real systems often requires a modification of the basic theory or sometimes introducing new terms to account for the special effects. The latter ones could be an external field, extending from a single particle to many particles or the interactions among system's components. Such effects become even more dramatic in the relativistic regime and greatly modify the behavior of the system. Let us take a system of interacting electrons as an example. The modelling of two spin-$\frac{1}{2}$ interacting electrons can be done with the classical theory with the aid of perturbations. Given that the interactions are small enough so that the coupling parameter is too small to be defined as a perturbation. But if such interactions between the electrons is greatly affecting the behavior of the systems, which results in a large coupling constant, then the results of perturbation methods are found to be significant from the observed measures. Even the attempts to extend the Schrödinger equations to the form of a nonlinear equation (NLS) was not a good option for many situations, where high energies are involved and the behavior of the systems is expected to alter the relativistic limits. So, the modelling of such system was realized to be possible by representing the electrons

with an interacting spinor fields with a nonlinear representations. Within this realm, The Dirac together with the field theories constitute the only satisfactory representations of such system, where the result was a set of coupled Dirac equations with an additional Kerr-nonlinearity terms and, consequently, leading to the development of the nonlinear Dirac equation (NLDE). Those nonlinearities are considered to be an indication or a signature for a solitonic behavior of the interacting fields.

The NLDE had been developed in different settings since the fifties of the last century [4, 5, 6, 7] and extended to describe various interacting quantum systems in many fields ranging from condensed matter [32] to high energy physics [8, 9, 10] in addition to describing different modes of coupling [7]. Even it could model a system of interacting fields for particles with zero rest mass like neutrinos [6].

All the points mentioned above worth at least a brief discussion as they constitute the considerations behind which NLDE had been developed. This chapter is devoted to this purpose. Next section shows the assumptions that led to the linear Dirac equation. The third section presents the development of NLDE and lists some of its forms available in the literature.

1.2   The linear Dirac equation

The Dirac equation represents the first accepted theory where quantum mechanics meets relativity. In addition to its implementation of a satisfactory modelling of relativistic quantum mechanical systems and a natural theoretical realization of particle's spin, it provided a deeper understanding of the sub-atomic world and the representation of their interactions by introducing new concepts like spinors, particle-like fields and antiparti-

cles. This section shows how the Dirac equation [11, 12, 13] had been developed in a covariant form depending on the axioms of quantum mechanics and special relativity.

The starting point is the relativistic energy of a free particle

$$-E^2 + \mathbf{P}^2 + m^2 = 0, \tag{1.1}$$

where we adopt the natural units ($c = \hbar = 1$), E is the particle's total energy, $\mathbf{P}$ denotes the spatial momentum and m is the mass.

Using the correspondence principle, where each observable can be replaced with it's corresponding operator, and employing the four-vector notation

$$p^\mu = (E, \mathbf{P}) = (i\frac{\partial}{\partial t}, -i\mathbf{\nabla}) = (i\frac{\partial}{\partial x_0}, -i\mathbf{\nabla}) = i\frac{\partial}{\partial x_\mu} = i\partial^\mu,$$

were $\mu = 0, 1, 2, 3$, Eq.1.1 becomes

$$(\partial_\mu \partial^\mu + m^2)\Psi = 0. \tag{1.2}$$

This last equation is the **Klein-Gordon equation**[11] for a free particle. It includes a second time-derivative, which has caused major difficulties when trying to analyze it and find its solutions. One of them is that it results in a non-positive definite probability density. Yet another one comes from the possibility of getting negative energies as well as positive ones for free particles. These difficulties has led Dirac [2] to think of another alternative equation, a one that is first order in space and time and

hermitian at the same time, i.e. of the form

$$\left(i\gamma^{\mu}\partial_{\mu} - m\right)\Psi = 0, \tag{1.3}$$

this is the **covariant form of Dirac equation**[1]. γ^{μ} are some unknowns which need to be determined so that Eq.1.3 is consistent with the relativistic energy-momentum relation (Eq.1.1) and hence results in the same energy eigenvalues that result from the Klein-Gordon equation (Eq.1.2). To fulfil this requirement, apply the operator $-(i\gamma^{\mu}\partial_{\mu} + m)$ on Eq.1.3:

$$\left(\gamma^{\mu}\partial_{\mu}\gamma^{\nu}\partial_{\nu} + m^2\right)\Psi = 0. \tag{1.4}$$

If Eq.1.4 is to be equivalent to eq.1.2, then

$\gamma^{\mu}\partial_{\mu}\gamma^{\nu}\partial_{\nu} = \gamma^{\mu}\gamma^{\nu}\partial_{\mu}\partial_{\nu} = \partial^{\mu}\partial_{\mu} = \eta^{\mu\nu}\partial_{\nu}\partial_{\mu}$, or

$$\frac{1}{2}\{\gamma^{\mu},\gamma^{\nu}\} = \eta^{\mu\nu}, \tag{1.5}$$

this last equation results in the following conditions on γ^{μ}:

$$\left(\gamma^0\right)^2 = 1, \tag{1.6}$$

$$\left(\gamma^i\right)^2 = -1; \qquad i = 1,2,3 \tag{1.7}$$

$$\gamma^{\mu}\gamma^{\nu} = -\gamma^{\nu}\gamma^{\mu}; \qquad \mu \neq \nu \tag{1.8}$$

---

[1]In the covariant form, spatial and time derivatives are multiplied by entities of the same kind. it is suitable in studying some symmetries of the equation. Other forms (not discussed here because they are out of the scope of this work) can be found in many texts on relativistic quantum mechanics [11, 14]

These last equations show that γ^μ are not just numbers. Instead they are traceless square matrices obeying Clifford algebra [15, 16]. γ^0 is hermitian while the other γ^i are anti-hermitian [14]. For them to be large enough to realize this kind of algebra, they have to be at least 4×4. So, Solutions $\Psi(\boldsymbol{x}, t)$ of the Dirac equation are at least four-dimensional vectors if it is to be represented in (3+1)-dimensional spacetime.

There is no unique choice of γ^μ. Here is the most common representation of them

$$\gamma^0 = \begin{pmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & -\mathbf{1} \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} \mathbf{0} & \sigma^i \\ -\sigma^i & \mathbf{0} \end{pmatrix}, \tag{1.9}$$

where $\mathbf{0}$ is a 2×2 null matrix, $\mathbf{1}$ is a 2×2 identity matrix. σ^i are the Pauli spin matrices

$$\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.10}$$

The interpretation of the solutions of Dirac equation has come up with some surprising results [13] that changed the visualization of subatomic world, like accepting the existence of antiparticles, and the concept of spinor field to represent particles. Physicists were ready to adopt those dramatic changes in their thoughts about matter, particularly upon the successful quantitative representation of some experimental observations that were unresolved until that time (late twenties of the last century) like the fine structure of the Hydrogen atom and the description of some properties that were already known of atomic and subatomic matter, like spin, without the need to any extra

principles. So, the Dirac equation has became a satisfactory Lorentz-invariant quantum theory of matter.

There is too much yet can be said about the Dirac equation and its solutions and interpretations. But that will not be closely related to our present field of discussion. Interested readers can check the bibliography [11, 13, 14, 12, 2, 17].

1.3 The NLDE

In this section, we proceed from the linear Dirac equation to NLDE and hence reach to the equations that acquire the interest of this work. To make the transition reasonable, at least from the logical point of view, I choose first to present a brief discussion of the issues which have led to the development of NLDE; namely, the interactions among the spinor fields. Then, I list some of the main forms of NLDE from literature to have the reader be aware of the fact that NLDE can be defined in various settings that differ from each other in the considerations taken into account, including the mode of coupling among the interacting fields and whether the particle representations of those fields are assumed to be massless or have some finite mass.

1.3.1 Interactions among spinor fields

The few decades followed the formulation of Dirac equation have witnessed a huge theoretical advances in physics. There are many reasons behind these advances. First, the new concepts introduced by the equation have encouraged theorists to make further investigations searching for more obvious image about the nature of matter and its behavior. Second, some experiments have been conducted in the thirties of the last century on nuclear radiation have resulted in observations lacked to a reasonable ex-

planation. They needed some new satisfactory modelling more than the single-particle Dirac equation. The path to such new modelling has appeared with the explanation of the beta-deacy presented by Fermi in 1933 [18] (Known as the Fermi theory of beta decay), in which he could justify that phenomenon by proposing an interactions among four spinor fields.

The above considerations has encouraged physicists to look for an extension to the Dirac equation that can cover a wider range of natural phenomena and real systems by including the interactions of the spinor fields in the new model. It was realized since the thirties of the last century that the interactions among the relativistic particles with intrinsic spin can have a limiting effect in the behavior of the systems and the reason for many experimental observations.

Starting from that point, various paths were followed and many different formulations were developed, where each of them had been built on its own different grounds and assumptions. Their aim was to develop a unified field theory of matter and its interactions. Ivanenko suggested in 1938 [19] a nonlinear generalization to the Dirac equation

$$[\gamma_\mu p^\mu - m + \lambda\left(\bar{\Psi}\Psi\right)]\Psi(\mathbf{x}) = 0, \tag{1.11}$$

where p^μ is the four-momentum operator, Ψ is the spinor wave function and $\bar{\Psi} = \Psi^\dagger \gamma_0$. $\Psi^\dagger$ is the hermitian conjugate of Ψ.

Heisenberg [20, 21] together with his collaborators [22] tried in a series of published papers to set a general unified theory by quantizing the field. They went for the

idea that any complete theory of interacting particles has to include nonlinearities to account for the self-interactions. Heisenberg [21, 23] proposed the equation

$$\gamma_\mu \frac{\partial \Psi}{\partial x_\mu} = A \gamma_\nu \Psi \left(\Psi^\dagger \gamma_\nu \Psi \right), \tag{1.12}$$

where A is a constant parameter with the dimension of square of a length.

On the other hand, Finkelstein *et al.* [24, 25], Schiff [26] and Malenka [27] have proposed a classical nonlinear spinor-field theory, where they also tried to find exact and approximate solutions [23] . Thirring [4] suggested that a nonlinear model of interaction can be soluble if it is formulated in one rather than three dimensions. He proposed in the same reference a Lagrangian density for the spinor field in (1+1)-dimensions (one time dimension and one spatial dimension) in which the self-interaction among spinor fields is represented by the term

$$\mathscr{L}' = \lambda \bar{\Psi} \Psi \bar{\Psi} \Psi, \tag{1.13}$$

where Ψ in the last equation represents a wave function with components representing the interacting spinor fields. Where this form of the Lagrangian density was suggested by Ivanenko [19, 28] but in three spatial dimensions.

The above nonlinear representations of self interactions among particles has formed the basis for various models of NLDE. Yet other formulations were developed assuming different paths and considerations in modelling interactions occur in nature, particularly in the fields of high energy physics, which are out of the scope of this thesis.

Interested readers can check the bibliography [13, 29].

1.3.2 Models of the NLDE

The attempts to develop a unified theory of matter through the nonlinear models mentioned in the previous discussion, i.e. Eqs. 1.11 and 1.12, have faced much difficulties [23, 30] inspite of some interesting inferences. Some of those problems were the inability to include gravitation, and the disagreement of some expected values of masses and charges with experiments. Furthermore, there was a problem in solving these equations completely due to geometrical complications [4]. These problems led the subsequent research to the direction of looking for a solvable models that can describe specific systems in condensed matter and particle physics. The start in such point of view came by Thirring [4, 31] who suggested that reducing the dimensionality of the equations can greatly simplify the problem and result in a solvable model. He depended on the (1+1)-dimensional Lagrangian density that he obtained in Eq. 1.13 to develop a set of coupled nonlinear equations describing the self interactions of spinor fields. Another form of NLDE was derived by M. Soler [5], where he showed that an equation of the form

$$i\gamma^{\mu}\partial_{\mu}\Psi - m\Psi + 2\lambda\left(\bar{\Psi}\Psi\right)\Psi = 0, \tag{1.14}$$

can have solutions interesting to physics.

Few years later, D. Gross and A. Neveu [6] used the Lagrangian density

$$\mathscr{L}_{\Psi} = \bar{\Psi}\left(i\gamma^{\mu}\partial_{\mu}\Psi\right) + \frac{1}{2}g^{2}\left(\bar{\Psi}\Psi\right)^{2}, \tag{1.15}$$

to describe interacting spinor fields in high-energy physics.

Another model of NLDE had been derived [32] by L.H. Haddad and L.D. Carr representing Bose-Einstein condensates in a honeycomb optical lattice. They started from a second-quantized Hamiltonian for a weakly interacting Boson gas to conclude a massless (2+1)-dimensional NLDE equation

$$\left(i\sigma^\mu \partial_\mu - U\bar{\Psi}A\Psi A - U\bar{\Psi}B\Psi B\right)\Psi = 0, \tag{1.16}$$

where $\Psi = \begin{pmatrix} \Psi_A & \Psi_B \end{pmatrix}^T$ is a field composed of the components Ψ_A and Ψ_B, the fields at the two sites A and B (degenerate sublattices) in the lattice unit cell, σ^i ($i = 1, 2, 3$) are Pauli matrices, U is the interaction energy and A and B are two constant matrices.

A.D. Alhaidari [7] has obtained a three-parameter massive NLDE in (1+1)-dimensions by applying variational techniques on a nonlinear self-interacting spinor Lagrangian. His model reads

$$
\begin{aligned}
i\partial_t \psi_+ =\ & m\psi_+ + \partial_x \psi_- \\
& + \left(\alpha_+ |\psi_+|^2 + \alpha_- |\psi_-|^2\right)\psi_+ + \alpha_W \left(\psi_+ \psi_-^* + \psi_+^* \psi_-\right)\psi_-,
\end{aligned}
\tag{1.17}
$$

$$
\begin{aligned}
i\partial_t \psi_- =\ & -m\psi_- - \partial_x \psi_+ \\
& + \left(\alpha_- |\psi_+|^2 + \alpha_+ |\psi_-|^2\right)\psi_- + \alpha_W \left(\psi_+ \psi_-^* + \psi_+^* \psi_-\right)\psi_+,
\end{aligned}
\tag{1.18}
$$

where $\psi_\pm(x,t)$ are the two components of the spinor field and m is the rest mass

of the particle. $\alpha_\pm = \alpha_V \pm \alpha_S$, where α_V, α_S and α_W are three dimensionless parameters defining the strengths of the different modes of coupling.

With introducing the three parameters in the last model, A.D. Alhaidari could generalize the model to cover and include some of the previously derived models and define new modes of coupling by setting the right values of those parameters. For example, the vector-vector coupling in the previous Eqs. 1.17 and 1.18 (which can be obtained by setting $\alpha_S = \alpha_W = 0$ and hence $\alpha_+ = \alpha_- = \alpha_V$) is equivalent to the massive Thirring [4] model (MTM)

$$i\partial_t \psi_+ = m\psi_+ + \partial_x \psi_- + \alpha_+ \left(|\psi_+|^2 + |\psi_-|^2 \right) \psi_+, \tag{1.19}$$

$$i\partial_t \psi_- = -m\psi_- - \partial_x \psi_+ + \alpha_+ \left(|\psi_+|^2 + |\psi_-|^2 \right) \psi_-, \tag{1.20}$$

whereas the scalar-scalar coupling which can be obtained by setting $\alpha_V = \alpha_W = 0$ and hence $\alpha_+ = -\alpha_- = \alpha_S$, is equivalent to the massive Gross-Neveu [6] model (MGNM)

$$i\partial_t \psi_+ = m\psi_+ + \partial_x \psi_- + \alpha_+ \left(|\psi_+|^2 - |\psi_-|^2 \right) \psi_+, \tag{1.21}$$

$$i\partial_t \psi_- = -m\psi_- - \partial_x \psi_+ - \alpha_+ \left(|\psi_+|^2 - |\psi_-|^2 \right) \psi_-. \tag{1.22}$$

In addition to the covering of the MTM and MGNM, Eqs. 1.17 and 1.18 can be used to define other new modes of coupling. Setting α_W to zero to get a generalized combination of the MTM and MGNM, while setting $\alpha_W = 0$ and $\alpha_S = \pm \alpha_V$ gives respectively the spin symmetric and pseudo-spin symmetric models [7].

In addition to the above mentioned models, various other models had been derived starting with different considerations and techniques. Worth noting here that apart

from the variety of the formulations, all the models have the same degree of nonlinearity, the Kerr-type nonlinearity, which has the solitonic behavior [33].

Starting from the next chapter, the focus of this thesis is devoted to MTM as defined in Eqs. 1.19 and 1.20.

Chapter 2: Conserved Quantities, Symmetries and Applications of NLDE

This chapter discusses some of the important physical properties of NLDE that can be deduced from its mathematical structure, in addition to a brief summary of recent experiments that has detected a behavior of some physical systems which can be best modelled with one of the forms of NLDE. The first section derives the continuity equation. The conservation of physical quantities is then concluded based upon the analysis of the components of the energy-momentum tensor. The second section considers the symmetries of the equations. Throughout these first two sections, the discussion is fully devoted to the MTM form of NLDE as defined by Eqs. 1.19 and 1.20

$$i\partial_t \psi_+ \;=\; m\psi_+ + \partial_x \psi_- + \alpha_+ \left(|\psi_+|^2 + |\psi_-|^2 \right) \psi_+, \tag{2.1}$$

$$i\partial_t \psi_- \;=\; -m\psi_- - \partial_x \psi_+ + \alpha_+ \left(|\psi_+|^2 + |\psi_-|^2 \right) \psi_-. \tag{2.2}$$

After that, the chapter is concluded with an abstract about some recent investigations of real systems in which the NLDE is found to be the most appropriate model for their experimental results.

2.1 The continuity equation and the energy-momentum tensor

2.1.1 The continuity equation for MTM model

The starting point is to write the (1+1)-dimensional MTM (Eqs. 2.1 and 2.2) in a compact form

$$i\gamma^\mu \partial_\mu \Psi - m\Psi - \alpha_+ \left(\bar{\Psi}\gamma^0\Psi\right)\gamma^0\Psi = 0, \tag{2.3}$$

where $\Psi = \left(\begin{array}{cc} \psi_+ & \psi_- \end{array} \right)^T$ have the components $\psi_\pm$, the two interacting spinor fields, and $\bar{\Psi} = (\Psi^*)^T . \gamma^0$, γ^μ $(\mu = 0, 1)$ are the two-dimensional Dirac matrices

$$\gamma^0 = \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \qquad \gamma^1 = i \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right). \tag{2.4}$$

The "adjoint" of Eq. 2.3, i.e. the equation for $\bar{\Psi}$ have the form

$$-i\partial_\mu \bar{\Psi}\gamma^\mu - m\bar{\Psi} - \alpha_+ \left(\bar{\Psi}\gamma^0\Psi\right)\bar{\Psi}\gamma^0 = 0. \tag{2.5}$$

Using this notation, the current vector j^μ can be defined as

$$j^\mu = \bar{\Psi}\gamma^\mu\Psi, \tag{2.6}$$

where the components

$$j^0 = \bar{\Psi}\gamma^0\Psi$$

$$= \begin{pmatrix} \psi_+^* & \psi_-^* \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^2 \cdot \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \qquad (2.7)$$

$$= \psi_+ \psi_+^* + \psi_- \psi_-^*$$

$$= |\psi_+|^2 + |\psi_-|^2,$$

and

$$j^1 = \bar{\Psi}\gamma^1\Psi$$

$$= \begin{pmatrix} \psi_+^* & \psi_-^* \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix} \qquad (2.8)$$

$$= i \left(\psi_+^* \psi_- - \psi_+ \psi_-^* \right),$$

Define respectively the density $\rho(x,t)$ and the current $j(x,t)$ of the self-interacting spinor fields

$$\rho(x,t) \quad = \quad j^0(x,t) = |\psi_+|^2 + |\psi_-|^2 \qquad (2.9)$$

$$j(x,t) \quad = \quad j^1(x,t) = i \left(\psi_+^* \psi_- - \psi_+ \psi_-^* \right). \qquad (2.10)$$

It can be easily shown that j^μ is conserved. First, re-write Eqs. 2.3 and 2.5

respectively in the forms

$$\gamma^\mu \partial_\mu \Psi \;=\; -i\left[m\Psi + \alpha_+ \left(\bar{\Psi}\gamma^0\Psi \right) \gamma^0 \Psi \right], \tag{2.11}$$

$$\left(\partial_\mu \bar{\Psi} \right) \gamma^\mu \;=\; i\left[m\bar{\Psi} + \alpha_+ \left(\bar{\Psi}\gamma^0\Psi \right) \bar{\Psi}\gamma^0 \right], \tag{2.12}$$

use the last two Eqs. 2.11 and 2.12 to expand $\partial_\mu j^\mu$

$$\begin{aligned}
\partial_\mu j^\mu &= \partial_\mu \left(\bar{\Psi}\gamma^\mu \Psi \right) \\[4pt]
&= \bar{\Psi}\partial_\mu \left(\gamma^\mu \Psi \right) + \left(\partial_\mu \bar{\Psi} \right) \gamma^m u\Psi \\[4pt]
&= -i\bar{\Psi}\left[m\Psi + \alpha_+ \left(\bar{\Psi}\gamma^0\Psi \right) \gamma^0\Psi \right] + i\left[m\bar{\Psi} + \alpha_+ \left(\bar{\Psi}\gamma^0\Psi \right) \bar{\Psi}\gamma^0 \right] \Psi \\[4pt]
&= 0.
\end{aligned} \tag{2.13}$$

The conservation of j^μ is recognized as a continuity equation. It can be written in the equivalent forms

$$\begin{aligned}
\partial_\mu j^\mu &= \partial_t j^0 + \partial_x j^1 \\[4pt]
&= \partial_t \left(\psi_+ \psi_+^* + \psi_- \psi_-^* \right) + i\partial_x \left(\psi_+^* \psi_- - \psi_-^* \psi_+ \right) \\[4pt]
&= 0,
\end{aligned} \tag{2.14}$$

or

$$\partial_t \rho(x,t) + \partial_x j(x,t) = 0, \tag{2.15}$$

which is the mathematical statement of the conservation of the current density $\rho(x,t)$.

2.1.2 A wave equation for the field density

A condition on the quantity $\partial_\nu j^\mu$, where $\nu \neq \mu$ can be derived for the MTM (Eqs. 2.1 and 2.2), in addition to the continuity equation $\partial_\mu j^\mu = 0$ defined above in Eqs. 2.13 and 2.14. A direct algebraic method is employed this time. The purpose from getting the second condition is to use it, together with the continuity equation, to obtain a wave equation for the density $\rho(x,t)$ defined in Eq. 2.9.

The starting point is the three-parameter model of NLDE derived by A.H. Alhaidari [7], Eqs. 1.17 and 1.18. Set $\alpha_W = 0$ to get a *combined model* of MTM and MGNM

$$i\partial_t \psi_+ = m\psi_+ + \partial_x \psi_- + \left(\alpha_+ |\psi_+|^2 + \alpha_- |\psi_-|^2\right)\psi_+, \tag{2.16}$$

$$i\partial_t \psi_- = -m\psi_- - \partial_x \psi_+ + \left(\alpha_- |\psi_+|^2 + \alpha_+ |\psi_-|^2\right)\psi_-, \tag{2.17}$$

These last two equations involve the MTM (set $\alpha_+ = \alpha_-$) and the MGNM (set $\alpha_+ = -\alpha_-$) in addition to other models defining different modes of couplings [7, 102] (refer to subsection 1.3.2).

The adjoint of Eqs. 2.16 and 2.17 are respectively

$$i\partial_t \psi_+^* = -m\psi_+^* - \partial_x \psi_-^* - \left(\alpha_+ |\psi_+|^2 + \alpha_- |\psi_-|^2\right)\psi_+^*, \tag{2.18}$$

$$i\partial_t \psi_-^* = m\psi_-^* + \partial_x \psi_+^* - \left(\alpha_- |\psi_+|^2 + \alpha_+ |\psi_-|^2\right)\psi_-^*, \tag{2.19}$$

multiply Eq. 2.16 by $-\psi_-^*$, Eq. 2.17 by ψ_+^*, Eq. 2.18 by ψ_- and Eq. 2.19 by

$-\psi_+$ to yield respectively the following four equations:

$$-i\psi_-^*\partial_t\psi_+ = -m\psi_-^*\psi_+ - \psi_-^*\partial_x\psi_-$$
$$- \left(\alpha_+|\psi_+|^2 + \alpha_-|\psi_-|^2\right)\psi_-^*\psi_+,$$

(2.20)

$$i\psi_+^*\partial_t\psi_- = -m\psi_+^*\psi_- - \psi_+^*\partial_x\psi_+$$
$$+ \left(\alpha_-|\psi_+|^2 + \alpha_+|\psi_-|^2\right)\psi_+^*\psi_-,$$

(2.21)

$$i\psi_-\partial_t\psi_+^* = -m\psi_-\psi_+^* - \psi_-\partial_x\psi_-^*$$
$$- \left(\alpha_+|\psi_+|^2 + \alpha_-|\psi_-|^2\right)\psi_-\psi_+^*,$$

(2.22)

$$-i\psi_+\partial_t\psi_-^* = -m\psi_+\psi_-^* - \psi_+\partial_x\psi_+^*$$
$$+ \left(\alpha_-|\psi_+|^2 + \alpha_+|\psi_-|^2\right)\psi_+\psi_-^*,$$

(2.23)

add Eqs. 2.20 through 2.23 to get upon simplifying

$$i\partial_t\left(\psi_+^*\psi_- - \psi_+\psi_-^*\right) = -\partial_x\left(|\psi_+|^2 + |\psi_-|^2\right)$$
$$- \left[2m + (\alpha_+ - \alpha_-)(|\psi_+|^2 - |\psi_-|^2)\right]\left(\psi_+^*\psi_- + \psi_+\psi_-^*\right).$$

(2.24)

Write the last equation in a more compact form by using Eqs. 2.9 and 2.10 and

introducing the new notation

$$\mu(x,t) \;=\; |\psi_+|^2 - |\psi_-|^2 \tag{2.25}$$

$$\tau(x,t) \;=\; \psi_+^* \psi_- + \psi_+ \psi_-^*, \tag{2.26}$$

to end with the following condition

$$\partial_t j(x,t) + \partial_x \rho(x,t) = -[2m + \mu(x,t)(\alpha_+ - \alpha_-)]\tau(x,t), \tag{2.27}$$

which can be written in terms of the components of j^μ as

$$\partial_t j^1 + \partial_x j^0 = -[2m + \mu(x,t)(\alpha_+ - \alpha_-)]\tau(x,t). \tag{2.28}$$

This last equation states the required condition on $\partial_\nu j^\mu, \nu \neq \mu$. Now we are ready to derive a wave equation for $\rho(x,t)$. Differentiate Eq. 2.15 with respect to t to get:

$$\partial_{tt}\rho(x,t) + \partial_{xt} j(x,t) = 0, \tag{2.29}$$

and Eq. 2.27 with respect to x to get:

$$\partial_{tx} j(x,t) + \partial_{xx}\rho(x,t) = -\partial_x\{[2m + \beta(x,t)(\alpha_+ - \alpha_-)]\tau(x,t)\}, \tag{2.30}$$

Now subtract Eq. 2.30 from Eq. 2.29 with keeping in mind that $\partial_{xt} j(x,t) =$

$\partial_{tx}j(x,t)$, we end with the equation

$$\partial_{tt}\rho(x,t) - \partial_{xx}\rho(x,t) = \partial_x\{[2m + \mu(x,t)(\alpha_+ - \alpha_-)]\tau(x,t)\}, \tag{2.31}$$

to get a wave equation for the density $\rho(x,t)$. It holds for both of the MTM, MGNM and any other model with $\alpha_W = 0$ in Eq. 1.17. As a special case of interest, for the MTM, where $\alpha_+ = \alpha_-$, the density wave equation becomes

$$\partial_{tt}\rho(x,t) - \partial_{xx}\rho(x,t) = \partial_x[2m\tau(x,t)], \tag{2.32}$$

2.1.3 Energy-momentum tensor for NLDE

Now we turn to the determination of some conserved quantities in a system composed of self-interacting spinor fields described by MTM. The field representation of such systems (a consequence of the equivalence between energy and mass in relativity, and the dual nature of subatomic entities according to quantum mechanics) implies the consideration of the Lagrangian density together with the energy-momentum tensor [12, 34] as the best tools to specify the conserved quantities.

The lagrangian density $\mathscr{L}$ of the field defined by Eq. 2.3 (or Eq. 2.5 as well) can be written as

$$\mathscr{L} = \frac{i}{2}\left(\bar{\Psi}\gamma^\mu\partial_\mu\Psi - \partial_\mu\bar{\Psi}\gamma^\mu\Psi\right) - m\bar{\Psi}\Psi - \frac{\alpha_+}{2}\left(\bar{\Psi}\gamma^0\Psi\right)\left(\bar{\Psi}\gamma^0\Psi\right), \tag{2.33}$$

from which the MTM in the form of Eq. 2.3 results as the Euler-Lagrange

equation

$$\partial_\mu \left[\frac{\partial \mathscr{L}}{\partial \left(\partial_\mu \bar{\Psi} \right)} \right] - \frac{\partial \mathscr{L}}{\partial \bar{\Psi}} = 0. \tag{2.34}$$

The form of $\mathscr{L}$ defined in Eq. 2.33 can be simplified by doing a trick to terminate the derivatives, $\gamma^\mu \partial_\mu \Psi$ and $\left(\partial_\mu \bar{\Psi} \right) \gamma^\mu$, from its expression by replacing them with their equivalent forms defined in Eqs. 2.11 and 2.12, respectively. Applying this trick would give $\mathscr{L}$ the form

$$\begin{aligned}
\mathscr{L} &= \frac{i}{2} \bar{\Psi} \{ -i \left[m\Psi + \alpha_+ \left(\bar{\Psi} \gamma^0 \Psi \right) \gamma^0 \Psi \right] \} \\
&\quad - \frac{i}{2} \{ i \left[m\bar{\Psi} + \alpha_+ \left(\bar{\Psi} \gamma^0 \Psi \right) \bar{\Psi} \gamma^0 \right] \} \Psi \\
&\quad - m\bar{\Psi}\Psi - \frac{\alpha_+}{2} \left(\bar{\Psi} \gamma^0 \Psi \right) \left(\bar{\Psi} \gamma^0 \Psi \right),
\end{aligned} \tag{2.35}$$

Which simplifies to

$$\mathscr{L} = \frac{\alpha_+}{2} \left(\bar{\Psi} \gamma^0 \Psi \right) \left(\bar{\Psi} \gamma^0 \Psi \right). \tag{2.36}$$

Now we develop an expression for the energy-momentum tensor $T^{\mu\nu}$ for MTM. For a complex field Φ, $T^{\mu\nu}$ is defined as [34]

$$T^{\mu\nu} = \frac{\partial \mathscr{L}}{\partial \left(\partial_\mu \Phi \right)} \partial^\nu \Phi + \frac{\partial \mathscr{L}}{\partial \left(\partial_\mu \bar{\Phi} \right)} \partial^\nu \bar{\Phi} - g^{\mu\nu} \mathscr{L}, \tag{2.37}$$

where the metric $g^{\mu\nu}$ is equivalent to γ^0. For the MTM (Eqs. 2.3 and 2.5), it

becomes

$$T^{\mu\nu} = \frac{i}{2}\left(\bar{\Psi}\gamma^\mu\partial^\nu\Psi - \partial^\nu\bar{\Psi}\gamma^\mu\Psi\right) - g^{\mu\nu}\frac{\alpha_+}{2}\left(\bar{\Psi}\gamma^0\Psi\right)\left(\bar{\Psi}\gamma^0\Psi\right). \tag{2.38}$$

There are here two points worth mentioning regarding the significance of $T^{\mu\nu}$. First, each element of it represents a physical quantity [12] of the system. In general, any element $T^{\mu\nu}$ represents the momentum flux (pressure) across a surface element $\sigma_{\mu\nu}$. Specifically for our (1+1)-dimensional system described by MTM, T^{00} is equivalent to the Hamiltonian (energy) density $\mathscr{H}$

$$\mathscr{H} = \frac{i}{2}\left(\bar{\Psi}\gamma^0\partial_t\Psi - \partial_t\bar{\Psi}\gamma^0\Psi\right) + \frac{\alpha_+}{2}\left(\bar{\Psi}\gamma^0\Psi\right)\left(\bar{\Psi}\gamma^0\Psi\right), \tag{2.39}$$

where the contravariant derivatives are switched into covariant ones using the relation

$$\partial^\nu = g^{\nu\lambda}\partial_\lambda. \tag{2.40}$$

The element T^{01} defines the momentum flux $\partial_x P$ in the x-direction

$$T^{01} = -\frac{i}{2}\left(\bar{\Psi}\gamma^0\partial_x\Psi - \partial_x\bar{\Psi}\gamma^0\Psi\right) = \partial_x P, \tag{2.41}$$

whereas the element T^{10} represents the time derivative of momentum flux $\partial_{tx}P$

$$T^{10} = \frac{i}{2}\left(\bar{\Psi}\gamma^1\partial_t\Psi - \partial_t\bar{\Psi}\gamma^1\Psi\right), \tag{2.42}$$

and the last element, T^{11}, represents the momentum flux, or the pressure

$$T^{11} = -\frac{i}{2}\left(\bar{\Psi}\gamma^1\partial_x\Psi - \partial_x\bar{\Psi}\gamma^1\Psi\right) + \frac{\alpha_+}{2}\left(\bar{\Psi}\gamma^0\Psi\right)\left(\bar{\Psi}\gamma^0\Psi\right). \tag{2.43}$$

Let us now turn to the second point. It is about the way of analyzing $T^{\mu\nu}$ in purpose to reveal the existence of any external forces affecting the considered system. In the case of absence of any external forces, $T^{\mu\nu}$ must be conserved, which is the present situation of our system composed of free self-interacting spinor fields. The mathematical assertion of this requirement can be achieved by verifying that $\partial_\mu T^{\mu\nu} = 0$. Here is a proof for that property.

Write $\partial_\mu T^{\mu\nu}$ in terms of $\partial_\mu\mathscr{L}$

$$\partial_\mu T^{\mu\nu} = \frac{i}{2}\partial_\mu\left(\bar{\Psi}\gamma^\mu\partial^\nu\Psi - \partial^\nu\bar{\Psi}\gamma^\mu\Psi\right) - g^{\mu\nu}\partial_\mu\mathscr{L}, \tag{2.44}$$

Using the symmetry of $g^{\mu\nu}$ ($g^{\mu\nu} = g^{\nu\mu}$) together with the property stated in Eq. 2.40, one can write

$$g^{\mu\nu}\partial_\mu = g^{\nu\mu}\partial_\mu = \partial^\nu, \tag{2.45}$$

apply it together with commutativity of derivatives ($\partial_\mu\partial^\nu = \partial^\nu\partial_\mu$) in expanding the form of $\partial_\mu T^{\mu\nu}$ in Eq. 2.44 as

$$\begin{aligned}
\partial_\mu T^{\mu\nu} = \frac{i}{2}\big[&\left(\partial_\mu\bar{\Psi}\right)\gamma^\mu\partial^\nu\Psi + \bar{\Psi}\partial_\nu\gamma^\mu\partial_\mu\Psi \\
&- \partial^\nu\left(\partial_\mu\bar{\Psi}\right)\gamma^\mu\Psi - \partial^\nu\bar{\Psi}\gamma^\mu\partial_\mu\Psi\big] - \partial^\nu\mathscr{L},
\end{aligned} \tag{2.46}$$

use Eqs. 2.11 and 2.12 to replace $\partial_\mu \bar{\Psi} \gamma^\mu$ and $\gamma^\mu \partial_\mu \Psi$ in Eq. 2.46 by their corresponding expressions and simplify to get

$$\partial_\mu T^{\mu\nu} = \frac{\alpha_+}{2} \{ - \left(\bar{\Psi} \gamma^0 \Psi \right) \bar{\Psi} \gamma^0 \partial^\nu \Psi + \bar{\Psi} \partial^\nu \left[\left(\bar{\Psi} \gamma^0 \Psi \right) \gamma^0 \Psi \right]$$
$$+ \partial^\nu \left[\left(\bar{\Psi} \gamma^0 \Psi \right) \bar{\Psi} \gamma^0 \right] \Psi - \left(\partial^\nu \bar{\Psi} \right) \left[\left(\bar{\Psi} \gamma^0 \Psi \right) \gamma^0 \Psi \right] \} - \partial^\nu \mathscr{L}, \tag{2.47}$$

or

$$\partial_\mu T^{\mu\nu} = \frac{\alpha_+}{2} \left[- \left(\bar{\Psi} \gamma^0 \Psi \right) \partial^\nu \left(\bar{\Psi} \gamma^0 \Psi \right) + 3 \left(\bar{\Psi} \gamma^0 \Psi \right) \partial^\nu \left(\bar{\Psi} \gamma^0 \Psi \right) \right] - \partial^\nu \mathscr{L}$$
$$= \frac{i}{2} \partial^\nu \left[\left(\bar{\Psi} \gamma^0 \Psi \right) \left(\bar{\Psi} \gamma^0 \Psi \right) \right] - \partial^\nu \mathscr{L} \tag{2.48}$$
$$= 0,$$

which proves the conservation of $T^{\mu\nu}$.

2.1.4 Integrals of the conserved quantities

In all throughout this section, many physical quantities were defined for the self-interacting spinor fields and their conservation was proved. The integration of those quantities results in turn in the evaluation of a corresponding conserved physical properties of the involved system. Here we summarize the integrals that define some of these properties as a quick reference, because we will come back to them in chapter 4, to evaluate the properties that can be deduced from the exact solutions we solve for.

The first of these properties is the total energy of the system E. It can be defined as the integral of the Hamiltonian density $\mathscr{H}$ that we already obtained in Eq.2.39

$$E = \int_{-\infty}^{+\infty} \mathscr{H}(x,t) dx, \tag{2.49}$$

whereas the current j^μ, that is defined in 2.6 and proved to be conserved in 2.13, can be used to obtain the conserved total charge Q by the integration of j^0

$$Q = \int_{-\infty}^{+\infty} j^0(x,t)dx. \tag{2.50}$$

The total momentum P is represented as the integration of T^{01} that is defined in Eq. 2.42

$$P = \int_{-\infty}^{+\infty} T^{01}(x,t)dx. \tag{2.51}$$

All of the above integrals were done in one spatial dimension according to the dimensionality of the MTM considered in this work.

2.2 The symmetry-breaking nature of NLDE

One of the major properties of MTM is the fact that it violates Lorentz symmetry. That will be evident by applying a Lorentz transformation on the Lagrangian density. Consider the Lorentz transformation

$$x^\mu \to x'^\nu = \Lambda_\mu^\nu x^\mu, \tag{2.52}$$

where x^μ is a vector in the spacetime and Λ_μ^ν is a matrix representation of the Lorentz transformation for vectors. On the other hand, the spinor Ψ and its adjoint $\bar{\Psi}$

transform respectively as

$$\Psi \rightarrow \Psi' = \Lambda_{\frac{1}{2}} \Psi, \tag{2.53}$$

$$\bar{\Psi} \rightarrow \bar{\Psi}' = \bar{\Psi} \Lambda_{\frac{1}{2}}^{-1}, \tag{2.54}$$

where $\Lambda_{\frac{1}{2}}$ is a Lorentz representation for spinors [35]. Let us find the transformation of the Lagrangian density defined in Eq. 2.36

$$\begin{aligned} \mathscr{L} &\rightarrow \mathscr{L}' \\ &\rightarrow \frac{\alpha_+}{2} \left(\bar{\Psi}' \gamma^0 \Psi' \right) \left(\bar{\Psi}' \gamma^0 \Psi' \right) \\ &\rightarrow \frac{\alpha_+}{2} \left(\bar{\Psi} \Lambda_{\frac{1}{2}}^{-1} \gamma^0 \Lambda_{\frac{1}{2}} \Psi \right) \left(\bar{\Psi} \Lambda_{\frac{1}{2}}^{-1} \gamma^0 \Lambda_{\frac{1}{2}} \Psi \right), \end{aligned} \tag{2.55}$$

using the property

$$\Lambda_{\frac{1}{2}}^{-1} \gamma^\mu \Lambda_{\frac{1}{2}} = \Lambda_\nu^\mu \gamma^\nu, \tag{2.56}$$

one finds that

$$\mathscr{L}' = \frac{\alpha_+}{2} \left(\bar{\Psi} \Lambda_\nu^0 \gamma^\nu \Psi \right) \left(\bar{\Psi} \Lambda_\nu^0, \gamma^\nu \Psi \right) \tag{2.57}$$

This last result shows that MTM violates Lorentz symmetry since $\mathscr{L}'$ is not equivalent to $\mathscr{L}$.

Relativistic symmetry breaking has became a common notion in many of the nonlinear interaction theories. Some theorists justify that notion by considering self-

interactions as local properties that not necessarily satisfy Lorentz invariance, while others assume that those interactions have to be included in a general relativistic formulation instead of special relativity.

For more on the symmetries of NLDE, the reader can check [32].

2.3 NLDE and the recent advances in condensed matter

Inspite of the fact that the nonlinear extensions of Dirac equation had been developed as early as only few years upon the development of Dirac equation itself, those nonlinear models have waited up until the last decade to get realized directly in experiments. They, of course, have been used during all these past decades in justifying and interpreting many experimental observations, especially in high energy physics, but what we seek here is a direct experimental investigation of real systems behaving according to NLDE.

That time-gap between theoretical settings and experimental realizations can be justified. The fundamental interactions that occur among particles on the sub-micro level are usually not detectable on the macro-scale, due to the fact that when considering large ensembles of particles, other properties of matter that add up, like mass, become responsible for the appearance of other dominant forces, like gravitation; while almost all interactions on the particle scale cancel out each other and hence vanish on the large scale. So, it has became evident at the present that detecting fundamental interactions and explaining natural phenomena among particles comprise a big challenge to encounter. On the other hand, Physicists were able a long time ago to push charged particles into relativistic limits in the lab. But that was accomplished inside a strong external fields, and again, those external fields in turn overcome and terminate any in-

teraction among those particles. Then, the difficulty was to generate a relativistic system of controllable and trackable self-interacting particles without the need to plunge that system inside a strong field.

The last two decades have witnessed an experimental advances, particularly in condensed matter and material science, that helped to get over the previous difficulties and to create real systems where the relativistic behavior together with the self-interactions can be investigated in a relatively simple experiments and instrumentations. In this section, two such experimental fields are briefly considered, namely, the electron transport in graphene [36] and the Bose-Einstein condensates [32], which are analogous to each other and even have been developed in-parallel so that the results of one field of them has pushed towards new results of the other. Also considered here a presentation of the results of a study on waveguide arrays [37, 38, 10, 39, 40] that shows that waves can behave according to NLDE.

2.3.1 Electron transport in graphene

In 1947 [41], physicists have started to argue the possibility of producing a two dimensional material from graphite as they have predicted theoretically extraordinary properties to make significant advances in industry. Until appropriate technical methods have developed, particularly those stem from nanoscience, the experimental trials to create such material have started in the early 90's [42] with achieving as low as 150 to 200 layers of atoms. In 2004, A. K. Geim, K. S. Novoselov and co-workers have succeeded in producing graphene with only few layers of carbon, the achievement upon which Geim and Novoselov were honoured the 2010 Nobel prize in physics.

Then, (later on became) Nobel prize winners proceeded to the investigation of the electrical properties of graphene through a series of published works in 2005 through 2007 [43, 44, 36] , where they ended up with a surprising observations and results. The major one was related to the electron transport, where they found that the electron's speed in graphene is comparable to the effective speed of light in graphene (around $10^6 m/s$), which means that a relativistic behavior of those electrons can be attained and investigated at a relatively low speeds. As expected, the data and observations have shown a behavior of the electrons as a massless (also had been expected theoretically before as a result of the linear dispersion relation [45]) fermions that can be described by the Dirac equation rather than the Schrödinger equation.

The work mentioned above has triggered a successive series of further experimental investigations in many research institutes on the electrical characteristics of graphene, focusing especially on the noticeably distinctive "ballistic" electron transport [46] and asserting the previously obtained results in addition to concluding that the interactions among electrons plays an important role that has to be involved in any modelling of that phenomenon together with introducing also an electron-phonon interaction in order to explain the noticed significant effect of temperature on the behavior of the electrons.

After almost a decade of the work on developing a satisfactory modelling to describe those interactions, no obvious complete theory has been adopted until this present time. There is a large amount of data and many factors need to be included in any modelling theory. Physicists in the field are following different paths [47, 48, 49, 50, 45], ranging from introducing correction factors to anticipating the principles of relativistic

classical and quantum field theories. A heavy work is still in progress.

2.3.2 Bose-Einstein condensation in hexagonal lattice

In analogy to the electron behavior in the hexagonal lattice structure of graphene discussed in the last section, it was verified experimentally [32, 51, 52] that when a boson gas is brought into the state of Bose-Einstein condensation and configured in a hexagonal optical lattice, its constituents show a behavior that can be described by NLDE. It has became evident that the interactions among the bosons significantly modulate their behavior. Few years ago [53], L.H. Haddad and L.D. Carr have investigated theorectically a system involving Bose-Einstein condensates of a boson gas in a honeycomb optical lattice, and derived an NLDE (Eq. 1.16) for the boson gas with Kerr-nonlinearity, then deduced numerical solutions for that NLDE they derived.

On the other hand, C. Kam and R. Liu [54], and A. Cappellaro and co-workers [56], have considered a nonlinear extension to the Schrödinger equation as a model to represent interactions among the constituents of the boson gas in the Bose-Einstein condensate. So, again, the variety of the factors considered, together with the significant variations among the results at different conditions, have reflected a variation in the formulations among the published works. Actually, the work still continues in this relatively new field.

2.3.3 NLDE in waveguide arrays

Another style is being adopted in studying and predicting the behavior of NLDE systems and self-interactions of spinor fields by simulating them with waveguide arrays [38, 10, 40]. That way of modelling physical systems has became common [39] in studying

solitary wave dynamics in lattice structures.

T. X. Tran, S. Longhi, and F. Biancalana [38] have presented a study of the non-linear effects of NLDE and the behavior of Dirac solitons by exposing a system of binary waveguide arrays numerically and analytically. In another article, A. Marini, S. Longhi and F. Biancalana [10] analyzed a coupled-mode equations to model the propagation of binary waveguide arrays with a modulated refractive index, as a simulation of a NLDE representing a system in high-energy physics; namely the neutrino oscillations. Their model has enabled them to consider the two cases of interacting and non-interacting neutrinos. Within the same field, J. Cuevas-Maraver [40] and co-workers have studied a NLDE by visualizing it as the continuum limit of a binary waveguide array.

The findings of the above optical models and the similar works can greatly help to provide a basis for further direct theoretical and experimental investigation of systems described by NLDE.

Chapter 3: Exact Techniques of Solving Nonlinear Evolution Equations

3.1 Introduction

The current understanding of natural phenomena implies obviously the realization of the nonlinear science to be vital to a vast range of theoretical and applied fields in mathematics, physics and engineering. Its importance to physicists is enhanced by the invention of many fundamental processes, especially during the last century, that are represented by nonlinear systems of equations. For instance, The explanation of the mechanism of beta decay in the thirties of the last century has came up with a nonlinear model in order to describe the Fermi four-spinor interaction theory. Another example can be taken from the solid state physics. The linear dispersion equation has failed to predict the right values of the heat capacity of solids, whereas the model that is recognized to fit well with the practically known values is the cubic nonlinear dispersion equation.

The general notion in scientific community is that the natural phenomena are represented by linear models. All the known fundamental laws of nature are described by linear mathematical formulations. But those fundamental laws can tell nothing about any interactions between the system constituents. If such interactions have an evident effect on the behavior of the system, then that effect must be taken into account while developing the model, which can be achieved by introducing nonlinear coupling terms to the original fundamental equation and hence converting it into a nonlinear equation.

Than notion has been already observed in many fields ranging from dynamics [59], to optics [60, 61, 62], to condensed matter and to high energy physics. In some special situations, those effects can be defined as a perturbation, with keeping in mind that such a technique fails when the coupling parameter is not small enough to perform such perturbation. The alternative in this case is to handle the nonlinear equation itself.

Finding an exact solution for a nonlinear equation can be (and in fact this is the case in many situations) a serious challenge. The source of that challenge lies mainly in the absence of a thorough classification of them, and then no general technique is identified for a specific classification of the equations. They do not obey the superposition principle, and so, no general solution can be identified. They can have -if they do exist- many different solutions. Even the solution of a nonlinear initial or boundary value problem can considerably change in form if one changes the initial or boundary conditions. Add to that, the specific exact technique that is being developed to solve a given nonlinear equation, will not necessarily work successfully for other equations, without at least modifying the procedure. So, generally speaking, the techniques of solving nonlinear equations exactly are usually related to a specific equation, rather than to a family of equations.

On the other hand, The study of nonlinear systems has enriched the natural sciences with new fields, like chaos and soliton theory, in addition to the development of a variety of the numerical approximation and exact analytical techniques, as well as the techniques of verifying the integrability of equations and the stability of their solutions.

Since the work in the whole thesis is concerned with exact analytical solutions

of NLDE, we devote our discussion to exact techniques, although one can find a great deal of numerical and approximation techniques. In the next section we present a quick review of the methods that we adopt in the next chapter when we discuss the solutions of NLDE. Readers interested to know more about other techniques can find detailed discussions in the bibliography [33, 63, 64].

3.2 Lax pair and Darboux transformations

This section presents the formulation of one of the very widely employed methodologies for analyzing nonlinear evolution equations, namely, the *Lax pair formulation*. Apart from its importance that it gained upon leading to a successful exact solving of many renowned equations, like the ***nonlinear Schrödinger equation (NLS)*** [65] and ***the Korteweg de-Vries equation (KDV)*** [66], we are giving this discussion as a basis for our work in the next chapter when we find the Lax pair for NLDE. Also in this section, we present the technique of finding exact solutions by using the *Darboux transformations*, which can be thought of as one of the possible choices to adopt as a technique of obtaining exact solutions form the Lax pair.

3.2.1 Lax formulation

In 1968, Peter Lax [67] proposed that a given nonlinear evolution equation can be related with some linear spectral problem. He suggested a spectral problem of the form

$$\mathscr{L}\phi = \lambda\phi \tag{3.1}$$

$$\partial_t\phi = \mathscr{A}\phi, \tag{3.2}$$

where $\mathscr{L}$ and $\mathscr{A}$ are two linear operators in the auxiliary function $\phi(x,t,\lambda)$.

λ is a complex spectral parameter. In order to simplify the treatment, he imposed a restriction on λ to be independent of x, i.e. to be either a constant or a function in time only. With this restriction, he could conclude a condition imposed on the operators in the above spectral problem. That condition, which is known as *Lax condition* or *compatibility condition*, can be understood as the already known requirement of the commutativity of the derivative (i.e. $\partial_{xt} = \partial_{tx}$). But inspite of its simplicity, it has led to a very useful condition as we show below. Let us first derive that compatibility condition. Differentiate both sides of Eq. 3.1 with respect to time

$$\partial_t\left(\mathscr{L}\phi\right) = \partial_t\left(\lambda\phi\right)$$

$$\rightarrow \left(\partial_t\mathscr{L}\right)\phi + \mathscr{L}\partial_t\phi = \left(\partial_t\lambda\right)\phi + \lambda\partial_t\phi \qquad (3.3)$$

$$\rightarrow \left(\partial_t\mathscr{L}\right)\phi + \mathscr{L}\mathscr{A}\phi = \left(\partial_t\lambda\right)\phi + \lambda\mathscr{A}\phi,$$

or

$$\{\partial_t\mathscr{L} + [\mathscr{L},\mathscr{A}]\}\phi = \left(\partial_t\lambda\right)\phi, \qquad (3.4)$$

where $[\mathscr{L},\mathscr{A}] := \mathscr{L}\mathscr{A} - \mathscr{A}\mathscr{L}$ is the commutator of $\mathscr{L}$ and $\mathscr{A}$. If we choose to consider λ to be independent of time

$$\{(\partial_t\mathscr{L}) + [\mathscr{L},\mathscr{A}]\}\phi = 0 \qquad ;\partial_t\lambda = 0. \qquad (3.5)$$

Eqs. 3.4 and 3.5 represent the compatibility condition or the Lax condition for the integrability of the spectral problem defined by Eqs. 3.1 and 3.2. This condi-

tion comprises the direct link between the spectral problem and the nonlinear evolution equation we want to solve. What one needs is to find the two operators $\mathscr{L}$ and $\mathscr{A}$ as defined in Eqs. 3.1 and 3.2 in such a way that their Lax condition is verified by the validity of the evolution equation.

For a more clarification, we mention the great work that has inspired Lax while developing this formulation. One year earlier -1967- [66], Gardner, Green, Kruskal and Miura have considered the KDV equation

$$\partial_t u - 6u\partial_x u + \partial_{xxx} u = 0, \tag{3.6}$$

where they made a connection between the solutions of the above equation and those for a linear spectral problem. According to Lax, he generalized their idea to the formulation defined above in Eqs. 3.1 and 3.2. He reconsidered the spectral problem for the KDV equation according to his generalization and defined the operators $\mathscr{L}$ and $\mathscr{A}$ as

$$\mathscr{L}\phi \;=\; (\partial_{xx} - u)\,\phi = \lambda\,\phi \tag{3.7}$$

$$\mathscr{A}\phi \;=\; [4\partial_{xxx} - 6u\partial_x - 3(\partial_x u)]\,\phi = \partial_t\phi. \tag{3.8}$$

Let us show that the KDV equation is equivalent to the compatibility condition for the operators defined in Eqs. 3.7 and 3.8

$$\partial_t\mathscr{L} = -\partial_t u, \tag{3.9}$$

and for $\mathscr{L}\mathscr{A}$

$$\mathscr{L}\mathscr{A} = 4\partial_{xxxxx} - 10u\partial_{xxx} - 15(\partial_x u)\partial_{xx}$$
$$+ \left(6u^2 - 12(\partial_{xx}u)\partial_x\right) + \left(3u\partial_x u - 3\partial_{xxx}\right), \tag{3.10}$$

while for $\mathscr{A}\mathscr{L}$

$$\mathscr{A}\mathscr{L} = 4\partial_{xxxxx} - 10u\partial_{xxx} - 15(\partial_x u)\partial_{xx}$$
$$+ \left(6u^2 - 12(\partial_{xx}u)\partial_x\right) + \left(9u\partial_x u - 4\partial_{xxx}\right), \tag{3.11}$$

so, evaluating the expression $\partial_t \mathscr{L} + [\mathscr{L}, \mathscr{A}]$ gives

$$\partial_t \mathscr{L} + [\mathscr{L}, \mathscr{A}] = -\left(\partial_t u - 6u\partial_x u + \partial_{xxx}u\right), \tag{3.12}$$

which shows that the validity of KDV equation, Eq. 3.6 implies the validity of the compatibility condition in Eq. 3.5.

Now, upon finding the Lax pair for a given nonlinear equation, which represents the corresponding linear spectral problem, one starts looking for a solutions to that spectral problem by following one of the techniques known in the literature, or even to develop another technique according to the nature of the problem. Regarding the repeatedly used exact techniques in literature, one can check for the ***inverse scattering transform (IST)*** [33], which has been used first to solve NLS equation [65]. It can be considered as the nonlinear version of Fourier transform. Another technique that proved its power in solving some of the equations is the ***Darboux transformation***, which is to be discussed in the last part of this present section. It is a special case of the more general

Bäcklund transformation [76, 77]. There are various other techniques in the literature, most of them had been developed within the last few decades, although some of them are based upon mathematical theories existed since the 19th century. The interested reader is forwarded to the bibliography [33, 63, 64].

In the occasion of talking about Lax pair in the literature, and in order to have reader be familiar with some terms that are frequently used, it worths mentioning that the solution of nonlinear equation (that one seeks from the whole story and denoted as $u(x,t)$ in the above Lax formulation) is some times known as *the potential of the spectral problem.*

We conclude this discussion with some remarks. First, there is no general structural method to construct the Lax pair. If one checks for them in the literature, one will find a very big variety. Some of them rely upon algebraic procedures [33], others involve the use of the relatively modern mathematical theories like groups, Lie algebras and manifolds [68]. Even the algebraic procedures are very different from each other in almost every single article. The way that a practitioner follows depends on his style and on the nature of the equation. Second, although the Lax pair had been successfully obtained for many of the equations, still many more are still without a Lax pair until this moment inspite of the attempts to calculate it. There is no such a test to confirm a priori whether a given equation has a Lax pair. Third, a common sense has appeared that the successful calculation of the Lax pair for a given equation is thought of as a strong evidence that the equation is square-integrable. This is not a theory, but it is of special concern in physics since the integrable equations are of special importance due to the physical meaning they could possess. Some equations are verified to have a Lax pair

but they don't have even a solution, others had been solved completely with physically acceptable solutions without having a Lax pair. The last note, if one could find a Lax pair for a given equation, one is likely able to find another ones for the same equation. In other words, a given equation can have in general many Lax pairs. There is more than a Lax pair for a given equation.

In the next part of the section we discuss a more general form of the Lax pair; constructed [33] by Ablowitz, Kaup, Newell and segur in 1974, based on Lax's formulation itself and on a successful work [65] done by Zakharov and Shabat (in 1972) in solving exactly NLS equation.

3.2.2   Matrix form of the Lax pair

According to the generalization done by Ablowitz and his coworkers, a spectral problem can be written in the form

$$\partial_x \Phi \;=\; \mathbf{U}\Phi \tag{3.13}$$

$$\partial_t \Phi \;=\; \mathbf{V}\Phi, \tag{3.14}$$

where Φ is a n-dimensional vector. $\mathbf{U}$ and $\mathbf{V}$ are n-dimensional square matrices.

The compatibility condition for this form can be derived by ensuring the requirement that $\partial_{xt}\Phi = \partial_{tx}\Phi$

$$\partial_x (\mathbf{V}\Phi) = \partial_t (\mathbf{U}\Phi)$$

$$\rightarrow (\partial_x \mathbf{V})\Phi + \mathbf{V}\partial_x \Phi = (\partial_t X)\Phi + \mathbf{U}\partial_t \Phi, \tag{3.15}$$

$$\rightarrow (\partial_x \mathbf{V})\Phi + \mathbf{V}\mathbf{U}\Phi = (\partial_t \mathbf{U})\Phi + \mathbf{U}\mathbf{V}\Phi$$

or

$$\partial_t \mathbf{U} - \partial_{\mathbf{x}} \mathbf{V} + [\mathbf{U}, \mathbf{V}] = \mathbf{0}. \tag{3.16}$$

The matrices $\mathbf{U}$ and $\mathbf{V}$ are also called Lax pair. This formulation written above had been first applied on the NLS equation

$$i\partial_t \psi + \partial_{xx} \psi + 2\sigma |\psi|^2 \psi = 0, \tag{3.17}$$

where $\sigma = \pm 1$ is a constant parameter. q and r are two complex potentials for the spectral problem. Zakharov and Shabat have defined the matrices $\mathbf{U}$ and $\mathbf{V}$ as

$$\mathbf{U} = \begin{pmatrix} i\lambda & iq \\ ir & -i\lambda \end{pmatrix} \tag{3.18}$$

$$\mathbf{V} = \begin{pmatrix} 2i\lambda^2 - iqr & 2i\lambda q + \partial_x q \\ 2i\lambda r - \partial_x r & -2i\lambda^2 + iqr \end{pmatrix}, \tag{3.19}$$

then applied them in the compatibility condition, Eq. 3.16, so they got

$$i\partial_t r + \partial_{xx} r + 2qr^2 = 0 \tag{3.20}$$

$$i\partial_t q + \partial_{xx} q - 2qr^2 = 0, \tag{3.21}$$

choosing $q = \pm r^*$ gives the two different cases $\sigma = \pm 1$ respectively of NLS Eq. 3.17. After that, they used the technique of inverse scattering transform (IST) to derive exact solutions. About two decades later [71], V. B. Matveev and A. M. Salle have considered the above spectral problem of the NLS equation using the Darboux

transformation, where they could find a series of multisoliton solutions, and hence they presented the advantage of that technique in obtaining a series of solutions as well be discussed in the next and last part of this section.

3.2.3 Darboux transformations

Following to the construction of the spectral problem (the Lax pair) defined by Eqs. 3.13 and 3.14 is the stage of finding its solutions (if they do exist), because the solution of that spectral problem involves the determination of its potential, $\psi(x,t)$, which is the solution of the original evolution equation we want to solve from the beginning. Solving such a problem can be challenging more than what it may seem. Even the procedure of the solution is almost always tedious and lengthy. Various techniques [64] are already developed to handle such spectral problems. Our interest here is to present a brief discussion of the formulations and tools to be used in the next chapter, which include the method of *Darboux transformation*.

Let us borrow some historical notes [77] in order to have the reader be aware of the logic that has guided to the technique of Darboux transformation in its form adopted in this thesis. The origin starts in 1882, when Darboux [74] has considered the covariance of Sturm-Liouville problems under gauge transformations and derived a transformation formula that generates a second solution from a first already-known one. His formulation waited for almost 70 years, until Crum [75] constructed an iterated Darboux transformations for the Sturm-Liouville problem. In 1975, Wadati et al. adopted Crum's method to solve some integrable nonlinear evolution equations and generate multi-soliton solutions. Another transformation technique was developed in parallel to Darboux transformation, namely Bäcklund transformation, which also had

been developed in 1882 by Bäcklund in differential geometry as a method to define pseudospherical surfaces with constant negative Gaussian curvatures, and then adopted in the seventies of the last century to find solutions of nonlinear equations.

The strength of Darboux and Bäcklund transformations become very obvious in nonlinear equations, since such equations can have infinitely many solutions, which can basically be constructed using iterative transformations. That justifies why both of the above transformations have waited for almost seven decades until the development of soliton theory in the course of analyzing nonlinear partial differential equations.

Now we develop the mathematical formulation of Darboux transformation. The spectral problem defined in Eq. 3.13 can be generalized [71] according to the form

$$\partial_x \Phi = \sum_n \sum_{k=1}^{N(n)} \mathbf{U}_{kn} \Phi \mathbf{M}_n^k + \sum_{j=0}^{N} \mathbf{V}_j \Phi \Lambda^j, \tag{3.22}$$

where Φ is a $m \times m$ matrix, Λ and $\mathbf{M}_n$ are diagonal matrices so that

$$\Lambda = \text{diag.}(\lambda_1, ..., \lambda_m),$$

$$\mathbf{M}_n = \text{diag.}((\lambda_1 - a_n)^{-1}, ..., (\lambda_m - a_n)^{-1}),$$

and $N(n) \leq n$.

It can be proved [71, 73, 77] that the above spectral problem, Eq. 3.22 is *covariant* under the transformation

$$\Phi[1] = \Phi \Lambda - \sigma \Phi, \qquad \sigma = \Phi_1 \Lambda_1 \Phi_1^{-1}, \tag{3.23}$$

(with Φ_1 is some known fixed solution of Eq. 3.22 at a given specified Λ_1) i.e. $\Phi[1]$ is a solution of the spectral problem

$$\partial_x \Phi[1] = \sum_n \sum_{k=1}^{N(n)} \mathbf{U}_{kn}[1]\Phi[1]\mathbf{M}_n^k[1] + \sum_{j=0}^{N} \mathbf{V}_j[1]\Phi[1]\Lambda^j, \tag{3.24}$$

Given that the transformed coefficients $\mathbf{U}_{kn}[1]$ and $\mathbf{V}_j[1]$ satisfy the following conditions

$$\mathbf{V}_N[1] = \mathbf{V}_N, \tag{3.25}$$

$$\mathbf{V}_{N-k}[1] = \mathbf{V}_{N-k} + \sum_{j=1}^{k} [\mathbf{V}_{N-k+j}, \sigma]\sigma^{j-1}; k = 1, ..., N \tag{3.26}$$

$$\mathbf{U}_{N(n),n}[1] = \sigma_n \mathbf{U}_{N(n),n}\sigma_n^{-1}, \tag{3.27}$$

$$\mathbf{U}_{N(n)-k,n}[1] = \sigma_n\{\mathbf{U}_{N(n)-k,n} + \sum_{j=1}^{k} [\mathbf{U}_{N(n)-k+j,n}, \sigma_n^{-1}]\}\sigma_n^{-1} \tag{3.28}$$

$$k = 1, 2, ..., N(n) - 1, \qquad \sigma_n = \sigma - a_n. \tag{3.29}$$

The definition of Darboux transformation according to Eq. 3.23 can be applied to get a new solution $\Phi[1]$ given that: the Lax pair is obtained for the evolution equation in hand and extended to the form of Eq. 3.22, in addition to having some known solution Φ_1 together with its fixed matrix of parameters Λ_1. This known solution can be considered as a set of column eigenfunctions ϕ_k corresponding to the eigenvalues λ_k. The Darboux transformation can be defined in equivalent different settings and assumptions, we adopt here in this section the formulation that will be employed in chapter four to solve MTM. The reader can check the bibliography for more details and other forms [71, 73, 76, 77].

Chapter 4: Exact Analytical Solutions of NLDE

4.1 Introduction

Now we reach the point of being ready to discuss the solutions obtained for NLDE after the preliminary material discussed in the previous chapters. We start with a survey from literature for the solutions of NLDE that had been derived before, in order to show the current status of the progress in solving NLDE. That will be covered in section 2. Section 3 shows a complete derivation of a stationary exact localized solutions for MTM and MGNM models, while section 4 derives a Lax pair for MTM, and then applies a Darboux transformation on that obtained Lax pair to obtain the same solutions done already before using analytical techniques. The chapter is then concluded with a discussion of the physical properties that can be inferred from the obtained solutions, including a calculation of the conserved quantities which comprise the components of the current vector and energy-momentum tensor identified in section 2.1.

4.2 A survey of the solutions in the literature

Before proceeding to the solutions obtained within the work on this thesis, where I find it as an essential step to do is to present a brief recap of the work which has been done so far in the aim of finding solutions of some of the NLDE models. The motivation here is the tremendous work (both in quantity and in diversity of the techniques) that many theorists and researchers teams have done in this topic.

The huge work done in the attempts to solve NLDE can be justified in many

ways. First, the early nonlinear models of unified field theory of matter and proposed mainly by Heisenberg [20, 21, 30] and his coworkers had run into difficulty as they resulted in a diverging integrals [78]. So, it was necessary at that time to resolve that problem in any models (namely the NLDE models) replacing the unified theory in practical fields of research. Second, as discussed in the previous chapters, the NLDE had been defined in different settings and considered whenever self-interacting systems are being involved in the fields of modern theoretical and applied physics, and its solutions can reveal much of the mystery about the behavior of such systems. Even the recent advances in experimental condensed matter has added more importance to the study of NLDE as well as the NLS model. Third, The Kerr nonlinearity included in NLDE models has converted them into a challenging problems to solve. That point in particular explains the diversity of the techniques and methods that one can find in the literature in solving NLDE models.

After all, the published work on solving NLDE is too vast to be covered in one section. Here we present specifically chosen works that show the techniques adopted so far. The reader is forwarded to the references of the works to be mentioned below for a deeper inspection.

The existence of localized solutions for NLDE had been targeted in many articles. Maria J. Esteban and Eric Séré [85, 86] proved the existence of some NLDE equations using general variational techniques. M. Balabane [82, 83, 80, 79, 84] and coworkers have proved the existence of infinitely many stationary states for NLDE. Thomas Bartsch and Yanheng Ding [87] proved, using variational methods, the existence of infinitely many geometrically different solutions of some classes of NLDE.

The stability of solutions had been extensively discussed by J. Werle [88, 89], who applied variable initial conditions on NLDE (and on nonlinear Klein-Gordon equation as well) to obtain restrictions which end with stable localized solutions. Furthermore, Pierre Mathieu and T.F. Morris [90] showed an existence of conditions for a localized stationary solutions of a general class of NLDE under the requirement that the nonlinearity approaches zero faster than the field itself.

Numerical solutions of NLDE can be found in the work of J. Xu, S. Shao, and H. Tang [94, 95] as well as in that of N. Bournaveas and G. E. Zourarisn [93]

Among the notable successes in obtaining exact solutions is that done by S. Y. Lee [96] *et al.*, who have defined a classical Lagrangian density for interacting massive fermions in (1+1)-dimensions. They proved that the energy-momentum tensor vanishes for such a model and concluded that it have to possess exact stationary solutions. They used direct analytic methods to obtain and write explicitly the solutions for different modes of interactions. L. H. Haddad , C. M. Weaver and L. D. Carr [53] found analytical and numerical solutions for a massless NLDE model. F. Cooper, A. Khare, and coworkers [97, 98, 99, 100, 101, 101] have considered the NLDE in a series of papers, where they provided an extensive analysis of different models, including the derivation of analytic stationary solutions and discussing their stability. U. Al Khawaja [102] has derived analytic solutions for NLDE in the cases of spin symmetric and pseudo-spin symmetric modes of interactions. Some of the derived solutions were localized while others were oscillatory.

4.3   New stationary solutions of MTM and MGNM

In this section, we derive new exact solutions of MTM (Eqs. 1.19 and 1.20) and MGNM (Eqs. 1.21 and 1.22). Analytic methods are employed here to find stationary localized solutions of the form

$$\psi_+(x,t) \;=\; \phi_+(x)e^{i\theta_+(t)}, \tag{4.1}$$

$$\psi_-(x,t) \;=\; \phi_-(x)e^{i\theta_-(t)}, \tag{4.2}$$

where $\phi_+(x)$, $\phi_-(x)$, $\theta_+(t)$, $\theta_-(t)$ are all real functions to be determined up to a constant factor dependent on initial and boundary conditions. Within the framework, all through this section, all the constant factors resulting from integrations will be scaled to zero since we are not considering a specific system.

There is something that can be done to help us have an idea about the form of the above four functions and hence reduce the ambiguity in them; which is to apply the above forms of the solutions in the continuity equation (Eq. 2.15) and in the condition on the expression $\partial_\nu j^\mu, \nu \neq \mu$ identified in Eq. 2.27.

4.3.1 Conditions imposed by the conservation of current

Apply the solution forms defined in Eqs. 4.1 and 4.2 in Eqs. 2.9, 2.10, 2.25 and 2.26 to evaluate respectively $\rho(x,t)$, $j(x,t)$, $\mu(x,t)$ and $\tau(x,t)$

$$\rho(x) = \phi_+^2(x) + \phi_-^2(x), \tag{4.3}$$

$$j(x,t) = 2\phi_+(x)\phi_-(x)sin[\theta_+(t) - \theta_-(t)], \tag{4.4}$$

$$\mu(x) = \phi_+^2(x) - \phi_-^2(x), \tag{4.5}$$

$$\tau(x,t) = 2\phi_+(x)\phi_-(x)cos[\theta_+(t) - \theta_-(t)]. \tag{4.6}$$

Apply the above forms in the continuity equation 2.15

$$\partial_x \left[2\phi_+(x)\phi_-(x)sin[\theta_+(t) - \theta_-(t)] \right] = 0, \tag{4.7}$$

which means that at least one of the following requirements must be satisfied: $\phi_+(x)\phi_-(x)$ is a constant independent of x, $\phi_+(x)$ is reciprocal to $\phi_-(x)$, or $sin[\theta_+(t) - \theta_-(t)]$.

The first option is excluded since it results in a non-localized wave solutions, which is irrelevant to the physical systems. The second option means that the localization of one of them leads to a second blowing up function and then it is also excluded. It remains for us to adopt the third option, which is equivalent to the requirement that the phase functions $\theta_+(t)$ and $\theta_-(t)$ are equivalent to each other or differ by multiples of π. This last requirement can result in a behavior of a physical system. For simplicity, let us assume that they are equivalent to each other and further, restrict the discussion to sys-

tems with energy equivalent to λ and hence $\theta_+(t) = \theta_-(t) = -\lambda t$. So, the components of the spinor field become

$$\psi_+(x,t) = \phi_+(x)e^{-i\lambda t}, \tag{4.8}$$

$$\psi_-(x,t) = \phi_-(x)e^{-i\lambda t}, \tag{4.9}$$

With the recognition of the last requirement, some of the previous forms will update. One major of them is that the current $j(x,t)$ identically vanishes. Let us list all of the forms down for a quick reference

$$\rho(x) = \phi_+^2(x) + \phi_-^2(x), \tag{4.10}$$

$$j = 0, \tag{4.11}$$

$$\mu(x) = \phi_+^2(x) - \phi_-^2(x), \tag{4.12}$$

$$\tau(x) = 2\phi_+(x)\phi_-(x). \tag{4.13}$$

Now let us see whether any other modification or relation could result from the condition on $\partial_\nu j^\mu, \nu \neq \mu$ which is implied by Eq. 2.27. Applying the above forms in it would yield

$$\frac{d}{dx}\rho(x) = -[2m + \mu(x)(\alpha_+ - \alpha_-)]\tau(x). \tag{4.14}$$

Now we are ready to proceed to the second stage together with the results and forms concluded so far.

4.3.2 Solutions of MTM using analytical methods

This part of the section is devoted to MTM. Apply the forms of $\psi_\pm(x,t)$ defined in Eqs. 4.8 and 4.9 in the MTM Eqs. 1.19 and 1.20

$$\phi_+(x)[m-\lambda+\alpha((\phi_+(x))^2+(\phi_-(x))^2)]+\phi'_-(x) \;=\; 0, \tag{4.15}$$

$$\phi_-(x)[m+\lambda-\alpha((\phi_+(x))^2+(\phi_-(x))^2)]+\phi'_+(x) \;=\; 0, \tag{4.16}$$

and multiply equation 4.15 by $2\phi_-(x)$ and equation 4.16 by $2\phi_+(x)$ yield

$$2\phi_+(x)\phi_-(x)\left[m-\lambda+\alpha\left(\phi_+^2(x)+\phi_-^2(x)\right)\right]$$
$$+2\phi_-(x)\phi'_-(x)=0, \tag{4.17}$$

$$2\phi_+(x)\phi_-(x)\left[m+\lambda-\alpha\left(\phi_+^2(x)+\phi_-^2(x)\right)\right]$$
$$+2\phi_+(x)\phi'_+(x)=0. \tag{4.18}$$

The last two equations can be replaced with another yet equivalent set of equations. Add Eqs. 4.17 and 4.18 one time, and subtract Eq. 4.17 from 4.18 respectively another time to get

$$4m\phi_+(x)\phi_-(x)+\frac{d}{dx}\left[\phi_+^2(x)+\phi_-^2(x)\right]=0, \tag{4.19}$$

$$4\phi_+(x)\phi_-(x)\{-\lambda + \alpha[\phi_+^2(x) + \phi_-^2(x)]\}$$
$$- \frac{d}{dx}[\phi_+^2(x) - \phi_-^2(x)] = 0. \tag{4.20}$$

It is easy to see that the last two equations can have simpler form if they can be written in terms of $\rho(x)$ and $\mu(x)$ and eliminate $\phi_\pm(x)$ from the system with Noting that

$2\phi_+(x)\phi_-(x) = \sqrt{\rho^2(x) - \mu^2(x)}$

$$2m\sqrt{\rho^2(x) - \mu^2(x)} + \rho'(x) = 0, \tag{4.21}$$
$$2\sqrt{\rho^2(x) - \mu^2(x)}(-\lambda + \alpha\rho(x)) - \mu'(x) = 0, \tag{4.22}$$

where the prime ($'$) means a first derivative. This last system can be decoupled. From equation 4.21: $2\sqrt{\rho^2(x) - \mu^2(x)} = -\rho'(x)/m$. Use it to remove the square root and its constituents from equation 4.22 then simplify

$$-\lambda\rho'(x) + \alpha\rho(x)\rho'(x) + m\mu'(x) = 0 \tag{4.23}$$

this last equation can be directly integrated to give

$$-\lambda\rho(x) + \frac{\alpha}{2}\rho^2(x) + m\mu(x) = C \tag{4.24}$$

where C is a constant of integration. Since we are not handling any special system, we can scale C to zero and proceed in the solving. So, let us solve equation 4.24

for $\mu(x)$ then square it with assuming $C = 0$

$$\mu^2(x) = \left(\frac{\alpha}{2m} \rho^2(x) - \frac{\lambda}{m} \rho(x) \right)^2, \tag{4.25}$$

but from equation 4.21

$$\mu^2(x) = \rho^2(x) - \frac{(\rho'(x))^2}{4m^2}, \tag{4.26}$$

then, we end with the following equation in the function $\rho(x)$ only

$$\rho^2(x) - \frac{(\rho'(x))^2}{4m^2} = \left(\frac{\alpha}{2m} \rho^2(x) - \frac{\lambda}{m} \rho(x) \right)^2, \tag{4.27}$$

re-write the last equation for $\rho'(x)$

$$\rho'(x) = \pm \rho(x) \sqrt{4m^2 - (\alpha \rho(x) - 2\lambda)^2}, \tag{4.28}$$

which can be solved analytically and completely. It has two different solutions:

One of them is valid for $\lambda = m$ and the other one holds for $\lambda \neq m$:

4.3.3 CASE I: $\lambda = m$

In this case, direct integration of equation 4.28 gives the solution:

$$\rho(x) = \frac{4m}{\alpha} \left(\frac{1}{4m^2 x^2 + 1} \right), \tag{4.29}$$

we have directly scaled the constant of integration to zero. Now apply the last expression in equation 4.24 to solve for $\mu(x)$

$$\mu(x) = \frac{4m}{\alpha} \left[\frac{4m^2x^2 - 1}{(4m^2x^2 + 1)^2} \right], \tag{4.30}$$

use equations 4.10, 4.12, 4.29 and 4.30 to solve for $\phi_{\pm}(x)$:

$$\phi_+(x) = \pm\sqrt{\frac{\rho(x) + \mu(x)}{2}} = \sqrt{\frac{m}{\alpha}} \frac{4mx}{4m^2x^2 + 1}, \tag{4.31}$$

$$\phi_-(x) = \pm\sqrt{\frac{\rho(x) - \mu(x)}{2}} = \sqrt{\frac{m}{\alpha}} \frac{2}{4m^2x^2 + 1}, \tag{4.32}$$

where the $\pm$ sign in front of the square root in the last two equations means that both of the signs are solutions given that the chosen sign is the same for both of the functions. So, one ends with the final solutions

$$\psi_+(x,t) = \sqrt{\frac{m}{\alpha}} \frac{4mx}{4m^2x^2 + 1} e^{-imt}, \tag{4.33}$$

$$\psi_-(x,t) = \sqrt{\frac{m}{\alpha}} \frac{2}{4m^2x^2 + 1} e^{-imt}, \tag{4.34}$$

which corresponds to a localized single-soliton solution. In case $\lambda = -m$, same solutions are retrieved but with interchanging the expressions of ψ_+ and ψ_-.

4.3.4   CASE II: $\lambda \neq m$

Both of the two possibilities imposed by the ($\pm$) signs in the equation 4.28 ended with the same solution in case I above. But here in case II, the (+) sign have ended with a solution apparently different from that which the ($\neg$) sign ended with. Here we considered the first solution and discarded the second one. The reason is that the second one

is diverging at some values of the position x while the first solution is converging and rather localized. So, our discussion from now on is restricted to the equation

$$\rho'(x) = \rho(x)\sqrt{4m^2 - (\alpha\rho(x) - 2\lambda)^2}, \tag{4.35}$$

let us make the following transformation

$$2m\beta(x) = \alpha\rho(x) - 2\lambda, \tag{4.36}$$

use it to re-write equation 4.35 in terms of $\beta(x)$:

$$\beta' = (2m\beta + 2\lambda)\sqrt{1 - \beta^2}, \tag{4.37}$$

where β' is the x-derivative of the real function β. Integrating the last equation gives the solution (with equating the constant of integration to zero):

$$\beta = -\frac{\lambda}{m} + \frac{m^2 - \lambda^2}{-\lambda m + m^2 \cosh\left(2x\sqrt{m^2 - \lambda^2}\right)}, \tag{4.38}$$

using this last solution in equation 4.36 and solving for ρ

$$\rho = \frac{2\left(m^2 - \lambda^2\right)}{\alpha\left[-\lambda + m\cosh\left(2x\sqrt{m^2 - \lambda^2}\right)\right]}, \tag{4.39}$$

now, from the last equation and equation 4.24, one could get μ:

$$\mu = -\frac{2\left(m^2 - \lambda^2\right)\left(m - \lambda\cosh\left(2x\sqrt{m^2 - \lambda^2}\right)\right)}{\alpha\left[\lambda - m\cosh\left(2x\sqrt{m^2 - \lambda^2}\right)\right]^2}, \tag{4.40}$$

finally, it remains to solve for ϕ_+ and ϕ_-:

$$\begin{aligned}
\phi_+(x) &= \pm\sqrt{\frac{\rho(x) + \mu(x)}{2}} \\
&= \pm\sqrt{\frac{m - \lambda}{\alpha}[-1 + \cosh(2x\sqrt{m^2 - \lambda^2})]\frac{m + \lambda}{\lambda - m\cosh(2x\sqrt{m^2 - \lambda^2})}},
\end{aligned} \tag{4.41}$$

$$\begin{aligned}
\phi_-(x) &= \pm\sqrt{\frac{\rho(x) - \mu(x)}{2}} \\
&= \pm\sqrt{\frac{m + \lambda}{\alpha}[1 + \cosh(2x\sqrt{m^2 - \lambda^2})]\frac{m - \lambda}{\lambda - m\cosh(2x\sqrt{m^2 - \lambda^2})}},
\end{aligned} \tag{4.42}$$

upon checking for the $\pm$ signs by applying in the original MTM Eqs. 1.19 and 1.20, we found that the signs for ϕ_+ and ϕ_- must be opposite. So, we can write

$$\begin{aligned}
&\phi_+(x) = \\
&\sqrt{\frac{m - \lambda}{\alpha}\left[-1 + \cosh\left(2x\sqrt{m^2 - \lambda^2}\right)\right]\frac{m + \lambda}{\lambda - m\cosh\left(2x\sqrt{m^2 - \lambda^2}\right)}},
\end{aligned} \tag{4.43}$$

$$\begin{aligned}
&\phi_-(x) = \\
&\sqrt{\frac{m + \lambda}{\alpha}\left[1 + \cosh\left(2x\sqrt{m^2 - \lambda^2}\right)\right]\frac{\lambda - m}{\lambda - m\cosh\left(2x\sqrt{m^2 - \lambda^2}\right)}},
\end{aligned} \tag{4.44}$$

and then the following final solutions are concluded

$$\psi_+(x,t) = \sqrt{\frac{m-\lambda}{\alpha/2}}\left[\frac{(m+\lambda)\sinh\left(x\sqrt{m^2-\lambda^2}\right)}{\lambda-m\cosh\left(2x\sqrt{m^2-\lambda^2}\right)}\right]e^{-i\lambda t}, \tag{4.45}$$

$$\psi_-(x,t) = \sqrt{\frac{m+\lambda}{\alpha/2}}\left[\frac{(\lambda-m)\cosh\left(x\sqrt{m^2-\lambda^2}\right)}{\lambda-m\cosh\left(2x\sqrt{m^2-\lambda^2}\right)}\right]e^{-i\lambda t}, \tag{4.46}$$

where the expressions under the square roots are positive valued regardless of the value of λ wether it is greater or less than m. The shape and properties of the solution varies according to the value of λ. It is localized when $|\lambda| < m$, while it becomes oscillatory for $|\lambda| > m$. For the latter case the solution can be rewritten as

$$\psi_+(x,t) = \sqrt{\frac{\lambda-m}{\alpha/2}}\left[\frac{(m+\lambda)\sin\left(x\sqrt{\lambda^2-m^2}\right)}{\lambda-m\cos\left(2x\sqrt{\lambda^2-m^2}\right)}\right]e^{-i\lambda t}, \tag{4.47}$$

$$\psi_-(x,t) = \sqrt{\frac{m+\lambda}{\alpha/2}}\left[\frac{(\lambda-m)\cos\left(x\sqrt{\lambda^2-m^2}\right)}{\lambda-m\cos\left(2x\sqrt{\lambda^2-m^2}\right)}\right]e^{-i\lambda t}. \tag{4.48}$$

As justified below in the Discussions section, the case of interest in physical applications is only that with $|\lambda| = m$. The other cases are physically unimportant: They are mentioned explicitly here just for the purpose of sake of completeness.

4.3.5 Solutions of MGNM using analytical methods

Following the procedure explained in the previous section, we obtain stationary solutions of MGNM equations defined in Eqs. 1.19 and 1.20, namely

$$\psi_+(x,t) = \sqrt{-\frac{m-\lambda}{\alpha/2}}\left[\frac{(m+\lambda)\cosh\left(x\sqrt{m^2-\lambda^2}\right)}{m+\lambda\cosh\left(2x\sqrt{m^2-\lambda^2}\right)}\right]e^{-i\lambda t}, \tag{4.49}$$

$$\psi_-(x,t) = \sqrt{-\frac{m+\lambda}{\alpha/2}}\left[\frac{(m-\lambda)\sinh\left(x\sqrt{m^2-\lambda^2}\right)}{m+\lambda\cosh\left(2x\sqrt{m^2-\lambda^2}\right)}\right]e^{-i\lambda t}. \tag{4.50}$$

The last solution shows that the wave functions totally vanish for $|\lambda| = m$. For $|\lambda| < m$, the parameter α must be negative to fulfil the condition assumed from the beginning that $\phi_\pm(x)$ are real, Whereas no solutions are defined for $|\lambda| > m$.

4.4 Lax pair and Darboux transformations of MTM

In this section, the same exact stationary solutions of Thirring model derived in the last section are to be re-addressed, but this time by following a different path, namely by obtaining a Lax pair for the model and then applying the Darboux transformation on it. The advantage gained here lies in the fact that basically a series of many other solutions can be concluded by repeatedly applying Darboux transformation.

4.4.1 Lax pair of MTM

In this section we represent derivatives by subscripts: $\partial_t \Phi := \Phi_t$ and $\partial_x \Phi := \Phi_x$ which saves space, in addition to the fact it's the most commonly used notation in this field.

We found that the coupled equations 1.19 and 1.20 admit the following Lax pair

$$\Phi_x = \mathbf{U_0}\Phi, \tag{4.51}$$

$$\Phi_t = \mathbf{V_0}\Phi, \tag{4.52}$$

where Φ is 4×4 matrix with arbitrary function components, and:

$$
\mathbf{U_0} = \begin{pmatrix}
-\frac{1}{2} \alpha J(x,t) & \psi_+(x,t) & 0 & 0 \\[2em]
m \alpha \psi_+^*(x,t) & -\frac{1}{2} \alpha J(x,t) & 0 & 0 \\[2em]
0 & 0 & \frac{1}{2} \alpha J(x,t) & \lambda \psi_-(x,t) \\[2em]
0 & 0 & -\frac{m\alpha}{\lambda} \psi_-(x,t)^* & \frac{1}{2} \alpha J(x,t)
\end{pmatrix}, \tag{4.53}
$$

$$
\mathbf{V_0} = \begin{pmatrix}
-\frac{1}{2}i \alpha \rho(x,t) - \frac{i}{2} m & -i \psi_-(x,t) & 0 & 0 \\[2em]
i m \alpha \psi_-(x,t)^* & \frac{1}{2}i \alpha \rho(x,t) + \frac{i}{2} m & 0 & 0 \\[2em]
0 & 0 & -\frac{1}{2}i \alpha \rho(x,t) + \frac{i}{2} m & i \lambda \psi_+(x,t) \\[2em]
0 & 0 & \frac{i m \alpha}{\lambda} \psi_+(x,t)^* & \frac{1}{2} i \alpha \rho(x,t) - \frac{i}{2} m
\end{pmatrix}. \tag{4.54}
$$

Assuming solutions of the stationary forms defined in Eqs. 4.8 and 4.9, the

Lax-pair takes the simpler form

$$
\mathbf{U_0} = \begin{pmatrix} \begin{pmatrix} 0 & \phi_+(x)\, e^{i\lambda t} \\[2em] \alpha\, m\, \phi_+(x)\, e^{-i\lambda t} & 0 \end{pmatrix} & \mathbf{0} \\[4em] \mathbf{0} & \begin{pmatrix} 0 & \lambda\, \phi_-(x) e^{i\lambda t} \\[2em] -\frac{\alpha m}{\lambda}\, \phi_-(x) e^{-i\lambda t} & 0 \end{pmatrix} \end{pmatrix} \tag{4.55}
$$

$$
\mathbf{V_0} = \begin{pmatrix} \begin{pmatrix} -\frac{1}{2} i\, \alpha\, \rho(x) - \frac{i}{2}\, m & -i\, \phi_-(x)\, e^{i\lambda t} \\[2em] i\, \alpha\, m\, \phi_-(x)\, e^{-i\lambda t} & \frac{1}{2} i\, \alpha\, \rho(x) + \frac{i}{2}\, m \end{pmatrix} & \mathbf{0} \\[4em] \mathbf{0} & \begin{pmatrix} \frac{i}{2}\, m - \frac{1}{2} i\alpha\rho(x) & i\, \lambda\, \phi_+(x)\, e^{i\lambda t} \\[2em] \frac{i\alpha m}{\lambda}\, \phi_+(x) e^{-i\lambda t} & \frac{1}{2} i\, \alpha\, \rho(x) - \frac{i}{2} m \end{pmatrix} \end{pmatrix} \,,
$$
$$\tag{4.56}$$

where, $\mathbf{0}$ is a 2×2 *zero* matrix. It is noted in the above Lax-pair that the 2×2 matrices on the diagonal are decoupled. Therefore, one can reduce the Lax-pair to the

following form

$$\mathbf{U_0} = \begin{pmatrix} 0 & \phi_+(x)\, e^{i\lambda t} \\ & \\ & \\ \alpha\, m\, \phi_+(x)\, e^{-i\lambda t} & 0 \end{pmatrix} \tag{4.57}$$

$$\mathbf{V_0} = \begin{pmatrix} -\frac{1}{2}i\, \alpha\, \rho(x) - \frac{i}{2}\, m & -i\phi_-(x)\, e^{i\lambda t} \\ & \\ & \\ i\, \alpha\, m\, \phi_-(x) e^{-i\lambda t} & \frac{1}{2}i\, \alpha\, \rho(x) + \frac{i}{2}m \end{pmatrix} \tag{4.58}$$

The compatibility condition $\mathbf{U_{0t}} - \mathbf{V_{0x}} + [\mathbf{U_0}, \mathbf{V_0}] = \mathbf{0}$ generates the following matrix,

$$\begin{pmatrix} 0 & \partial_x \phi_-(x) + \phi_+(x)[m + \lambda + \alpha\rho(x)] \\ & \\ & \\ \partial_x \phi_+(x) + \phi_-(x)[m + \lambda + \alpha\rho(x)] & 0 \end{pmatrix} = 0 \tag{4.59}$$

leading to the model equations 1.19 and 1.20 in stationary form, namely

$$\partial_x \phi_-(x) + \phi_+(x)[m + \lambda + \alpha\rho(x)] = 0, \tag{4.60}$$

$$\partial_x \phi_+(x) + \phi_-(x)[m + \lambda + \alpha\rho(x)] = 0. \tag{4.61}$$

Applying the Darboux transformation requires a linear system associated with the NLDE through the compatibility condition. A linear system with spectral parameters is found here by inspection

$$\Phi_x = \mathbf{U_0}\Phi + \mathbf{U_1}\Phi\Lambda, \tag{4.62}$$

$$\Phi_t = \mathbf{V_0}\Phi + \mathbf{V_1}\Phi\Lambda + \mathbf{V_2}\Phi\Lambda^2. \tag{4.63}$$

The consistency condition $\Phi_{xt} = \Phi_{tx}$ requires the following condition on the matrices

U$_{\mathbf{0,1}}$ and **V**$_{\mathbf{0,1,2}}$

$$\mathbf{U}_{\mathbf{0}x} - \mathbf{V}_{\mathbf{0}t} + \left[\mathbf{U_0}, \mathbf{V_0}\right] = 0, \tag{4.64}$$

$$\mathbf{U}_{\mathbf{1}t} - \mathbf{V}_{\mathbf{1}x} + \left[\mathbf{U_0}, \mathbf{V_1}\right] + \left[\mathbf{U_1}, \mathbf{V_0}\right] = 0, \tag{4.65}$$

$$\mathbf{U}_{\mathbf{2}t} - \mathbf{V}_{\mathbf{2}x} + \left[\mathbf{U_0}, \mathbf{V_2}\right] + \left[\mathbf{U_1}, \mathbf{V_1}\right] + \left[\mathbf{U_2}, \mathbf{V_0}\right] = 0, \tag{4.66}$$

$$\left[\mathbf{U_1}, \mathbf{V_2}\right] + \left[\mathbf{U_2}, \mathbf{V_1}\right] = 0, \tag{4.67}$$

$$\left[\mathbf{U_2}, \mathbf{V_2}\right] = 0. \tag{4.68}$$

The specific forms of $\mathbf{U_1}$ and $\mathbf{V_{1,2}}$ will be determined only for the zero seed in the next section.

4.4.2   Darboux transformations of MTM

Applying the Darboux transformation, as defined below, on Φ transforms it into another field $\Phi[1]$. For the transformed field $\Phi[1]$ to be a solution of the linear system, the Lax pair must also be transformed in a certain manner. The transformed Lax pair will be a functional of a new solution of the same differential equation.

Practically, this is performed as follows. First, we find the Lax pair and an exact solution of the differential equation, known as the seed solution. Fortunately, the trivial solution can be used as a seed, leading to nontrivial solutions. Using the Lax pair and the seed solution, the linear system is then solved and the components of Φ are found. The new solution is expressed in terms of these components and the seed solution. The following detailed derivation of the new solution clarifies this procedure further. For

zero seed ($\phi_+(x) = \phi_-(x) = 0$), the Lax-pair matrices take the following form

$$\Lambda = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \tag{4.69}$$

$$\mathbf{U_0} = \mathbf{0}, \tag{4.70}$$

$$\mathbf{U_1} = \begin{pmatrix} a'(x) + b(x) & c_3 c_4 a'(x) e^{i\lambda t - \frac{i}{c_1}\sqrt{\frac{\alpha}{m}}(m+\lambda)a(x)} \\ c_4 a'(x) e^{-i\lambda t + \frac{i}{c_1}\sqrt{\frac{\alpha}{m}}(m+\lambda)a(x)} & b(x) \end{pmatrix}, \tag{4.71}$$

$$\mathbf{V_0} = \frac{i}{2} \begin{pmatrix} \lambda + \sqrt{\frac{\alpha}{m}}(m+\lambda) & 0 \\ 0 & -\lambda - \sqrt{\frac{\alpha}{m}}(m+\lambda) \end{pmatrix}, \tag{4.72}$$

$$\mathbf{V_1} = \begin{pmatrix} c_1 + c_2 & c_4 c_1 c_3 e^{i\lambda t - \frac{i}{c_1}\sqrt{\frac{\alpha}{m}}(m+\lambda)a(x)} \\ c_4 c_1 e^{-i\lambda t - \frac{i}{c_1}\sqrt{\frac{\alpha}{m}}(m+\lambda)a(x)} & c_2 \end{pmatrix}, \tag{4.73}$$

$$\mathbf{V_2} = \mathbf{0}. \tag{4.74}$$

Considering the following version of DT

$$\Phi[1] = \Phi.\Lambda - \sigma\Phi, \tag{4.75}$$

where, $\Phi[1]$ is the transformed field and $\sigma = \Phi_0.\Lambda.\Phi_0^{-1}$.

Here,

$$\Phi_0 = \begin{pmatrix} \psi_1(x,t)e^{i\lambda t/2} & \psi_2(x,t)e^{i\lambda t/2} \\ \varphi_1(x,t)e^{-i\lambda t/2} & \varphi_2(x,t)e^{-i\lambda t/2} \end{pmatrix} \tag{4.76}$$

where,

$$\psi_1(x,t) = \frac{ic_2\,2\,x^{c_0}\lambda_1 C_1(t)}{m\sqrt{\frac{\alpha}{m}} + \sqrt{\frac{\alpha}{m}}\lambda - 2ic_1\lambda_1} + x^{\frac{2ic_0c_1\lambda_1}{\sqrt{\frac{\alpha}{m}}(m+\lambda)}} C_2(t), \tag{4.77}$$

$$\psi_2(x,t) = \frac{ic_2\,2\,x^{c_0}\lambda_1 C_3(t)}{m\sqrt{\frac{\alpha}{m}} + \sqrt{\frac{\alpha}{m}}\lambda - 2ic_1\lambda_1} + x^{\frac{2ic_0c_1\lambda_1}{\sqrt{\frac{\alpha}{m}}(m+\lambda)}} C_4(t), \tag{4.78}$$

$$\varphi_1(x,t) = C_1(t), \tag{4.79}$$

$$\varphi_2(x,t) = C_3(t), \tag{4.80}$$

is a known solution of the linear system 4.62 and 4.63. For the transformed field $\Phi[1]$ to be a solution of the linear system, the matrix $\mathbf{U_0}$ for instance must be transformed as

$$\mathbf{U_0}[1] = \sigma \mathbf{U}\sigma^{-1} + \sigma_x\sigma^{-1} \tag{4.81}$$

where, σ^{-1} is the inverse of σ. This equation gives the new solution of the NLDE in terms of the seed solutions of the stationary version of the model given by Eqs. 4.60 and 4.61. By following the procedure given above we start from trivial zero seed and obtain the auxiliary field variable of the following form. We obtain exact soliton solution of the following forms and we classify them into four cases, namely

Case I: If $\lambda = m$

$$\phi_+(x) = \left[\frac{4m^{3/2}x}{(1+4m^2x^2)\alpha^{1/2}}\right], \tag{4.82}$$

$$\phi_-(x) = \left[\frac{2m^{1/2}}{(1+4m^2x^2)\alpha^{1/2}}\right], \tag{4.83}$$

with

$$\rho = \frac{4m}{\alpha + 4m^2 x^2 \alpha}, \quad (\alpha > 0), \tag{4.84}$$

$$J = \frac{4m(-1 + 4m^2 x^2)}{(1 + 4m^2 x^2)^2 \alpha}. \tag{4.85}$$

Case II: If $\lambda = -m$

$$\phi_-(x) = \left[\frac{4m^{3/2} x}{(1 + 4m^2 x^2)\beta} \right], \tag{4.86}$$

$$\phi_+(x) = \left[\frac{2m^{1/2}}{(1 + 4m^2 x^2)\beta} \right], \tag{4.87}$$

$$\rho = -\frac{4m}{\alpha + 4m^2 x^2 \alpha}, \quad (\alpha < 0), \tag{4.88}$$

$$J = \frac{4m(-1 + 4m^2 x^2)}{(1 + 4m^2 x^2)^2 \alpha}. \tag{4.89}$$

Case III: If $\lambda > m$

$$\phi_+(x) = \sqrt{\frac{\lambda - m}{\alpha} \left[1 - \cos\left(2x\sqrt{\lambda^2 - m^2} \right) \right]} \frac{m + \lambda}{\lambda - m\cos\left(2x\sqrt{\lambda^2 - m^2} \right)}, \tag{4.90}$$

$$\phi_-(x) = \sqrt{\frac{m + \lambda}{\alpha} \left[1 + \cos\left(2x\sqrt{\lambda^2 - m^2} \right) \right]} \frac{\lambda - m}{\lambda - m\cos\left(2x\sqrt{\lambda^2 - m^2} \right)}. \tag{4.91}$$

Case IV: If $\lambda < m$

$$\phi_+(x) = \sqrt{\frac{m - \lambda}{\alpha} \left[-1 + \cosh\left(2x\sqrt{m^2 - \lambda^2} \right) \right]} \frac{m + \lambda}{\lambda - m\cosh\left(2x\sqrt{m^2 - \lambda^2} \right)}, \tag{4.92}$$

$$\phi_-(x) = \sqrt{\frac{m + \lambda}{\alpha} \left[1 + \cosh\left(2x\sqrt{m^2 - \lambda^2} \right) \right]} \frac{\lambda - m}{\lambda - m\cosh\left(2x\sqrt{m^2 - \lambda^2} \right)}. \tag{4.93}$$

4.5 Discussion of the results

4.5.1 Results relevant to physics

According to MTM, the solutions which are physically important are those for $|\lambda| = m$: i.e. those shown in Eqs. 4.33 and 4.34. The reason behind discarding the other solutions comes from the definition of relativistic energy (in natural units) $E = \pm\sqrt{m^2 + |p|^2}$ and keeping in mind that we are discussing in this present work only stationary solutions and hence the momentum is assumed to be zero: $p = 0$; So it is supposed that the energy $|\lambda| = |E| = m$. All other cases with $|\lambda| \neq m$ are not to be considered from now and on.

The above discussion applies also to MGNM equations. But we show above in section 4.3.5 that the stationary solutions already vanish in case $|\lambda| = m$ for MGNM and then no physical significance is recognized for their solutions.

So, in the remaining of this chapter; our discussion will be devoted to Eqs. 4.33 and 4.34; the solutions of MTM for the case $|\lambda| = m$.

4.5.2 Locality of the results

The solutions which are of concern to us (Eqs. 4.33 and 4.34) satisfy the most important requirement in any representation of physical systems, namely the requirements of being localized in space, an then they are square-integrable. To show that they are localized, let us check first for a plot of their envelope in the following figures 4.1 and 4.2. Here we scale both of m and α_+ to 1

Also, one can check directly for the integrability by evaluating the integral of $\rho = |\psi_+|^2 + |\psi_-|^2$ on the whole space. This is already done in the next section, where
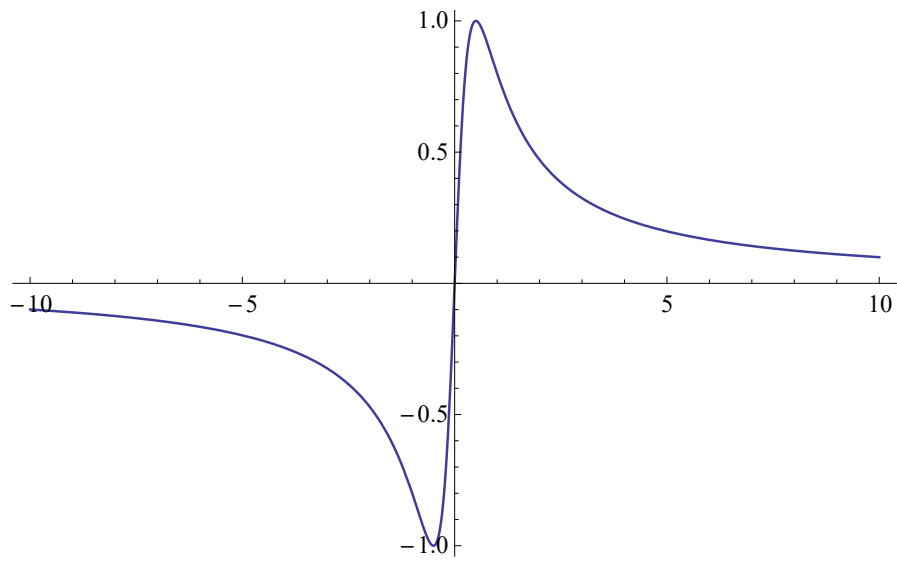
Figure 4.1: ψ_+ for $m = \alpha_+ = 1$
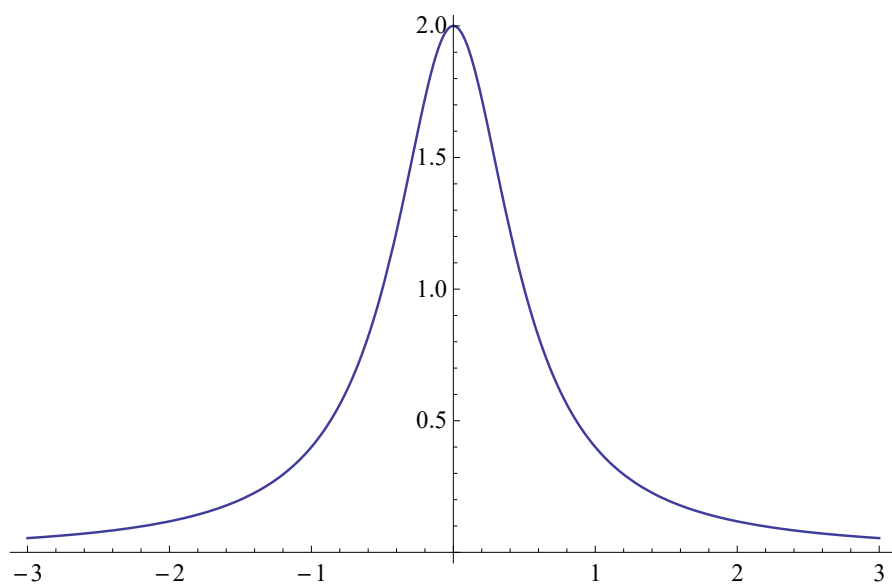
Figure 4.2: ψ_- for $m = \alpha_+ = 1$

we calculate the integral of $j^0 = \rho$ in order to evaluate the total charge, and find that it

is equivalent to $\frac{2\pi}{\alpha_+}$.

4.5.3 Calculations of the conserved quantities

Now, we make use of the results and formulations of the Lagrangian density and the

energy-momentum tensor that are already derived in section 2.1.3 and 2.1.4, to calculate

some of the conserved quantities of a system represented by the solution defined in Eqs.

4.33 and 4.34. Let us first write the Spinor field and its adjoint for that solution

$$\Psi = \sqrt{\frac{m}{\alpha}} \begin{pmatrix} \frac{4mx}{4m^2x^2+1} \\ \frac{2}{4m^2x^2+1} \end{pmatrix} e^{-imt}, \tag{4.94}$$

$$\bar{\Psi} = \sqrt{\frac{m}{\alpha}} \begin{pmatrix} \frac{4mx}{4m^2x^2+1} & -\frac{2}{4m^2x^2+1} \end{pmatrix} e^{imt}. \tag{4.95}$$

Total energy

Apply the spinor field solution (Eqs. 4.94 and 4.95) in the Hamiltonian density as

defined in Eq. 2.39

$$\mathscr{H} = \frac{1}{2}\alpha_+ \left(\frac{4m}{\alpha_+ + 4\alpha_+ m^2 x^2} \right)^2, \tag{4.96}$$

then the total energy can be calculated using Eq. 2.49

$$E = \int_{-\infty}^{+\infty} [\frac{1}{2}\alpha_+ \left(\frac{4m}{\alpha_+ + 4\alpha_+ m^2 x^2} \right)^2]dx = \frac{2m\pi}{\alpha_+}. \tag{4.97}$$

The last result shows that the total energy is a constant conserved quantity as

expected for a stationary system.

Total charge

Use Eq. 2.7 to calculate the probability density $j^0 = \rho = \bar{\Psi}\gamma^0\Psi$ for the spinor field defined in Eq. 4.94

$$j^0 = \bar{\Psi}\gamma^0\Psi = \frac{4m}{\alpha_+ + 4\alpha_+ m^2 x^2}, \tag{4.98}$$

this last expression can be integrated according to Eq. 2.50 to give the total charge

$$Q = \int_{-\infty}^{+\infty}(j^0)dx = \int_{-\infty}^{+\infty}(\frac{4m}{\alpha_+ + 4\alpha_+ m^2 x^2})dx = \frac{2\pi}{\alpha_+}, \tag{4.99}$$

which is also independent of x nor t and hence becomes a conserved physical property.

Total momentum

According to Eq. 2.51 total momentum P is considered to be the spatial integration of $P = \int_{-\infty}^{+\infty} T^{01}(x,t)dx$. $T^{01}(x,t)$ is already stated in Eq. 2.41. Apply the spinor field solution forms in that equation to get

$$T^{01} = -\frac{i}{2}\left(\bar{\Psi}\gamma^0\partial_x\Psi - \partial_x\bar{\Psi}\gamma^0\Psi\right) = \partial_x P = 0, \tag{4.100}$$

And hence ***the total momentum vanishes***. Such a result is the only logically consistent one with our assumption of a stationary system.

Chapter 5: Conclusions and Remarks for Future Work

The accomplishment of this present thesis work has came up with some conclusions, in addition to other remarks that may be helpful for future work on NLDE. Here down listed are both of them in two separate sections.

5.1 Conclusions

- The mathematical modelling of the self-interactions among spinor fields in the relativistic classical field theory is the NLDE with all of its various formulations.

  All of those models share the common property that they have the cubic Kerr-nonlinearity. The formulation of NLDE has mainly came as a replacement of the earlier unified field theory of matter. The major differences between a specific NLDE model and another, involve the mode of interaction, and whether their is an effective mass of the field particles or they are considered to be massless.

- From the formulations discussed in chapter two, we find that MTM possesses some interesting characteristics: Its current density $\rho(x,t)$ satisfies a wave equation (Eq. 2.32). Whereas, both the current vector and energy momentum tensor were proved to be conserved, which means that MTM represents a conservative stable model, and then it can describe an existing real systems. This notion is supported by the derivation of exact localized smooth solutions for a stationary systems.

- In chapter four, we have analyzed the MTM and derived many of the properties

from its mathematical structure. We have verified that it has stationary solutions by solving the model and find those solutions explicitly in exact closed form. Also we obtained a Lax pair of MTM and concluded that in general, it can have many Lax pairs. Then, upon applying the Darboux transformation on the obtained Lax pair, we could recover the same stationary solutions again.

According to my present knowledge, there was no published exact stationary solutions for MTM that are written in closed form before this present one derived in this thesis. Also, no Lax pair has been derived before for any NLDE model. Those findings, in addition to other details about the Darboux transformation, were organized in a manuscript [103] in collaboration with the Professor Usama Al Khawaja and P.S. Vinayagam, and submitted to the journal *communications in nonlinear science and numerical analysis*.

As we found it stated in other previous published articles in the literature (Discussed in section 4.2), and concluded it as well from achieving the derivation of Lax pair for MTM, such a model can generally have infinitely many solutions, because the repetitive application of Darboux transformation on the Obtained Lax pair can result in a series of different solutions.

- The MGNM possess stationary analytic solutions. We have obtained these solutions, although no physical relevance is realized yet to them.

5.2 Remarks for future work

The work on this thesis is considered to be a continuation to the general analytic study of NLDE models, and focusing particularly on MTM. Inspite of the huge work done

and the progress had been achieved so far in this field, there still much more open issues that need to be resolved. The following is a brief listing of some of them as remarks for future research.

- The major issue is related to the extension of the set of solutions of NLDE. The solutions derived so far are either numerical or exact stationary ones. It will be more useful and even interesting to look for traveling wave solutions, especially upon the successful derivation of the stationary solutions.

  In relativistic theory, the energy relation $E^2 = m^2c^4 + p^2c^2$ leads to restricting stationary solutions to only one case of the possible value of the energy, namely $E = m$. That restriction does not exist in non-relativistic models, where a stationary model can describe a wide range of the possible energies. That crucial difference makes it obligated to look for a travelling wave solutions for relativistic models, including NLDE, if one is interested in cases other than $E = m$. One of the options that one can think of is the Darboux transformation, since it can result in other solutions from the existing ones.

- The dimensionality of the solved MTM in this work is a one spatial dimension in addition to that representing time. When W. Thirring has restricted this dimensionality to his model, he aimed to resolve the problem of the diverging integrals in the (3+1)-dimensional nonlinear field theory, since he suggested that the dimensionality is the reason behind that non-integrability. But yet a model with more dimensions is required in many practical fields, as in nano-science and the condensates of ultra-cold atoms, where the two dimensional spatial system is very useful. Obtaining solvable models in two or three spatial dimensions will find

wide range of applications in such practical fields.

- Finally, I see it worth's to extend the space of applying the techniques adopted in this thesis, which worked well for the MTM and MGNM, on other forms of NLDE.

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