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THE PANTOGRAPH EQUATION IN QUANTUM CALCULUS

by

THOMAS GRIEBEL

A THESIS

Presented to the Graduate Faculty of the

MISSOURI UNIVERSITY OF SCIENCE AND TECHNOLOGY

In Partial Fulfillment of the Requirements for the Degree

MASTER OF SCIENCE

in

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Approved by

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Dr. Elvan Akin
Dr. Gregory Gelles

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ABSTRACT

In this thesis, the pantograph equation in quantum calculus is investigated. The pantograph equation is a famous delay differential equation that has been known since 1971. Till the present day, the continuous and the discrete cases of the pantograph equation are well studied. This thesis deals with different pantograph equations in quantum calculus. An explicit solution representation and the exponential behavior of solutions of a pantograph equation are proved. Furthermore, several pantograph equations regarding asymptotic stability are considered. In fact, conditions for the asymptotic stability of the zero solution are derived and subsequently illustrated by examples. Moreover, an explicit solution in terms of the exponential function for a special pantograph equation is obtained.

ACKNOWLEDGMENTS

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1. INTRODUCTION

In the 1960s, the British Railways wanted to make the electric locomotive faster. An important construct was the pantograph, which collects current from an overhead wire. Therefore, J. R. Ockendon and A. B. Tayler studied the motion of the pantograph head on an electric locomotive in [16]. In the solution procedure of this problem, they came across a special delay differential equation of the form

$$x'(t) = ax(t) + bx(\lambda t), \quad t > 0,$$

where a, b are real constants and $0 < \lambda < 1$ for $\lambda \in \mathbb{R}$. When the article was published in 1971, this kind of delay differential equation was called pantograph equation.

In the following years, the pantograph equation became a prime example for a delay differential equation. The continuous and discrete cases of the pantograph equation have been well studied over the last several decades. The continuous and discrete cases denote different examples of time scales $\mathbb{T}$ that are arbitrary nonempty closed subsets of real numbers. The general theory of calculus on time scales is described, e.g., in [3, 4]. In the continuous case, which means the time scale of the real numbers $\mathbb{T} = \mathbb{R}$, the pantograph equation is presented as a differential equation. This is studied, e.g., in [7, 10, 11, 14]. In the discrete case, which means the time scale of the integers $\mathbb{T} = \mathbb{Z}$, here especially the nonnegative integers $\mathbb{T} = \mathbb{N}_0$, the pantograph equation is presented as a difference equation, which is studied, e.g., in [11, 12, 15]. The present study focuses on the pantograph equation in the quantum case, which is the time scale $\mathbb{T} = q^{\mathbb{N}_0}$ for $q > 1$. The general theory of calculus on the time scale $\mathbb{T} = q^{\mathbb{N}_0}$ is also called quantum calculus.

This thesis is structured as follows. Section 2 describes the origin of the pantograph equation. For this purpose, the work of [16] is summarized. Section 3 introduces some basics of stability theory for differential and difference equations. In Section 4, results of the pantograph equation in the continuous case are discussed. Section 5 deals with findings of the pantograph equation in the discrete case. Section 6 presents a short introduction to quantum calculus. In Section 7, the pantograph equation in quantum calculus is studied. First, an explicit solution representation and the exponential behavior of the solution of a pantograph equation are proved. Second, asymptotic stability conditions of the zero solution for different pantograph equations are derived. Third, an explicit solution for a special pantograph equation is obtained. Section 8 concludes the thesis work with a brief summary and an outlook on further research about this topic.

2. ORIGIN OF THE PANTOGRAPH EQUATION

This section wishes to motivate the consideration of the pantograph equation. For this purpose, the origin of the pantograph equation is described and summarized, which is, in fact, the paper [16] from J. R. Ockendon and A. B. Tayler.

The British Railways wanted to develop a new type of electric locomotive in the 1960s. The goal was to make the trains faster. An important component for the new high speed electric locomotive was the *pantograph*. The purpose of the pantograph is to collect current from an overhead wire, which is necessary for the locomotive to be able to move; see Figure 2.1.

(a) Pantograph [20] (b) Current collection system [18]

Figure 2.1. Pantograph and overhead wire system.

To make sure that the electric locomotive can move smoothly with high speed, it is necessary that there are no interruptions in the current collection system. Therefore, the pantograph should stay in contact with the overhead wire for the whole time, particularly when the pantograph passes the supports of the overhead wire, which is a critical passage. With this motivation and other questions about the pantograph, J. R. Ockendon and A. B.

Tayler studied the problem to determine the *motion of a pantograph head on an electric locomotive* in [16]. To this end, the goal is to build up a mathematical model of the current collection system and find the solution of the motion of the pantograph head.

In the following, the mathematical model is based on the description in [1, 9, 16]. The mathematical model of this system includes modeling the motion of the wire connected with the dynamics of the supports and modeling the dynamics of the pantograph. In Section 2.5, the results are based on the description in [16]. In the process to find the solution, the authors come across a delay differential equation, and from that moment, this kind of delay differential equation is known as the *pantograph equation*, which is still a research topic of interest.

2.1. MATHEMATICAL MODEL

In the following, the system where the pantograph is collecting current from the overhead wire is modeled to determine the motion of the pantograph head. First, the overhead trolley wire is modeled. Second, the model of the pantograph is considered. Lastly, a simplified formulation of the whole problem is derived. In this discussion, it is assumed that the electric locomotive moves with constant speed U and passes the supports of the overhead trolley wire as in Figure 2.2.

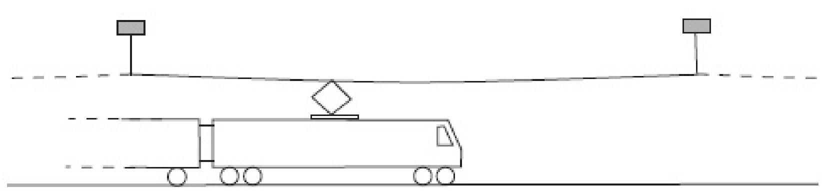

Figure 2.2. The movement of the electric locomotive [9].

2.2. MODEL OF THE OVERHEAD TROLLEY WIRE

The model of the overhead trolley wire is shown in Figure 2.3. The overhead wire

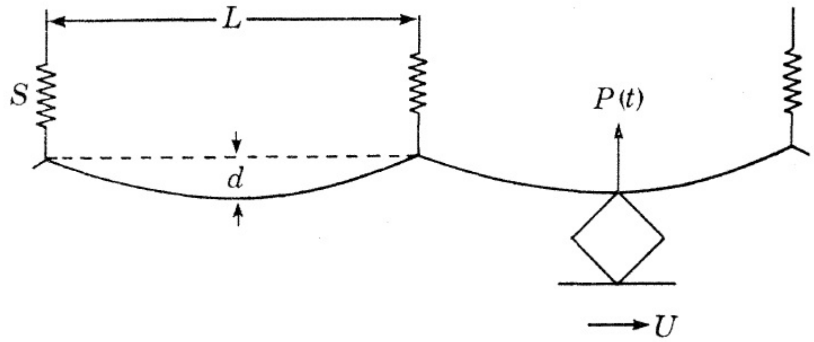

Figure 2.3. Model of the overhead trolley wire [16].

is rigidly attached to the supports, which are modeled as equally spaced stiff springs with spring constant S at distance L. The wire is stretched under constant tension T. The contact force in the vertical direction between the pantograph, which is the rhombus in Figure 2.3, and the overhead wire is $P(t)$. Moreover, the contact force $P(t)$ acts in the vertical direction upwards, which means it is positive when the head of the pantograph and the wire are in contact. Since it is assumed that the train moves with constant speed U, the pantograph and the contact force do so as well.

The motion of the head of the pantograph is determined by the function $Y(x,t)$, which is the vertical displacement of the wire at position x and time t. It is assumed that the overhead wire behaves as a thin beam under tension and the displacement is small and only in the vertical direction. For this reason, we can use the Euler–Bernoulli beam theory with the famous Euler–Bernoulli equation, namely

$$\rho\frac{\partial^2 Y}{\partial t^2} + EI\frac{\partial^4 Y}{\partial x^4} = q,$$

where ρ is the linear density (mass per unit length), EI is the flexural rigidity of the wire, and q is the external force. Under consideration that the wire is under constant tension T, the equation

$$\rho\frac{\partial^2 Y}{\partial t^2} + EI\frac{\partial^4 Y}{\partial x^4} - T\frac{\partial^2 Y}{\partial x^2} = q$$

follows; see [17]. If the flexural rigidity of the wire was neglected, the equation would become the inhomogeneous wave equation. The external forces consist of, first, the gravitational force ρg. Second, the contact force $P(t)$ acts at the point $x = Ut$ where the pantograph head passes the supports. Third, the spring force terms at the supports result from the springs with spring constant S. Since this equation is in terms of the vertical force per unit length of the wire, the point force is modeled by the Dirac delta function so that the contact force term $P(t)\delta(x - Ut)$ and the spring force term $S \sum_{n=-\infty}^{\infty} Y\delta(x - nL)$ are connected with the Dirac delta function:

$$\rho \frac{\partial^2 Y}{\partial t^2} + EI \frac{\partial^4 Y}{\partial x^4} - T \frac{\partial^2 Y}{\partial x^2} = P(t)\delta(x - Ut) - \rho g - S \sum_{n=-\infty}^{\infty} Y\delta(x - nL).$$

In consideration of damping in the wire with damping coefficient η, the damping term $\eta \frac{\partial Y}{\partial t}$ appears. Therefore, the vertical motion $Y(x, t)$ of the wire is described by the equation

$$\rho \left(\frac{\partial^2 Y}{\partial t^2} + g \right) + EI \frac{\partial^4 Y}{\partial x^4} - T \frac{\partial^2 Y}{\partial x^2} + \eta \frac{\partial Y}{\partial t} + S \sum_{n=-\infty}^{\infty} Y\delta(x - nL) = P(t)\delta(x - Ut). \qquad (2.1)$$

2.3. MODEL OF THE PANTOGRAPH

The model of the pantograph is displayed in Figure 2.4 Although the geometry of

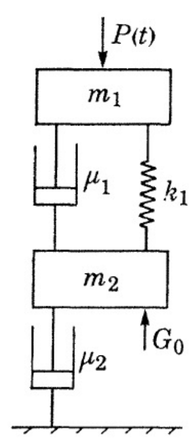

Figure 2.4. Model of the pantograph [16].

the pantograph is complex, the pantograph can be simply modeled as a mass-spring-damper system, where m_1 is the mass of the pantograph head and m_2 is the mass of the pantograph frame. These masses (m_1 and m_2) are connected with each other by a spring with spring constant k_1 and a velocity damper with damping coefficient μ_1. A velocity damper with damping coefficient μ_2 connects m_2 to the train roof. It is assumed here that the vertical motion of the train roof can be neglected. The constant force G_0 represents stiff springs attached between the train roof and the pantograph frame (m_2). The contact force $P(t)$, which results from the contact between the overhead wire and the pantograph head, acts downwards on the pantograph head. The vertical displacement from m_1 is defined by $y(t)$, the vertical displacement from m_2 is $z(t)$.

To figure out the equation for the motion of the pantograph head, a free-body diagram around m_1 is considered. The following forces are included: the spring force $k_1(y - z)$ that keeps the pantograph in contact with the wire, the damping force $\mu_1(\frac{dy}{dt} - \frac{dz}{dt})$, the dynamic force (mass × acceleration) $m_1 \frac{d^2y}{dt^2}$, the gravitational force $m_1 g$, and the contact force $P(t)$. Using this free-body diagram and Newton's first law of motion, y and z satisfy the equation

$$m_1 \frac{d^2y}{dt^2} + \mu_1 \left(\frac{dy}{dt} - \frac{dz}{dt} \right) + k_1(y - z) + P(t) + m_1 g = 0. \tag{2.2}$$

Similarly, the equation for the motion of the pantograph frame m_2 can be derived as

$$m_2 \frac{d^2z}{dt^2} + \mu_2 \frac{dz}{dt} + m_2 g = G_0 + \mu_1 \left(\frac{dy}{dt} - \frac{dz}{dt} \right) + k_1(y - z) + P(t). \tag{2.3}$$

Equation (2.3) can be transformed by using (2.2) so that the system of second-order differential equations follows:

$$m_1 \frac{d^2y}{dt^2} + \mu_1 \left(\frac{dy}{dt} - \frac{dz}{dt} \right) + k_1(y - z) + P + m_1 g = 0 \tag{2.4}$$

$$m_1 \frac{d^2y}{dt^2} + m_2 \frac{d^2z}{dt^2} + \mu_2 \frac{dz}{dt} + m_1 g + m_2 g + P = G_0. \tag{2.5}$$

2.4. SIMPLIFIED FORMULATION OF THE PROBLEM

To complete the model, the condition that the pantograph and the wire are in contact is considered; that is $Y(Ut,t) = y(t)$ with $P(t) > 0$ and the initial conditions at $t = 0$, which are not further described.

In the following, a few simplifications are done because small parameters are involved after nondimensionalizing the problem. The values of the parameter are given from the British Railways in [16]. Subtracting the static displacement $Y_s(x)$ from $Y(x,t)$ and nondimensionalizing the involved variables, the equations below for the static displacement Y_s and the dynamic displacement Y, which is measured relative to the static position of the wire, follow from (2.1):

$$\frac{EI}{L^2T}\frac{d^4Y_s}{dx^4} - \frac{d^2Y_s}{dx^2} + \frac{\rho g L^2}{Td} = 0 \tag{2.6}$$

$$\frac{\rho U^2}{T}\frac{\partial^2 Y}{\partial t^2} + \frac{EI}{L^2T}\frac{\partial^4 Y}{\partial x^4} - \frac{\partial^2 Y}{\partial x^2} + \frac{\eta L U}{T}\frac{\partial Y}{\partial t} + \frac{LS}{T}\sum_{n=-\infty}^{\infty} Y\delta(x-n) = P(t)\delta(x-t). \tag{2.7}$$

Now, only the interval $-1 < x < 1$ is considered. The flexural rigidity can be ignored in both equations because the parameter in front of the terms $\frac{d^4Y_s}{dx^4}$ and $\frac{\partial^4 Y}{\partial x^4}$ is $O(10^{-6})$, which is small compared to the other terms. Therefore, from (2.6) follows a second-order differential equation, and the solution is

$$Y_s = 4(x^2 - |x|) + Y_{s_0} + O\left(10^{-6}\frac{d^4Y_s}{dx^4}\right), \tag{2.8}$$

where $Y_{s_0} = -8T/LS$ is the displacement of a support in equilibrium.

For (2.7), in addition to the flexural rigidity term, the damping of the wire can be neglected because the parameter connected to $\frac{\partial Y}{\partial t}$ is also small compared to the other parameters. Moreover, assuming $x \neq 0$, the spring force term $\frac{LS}{T}\sum_{n=-\infty}^{\infty} Y\delta(x-n)$ is 0 on the interval $-1 < x < 1$. Furthermore, with $c = (T/\rho U^2)^{\frac{1}{2}}$, (2.7) leads to an inhomogeneous wave equation

$$\frac{1}{c^2}\frac{\partial^2 Y}{\partial t^2} - \frac{\partial^2 Y}{\partial x^2} = P(t)\delta(x-t) + O\left(10^{-6}\frac{\partial^4 Y}{\partial x^4}, 10^{-3}\frac{\partial Y}{\partial t}\right). \tag{2.9}$$

With the nondimensionalization of the variables from (2.4) and (2.5) and several parameters with similar size listed in [16], the pantograph equation becomes

$$P = P_0 - b_1\frac{dz}{dt} - \varepsilon\left(\frac{d^2 z}{dt^2} + a_1\frac{d^2 y}{dt^2}\right) \tag{2.10}$$

$$y - z = -\varepsilon\left(a_2 P + b_2\frac{dy}{dt} - b_2\frac{dz}{dt}\right) - a_1 a_2 \epsilon^2 \frac{d^2 y}{dt^2}, \tag{2.11}$$

where $\varepsilon = O(10^{-2})$ and all the other coefficients are almost of similar size $O(1)$.

The contact condition with nondimensional variables is

$$Y(t,t) = y(t) - Y_s(t). \tag{2.12}$$

It is assumed that Y is continuous, particularly at $x = 0$. At the point $x = 0$, the term $\frac{\partial^2 Y}{\partial x^2}$ is equal to the spring force term with the Dirac delta function, so that $\frac{\partial Y}{\partial x}$ has a jump with this magnitude at $x = 0$. Therefore, the boundary condition at the support $x = 0$ is

$$K\varepsilon\left[\frac{\partial Y}{\partial x}\right]_{x=0-}^{x=0+} = Y, \tag{2.13}$$

where $K = -Y_{s_0}/8\varepsilon$. Moreover, it is assumed that the pantograph motion is continuous and cannot reach an infinite acceleration, so that y, z and the velocities $\frac{dy}{dt}$, $\frac{dz}{dt}$ have no jumps. Particularly at $t = 0$, the conditions

$$[y]_{0-}^{0+} = [z]_{0-}^{0+} = \left[\frac{dy}{dt}\right]_{0-}^{0+} = \left[\frac{dz}{dt}\right]_{0-}^{0+} = 0 \tag{2.14}$$

follow.

Equations (2.8) to (2.14) with the initial conditions, which are not elaborated in further detail, complete the model.

2.5. SOLUTION PROCEDURE

After completing the formulation of the problem, the goal is to find a solution for the vertical displacement of the pantograph and the contact force. In the following, the solution techniques are roughly described; more details can be found in [16].

This problem is a singular perturbation problem. Therefore, the first step is to consider the solution for $\varepsilon = 0$; this solution is called the outer solution. The state is reached as soon as the pantograph passed one support. That means this is a valid solution for the case that the pantograph is not near a support. If near a support, it is not a uniformly valid first approximation. The vertical displacement of the pantograph in each span, the area between two supports, is independent of the vertical displacement in other spans. For that reason, the general solution in this case is a periodic solution, and it is enough to consider only the solution in one span, e.g., $0 < x < 1$. The solution has different forms in different areas of the x, t plane. These regions are bounded by the train path $x = t$, the so-called characteristic $x = ct$, where c is the wavespeed (the motion can not travel faster than the wavespeed), and the reflexions between the next support at $x = 1$, the previous support at $x = 0$ and the train path $x = t$. These regions are shown in Figure 2.5.

The physical interpretation for this case is that the rigidity of the overhead wire and the elasticity of the supports are neglected and the pantograph dynamics is simplified. That is okay for this case, but these effects have to be considered when the pantograph is near a support.

The second step is to find the solution for the motion of the pantograph near a support. Therefore, it is considered $\varepsilon \neq 0$. This solution is called the inner solution. For convenience, J. R. Ockendon and A. B. Tayler considered the inner solution in two parts; see [16]. The process of finding the solution leads to a problem of solving a system of four

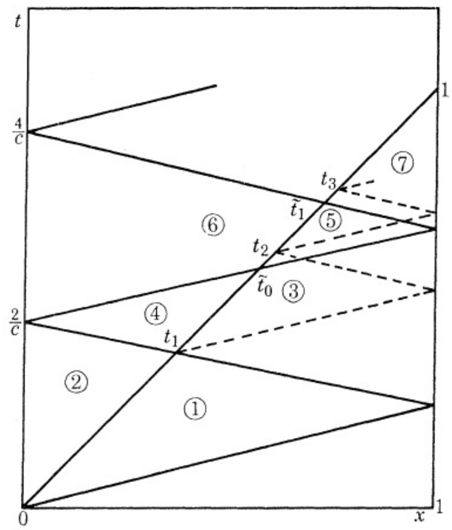

Figure 2.5. The bounded regions of the x, t plane where the forms of the solutions are different [16].

linear differential equations including one term with a stretched argument, which is a kind of delay differential equation. At this point the so-called *pantograph equation* was born, which is known as

$$y'(t) = ay(t) + by(\lambda t),$$

where a, b are real constants and $0 < \lambda < 1$ for $\lambda \in \mathbb{R}$. In the following years, more versions and modifications of the pantograph equation were created and researched. After this discovery, much research concerning the pantograph equation has been completed.

Back to the process of finding the solution, using the matching condition, the perturbation solution is obtained, which describes the motion of the pantograph head. This solution can be used for pantograph problems where the physical parameters are not significantly different from the parameters listed in [16]. Under this restriction, the perturbation solution can be applied for many practical situations connected with the motion of the pantograph, and the effects when the parameters are slightly changed can be observed.

3. STABILITY THEORY

In this section, some basic concepts and ideas about stability of differential and difference equations are introduced. The goal of stability theory is to understand the behavior of the solution of differential or difference equations under small variations of the initial conditions without computing the solution.

3.1. STABILITY THEORY OF DIFFERENTIAL EQUATIONS

First, we consider differential equations. The following is based on [8]. A scalar autonomous differential equation is considered, which is given by the differential equation

$$x'(t) = f\left(x(t)\right), \tag{3.1}$$

where f is a given real-valued function $f : \mathbb{R} \to \mathbb{R}$, $x \mapsto f(x)$ and x a real-valued differentiable function $x : I \to \mathbb{R}$, $t \mapsto x(t)$ with I an open interval of $\mathbb{R}$. Equation (3.1) is called autonomous because f does not depend on t. Moreover, the differential equation (3.1) together with the initial condition at the initial time t_0,

$$x(t_0) = x_0, \tag{3.2}$$

is called an initial value problem.

The function x is a solution of the initial value problem if $x'(t) = f\left(x(t)\right)$ for all $t \in I$ and $x(t_0) = x_0$. There is no loss of generality if it is assumed that $t_0 = 0$; see [8]. The first important point to think about is the existence and uniqueness of solutions.

Theorem 3.1.1. *(i) If $f : \mathbb{R} \to \mathbb{R}$ is continuous, then, for any $x_0 \in \mathbb{R}$, there is an interval (possible infinite) $I_{x_0} = (\alpha_{x_0}, \beta_{x_0})$ containing $t_0 = 0$ and a solution x of the initial value problem*

$$x'(t) = f\left(x(t)\right), \quad x(0) = x_0,$$

defined for all $t \in I_{x_0}$. Also, if α_{x_0} is finite, then

$$\lim_{t \to \alpha_{x_0}^+} |x(t)| = +\infty$$

or, if β_{x_0} is finite, then

$$\lim_{t \to \beta_{x_0}^-} |x(t)| = +\infty.$$

(ii) If, in addition, the function f is continuously differentiable, meaning f is differentiable and its first derivative is continuous, then the solution x is unique on I_{x_0} and x is continuous in (t, x_0) together with its derivative, meaning x is also continuously differentiable.

An important role in stability theory is played by the so-called equilibrium points.

Definition 3.1.2. *A point $\bar{x} \in \mathbb{R}$ is called an equilibrium point (also critical point, steady state solution, etc.) of $x'(t) = f(x(t))$ if $f(\bar{x}) = 0$.*

Now, the concept of stability of an equilibrium point is considered. As stated in [8], roughly speaking, we can say that an equilibrium point is stable if all solutions starting near the equilibrium point $\bar{x}$ stay near $\bar{x}$. Moreover, if nearby solutions tend to $\bar{x}$ as $t \to \infty$, then $\bar{x}$ is called asymptotically stable. The definitions for stable, unstable, and asymptotically stable equilibrium points are given below.

Definition 3.1.3. *(i) An equilibrium point $\bar{x}$ of* (3.1) *is said to be stable if, for any given $\varepsilon > 0$, there exists $\delta(\varepsilon) > 0$, such that, for every x_0, for which $|x_0 - \bar{x}| < \delta$, the solution x of* (3.1) *through x_0 at* 0 *satisfies the inequality $|x(t) - \bar{x}| < \varepsilon$ for all $t \geq 0$.*

(ii) *An equilibrium point $\bar{x}$ of* (3.1) *is said to be unstable if it is not stable.*

(iii) *An equilibrium point $\bar{x}$ of* (3.1) *is said to be asymptotically stable if it is stable and, in addition to that, there is an $r > 0$ such that the solution x of* (3.1) *through x_0 satisfies $|x(t) - \bar{x}| \to 0$ as $t \to +\infty$ whenever $|x_0 - \bar{x}| < r$.*

3.2. STABILITY THEORY OF DIFFERENCE EQUATIONS

After considering differential equations, we look at difference equations. The following introduction into stability theory of difference equations is based on [6].

The following system of linear difference equations is given:

$$x_{n+1} = Ax_n, \quad x(n_0) = x_0, \tag{3.3}$$

where $x_n \in \mathbb{R}^k$ and A is a $k \times k$ nonsingular matrix. Equation (3.3) is called autonomous because A does not depend on n, similar to the continuous case. The first question that comes up about the difference equation (3.3) is the existence and uniqueness of solutions.

Theorem 3.2.1. *For each $x_0 \in \mathbb{R}^k$ and $n_0 \in \mathbb{N}_0$, there exists a unique solution x_n of difference equation* (3.3) *with $x(n_0) = x_0$.*

Simultaneous to the stability theory of differential equations, we define equilibrium points for difference equations.

Definition 3.2.2. *A point $\bar{x} \in \mathbb{R}^k$ is called an equilibrium point of difference equation* (3.3) *if $A\bar{x} = \bar{x}$.*

It is often assumed that $\bar{x}$ is the origin 0, called the zero solution. In the following definition, stable, unstable, and asymptotically stable equilibrium points are defined.

Definition 3.2.3. *An equilibrium point $\bar{x}$ of difference equation* (3.3) *is said to be* *(i)* *stable if, given* $\varepsilon > 0$ *and* $n_0 \geq 0$, *there exists* $\delta = \delta(\varepsilon, n_0)$, *such that* $|x_0 - \bar{x}| < \delta$ *implies* $|x_n - \bar{x}| < \varepsilon$ *for all* $n \geq n_0$.

(ii) *unstable if it is not stable.*

(iii) *attracting if there exists* $\mu = \mu(n_0)$, *such that* $|x_0 - \bar{x}| < \mu$ *implies* $\lim_{n\to\infty} x_n = \bar{x}$.

(iv) *asymptotically stable if it is stable and attracting.*

For the linear autonomous system like (3.3), there exist the following stability results.

Theorem 3.2.4. *The following statements hold:*

(i) *The zero solution of* (3.3) *is stable if and only if* $\rho(A) \leq 1$, *where* $\rho(A)$ *is the spectral radius of matrix A, and the eigenvalues of unit modulus are semisimple; that is, the corresponding Jordan block is diagonal.*

(ii) *The zero solution of* (3.3) *is asymptotically stable if and only if* $\rho(A) < 1$.

If it is said that the zero solution of difference equation (3.3) is asymptotically stable that means that all solutions of (3.3) converge to 0.

Consider the linear scalar difference equation of kth order

$$x_{n+k} + a_1 x_{n+k-1} + a_2 x_{n+k-2} + \cdots + a_k x_n = 0, \tag{3.4}$$

where a_i, $i = 1, 2, \ldots, k$ are real numbers. For this case, there exists a famous criterion to see if the zero solution is asymptotically stable, that is, the Schur–Cohn criterion; see [6].

Theorem 3.2.5 (Schur–Cohn criterion)**.** *The zeros of the characteristic polynomial*

$$P(\nu) = \nu^k + a_1 \nu^{k-1} + \cdots + a_{k-1}\nu + a_k$$

of the difference equation (3.4) *lie inside the unit disk if and only if the following conditions hold:*

(i) $P(1) > 0$,

(ii) $(-1)^k P(-1) > 0$,

(iii) *the* $(k-1) \times (k-1)$ *matrices*

$$M_{k-1}^{\pm} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ a_1 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ a_{k-3} & & \ddots & 1 & 0 \\ a_{k-2} & a_{k-3} & \cdots & a_1 & 1 \end{pmatrix} \pm \begin{pmatrix} 0 & \cdots & \cdots & 0 & a_k \\ \vdots & & \iddots & a_k & a_{k-1} \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ 0 & a_k & \iddots & & a_3 \\ a_k & a_{k-1} & \cdots & a_3 & a_2 \end{pmatrix}$$

are positive innerwise; that is, the determinants of all its inners are positive.

Using the Schur–Cohn criterion (Theorem 3.2.5), necessary and sufficient conditions for asymptotic stability of the zero solution of difference equation (3.4) may be derived.

4. THE PANTOGRAPH EQUATION - CONTINUOUS CASE

The pantograph equation has been well studied in the continuous case over the last several decades. In this section, an overview of important results is given regarding the pantograph equation as a differential equation. The goal is to consider some results regarding asymptotic stability and asymptotic behavior of the pantograph equation.

The pantograph equation that is considered in the continuous case is

$$x'(t) = ax(t) + bx(\lambda t), \quad t > 0, \tag{4.1}$$

where a, b are real or in some cases even complex constants and $0 < \lambda < 1$ for $\lambda \in \mathbb{R}$. The differential equation (4.1) together with the initial condition (4.2) is an initial value problem:

$$x(0) = 1. \tag{4.2}$$

The first important thing when considering an initial value problem is to make sure that it is well posed. For this purpose, the following results about the existence and uniqueness of solutions of the pantograph equation are given below; see [14].

Theorem 4.0.1. *If* $0 < \lambda < 1$*, then, given any* $\delta > 0$*, the problem defined by* (4.1) *and* (4.2)*, where a is a possibly complex constant, b is a real constant, has one and only one solution in* $[0, \delta]$*, and this solution is in fact a solution for all t and is an integral function of t.*

The proof is done in [14] with Picard's method of successive approximations. In fact, the theorem can be extended to complex constants a and b, which is proved in [10]. Hence, the initial value problem is also well posed for complex constants a and b. In addition to that, it is shown in [14] that the initial value problem given by (4.1) and (4.2) is not well posed if $\lambda > 1$.

Theorem 4.0.2. *Assume $\lambda > 1$, a is a possibly complex constant, and b is a real constant. There is no solution of* (4.1) *and* (4.2) *that is analytic in a neighborhood of $x = 0$. However, there are an infinite number of solutions, even of infinitely differentiable solutions, of* (4.1) *and* (4.2).

After considering the existence and uniqueness of solutions of (4.1) and (4.2), we study the asymptotic stability of solutions of the pantograph equation. Some results can be found in [11] for complex constants a and b. The first result is a rough estimation of the domain for asymptotic stability. It is proved in [11] with classical techniques of direct upper bounds.

Lemma 4.0.3. *The solution of the pantograph equation* (4.1) *and* (4.2) *tends to* 0 *if* $\mathrm{Re}(a) < 0$ *and* $|b| < -\mathrm{Re}(a)$.

Furthermore, there is also a result in [11] for the general stability domain, which is more precise.

Theorem 4.0.4. *The zero solution of* (4.1) *and* (4.2) *is asymptotically stable if* $\mathrm{Re}(a) < 0$ *and* $|b| < |a|$.

The asymptotic behavior of perturbations solutions of the pantograph equation is studied in [7], which means we consider a specific kind of pantograph equation with the assumption that λ is very close to 1. To do so, we let $\lambda = 1 - \varepsilon$, where $\varepsilon > 0$ is a very small real number. The following pantograph equation is considered:

$$x'(t) = ax(t) + bx\left((1-\varepsilon)t\right), \quad t > 0, \tag{4.3}$$

$$x(0) = 1. \tag{4.4}$$

During the proof of the following theorem about the asymptotic behavior of solutions of (4.3) and (4.4), there are some asymptotic expansions and approximations done; see [7]. For this reason, the behavior when $t = O(1/\varepsilon)$ is considered by putting $t = \tau/\varepsilon$. In the derivation, it is found that there exists a real solution for all $\tau > 0$ as long as $a > 0$ or $a + b > 0$.

Theorem 4.0.5. *Let c be a real constant. The asymptotic behavior of the solution of* (4.3) *and* (4.4) *as* $\tau \longrightarrow \infty$ *is:*

(i) $b > 0$ *and* $a > 0$*, then* $x \sim c \exp(a\tau/\varepsilon)$.

(ii) $b > 0$ *and* $a = 0$*, then* $x \sim c \exp\left(\frac{(\log \tau)^2}{2\varepsilon}\right)$.

(iii) $b > 0$ *and* $a < 0$*, then* $x \sim c\tau^{-(1/\varepsilon)\log(-a/b)}$.

(iv) $b < 0$ *and* $a + b > 0$*, then* $x \sim c \exp(a\tau/\varepsilon)$.

(v) $a > 0 > a + b$*, then* $x \sim c \exp(a\tau/\varepsilon)$.

The following example confirms the result from Theorem 4.0.5, which is said to be the favorite example of Leslie Fox, one of the authors in [7].

Example 4.0.6. *It is considered that* $a = 0.95$*,* $b = -1$*, and* $\lambda = 1 - \varepsilon = 0.99$*, so that we get the initial value problem*

$$x'(t) = 0.95x(t) - x(0.99t), \quad t > 0, \tag{4.5}$$

$$x(0) = 1. \tag{4.6}$$

The results of the numerical simulation in [7] are shown in Figure 4.1.

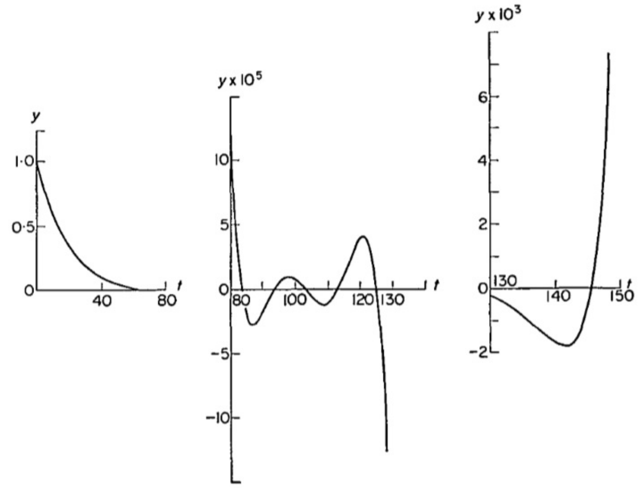

Figure 4.1. The simulation results of (4.5) and (4.6) in [7].

Example 4.0.6 shows that it is dangerous to stop the simulation too early because it seems that the solution tends to 0 as $t \to \infty$, but in fact, it grows exponentially after a certain point t^*, which can be really large. The behavior for large t is exponential as stated by the analytical result in Theorem 4.0.5.

5. THE PANTOGRAPH EQUATION - DISCRETE CASE

In this section, we consider discretizations of the pantograph equation (4.1). The discrete case was a topic of interest in the last 20 years and is well studied. Important results regarding asymptotic behavior and asymptotic stability are considered.

In the following, the results of [15] are shown. The initial value problem given by (4.1) and (4.2) is considered with the assumptions

$$|a| + b < 0 \quad \text{and} \quad 0 < \lambda < 1, \tag{5.1}$$

where λ is very close to 1. In the previous section, we considered in Theorem 4.0.5 the case with the assumption $a > 0 > a + b$ and the result that the asymptotic behavior of the solution is $x \sim c\exp(a\tau/\varepsilon)$, which confirmed the result in Example 4.0.6. With the assumption (5.1), we are in this case. In fact,

$$\begin{aligned}
&a > 0 > a + b \\
&\text{with} \quad a > 0 \Rightarrow |a| = a \\
&\text{with} \quad a + b < 0 \Rightarrow |a| + b < 0.
\end{aligned}$$

For this reason, it is expected to get similar results for the discretization as for the specific pantograph equation (4.3). The discretization is done with the forward Euler method, which is one of the basic but also most famous numerical methods to discretize a differential equation:

$$\begin{aligned}
&x'(t) = ax(t) + bx(\lambda t) \\
\Longrightarrow \quad &\frac{x_{n+1} - x_n}{h} = ax_n + bx_{\lfloor \lambda n \rfloor}
\end{aligned}$$

$$\implies x_{n+1} = (1 + ah)x_n + bhx_{\lfloor \lambda n \rfloor}, \quad n \in \mathbb{N}_0, \tag{5.2}$$

where $h > 0$ is the stepsize, $\lfloor x \rfloor$ is the integer floor function, $x_0 = 1$, and x_n is the numerical approximation of $x(hn)$. Hence, we get a difference equation with constant coefficients but of variable order because of the floor function. Indeed, the order of the difference equation (5.2) is increasing. For certain intervals the order is fixed. As stated in [15], for

$$n \in \left(\frac{m + \lambda - 1}{1 - \lambda}, \frac{m + \lambda}{1 - \lambda} \right), \quad m \in \mathbb{N}_0$$

(5.2) is given by

$$x_{n+1} = (1 + ah)x_n + bhx_{n-m}, \quad n \in \mathbb{N}_0 \tag{5.3}$$

and is of constant order $m + 1$. More explanation about the increasing order is given in Example 5.0.2. The characteristic polynomial of (5.3) is

$$P_m(\nu) = \nu^{m+1} - (1 + ah)\nu^m - bh.$$

If all the zeros of the corresponding characteristic polynomial are inside the open unit disk, then the characteristic polynomial is of Schur type. Moreover, the solution x_n tends to 0 as $n \to \infty$ for any initial values $x_0, x_1, \ldots, x_m$ if and only if the characteristic polynomial is of Schur type.

It is shown in [15] that there is a critical value m^* for the order of the difference equation. Up to this order, the solution tends to 0 but while the order is increasing and reaches the critical order m^*, the solution blows up. The following result is proved in [15].

Proposition 5.0.1. *Assume $|a| + b < 0$ and $h < 1/(|a| + |b|)$. Then, there exists a critical value $m^* = m^*(h) \in \mathbb{N}_0$ for the order of the difference equation* (5.3) *such that $P_m(\nu)$ is of Schur type if and only if $m \leq m^*$. Furthermore,*

$$\lim_{h\to 0} h\, m^*(h) = \begin{cases} \min\left\{\frac{1}{(b^2-a^2)^{1/2}} \arctan \frac{(b^2-a^2)^{1/2}}{a}, \frac{1}{a}\right\}, & a > 0, \\ \frac{\pi}{2|b|}, & a = 0, \\ \frac{1}{(b^2-a^2)^{1/2}} \left(\pi + \arctan \frac{(b^2-a^2)^{1/2}}{a}\right), & a < 0. \end{cases}$$

Therefore, to see the correct asymptotic behavior of the solution, we cannot stop the simulation too early. Because if it stops too early, we could conclude that the solution tends to 0 as $t \to \infty$, which is wrong.

After this discovery was made, further research was done in [12] about this phenomenon that the solution tends to 0 but after reaching a certain point, the solution blows up. The result in [12] improves the result in [15] and shows a more general approach. In the following, the approach of [12] is described. The discretization is done through the backward Euler method:

$$\begin{aligned} & x'(t) = ax(t) + bx(\lambda t) \\ \implies\ & \frac{x_{n+1} - x_n}{h} = ax_{n+1} + bx_{\lfloor \lambda(n+1) \rfloor} \\ \implies\ & (1 - ah)x_{n+1} - x_n - bhx_{\lfloor \lambda(n+1) \rfloor} = 0 \\ \implies\ & x_{n+1} - \frac{1}{(1 - ah)} x_n + \frac{-bh}{(1 - ah)} x_{\lfloor \lambda(n+1) \rfloor} = 0, \quad n \in \mathbb{N}_0 \end{aligned} \tag{5.4}$$

with the integer floor function $\lfloor x \rfloor$, the stepsize $h > 0$, $x_0 = 1$, and x_n as the numerical approximation of $x(hn)$. Due to $\lfloor \lambda(n + 1) \rfloor$, the order of the difference equation (5.4) increases as n increases. An interval is considered, where the difference equation is of fixed order. For

$$n \in \left(\frac{m + \lambda - 1}{1 - \lambda}, \frac{m + \lambda}{1 - \lambda} \right], \quad m \in \mathbb{N}_0,$$

(5.4) is given by

$$x_{n+1} - \alpha x_n + \beta x_{n-m} = 0 \tag{5.5}$$

with

$$\alpha = \frac{1}{(1 - ah)}, \quad \beta = \frac{-bh}{(1 - ah)}. \tag{5.6}$$

Thus, the order of the difference equation (5.5) is $m + 1$ for the interval above.

Example 5.0.2. *This example is given to illustrate that the order of the difference equation* (5.4) *increases. Consider $\lambda = 0.99$. Then, for $m = 0$ and*

$$n \in (-1, 99],$$

(5.4) *is given by*

$$x_{n+1} - \frac{1 + bh}{(1 - ah)} x_n = 0$$

and is of order $m + 1 = 1$. For $m = 1$ and

$$n \in (99, 199],$$

(5.4) *is given by*

$$x_{n+1} - \frac{1}{(1 - ah)} x_n + \frac{-bh}{(1 - ah)} x_{n-1} = 0$$

and is of order $m + 1 = 2$. All in all, if n is increasing, the order of the difference equation is also increasing.

In [12], the Schur–Cohn criterion is used to find out the critical order m^* of the difference equation where it changes from the conditions for asymptotic stability holds to the conditions does not hold any more. The Schur–Cohn criterion (see Theorem 3.2.5) is especially formulated in [12] for the difference equation (5.5).

Theorem 5.0.3. *The zeros of the characteristic polynomial*

$$P_m(\nu) = \nu^{m+1} - \alpha\nu^m + \beta$$

of the difference equation (5.5) *with* α *and* β *from* (5.6) *lie inside the unit disk if and only if the following conditions hold:*

(i) $P_m(1) > 0$,

(ii) $(-1)^{m+1}P_m(-1) > 0$,

(iii) the $m \times m$ *matrices*

$$M_m^{\pm} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ -\alpha & 1 & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 & 0 \\ 0 & \cdots & 0 & -\alpha & 1 \end{pmatrix} \pm \begin{pmatrix} 0 & \cdots & \cdots & 0 & \beta \\ \vdots & & \iddots & \beta & 0 \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ 0 & \beta & \iddots & & \vdots \\ \beta & 0 & \cdots & \cdots & 0 \end{pmatrix}$$

are positive innerwise.

With condition (i) and (ii) of Theorem 5.0.3, some conditions of the stepsize are derived. Summarizing these conditions, only one condition remains, which is $h < 2/(a + |b|)$, and makes sure the condition (i) and (ii) from Theorem 5.0.3 are satisfied. With condition (iii) from Theorem 5.0.3, it is found the critical order m^* of the difference equation up to that the condition holds and the solution tends to 0 but it is not valid anymore for $m > m^*$ and this tendency disappears. All in all, the following theorem is proved in [12].

Theorem 5.0.4. *Let* $|a| + b < 0$*,* $h < \frac{2}{(a+|b|)}$*, and let the values* α *and* β *be given by* (5.6)*. Then, all roots of the characteristic polynomial in Theorem 5.0.3 lie inside the unit disk if and only if*

$$m \leq m^* := \begin{cases} \lfloor z \rfloor, & z \notin \mathbb{N}_0, \\ z - 1, & z \in \mathbb{N}_0, \end{cases}$$

where

$$z = 2 \arctan\left(-\frac{\sqrt{4\alpha^2 - (1 + \alpha^2 - \beta^2)^2}}{1 - \beta^2 - \alpha^2 - 2\alpha\beta}\right) \Big/ \arcsin\left(\frac{\sqrt{4\alpha^2 - (1 + \alpha^2 - \beta^2)^2}}{2\alpha}\right).$$

6. QUANTUM CALCULUS

Before we start to consider several pantograph equations in quantum calculus in the next section, we want to give a short introduction to quantum calculus, which is based on [13]. The quantum case, which means the set $\mathbb{T} = q^{\mathbb{N}_0}$ for $q > 1$, is in addition to the continuous and discrete cases an important and famous example of a time scale (an arbitrary nonempty closed subset of real numbers). The general theory of calculus on time scales can be found, e.g., in [3, 4].

From now on, the following set is considered.

Definition 6.0.1. *Let $q > 1$ be a real number and*

$$\mathbb{T} = q^{\mathbb{N}_0} = \{q^n : n \in \mathbb{N}_0\} = \left\{1, q, q^2, \dots\right\}.$$

The q-derivative is defined as follows.

Definition 6.0.2. *The q-derivative of a function $f : q^{\mathbb{N}_0} \to \mathbb{R}$ at t is given by*

$$f^{\Delta}(t) = \frac{f(qt) - f(t)}{(q-1)t} \quad \text{for all} \quad t \in q^{\mathbb{N}_0}.$$

Note that

$$\lim_{q \to 1} f^{\Delta}(t) = f'(t),$$

if $f : \mathbb{R} \to \mathbb{R}$ is differentiable.

Theorem 6.0.3. *Assume that $f, g : q^{\mathbb{N}_0} \to \mathbb{R}$. The derivative of the product of f and g at t is then given by the product rule in quantum calculus, which is*

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(qt)g^{\Delta}(t).$$

There exist multiple ways for the expression of the q-exponential function; see [13]. Here, we consider the definition of the q-exponential function in [4].

Definition 6.0.4. *Let $a \in \mathbb{R}$. The q-exponential function $e_a(t)$ is defined by*

$$e_a(t) = \prod_{i=0}^{\log_q(t)-1} \left(1 + a(q-1)q^i\right) \quad \textit{for all} \quad t \in q^{\mathbb{N}_0}.$$

The integral in quantum calculus is defined in the following way; see [4].

Definition 6.0.5. *Let $m, n \in \mathbb{N}_0$ with $m < n$ and $f : q^{\mathbb{N}_0} \to \mathbb{R}$. The q-integral is defined as*

$$\int_{q^m}^{q^n} f(t)\Delta t = (q-1)\sum_{k=m}^{n-1} q^k f\left(q^k\right).$$

7. THE PANTOGRAPH EQUATION - QUANTUM CASE

In this section, we discuss the pantograph equation in quantum calculus. There are multiple ways to display the differential pantograph equation (4.1) in quantum calculus. The main goal is to do research on these pantograph equations in quantum calculus regarding asymptotic stability and asymptotic behavior of their solutions.

7.1. ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF THE PANTOGRAPH EQUATION IN QUANTUM CALCULUS

In this section, we consider the pantograph equation in quantum calculus in the form

$$x^{\Delta}(t) = ax(t) + bx\left(\frac{t}{q^N}\right), \quad t \in q^{\mathbb{N}_0}, \quad N \in \mathbb{N} \tag{7.1}$$

with the initial function

$$x(t) = f(t), \quad t \in \Omega := \{q^{-N}, q^{-N+1}, \ldots, q^{-1}, 1\}, \tag{7.2}$$

where a and b are real numbers. The delay term $x(\lambda t)$ in the differential equation (4.1) is represented in the probably most intuitive way in quantum calculus, in fact $x\left(\frac{t}{q^N}\right)$, where $N \in \mathbb{N}$. Thus, λ, which is a real number between 0 and 1, is presented by $\frac{1}{q^N}$ in quantum calculus, so that the delay term $\frac{t}{q^N} \in q^{\mathbb{N}_0} \cup \Omega$ for $t \in q^{\mathbb{N}_0}$. Therefore, a floor function is not needed in (7.1). With the definition of the q-derivative (see Definition 6.0.2), we derive a difference equation from the q-difference equation (7.1):

$$\begin{aligned} x^{\Delta}(t) &= ax(t) + bx\left(\frac{t}{q^N}\right) \\ \Longrightarrow \quad \frac{x(qt) - x(t)}{(q-1)t} &= ax(t) + bx\left(\frac{t}{q^N}\right) \end{aligned}$$

$$\implies x(qt) = (1 + a(q-1)t)\, x(t) + b(q-1)t x\left(\frac{t}{q^N}\right).$$

If we put $t_n := q^n$ and $t_{n+1} := t_n q = q^n q = q^{n+1}$, then

$$x(t_{n+1}) = (1 + a(q-1)t_n)\, x(t_n) + b(q-1)t_n x\left(\frac{t_n}{q^N}\right).$$

It also holds that $\frac{t_n}{q^N} = \frac{q^n}{q^N} = q^{n-N} = t_{n-N}$. Thus,

$$x(t_{n+1}) = (1 + a(q-1)t_n)\, x(t_n) + b(q-1)t_n x\left(t_{n-N}\right).$$

Define $x_n := x(t_n)$, $x_{n+1} := x(t_{n+1})$, and $x_{n-N} := x(t_{n-N})$. Hence,

$$x_{n+1} = (1 + a(q-1)t_n)\, x_n + b(q-1)t_n x_{n-N}.$$

Therefore, (7.1) reduces to the difference equation of order $N+1$ and variable coefficients

$$x_{n+1} = \alpha_n x_n + \beta_n x_{n-N}, \quad n \in \mathbb{N}_0, \tag{7.3}$$

where

$$\alpha_n = 1 + a(q-1)t_n, \quad \beta_n = b(q-1)t_n, \tag{7.4}$$

and the values from the initial function (7.2)

$$x_{-N}, x_{-N+1}, \ldots, x_{-1}, x_0 \tag{7.5}$$

complete the initial value problem.

7.1.1. An Explicit Solution Representation in Matrix Form. In this section, we are interested in an explicit solution form of q-difference equation (7.1) and the initial function (7.2). With (7.3), (7.4), and the initial values (7.5), the solution can be calculated recursively.

It turns out that an explicit solution in matrix form can be obtained using the results in [2]. In the following, the case $N = 1$ for (7.3) is considered, i.e.,

$$x_{n+1} = \alpha_n x_n + \beta_n x_{n-1}, \tag{7.6}$$

where α_n and β_n from (7.4) and with (7.5) the initial values

$$x_{-1}, x_0. \tag{7.7}$$

Theorem 7.1.1. *The solution of* (7.6) *and* (7.7) *can be represented as*

$$\begin{pmatrix} x_{2n-1} \\ x_{2n} \end{pmatrix} = A_{2n-1}A_{2n-3}\ldots A_1 \begin{pmatrix} x_{-1} \\ x_0 \end{pmatrix}, \quad n \in \mathbb{N},$$

where

$$A_n = \begin{pmatrix} \beta_n & \alpha_n \\ \alpha_{n+1}\beta_n & \beta_{n+1} + \alpha_{n+1}\alpha_n \end{pmatrix}.$$

Proof. The proof follows by using [2, Theorem 4.1]. □

Example 7.1.2. *To verify the solution representation in Theorem 7.1.1, we first calculate* x_1, x_2, x_3, x_4 *with the recursive relation* (7.6) *and second with the explicit solution representation in matrix form in Theorem 7.1.1:*

$$x_1 = \alpha_1 x_0 + \beta_1 x_{-1},$$

$$x_2 = \alpha_2 x_1 + \beta_2 x_0 = \alpha_2(\alpha_1 x_0 + \beta_1 x_{-1}) + \beta_2 x_0 = [\alpha_2\alpha_1 + \beta_2]\, x_0 + \alpha_2\beta_1 x_{-1},$$

$$
\begin{aligned}
x_3 &= \alpha_3 x_2 + \beta_3 x_1 = \alpha_3 \Big\{ [\alpha_2\alpha_1 + \beta_2]\, x_0 + \alpha_2\beta_1 x_{-1} \Big\} + \beta_3 \Big\{ \alpha_1 x_0 + \beta_1 x_{-1} \Big\} \\
&= \Big\{ \alpha_3 (\alpha_2\alpha_1 + \beta_2) + \beta_3\alpha_1 \Big\} x_0 + \Big\{ \alpha_3\alpha_2\beta_1 + \beta_3\beta_1 \Big\} x_{-1}, \\
x_4 &= \alpha_4 x_3 + \beta_4 x_2 = \alpha_4 \Big\{ [\alpha_3 (\alpha_2\alpha_1 + \beta_2) + \beta_3\alpha_1]\, x_0 + [\alpha_3\alpha_2\beta_1 + \beta_3\beta_1]\, x_{-1} \Big\} + \\
&\beta_4 \Big\{ [\alpha_2\alpha_1 + \beta_2]\, x_0 + \alpha_2\beta_1 x_{-1} \Big\} = \Big\{ \alpha_4 [\alpha_3 (\alpha_2\alpha_1 + \beta_2) + \beta_3\alpha_1] + \beta_4 [\alpha_2\alpha_1 + \beta_2] \Big\} x_0 \\
&+ \Big\{ \alpha_4 [\alpha_3\alpha_2\beta_1 + \beta_3\beta_1] + \beta_4\alpha_2\beta_1 \Big\} x_{-1}.
\end{aligned}
$$

With Theorem 7.1.1, the values x_1, x_2, x_3, x_4 are calculated as

$$
\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = A_1 \begin{pmatrix} x_{-1} \\ x_0 \end{pmatrix} = \begin{pmatrix} \beta_1 & \alpha_1 \\ \alpha_2\beta_1 & \beta_2 + \alpha_2\alpha_1 \end{pmatrix} \begin{pmatrix} x_{-1} \\ x_0 \end{pmatrix} = \begin{pmatrix} \beta_1 x_{-1} + \alpha_1 x_0 \\ \alpha_2\beta_1 x_{-1} + [\beta_2 + \alpha_2\alpha_1]\, x_0 \end{pmatrix},
$$

$$
\begin{pmatrix} x_3 \\ x_4 \end{pmatrix} = A_3 A_1 \begin{pmatrix} x_{-1} \\ x_0 \end{pmatrix} = \begin{pmatrix} \beta_3 & \alpha_3 \\ \alpha_4\beta_3 & \beta_4 + \alpha_4\alpha_3 \end{pmatrix} \begin{pmatrix} \beta_1 & \alpha_1 \\ \alpha_2\beta_1 & \beta_2 + \alpha_2\alpha_1 \end{pmatrix} \begin{pmatrix} x_{-1} \\ x_0 \end{pmatrix} =
$$

$$
\begin{pmatrix} \alpha_3\alpha_2\beta_1 + \beta_3\beta_1 & \alpha_3 (\alpha_2\alpha_1 + \beta_2) + \beta_3\alpha_1 \\ \alpha_4 [\alpha_3\alpha_2\beta_1 + \beta_3\beta_1] + \beta_4\alpha_2\beta_1 & \alpha_4 [\alpha_3 (\alpha_2\alpha_1 + \beta_2) + \beta_3\alpha_1] + \beta_4 [\alpha_2\alpha_1 + \beta_2] \end{pmatrix} \begin{pmatrix} x_{-1} \\ x_0 \end{pmatrix}.
$$

We see that the solution expressions are the same.

7.1.2. The Exponential Behavior of Solutions. It is not only possible to observe that the solution of (7.1), (7.2) has, under certain conditions on the constants a and b, a turn around point; that is, the solution decreases and tends to 0 but after reaching the turn around point, the solution blows up. Moreover, it is also possible to prove that the solution of the pantograph equation (7.1) in quantum calculus behaves like a q-exponential function as $t \to \infty$. To show this behavior, we follow the idea and work in [5].

Consider the q-difference equation

$$
x^{\Delta}(t) = a x(t), \tag{7.8}
$$

which has no delay term. The solution e_a of (7.8) satisfying $e_a(1) = 1$ is given by the q-exponential function (see Definition 6.0.4)

$$e_a(t) = \prod_{i=0}^{\log_q(t)-1} \left(1 + a(q-1)q^i\right), \quad t \in q^{\mathbb{N}_0}. \tag{7.9}$$

Theorem 7.1.3. *Let be given the q-difference equation*

$$x^{\Delta}(t) = ax(t) + bx\left(\frac{t}{q^N}\right), \quad t \in q^{\mathbb{N}_0}, \quad N \in \mathbb{N},$$

where a and b are nonzero constants, and the initial function (7.2)*. Then, there exists a constant C such that for any solution x of* (7.1)*, we have*

$$\lim_{t\to\infty} \frac{x(t)}{e_a(t)} = C.$$

Proof. Assume that x is a solution of (7.1). Then, we define

$$y(t) = \frac{x(t)}{e_a(t)}, \quad \text{for} \quad t \in q^{\mathbb{N}_0}. \tag{7.10}$$

With the product rule in quantum calculus (see Theorem 6.0.3), i.e.,

$$(fg)^{\Delta}(t) = f^{\Delta}(t)g(t) + f(qt)g^{\Delta}(t)$$

and using (7.10), $x^{\Delta}(t)$ becomes

$$\begin{aligned} x^{\Delta}(t) &= e_a^{\Delta}(t)y(t) + e_a(qt)y^{\Delta}(t) = ae_a(t)y(t) + e_a(qt)y^{\Delta}(t) \\ &= ae_a(t)\frac{x(t)}{e_a(t)} + e_a(qt)y^{\Delta}(t) = ax(t) + e_a(qt)y^{\Delta}(t). \end{aligned}$$

Hence, the q-difference equation

$$x^{\Delta}(t) = ax(t) + bx\left(\frac{t}{q^N}\right)$$

can be transformed to

$$\begin{aligned}
&x^{\Delta}(t) = ax(t) + bx\left(\frac{t}{q^N}\right)\\
&\implies ax(t) + e_a(qt)y^{\Delta}(t) = ax(t) + bx\left(\frac{t}{q^N}\right)\\
&\implies y^{\Delta}(t) = \frac{b}{e_a(qt)}e_a\left(\frac{t}{q^N}\right)y\left(\frac{t}{q^N}\right)\\
&\implies y^{\Delta}(t) = b\frac{e_a\left(\frac{t}{q^N}\right)}{e_a(qt)}y\left(\frac{t}{q^N}\right), \quad t \geq q^N.
\end{aligned} \tag{7.11}$$

Define $t_k := q^{k+N}$ and $S_k := \sup\{|y(t_k q^{-p})|, p = 0, 1, \ldots, k\}$ for $k \in \mathbb{N}_0$. Consider (7.11)

$$\begin{aligned}
&y^{\Delta}(t_k) = b\frac{e_a\left(\frac{t_k}{q^N}\right)}{e_a(qt_k)}y\left(\frac{t_k}{q^N}\right)\\
&\implies \frac{y(t_{k+1}) - y(t_k)}{(q-1)t_k} = b\frac{e_a\left(\frac{t_k}{q^N}\right)}{e_a(qt_k)}y\left(\frac{t_k}{q^N}\right)\\
&\implies y(t_{k+1}) = y(t_k) + (q-1)t_k b\left(\frac{e_a\left(\frac{t_k}{q^N}\right)}{e_a(t_{k+1})}y\left(\frac{t_k}{q^N}\right)\right).
\end{aligned}$$

Using the definition of S_k, we get

$$|y(t_{k+1})| \leq S_k\left(1 + (q-1)t_k\,|b|\left|\frac{e_a\left(\frac{t_k}{q^N}\right)}{e_a(t_{k+1})}\right|\right).$$

Now, consider the exponential term and use the expression in (7.9) to obtain

$$\left|\frac{e_a\left(\frac{t_k}{q^N}\right)}{e_a(t_{k+1})}\right| = \left|\frac{e_a\left(q^k\right)}{e_a(q^{k+1+N})}\right| = \left|\frac{\prod_{i=0}^{k-1}\left(1+a(q-1)q^i\right)}{\prod_{i=0}^{k+N}\left(1+a(q-1)q^i\right)}\right|$$
$$= \prod_{p=0}^{N}\frac{1}{\left|1+a(q-1)q^{k+p}\right|} = O\left(q^{-k(N+1)}\right) \quad \text{as} \quad k\to\infty. \tag{7.12}$$

Sine $N+1>1$ because $N\in\mathbb{N}$, we get

$$|y(t_{k+1})| \le S_k\left(1+O\left(q^{-rk)}\right)\right) \quad \text{as} \quad k\to\infty,$$

where $r>0$ is a suitable constant. Thus,

$$S_{k+1} \le S_k\left(1+O\left(q^{-rk)}\right)\right) \quad \text{as} \quad k\to\infty.$$

Therefore, the sequence (S_k) is bounded as $k\to\infty$ and so $|y(t)|\le M$ for a suitable positive real number M and $t\in q^{\mathbb{N}_0}$.

Now, we finally prove the statement of Theorem 7.1.3 by showing that y has a (finite) limit. Let $\varepsilon>0$ and let $t_l, t_n\in q^{\mathbb{N}_0}$, $q^N<t_l<t_n$. Consider (7.11) and integrate both sides

$$\begin{aligned}
& y^\Delta(t) = b\frac{e_a\left(\frac{t}{q^N}\right)}{e_a(qt)}y\left(\frac{t}{q^N}\right) \\
\implies & \int_{t_l}^{t_n} y^\Delta(t)\Delta t = \int_{t_l}^{t_n} b\frac{e_a\left(\frac{t}{q^N}\right)}{e_a(qt)}y\left(\frac{t}{q^N}\right)\Delta t \\
\implies & y(t_n)-y(t_l) = \int_{t_l}^{t_n} b\frac{e_a\left(\frac{t}{q^N}\right)}{e_a(qt)}y\left(\frac{t}{q^N}\right)\Delta t.
\end{aligned}$$

With the observations above,

$$|y(t_n) - y(t_l)| \leq M\,|b| \int_{t_l}^{t_n} \left| \frac{e_a\left(\frac{t}{q^N}\right)}{e_a(qt)} \right| \Delta t.$$

Using (7.12), we can estimate the integral and observe its convergence as $t_n \to \infty$. For t_l, t_n large enough

$$|b| \int_{t_l}^{t_n} \left| \frac{e_a\left(\frac{t}{q^N}\right)}{e_a(qt)} \right| \Delta t \leq \frac{\varepsilon}{M}.$$

Therefore, $|y(t_n) - y(t_l)| \leq \varepsilon$ and y is convergent to a finite limit C, which completes the proof. □

Example 7.1.4. *Consider the pantograph equation*

$$x^{\Delta}(t) = 3x(t) + 0.5x\left(\frac{t}{q}\right)$$

with the initial values

$$x_{-1} = 1, \quad x_0 = 1.$$

With Theorem 7.1.3, the asymptotic behavior of the solution is exponential. Using the programming language Matlab *and let $q = 1.01$, the solution of the pantograph equation together with the q-exponential function is illustrated in Figure 7.1. It seems that the solution of the pantograph equation behaves exponential, which supports the result from Theorem 7.1.3. The x-axis is in terms of n instead of q^n because the result is better visible in this way.*

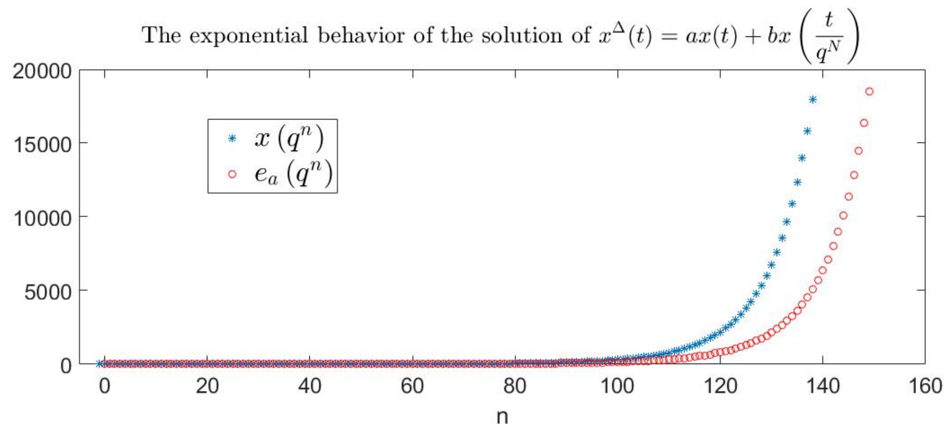

Figure 7.1. The exponential behavior of solutions of the pantograph equation (7.1) with $N = 1$, $a = 3$, $b = 0.5$, and $q = 1.01$.

7.2. STABILITY ANALYSIS OF DIFFERENT PANTOGRAPH EQUATIONS IN QUANTUM CALCULUS

This section discusses a different type of the pantograph equation in quantum calculus. The goal in this section is to find some conditions for the asymptotic stability of the zero solution for this kind of pantograph equation. Three variants of this kind of pantograph equation are considered, they are only different in the delay term.

7.2.1. The Pantograph Equation of the Form $x^{\Delta}(t) = \frac{a}{t}x(t) + \frac{b}{t}x\left(\frac{t}{q^N}\right)$. The coefficients a and b in the pantograph equation (4.1) are constants. To get also constant coefficients for the difference equation that is derived from the corresponding q-difference equation, we consider a pantograph equation in quantum calculus with variable coefficients. For this purpose, we consider the pantograph equation

$$x^{\Delta}(t) = \frac{a}{t}x(t) + \frac{b}{t}x\left(\frac{t}{q^N}\right), \quad t \in q^{\mathbb{N}_0}, \quad N \in \mathbb{N} \tag{7.13}$$

with the same initial function as in (7.2)

$$x(t) = f(t), \quad t \in \Omega.$$

With the q-derivative (see Definition 6.0.2), a corresponding difference equation is derived as

$$\begin{aligned} x^{\Delta}(t) &= \frac{a}{t}x(t) + \frac{b}{t}x\left(\frac{t}{q^N}\right) \\ \implies \frac{x(qt) - x(t)}{(q-1)t} &= \frac{a}{t}x(t) + \frac{b}{t}x\left(\frac{t}{q^N}\right) \\ \implies x(qt) &= (1 + a(q-1))\,x(t) + b(q-1)x\left(\frac{t}{q^N}\right). \end{aligned}$$

If we put $t_n := q^n$ and $t_{n+1} := t_n q = q^n q = q^{n+1}$, then

$$x(t_{n+1}) = (1 + a(q-1))\,x(t_n) + b(q-1)x\left(\frac{t_n}{q^N}\right).$$

It also follows that $\frac{t_n}{q^N} = \frac{q^n}{q^N} = q^{n-N} = t_{n-N}$. Hence

$$x(t_{n+1}) = (1 + a(q-1))\,x(t_n) + b(q-1)x\left(t_{n-N}\right).$$

Define $x_n := x(t_n)$, $x_{n+1} := x(t_{n+1})$, and $x_{n-N} := x(t_{n-N})$. Then,

$$x_{n+1} = (1 + a(q-1))\,x_n + b(q-1)x_{n-N}.$$

Therefore, (7.13) reduces to the difference equation of order $N+1$ and constant coefficients

$$x_{n+1} = \alpha x_n + \beta x_{n-N}, \quad n \in \mathbb{N}_0, \tag{7.14}$$

where

$$\alpha = 1 + a(q-1), \quad \beta = b(q-1). \tag{7.15}$$

The given values $x_{-N}, x_{-N+1}, \ldots, x_{-1}, x_0$ from the initial function in (7.2) complete the initial value problem.

Because the difference equation (7.14) has constant coefficients and order $N+1$, the Schur–Cohn criterion (see Theorem 3.2.5) can be applied to look for asymptotic stability conditions. For completeness, we state the following theorem, that is, the Schur–Cohn criterion applied to the difference equation (7.14) with (7.15).

Theorem 7.2.1. *The zeros of the characteristic polynomial*

$$P_N(v) = v^{N+1} - \alpha v^N - \beta \tag{7.16}$$

with

$$\alpha = 1 + a(q-1), \quad \beta = b(q-1)$$

of the difference equation (7.14) *lie inside the unit disk if and only if the following conditions hold:*

(i) $P_N(1) > 0$*,*

(ii) $(-1)^{N+1} P_N(-1) > 0$*,*

(iii) the $N \times N$ *matrices*

$$M_N^{\pm} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ -\alpha & 1 & \ddots & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 1 & 0 \\ 0 & \cdots & 0 & -\alpha & 1 \end{pmatrix} \pm \begin{pmatrix} 0 & \cdots & \cdots & 0 & -\beta \\ \vdots & & \iddots & -\beta & 0 \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ 0 & -\beta & \iddots & & \vdots \\ -\beta & 0 & \cdots & \cdots & 0 \end{pmatrix}$$

are positive innerwise.

First, the case $N = 1$ is considered. From now on, assume that $|a| + b < 0$. The difference equation (7.14) becomes

$$x_{n+1} = \alpha x_n + \beta x_{n-1}, \quad n \in \mathbb{N}_0$$

with α and β in (7.15). From Condition (i) of Theorem 7.2.1, we obtain

$$\begin{aligned}
&P_1(1) > 0 \\
&\implies 1 - \alpha - \beta > 0 \\
&\implies 1 - (1 + a(q-1)) - b(q-1) > 0 \\
&\implies -(a+b)(q-1) > 0 \\
&\implies a + b < 0, \quad \text{since} \quad q - 1 > 0.
\end{aligned}$$

This is satisfied by the assumption $|a| + b < 0$. Condition (ii) of Theorem 7.2.1 leads to

$$\begin{aligned}
&(-1)^2 P_1(-1) > 0 \\
&\implies 1 + \alpha - \beta > 0 \\
&\implies 1 + (1 + a(q-1)) - b(q-1) > 0 \\
&\implies 2 + (a-b)(q-1) > 0 \\
&\implies q - 1 > \frac{-2}{a-b}, \quad \text{since} \quad a - b > 0 \quad \text{from} \quad |a| + b < 0,
\end{aligned}$$

which is always true because the right-hand side is negative. Condition (iii) of Theorem 7.2.1 implies

$$\begin{aligned}
&\det\left(M_1^{\pm}\right) = \det(1 \pm (-\beta)) = 1 \pm \beta > 0 \\
&\implies 1 + \beta > 0 \quad \text{and} \quad 1 - \beta > 0.
\end{aligned}$$

From the first inequality $1 + \beta > 0$, we get

$$\begin{aligned}
&1 + \beta > 0 \\
&\implies 1 + b(q-1) > 0 \\
&\implies q - 1 < \frac{-1}{b}, \quad \text{since} \quad b < 0 \quad \text{from} \quad |a| + b < 0
\end{aligned}$$

$$\implies q < 1 - \frac{1}{b}.$$

Since $b < 0$, the right-hand side is always greater than 1. The second inequality $1 - \beta > 0$ leads to

$$\begin{aligned}
&1 - \beta > 0 \\
&\implies 1 - b(q-1) > 0 \\
&\implies q - 1 > \frac{1}{b}, \quad \text{since} \quad b < 0 \quad \text{from} \quad |a| + b < 0.
\end{aligned}$$

This condition is always true because $b < 0$. All in all, we only need the condition $q < 1 - \frac{1}{b}$ to obtain the asymptotic stability of the zero solution for $N = 1$.

Theorem 7.2.2. *Let $|a| + b < 0$ and α, β be given by* (7.15). *Then, the zeros of the characteristic polynomial* (7.16) *for $N = 1$ lie inside the unit disk if and only if*

$$q < 1 - \frac{1}{b}.$$

Proof. The proof is given as derivation right before this theorem by using Theorem 7.2.1. □

Example 7.2.3. *Let $N = 1$, $a = 0.1$, and $b = -0.5$. Thus, consider the pantograph equation*

$$x^{\Delta}(t) = \frac{0.1}{t}x(t) - \frac{0.5}{t}x\left(\frac{t}{q}\right)$$

with the initial values

$$x_{-1} = 1, \quad x_0 = 1.$$

With Theorem 7.2.2, the zero solution is asymptotically stable if and only if

$$q < 1 - \frac{1}{b} = 3.$$

First, we consider $q = 2.9$ *meaning that, by Theorem 7.2.2, the zero solution is asymptotically stable. Using the programming language* Matlab *(see appendix, Listing 9.1), we get Figure 7.2, which supports our result in Theorem 7.2.2. The x-axis is in terms of n instead of* q^n *because the result is better visible in this way.*

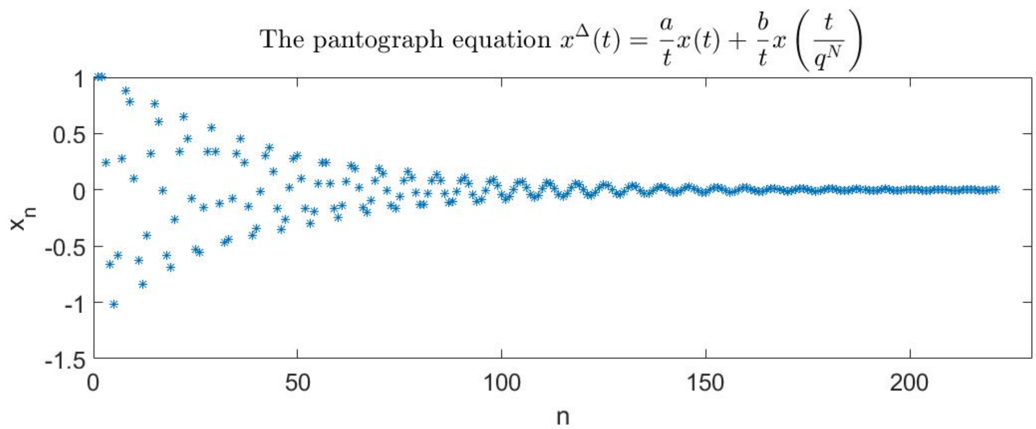

Figure 7.2. The solution of the pantograph equation (7.13) with $N = 1$, $a = 0.1$, $b = -0.5$, and $q = 2.9$.

Now, we consider $q = 3.1$. *Therefore, the zero solution is not asymptotically stable anymore with Theorem 7.2.2. This is also illustrated in Figure 7.3 using* Matlab *(see appendix, Listing 9.1). The solution is oscillating with an increasing amplitude as n is increasing.*

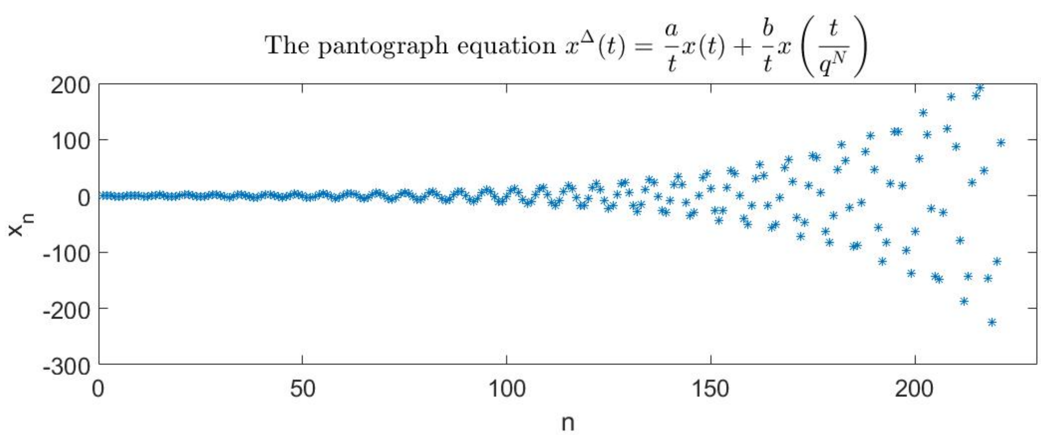

Figure 7.3. The solution of the pantograph equation (7.13) with $N = 1$, $a = 0.1$, $b = -0.5$, and $q = 3.1$.

Lastly, we consider $q = 3$. That is exactly the critical value of q for which, with Theorem 7.2.2, the statement changes from the zero solution is asymptotically stable to the zero solution is not asymptotically stable. Hence, the zero solution could be stable or unstable for $q = 3$. It seems that the zero solution is stable, which is illustrated in Figure 7.4 using Matlab *(see appendix, Listing 9.1).*

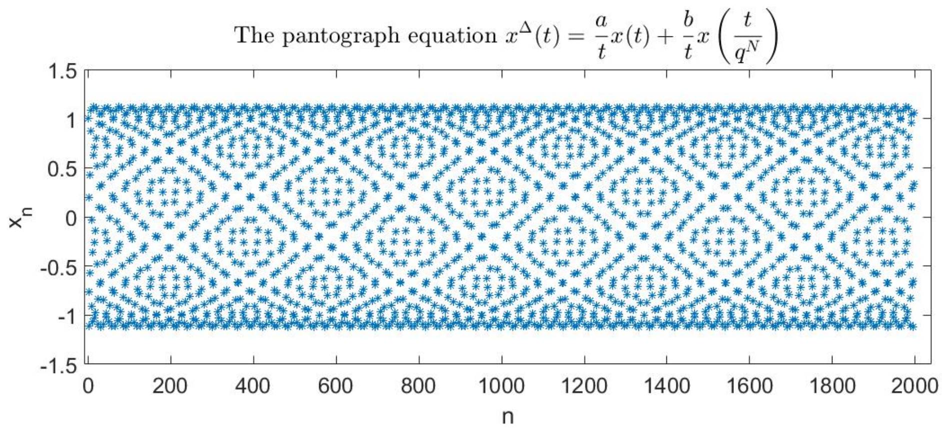

Figure 7.4. The solution of the pantograph equation (7.13) with $N = 1$, $a = 0.1$, $b = -0.5$, and $q = 3$.

Second, consider the case $N = 2$. Furthermore, it is still assumed that $|a| + b < 0$. The difference equation (7.14) becomes

$$x_{n+1} = \alpha x_n + \beta x_{n-2}, \quad n \in \mathbb{N}_0$$

with α and β in (7.15). Condition (i) of Theorem 7.2.1 is the same as in the previous case for $N = 1$, i.e.,

$$\begin{aligned} &P_2(1) > 0 \\ &\implies 1 - \alpha - \beta > 0 \\ &\implies a + b < 0, \end{aligned}$$

which is satisfied by the assumption $|a| + b < 0$. Condition (ii) of Theorem 7.2.1 implies

$$
\begin{aligned}
&(-1)^3 P_2(-1) > 0 \\
&\implies 1 + \alpha + \beta > 0 \\
&\implies 1 + (1 + a(q-1)) + b(q-1) > 0 \\
&\implies 2 + (a+b)(q-1) > 0 \\
&\implies q - 1 < \frac{-2}{a+b}, \quad \text{since} \quad a + b < 0 \quad \text{from} \quad |a| + b < 0 \\
&\implies q < 1 - \frac{2}{a+b}.
\end{aligned}
$$

Since $a + b < 0$, the right-hand side is always greater than 1. Condition (iii) of Theorem 7.2.1 leads to

$$
\begin{aligned}
&\det\left(M_2^+\right) = \det\begin{pmatrix} 1 & -\beta \\ -\alpha - \beta & 1 \end{pmatrix} > 0 \\
&\implies 1 - \beta(\alpha + \beta) > 0
\end{aligned}
$$

and

$$
\begin{aligned}
&\det\left(M_2^-\right) = \det\begin{pmatrix} 1 & \beta \\ -\alpha + \beta & 1 \end{pmatrix} > 0 \\
&\implies 1 + \beta(\alpha - \beta) > 0.
\end{aligned}
$$

The first inequality $1 - \beta(\alpha + \beta) > 0$ becomes

$$
\begin{aligned}
&1 - \beta(\alpha + \beta) > 0 \\
&\implies 1 - b(q-1)\Big(1 + a(q-1) + b(q-1)\Big) > 0 \\
&\implies 1 - b(q-1) - (ab + b^2)(q-1)^2 > 0.
\end{aligned}
$$

If we put $z = q - 1$, then a quadratic equation is given on the left-hand side. Set the left-hand side equal to 0, find the roots of this equation, and therefore find the critical value q^* up to which the inequality holds, i.e.,

$$\begin{aligned} &1 - bz - (ab + b^2)z^2 = 0 \\ \implies \quad & z_{1,2} = \frac{b \pm \sqrt{b(5b + 4a)}}{-2b(a + b)}. \end{aligned}$$

Since $q > 1$ implies $q-1 > 0$ and for $q \to 1$, we have $z \to 0$ and thus $1-bz-(ab+b^2)z^2 \to 1$. That means the quadratic equation is positive for q close to 1, and since the quadratic map is continuous, the smallest root greater than 0 is the critical value q^* up to which the condition holds. Moreover, $-(ab + b^2) = -b(a + b) < 0$ because $|a| + b < 0$. For this reason, the quadratic equation is a parabola that opens downwards. This implies that one root is smaller than 0 and one root is greater than 0. For our purpose, only the root greater than 0 is relevant. Therefore, we consider only the inequality

$$q < \frac{b - \sqrt{b(5b + 4a)}}{-2b(a + b)} + 1.$$

From the second inequality $1 + \beta(\alpha - \beta) > 0$, we obtain

$$\begin{aligned} &1 + \beta(\alpha - \beta) > 0 \\ \implies \quad & 1 + b(q - 1)\Big(1 + a(q - 1) - b(q - 1)\Big) > 0 \\ \implies \quad & 1 + b(q - 1) + (ab - b^2)(q - 1)^2 > 0. \end{aligned}$$

If we put $z = q - 1$, then a quadratic equation on the left-hand side is given. Set the left-hand side equal to 0 again, find the roots of this equation, and thus find the critical value q^* up to which the inequality holds, i.e.,

$$1 + bz + (ab - b^2)z^2 = 0$$

$$\implies z_{1,2} = \frac{-b \pm \sqrt{b(5b-4a)}}{2b(a-b)}.$$

With the same explanation as before because the leading coefficient is negative, that is $b(a-b) < 0$, only the root greater than 0 is relevant. Therefore, we consider only the inequality

$$q < \frac{-b - \sqrt{b(5b-4a)}}{2b(a-b)} + 1.$$

Now, we can state the theorem about the asymptotic stability of the zero solution for $N = 2$.

Theorem 7.2.4. *Let* $|a| + b < 0$ *and* α, β *be given by* (7.15). *Then, the zeros of the characteristic polynomial* (7.16) *for* $N = 2$ *lie inside the unit disk if and only if*

$$q < \min\left\{1 - \frac{2}{a+b}, \frac{b - \sqrt{b(5b+4a)}}{-2b(a+b)} + 1, \frac{-b - \sqrt{b(5b-4a)}}{2b(a-b)} + 1\right\}.$$

Proof. The proof is given as derivation right before the theorem by using Theorem 7.2.1. □

Example 7.2.5. *Let* $N = 2$, $a = 0.45$, *and* $b = -0.5$. *We consider*

$$x^{\Delta}(t) = \frac{0.45}{t}x(t) - \frac{0.5}{t}x\left(\frac{t}{q^2}\right)$$

with the initial values

$$x_{-2} = 1, \quad x_{-1} = 1, \quad x_0 = 1.$$

Theorem 7.2.4 says that the zero solution for this equation above is asymptotically stable if and only if

$$q < \min\{41,\ 22.8322,\ 2.0171\} = 2.0171.$$

Let us consider $q = 2.01$. *With Theorem 7.2.4, the zero solution is asymptotically stable, which is illustrated in Figure 7.5 using* Matlab *(see appendix, Listing 9.1).*

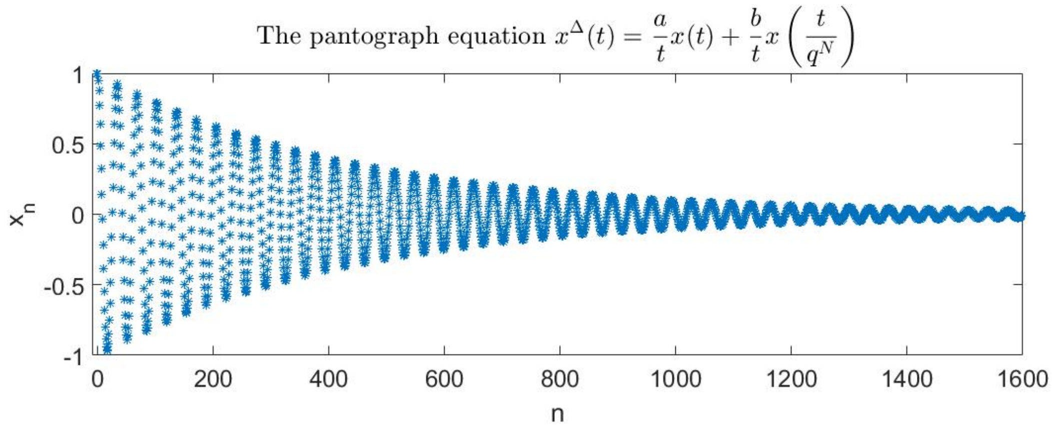

Figure 7.5. The solution of the pantograph equation (7.13) with $N = 2$, $a = 0.45$, $b = -0.5$, and $q = 2.01$.

If we consider $q = 2.02$*, which is a little bit greater than* 2.0171*, the zero solution is, by Theorem 7.2.2, not asymptotically stable anymore. This is also supported in Figure 7.6 using* Matlab *(see appendix, Listing 9.1). The solution is oscillating with an increasing amplitude as n is increasing.*

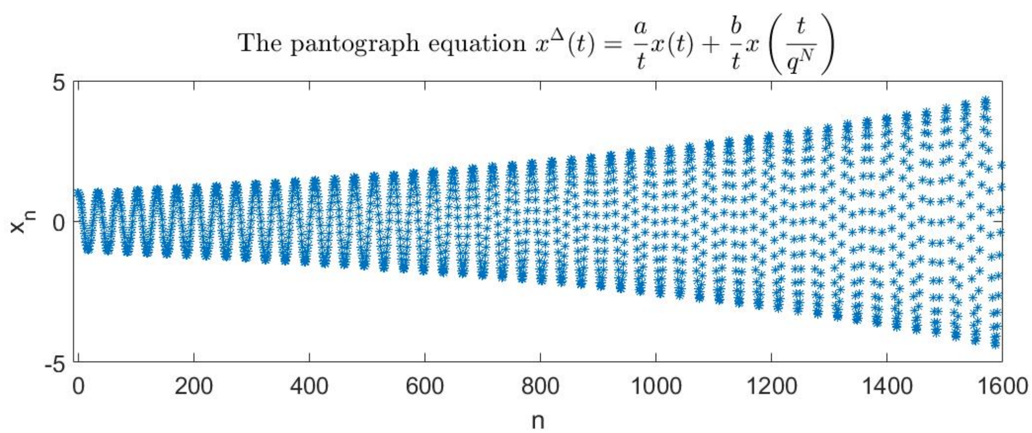

Figure 7.6. The solution of the pantograph equation (7.13) with $N = 2$, $a = 0.45$, $b = -0.5$, and $q = 2.02$.

Third, consider the case $N = 3$. It is still assumed that $|a| + b < 0$. The difference equation (7.14) becomes

$$x_{n+1} = \alpha x_n + \beta x_{n-3}, \quad n \in \mathbb{N}_0$$

with α and β in (7.15). Condition (i) of Theorem 7.2.1 is the same as in the two previous cases for $N = 1, 2$, namely

$$\begin{aligned} &P_3(1) > 0 \\ &\implies 1 - \alpha - \beta > 0 \\ &\implies a + b < 0. \end{aligned}$$

This is satisfied by the assumption $|a| + b < 0$. From Condition (ii) of Theorem 7.2.1, we get

$$\begin{aligned} &(-1)^4 P_3(-1) > 0 \\ &\implies 1 + \alpha - \beta > 0 \\ &\implies 1 + (1 + a(q-1)) - b(q-1) > 0 \\ &\implies 2 + (a-b)(q-1) > 0 \\ &\implies q - 1 > \frac{-2}{a-b}, \quad \text{since} \quad a - b > 0 \quad \text{from} \quad |a| + b < 0. \end{aligned}$$

Since the right-hand side is negative, this condition is always satisfied. Condition (iii) of Theorem 7.2.1, which says that all determinants of the matrix inners have to be positive, implies

$$\begin{aligned} &\det\left(M_3^+\right) = \det\begin{pmatrix} 1 & 0 & -\beta \\ -\alpha & 1-\beta & 0 \\ -\beta & -\alpha & 1 \end{pmatrix} > 0 \\ &\implies 1 - \beta - \alpha^2\beta - \beta^2(1-\beta) > 0 \end{aligned}$$

and

$$
\det\left(M_3^-\right) = \det\begin{pmatrix} 1 & 0 & \beta \\ -\alpha & 1+\beta & 0 \\ \beta & -\alpha & 1 \end{pmatrix} > 0
$$

$$
\implies 1 + \beta + \alpha^2\beta + \beta^2(1-\beta) > 0.
$$

In addition to that, the determinant of the matrix inners leads to the inequalities

$$
\det(1 \pm \beta) > 0
$$

$$
\implies 1 + \beta > 0 \quad \text{and} \quad 1 - \beta > 0.
$$

The last two inequalities are the same as in the case $N = 1$. Thus, the inequality $1 + \beta > 0$ becomes

$$
1 + \beta > 0
$$

$$
\implies q < 1 - \frac{1}{b}.
$$

Since $b < 0$, the right-hand side is always greater than 1. From the other inequality $1-\beta > 0$, we get

$$
1 - \beta > 0
$$

$$
\implies q - 1 > \frac{1}{b},
$$

which is always true because $b < 0$. All in all, we get the condition $q < 1 - \frac{1}{b}$ from the determinant of the matrix inners. Moreover, the first inequality $1-\beta-\alpha^2\beta-\beta^2(1-\beta) > 0$ leads to

$$\begin{aligned} &1-\beta-\alpha^2\beta-\beta^2(1-\beta) > 0 \\ &\implies 1-2b(q-1)-(2ab+b^2)(q-1)^2+(b^3-ba^2)(q-1)^3 > 0. \end{aligned}$$

If we put $z = q-1$, then it is given a cubic equation on the left-hand side. Set the left-hand side equal to 0, find the roots of this equation with Vieta's substitution, and thus find the critical value q^* up to which the inequality holds, namely

$$1-2bz-(2ab+b^2)z^2+(b^3-ba^2)z^3 = 0.$$

Define the coefficients as $c_1 := b^3 - ba^2$, $c_2 := -2ab - b^2$, $c_3 := -2b$, $c_4 := 1$. Then, we follow the idea of Vieta's substitution, which can be found in [19]. Dividing the equation by c_1 and substituting $z = y - \frac{c_2}{3c_1}$, we get the so-called depressed cubic equation

$$y^3 + py + q = 0,$$

where

$$\begin{aligned} p &= \frac{3c_1c_3 - c_2^2}{3c_1^2}, \\ q &= \frac{2c_2^3 - 9c_1c_2c_3 + 27c_1^2c_4}{27c_1^3}. \end{aligned}$$

Then, make the substitution $y = w - \frac{p}{3w}$, which is known as Vieta's substitution, i.e.,

$$w^3 + q - \frac{p^3}{27w^3} = 0.$$

Multiplying this equation by w^3, it follows that

$$\begin{aligned} & w^6 + qw^3 - \frac{p^3}{27} = 0 \\ \Longrightarrow\ & \left(w^3\right)^2 + q\left(w^3\right) - \frac{p^3}{27} = 0. \end{aligned}$$

Solve this quadratic equation now for w^3 to obtain

$$w_{1,2}^3 = \frac{-q \pm \sqrt{q^2 + \frac{4}{27}p^3}}{2}.$$

Choose one of the roots, here

$$w_1^3 = \frac{-q + \sqrt{q^2 + \frac{4}{27}p^3}}{2}$$

and get

$$w_1 = \sqrt[3]{\frac{-q + \sqrt{q^2 + \frac{4}{27}p^3}}{2}}.$$

Then, the three roots can be expressed with $\xi = -\frac{1}{2} + \frac{1}{2}\sqrt{3}i$, so that

$$y_1 = w_1 - \frac{p}{3w_1}, \quad y_2 = \xi w_1 - \frac{p}{3\xi w_1}, \quad y_3 = \xi^2 w_1 - \frac{p}{3\xi^2 w_1},$$

and therefore

$$z_1 = y_1 - \frac{c_2}{3c_1}, \quad z_2 = y_2 - \frac{c_2}{3c_1}, \quad z_3 = y_3 - \frac{c_2}{3c_1}.$$

Finally, we get the three inequalities

$$q < z_1 + 1, \quad q < z_2 + 1, \quad q < z_3 + 1$$

but not every inequality is relevant. Only the inequalities where the roots are real numbers and the right-hand side is greater than 1 are relevant. Therefore, we consider only the inequalities

$$q < z_i + 1, \quad \text{where} \quad z_i \in \mathbb{R} \quad \text{and} \quad z_i + 1 > 1 \quad \text{for} \quad i = 1, 2, 3$$

and define

$$\eta := \min\{z_i + 1 \,:\, z_i \in \mathbb{R} \quad \text{and} \quad z_i + 1 > 1 \quad \text{for} \quad i = 1, 2, 3\}\,. \tag{7.17}$$

Since $q > 1$ implies $q - 1 > 0$ and for $q \to 1$, we have $z \to 0$ and hence $1 - 2b(q-1) - (2ab + b^2)(q-1)^2 + (b^3 - ba^2)(q-1)^3 \to 1$. That means the cubic equation is positive for q close to 1, and since the cubic map is continuous, the smallest real root greater than 0 is the critical value q^* up to which the condition holds. From the second equation $1 + \beta + \alpha^2\beta + \beta^2(1-\beta) > 0$, we obtain

$$\begin{aligned} &1 + \beta + \alpha^2\beta + \beta^2(1-\beta) > 0 \\ &\implies 1 + 2b(q-1) + (2ab - b^2)(q-1)^2 + (a^2b - b^3)(q-1)^3 > 0. \end{aligned}$$

If we put $\tilde{z} = q - 1$, then a cubic equation on the left-hand side is given. Set the left-hand side equal to 0, find the roots of this equation with Vieta's substitution, and thus find the critical value q^* up to which the inequality holds, namely

$$1 + 2b\tilde{z} + (2ab - b^2)\tilde{z}^2 + (a^2b - b^3)\tilde{z}^3 = 0.$$

Define the coefficients as $d_1 := a^2b - b^3$, $d_2 := 2ab - b^2$, $d_3 := 2b$, $d_4 := 1$. Then, we follow again the idea of Vieta's substitution and do the same procedure as above. Dividing the equation by d_1 and substituting $\tilde{z} = y - \frac{d_2}{3d_1}$, we get the depressed cubic equation

$$y^3 + \tilde{p}y + \tilde{q} = 0,$$

where

$$\begin{aligned}\tilde{p} &= \frac{3d_1d_3 - d_2^2}{3d_1^2},\\ \tilde{q} &= \frac{2d_2^3 - 9d_1d_2d_3 + 27d_1^2d_4}{27d_1^3}.\end{aligned}$$

As a next step, make the substitution $y = w - \frac{\tilde{p}}{3w}$ to obtain

$$w^3 + \tilde{q} - \frac{\tilde{p}^3}{27w^3} = 0.$$

Multiplying this equation by w^3 leads to

$$\begin{aligned}&w^6 + \tilde{q}w^3 - \frac{\tilde{p}^3}{27} = 0\\ \implies &\left(w^3\right)^2 + \tilde{q}\left(w^3\right) - \frac{\tilde{p}^3}{27} = 0.\end{aligned}$$

Solve the quadratic equation now for w^3 to get

$$w_{1,2}^3 = \frac{-\tilde{q} \pm \sqrt{\tilde{q}^2 + \frac{4}{27}\tilde{p}^3}}{2}.$$

Choose one of the roots, here

$$w_1^3 = \frac{-\tilde{q} + \sqrt{\tilde{q}^2 + \frac{4}{27}\tilde{p}^3}}{2}$$

and get

$$w_1 = \sqrt[3]{\frac{-\tilde{q} + \sqrt{\tilde{q}^2 + \frac{4}{27}\tilde{p}^3}}{2}}.$$

Then, the three roots can be expressed with $\xi = -\frac{1}{2} + \frac{1}{2}\sqrt{3}i$, so that

$$y_1 = w_1 - \frac{\tilde{p}}{3w_1}, \quad y_2 = \xi w_1 - \frac{\tilde{p}}{3\xi w_1}, \quad y_3 = \xi^2 w_1 - \frac{\tilde{p}}{3\xi^2 w_1},$$

and therefore

$$\tilde{z}_1 = y_1 - \frac{d_2}{3d_1}, \quad \tilde{z}_2 = y_2 - \frac{d_2}{3d_1}, \quad \tilde{z}_3 = y_3 - \frac{d_2}{3d_1}.$$

We get the three inequalities

$$q < \tilde{z}_1 + 1, \quad q < \tilde{z}_2 + 1, \quad q < \tilde{z}_3 + 1$$

but only the inequalities where the roots are real numbers and the right-hand side is greater than 1 are relevant. Therefore, we consider only the inequalities

$$q < \tilde{z}_i + 1, \quad \text{where} \quad \tilde{z}_i \in \mathbb{R} \quad \text{and} \quad \tilde{z}_i + 1 > 1 \quad \text{for} \quad i = 1, 2, 3$$

and define

$$\vartheta := \min\left\{\tilde{z}_i + 1 \;:\; \tilde{z}_i \in \mathbb{R} \quad \text{and} \quad \tilde{z}_i + 1 > 1 \quad \text{for} \quad i = 1, 2, 3\right\}. \tag{7.18}$$

Since $q > 1$ implies $q - 1 > 0$ and for $q \to 1$, we have $\tilde{z} \to 0$ and hence $1 + 2b(q-1) + (2ab - b^2)(q-1)^2 + (a^2b - b^3)(q-1)^3 \to 1$. That means the cubic equation is positive for q close to 1, and since it is continuous, the smallest real root greater than 0 is the critical value q^* up to which the condition holds.

Finally, we can formulate the theorem about the asymptotic stability of the zero solution for $N = 3$.

Theorem 7.2.6. *Let $|a| + b < 0$ and α, β be given by* (7.15). *Then, the zeros of the characteristic polynomial* (7.16) *for $N = 3$ lie inside the unit disk if and only if*

$$q < \min \left\{ 1 - \frac{1}{b}, \eta, \vartheta \right\}$$

with η from (7.17) *and ϑ from* (7.18).

Proof. The proof is given as derivation right before the theorem by using Theorem 7.2.1. □

Example 7.2.7. *Let $N = 3$, $a = 0.95$, and $b = -1$. We consider the pantograph equation*

$$x^{\Delta}(t) = \frac{0.95}{t} x(t) - \frac{1}{t} x\left(\frac{t}{q^3}\right)$$

with the initial values

$$x_{-3} = 1, \quad x_{-2} = 1, \quad x_{-1} = 1, \quad x_0 = 1.$$

Theorem 7.2.4 says that the zero solution for this equation is asymptotically stable if and only if

$$q < \min \left\{ 1 - \frac{1}{b}, \eta, \vartheta \right\} = \min \{2,\ 12.1525,\ 1.3371\} = 1.3371.$$

First, consider $q = 1.33$. With Theorem 7.2.6, the zero solution is asymptotically stable, which is supported in Figure 7.7 using Matlab *(see appendix, Listing 9.1).*

Second, consider $q = 1.34$, which is a little bit greater than 1.3371. The zero solution is not asymptotically stable anymore with Theorem 7.2.6, which is also illustrated in Figure 7.8 using Matlab *(see appendix, Listing 9.1). The solution is oscillating with an increasing amplitude as n is increasing.*

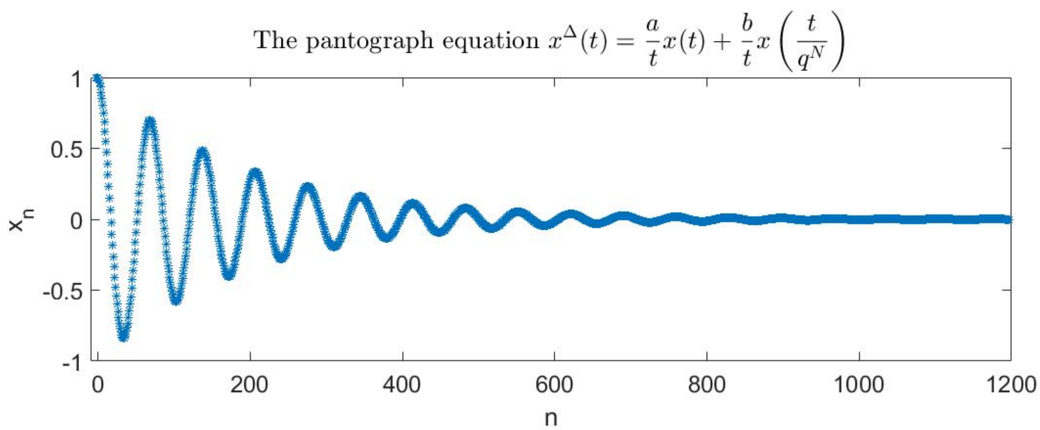

Figure 7.7. The solution of the pantograph equation (7.13) with $N = 3$, $a = 0.95$, $b = -1$, and $q = 1.33$.

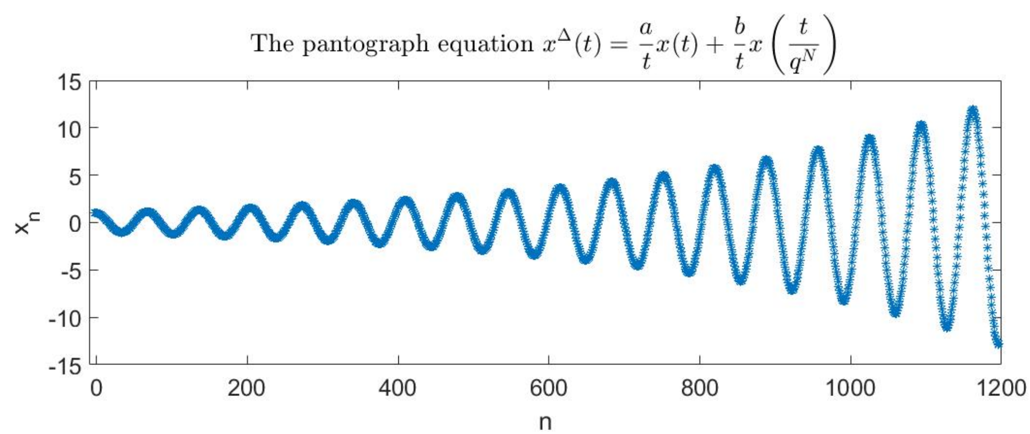

Figure 7.8. The solution of the pantograph equation (7.13) with $N = 3$, $a = 0.95$, $b = -1$, and $q = 1.34$.

7.2.2. The Pantograph Equation of the Form $x^{\Delta}(t) = \frac{a}{t}x(t) + \frac{b}{t}x\left(\left\lfloor \frac{t}{q^{\lambda}} \right\rfloor_q\right)$.

In the previous section, the delay term, in particular, the delay parameter $0 < \lambda < 1$ for the continuous case is presented as $1/q^N$ in the quantum case, which makes sure that $t/q^N \in q^{\mathbb{N}_0} \cup \Omega$ for $t \in q^{\mathbb{N}_0}$. In this case, the delay term t/q^N is somehow "smooth" compared to the discrete case, where the delay term is presented by the floor function. Now, we want to generalize the delay term and thus consider the delay parameter $1/q^{\lambda}$ with $\lambda > 0$ for $\lambda \in \mathbb{R}$. But for this delay parameter, it does not have to be true that $t/q^{\lambda} \in q^{\mathbb{N}_0} \cup \Omega$ for

$t \in q^{\mathbb{N}_0}$. Hence, we define the floor function in quantum calculus as

$$\lfloor s \rfloor_q := \sup\left\{t \in q^{\mathbb{N}_0} : t \leq s\right\} \quad \text{for} \quad s \in \mathbb{R}. \tag{7.19}$$

We consider the q-difference equation

$$x^{\Delta}(t) = \frac{a}{t}x(t) + \frac{b}{t}x\left(\left\lfloor \frac{t}{q^{\lambda}} \right\rfloor_q\right), \quad t \in q^{\mathbb{N}_0}, \quad \lambda > 0. \tag{7.20}$$

With the definition of the q-derivative (see Definition 6.0.2), a corresponding difference equation is derived as

$$\begin{aligned}
&x^{\Delta}(t) = \frac{a}{t}x(t) + \frac{b}{t}x\left(\left\lfloor \frac{t}{q^{\lambda}} \right\rfloor_q\right) \\
\Longrightarrow\ & \frac{x(qt) - x(t)}{(q-1)t} = \frac{a}{t}x(t) + \frac{b}{t}x\left(\left\lfloor \frac{t}{q^{\lambda}} \right\rfloor_q\right) \\
\Longrightarrow\ & x(qt) = (1 + a(q-1))\, x(t) + b(q-1)x\left(\left\lfloor \frac{t}{q^{\lambda}} \right\rfloor_q\right).
\end{aligned}$$

If we put $t_n := q^n$ and $t_{n+1} := t_n q = q^n q = q^{n+1}$, then

$$x(t_{n+1}) = (1 + a(q-1))\, x(t_n) + b(q-1)x\left(\left\lfloor \frac{t_n}{q^{\lambda}} \right\rfloor_q\right).$$

$x\left(\left\lfloor \frac{t_n}{q^{\lambda}} \right\rfloor_q\right) = x\left(\left\lfloor q^{n-\lambda} \right\rfloor_q\right) = x\left(q^{\lfloor n-\lambda \rfloor}\right)$, where $\lfloor n - \lambda \rfloor$ is the normal integer floor function $\left(\lfloor x \rfloor = \max\{m \in \mathbb{Z} : m \leq x\} \text{ for } x \in \mathbb{R}\right)$. Thus,

$$x(t_{n+1}) = (1 + a(q-1))\, x(t_n) + b(q-1)x\left(q^{\lfloor n-\lambda \rfloor}\right).$$

Define $x_n := x(t_n)$, $x_{n+1} := x(t_{n+1})$ and with $q^{\lfloor n-\lambda \rfloor} = t_{\lfloor n-\lambda \rfloor}$, we get $x\left(t_{\lfloor n-\lambda \rfloor}\right) = x_{\lfloor n-\lambda \rfloor} = x_{n-\lceil \lambda \rceil}$ because $n \in \mathbb{N}$ and $\lambda > 0$ for $\lambda \in \mathbb{R}$, where $\lceil x \rceil$ is the integer ceiling function $\left(\lceil x \rceil = \min\{m \in \mathbb{Z} : m \geq x\} \text{ for } x \in \mathbb{R}\right)$. If we put $N := \lceil \lambda \rceil$, then $x_{n-\lceil \lambda \rceil} = x_{n-N}$ and we end up for the pantograph equation (7.20) in the same case as for (7.13)

$$x_{n+1} = \alpha x_n + \beta x_{n-N}, \quad n \in \mathbb{N}_0, \tag{7.21}$$

where

$$N = \lceil \lambda \rceil, \quad \alpha = 1 + a(q-1), \quad \beta = b(q-1). \tag{7.22}$$

The initial values $x_{-N}, x_{-N+1}, \dots, x_{-1}, x_0$ complete the initial value problem.

Example 7.2.8. *We consider*

$$x^{\Delta}(t) = -\frac{0.1}{t}x(t) - \frac{1}{t}x\left(\left\lfloor \frac{t}{q^{2.1}} \right\rfloor_q\right)$$

with the initial values

$$x_{-3} = 1, \quad x_{-2} = 1, \quad x_{-1} = 1, \quad x_0 = 1$$

because

$$\lambda = 2.1 \implies N = \lceil \lambda \rceil = 3.$$

Apply the asymptotic stability conditions from the previous case for (7.13) *with* $N = 3$. *With Theorem 7.2.6, the zero solution is asymptotically stable if and only if*

$$q < \min\left\{1 - \frac{1}{b}, \eta, \vartheta\right\} = \min\{2,\ 2.1924,\ 1.4634\} = 1.4634.$$

For $q = 1.05$*, the solution is given in Figure 7.9 using Matlab (see appendix, Listing 9.1).*

For $q = 1.5$*, the zero solution is not asymptotically stable anymore with Theorem 7.2.6, which is illustrated in Figure 7.10 using Matlab (see appendix, Listing 9.1).*

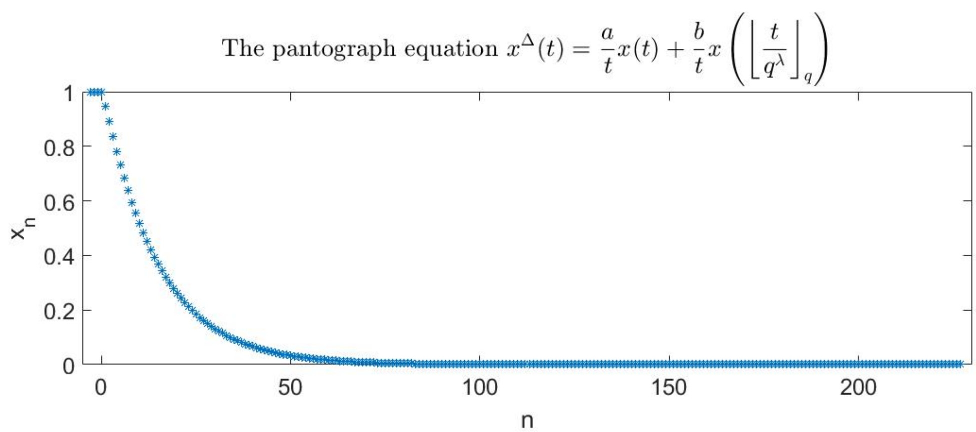

Figure 7.9. The solution of the pantograph equation (7.20) with $\lambda = 2.1$, $a = -0.1$, $b = -1$, and $q = 1.05$.

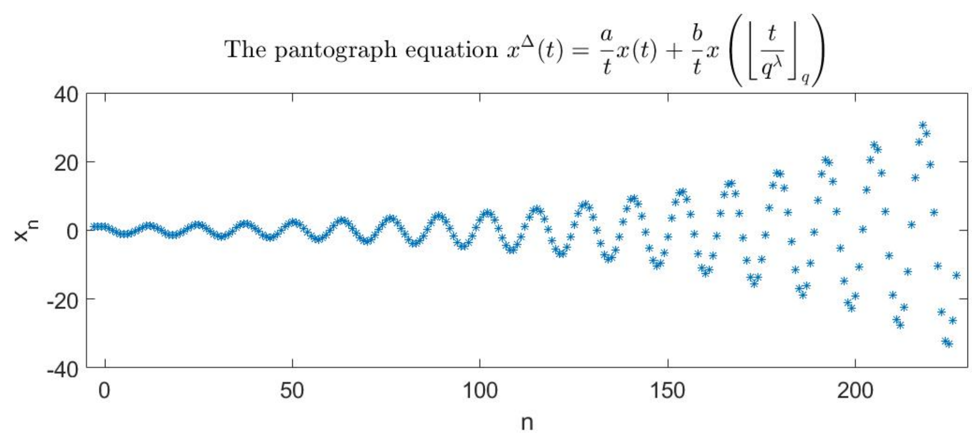

Figure 7.10. The solution of the pantograph equation (7.20) with $\lambda = 2.1$, $a = -0.1$, $b = -1$, and $q = 1.5$.

7.2.3. The Pantograph Equation of the Form $x^{\Delta}(t) = \frac{a}{t}x(t) + \frac{b}{t}x\left(\lfloor \lambda t \rfloor_q\right)$. In Section 7.2.2, we discussed the delay term $\left\lfloor \frac{t}{q^{\lambda}} \right\rfloor_q$. To make it even more general and similar to the continuous case, the delay term $\lfloor \lambda t \rfloor_q$ with $0 < \lambda < 1$ for $\lambda \in \mathbb{R}$ is considered, where $\lfloor x \rfloor_q$ is again the floor function in quantum calculus from (7.19). We consider the q-difference equation with the most general delay term of the form

$$x^{\Delta}(t) = \frac{a}{t}x(t) + \frac{b}{t}x\left(\lfloor \lambda t \rfloor_q\right), \quad t \in q^{\mathbb{N}_0}, \quad 0 < \lambda < 1. \tag{7.23}$$

Using the definition of the q-derivative (see Definition 6.0.2), a corresponding difference equation is derived as

$$\begin{aligned} x^{\Delta}(t) &= \frac{a}{t}x(t) + \frac{b}{t}x\left(\lfloor \lambda t \rfloor_q\right) \\ \implies \frac{x(qt) - x(t)}{(q-1)t} &= \frac{a}{t}x(t) + \frac{b}{t}x\left(\lfloor \lambda t \rfloor_q\right) \\ \implies x(qt) &= (1 + a(q-1))\, x(t) + b(q-1)x\left(\lfloor \lambda t \rfloor_q\right). \end{aligned}$$

If we put $t_n := q^n$ and $t_{n+1} := t_n q = q^n q = q^{n+1}$, then

$$x(t_{n+1}) = (1 + a(q-1))\, x(t_n) + b(q-1)x\left(\lfloor \lambda t_n \rfloor_q\right). \tag{7.24}$$

Consider $x\left(\lfloor \lambda t_n \rfloor_q\right)$ or in particular $\lfloor \lambda t_n \rfloor_q = \lfloor \lambda q^n \rfloor_q$. First, we want to know for which condition λq^n is still greater than or equal to the next smaller number of q^n in $q^{\mathbb{N}_0}$, which is q^{n-1}:

$$\begin{aligned} &\lambda q^n \geq q^{n-1} \\ &\implies \lambda q \geq 1 \\ &\implies q \geq \lambda^{-1} \\ &\implies \log(q) \geq \log\left(\lambda^{-1}\right) \\ &\implies 1 \geq \frac{-\log(\lambda)}{\log(q)} = -\log_q(\lambda) \end{aligned}$$

The right-hand side is positive because $0 < \lambda < 1$ and thus $-\log(\lambda) > 0$. Note that $\lambda q^n < q^n$ is always true. Thus, in this case, $x\left(\lfloor \lambda t_n \rfloor_q\right) = x_{n-1}$. In the following, we see why it is better to use the logarithmic function for the inequality above. Now, we look for which condition λq^n is still greater than or equal to the second next smaller number of q^n

in $q^{\mathbb{N}_0}$, which is q^{n-2}:

$$\begin{aligned}
&\lambda q^n \geq q^{n-2} \\
&\implies \lambda q^2 \geq 1 \\
&\implies q^2 \geq \lambda^{-1} \\
&\implies \log\left(q^2\right) \geq \log\left(\lambda^{-1}\right) \\
&\implies 2 \geq \frac{-\log(\lambda)}{\log(q)} = -\log_q(\lambda)
\end{aligned}$$

and

$$\begin{aligned}
&\lambda q^n < q^{n-1} \\
&\implies \lambda q < 1 \\
&\implies q < \lambda^{-1} \\
&\implies \log(q) < \log\left(\lambda^{-1}\right) \\
&\implies 1 < \frac{-\log(\lambda)}{\log(q)} = -\log_q(\lambda).
\end{aligned}$$

Hence, $x\left(\lfloor \lambda t_n \rfloor_q\right) = x_{n-2}$ if $2 \geq -\log_q(\lambda) > 1$. Now, consider the general case $q^{n-N} \leq \lambda q^n < q^{n-N+1}$ for $N < n$, $N \in \mathbb{N}$:

$$\begin{aligned}
&\lambda q^n \geq q^{n-N} \\
&\implies \lambda q^N \geq 1 \\
&\implies q^N \geq \lambda^{-1} \\
&\implies \log\left(q^N\right) \geq \log\left(\lambda^{-1}\right) \\
&\implies N \geq \frac{-\log(\lambda)}{\log(q)} = -\log_q(\lambda)
\end{aligned}$$

and

$$
\begin{aligned}
&\lambda q^n < q^{n-N+1} \\
&\implies \lambda q^{N-1} < 1 \\
&\implies q^{N-1} < \lambda^{-1} \\
&\implies \log\left(q^{N-1}\right) < \log\left(\lambda^{-1}\right) \\
&\implies N - 1 < \frac{-\log(\lambda)}{\log(q)} = -\log_q(\lambda).
\end{aligned}
$$

Therefore, $x\left(\lfloor \lambda t_n \rfloor_q\right) = x_{n-N}$ if $N \geq -\log_q(\lambda) > N - 1$ or $N = \lceil -\log_q(\lambda) \rceil$.

All in all, (7.24) becomes

$$
x_{n+1} = \alpha x_n + \beta x_{n-N}, \quad n \in \mathbb{N}_0, \tag{7.25}
$$

where

$$
N = \lceil -\log_q(\lambda) \rceil, \quad \alpha = 1 + a(q-1), \quad \beta = b(q-1). \tag{7.26}
$$

The initial values $x_{-N}, x_{-N+1}, \ldots, x_{-1}, x_0$ complete the initial value problem. Moreover, we end up again for the pantograph equation (7.23) in the same case as for (7.13). Thus, we can apply the results from Section 7.2.1 to determine if the zero solution is asymptotically stable or not. But in this case, N depends also on q, which means if we change q, it is possible that the order of the difference equation (7.25) changes and therefore we have to consider the conditions for the new order of the difference equation to determine the asymptotic stability of the zero solution.

Example 7.2.9. *We consider the pantograph equation*

$$
x^{\Delta}(t) = \frac{0.9}{t} x(t) - \frac{1}{t} x\left(\lfloor 0.9\, t \rfloor_q\right)
$$

with the initial values

$$x_{-2} = 1, \quad x_{-1} = 1, \quad x_0 = 1$$

because

$$\lambda = 0.9, \quad q = 1.1 \implies N = \left\lceil -\log_q(\lambda) \right\rceil = \lceil 1.1055 \rceil = 2.$$

Apply the asymptotic stability conditions from the previous case for (7.13) *with* $N = 2$. *Theorem 7.2.4 says that the zero solution for this pantograph equation is asymptotically stable if and only if*

$$q < \min \{21, \ 11.9161, \ 1.5086\} = 1.5086.$$

Since $q = 1.1$*, the zero solution is asymptotically stable, which is supported in Figure 7.11 using* MATLAB *(see appendix, Listing 9.1).*

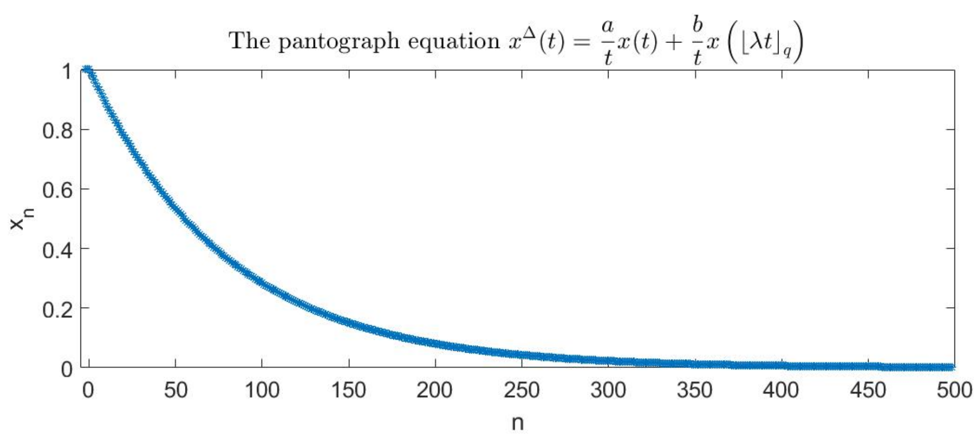

Figure 7.11. The solution of the pantograph equation (7.23) with $a = 0.9$, $b = -1$, $\lambda = 0.9$, $q = 1.1$, and thus $N = 2$.

Now, consider $q = 2.1$. *Since we change q, it is also possible that the order of the difference equation changes:*

$$\lambda = 0.9, \quad q = 2.1 \implies N = \left\lceil -\log_q(\lambda) \right\rceil = \lceil 0.1420 \rceil = 1.$$

With the results from Section 7.2.1, particularly from Theorem 7.2.2, the zero solution is asymptotically stable if and only if

$$q < 1 - \frac{1}{b} = 2.$$

Therefore, the zero solution is not asymptotically stable, which is illustrated in Figure 7.12 using MATLAB (see appendix, Listing 9.1).

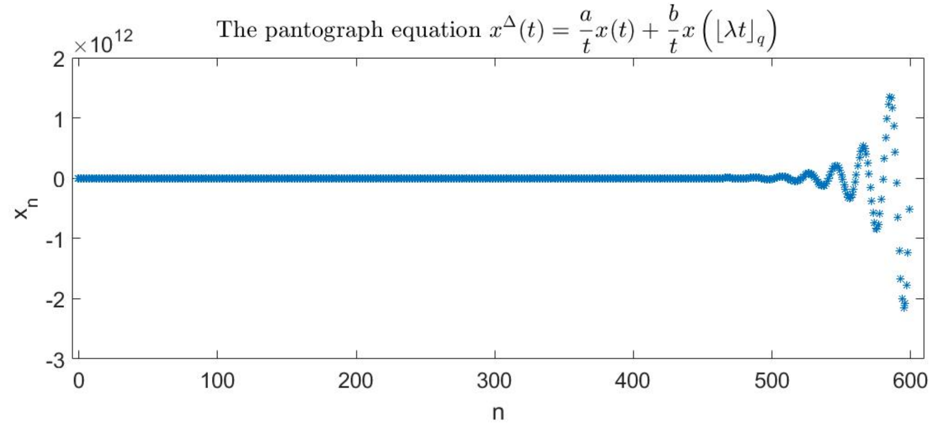

Figure 7.12. The solution of the pantograph equation (7.23) with $a = 0.9$, $b = -1$, $\lambda = 0.9$, $q = 2.1$, and thus $N = 1$.

7.3. AN EXPLICIT SOLUTION FOR A SPECIAL PANTOGRAPH EQUATION IN QUANTUM CALCULUS

In Section 7.1 and 7.2, we considered the asymptotic behavior of solutions and derived asymptotic stability conditions of the zero solution for different pantograph equations. With these results, we know something about the solution without computing the solution. Since it is often not easy or even not possible to find an explicit analytic solution, we can use the results from Section 7.1 and 7.2 to discuss the behavior of solutions. But sometimes it is possible to find a nice explicit solution, which is even better.

In this section, we consider a special kind of pantograph equation in quantum calculus, where the coefficient of the delay term is modified. For this pantograph equation, we obtain an explicit solution.

For the following consideration, we use the q-exponential function from Definition 6.0.4. For $t = q^n$ and $\lambda \in \mathbb{R}$, we get

$$e_\lambda(t) = e_\lambda(q^n) = \prod_{k=0}^{n-1} \left(1 + \lambda(q-1)q^k\right). \tag{7.27}$$

Assuming $x(t) = e_\lambda(t)$ and using (7.27), we obtain

$$\begin{aligned} x^\Delta(t) &= \frac{x(qt) - x(t)}{(q-1)t} = \frac{x(q^{n+1}) - x(q^n)}{(q-1)q^n} \\ &= \frac{\prod_{k=0}^{n} \left(1 + \lambda(q-1)q^k\right) - \prod_{k=0}^{n-1} \left(1 + \lambda(q-1)q^k\right)}{(q-1)q^n} \\ &= \frac{(1 + \lambda(q-1)q^n - 1) \prod_{k=0}^{n-1} \left(1 + \lambda(q-1)q^k\right)}{(q-1)q^n} \\ &= \frac{\lambda(q-1)q^n}{(q-1)q^n} \prod_{k=0}^{n-1} \left(1 + \lambda(q-1)q^k\right) \\ &= \lambda \prod_{k=0}^{n-1} \left(1 + \lambda(q-1)q^k\right) = \lambda x(t). \end{aligned} \tag{7.28}$$

We claim that for a special kind of pantograph equation in quantum calculus we know an explicit solution representation in terms of the exponential function.

Theorem 7.3.1. *Let $N \in \mathbb{N}$. For the pantograph equation*

$$x^\Delta(t) = ax(t) + b\left(\prod_{j=1}^{N} \left(1 + (a+b)(q-1)\frac{t}{q^j}\right)\right) x\left(\frac{t}{q^N}\right), \quad t = q^n, \tag{7.29}$$

the solution is given by

$$x(t) = e_{a+b}(t).$$

Proof. Plug $x(t) = e_{a+b}(t)$ with $t = q^n$ into (7.29) and use (7.27) together with the property (7.28). Then,

$$
\begin{aligned}
& x^{\Delta}(t) = ax(t) + b\left(\prod_{j=1}^{N}\left(1 + (a+b)(q-1)\frac{t}{q^j}\right)\right) x\left(\frac{t}{q^N}\right) \\
\Longrightarrow\ & x^{\Delta}(t) = ax(t) + b\left(\prod_{j=n-N}^{n-1}\left(1 + (a+b)(q-1)q^j\right)\right)\left(\prod_{k=0}^{n-(N+1)}\left(1 + (a+b)(q-1)q^k\right)\right) \\
\Longrightarrow\ & (a+b)x(t) = ax(t) + b\prod_{k=0}^{n-1}\left(1 + (a+b)(q-1)q^k\right) \\
\Longrightarrow\ & (a+b)x(t) = ax(t) + bx(t),
\end{aligned}
$$

which completes the proof. □

For the case $N = 1$, we obtain the following result from Theorem 7.3.1.

Corollary 7.3.2. *For the pantograph equation*

$$x^{\Delta}(t) = ax(t) + b\left(1 + (a+b)(q-1)\frac{t}{q}\right) x\left(\frac{t}{q}\right), \quad t = q^n,$$

the solution is given by

$$x(t) = e_{a+b}(t).$$

All in all, for the special pantograph equation (7.29) it is possible to find a nice explicit solution, which is the q-exponential function. But also for pantograph equations, where it is not easy or even not possible to find an explicit analytic solution, we obtained some results about the asymptotic behavior and the asymptotic stability of solutions in this section.

8. CONCLUSION

The pantograph equation is a well-studied equation in the continuous and discrete cases. Some important results about the pantograph equation in these cases regarding asymptotic behavior and asymptotic stability are presented in Section 4 and 5. After considering what is already done in the continuous and discrete cases, we investigated the pantograph equation in quantum calculus in Section 7.

A topic of interest was how to present the coefficients of the pantograph equation in quantum calculus. Using constant coefficients for the q-difference equation, we derived a difference equation with fixed order but variable coefficients. In this case, we found an explicit solution representation in matrix form and, moreover, we proved the exponential behavior of solutions for this pantograph equation in Section 7.1.

Another possibility is to use variable coefficients for the pantograph equation in quantum calculus. For this case, we also discussed an additional interesting question regarding the pantograph equation in quantum calculus: How does the delay term look like in quantum calculus? For this purpose, we considered different pantograph equations with different delay terms and derived asymptotic stability conditions of the zero solution for special cases in Section 7.2. The general idea through this discovery was to derive a difference equation from the the q-difference equation that gave us an equivalent initial value problem. Using the Schur–Cohn criterion for the difference equation, conditions for the asymptotic stability of the zero solution of the q-difference equation were derived.

It is always our purpose to find an explicit solution for an equation. To this end, we considered in Section 7.3 a pantograph equation in quantum calculus with a modified variable coefficient of the delay term. Moreover, we found an explicit solution for this pantograph equation, which is the q-exponential function.

Further research should be done to generalize the results about the pantograph equation to the time scales theory. Combining the continuous, the discrete, and the quantum cases is an challenging task for the future. In addition to that, generalizing the results in Section 7.2 about the asymptotic stability of the zero solution of pantograph equations in quantum calculus is also an research topic for the future. In particular, to generalize the results for an arbitrary N and for more general variable coefficients is an interesting research topic for the future.

APPENDIX

MATLAB CODE TO COMPUTE THE SOLUTION OF THE PANTOGRAPH EQUATION

The following code is written in the programming language MATLAB. The purpose of this code is to compute the solution of pantograph equations in quantum calculus, which is used to plot the solutions and to illustrate the analytical results.

Listing 9.1. Solution of the pantograph equation in quantum calculus.

```
%pantograph equation in quantum calculus
 clear all; close all; clc;

%% Example 1
 a = 0.1;
 b = -0.5;
 q = 2.9; %2.9 %3.1
N = 1;

 iter = 220;

%% Example 2
% a = 0.45;
% b = -0.5;
% q = 2.02; $2.01 %2.02
% N = 2;
```

```
%
% iter = 1600;

%% Example 3
% a = 0.95;
% b = -1;
% q = 1.33; %1.33 %1.34
% N = 3;
%
% iter = 1200;
%
%% Example 4
% a = -0.1;
% b = -1;
% q = 1.05; %1.05 %1.5
% N = 3; %lambda =2.1
%
% iter = 230;

%% Example 5
% a = 0.9;
% b = -1;

% q = 1.1; %lambda =0.9
% %N depends on q and lambda, here we get N=2
% N = 2;
% iter = 500;
```

```

% q = 2.1; %lambda=0.9
% %N depends on q and lambda, here we change q and therefore the order of the difference equation also changes N=1
% N = 1;
% iter = 600;

%% Conditions on q for asymptotic stability of the zero solution


%conditions for N=1
crit_value_q_N1 = (-1/(b))+1;


%conditions for N=2
crit_value_q_N2_cond_ii = (-2/(a+b))+1;

crit_value_q_N2_cond_iii_1 = (b+sqrt(b*(5*b+4*a)))/(-2*b*(a+b))+1;
crit_value_q_N2_cond_iii_2 = (b-sqrt(b*(5*b+4*a)))/(-2*b*(a+b))+1;
crit_value_q_N2_cond_iii_3 = (-b+sqrt(b*(5*b-4*a)))/(2*b*(a-b))+1;
crit_value_q_N2_cond_iii_4 = (-b-sqrt(b*(5*b-4*a)))/(2*b*(a-b))+1;

```

```
%conditions for N=3
crit_value_q_N3 = (-1/(b))+1;

first_coef_N3_1 = b^3-a^2*b;
second_coef_N3_1 = -2*a*b-b^2;
third_coef_N3_1 = -2*b;
fourth_coef_N3_1 = 1;
r_N3_1 = roots([first_coef_N3_1 second_coef_N3_1
    third_coef_N3_1 fourth_coef_N3_1]);

crit_value_q_N3_1_1 = r_N3_1(1)+1;
crit_value_q_N3_1_2 = r_N3_1(2)+1;
crit_value_q_N3_1_3 = r_N3_1(3)+1;

first_coef_N3_2 = -b^3+a^2*b;
second_coef_N3_2 = 2*a*b-b^2;
third_coef_N3_2 = 2*b;
fourth_coef_N3_2 = 1;
r_N3_2 = roots([first_coef_N3_2 second_coef_N3_2
    third_coef_N3_2 fourth_coef_N3_2]);

crit_value_q_N3_2_1 = r_N3_2(1)+1;
crit_value_q_N3_2_2 = r_N3_2(2)+1;
crit_value_q_N3_2_3 = r_N3_2(3)+1;

%% soltuion of the pantograph equation
```

```

x = zeros(iter,1);
t = zeros(iter,1);

%inital values
for k=1:N+1
    t(k) = 1/q^(N+1-k);
    x(k) = 1;
end

%recursive solution calculation
for k=N+1:iter-1
    x(k+1) = (1+a*(q-1))*x(k) + b*(q-1)*x(k-N);
    t(k+1) = q*t(k);
end

figure(1)
plot(-N:iter-N-1,x,'*');
title('The pantograph equation $$x^{\Delta} (t)= \frac{a}{t} x(t) + \frac{b}{t}x \left(\frac{t}{q^N}\right)$$','interpreter','latex','FontSize',24)
xlabel('n','FontSize',28)
ylabel('x_n','FontSize',28)

figure(2)
plot(t,x,'*');
```

```
title('The pantograph equation $$x^{\Delta} (t)= \frac{a}{t} x(t) + \frac{b}{t}x \left(\frac{t}{q^N}\right)$$','interpreter','latex','FontSize',24)
xlabel('q^n','FontSize',28)
ylabel('x_n','FontSize',28)
```

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VITA

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