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Closed-form solutions to discrete-time portfolio optimization problems

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CLOSED-FORM SOLUTIONS TO DISCRETE-TIME PORTFOLIO OPTIMIZATION PROBLEMS

by

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ABSTRACT

In this work, we study some discrete time portfolio optimization problems. After a brief introduction of the corresponding continuous time models, we introduce the discrete time financial market model. The change in asset prices is modeled in contrast to the continuous time market by stochastic difference equations. We provide solutions for these stochastic difference equations. Then we introduce the discrete time risk-measure and the portfolio optimization problems. We provide closed form solutions to the discrete time problems. The main contribution of this thesis are the closed form solutions to the discrete time portfolio models. For simulation purposes the discrete time financial market is better. Examples illustrating our theoretical results are provided.

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1. INTRODUCTION

Portfolio-optimization is very important for companies and for private persons. The historic event in modern finance was the development of Markowitz' so-called mean-variance optimization problem. This one-period model was published 1952 [8]. From this point on financial institutions are able to calculate the so-called efficient frontier. Markowitz solved the following optimization problems. There are two equivalent versions.

The first version minimizes the variance of the portfolio and guarantees a given level of expected return. The second version maximizes the expected return for a given level of variance. The solution of this problem was a mile-stone in finance. It is frequently used in practice, because it is easy to solve, for example with excel solver or with the Matlab financial toolbox. Basically it is only a quadratic programming problem.

With the development of better computer technology, mathematicians started to find a better approach to the optimization problem. One of the main disadvantages of the Markowitz model is the fact that this is a one-period model and there is no uncertainty included. That means a company has to estimate expected returns, variances and correlations and they are now constant over the time and after the optimization they know how to allocate their portfolios. In reality, a company is able to re-allocate the portfolio whenever it wants. It seems realistic that a young person should have more stocks and then over time they should re-allocate the portfolio into more risk-less assets. With the development of stochastic-dynamic optimization, people began to solve dynamic portfolio optimization problems.

In this thesis we are considering a constant rebalanced portfolio process. Therefore the weights of the asset are constant over time. This simplifies the calculation. With non-constant portfolio-weights we have to deal with stochastic-dynamic programming methods. In [2], the authors developed the mathematical background in optimization and provided solutions to standard portfolio optimization problems.

A further question in multistage portfolio optimization is which risk measure a company should use. In the one-period model of Markowitz, variance is used, but in recent years, other risk measures are getting more important. The development of the portfolio is a random variable and therefore we can use other risk measures than the variance, because variance is only the quadratic deviation from the expectation. A positive deviation of the portfolio from the expectation is good. Therefore mathematicians started to minimize losses according to the quantile of the assumed distribution. The distribution assumption simplifies the calculation a lot.

In Section 2 we review the basics from probability. We give the definition of the stochastic process Brownian motion, which we use to model the discrete time financial market. Then we discuss the continuous time models [6] in Section 3. We provide an example and the Matlab implementation of the continuous time portfolio optimization problem. In Section 4 we introduce the discrete time financial market. We choose a discrete time financial market, because it describes real life more accurate. We do not trade continuously. We introduce the price processes of the assets and provide the corresponding Matlab implementation. Afterwards we introduce the portfolio and its properties.

In Section 5 we introduce the risk measures which are used in this thesis. In Section 6 we solve the discrete time portfolio optimization problems for the risk measures and the multi-period mean-variance portfolio optimization problem in discrete time. We provide examples and the Matlab implementation for all optimization problems.

2. PRELIMINARIES FROM PROBABILITY

In this section we review the basics from probability and financial mathematics which we need in this thesis.

To define the price process of the risky assets we need a special type of stochastic process. This process is called Brownian motion. This is the basic process used in the continuous and discrete time portfolio model.

Definition 2.1 (Brownian motion, see [1, Definition 5.3.1])**.** A stochastic process $X = (X(t))_{t\geq 0}$ is a standard (one-dimensional) Brownian motion $B_i(t)$, on some probability space $(\Omega, \mathbb{F}, \mathbb{P})$, if

1. $X(0) = 0$ a.s.,
2. X has independent increments: $X(t+u) - X(t)$ is independent of $\sigma(X(s) : s \leq t)$ for $u \geq 0$,
3. X has stationary increments: the law of $X(t+u) - X(t)$ depends only on u,
4. X has Gaussian increments: $X(t+u) - X(t)$ is normally distributed with mean 0 and variance u, i.e., $X(t+u) - X(t) \sim \mathrm{N}(0, u)$,
5. X has continuous paths: $X(t)$ is a continuous function of t, i.e., $t \to X(t, \omega)$ is continuous in t for all $\omega \in \Omega$.

A standard Brownian motion in n dimensions is then defined as $B(t) := (B_1(t), \ldots, B_n(t))$, where $B_1(t), \ldots, B_n(t)$ are independent standard one-dimensional Brownian motions. From the definition it follows that $B_i(t) \sim \mathrm{N}(0, t)$.

Furthermore we need the definition of expectation and some basic calculation rules.

Definition 2.2 (Expectation, see [1, Definition 2.3.3])**.** The *expectation* $\mathbb{E}$ of a random variable X on $(\Omega, \mathbb{F}, \mathbb{P})$ is defined by

$$\mathbb{E}(X) := \int_\Omega X \mathrm{d}\mathbb{P}.$$

From the definition of the expectation we know that expectations are linear. In the next lemma (see [1]) we give the expectation of a product of random variables.

Lemma 2.1 (Expectation of a product, see [1, Theorem 2.3.1])**.** *If* $X_1, \dots, X_n$ *are independent and* $\mathbb{E}(|X_i|) < \infty$ *for all* $i = 1, \dots, n$*, then*

$$\mathbb{E}\left(\prod_{i=1}^{n} X_i\right) = \prod_{i=1}^{n} \mathbb{E}(X_i).$$

This lemma is very important when calculating expectation and variance of the portfolio wealth, where we have to deal with a product of n independent Brownian motions.

Now we introduce the variance.

Definition 2.3 (Variance of a random variable, see [1, Definition 2.3.3])**.** The *variance* $\mathbb{V}\mathrm{ar}$ of a random variable X on $(\Omega, \mathbb{F}, \mathbb{P})$ is defined by

$$\mathbb{V}\mathrm{ar}(X) := \mathbb{E}\left[(X - \mathbb{E}(X))^2\right] = \mathbb{E}(X^2) - (\mathbb{E}(X))^2.$$

Now we recall expectations and variances of a standard one-dimensional Brownian motion.

Lemma 2.2 (Expectation and variance of a standard Brownian motion, see [1, Chapter 5.3.2])**.** *Let* $B(t)$ *be a standard one-dimensional Brownian motion. Then*

$$\mathbb{E}(B(t)) = 0, \qquad \mathbb{V}\mathrm{ar}(B(t)) = \mathbb{E}(B^2(t)) = t.$$

Finally we need the distribution of the sum of normally distributed random variables.

Lemma 2.3 (Distribution of a sum of n normally distributed random variables)**.** *If* $X_1, \dots, X_n$ *are independent normally distributed random variables with mean* μ_i *and variance* σ_i^2*, then*

$$\sum_{i=1}^{n} X_i \sim \mathrm{N}\left(\sum_{i=1}^{n} \mu_i, \sum_{i=1}^{n} \sigma_i^2\right).$$

3. SUMMARY OF THE CONTINUOUS TIME PROBLEM

In this section we give a short summary of the paper [7]. The same method can also be found in [6]. In [7], the authors solved the continuous time multi-period Earnings-at-Risk and the multi-period mean-variance optimization problem with a constant rebalanced portfolio.

3.1. FINANCIAL MARKET

The authors in [7] considered a standard Black–Scholes financial market with $n+1$ assets. The assets are traded continuously in $[0, T]$.

The price of the risk-free asset at time t is denoted by $P_0(t)$. The prices of the other assets are denoted by $P_i(t)$, $i = 1, \ldots, n$. The price of the risk-free asset follows the differential equation

$$\mathrm{d}P_0(t) = rP_0(t)\mathrm{d}t, \qquad P_0(0) = 1.$$

In [7], r stands for the interest rate, which is assumed to be constant over time.

The asset prices $P_1(t), \ldots, P_n(t)$ are following the stochastic differential equations (for more detailed theory we refer to [1, 5])

$$\mathrm{d}P_i(t) = P_i(t)\left(b_i\mathrm{d}t + \sum_{j=1}^{n} \sigma_{ij}\mathrm{d}B_j(t)\right), \qquad P_i = p_i, \qquad i = 1, \ldots, n.$$

Here, $b = (b_1, \ldots, b_n)'$ denotes the vector of stock-appreciation rates, $\sigma = (\sigma_{ij})_{n\times n}$ is the matrix of stock volatilities and $B(t) = (B_1(t), \ldots, B_n(t))'$ is a standard n-dimensional Brownian motion.

3.2. PORTFOLIO CONSTRUCTION AND PROPERTIES

In [7], $\Pi_i(t)$ denotes the fraction of the wealth $W^\pi(t)$ invested in asset i at time t, and $\Pi(t) = (\Pi_1(t), \ldots, \Pi_n(t))' \in \mathbb{R}^n$. With Π_0, the weight of the risk-free investment is denoted, which can be calculated by $\Pi_0(t) = 1 - \Pi(t)'\mathbf{1}$, where $\mathbf{1} = (1, \ldots, 1)'$. The strategy with $\Pi_0(t) = 1$ is called the pure bond strategy.

With the previous definitions, the numbers of shares of each asset are then given by

$$N_0(t) = W^{\Pi}(t)\frac{1 - \Pi(t)'\mathbf{1}}{P_0(t)}, \qquad N_i(t) = W^{\Pi}(t)\frac{\Pi_i(t)}{P_i(t)}, \qquad i = 1, \ldots, n.$$

Using these equations, the wealth of the portfolio at time t can be calculated as

$$W^{\Pi}(t) = \sum_{i=0}^{n} N_i(t)P_i(t).$$

In [7], the authors assume that there are no transaction costs and no consumption and that the portfolio strategy Π is self financing. The change in the portfolio wealth follows then a stochastic differential equation of the form

$$\mathrm{d}W^{\Pi}(t) = W^{\Pi}(t)\left[\left((1 - \Pi(t)'\mathbf{1})r + \Pi(t)'b)\right)\mathrm{d}t + \Pi(t)'\sigma \mathrm{d}B(t)\right]. \tag{1}$$

In [7], the portfolio starts with a positive initial wealth $W^{\Pi}(0) = w$. Also the authors consider a constant rebalanced portfolio strategy, which means that the portfolio vector is constant over time. To keep the fraction constant in the portfolio we have to trade continuously.

The solution to this stochastic differential equation (1) and the properties such as expectation and variance of the solution are provided in [7]. The solutions are stated in the following equations:

$$W^{\Pi}(t) = w \exp\left((\Pi'(b - r\mathbf{1}) + r - \left\|\Pi'\sigma\right\|^2/2)t + \Pi'\sigma B(t)\right),$$

$$\mathbb{E}(W^{\Pi}(t)) = w \exp\left((\Pi'(b - r\mathbf{1}) + r)t\right),$$

$$\mathbb{V}\mathrm{ar}(W^{\Pi}(t)) = w^2 \exp\left(2(\Pi'(b - r\mathbf{1}) + r)t\right)\left(\exp\left(\left\|\Pi'\sigma\right\|^2 t\right) - 1\right).$$

The authors in [7] introduced now the relevant risk measure for the optimization.

Definition 3.1 (Earnings-at-Risk, see [7, Definition 1])**.** The difference between the mean terminal wealth and the risk measure p with the same portfolio Π

$$\mathrm{EaR}(\Pi) := \mathbb{E}(W^{\Pi}(t)) - p(\Pi),$$

is called the *Earnings-at-Risk* (EaR) of the portfolio Π.

With the definition of EaR and the closed form equation of $W^{\Pi}(t)$, the authors of [7] found the closed expression to calculate EaR,

$$\mathrm{EaR}(\Pi) = w \exp\left((\Pi^{'}(b - r\mathbf{1}) + r)T\right)\left[1 - \exp\left(z_{\alpha}\left\|\Pi^{'}\sigma\right\|\sqrt{T} - \left\|\Pi^{'}\sigma\right\|^{2} T/2\right)\right].$$

In [7], z_{α} denotes the α-quantile of the standard normal distribution. With that expression the authors of [7] are able to introduce the optimization problem. In [7] the authors minimize the EaR of the terminal wealth under a given level of expected terminal wealth. This is stated in the optimization problem

$$\begin{cases} \min\limits_{\Pi \in \mathbb{R}^n} \ \mathrm{EaR}(\Pi) \\ \text{s.t. } \mathbb{E}(\mathrm{W}^{\Pi}(\mathrm{T})) \geq \mathrm{C}. \end{cases} \tag{2}$$

3.3. MULTI-PERIOD MEAN-EARNINGS-AT-RISK PROBLEM IN CONTINUOUS TIME

Theorem 3.1 (Closed-form solution, see [7, Theorem 3.2])**.** *Assume that* $b \neq r\mathbf{1}$*. Then the unique optimal policy for the* EaR *problem, stated in* (2)*, is given by*

$$\Pi^{*} = \varepsilon^{*}\frac{(\sigma\sigma^{'})^{-1}(b - r\mathbf{1})}{\|\sigma^{-1}(b - r\mathbf{1})\|},$$

where ε^{*} *is given by*

$$\varepsilon^{*} = \frac{\ln(\frac{C}{w}) - rT}{\|\sigma^{-1}(b - r\mathbf{1})\|\, T}.$$

The corresponding terminal wealth is $\mathbb{E}(W^{\Pi^*}(T)) = C$ *with Earnings-at-Risk*

$$\mathrm{EaR}(\Pi^*) = C\left[1 - \exp\left(z_\alpha \varepsilon^* \sqrt{T} - (\varepsilon^*)^2 T/2\right)\right].$$

From this theorem, the authors got the result that the optimal Earnings-at-Risk $\mathrm{EaR}(\Pi^*)$ is a function of the expected terminal wealth, i.e.,

$$\mathrm{EaR}(\xi) = \xi\left[1 - \exp\left(z_\alpha \frac{\ln(\frac{\xi}{w}) - rT}{\|\sigma^{-1}(b - r\mathbf{1})\|\sqrt{T}} - \frac{(\ln(\frac{\xi}{w}) - rT)^2}{2\,\|\sigma^{-1}(b - r\mathbf{1})\|^2\, T}\right)\right],$$

where $\xi := \mathbb{E}(W^{\Pi^*}(T))$. With that expression we can plot now the so-called efficient frontier of the mean-EaR problem.

3.4. MULTI-PERIOD MEAN-VARIANCE MODEL IN CONTINUOUS TIME

In this subsection we review the solution of the multi-period mean-variance model in continuous time from [7],

$$\begin{cases} \min\limits_{\Pi \in \mathbb{R}^n} \ \mathbb{V}\mathrm{ar}(\Pi) \\ \text{s.t. } \mathbb{E}(\mathrm{W}^\Pi(\mathrm{T})) \geq \mathrm{C}. \end{cases} \tag{3}$$

The solution of this problem is given in the following theorem.

Theorem 3.2 (Closed-form solution, see [7, Theorem 3.3])**.** *Assume that* $b \neq r\mathbf{1}$. *Then the unique optimal policy for the mean-variance problem, stated in* (3) *is given by*

$$\Pi^* = \varepsilon^* \frac{(\sigma\sigma')^{-1}(b - r\mathbf{1})}{\|\sigma^{-1}(b - r\mathbf{1})\|},$$

where ε^* *is given by*

$$\varepsilon^* = \frac{\ln(\frac{C}{w}) - rT}{\|\sigma^{-1}(b - r\mathbf{1})\|\, T}.$$

The corresponding terminal wealth is $\mathbb{E}(W^{\Pi^*}(T)) = C$ *with variance*

$$\mathbb{V}\mathrm{ar}(\Pi^*) = C^2 \left[\exp((\varepsilon^*)^2 T) - 1\right].$$

The $\mathbb{V}\mathrm{ar} := \mathbb{V}\mathrm{ar}(W^{\Pi^*}(T))$ is a function of the expected terminal wealth $\xi := \mathbb{E}(W^{\Pi^*(T)})$. With that notation we can plot the efficient frontier for the mean-variance problem in the mean-variance space.

$$\mathbb{V}\mathrm{ar}(\xi) = \xi^2 \left[\exp\left(\frac{(\ln(\frac{\xi}{w}) - rT)^2}{\left\|\sigma^{-1}(b - r\mathbf{1})\right\|^2 T}\right) - 1\right].$$

In this thesis we solve these optimization problems in discrete time settings, with a discrete time financial market.

We finish this section with two examples, where we solve a multi-period mean-EaR and a multi-period mean-variance problem.

3.5. EXAMPLE FOR THE CONTINUOUS MEAN-EARNINGS-AT-RISK PROBLEM

In this section we calculate an example for the mean-EaR problem and calculate the corresponding portfolio strategy and check the results of the optimization.

First we present a graph for the so-called efficient frontier. In Figure 3.1 we can see the plot of the mean-EaR efficient frontier.

Now we start with the example. Let

$$b = (0.1, 0.2, 0.3)',$$

$$r = 0.05,$$

$$\sigma = \begin{pmatrix} 0.2 & 0.01 & 0.03 \\ 0.1 & 0.3 & 0.04 \\ 0.05 & 0.03 & 0.1 \end{pmatrix},$$

$$w = 1000, \quad C = 1060, \quad z_\alpha = -1.64, \quad T = 1.$$

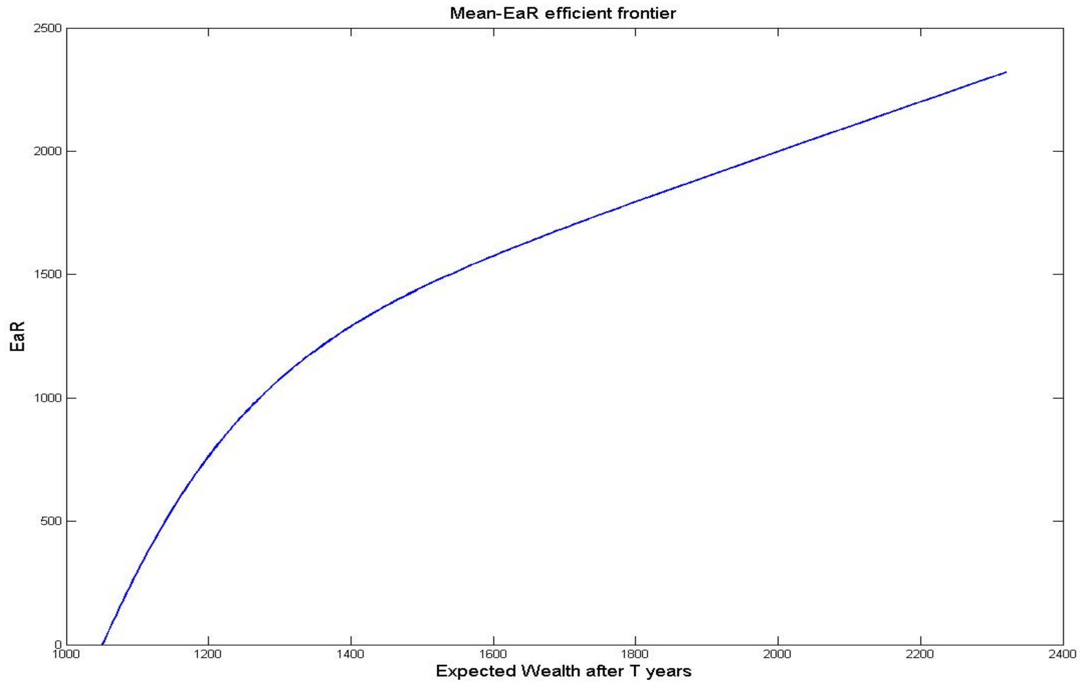

Figure 3.1. Continuous time multi-period mean-EaR efficient frontier

Now we calculate

$$\begin{aligned}
\varepsilon^* &= \frac{\ln(\frac{C}{w}) - rT}{\left\| \sigma^{-1}(b - r\mathbf{1}) \right\| T} = \frac{\ln(\frac{1060}{1000}) - 0.05}{\left\| \begin{pmatrix} 0.2 & 0.01 & 0.03 \\ 0.1 & 0.3 & 0.04 \\ 0.05 & 0.03 & 0.1 \end{pmatrix}^{-1} \begin{pmatrix} 0.1 - 0.05 \\ 0.2 - 0.05 \\ 0.3 - 0.05 \end{pmatrix} \right\|} \\
&= \frac{0.0083}{2.5173} = 0.0033.
\end{aligned}$$

With that ε^* we calculate the Earnings-at-Risk for our portfolio with an expected terminal wealth of C

$$\begin{aligned}
\mathrm{EaR}(\Pi^*) &= C\left[1 - \exp\left(z_\alpha \varepsilon^* \sqrt{T} - \left\| \Pi' \sigma \right\|^2 T/2\right)\right] \\
&= 1060\left[1 - \exp\left(-1.64 \cdot 0.0033 - \frac{(0.0033)^2}{2}\right)\right] = 5.7007.
\end{aligned}$$

This is the minimal Earnings-at-Risk for the portfolio with an expected terminal wealth of

1060 at time 1. Now we calculate the investment strategy, because we want to know the weights of the assets:

$$\begin{aligned}
\Pi^* &= \varepsilon^* \frac{(\sigma\sigma')^{-1}(b - r\mathbf{1})}{\|\sigma^{-1}(b - r\mathbf{1})\|} \\
&= 0.033 \cdot \frac{\left(\begin{pmatrix} 0.2 & 0.01 & 0.03 \\ 0.1 & 0.3 & 0.04 \\ 0.05 & 0.03 & 0.1 \end{pmatrix} \begin{pmatrix} 0.2 & 0.01 & 0.03 \\ 0.1 & 0.3 & 0.04 \\ 0.05 & 0.03 & 0.1 \end{pmatrix}'\right)^{-1} \begin{pmatrix} 0.1 - 0.05 \\ 0.2 - 0.05 \\ 0.3 - 0.05 \end{pmatrix}}{\left\| \begin{pmatrix} 0.2 & 0.01 & 0.03 \\ 0.1 & 0.3 & 0.04 \\ 0.05 & 0.03 & 0.1 \end{pmatrix}^{-1} \begin{pmatrix} 0.1 - 0.05 \\ 0.2 - 0.05 \\ 0.3 - 0.05 \end{pmatrix} \right\|} \\
&= \begin{pmatrix} -0.0087 \\ -0.0024 \\ 0.0363 \end{pmatrix}.
\end{aligned}$$

This means $1 + 0.0087 + 0.0024 - 0.0363 = 97.48\%$ are invested risk free and 3.63% in asset 3. The weight of asset 1 is -0.87%. That means we have to short-sell the asset.

Now we check if the expected wealth at time 1 is really 1060. To do so, we calculate the expectation of our portfolio strategy.

$$\begin{aligned}
\mathbb{E}(W^{\Pi}(1)) &= w \exp\left(\Pi'(b - r\mathbf{1}) + r\right) \\
&= 1000 \exp\left((-0.0087, -0.0024, 0.0363) \begin{pmatrix} 0.1 - 0.05 \\ 0.2 - 0.05 \\ 0.3 - 0.05 \end{pmatrix} + 0.05 \right) = 1060.
\end{aligned}$$

In Figure 3.2 we see the Matlab implementation for the closed form solution of the multi-period mean-EaR optimization problem. With that implementation we are able to easily calculate efficient portfolios.

```
%Solution of the continuous time multi-period mean-EaR problem.

function[epsstar,EaR,Policy]=contEaR(C,w,r,b,sigma,za,T)

j=length(C);
k=length(b);
epsstar=zeros(j,1);
EaR=zeros(j,1);
Policy=zeros(j,k);

for i=1:j
epsstar(i) =(log(C(i)/w)-r*T)/(norm(inv(sigma)*(b-r))*T);

EaR(i)=C(i)*(1-exp(za*epsstar(i)*sqrt(T)-epsstar(i)^2*T/2));

Policy(i,:)= epsstar(i)*(inv((sigma*sigma'))*(b-r))/norm(inv(sigma)*(b-r));
end
```

Figure 3.2. Matlab implementation of the continuous time multi-period mean-EaR problem

3.6. EXAMPLE FOR THE CONTINUOUS TIME MULTI-PERIOD MEAN VARIANCE PROBLEM

In this section we calculate an example for the mean-$\mathbb{V}$ar problem and calculate the corresponding portfolio strategy and check the results of the optimization. Let

$$b = (0.1, 0.2, 0.3)',$$

$$r = 0.05,$$

$$\sigma = \begin{pmatrix} 0.2 & 0.01 & 0.03 \\ 0.1 & 0.3 & 0.04 \\ 0.05 & 0.03 & 0.1 \end{pmatrix},$$

$$w = 1000, \quad C = 1110, \quad T = 2.$$

Now we calculate

$$\begin{aligned}\varepsilon^* &= \frac{\ln(\frac{C}{w}) - rT}{\|\sigma^{-1}(b - r\mathbf{1})\|\, T} = \frac{\ln(\frac{1110}{1000}) - 0.05 \cdot 2}{\left\| \begin{pmatrix} 0.2 & 0.01 & 0.03 \\ 0.1 & 0.3 & 0.04 \\ 0.05 & 0.03 & 0.1 \end{pmatrix}^{-1} \begin{pmatrix} 0.1 - 0.05 \\ 0.2 - 0.05 \\ 0.3 - 0.05 \end{pmatrix} \right\| \cdot 2} \\ &= \frac{0.00436}{5.0346} = 0.00086601.\end{aligned}$$

With that ε^* we calculate the variance for our portfolio with an expected terminal wealth of C

$$\begin{aligned}\mathbb{V}\text{ar}(\Pi^*(2)) &= C^2 \left[\exp(2 \cdot (\varepsilon^*)^2) - 1\right] \\ &= 1110^2 \left[\exp(0.00086601^2 \cdot 2) - 1\right] = 1.8481.\end{aligned}$$

This is the minimal variance for the portfolio with an expected terminal wealth of 1110 at time 2. Now we calculate the investment strategy, because we want to know the weights of the assets:

$$\begin{aligned}\Pi^* &= \varepsilon^* \frac{(\sigma\sigma')^{-1}(b - r\mathbf{1})}{\|\sigma^{-1}(b - r\mathbf{1})\|} \\ &= 0.0009 \cdot \frac{\left(\begin{pmatrix} 0.2 & 0.01 & 0.03 \\ 0.1 & 0.3 & 0.04 \\ 0.05 & 0.03 & 0.1 \end{pmatrix} \begin{pmatrix} 0.2 & 0.01 & 0.03 \\ 0.1 & 0.3 & 0.04 \\ 0.05 & 0.03 & 0.1 \end{pmatrix}' \right)^{-1} \begin{pmatrix} 0.1 - 0.05 \\ 0.2 - 0.05 \\ 0.3 - 0.05 \end{pmatrix}}{\left\| \begin{pmatrix} 0.2 & 0.01 & 0.03 \\ 0.1 & 0.3 & 0.04 \\ 0.05 & 0.03 & 0.1 \end{pmatrix}^{-1} \begin{pmatrix} 0.1 - 0.05 \\ 0.2 - 0.05 \\ 0.3 - 0.05 \end{pmatrix} \right\|} \\ &= \begin{pmatrix} -0.0023 \\ -0.0006 \\ 0.0096 \end{pmatrix}.\end{aligned}$$

```
%Solution of the continuous time multi-period mean-variance problem.

function[epsstar,VaR,Policy]=contVaR(C,w,r,b,sigma,T)

j=length(C);
k=length(b);
epsstar=zeros(j,1);
VaR=zeros(j,1);
Policy=zeros(j,k);

for i=1:j
epsstar(i) =(log(C(i)/w)-r*T)/(norm(inv(sigma)*(b-r))*T);

VaR(i)=C(i)^2*(exp((epsstar(i)^2)*T)-1);

Policy(i,:)= epsstar(i)*(inv((sigma*sigma'))*(b-r))/norm(inv(sigma)*(b-r));
end
```

Figure 3.3. Matlab implementation of the continuous time multi-period mean-variance problem

As a result, we have to invest 0.96 % of the initial wealth in asset 3. The rest is invested risk free.

Now we check if the expected wealth at time 2 really is 1110. To do so, we calculate

$$\begin{aligned}\mathbb{E}(W^{\Pi}(2)) &= w\exp\Big(2\cdot(\Pi'(b-r\mathbf{1})+r)\Big)\\ &= 1000\exp^2\left((-0.0023,-0.0006,0.0096)\begin{pmatrix}0.1-0.05\\0.2-0.05\\0.3-0.05\end{pmatrix}+0.05\right)=1110.\end{aligned}$$

In Figure 3.3 we see the Matlab implementation for the closed form solution of the multi-period mean-variance optimization problem. With that implementation we are able to easily calculate efficient portfolios. In Figure 3.4 we can see the plot of the so-called efficient frontier.

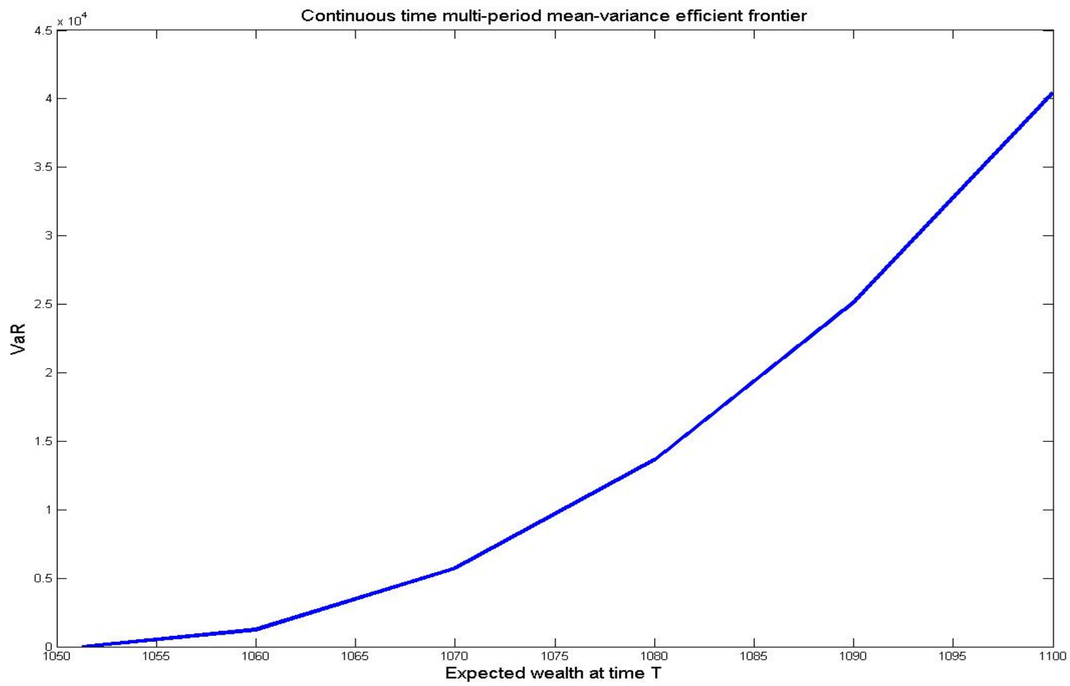

Figure 3.4. Multi-period mean-variance efficient frontier

4. PORTFOLIO MODEL

We construct our portfolio with $n+1$ assets. In our model we are considering discrete trading times on $[0,T]\cap\mathbb{N}$, where $T\in\mathbb{N}$.

Let us denote the price of asset i at time t with $P_i(t)$ for $i=0,\dots,n$. We have one risk-free asset in our model. Without loss of generality it is asset $i=0$.

4.1. PRICE PROCESS OF THE RISK-FREE ASSET

The risk-free asset is the bank account which pays interest with rate r every year. This asset is assumed to be risk free. After the financial crisis, this assumption is questionable. We assume that the interest rate r is constant on $[0,T]$. Denote by $P_0(t)$ the price of the risk-free asset at time t. Then the price of the risk-free asset follows the equation

$$P_0(t+1)-P_0(t)=P_0(t)r. \tag{4}$$

This means that the change of the price of the risk-free asset is exactly the interest payment received at time $t+1$. From (4) it follows that

$$P_0(t+1)=(1+r)P_0(t). \tag{5}$$

With (5) we can calculate the price of the risk-free asset at each time t using the price at time 0.

Lemma 4.1. *The solution of* (4) *is given by*

$$P_0(t)=P_0(0)(1+r)^t, \qquad t\in\mathbb{N}_0. \tag{6}$$

Proof. The proof is done by induction: Clearly,

$$P_0(0)(1+r)^0=P_0(0),$$

so (6) holds for $t = 0$. Assume (6) holds for some $t = k \in \mathbb{N}_0$. Then

$$P_0(0)(1+r)^{k+1} = P_0(0)(1+r)^k(1+r) = P_0(k)(1+r) = P_0(k+1),$$

so (6) holds for $t = k + 1$. □

To simulate the price process of the risk-free asset, we can use the Matlab function from Figure 4.1. With that Matlab function, we can calculate the price process for every initial wealth x, interest rate r and time period t. If we choose $P_0(0) = 1$, then we are

```
% Price Process of the risk-free asset

function[time,price]=riskfree(x,r,t)

time=zeros(t+1,1)
price=zeros(t+1,1)
price(1)=x
time(1)=0

for i=2:t+1
     price(i)=price(i-1)*(1+r);
     time(i)=i-1;
end

stem(time,price)
```

Figure 4.1. Matlab implementation of the discrete-time price process of the risk-free asset

only interested in the percentage performance of the risk-free asset.

If we want to calculate the monetary change, we have to multiply the percentage change with the initial amount of money. In Figure 4.2 we see the plot of the price process of the risk-free asset.

Example 4.1. Let $r = 10\%$, $t = 5$ and $P_0(0) = 1$. Assume our initial wealth is 1,000. With equation (6) we get

$$P_0(5) = (1 + 0.1)^5 = 1.61051.$$

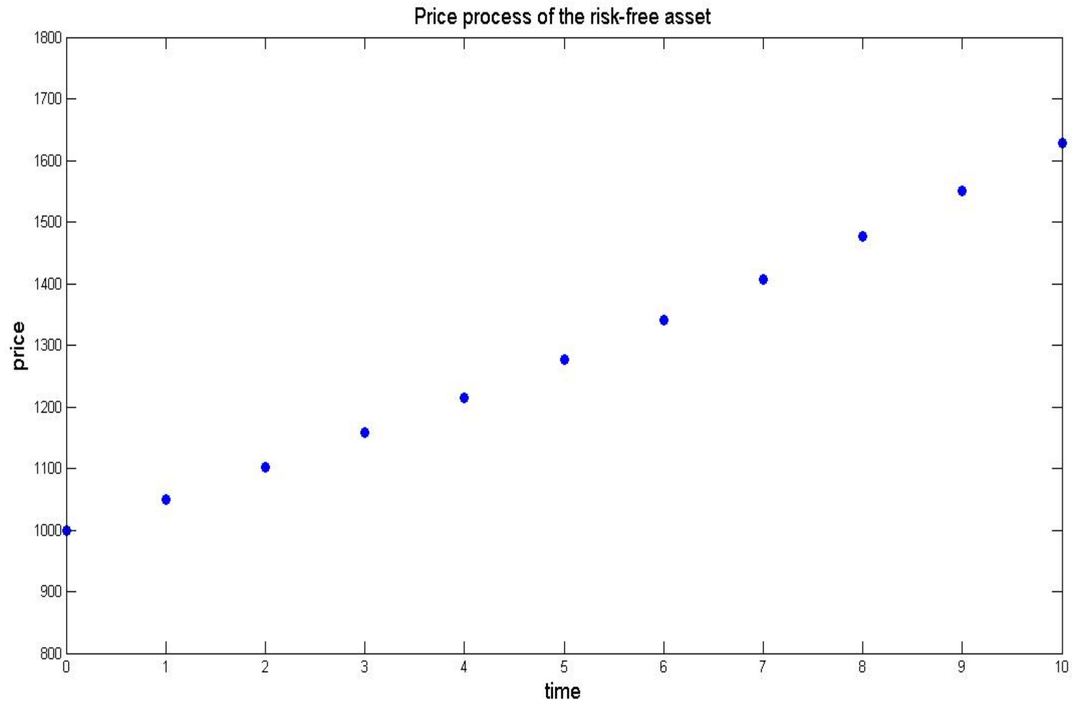

Figure 4.2. Price process of the risk-free asset

The performance of the risk-free asset in this example is $1.61051 - 1 = 0.61951 = 61.05\%$. The value of our portfolio at the end of period 5 is

$$P_0(5) \cdot 1000 = 1.61051 \cdot 1000 = 1,6110.51.$$

4.2. PRICE PROCESS OF THE RISKY ASSETS

In this subsection we introduce the price processes of the risky assets. These are described by stochastic difference equations. First we need some notation to define the price processes of the risky assets.

Let $b = (b_1, ..., b_n)'$ be the vector with the expected returns of the individual assets, and denote by $\sigma = (\sigma_{ij})_{n \times n}$ the $n \times n$ matrix with the stock volatilities. To simplify the calculations, b and σ are assumed to be constant over the time. Furthermore we assume that σ is invertible and $b_i > r$ for all $i \in \{1, \ldots, n\}$. If the expected return of a stock would be smaller than or equal to the risk free rate, why should we buy the risky asset?

We expect a premium for the risk.

Now we can introduce the stochastic difference equations for the risky assets as

$$P_i(t{+}1){-}P_i(t) = P_i(t)\left(b_i + \sum_{j=1}^{n} \sigma_{ij}\left(B_j(t+1) - B_j(t)\right)\right), \qquad P_i(0) = p_i, \qquad i = 1, \ldots, n. \tag{7}$$

Here, b_i is the drift of the risky asset i, and the volatility of the risky asset is then described by a sum of n independent Brownian motions multiplied by the volatility terms.

Now we are interested in calculating the price of asset i at time t explicitly. To do so, we re-arrange (7) as

$$P_i(t+1) = P_i(t)\left[1 + b_i + \sum_{j=1}^{n} \sigma_{ij}\left(B_j(t+1) - B_j(t)\right)\right].$$

Before we solve the equation, we provide the graph for the price process of the risky asset. In Figure 4.3 we see the price process for different simulations of the risky asset.

Now we can express the price of asset i at time t in the following way. We can find the solution by backwards recursion.

Lemma 4.2 (Solution of equation (7))**.** *The solution of* (7) *is given by*

$$P_i(t) = p_i\left[\prod_{a=0}^{t-1}\left(1 + b_i + \sum_{j=1}^{n} \sigma_{ij}\left(B_j(a+1) - B_j(a)\right)\right)\right], \qquad t \in \mathbb{N}_0. \tag{8}$$

Proof. The proof is done by induction. Since $P_i(0) = p_i$ and the empty product is 1 by definition, (8) holds for $t = 0$. Since

$$\begin{aligned} P_i(1) &= P_i(0)\left(1 + b_i + \sum_{j=1}^{n} \sigma_{ij}(B_j(1) - B_j(0))\right) \\ &= p_i\left[\prod_{a=0}^{0}\left(1 + b_i + \sum_{j=1}^{n} \sigma_{ij}\left(B_j(a+1) - B_j(a)\right)\right)\right], \end{aligned}$$

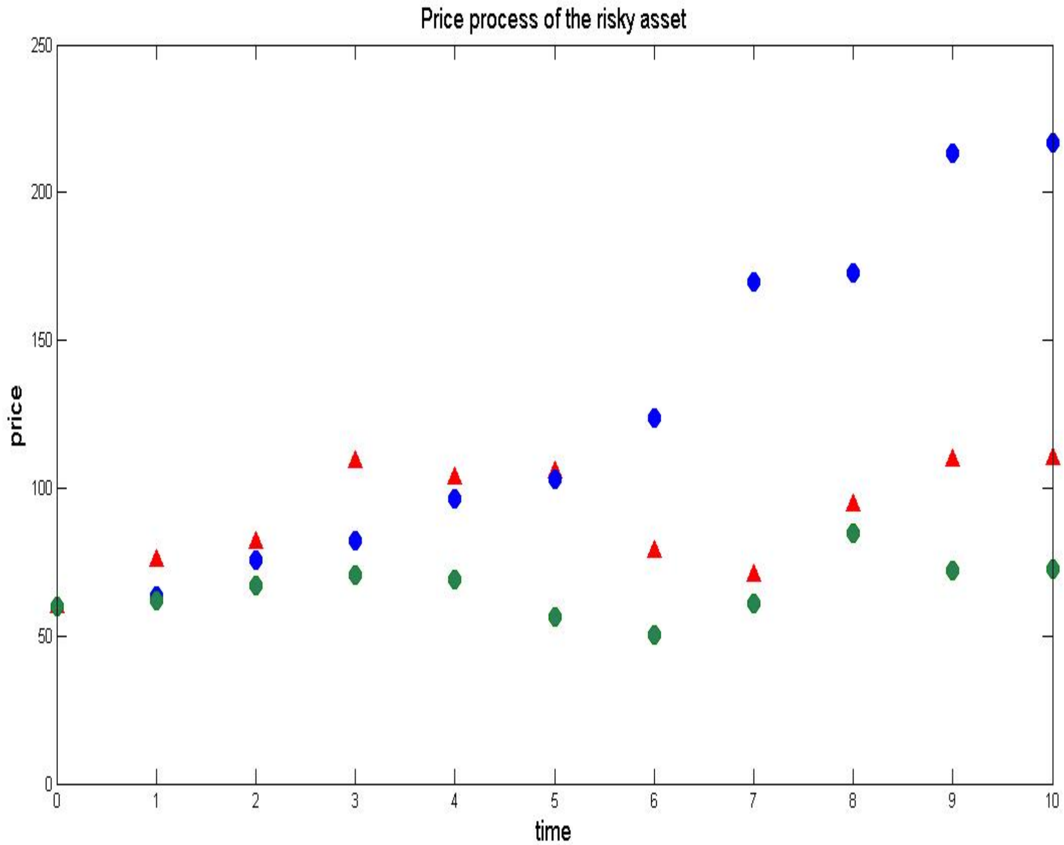

Figure 4.3. Price process of the risky asset

(8) also holds for $t = 1$. Now we assume that (8) holds for some $t = k \in \mathbb{N}$. Then

$$P_i(k+1) = P_i(k)\left[1 + b_i + \sum_{j=1}^{n} \sigma_{ij}\left(B_j(k+1) - B_j(k)\right)\right]$$

$$= \quad p_i\left[1 + b_i + \sum_{j=1}^{n} \sigma_{ij}\left(B_j(k+1) - B_j(k)\right)\right]\prod_{a=0}^{k-1}\left[1 + b_i + \sum_{j=1}^{n} \sigma_{ij}\left(B_j(a+1) - B_j(a)\right)\right]$$

$$= \quad p_i\prod_{a=0}^{k}\left[1 + b_i + \sum_{j=1}^{n} \sigma_{ij}\left(B_j(a+1) - B_j(a)\right)\right],$$

so (8) holds for $t = k + 1$. □

As a next step we provide the Matlab implementation for the price process of the risky asset. In Figure 4.4 we see the Matlab code. With that Matlab function, we can

```
% Price Process of the risky asset
function[time,price]=riskyasset(p,r,b,sigma,t,simul)
time=zeros(t+1,1);
price=zeros(simul,t+1);

for i=1:simul
    price(i,1)=p;
end

time(1)=0;
z=simul;

for j=1:z
    for i=2:t+1
            price(j,i)=price(j,i-1)*(1+b+sigma*randn());
            time(i)=i-1;
    end
end
plot(time,price)
```

Figure 4.4. Matlab implementation of the price process of the risky asset

simulate different paths of the price process.

4.3. ESTIMATION OF THE PARAMETERS FOR THE PRICE PROCESS OF THE RISKY-ASSET

A very important step in the simulation of the price process of the risky asset is the estimation of the input data. We need basically two parameters to simulate the price process of the risky asset, namely the stock appreciation rate and the volatility. We can estimate the stock appreciation rate from historical data by calculating the average return over a period. One way to estimate the stock appreciation rate is given in [4] by

$$b = \frac{1}{T} \ln \left(\frac{S_t}{S_0} \right).$$

This estimate is an overestimate of the actual expectation. The right estimate would be

$$b = \mu - \frac{\sigma^2}{2}.$$

For the volatilities this is more difficult. There are two options. The first option is to estimate volatility from historical data. The problem with historical data is that we have to assume that the future behaves in average like the past. The formula to estimate volatility is given as follows.

Lemma 4.3 (Estimating volatility from historical data, see [4, Section 13.4]). *Assume we have $n+1$ observations of the stock. Let S_i be the price of the stock at the end of the i-th interval, with $i = 1, 2, \ldots, n$. Assume τ is the length of the time interval in years. Let*

$$u_i = \ln\left(\frac{S_i}{S_{i-1}}\right) \qquad \textit{for} \qquad i = 1, 2, \ldots, n.$$

The usual estimate for the standard deviation of the u_i is given by

$$s = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(u_i - \bar{u})^2},$$

where $\bar{u}$ is the mean of the u_i. The variable s is therefore an estimate of $\sigma\sqrt{\tau}$. From that it follows that we can estimate σ by

$$\hat{\sigma} = \frac{s}{\sqrt{\tau}}.$$

Another approach which is widely used in practice is the method of the so-called implied volatilities. The idea behind this method is to calculate the volatility which is implied by option prices which can be easily observed in the market. The value of an European call on a non-dividend paying stock, which can be found in [4], is given by

$$c = S_0 N(d_1) - Ke^{-rt} N(d_2),$$

where

$$d_1 = \frac{\ln(\frac{S_0}{K}) + (r + \frac{\sigma^2}{2})T}{\sigma\sqrt{T}} \quad \text{and} \quad d_2 = \frac{\ln(\frac{S_0}{K}) + (r - \frac{\sigma^2}{2})T}{\sigma\sqrt{T}}.$$

We can observe all the data in the previous equation. The implied volatility is that σ which gives us the market value of the option in the Black–Scholes formula. Basically we

```
%Value of an European Call option in Black-Scholes setting
%Calculation of the implied volatility

function value=option(stock,K,r,T,call)
value=@f
impliedvol=fzero(@f,[-0.1 1]) %calculates the implied volatility of the stock
function y=f(x)

y=(stock*normcdf((log(stock/K)+(r+(x^2)/2)*T)/(x*sqrt(T)))
-K*exp(-r*T)*normcdf((log(stock/K)+(r-x^2/2)*T)/(x*sqrt(T))))-call
end
end
```

Figure 4.5. Matlab implementation of the implied volatility method

can rewrite the problem in finding a zero. This can be solved with numerical methods like Newton's method. But how can we simulate now the stock prices? There are some general steps:

1. Estimate the expected returns for the stock from historical data.

2. Estimate the volatility from historical data or use the implied volatility method.

3. Calculate the product from the Lemma 4.2 with different scenarios for $B(t+1)-B(t)$. Note: $B(t+1) - B(t) \sim N(0,1)$.

4.4. PORTFOLIO CONSTRUCTION

In this subsection we define the portfolio for our model. With $X^{\varphi}(t)$ we denote the total wealth at time t, and $\varphi_i(t)$ is the fraction of $X^{\varphi}(t)$ invested in asset i at time t. The vector $\varphi(t) = (\varphi_1(t), \ldots, \varphi_n(t))' \in \mathbb{R}^n$ is called the portfolio construction process, and $X^{\varphi}(t)$ is called the wealth process of the portfolio. We can calculate the weight of the risk-free asset in the portfolio by

$$\varphi_0(t) = 1 - \varphi(t)'\mathbf{1}, \quad \text{where} \quad \mathbf{1} = (1, \ldots, 1)'.$$

If $\varphi_0(t) = 1$, then the entire wealth is invested in the risk-free asset. Such a strategy is called a pure bond strategy.

Now we are able to calculate the number of shares of asset i at time t. The number of shares of the risk-free asset in our portfolio is

$$N_0(t) = X^{\varphi}(t)\tfrac{1-\varphi(t)'\mathbf{1}}{P_0(t)}. \tag{9}$$

The numbers of shares of the risky assets are

$$N_i(t) = X^{\varphi}(t)\frac{\varphi_i(t)}{P_i(t)}, \qquad i = 1, \ldots, n. \tag{10}$$

Now we are interested in calculating the wealth of the portfolio at time t.

Lemma 4.4 (Total wealth of the portfolio)**.** *The wealth of the portfolio at time t is given by*

$$X^{\varphi}(t) = \sum_{i=0}^{n} N_i(t)P_i(t).$$

Proof. We calculate

$$\begin{aligned}
\sum_{i=0}^{n} N_i(t)P_i(t) &= N_0(t)P_0(t) + \sum_{i=1}^{n} N_i(t)P_i(t) \\
&\overset{(10),(9)}{=} \frac{X^{\varphi}(t)}{P_0(t)}(1-\varphi(t)'\mathbf{1})P_0(t) + \sum_{i=1}^{n} \frac{X^{\varphi}(t)}{P_i(t)}\varphi_i(t)P_i(t) \\
&= X^{\varphi}(t)(1-\varphi(t)'\mathbf{1}) + \sum_{i=1}^{n} X^{\varphi}(t)\varphi_i(t) \\
&= X^{\varphi}(t)\left[(1-\varphi(t)'\mathbf{1}) + \sum_{i=1}^{n} \varphi_i(t)\right] \\
&= X^{\varphi}(t).
\end{aligned}$$

The proof is complete. □

With Lemma 4.4 we can easily calculate the wealth at time t. We need only the prices and the numbers of shares of all assets at time t.

The assumptions in this thesis are: We have no transaction costs, no consumption over time and a self-financing portfolio strategy.

A self-financing strategy is a strategy where we start with an initial amount of money and we can only gain or loose from the assets. There is no consumption or investment over time.

As a next step we want to know how the portfolio wealth changes over one period. As a solution we get a stochastic difference equation.

Lemma 4.5 (Change in portfolio wealth over one period)**.** *We have*

$$X^{\varphi}(t+1) - X^{\varphi}(t) = X^{\varphi}(t)\left((1-\varphi(t)'\mathbf{1})r + \varphi(t)'b + \varphi(t)'\sigma(B(t+1)-B(t))\right).$$

Proof. First we recall that we have a self-financing portfolio strategy. The change in the portfolio wealth is calculated before trading at $t+1$. We use Lemma 4.4 to calculate

$$\begin{aligned}
X^{\varphi}(t+1) - X^{\varphi}(t) &= \sum_{i=0}^{n} N_i(t)(P_i(t+1) - P_i(t)) \\
&= N_0(t)(P_0(t+1) - P_0(t)) + \sum_{i=1}^{n} N_i(t)(P_i(t+1) - P_i(t)) \\
&\overset{(4),(7)}{=} N_0(t)rP_0(t) + \sum_{i=1}^{n} N_i(t)b_iP_i(t) + \sum_{i=1}^{n} N_i(t)P_i(t)\sum_{j=1}^{n}\sigma_{ij}(B_j(t+1) - B(t)) \\
&\overset{(9),(10)}{=} rX^{\varphi}(t)(1-\varphi(t)'\mathbf{1}) + \sum_{i=1}^{n} X^{\varphi}(t)\varphi_i(t)b_i + \sum_{i=1}^{n} X^{\varphi}(t)\varphi_i(t)\sum_{j=1}^{n}\sigma_{ij}(B_j(t+1) - B(t)) \\
&= X^{\varphi}(t)\left((1-\varphi(t)'\mathbf{1})r + \varphi(t)'b + \varphi(t)'\sigma(B(t+1)-B(t))\right).
\end{aligned}$$

This completes the proof. □

In this thesis we only consider a so-called constant rebalanced investment portfolio strategy. That means that we have the same $\varphi(t) = \varphi = (\varphi_1(t), \ldots, \varphi_n(t))'$ at each time $t \in [0, T] \cap \mathbb{Z}$. To keep the weights of the single assets constant over time, we have to trade after each step from t to $t+1$.

With Lemma 4.5, we have everything to calculate the wealth of our portfolio at time t directly, that means without knowing the wealth at time $t-1$. We may rewrite the formula from Lemma 4.5 as

$$X^{\varphi}(t+1) = X^{\varphi}(t)\left[1 + (1-\varphi(t)'\mathbf{1})r + \varphi(t)'b + \varphi(t)'\sigma(\Delta B(t))\right], \qquad t \in \mathbb{N}_0. \tag{11}$$

We can solve (11) with backwards induction. The closed formula for the wealth of our portfolio at t is given in the next lemma.

Lemma 4.6 (Solution of (11))**.** *We have*

$$X^{\varphi}(t) = X^{\varphi}(0)\prod_{a=0}^{t-1}\left[1 + (1-\varphi'\mathbf{1})r + \varphi'b + \varphi'\sigma\Delta B(a)\right], \qquad t \in \mathbb{N}_0. \tag{12}$$

Proof. The proof is done by induction. Clearly (12) holds for $t = 0$. Next,

$$\begin{aligned} X^{\varphi}(1) &\overset{(11)}{=} X^{\varphi}(0)\left[1 + (1-\varphi'\mathbf{1})r + \varphi'b + \varphi'\sigma\Delta B(0)\right] \\ &= X^{\varphi}(0)\prod_{a=0}^{0}\left[1 + (1-\varphi'\mathbf{1})r + \varphi'b + \varphi'\sigma\Delta B(a)\right], \end{aligned}$$

so (12) holds for $t = 1$. Now assume that (12) holds for some $t = k \in \mathbb{N}$. Then,

$$\begin{aligned} X^{\varphi}(k+1) &\overset{(11)}{=} X^{\varphi}(k)\left(1 + (1-\varphi'\mathbf{1})r + \varphi'b + \varphi'\sigma\Delta B(k)\right) \\ &= X^{\varphi}(0)\left(1 + (1-\varphi'\mathbf{1})r + \varphi'b + \varphi'\sigma\Delta B(k)\right)\prod_{a=0}^{k-1}\left[1 + (1-\varphi'\mathbf{1})r + \varphi'b + \varphi'\sigma\Delta B(a)\right] \\ &= X^{\varphi}(0)\prod_{a=0}^{k}\left[1 + (1-\varphi'\mathbf{1})r + \varphi'b + \varphi'\sigma\Delta B(a)\right], \end{aligned}$$

so (12) holds for $t = k + 1$. □

We use the closed formula (12) for $X^{\varphi}(t)$ to calculate expectation and variance of the portfolio. Before we start to calculate the expectation and variance of the wealth process we introduce for convenience

$$\alpha = (1-\varphi'\mathbf{1})r + \varphi'b = \varphi'(b - r\mathbf{1}) + r \quad \text{and} \quad c = \sigma'\varphi$$

so that (12) may be rewritten as

$$X^{\varphi}(t) = X^{\varphi}(0) \prod_{a=0}^{t-1} \left[1 + \alpha + c^{'} \Delta B(a) \right]. \tag{13}$$

With $X^{\varphi}(0) = x > 0$ we denote the initial wealth of the portfolio.

Lemma 4.7 (Expectation of the wealth process)**.** *The expectation is given by*

$$\mathbb{E}(X^{\varphi}(t)) = X^{\varphi}(0)(1+\alpha)^t = x(1+\alpha)^t. \tag{14}$$

Proof. We use (13) to find

$$\begin{aligned}
\mathbb{E}\left(X^{\varphi}(t)\right) &= \mathbb{E}\left(X^{\varphi}(0) \prod_{a=0}^{t-1} \left[1 + \alpha + c^{'} \Delta B(a) \right] \right) \\
&\overset{\Delta B(0),\ldots,\Delta B(t-1) \text{ indep.}}{=} X^{\varphi}(0) \prod_{a=0}^{t-1} \mathbb{E}\left(1 + \alpha + c^{'} \Delta B(a) \right) \\
&\overset{\text{linearity of } \mathbb{E}}{=} X^{\varphi}(0) \prod_{a=0}^{t-1} \left(\mathbb{E}\left(1+\alpha\right) + c^{'} \underbrace{\mathbb{E}\left(\Delta B(a)\right)}_{=0} \right) \\
&= X^{\varphi}(0) \prod_{a=0}^{t-1} (1+\alpha) \\
&= X^{\varphi}(0)(1+\alpha)^t = x(1+\alpha)^t.
\end{aligned}$$

This shows (14). □

Lemma 4.8 (Variance of the wealth process)**.** *The variance is given by*

$$\mathbb{V}\text{ar}(X^{\varphi}(t)) = x^2 \left[((1+\alpha)^2 + c^{'} c)^t - (1+\alpha)^{2t} \right]. \tag{15}$$

Proof. First note that $\mathbb{E}\left(\Delta B_i(a) \Delta B_i(a)\right) = \mathbb{E}\left((\Delta B_i(a))^2\right) = 1$. If $i \neq j$, then $\Delta B_i(a)$ and $\Delta B_j(a)$ are independent and we can write the expectation of the product as a product of expectations. But every $\Delta B_i(a)$ follows a normal distribution with mean 0 and variance

1. Therefore $\mathbb{E}\left(\Delta B_i(a)\Delta B_j(a)\right) = 0$ if $i \neq j$. Using this we find

$$\begin{aligned}
\mathbb{E}\left(\left(\sum_{i=1}^{n} c_i \Delta B_i(a)\right)^2\right) &= \mathbb{E}\left(\sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j \Delta B_i(a)\Delta B_j(a)\right) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} c_i c_j \mathbb{E}\left(\Delta B_i(a)\Delta B_j(a)\right) \\
&= \sum_{i=1}^{n} c_i c_i 1 = \sum_{i=1}^{n} c_i^2 = c^{'} c.
\end{aligned}$$

Using this and (13), we find

$$\begin{aligned}
\mathbb{E}((X^{\varphi}(t))^2) &= \mathbb{E}\left(x^2 \prod_{a=0}^{t-1}\left[1+\alpha+c^{'}\Delta B(a)\right]^2\right) \\
&= x^2\mathbb{E}\left(\prod_{a=0}^{t-1}\left((1+\alpha)^2 + 2(1+\alpha)c^{'}\Delta B(a) + \left(\sum_{i=1}^{n} c_i \Delta B_i(a)\right)\right)^2\right) \\
&= x^2 \prod_{a=0}^{t-1}\left((1+\alpha)^2 + 2(1+\alpha)c^{'}\underbrace{\mathbb{E}(\Delta B(a))}_{=0} + \mathbb{E}\left(\left(\sum_{i=1}^{n} c_i \Delta B_i(a)\right)^2\right)\right) \\
&= x^2 \prod_{a=0}^{t-1}\left((1+\alpha)^2 + c^{'}c\right) = x^2\left((1+\alpha)^2 + c^{'}c\right)^t.
\end{aligned}$$

Hence, by Definition 2.3 and Lemma 4.7, we arrive at

$$\begin{aligned}
\mathbb{V}\mathrm{ar}(X^{\varphi}(t)) &= \mathbb{E}\left((X^{\varphi}(t))^2\right) - \left(\mathbb{E}(X^{\varphi}(t))\right)^2 \\
&= x^2\left((1+\alpha)^2 + c^{'}c\right)^t - x^2(1+\alpha)^{2t} \\
&= x^2\left[\left((1+\alpha)^2 + c^{'}c\right)^t - (1+\alpha)^{2t}\right] \\
&= x^2(1+\alpha)^{2t}\left[\left(1+\frac{c^{'}c}{(1+\alpha)^2}\right)^t - 1\right].
\end{aligned}$$

This shows (15). $\square$

5. DEFINITION OF THE RISK MEASURES

In this section we introduce the risk measures used in this thesis.

We consider only one-period risk measures. The change in portfolio wealth is a random variable. For a portfolio φ with wealth $X^{\varphi}(1)$ we define $\mu(\varphi)$ as a risk measure corresponding to the β-quantile of $X^{\varphi}(1)$,

$$\mathbb{P}(X^{\varphi}(1) \leq \mu(\varphi)) = \beta, \qquad \beta \in (0,1). \tag{16}$$

The strategy is to find a closed expression for $\mu(\varphi)$ for a given β. Usually β is a small number, because we are interested in knowing that the portfolio wealth is bigger than $\mu(\varphi)$ with probability $1-\beta$. With z_β we denote the β-quantile of the standard normal distribution.

Lemma 5.1. *$\mu(\varphi)$ in (16) is given by*

$$\mu(\varphi) = (z_\beta \|c\| + 1 + \alpha)x. \tag{17}$$

Proof. First we use (16) and (13) with $t=1$ to obtain

$$\begin{aligned}
\beta &= \mathbb{P}(X^{\varphi}(1) \leq \mu(\varphi)) \\
&= \mathbb{P}\left(x\left[1+\alpha+c^{'}\Delta B(0)\right] \leq \mu(\varphi)\right) \\
&= \mathbb{P}\left(1+\alpha+c^{'}\Delta B(0) \leq \frac{\mu(\varphi)}{x}\right) \\
&= \mathbb{P}\left(c^{'}\Delta B(0) \leq \frac{\mu(\varphi)}{x} - 1 - \alpha\right) \\
&= \mathbb{P}\left(\frac{c^{'}\Delta B(0)}{\|c\|} \leq \left(\frac{\mu(\varphi)}{x} - 1 - \alpha\right)\frac{1}{\|c\|}\right).
\end{aligned}$$

Since $\frac{c^{'}\Delta B(0)}{\|c\|} \sim N(0,1)$, we find that

$$\left(\frac{\mu(\varphi)}{x} - 1 - \alpha\right)\frac{1}{\|c\|} = z_\beta. \tag{18}$$

Finally we solve (18) for $\mu(\varphi)$ to get (17). □

In this thesis we assume that $\beta < 0.5$. From this assumption it follows that $z_\beta < 0$.

5.1. EARNINGS-AT-RISK

The difference between the expected wealth after one period and the risk measure $\mu(\varphi)$ with the same portfolio φ is called *Earnings-at-Risk*.

Definition 5.1 (Earnings-at-Risk)**.** $\mathrm{EaR}(\varphi) := \mathbb{E}(X^\varphi(1)) - \mu(\varphi)$.

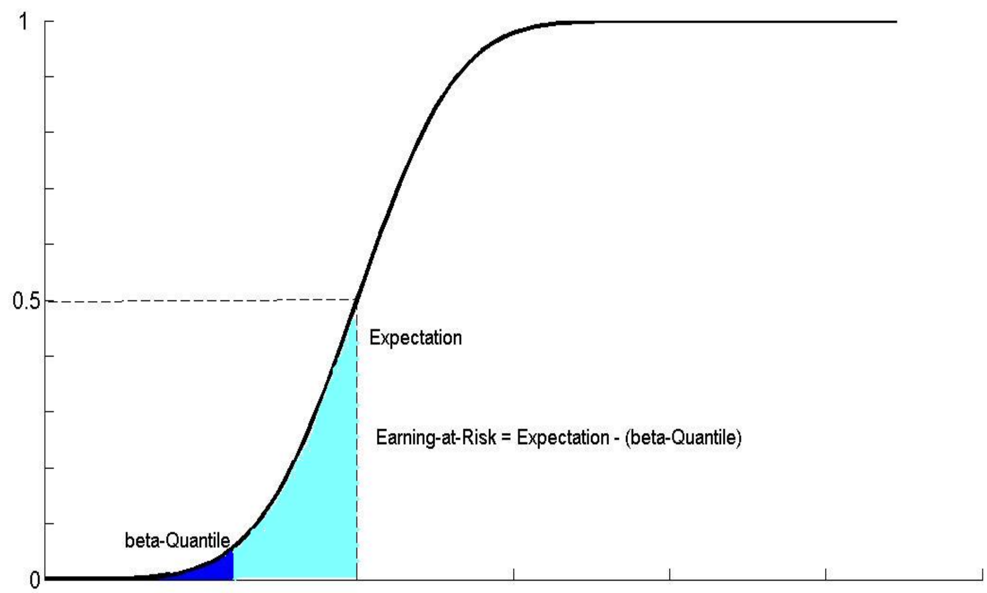

Figure 5.1. Illustration of the risk measure Earnings-at-Risk

Lemma 5.2. *We have*

$$\mathrm{EaR}(\varphi) = -x z_\beta \|c\| . \tag{19}$$

Proof. In Lemma 5.1 we calculated already $\mu(\varphi)$. By Lemma 4.7, we also know the expected wealth over one period. With Definition 5.1, (14) and (17), we get

$$\begin{aligned}\mathrm{EaR}(\varphi) &= x(1+\alpha) - x(z_\beta \|c\| + 1 + \alpha) \\ &= x(1+\alpha - z_\beta \|c\| - 1 - \alpha) \\ &= -x z_\beta \|c\| .\end{aligned}$$

This shows (19). □

From the assumption that $\beta < 0.5$ and the fact that $z_\beta < 0$, it follows that EaR is positive over one period.

In the next lemma we give some important properties of the EaR.

Lemma 5.3. *We have*

$$\min_{\varphi \in \mathbb{R}^n} \mathrm{EaR}(\varphi) = 0, \tag{20}$$

$$\mathrm{EaR}(\varphi) = 0 \quad \textit{iff} \quad \varphi = \mathfrak{o}, \quad \textit{where} \quad \mathfrak{o} = (0, \ldots, 0)' \tag{21}$$

and

$$\sup_{\varphi \in \mathbb{R}^n} \mathrm{EaR}(\varphi) = \infty. \tag{22}$$

Proof. We use formula (19). Since $\mathrm{EaR}(\mathfrak{o}) = 0$ and $\mathrm{EaR}(\varphi) \geq 0$ for all $\varphi \in \mathbb{R}^n$, we see that in (20) holds. Moreover, if $\mathrm{EaR}(\varphi) = 0$, then $c = 0$ and, since σ is invertible, then $\varphi = \mathfrak{o}$. This shows (21). Now let $N \in \mathbb{N}$ and define

$$\varphi = \varphi_N = \frac{N}{-x z_\beta \sqrt{n}} \left(\sigma^{-1}\right)' \mathbb{1}.$$

Then

$$\|c\| = \left\|\sigma' \varphi\right\| = \frac{N}{-x z_\beta \sqrt{n}} \|\mathbb{1}\| = \frac{N}{-x z_\beta},$$

and

$$\mathrm{EaR}(\varphi) = -xz_\beta \|c\| = N.$$

Hence, for each $N \in \mathbb{N}$, there exists $\varphi_N \in \mathbb{R}^n$ with $\mathrm{EaR}(\varphi_N) = N$. Thus (22) holds. □

From Lemma 5.3 we can see that the Earnings-at-Risk of a pure bond strategy is zero.

5.2. CAPITAL-AT-RISK

In this subsection we introduce the risk measure *Capital-at-Risk*. This risk measure is defined as follows.

Definition 5.2 (Capital-at-Risk)**.** $\mathrm{CaR}(\varphi)$ is defined as the possible risk-free profit minus $\mu(\varphi)$.

Lemma 5.4. *We have*

$$\mathrm{CaR}(\varphi) = x(-z_\beta \|c\| - \varphi^{'}(b - r\mathbf{1})). \tag{23}$$

Proof. By Definition 5.2 and (17), we obtain

$$\begin{aligned} \mathrm{CaR}(\varphi) &= x(1+r) - x(z_\beta \|c\| + 1 + \alpha) \\ &= x\left(1 + r - (z_\beta \|c\| + 1 + \varphi^{'}(b - r\mathbf{1}) + r)\right) \\ &= x\left(-z_\beta \|c\| - \varphi^{'}(b - r\mathbf{1})\right). \end{aligned}$$

This shows (23). □

5.3. VALUE-AT-RISK

In this subsection we define the risk measure Value-at-Risk. There are several different definitions for the Value-at-Risk. We define *Value-at-Risk* as follows.

Definition 5.3 (Value-at-Risk)**.** $\mathrm{VaR}_\beta(\varphi) = \mu(\varphi)$.

Example 5.1. Consider a portfolio with the following properties. Let $\beta = 0.05$ and $\mu(\varphi) = 1000$. Then the Value-at-Risk of the portfolio is 1000, and the economic interpretation is that the value of the portfolio is bigger than 1000 with probability 0.95.

In our case the Value-at-Risk is an absolute measure. If we want to measure losses we have to subtract VaR from our initial wealth.

We see that $\mathrm{EaR}, \mathrm{CaR}$ and VaR are all closely related.

6. DISCRETE-TIME PORTFOLIO OPTIMIZATION PROBLEMS

In this section we want to provide closed form solutions to some portfolio optimization problems. For the rest of this thesis we assume

$$b \neq r\mathbf{1}.$$

We let $\lambda \in \mathbb{R}$ and define

$$\varphi^* = \frac{\lambda(\sigma\sigma')^{-1}(b - r\mathbf{1})}{\|\sigma^{-1}(b - r\mathbf{1})\|} \tag{24}$$

and

$$\Theta = \left\|\sigma^{-1}(b - r\mathbf{1})\right\|. \tag{25}$$

First we state some very useful lemmas, which are needed for the proofs of the closed form solutions of the portfolio optimization problems.

Lemma 6.1. *We have*

$$(\varphi^*)'(b - r\mathbf{1}) = \lambda\Theta. \tag{26}$$

Proof. Using the definitions (24) and (25), we get

$$\begin{aligned}
(\varphi^*)'(b - r\mathbf{1}) &= \lambda\frac{(b - r\mathbf{1})'(\sigma\sigma')^{-1}}{\|\sigma^{-1}(b - r\mathbf{1})\|}(b - r\mathbf{1}) \\
&= \lambda\frac{(b - r\mathbf{1})'(\sigma')^{-1}\sigma^{-1}(b - r\mathbf{1})}{\|\sigma^{-1}(b - r\mathbf{1})\|} \\
&= \lambda\frac{(\sigma^{-1}(b - r\mathbf{1}))'\sigma^{-1}(b - r\mathbf{1})}{\|\sigma^{-1}(b - r\mathbf{1})\|} \\
&= \lambda\frac{\|\sigma^{-1}(b - r\mathbf{1})\|^2}{\|\sigma^{-1}(b - r\mathbf{1})\|} \\
&= \lambda\left\|\sigma^{-1}(b - r\mathbf{1})\right\|,
\end{aligned}$$

$$= \lambda\Theta,$$

which completes the proof. □

Lemma 6.2. *We have*

$$\left\|\sigma^{'}\varphi^{*}\right\| = |\lambda|. \tag{27}$$

Proof. Using definitions (24) and (25), we get

$$\begin{aligned}
\left\|\sigma^{'}\varphi^{*}\right\| &= \left\|\sigma^{'}\frac{\lambda(\sigma\sigma^{'})^{-1}(b - r\mathbf{1})}{\Theta}\right\| \\
&= \left\|\lambda\frac{\sigma^{'}(\sigma^{'})^{-1}\sigma^{-1}(b - r\mathbf{1})}{\Theta}\right\| \\
&= \|\lambda\|\frac{\Theta}{\Theta} = |\lambda|1 = |\lambda|.
\end{aligned}$$

This completes the proof. □

Lemma 6.3. *We have*

$$|\varphi^{'}(b - r\mathbf{1})| \leq \left\|\sigma^{'}\varphi\right\|\Theta. \tag{28}$$

Proof. Using definitions (24) and (25), we get with the help of Cauchy-Schwarz' inequality

$$|\varphi^{'}(b - r\mathbf{1})| = |\varphi^{'}\sigma\sigma^{-1}(b - r\mathbf{1})| = |(\sigma^{'}\varphi)^{'}(\sigma^{-1}(b - r\mathbf{1}))| \leq \left\|\sigma^{'}\varphi\right\| \left\|\sigma^{-1}(b - r\mathbf{1})\right\| = \left\|\sigma^{'}\varphi\right\|\Theta.$$

This completes the proof. □

6.1. CLOSED FORM SOLUTION OF THE ONE-PERIOD MEAN-EARNINGS-AT-RISK OPTIMIZATION PROBLEM

In this subsection we introduce the discrete time one-period mean-Earnings-at-Risk problem and provide a closed form solution. As in Markowitz' classical mean-variance problem, we want to minimize the Earnings-at-Risk under a given level of expected return. The difference is only that we use a different risk measure.

We solve the optimization problem

$$\begin{cases} \min\limits_{\varphi \in \mathbb{R}^n} \ \mathrm{EaR}(\varphi) \\ \text{s.t. } \mathbb{E}(X^{\varphi}(1)) \geq C, \end{cases} \tag{29}$$

where C is the expected terminal wealth at time $T = 1$. If we expect only the return of the risk-free asset, we get an Earnings-at-Risk of zero, as we can see from (21).

Theorem 6.1 (Closed form solution of the discrete time one-period mean-EaR optimization problem)**.** *The closed form solution of the one-period mean-Earnings-at-Risk problem* (29) *is given by*

$$\varphi^* = \frac{\lambda(\sigma\sigma^{'})^{-1}(b - r\mathbf{1})}{\Theta},$$

where λ is given as

$$\lambda = \frac{\left(\frac{C}{x} - 1 - r\right)^{+}}{\Theta} \qquad \textit{where} \quad z^{+} = \begin{cases} z & \textit{if} \quad z \geq 0 \\ 0 & \textit{if} \quad z < 0. \end{cases}$$

The expected wealth after one period is C with Earnings-at-Risk $-xz_{\beta}\lambda$.

Proof. Using (19) and (14) for $t = 1$, we rewrite (29) as

$$\begin{cases} \min\limits_{\varphi \in \mathbb{R}^n} \ (-xz_{\beta}\, \|c\|) \\ \text{s.t. } x(1 + \alpha) \geq C. \end{cases} \tag{30}$$

The feasible set of the problem (30) is given by

$$A = \left\{ \varphi \in \mathbb{R}^n : \ \varphi^{'}(b - r\mathbf{1}) \geq \frac{C}{x} - 1 - r \right\}.$$

From the definition of c and α, we may rewrite (30) as

$$\min_{\varphi\in A} g(\varphi), \qquad \text{where} \quad g(\varphi) = -xz_\beta \left\|\sigma^{'}\varphi\right\|.$$

To prove the theorem we show that

1. $g(\varphi^*) = -xz_\beta\lambda$,

2. $g(\varphi) \geq g(\varphi^*)$ for all $\varphi \in A$,

3. $\varphi^* \in A$.

As a first step we show that

$$g(\varphi^*) = -xz_\beta \left\|\sigma^{'}\varphi^*\right\| \overset{(27)}{=} -xz_\beta|\lambda| \overset{\lambda\geq 0}{=} -xz_\beta\lambda. \tag{31}$$

Next, let $\varphi \in A$. Then

$$\begin{aligned} g(\varphi) \quad &= \quad g(\varphi)\frac{\Theta}{\Theta} = \frac{-xz_\beta}{\Theta}\left\|\sigma^{'}\varphi\right\|\Theta \overset{(28)}{\geq} \frac{-xz_\beta}{\Theta}|\varphi^{'}(b-r\mathbf{1})| \geq \frac{-xz_\beta}{\Theta}\varphi^{'}(b-r\mathbf{1}) \\ &\overset{\varphi\in A, g(\varphi)\geq 0}{\geq} \quad \frac{-xz_\beta}{\Theta}\left(\frac{C}{x}-1-r\right)^+ = -xz_\beta\lambda = g(\varphi^*). \end{aligned}$$

Finally we show that $\varphi^* \in A$. To do so, we calculate

$$(\varphi^*)^{'}(b-r\mathbf{1}) \overset{(26)}{=} \lambda\Theta = \frac{\left(\frac{C}{x}-1-r\right)^+}{\Theta}\Theta = \left(\frac{C}{x}-1-r\right)^+ \geq \frac{C}{x}-1-r.$$

This completes the proof. □

As an immediate consequence of Theorem 6.1 we get that the optimal Earnings-at-Risk is a function of the expected terminal wealth. An investor is now able to plot a graph for different expected terminal wealths. Since the supremum of EaR is infinity and the constraint of (29) is unbounded from above, the solution of the corresponding maximum problem is infinity.

We denote with $\omega := \mathbb{E}(X^{\varphi}(1))$ the expected wealth after 1-year. We plug ω into λ given by Theorem 6.1. Using (31), we get

$$\mathrm{EaR}(\omega) = -xz_\beta\lambda = -xz_\beta \frac{\left(\frac{\omega}{x} - 1 - r\right)^+}{\|\sigma^{-1}(b - r\mathbf{1})\|}.$$

If we know our desirable expected return over one period, then we can calculate λ and the portfolio construction strategy.

Another way is that we accept a certain amount as Earnings-at-Risk and then we calculate ω and set this equal to C. Then we are able to calculate the optimal portfolio.

6.2. EXAMPLE FOR THE DISCRETE MEAN-EARNINGS-AT-RISK PROBLEM

In this section we calculate an example for the one-period mean-EaR problem and calculate the corresponding portfolio strategy and check the results of the optimization. Before we start with the example, we provide the graph of the mean-EaR efficient frontier. The plot for the mean-EaR efficient frontier is given in Figure 6.1. With the Matlab function in Figure 6.2, we are able to easily calculate efficient portfolios. Let

$$b = (0.1, 0.2, 0.3)',$$

$$r = 0.05,$$

$$\sigma = \begin{pmatrix} 0.2 & 0.01 & 0.03 \\ 0.1 & 0.3 & 0.04 \\ 0.05 & 0.03 & 0.1 \end{pmatrix},$$

$$x = 1000, \quad C = 1056, \quad z_\beta = -1.64.$$

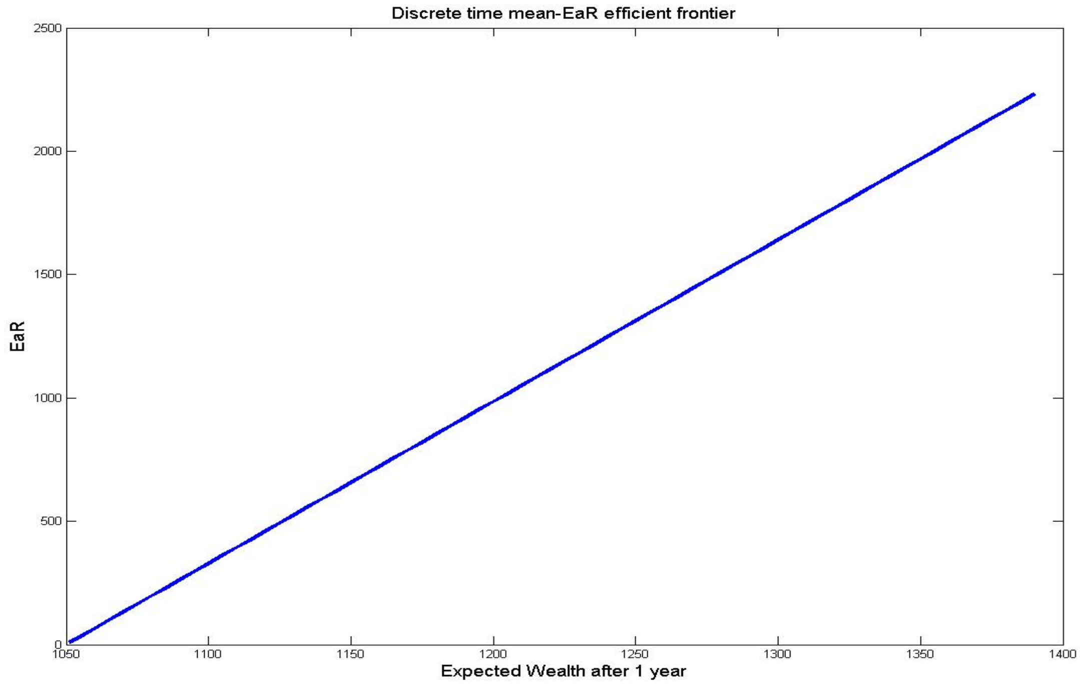

Figure 6.1. Discrete time one-period mean-EaR efficient frontier

```
%Solution of the discrete time one-period mean-EaR problem.

function[epsstar,EaR,Policy]=discreteEaR(C,w,r,b,sigma,za)

j=length(C);
k=length(b);
epsstar=zeros(j,1);
EaR=zeros(j,1);
Policy=zeros(j,k);

for i=1:j
epsstar(i) =((C(i)/w)-r-1)/norm(inv(sigma)*(b-r));

EaR(i)=-w*za*epsstar(i);

Policy(i,:)= epsstar(i)*(inv((sigma*sigma'))*(b-r))/norm(inv(sigma)*(b-r));
end
```

Figure 6.2. Matlab implementation of the discrete time one-period mean-EaR problem

Now we calculate

$$\lambda = \frac{\frac{C}{x} - 1 - r}{\|\sigma^{-1}(b - r\mathbf{1})\|} = \frac{\frac{1056}{1000} - 1 - 0.05}{\left\| \begin{pmatrix} 0.2 & 0.01 & 0.03 \\ 0.1 & 0.3 & 0.04 \\ 0.05 & 0.03 & 0.1 \end{pmatrix}^{-1} \begin{pmatrix} 0.1 - 0.05 \\ 0.2 - 0.05 \\ 0.3 - 0.05 \end{pmatrix} \right\|}$$

$$= \frac{0.0060}{2.5173} = 0.00238.$$

With that λ we calculate the Earnings-at-Risk for our portfolio with an expected terminal wealth of C as

$$\begin{aligned}\text{EaR}(\varphi^*) &= -xz_\beta\lambda \\ &= (-1000)\cdot(-1.64)\cdot 0.00238 = 3.9089.\end{aligned}$$

This is the minimal Earnings-at-Risk for the portfolio with an expected terminal wealth of 1056 at time 1.

$$\begin{aligned}\varphi^* &= \lambda\frac{(\sigma\sigma')^{-1}(b-r\mathbf{1})}{\|\sigma^{-1}(b-r\mathbf{1})\|} \\ &= 0.006\cdot\frac{\left(\begin{pmatrix}0.2 & 0.01 & 0.03\\ 0.1 & 0.3 & 0.04\\ 0.05 & 0.03 & 0.1\end{pmatrix}\begin{pmatrix}0.2 & 0.01 & 0.03\\ 0.1 & 0.3 & 0.04\\ 0.05 & 0.03 & 0.1\end{pmatrix}'\right)^{-1}\begin{pmatrix}0.1-0.05\\ 0.2-0.05\\ 0.3-0.05\end{pmatrix}}{\left\|\begin{pmatrix}0.2 & 0.01 & 0.03\\ 0.1 & 0.3 & 0.04\\ 0.05 & 0.03 & 0.1\end{pmatrix}^{-1}\begin{pmatrix}0.1-0.05\\ 0.2-0.05\\ 0.3-0.05\end{pmatrix}\right\|} \\ &= \begin{pmatrix}-0.0063\\ -0.0018\\ 0.0263\end{pmatrix}.\end{aligned}$$

This means 2.63% are invested in asset 3 and the rest is invested risk free.

Now we check if the expected wealth at time 1 really is 1056. To do so, we calculate

$$\begin{aligned}\mathbb{E}(X^\varphi(1)) &= x(1+\varphi'(b-r\mathbf{1})+r) \\ &= 1000\left(1+(-0.0063,-0.0018,0.0263)\begin{pmatrix}0.1-0.05\\ 0.2-0.05\\ 0.3-0.05\end{pmatrix}+0.05\right) = 1056.\end{aligned}$$

6.3. CLOSED FORM SOLUTION OF THE ONE-PERIOD CAPITAL-AT-RISK PROBLEM

In this subsection we introduce the discrete time one-period mean-Capital-at-Risk problem and provide a closed form solution. The solution of the continuous time optimization problem can be found in [3]. We accept a certain amount as Capital-at-Risk and we want to maximize the expected return. We solve the optimization problem

$$\begin{cases} \max\limits_{\varphi \in \mathbb{R}^n} \mathbb{E}(X^{\varphi}(1)) \\ \text{s.t. } \operatorname{CaR}(\varphi) = C, \end{cases} \tag{32}$$

i.e., using (23) and (14) for $t = 1$,

$$\begin{cases} \max\limits_{\varphi \in \mathbb{R}^n} x(1+\alpha) \\ \text{s.t. } x(-z_{\beta}\left\|c\right\| - \varphi^{'}(b - r\mathbf{1})) = C, \end{cases} \tag{33}$$

where C is the CaR at time $T = 1$. From the definition of c and α, we may rewrite (33) as

$$\max_{\varphi \in A} g(\varphi), \quad \text{where} \quad g(\varphi) = x\left(1 + \varphi^{'}(b - r\mathbf{1}) + r\right), \tag{34}$$

where

$$A = \left\{\varphi \in \mathbb{R}^n : \ \varphi^{'}(b - r\mathbf{1}) = -\frac{C}{x} - z_{\beta}\left\|\sigma^{'}\varphi\right\|\right\} \tag{35}$$

is the feasible set of the problem. We also consider the problem

$$\begin{cases} \min\limits_{\varphi \in \mathbb{R}^n} \mathbb{E}(X^{\varphi}(1)) \\ \text{s.t. } \operatorname{CaR}(\varphi) = C, \end{cases} \tag{36}$$

i.e., using (23) and (14) for $t = 1$,

$$\begin{cases} \min\limits_{\varphi \in \mathbb{R}^n} x(1+\alpha) \\ \text{s.t. } x(-z_{\beta}\left\|c\right\| - \varphi^{'}(b - r\mathbf{1})) = C, \end{cases} \tag{37}$$

where C is the CaR at time $T = 1$. From the definition of c and α, we may rewrite (37) as

$$\min_{\varphi \in A} g(\varphi), \tag{38}$$

where g and A are as in (34) and (35). Before we start with the theorems we give an overview of the solutions in this subsection.

Table 6.1. Overview mean-Capital-at-Risk problem

C	$\Theta + z_\beta$	Result	See
> 0	< 0	Found max and min	Th. 6.2
> 0	> 0	Found min	Th. 6.3
< 0	> 0	Found min	Th. 6.4
< 0	< 0	$A = \emptyset$	Th. 6.5

As a next step, we state two important lemmas for the proofs of the mean-CaR optimization problem.

Lemma 6.4. *If $\varphi \in A$, then*

$$\varphi^{'}(b - r\mathbf{1}) \geq -\frac{C}{x}\frac{\Theta}{\Theta - z_\beta}. \tag{39}$$

Proof. We calculate

$$\varphi^{'}(b - r\mathbf{1}) \geq -|\varphi^{'}(b - r\mathbf{1})| \overset{(28)}{\geq} - \left\|\sigma^{'}\varphi\right\| \Theta \overset{\varphi \in A}{=} \frac{\varphi^{'}(b - r\mathbf{1}) + \frac{C}{x}}{z_\beta}\Theta. \tag{40}$$

We re-arrange (40) to get

$$-z_\beta \varphi^{'}(b - r\mathbf{1}) \geq -\Theta\varphi^{'}(b - r\mathbf{1}) - \frac{C}{x}\Theta,$$

hence

$$(\Theta - z_\beta)\varphi^{'}(b - r\mathbf{1}) \geq -\frac{C}{x}\Theta,$$

and thus, since $\Theta - z_\beta > 0$,

$$\varphi^{'}(b - r\mathbf{1}) \geq -\frac{C}{x}\frac{\Theta}{\Theta - z_\beta}.$$

The proof is complete. □

Lemma 6.5. *If $\varphi \in A$, then*

$$-(\Theta + z_\beta)\varphi^{'}(b - r\mathbf{1}) \leq \frac{C}{x}\Theta. \tag{41}$$

Proof. We calculate

$$\varphi^{'}(b - r\mathbf{1}) \leq |\varphi^{'}(b - r\mathbf{1})| \overset{(28)}{\leq} \left\|\sigma^{'}\varphi\right\|\Theta \overset{\varphi \in A}{=} \frac{\varphi^{'}(b - r\mathbf{1}) + \frac{C}{x}}{-z_\beta}\Theta. \tag{42}$$

We re-arrange (42) to get

$$-z_\beta\varphi^{'}(b - r\mathbf{1}) \leq \Theta\varphi^{'}(b - r\mathbf{1}) + \frac{C}{x}\Theta,$$

hence

$$-(\Theta + z_\beta)\varphi^{'}(b - r\mathbf{1}) \leq \frac{C}{x}\Theta.$$

The proof is complete. □

Theorem 6.2 (Closed form solution to the discrete time one-period mean-CaR optimization problem, part 1)**.** *We assume that*

$$z_\beta + \Theta < 0 \quad \textit{and} \quad C > 0.$$

The closed form solution of the one-period mean-Capital-at-Risk problem (32) *is given by*

$$\varphi^* = \frac{\lambda(\sigma\sigma')^{-1}(b - r\mathbf{1})}{\|\sigma^{-1}(b - r\mathbf{1})\|},$$

where λ *is given as*

$$\lambda = -\frac{\frac{C}{x}}{\Theta + z_\beta}.$$

The expected wealth after one period is then given by

$$\mathbb{E}(X^{\varphi^*}(1)) = x\,(1 + \lambda\Theta + r) \tag{43}$$

with $\mathrm{CaR}(\varphi^*) = C$*. The closed-form solution of problem* (36) *is given by*

$$\varphi^{**} = \frac{\mu(\sigma\sigma')^{-1}(b - r\mathbf{1})}{\|\sigma^{-1}(b - r\mathbf{1})\|},$$

where μ *is given as*

$$\mu = -\frac{\frac{C}{x}}{\Theta - z_\beta}.$$

The expected wealth after one period is then given by

$$\mathbb{E}(X^{\varphi^{**}}(1)) = x\,(1 + \mu\Theta + r) \tag{44}$$

with $\mathrm{CaR}(\varphi^{**}) = C$*.*

Proof. We show that

1. $g(\varphi^*) = x(1 + \lambda\Theta + r) \quad \text{and} \quad g(\varphi^{**}) = x(1 + \mu\Theta + r)$,
2. $g(\varphi^{**}) \le g(\varphi) \le g(\varphi^*)$ for all $\varphi \in A$,
3. $\varphi^*, \varphi^{**} \in A$.

To prove that $g(\varphi^*)$ and $g(\varphi^{**})$ have the stated form follows directly from Lemma 6.1. Next, let $\varphi \in A$. We now show that $g(\varphi) \geq g(\varphi^{**})$. From (39) we get

$$\varphi^{'}(b - r\mathbf{1}) \geq -\frac{C}{x}\frac{\Theta}{\Theta - z_\beta} = \mu\Theta. \tag{45}$$

Therefore,

$$g(\varphi) \;=\; x\left(1 + \varphi^{'}(b - r\mathbf{1}) + r\right) \overset{(45)}{\geq} x\left(1 + \mu\Theta + r\right) = g(\varphi^{**}).$$

As a next step we show that $g(\varphi) \leq g(\varphi^*)$. From (41) we get

$$-(\Theta + z_\beta)\varphi^{'}(b - r\mathbf{1}) \leq \frac{C}{x}\Theta,$$

and thus, since $\Theta + z_\beta < 0$,

$$\varphi^{'}(b - r\mathbf{1}) \leq -\frac{C}{x}\frac{\Theta}{\Theta + z_\beta} = \lambda\Theta. \tag{46}$$

Therefore,

$$g(\varphi) \;=\; x\left(1 + \varphi^{'}(b - r\mathbf{1}) + r\right) \overset{(46)}{\leq} x\left(1 + \lambda\Theta + r\right) = g(\varphi^{*}).$$

As a last step we show that $\varphi^*, \varphi^{**} \in A$. To do so, we calculate

$$\begin{aligned}
-z_\beta \left\|\sigma'\varphi^*\right\| - (\varphi^*)^{'}(b - r\mathbf{1}) \;&\overset{(27),(26)}{=}\; -z_\beta|\lambda| - \lambda\Theta \\
&=\; -z_\beta\left|-\frac{\frac{C}{x}}{\Theta + z_\beta}\right| + \frac{\frac{C}{x}}{\Theta + z_\beta}\Theta \\
&=\; z_\beta\frac{\frac{C}{x}}{\Theta + z_\beta} + \frac{\frac{C}{x}}{\Theta + z_\beta}\Theta \\
&=\; \frac{\frac{C}{x}}{\Theta + z_\beta}\left(z_\beta + \Theta\right) = \frac{C}{x}.
\end{aligned}$$

Therefore $\varphi^* \in A$. Now we calculate

$$
\begin{aligned}
-z_\beta \left\|\sigma'\varphi^{**}\right\| - (\varphi^{**})'(b - r\mathbf{1}) &\overset{(27),(26)}{=} -z_\beta|\mu| - \mu\Theta \\
&= -z_\beta\left|-\frac{\frac{C}{x}}{\Theta - z_\beta}\right| + \frac{\frac{C}{x}}{\Theta - z_\beta}\Theta \\
&= -z_\beta\frac{\frac{C}{x}}{\Theta - z_\beta} + \frac{\frac{C}{x}}{\Theta - z_\beta}\Theta \\
&= \frac{\frac{C}{x}}{\Theta - z_\beta}(-z_\beta + \Theta) = \frac{C}{x}.
\end{aligned}
$$

Therefore $\varphi^{**} \in A$. This completes the proof. □

Theorem 6.3 (Closed form solution to the discrete time one-period mean-CaR optimization problem, part 2)**.** *We assume that*

$$z_\beta + \Theta > 0 \quad \textit{and} \quad C > 0.$$

The closed form solution of problem (36) *is given by*

$$\varphi^{**} = \frac{\mu(\sigma\sigma')^{-1}(b - r\mathbf{1})}{\left\|\sigma^{-1}(b - r\mathbf{1})\right\|},$$

where μ *is given as*

$$\mu = -\frac{\frac{C}{x}}{\Theta - z_\beta}.$$

The expected wealth after one period is then given by

$$\mathbb{E}(X^{\varphi^{**}}(1)) = x\left(1 + \mu\Theta + r\right) \tag{47}$$

with $\mathrm{CaR}(\varphi^{**}) = C$*.*

Proof. We show that

1. $g(\varphi^{**}) = x(1 + \mu\Theta + r)$,

2. $g(\varphi^{**}) \leq g(\varphi)$ for all $\varphi \in A$,

3. $\varphi^{**} \in A$.

To prove that $g(\varphi^{**})$ has the stated form follows directly from Lemma 6.1. Next, let $\varphi \in A$. We now show that $g(\varphi) \geq g(\varphi^{**})$. From (39) we get

$$\varphi^{'}(b - r\mathbf{1}) \geq -\frac{C}{x}\frac{\Theta}{\Theta - z_\beta} = \mu\Theta. \tag{48}$$

Therefore,

$$g(\varphi) \;=\; x\left(1 + \varphi^{'}(b - r\mathbf{1}) + r\right) \stackrel{(48)}{\geq} x\,(1 + \mu\Theta + r) = g(\varphi^{**}).$$

As a last step we show that $\varphi^{**} \in A$. To do so, we calculate

$$\begin{aligned}
-z_\beta\,\|\sigma'\varphi^{**}\| - (\varphi^{**})^{'}(b - r\mathbf{1}) &\stackrel{(27),(26)}{=} -z_\beta|\mu| - \mu\Theta \\
&= -z_\beta\left|-\frac{\frac{C}{x}}{\Theta - z_\beta}\right| + \frac{\frac{C}{x}}{\Theta - z_\beta}\Theta \\
&= -z_\beta\frac{\frac{C}{x}}{\Theta - z_\beta} + \frac{\frac{C}{x}}{\Theta - z_\beta}\Theta \\
&= \frac{\frac{C}{x}}{\Theta - z_\beta}(-z_\beta + \Theta) = \frac{C}{x}.
\end{aligned}$$

Therefore $\varphi^{**} \in A$. This completes the proof. □

Theorem 6.4 (Closed form solution to the discrete time one-period mean-CaR optimization problem, part 3)**.** *We assume that*

$$z_\beta + \Theta > 0 \quad \textit{and} \quad C < 0.$$

The closed form solution of problem (36) *is given by*

$$\varphi^* = \frac{\lambda(\sigma\sigma^{'})^{-1}(b - r\mathbf{1})}{\|\sigma^{-1}(b - r\mathbf{1})\|},$$

where λ *is given as*

$$\lambda = -\frac{\frac{C}{x}}{\Theta + z_\beta}.$$

The expected wealth after one period is then given by

$$\mathbb{E}(X^{\varphi^*}(1)) = x\,(1 + \lambda\Theta + r) \tag{49}$$

with $\mathrm{CaR}(\varphi^*) = C$.

Proof. We show that

1. $g(\varphi^*) = x(1 + \lambda\Theta + r)$,
2. $g(\varphi^*) \leq g(\varphi)$ for all $\varphi \in A$,
3. $\varphi^* \in A$.

To prove that $g(\varphi^*)$ has the stated form follows directly from Lemma 6.1. Next, let $\varphi \in A$. We show that $g(\varphi) \geq g(\varphi^*)$. From (41) we get

$$-(\Theta + z_\beta)\varphi^{'}(b - r\mathbf{1}) \leq \frac{C}{x}\Theta,$$

and thus, since $\Theta + z_\beta > 0$,

$$\varphi^{'}(b - r\mathbf{1}) \geq -\frac{C}{x}\frac{\Theta}{\Theta + z_\beta} = \lambda\Theta. \tag{50}$$

Therefore,

$$g(\varphi) \;\; = \;\; x\left(1 + \varphi^{'}(b - r\mathbf{1}) + r\right) \overset{(50)}{\geq} x\,(1 + \lambda\Theta + r) = g(\varphi^*).$$

As a last step we show that $\varphi^* \in A$. To do so, we calculate

$$-z_\beta\,\|\sigma'\varphi^*\| - (\varphi^*)^{'}(b - r\mathbf{1}) \overset{(27),(26)}{=} -z_\beta|\lambda| - \lambda\Theta$$

$$= -z_\beta \left| -\frac{\frac{C}{x}}{\Theta + z_\beta} \right| + \frac{\frac{C}{x}}{\Theta + z_\beta}\Theta$$

$$= z_\beta \frac{\frac{C}{x}}{\Theta + z_\beta} + \frac{\frac{C}{x}}{\Theta + z_\beta}\Theta$$

$$= \frac{\frac{C}{x}}{\Theta + z_\beta}(z_\beta + \Theta) = \frac{C}{x}.$$

Therefore $\varphi^* \in A$. This completes the proof. □

Theorem 6.5 (Closed form solution to the discrete time one-period mean-CaR optimization problem, part 4)**.** *We assume that*

$$z_\beta + \Theta < 0 \quad \textit{and} \quad C < 0.$$

Then both (36) *and the mean-Capital-at-Risk problem* (32) *are unsolvable.*

Proof. Let $\varphi \in A$. From (39) we know that

$$\varphi^{'}(b - r\mathbf{1}) \geq -\frac{C}{x}\frac{\Theta}{\Theta - z_\beta} > 0. \tag{51}$$

We also know from (41) that

$$-(\Theta + z_\beta)\varphi^{'}(b - r\mathbf{1}) \leq \frac{C}{x}\Theta,$$

and thus, since $\Theta + z_\beta < 0$,

$$\varphi^{'}(b - r\mathbf{1}) \leq -\frac{C}{x}\frac{\Theta}{\Theta + z_\beta} < 0. \tag{52}$$

If we combine (51) and (52), we get a contradiction. Therefore the feasible set is empty, i.e., $A = \emptyset$, and both problems (32) and (36) are unsolvable. This completes the proof. □

6.4. EXAMPLES FOR THE DISCRETE MEAN-CAPITAL-AT-RISK PROBLEM

In this section we calculate an example for the one-period mean-CaR problem and calculate the corresponding portfolio strategy and check the results of the optimization. In the first example we calculate the minimal expected wealth with $\mathrm{CaR} = C$. Let

$$b = (0.1, 0.2, 0.3)',$$

$$r = 0.05, \quad z_\beta = -1.64.$$

$$\sigma = \begin{pmatrix} 0.2 & 0.01 & 0.03 \\ 0.1 & 0.3 & 0.04 \\ 0.05 & 0.03 & 0.1 \end{pmatrix},$$

$$\left\|\sigma^{-1}(b - r\mathbf{1})\right\| + z_\beta = \left\| \begin{pmatrix} 0.2 & 0.01 & 0.03 \\ 0.1 & 0.3 & 0.04 \\ 0.05 & 0.03 & 0.1 \end{pmatrix}^{-1} \begin{pmatrix} 0.1 - 0.05 \\ 0.2 - 0.05 \\ 0.3 - 0.05 \end{pmatrix} \right\| - 1.64 = 0.8773.$$

Therefore C should be negative. Let

$$x = 1000, \quad C = -100.$$

Now we calculate

$$\begin{aligned} \lambda &= -\frac{\frac{C}{x}}{\|\sigma^{-1}(b - r\mathbf{1})\| + z_\beta} = \frac{\frac{100}{1000}}{\left\| \begin{pmatrix} 0.2 & 0.01 & 0.03 \\ 0.1 & 0.3 & 0.04 \\ 0.05 & 0.03 & 0.1 \end{pmatrix}^{-1} \begin{pmatrix} 0.1 - 0.05 \\ 0.2 - 0.05 \\ 0.3 - 0.05 \end{pmatrix} \right\| - 1.64} \\ &= \frac{0.1}{0.8773} = 0.11399. \end{aligned}$$

Now we calculate the investment strategy, because we want to know the weights of the assets:

$$\varphi^* = \lambda \frac{(\sigma\sigma')^{-1}(b - r\mathbf{1})}{\|\sigma^{-1}(b - r\mathbf{1})\|}$$

$$= 0.1140 \cdot \frac{\left(\begin{pmatrix} 0.2 & 0.01 & 0.03 \\ 0.1 & 0.3 & 0.04 \\ 0.05 & 0.03 & 0.1 \end{pmatrix} \begin{pmatrix} 0.2 & 0.01 & 0.03 \\ 0.1 & 0.3 & 0.04 \\ 0.05 & 0.03 & 0.1 \end{pmatrix}'\right)^{-1} \begin{pmatrix} 0.1 - 0.05 \\ 0.2 - 0.05 \\ 0.3 - 0.05 \end{pmatrix}}{\left\| \begin{pmatrix} 0.2 & 0.01 & 0.03 \\ 0.1 & 0.3 & 0.04 \\ 0.05 & 0.03 & 0.1 \end{pmatrix}^{-1} \begin{pmatrix} 0.1 - 0.05 \\ 0.2 - 0.05 \\ 0.3 - 0.05 \end{pmatrix} \right\|}$$

$$= \begin{pmatrix} -0.30363 \\ -0.083839 \\ 1.2588 \end{pmatrix}.$$

This means 125.89% are invested in asset 3.

Now we calculate the expected return of this strategy. To do so, we calculate

$$\begin{aligned} \mathbb{E}(X^{\varphi}(1)) &= x\left(1 + \varphi'(b - r\mathbf{1}) + r\right) \\ &= 1000 \cdot \left(1 + (-0.30363, -0.083839, 1.2588) \begin{pmatrix} 0.1 - 0.05 \\ 0.2 - 0.05 \\ 0.3 - 0.05 \end{pmatrix} + 0.05 \right) = 1336.9. \end{aligned}$$

As a next step we check if the CaR of this strategy really is -100.

$$\mathrm{CaR}(\varphi^*) = x(-z_\beta \left\|\sigma'\varphi\right\| - \varphi'(b - r\mathbf{1})) = 1000 \cdot (1.64 \cdot 0.114 - 0.286973) = -100. \quad (53)$$

This is an example where we calculated the minimal expectation with CaR $= -100$. We provide a second example where C is positive. In this example we calculate the maximal expected wealth with CaR $= C$. Let

$$b = (0.1, 0.2, 0.3)',$$

$$r = 0.05, \quad z_\beta = -1.64.$$

$$\sigma = \begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.2 \end{pmatrix},$$

$$\left\|\sigma^{-1}(b - r\mathbf{1})\right\| + z_\beta = \left\| \begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.2 \end{pmatrix}^{-1} \begin{pmatrix} 0.1 - 0.05 \\ 0.2 - 0.05 \\ 0.3 - 0.05 \end{pmatrix} \right\| - 1.64 = -0.20386.$$

Therefore C should be positive.

$$x = 1000, \quad C = 20.$$

Now we calculate

$$\begin{aligned} \lambda &= -\frac{\frac{C}{x}}{\|\sigma^{-1}(b - r\mathbf{1})\| + z_\beta} = -\frac{\frac{20}{1000}}{\left\| \begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.2 \end{pmatrix}^{-1} \begin{pmatrix} 0.1 - 0.05 \\ 0.2 - 0.05 \\ 0.3 - 0.05 \end{pmatrix} \right\| - 1.64} \\ &= -\frac{0.02}{-0.20386} = 0.098107. \end{aligned}$$

Now we calculate the investment strategy, because we want to know the weights of the assets:

$$\begin{aligned} \varphi^* &= \lambda \frac{(\sigma\sigma')^{-1}(b - r\mathbf{1})}{\|\sigma^{-1}(b - r\mathbf{1})\|} \\ &= 0.06299 \cdot \frac{\left(\begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.2 \end{pmatrix} \begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.2 \end{pmatrix}' \right)^{-1} \begin{pmatrix} 0.1 - 0.05 \\ 0.2 - 0.05 \\ 0.3 - 0.05 \end{pmatrix}}{\left\| \begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.2 \end{pmatrix}^{-1} \begin{pmatrix} 0.1 - 0.05 \\ 0.2 - 0.05 \\ 0.3 - 0.05 \end{pmatrix} \right\|} \end{aligned}$$

$$= \begin{pmatrix} 0.34156 \\ 0.11385 \\ 0.42695 \end{pmatrix}.$$

This means 34.156% are invested in asset 1, 11.385% are invested in asset 2 and 42.695% are invested in asset 3.

Now we calculate the expected return of this strategy. To do so, we calculate

$$\begin{aligned} \mathbb{E}(X^{\varphi}(1)) &= x\left(1 + \varphi^{'}(b - r\mathbf{1}) + r\right) \\ &= 1000 \cdot \left(1 + (0.34156, 0.11385, 0.42695) \begin{pmatrix} 0.1 - 0.05 \\ 0.2 - 0.05 \\ 0.3 - 0.05 \end{pmatrix} + 0.05 \right) = 1190.9. \end{aligned}$$

As a next step we check if the CaR of this strategy really is 20.

$$\text{CaR}(\varphi^{*}) = x(-z_{\beta} \left\| \sigma^{'}\varphi \right\| - \varphi^{'}(b - r\mathbf{1})) = 1000 \cdot (1.64 \cdot 0.098107 - 0.14089) = 20.$$

In this example we calculated the maximal expectation with CaR $= 20$.

6.5. CLOSED FORM SOLUTION OF THE ONE-PERIOD VALUE-AT-RISK PROBLEM

In this subsection we introduce the discrete time one-period mean-Value-at-Risk problem and provide a closed form solution.

We accept a certain amount as Value-at-Risk and we want to find the portfolio strategy which maximizes our expected wealth. We solve the optimization problem

$$\begin{cases} \max\limits_{\varphi \in \mathbb{R}^n} \mathbb{E}(X^{\varphi}(1)) \\ \text{s.t. } \text{VaR}(\varphi) = C, \end{cases} \tag{54}$$

i.e., using (17) and (14) with $t = 1$,

$$\begin{cases} \max\limits_{\varphi \in \mathbb{R}^n} x(1+\alpha) \\ \text{s.t. } \left(z_\beta \|c\| + 1 + \alpha\right) x = C, \end{cases} \tag{55}$$

where C is the VaR at time $T = 1$. From the definition of c and α, we may rewrite (55) as

$$\max_{\varphi \in A} g(\varphi), \quad \text{where} \quad g(\varphi) = x(1 + \varphi^{'}(b - r\mathbf{1}) + r), \tag{56}$$

where

$$A = \left\{ \varphi \in \mathbb{R}^n : \ \varphi^{'}(b - r\mathbf{1}) = \frac{C}{x} - 1 - r - z_\beta \|c\| \right\} \tag{57}$$

is the feasible set of the problem. We also consider the problem

$$\begin{cases} \min\limits_{\varphi \in \mathbb{R}^n} \mathbb{E}(X^\varphi(1)) \\ \text{s.t. } \mathrm{VaR}(\varphi) = C, \end{cases} \tag{58}$$

i.e., using (17) and (14) with $t = 1$,

$$\begin{cases} \min\limits_{\varphi \in \mathbb{R}^n} x(1+\alpha) \\ \text{s.t. } \left(z_\beta \|c\| + 1 + \alpha\right) x = C, \end{cases} \tag{59}$$

where C is the VaR at time $T = 1$. From the definition of c and α, we may rewrite (59) as

$$\min_{\varphi \in A} g(\varphi), \tag{60}$$

where g and A are as in (56) and (57). Before we state the theorems we give an overview of the solutions in this subsection in a table. As a next step, we state two important lemmas for the proofs of the mean-VaR optimization problem.

Table 6.2. Overview mean-Value-at-Risk problem

$\frac{C}{x}-1-r$	$\Theta+z_\beta$	Result	See
<0	<0	Found max and min	Th. 6.6
>0	>0	Found min	Th. 6.7
<0	>0	Found min	Th. 6.8
>0	<0	$A=\emptyset$	Th. 6.9

Lemma 6.6. *If* $\varphi \in A$, *then*

$$\varphi^{'}(b-r\mathbf{1}) \geq \frac{\frac{C}{x}-1-r}{\Theta - z_\beta}\Theta. \tag{61}$$

Proof. We calculate

$$\varphi^{'}(b-r\mathbf{1}) \geq -|\varphi^{'}(b-r\mathbf{1})| \overset{(28)}{\geq} -\left\|\sigma^{'}\varphi\right\|\Theta \overset{\varphi\in A}{=} \frac{\varphi^{'}(b-r\mathbf{1})-\frac{C}{x}+1+r}{z_\beta}\Theta. \tag{62}$$

We re-arrange (62) to get

$$-z_\beta\varphi^{'}(b-r\mathbf{1}) \geq -\Theta\varphi^{'}(b-r\mathbf{1}) + \left(\frac{C}{x}-1-r\right)\Theta,$$

hence

$$(\Theta - z_\beta)\varphi^{'}(b-r\mathbf{1}) \geq \left(\frac{C}{x}-1-r\right)\Theta.$$

and thus, since $\Theta - z_\beta > 0$,

$$\varphi^{'}(b-r\mathbf{1}) \geq \frac{\frac{C}{x}-1-r}{\Theta - z_\beta}\Theta.$$

The proof is complete. □

Lemma 6.7. *If $\varphi \in A$, then*

$$(z_\beta + \theta)\,\varphi^{'}(b - r\mathbf{1}) \geq \left(\frac{C}{x} - 1 - r\right)\Theta. \tag{63}$$

Proof. We calculate

$$\varphi^{'}(b - r\mathbf{1}) \leq |\varphi^{'}(b - r\mathbf{1})| \overset{(28)}{\leq} \Theta\,\|\sigma'\varphi\| \overset{\varphi\in A}{=} \Theta\left(\frac{-\varphi^{'}(b - r\mathbf{1}) + \frac{C}{x} - 1 - r}{z_\beta}\right). \tag{64}$$

We re-arrange (64) to get

$$(z_\beta + \theta)\,\varphi^{'}(b - r\mathbf{1}) \geq \left(\frac{C}{x} - 1 - r\right)\Theta.$$

The proof is complete. □

Theorem 6.6 (Closed form solution to the discrete time one-period mean-VaR optimization problem, part 1)**.** *We assume that*

$$\theta + z_\beta < 0 \quad \text{and} \quad C < x(1 + r). \tag{65}$$

The closed form solution of problem (54) *is given by*

$$\varphi^* = \frac{\lambda(\sigma\sigma^{'})^{-1}(b - r\mathbf{1})}{\|\sigma^{-1}(b - r\mathbf{1})\|},$$

where λ is given by

$$\lambda = \frac{\frac{C}{x} - 1 - r}{\Theta + z_\beta}.$$

The expected wealth after one period is then given by

$$\mathbb{E}(X^{\varphi^*}(1)) = x\,(1 + \lambda\Theta + r)$$

with $\mathrm{VaR}(\varphi^*) = C$. *The closed-form solution of problem* (58) *is given by*

$$\varphi^{**} = \frac{\mu(\sigma\sigma^{'})^{-1}(b - r\mathbf{1})}{\|\sigma^{-1}(b - r\mathbf{1})\|},$$

where μ *is given by*

$$\mu = \frac{\frac{C}{x} - 1 - r}{\Theta - z_\beta}.$$

The expected wealth after one period is then given by

$$\mathbb{E}(X^{\varphi^{**}}(1)) = x\,(1 + \mu\Theta + r)$$

with $\mathrm{VaR}(\varphi^*) = C$.

Proof. We show that

1. $g(\varphi^*) = x(1 + \lambda\Theta + r),\ g(\varphi^{**}) = x(1 + \mu\Theta + r)$,
2. $g(\varphi^{**}) \le g(\varphi) \le g(\varphi^*)$ for all $\varphi \in A$,
3. $\varphi^*, \varphi^{**} \in A$.

To prove that $g(\varphi^*), g(\varphi^{**})$ have the stated form follows directly from Lemma 6.1. Next, let $\varphi \in A$. We show that $g(\varphi) \le g(\varphi^*)$. From (63) we get

$$(z_\beta + \theta)\,\varphi^{'}(b - r\mathbf{1}) \ge \left(\frac{C}{x} - 1 - r\right)\Theta,$$

and thus, since $\Theta + z_\beta < 0$,

$$\varphi^{'}(b - r\mathbf{1}) \le \frac{\left(\frac{C}{x} - 1 - r\right)}{z_\beta + \Theta}\Theta = \lambda\Theta. \tag{66}$$

Now,

$$g(\varphi) = x(1 + \varphi^{'}(b - r\mathbf{1}) + r) \overset{(66)}{\le} x(1 + \lambda\Theta + r) = g(\varphi^*).$$

As next step we show that $g(\varphi) \geq g(\varphi^{**})$. From (61) we get

$$\varphi^{'}(b - r\mathbf{1}) \geq \frac{\frac{C}{x} - 1 - r}{\Theta - z_\beta}\Theta = \mu\Theta. \tag{67}$$

Therefore,

$$g(\varphi) = x(1 + \varphi^{'}(b - r\mathbf{1}) + r) \overset{(67)}{\geq} x(1 + \mu\Theta + r) = g(\varphi^{**}).$$

As a last step we show that $\varphi^*, \varphi^{**} \in A$.

$$\begin{aligned}
x(z_\beta \|\sigma'\varphi^*\| + 1 + (\varphi^*)^{'}(b - r\mathbf{1}) + r) &\overset{(27),(26)}{=} x\left(z_\beta \left|\frac{\frac{C}{x} - 1 - r}{\Theta + z_\beta}\right| + 1 + \frac{\frac{C}{x} - 1 - r}{\Theta + z_\beta}\Theta + r\right) \\
&= x\left(z_\beta \frac{\frac{C}{x} - 1 - r}{\Theta + z_\beta} + 1 + \frac{\frac{C}{x} - 1 - r}{\Theta + z_\beta}\Theta + r\right) \\
&= x(1 + r) + x\left(\frac{\frac{C}{x} - 1 - r}{\Theta + z_\beta}(\Theta + z_\beta)\right) \\
&= x(1 + r) + x\left(\frac{C}{x} - 1 - r\right) = C.
\end{aligned}$$

Therefore $\varphi^* \in A$. Now we calculate

$$\begin{aligned}
x(z_\beta \|\sigma'\varphi^{**}\| + 1 + (\varphi^{**})^{'}(b - r\mathbf{1}) + r) &\overset{(27),(26)}{=} x\left(z_\beta \left|\frac{\frac{C}{x} - 1 - r}{\Theta - z_\beta}\right| + 1 + \frac{\frac{C}{x} - 1 - r}{\Theta - z_\beta}\Theta + r\right) \\
&= x\left(-z_\beta \frac{\frac{C}{x} - 1 - r}{\Theta - z_\beta} + 1 + \frac{\frac{C}{x} - 1 - r}{\Theta - z_\beta}\Theta + r\right) \\
&= x(1 + r) + x\left(\frac{\frac{C}{x} - 1 - r}{\Theta - z_\beta}(\Theta - z_\beta)\right) \\
&= x(1 + r) + x\left(\frac{C}{x} - 1 - r\right) = C.
\end{aligned}$$

Therefore $\varphi^{**} \in A$. The proof is complete. □

Theorem 6.7 (Closed form solution to the discrete time one-period mean-VaR optimization problem, part 2)**.** *We assume that*

$$\theta + z_\beta > 0 \quad \textit{and} \quad C > x(1 + r). \tag{68}$$

The closed form solution of problem (58) *is given by*

$$\varphi^* = \frac{\lambda(\sigma\sigma^{'})^{-1}(b - r\mathbf{1})}{\|\sigma^{-1}(b - r\mathbf{1})\|},$$

where λ *is given by*

$$\lambda = \frac{\frac{C}{x} - 1 - r}{\Theta + z_\beta}.$$

The expected wealth after one period is then given by

$$\mathbb{E}(X^{\varphi^*}(1)) = x\,(1 + \lambda\Theta + r)$$

with $\mathrm{VaR}(\varphi^*) = C$*.*

Proof. We show that

1. $g(\varphi^*) = x(1 + \lambda\Theta + r)$,

2. $g(\varphi) \geq g(\varphi^*)$ for all $\varphi \in A$,

3. $\varphi^* \in A$.

To prove that $g(\varphi^*)$ has the stated form follows directly from Lemma 6.1. Next, let $\varphi \in A$. We show that $g(\varphi) \leq g(\varphi^*)$. From (63) we get

$$(z_\beta + \theta)\,\varphi^{'}(b - r\mathbf{1}) \geq \left(\frac{C}{x} - 1 - r\right)\Theta,$$

and thus, since $\Theta + z_\beta > 0$,

$$\varphi^{'}(b - r\mathbf{1}) \geq \frac{\left(\frac{C}{x} - 1 - r\right)}{z_\beta + \Theta}\Theta = \lambda\Theta. \tag{69}$$

Now

$$g(\varphi) = x(1 + \varphi^{'}(b - r\mathbf{1}) + r) \overset{(69)}{\geq} x(1 + \lambda\Theta + r) = g(\varphi^*).$$

As a last step we show that $\varphi^* \in A$. To do so, we calculate

$$\begin{aligned} x(z_\beta \|\sigma'\varphi^*\| + 1 + (\varphi^*)'(b - r\mathbf{1}) + r) &\overset{(27),(26)}{=} x\left(z_\beta \left|\frac{\frac{C}{x} - 1 - r}{\Theta + z_\beta}\right| + 1 + \frac{\frac{C}{x} - 1 - r}{\Theta + z_\beta}\Theta + r\right) \\ &= x\left(z_\beta \frac{\frac{C}{x} - 1 - r}{\Theta + z_\beta} + 1 + \frac{\frac{C}{x} - 1 - r}{\Theta + z_\beta}\Theta + r\right) \\ &= x(1+r) + x\left(\frac{\frac{C}{x} - 1 - r}{\Theta + z_\beta}(\Theta + z_\beta)\right) \\ &= x(1+r) + x\left(\frac{C}{x} - 1 - r\right) = C. \end{aligned}$$

Therefore $\varphi^* \in A$. The proof is complete. □

Theorem 6.8 (Closed form solution to the discrete time one-period mean-VaR optimization problem, part 3)**.** *We assume that*

$$\theta + z_\beta > 0 \quad \textit{and} \quad C < x(1+r). \tag{70}$$

The closed form solution of problem (58) *is given by*

$$\varphi^* = \frac{\mu(\sigma\sigma')^{-1}(b - r\mathbf{1})}{\|\sigma^{-1}(b - r\mathbf{1})\|},$$

where μ *is given by*

$$\mu = \frac{\frac{C}{x} - 1 - r}{\Theta - z_\beta}.$$

The expected wealth after one period is then given by

$$\mathbb{E}(X^{\varphi^*}(1)) = x\,(1 + \mu\Theta + r) \tag{71}$$

with $\mathrm{VaR}(\varphi^*) = C$.

Proof. We show that

1. $g(\varphi^*) = x(1 + \mu\Theta + r)$,

2. $g(\varphi) \geq g(\varphi^*)$ for all $\varphi \in A$,

3. $\varphi^* \in A$.

To prove that $g(\varphi^*)$ has the stated form follows directly from Lemma 6.1. Next, let $\varphi \in A$. We show that $g(\varphi) \geq g(\varphi^*)$. From (61) we get

$$\varphi^{'}(b - r\mathbf{1}) \geq \frac{\frac{C}{x} - 1 - r}{\Theta - z_\beta}\Theta = \mu\Theta \tag{72}$$

Therefore,

$$g(\varphi) \quad = \quad x\left(1 + \varphi^{'}(b - r\mathbf{1}) + r\right) \overset{(72)}{\geq} x\left(1 + \mu\Theta + r\right) = g(\varphi^*).$$

As a last step we show that $\varphi^* \in A$.

$$\begin{aligned} x(z_\beta \|\sigma'\varphi^*\| + 1 + (\varphi^*)^{'}(b - r\mathbf{1}) + r) &\overset{(27),(26)}{=} x\left(z_\beta \left|\frac{\frac{C}{x} - 1 - r}{\Theta - z_\beta}\right| + 1 + \frac{\frac{C}{x} - 1 - r}{\Theta - z_\beta}\Theta + r\right) \\ &= x\left(-z_\beta \frac{\frac{C}{x} - 1 - r}{\Theta - z_\beta} + 1 + \frac{\frac{C}{x} - 1 - r}{\Theta - z_\beta}\Theta + r\right) \\ &= x(1 + r) + x\left(\frac{\frac{C}{x} - 1 - r}{\Theta - z_\beta}(\Theta - z_\beta)\right) \\ &= x(1 + r) + x\left(\frac{C}{x} - 1 - r\right) = C. \end{aligned}$$

The proof is completed. □

Theorem 6.9 (Closed form solution to the discrete time one-period mean-VaR optimization problem, part 4)**.** *We assume that*

$$\theta + z_\beta < 0 \quad \textit{and} \quad C > x(1 + r). \tag{73}$$

Then both (58) *and the mean-Value-at-Risk problem* (54) *are unsolvable.*

Proof. From (61) we know that

$$\varphi^{'}(b - r\mathbf{1}) \geq \frac{\frac{C}{x} - 1 - r}{\Theta - z_\beta}\Theta > 0. \tag{74}$$

We also know from (63) and the fact that $\Theta + z_\beta < 0$ that

$$\varphi^{'}(b - r\mathbf{1}) \leq \left(\frac{\frac{C}{x} - 1 - r}{\Theta + z_\beta} \right) \Theta < 0. \tag{75}$$

If we combine (74) and (75), we get a contradiction. Therefore the feasible set is empty, i.e., $A = \emptyset$, and both problems (54) and (58) are unsolvable. This completes the proof. □

6.6. EXAMPLES FOR THE DISCRETE MEAN-VALUE-AT-RISK PROBLEM

In this section we calculate an example for the one-period mean-CaR problem and calculate the corresponding portfolio strategy and check the results of the optimization. In the first example we calculate the minimum expected wealth with VaR $= C$. Let

$$b = (0.1, 0.2, 0.3)^{'},$$

$$r = 0.05, \quad z_\beta = -1.64.$$

$$\sigma = \begin{pmatrix} 0.2 & 0.01 & 0.03 \\ 0.1 & 0.3 & 0.04 \\ 0.05 & 0.03 & 0.1 \end{pmatrix},$$

$$\left\|\sigma^{-1}(b - r\mathbf{1})\right\| + z_\beta = \left\| \begin{pmatrix} 0.2 & 0.01 & 0.03 \\ 0.1 & 0.3 & 0.04 \\ 0.05 & 0.03 & 0.1 \end{pmatrix}^{-1} \begin{pmatrix} 0.1 - 0.05 \\ 0.2 - 0.05 \\ 0.3 - 0.05 \end{pmatrix} \right\| - 1.64 = 0.8773.$$

Therefore $C > x(1 + r)$.

$$x = 1000, \quad C = 1060.$$

Now we calculate

$$\lambda = \frac{\frac{C}{x} - 1 - r}{\|\sigma^{-1}(b - r\mathbf{1})\| + z_\beta} = \frac{\frac{1060}{1000} - 1 - 0.05}{\left\| \begin{pmatrix} 0.2 & 0.01 & 0.03 \\ 0.1 & 0.3 & 0.04 \\ 0.05 & 0.03 & 0.1 \end{pmatrix}^{-1} \begin{pmatrix} 0.1 - 0.05 \\ 0.2 - 0.05 \\ 0.3 - 0.05 \end{pmatrix} \right\| - 1.64}$$

$$= \frac{0.01}{0.8773} = 0.011399.$$

Now we calculate the investment strategy, because we want to know the weights of the assets:

$$\varphi^* = \lambda \frac{(\sigma\sigma^{'})^{-1}(b - r\mathbf{1})}{\|\sigma^{-1}(b - r\mathbf{1})\|}$$

$$= 0.011399 \cdot \frac{\left(\begin{pmatrix} 0.2 & 0.01 & 0.03 \\ 0.1 & 0.3 & 0.04 \\ 0.05 & 0.03 & 0.1 \end{pmatrix} \begin{pmatrix} 0.2 & 0.01 & 0.03 \\ 0.1 & 0.3 & 0.04 \\ 0.05 & 0.03 & 0.1 \end{pmatrix}^{'} \right)^{-1} \begin{pmatrix} 0.1 - 0.05 \\ 0.2 - 0.05 \\ 0.3 - 0.05 \end{pmatrix}}{\left\| \begin{pmatrix} 0.2 & 0.01 & 0.03 \\ 0.1 & 0.3 & 0.04 \\ 0.05 & 0.03 & 0.1 \end{pmatrix}^{-1} \begin{pmatrix} 0.1 - 0.05 \\ 0.2 - 0.05 \\ 0.3 - 0.05 \end{pmatrix} \right\|}$$

$$= \begin{pmatrix} -0.030363 \\ -0.0083839 \\ 0.12588 \end{pmatrix}.$$

This means 12.588% are invested in asset 3.

Now we calculate the expected return of this strategy. To do so, we calculate

$$\mathbb{E}(X^\varphi(1)) = x\left(1 + \varphi^{'}(b - r\mathbf{1}) + r\right)$$

$$= 1000 \cdot \left(1 + (-0.030363, -0.0083839, 0.12588) \begin{pmatrix} 0.1 - 0.05 \\ 0.2 - 0.05 \\ 0.3 - 0.05 \end{pmatrix} + 0.05 \right)$$

$$= \quad 1078.7.$$

As a next step we check if the VaR of this strategy really is 1060.

$$\begin{aligned}
\mathrm{VaR}(\varphi^*) &= (z_\beta \|c\| + 1 + \alpha)x \\
&= 1000 \cdot \left(-1.64 \cdot 0.011399 + 1 + (-0.030363, -0.0083839, 0.12588) \begin{pmatrix} 0.1 - 0.05 \\ 0.2 - 0.05 \\ 0.3 - 0.05 \end{pmatrix} + 0.05 \right) \\
&= 1060.
\end{aligned}$$

We provide a second example when $C < x(1+r)$. In this example we calculate the maximal expected wealth with $VaR = C$. Let

$$b = (0.1, 0.2, 0.3)',$$

$$r = 0.05, \quad z_\beta = -1.64.$$

$$\sigma = \begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.2 \end{pmatrix},$$

$$\left\|\sigma^{-1}(b - r\mathbf{1})\right\| + z_\beta = \left\| \begin{pmatrix} 0.1 & 0 & 0 \\ 0 & 0.3 & 0 \\ 0 & 0 & 0.2 \end{pmatrix}^{-1} \begin{pmatrix} 0.1 - 0.05 \\ 0.2 - 0.05 \\ 0.3 - 0.05 \end{pmatrix} \right\| - 1.64 = -0.20386.$$

Therefore $C < x(1 + r)$.

$$x = 1000, \quad C = 1030.$$

Now we calculate

$$\begin{aligned}\lambda &= \frac{\frac{C}{x}-1-r}{\|\sigma^{-1}(b-r\mathbf{1})\|+z_\beta} = \frac{\frac{1030}{1000}-1-0.05}{\left\|\begin{pmatrix}0.1 & 0 & 0\\ 0 & 0.3 & 0\\ 0 & 0 & 0.2\end{pmatrix}^{-1}\begin{pmatrix}0.1-0.05\\ 0.2-0.05\\ 0.3-0.05\end{pmatrix}\right\| - 1.64}\\ &= \frac{-0.02}{-0.20386} = 0.098107.\end{aligned}$$

Now we calculate the investment strategy, because we want to know the weights of the assets:

$$\begin{aligned}\varphi^* &= \lambda\frac{(\sigma\sigma^{'})^{-1}(b-r\mathbf{1})}{\|\sigma^{-1}(b-r\mathbf{1})\|}\\ &= 0.098107\cdot\frac{\left(\begin{pmatrix}0.1 & 0 & 0\\ 0 & 0.3 & 0\\ 0 & 0 & 0.2\end{pmatrix}\begin{pmatrix}0.1 & 0 & 0\\ 0 & 0.3 & 0\\ 0 & 0 & 0.2\end{pmatrix}^{'}\right)^{-1}\begin{pmatrix}0.1-0.05\\ 0.2-0.05\\ 0.3-0.05\end{pmatrix}}{\left\|\begin{pmatrix}0.1 & 0 & 0\\ 0 & 0.3 & 0\\ 0 & 0 & 0.2\end{pmatrix}^{-1}\begin{pmatrix}0.1-0.05\\ 0.2-0.05\\ 0.3-0.05\end{pmatrix}\right\|}\\ &= \begin{pmatrix}0.34156\\ 0.11385\\ 0.42695\end{pmatrix}.\end{aligned}$$

This means 34.156% are invested in asset 1, 11.385% are invested in asset 2 and 42.695% are invested in asset 3.

Now we calculate the expected return of this strategy. To do so, we calculate

$$\mathbb{E}(X^{\varphi}(1)) = x\Big(1+\varphi^{'}(b-r\mathbf{1})+r\Big)$$

$$= \quad 1000 \cdot \left(1 + (0.34156, 0.11385, 0.42695) \begin{pmatrix} 0.1 - 0.05 \\ 0.2 - 0.05 \\ 0.3 - 0.05 \end{pmatrix} + 0.05 \right) = 1190.9.$$

As a next step we check if the VaR of this strategy really is 1030.

$$\begin{aligned} \mathrm{VaR}(\varphi^*) &= (z_\beta \|c\| + 1 + \alpha)x \\ &= 1000 \cdot \left(-1.64 \cdot 0.098107 + 1 + (0.34156, 0.11385, 0.42695) \begin{pmatrix} 0.1 - 0.05 \\ 0.2 - 0.05 \\ 0.3 - 0.05 \end{pmatrix} + 0.05 \right) \\ &= 1030. \end{aligned}$$

6.7. DISCRETE TIME MULTI-PERIOD MEAN-VARIANCE PROBLEM

In this section we introduce the multi-period mean-variance problem and provide a closed form solution. Like in Markowitz' classical mean-variance problem we want to minimize the variance under a given level of expected return. The difference is only that we solve it as a discrete time multi-period model. We solve the optimization problem

$$\begin{cases} \min\limits_{\varphi \in \mathbb{R}^n} \ \mathbb{V}\mathrm{ar}(X^\varphi(T)) \\ \text{s.t. } \mathbb{E}(X^\varphi(T)) \geq C, \end{cases} \tag{76}$$

where C is the expected terminal wealth at time T. We assume that the expected return of the investor is greater than the return of the risk-free asset. If we expect only the return of the risk free asset, we should get a variance of zero because we invest everything risk free. In the solution we have to check if we get a variance of zero for the pure bond strategy. Formally our assumption is

$$C \geq x(1 + r)^T.$$

If $C < x(1 + r)^T$, then we invest everything risk free and get a variance of zero.

Theorem 6.10 (Closed form solution of the discrete time multi-period mean-variance optimization problem)**.** *The closed-form solution of the multi-period mean-variance problem* (76) *is given by*

$$\varphi^* = \frac{\lambda(\sigma\sigma')^{-1}(b - r\mathbf{1})}{\|\sigma^{-1}(b - r\mathbf{1})\|},$$

where λ *is given as*

$$\lambda = \frac{\sqrt[T]{\frac{C}{x}} - 1 - r}{\Theta}.$$

The expected wealth after T *periods is then* C *with variance*

$$\mathbb{V}\text{ar}(X^\varphi(T)) = x^2\left(\left[\left(\frac{C}{x}\right)^{\frac{2}{T}} + \lambda^2\right]^T - \left(\frac{C}{x}\right)^2\right). \tag{77}$$

Proof. Using (14) and (15) for $t = T$, we rewrite problem (76) as

$$\begin{cases} \min\limits_{\varphi\in\mathbb{R}^n} x^2\left[\left((1+\alpha)^2 + c'c\right)^T - (1+\alpha)^{2T}\right] \\ \qquad\qquad \text{s.t. } x(1+\alpha)^T \geq C. \end{cases} \tag{78}$$

which is the same, when c and α are substituted, as

$$\min_{\varphi\in A} g(\varphi),$$

where

$$g(\varphi) = x^2\left[\left((1+\varphi'(b - r\mathbf{1}) + r)^2 + \varphi'\sigma\sigma'\varphi\right)^T - (1+\varphi'(b - r\mathbf{1}) + r)^{2T}\right],$$

and where the feasible set of problem (78) is given by

$$A = \left\{\varphi \in \mathbb{R}^n : \ \varphi'(b - r\mathbf{1}) \geq \sqrt[T]{\frac{C}{x}} - 1 - r\right\}.$$

To prove the theorem we show that

1. $g(\varphi^*) = x^2\left[\left(\left(\frac{C}{x}\right)^{\frac{2}{T}} + \lambda^2\right)^T - \left(\frac{C}{x}\right)^2\right]$,

2. $g(\varphi) \geq g(\varphi^*)$ for all $\varphi \in A$,

3. $\varphi^* \in A$.

As a first step we show that

$$\begin{aligned}
g(\varphi^*) &= x^2\left[\left((1+(\varphi^*)'(b-r\mathbf{1})+r)^2+(\varphi^*)'\sigma\sigma'\varphi^*\right)^T-(1+(\varphi^*)'(b-r\mathbf{1})+r)^{2T}\right]\\
&\overset{(26),(27)}{=} x^2\left[\left((1+\lambda\Theta+r)^2+\lambda^2\right)^T-(1+\lambda\Theta+r)^{2T}\right]\\
&= x^2\left[\left(\left(1+\sqrt[T]{\frac{C}{x}}-1-r+r\right)^2+\lambda^2\right)^T-\left(1+\sqrt[T]{\frac{C}{x}}-1-r+r\right)^{2T}\right]\\
&= x^2\left[\left(\left(\sqrt[T]{\frac{C}{x}}\right)^2+\lambda^2\right)^T-\left(\sqrt[T]{\frac{C}{x}}\right)^{2T}\right]\\
&= x^2\left[\left(\left(\frac{C}{x}\right)^{\frac{2}{T}}+\lambda^2\right)^T-\left(\frac{C}{x}\right)^2\right].
\end{aligned}$$

Next, let $\varphi \in A$. We show that $g(\varphi) \geq g(\varphi^*)$. To do so, we calculate

$$\left\|\sigma'\varphi\right\|^2 \overset{(28)}{\geq} \frac{|\varphi'(b-r\mathbf{1})|^2}{\Theta^2} \geq \left|\frac{\varphi'(b-r\mathbf{1})}{\Theta}\right|^2 \geq \lambda^2, \tag{79}$$

and thus

$$\begin{aligned}
g(\varphi) &= x^2\left[\left((1+\varphi'(b-r\mathbf{1})+r)^2+\varphi'\sigma\sigma'\varphi\right)^T-(1+\varphi'(b-r\mathbf{1})+r)^{2T}\right]\\
&\overset{(79)}{\geq} x^2\left[\left((1+\varphi'(b-r\mathbf{1})+r)^2+\lambda^2\right)^T-(1+\varphi'(b-r\mathbf{1})+r)^{2T}\right]\\
&\overset{\varphi\in A}{\geq} x^2\left[\left(\left(1+\sqrt[T]{\frac{C}{x}}-1-r+r\right)^2+\lambda^2\right)^T-\left(1+\sqrt[T]{\frac{C}{x}}-1-r+r\right)^{2T}\right]\\
&= x^2\left[\left(\left(\sqrt[T]{\frac{C}{x}}\right)^2+\lambda^2\right)^T-\left(\sqrt[T]{\frac{C}{x}}\right)^{2T}\right]
\end{aligned}$$

$$= \quad x^2 \left[\left(\left(\frac{C}{x} \right)^{\frac{2}{T}} + \lambda^2 \right)^T - \left(\frac{C}{x} \right)^2 \right] = g(\varphi^*),$$

where in the second inequality sign we have used the subsequent Lemma 6.8. As a last step we show that φ^* is in the feasible set. To do so, we calculate

$$\begin{aligned} x\left(1 + (\varphi^*)'(b - r\mathbf{1}) + r\right)^T &\overset{(26)}{=} x(1 + \lambda\Theta + r)^T \\ &= x\left(1 + \frac{\sqrt[T]{\frac{C}{x}} - 1 - r}{\Theta}\Theta + r\right)^T = x\frac{C}{x} = C. \end{aligned}$$

Hence $\varphi^* \in A$. This completes the proof. □

Lemma 6.8. *Let* $c_1, c_2 > 0$ *and* $T \in \mathbb{N}$ *and define* $f : [0, \infty) \mapsto \mathbb{R}$ *by*

$$f(x) = \left((c_1 + x)^2 + c_2\right)^T - (c_1 + x)^{2T}.$$

Then f *is increasing.*

Proof. We let $x \geq 0$ and calculate

$$\begin{aligned} f'(x) &= T\left((c_1 + x)^2 + c_2\right)^{T-1} 2(c_1 + x) - 2T(c_1 + x)^{2T-1} \\ &= 2T(c_1 + x)\left[\left((c_1 + x)^2 + c_2\right)^{T-1} - (c_1 + x)^{2T-2}\right] \\ &\geq 2T(c_1 + x)\left[((c_1 + x)^2)^{T-1} - (c_1 + x)^{2T-2}\right] \\ &= 0, \end{aligned}$$

which completes the proof. □

As an immediate consequence of Theorem 6.10 we get that the mean-variance is a function of the expected terminal wealth. An investor is now able to plot a graph for different expected terminal wealths.

Let us denote with $\omega := \mathbb{E}(X^{\varphi}(T))$ the expected wealth after T-periods. Now we can plug it into the result of Theorem 6.10 to get

$$\mathbb{V}\mathrm{ar}(\omega) = x^2 \left(\left[\left(\frac{\omega}{x}\right)^{\frac{2}{T}} + \left(\frac{\sqrt[T]{\frac{\omega}{x}} - 1 - r}{\Theta} \right)^2 \right]^T - \left(\frac{\omega}{x}\right)^2 \right).$$

If we know our desirable expected terminal wealth, then we can calculate λ and the portfolio construction strategy.

Another way is that we accept a certain amount as variance and then we calculate ω and set this equal to C. Then we are able to calculate the optimal portfolio.

6.8. EXAMPLE FOR THE DISCRETE MULTI-PERIOD MEAN-VARIANCE PROBLEM

In this subsection we calculate an example for the one-period mean-$\mathbb{V}$ar problem and calculate the corresponding portfolio strategy and check the results of the optimization. Let

$$b = (0.1, 0.2, 0.3)',$$

$$r = 0.05,$$

$$T = 2,$$

$$\sigma = \begin{pmatrix} 0.2 & 0.01 & 0.03 \\ 0.1 & 0.3 & 0.04 \\ 0.05 & 0.03 & 0.1 \end{pmatrix},$$

$$x = 1000, \quad C = 1110.$$

Now we calculate

$$\lambda = \frac{\sqrt[T]{\frac{C}{x}} - 1 - r}{\|\sigma^{-1}(b - r\mathbf{1})\|} = \frac{\sqrt[T]{\frac{1110}{1000}} - 1 - 0.05}{\left\| \begin{pmatrix} 0.2 & 0.01 & 0.03 \\ 0.1 & 0.3 & 0.04 \\ 0.05 & 0.03 & 0.1 \end{pmatrix}^{-1} \begin{pmatrix} 0.1 - 0.05 \\ 0.2 - 0.05 \\ 0.3 - 0.05 \end{pmatrix} \right\|}$$

$$= \frac{0.0035654}{2.5173} = 0.0014163.$$

With that λ we calculate the variance of our portfolio with an expected terminal wealth of C

$$\mathbb{V}\text{ar}(\varphi^*) = x^2 \left(\left[\left(\frac{C}{x}\right)^{\frac{2}{T}} + \lambda^2 \right]^T - \left(\frac{C}{x}\right)^2 \right)$$

$$= 1000^2 \left(\left[\left(\frac{1110}{1000}\right)^{\frac{2}{2}} + (0.0014163)^2 \right]^2 - \left(\frac{1110}{1000}\right)^2 \right) = 4.4534.$$

That is the minimal variance for the portfolio with an expected terminal wealth of 1110 at time 2. Now we calculate the investment strategy, because we want to know the weights of the assets:

$$\varphi^* = \lambda \frac{(\sigma\sigma')^{-1}(b - r\mathbf{1})}{\|\sigma^{-1}(b - r\mathbf{1})\|}$$

$$= 0.0014 \cdot \frac{\left(\begin{pmatrix} 0.2 & 0.01 & 0.03 \\ 0.1 & 0.3 & 0.04 \\ 0.05 & 0.03 & 0.1 \end{pmatrix} \begin{pmatrix} 0.2 & 0.01 & 0.03 \\ 0.1 & 0.3 & 0.04 \\ 0.05 & 0.03 & 0.1 \end{pmatrix}' \right)^{-1} \begin{pmatrix} 0.1 - 0.05 \\ 0.2 - 0.05 \\ 0.3 - 0.05 \end{pmatrix}}{\left\| \begin{pmatrix} 0.2 & 0.01 & 0.03 \\ 0.1 & 0.3 & 0.04 \\ 0.05 & 0.03 & 0.1 \end{pmatrix}^{-1} \begin{pmatrix} 0.1 - 0.05 \\ 0.2 - 0.05 \\ 0.3 - 0.05 \end{pmatrix} \right\|}$$

$$= \begin{pmatrix} -0.0038 \\ -0.0010 \\ 0.0156 \end{pmatrix}.$$

This means we invest 1.56% of our initial wealth in asset 3. The rest is invested risk free.

Now we check if the expected wealth at time 2 really is 1110. To do so, we calculate

$$\begin{aligned}\mathbb{E}(X^{\varphi}(2)) &= x(1+\varphi^{'}(b-r\mathbf{1})+r)^2 \\ &= 1000\cdot\left(1+(-0.0038,-0.0010,0.0156)\begin{pmatrix}0.1-0.05\\0.2-0.05\\0.3-0.05\end{pmatrix}+0.05\right)^2 = 1110.\end{aligned}$$

The discrete time multi-period efficient frontier is provided in Figure 6.3

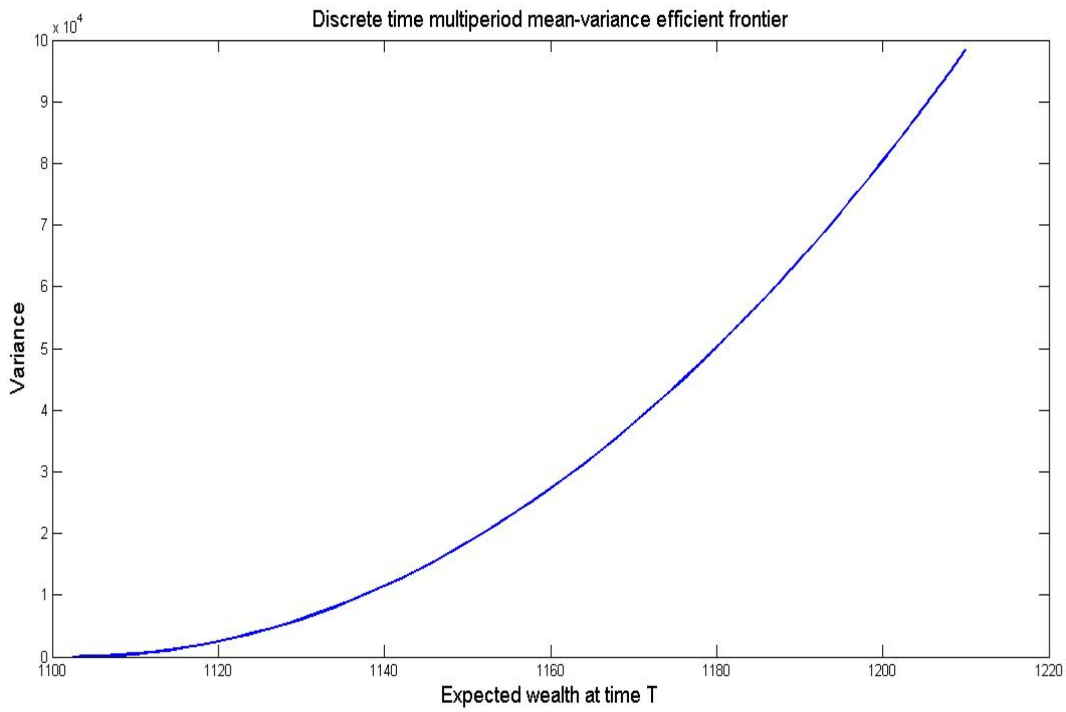

Figure 6.3. Discrete time multi-period mean-variance efficient frontier

In Figure 6.4 we provide the Matlab implementation for the closed form solution of the discrete time multi-period mean-variance optimization problem. With that implementation we are able to easily calculate efficient portfolios.

```
%Solution of the discrete time multi-period mean-variance problem

function[epsstar,Var,Policy]=discretemv(C,w,r,b,sigma,T)

j=length(C);
k=length(b);
epsstar=zeros(j,1);
Var=zeros(j,1);
Policy=zeros(j,k);

for i=1:j
epsstar(i)=(((C(i)/w))^(1/T)-r-1)/norm(inv(sigma)*(b-r));

Var(i)=w^2*((((C(i)/w)^(1/T))^2+epsstar(i)^2)^T-(C(i)/w)^2);

Policy(i,:)= epsstar(i)*(inv((sigma*sigma'))*(b-r))/norm(inv(sigma)*(b-r));
end
```

Figure 6.4. Matlab implementation for the discrete time multi-period mean-variance problem

7. CONCLUSIONS

In this thesis we introduced a discrete time financial market model in order to describe real life more accurate. In the continuous time models we assume that we trade continuously over a given time period. In practice we do not trade continuously. For example at weekends we have no trading. Also if we trade in very small time intervals it is still a discrete time model. Therefore we introduced a discrete time market model. A possible extension for the discrete time market model would be to allow jumps in time. As we saw in Section 4, the discrete time financial market model is very good for simulation purposes. We need only to simulate standard normal distributed random variables and calculate the product from Lemma 4.2. We also introduced the risk measure Earning-at-Risk, Capital-at-Risk and Value-at-Risk for the discrete time financial market. For the risk measures we restricted ourself to one period. Otherwise we would have to deal with a product of normally distributed random variables, which follows an unnamed distribution. A possible extension for our model would be to extend it to a multi-period model and calculate the quantiles of the distribution numerically.

For the one-period risk measures we found closed-form expressions to calculate the optimal portfolios. With our method to prove the theorems we could simplify the continuous time proofs from [6, 7]. For the multi-period mean-variance problem we found a closed-form solution to find optimal portfolios. With the portfolio weights we can easily calculate the minimal variance for the portfolio.

An interesting insight of this thesis is that the optimal portfolio does not depend on the number of assets. Like in the continuous time models it only depends on $\|\sigma^{-1}(b - r\mathbf{1})\|$, which can be seen as kind of a mutual fund theorem.

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