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Pricing of geometric Asian options in general affine stochastic volatility models

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PRICING OF GEOMETRIC ASIAN OPTIONS IN GENERAL AFFINE STOCHASTIC VOLATILITY MODELS

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ABSTRACT

In this thesis, we look at the general affine pricing model introduced in [11]. This model allows to price geometric Asian options, which are of big interest due to their lower volatility in comparison to, for example, European options. Because of their structure and in order to be able to price these options, we look at the basic theory of Lévy processes and stochastic calculus. Furthermore, we provide the detailed description of the parameters of the pricing formulas for some popular specific single-factor stochastic volatility models with jumps and generalize the approach of [11] to multi-factor models.

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INTRODUCTION

In this thesis, we look at [11] introduced by Hubalek, Keller-Ressel and Sgarra. In their paper, they present a general model for the prices of geometric Asian options in affine stochastic volatility models with jumps. There is a growing interest in Asian options because in difference to European options, they take the average price instead of the price at some point in time into account, which as a result reduces the volatility of these options, i.e., the risk. Furthermore, affine models are quite easy to handle but can still approximate the real world very well, which makes [11] an interesting paper. For a more general overview regarding affine models, we refer to [8] and [9].

This thesis is structured as follows. In Section 1, we discuss the basic theory of Lévy processes and noncontinuous stochastic calculus. In the second section, we turn to the pricing model given in [11] and provide the proofs for the statements that are made there in order to get a further understanding of the concepts. At the end of that section, we state and prove formulas for the prices of the geometric average price and the geometric average strike Asian option, for which a system of differential equations involving the so-called functional characteristics has to be solved. In Section 3, we look at all specific stochastic volatility models with jumps that were discussed in [11] and present further models and again provide all the proofs in detail. In the fourth section, we generalize the approach as given in [11] from single-factor models to multi-factor models.

1 PRELIMINARIES

This section introduces some basic concepts regarding noncontinuous stochastic calculus. We focus especially on Lévy processes, which we will also introduce. To come up with the main result regarding Lévy processes, i.e., the Lévy–Itô decomposition, we first look at infinitely divisible distributions. Here we closely follow [2].

1.1 INFINITE DIVISIBILITY

Definition 1.1.1. A random vector $X : \Omega \mapsto \mathbb{R}^d$ is called infinitely divisible if for every $n \in \mathbb{N}$, there are iid random vectors $Y_1, \dots, Y_n$ such that

$$X \overset{d}{=} Y_1 + \ldots + Y_n.$$

Remark 1.1.2. It is well known that X is infinitely divisible if and only if for its characteristic function ϕ_X and for every $n \in \mathbb{N}$, there is a characteristic function ϕ_Y such that

$$\phi_X = (\phi_Y)^n .$$

Some examples of infinitely divisible distributions are the normal distribution as well as the (compound) Poisson distribution.

For the next result, we need one further definition.

Definition 1.1.3. Let ν be a Borel measure on $\mathbb{R}^d \backslash \{0\}$. Then ν is called a Lévy measure provided

$$\int_{\mathbb{R}^d \backslash \{0\}} (|y|^2 \wedge 1)\nu(dy) < \infty.$$

Theorem 1.1.4 (Lévy–Khintchine formula). Let μ be a d-dimensional Borel measure. Then μ is infinitely divisible if and only if there are $b \in \mathbb{R}^d$, $A \in \mathbb{R}^{d\times d}$ symmetric positive definite and a Lévy measure ν on $\mathbb{R}^d \backslash \{0\}$ such that for every $u \in \mathbb{R}^d$

$$\phi_\mu(u) = \exp\left\{ i(b,u) - \frac{1}{2}(u, Au) + \int_{\mathbb{R}^d \backslash \{0\}} \left(e^{i(u,y)} - 1 - i(u,y)\chi_{|y|<1} \right) \nu(dy) \right\}.$$

This result is amazing. It means that we can uniquely represent every characteristic function of an infinitely divisible distribution by this expression.

In the next subsection, we will see that we have a similar result for Lévy processes.

1.2 LÉVY PROCESSES

We begin with the definition of a Lévy process.

Definition 1.2.1. A stochastic process X is called a Lévy process if

(L1) $X(0) = 0$,

(L2) X has independent and stationary increments,

(L3) X is stochastically continuous, i.e.,

$$\lim_{t\to s} \mathbb{P}[|X_t - X_s| > \varepsilon] = 0 \text{ for all } \varepsilon > 0.$$

Observe that $(L3)$ due to $(L1)$ and $(L2)$ is equivalent to

$$\lim_{t\searrow 0}\mathbb{P}[|X_t|>\varepsilon]=0 \text{ for all } \varepsilon>0.$$

Lemma 1.2.2. Every Lévy process is infinitely divisible.

Proof. Let $(X_t)_{t\geq 0}$ be a Lévy process and let $t>0$. Then, due Definition 1.2.1, we have

$$X_t \overset{d}{=} \sum_{k=1}^{n} X_{t\frac{k}{n}} - X_{t\frac{k-1}{n}},$$

where the terms are all independent and identically distributed. Hence, denoting $Y_k := X_{t\frac{k}{n}} - X_{t\frac{k-1}{n}}$, we have shown the claim. □

With this knowledge, we note that we can use the results obtained in Subsection 1.1 and hence get the Lévy–Khintchine formula for Lévy processes.

Theorem 1.2.3 (Lévy–Khintchine for Lévy processes). Let $(X_t)_{t\geq 0}$ be a Lévy process. Then

$$\mathbb{E}[e^{i(u,X_t)}]=\exp\left\{t\left[i(b,u)-\frac{1}{2}(u,Au)+\int_{\mathbb{R}^d\setminus\{0\}}\left(e^{i(u,y)}-1-i(u,y)\chi_{|y|<1}\right)\nu(dy)\right]\right\},$$

where (b,A,ν) are called the characteristics of X_1.

Remark 1.2.4. Observe that, due to the uniqueness of the characteristic function, the distribution of a Lévy process is uniquely determined by the distribution of the process at one point in time. For simplicity, it is convenient to choose X_1.

1.3 POISSON INTEGRATION

We introduce the Poisson integration in several steps. First of all, we count the jumps of a Lévy process X until time $t \geq 0$.

$$N(t, A) := \# \{0 \leq s \leq t : \Delta X_s(\omega) \in A\} = \sum_{0 \leq s \leq t} \chi_A \left(\Delta X_s(\omega)\right),$$

where ΔX is the jump process, i.e., $\Delta X_t = X_t - X_{t-}$. Further, we have

$$\mathbb{E}[N(t, A)] = \int N(t, A) d\mathbb{P}(\omega)$$

and we call $\mu(\cdot) = \mathbb{E}[N(1, \cdot)]$ the intensity measure.

Theorem 1.3.1. If A is bounded below, i.e., $0 \notin \bar{A}$, then $N(t, A) < \infty$ a.s. for all $t \geq 0$. Furthermore $\mu(A) < \infty$ and $(N(t, A))_{t \geq 0}$ is a Poisson process with intensity $\mu(A)$.

Before we get to the Poisson integration, we introduce the following.

Definition 1.3.2. Let A be bounded below and $t \geq 0$. Then we define the compensated Poisson random measure $\widetilde{N}(t, A)$ by

$$\widetilde{N}(t, A) = N(t, A) - t\mu(A).$$

Observe that $\widetilde{N}(t, A)$ is a martingale valued measure, i.e., for all fixed A bounded below, $(\widetilde{N}(t, A))_{t \geq 0}$ is a martingale.

Now, we can finally define the Poisson integral.

Definition 1.3.3. Let N be a Poisson random measure associated to some Lévy process X. Let further $f : \mathbb{R}^d \mapsto \mathbb{R}^d$ be Borel measurable and A be bounded below. Then for all $t > 0$ and $\omega \in \Omega$, the Poisson integral of f is defined to be

$$\int_A f(x)N(dt,dx)(\omega) = \sum_{x\in A} f(x)N(t,\{x\})(\omega) = \sum_{0\le u\le t} f(\Delta X_u)\chi_A(\Delta X_u).$$

Theorem 1.3.4. Let A be bounded below. Then for every $t \geq 0$, $\int_A f(x)N(t,dx)$ has a compound Poisson distribution, i.e.,

$$\mathbb{E}\left[\exp\left\{i\left(u, \int_A f(x)N(t,dx)\right)\right\}\right] = \exp\left\{t\int_{\mathbb{R}^d}\left(e^{i(u,x)}-1\right)\mu_{f,A}(dx)\right\}.$$

Furthermore, we can define the following.

Definition 1.3.5. Let $f \in L^1(\Omega, A, \mu_A)$ and $t \geq 0$. Then the compensated Poisson integral is given by

$$\int_A f(x)\widetilde{N}(t,dx) = \int_A f(x)N(t,dx) - t\int_A f(x)\mu(dx).$$

Similarly to Theorem 1.3.4, we have

$$\mathbb{E}\left[\exp\left\{i\left(u, \int_A f(x)\widetilde{N}(t,dx)\right)\right\}\right] = \exp\left\{t\int_{\mathbb{R}^d}\left(e^{i(u,x)}-1-i(u,x)\right)\mu_{f,A}(dx)\right\}.$$

1.4 THE LÉVY–ITÔ DECOMPOSITION

Now we are ready to state the following fantastic result.

Theorem 1.4.1 (Lévy–Itô decomposition). Let X be a Lévy process. Then, there exist $b \in \mathbb{R}^d$, a Brownian motion B_A with covariance matrix A and an independent Poisson random measure N on $(\mathbb{R}^+ \times \mathbb{R}^d\backslash\{0\})$ such that for all $t \geq 0$

$$X_t = bt + B_A(t) + \int_{|x|<1} x\widetilde{N}(t,dx) + \int_{|x|\ge 1} xN(t,dx).$$

This means we can write any Lévy process as the independent combination of a Brownian motion with some drift b and covariance matrix A and some Poisson random measure N, i.e., each Lévy process can be split up in its continuous and its jump part. Furthermore, the jump and the continuous part are independent of each other and the decomposition is unique. This makes Lévy processes a very good instrument to, for example, improve pure jump or pure continuous models.

1.5 MARKOV PROCESSES

A wider space of stochastic processes are the so called Markov processes, which are characterized through the following.

Definition 1.5.1. Let $(\mathcal{F}_t)_{t\geq 0}$ be some filtration. An adapted process is a Markov process if for every $f \in \mathcal{B}_b(\mathbb{R}^d)$, i.e., all functions that are bounded and Borel measurable on $\mathbb{R}^d$, and $0 \leq s \leq t < \infty$

$$\mathbb{E}[f(X_t)|\mathcal{F}_s] = \mathbb{E}[f(X_t)|X_s] \text{ a.s.},$$

i.e., the process only depends on the present, not on the past. Define further the operator $T_{s,t}$ by

$$\left(T_{s,t}f\right)(x) = \mathbb{E}[f(X_t)|X_s = x].$$

Definition 1.5.2. A Markov process is called normal if

$$T_{s,t}\left(\mathcal{B}_b(\mathbb{R}^d)\right) \subset \mathcal{B}_b(\mathbb{R}^d) \text{ for all } s,t.$$

Theorem 1.5.3. Let X be a normal Markov process. Then

(i) $T_{s,t}$ is linear,

(ii) $T_{s,s} = 1$,

(iii) $T_{r,s}T_{s,t} = T_{r,t}$.

Definition 1.5.4. A Markov process is (time-)homogeneous if $T_{s,t} = T_{0,t-s}$.

Remark 1.5.5. A family of linear operators on a Banach space that fulfills $T_{s+t} = T_s T_t$ is called a semigroup.

Definition 1.5.6. A homogeneous Markov process is a Feller process if

(i) $T_t(C_0(\mathbb{R}^d)) \subset C_0(\mathbb{R}^d)$ for all $t \geq 0$

(ii) $\lim_{t\to 0} \|T_t f - f\|_\infty = 0$ for all $f \in C_0(\mathbb{R}^d)$.

Theorem 1.5.7. Every Lévy process is a Feller process.

Definition 1.5.8. Let $(T_t)_{t\geq 0}$ be an arbitrary semigroup on some Banach space B. Define

$$D_A = \left\{ \psi \in B : \text{ There exists } \phi_\psi \in B \text{ s.t. } \lim_{t\searrow 0} \left\| \frac{T_t\psi - \psi}{t} - \phi_\psi \right\| = 0 \right\}$$

and let $A\psi = \phi_\psi$. Then, $A\psi = \lim_{t\searrow 0} \frac{T_t\psi-\psi}{t}$ and A is called the (infinitesimal) generator of $(T_t)_{t\geq 0}$.

Remark 1.5.9. If $(T_t)_{t\geq 0}$ is a Feller semigroup associated to a Feller process X, then A is called the generator of X.

Theorem 1.5.10. We have

$$T_t A\psi = AT_t\psi.$$

1.6 STOCHASTIC INTEGRATION FOR LÉVY PROCESSES

Now we go back to Lévy processes and state Itô's formula for general Lévy processes. We do this in three steps. First we recall the Itô formula for the continuous case and then we go on to the Poisson case and finally end up with Itô's lemma for general Lévy processes. To do this, we have to introduce the following.

Definition 1.6.1. (i) The quadratic variation between two Brownian integrals $X_t = \int_0^t F_s^1 dB_s + \int_0^t G_s^1 ds$ and $Y_t = \int_0^t F_s^2 dB_s + \int_0^t G_s^2 ds$ is given by

$$[X,Y]_t = \int_0^t F_s^1 F_s^2 ds,$$

i.e., $d[B,B]_t = dt$ where B is a standard Brownian motion.

(ii) Let now X, Y be Lévy processes, i.e.,

$$X_t = X_0 + \int_0^t F_s^1 dB_s + \int_0^t G_s^1 ds + \int_0^t \int_A K_s^1(x) N(ds,dx)$$

and Y accordingly. Then we have

$$[X,Y]_t = \int_0^t F_s^1 F_s^2 ds + \sum_{0 \le s \le t} \Delta X_s \Delta Y_s,$$

i.e., the discontinuous Poisson part is added to the quadratic variation. Furthermore, we have that $d[N,N]_t = dN_t$.

With this definition, we can now give the first version of Itô's lemma.

Theorem 1.6.2 (Itô's lemma for continuous stochastic integrals). Let $f \in C^2(\mathbb{R}^d)$. Let further $X_t = \int_0^t F_s dB_s + \int_0^t G_s ds$. Then

$$f(X_t) - f(X_0) = \sum_{i=1}^d \int_0^t \frac{\partial}{\partial x_i} f(X_s) dX_s^i + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \frac{\partial^2}{\partial x_i \partial x_j} f(X_s) d[X^i, X^j]_s.$$

For pure Poisson stochastic integrals, i.e., $X_t = X_0 + \int_0^t \int_A K_s(x)N(ds, dx)$, we have the following.

Theorem 1.6.3 (Itô's lemma for Poisson stochastic integrals). Let X be a Poisson stochastic integral and let $f \in C(\mathbb{R}^d)$. Then

$$f(X_t) - f(X_0) = \int_0^t \int_A \left[f(X_{s-} + K_s(x)) - f(X_{s-}) \right] N(ds, dx).$$

For simple Lévy processes with A bounded below, i.e.,

$$X_t = X_0 + \int_0^t G_s ds + \int_0^t F_s dB_s + \int_0^t \int_A K_s(x)N(ds, dx), \tag{1.1}$$

we have the following:

Lemma 1.6.4 (Itô's lemma). Let X be as in (1.1) and let $f \in C^2(\mathbb{R}^d)$. Then

$$\begin{aligned} f(X_t) - f(X_0) = & \sum_{i=1}^d \int_0^t \frac{\partial}{\partial x_i} f(X_{s-}) dX_c^i(s) + \frac{1}{2} \sum_{i,j=1}^d \int_0^t \frac{\partial^2}{\partial x_i \partial x_j} f(X_{s-}) d[X_c^i, X_c^j]_s \\ & + \int_0^t \int_A \left[f(X_{s-} + K_s(x)) - f(X_{s-}) \right] N(ds, dx), \end{aligned}$$

where $X_c(s)$ denotes the continuous part at time s, i.e., $dX_c(s) = G_s ds + F_s dB_s$.

By the Lévy–Itô decomposition (Theorem 1.4.1), we know that we can display any Lévy process as

$$dX_t = G_t dt + F_t dB_t + \int_{|x|<1} H_t(x) \widetilde{N}(dt, dx) + \int_{|x|\geq 1} K_t(x) N(dt, dx), \tag{1.2}$$

such that we now can state the most general version of Itô's lemma for Lévy processes.

Theorem 1.6.5 (Itô's formula for general Lévy processes). Let X be a Lévy process as in (1.2) and let $f \in C^2(R^d)$. Then

$$f(X_t) - f(X_0) = \sum_{i=1}^{d} \int_0^t \frac{\partial}{\partial x_i} f(X_{s-}) dX_c^i(s) + \frac{1}{2} \sum_{i,j=1}^{d} \int_0^t \frac{\partial^2}{\partial x_i \partial x_j} f(X_{s-}) d[X_c^i, X_c^j]_s$$
$$+ \int_0^t \int_{|x|\geq 1} \left[f(X_{s-} + K_s(x)) - f(X_{s-}) \right] N(ds, dx)$$
$$+ \int_0^t \int_{|x|<1} \left[f(X_{s-} + H_s(x)) - f(X_{s-}) \right] \widetilde{N}(ds, dx)$$
$$+ \int_0^t \int_{|x|<1} \left[f(X_{s-} + H_s(x)) - f(X_{s-}) - \sum_{i=1}^{d} H_s^i(x) \frac{\partial}{\partial x_i} f(X_{s-}) \right] \nu(dx, ds).$$

In [2], it is furthermore shown that this result can be simplified to the following.

Theorem 1.6.6 (Itô's formula—2nd version). Let X be a Lévy-type stochastic integral and let $f \in C^2(\mathbb{R}^d)$. Then

$$f(X_t) - f(X_0) = \sum_{i=1}^{d} \int_0^t \frac{\partial}{\partial x_i} f(X_{s-}) dX_s^i + \frac{1}{2} \sum_{i,j=1}^{d} \int_0^t \frac{\partial^2}{\partial x_i \partial x_j} f(X_{s-}) d[X_c^i, X_c^j]_s$$
$$+ \sum_{0 \leq s \leq t} \left[f(X_s) - f(X_{s-}) - \sum_{i=1}^{d} \Delta X_s^i \frac{\partial}{\partial x_i} f(X_{s-}) \right].$$

Sometimes it is more convenient to work with the first version, and that is why we have mentioned both here.

2 THE MODEL

In this section, we deal with the model introduced in [11]. In [11], there are mainly statements without proofs. We go over most of these statements and give proofs in order to better understand the model and to confirm the results.

2.1 MODEL SETUP

Fix $T > 0$ and let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$ be a stochastic basis satisfying the usual hypotheses and such that all processes, that we introduce, exist.

Definition 2.1.1. A stochastically continuous time-homogeneous Markov process $(X_t, \mathbb{P}^x)_{t \geq 0, x \in D}$ with state space $D = \mathbb{R}_+^m \times \mathbb{R}^n$ is called an affine process if there are functions $\phi : \mathbb{R}_+ \times \mathcal{U} \mapsto \mathbb{C}_-$ and $\psi : \mathbb{R}_+ \times \mathcal{U} \mapsto \mathcal{U}$ such that

$$\log \mathbb{E}\left[e^{uX_t} | X_0\right] = \phi(t, u) + X_0 \psi(t, u) \text{ for all } (t, u) \in \mathbb{R}_+ \times \mathcal{U},$$

where

$$\mathbb{C}_- := \{u \in \mathbb{C} : \Re(u) < 0\}, \ \mathcal{U} := \mathbb{C}_-^m \times i\mathbb{R}^n.$$

The logarithm above denotes the distinguished logarithm in the complex plane such that ϕ and ψ are jointly continuous.

Since $(X_t, \mathbb{P}^x)$ is a time-homogeneous Markov process, Definition 2.1.1 can be extended to general $0 \leq s \leq t$, i.e., we have the following.

Lemma 2.1.2.

$$\log \mathbb{E}\left[e^{uX_t} | X_s\right] = \phi(t - s, u) + X_s \psi(t - s, u) \text{ for all } 0 \leq s \leq t.$$

Proof. Let $0 \leq s \leq t$ and let $f_u(x) = e^{ux}$ for $u \in \mathcal{U}$. Observe that $f_u \in B_b(D)$. Denote by $T_{s,t} = \mathbb{E}[f_u(X_t)|X_s]$ the associated operator of the Markov process $X = (X_t, \mathbb{P}^x)$. Since X is time-homogeneous, we have $T_{s,t} = T_{0,t-s}$, and with $x \in D$, we get

$$\begin{aligned}\mathbb{E}[f_u(X_t)|X_s = x] = T_{s,t}(x) = T_{0,t-s}(x) = \mathbb{E}[f_u(X_{t-s})|X_0 = x] \\ = \phi(t-s,u) + x\psi(t-s,u).\end{aligned}$$

Since $x \in D$ was chosen arbitrarily, we get the claim. □

Definition 2.1.3. An affine process is regular if

$$F(u) := \frac{\partial \phi}{\partial t}(t,u)|_{t=0^+}, \ R(u) := \frac{\partial \psi}{\partial t}(t,u)|_{t=0^+}$$

exist for all $u \in \mathcal{U}$ and are continuous at $u = 0$. $(F(u), R(u))$ are called functional characteristics of (X_t).

Remark 2.1.4. It has been shown in [14] that any affine process as defined in Definition 2.1.3 is regular, i.e., $F(u)$ and $R(u)$ are well defined.

We are interested in stock prices given by $S_t = \exp\{(r-q)t + X_t\}$, where r is the risk-free return and q denotes the dividend yield. Hence X_t is the discounted dividend-corrected log price, where dividend corrected in this context means that we add up the dividends. For more details on dividend models, c.f. [5].

Furthermore, we model the volatility of the price process. Let $(V_t)_{t\geq 0}$, $V_t : \Omega \mapsto \mathbb{R}_+$, $V_0 > 0$, be such that (X_t, V_t) is a stochastically continuous time-homogeneous Markov process.

Definition 2.1.5. (X_t, V_t) is called an affine stochastic volatility (ASV) model if

$$\log \mathbb{E}\left[e^{uX_t + wV_t}|X_0, V_0\right] = \phi(t,u,w) + X_0 u + V_0 \psi(t,u,w).$$

Observe that a bivariate model has functional characteristics $F, R = (R_1, R_2)$. For an ASV, we have $R_1 = 0$. Hence we denote the functional characteristics by F and $R = R_2$.

Theorem 2.1.6. [11, Theorem 1] Let $(X_t, V_t)_{t\geq 0}$ be a regular ASV process. Then there exist positive semidefinite matrices $a, \alpha,\ b, \beta \in \mathbb{R}^2$, $c, \gamma \geq 0$ and Lévy measures m, μ on $\mathbb{R}^2$ such that

$$\begin{aligned} F(u,w) =& \frac{1}{2}(u,w)a(u,w)^T + b(u,w)^T - c \\ &+ \int_{D\setminus\{0\}} \left(e^{ux+wy} - 1 - h_F(x,y)(u,w)^T\right) m(dx,dy), \\ R(u,w) =& \frac{1}{2}(u,w)\alpha(u,w)^T + \beta(u,w)^T - \gamma \\ &+ \int_{D\setminus\{0\}} \left(e^{ux+wy} - 1 - h_R(x,y)(u,w)^T\right) \mu(dx,dy), \end{aligned}$$

where h_F and h_R are suitable truncation functions. Moreover ϕ and ψ fulfill the generalized Riccati equations

$$\partial_t\phi(t,u,w) = F\left(u,\psi(t,u,w)\right), \qquad \phi(0,u,w) = 0, \tag{2.1}$$

$$\partial_t\psi(t,u,w) = R\left(u,\psi(t,u,w)\right), \qquad \psi(0,u,w) = w. \tag{2.2}$$

The first part follows by showing that R and F are the logarithms of characteristic functions of weakly infinitely divisible processes, and then the statement follows by the Lévy–Khintchine formula (Theorem 1.1.4).

For the second part, we need the following lemma:

Lemma 2.1.7. Let ϕ and ψ be as in (2.1) and (2.2). Then, we have the semiflow property

$$\begin{aligned} \phi(t+s,u,w) &= \phi(t,u,w) + \phi(s,u,\psi(t,u,w)), \\ \psi(t+s,u,w) &= \psi(s,u,\psi(t,u,w)). \end{aligned}$$

Proof. Let $f_{(u,w)}(x,y) = e^{ux+wy}$ and $T_{s,t}f(x,v) = \mathbb{E}[f(X_t,V_t)|X_s = x, V_s = v]$. Denoting $T_t = T_{0,t}$, we have

$$T_{s+t}f_{(u,w)}(x,v) = e^{\phi(s+t,u,w)} f_{(u,\psi(s+t,u,w))}(x,v).$$

On the other hand, we have

$$\begin{aligned} T_{s+t}f_{(u,w)}(x,v) &= T_{t,s+t} \circ T_t f_{(u,w)}(x,v) \\ &= T_{t,s+t}(e^{\phi(t,u,w)} f_{(u,\psi(t,u,w))}(x,v)) \\ &= T_s(e^{\phi(t,u,w)} f_{(u,\psi(t,u,w))}(x,v)) \\ &= e^{\phi(t,u,w)+\phi(s,u,\psi(t,u,w))} f_{(u,\psi(s,u,\psi(t,u,w)))}(x,v). \end{aligned}$$

Now taking logs and comparing the two approaches, we get

$$\begin{aligned} \phi(t+s,u,w) &= \phi(t,u,w) + \phi(s,u,\psi(t,u,w)), \\ \psi(t+s,u,w) &= \psi(s,u,\psi(t,u,w)), \end{aligned}$$

i.e., the claim follows. □

Now we are able to prove the second part of Theorem 2.1.6.

Proof. The boundary conditions $\phi(0,u,w) = 0$ and $\psi(0,u,w) = w$ hold by definition. So it is enough to prove, using Lemma 2.1.7, the differential equations (2.1) and (2.2):

$$\begin{aligned} \partial_t \phi(t,u,w) &= \lim_{s \searrow 0} \frac{\phi(t+s,u,w) - \phi(t,u,w)}{s} \\ &= \lim_{s \searrow 0} \frac{\phi(s,u,\psi(t,u,w)) - \phi(0,u,\psi(t,u,w))}{s} \\ &= \partial_s \phi(s,u,\psi(t,u,w))|_{s=0^+} = F(u,\psi(t,u,w)), \end{aligned}$$

where the second equality holds due to $\phi(0, u, \psi(t, u, w)) = 0$. Similarly, we have

$$\begin{aligned}\partial_t \psi(t, u, w) &= \lim_{s \searrow 0} \frac{\psi(t+s, u, w) - \psi(t, u, w)}{s} \\ &= \lim_{s \searrow 0} \frac{\psi(s, u, \psi(t, u, w)) - \psi(0, u, \psi(t, u, w))}{s} \\ &= \partial_s \psi(s, u, \psi(t, u, w))|_{s=0^+} = R(u, \psi(t, u, w)).\end{aligned}$$

This completes the proof. □

Proposition 2.1.8. [13, Corollary 2.7] Let (X_t, V_t) be a regular ASV process and let $\chi(u) := \frac{\partial R}{\partial w}(u, w)|_{w=0}$.

(i) Suppose $F(0,0) = R(0,0) = 0$ and $\chi(0) < \infty$. Then $(e^{X_t})_{t \geq 0}$ is conservative, i.e., there are no moment explosion and no killing rates ($c = \gamma = 0$ in Theorem 2.1.6).

(ii) Suppose $(e^{X_t})_{t \geq 0}$ is conservative. Then $(e^{X_t})_{t \geq 0}$ is a martingale if and only if $F(1,0) = R(1,0) = 0$ and $\chi(1) < \infty$.

Proof. We just state the proof for the statement that was given in [11], i.e., "$\Leftarrow$" of (ii). The rest can be found in [13]. We have $V_0 > 0$ a.s. Since (X_t, V_t) is a Markov process, we get

$$\mathbb{E}\left[e^{X_t} | \mathcal{F}_s\right] = \mathbb{E}\left[e^{X_t} | V_s, X_s\right] = \exp\{\phi(t-s, 1, 0) + V_s \psi(t-s, 1, 0) + X_s\}.$$

Hence, $\left(e^{X_t}\right)_{t \geq 0}$ is a martingale if and only if $(X_t)_{t \geq 0}$ is conservative and $\phi(t, 1, 0) = \psi(t, 1, 0) = 0$ for every $t \geq 0$. Let (e^{X_t}) be conservative, then also $(X_t)_{t \geq 0}$ is conservative. Let further $F(1,0) = R(1,0) = 0$. By Theorem 2.1.6 we know that ψ satisfies the differential equation

$$\partial_t \psi(t, 1, w) = R(1, \psi(t, 1, w)), \quad \psi(0, 1, w) = w.$$

Observe that $\widetilde{\psi}(t,1,0) = 0$ is a solution to this ODE with $w = 0$. Since $R(1,w)$ is continuously differentiable on $(-\infty, 0)$ by assumption, it is also Lipschitz on this interval. Since $\phi(1) < \infty$, $R(1,w)$ is Lipschitz on $(-\infty, 0]$. Therefore the ODE omits a unique solution, i.e., the trivial solution is the only solution and hence we have $\psi(t,1,0) = 0$. A similar argument shows that $\phi(t,1,0) = 0$. Hence we have shown that $(e^{X_t})_{t\geq 0}$ is a martingale. □

2.2 INTEGRAL FUNCTIONALS FOR ASV MODELS

Define the associated integral processes of X and V by

$$Y_t = \int_0^t X_s ds, \qquad\qquad Z_t = \int_0^t V_s ds.$$

Now we can state the following proposition.

Proposition 2.2.1. [11, Proposition 2] Let (X_t, V_t) be an ASV model with functional characteristics (F, R). Then the joint law of (X_t, V_t, Y_t, Z_t) is described by

$$\begin{aligned} \log \mathbb{E}\left[e^{u_1 X_t + u_2 V_t + u_3 Y_t + u_4 Z_t} | X_0, Y_0\right] =& \Phi(t, u_1, u_2, u_3, u_4) + (u_1 + u_3 t) X_0 \\ &+ \Psi(t, u_1, u_2, u_3, u_4) V_0, \end{aligned}$$

where

$$\begin{aligned} \dot{\Phi} &= F(u_1 + u_3 t, \Psi), \qquad\qquad & \Phi(0) = 0, \\ \dot{\Psi} &= R(u_1 + u_3 t, \Psi), & \Psi(0) = u_2. \end{aligned}$$

For the proof of Proposition 2.2.1, we need the following theorem, which we state without proof.

Theorem 2.2.2. [12, Theorem 4.10] Let $(X_t, \mathbb{P}^x)_{t\geq 0, x\in D}$ be a càdlàg regular affine process on the state space D with functional characteristics (F,R). Let further $Y_t = \int_0^t X_s ds$, $\mathbb{P}^{(x,y)} = \mathbb{P}^x \circ \theta_y^{-1}$, where θ_y is a shift operator. Then, $(X_t, Y_t)_{t\geq 0}$ is a regular affine process on D^2 under $\mathbb{P}^{(x,y)}$ with functional characteristics

$$\widetilde{F}(u_x, u_y) = F(u, x), \qquad \widetilde{R}(u_x, u_y) = \begin{pmatrix} R(u_x) + u_y \\ 0 \end{pmatrix}.$$

Proof of Proposition 2.2.1. Applying Theorem 2.2.2 to $(X_t, V_t)_{t\geq 0}$ yields the affinity of (X_t, V_t, Y_t, Z_t), i.e.,

$$\begin{aligned}\log \mathbb{E}\left[e^{u_1 X_t + u_2 V_t + u_3 Y_t + u_4 Z_t} | X_0, Y_0, Y_0, Z_0\right] =& \Phi(t) + \psi_1(t) X_0 + \psi_2(t) V_0 \\ &+ \psi_3(t) Y_0 + \psi_4(t) Z_0.\end{aligned}$$

Furthermore, Theorem 2.2.2 gives

$$\begin{aligned}\dot{\Phi}(t) = \widetilde{F}(\psi_1, \psi_2, \psi_3, \psi_4) = F(\psi_1, \psi_2), \qquad & \Phi(0) = 0, \\ \widetilde{R}(\psi_1, \psi_2, \psi_3, \psi_4) = \begin{pmatrix} R(\psi_1, \psi_2) + \begin{pmatrix} \psi_3 \\ \psi_4 \end{pmatrix} \\ 0 \end{pmatrix}, \qquad & \psi_i(0) = u_i, \ i = 1, 2, 3, 4.\end{aligned}$$

Noting that the functional characteristics of an ASV are F and $(0, R)$ yields

$$\begin{aligned}\dot{\psi}_1(t) &= \psi_3(t), & \psi_1(0) &= u_1, \\ \dot{\psi}_2(t) &= R(\psi_1(t), \psi_2(t)) + \psi_4(t), & \psi_1(0) &= u_2, \\ \dot{\psi}_3(t) &= 0, & \psi_1(0) &= u_3, \\ \dot{\psi}_4(t) &= 0, & \psi_1(0) &= u_4.\end{aligned}$$

We see that $\psi_3(t) = u_3,\ \psi_4(t) = u_4$. Hence $\psi_1(t) = u_1 + u_3 t$ and finally

$$\dot{\psi}_2(t) = R(u_1 + u_3 t, \psi_2(t)) + u_4.$$

Noting that $Y_0 = Z_0 = 0$ and denoting $\Psi = \psi_2$ gives the claim. □

2.3 CHANGE OF NUMÉRAIRE FOR ASV MODELS

In [11], the authors change the numéraire from the dividend corrected bond $(B_t)_{t\geq 0} = \left(e^{(r-q)t}\right)_{t\geq 0}$ to the stock $(S_t)_{t\geq 0}$. Also they denote the martingale measure w.r.t. $(B_t)_{t\geq 0}$ by $\mathbb{Q}^0$ and w.r.t. the stock by $\mathbb{Q}^1$. The associated expectations are denoted by $\mathbb{E}^0$ and $\mathbb{E}^1$. Furthermore

$$\log \mathbb{E}^0\left[e^{u_1 X_t + u_2 V_t} | X_0, V_0\right] = \phi^0(t, u_1, u_2) + X_0 \phi_1^0(t, u_1, u_2) + V_0 \phi_2^0(t, u_1, u_2).$$

The density process is given by

$$\frac{d\mathbb{Q}^1}{d\mathbb{Q}^0}(t) = \frac{d\mathbb{Q}^1}{d\mathbb{Q}^0}\Big|_{\mathcal{F}_t} = \frac{S(T)B(0)}{B(T)S(0)}\Big|_{\mathcal{F}_t} = \frac{S(t)}{B(t)S(0)} = e^{X_t - X_0}.$$

To get further insight, the authors give the following lemma.

Lemma 2.3.1. [11, Lemma 1] Let (X,V) be affine under $\mathbb{Q}^0$ with functional characteristics F^0 and R^0. Then (X, V) is affine under $\mathbb{Q}^1$ with functional characteristics

$$F^1(u, w) = F^0(u + 1, w), \qquad R^1(u, w) = R(u + 1, w).$$

Proof. To see this, we use Bayes' rule, i.e., $\mathbb{E}^{\mathbb{Q}}\left[X|\mathcal{F}_t\right] = \frac{\mathbb{E}^{\mathbb{P}}\left[X \frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_T}|\mathcal{F}_t\right]}{\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_t}}$ for all $T \geq t$ and equivalent measures $\mathbb{Q}, \mathbb{P}$. We have

$$\log \mathbb{E}^1\left[e^{uX_t + wV_t}|X_0, V_0\right] = \log\left(\mathbb{E}^0\left[e^{uX_t + wV_t}\frac{d\mathbb{Q}^1}{d\mathbb{Q}^0}\Big|_{\mathcal{F}_t}\Big| X_0, V_0\right] \cdot \frac{1}{\frac{d\mathbb{Q}^1}{d\mathbb{Q}^0}|_{\mathcal{F}_0}}\right)$$

$$
\begin{aligned}
&= \log \mathbb{E}^0 \left[e^{uX_t + wV_t} e^{X_t - X_0} | X_0, V_0 \right] \\
&= \log \mathbb{E}^0 \left[e^{(u+1)X_t + wV_t} | X_0, V_0 \right] - X_0 \\
&= \phi^0(t, u+1, w) + X_0 \left(\psi_1^0(t, u+1, w) - 1 \right) + V_0 \psi_2^0(t, u+1, w) \\
&=: \phi^1(t, u, w) + X_0 \psi_1^1(t, u, w) + V_0 \psi_2^1(t, u, w).
\end{aligned}
$$

Hence, we get $F^1(u, w) = F^0(u+1, w)$ and $R^1(u, w) = R^0(u+1, w)$. □

If we assume that $(e^{X_t})_{t \geq 0}$ is a $\mathbb{Q}^0$ martingale, then by Proposition 2.1.8 (ii), we have $F^0(1, 0) = R^0(1, 0) = 0$ and hence $F^1(0, 0) = R^1(0, 0) = 0$. The next lemma describes the connection between X and its associated integral process Y.

Lemma 2.3.2. [11, Lemma 2] Let (X, V) be an ASV model. Then the joint law of (X, Y) under $\mathbb{Q}^1$ is described by

$$
\log \mathbb{E}^1 \left[e^{uX_t + wY_t} | X_0, V_0 \right] = \Phi(t, u, w) + (u + wt) X_0 + \Psi(t, u, w) V_0,
$$

where

$$
\begin{aligned}
\dot{\Phi} &= F(u + 1 + wt, \Psi), \qquad && \Phi(0) = 0, \\
\dot{\Psi} &= R(u + 1 + wt, \Psi), \qquad && \Psi(0) = 0.
\end{aligned}
$$

Proof. Applying Proposition 2.2.1 yields

$$
\log \mathbb{E}^1 \left[e^{uX_t + wY_t} | X_0, V_0 \right] = \Phi(t, u, w) + (u + wt) X_0 + \Psi(t, u, w) V_0,
$$

where

$$
\begin{aligned}
\dot{\Phi} &= F^1(u + wt, \Psi), \qquad && \Phi(0) = 0, \\
\dot{\Psi} &= R^1(u + wt, \Psi), \qquad && \Psi(0) = 0.
\end{aligned}
$$

Now, by Lemma 2.3.1, we get

$$F^1(u+wt,\Psi) = F(u+1+wt,\Psi) \text{ and } R^1(u+wt,\Psi) = R(u+1+wt,\Psi).$$

This concludes the proof. $\square$

2.4 GENERAL RESULTS FOR GEOMATRIC ASIAN OPTIONS

Denote the geometric average log return process (until time T) by

$$\bar{X}_t = r - q + \frac{1}{T}\int_0^t X_s ds = r - q + \frac{Y_t}{T}$$

and let the geometric average stock price process (until time T) be given by

$$\widehat{S}_t = e^{\bar{X}_t} = \exp\left\{r - q + \frac{1}{T}\int_0^t X_s ds\right\} = \exp\left\{r - q + \frac{Y_t}{T}\right\}.$$

Remark 2.4.1. Note that these processes only display the average at $t = T$. This is good enough for our purposes since we are only interested in this point in time.

To get the formula for the average price Asian option, we need the following corollary which follows immediately from Proposition 2.2.1.

Corollary 2.4.2. [11, Corollary 2] Let (X, V) be an ASV model. Then the law of $Y_t = \int_0^t X_s ds$ is described by

$$\log \mathbb{E}\left[e^{wY_t} | X_0, V_0\right] = \Phi(t,w) + wtX_0 + V_0\Psi(t,w),$$

where

$$\dot{\Phi} = F(wt,\Psi), \qquad \Phi(0) = 0,$$

$$\dot{\Psi} = R(wt,\Psi), \qquad \Psi(0) = 0.$$

Before we give the formulas for the geometric Asian options, we recall that the payoff at maturity T of an

(i) average price Asian call option is given by $\left(\widehat{S}_T - K\right)_+$,

(ii) average strike Asian call option is given by $\left(S_T - \widehat{S}_T\right)_+$.

Here K denotes the strike price, S_T is the price at time T and $\widehat{S}_T$ is the (here: geometric) average price over the period $[0, T]$. Furthermore recall that the price at time $0 \leq t \leq T$ of an option is given by the present value of its future cash flows.

Now we are ready to state the first result.

Theorem 2.4.3. [11, Theorem 2] Suppose there exists $a > 1$ such that

$$\mathbb{E}\left[e^{a\bar{X}_T}\right] < \infty.$$

Then the time zero value of a geometric average price Asian call option is given by

$$\mathbb{E}\left[e^{-rT}\left(\widehat{S}_T - K\right)_+ \Big| X_0, V_0\right] = \frac{e^{-rT}}{2\pi i}\int_{a-i\infty}^{a+i\infty}\left(\frac{1}{K}\right)^u \frac{K}{u(u-1)} e^{\kappa(T,u)} du$$

with cumulant function $\kappa(t, u) = \log \mathbb{E}\left[e^{u\bar{X}_t} | X_0, V_0\right]$. Furthermore,

$$\kappa(T, u) = u(r - q) + \phi(T, u) + uX_0 + \psi(T, u)V_0,$$

where

$$\dot{\phi}(t, u) = F\left(\frac{ut}{T}, \psi\right), \qquad \phi(0, u) = 0,$$

$$\dot{\psi}(t, u) = R\left(\frac{ut}{T}, \psi\right), \qquad \psi(0, u) = 0.$$

Proof. First observe that a standard inverse Laplace transform gives

$$\left(\widehat{S}_T - K\right)_+ = \left(e^{\bar{X}_T} - K\right)_+ = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\left(\frac{1}{K}\right)^u \frac{K}{u(u-1)} e^{u\bar{X}_T} du.$$

Hence, using Fubini's theorem, we get

$$\begin{aligned}\mathbb{E}\left[e^{-rT}(\widehat{S}_T - K)_+ | X_0, V_0\right] &= \mathbb{E}\left[\frac{e^{-rT}}{2\pi i}\int_{a-i\infty}^{a+i\infty}\left(\frac{1}{K}\right)^u \frac{K}{u(u-1)} e^{u\bar{X}_T} du \middle| X_0, V_0\right] \\ &= \frac{e^{-rT}}{2\pi i}\int_{a-i\infty}^{a+i\infty}\left(\frac{1}{K}\right)^u \frac{K}{u(u-1)} \mathbb{E}\left[e^{u\bar{X}_T} \middle| X_0, V_0\right] du \\ &= \frac{e^{-rT}}{2\pi i}\int_{a-i\infty}^{a+i\infty}\left(\frac{1}{K}\right)^u \frac{K}{u(u-1)} e^{\kappa(T,u)} du.\end{aligned}$$

We still have to verify the formula for $\kappa(T,u)$. By Corollary 2.4.2, we have

$$\begin{aligned}\kappa(t,u) &= \log \mathbb{E}\left[e^{u\bar{X}_t} \middle| X_0, V_0\right] = \log \mathbb{E}\left[e^{u(r-q)+\frac{u}{T}Y_t} | X_0, V_0\right] \\ &= u(r-q) + \Phi\left(t, \frac{u}{T}\right) + \frac{ut}{T}X_0 + V_0\Psi\left(t, \frac{u}{T}\right),\end{aligned}$$

where

$$\begin{aligned}\dot{\Phi} &= F\left(\frac{ut}{T}, \Psi\right), & \Phi(0) &= 0, \\ \dot{\Psi} &= R\left(\frac{ut}{T}, \Psi\right), & \Psi(0) &= 0.\end{aligned}$$

Let $\phi(t,u) = \Phi\left(t, \frac{u}{T}\right),\ \psi(t,u) = \Psi\left(t, \frac{u}{T}\right)$. Then

$$\begin{aligned}\dot{\phi} &= F\left(\frac{ut}{T}, \psi\right), & \phi(0) &= 0, \\ \dot{\psi} &= R\left(\frac{ut}{T}, \psi\right), & \psi(0) &= 0.\end{aligned}$$

Now, letting $t \to T$ gives

$$\begin{aligned}\log \mathbb{E}\left[e^{\frac{u}{T}Y_T}|X_0,V_0\right] &= \log \mathbb{E}\left[\lim_{t\to T} e^{\frac{u}{T}Y_t}|X_0,V_0\right] = \lim_{t\to T}\log \mathbb{E}\left[e^{\frac{u}{T}Y_t}|X_0,V_0\right] \\ &= \lim_{t\to T}\left(\phi(t,u)+\frac{ut}{T}X_0+\psi(t,u)V_0\right) \\ &= \phi(T,u)+uX_0+\psi(T,u)V_0.\end{aligned}$$

This completes the proof. □

For the second result, we need a slightly different setting, namely

$$\bar{X}_t = \frac{1}{T}\int_0^t X_s = \frac{1}{T}V_t, \quad \widehat{S}_t = e^{(r-q)t+\bar{X}_t}.$$

Theorem 2.4.4. [11, Theorem 3] If there exists $b < 0$ such that

$$\mathbb{E}\left[e^{b\bar{X}_T}\right] < \infty,$$

then the time zero value of a geometric average strike Asian call option is given by

$$\mathbb{E}^0\left[e^{-rT}(S_T-\widehat{S}_T)_+|X_0,V_0\right] = \frac{e^{-qT}}{2\pi i}\int_{b-i\infty}^{b+i\infty}\frac{1}{u(u-1)}e^{\kappa(T,u)}du$$

with cumulant function $\kappa(t,u) = \log \mathbb{E}^0\left[e^{u\bar{X}_t+(1-u)X_t}|X_0,V_0\right]$. Furthermore,

$$\kappa(T,u) = \phi(T,u)+V_0\psi(T,u)+X_0,$$

where

$$\begin{aligned}\dot{\phi}(t,u) &= F\left(\frac{ut}{T}+1-u,\psi\right), & \phi(0,u) &= 0,\\ \dot{\psi}(t,u) &= R\left(\frac{ut}{T}+1-u,\psi\right), & \psi(0,u) &= 0.\end{aligned}$$

Proof. First of all observe that

$$
\begin{aligned}
&\mathbb{E}^0\left[\left(S_T-\widehat{S}_T\right)_+|X_0,V_0\right]=\mathbb{E}^1\left[\left(S_T-\widehat{S}_T\right)_+e^{X_0-X_T}|X_0,V_0\right]\\
&=\mathbb{E}^1\left[\left(e^{(r-q)T+X_T}-e^{(r-q)T+\bar{X}_T}\right)_+e^{X_0-X_T}|X_0,V_0\right]\\
&=S_0e^{(r-q)T}\mathbb{E}^1\left[\left(1-e^{\bar{X}_T-X_T}\right)_+|X_0,V_0\right].
\end{aligned}
$$

Hence, using a standard inverse Laplace transform and Fubini's theorem yields

$$
\begin{aligned}
&\mathbb{E}^0\left[e^{-rT}\left(S_T-\widehat{S}_T\right)_+|X_0,V_0\right]=S_0e^{-qT}\mathbb{E}^1\left[\left(1-e^{\bar{X}_T-X_T}\right)_+|X_0,V_0\right]\\
&=\mathbb{E}^1\left[S_0\frac{e^{-qT}}{2\pi i}\int_{b-i\infty}^{b+i\infty}\frac{1}{u(u-1)}e^{u\left(\bar{X}_T-X_T\right)}du|X_0,V_0\right]\\
&=\frac{e^{-qT}}{2\pi i}\int_{b-i\infty}^{b+i\infty}\frac{1}{u(u-1)}\mathbb{E}^1\left[S_0e^{u\left(\bar{X}_T-X_T\right)}|X_0,V_0\right]du\\
&=\frac{e^{-qT}}{2\pi i}\int_{b-i\infty}^{b+i\infty}\frac{1}{u(u-1)}\mathbb{E}^0\left[e^{u\bar{X}_T+(1-u)X_T}|X_0,V_0\right]du\\
&=\frac{e^{-qT}}{2\pi i}\int_{b-i\infty}^{b+i\infty}\frac{1}{u(u-1)}e^{\kappa(T,u)}du.
\end{aligned}
$$

It remains to show the form of $\kappa(T,u)$. We have

$$
\begin{aligned}
\kappa(t,u)&=\log\mathbb{E}^0\left[e^{u\bar{X}_t+(1-u)X_t}|X_0,V_0\right]=\log\mathbb{E}^1\left[e^{u\left(\bar{X}_t-X_t\right)+X_0}|X_0,V_0\right]\\
&=\log\mathbb{E}^1\left[e^{-uX_t+\frac{u}{T}V_t}|X_0,V_0\right]+X_0,
\end{aligned}
$$

such that applying Lemma 2.3.2 gives

$$
\kappa(t,u)=\Phi\left(t,-u,\frac{u}{T}\right)+\left(-u+\frac{ut}{T}\right)X_0+\Psi\left(t,-u,\frac{u}{T}\right)V_0+X_0,
$$

where

$$
\dot{\Phi}=F\left(\frac{ut}{T}+1-u,\Psi\right),\qquad\qquad \Phi(0)=0,
$$

$$\dot{\Psi} = R\left(\frac{ut}{T} + 1 - u, \Psi\right), \qquad \Psi(0) = 0.$$

Taking $\phi(t,u) = \Phi\left(t, -u, \frac{u}{T}\right)$ and $\psi(t,u) = \Psi\left(t, -u, \frac{u}{T}\right)$ yields

$$\dot{\phi} = F\left(\frac{ut}{T} + 1 - u, \psi\right) \qquad \phi(0) = 0$$
$$\dot{\psi} = R\left(\frac{ut}{T} + 1 - u, \psi\right) \qquad \psi(0) = 0.$$

Now, letting $t \to T$ gives

$$\begin{aligned}
&\log \mathbb{E}^0\left[e^{u\bar{X}_T + (1-u)X_T} | X_0, V_0\right] = \log \mathbb{E}^1\left[e^{u\left(\bar{X}_T - X_T\right) + X_0} | X_0, V_0\right] \\
&= \log \mathbb{E}^1\left[\lim_{t\to T} e^{u\left(\bar{X}_t - X_t\right) + X_0} | X_0, V_0\right] \\
&= \lim_{t\to T} \log \mathbb{E}^1\left[e^{u\left(\bar{X}_t - X_t\right) + X_0} | X_0, V_0\right] \\
&= \lim_{t\to T}\left(\phi(t,u) + \left(u\left(\frac{t}{T} - 1\right) + 1\right) X_0 + \psi(t,u)V_0\right) \\
&= \phi(T,u) + \psi(T,u)V_0 + X_0.
\end{aligned}$$

This completes the proof. □

3 SINGLE-FACTOR MODELS

In this section, we study all models discussed in [11] as well as some selected further models. For the models in [11], we provide proofs to the statements that are made there. Furthermore we work out the details. We will work as follows. Firstly, we obtain the form of the functional characteristics. Then we give the Riccati equations for the average price and the average price. Finally, we solve the Riccati equations.

3.1 HESTON MODEL WITH PERFECT POSITIVE/NEGATIVE CORRELATION

The dynamics of the Heston model are given by

$$dX_t = \left(-\frac{1}{2}V_t\right)dt + \sqrt{V_t}dW_t^1,$$
$$dV_t = \lambda(\theta - V_t)dt + \zeta\sqrt{V_t}dW_t^2,$$

where $\lambda, \theta, \zeta > 0$. We assume here that the correlation between the SBMs W^1 and W^2 is given by $\rho = \pm 1$, i.e., $W^1 \overset{d}{=} \pm W^2$. Using Itô's lemma (Theorem 1.6.2), one can obtain that the functional characteristics are given by

$$F(u,w) = \lambda\theta w, \qquad R(u,w) = \frac{1}{2}(u^2 - u) + \frac{w^2\zeta^2}{2} - \lambda w \pm uw\zeta. \tag{3.1}$$

We give a more general proof for (3.1) in the next subsection. Furthermore, we have the Riccati equations of the average price

$$\dot{\phi} = \lambda\theta\psi, \qquad \phi(0) = 0, \tag{3.2}$$
$$\dot{\psi} = \frac{\zeta^2}{2}\psi^2 - \left(\lambda \mp \zeta\frac{ut}{T}\right)\psi + \frac{1}{2}\frac{ut}{T}\left(\frac{ut}{T} - 1\right), \qquad \psi(0) = 0. \tag{3.3}$$

Using the standard substitution for Riccati equations $\psi(t) = -\frac{2}{\zeta^2}\frac{y'(t)}{y(t)}$, we obtain that (3.3) is equivalent to

$$y'' + \left(\lambda \mp \zeta \frac{ut}{T}\right) y' + \frac{\zeta^2}{4}\frac{ut}{T}\left(\frac{ut}{T} - 1\right) y = 0. \tag{3.4}$$

Theorem 3.1.1. The solution to (3.4) is given by

$$y(t) = c_1 y_1(t) + c_2 y_2(t) = c_1 e^{\pm \frac{t\left(\lambda \mp \zeta \frac{ut}{2T}\right)}{2}} \operatorname{Ai}\left(x(t)\right) + c_2 e^{\pm \frac{t\left(\lambda \mp \zeta \frac{ut}{2T}\right)}{2}} \operatorname{Bi}\left(x(t)\right),$$

where $\operatorname{Ai}(x)$ and $\operatorname{Bi}(x)$ are the Airy functions of the first and second kind, respectively, and

$$x(t) = \frac{T\lambda^2 \mp 2tu\zeta\lambda + u\zeta\left(t\zeta - 2\right)}{2^{\frac{2}{3}} T \left(\frac{u\zeta(\zeta - 2\lambda)}{T}\right)^{\frac{2}{3}}}.$$

Proof. Assume $\rho = 1$ (the case $\rho = -1$ follows analogously). The Airy functions fulfill the Stokes equation (c.f. [1, page 446])

$$\frac{d^2 w}{dx^2} - xw = 0. \tag{3.5}$$

By letting $v(t) = w(x(t))$, we find

$$v'(t) = w'(x(t))x'(t)$$

and

$$\begin{aligned} v''(t) &= w''(x(t))(x'(t))^2 + w'(x(t))x''(t) \\ &= x(t)v(t)(x'(t))^2 + \frac{v'(t)}{x'(t)}x''(t), \end{aligned}$$

which yields that (3.5) is equivalent to

$$\frac{d^2v}{dt^2}\frac{1}{x'(t)^2} - \frac{x''(t)}{x'(t)^3}\frac{dv}{dt} - x(t)v = 0.$$

Now, using the integrating factor $e^{\frac{t\left(\lambda-\zeta\frac{ut}{2T}\right)}{2}}$, i.e., letting $y(t) = e^{-\frac{t\left(\lambda-\zeta\frac{ut}{2T}\right)}{2}}v(t)$, yields

$$\begin{aligned} & y'' + y'\left(\lambda - \zeta\frac{ut}{T} - \frac{x''(t)}{x'(t)}\right) \\ & \quad + y\left(\frac{\left(\lambda-\zeta\frac{ut}{T}\right)^2}{4} + \zeta\frac{u}{2T} - x(t)x'(t)^2 - \frac{1}{2}\left(\lambda-\zeta\frac{ut}{T}\right)\frac{x''(t)}{x'(t)}\right) = 0. \end{aligned} \tag{3.6}$$

Furthermore, we have

$$x'(t) = \frac{\zeta u\left(\zeta - 2\lambda\right)}{2^{\frac{2}{3}}T\left(\frac{u\zeta(\zeta-2\lambda)}{T}\right)^{\frac{2}{3}}}, \qquad x''(t) = 0.$$

Hence, plugging this into (3.6) gives

$$y''(t) + y'(t)\left(\lambda - \zeta\frac{ut}{T}\right) + y(t)\left(\frac{\left(\lambda-\zeta\frac{ut}{T}\right)^2}{4} - \zeta\frac{u}{2T} - x(t)x'(t)^2\right) = 0,$$

i.e.,

$$\begin{aligned} & y''(t) + y'(t)\left(\lambda - \zeta\frac{ut}{T}\right) \\ & + y(t)\left(\frac{\lambda^2}{4} - \frac{1}{2}\lambda\zeta\frac{ut}{T} + \frac{1}{4}\left(\zeta\frac{ut}{T}\right)^2 - \zeta\frac{u}{2T} - \frac{\lambda^2}{4} + \frac{1}{2}\lambda\zeta\frac{ut}{T} + \frac{\zeta^2}{4}\frac{ut}{T} + \zeta\frac{u}{2T}\right) = 0, \end{aligned}$$

i.e.,

$$y''(t) + y'(t)\left(\lambda - \zeta\frac{ut}{T}\right) + y(t)\frac{\zeta^2}{4}\frac{ut}{T}\left(\frac{ut}{T} - 1\right) = 0.$$

Hence, we obtained (3.4), which concludes the proof. □

Moreover, noting that we have the condition $\psi(0) = 0$ in (3.3), we have the following:

Corollary 3.1.2. The solution to (3.3) is given by

$$\psi(t) = -\frac{2}{\zeta^2}\frac{y_2'(0)y_1'(t) - y_1'(0)y_2'(t)}{y_2'(0)y_1(t) - y_1'(0)y_2(t)}.$$

Furthermore, we get the solution to (3.2) as

$$\phi(t) = -\lambda\theta\frac{2}{\zeta^2}\ln\left\{\frac{y_2'(0)y_1(t) - y_1'(0)y_2(t)}{y_2'(0)y_1(0) - y_1'(0)y_2(0)}\right\}.$$

For the Riccati equations of the average strike, one can argue accordingly.

3.2 GENERAL HESTON MODEL

The Heston model was introduced in [10]. Here, the dynamics are given by

$$dX_t = \left(-\frac{1}{2}V_t\right)dt + \sqrt{V_t}dW_t^1,$$
$$dV_t = \lambda(\theta - V_t)dt + \zeta\sqrt{V_t}dW_t^2,$$

where $\lambda, \theta, \zeta > 0$ and W^1, W^2 are SBMs having correlation $\rho \in (-1, 1)$. It is well known that V, which is given as a CIR-type SDE remains strictly positive if $\zeta^2 < 2\lambda\theta$. To relate the Heston model to the general approach as given in Section 2, our first goal is to confirm [11] and show the following.

Theorem 3.2.1. The functional characteristics for the general Heston model are given by

$$F(u, w) = \lambda\theta w, \qquad R(u, w) = \frac{w^2\zeta^2}{2} - (\lambda - u\rho\zeta)\,w + \frac{1}{2}(u^2 - u). \tag{3.7}$$

Proof. Taking $f(x, v) = e^{ux+wv}$ and applying Itô's formula (Theorem 1.6.2), we obtain

$$de^{uX_t+wV_t} = e^{uX_t+wV_t}\left(udX_t + wdV_t\right)$$

$$
\begin{aligned}
&+ e^{uX_t+wV_t}\left(\frac{1}{2}u^2 d[X,X]_t + uwd[X,V]_t + \frac{1}{2}w^2 d[V,V]_t\right)\\
=&e^{uX_t+wV_t}\left(-\frac{1}{2}uV_t dt + u\sqrt{V_t}dW_t^1 + w\lambda\theta dt - w\lambda V_t dt + w\zeta\sqrt{V_t}dW_t^2\right)\\
&+ e^{uX_t+wV_t}\left(\frac{1}{2}u^2 V_t dt + \zeta V_t uw\rho dt + \frac{1}{2}w^2\zeta^2 V_t dt\right)\\
=&e^{uX_t+wV_t}\left(V_t\left[\frac{1}{2}(u^2-u) + \frac{w^2\zeta^2}{2} - \lambda w + \rho uw\zeta\right] + w\lambda\theta\right)dt + \text{MART},
\end{aligned}
$$

where MART are integrals with respect to the Brownian motion which are martingales and hence will be dropped when we take the expected value to obtain

$$
\begin{aligned}
&\mathbb{E}\left[e^{uX_t+wV_t} - e^{uX_0+wV_0}|X_0,V_0\right]\\
&\quad = \mathbb{E}\left[\int_0^t e^{uX_s+wV_s}\left(V_s\left[\frac{1}{2}(u^2-u) + \frac{w^2\zeta^2}{2} - \lambda w + \rho uw\zeta\right] + w\lambda\theta\right)ds\middle|X_0,V_0\right].
\end{aligned}
$$

Hence, we have

$$
\begin{aligned}
\frac{\partial}{\partial t}\mathbb{E}\left[e^{uX_t+wV_t}|X_0,V_0\right]|_{t=0} =&\mathbb{E}\left[e^{uX_0+wV_0}V_0|X_0,V_0\right]\left[\frac{1}{2}(u^2-u) + \frac{w^2\zeta^2}{2} - \lambda w + \rho uw\zeta\right]\\
&+ \mathbb{E}\left[e^{uX_0+wV_0}|X_0,V_0\right]w\lambda\theta\\
=&e^{uX_0+wV_0}V_0\left[\frac{1}{2}(u^2-u) + \frac{w^2\zeta^2}{2} - \lambda w + \rho uw\zeta\right]\\
&+ e^{uX_0+wV_0}w\lambda\theta,
\end{aligned}
$$

i.e.,

$$
\frac{\frac{\partial}{\partial t}\mathbb{E}\left[e^{uX_t+wV_t}|X_0,V_0\right]|_{t=0}}{e^{uX_0+wV_0}} = V_0\left[\frac{1}{2}(u^2-u) + \frac{w^2\zeta^2}{2} - \lambda w + \rho uw\zeta\right] + w\lambda\theta.
$$

Now, noting that (c.f. [12, (2.8)–(2.10)])

$$
\frac{\frac{\partial}{\partial t}\mathbb{E}\left[e^{uX_t+wV_t}|X_0,V_0\right]|_{t=0}}{e^{uX_0+wV_0}} = F(u,w) + ((X_0,V_0),R(u,w)),
$$

we finally get

$$F(u,w) = \lambda\theta w, \qquad R(u,w) = \frac{w^2\zeta^2}{2} - (\lambda - u\rho\zeta)\, w + \frac{1}{2}(u^2 - u).$$

Hence, the result is confirmed. □

3.2.1 Average Price. To calculate the value of an average price Asian option, we combine this result now with the Riccati equations obtained in Theorem 2.4.3, i.e., substituting $w := \psi,\ u := \frac{ut}{T}$ into (3.7). We get

$$\dot{\phi} = \lambda\theta\psi, \qquad \phi(0) = 0, \tag{3.8}$$

$$\dot{\psi} = \frac{\zeta^2}{2}\psi^2 - \left(\lambda - \rho\zeta\frac{ut}{T}\right)\psi + \frac{1}{2}\frac{ut}{T}\left(\frac{ut}{T} - 1\right), \qquad \psi(0) = 0. \tag{3.9}$$

Lemma 3.2.2. Using the standard substitution

$$\psi(t) = -\frac{2}{\zeta^2}\frac{y'(t)}{y(t)}, \tag{3.10}$$

(3.9) can be transformed into

$$y'' + \left(\lambda - \rho\zeta\frac{ut}{T}\right)y' + \frac{\zeta^2}{4}\frac{ut}{T}\left(\frac{ut}{T} - 1\right)y = 0. \tag{3.11}$$

Proof. Using the substitution (3.10), we get

$$\frac{d}{dt}\left(-\frac{2}{\zeta^2}\frac{y'(t)}{y(t)}\right) = \frac{\zeta^2}{2}\left(-\frac{2}{\zeta^2}\frac{y'(t)}{y(t)}\right)^2 + \left(\lambda - \rho\zeta\frac{ut}{T}\right)\frac{2}{\zeta^2}\frac{y'(t)}{y(t)} + \frac{1}{2}\frac{ut}{T}\left(\frac{ut}{T} - 1\right),$$

i.e.,

$$\frac{2}{\zeta^2}\frac{y'(t)^2 - y(t)y''(t)}{y(t)^2} = \frac{2}{\zeta^2}\frac{y'(t)^2}{y(t)^2} + \left(\lambda - \rho\zeta\frac{ut}{T}\right)\frac{2}{\zeta^2}\frac{y'(t)}{y(t)} + \frac{1}{2}\frac{ut}{T}\left(\frac{ut}{T} - 1\right),$$

i.e.,

$$\frac{y''(t)}{y(t)} + \left(\lambda - \rho\zeta\frac{ut}{T}\right)\frac{y'(t)}{y(t)} + \frac{\zeta^2}{4}\frac{ut}{T}\left(\frac{ut}{T} - 1\right) = 0,$$

i.e.,

$$y''(t) + \left(\lambda - \rho\zeta\frac{ut}{T}\right)y'(t) + \frac{\zeta^2}{4}\frac{ut}{T}\left(\frac{ut}{T} - 1\right)y(t) = 0,$$

i.e., (3.11). □

To solve this equation we use a numerical approach. We use Wolfram Alpha to find a solution and check it by hand.

Theorem 3.2.3. One solution of (3.11) can be expressed in terms of Kummer's confluent hypergeometric function $M(a, b, c(t)) = {}_1F_1(a, b, c(t))$ with parameters

$$\begin{aligned} a &= \frac{-\lambda^2 + \rho\zeta(\lambda - 2\frac{u}{T}\xi) - \frac{\zeta^2}{4}}{8\xi^{\frac{3}{2}}\frac{u}{T}\zeta} + \frac{1}{4}, \\ b &= \frac{1}{2}, \\ c(t) &= \frac{1}{2}\frac{((\frac{1}{2} + \xi u\frac{t}{T})\zeta - \lambda\rho)^2}{\zeta\frac{u}{T}\xi^{\frac{3}{2}}}. \end{aligned}$$

Furthermore, we have $\xi = \rho^2 - 1$ and we need the integrating factor

$$A(t) = \exp\left\{-\frac{1}{4}t\left(2\lambda - \zeta\rho u\frac{t}{T} + \frac{\rho^2\zeta u\frac{t}{T} - 2\lambda\rho + \zeta(1 - u\frac{t}{T})}{\xi^{\frac{1}{2}}}\right)\right\},$$

i.e., a solution is given by $y(t) = A(t)M(a, b, c(t))$.

To show this, we first state the following lemma.

Lemma 3.2.4. Kummer's confluent hypergeometric function ${}_1F_1(a,b,z)$ satisfies the differential equation

$$z\frac{d^2w}{dz^2}+(b-z)\frac{dw}{dz}-aw=0. \tag{3.12}$$

Taking $z=c(t)$, (3.12) can be transformed into

$$\begin{aligned}&A^{-1}(t)y''(t)+\left[2(A^{-1})'(t)+\left(\frac{bc'(t)}{c(t)}-c'(t)-\frac{c''(t)}{c'(t)}\right)A^{-1}(t)\right]y'(t)\\&\quad+\left[(A^{-1})''(t)+\left(\frac{bc'(t)}{c(t)}-c'(t)-\frac{c''(t)}{c'(t)}\right)(A^{-1})'(t)-a\frac{c'(t)^2}{c(t)}A^{-1}(t)\right]y(t)=0.\end{aligned} \tag{3.13}$$

Proof. We have

$$\begin{aligned}\frac{d^2w}{dz^2}&=\frac{d}{dz}\left(\frac{dw}{dt}\frac{dt}{dz}\right)=\frac{d}{dz}\left(\frac{dw}{dt}\frac{1}{\frac{dz}{dt}}\right)+\frac{d}{dz}\left(\frac{dw}{dt}\frac{1}{c'(t)}\right)\\&=\frac{dw}{dt}\frac{d}{dz}\left(\frac{1}{c'(t)}\right)+\frac{d}{dz}\left(\frac{dw}{dt}\right)\frac{1}{c'(t)}.\end{aligned}$$

The relationship between the operator and $\frac{d}{dz}$ and $\frac{d}{dt}$ is given by

$$\frac{df}{dz}=\frac{df}{dt}\frac{dt}{dz}=\frac{df}{dt}\frac{1}{c'(t)}$$

such that we get

$$\frac{d^2w}{dz^2}=\frac{d^2w}{dt^2}\frac{1}{(c'(t))^2}-\frac{c''(t)}{(c'(t))^3}\frac{dw}{dt}.$$

Hence we can write (3.12) as

$$\frac{c(t)}{(c'(t))^2}\frac{d^2w}{dt^2}+\left(\frac{b-c(t)}{c'(t)}-\frac{c''(t)c(t)}{(c'(t))^3}\right)\frac{dw}{dt}-aw=0. \tag{3.14}$$

Now, taking into account the integrating factor $A(t)$, i.e., $y(t) = A(t)w(t)$, i.e., $w(t) = \frac{y(t)}{A(t)} =: A^{-1}(t)y(t)$, we have

$$w'(t) = (A^{-1})'(t)y(t) + A^{-1}(t)y'(t),$$
$$w''(t) = (A^{-1})''(t)y(t) + 2(A^{-1})'(t)y'(t) + A^{-1}(t)y''(t).$$

Putting this into (3.14) gives

$$\frac{c(t)}{c'(t)^2}\left[(A^{-1})''(t)y(t) + 2(A^{-1})'(t)y'(t) + A^{-1}(t)y''(t)\right] + \left(\frac{b-c(t)}{c'(t)} - \frac{c''(t)c(t)}{c'(t)^3}\right)\left[(A^{-1})'(t)y(t) + A^{-1}(t)y'(t)\right] - aA^{-1}(t)y(t) = 0.$$

In the next step, we sort the arguments and end up with

$$y''(t)A^{-1}(t)\frac{c(t)}{c'(t)^2} + y'(t)\left[2(A^{-1})'(t)\frac{c(t)}{(c'(t))^2} + \left(\frac{b-c(t)}{c'(t)} - \frac{c''(t)c(t)}{(c'(t))^3}\right)A^{-1}(t)\right] + y(t)\left[\frac{c(t)}{(c'(t))^2}(A^{-1})''(t) + \left(\frac{b-c(t)}{c'(t)} - \frac{c''(t)c(t)}{(c'(t))^3}\right)(A^{-1})'(t) - aA^{-1}(t)\right] = 0.$$

Finally, dividing by $\frac{c(t)}{(c'(t))^2}$ gives the result. $\square$

Now we are ready to prove Theorem 3.2.3.

Proof of Theorem 3.2.3. By Lemma 3.2.4, we have that the time depended version of (3.12) is given by (3.13), i.e.,

$$A^{-1}(t)y''(t) + \left[2(A^{-1})'(t) + \left(\frac{bc'(t)}{c(t)} - c'(t) - \frac{c''(t)}{c'(t)}\right)A^{-1}(t)\right]y'(t) + \left[(A^{-1})''(t) + \left(\frac{bc'(t)}{c(t)} - c'(t) - \frac{c''(t)}{c'(t)}\right)(A^{-1})'(t) - a\frac{(c'(t))^2}{c(t)}A^{-1}(t)\right]y(t) = 0.$$

Observe that we will need the derivatives of c and A^{-1}.

$$A^{-1}(t) = \exp\left\{\frac{1}{4}t\left(2\lambda - \zeta\rho u\frac{t}{T} + \frac{\rho^2\zeta u\frac{t}{T} - 2\lambda\rho + \zeta(1-u\frac{t}{T})}{\xi^{\frac{1}{2}}}\right)\right\},$$

$$
\begin{aligned}
(A^{-1})'(t) &= A^{-1}(t) \cdot \frac{1}{4}\left(2\lambda - 2\zeta\rho u\frac{t}{T} + \frac{2\rho^2\zeta u\frac{t}{T} - 2\rho + \zeta(1-2u\frac{t}{T})}{\xi^{\frac{1}{2}}}\right) \\
&= A^{-1}(t) \cdot \frac{1}{2}\left(\lambda - \zeta\rho u\frac{t}{T} + c'(t)\right), \\
(A^{-1})''(t) &= A^{-1}(t)\Bigg\{\frac{1}{4}\left(\lambda^2 - 2\zeta\lambda\rho u\frac{t}{T} + \zeta^2\rho^2u^2\frac{t^2}{T^2} + 2\left(\lambda - \zeta\rho u\frac{t}{T}\right)c'(t) + (c'(t))^2\right) \\
&\qquad - \frac{1}{2}\zeta\rho\frac{u}{T} + \frac{1}{2}c''(t)\Bigg\}, \\
c(t) &= \frac{1}{2}\frac{\left(\left(\frac{1}{2} + \xi u\frac{t}{T}\right)\zeta - \lambda\rho\right)^2}{\zeta\frac{u}{T}\xi^{\frac{3}{2}}}, \\
c'(t) &= \frac{\left(\frac{1}{2} + \xi u\frac{t}{T}\right)\zeta - \lambda\rho}{\xi^{\frac{1}{2}}}, \\
c''(t) &= \xi^{\frac{1}{2}}\zeta\frac{u}{T}.
\end{aligned}
$$

For the argument in front of $y'(t)$, we have

$$
\begin{aligned}
&2(A^{-1})'(t) + \left(\frac{bc'(t)}{c(t)} - c'(t) - \frac{c''(t)}{c'(t)}\right)A^{-1}(t) \\
=&A^{-1}(t)\left(\lambda - \zeta\rho u\frac{t}{T} + c'(t) + \frac{1}{2}\frac{c'(t)}{c(t)} - c'(t) - \frac{c''(t)}{c'(t)}\right) \\
=&A^{-1}(t)\left(\lambda - \zeta\rho u\frac{t}{T}\right),
\end{aligned}
$$

since

$$
\frac{1}{2}\frac{c'(t)}{c(t)} = \frac{\zeta\frac{u}{T}\xi}{\left(\frac{1}{2} + \xi u\frac{t}{T}\right)\zeta - \lambda\rho} = \frac{c''(t)}{c'(t)}.
$$

For the argument in front of $y(t)$, we get

$$
\begin{aligned}
&(A^{-1})''(t) + \left(\frac{bc'(t)}{c(t)} - c'(t) - \frac{c''(t)}{c'(t)}\right)(A^{-1})'(t) - a\frac{(c'(t))^2}{c(t)}A^{-1}(t) \\
=&(A^{-1})''(t) - c'(t)(A^{-1})'(t) - a\frac{(c'(t))^2}{c(t)}A^{-1}(t) \\
=&A^{-1}(t)\Bigg\{\frac{1}{4}\left(\lambda^2 - 2\zeta\lambda\rho u\frac{t}{T} + \zeta^2\rho^2u^2\frac{t^2}{T^2} + 2\left(\lambda - \zeta\rho u\frac{t}{T}\right)c'(t) + (c'(t))^2\right)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{1}{2}\zeta\rho\frac{u}{T}+\frac{1}{2}c''(t)\Bigg\}-c'(t)(A^{-1})(t)\cdot\frac{1}{2}\left(\lambda-\zeta\rho u\frac{t}{T}+c'(t)\right)-a\frac{(c'(t))^2}{c(t)}(A^{-1})(t)\\
=&A^{-1}(t)\Bigg\{\frac{1}{4}\left(\lambda^2-2\zeta\lambda\rho u\frac{t}{T}+\zeta^2\rho^2u^2\frac{t^2}{T^2}+2\left(\lambda-\zeta\rho u\frac{t}{T}\right)c'(t)+(c'(t))^2\right)\\
&-\frac{1}{2}\zeta\rho\frac{u}{T}+\frac{1}{2}c''(t)-c'(t)\frac{1}{2}\left(\lambda-\zeta\rho u\frac{t}{T}+c'(t)\right)-a\frac{(c'(t))^2}{c(t)}\Bigg\}\\
=&A^{-1}(t)\frac{\zeta^2}{4}u\frac{t}{T}\left(u\frac{t}{T}-1\right),
\end{aligned}
$$

which is exactly what we wanted. For the interested reader, we give the details of the algebra:

$$
\begin{aligned}
&\frac{1}{4}\left[\lambda^2-2\zeta\lambda\rho u\frac{t}{T}+\zeta^2\rho^2u^2\frac{t^2}{T^2}+2\left(\lambda-\zeta\rho u\frac{t}{T}\right)c'(t)+(c'(t))^2\right]\\
&-\frac{1}{2}\zeta\rho\frac{u}{T}+\frac{1}{2}c''(t)-c'(t)\frac{1}{2}\left(\lambda-\zeta\rho u\frac{t}{T}+c'(t)\right)-a\frac{(c'(t))^2}{c(t)}\\
=&\frac{1}{4}\left(\lambda-\zeta\rho u\frac{t}{T}\right)^2-\frac{1}{2}\left(\lambda-\zeta\rho u\frac{t}{T}\right)c'(t)+\frac{1}{4}(c'(t))^2\\
&-\frac{1}{2}\zeta\rho\frac{u}{T}+\frac{1}{2}c''(t)-\frac{1}{2}c'(t)\left(\lambda-\zeta\rho u\frac{t}{T}\right)-\frac{1}{2}(c'(t))^2-a\frac{(c'(t))^2}{c(t)}\\
=&\frac{1}{4}\left(\lambda-\zeta\rho u\frac{t}{T}\right)^2-\frac{1}{2}\zeta\rho\frac{u}{T}+\frac{1}{2}\xi^{\frac{1}{2}}\zeta\frac{u}{T}-\frac{1}{4}\frac{\left(\left(\frac{1}{2}+\xi u\frac{t}{T}\right)\zeta-\lambda\rho\right)^2}{\xi}-2a\zeta\frac{u}{T}\xi^{\frac{1}{2}}\\
=&\frac{1}{4}\left(\lambda-\zeta\rho u\frac{t}{T}\right)^2-\frac{1}{2}\zeta\rho\frac{u}{T}-\frac{1}{4}\frac{\left(\left(\frac{1}{2}+\xi u\frac{t}{T}\right)\zeta-\lambda\rho\right)^2}{\xi}-\frac{-\lambda^2+\rho\zeta\left(\lambda-2\frac{u}{T}\xi\right)-\frac{\zeta^2}{4}}{4\xi}\\
=&\frac{1}{4}\left[\left(\lambda-\zeta\rho u\frac{t}{T}\right)^2-\frac{\left(\left(\frac{1}{2}+\xi u\frac{t}{T}\right)\zeta-\lambda\rho\right)^2-\lambda^2+\rho\zeta\lambda-\frac{\zeta^2}{4}}{\xi}\right]\\
=&\frac{1}{4}\left[\lambda^2-2\lambda\zeta\rho u\frac{t}{T}+\zeta^2\rho^2u^2\frac{t^2}{T^2}-u\frac{t}{T}\zeta^2-\xi u^2\frac{t^2}{T^2}\zeta^2+2u\frac{t}{T}\zeta\lambda\rho-\lambda^2\right]\\
=&\frac{\zeta^2}{4}u\frac{t}{T}\left(u\frac{t}{T}-1\right),
\end{aligned}
$$

where the last line is due to $\xi=\rho^2-1$. $\square$

According to [18], the second independent solution to (3.11) is given by

$$U(a,b,c(t)) = \frac{\Gamma(1-b)}{\Gamma(a-b+1)}M(a,b,c(t)) + \frac{\Gamma(b-1)}{\Gamma(a)}c(t)^{1-b}M(a-b+1,2-b,c(t)). \tag{3.15}$$

Hence, we immediately have the following.

Corollary 3.2.5. The solution to (3.11) can be expressed as a linear combination of

$$y_1(t) = A(t) \cdot M\left(a, \frac{1}{2}, c(t)\right)$$

and

$$y_2(t) = A(t) \cdot B(t) \cdot M\left(a + \frac{1}{2}, \frac{3}{2}, c(t)\right),$$

where

$$B(t) = \left(\frac{1}{2} + \xi u \frac{t}{T}\right)\zeta - \lambda\rho.$$

Now, going back to (3.9), i.e.,

$$\dot{\psi} = \frac{\zeta^2}{2}\psi^2 - \left(\lambda - \rho\zeta u \frac{t}{T}\right)\psi + \frac{1}{2}u\frac{t}{T}\left(u\frac{t}{T} - 1\right),$$

and recalling that we have the initial condition $\psi(0) = 0$, we have the following.

Corollary 3.2.6. The solution to (3.9) is given by

$$\psi(t) = -\frac{2}{\zeta^2}\frac{y_2'(0)y_1'(t) - y_1'(0)y_2'(t)}{y_2'(0)y_1(t) - y_1'(0)y_2(t)}.$$

Furthermore, we get the solution to (3.8) as

$$\phi(t) = -\lambda\theta\frac{2}{\zeta^2}\ln\left\{\frac{y_2'(0)y_1(t) - y_1'(0)y_2(t)}{y_2'(0)y_1(0) - y_1'(0)y_2(0)}\right\}.$$

3.2.2 Average Strike. For the average strike, we obtain the Riccati equations

$$\dot{\phi} = \lambda\theta\psi, \qquad \phi(0) = 0, \tag{3.16}$$

$$\begin{aligned}\dot{\psi} &= \frac{\zeta^2}{2}\psi^2 - \left(\lambda - \rho\zeta\left(u\left(\frac{t}{T} - 1\right) + 1\right)\right)\psi \\ &\quad + \frac{1}{2}\left[u^2\left(\frac{t}{T} - 1\right)^2 + u\left(\frac{t}{T} - 1\right)\right], \psi(0) = 0,\end{aligned} \tag{3.17}$$

which are essentially the same differential equations, and they can be solved in the same fashion as described above, i.e., we can use the transformation (3.10) to obtain that (3.17) is equivalent to

$$y'' + \left(\lambda - \rho\zeta\left(u\left(\frac{t}{T} - 1\right) + 1\right)\right)y' + \frac{\zeta^2}{4}\left[u^2\left(\frac{t}{T} - 1\right)^2 + u\left(\frac{t}{T} - 1\right)\right]y = 0. \tag{3.18}$$

Theorem 3.2.7. (i) One solution to (3.18) can be expressed in terms of Kummer's confluent hypergeometric function $M(a, b, c(t)) = {}_1F_1(a, b, c(t))$ with parameters

$$\begin{aligned} a &= \frac{-\lambda^2 + \rho\zeta\left(\lambda - 2\frac{u}{T}\xi\right) - \frac{\zeta^2}{4}}{8\xi^{\frac{3}{2}}\frac{u}{T}\zeta} + \frac{1}{4}, \\ b &= \frac{1}{2}, \\ c(t) &= \frac{1}{2}\frac{\left(\zeta\left(\xi\left(u\left(\frac{t}{T} - 1\right) + 1\right) + \frac{1}{2}\right) - \lambda\rho\right)^2}{\xi^{\frac{3}{2}}\frac{u}{T}\zeta}. \end{aligned}$$

Furthermore, we have $\xi = \rho^2 - 1$ and we need the integrating factor

$$A(t) = \exp\Bigg\{ -\frac{1}{4}t\Bigg(2\lambda - \zeta\rho\left(u\frac{t}{T} + 2(1-u)\right) \\ + \frac{\xi\zeta\left(u\frac{t}{T} + 2(1-u)\right) - 2\lambda\rho + \zeta}{\xi^{\frac{1}{2}}}\Bigg)\Bigg\}.$$

(ii) Recalling (3.15), the second solution to (3.18) can be expressed as a linear combination of

$$y_1(t) = A(t) \cdot M\left(a, \frac{1}{2}, c(t)\right)$$

and

$$y_2(t) = A(t) \cdot B(t) \cdot M\left(a + \frac{1}{2}, \frac{3}{2}, c(t)\right),$$

where

$$B(t) = \zeta\left(\xi\left(u\left(\frac{t}{T} - 1\right) + 1\right) + \frac{1}{2}\right) - \lambda\rho.$$

Proof. Observe that

$$\begin{aligned}
c'(t) &= \frac{\zeta\left(\xi\left(u\left(\frac{t}{T} - 1\right) + 1\right) + \frac{1}{2}\right) - \lambda\rho}{\xi^{\frac{1}{2}}} \\
c''(t) &= \zeta\frac{u}{T}\xi^{\frac{1}{2}} \\
(A^{-1})'(t) &= A^{-1}(t)\frac{1}{4}\left[2\lambda - 2\zeta\rho\left(u\frac{t}{T} + 1 - u\right) + \frac{2\zeta\xi\left(u\frac{t}{T} + 1 - u\right)}{\xi^{\frac{1}{2}}}\right] \\
&= A^{-1}(t)\frac{1}{2}\left[\lambda - \zeta\rho\left(u\frac{t}{T} + 1 - u\right) + c'(t)\right] \\
(A^{-1})''(t) &= A^{-1}(t)\Bigg\{\frac{1}{4}\left(\lambda - \zeta\rho\left(u\frac{t}{T} + 1 - u\right) + (c'(t))^2\right)^2 \\
&\qquad - \frac{1}{2}\zeta\rho\frac{u}{T} + \frac{1}{2}c''(t)\Bigg\},
\end{aligned}$$

such that using (3.13), i.e.,

$$A^{-1}(t)y''(t) + \left[2(A^{-1})'(t) + \left(\frac{bc'(t)}{c(t)} - c'(t) - \frac{c''(t)}{c'(t)}\right) A^{-1}(t)\right] y'(t)$$
$$+ \left[(A^{-1})''(t) + \left(\frac{bc'(t)}{c(t)} - c'(t) - \frac{c''(t)}{c'(t)}\right)(A^{-1})'(t) - a\frac{(c'(t))^2}{c(t)} A^{-1}(t)\right] y(t) = 0,$$

gives that the factor in front of $y'(t)$ is given by

$$\lambda - \zeta\rho\left(u\frac{t}{T} + 1 - u\right),$$

since we still have that $\frac{bc'(t)}{c(t)} = \frac{c''(t)}{c'(t)}$.

For the factor in front of $y(t)$, we have

$$\begin{aligned}
&\frac{1}{4}\left(\lambda - \zeta\rho\left(u\frac{t}{T} + 1 - u\right) + (c'(t))^2\right)^2 - \frac{1}{2}\zeta\rho\frac{u}{T} \\
&+ \frac{1}{2}c''(t) - \frac{1}{2}c'(t)\left[\lambda - \zeta\rho\left(u\frac{t}{T} + 1 - u\right) + c'(t)\right] - a\frac{(c'(t))^2}{c(t)} \\
=&\frac{1}{4}\left(\lambda - \zeta\rho\left(u\frac{t}{T} + 1 - u\right)\right)^2 - \frac{1}{2}\zeta\rho\frac{u}{T} - \frac{1}{4}c'(t) - a\frac{(c'(t))^2}{c(t)} \\
=&\frac{1}{4}\left(\lambda - \zeta\rho\left(u\frac{t}{T} + 1 - u\right)\right)^2 - \frac{1}{2}\zeta\rho\frac{u}{T} \\
&+ \frac{1}{4\xi}\left\{\lambda^2 - \rho\zeta\lambda + \frac{\zeta^2}{4} - \zeta^2\left[\xi^2\left(u\left(\frac{t}{T} - 1\right) + 1\right)^2 + \xi\left(u\left(\frac{t}{T} - 1\right) + 1\right) + \frac{1}{4}\right]\right. \\
&\left. + 2\zeta\rho\lambda\left(\xi\left(u\left(\frac{t}{T} - 1\right) + 1\right) + \frac{1}{2}\right) - \lambda^2\rho^2\right\} \\
=&\frac{1}{4}\left\{\left(\lambda - \zeta\rho\left(u\frac{t}{T} + 1 - u\right)\right)^2 - \frac{1}{2}\zeta\rho\frac{u}{T} - \lambda^2 - \zeta^2\left(u\left(\frac{t}{T} - 1\right) + 1\right)\right. \\
&\left. + 2\lambda\rho\zeta\left(u\left(\frac{t}{T} - 1\right) + 1\right) - \xi\left(u\left(\frac{t}{T} - 1\right) + 1\right)^2\zeta^2\right\} \\
=&\frac{\zeta^2}{4}\left[\left(u\left(\frac{t}{T} - 1\right) + 1\right)^2 - \left(u\left(\frac{t}{T} - 1\right) + 1\right)\right].
\end{aligned}$$

Hence, we have the claim. ∎

Remark 3.2.8. Our results for the Riccati equations of both the average price and the average strike do not match the results as given in [11].

3.3 THE BATES MODEL

For the Bates model, as introduced in [4], we add a jump process to the general Heston model. In particular, we take a compound Poisson process, which is given by

$$Z = (Z_t)_{t\geq 0}, \quad Z_t = \sum_{k=1}^{N_t} J_k,$$

where $(N_t)_{t\geq 0}$ is a Poisson process with intensity ν, i.e., $\mathbb{E}[N_t] = \nu t$ and $J_k \overset{iid}{\sim} \mathcal{N}(\gamma, \delta^2)$. Furthermore the Poisson process is independent of J_k.

Lemma 3.3.1. The Lévy measure of Z is given by

$$U(dx) = \frac{\nu}{\delta\sqrt{2\pi}} \exp\left\{-\frac{(x-\gamma)^2}{2\delta^2}\right\},$$

and its cumulant moment generating function is given by

$$\kappa(u) = \log \mathbb{E}\left[e^{uZ_1}\right] = \nu\left(e^{\gamma u + \frac{1}{2}\delta^2 u^2} - 1\right).$$

Proof. Let us denote the characteristic function of the distribution $\mathcal{N}(\gamma, \delta^2)$ by ϕ_J. Then

$$\begin{aligned}
\phi_{Z_t}(u) &= \mathbb{E}\left[e^{iuZ_t}\right] = \mathbb{E}\left[e^{iu\sum_{k=1}^{N_t} J_k}\right] = \sum_{n=0}^{\infty} \mathbb{E}\left[e^{iu\sum_{k=1}^{n} J_k}\right] \mathbb{P}\left[N_t = n\right] \\
&= e^{-\nu t} \sum_{n=0}^{\infty} \frac{(\nu t)^n}{n!} \prod_{k=1}^{n} \mathbb{E}\left[e^{iuJ_k}\right] = e^{-\nu t} \sum_{n=0}^{\infty} \frac{(\nu t)^n}{n!} \mathbb{E}\left[e^{iuJ_1}\right]^n \\
&= e^{-\nu t} \sum_{n=0}^{\infty} \frac{(\nu t \phi_J(u))^n}{n!} = e^{\nu t(\phi_J(u)-1)}
\end{aligned}$$

$$= \exp\left\{ t \int_{\mathbb{R}} \left(e^{iux} - 1\right) \frac{\nu}{\delta\sqrt{2\pi}} \exp\left\{ -\frac{(x-\gamma)^2}{2\delta^2} \right\} dx \right\}.$$

The Lévy measure is uniquely defined such that it is given by $U(dx)$. Furthermore, we have

$$\begin{aligned}
\kappa(u) &= \log \mathbb{E}\left[e^{uZ_1}\right] = \log \mathbb{E}\left[e^{u\sum_{k=1}^{N_1} J_k}\right] = \log\left(\sum_{n=0}^{\infty} \mathbb{E}\left[e^{u\sum_{k=1}^{n} J_k}\right] \mathbb{P}\left[N_1 = n\right] \right) \\
&= \log\left(e^{-\nu} \sum_{n=0}^{\infty} \frac{\nu^n}{n!} \mathbb{E}\left[e^{uJ_1}\right]^n \right) = \log\left(e^{-\nu} \sum_{n=0}^{\infty} \frac{\left(\nu \mathbb{E}\left[e^{uJ_1}\right]\right)^n}{n!} \right) \\
&= \nu\left(e^{u\gamma + \frac{1}{2}u^2\delta^2} - 1 \right).
\end{aligned}$$

Hence, we have the result. □

The dynamics in the Bates model are given by

$$\begin{aligned}
dX_t &= \left(-\kappa(1) - \frac{1}{2}V_t \right) dt + \sqrt{V_t} W_t^1 + dZ_t, \\
dV_t &= \lambda(\theta - V_t)dt + \zeta\sqrt{V_t} dW_t^2.
\end{aligned}$$

Before we go on to obtain the functional characteristics, we recall some theory regarding stochastic integration with respect to Lévy processes.

Definition 3.3.2. Let $(N_t)_{t\geq 0}$ be a Poisson process with intensity λ. Then we denote the compensated Poisson process by

$$\left(\widetilde{N}_t\right)_{t\geq 0}, \quad \widetilde{N}_t = N_t - \mathbb{E}N_t = N_t - \lambda t.$$

Remark 3.3.3. Observe that $\left(\widetilde{N}_t\right)_{t\geq 0}$ is a martingale and hence, integrating a L^2-function w.r.t. $\widetilde{N}_t$ also yields a martingale.

Theorem 3.3.4. (Weak version of Itô's lemma for Lévy processes) Let Y be a Lévy type integral, i.e.,

$$Y_t = Y_0 + \int_0^t G(s)ds + \int_0^t F(s)dW_s + \int_0^t K(s)dN_s.$$

Then for $f \in C^2(\mathbb{R}^n),\ t \geq 0$, we have

$$\begin{aligned} f(Y_t) - f(Y_0) = &\sum_{i=1}^{n} \int_0^t \partial_{x_i} f(Y_{s-}) dY_i^c(s) + \sum_{i,j=1}^{n} \frac{1}{2} \int_0^t \partial_{x_i x_j} f(Y_{s-}) d[Y_i^c, Y_j^c]_s \\ &+ \int_0^t \left[f(Y_{s-} + K(s)) - f(Y_{s-}) \right] dN_s, \end{aligned}$$

where Y^c denotes the continuous part of the process Y.

Remark 3.3.5. For the compound Poisson process defined above, observe that since $Z_0 = N_0 = 0$ a.s., it follows that

$$Z_t = \sum_{k=1}^{N_t} J_k = \int_0^t J_{N_s} dN_s$$

and hence

$$dZ_t = J_{N_t} dN_t.$$

Theorem 3.3.6. The functional characteristics are, as stated in [11], given by

$$F(u, w) = \lambda\theta w + \kappa(u) - u\kappa(1) \qquad R(u, w) = \frac{1}{2}(u^2 - u) + \frac{\zeta^2}{2} w^2 - \lambda w + uw\rho\zeta. \tag{3.19}$$

Proof. Let $f(x,v) = e^{ux+wv}$. Then, by the weak version of Itô's lemma (Theorem 3.3.4), we have

$$
\begin{aligned}
e^{uX_t+wV_t} - e^{uX_0+wV_0} &= \int_0^t e^{uX_{s-}+wV_{s-}} \left[udX_s^c + wY_s^c\right] \\
&+ \int_0^t e^{uX_{s-}+wV_{s-}} \frac{1}{2}\big(u^2 d[X^c,X^c]_s + 2uwd[X^c,V^c]_s + w^2 d[V^c,V^c]_s\big) \\
&+ \int_0^t \left[f(X_{s-}+J_{N_s},V_{s-}) - f(X_{s-},V_{s-})\right] dN_s \\
&= \int_0^t e^{uX_{s-}+wV_{s-}} \left[w\lambda\theta - u\kappa(1)\right] ds \\
&+ \int_0^t V_s \left(\frac{1}{2}(u^2-u) + \frac{\zeta^2}{2}w^2 - \lambda w + \rho uw\zeta\right) ds + \text{MART} \\
&+ \int_0^t \left[f(X_{s-}+J_{N_s},V_{s-}) - f(X_{s-},V_{s-})\right] dN_s,
\end{aligned}
$$

where MART are integrals with respect to the Brownian motion which are martingales and hence will be dropped when we take the expected value to obtain

$$
\begin{aligned}
\mathbb{E}\left[e^{uX_t+wV_t} - e^{uX_0+wV_0}|X_0,V_0\right] &= \int_0^t \mathbb{E}\left[e^{uX_{s-}+wV_{s-}}\left(-u\kappa(1)+w\lambda\theta\right)|X_0,V_0\right] ds \\
&+ \int_0^t \mathbb{E}\left[e^{uX_{s-}+wV_{s-}} V_s \left(\frac{1}{2}(u^2-u) + \frac{\zeta^2}{2}w^2 - \lambda w + \rho uw\zeta\right) \Big| X_0,V_0\right] ds \\
&+ \mathbb{E}\left[\int_0^t [f(X_{s-}+J_{N_s},V_{s-}) - f(X_{s-},V_{s-})]dN_s \Big| X_0,V_0\right].
\end{aligned}
$$

Now, since $\widetilde{N}_t$ is a martingale, we get

$$
\begin{aligned}
&\mathbb{E}\left[\int_0^t \left[f(X_{s-}+J_{N_s},V_{s-}) - f(X_{s-},V_{s-})\right] dN_s|X_0,V_0\right] \\
&= \mathbb{E}\left[\int_0^t \left[f(X_{s-}+J_{N_s},V_{s-}) - f(X_{s-},V_{s-})\right] d(\widetilde{N}_s+\nu s)|X_0,V_0\right] \\
&= \mathbb{E}\left[\int_0^t \left[f(X_{s-}+J_{N_s},V_{s-}) - f(X_{s-},V_{s-})\right] \nu ds|X_0,V_0\right] \\
&= \int_0^t \mathbb{E}\left[e^{uX_{s-}+wV_{s-}}\left(e^{uJ_{N_s}}-1\right)\nu|X_0,V_0\right] ds \\
&= \int_0^t \mathbb{E}\left[e^{uX_{s-}+wV_{s-}}|X_0,V_0\right] \mathbb{E}\left[\left(e^{uJ_1}-1\right)\nu|X_0,V_0\right] ds
\end{aligned}
$$

$$= \int_0^t \mathbb{E}\left[e^{uX_{s-}+wV_{s-}}|X_0, V_0\right]\kappa(u)ds.$$

Similarly to the proof in the Heston model, we have

$$\begin{aligned}\frac{\frac{\partial}{\partial t}\mathbb{E}\left[e^{uX_t+wY_t}|X_0, V_0\right]|_{t=0}}{e^{uX_0+wY_0}} =& \lambda\theta w + \kappa(u) - u\kappa(1)\\ &+ V_0\left(\frac{1}{2}(u^2-u) + \frac{\zeta^2}{2}w^2 - \lambda w + \rho u w \zeta\right).\end{aligned}$$

In conclusion, we get

$$F(u,w) = \lambda\theta w + \kappa(u) - u\kappa(1), \qquad R(u,w) = \frac{1}{2}(u^2-u) + \frac{\zeta^2}{2}w^2 - w\lambda + \zeta u w \rho,$$

i.e., the claim holds. □

3.3.1 Average Price. Having the functional characteristics, we can now, recalling that $u := \frac{ut}{T}$, $w := \psi$, state the Riccati equations for the average price as

$$\begin{aligned}\dot{\phi} &= \lambda\theta\psi + \kappa\left(\frac{ut}{T}\right) - \frac{ut}{T}\kappa(1), && \phi(0) = 0,\\ \dot{\psi} &= \frac{\zeta^2}{2}\psi^2 - \left(\lambda - \rho\zeta\frac{ut}{T}\right)\psi + \frac{1}{2}\frac{ut}{T}\left(\frac{ut}{T} - 1\right), && \psi(0) = 0.\end{aligned}$$

Observe that the equation for ψ in the Bates model coincides with the equation for ψ in the Heston model and that the equation for ϕ in the Bates model has just two additional terms that depend on neither ϕ nor ψ. Thus we have

$$\begin{aligned}\phi_{\text{Bates}}(t) &= \phi_{\text{Heston}}(t) + \int_0^t \kappa\left(\frac{us}{T}\right)ds - \frac{ut^2}{2T}\kappa(1)\\ \psi_{\text{Bates}}(t) &= \psi_{\text{Heston}}(t).\end{aligned}$$

3.3.2 Average Strike. For the average strike, the Riccati equations are given by

$$\begin{aligned}
\dot{\phi} =& \lambda\theta\psi + \kappa\left(u\left(\frac{t}{T}-1\right)+1\right) - \left(u\left(\frac{t}{T}-1\right)+1\right)\kappa(1), \qquad \phi(0)=0,\\
\dot{\psi} =& \frac{\zeta^2}{2}\psi^2 - \left(\lambda - \rho\zeta\left(u\left(\frac{t}{T}-1\right)+1\right)\right)\psi\\
&+\frac{1}{2}\left(u^2\left(\frac{t}{T}-1\right)^2 + u\left(\frac{t}{T}-1\right)\right), \qquad \psi(0)=0.
\end{aligned}$$

Again, the equation for ψ is the same as in the Heston model. Hence we get

$$\begin{aligned}
\phi_{\text{Bates}}(t) &= \phi_{\text{Heston}}(t) + \int_0^t \kappa\left(u\left(\frac{s}{T}-1\right)+1\right)ds - t\left(u\left(\frac{t}{2T}-1\right)+1\right)\kappa(1)\\
\psi_{\text{Bates}}(t) &= \psi_{\text{Heston}}(t).
\end{aligned}$$

3.4 THE TURBO-BATES MODEL

To refine the Bates model, the jump intensity is now assumed to be state dependent. In [11], a simplified version of the turbo-Bates model is considered, where the dynamics are given by

$$\begin{aligned}
dX_t &= \left(-\nu_0\kappa(1) - \left(\frac{1}{2}+\nu_1\kappa(1)\right)V_t\right)dt + \sqrt{V_t}dW_t^1 + \int_D x\widetilde{N}(V_t, dt, dx),\\
dV_t &= -\lambda(V_t-\theta)dt + \zeta\sqrt{V_t}dW_t^2,
\end{aligned}$$

where $\widetilde{N}(V_t, dt, dx) = N(V_t, dt, dx) - \mu(V_t, dt, dx)$ and $N(V_t, dt, dx)$ is a Poisson random measure with predictable compensator $\mu(V_t, dt, dx) = (\nu_0 + \nu_1 V_t)F(dx)dt$ and F being some fixed jump size distribution. Furthermore $\kappa(u)$ denotes the cumulant generating function of F.

Theorem 3.4.1. The functional characteristics are given by

$$F(u,w) = \nu_0\kappa(u) - u\nu_0\kappa(1) + \lambda\theta w,$$
$$R(u,w) = \frac{1}{2}(u^2-u) + \frac{\zeta^2}{2}w^2 - \lambda w + \rho\zeta uw + \nu_1\kappa(u) - u\nu_1\kappa(1),$$

Proof. Let $f(x,v) = e^{ux+vw}$. Then, by the Itô formula for general Lévy processes (Theorem 1.6.5), we get

$$\begin{aligned}
e^{uX_t+wV_t} - e^{uX_0+wV_0} &= \int_0^t e^{uX_{s-}+wV_{s-}}(udX_s^c + wdV_s^c) \\
&+ \int_0^t e^{uX_{s-}+wV_{s-}}\left(\frac{1}{2}u^2 d[X^c,X^c]_s + uwd[X^c,V^c]_s + \frac{1}{2}w^2 d[V^c,V^c]_s\right) \\
&+ \int_0^t \int_D \left(f(X_{s-}+x,V_{s-}) - f(X_{s-},V_{s-})\right) d\widetilde{N}(V_s,ds,dx) \\
&+ \int_0^t \int_D \left(f(X_{s-}+x,V_{s-}) - f(X_{s-},V_{s-}) - uf'(X_{s-},V_{s-})\right)\mu(V_s,ds,dx) \\
&= \int_0^t e^{uX_{s-}+wV_{s-}}\left(\left[-u\nu_0\kappa(1) - u\left(\frac{1}{2}+\nu_1\kappa(1)\right)V_s\right]ds + u\sqrt{V_s}dW_s^1\right) \\
&+ \int_0^t e^{uX_{s-}+wV_{s-}}\left((\lambda\theta w - \lambda wV_s)ds + w\zeta\sqrt{V_s}dW_s^2\right) \\
&+ \int_0^t e^{uX_{s-}+wV_{s-}}\left(\frac{1}{2}u^2V_s + \rho uw\zeta V_s + \frac{1}{2}w^2\zeta^2V_s\right)ds \\
&+ \int_0^t \int_D e^{uX_{s-}+wV_{s-}}\left(e^{ux}-1\right) d\widetilde{N}(V_s,ds,dx) \\
&+ \int_0^t \int_D e^{uX_{s-}+wV_{s-}}\left(e^{ux}-ux-1\right)\mu(V_s,ds,dx).
\end{aligned}$$

Taking expectations and recalling that the integrals w.r.t. W_t^1, W_t^2 and also w.r.t. $\widetilde{N}(V_t,dt,dx)$ are martingales gives

$$\begin{aligned}
\mathbb{E}\left[e^{uX_t+wV_t} - e^{uX_0+wV_0}|X_0,V_0\right] &= \mathbb{E}\left[\int_0^t e^{uX_{s-}+wV_{s-}}\left(-u\nu_0\kappa(1)+\lambda\theta w\right)ds\Big|X_0,V_0\right] \\
&+ \mathbb{E}\left[\int_0^t e^{uX_{s-}+wV_{s-}}\left(\frac{1}{2}(u^2-u) - \nu_1\kappa(1) - \lambda w + \rho uw\zeta + \frac{\zeta^2}{2}w^2\right)V_s ds\Big|X_0,V_0\right] \\
&+ \mathbb{E}\left[\int_0^t \int_D e^{uX_{s-}+wV_{s-}}(e^{ux}-1-ux)\mu(V_s,ds,dx)\Big|X_0,V_0\right].
\end{aligned}$$

Inspecting the last line, we find

$$\begin{aligned}
&\mathbb{E}\left[\int_0^t \int_D e^{uX_{s-}+wV_{s-}}(e^{ux}-ux-1)\mu(V_s,ds,dx)\Big|X_0,V_0\right] \\
&= \mathbb{E}\left[\int_0^t \int_D e^{uX_{s-}+wV_{s-}}(e^{ux}-ux-1)(\nu_0+\nu_1 V_s)F(dx)ds\Big|X_0,V_0\right] \\
&= \mathbb{E}\left[\int_0^t e^{uX_{s-}+wV_{s-}}(\nu_0+\nu_1 V_s)\int_D (e^{ux}-1-ux)F(dx)ds\Big|X_0,V_0\right] \\
&= \int_0^t \mathbb{E}\left[e^{uX_{s-}+wV_{s-}}(\nu_0+\nu_1 V_s)\int_D (e^{ux}-1-ux)F(dx)\Big|X_0,V_0\right] ds \\
&= \int_0^t \mathbb{E}\left[e^{uX_{s-}+wV_{s-}}(\nu_0+\nu_1 V_s)\kappa(u)|X_0,V_0\right] ds,
\end{aligned}$$

where the last step is a consequence of the Lévy–Khintchine formula for Lévy processes (Theorem 1.2.3). Hence using the same arguments as in the previous models gives the claim, i.e.,

$$\begin{aligned}
F(u,w) &= \nu_0\kappa(u) - u\nu_0\kappa(1) + \lambda\theta w, \\
R(u,w) &= \frac{1}{2}(u^2-u) + \frac{\zeta^2}{2}w^2 - \lambda w + \rho\zeta uw + \nu_1\kappa(u) - u\nu_1\kappa(1).
\end{aligned}$$

This concludes the proof. □

Using this knowledge, the Riccati equations for the average price are given by

$$\dot{\phi} = \lambda\theta\psi + \nu_0\kappa\left(\frac{ut}{T}\right) - \frac{ut}{T}\nu_0\kappa(1), \qquad \phi(0)=0, \tag{3.20}$$

$$\dot{\psi} = \frac{\zeta^2}{2}\psi^2 - \left(\lambda - \rho\zeta\frac{ut}{T}\right)\psi + \frac{1}{2}\frac{ut}{T}\left(\frac{ut}{T}-1\right) + \nu_1\kappa\left(\frac{ut}{T}\right) - \frac{ut}{T}\nu_1\kappa(1), \quad \psi(0)=0. \tag{3.21}$$

For the average strike, we get

$$\dot{\phi} = \lambda\theta\phi + \nu_0\kappa\left(\frac{ut}{T}+1-u\right) - \left(\frac{ut}{T}+1-u\right)\nu_0\kappa(1), \qquad \phi(0)=0,$$

$$\dot{\psi} = \frac{\zeta^2}{2}\psi^2 + \nu_1\kappa\left(\frac{ut}{T}+1-u\right) - \left(\frac{ut}{T}+1-u\right)\nu_1\kappa(1)$$
$$- \left(\lambda - \rho\zeta\left(\frac{ut}{T}+1-u\right)\right)\psi + \frac{1}{2}\left(u^2\left(\frac{t}{T}-1\right)^2 + u\left(\frac{t}{T}-1\right)\right), \quad \psi(0) = 0.$$

3.5 BARNDORFF-NIELSEN–SHEPHARD MODEL

This model was introduced in [3] by Ole Barndorff-Nielsen and Neil Shephard. It is constructed from a subordinator, i.e., a Lévy process that is nondecreasing a.s., which is called background driving Lévy process, BDLP for short. We assume that the cumulant function of the subordinator $\kappa(\theta) = \log \mathbb{E}[e^{\theta Z(1)}]$ exists for all $\Re(\theta) < l$ for some $l > 0$. The dynamics in this model are given as

$$dX_t = \left(-\kappa(\rho) - \frac{1}{2}V_{t-}\right)dt + \sqrt{V_{t-}}dW_t + \rho dZ_\lambda(t),$$
$$dV_t = -\lambda V_{t-}dt + dZ_\lambda(t),$$

where $V_0 > 0,\ \lambda > 0$ and $\rho \leq 0$.

Theorem 3.5.1. The functional characteristics are in this case given by

$$F(u,w) = \lambda\kappa(w+\rho u) - u\lambda\kappa(\rho),$$
$$R(u,w) = \frac{1}{2}(u^2-u) - \lambda w.$$

Proof. First observe that, since Z_λ is a subordinator, according to [15], it has the form

$$Z_\lambda(t) = Ct + \int_0^\infty \int_0^t xN(ds,dx),$$

where $N(ds,dx)$ is a Poisson random measure such that $N(ds,dx) = \lambda\nu(dx)ds$ with $\nu(dx)$ being the Lévy measure of Z_λ. Furthermore, we have $C = 0$ (c.f. [16]). Let now $f(x,v) = e^{ux+wv}$. Then, by Itô's formula (Theorem 1.6.5), we get

$$\begin{aligned} e^{uX_{t-}+wV_{t-}} - e^{uX_0+wV_0} &= \int_0^t e^{uX_{s-}+wV_{s-}} \left(udX_s^c + wdV_s^c\right) \\ &+ \int_0^t e^{uX_{s-}+wV_{s-}} \left(\frac{1}{2}u^2 d[X^c,X^c]_s + uwd[X^c,V^c]_s + \frac{1}{2}w^2 d[V^c,V^c]_s\right) \\ &+ \int_0^t \int_0^\infty \left(f(X_{s-}+x, V_{s-}+x) - f(X_{s-},V_{s-})\right) \lambda\nu(dx)ds \\ &= \int_0^t e^{uX_{s-}+wV_{s-}} \left(-u\kappa(\rho)ds + \left(\frac{1}{2}(u^2-u) - \lambda w\right) V_{s-}ds + u\sqrt{V_{s-}}dW_s\right) \\ &+ \int_0^t e^{uX_{s-}+wV_{s-}} \int_0^\infty \left(e^{(w+u\rho)x} - 1\right) \lambda\nu(dx)ds. \end{aligned}$$

Hence, taking expectations yields

$$\begin{aligned} &\mathbb{E}\left[e^{uX_{t-}+wV_{t-}} - e^{uX_0+wV_0} \middle| X_0, V_0\right] \\ =&\mathbb{E}\left[\int_0^t e^{uX_{s-}+wV_{s-}} \left(-u\kappa(\rho)ds + \left(\frac{1}{2}(u^2-u) - \lambda w\right) V_{s-}ds\right) \middle| X_0, V_0\right] \\ &+ \mathbb{E}\left[\int_0^t e^{uX_{s-}+wV_{s-}} \lambda\kappa(w+\rho u)ds \middle| X_0, V_0\right], \end{aligned}$$

where the last line is due to the Lévy–Khintchine formula for Lévy processes (Theorem 1.2.3). Following the proofs regarding the functional characteristics of the previous models yields

$$\begin{aligned} F(u,w) &= \lambda\kappa(w+\rho u) - u\lambda\kappa(\rho), \\ R(u,w) &= \frac{1}{2}(u^2-u) - \lambda w, \end{aligned}$$

which concludes the proof. □

3.5.1 Average Price. The Riccati equations for the average price are given by

$$\dot{\phi} = \lambda\kappa\left(\psi + \rho\frac{ut}{T}\right) - \frac{ut}{T}\lambda\kappa(\rho), \qquad \phi(0) = 0, \tag{3.22}$$

$$\dot{\psi} = \frac{1}{2}\frac{ut}{T}\left(\frac{ut}{T} - 1\right) - \lambda\psi, \qquad\qquad \psi(0) = 0. \tag{3.23}$$

Lemma 3.5.2. The solution to (3.23) is given by

$$\psi(t) = \frac{u^2}{2T^2}f_2(t) - \frac{u}{2T}f_1(t),$$

where

$$f_0(t) = \frac{1-e^{-\lambda t}}{\lambda}, \;\; f_1(t) = \frac{t}{\lambda} - \frac{1-e^{-\lambda t}}{\lambda^2}, \;\; f_2(t) = \frac{t^2}{\lambda} - \frac{2t}{\lambda^2} + \frac{2(1-e^{-\lambda t})}{\lambda^3}.$$

Proof. Observe that (3.23) is a linear first-order differential equation in one variable. Hence we take the integrating factor $e^{\int_0^t \lambda dx} = e^{\lambda t}$ and by basic theory of ODEs, the general solution is given by

$$\begin{aligned}\phi(t) &= e^{-\lambda t}\left[C + \int_0^t \frac{1}{2}\frac{us}{T}\left(\frac{us}{T} - 1\right)e^{\lambda s}ds\right]\\ &= e^{-\lambda t}\left[C + \frac{u^2}{2T^2}\int_0^t s^2 e^{\lambda s}ds - \frac{u}{2T}\int_0^t se^{\lambda s}ds\right]\\ &= e^{-\lambda t}\left[C + \frac{u^2}{2T^2}\left(\frac{e^{\lambda t}(\lambda^2t^2 - 2\lambda t + 2) - 2}{\lambda^3}\right) + \frac{u^2}{2T}\left(\frac{e^{\lambda t}(\lambda t - 1) + 1}{\lambda^2}\right)\right]\\ &= \frac{u^2}{2T^2}f_2(t) - \frac{u}{2T}f_1(t),\end{aligned}$$

noting that $\psi(0) = 0$ implies $C = 0$. □

Furthermore, we have the following immediate result.

Corollary 3.5.3. The solution to (3.22) is given by

$$\phi(t) = \int_0^t \lambda\kappa\left(\psi(s) + \rho\frac{us}{T}\right)ds - \frac{ut^2}{2T}\lambda\kappa(\rho).$$

3.5.2 Average Strike. The Riccati equations for the average strike are given by

$$\dot{\phi} = \lambda\kappa \left(\psi + \rho \left(\frac{ut}{T} + 1 - u\right)\right) - \left(u \left(\frac{t}{T} - 1\right) + 1\right) \lambda\kappa(\rho), \qquad \phi(0) = 0,$$
$$\dot{\psi} = \frac{1}{2} \left(u^2 \left(\frac{t}{T} - 1\right)^2 + u \left(\frac{t}{T} - 1\right)\right) - \lambda\psi, \qquad \psi(0) = 0.$$

Analogously as for the Riccati equations of the average price, one can show that

$$\psi(t) = \frac{u^2}{2T^2} f_2(t) + \frac{u}{T} \left(\frac{1}{2} - u\right) f_1(t) + \frac{1}{2}(u^2 - u) f_0(t)$$

as well as

$$\phi(t) = \int_0^t \lambda\kappa \left(\psi(s) + \rho \left(u \left(\frac{s}{T} - 1\right) + 1\right)\right) ds - t \left(u \left(\frac{t}{2T} - 1\right) + 1\right) \lambda\kappa(\rho).$$

3.6 OU TIME-CHANGED LÉVY PROCESSES

Time-changed Lévy processes were first introduced in [6]. They are able to reflect effects that do not follow the normal timeline. Clearly there are events that affect the price process faster and others that affect it slower, which makes time-changed models a good instrument to deal with these differences.

Let L be a Lévy process with cumulant function $\theta(u) = \log \mathbb{E}[e^{uL(1)}]$. Define

$$X_t := L\left(\Gamma(t)\right),$$

where $\Gamma(t)$ is a nonnegative increasing process that is independent of L. A popular time change is given as an integrated Ornstein–Uhlenbeck process.

Definition 3.6.1. The Ornstein–Uhlenbeck time change is given as

$$\Gamma(t) = \int_0^t V(s)ds,$$

where V is given by

$$dV(t) = -\lambda V_t dt + dU_t$$

with U being a pure jump subordinator and $\kappa(u)$ its cumulant function.

Theorem 3.6.2. Assuming the time change as introduced in Defintion 3.6.1, the functional characteristics are given by

$$F(u,w) = \lambda\kappa(w), \qquad R(u,w) = -\lambda w + \theta(u). \tag{3.24}$$

Proof. First of all, observe that according to [6], X is a pure jump process, such that taking $f(x,v) = e^{ux+wv}$, the Itô formula (Theorem 1.6.4) yields

$$\begin{aligned} e^{uX_t+wV_t} - e^{uX_0+wV_0} &= \int_0^t e^{uX_{s-}+wV_{s-}} w dV_s^c \\ &+ \int_0^t e^{uX_{s-}+wV_{s-}} \left(e^{ux}-1\right) dX_s + \int_0^t e^{uX_{s-}+wV_{s-}} \left(e^{wx}-1\right) dU_s \\ &= \int_0^t e^{uX_{s-}+wV_{s-}} \left(-\lambda w V_s\right) ds + \int_0^t e^{uX_{s-}+wV_{s-}} \theta(u) V_s ds \\ &+ \int_0^t e^{uX_{s-}+wV_{s-}} \lambda\kappa(w) ds, \end{aligned}$$

where the second equality is due to the independence of $\Gamma(t)$ and L, and due to the fact that V satisfies an Ornstein–Uhlenbeck equation. Hence, combining these integrals and doing the same things as before, we get

$$F(u,w) = \lambda\kappa(w), \qquad R(u,w) = -\lambda w + \theta(u).$$

This is the claim. □

3.6.1 Average Price. The Riccati equations for the average price are given by

$$\dot{\phi} = \lambda\kappa(\psi), \qquad \phi(0) = 0, \tag{3.25}$$

$$\dot{\psi} = -\lambda\psi + \theta\left(\frac{ut}{T}\right), \qquad \psi(0) = 0. \tag{3.26}$$

As an example, in [11] the Kou double exponential Lévy process with time change implied by an integrated OU process as in Definition 3.6.1 is proposed. The cumulant function θ is given by

$$\theta(u) = \nu u\left[\frac{p}{\alpha_+ - u} - \frac{1-p}{\alpha_- + u}\right],$$

where ν is the intensity of the jump process and α_+, α_- describe the exponential tails.

Theorem 3.6.3. The solution to (3.26) is given by

$$\begin{aligned}\phi(t) =& \frac{\nu\alpha_+ pT}{u} e^{\lambda\left(\frac{T\alpha_+}{u} - t\right)}\left[\mathrm{Ei}\left(-\lambda\frac{\alpha_+ T}{u}\right) - \mathrm{Ei}\left(\lambda\left(t - \frac{\alpha_+ T}{u}\right)\right)\right] \\ &+ \frac{\nu\alpha_-(p-1)T}{u} e^{-\lambda\left(\frac{\alpha_- T}{u} + t\right)}\left[\mathrm{Ei}\left(\lambda\left(\frac{\alpha_- T}{u}\right)\right) - \mathrm{Ei}\left(\lambda\left(t + \frac{\alpha_- T}{u}\right)\right)\right] \\ &+ \frac{\nu}{\lambda}\left(e^{-\lambda t} - 1\right).\end{aligned}$$

Proof. Observe that the Riccati equations are first-order differential equations, and hence, we can easily solve them explicitly. We use the integrating factor $e^{\lambda t}$ and get

$$\begin{aligned}\psi(t) =& e^{-\lambda t}\left[C + \int_0^t \nu\frac{us}{T}\left(\frac{p}{\alpha_+ - \frac{us}{T}} - \frac{1-p}{\alpha_- + \frac{us}{T}}\right) e^{\lambda s} ds\right] \\ =& e^{-\lambda t}\Bigg[C - \frac{\nu}{\lambda u} e^{-\frac{\alpha_- \lambda T}{u}}\Bigg(\alpha_+ \lambda p T e^{\frac{\lambda T(\alpha_+ + \alpha_-)}{u}}\,\mathrm{Ei}\left(\lambda\left(t - \frac{\alpha_+ T}{u}\right)\right) \\ &+ \alpha_- \lambda(p-1)T\,\mathrm{Ei}\left(\lambda\left(t + \frac{\alpha_- T}{u}\right)\right) + u e^{\lambda\left(\frac{\alpha_- T}{u} + t\right)}\Bigg)\Bigg],\end{aligned}$$

where $\mathrm{Ei}(x) = -\int_{-x}^{\infty} \frac{1}{w} e^{-w} dw$ is the exponential integral. Using the initial condition $\psi(0) = 0$ yields

$$C = \frac{\nu}{\lambda u} e^{-\frac{\alpha_- \lambda T}{u}} \left(\alpha_+ \lambda p T e^{\frac{\lambda T(\alpha_+ + \alpha_-)}{u}} \operatorname{Ei}\left(-\lambda \frac{\alpha_+ T}{u} \right) \right. \\ \left. + \alpha_- \lambda (p-1) T \operatorname{Ei}\left(\lambda \left(\frac{\alpha_- T}{u} \right) \right) + u e^{\frac{\lambda \alpha_- T}{u}} \right),$$

and hence we conclude

$$\begin{aligned} \psi(t) = & \frac{\nu}{\lambda u} e^{-\frac{\alpha_- \lambda T}{u} - \lambda t} \left[\alpha_+ \lambda p T e^{\frac{\lambda T(\alpha_+ + \alpha_-)}{u}} \operatorname{Ei}\left(-\lambda \frac{\alpha_+ T}{u} \right) \right. \\ & + \alpha_- \lambda (p-1) T \operatorname{Ei}\left(\lambda \left(\frac{\alpha_- T}{u} \right) \right) + u e^{\frac{\lambda \alpha_- T}{u}} \\ & - \alpha_+ \lambda p T e^{\frac{\lambda T(\alpha_+ + \alpha_-)}{u}} \operatorname{Ei}\left(\lambda \left(t - \frac{\alpha_+ T}{u} \right) \right) \\ & \left. - \alpha_- \lambda (p-1) T \operatorname{Ei}\left(\lambda \left(t + \frac{\alpha_- T}{u} \right) \right) - u e^{\lambda \left(\frac{\alpha_- T}{u} + t \right)} \right] \\ = & \frac{\nu \alpha_+ p T}{u} e^{\lambda \left(\frac{T \alpha_+}{u} - t \right)} \left[\operatorname{Ei}\left(-\lambda \frac{\alpha_+ T}{u} \right) - \operatorname{Ei}\left(\lambda \left(t - \frac{\alpha_+ T}{u} \right) \right) \right] \\ & + \frac{\nu \alpha_- (p-1) T}{u} e^{-\lambda \left(\frac{\alpha_- T}{u} + t \right)} \left[\operatorname{Ei}\left(\lambda \left(\frac{\alpha_- T}{u} \right) \right) - \operatorname{Ei}\left(\lambda \left(t + \frac{\alpha_- T}{u} \right) \right) \right] \\ & + \frac{\nu}{\lambda} \left(e^{-\lambda t} - 1 \right). \end{aligned}$$

□

3.6.2 Average Strike.

For the average strike we have

$$\begin{aligned} \dot{\phi} &= \lambda \kappa(\psi) & \phi(0) &= 0 \\ \dot{\psi} &= -\lambda \psi + \theta \left(u \left(\frac{t}{T} - 1 \right) + 1 \right) & \psi(0) &= 0. \end{aligned}$$

Here, we can process accordingly and get

$$\psi(t) = \nu \alpha_+ p T e^{\lambda \left(\frac{(\alpha_+ + u - 1) T}{u} - t \right)} \frac{\operatorname{Ei}\left(-\frac{\lambda T(\alpha_+ + u - 1)}{u} \right) - \operatorname{Ei}\left(\frac{\lambda (tu - T(\alpha_+ + u - 1))}{u} \right)}{u}$$

$$+ \nu\alpha_-(p-1)Te^{-\lambda\left(\frac{(\alpha_- - u+1)T}{u}+t\right)} \frac{\mathrm{Ei}\left(\frac{\lambda T(\alpha_- - u+1)}{u}\right) - \mathrm{Ei}\left(\frac{\lambda(T(\alpha_- - u+1)+tu)}{u}\right)}{u}$$
$$+ \frac{\nu}{\lambda}\left(e^{-\lambda t}-1\right).$$

3.7 OU TIME-CHANGED GAMMA PROCESS

A Gamma process is a pure jump Lévy process with cumulant function

$$\theta(u) = -\gamma \log\left(1-\frac{u}{\eta}\right),$$

where η is a scaling parameter and γ is the rate of jump arrivals. Recall that the functional characteristics for a general Lévy process are given by (3.24), and thus we have

$$F(u,w) = \lambda\kappa(w), \qquad R(u,w) = -\lambda w - \gamma \log\left(1-\frac{u}{\eta}\right).$$

The Riccati equations for the average price are hence given by

$$\dot{\phi} = \lambda\kappa(\psi), \qquad \phi(0)=0, \tag{3.27}$$
$$\dot{\psi} = -\lambda\psi - \gamma\log\left(1-\frac{ut}{T\eta}\right), \qquad \psi(0)=0. \tag{3.28}$$

As before, we only have to solve (3.28) and then integration yields the solution to (3.27).

Theorem 3.7.1. The solution to (3.28) is given by

$$\psi(t) = \frac{\gamma}{\lambda}e^{\lambda\left(\frac{\eta T}{u}-t\right)}\left[\mathrm{Ei}\left(\lambda\left(t-\frac{\eta T}{u}\right)\right) - \mathrm{Ei}\left(-\lambda\frac{\eta T}{u}\right)\right] - \frac{\gamma}{\lambda}\log\left(1-\frac{ut}{T\eta}\right).$$

Proof. Observe that (3.28) is of first order. Hence we use the integrating factor $e^{\lambda t}$ to obtain

$$\frac{d}{dt}\left(e^{\lambda t}\psi(t)\right) + \gamma \log\left(1 - \frac{ut}{T\eta}\right) e^{\lambda t} = 0.$$

Equivalently, with $C \in \mathbb{R}$, we have

$$\psi(t) = e^{-\lambda t}\left[C - \gamma \int_0^t \log\left(1 - \frac{us}{T\eta}\right) e^{\lambda s} ds\right].$$

Using integration by parts yields

$$\begin{aligned}\int_0^t \log\left(1 - \frac{us}{T\eta}\right) e^{\lambda s} ds &= \left(\log\left(1 - \frac{us}{T\eta}\right)\frac{e^{\lambda s}}{\lambda}\right)\Bigg|_0^t - \int_0^t \frac{u}{us - \eta T}\frac{e^{\lambda s}}{\lambda} ds \\ &= \frac{1}{\lambda} e^{\lambda t} \log\left(1 - \frac{ut}{T\eta}\right) - \frac{1}{\lambda}\int_0^t \frac{1}{s - \frac{\eta T}{u}} e^{\lambda s} ds.\end{aligned}$$

Now, using the substitution $w = -\lambda\left(s - \frac{T\eta}{u}\right)$, we obtain further

$$\frac{1}{\lambda}\int_0^t \frac{1}{s - \frac{\eta T}{u}} e^{\lambda s} ds = -e^{\frac{\lambda \eta T}{u}} \int_{-\lambda\left(t - \frac{\eta T}{u}\right)}^{\lambda \frac{\eta T}{u}} \frac{1}{w} e^{-w} dw.$$

Recalling that the exponential integral is given by $\mathrm{Ei}(x) = -\int_{-x}^{\infty} \frac{1}{w} e^{-w} dw$, we get

$$\psi(t) = e^{-\lambda t}\left[C + \frac{\gamma}{\lambda} e^{\frac{\lambda \eta T}{u}}\left[\mathrm{Ei}\left(\lambda\left(t - \frac{\eta T}{u}\right)\right) - \mathrm{Ei}\left(-\lambda \frac{\eta T}{u}\right)\right] - \frac{\gamma}{\lambda} e^{\lambda t} \log\left(1 - \frac{ut}{T\eta}\right)\right].$$

Furthermore, $\psi(0) = 0$ implies $C = 0$, and hence the solution to (3.28) is given by

$$\psi(t) = \frac{\gamma}{\lambda} e^{\lambda\left(\frac{\eta T}{u} - t\right)}\left[\mathrm{Ei}\left(\lambda\left(t - \frac{\eta T}{u}\right)\right) - \mathrm{Ei}\left(-\lambda \frac{\eta T}{u}\right)\right] - \frac{\gamma}{\lambda} \log\left(1 - \frac{ut}{T\eta}\right).$$

□

For the average strike, one can argue accordingly.

3.8 CIR TIME-CHANGED LÉVY PROCESSES

Here, [11] assumes a time change, i.e., $X(t) = L\left(\int_0^t V_s ds\right)$, where the volatility $(V_t)_{t\geq 0}$ is now given as a CIR-type SDE

$$dV_t = -\lambda(V_t - \theta)dt + \eta\sqrt{V_t}dW_t.$$

Further, let $\kappa(u)$ be the cumulant function of L. Then, by exactly the same arguments as in the OU time-changed case, we find the expressions for the functional characteristics as

$$F(u,w) = \lambda\theta w, \qquad R(u,w) = \frac{\eta^2}{2}w^2 - \lambda w + \kappa(u).$$

The Riccati equations for the average price are given by

$$\dot{\phi} = \lambda\theta\psi, \qquad \phi(0) = 0, \tag{3.29}$$

$$\dot{\psi} = \frac{\eta^2}{2}\psi^2 - \lambda\psi + \kappa\left(\frac{ut}{T}\right), \qquad \psi(0) = 0. \tag{3.30}$$

Also here, we want to conclude with an example. As before, [11] proposes the Kou double exponential Lévy process, i.e.,

$$\kappa(u) = \nu u\left[\frac{p}{\alpha_+ - u} - \frac{1-p}{\alpha_- + u}\right]$$

with ν, α_+, α_- as in Subsection 3.6. We further assume a symmetric jump distribution, i.e., $p = \frac{1}{2}$ and $a = \alpha_+ = \alpha_-$.

As a first step, we have to following.

Lemma 3.8.1. (3.30) can be transformed into

$$y''(t) + \lambda y'(t) + \frac{\eta^2}{2}\nu \frac{\left(\frac{ut}{T}\right)^2}{a^2 - \left(\frac{ut}{T}\right)^2} y(t) = 0, \tag{3.31}$$

using $\psi(t) = -\frac{2}{\eta^2}\frac{y'(t)}{y(t)}$.

Proof. We have

$$\begin{aligned}\dot{\psi}(t) &= \frac{2}{\eta^2}\frac{(y'(t))^2 - y''(t)y(t)}{(y(t))^2} \\ &= \frac{2}{\eta^2}\frac{(y'(t))^2}{(y(t))^2} + \lambda\frac{2}{\eta^2}\frac{y'(t)}{y(t)} + \kappa\left(\frac{ut}{T}\right) = \frac{\eta^2}{2}\psi^2 - \lambda\psi + \kappa\left(\frac{ut}{T}\right),\end{aligned}$$

so that simplifying and multiplying by $\frac{\eta^2}{2}y(t)$ gives the claim. □

Before we go on to the solution of (3.31), we need the following lemma.

Lemma 3.8.2. The confluent Heun equation is given by

$$\frac{d^2w}{dz^2} + \left(\alpha + \frac{\beta}{z} + \frac{\gamma}{z-1}\right)\frac{dw}{dz} + \frac{\delta z - \varepsilon}{z(z-1)}w = 0. \tag{3.32}$$

It can be transformed into

$$\begin{aligned}&\frac{d^2y}{dt^2} + \left(\lambda + \frac{4\left(\frac{u}{T}\right)^2 t}{a^2 - \left(\frac{ut}{T}\right)^2} + z'(t)\left[\alpha + \frac{\beta}{z(t)} + \frac{\gamma}{z(t)-1}\right] - \frac{z''(t)}{z'(t)}\right)\frac{dy}{dt} \\ &\quad + \left\{\frac{\lambda^2}{4} + \frac{2\left(\frac{u}{T}\right)^2(1+t\lambda)}{a^2 - \left(\frac{ut}{T}\right)^2} + \frac{8\left(\frac{u}{T}\right)^4 t^2}{(a^2 - \left(\frac{ut}{T}\right)^2)^2} + (z'(t))^2\frac{\delta z - \varepsilon}{z(t)(z(t)-1)}\right. \\ &\quad \left. + \left(z'(t)\left[\alpha + \frac{\beta}{z(t)} + \frac{\gamma}{z(t)-1}\right] - \frac{z''(t)}{z'(t)}\right)\left(\frac{\lambda}{2} + \frac{2\left(\frac{u}{T}\right)^2 t}{a^2 - \left(\frac{ut}{T}\right)^2}\right)\right\}y(t) = 0.\end{aligned} \tag{3.33}$$

Proof. Recall from the Heston model that $\frac{df}{dz} = \frac{df}{dt}\frac{1}{z'(t)}$ and hence

$$\frac{dw}{dz} = \frac{dw}{dt}\frac{1}{z'(t)}, \quad \frac{d^2w}{dz^2} = \frac{d^2w}{dt^2}\frac{1}{(z'(t))^2} - \frac{dw}{dt}\frac{z''(t)}{(z'(t))^3}.$$

Thus, in the first step, we can rewrite (3.32) as

$$\frac{d^2w}{dt^2} + \left[z'(t)\left(\alpha + \frac{\beta}{z(t)} + \frac{\gamma}{z(t)-1}\right) - \frac{z''(t)}{z'(t)}\right]\frac{dw}{dt} + (z'(t))^2\frac{\delta z(t) - \varepsilon}{z(t)(z(t)-1)}w(t) = 0. \tag{3.34}$$

Now let $y(t) = e^{-\frac{\lambda t}{2}}\left(a^2 - \left(\frac{ut}{T}\right)^2\right)w(t)$. Then we have

$$\begin{aligned}
\frac{dw}{dt} =& \frac{e^{\frac{\lambda t}{2}}}{a^2 - \left(\frac{ut}{T}\right)^2}\left[y'(t) + y(t)\left(\frac{\lambda}{2} + \frac{2\left(\frac{u}{T}\right)^2 t}{a^2 - \left(\frac{ut}{T}\right)^2}\right)\right] \\
\frac{d^2w}{dt^2} =& \frac{e^{\frac{\lambda t}{2}}}{a^2 - \left(\frac{ut}{T}\right)^2}\left[y''(t) + y'(t)\left(\lambda + \frac{4\left(\frac{u}{T}\right)^2 t}{a^2 - \left(\frac{ut}{T}\right)^2}\right)\right. \\
&\left. + y(t)\left(\frac{\lambda^2}{4} + \frac{2\left(\frac{u}{T}\right)^2(1+t\lambda)}{a^2 - \left(\frac{ut}{T}\right)^2} + \frac{8\left(\frac{u}{T}\right)^4 t^2}{(a^2 - \left(\frac{ut}{T}\right)^2)^2}\right)\right].
\end{aligned}$$

Plugging this into (3.34) yields the transformed version, i.e.,

$$\begin{aligned}
&\frac{d^2y}{dt^2} + \left(\lambda + \frac{4\left(\frac{u}{T}\right)^2 t}{a^2 - \left(\frac{ut}{T}\right)^2} + z'(t)\left[\alpha + \frac{\beta}{z(t)} + \frac{\gamma}{z(t)-1}\right] - \frac{z''(t)}{z'(t)}\right)\frac{dy}{dt} \\
&+ \left\{\frac{\lambda^2}{4} + \frac{2\left(\frac{u}{T}\right)^2(1+t\lambda)}{a^2 - \left(\frac{ut}{T}\right)^2} + \frac{8\left(\frac{u}{T}\right)^4 t^2}{(a^2 - \left(\frac{ut}{T}\right)^2)^2} + (z'(t))^2\frac{\delta z - \varepsilon}{z(t)(z(t)-1)}\right. \\
&\left. + \left(z'(t)\left[\alpha + \frac{\beta}{z(t)} + \frac{\gamma}{z(t)-1}\right] - \frac{z''(t)}{z'(t)}\right)\left(\frac{\lambda}{2} + \frac{2\left(\frac{u}{T}\right)^2 t}{a^2 - \left(\frac{ut}{T}\right)^2}\right)\right\}y(t) = 0.
\end{aligned}$$

□

Theorem 3.8.3. One solution to (3.34) can be expressed in terms of Heun's confluent function (c.f. [17]) with parameters

$$\begin{aligned}
&\alpha = 0, \quad \beta = \frac{1}{2}, \quad \gamma = 2, \\
&\delta = -\frac{a^2(\lambda^2 + 2\nu\eta^2)}{16\left(\frac{u}{T}\right)^2}, \\
&\varepsilon = -\frac{\lambda^2 a^2 + 8\left(\frac{u}{T}\right)^2}{16\left(\frac{u}{T}\right)^2},
\end{aligned}$$

$$z(t) = \frac{\left(\frac{ut}{T}\right)^2}{a^2}.$$

Proof. By (3.33), the factor in front of $y'(t)$ is given by

$$\begin{aligned}
&\lambda + \frac{4\left(\frac{u}{T}\right)^2 t}{a^2 - \left(\frac{ut}{T}\right)^2} + z'(t)\left[\alpha + \frac{\beta}{z} + \frac{\gamma}{z-1}\right] - \frac{z''(t)}{z'(t)} \\
=&\lambda + \frac{4\left(\frac{u}{T}\right)^2 t}{a^2 - \left(\frac{ut}{T}\right)^2} + \frac{2t\left(\frac{u}{T}\right)^2}{a^2}\left[0 + \frac{a^2}{2\left(\frac{ut}{T}\right)^2} + \frac{2a^2}{\left(\frac{ut}{T}\right)^2 - a^2}\right] - \frac{1}{t} \\
=&\lambda + \frac{4\left(\frac{u}{T}\right)^2 t}{a^2 - \left(\frac{ut}{T}\right)^2} + \frac{1}{t} - \frac{4\left(\frac{u}{T}\right)^2 t}{a^2 - \left(\frac{ut}{T}\right)^2} - \frac{1}{t} = \lambda.
\end{aligned}$$

For the factor in front of $y(t)$ we get

$$\begin{aligned}
&\frac{\lambda^2}{4} + \frac{2\left(\frac{u}{T}\right)^2(1+t\lambda)}{a^2 - \left(\frac{ut}{T}\right)^2} + \frac{8\left(\frac{u}{T}\right)^4 t^2}{(a^2 - \left(\frac{ut}{T}\right)^2)^2} + (z'(t))^2\frac{\delta z - \varepsilon}{z(t)(z(t)-1)} \\
&+ \left(z'(t)\left[\alpha + \frac{\beta}{z(t)} + \frac{\gamma}{z(t)-1}\right] - \frac{z''(t)}{z'(t)}\right)\left(\frac{\lambda}{2} + \frac{2\left(\frac{u}{T}\right)^2 t}{a^2 - \left(\frac{ut}{T}\right)^2}\right) \\
=&\frac{\lambda^2}{4} + \frac{2\left(\frac{u}{T}\right)^2(1+t\lambda)}{a^2 - \left(\frac{ut}{T}\right)^2} + \frac{8\left(\frac{u}{T}\right)^4 t^2}{(a^2 - \left(\frac{ut}{T}\right)^2)^2} - \frac{4\frac{u^4t^2}{T^4a^2}\delta - 4\left(\frac{u}{T}\right)^2\varepsilon}{a^2 - \left(\frac{ut}{T}\right)^2} \\
&- \frac{4\left(\frac{u}{T}\right)^2 t}{a^2 - \left(\frac{ut}{T}\right)^2}\left(\frac{\lambda}{2} + \frac{2\left(\frac{u}{T}\right)^2 t}{a^2 - \left(\frac{ut}{T}\right)^2}\right) \\
=&\frac{\lambda^2}{4} + \frac{1}{a^2 - \left(\frac{ut}{T}\right)^2}\left[2\left(\frac{u}{T}\right)^2 - 4\frac{u^4t^2}{T^4a^2}\delta + 4\left(\frac{u}{T}\right)^2\varepsilon\right] \\
=&\frac{\eta^2}{2}\nu\frac{\left(\frac{ut}{T}\right)^2}{a^2 - \left(\frac{ut}{T}\right)^2}.
\end{aligned}$$

Hence the parametrization works. □

Lemma 3.8.4. A second solution to (3.33) is given by the parametrization

$$\tilde{\alpha} = 0, \quad \tilde{\beta} = 2, \quad \tilde{\gamma} = \frac{1}{2},$$
$$\tilde{\delta} = -\frac{a^2\left(\lambda^2 + 2\nu\eta^2\right)}{16\left(\frac{u}{T}\right)^2},$$

$$\tilde{\varepsilon} = -\frac{\lambda^2 a^2 + 8\left(\frac{u}{T}\right)^2}{16\left(\frac{u}{T}\right)^2},$$
$$\tilde{z}(t) = 1 - \frac{\left(\frac{ut}{T}\right)^2}{a^2}.$$

Proof. To see that this really is a solution, observe that for the factor in front of $y(t)$, there is no change and for the factor in front of $y'(t)$, the constants are altered in the right way. □

Furthermore, these two solutions are independent, such that, recalling that $\psi(0) = 0$, analogously to the Heston model, we have the following corollary.

Corollary 3.8.5. The solution to (3.30) is given by

$$\psi(t) = -\frac{2}{\eta^2}\frac{y_2'(0)y_1'(t) - y_1'(0)y_2'(t)}{y_2'(0)y_1(t) - y_1'(0)y_2(t)},$$

where

$$y_1(t) = \exp\left\{-\frac{\lambda t}{2}\right\}\left(a^2 - \left(\frac{ut}{T}\right)^2\right) C(\alpha, \beta, \gamma, \delta, \varepsilon, z(t)),$$
$$y_2(t) = \exp\left\{-\frac{\lambda t}{2}\right\}\left(a^2 - \left(\frac{ut}{T}\right)^2\right) C(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{\delta}, \tilde{\varepsilon}, \tilde{z}(t)).$$

Here C is the confluent Heun function or CHF for short.

For the average strike, one can argue accordingly, and obtain similar results.

Remark 3.8.6. We want to point out that the results we obtained here differ from the ones given in [11].

4 MULTI-FACTOR MODELS

In Section 3, we saw some of the most popular single-factor models. Now we want to extend the theory given in [11]. We will see that we can easily use the pricing model for models with more than one factor. The significance of multi-factor models was addressed in [7]. One of the main aspects of multi-factor models is that they can explain the volatility smile in a more advanced way. We work as follows. First we study the most important details of the general pricing model and adjust the definitions and theorems to fit the new format and then determine the details for several specific models.

4.1 THE GENERAL MODEL

Let $(X_t)_{t\geq 0}$ be an affine process as defined in Definition 2.1.1 and let $(V_t)_{t\geq 0}$, $V_t : \Omega \mapsto \mathbb{R}_+^n$ be such that $V_0 > 0$ componentwise and $(X_t, V_t)_{t\geq 0}$ is a stochastically continuous time-homogeneous Markov process.

Definition 4.1.1 (Analogue to Definition 2.1.5). Let $w = (w_1, \dots, w_n)$. (X_t, V_t) is a affine stochastic volatility (ASV) model if

$$\log \mathbb{E}\left[e^{uX_t + w\cdot V_t} | X_0, V_0\right] = \phi(t, u, w) + X_0 u + \sum_{i=1}^{n} V_0^i \psi_i(t, u, w),$$

where V_t^i denotes the ith component of V_t.

Observe that Definition 4.1.1 is the same as Definition 2.1.5, where we only interpret $w \cdot V_t$ as a dot product instead of a simple multiplication. In contrast to the original model, we have not only 2 but $n+1$ functional characteristics, $F, R_1, \dots, R_n$.

Furthermore, analogue to Theorem 2.1.6, we have the following result.

Lemma 4.1.2. ϕ and ψ_i, $1 \leq i \leq n$ fulfill the generalized Riccati equations

$$\partial_t \phi(t,u,w) = F\left(u, \psi_1(t,u,w), \ldots, \psi_n(t,u,w)\right), \qquad \phi(0,u,w) = 0,$$
$$\partial_t \psi_i(t,u,w) = R_i\left(u, \psi_1(t,u,w), \ldots, \psi_n(t,u,w)\right), \qquad \psi_i(0,u,w) = 0.$$

Proof. This follows immediately from the proof of Theorem 2.1.6 applied to each component. □

Define $Y_t = \int_0^t X_s ds$ and $Z_t = \int_0^t V_s ds$ componentwise.

Theorem 4.1.3 (Analogue to Proposition 2.2.1). Let (X_t, V_t) be an ASV model with functional characteristics $(F, R_1, \ldots, R_n)$. Then the joint law of (X_t, V_t, Y_t, Z_t) is described by

$$\begin{aligned}\log \mathbb{E}\left[e^{u_1 X_t + u_2 \cdot V_t + u_3 Y_t + u_4 \cdot Z_t} | X_0, V_0\right] =& \Phi(t, u_1, u_2, u_3, u_4) + (u_1 + u_3 t) X_0 \\ &+ \sum_{i=1}^{n} \Psi_i(t, u_1, u_2, u_3, u_4) V_0^i,\end{aligned}$$

where for $1 \leq i \leq n$

$$\dot{\Phi} = F(u_1 + u_3 t, \Psi_1, \ldots, \Psi_n), \qquad \Phi(0) = 0,$$
$$\dot{\Psi}_i = R_i(u_1 + u_3 t, \Psi_i) \qquad \Psi_i(0), = u_2^i.$$

Proof. This follows immediately from the proof of Proposition 2.2.1 applied to each component. □

Furthermore, we get the next two results, which also follow by looking at the components and then applying the methods that were used in the proofs of the original model.

Lemma 4.1.4 (Analogue to Lemma 2.3.2). Let $(X_t, V_t)_{t\geq 0}$ be an ASV model. Then the joint law of $(X_t, V_t)_{t\geq 0}$ under the martingale measure associated to the stock being the numéraire, $\mathbb{Q}^1$, is described by

$$\log \mathbb{E}^1\left[e^{uX_t+wY_t}|X_0\right] = \Phi(t,u,w) + (u+wt)X_0 + \sum_{i=1}^{n} \Psi_i(t,u,w)V_0^i,$$

where for $1 \leq i \leq n$

$$\begin{aligned} \dot{\Phi} &= F(u+1+wt, \Psi_1, \ldots, \Psi_n), && \Phi(0) = 0, \\ \dot{\Psi}_i &= R_i(u+1+wt, \Psi_i), && \Psi_i(0) = 0. \end{aligned}$$

Lemma 4.1.5 (Analogue to Corollary 2.4.2). Let $(X_t, V_t)_{t\geq 0}$ be an ASV model. The law of Y_t is described by

$$\log \mathbb{E}\left[e^{wY_t}|X_0, V_0\right] = \Phi(t,w) + wtX_0 + \sum_{i=1}^{n} V_0^i \Psi_i(t,w),$$

where for $1 \leq i \leq n$

$$\begin{aligned} \dot{\Phi} &= F(wt, \Psi_1, \ldots, \Psi_n) && \Phi(0) = 0, \\ \dot{\Psi}_i &= R_i(wt, \Psi_i), && \Psi_i(0) = 0. \end{aligned}$$

Finally we have the following pricing formulas.

Theorem 4.1.6 (Analogue to Theorem 2.4.3). Suppose there exists $a > 1$ such that

$$\mathbb{E}\left[e^{a\bar{X}_T}\right] < \infty.$$

Then the time zero value of a geometric average price Asian call option is given by

$$\mathbb{E}\left[e^{-rT}\left(\widehat{S}_T - K\right)_+ \Big| X_0, V_0\right] = \frac{e^{-rT}}{2\pi i}\int_{a-i\infty}^{a+i\infty}\left(\frac{1}{K}\right)^u \frac{K}{u(u-1)}e^{\kappa(T,u)}du$$

with cumulant function $\kappa(t,u) = \log \mathbb{E}\left[e^{u\bar{X}_t}|X_0, V_0\right]$. Furthermore,

$$\kappa(T,u) = u(r-q) + \phi(T,u) + uX_0 + \sum_{i=1}^{n}\psi_i(T,u)V_0^i,$$

where for $1 \leq i \leq n$

$$\dot{\phi}(t,u) = F\left(\frac{ut}{T}, \psi_1, \ldots, \psi_n\right), \qquad \phi(0,u) = 0,$$

$$\dot{\psi}_i(t,u) = R_i\left(\frac{ut}{T}, \psi_i\right), \qquad \psi_i(0,u) = 0.$$

Theorem 4.1.7 (Analogue to Theorem 2.4.4). If there is $b < 0$ such that

$$\mathbb{E}\left[e^{b\bar{X}_T}\right] < \infty,$$

then the time zero value of a geometric average strike Asian call option is given by

$$\mathbb{E}^0\left[e^{-rT}(S_T - \widehat{S}_T)_+|X_0, V_0\right] = \frac{e^{-qT}}{2\pi i}\int_{b-i\infty}^{b+i\infty}\frac{1}{u(u-1)}e^{\kappa(T,u)}du$$

with cumulant function $\kappa(t,u) = \log \mathbb{E}^0\left[e^{u\bar{X}_t + (1-u)X_t}|X_0, V_0\right]$. Furthermore,

$$\kappa(T,u) = \phi(T,u) + X_0 + \sum_{i=1}^{n}V_0^i\psi_i(T,u),$$

where for $1 \leq i \leq n$

$$\dot{\phi}(t,u) = F\left(\frac{ut}{T} + (1-u), \psi_1, \ldots, \psi_n\right), \qquad \phi(0,u) = 0,$$

$$\dot{\psi}_i(t,u) = R_i\left(\frac{ut}{T} + (1-u), \psi_i\right), \qquad \psi_i(0,u) = 0.$$

Remark 4.1.8. Note that due to the affine structure of the model and the independence of the volatility factors, there are not many changes to the original model. Instead of wV_t we have the dot product and we end up having a bigger system of Riccati equations.

4.2 SPECIFIC MODELS

The first model we are looking at is the double Heston model, i.e., $n = 2$.

4.2.1 Double Heston Model. According to [7], the dynamics of this model are given by

$$\begin{aligned} dX_t &= \left(-\frac{1}{2}\left(V_t^1 + V_t^2\right)\right)dt + \sqrt{V_t^1}dW_t^1 + \sqrt{V_t^2}dW_t^2, \\ dV_t^1 &= \lambda_1\left(\theta_1 - V_t^1\right)dt + \zeta_1\sqrt{V_t^1}dZ_t^1, \\ dV_t^2 &= \lambda_2\left(\theta_2 - V_t^2\right)dt + \zeta_2\sqrt{V_t^2}dZ_t^2, \end{aligned}$$

where Z_t^i and W_t^i are SBMs and the dependence structure between them is given as

$$\begin{aligned} &\mathrm{Corr}(W_t^1, W_t^2) = \mathrm{Corr}(W_t^1, Z_t^2) = \mathrm{Corr}(W_t^2, Z_t^1) = 0, \\ &\mathrm{Corr}(W_t^1, Z_t^1) = \rho_1, \qquad \mathrm{Corr}(W_t^2, Z_t^2) = \rho_2, \end{aligned}$$

which means that we have two independent volatility factors influencing the price process.

Theorem 4.2.1. The functional characteristics are given by

$$F(u, w_1, w_2) = \lambda_1\theta_1 w_1 + \lambda_2\theta_2 w_2, \tag{4.1}$$

$$R_1(u,w_1) = \frac{\zeta_1^2}{2} w_1^2 - (\lambda_1 - u\zeta_1\rho_1)\, w_1 + \frac{1}{2}\left(u^2 - u\right), \tag{4.2}$$
$$R_2(u,w_2) = \frac{\zeta_2^2}{2} w_2^2 - (\lambda_2 - u\zeta_2\rho_2)\, w_2 + \frac{1}{2}\left(u^2 - u\right). \tag{4.3}$$

Proof. There are no jump parts in the double Heston model, so that we can apply Itô's lemma for continuous stochastic integrals (Theorem 1.6.2) to $(X_t, V_t)_{t\geq 0}$ and $f(x, v_1, v_2) = e^{ux+w_1v_1+w_2v_2}$ to obtain the result, i.e.,

$$\begin{aligned} de^{uX_t+w_1V_t^1+w_2V_t^2} = e^{uX_t+w_1V_t^1+w_2V_t^2}\Big[& udX_t + w_1dV_t^1 + w_2dV_t^2 + \frac{1}{2}u^2 d[X,X]_t \\ & + uw_1 d[X,V^1]_t + uw_2 d[X,V^2]_t + w_1w_2 d[V^1,V^2]_t \\ & + \frac{1}{2}w_1^2 d[V^1,V_1]_t + \frac{1}{2}w_2^2 d[V^2,V^2]_t\Big]. \end{aligned}$$

We have

$$\begin{aligned} d[X,X]_t &= \left(V_t^1 + V_t^2\right) dt, \\ d[X,V^i]_t &= \zeta_i\rho_i V_t^i dt, \\ d[V^1,V^2]_t &= 0, \\ d[V^i,V^i]_t &= \zeta_i^2 V_t^i dt. \end{aligned}$$

Letting $w \cdot V_t = w_1V_t^1 + w_2V_t^2$, we get

$$\begin{aligned} de^{uX_t+w\cdot V_t} = e^{uX_t+w\cdot V_t}\Big[& -\frac{1}{2}u\left(V_t^1+V_t^2\right)dt + u\left(\sqrt{V_t^1}dW_t^1 + \sqrt{V_t^2}dW_t^2\right) \\ & + w_1\lambda_1(\theta_1 + V_t^1)dt + w_1\zeta_1\sqrt{V_t^1}dZ_t^1 \\ & + w_2\lambda_2(\theta_2 + V_t^2)dt + w_2\zeta_2\sqrt{V_t^2}dZ_t^2 \\ & + \frac{1}{2}u^2\left(V_t^1+V_t^2\right)dt + uw_1\zeta_1\rho_1V_t^1dt + uw_2\zeta_2\rho_2V_t^2dt \\ & + \frac{1}{2}w_1^2\zeta_1^2dt + \frac{1}{2}w_2^2\zeta_2^2dt\Big]. \end{aligned}$$

Sorting the terms, we obtain

$$
\begin{aligned}
de^{uX_t+w\cdot V_t} = e^{uX_t+w\cdot V_t}\Bigg[& (w_1\lambda_1\theta_1 + w_2\lambda_2\theta_2)\,dt \\
& + V_t^1\left(\frac{\zeta_1^2}{2}w_1^2 - (\lambda_1 - \zeta_1\rho_1 u)\,w_1 + \frac{1}{2}(u^2-u)\right)dt \\
& + V_t^2\left(\frac{\zeta_2^2}{2}w_2^2 - (\lambda_2 - \zeta_2\rho_2 u)\,w_2 + \frac{1}{2}(u^2-u)\right)dt \\
& + \mathrm{MART}\Bigg],
\end{aligned}
$$

where MART denotes integrals with respect to the Brownian motion. Hence taking expectations and working similarly as in the other models, we get the result. □

Remark 4.2.2. Note that the functional characteristics (4.2) and (4.3) agree with the one for the single-factor Heston model. Only (4.1) is different and is obtained by summation. Hence, the general form of the functional characteristics, i.e., for $n \in \mathbb{N}$, will have n similar equations for R_i, and F is given by $F(u,w) = \sum_{i=1}^{n} \lambda_i\theta_i w_i$.

The Riccati equations for the average price are given by

$$\dot{\phi} = \lambda_1\theta_1 w_1 + \lambda_2\theta_2 w_2, \qquad \phi(0) = 0, \tag{4.4}$$

$$\dot{\psi}_1 = \frac{\zeta_1^2}{2}w_1^2 - \left(\lambda_1 - \zeta_1\rho_1\frac{ut}{T}\right)w_1 + \frac{1}{2}\left(\left(\frac{ut}{T}\right)^2 - \frac{ut}{T}\right), \qquad \psi_1(0) = 0, \tag{4.5}$$

$$\dot{\psi}_2 = \frac{\zeta_2^2}{2}w_2^2 - \left(\lambda_2 - \zeta_2\rho_2\frac{ut}{T}\right)w_2 + \frac{1}{2}\left(\left(\frac{ut}{T}\right)^2 - \frac{ut}{T}\right), \qquad \psi_2(0) = 0. \tag{4.6}$$

Because of the similarities to the single-factor Heston model, we can solve (4.5) and (4.6) as before. The solution to each of them is given as a linear combination of confluent hypergeometric Kummer functions. Also as before, ϕ can be obtained by integration. Analogously, one can obtain the results for the Riccati equations of the average strike.

4.2.2 Other Models.

(i) Let us assume we have the dynamics

$$\begin{aligned}dX_t =& \left(-(\kappa_1(\rho_1)+\kappa_2(\rho_2)) - \frac{1}{2}\left(V_{t-}^1 + V_{t-}^2\right)\right)dt + \sqrt{V_{t-}^1}dW_t^1\\ &+ \sqrt{V_{t-}^2}dW_t^2 + \rho_1 dZ_{\lambda_1}(t) + \rho_2 dZ_{\lambda_2}(t),\\ dV_t^1 =& -\lambda_1 V_{t-}^1 dt + dZ_{\lambda_1}(t),\\ dV_t^2 =& -\lambda_2 V_{t-}^2 dt + dZ_{\lambda_2}(t),\end{aligned}$$

where W^i are SBMs and Z_{λ_i} are BDLPs with cumulant functions $\kappa_i(u)$, $\lambda_i > 0$ and $\rho_i \leq 0$. Then clearly the functional characteristics are given by

$$\begin{aligned}F(u,w_1,w_2) &= \lambda\left(\kappa_1(w_1+\rho_1 u)+\kappa_2(w_2+\rho_2 u)\right) - u\left(\lambda_1\kappa_1(\rho_1)+\lambda_2\kappa_2(\rho_2)\right),\\ R_i(u,w_i) &= \frac{1}{2}(u^2-u) - \lambda w_i,\end{aligned}$$

what can be seen analogously to the proof given in the single-factor Barndorff-Nielsen–Shephard model.

(ii) Let us now assume that we have a mixture of CIR-type and OU-type stochastic volatility, i.e., assume the dynamics

$$\begin{aligned}dX_t &= \left(-\kappa(\rho_1) - \frac{1}{2}\left(V_{t-}^1 + V_{t-}^2\right)\right)dt + \sqrt{V_{t-}^1}dW_t^1 + \sqrt{V_{t-}^2}dW_t^2 + \rho_1 dZ_\lambda(t),\\ dV_t^1 &= -\lambda V_{t-}^1 dt + dZ_\lambda(t),\\ dV_t^2 &= \eta\left(\theta - V_t^2\right)dt + \zeta\sqrt{V_t^2}dZ_t,\end{aligned}$$

where W^i and Z are SBMs with $\mathrm{Corr}(W^1,W^2) = \mathrm{Corr}(W^1,Z^2) = 0$ and $\mathrm{Corr}(W^i,Z^i) = \rho_i$ and Z_λ is a BDLP with cumulant function $\kappa(u)$ as above.

Then the functional characteristics are given by

$$\begin{aligned} F(u, w_1, w_2) &= \lambda\kappa_1(w_1 + \rho_1 u) - u\lambda\kappa(\rho_1) + \eta\theta w_2, \\ R_1(u, w_1) &= \frac{1}{2}(u^2 - u) - \lambda w_1, \\ R_2(u, w_2) &= \frac{\zeta^2}{2} w_2^2 - (\eta - u\zeta\rho_2)\, w_2 + \frac{1}{2}\left(u^2 - u\right). \end{aligned}$$

We see that we have to deal with the same equations as in the single-factor models. Noting that Riccati equations have the nice property that the equation for ϕ is solved by simply integrating the right hand side, we can solve the Riccati equations for the average price and average strike by applying the results obtained in the single-factor models on each factor.

So in conclusion, we can easily extend the model introduced in [11] to determine the average price and the average strike of geometric Asian options for multi-factor models. Furthermore, the formulas can be obtained from the results in the single-factor models.

CONCLUSION

In this thesis, we first introduced some concepts of stochastic calculus that were important in order to deal with the specific models in Section 3. Then, in the second section, we stated the model as given in [11] and saw that there are a pricing formulas for both the geometric average price and the geometric average strike Asian options for general affine processes, which involve the inverse Laplace transform and so-called generalized Riccati equations. In the third section, we discussed the solutions to these systems of ordinary differential equations for popular models, and the fourth section addressed the generalization of the pricing model to multi-factor models. We saw that due to linearity and the indepenence of the volatility factors, one can deal with the multi-factor models in the same manner as the single-factor models using the results of the latter. In conclusion, all information regarding the pricing formulas for the popular models was provided and proven in detail.

In order to get real prices, one needs to apply the Laplace transform. We saw that for some models, the solution to the Riccati equations is given by special functions like the hypergeometric functions, which makes the Laplace transform a hard problem to face. Further research should be done in regard of solving this issue.

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