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Boundary control of parabolic PDE using adaptive dynamic programming

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BOUNDARY CONTROL OF PARABOLIC PDE USING ADAPTIVE DYNAMIC PROGRAMMING

by

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Paper II, pages: 45-84, B. Talaei, S. Jagannathan and J. Singler, “Boundary Control of Linear Uncertain One-Dimensional Parabolic PDE Using Approximate Dynamic Programming,” Under review in IEEE Transactions on Neural Networks.

Paper III, pages: 85-118, B. Talaei, S. Jagannathan and J. Singler, “Output Feedback Boundary Control of Uncertain Coupled Semi-Linear Parabolic PDE Using Neuro Dynamic Programming,” Under review in IEEE Transactions on Neural Networks.

Paper IV, pages: 119-148, B. Talaei, S. Jagannathan and J. Singler, “Boundary Control of Two Dimensional Burgers PDE Using Approximate Dynamic Programming,” Under review in IEEE Transactions on Neural Networks.

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ABSTRACT

In this dissertation, novel adaptive/approximate dynamic programming (ADP) based state and output feedback control methods are presented for distributed parameter systems (DPS) which are expressed as uncertain parabolic partial differential equations (PDEs) in one and two dimensional domains. In the first step, the output feedback control design using an early lumping method is introduced after model reduction. Subsequently controllers were developed in four stages; Unlike current approaches in the literature, state and output feedback approaches were designed without utilizing model reduction for uncertain linear, coupled nonlinear and two-dimensional parabolic PDEs, respectively. In all of these techniques, the infinite horizon cost function was considered and controller design was obtained in a forward-in-time and online manner without solving the algebraic Riccati equation (ARE) or using value and policy iterations techniques.

Providing the stability analysis in the original infinite dimensional domain was a major challenge. Using Lyapunov criterion, the ultimate boundedness (UB) result was demonstrated for the regulation of closed-loop system using all the techniques developed herein. Moreover, due to distributed and large scale nature of state space, pure state feedback control design for DPS has proven to be practically obsolete. Therefore, output feedback design using limited point sensors in the domain or at boundaries are introduced. In the final two papers, the developed state feedback ADP control method was extended to regulate multi-dimensional and more complicated nonlinear parabolic PDE dynamics.

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1. INTRODUCTION

In control theory, the variables under study (temperature, displacement, concentration, velocity, etc.) are usually referred to as the states. For these physical systems, the state equation may be of one of the following types: ordinary differential equation (ODE), partial differential equation, functional differential equation, integro-differential equation, or abstract evolution equations. Of special interest is the case where system dynamics are modeled by partial differential equations (PDE).

Many different industrial systems are inherently distributed in space and their dynamics depend upon spatial position and time; therefore, they are modeled by PDE. For example, PDE dynamics can be seen in fluid flows in aerodynamics and propulsion applications, plasma in lasers, fusion reactors, hypersonic vehicles, liquid metals in cooling systems for tokamaks and computers as well as in welding and metal casting processes, acoustic waves and water waves irrigation systems. Flexible structures in civil engineering applications including aircraft wings and helicopter rotors, astronomical telescopes, and nanotechnology devices such as the atomic force microscope, electromagnetic waves and quantum mechanical systems, waves and “ripple” instabilities in thin film manufacturing. These devices are all usually referred to as distributed parameter systems (DPS). Other examples are flame dynamics, chemical processes and internal combustion engines. DPS is in contrast to lumped parameter systems (LPS), where the dynamics are described by ordinary differential equations (ODE).

On the other hand, optimal control, is often seen as a major design approach in modern control theory. Approximate dynamic programming (ADP), as part of optimal control, seeks computationally feasible solutions for the cases where the state space is considerably large and the dynamics are nonlinear and uncertain. These are common problems in DPS control design.

Generally, the control design methodologies for DPS can be divided into two categories: early and late lumping. In the first approach, referred to as early lumping, a finite dimensional state representation is first obtained and the controller is designed subsequently. Different methods in this category employ model reduction in different ways to extract a finite-dimensional subsystem. This subsystem can be controlled and even show a robustness when neglecting the remaining infinite dimensional dynamics in the design. The obvious benefit of this approach is the possibility of applying various control methods from LPS to DPS. This feature is particularly useful for nonlinear or semi-linear PDE, since control design approaches in the late lumping category for nonlinear PDE are still in early stages of development.

In the late lumping methods, however, the controller design is performed in the original infinite dimensional domain, and the control input is subsequently approximated for implementation. Therefore, the controllers in this category are more accurate even though dealing with PDE dynamics in original infinite dimensional domain makes their design and performance analysis much more difficult. The control design techniques pursued in this research include both methods but with more emphasis on the latter, since they have rarely been considered in ADP literature.

In another aspect, control of PDE is classified roughly into two types depending upon where the actuators are located: "in domain" control, where the actuation penetrates inside the domain of the PDE system or is evenly distributed everywhere in the domain (likewise with sensing) and "boundary" control, where the actuation is applied only through the boundary conditions. Boundary control is generally considered to be physically more realistic because actuation is nonintrusive (for example, think of a fluid flow where actuation would normally be from the walls of the flow domain). However, it is harder from both practical and theoretical points of view since one has at least an order of magnitude fewer control inputs than states.

This dissertation deals exclusively with boundary control. Basically, more than one option exists when it comes to boundary actuation. In certain applications, it is necessary to actuate the boundary value of the state variable of the PDE which is referred to as the Dirichlet actuation. This is the case, for example, in flow control where microjets are used to actuate the boundary values of the velocity at the wall. In other applications, it is only natural to actuate the boundary value of the gradient of the state variable of the PDE which is referred to as the Neumann actuation. This is the case, for example, in thermal problems where one can actuate the heat flux or temperature gradient but not the temperature itself. The developed control designs are implemented using both Dirichlet and Neumann actuation.

The PDE control problems are complex enough even in one-dimensional cases, such as found in a string, a beam, a chemical tubular reactor, and so on. Each component can be unstable with a large number of unstable eigenvalues, and can be highly nontrivial for control. However, many physical DPS can evolve in two and three dimensions. It is true that a few of them are dominated by phenomena evolving in one coordinate direction, while the phenomena in the other directions are stable and slow. However, inherent multi-dimensional systems still exist. This is particularly the case with Navier–Stokes or Burgers equations, where the full realism of turbulent fluid behavior is exhibited only in two or three dimensions. In Papers IV and V, we present such problems: state and output feedback ADP boundary control design for two dimensional nonlinear parabolic PDE equation when the domain shape is square.

1.1. LITERATURE REVIEW

The study of optimal control theory for infinite dimensional systems can be traced back to the beginning of the 1960s. The main goal of such a theory was either using the achieved results from optimal control of LPS through early lumping or to establish the infinite dimensional version of the fundamental theory in the late lumping category.

In the early lumping approaches, either Galerkin [1], [2], or finite difference approximation methods (FDM) [3],[4] were used to convert the PDE into a set of ODE. In Galerkin methods, which uses proper orthogonal decomposition (POD) or finite elements [5], the reduced order model was obtained by transforming the PDE into “weak integral form” through appropriate spatial basis functions that approximate the solution of PDE. In contrast, in FDM, the reduced order model is extracted by approximating the PDE dynamics at specific points in space based on a structured mesh. One of the main benefits of using a structured mesh in DPS control is the easy observation of finite dimensional states by means of physical sensors in the spatial domain [6]. This is generally a difficult task in spectral Galerkin methods [6], [7] . There is extensive literature in convergence behavior and applicability of the control approaches developed based on these numerical methods for parabolic PDE [9]–[11].

In contrast, the promising results gathered from the linear optimal control of LPS encouraged researchers in the late lumping category to develop the operator theory [12] for optimal control of DPS. This work extended further to boundary control [13] where the design was performed in the infinite dimensional setting. However, a closed-form solution requires solving the operator Riccati equations in a backward-in-—time calculation. This process is significantly more time consuming in the infinite dimensional state space for DPS.

Subsequently, various other approaches [14] were proposed. In particular, boundary actuation received special attention in the previous decade due to the backstepping approach [15]–[17] wherein the controller was developed through conventional calculus instead of the operator theory. Although this approach is not optimal, it can achieve "inverse optimality" [17]. The backstepping approach further enhanced the stability study through conventional calculus in the original PDE domain.

The advances in ADP and its successful forward-in-time implementation of optimal control policy for LPS [18] led to further investigations of its application to DPS [19]–[24]. However, as mentioned earlier, due to the difficulty of designing the controller in an infinite dimensional domain and with PDE dynamics [24], the DPS system was usually discretized into an approximate finite dimensional state space [21]; subsequently, the well-developed ADP algorithms [25] were utilized for this reduced-order model. The benefit of this design approach was the possibility of using either the state-of-the-art ADP or other suboptimal feedback control schemes. However, a degraded performance was observed due to the reduced order finite dimensional model in the control design.

1.2. ORGANIZATION OF THE DISSERATION

This dissertation includes five papers which introduce novel state-of-the-art ADP control approaches for uncertain parabolic PDE in one and two dimensions. The outline of this dissertation is summarized in Figure 1.1.

The main purpose of the first stage of this research was the design and analysis of an online output feedback near optimal boundary controller for DPS in the finite dimensional domain. In particular, an ADP-based output feedback controller was designed for DPS modeled by one-dimensional semi-linear parabolic PDE with input constraints. The nonlinearity in the dynamics is considered unknown but satisfies the locally Lipschitz continuous condition. The FDM is mainly used to find the approximate finite dimensional state space. Even reduced order models of the DPS pure state feedback control method seem inappropriate for practical applications due to the need for full state measurability. Therefore, the output feedback control design methodologies have gained considerable attention [26]–[30]. Accordingly, in this paper, an output feedback ADP control method with guaranteed closed-loop stability is illustrated for situations where very few sensors are available in the spatial domain and system dynamics are uncertain.

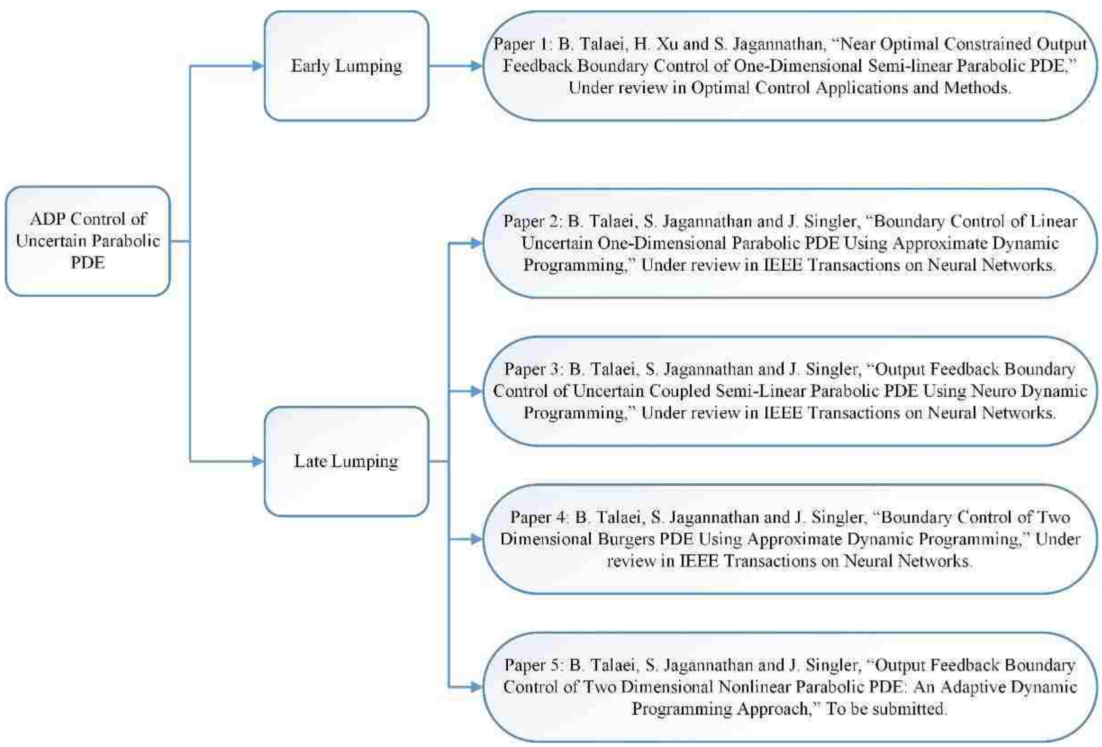

Figure 1.1. Outline of five research stages.

In contrast, Stage 2 of this research is introduced in second paper, which shows how a new ADP-based boundary control was developed for uncertain DPS described by linear parabolic PDEs without model reduction. The boundary control problem is specifically examined, and both Neumann and Dirichlet boundary actuation conditions are considered. The controller’s development is novel and based on a new definition for the value functional as a surface integral. Consequently, the Hamilton Jacobi Bellman (HJB) equation and the optimal boundary control policy are derived through calculus in the infinite dimensional domain. The value functional is further approximated without iteration and as forward-in-time in order to design a suboptimal adaptive controller whereas the PDE dynamics are uncertain. The proposed suboptimal boundary controller is the first to be designed for a DPS according to the forward-in-time ADP without any model reduction and with provision of a closed-loop stability proof.

Stage 3 of this research is introduced in Paper III, which addresses an output feedback boundary control scheme using ADP for uncertain DPS expressed as coupled semi-linear parabolic PDE. Once again, model reduction was not necessary. Similar to the second stage of research introduced in Paper II, the optimal control problem is formulated in the original PDE domain and solved forward-in-time without using any finite dimensional model approximation prior to the control design. Moreover, a neural network (NN) observer is proposed for the online estimation of states of the coupled PDE when the system dynamics are partially uncertain so that the need for availability of system states and accurate dynamics are relaxed.

Stage 4 of this research is introduced in Paper IV, which addresses an ADP-based boundary control of two-dimensional (2D) uncertain Burgers equation without using model reduction. The optimal control problem again is formulated in the original PDE domain and solved forward-in-time without using any finite dimensional model approximation prior to the control design. After defining an appropriate value functional for 2D geometry, the HJB equation was derived in the infinite dimensional space and the optimal control policy was obtained based on the necessary conditions of optimality.

The final Paper designs and analyzes an output feedback ADP control method for 2D uncertain nonlinear reaction diffusion PDE. The design extends the developed approach for one dimensional setting which does not utilize model reduction. Hence, the HJB is derived and an adaptive algorithm is developed to approximate its solution in the PDE domain forward-in-time and without using policy or value iterations. A PDE observer is designed to estimate the unavailable state in the two dimensional domain. Moreover, the stability analysis is also carried in the original infinite dimensional domain using calculus. The boundary control problem which is more theoretically challenging and practically relevant is addressed. Since abstract operator theory is avoided, the Paper is comprehensible for majority of engineers that are not quite familiar with functional analysis. Simulation results confirm that the presented output feedback control method has good convergence and control performance on an uncertain unstable 2D diffusion-reaction process.

1.3. CONTRIBUTIONS

Generally, the contributions of this research are two-fold: First and foremost, a novel design is presented of an approximately optimal controller based on ADP in forward-in-time manner for DPS governed by uncertain parabolic equations without model reduction. This further enhances the stability analysis in the original infinite dimensional domain where an ultimate boundedness guarantee is obtained for the closed-loop system. Second, existing ADP approaches [19]–[24] for control of DPS systems rely on availability of system states. However, the spatially distributed nature of these systems makes output feedback control approaches significantly more useful and practical [26]–[30]. Therefore, relaxing the requirement of sensing all states in Paper I for early lumping and Papers III and IV for late lumping category is another considerable contribution of this research. Specifically, the contributions of each Paper are given as follows.

The main contributions of the Paper I include proposing an output feedback input constrained NDP-based optimal control for uncertain high dimensional nonlinear continuous-time systems and analyzing its applicability to boundary control of DPS modeled as one-dimensional parabolic PDE by using FDM model reduction.

Accordingly, in Paper II, two novel NN frameworks are proposed for identifying the unknown PDE dynamics and approximating the value function in original infinite dimensional domain, respectively. The tuning law for the identifier has a special design using basis functions as integral kernels such that the boundedness of identifier approximation error in the original PDE domain is guaranteed. This is substantially different from ODE-based NN identifiers [31] where stability is solely assured for finite dimensional models. In addition, a novel weight update law is proposed for the near optimal value function that minimizes the HJB approximation error in a forward-in-time manner. This in turn provides the system closed-loop stability in PDE domain, which is different from finite dimensional ADP approaches where stability is only guaranteed for reduced order models.

The research introduced in Paper III was motivated by the fact that system states are not available in the entire spatial domain and the nonlinearity in coupled PDE dynamics can be highly uncertain; hence, an NN observer is proposed to estimate the states online when the nonlinearity in the coupled PDE dynamics is unknown. Combined with a novel NN adaptive boundary control method that minimizes the HJB approximation error forward-in-time in the PDE domain, the proposed observer-controller framework will again provide the closed-loop stability guarantee in original infinite dimensional domain.

Paper IV emphasizes the importance of higher dimensional PDE in various applications such as fluid flow and lack of ADP controllers in literature for these kind of DPS. This lack of ADP controllers motivated the authors to solve the boundary control of a multi-dimensional nonlinear PDE like 2D Burgers equation with unknown nonlinearity in PDE dynamics. Similar to Papers IV and V, the boundary controller is obtained without model reduction prior to controller synthesis.

Finally, the fifth Paper designs the ADP control method for 2D uncertain nonlinear reaction diffusion PDE when only output is available. A PDE observer is designed to estimate the unavailable two dimensional state. Subsequently, the HJB is derived and an adaptive algorithm is developed to approximate its solution in the PDE domain. The real-world applications of PDE and in particular nonlinear reaction diffusion systems are almost in multi-dimensional domains. Although the design and analysis is much more challenging in two or three dimensions, very few ADP approaches were designed in these settings. Moreover, since the system is spatially distributed, it is necessary that the controller only rely on availability of few measurable outputs rather than the state on whole spatial domain. However, most of developed ADP approaches in the literature, design and analyze the controller based on the assumption that state is available throughout the space.

PAPER

I. NEAR OPTIMAL CONSTRAINED OUTPUT FEEDBACK BOUNDARY CONTROL OF ONE-DIMENSIONAL SEMI-LINEAR PARABOLIC PDE

B. Talaei, H. Xu and S. Jagannathan

This paper develops a novel neuro dynamic programming (NDP) based output feedback boundary control of distributed parameter systems (DPS) governed by one-dimensional semi-linear parabolic partial differential equations (PDE) in the presence of control constraints and uncertain system dynamics. First, finite difference method (FDM) is utilized to obtain a reduced order model of parabolic PDE. Subsequently, a near optimal control scheme is proposed for the reduced order finite dimensional system by using neural network (NN) time-based approximate dynamic programming (ADP) when only outputs are available for measurement. In other words, the proposed ADP scheme relaxes the policy or value iterations and availability of system states. A NN is utilized to estimate unavailable states under unknown nonlinear dynamics. Moreover, a second NN is proposed to estimate a non-quadratic value function online. Subsequently, by using the identified states and dynamics from the observer and estimated value function, the optimal control input that inherently falls within actuator limits is obtained. Local uniformly ultimately boundedness (UUB) of the closed-loop system is verified by using standard Lyapunov theory. The performance of presented NDP control scheme is successfully verified by simulation on a diffusion reaction process.

1. INTRODUCTION

Distributed parameter systems (DPS) are a major part of dynamical systems with wide range of industrial applications such as heat and mass transfer [1],[2], bioengineering [3], flexible structures [4],[5] and multi agent systems [6]. Since these systems are distributed in space, their states change with both position and time. As a consequence, their dynamics in state space are governed by partial differential equations (PDE) rather than ordinary differential equations (ODE) that describe the behavior of lumped parameter systems (LPS). In particular, parabolic PDEs model a broad range of DPS with applications in reaction-diffusion processes[7],[8] and energy management of buildings [9].

In one aspect, depending on the location of actuators, control of DPS can be categorized into in-domain and boundary control [10]. Boundary control [11],[12] is important in DPS because of lower cost and convenience of actuator placement. In fact, in many DPS governed by parabolic PDE, such as heat or fluid flow, placing an actuator in the spatial domain is a quite difficult or impossible feat 0.

From another aspect, the control design methodologies for DPS can be split into two general categories: late and early lumping. In the late lumping methods [14]-[18], the controller design is performed in the original infinite dimensional domain and the control input is approximated afterwards [18] for implementation. Despite the accuracy of control design approaches in this category [17],[18], many of them deal with PDE dynamics through abstract representation formulated in functional analysis [19] and therefore attract limited interest in engineering practice. The second approach is referred to as early lumping [7],[23], [24],[46] where a finite dimensional state representation is first obtained and a controller is designed subsequently [20]. The obvious benefit of this approach is the possibility of applying various control methods of LPS to DPS. This feature is particularly useful for nonlinear or semi-linear PDE, since control design approaches in the late lumping category are still in early stages of development [21].

In the early lumping approaches, either Galerkin [25],[46] or finite difference approximation methods (FDM) [27],[33] are used to convert the PDE into a set of ODE.

In Galerkin methods, which uses proper orthogonal decomposition (POD) [25] or finite elements [26], the reduced order model is obtained by transforming the PDE into "weak integral form" through appropriate spatial basis functions that approximate the solution of PDE. In contrast, in FDM, the reduced order model is extracted by approximating the PDE dynamics at specific points in space based on a structured mesh. One of the main benefits of using a structured mesh in DPS control is the easy observation of finite dimensional states by means of physical sensors in spatial domain [27]. This is generally a difficult task in spectral Galerkin methods [28],[37]. There is extensive literature in convergence behavior and applicability of the control approaches developed based on these numerical methods for parabolic PDE. See for example [6],[27],[44]-[46],[49] and the references therein.

Optimal control, on the other hand, has emerged [29], and remained [14],[23],[30], as one of popular methods of controlling a DPS. However, the real-time computation of optimal control [31] and the curse of dimensionality are significantly increased for DPS because of large scale nature of state space representation [30],[18]. In the recent decade, neuro-dynamic programming (NDP) was proposed [32] and established [41] as a new approach to find approximate but tractable solutions for optimal control of large scale and possibly uncertain systems. The design and analysis of NDP based controllers for DPS subsequently appears to be a promising field of research [22],[23],[33] with enormous potentials that have yet to be discovered.

The main purpose of this paper is design and analysis of an online output feedback near optimal boundary controller for DPS in the finite dimensional domain. In particular, an NDP based output feedback controller is designed for DPS modeled by one-dimensional semi-linear parabolic PDE with input constraints. The nonlinearity in the dynamics is considered to be unknown but satisfies locally Lipschitz condition. The FDM is mainly used to find the approximate finite dimensional state space. Even with reduced order models of DPS pure state feedback control method seems inappropriate for practical applications due to the need for full state measurability. Therefore, the output feedback control design methodologies have gained considerable attention [35]-[39]. Accordingly, in this paper, an output feedback NDP control method with guaranteed closed-loop stability is illustrated for situations where only very few sensors are available in the spatial domain and system dynamics are uncertain.

Despite various iterative NDP schemes in the literature [41] to obtain solution to the Bellman or Hamilton-Jacobi-Bellman equation, the large number of iterations within a sampling interval still makes their online implementation and stability analysis difficult [42]. However, in this paper a relatively new NDP scheme is developed based on conventional adaptive control method [43]. Moreover, by introducing a non-quadratic value function [47], it will be shown that the control design can adapt to actuators with magnitude constraints [48]. Boundedness of the error between the proposed control input and its truly optimal value will be proven for both state and output feedback schemes where this bound can be reduced through design parameters. Simulation results confirm that the combination of near optimal update law and the non-quadratic cost function can produce a satisfactory boundary control policy for regulation of unstable one-dimensional semi-linear parabolic PDE.

Therefore, the main contributions of the paper include proposing an output feedback input constrained NDP-based optimal control for uncertain high dimensional nonlinear continuous-time systems and analyzing its applicability to boundary control of DPS modeled as one-dimensional parabolic PDE by using FDM model reduction.

The rest of the paper is organized as follows. In Section 0 the class of DPS under consideration is described and the finite dimensional state space is derived. Subsequently, the NN output feedback near optimal controller for the uncertain finite dimensional system is designed. In Section 3, the stability of proposed method in the finite dimensional domain is addressed. Section 4 demonstrates the simulation results and Section 0 provides the conclusions of the paper.

2. FINITE DIMENSIONAL REPRESENTATION OF DPS MODELED BY PARABOLIC PDE WITH BOUNDARY CONTROL INPUT

In this section, after introducing the class of DPS, the FDM is utilized to obtain the reduced order model.

2.1. CLASS OF DISTRIBUTED PARAMETER SYSTEMS

Consider a DPS modeled by one-dimensional semi-linear parabolic PDE with the following dynamics [11],[12] given by

$$\begin{aligned} & x_t(z,t) = a x_{zz}(z,t) + f(x,z) \qquad z \in \Omega = [0,l] \\ & x(0,t) = g_1 u_1, \\ & x(l,t) = g_2 u_2, \qquad -\rho_M \le u_i \le \rho_M, i = 1,2 \\ & x(z,0) = x_0(z) \end{aligned} \tag{1}$$

where $x(z,t) \in H^2(\Omega)$ is the system state with $H^2(\Omega)$ being the Sobolev space of second order on domain Ω, x_t stands for the derivative of x with regard to t, x_{zz} is the second derivative of state x with respect to z, $t \in [0,\infty)$ represents time, $z \in \Omega$ represents the spatial variable and $l > 0$ is a positive scalar. Here, $a > 0$ is the known diffusion constant and $f(x,z): H^2(\Omega)\times\Omega \to C^1$ represents an unknown nonlinear function with C^1 being the space of continuously differentiable functions, $u_i(t)$, $1 \le i \le 2$, are the continuous boundary control inputs, $g_1, g_2 \in \mathbb{R}$ are known constants, $\rho_M > 0$ represents the saturation limit for the actuators and $x_0(z) \in C^2(\Omega)$ is the initial state of the system. Next, the following assumption is necessary to proceed.

Assumption 1 [50]: The function $f(x,z)$ is locally Lipschitz continuous with respect to x that is

$$\| f(x_1,z) - f(x_2,z) \| \le L_f \|x_1 - x_2\|, \tag{2}$$

where L_f is an unknown Lipschitz constant. This assumption is satisfied by practical DPS modeled by semi-linear parabolic PDEs in the literature [6],[17] ,[66]. Next, the finite dimensional representation of system will be derived.

2.2. FINITE DIFFERENCE APPROXIMATE REPRESENTATION

Keeping the time continuous, a set of points $\bar{z}_i; i = 0,\ldots,N$ in the domain Ω are chosen where $\bar{z}_0 = 0$ and $\bar{z}_N = l$. The state of the system at these points is shown by vector

$\bar{x}$ in this paper. For simplicity, it is assumed that these points are equally spaced along the domain. Define $\Delta z = \bar{z}_i - \bar{z}_{i-1}$ and $\bar{x}_i$ to represent the state of the DPS at all points in the local neighborhood $\Omega_i = (\bar{x}_i - \Delta z/2, \bar{x}_i + \Delta z/2)$ $(1 \le i \le N-1)$. Substituting the second order accurate formula for x_{zz} at $\bar{z}_i; i = 0,..., N$ [27] as

$$x_{zz}\,|_i = \frac{\bar{x}_{i+1} - 2\bar{x}_i + \bar{x}_{i-1}}{2\Delta z^2} + O(\Delta z^2), \tag{3}$$

in system dynamics (1) and combining the system of equations in the matrix format, the approximated dynamics for the DPS can be obtained as

$$\dot{\bar{x}} = \begin{bmatrix} \frac{d\bar{x}_1}{dt} \\ \frac{d\bar{x}_2}{dt} \\ \vdots \\ \frac{d\bar{x}_n}{dt} \end{bmatrix} = \overbrace{\frac{a}{\Delta z^2}\begin{bmatrix} -2 & 1 & ... & 0 \\ 1 & -2 & 1... & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & ... & 1 & -2 \end{bmatrix}\begin{bmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \vdots \\ \bar{x}_n \end{bmatrix}}^{A\bar{x}} + \bar{f}(\bar{x},\bar{z}) + \overbrace{\begin{bmatrix} \frac{ag_1}{\Delta z^2} & 0 \\ 0 & \vdots \\ \vdots & 0 \\ 0 & \frac{ag_2}{\Delta z^2} \end{bmatrix}}^{\bar{g}}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \tag{4}$$

where $n = N-1$, $\bar{x}(t) \in \mathbb{R}^n$ represents the discretized state, $\dot{\bar{x}}$ is its time derivative, $A \in \mathbb{R}^{n\times n}$ is a constant matrix with the special structure as shown above, $\bar{f}(\bar{x},\bar{z}) \in \mathbb{R}^n$ is discretized version of $f(x,z)$ [27] and $\bar{g} \in \mathbb{R}^{n\times 2}$. Therefore, the reduced order model for DPS (1) can be represented as

$$\dot{\bar{x}} = A\bar{x} + \bar{f}(\bar{x},\bar{z}) + \bar{g}u = \bar{h}(\bar{x},\bar{z}) + \bar{g}u, \tag{5}$$

where $\bar{h}(\bar{x},\bar{z}) = A\bar{x} + \bar{f}(\bar{x},\bar{z})$.

Remark 1: Using Galerkin approach, a finite dimensional approximated dynamics for system (1) can be obtained in affine form [56] and be used in the optimal boundary control design. The Galerkin approach provides a finite modal space with lower dimensions [7]. However, more in-depth knowledge of DPS dynamics is required to extract the dominant modes of the PDE which should be obtained either analytically [56] or through experiments and simulations [34]. Besides, for the purpose of observability of the reduced order model in output feedback control, stronger assumptions are required on the number and placement of sensors in Galerkin approach [37]. As described in the next section , these assumptions can be relaxed when FDM is utilized for model reduction.

Considering the size of state space for system dynamics (5) and the cost of sensor placement, a practical control approach for DPS should only rely on fewer physical sensors in the spatial domain. Therefore, in the next section, the proposed output feedback NDP controller design will be explained by using the finite dimensional reduced order model for situations when all states are not available for measurement. Here, the input matrix $\bar{g}$ is considered bounded such that $\|\bar{g}\| \le g_M$. Moreover, without loss of generality, it is assumed that the chosen mesh size Δz is such that the controllability of the reduced order system is preserved after approximation, in the sense that there exists a continuous control policy within constraint bounds that stabilizes the system, with $\bar{x} = 0$ being a unique equilibrium point on the compact set $\chi \subseteq \mathbb{R}^n$.

2.3. OUTPUT FEEDBACK CONSTRAINED NDP CONTROL DESIGN

Consider the following functional over infinite horizon given by

$$V(\bar{x},u) = \int_{t_0}^{\infty} (Q(\bar{x}) + N(u))dt, \tag{6}$$

where $V(\bar{x},u)$ denotes the cost function, $Q(\bar{x})$ is a positive definite function penalizing the state $\bar{x}$ and $N(u)$ is also a positive definite function that penalizes the control input u. In order to take into account input constraints as a priori in the design, if $\rho(.)$ is a monotone saturation function with the bound ρ_M i.e. $-\rho_M \le \rho(\upsilon) \le \rho_M$, $\forall \upsilon \in (-\infty, \infty)$, $N(u)$ is defined as [47]

$$N(u) = \sum_{i=1}^{2} 2r_i \int_0^{u_i} \rho^{-1}(v)dv, \tag{7}$$

where $r_i > 0$ for $1 \le i \le 2$ are positive scalars and v is the integral variable. Next, the following standard assumption is needed.

Assumption 2 [40]: It is assumed that saturation function $\rho(.)$ is Lipschitz continuous and its inverse function $\rho^{-1}(.)$ also satisfies the Lipschitz continuous condition as

$$\left\| \int_{u_1}^{u_2} \rho^{-1}(v)dv \right\| = L_I \left\| u_1 - u_2 \right\|, \tag{8}$$

where L_I is the Lipschitz constant. It should be noted that many saturation functions such as $\tanh(.)$ and $\mathrm{tansig}(.)$ that has quick convergence to ρ_M satisfy this condition. The objective of the control design is to determine a continuous stabilizing policy that minimizes the value function (6). Under the assumption that V is differentiable with regard to $\bar{x}$, an infinitesimal equivalent to (6) is given by [53]

$$0 = Q(\bar{x}) + N(u) + V_{\bar{x}}^T[\bar{h}(\bar{x}, z) + \bar{g}u]. \tag{9}$$

where $V_{\bar{x}}$ is the derivative of V with regard to $\bar{x}$. Subsequently, the Hamiltonian is expressed as

$$H(\bar{x}, u) = Q(\bar{x}) + \sum_{i=1}^{2} 2r_i \int_0^{u_i} \rho^{-1}(v)dv + V_{\bar{x}}^T[\bar{h}(\bar{x}, z) + \bar{g}u], \tag{10}$$

and the optimal control policy is obtained by using stationary condition $\frac{\partial H}{\partial u_i}$ which is given by

$$u_i^*(\bar{x}) = -\rho(\frac{1}{2} r_i^{-1} row_i(\bar{g}^T)V_{\bar{x}}^*), \tag{11}$$

where V^* represents the optimal value function and $row_i(\bar{g}^T)$ denotes the i th ($1 \le i \le 2$) row of matrix $\bar{g}^T$. Equivalently, by defining the positive definite matrix R as

$$R = \begin{bmatrix} r_1 & 0 \\ 0 & r_2 \end{bmatrix}, \tag{12}$$

the optimal control input u^* can be expressed as

$$u^*(\bar{x}) = -\rho(\frac{1}{2} R^{-1}\bar{g}^T V_{\bar{x}}^*). \tag{13}$$

Note that the non-quadratic value function (6) results in the optimal control policy (13) which satisfies the actuator limits. This is completely different from non-constraint approximate dynamic programming (ADP) control [43] where the control constraints are not asserted. Substituting (13) into (10) yields the non-quadratic HJB equation in the form

$$V_{\bar{x}}^{*T}\bar{h}(\bar{x}, z) + Q(\bar{x}) + \sum_{i=1}^{2} 2r_i \int_0^{u_i^*} \rho^{-1}(v)dv - V_{\bar{x}}^{*T}\bar{g}\rho(\frac{1}{2} R^{-1}\bar{g}^T V_{\bar{x}}^*) = 0. \tag{14}$$

Before proceeding, the following mild assumptions are necessary.

Assumption 3 [43],[23]: There exists a Lyapunov function $G(\bar{x})$ for $\forall \bar{x} \in \chi$, such that

$$G_{\bar{x}}(\bar{h}(\bar{x}, z) + \bar{g}u^*) \le -\delta \|G_{\bar{x}}\|^2, \tag{15}$$

where $G_{\bar{x}}$ denotes the derivative of G with respect to $\bar{x}$ and δ is a positive constant. Note that since u^* makes the system uniformly asymptotically stable, the existence of a class K function $\beta(\|\bar{x}\|)$ such that for $\forall \bar{x} \in \chi$

$$G_{\bar{x}}(\bar{h}(\bar{x}, z) + \bar{g}u^*) \leq -\beta(\|\bar{x}\|), \tag{16}$$

is guaranteed by Lyapunov converse theorem [64]. Consequently, knowing that every Lyapunov function $G(x)$ is the value function for some meaningful cost function [54],[55], the assumption that (15) holds for $G(x)$ can be satisfied by selecting appropriate $Q(.)$ and $N(.)$ in cost function (6) such that for $\forall \bar{x} \in \chi$ one has

$$Q(\bar{x}) + N(u^*) \geq \delta \|G_{\bar{x}}\|^2 . \tag{17}$$

It follows from HJB equation (9) that

$$G_{\bar{x}}(\bar{h}(\bar{x}, z) + \bar{g}u^*) \leq -Q(\bar{x}) - N(u^*) \leq -\delta \|G_{\bar{x}}\|^2 . \tag{18}$$

Assumption 4: For $\forall \bar{x} \in \chi$, $Q(\bar{x})$ in the cost function (6) is Lipschitz continuous i.e.

$$\|Q(\bar{x}_1) - Q(\bar{x}_2)\| \leq L_Q \|\bar{x}_1 - \bar{x}_2\|, \tag{19}$$

where L_Q is the Lipschitz constant. It should be noted that $Q(\bar{x})$ with the quadratic or any other polynomial structure satisfies this condition. Moreover, the $\bar{x}$-derivative of basis function $\phi_{\bar{x}}(\bar{x})$ is also assumed to be Lipschitz continuous for $\forall \bar{x} \in \chi$, i.e.

$$\|\phi_{\bar{x}}(\bar{x}_1) - \phi_{\bar{x}}(\bar{x}_2)\| \leq L_\phi \|\bar{x}_1 - \bar{x}_2\|, \tag{20}$$

where L_ϕ is the Lipschitz constants.

Assumption 5: It is assumed that there exists $m \leq n$ sensors placed in the spatial domain and $\bar{y} = C\bar{x}$ where $C \in \mathbb{R}^{m \times n}$. Moreover, the pair (A, C) is observable.

As a practical example, C can be defined as

$$C_{ij} = \begin{cases} 1 & \text{if } \bar{y}_i = \bar{x}_j \\ 0 & \text{otherwise} \end{cases}. \tag{21}$$

The assumption that pair (A, C) is observable can be justified with following lemma.

Lemma 1: Suppose that there only exists a sensor present at the boundary $\bar{z}_1$. Then the pair (A, C) is observable.

Proof: If the sensor is placed at the boundary, the matrix $C \in \mathbb{R}^{1\times n}$ can be expressed as

$$C = [1, \underbrace{0, ..., 0}_{n-1}]. \tag{22}$$

Recalling the structure of matrix A in(4), it will be proven by induction that the observability matrix $O(A,C)$ is lower triangular and therefore has the rank of n. For this purpose, it will be shown that for $1 \le i \le n$, the i th entry of i th row of $O(A,C)$ is nonzero and for $i+1 \le j \le n$, the j th entry of the same row is zero and therefore the $O(A,C)$ has n independent rows.

Since the i th row of $O(A,C)$ is CA^{i-1}, according to definition of C, it is the first row of matrix A^{i-1}. For $i=1$, it is clear that the first element of first row of identity matrix I is nonzero and the rest of entries are zero. Now suppose that the $i-1$th row of $O(A,C)$ which is equivalently the first row of matrix A^{i-2}, namely $Row_1\{A^{i-2}\}$, has its $i-1$ th entry as nonzero i.e. $A^{i-2}[1,i-1] \neq 0$, and the rest entries on its right hand side i.e. $A^{i-2}[1,j]$ for $i \le j \le n$ are equal to zero. The i th entry of the i th row of $O(A,C)$, namely $A^{i-1}[1,i]$, can be calculated as

$$\begin{aligned} &A^{i-1}[1,i] = row_1\{A^{i-2}\} \times col_i\{A\} = row_1\{A^{i-2}\} \times [\underbrace{0,...,0}_{i-2}, \frac{a}{\Delta z^2}, ...]^T \\ &= \frac{a}{\Delta z^2} A^{i-2}[1,i-1] \neq 0. \end{aligned} \tag{23}$$

Its j th entry for $i+1 \le j \le n$ can be determined as

$$A^{i-1}[1,j] = row_1\{A^{i-2}\} \times col_j\{A\} = [..., \underbrace{0,...,0}_{n-i+1}] \times [\underbrace{0,...,0}_{j-2}, \underbrace{\frac{a}{\Delta z^2}, ...}_{n-j+2}]^T = 0. \tag{24}$$

It follows that the matrix $O(A,C)$ is lower triangular for arbitrary value of $n \in \mathbb{N}$ and subsequently has full rank and therefore the pair (A,C) is observable. By increasing the number of sensors, the chance of full observability of the system increases, and the assumption that for a general matrix C, the pair (A,C) is observable is reasonable. Therefore, in the next section an observer will be developed for the system in the finite dimensional setting.

2.4. OBSERVER DESIGN

The objective in this section is to introduce an observer for the estimation of system state in the presence of unknown dynamics when very few states are available for measurement. The estimated system state and unknown dynamics can be subsequently utilized in the controller design. If the estimation of state vector $\bar{x}$ is denoted by $\hat{\bar{x}}$, consider the observer with dynamics

$$\dot{\hat{\bar{x}}} = A\hat{\bar{x}} + \hat{\bar{f}}(\hat{\bar{x}}, \bar{z}) + \Gamma(\bar{y} - C\hat{\bar{x}}) + \bar{g}u, \tag{25}$$

where $\Gamma \in R^{n \times m}$ is the observer gain matrix. Subsequently, $\bar{f}(\bar{x}, \bar{z})$ can be represented in NN approximated form on the compact set χ as

$$\bar{f}(\bar{x}, \bar{z}) = W_f^T \sigma_f(\bar{x}, \bar{z}) + \varepsilon_f(\bar{x}, \bar{z}), \tag{26}$$

where $\sigma_f(\bar{x}, \bar{z}) : \chi \times \Omega \to \mathbb{R}^l$ denotes the bounded and Lipschitz activation function, $W_f \in \mathbb{R}^{l \times n}$ represents NN identifier target weight matrix with the bound W_{fM} [61],[62] on compact set $\chi \times \Omega$, and ε_f is the bounded approximation reconstruction error. The bound and Lipschitz constant for σ_f are represented by σ_{fM} and L_σ, respectively. Therefore, the identifier can be introduced with the following structure given by

$$\hat{f}(\hat{\bar{x}}, \bar{z}) = \hat{W}_f \sigma_f(\hat{\bar{x}}, \bar{z}). \tag{27}$$

In order to make the NN identifier weight matrix close to its target value, the NN tuning law for weight matrix $\hat{W}_f$ is provided as

$$\dot{\hat{W}}_f = -\alpha_f \hat{W}_f + \sigma_f(\hat{\bar{x}}, \bar{z})\tilde{\bar{y}}^T J^T, \tag{28}$$

with α_f being a design constant and the matrix $J \in R^{n \times m}$ being introduced to match the dimension of output error $\tilde{\bar{y}}$ with $\hat{W}_f$. Therefore, the error dynamics for the observer can be expressed as

$$\begin{aligned}
\dot{\tilde{\bar{x}}} = \dot{\bar{x}} - \dot{\hat{\bar{x}}} &= (A - \Gamma C)\tilde{\bar{x}} + W_f^T \sigma_f(\bar{x}, \bar{z}) - \hat{W}_f \sigma_f(\hat{\bar{x}}, \bar{z}) + \varepsilon_f \\
&= (A - \Gamma C)\tilde{\bar{x}} + W_f^T \sigma_f(\bar{x}, \bar{z}) + \{-W_f^T \sigma_f(\hat{\bar{x}}, \bar{z}) + W_f^T \sigma_f(\hat{\bar{x}}, \bar{z})\} - \hat{W}_f \sigma_f(\hat{\bar{x}}, \bar{z}) + \varepsilon_f \\
&= (A - \Gamma C)\tilde{\bar{x}} + W_f^T (\sigma_f(\bar{x}, \bar{z}) - \sigma_f(\hat{\bar{x}}, \bar{z})) + \tilde{W}_f^T \sigma_f(\hat{\bar{x}}, \bar{z}) + \varepsilon_f,
\end{aligned} \tag{29}$$

whereas the error dynamics for the identifier weights, $\tilde{W}_f = W_f - \hat{W}_f$, are given by

$$\dot{\tilde{W}}_f = \alpha_f \hat{W}_f - \sigma_f(\hat{\bar{x}}, \bar{z})\tilde{\bar{y}}^T J^T. \tag{30}$$

Under the update laws (25) and (28) for the observer, it will be shown in Section 3 that in the presence of bounded inputs, the state estimation error $\tilde{x}$ and weight estimation error $\tilde{W}_f$ will be uniformly ultimately bounded. In the next section, the output feedback optimal control design by using proposed observer will be illustrated by relaxing the need for boundedness of the control input.

2.5. OUTPUT FEEDBACK BASED NEAR OPTIMAL CONTROLLER

From the control policy (13), it is clear that in order to implement the truly optimal controller, the optimal value function $V^*(\bar{x})$ should be obtained by solving the non-quadratic HJB equation (14). Since solving the HJB equation even for nonlinear systems with state measurements is very burdensome, traditional dynamic programming based optimal schemes [41] utilize two NNs, one for the value function, referred to as "critic" network and the second for the control input referred as "action" network in order to provide a near optimal control input [59]. However, in this paper, the NDP control scheme is realized online by using only a single NN.

The value function $V^*(\bar{x})$ can be expressed by using a NN on the compact set χ in the form

$$V^*(\bar{x}) = W_V^T \phi(\bar{x}) + \varepsilon_V(\bar{x}), \tag{31}$$

where $W_V \in \mathbb{R}^r$ is the target NN weight vector, $\phi(\bar{x}) : \mathbb{R}^n \to \mathbb{R}^r$ is the bounded activation function and $\varepsilon_V(\bar{x})$ is the NN reconstruction error. Subsequently, its partial derivative with regard to $\bar{x}$ is given by

$$V_{\bar{x}}^* = \phi_{\bar{x}}^T(\bar{x}) W_V + \varepsilon_{V\bar{x}}(\bar{x}), \tag{32}$$

where it is assumed that the NN reconstruction error $\varepsilon_V(\bar{x})$ and its gradient with respect to $\bar{x}$ are bounded above i.e. $\varepsilon_V(\bar{x}) \le \varepsilon_{VM}$ and $\varepsilon_{V\bar{x}}(\bar{x}) \le \varepsilon_{VxM}$, respectively [59],[60]. Substituting the NN approximation of $V_{\bar{x}}^*$ in the Hamiltonian yields

$$H = 0 = Q(\bar{x}) + \sum_{i=1}^{2} 2r_i \int_0^{u_i^*} \rho^{-1}(v)dv + W_V^T \phi_{\bar{x}}(\bar{x})[\bar{h}(\bar{x}, z) - \bar{g}\rho(\frac{1}{2} R^{-1} \bar{g}^T (\phi_{\bar{x}}(\bar{x}) W_V + \varepsilon_V(\bar{x})))] + \varepsilon_H, \tag{33}$$

where $\varepsilon_H = \varepsilon_{V\bar{x}}^T(\bar{x})[\bar{h}(\bar{x}, z) - \bar{g}\rho(\frac{1}{2} R^{-1} \bar{g}^T \phi_{\bar{x}}(\bar{x}) W_V + \varepsilon_V(\bar{x}))]$.

Similar to state feedback method described in [51], the general output feedback control framework is to approximate the value function and HJB equation by using the state estimation $\hat{x}$ and approximated dynamics $\hat{f}(\hat{\bar{x}},\bar{z})$, and providing an update law that insures closed loop stability of the system. In order to achieve this result, the value function can be approximated as

$$\hat{V}(\hat{\bar{x}},u) = \hat{W}_V \phi_{\bar{x}}(\hat{\bar{x}}), \tag{34}$$

where $\hat{W}_V \in \mathbb{R}^r$ is the approximated weight matrix and $\phi(\hat{\bar{x}}):\mathbb{R}^n \to \mathbb{R}^r$ is the bounded activation function which depends on state estimation $\hat{\bar{x}}$. Subsequently, by using the estimated state $\hat{\bar{x}}$ and identified system dynamics $\hat{\bar{h}}(\hat{\bar{x}},\bar{z})$ the approximated Hamiltonian can be expressed as

$$\begin{aligned} \hat{H} &= Q(\hat{\bar{x}}) + \sum_{i=1}^{2} 2r_i \int_0^{\hat{u}_i} \rho^{-1}(v)dv + \hat{V}_{\bar{x}}^T(\hat{\bar{x}})[\hat{\bar{h}}(\hat{\bar{x}},\bar{z}) + \bar{g}\hat{u}] = Q(\hat{\bar{x}}) + \sum_{i=1}^{2} 2r_i \int_0^{\hat{u}_i} \rho^{-1}(v)dv \\ &+ \hat{W}_V^T \phi_{\bar{x}}(\hat{\bar{x}})[\hat{\bar{h}}(\hat{\bar{x}},\bar{z}) + \bar{g}\rho(\frac{1}{2}R^{-1}\bar{g}^T \phi_{\bar{x}}^T(\hat{\bar{x}})\hat{W}_V)]. \end{aligned} \tag{35}$$

Since $H = 0$, $\hat{H}$ can be represented as

$$\begin{aligned} \hat{H} &= \hat{H} - H = Q(\hat{\bar{x}}) - Q(\bar{x}) + \sum_{i=1}^{2} 2r_i \int_{u_i^*}^{\hat{u}_i} \rho^{-1}(v)dv + \hat{W}_V^T \phi_{\bar{x}}(\hat{\bar{x}})[\hat{\bar{h}}(\hat{\bar{x}},\bar{z}) + \bar{g}\hat{u}] - W_V^T \phi_{\bar{x}}(\bar{x}) \\ &[\bar{h}(\bar{x},\bar{z}) + \bar{g}u^*] = Q(\hat{\bar{x}}) - Q(\bar{x}) + \sum_{i=1}^{2} 2r_i \int_{u_i^*}^{\hat{u}_i} \rho^{-1}(v)dv \underbrace{+ \hat{W}_V^T \phi_{\bar{x}}(\hat{\bar{x}})\hat{\bar{h}}(\hat{\bar{x}},\bar{z}) - W_V^T \phi_{\bar{x}}(\bar{x})\bar{h}(\bar{x},\bar{z})}_{T_1} \\ &\underbrace{+ \hat{W}_V^T \phi_{\bar{x}}(\hat{\bar{x}})\bar{g}\hat{u} - W_V^T \phi_{\bar{x}}(\bar{x})\bar{g}u^*}_{T_2} - \varepsilon_H, \end{aligned} \tag{36}$$

where the terms T_1 and T_2 can be further expanded as

$$\begin{aligned} T_1 &= \tilde{W}_V^T \phi_{\bar{x}}(\hat{\bar{x}})\hat{\bar{h}}(\hat{\bar{x}},\bar{z}) - W_V^T \phi_{\bar{x}}(\hat{\bar{x}})\tilde{\bar{h}}(\hat{\bar{x}},\bar{z}) + W_V^T \{\phi_{\bar{x}}(\hat{\bar{x}}) - \phi_{\bar{x}}(\bar{x})\}\bar{h}(\hat{\bar{x}},\bar{z}) \\ &+ W_V^T \phi_{\bar{x}}(\bar{x})\{\bar{h}(\hat{\bar{x}},\bar{z}) - \bar{h}(\bar{x},\bar{z})\}, \end{aligned} \tag{37}$$

and

$$T_2 = \tilde{W}_V^T \phi_{\bar{x}}(\hat{\bar{x}})\bar{g}\hat{u} + W_V^T \{\phi_{\bar{x}}(\hat{\bar{x}}) - \phi_{\bar{x}}(\bar{x})\}\bar{g}\hat{u} - W_V^T \phi_{\bar{x}}(\bar{x})\bar{g}\tilde{u}. \tag{38}$$

Therefore substituting (37) and (38) into (36) and defining $\hat{\omega} = \phi_{\bar{x}}(\hat{\bar{x}})(\hat{\bar{h}}(\hat{\bar{x}},\bar{z}) - \bar{g}\rho(\frac{1}{2}R^{-1}\bar{g}^T \phi_{\bar{x}}^T(\hat{\bar{x}})\hat{W}_V))$ yields

$$\hat{H} = \tilde{W}_V^T \hat{\omega} + Q(\hat{\bar{x}}) - Q(\bar{x}) + \sum_{i=1}^{2} 2r_i \int_{u_i^*}^{\hat{u}_i} \rho^{-1}(v)dv - W_V^T \phi_{\bar{x}}(\hat{\bar{x}})\tilde{\bar{h}}(\hat{\bar{x}}, \bar{z}) + W_V^T \{\phi_{\bar{x}}(\hat{\bar{x}})$$
$$-\phi_{\bar{x}}(\bar{x})\}\bar{h}(\hat{\bar{x}}, \bar{z}) + W_V^T \phi_{\bar{x}}(\bar{x})\{\bar{h}(\hat{\bar{x}}, \bar{z}) - \bar{h}(\bar{x}, \bar{z})\} + W_V^T \{\phi_{\bar{x}}(\hat{\bar{x}}) - \phi_{\bar{x}}(\bar{x})\}\bar{g}\hat{u} \tag{39}$$
$$-W_V^T \phi_{\bar{x}}(\bar{x})\bar{g}\tilde{u} - \varepsilon_H .$$

The objective here is to minimize the estimated Hamiltonian (45) along the system trajectory, such that approximate optimality can be achieved. Therefore, the update law for tuning the NN weights is found by minimizing $\hat{H}$ via normalized gradient descent scheme as

$$\dot{\hat{W}}_V = -\alpha \frac{\hat{\omega}}{(1 + \|\hat{\bar{x}}\| + \|\hat{W}_V\|)^2} \hat{H}(\hat{\bar{x}}, \hat{u}) - \eta \hat{W}_V , \tag{40}$$

where α and η are positive design constants. Finally, the control input can be expressed as

$$\hat{u} = -\rho(\frac{1}{2} R^{-1} \bar{g}^T \phi_{\bar{x}}^T (\hat{\bar{x}}) \hat{W}_V). \tag{41}$$

The flowchart of proposed output feedback control scheme is shown in Figure 2.1.

Remark: *(State feedback optimal control design)* [51]: The state feedback version of proposed control approach can also be obtained by making some modifications to presented output feedback method. Assuming all system states are measurable, the identifier dynamics are provided as

$$\hat{\bar{f}}(\bar{x}, \bar{z}) = \hat{W}_f^T \sigma_f (\bar{x}, \bar{z}). \tag{42}$$

In order to find the update law for $\hat{W}_f$, the observer (25) is first substituted by the following state estimator as

$$\dot{\hat{\bar{x}}} = A\bar{x} + \hat{\bar{f}}(\bar{x}, z) + \bar{g}u + K\tilde{\bar{x}}, \tag{43}$$

where $\tilde{x} = \bar{x} - \hat{\bar{x}}$ is the state estimation error, $K \in \mathbb{R}^{n \times n}$ is a positive definite matrix. The NN update law for weight matrix $\hat{W}_f$ is provided by

$$\dot{\hat{W}}_f = -\alpha_f \hat{W}_f + \sigma_f (\bar{x}, \bar{z}) \tilde{\bar{x}}^T , \tag{44}$$

where $\alpha_f > 0$ is the design parameter. Therefore, the estimated Hamiltonian is expressed as

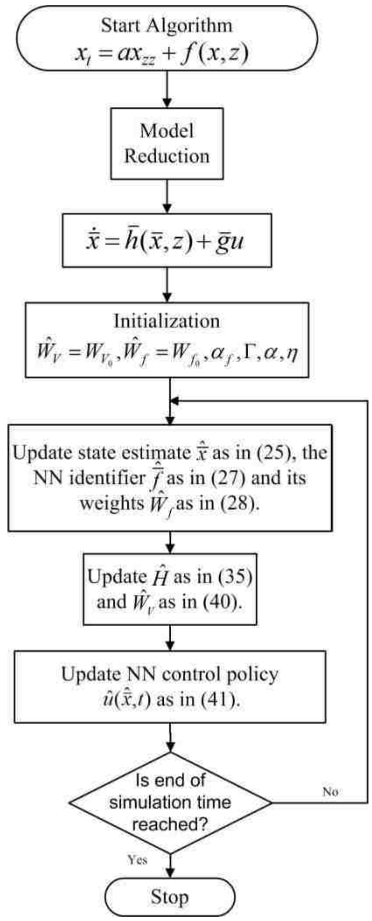

Figure 2.1. Flowchart of proposed output feedback controller.

$$\hat{H} = Q(\bar{x}) + \sum_{i=1}^{2} 2r_i \int_0^{\hat{u}_i} \rho^{-1}(v)dv + \hat{W}_V^T \phi_{\bar{x}}(\bar{x})[\hat{\bar{h}}(\bar{x},z) - \bar{g}\rho(\frac{1}{2}R^{-1}\bar{g}^T\phi_{\bar{x}}(\bar{x})\hat{W}_V)], \tag{45}$$

where $\hat{W}_V \in \mathbb{R}^r$ is the estimated weight matrix for value function with the update law for tuning the value function NN weights as

$$\dot{\hat{W}}_V = -\alpha \frac{\hat{\omega}}{1+\hat{\omega}^T\hat{\omega}} \hat{H}(\bar{x},\hat{u}) - \eta \hat{W}_V, \tag{46}$$

with $\hat{\omega} = \phi_{\bar{x}}(\bar{x})(\hat{\bar{h}}(\bar{x},\bar{z}) - \bar{g}\rho(\frac{1}{2}R^{-1}\bar{g}^T\phi_{\bar{x}}^T(\bar{x})\hat{W}_V))$ and α and η being positive scalars representing the tuning parameters. Finally, the actual control input is obtained as

$$\hat{u} = -\rho(\frac{1}{2}R^{-1}\bar{g}^T\phi_{\bar{x}}^T(\bar{x})\hat{W}_V). \tag{47}$$

For stability proof of state feedback control design refer to [51]. Next the stability analysis for the presented output feedback control policy with proposed update laws is introduced.

3. STABILITY ANALYSIS

Lemma 2: The matrix $A_{n\times n}$ defined as in (4) is negative definite.

Proof: Refer to Appendix.

Theorem 1: *(Boundedness of observer):* Let the initial NN observer error $\tilde{x}$ and observer weight estimation error $\tilde{W}_f$ be residing in a compact set $\mho_1$ and the proposed NN observer and its weight update law be provided as in (25) and (28), respectively. In the presence of bounded inputs, there exists a positive definite observer gain Γ and positive tuning parameter $\alpha_f > 1$ for the observer weight update law such that state estimation error $\tilde{x}$ and weight estimation error $\tilde{W}_f$ are all uniformly ultimately bounded (*UUB*).

Proof: Refer to the Appendix.

Lemma 3: The following inequality

$$\varepsilon_H \leq c_{H1}\left\|\hat{\bar{x}}\right\| + c_{H2}\left\|\tilde{\bar{x}}\right\| + c_{H3}, \tag{48}$$

holds where c_{H1}, c_{H2} and c_{H3} are positive constants.

Proof: Refer to Appendix.

Theorem 2: *(Boundedness of closed-loop system states under approximate control input):* Consider system (5) and let the NN observer, and its weight update law be provided as in (25), and (28) and tuning law for value function weights and control input be provided by (40) and (41), respectively with $\alpha > 0$ and $\eta > \eta_0$. Under the Assumptions 1 to 5, the system state x, state estimation error $\tilde{x}$ and weight estimation errors $\tilde{W}_f$ and $\tilde{W}_V$ are all uniformly ultimately bounded (UUB). Moreover, the approximation error between the actual control input and truly optimal one is also bounded where the ultimate bound depends on estimation errors $\tilde{x}$ and $\tilde{W}_V$ and reconstruction error ε_{VM} .

Proof: Refer to Appendix.

4. SIMULATION RESULTS

In this section the performance of proposed ADP based NN controller is examined by simulation on a semi-linear diffusion reaction process. Consider the plant with PDE dynamics given by

$$x_t = \beta_1 x_{zz} + \frac{\beta_2}{1+\beta_3(z-z_0)^2} x + \beta_4 e^{\frac{-1}{1+(z-z_0)^2}} x^2, \qquad 0 \le z \le 1 \tag{49}$$

with boundary and initial conditions given by

$$x(0,t) = 0, \qquad x(1,t) = g_1 u(t), \qquad x(z,0) = \sin(\pi z). \tag{50}$$

This example is motivated by chemical reaction-diffusion processes where due to effect of the catalyst, the reaction takes place only at a local site [64]. The following values are selected as the model parameters: $\beta_1 = 0.25$, $\beta_2 = 15$, $\beta_3 = 4$, $\beta_4 = 10$, $g_1 = 1$ and $z_0 = 0.5$. It is also assumed that the control input is subject to hard constraints with $u \le \rho_M = 10$. For these values, it was verified that open loop system state blows up in finite time. In order to find the FDM reduced order state space model of the system for control synthesis as explained in Section 0, the spatial domain of system ($[0,1]$) is discretized into 10 intervals and therefore $\Delta z = 0.1$. The controller is subsequently implemented on the system with $\Delta z = 0.05$ using MATLAB which uses the implicit method of lines [63] for numerical solution of one dimensional PDE. The controller is updated and applied in real time setting with step size Δt of 1 msec in order to have acceptable performance and computation rate. In the following, the simulation results are shown and explained separately for output feedback and state feedback control methods.

4.1. OUTPUT FEEDBACK CONTROL

In this section the performance of output feedback controller is evaluated. It is assumed that three sensors exist in the spatial domain, two at the boundaries and one at $z = 0.5$. The observer gain $\Gamma_{20\times3}$ is calculated such that $\lambda_{\max}(A-\Gamma C) \simeq -200$. The observer is designed with $\alpha_f = 10$ and $J_{20\times3}$ in update law (30) is selected as

α

$$J=\begin{bmatrix} \overset{1\times 6}{1,\ldots,1} & \overset{1\times 7}{0,\ldots,0} & \overset{1\times 7}{0,\ldots,0} \\ 0,\ldots,0 & 1,\ldots,1 & 0,\ldots,0 \\ 0,\ldots,0 & 0,\ldots,0 & 1,\ldots,1 \end{bmatrix}. \tag{51}$$

For simulation of optimal control design, the following infinite horizon performance index

$$V(\bar{x})=\int_0^\infty(\bar{x}^T\bar{x}+2\int_0^u \rho^{-1}(v)dv)dt, \tag{52}$$

is considered where the chosen nonquadratic constrained function for control input is expressed as $\rho(.)=10\tanh(./10)$. In order to implement the identifier, the NN activation function for $f(\hat{\bar{x}}_k)$ $(1\le k\le n)$ is selected as: $\sigma(\hat{\bar{x}}_k,\bar{z})=\frac{1}{\theta(\bar{z}-\bar{z}_0)^2}\rho([1,\hat{\bar{x}}_k,\hat{\bar{x}}_k^2,\hat{\bar{x}}_k^3,\hat{\bar{x}}_k^4])^T$ where $\rho(.)=\tanh(.)$ and $\theta=4$. Further, a combination of six radial basis functions $\phi_l(1\le l\le 6)$ are used to estimate $V^*(\hat{\bar{x}})$ with the inverse quadratic structure as

$$\phi_l=\sum_{i=\lambda_{l-1}}^{\lambda_l}\hat{\bar{x}}_i\sum_{j=1}^{n}\frac{\theta_1}{1+\theta_2(\bar{z}_j-\bar{z}_l)^2}P_l(\hat{\bar{x}}_j), \tag{53}$$

where $P_l(x_j)\in\{\hat{\bar{x}}_j,\hat{\bar{x}}_j^3\}$, $z_l\in\{0.2,0.5,0.8\}$, $\lambda_l\in\{1,3,7,10\}$, $\theta_1=.001$ and $\theta_2=30$.

The parameters for the controller and its update law are chosen as $\rho_M=10$, $\alpha=1$ and $\eta=20$. Note that the parameter b_{WV} in the proof can be calculated as $b_{WV}\cong 5$ and therefore $\eta>b_{WV}$. The initial conditions for observer weights are chosen as $\hat{W}_{fk}(0)=[1,1,1,1,1]^T$ and for value function weights as $W_V(0)=10^2\times[1,1,1,0,0,0]^T$. Figure 4.1 shows the performance of controller in stabilizing system state $x(z,t)$ at origin. The control input is also shown in the state trajectory as $x(1,t)$.

In order to compare the convergence behavior of HJB error with control input and verify the robustness of the controller to design parameters, simulation is repeated for different values of gain, control constraint, the robust gain and observer gain.

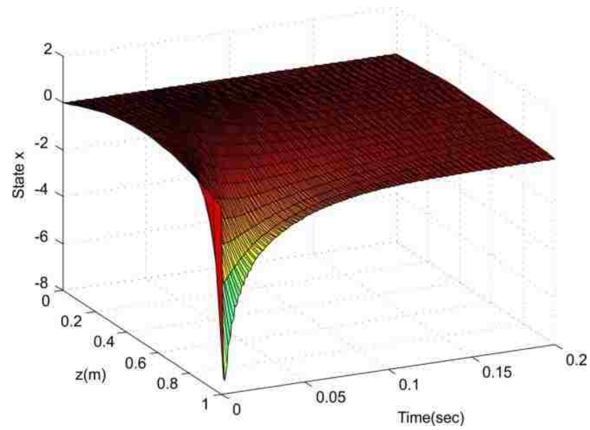

Figure 4.1. Closed loop state history of system under output feedback control law.

Figure 4.2 shows that increasing the value of α will increase the convergence rate for HJB error, but it may result in overshoots in control input. As shown in Figure 4.3, reducing the control input constraint would smoothen and decrease the control input peak value. However, the smaller the control constraint, the longer it takes for control input and HJB error to converge. Subsequently, Figure 4.4 shows that increasing the update law parameter η would reduce the control input but it has the adverse effect of slower convergence rate for HJB error.

Figure 4.5 shows the robustness of controller to variations in the observer gain Γ by depicting the control input and HJB error when Γ is reduced by half or to 0.1 of its original value. As shown, decreasing the observer gain would reduce the initial peak value of control input and HJB error but it makes the HJB error and control input converge slower to zero consequently. It should be noted that faster convergence rate for HJB error compared to control input in all simulations confirms the capability of update law to obtain local optimality along system trajectories.

4.2. STATE FEEDBACK CONTROL

In this section the performance of state feedback controller is studied and compared with the output feedback control scheme. The optimal performance cost function is selected as to compare with (52) and the parameters for the controller and its update law are also chosen similar to output feedback case. The initial conditions for identifier weights are chosen as [1,1,1,1,1] and for value function weights as $10^2 \times [1,1,1,0,0,0]$. The tuning parameters for adaptive update laws are chosen to be $\alpha = 1$, $\eta = 20$, $\alpha_f = 10$ and $K = 100$. Figure 4.6 clearly shows that the controller stabilizes the system state at zero in 200 msec.

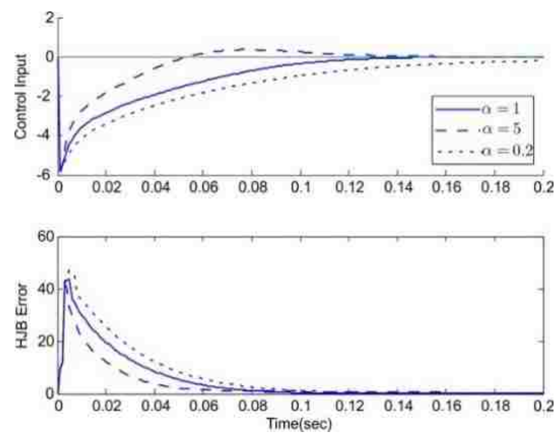

Figure 4.2. Effect of design parameters on state feedback control input and HJB error convergence.

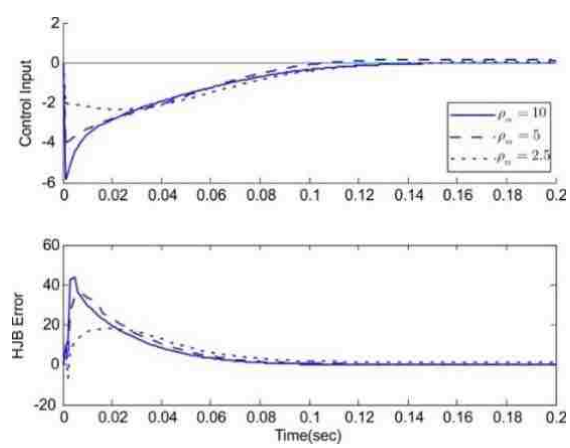

Figure 4.3. Effect of control constraint on control input and HJB error.

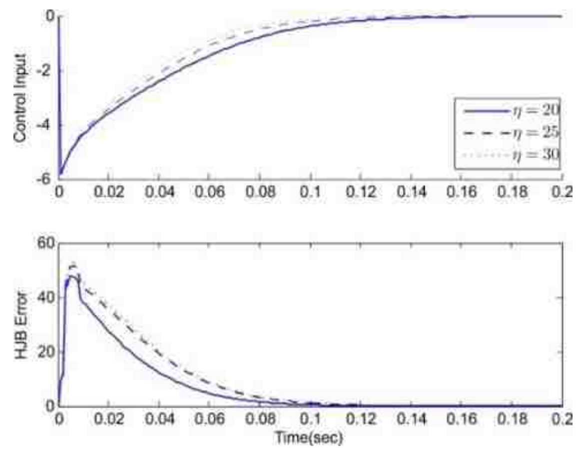

Figure 4.4. Effect of design parameters on output feedback control input and HJB error convergence.

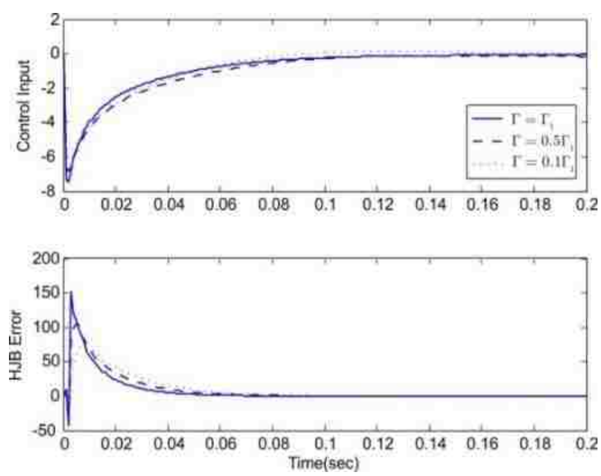

Figure 4.5. Effect of observer gain on control input and HJB error convergence.

By comparing Figures 4.6 and 4.1, the output feedback control input turns to have larger magnitude at the beginning because of observer initial conditions. However, the output feedback control policy is able to provide closed-loop performance similar to that of the state feedback design afterwards. This indicates the capability of observer in providing an acceptable estimate of system state within short period of time.

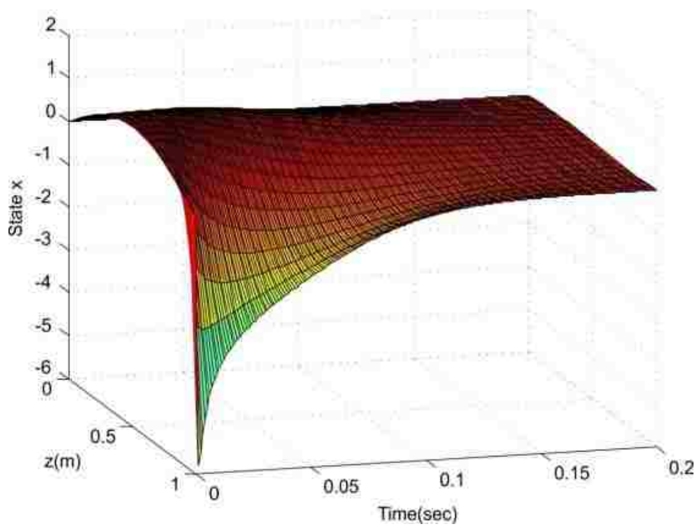

Figure 4.6. Closed loop state history of system under state feedback control law.

5. CONCLUSIONS

In this paper, an output feedback ADP near optimal NN boundary controller was designed for DPS which is described by one-dimensional semi-linear parabolic PDE with control input constraints and unknown nonlinearity in dynamics. For controller synthesis, a discretized model of DPS by using finite difference approximation (FDM) appeared to provide satisfactory results. The FDM approach lead to an affine nonlinear finite dimensional dynamical representation of DPS where the boundary control input could be designed based on optimal control method for finite dimensional systems. Then a novel ADP scheme was provided by using two NNs to estimate unavailable states under uncertain system dynamics and approximate the optimal value function when only a few states were available for measurement in spatial domain. Since the input constraints were incorporated as a priori in the design, the control input lay inherently within the actuator limits. Uniformly ultimately boundedness (UUB) of the closed-loop system was successfully verified by using standard Lyapunov theory. Finally, Simulation results confirmed the effectiveness of output feedback controller on a diffusion reaction process.

APPENDIX

Proof of Lemma 2

Let $\bar{x} = [\bar{x}_1, ..., \bar{x}_n]^T \neq 0$ and suppose $\bar{x}_i$ is its first nonzero element. Subsequently

$$\begin{aligned} \bar{x}^T A\bar{x} &= \frac{a}{\Delta z^2}(-2\bar{x}_1^2 + 2\bar{x}_1\bar{x}_2 - 2\bar{x}_2^2 + 2\bar{x}_2\bar{x}_3 ... + 2\bar{x}_{n-1}\bar{x}_n - 2\bar{x}_n^2) \\ &\overset{(\bar{x}_1 = ... = \bar{x}_{i-1} = 0)}{=} \frac{a}{\Delta z^2}(-2\bar{x}_i^2 + 2\bar{x}_i\bar{x}_{i+1} - 2\bar{x}_{i+1}^2 + ...), \end{aligned} \tag{54}$$

Using young inequality, one can write $2\bar{x}_j\bar{x}_{j+1} \leq \bar{x}_j^2 + \bar{x}_{j+1}^2$ for $i \leq j \leq n-1$. Therefore

$$\bar{x}^T A\bar{x} \leq \frac{a}{\Delta z^2}(-2\bar{x}_i^2 + \bar{x}_i^2 + \bar{x}_{i+1}^2 - 2\bar{x}_{i+1}^2 + \bar{x}_{i+1}^2 ... + \bar{x}_n^2 - 2\bar{x}_n^2) \leq \frac{-a}{\Delta z^2}\bar{x}_i^2, \tag{55}$$

which shows that matrix A is negative definite.

Proof of Theorem 1

In order to prove the stability of the observer, consider the Lyapunov function given by

$$L_c = \frac{1}{2}\tilde{x}^T\tilde{x} + \frac{1}{2}tr(\tilde{W}_f^T\tilde{W}_f). \tag{56}$$

Subsequently, by taking its derivative $\dot{L}_c$ one has

$$\begin{aligned} \dot{L}_c &= \tilde{\bar{x}}^T\dot{\tilde{\bar{x}}} + tr(\tilde{W}_f^T\dot{\tilde{W}}_f) = \tilde{\bar{x}}^T((A - \Gamma C)\tilde{\bar{x}} + W_f^T(\sigma_f(\bar{x}, \bar{z}) - \sigma_f(\hat{\bar{x}}, \bar{z})) \\ &+ \tilde{W}_f^T\sigma_f(\hat{\bar{x}}, \bar{z})) + \tilde{\bar{x}}^T\varepsilon_f + tr(\tilde{W}_{If}^T(\alpha_{If}\hat{W}_{If} - \sigma_{If}(\hat{\bar{x}}, \bar{z})\tilde{\bar{y}}^T J^T)) \end{aligned} \tag{57}$$

$$\begin{aligned} &= \tilde{\bar{x}}^T(A - \Gamma C)\tilde{\bar{x}} + \tilde{\bar{x}}^T W_f^T(\sigma_f(\bar{x}, \bar{z}) - \sigma_{If}(\hat{\bar{x}}, \bar{z})) + \tilde{\bar{x}}^T\tilde{W}_f^T\sigma_f(\hat{\bar{x}}, \bar{z}) + \tilde{\bar{x}}^T\varepsilon_f \\ &+ tr(\tilde{W}_f^T\alpha_f(W_f - \tilde{W}_f)) - tr(\tilde{W}_f^T\sigma_f(\hat{\bar{x}}, \bar{z})\tilde{\bar{y}}^T J^T). \end{aligned}$$

Since the activation function σ_f is bounded on $\chi \times \Omega$, using Cauchy Schwarz inequality yields

$$\tilde{\bar{x}}^T W_f^T(\sigma_f(\bar{x}, \bar{z}) - \sigma_f(\hat{\bar{x}}, \bar{z})) \leq 2\|W_f\|\sigma_{fM}\|\tilde{\bar{x}}\|. \tag{58}$$

Moreover, by using Young's inequality one has

$$\tilde{\bar{x}}^T\tilde{W}_f^T\sigma_f(\hat{\bar{x}}, \bar{z}) \leq \|\tilde{\bar{x}}\|\|\tilde{W}_f\|\sigma_{fM} \leq \sigma_{fM}^2\|\tilde{\bar{x}}\|^2 + \frac{1}{4}\|\tilde{W}_f\|^2, \tag{59}$$

and

$$\left\|\tilde{\bar{x}}^T \varepsilon_f\right\| \le \left\|\tilde{\bar{x}}\right\| \varepsilon_{fM} \le \frac{1}{2}\varepsilon_{fM}^2 + \frac{1}{2}\left\|\tilde{\bar{x}}\right\|^2 . \tag{60}$$

Therefore

$$\begin{aligned}
\dot{L}_c &\le (\lambda_{\max}(A-\Gamma C))\left\|\tilde{\bar{x}}\right\|^2 + 2\sigma_{fM}^2\left\|\tilde{\bar{x}}\right\|^2 + \frac{1}{2}\left\|W_f\right\|^2 + \sigma_{fM}^2\left\|\tilde{\bar{x}}\right\|^2 + \frac{1}{4}\left\|\tilde{W}_f\right\|^2 + \frac{1}{2}\varepsilon_{fM}^2 + \frac{1}{2}\left\|\tilde{\bar{x}}\right\|^2 - \frac{1}{2}\alpha_f\left\|\tilde{W}_f\right\|^2 \\
&+ \frac{1}{2}\alpha_f W_{fM}{}^2 + \frac{1}{4}\left\|\tilde{W}_f\right\|^2 + \sigma_{fM}^2\left\|\tilde{\bar{y}}\right\|^2\left\|J\right\|^2 \le (\lambda_{\max}(A-\Gamma C) + 1 + \sigma_{fM}^2(3+\left\|C\right\|^2\left\|J\right\|^2)) \\
&\left\|\tilde{\bar{x}}\right\|^2 + (-\frac{1}{2}\alpha_f + \frac{1}{2})\left\|\tilde{W}_f\right\|^2 + \underbrace{\frac{1}{2}\varepsilon_{fM}^2 + W_{fM}^2(\frac{1}{2}\alpha_f + \frac{1}{2})}_{\varepsilon_c} = \kappa\left\|\tilde{\bar{x}}\right\|^2 + (-\frac{1}{2}\alpha_f + \frac{1}{2})\left\|\tilde{W}_f\right\|^2 + \varepsilon_c .
\end{aligned} \tag{61}$$

where $\lambda_{\max}(A-\Gamma C)$ is the maximum eigenvalue of matrix $A-\Gamma C$, $\kappa = \lambda_{\max}(A-\Gamma C) + 1 + \sigma_{fM}^2(3+\left\|C\right\|^2\left\|J\right\|^2)$ and $\varepsilon_c = \frac{1}{2}\varepsilon_{fM}^2 + W_{fM}^2(\frac{1}{2}\alpha_f + \frac{1}{2})$.

Therefore since the system is observable, one can arbitrarily choose Γ such that $\kappa < 0$. Consequently, by choosing $\alpha_f > 1$, the observer dynamics are UUB [64].

Proof of Lemma 3

ε_H can be expressed as

$$\begin{aligned}
\varepsilon_H &= \varepsilon_{V\bar{x}}^T(\bar{x})[\bar{h}(\bar{x},z) - \bar{g}\rho(\frac{1}{2}R^{-1}\bar{g}^T\phi_{\bar{x}}(\bar{x})W_V + \varepsilon_V(\bar{x}))] \\
&\le \varepsilon_{V\bar{x}M}[\left\|A\right\|\left\|\bar{x}\right\| + W_{fM}\sigma_{fM} + \varepsilon_{fM} + \bar{g}_M\rho_M] \le \varepsilon_{V\bar{x}M}\left\|A\right\|\left\|\bar{x}\right\| + \varepsilon_{V\bar{x}M}[W_{fM}\sigma_{fM} + \varepsilon_{fM} + \bar{g}_M\rho_M] \\
&\le \varepsilon_{V\bar{x}M}\left\|A\right\|(\left\|\hat{\bar{x}}\right\| + \left\|\tilde{\bar{x}}\right\|) + \varepsilon_{V\bar{x}M}[W_{fM}\sigma_{fM} + \varepsilon_{fM} + \bar{g}_M\rho_M] \\
&\le c_{H1}\left\|\hat{\bar{x}}\right\| + c_{H2}\left\|\tilde{\bar{x}}\right\| + c_{H3},
\end{aligned} \tag{62}$$

where $c_{H1} = \varepsilon_{V\bar{x}M}\left\|A\right\|$, $c_{H2} = \varepsilon_{V\bar{x}M}\left\|A\right\|$ and $c_{H3} = \varepsilon_{V\bar{x}M}[W_{fM}\sigma_{fM} + \varepsilon_{fM} + \bar{g}_M\rho_M]$.

Proof of Theorem 2

If the Lyapunov function L_b is defined as $L_b = \frac{1}{2}\tilde{W}_V^T\tilde{W}_V$, taking its derivative $\dot{L}_b$ and substituting update law (40) yields

$$\begin{aligned}
\dot{L}_b &= \tilde{W}_V^T\dot{\tilde{W}}_V = -\alpha\frac{\hat{\omega}}{(1+\left\|\hat{\bar{x}}\right\|+\left\|\hat{W}_V\right\|)^2}\tilde{W}_V^T\hat{H}(\hat{\bar{x}},\hat{u}) - \eta\tilde{W}_V^T(W_V + \tilde{W}_V) = -\alpha\frac{\tilde{W}_V^T\hat{\omega}}{(1+\left\|\hat{\bar{x}}\right\|+\left\|\hat{W}_V\right\|)^2} \\
&[\tilde{W}_V^T\hat{\omega} + Q(\hat{\bar{x}}) - Q(\bar{x}) + \sum_{i=1}^{2}2r_i\int_{u_i^*}^{\hat{u}_i}\rho^{-1}(v)dv - W_V^T\phi_{\bar{x}}(\hat{\bar{x}})\tilde{\bar{h}}(\hat{\bar{x}},\bar{z}) + W_V^T\{\phi_{\bar{x}}(\hat{\bar{x}}) - \phi_{\bar{x}}(\bar{x})\}\bar{h}(\hat{\bar{x}},\bar{z}) \\
&+ W_V^T\phi_{\bar{x}}(\bar{x})\{\bar{h}(\hat{\bar{x}},\bar{z}) - \bar{h}(\bar{x},\bar{z})\} + W_V^T\{\phi_{\bar{x}}(\hat{\bar{x}}) - \phi_{\bar{x}}(\bar{x})\}\bar{g}\hat{u} - W_V^T\phi_{\bar{x}}(\bar{x})\bar{g}\tilde{u} - \varepsilon_H] - \eta\tilde{W}_V^T(W_V + \tilde{W}_V).
\end{aligned} \tag{63}$$

Consequently:

$$\dot{L}_b \le -\alpha \frac{\tilde{W}_V^T \hat{\omega}\hat{\omega}^T \tilde{W}_V}{(1+\|\hat{\tilde{x}}\|+\|\hat{W}_V\|)^2} + \alpha \frac{\|\tilde{W}_V^T \hat{\omega}\|}{(1+\|\hat{\tilde{x}}\|+\|\hat{W}_V\|)^2} \{\|Q(\hat{\tilde{x}})-Q(\tilde{x})\| + \left\|\sum_{i=1}^{2} 2r_i \int_{u_i^*}^{\hat{u}_i} \rho^{-1}(v)dv\right\|$$
$$\|W_V^T \phi_{\bar{x}}(\hat{\bar{x}})\tilde{\bar{h}}(\hat{\bar{x}},\bar{z})\| + \|W_V^T\{\phi_{\bar{x}}(\hat{\bar{x}})-\phi_{\bar{x}}(\bar{x})\}\bar{h}(\hat{\bar{x}},\bar{z})\| + \|W_V^T \phi_{\bar{x}}(\bar{x})\{\bar{h}(\hat{\bar{x}},\bar{z})-\bar{h}(\bar{x},\bar{z})\}\| \tag{64}$$
$$+\|W_V^T\{\phi_{\bar{x}}(\hat{\bar{x}})-\phi_{\bar{x}}(\bar{x})\}\bar{g}\hat{u}\| + \|W_V^T \phi_{\bar{x}}(\bar{x})\bar{g}\tilde{u}\| + \|\varepsilon_H\|\} - \eta \tilde{W}_V^T \tilde{W}_V + \eta \|\tilde{W}_V^T W\|.$$

Based on assumption 4 one has

$$\|Q(\hat{\bar{x}})-Q(\bar{x})\| \le L_Q \|\tilde{\bar{x}}\|, \tag{65}$$

and since $-\rho_M \le u^*, \hat{u} \le \rho_M$, based on assumption 2

$$\left\|\sum_{i=1}^{2} 2r_i \int_{u_i^*}^{\hat{u}_i} \rho^{-1}(v)dv\right\| \le 2L_I \sum_{i=1}^{2} r_i \|\tilde{u}_i\| \le 2L_I \|R\|\|\tilde{u}\|. \tag{66}$$

Noting that function $\rho(.)$ satisfies the Lipschitz condition with Lipschitz constant L_ρ and $\|W_V\| \le W_{VM}$, $\|\tilde{u}\|$ satisfies

$$\|\tilde{u}\| = \left\|\rho(\frac{1}{2}R^{-1}\bar{g}^T \phi_{\bar{x}}^T(\hat{\bar{x}})\hat{W}_V) - \rho(\frac{1}{2}R^{-1}\bar{g}^T(\phi_{\bar{x}}^T(\bar{x})W_V + \varepsilon_V))\right\| \le \frac{1}{2}L_\rho \|R^{-1}\| \bar{g}_M$$
$$\|\phi_{\bar{x}}^T(\hat{\bar{x}})\hat{W}_V - (\phi_{\bar{x}}^T(\bar{x})W_V + \varepsilon_V)\| \le \frac{1}{2}L_\rho \|R^{-1}\| \bar{g}_M (\phi_{\bar{x}M}\|\tilde{W}_V\| + L_\phi W_{VM}\|\tilde{x}\| + \varepsilon_{VM}). \tag{67}$$

Therefore,

$$\left\|\sum_{i=1}^{2} 2r_i \int_{u_i^*}^{\hat{u}_i} \rho^{-1}(v)dv\right\| \le L_I L_\rho \|R\|\|R^{-1}\| \bar{g}_M (\phi_{\bar{x}M}\|\tilde{W}_V\| + L_\phi W_{VM}\|\tilde{x}\| + \varepsilon_{VM}). \tag{68}$$

Besides, since $\phi_{\bar{x}}(.) \le \phi_{\bar{x}M}$,

$$\|W_V^T \phi_{\bar{x}}(\hat{\bar{x}})\tilde{\bar{h}}(\hat{\bar{x}},\bar{z})\| \le W_{VM}\phi_{\bar{x}M}(\sigma_{fM}\|\tilde{W}_f\| + \varepsilon_{fM}). \tag{69}$$

Further, based on Assumption 3, since $\phi_{\bar{x}}(.)$ is Lipschitz one has

$$\|W_V^T\{\phi_{\bar{x}}(\hat{\bar{x}})-\phi_{\bar{x}}(\bar{x})\}\bar{h}(\hat{\bar{x}},\bar{z})\| \le \|W_V^T\{\phi_{\bar{x}}(\hat{\bar{x}})-\phi_{\bar{x}}(\bar{x})\}\|\|A\hat{\bar{x}} + W_f^T\sigma_f(\hat{\bar{x}},\bar{z}) + \varepsilon_f(\hat{\bar{x}},\bar{z})\|$$
$$\le W_{VM}L_\phi \|A\|\|\tilde{\bar{x}}\|\|\hat{\bar{x}}\| + W_{VM}L_\phi(W_{fM}\sigma_{fM} + \varepsilon_{fM})\|\tilde{\bar{x}}\|. \tag{70}$$

Accordingly,

$$\|W_V^T \phi_{\bar{x}}(\bar{x})\{\bar{h}(\hat{\bar{x}},\bar{z}) - \bar{h}(\bar{x},\bar{z})\}\| \le W_{VM}\phi_{\bar{x}M}L_h\|\tilde{x}\|. \tag{71}$$

Since $\hat{u} \le \rho_M$,

$$\|W_V^T\{\phi_{\bar{x}}(\hat{\bar{x}})-\phi_{\bar{x}}(\bar{x})\}\bar{g}\hat{u}\| \le W_{VM}L_\phi \bar{g}_M \rho_M \|\tilde{x}\|. \tag{72}$$

In addition since

$$\|\tilde{u}\| = \|u^* - \hat{u}\| = \left\|\rho(\frac{1}{2}R^{-1}\bar{g}^T(\phi_{\bar{x}}^T(\bar{x})W_V + \varepsilon_V)) - \rho(\frac{1}{2}R^{-1}\bar{g}^T\phi_{\bar{x}}^T(\hat{\bar{x}})\hat{W}_V)\right\| \le 2\rho_M, \tag{73}$$

and alternatively using (67)

$$\begin{aligned}\|W_V^T\phi_{\bar{x}}(\bar{x})\bar{g}\tilde{u}\| &\le \|(\hat{W}_V - \tilde{W}_V)^T\phi_{\bar{x}}(\bar{x})\bar{g}\tilde{u}\| \le 2\|\tilde{W}_V\|\phi_{\bar{x}M}\bar{g}_M\rho_M \\ &+ \|\hat{W}_V\|\phi_{\bar{x}M}\bar{g}_M^2 L_\rho\|R^{-1}\|(\phi_{\bar{x}M}\|\tilde{W}_V\| + L_\phi W_{VM}\|\tilde{\bar{x}}\| + \varepsilon_{VM}).\end{aligned} \tag{74}$$

Therefore,

$$\begin{aligned}\dot{L}_b \le& -\alpha\frac{\tilde{W}_V^T\hat{\omega}\hat{\omega}^T\tilde{W}_V}{(1+\|\hat{\bar{x}}\|+\|\hat{W}_V\|)^2} + \alpha\frac{\|\tilde{W}_V^T\hat{\omega}\|}{(1+\|\hat{\bar{x}}\|+\|\hat{W}_V\|)^2}\{(L_Q + W_{VM}L_\phi(W_{fM}\sigma_{fM} + \varepsilon_{fM}) + W_{VM}\phi_{\bar{x}M}L_h + W_{VM}L_\phi\bar{g}_M\rho_M \\ &+ L_I L_\rho\|R\|\|R^{-1}\|\bar{g}_M L_\phi W_{VM} + c_{H2})\|\tilde{\bar{x}}\| + L_\rho\phi_{\bar{x}M}L_\phi W_{VM}\|\hat{W}_V\|\bar{g}_M^2\|R^{-1}\|\|\tilde{\bar{x}}\| + W_{VM}L_\phi\|A\|\|\hat{\bar{x}}\|\|\tilde{\bar{x}}\| \\ &+ W_{VM}\phi_{\bar{x}M}\sigma_{fM}\|\tilde{W}_f\| + c_{H1}\|\hat{\bar{x}}\| + \|\hat{W}_V\|\phi_{\bar{x}M}\bar{g}_M^2 L_\rho\|R^{-1}\|\varepsilon_{VM} + \|\hat{W}_V\|\phi_{\bar{x}M}\bar{g}_M^2 L_\rho\|R^{-1}\|\phi_{\bar{x}M}\|\tilde{W}_V\| \\ &+ (2\phi_{\bar{x}M}\bar{g}_M\rho_M + L_I L_\rho\|R\|\|R^{-1}\|\bar{g}_M\phi_{\bar{x}M})\|\tilde{W}_V\| + (L_I L_\rho\|R\|\|R^{-1}\|\bar{g}_M\varepsilon_{VM} + W_{VM}\phi_{\bar{x}M}\varepsilon_{fM} + c_{H3})\} \\ &- \eta\tilde{W}_V^T\tilde{W}_V + \eta\|\tilde{W}_V^T W\|.\end{aligned} \tag{75}$$

Therefore since $1, \|\hat{\bar{x}}\|, \|\hat{W}_V\| \le 1 + \|\hat{\bar{x}}\| + \|\hat{W}_V\|$, using Cauchy-Schwarz and Young inequalities ($ab \le \frac{1}{2\theta}a^2 + \frac{\theta}{2}b^2$) with $\theta = 5$ yields

$$\begin{aligned}\dot{L}_b \le& -\alpha\frac{\tilde{W}_V^T\hat{\omega}\hat{\omega}^T\tilde{W}_V}{10(1+\|\hat{\bar{x}}\|+\|\hat{W}_V\|)^2} - \frac{1}{2}\eta\tilde{W}_V^T\tilde{W}_V + \frac{5}{2}\alpha(L_Q + W_{VM}L_\phi(W_{fM}\sigma_{fM} + \varepsilon_{fM}) + W_{VM}\phi_{\bar{x}M}L_h \\ &+ W_{VM}L_\phi\bar{g}_M\rho_M + L_I L_\rho\|R\|\|R^{-1}\|\bar{g}_M L_\phi W_{VM} + c_{H2})^2\|\tilde{\bar{x}}\|^2 + \frac{5}{2}\alpha(W_{VM}L_\phi\|A\|)^2\|\tilde{\bar{x}}\|^2 \\ &+ \frac{5}{2}\alpha(L_\rho\phi_{\bar{x}M}L_\phi W_{VM}\bar{g}_M^2\|R^{-1}\|)^2\|\tilde{\bar{x}}\|^2 + \frac{5}{2}\alpha W_{VM}^2\phi_{\bar{x}M}^2\sigma_{fM}^2\|\tilde{W}_f\|^2 + \frac{5}{2}\alpha(\phi_{\bar{x}M}^2\bar{g}_M^2 L_\rho\|R^{-1}\|)^2\|\tilde{W}_V\|^2 \\ &+ \frac{5}{2}\alpha(2\phi_{\bar{x}M}\bar{g}_M\rho_M + L_I L_\rho\|R\|\|R^{-1}\|\bar{g}_M\phi_{\bar{x}M})^2\|\tilde{W}_V\|^2 + \frac{5}{2}\alpha {c_{H1}}^2 + \frac{5}{2}\alpha(W_{VM}\phi_{\bar{x}M}\varepsilon_{fM} + c_{H3})^2 \\ &+ \frac{5}{2}\alpha(\phi_{\bar{x}M}\bar{g}_M^2 L_\rho\|R^{-1}\|\varepsilon_{VM})^2 + \frac{1}{2}\eta W_{VM}^2 \\ \le& -\alpha\frac{\tilde{W}_V^T\hat{\omega}\hat{\omega}^T\tilde{W}_V}{10(1+\|\hat{\bar{x}}\|+\|\hat{W}_V\|)^2} - \frac{1}{2}\eta\|\tilde{W}_V\|^2 + b_{\bar{x}}\|\tilde{\bar{x}}\|^2 + b_{Wf}\|\tilde{W}_f\|^2 + b_{WV}\|\tilde{W}_V\|^2 + \varepsilon_b,\end{aligned} \tag{76}$$

where

$$b_{\bar{x}} = \frac{5}{2}\alpha(L_Q + W_{VM}L_\phi(W_{fM}\sigma_{fM} + \varepsilon_{fM}) + L_I L_\rho \|R\|\|R^{-1}\|\bar{g}_M L_\phi W_{VM} + W_{VM}\phi_{\bar{x}M}L_h$$
$$+W_{VM}L_\phi \bar{g}_M\rho_M + c_{H2})^2 + \frac{5}{2}\alpha(W_{VM}L_\phi\|A\|)^2 + \frac{5}{2}\alpha(L_\rho\phi_{\bar{x}M}L_\phi W_{VM}\bar{g}_M^2\|R^{-1}\|)^2,$$
$$b_{Wf} = \frac{5}{2}\alpha W_{VM}^2\phi_{\bar{x}M}^2\sigma_{fM}^2,$$
$$b_{WV} = \frac{5}{2}\alpha(\phi_{\bar{x}M}^2\bar{g}_M^2 L_\rho\|R^{-1}\|)^2 + \frac{5}{2}\alpha(2\phi_{\bar{x}M}\bar{g}_M\rho_M + L_I L_\rho\|R\|\|R^{-1}\|\bar{g}_M\phi_{\bar{x}M})^2, \quad (77)$$

and $\varepsilon_b = \frac{11}{4}\alpha c_{H1}^2 + \frac{11}{4}\alpha(W_{VM}\phi_{\bar{x}M}\varepsilon_{fM} + c_{H3})^2 + \frac{11}{4}(\phi_{\bar{x}M}\bar{g}_M^2 L_\rho\|R^{-1}\|\varepsilon_{VM})^2 + \frac{1}{2}\eta W_{VM}^2$. If $\eta > \eta_0 = b_{WV}$, define $\gamma = \eta - \eta_0$. Moreover, consider the Lyapunov function $L_a = G(\bar{x})$. Taking its derivative $\dot{L}_a$ yields

$$\dot{L}_a = G_{\bar{x}}(\bar{f}(\bar{x}) + \bar{g}\hat{u}) = G_{\bar{x}}\left(\bar{f}(\bar{x}) + \bar{g}u^*\right) - G_{\bar{x}}\bar{g}\tilde{u} \le -\delta\|G_{\bar{x}}\|^2 + \|G_{\bar{x}}\bar{g}\tilde{u}\|. \quad (78)$$

Therefore substituting (67) into (78) and using Young inequality yield

$$\dot{L}_a \le -\delta\|G_{\bar{x}}\|^2 + \|G_{\bar{x}}\|\|\bar{g}\tilde{u}\| \le -\delta\|G_{\bar{x}}\|^2 + \frac{1}{2}(\tau_1 + \tau_2 + \tau_3)\|G_{\bar{x}}\|^2 + \frac{1}{2\tau_1}L_\rho^2\|R^{-1}\|^2\bar{g}_M^2\phi_{\bar{x}M}^2\|\tilde{W}_V\|^2$$
$$+\frac{1}{2\tau_2}L_\rho^2\|R^{-1}\|^2\bar{g}_M^2 L_\phi^2 W_{VM}^2\|\tilde{\bar{x}}\|^2 + \frac{1}{2\tau_3}L_\rho^2\|R^{-1}\|^2\bar{g}_M^2\varepsilon_{VM}^2. \quad (79)$$

Taking $\tau_1 + \tau_2 + \tau_3 < \delta$, one has

$$\dot{L}_a \le -\frac{1}{2}\delta\|G_{\bar{x}}\|^2 + a_{WV}\|\tilde{W}_V\|^2 + a_{\bar{x}}\|\tilde{\bar{x}}\|^2 + \varepsilon_a, \quad (80)$$

where

$$a_{WV} = \frac{1}{2\tau_1}L_\rho^2\|R^{-1}\|^2\bar{g}_M^2\phi_{\bar{x}M}^2,$$
$$a_{\bar{x}} = \frac{1}{2\tau_2}L_\rho^2\|R^{-1}\|^2\bar{g}_M^2 L_\phi^2 W_{VM}^2, \quad (81)$$

and $\varepsilon_a = \frac{1}{2\tau_3}L_\rho^2\|R^{-1}\|^2\bar{g}_M^2\varepsilon_{VM}^2$.

Closed loop stability proof: Consider the Lyapunov function: $L = \mu_a L_a + \mu_b L_b + \mu_c L_c$, where μ_a, μ_b and μ_c are positive constants. Taking the time derivative of L yields

$$\dot{L} = \mu_a\dot{L}_a + \mu_b\dot{L}_b + \mu_c\dot{L}_c \le -\frac{1}{2}\mu_a\delta\|G_{\bar{x}}\|^2 + \mu_a a_{WV}\|\tilde{W}_V\|^2 + \mu_a a_{\bar{x}}\|\tilde{\bar{x}}\|^2 + \mu_a\varepsilon_a$$
$$-\frac{1}{2}\mu_b\gamma\|\tilde{W}_V\|^2 + \mu_b b_{\bar{x}}\|\tilde{\bar{x}}\|^2 + \mu_b b_{Wf}\|\tilde{W}_f\|^2 + \mu_b\varepsilon_b + \mu_c\kappa\|\tilde{\bar{x}}\|^2 + \mu_c(-\frac{1}{2}\alpha_{If} + \frac{1}{2})\|\tilde{W}_f\|^2 + \mu_c\varepsilon_c. \quad (82)$$

Subsequently,

$$\begin{aligned}\dot{L} \le & -\frac{1}{2}\mu_a\delta\|G_{\bar{x}}\|^2 - \frac{1}{4}\mu_b\gamma\|\tilde{W}_V\|^2 \underbrace{-\frac{1}{4}\mu_b\gamma\|\tilde{W}_V\|^2 + \mu_a a_{WV}\|\tilde{W}_V\|^2}_{\text{this term should be<0}} \\ & \underbrace{+[\frac{\mu_c\kappa}{2} + \mu_a a_{\bar{x}} + \mu_b b_{\bar{x}}]\|\tilde{\bar{x}}\|^2}_{\text{this term should be<0}} + \frac{\mu_c}{2}(-\frac{1}{2}\alpha_f + \frac{1}{2})\|\tilde{W}_f\|^2 \\ & + \underbrace{\frac{\mu_c}{2}(-\frac{1}{2}\alpha_f + \frac{1}{2})\|\tilde{W}_f\|^2 + \mu_b b_{Wf}\|\tilde{W}_f\|^2}_{\text{this term should be<0}} + \varepsilon_L.\end{aligned} \tag{83}$$

where $\varepsilon_L = \mu_a\varepsilon_a + \mu_b\varepsilon_b + \mu_c\varepsilon_c$. Therefore if $\mu_b > \frac{4}{\gamma}\mu_a a_{WV}$ and $\mu_c > \max\{\frac{2}{|\kappa|}[\mu_a a_{\bar{x}} + \mu_b b_{\bar{x}}], \frac{4\mu_b b_{Wf}}{(\alpha_f - 1)}\}$,

the following inequality for $\dot{L}$ holds as

$$\dot{L} \le -\frac{1}{2}\mu_a\delta\|G_{\bar{x}}\|^2 - \frac{1}{4}\mu_b\gamma\|\tilde{W}_V\|^2 + \frac{1}{2}\mu_c\kappa\|\tilde{\bar{x}}\|^2 + \frac{\mu_c}{2}(-\frac{1}{2}\alpha_f + \frac{1}{2})\|\tilde{W}_f\|^2 + \varepsilon_L. \tag{84}$$

Note that according to Theorem 3, $\alpha_f > 1$ and $\kappa < 0$. Therefore $\dot{L}$ is less than zero provided that

$$\|G_{\bar{x}}\| > \sqrt{\frac{2\varepsilon_L}{\mu_a\delta}} \text{ or } \|\tilde{W}_V\| > \sqrt{\frac{4\varepsilon_L}{\mu_b\gamma}} \text{ or } \|\tilde{W}_f\| > \sqrt{\frac{4\varepsilon_L}{\mu_c(\alpha_f - 1)}} \text{ or } \|\tilde{\bar{x}}\| > \sqrt{\frac{2\varepsilon_L}{|\kappa|\mu_c}}.$$

Note also that the bounds for $\bar{x}, \tilde{W}_V, \tilde{W}_f$ and $\tilde{\bar{x}}$ can be tuned by changing design parameters δ, γ (by changing η), α_f and κ (by changing observer gain Γ), respectively. Accordingly, the closed loop system is UUB [64]. The simulation results demonstrate that the bounds are reasonably small through appropriate selection of these design parameters. Besides, using (67) yields

$$\begin{aligned}\|\tilde{u}\| = \|u^* - \hat{u}\| & \le \frac{1}{2}L_\rho\|R^{-1}\|\bar{g}_M(\phi_{\bar{x}M}\|\tilde{W}_V\| + L_\phi W_{VM}\|\tilde{x}\| + \varepsilon_{VM}) \\ & \le \frac{1}{2}L_\rho\|R^{-1}\|\bar{g}_M(\phi_{\bar{x}M}\sqrt{\frac{4\varepsilon_L}{\mu_b\eta}} + L_\phi W_{VM}\sqrt{\frac{2\varepsilon_L}{|\kappa|\mu_c}} + \varepsilon_{VM}).\end{aligned} \tag{85}$$

Consequently, the approximation error between the actual control and truly optimal policy is also ultimately bounded with the ultimate bound dependent on estimation errors $\tilde{x}, \tilde{W}_V$ and reconstruction error ε_{VM}. Furthermore, the ultimate bound can be tuned by changing the design parameters η, Γ and by increasing the number of neurons.

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II. BOUNDARY CONTROL OF LINEAR UNCERTAIN ONE DIMENSIONAL PARABOLIC PDE USING APPROXIMATE DYNAMIC PROGRAMMING

B. Talaei , S. Jagannathan and J. Singler

This paper develops a near optimal boundary control policy for distributed parameter systems (DPS) governed by uncertain linear one-dimensional parabolic partial differential equations (PDE) under Neumann or Dirichlet boundary control conditions by using approximate dynamic programming (ADP). A quadratic surface integral is proposed to express the optimal cost functional for the infinite dimensional state space as extension of its representation for lumped parameter systems (LPS). Accordingly, the Hamilton-Jacobi-Bellman (HJB) equation is formulated in the infinite dimensional domain without using any model reduction. Subsequently, a neural network (NN) identifier is developed to estimate the unknown spatially varying coefficient in PDE dynamics. Novel tuning law is proposed using basis functions as integral kernels to guarantee the boundedness of identifier approximation error in PDE domain. Since solving the HJB equation for the exact optimal value functional is burdensome, a radial basis network (RBN) is subsequently proposed to generate a computationally feasible approximate solution for the optimal surface kernel function online and in a forward-in-time manner. The tuning law for near optimal RBN weights is also newly created such that the HJB equation error is minimized while the dynamics are identified and closed-loop system remains stable in PDE domain. Consequently, the near optimal integral boundary control policy is derived. Ultimate boundedness (UB) of the closed-loop system is verified by using the Lyapunov theory. The performance of proposed controller is successfully confirmed by simulation on an unstable diffusion reaction process under various boundary conditions.

1. INTRODUCTION

Distributed parameter systems (DPS) are a major part of dynamical systems with wide range of applications [1]–[7]. As the name suggests, DPS arise in environments (e.g. heat and mass transfer [1], diffusion-reaction processes [2],[3], flexible structures [5], wave equation [6], swarm formation control [7], etc.) where system behavior changes continuously throughout the space . Similar to the case of lumped parameter systems (LPS), the attributes of a controller design for DPS should include: 1. a simple and reliable design; 2. feasible real-time implementation; 3. being robust to disturbances and modeling errors; and 4. closed-loop stability. However, the major challenge in the control of DPS when compared to a LPS is the infinite dimensional nature of state space modeled by partial differential equations (PDE). This characteristic makes control design difficult in contrast to a finite set of ordinary differential equations (ODE) in LPS.

The promising results gathered from the linear optimal control of LPS encouraged researchers to develop the operator theory [8] for optimal control of DPS. This work was extended further to boundary control [9] where the design is performed in the infinite dimensional setting. However, a closed-form solution requires solving the operator Riccati equations backward in time. This is significantly more time consuming in the infinite dimensional state space for DPS. Subsequently, various other approaches under the general category of *optimize-then-discretize* control [10],[11] were proposed. In particular, boundary actuation received special attention in the previous decade due to *backstepping* approach [12]–[14] wherein the controller is developed through conventional calculus instead of operator theory. The backstepping approach further enhances the stability study through conventional calculus in the original PDE domain.

The advances in approximate dynamic programming (ADP) and its successful forward-in-time implementation of optimal control policy for LPS [15], motivated researchers to investigate its application to DPS as well [16]-[21]. However, due to the difficulty of designing the controller in infinite dimensional domain and with PDE dynamics [21], the DPS system was usually discretized into an approximate finite dimensional state space [22] and subsequently the well-developed ADP algorithms [24] were utilized for this reduced order model. Design of a controller in a finite dimensional state space is broadly categorized as *discretize-then-optimize* [23] method. The benefit of this design approach is the possibility of using either the state of the art ADP or other feedback suboptimal control schemes. However, its limitation is the possible degraded performance due to the reduced order finite dimensional model in control design.

In contrast, in this paper, a new ADP based boundary control of DPS governed by uncertain linear parabolic partial differential equations is introduced. The boundary control problem is specifically examined and both Neumann and Dirichlet boundary actuation conditions are considered. Unlike aforementioned ADP based control methods for DPS, no model reduction is utilized prior to the control design. The controller's development is novel and based on a new definition for the value functional as a surface integral. Consequently, the Hamilton Jacobi Bellman (HJB) equation and the optimal boundary control policy are derived through calculus in the infinite dimensional domain. The value functional is further approximated iteration-free and forward-in-time in order to design a suboptimal adaptive controller whereas the PDE dynamics are uncertain. The proposed suboptimal boundary controller is the first to be designed for a DPS according to the forward-in-time ADP without any model reduction and with providing closed-loop stability proof.

Many recent ADP algorithms use policy or value iterations to find the approximate optimal control [24]. As their name suggests, these methods begin with an initial control policy and reach to the optimal one by iterating value or policy evaluation and improvement cycles at each time step. In order to reduce the computational burden, new ADP algorithms [25] have suggested adaptive laws to satisfy the HJB equation gradually along system trajectories. This advantage comes with the cost of less accuracy in finding the optimal control. However, for DPS control design, reducing the computations is a major priority since the number of system states is theoretically infinity. Therefore, the proposed boundary control policy in this paper is based on an iteration-free adaptive scheme with novel update laws that can achieve system stability and local optimality along trajectories whereas the computations are significantly reduced. Lyapunov analysis demonstrates that provided an initial admissible control policy [26], the error between the actual and truly optimal control will always remain bounded with tunable bounds whereas all approximation reconstruction errors are considered. Simulation results confirm that the presented approach has good convergence and control performance on parabolic PDE dynamics.

The development of the controller is as follows. First, it will be shown that a surface integral is a good representation for the quadratic value functional in PDE state space. Accordingly, the HJB equation is formulated in the infinite dimensional setting. Subsequently, the optimal boundary control policy is derived by using necessary conditions of optimality. Since the system dynamics are continuous, solving the HJB equation requires the PDE dynamics to be known [24]. Therefore, a neural network (NN) method is proposed based on a PDE state estimator and an unconventional weight update law to identify the unknown spatially varying coefficients in the PDE model. Subsequently, the surface kernel function is approximated as infinite dimensional array of parameters to construct the optimal value functional by using a radial basis network (RBN). Consequently, the approximated optimal control policy is derived by using the estimation of value functional and identified system dynamics. A diffusion-reaction process is considered to assess the performance of control approach through simulation.

Accordingly, in this paper two novel NN frameworks are proposed for identifying the unknown PDE dynamics and approximating the surface kernel function, respectively. The tuning law for identifier has a special design using basis functions as integral kernels such that the boundedness of identifier approximation error in original PDE domain is guaranteed. This is substantially different from ODE-based NN identifiers [19] where stability is solely assured for finite dimensional model. In addition, a novel weight update law is proposed for the RBN near optimal surface kernel that minimizes the HJB approximation error in a forward-in-time manner. This in turn will provide the system closed-loop stability in PDE domain which is different from finite dimensional ADP approaches where stability is only guaranteed for reduced order model.

Notations: Throughout the paper, $\|.\|$ stands for Euclidean norm for vectors or Frobenius norm for matrices. We recall the inequality $\|.\|_2 \le \|.\|_F$ where $\|.\|_2$ and $\|.\|_F$ represent the induced 2 norm and the Frobenius norm, respectively. The $\mathcal{L}_2$ norm is also defined as $\|.\|_{\mathcal{L}_2} = (\int_0^l \|.\|^2 \, dz)^{\frac{1}{2}}$.

The rest of the paper is organized as follows. In Section 2, the class of DPS under consideration is described and the state feedback optimal control approaches are explained separately for different boundary conditions. Section 4 demonstrates the simulation results and Section 5 provides the conclusions of the paper.

2. APPROXIMATE OPTIMAL CONTROL OF PARABOLIC PDE

In this paper, a DPS governed by an uncertain linear one-dimensional parabolic PDE

$$\frac{\partial x(z,t)}{\partial t} = \frac{\partial^2 x(z,t)}{\partial z^2} + \lambda(z)x(z,t), \tag{1}$$

is considered, where $x(.,t) \in \mathcal{H}_2[0,l]$ is the system state with $z \in [0,l]$ being the spatial variable, $l > 0$, t representing time and $\mathcal{H}_2$ being the Sobolev space of second order; $\frac{\partial x}{\partial t}$ is the time derivative of x, $\frac{\partial^2 x}{\partial z^2}$ is its second spatial derivative and $\lambda(.) \in C_1([0,l])$ is an unknown spatially varying coefficient with C_1 being the space of continuously differentiable functions. Considering the above PDE dynamics and assuming that measurement of system state is available throughout the spatial domain, in the following two subsections the ADP controller is designed for Neumann and Dirichlet boundary conditions, respectively.

2.1. NEUMANN BOUNDARY CONTROL

Consider the linear DPS (1) with Neumann boundary control at $z = l$ and general Robin boundary condition at $z = 0$ as

$$\frac{\partial x}{\partial z}|_{z=l} = gu, \qquad \frac{\partial x}{\partial z}|_{z=0} = -hx(0), \tag{2}$$

where u is the control input and $g, h \in \Re$ are known constants. The objective is to design a controller to minimize the following infinite horizon cost functional $\bar{V}$ given by

$$\bar{V}(x,u) = \int_{t_0}^{\infty} \left\{Q(x) + ru^2\right\} dt, \tag{3}$$

where r is a positive constant and $Q(x)$ is a positive definite function. If the system state were a finite dimensional $n \times 1$ vector x_f, it is well-known that $Q(x_f)$ could be defined in quadratic form as

$$Q_L(x_f) = x_f^T Q_l x_f = \sum_{i=1}^{n}\sum_{j=1}^{n} x_{f_i} Q_{l_{ij}} x_{f_j} \tag{4}$$

where Q_l is a positive definite $n \times n$ kernel matrix, x_{f_i} is the i th element of vector x_f and $Q_{l_{ij}}$ is the entry of matrix Q_l at the i th row and j th column. However, in the case of DPS, since there are infinitely many states $[x(z_0), x(z_1), ..., x(z_k), ...]^T \quad z_i \in [0, l]$ [29] that are continuous in the spatial domain, the finite dimensional summations in (4) should be substituted by integrals. Therefore, intuitively, taking $s, z \in [0, l]$ as continuous spatial variables for a surface kernel function $q(s, z)$ which resemble discrete variables i, j as rows and columns of matrix Q_l in ((4)), $Q(x)$ for DPS can be specified equivalent to ((4)) in the following surface integral form as

$$Q(x) = \int_0^l \int_0^l x(s) q(s, z) x(z) ds dz, \tag{5}$$

where $q(.,.) \in C_2([0, l], [0, l])$ and $x(z)$ is short form of $x(z, t)$. Note that $q(.,.)$ is a two dimensional continuous kernel function that has the same role of the kernel matrix Q_l in finite dimensional definition (4).

Remark 1: In order to further clarify the definition of $Q(x)$ in ((5)) as extension of finite dimensional definition (4), take $0 = z_0 \leq z_1 \leq ... \leq z_{n-1} \leq z_n = l$ and $0 = s_0 \leq s_1 \leq ... \leq s_{n-1} \leq s_n = l$ as two partitions for $[0, l]$. Assigning $x_f = [x(z_1), ..., x(z_i), ..., x(z_n)]^T$ as a $n \times 1$ finite dimensional subset of x and defining $\Delta z_i = z_i - z_{i-1}$ and $\Delta s_i = s_i - s_{i-1}$ for $1 \leq i \leq n$, one can express $Q(x)$ in (5) as

$$\begin{aligned} Q(x) &= \int_0^l \int_0^l x(s) q(s, z) x(z) ds dz \\ &\approx \sum_{i=1}^n \sum_{j=1}^n x_{f_i} q(s_i, z_j) x_{f_j} \Delta s_i \Delta z_j \end{aligned} \tag{6}$$

where the Riemann approximated definition [37] of integrals is used. By defining $Q_{l_{ij}} = q(s_i, z_j) \Delta s_i \Delta z_j$, equation (6) can be viewed analogous to definition of Q_L in (4). This implies that definition of $Q(x)$ in ((5)) reduces to (4) for conventional finite dimensional state spaces. In order to proceed, the following assumption is necessary:

Assumption 1: The function $q(.,.)$ is symmetric, i.e. $q(s, z) = q(z, s)$, and $Q(x)$ is *positive definite*, i.e. $Q(x) \geq q_{min} \| x \|_{\mathcal{L}_2}^2$ with $q_{min} \geq b_q$ where b_q is a positive constant that will be defined later in the paper.

Below, an HJB equation is derived for the Neumann boundary control problem with cost functional $\bar{V}$ as in (3). To our knowledge, thorough results for parabolic PDE boundary control problems with the infinite horizon cost functional (3) are not available; rigorous results do exist when the infinite horizon cost functional contains an exponential weight as in [30] and [31]. Instead of a fully thorough derivation of the HJB equation here, we proceed by extending a formal derivation inspired from finite dimensional dynamic programming.

The optimal cost functional is represented by $\bar{V}^*(x,t)$. Similar to ADP control design of linear LPS with quadratic cost function [27], if $p(s,z) \in C_1([0,l],[0,l])$ is a symmetric kernel function, $\bar{V}^*(x,t)$ can be represented by a surface integral as

$$\bar{V}^*(x,t) = \frac{1}{2}\int_0^l V^*(x(z),z)dz = \frac{1}{2}\int_0^l V^*(x(s),s)ds. \tag{7}$$

where $V^*(x(z),z) = \int_0^l x(s)p(s,z)x(z)ds$ or $V^*(x(s),s) = \int_0^l x(s)p(s,z)x(z)dz$. Note that the last equality in (7) is obtained by changing the order of integration.

Remark 2: Although the representation (7) was mainly inspired from finite dimensional ADP controller designs, it can also be derived more rigorously using operator theory. The optimal cost function $\bar{V}^* : \mathcal{L}^2(0,l) \to \Re$ is known to be expressed as [9]

$$\bar{V}^*(x) = \int_0^l x(z)[\Pi x](z)dz, \tag{8}$$

where $\Pi : \mathcal{L}^2 \to \mathcal{L}^2$ is a bounded linear operator that solves the operator algebraic Riccati equation. If Π is a Hilbert-Schmidt operator, there exists a kernel function $p(z,s)$ such that [38]

$$[\Pi x](z) = \int_0^l p(z,s)x(s)ds. \tag{9}$$

Substituting (9) into (8) yields to representation (7) for $\bar{V}^*(x)$. It has been shown that the operator Π for the type of control problem discussed here is indeed Hilbert-Schmidt [9]. This will further prove the existence of integral boundary control policies derived in the following.

By taking the current time interval $[t, t+\delta t)$, $\bar{V}$ in (3) can be represented in the recursive form as

$$\bar{V}(x,u,t) = \int_{t}^{t+\delta t} \left\{Q(x)+ru^2\right\} dt + \bar{V}(x,u,t+\delta t), \tag{10}$$

where $\bar{V}(x,u,t+\delta t)$ is the cost to go from time $t+\delta t$ to ∞. Hence, the optimal value functional can be represented as

$$\bar{V}^*(x,t) = \min_u \{ \int_{t}^{t+\delta t} \left\{Q(x)+ru^2\right\} dt + \bar{V}(x,t+\delta t)\}. \tag{11}$$

Now by invoking the principle of optimality, Equation (11) becomes

$$\bar{V}^*(x,t) = \min_u \{ \int_{t}^{t+\delta t} \left\{Q(x)+ru^2\right\} dt\} + \bar{V}^*(x,t+\delta t). \tag{12}$$

It is assumed that $\bar{V}^*(x,t)$ is Gáteaux analytic [32], i.e. its differential with respect to infinitesimal change of state x exists in the direction of system trajectory. If an integral functional $Y(x) = \int_0^l y(x)dz$ with $y(x)$ being a function of x, is Gáteaux analytic, according to calculus of variations [32], $Y(x+\delta x)$ with δx being an infinitesimal change in x can be represented by its first order approximation as

$$Y(x+\delta x) \approx Y(x) + \int_0^l y_x \delta x dz \tag{13}$$

where $y_x = \dfrac{\partial y}{\partial x}$. Moreover, according to (7), we get

$$\frac{1}{2}\int_0^l V^*_{x(s)} \delta x(s) ds = \frac{1}{2}\int_0^l V^*_{x(z)} \delta x(z) dz. \tag{14}$$

Therefore, revisiting (12), $\bar{V}^*(x,t+\delta t)$ can be expressed in its first order approximation form as

$$\begin{aligned} &\bar{V}^*(x,t+\delta t) \approx \bar{V}^*(x,t) + \frac{1}{2}\int_0^l V^*_{x(s)} \delta x(s) ds \\ &+ \frac{1}{2}\int_0^l V^*_{x(z)} \delta x(z) dz + \frac{\partial \bar{V}^*}{\partial t}\delta t = \bar{V}^*(x,t) \\ &+ \int_0^l V^*_{x(z)} \delta x(z) dz + \frac{\partial \bar{V}^*}{\partial t}\delta t, \end{aligned} \tag{15}$$

where δx is an infinitesimal variation in x as a consequence of δt change in time, $V^*_{x(s)} = \dfrac{\partial V^*}{\partial x(s)}$ and $V^*_{x(z)} = \dfrac{\partial V^*}{\partial x(z)}$ are partial derivatives of V^* with respect to $x(s)$ and $x(z)$, respectively and $\frac{\partial \bar{V}^*}{\partial t}$ is partial derivative of $\bar{V}^*$ with respect to t. Substituting approximation (15) into equation (12) and canceling $\bar{V}^*(x,t)$ on both sides yields

$$0 = \min_u \{Q(x) + ru^2 + \int_0^l V^*_{x(z)} \delta x(z) dz + \frac{\partial \bar{V}^*}{\partial t} \delta t\}. \tag{16}$$

Dividing through out by δt, letting $\delta t \to 0$ and substituting dynamics (1) gives

$$\begin{aligned} 0 &= \min_u \{Q(x) + ru^2 + \int_0^l V^*_{x(z)} \frac{\partial x}{\partial t} dz\} = \min_u \{Q(x) \\ &+ ru^2 + \int_0^l V^*_{x(z)} [\frac{\partial^2 x}{\partial z^2} + \lambda(z) x(z,t)] dz + \frac{\partial \bar{V}^*}{\partial t}\}. \end{aligned} \tag{17}$$

Since due to the infinite time horizon, the cost functional $\bar{V}^*$ as defined in (7) is only state dependent and not explicitly dependent on time, $\frac{\partial \bar{V}^*}{\partial t} = 0$ [27]. Therefore, the Hamilton Jacobi Bellman (HJB) equation can be represented by

$$\begin{aligned} H^* = 0 &= \min_u \{Q(x) + ru^2 + \int_0^l V^*_{x(z)} [\frac{\partial^2 x}{\partial z^2} \\ &+ \lambda(z) x(z,t)] dz\}. \end{aligned} \tag{18}$$

In [28] a similar result but with using a different approach is derived for parabolic semi-linear PDE. Subsequently, the Hamiltonian is defined as

$$H = Q(x) + ru^2 + \int_0^l V^*_{x(z)} [\frac{\partial^2 x}{\partial z^2} + \lambda(z) x(z,t)] dz. \tag{19}$$

Using integration by parts, one has

$$\begin{aligned} H &= Q(x) + ru^2 + \int_0^l V^*_{x(z)} [\lambda(z) x(z,t)] dz \\ &+ \int_0^l V^*_{x(z)} \frac{\partial^2 x}{\partial z^2} dz = Q(x) + ru^2 + \int_0^l V^*_x [\lambda(z) x(z,t)] dz \\ &- \int_0^l \frac{\partial V^*_x}{\partial z} \frac{\partial x}{\partial z} dz + g V^*_x(x,l) u + h V^*_x(x,0) x(0,t), \end{aligned} \tag{20}$$

where $\frac{\partial V^*_x}{\partial z}$ is derivative of $V^*_{x(z)}$ with respect to z defined by

$$\frac{\partial V^*_x}{\partial z} = \lim_{\Delta z \to 0} \frac{V^*_{x(z)}(x, z + \Delta z) - V^*_{x(z)}(x,z)}{\Delta z}. \tag{21}$$

Based on necessary conditions of optimality, in order for the control input to be minimizing for Hamiltonian (20), the Fréchet derivative of this equation with respect to u should be zero [32]. Therefore, the control policy can be obtained as

$$\frac{dH}{du} = 0 \Rightarrow u^* = -\frac{1}{2r} g V^*_{x(z)}(x,l). \tag{22}$$

Substituting the optimal control (22) in equation (20), one has the HJB equation in the form

$$\begin{aligned} H^* &= 0 = Q(x) + \int_0^l V_x^*[\lambda(z)x(z)]dz \\ &-\int_0^l V_{xz}^* x_z dz - \frac{1}{4r} g^2 V_x^{*2}(l) + hV_x^*(0)x(0), \end{aligned} \tag{23}$$

where $x(z,t)$ and $V^*_{x(z)}(x,z)$ are represented by $x(z)$ and $V_x^*(z)$ for brevity. This convention will be used throughout the paper. Since the reaction coefficient $\lambda(z)$ in system dynamics (1) may be uncertain, the controller should be capable of identifying it online. Therefore, in the next section an identifier will be developed for this unknown spatially varying coefficient.

2.1.1. Identifier Design. Since $\lambda(.) \in C_1(0,l)$, by choosing a set of smooth bounded basis functions $\sigma_{\lambda i}(z) \in C_1(0,l)$, $1 \le i \le m$ the function $\lambda(z)$ can be represented as [39]

$$\lambda(z) = W_\lambda^T \sigma_\lambda(z) + \varepsilon_\lambda(z) \tag{24}$$

where $W_\lambda \in \Re^m$ represent NN identifier target weight vector with bound $\|W_\lambda\| \leqslant W_{\lambda M}$ and $\varepsilon_\lambda(z)$ is the approximation error which is also assumed to be bounded such that $\|\varepsilon_\lambda(z)\| \leqslant \varepsilon_{\lambda M}$. The estimation error bound $\varepsilon_{\lambda M}$ can be made arbitrarily small by increasing the number of basis functions [40].

As might be expected, the identifier dynamics can be provided by

$$\hat{\lambda}(z) = \hat{W}_\lambda^T \sigma_\lambda(z). \tag{25}$$

In order to find the tuning law for $\hat{W}_\lambda$, a state estimator with following PDE dynamics is initially considered as

$$\frac{\partial \hat{x}}{\partial t} = \frac{\partial^2 \hat{x}}{\partial z^2} + \hat{\lambda}(z)x + \psi \tilde{x},$$
$$\frac{\partial \hat{x}}{\partial z}(l) = gu, \frac{\partial \hat{x}}{\partial z}(0) = -hx(0), \tag{26}$$

where $\tilde{x} = x - \hat{x}$ is the state estimation error and ψ is a positive constant. Accordingly, the NN tuning law for $\hat{W}_\lambda$ is provided by

$$\dot{\hat{W}}_\lambda = -\alpha_\lambda \hat{W}_\lambda + \int_0^l \sigma_\lambda(z) x\tilde{x} dz, \tag{27}$$

where $\alpha_\lambda > 0$ is the design parameter.

Remark 3: The second term in update law (27) integrates the PDE state estimation error $\tilde{x}$ with different basis functions $\sigma_{\lambda i}$ as integral kernels. Therefore, if estimation $\hat{\lambda}$ and consequently $\hat{x}$ in (26) is unacceptable, the second term in (27) will adjust the weights associated with $\sigma_{\lambda i}$ s so that estimation error $\tilde{x}$ is minimized. By using the proposed update law, it will be shown in Section 3 that in presence of bounded input, the NN identifier weights and PDE state estimate will remain UB.

Accordingly, the dynamics of $\tilde{x}$ will be governed by

$$\frac{\partial \tilde{x}}{\partial t} = \frac{\partial^2 \tilde{x}}{\partial z^2} + \tilde{W}_\lambda^T \sigma_f(z)x + \varepsilon_\lambda(z) - \psi \tilde{x},$$
$$\left.\frac{\partial \tilde{x}}{\partial z}\right|_{z=l} = 0, \left.\frac{\partial \tilde{x}}{\partial z}\right|_{z=0} = 0, \tag{28}$$

where the weight estimation error $\tilde{W}_\lambda = W_\lambda - \hat{W}_\lambda$. Noting that $\dot{\tilde{W}}_\lambda = -\dot{\hat{W}}_\lambda$, the dynamics of NN identifier weight estimation error can be represented as

$$\dot{\tilde{W}}_\lambda = \alpha_\lambda \hat{W}_\lambda - \int_0^l \sigma_\lambda(x,z) x\tilde{x} dz. \tag{29}$$

Since an estimate of system dynamics is now available, in the next section the ADP approximate optimal control will be addressed.

2.1.2. Approximate Optimal Control Design. Since solving partial integro-differential equation (PIDE) ((23)) for the exact $V^*(z)$ is too difficult and time consuming [33], the objective is to find a suitable structure for estimation of $V^*(z)$ in (23). Unlike ADP control designs in finite dimensions, the continuous function $p(s,z)$ in (7) can be interpreted as an *infinite dimensional* array of unknown parameters to be approximated. It is well-known that radial basis networks (RBN) can estimate an unknown continuous multi-variable function [34]. $p(s,z)$ can be represented in RBN approximated form as

$$p(s,z)=W_V^T\phi(s,z)+\varepsilon_p(s,z), \tag{30}$$

where $\phi(s,z):[0,l]\times[0,l]\to[C_2]^{n_p}$ is a vector of n_p radial basis functions $\varphi_j((s,z)-(s_{0j},z_{0j})),1\le j\le n_p$, $W_V\in\Re^{n_p}$ and $\varepsilon_p(s,z)$ is the estimation error. It is assumed that the norm of estimation error and its first and second spatial derivatives ε_{pz}, ε_{pzz} are bounded with the bounds ε_{pM}, ε_{pzM} and ε_{pzzM}, respectively. Note that the estimation error ε_p can be reduced by increasing the number of neurons. Hence, $V^*(x,z)$ can be represented in approximated form as

$$\begin{aligned}V^*(x,z)&=\int_0^l x(s)W_V^T\phi(s,z)x(z)ds+\varepsilon_V(x,z)\\&=W_V^T\Phi(x,z)+\varepsilon_V(x,z),\end{aligned} \tag{31}$$

where $\Phi(x,z)$ is a $n_p\times 1$ vector defined as

$$\Phi(x,z)=\int_0^l x(s)\phi(s,z)x(z)ds, \tag{32}$$

and $\varepsilon_V(x,z)=\int_0^l x(s)\varepsilon_p(s,z)x(z)ds$. Subsequently, the optimal value functional can be expressed as

$$U^*(x)=W_V^T\int_0^l\Phi(x,z)dz+\varepsilon_U, \tag{33}$$

with $\varepsilon_U=\int_0^l\varepsilon_V dz$. Consequently, the optimal control policy can be represented as

$$u^*=\frac{-1}{2r}g\Phi_x(l)^T W_V+\varepsilon_u, \tag{34}$$

with $\varepsilon_u=\frac{-1}{2r}g\varepsilon_{V_x}(l)$. Finally, the HJB equation ((23)) can be represented in the new form as

$$H^* = Q(x) + \int_0^l W_V^T \Phi_x(z)[\lambda(z)x(z)]dz$$
$$-\int_0^l W_V^T \frac{\partial \Phi_x}{\partial z}\frac{\partial x}{\partial z}dz - \frac{1}{4r}W_V^T \Phi_x(l) g^2 \Phi_x(l)^T W_V \tag{35}$$
$$+h\Phi_x(0)^T W_V x(0) + \varepsilon_H,$$

where ε_H is derived as

$$\varepsilon_H = \int_0^l \varepsilon_{V_x}(z)[\lambda(z)x(z)]dz - \int_0^l \frac{\partial \varepsilon_{V_x}}{\partial z}\frac{\partial x}{\partial z}dz$$
$$-\frac{1}{4r}g^2\varepsilon_{V_x}^2(l) - \frac{1}{2r}W_V^T \Phi_x(l) g^2 \varepsilon_{V_x}(l) + h\varepsilon_{V_x}(0)x(0). \tag{36}$$

If the value functional is approximated as

$$\hat{U}^*(x) = \int_0^l \hat{V}^*(z)dz = \hat{W}_V^T \int_0^l \Phi(x,z)dz, \tag{37}$$

then the approximated HJB can be represented by

$$\hat{H} = Q(x) + \int_0^l \hat{W}_V^T \Phi_x(z)[\hat{\lambda}(z)x(z)]dz$$
$$-\int_0^l \hat{W}_V^T \frac{\partial \Phi_x}{\partial z}\frac{\partial x}{\partial z}dz - \frac{1}{4r}\hat{W}_V^T \Phi_x(l) g^2 \Phi_x^T(l)\hat{W}_V \tag{38}$$
$$+h\Phi_x^T(0)\hat{W}_V x(0),$$

and the control policy would be

$$\hat{u} = \frac{-1}{2r} g \Phi_x(l)^T \hat{W}_V. \tag{39}$$

In order to update the RBN weights online, the following tuning law is proposed

$$\dot{\hat{W}}_V = -\alpha_1 \frac{\omega_N}{\zeta_N^2}\hat{H} - \alpha_2 \hat{W}_V - \alpha_3 \hat{W}_V \|\hat{W}_V\|^2 - \alpha_4 \|x\|_{\mathcal{L}_2}^2 \hat{W}_V, \tag{40}$$

where

$$\omega_N = \int_0^l \Phi_x(z)[\hat{\lambda}(z)x(z)]dz - \int_0^l \frac{\partial \Phi_x}{\partial z}\frac{\partial x}{\partial z}dz$$
$$+h\Phi_x(0)x(0) - \frac{1}{2r}\Phi_x(l) g^2 \Phi_x^T(l)\hat{W}_V, \tag{41}$$

$\zeta_N = c_1 \|x\|_{\mathcal{L}_2}^2 + c_2 x(0)^2 + c_3 x(l)^2 + c_4$ and $c_1,...,c_4$ and $\alpha_1,...,\alpha_4$ are appropriate positive design parameters.

Remark 4: The first term in update law (40), minimizes the approximated Hamiltonian whereas the other terms are necessary to insure the closed loop stability as will be explained in the proof. Under an initial admissible control policy, it will be shown in Section 3 that the update law (40) and control policy (22) along with developed identifier in Section 2.1.1, cause the system state vector x and the weights estimation errors $\tilde{W}_V$, $\tilde{W}_\lambda$ to be ultimately bounded (UB).

Next, the near optimal control design for Dirichlet boundary control condition will be illustrated.

2.2. DIRICHLET BOUNDARY CONTROL

Consider the linear DPS (1) but under Dirichlet boundary control at $z=l$ and the same general Robin boundary condition at $z=0$ as

$$x(l) = gu, \quad \frac{\partial x}{\partial z}|_{z=0} = -hx(0). \tag{42}$$

Using integration by parts twice in this case, the Hamiltonian can be represented as

$$\begin{aligned} H &= Q(x) + ru^2 + \int_0^l V_x^*[\lambda(z)x]dz \\ &+V_x^*(l)\frac{\partial x}{\partial z}|_{z=l} - V_x^*(0)\frac{\partial x}{\partial z}|_{z=0} - \{\frac{\partial V_x^*}{\partial z}|_{z=l}\; gu \\ &-\frac{\partial V_x^*}{\partial z}|_{z=0}\; x(0) - \int_0^l \frac{\partial^2 V_x^*}{\partial z^2} xdz\}, \end{aligned} \tag{43}$$

where $\frac{\partial V_x^*}{\partial z}$ and $\frac{\partial^2 V_x^*}{\partial z^2}$ are the first and second spatial derivative of V_x^* with respect to z. Requiring the Fréchet derivative of this equation with respect to u equaling zero yields

$$\frac{\partial H}{\partial u} = -g\frac{\partial V_x^*}{\partial z}|_{z=l} + 2ru = 0 \Rightarrow u^* = \frac{1}{2r} g \frac{\partial V_x^*}{\partial z}|_{z=l} . \tag{44}$$

Substituting the optimal control in equation (43), the HJB equation for DPS ((1)) under Dirichlet boundary condition (42) can be represented in the form,

$$\begin{aligned} H^* &= 0 = Q(x) + \int_0^l V_x^*[\lambda(z)x(z)]dz + V_x^*(l)\frac{\partial x}{\partial z}|_{z=l} \\ &-V_x^*(0)\frac{\partial x}{\partial z}|_{z=0} + \int_0^l \frac{\partial^2 V_x^*}{\partial z^2} xdz + \frac{\partial V_x^*}{\partial z}|_{z=0}\; x(0) \\ &-\frac{1}{4r} g^2 \frac{\partial V_x^*}{\partial z}^2 |_{z=l} . \end{aligned} \tag{45}$$

Similarly, if V^* is represented in approximated form of (31), then the optimal control would be

$$u^* = \frac{1}{2r} g \frac{\partial \Phi_x^T}{\partial z}|_{z=l} W_V + \varepsilon_u, \tag{46}$$

with $\varepsilon_u = \frac{1}{2r} g \varepsilon_{V_{xz}} |_{z=l}$. Hence, the HJB equation becomes

$$\begin{aligned} H^* &= Q(x) + \int_0^l W_V^T \Phi_x [\lambda(z) x(z)] dz + W_V^T \Phi_x(l) \frac{\partial x}{\partial z}|_{z=l} \\ &- W_V^T \Phi_x(0) \frac{\partial x}{\partial z}|_{z=0} + \int_0^l W_V^T \frac{\partial^2 \Phi_x}{\partial z^2} x dz + \frac{\partial \Phi_x^T}{\partial z}|_{z=0} \\ & W_V x(0) - \frac{1}{4r} W_V^T \frac{\partial \Phi_x}{\partial z}|_{z=l} g^2 \frac{\partial \Phi_x^T}{\partial z}|_{z=l} W_V + \varepsilon_H, \end{aligned} \tag{47}$$

where $\Phi_x(z)$ denotes $\Phi_x(x,z)$ for brevity and ε_H is derived as

$$\begin{aligned} \varepsilon_H &= \int_0^l \varepsilon_{V_x}(z)[\lambda(z) x(z)] dz + \varepsilon_{V_x}(l) \frac{\partial x}{\partial z}|_{z=l} \\ &- \varepsilon_{V_x}(0) \frac{\partial x}{\partial z}|_{z=0} + \int_0^l \frac{\partial^2 \varepsilon_{V_x}}{\partial z^2} x dz + \frac{\partial \varepsilon_{V_x}}{\partial z}|_{z=0} x(0) \\ &- \frac{1}{4r} g^2 (\frac{\partial \varepsilon_{V_x}}{\partial z}|_{z=l})^2 - \frac{1}{2r} W_V^T \frac{\partial \Phi_x}{\partial z}|_{z=l} g^2 \frac{\partial \varepsilon_{V_x}}{\partial z}|_{z=l}. \end{aligned} \tag{48}$$

If the value functional is estimated as

$$\hat{\bar{V}}^*(x) = \int_0^l \hat{V}^*(x,z) dz = \hat{W}_V^T \int_0^l \Phi(x,z) dz, \tag{49}$$

the approximated HJB would be

$$\begin{aligned} \hat{H} &= Q(x) + \int_0^l \hat{W}_V^T \Phi_x [\hat{\lambda}(z) x(z)] dz + \hat{W}_V^T \Phi_x(l) \frac{\partial x}{\partial z}|_{z=l} \\ &- \hat{W}_V^T \Phi_x(0) \frac{\partial x}{\partial z}|_{z=0} + \int_0^l \hat{W}_V^T \frac{\partial^2 \Phi_x}{\partial z^2} x dz \\ &+ \frac{\partial \Phi_x^T}{\partial z}|_{z=0} \hat{W} x(0) - \frac{1}{4r} \hat{W}_V^T \frac{\partial \Phi_x}{\partial z}|_{z=l} g^2 \frac{\partial \Phi_x^T}{\partial z}|_{z=l} \hat{W}_V, \end{aligned} \tag{50}$$

where the same identifier as in Section 0 is used to find $\hat{\lambda}(z)$. Therefore, the control input can be represented by

$$\hat{u} = \frac{1}{2r} g \frac{\partial \Phi_x^T}{\partial z}|_{z=l} \hat{W}_V. \tag{51}$$

The value functional weight tuning law for Dirichlet boundary control is defined as

$$\dot{\hat{W}}_V = -\alpha_1 \frac{\omega_D}{\zeta_D^2} \hat{H} - \alpha_2 \hat{W}_V - \alpha_3 \hat{W}_V \| \hat{W}_V \|^2 - \alpha_4 \| x \|^2 \hat{W}_V , \tag{52}$$

where

$$\begin{aligned} \omega_D &= \int_0^l \Phi_x [\hat{\lambda}(z) x(z)] dz + \int_0^l \frac{\partial^2 \Phi_x}{\partial z^2} x dz \\ &+ \Phi_x(l) \frac{\partial x}{\partial z}|_{z=l} - \Phi_x(0) \frac{\partial x}{\partial z}|_{z=0} + \frac{\partial \Phi_x}{\partial z}|_{z=0} x(0) \\ &- \frac{1}{2r} \frac{\partial \Phi_x}{\partial z}|_{z=l} \, g^2 \frac{\partial \Phi_x}{\partial z}^T |_{z=l} \hat{W}_V , \end{aligned} \tag{53}$$

and $\zeta_D = c_1 \| x \|_{\mathcal{L}_2}^2 + c_2 x(0)^2 + c_3 (\frac{\partial x}{\partial z}|_{z=l})^2 + c_4$ with $c_1, ..., c_4$ and $\alpha_1, ..., \alpha_4$ being appropriate positive design parameters. The flowchart of proposed control scheme for different boundary conditions is shown in Figure 2.1.

In the next section, the authors will illustrate the ultimate boundedness of the closed-loop system with the developed boundary control policies for Neumann and Dirichlet boundary conditions, respectively.

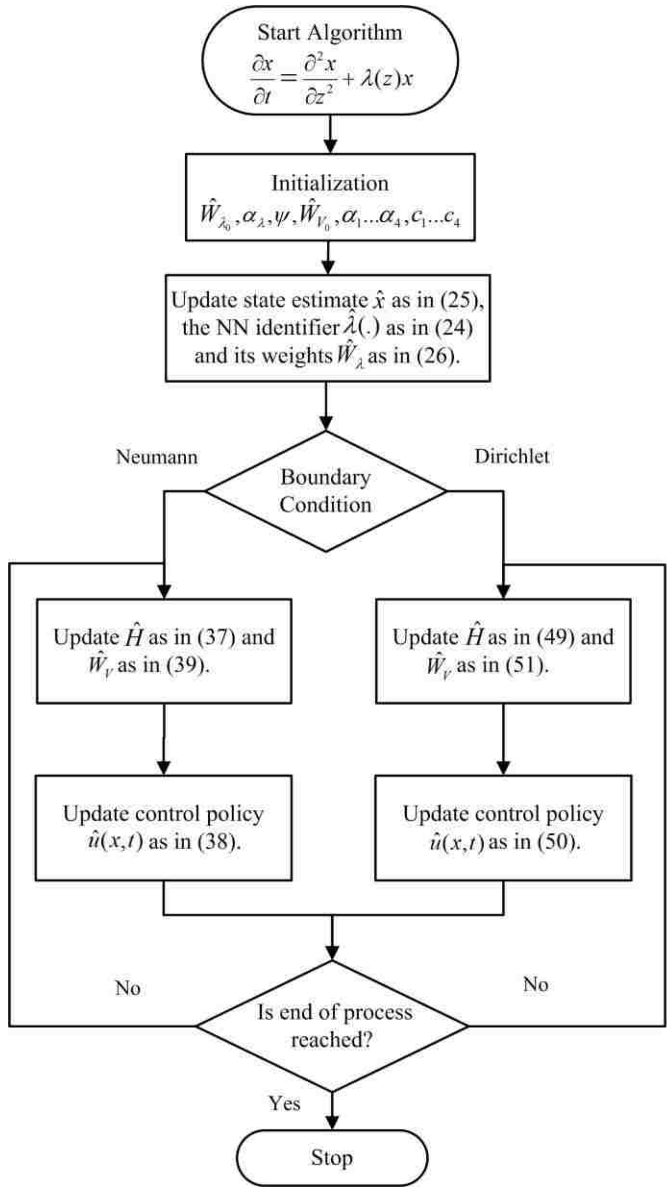

Figure 2.1. Flowchart of proposed controllers.

3. STABILITY ANALYSIS

System closed-loop stability will be examined by using Lyapunov criterion. First, it will be shown in Theorem 1 that in presence of bounded input, the identifier dynamics and estimation error will be bounded. In Lemma 1, a bound will be found for Hamiltonian reconstruction error ε_H when Neumann boundary control policy is implemented. This bound will be used later in closed-loop stability proof. Consequently, Theorem 2 will address ultimately boundedness (UB) of all closed-loop states for Neumann boundary control condition. Finally, Lemma 2 and Theorem 3 will state similar results when Dirichlet boundary control policy is pursued.

Theorem 1 (Boundedness of NN identifier): Let the initial NN identifier weight estimation error $\tilde{W}_\lambda$ and state estimation error $\tilde{x}$ be residing in compact sets $\mho_1$ and $\mho_2$, and the proposed NN identifier, the state estimator and NN weight tuning law be provided by (25), (26) and (27), respectively. In the presence of bounded inputs, there exists a positive constant $\psi > \frac{1}{2}$ and tuning parameter $\alpha_\lambda > 0$ for the identifier weight update law such that the state estimation error $\tilde{x}$ and weight estimation error are all UB.

Lemma 1: For Neumann control policy, the following inequality

$$|\varepsilon_H| \leqslant c_{H1} \|x\|_{\mathcal{L}_2}^2 + c_{H2} x(0)^2 + c_{H3} x(l)^2, \tag{54}$$

holds where c_{H1}, ..., c_{H3} are positive constants. Moreover $|\varepsilon_H| \leq \xi_{NM} \zeta_N$ with ζ_N defined as in update law (40) and ξ_{NM} being a positive constant.

Proof: Refer to Appendix.

Theorem 2: Consider the DPS system with PDE dynamics (1) under boundary conditions (2). Let NN identifier, the state estimator and NN weight tuning law be provided by (25), (26) and (27), respectively. Then, under Assumption 1, an initial admissible boundary control and the control policy (39) in order to reduce the infinite-horizon cost functional (3), and under update law (40) for approximated value functional weights with $0 < \alpha_1 < 1$, $\alpha_2 \geq \alpha_3$ and $\alpha_3 \geq \theta_{\alpha_3}^N$ where $\theta_{\alpha_2}^N$ is a positive constant, the DPS system state x and weight approximation error $\tilde{W}_V$ will remain UB. Moreover, the actual control input will be bounded close to its optimal value.

Proof: Refer to Appendix.

Lemma 2: For Dirichlet boundary control policy, the following inequality

$$|\varepsilon_H| \leqslant c_{H1} \|x\|_{\mathcal{L}_2}^2 + c_{H2} x(0)^2 + c_{H3} (\frac{\partial x}{\partial z}|_{z=l})^2, \tag{55}$$

holds where c_{H1}, ..., c_{H3} are positive constants. Moreover $|\varepsilon_H| \leq \xi_{DM} \zeta_D$ with ζ_D defined as in update law (52) and ξ_{DM} being a positive constant.

Proof: Refer to Appendix.

Theorem 3: Consider the DPS system with PDE dynamics (1) and boundary conditions (42). Let NN identifier, the state estimator and NN weight tuning law be provided by (25), (26) and (27), respectively. Under assumption 1, an initial admissible boundary control input and the control policy (51) in order to reduce the infinite-horizon cost functional (3), the DPS system state x and estimation error $\tilde{W}_V$ will remain UB under the update law (52) where $0 < \alpha_1 < 1$, $\alpha_2 > \alpha_3$ and $\alpha_3 > \theta_{\alpha_3}^D$ with $\theta_{\alpha_3}^D$ being a positive constant. Moreover, the actual control policy will be bounded close to its optimal value.

Proof: Refer to Appendix.

Remark 5: The presented control design is also applicable to more general linear PDE dynamics such as [12]

$$\begin{aligned}\frac{\partial x}{\partial t}(z,t) &= \frac{\partial^2 x}{\partial z^2}(z,t) + b(z)\frac{\partial x(z,t)}{\partial z} + \lambda(z)x(z,t) \\ &+ c(z)x(0,t) + \int_0^{\rho} f(z,w)x(w,t)dw,\end{aligned} \tag{56}$$

where $b(.), c(.) \in C_1([0,l])$, $f(.,.) \in C_1([0,l],[0,l])$, $0 \leq \rho \leq l$ and $w \in [0,\rho]$ is the integral variable. The basic PDE dynamics ((1)) are primarily chosen in this paper to simplify the illustrations. Moreover, it has been shown [12] that most widely applied DPS modeled by linear parabolic PDE are either in the form of or can be transformed into (1).

In the following section, numerical implementation of the presented controllers will be illustrated through simulation on a reaction-diffusion process.

4. SIMULATIONS

In this section, in order to verify the performance of proposed controllers, simulation examples are provided for both Neumann and Dirchlet boundary actuation conditions. A typical reaction-diffusion system [12] with following linear parabolic PDE dynamics was considered:

$$\frac{\partial x(z,t)}{\partial t} = \frac{\partial^2 x(z,t)}{\partial z^2} + \frac{\beta}{1+4(z-0.5)^2} x(z,t) \tag{57}$$

where $\beta > 0$, $z \in [0,1]$ and control input was only present at $z = 1$. The MATLAB *pdepe* function which uses the method of lines [41] for numerical solution of one dimensional PDE was used for simulation of dynamics in a real-time control setting with $dz = 0.05$.

4.1. NEUMANN BOUNDARY CONTROL

In the first simulation the performance of state feedback Neumann controller was evaluated. The chosen boundary conditions are expressed as

$$x_z(0) = 0, \;\; x_z(1) = gu \;\; x(z,0) = x_0(z), \tag{58}$$

and sampling time for updating the control input was chosen as $t_s = 10 msec$. In this simulation the chosen process parameters are $\beta = 8$ and $g = 1$. By setting $u = 0$, it was verified that the system open loop response is highly unstable and blows up very fast. An initial admissible control policy was found to be $u_0 = -21$ by using pole placement for approximate dynamics obtained from finite difference method. The chosen basis functions $\sigma_{\lambda i}$, $1 \le i \le 5$ to identify $\lambda(z)$ are expressed as

$$\sigma_{\lambda i} = \frac{1}{1+12(z-z_i)^2} \tag{59}$$

with $z_i \in \{0.1, 0.3, 0.5, 0.7, 0.9\}$. The chosen $q(s,z)$ in cost functional ((5)) can be expressed as

$$q(s,z) = L_j(distance(s_0 + z_0 - l = 0, (s,z))) \tag{60}$$

with $L_j(.)$ being the Landau kernel [36], which is a continuous approximation for Dirac delta function, with $j = 500$, and the function $distance(.,.)$ calculates the distance between the diameter $s_0 + z_0 - l = 0$ and point $(s,z) \in [0,l] \times [0,l]$. The motivation behind choosing this $q(s,z)$ is that it resembles the finite dimensional identity matrix in infinite dimensional cost functional ((5)). Thirty six radial basis functions were chosen as ϕ_i s, $1 \le i \le 36$, to approximate $p(s,z)$ with the structure expressed as

$$\phi_i(s,z) = \frac{k}{1+\nu \| (s,z)-(s_i,z_i) \|^2} \tag{61}$$

where $\nu = 40, k = 0.4$ and s_i s and z_i s were chosen from the set $S = \{0.1, 0.25, 0.4, 0.55, 0.7, 1\}$. However, since $p(s,z)$ is symmetric, only 21 weights were needed to be updated and other weights could be found based on symmetry. The identifier and controller parameters were chosen to be $\psi = 10, \alpha_\lambda = 1$, $\alpha_1 = 0.1, \alpha_2 = 0.3, \alpha_3 = 0.25, \alpha_4 = 0.01, c_1 = 2$, $c_2 = 0.02$, $c_2 = c_3 = 0.01$ and $W_0 = 0.1 \times Ones_{36\times1}$. Note that the necessary design conditions, i.e. $\alpha_2 > \alpha_3$ and $\alpha_3 > \theta^N_{\alpha_3} = \frac{\| \phi_l \|^4_{\mathcal{L}_2} g^4}{8c_1^2 r^2} \simeq 0.2$ are satisfied for these parameters. As Figure 4.1 confirms, the controller was able to stabilize the system without any overshoots. Figure 4.2 shows that the control input was smooth. Moreover, HJB error converged faster than control input and increasing α_1 would accelerate its convergence rate. This shows that update law ((40)) was effective in finding a near optimal controller. Finally, Figure 4.3 shows the estimated $\hat{p}(s,z)$ at the end of simulation.

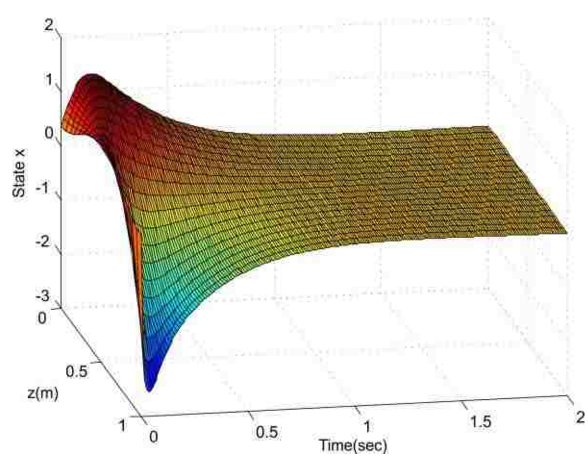

Figure 4.1. Closed loop state evolution under Neumann boundary condition.

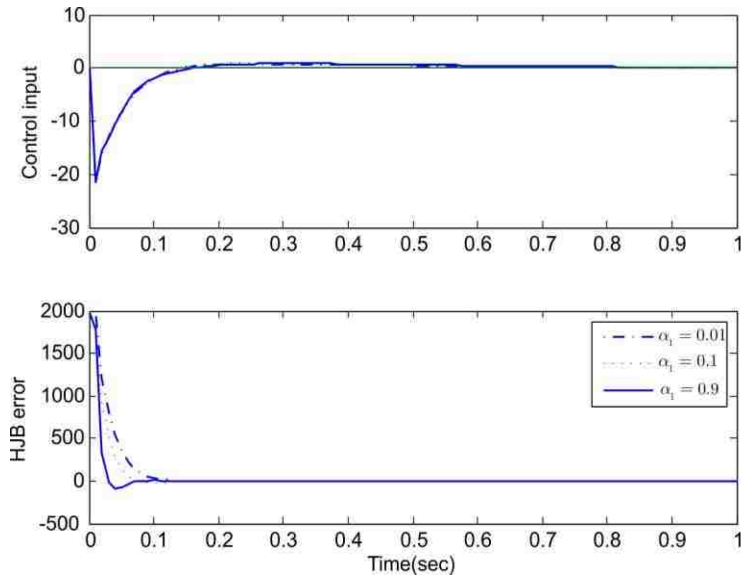

Figure 4.2. Control input and HJB error for Neumann boundary controller.

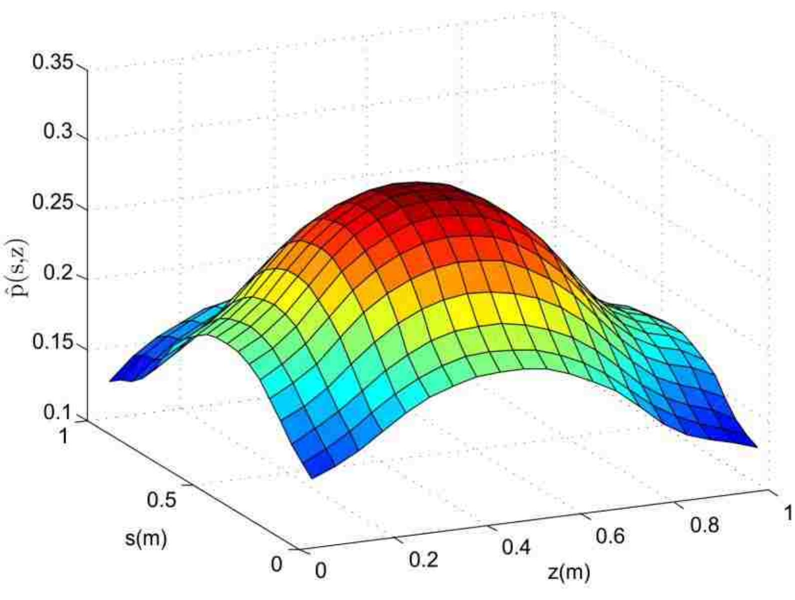

Figure 4.3. Estimated $\hat{p}(s,z)$ for Neumann boundary controller.

4.2. DIRICHLET BOUNDARY CONTROL

In this case, the boundary and initial conditions are expressed as

$$x(0) = 0,\ \ x(1) = gu,\ \ x(z,0) = sin(\pi z). \tag{62}$$

For this example, β was chosen to be 17 and $g=1$. The sampling time for updating control input was $t_s = 2msec$. The initial admissible control policy was chosen to be $u_0 = -7.8$. The function $q(s,z)$ was expressed as in ((60)) and the identifier basis functions $\sigma_{\lambda i}$, $1 \le i \le 5$ and value function basis functions ϕ_js $1 \le j \le 36$ were chosen so as to compare with ((59)) and ((61)), respectively. The update law parameters were expressed as $\psi = 10, \alpha_\lambda = 1$, $\alpha_1 = 0.1, \alpha_2 = 0.2, \alpha_3 = 0.1, \alpha_4 = 0.01, c_1 = 2$, $c_2 = 0.02$, $c_2 = c_3 = 0.01$ and W_0 was a random vector with positive entries. It can be easily verified that the necessary design conditions, i.e. $\alpha_2 > \alpha_3$ and $\alpha_3 > \theta^D_{\alpha_3} = \frac{\| \phi_{lz} \|^4_{\mathcal{L}_2}\ g^4}{8c_1^2 r^2} \simeq 0.08$ are satisfied for these parameters. Figure 4.4 shows the performance of controller in regulation of the DPS state. The smoothness of control input and fast convergence of HJB error are also shown in Figure 4.5. In addition, similar to Neumann boundary controller, increasing the update law parameter α_1 would speed up HJB error convergence rate. Figure 4.6 shows the estimated $\hat{p}(s,z)$ at the end of simulation. Finally, Figure 4-8 depicts the control gain kernels for Neumann boundary condition $K_{uN}(s) = |\hat{W}_V{}^T \phi(s,l)|$ and Dirichlet boundary condition $K_{uD}(s) = |\hat{W}_V{}^T \frac{\partial \phi(s,z)}{\partial z}|_{z=l}|$, $0 \le s \le l$ corresponding to estimated $\hat{p}(s,z)$ as shown in Figure 4.3 and Figure 4.6, respectively. Qualitatively, Figure 4.7 shows that feedback from the middle of spatial domain is significantly more important for system stabilization than places near the boundaries.

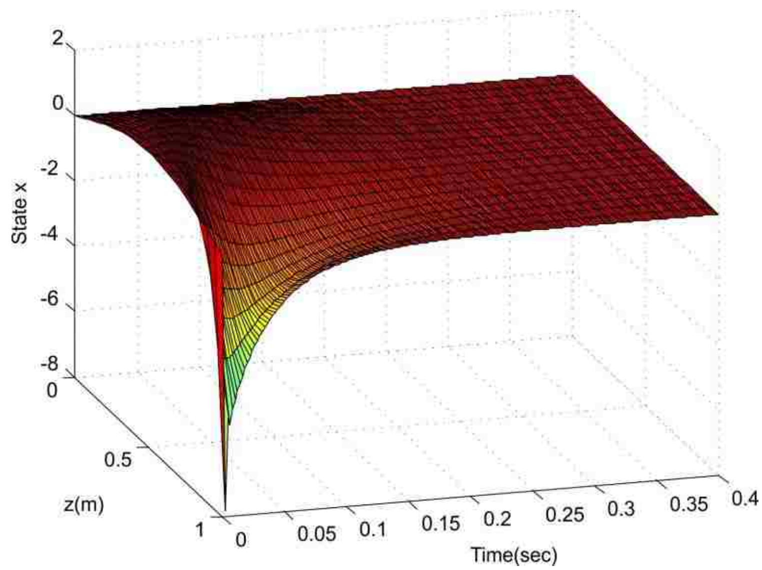

Figure 4.4. Closed loop state evolution under Dirichlet boundary condition.

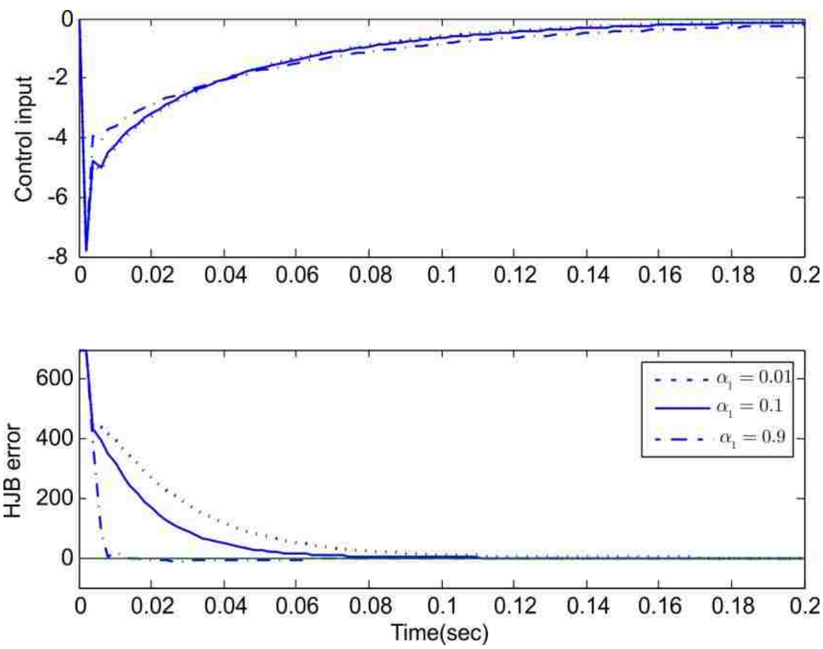

Figure 4.5. Control input and HJB error for Dirichlet boundary controller.

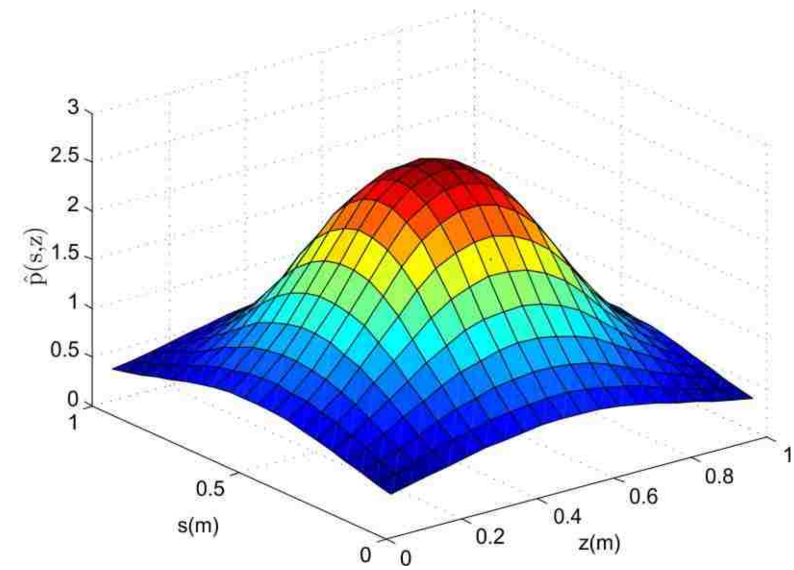

Figure 4.6. Estimated $\hat{p}(s,z)$ for Dirichlet boundary controller.

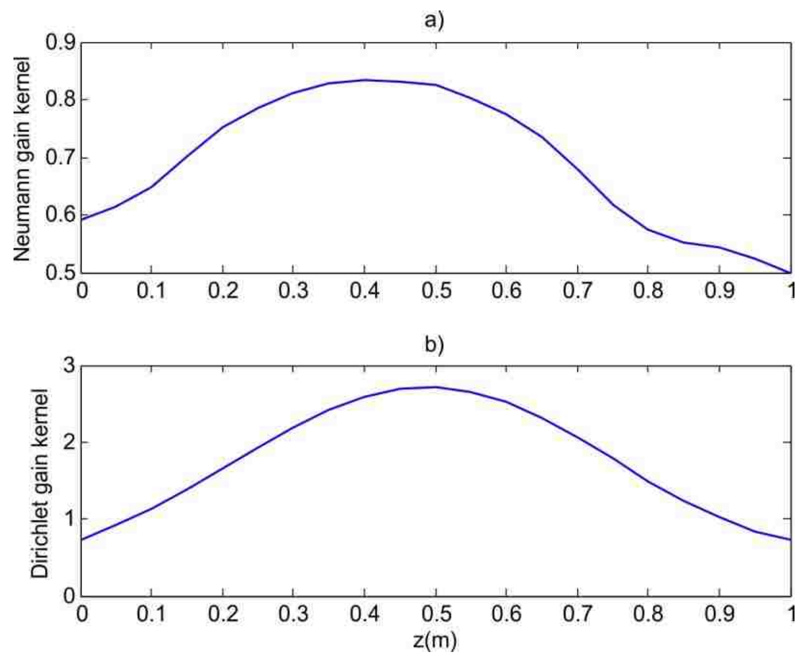

Figure 4.7. Feedback gain kernels a) K_{uN} for Neumann boundary condition, b) K_{uD} for Dirichlet boundary condition.

5. CONCLUSIONS

This paper developed an ADP-based near optimal boundary control scheme for DPS governed by uncertain linear one-dimensional parabolic PDE under both Neumann and Dirichlet actuation conditions without any finite dimensional model approximation prior to control design. By defining the value functional as the extension of its definition from linear LPS optimal control design, the HJB equation was derived in original infinite dimensional state space. The proposed identifier was effective in estimating the unknown coefficient over the space in system dynamics. Based on defined structure for the value functional as a surface integral, a RBN was proposed to estimate its unknown parameters as a continuous two-variable kernel function. The update law for RBN unknown weights was defined to reduce the HJB error effectively and insuring system stability whereas PDE dynamics was uncertain. Ultimate boundedness of the closed-loop system was verified by using the standard Lyapunov theory with consideration of all approximation reconstruction errors. Since model reduction was not utilized in control development, the design is more reliable and can be applied to achieve accurate control and closed loop stability of the original infinite dimensional system. The performance of proposed control method was successfully verified on a diffusion reaction process.

APPENDIX

In order to avoid long derivations in the proof, we first define $x_z = \frac{\partial x}{\partial z}$, $x_{zz} = \frac{\partial^2 x}{\partial z^2}$, $\Phi_x = \frac{\partial \Phi}{\partial x}$, $\Phi_{xz} = \frac{\partial \Phi_x}{\partial z}$, $\Phi_{xzz} = \frac{\partial^2 \Phi_x}{\partial z^2}$, $\varepsilon_{V_{xz}} = \frac{\partial \varepsilon_{V_x}}{\partial z}$ and $\varepsilon_{V_{xzz}} = \frac{\partial^2 \varepsilon_{V_x}}{\partial z^2}$. First the proof for Theorem 1 will be provided:

Proof of Theorem 1: In order to prove stability of state feedback identifier, consider the Lyapunov function

$$L_c = \frac{1}{2}\mu_c \int_0^l \tilde{x}^2 dz + \frac{1}{2}\mu_c \tilde{W}_\lambda^T \tilde{W}_\lambda, \tag{63}$$

where μ_c is a positive scalar. Taking its derivative $\dot{L}_c$ and substituting error dynamics (28) and (29) yields

$$\begin{aligned} \dot{L}_c &= \mu_c \int_0^l \tilde{x}(\tilde{x}_{zz} + \tilde{W}_\lambda^T \sigma_\lambda(z) x + \varepsilon_\lambda(z) - \psi \tilde{x}) dz \\ &+ \mu_c \tilde{W}_\lambda^T (\alpha_\lambda \hat{W}_\lambda - \int \sigma_\lambda(z) x \tilde{x} dz) = -\mu_c \int_0^l \psi \tilde{x}^2 dz \\ &+ \mu_c \int_0^l \tilde{x}\tilde{x}_{zz} dz + \mu_c \int_0^l \tilde{x} \varepsilon_\lambda(z) dz + \mu_c \int_0^l \tilde{x} \tilde{W}_\lambda^T \sigma_\lambda(z) x dz \\ &+ \mu_c \tilde{W}_\lambda^T \alpha_\lambda (W_\lambda - \tilde{W}_\lambda) - \mu_c \int_0^l \tilde{W}_\lambda^T \sigma_\lambda(z) x \tilde{x} dz. \end{aligned} \tag{64}$$

Taking integration by parts and substituting PDE dynamics (28) for state estimation error $\tilde{x}$ yields

$$\begin{aligned} \int_0^l \tilde{x}\tilde{x}_{zz} dz &= \tilde{x}(l)\tilde{x}_z\big|_{z=l} - \tilde{x}(0)\tilde{x}_z\big|_{z=0} - \int_0^l \tilde{x}_z^2 dz \\ &= -\int_0^l \tilde{x}_z^2 dz \leqslant 0. \end{aligned} \tag{65}$$

Therefore, using Young's inequality ($\beta_1\beta_2 \leqslant \frac{1}{2\gamma}\beta_1^2 + \frac{\gamma}{2}\beta_2^2$)

$$\dot{L}_c \leqslant -\mu_c(\psi - \frac{1}{2})\|\tilde{x}\|_{\mathcal{L}_2}^2 - \frac{1}{2}\mu_c \alpha_\lambda \|\tilde{W}_\lambda\|^2 + \varepsilon_c, \tag{66}$$

where $\varepsilon_c = \frac{1}{2}\mu_c \alpha_\lambda W_{\lambda M}^2 + \frac{l}{2}\mu_c \varepsilon_{\lambda M}^2$. Therefore by selecting $\psi > \frac{1}{2}$, the identifier dynamics are UB.

Proof of Lemma 1: Using integration by parts one will get

$$\begin{aligned} \varepsilon_H &= \int_0^l \varepsilon_{V_x}(z)[\lambda(z)x(z)]dz - \varepsilon_{V_{xz}}\,|_{z=l}\; x(l) \\ &\varepsilon_{V_{xz}}\,|_{z=0}\; x(0) + \int_0^l \varepsilon_{V_{xzz}} x dz - \frac{1}{4r} g^2 \varepsilon_{V_x}^2(l) \\ &-\frac{1}{2r} W_V^T \Phi_x(l) g^2 \varepsilon_{V_x}(l) + h\varepsilon_{V_x}(0)x(0). \end{aligned} \tag{67}$$

Therefore

$$\begin{aligned} |\varepsilon_H| \leqslant & |\varepsilon_{pM}\int_0^l x(s)ds\int_0^l[\lambda(z)x(z)]dz| \\ &+|\varepsilon_{V_{xz}}(l)x(l)| + |\varepsilon_{V_{xz}}(0)x(0)| \\ &+|\varepsilon_{pzzM}\int_0^l x(s)ds\int_0^l x(z)dz| + |\frac{1}{4r}g^2\varepsilon_{V_x}^2(l)| \\ &+|\frac{1}{2r}W_V^T\Phi_x(l)g^2\varepsilon_{V_x}(l)| + |h\varepsilon_{V_x}(0)x(0)|. \end{aligned} \tag{68}$$

According to Hölder's inequality [37] ($\int a(z)b(z)dz \le \| a \|_{\mathcal{L}_2} \| b \|_{\mathcal{L}_2}$)

$$\int_0^l x(z)dz \le \sqrt{l}\, \| x \|_{\mathcal{L}_2} . \tag{69}$$

Thus, according to Young's inequality

$$\begin{aligned} |\varepsilon_{V_{xz}}(l)x(l)| &\le \sqrt{l}\varepsilon_{pzM}\, \| x \|_{\mathcal{L}_2} |x(l)| \\ &\le \frac{1}{2} l\varepsilon_{pzM}^2 \| x \|_{\mathcal{L}_2}^2 + \frac{1}{2}|x(l)|^2 . \end{aligned} \tag{70}$$

Consequently,

$$\begin{aligned} |\varepsilon_H| \leqslant & l\varepsilon_{pM}\lambda_M \|x\|_{\mathcal{L}_2}^2 + l\varepsilon_{pzzM}\|x\|_{\mathcal{L}_2}^2 + \frac{1}{4r} lg^2\varepsilon_{pM}^2 \|x\|_{\mathcal{L}_2}^2 \\ &+\frac{1}{2r}\sqrt{l}g^2 \|\phi_l\|_{\mathcal{L}_2} \|W_V\| \varepsilon_{pM} \|x\|_{\mathcal{L}_2}^2 + |h| \, |x(0)|\sqrt{l}\varepsilon_{pM} \\ &\|x\|_{\mathcal{L}_2} + |x(0)|\sqrt{l}\varepsilon_{pzM}\|x\|_{\mathcal{L}_2} + \frac{1}{2} l\varepsilon_{pzM}^2 \|x\|_{\mathcal{L}_2}^2 + \frac{1}{2}|x(l)|^2 \\ &= c_{H1}\|x\|_{\mathcal{L}_2}^2 + c_{H2}x(0)^2 + c_{H3}|x(l)|^2, \end{aligned} \tag{71}$$

where λ_M is maximum of continuous function $\lambda(.)$ over $[0,l]$ and $\|\phi_l\|_{\mathcal{L}_2}$ is abbreviation for $\|\phi(s,l)\|_{\mathcal{L}_2}$. Hence, $c_{H1} = l\varepsilon_{pM}(\lambda_M + \frac{1}{4r}g^2(\varepsilon_{pM} + \frac{2}{\sqrt{l}}\|\phi_l\|_{\mathcal{L}_2}\|W_V\|))$ $+l\varepsilon_{pzzM} + \frac{1}{2}l\varepsilon_{pzM}^2 + \frac{1}{2}l(|h|\varepsilon_{pM} + \varepsilon_{pzM})^2$ and $c_{H2} = c_{H3} = \frac{1}{2}$.

Moreover, recalling $\zeta_N = c_1 \| x \|_{\mathcal{L}_2}^2 + c_2 x(0)^2 + c_3 x(l)^2 + c_4$, $|\varepsilon_H|$ can be expressed as

$$\begin{aligned} |\varepsilon_H| &\le (\frac{c_{H1}}{c_1}) c_1 \|x\|_{\mathcal{L}_2}^2 + (\frac{c_{H2}}{c_2}) c_2 x(0)^2 \\ &+ (\frac{c_{H3}}{c_3}) c_3 x(l)^2 + c_4 \leqslant \xi_{NM} \zeta_N, \end{aligned} \tag{72}$$

where $\xi_{NM} = \max\{\frac{c_{H1}}{c_1}, \frac{c_{H2}}{c_2}, \frac{c_{H3}}{c_3}, 1\}$.

Proof of theorem 2: Consider the following Lyapunov function

$$L = \underbrace{\frac{1}{2}\mu_a \tilde{W}_V^T \tilde{W}_V}_{L_a} + \underbrace{\int_0^l V^*(x)dz}_{L_b} + L_c, \tag{73}$$

where μ_a is a positive tuning constant and $\tilde{W}_V = \hat{W}_V - W_V$. Note that the first term L_a is added to insure the boundedness of value functional weight estimation error whereas the second term L_b is added to guarantee the boundedness of system state x. Since $H^* = 0$, $\hat{H}$ can be represented according to (47) and (50) as

$$\begin{aligned} \hat{H} &= \hat{H} - H^* = \int_0^l \tilde{W}_V \Phi_x(z)\{\lambda(z)x(z)\}dz \\ &- \int_0^l \tilde{W}_V \Phi_{xz}(z) x_z dz + h\Phi_x^T(0)\tilde{W}_V x(0) - \int_0^l W_V^T \Phi_x(z) \\ &(\tilde{W}_\lambda \sigma(z) + \varepsilon_\lambda) x(z) dz + \frac{1}{4r} W_V^T \Phi_x(l) g^2 \Phi_x^T(l) W_V + \\ &- \underbrace{\frac{1}{4r}(W_V + \tilde{W}_V)^T \Phi_x(l) g^2 \Phi_x^T(l)(W_V + \tilde{W}_V)}_{expand} \\ &\underbrace{\frac{1}{2r}\{\tilde{W}_V^T \Phi_x(l) g^2 \Phi_x^T(l)\hat{W}_V - \tilde{W}_V^T \Phi_x(l) g^2 \Phi_x^T(l)\hat{W}_V\}}_{add \,\&\, subtract} \\ &= \tilde{W}_V^T \omega_N + \frac{1}{4r}\tilde{W}_V^T \Phi_x(l) g^2 \Phi_x{}^T(l)\tilde{W}_V - \varepsilon_H \\ &- \int_0^l W_V^T \Phi_x(z)[(\tilde{W}_\lambda \sigma(z) + \varepsilon_\lambda) x(z)] dz. \end{aligned} \tag{74}$$

Therefore, taking the derivative of L_a and using update law (40) yields

$$\dot{L}_a = -\mu_a\alpha_1\frac{\tilde{W}_V^T\omega_N}{\zeta_N^2}\left\{\begin{array}{l}\tilde{W}_V^T\omega_N + \frac{1}{4r}\tilde{W}_V^T\Phi_x(l)g^2 \\ \Phi_x^T(l)\tilde{W}_V - \varepsilon_H - \int_0^l W_V^T \\ \Phi_x(\tilde{W}_\lambda\sigma(z)+\varepsilon_\lambda)x(z)dz\end{array}\right\}$$

$$-\mu_a\alpha_2\tilde{W}_V^T(W_V+\tilde{W}_V) - \mu_a\alpha_3\tilde{W}_V^T\hat{W}_V\|\hat{W}_V\|^2 \quad (75)$$

$$-\mu_a\alpha_4\|x\|_{\mathcal{L}_2}^2\ \tilde{W}_V^T\hat{W}_V,$$

subsequently, expanding $-\tilde{W}_V^T\hat{W}_V\left\|\hat{W}_V\right\|^2$ and using Young's and Cauchy-Schwarz inequalities yields

$$\begin{aligned}-\tilde{W}_V^T\hat{W}_\Phi\left\|\hat{W}_\Phi\right\|^2 &= -\tilde{W}_V^T(W_V+\tilde{W}_V)(W_V+\tilde{W}_V)^T \\ (W_V+\tilde{W}_V) &= -\tilde{W}_V^T W_V\left\|W_V\right\|^2 - 2(\tilde{W}_V^T W_V)^2 \\ &-\left\|\tilde{W}_V\right\|^2\left\|W_V\right\|^2 - 3\tilde{W}_V^T W_V\left\|\tilde{W}_V\right\|^2 - \left\|\tilde{W}_V\right\|^4 \\ &\leqslant -\frac{1}{4}\left\|\tilde{W}_V\right\|^4 + \frac{1}{2}\left\|\tilde{W}_V\right\|^2 + \frac{81}{4}\left\|W_V\right\|^4 + \frac{1}{2}\left\|W_V\right\|^6,\end{aligned} \quad (76)$$

According to Young's Inequality

$$\begin{aligned}&-(\mu_a\alpha_1\frac{\tilde{W}_V^T\omega_N}{\zeta_N^2})(\frac{1}{4r}\tilde{W}_V^T\Phi_x(l)g^2\Phi_x^T(l)\tilde{W}_V) \\ &\le \mu_a\alpha_1^2\frac{\tilde{W}_V^T\omega_N\omega_N^T\tilde{W}_V}{3\zeta_N^2} + \mu_a\frac{3\|\phi_l\|_{\mathcal{L}_2}^4\ g^4\|x\|_{\mathcal{L}_2}^4}{64r^2\zeta_N^2}\|\tilde{W}_V\|^4.\end{aligned} \quad (77)$$

Moreover,

$$\begin{aligned}&(\mu_a\alpha_1\frac{\tilde{W}_V^T\omega_N}{\zeta_N^2})(\int_0^l W_V^T\Phi_x(z)(\tilde{W}_\lambda\sigma(z)+\varepsilon_\lambda)x(z)dz) \\ &\le \mu_a\alpha_1^2\frac{\tilde{W}_V^T\omega_N\omega_N^T\tilde{W}_V}{3\zeta_N^2} \\ &+\mu_a\frac{3W_{VM}^2(\|\tilde{W}_\lambda\|^2\ \sigma_{\lambda M}^2+\varepsilon_\lambda^2)\|\phi\|_{\mathcal{L}_2}^2\|x\|_{\mathcal{L}_2}^4}{2\zeta_N^2}.\end{aligned} \quad (78)$$

where $\|\phi\|_{\mathcal{L}_2} = (\int_0^l\int_0^l\|\phi(s,z)\|^2\ dsdz)^{\frac{1}{2}}$. And

$$(\mu_a\alpha_1\frac{\tilde{W}_V^T\omega_N}{\zeta_N^2})\varepsilon_H \le \mu_a\alpha_1^2\frac{\tilde{W}_V^T\omega_N\omega_N^T\tilde{W}_V}{3\zeta_N^2} + \mu_a\frac{3\varepsilon_H^2}{4\zeta_N^2}. \quad (79)$$

By multiplying the terms in (75) and using (76)-(79), $\dot{L}_a$ can be derived as

$$
\begin{aligned}
\dot{L}_a \leqslant\, & \mu_a(-\alpha_1+\alpha_1^2)\frac{\tilde{W}_V^T\omega_N\omega_N^T\tilde{W}_V}{\zeta_N^2} \\
& +\mu_a\frac{3\varepsilon_H^2}{4\zeta_N^2}+\mu_a\frac{3\|\phi_l\|_{\mathcal{L}_2}^4\ g^4\ \|x\|_{\mathcal{L}_2}^4}{64r^2\zeta_N^2}\|\|\tilde{W}_V\|^4 \\
& +\mu_a\frac{3W_{VM}^2(\|\tilde{W}_\lambda\|^2\ \sigma_{\lambda M}^2+\varepsilon_\lambda^2)\|\phi\|_{\mathcal{L}_2}^2\|x\|_{\mathcal{L}_2}^4}{2\zeta_N^2} \\
& +\mu_a(-\frac{\alpha_2}{2}+\frac{\alpha_3}{2})\|\tilde{W}_V\|^2-\frac{\mu_a\alpha_3}{4}\|\tilde{W}_V\|^4 \\
& +\frac{\mu_a\alpha_2}{2}W_{VM}^2+\frac{81\mu_a\alpha_3}{4}W_{VM}^4+\frac{\mu_a\alpha_3}{2}W_{VM}^6 \\
& +\frac{\mu_a\alpha_2}{2}W_{VM}^2+\frac{81\mu_a\alpha_3}{4}W_{VM}^4+\frac{\mu_a\alpha_3}{2}W_{VM}^6 \\
& -\frac{\mu_a\alpha_4}{2}\|x\|_{\mathcal{L}_2}^2\|\tilde{W}_V\|^2+\frac{\mu_a\alpha_4}{2}\|x\|_{\mathcal{L}_2}^2\ W_{VM}^2.
\end{aligned}
\tag{80}
$$

By observing the fact that $\frac{\|x\|_{\mathcal{L}_2}^4}{\zeta_N^2}\leqslant\frac{1}{c_1^2}$ for all $x\in\mathcal{L}_2[0,l]$, if $\alpha_2>\alpha_3$ and $\alpha_3>\theta_{\alpha_3}^N=\frac{\|\phi_l\|^4\ g^4}{8c_1^2r^2}$,

define $\eta=\frac{1}{2}(\alpha_2-\alpha_3)$. Therefore, if $0<\alpha_1<1$ then

$$
\begin{aligned}
\dot{L}_a \leqslant & -\mu_a\eta\left\|\tilde{W}_V\right\|^2-\frac{\mu_a\alpha_4}{2}\|x\|_{\mathcal{L}_2}^2\|\tilde{W}_V\|^2+b_1\|x\|_{\mathcal{L}_2}^2 \\
& +\mu_a\frac{3W_{VM}^2\ \|\tilde{W}_\lambda\|^2\ \sigma_{\lambda M}^2\ \|\phi\|_{\mathcal{L}_2}^2}{2c_1^2}+\varepsilon_a,
\end{aligned}
\tag{81}
$$

where $b_1=\frac{\mu_a\alpha_4}{2}W_{VM}^2$ and by using Lemma 1, $\varepsilon_a=\frac{3\mu_a}{4}\xi_{NM}^2+\mu_a\frac{3W_{VM}^2\varepsilon_\lambda^2\|\phi\|_{\mathcal{L}_2}^2}{2c_1^2}+\frac{\mu_a\alpha_2}{2}W_{VM}^2+\frac{81\mu_a\alpha_3}{4}W_{VM}^4+\frac{\mu_a\alpha_3}{2}W_{VM}^6$. Subsequently for $\dot{L}_b$ one has

$$\begin{aligned}
\dot{L}_b &= \int_0^l V_x^*[x_{zz} + \lambda(z)x(z)]dz \\
&= \int_0^l V_x^*(\lambda(z)x(z))dz + \int_0^l V_x^* x_{zz} dz \\
&= \int_0^l V_x^*[\lambda(z)x(z)]dz - \int_0^l V_{xz}^* x_z dz + gV_x^*(l)(u^* + \tilde{u}) \\
&\le -q_{min} \| x \|^2 - \frac{1}{2r} V_x^*(l) g^2 \Phi_x^T(l)\tilde{W}_V - gV_x^*(l)\varepsilon_u \\
&\le -q_{min} \| x \|^2 - \frac{1}{2r} V_x^*(l) g^2 \Phi_x^T(l)\tilde{W}_V + \\
&\frac{1}{2r} g^2 (\Phi_x^T(l)W_V + \int_0^l x(s)\varepsilon_p(s,z)ds)(\int_0^l x(s)\varepsilon_p(s,z)ds).
\end{aligned} \tag{82}$$

where q_{min} is defined in Assumption 1. By using Cauchy-Schwartz inequality, one has

$$\frac{1}{2r} V_x^*(l) g^2 \Phi_x^T(l)\tilde{W}_V \le b_2 \| x \|_{\mathcal{L}_2}^2 \| \tilde{W}_V \| \tag{83}$$

where $b_2 = \frac{1}{2r} g^2 (\| \phi_l \|_{\mathcal{L}_2} W_{VM} + \sqrt{l}\varepsilon_{pM}) \| \phi_l \|_{\mathcal{L}_2}$, and

$$\begin{aligned}
&\frac{1}{2r} g^2 (\Phi_x^T(l)W_V + \int_0^l x(s)\varepsilon_p(s,z)ds)(\int_0^l x(s) \\
&\varepsilon_p(s,z)ds) \le b_3 \| x \|_{\mathcal{L}_2}^2 .
\end{aligned} \tag{84}$$

where $b_3 = \frac{1}{2r} g^2 (\| \phi_l \|_{\mathcal{L}_2} W_{VM} + \sqrt{l}\varepsilon_{pM})\sqrt{l}\varepsilon_{pM}$. Therefore,

$$\dot{L}_b \le -q_{min} \| x \|_{\mathcal{L}_2}^2 + b_2 \| x \|_{\mathcal{L}_2}^2 \| \tilde{W}_V \| + b_3 \| x \|_{\mathcal{L}_2}^2 . \tag{85}$$

Combining $\dot{L}_a$, $\dot{L}_b$ and $\dot{L}_c$ yields

$$
\begin{aligned}
&\dot{L} \le -q_{min} \| x \|_{\mathcal{L}_2}^2 - \mu_a \eta \| \tilde{W}_V \|^2 - \frac{\mu_a \alpha_4}{2} \| x \|_{\mathcal{L}_2}^2 \\
&(\| \tilde{W}_V \|^2 - \frac{2b_2}{\mu_a \alpha_4} \| \tilde{W}_V \|) + (b_1 + b_3) \| x \|_{\mathcal{L}_2}^2 + \varepsilon_a \\
&+\mu_a \frac{3W_{VM}^2 \| \tilde{W}_\lambda \|^2 \sigma_{\lambda M}^2 \| \phi \|_{\mathcal{L}_2}^2}{2c_1^2} - \mu_c (\psi - \frac{1}{2}) \|\tilde{x}\|_{\mathcal{L}_2}^2 \\
&-\frac{1}{2} \mu_c \alpha_\lambda \left\|\tilde{W}_\lambda\right\|^2 + \varepsilon_c \\
&\le -q_{min} \| x \|_{\mathcal{L}_2}^2 - \frac{\mu_a \alpha_4}{2} \| x \|_{\mathcal{L}_2}^2 (\| \tilde{W}_V \| - \frac{b_2}{\mu_a \alpha_4})^2 \\
&+(b_1 + \frac{b_2^2}{2\mu_a \alpha_4} + b_3) \| x \|_{\mathcal{L}_2}^2 - \mu_a \eta \| \tilde{W}_V \|^2 + \varepsilon_a \\
&+\mu_a \frac{3W_{VM}^2 \| \tilde{W}_\lambda \|^2 \sigma_{\lambda M}^2 \| \phi \|_{\mathcal{L}_2}^2}{2c_1^2} - \mu_c (\psi - \frac{1}{2}) \|\tilde{x}\|_{\mathcal{L}_2}^2 \\
&-\frac{1}{2} \mu_c \alpha_\lambda \left\|\tilde{W}_\lambda\right\|^2 + \varepsilon_c \\
&\le -q_{min} \| x \|_{\mathcal{L}_2}^2 - \mu_a \eta \| \tilde{W}_V \|^2 + b_4 \| x \|_{\mathcal{L}_2}^2 \\
&+\mu_a \frac{3W_{VM}^2 \| \tilde{W}_\lambda \|^2 \sigma_{\lambda M}^2 \| \phi \|_{\mathcal{L}_2}^2}{2c_1^2} - \mu_c (\psi - \frac{1}{2}) \|\tilde{x}\|_{\mathcal{L}_2}^2 \\
&-\frac{1}{2} \mu_c \alpha_\lambda \left\|\tilde{W}_\lambda\right\|^2 + \varepsilon,
\end{aligned}
\tag{86}
$$

where $b_4 = b_1 + \frac{b_2^2}{2\mu_a \alpha_4} + b_3$ and $\varepsilon = \varepsilon_a + \varepsilon_c$. If we define $b_q = 2b_4$ and q_{min} as defined in Assumption 1 satisfies $q_{min} \ge b_q$ and $\mu_c > \mu_a \frac{6W_{VM}^2 \sigma_{\lambda M}^2 \| \phi \|_{\mathcal{L}_2}^2}{c_1^2 \alpha_\lambda}$ then

$$
\begin{aligned}
&\dot{L} \le -\frac{q_{min}}{2} \| x \|_{\mathcal{L}_2}^2 - \mu_a \eta \| \tilde{W}_V \|^2 - \mu_c (\psi - \frac{1}{2}) \|\tilde{x}\|_{\mathcal{L}_2}^2 \\
&-\frac{1}{4} \mu_c \alpha_\lambda \left\|\tilde{W}_\lambda\right\|^2 + \varepsilon.
\end{aligned}
\tag{87}
$$

Therefore $\dot{L}$ is always less than zero if

$\| x \|_{\mathcal{L}_2} > \sqrt{\frac{2\varepsilon}{q_{min}}}$ or $\| \tilde{W}_V \| > \sqrt{\frac{\varepsilon}{\mu_a \eta}}$ or $\| \tilde{W}_\lambda \| > \sqrt{\frac{4\varepsilon}{\mu_c \alpha_\lambda}}$ or $\| \tilde{x} \|_{\mathcal{L}_2} > \sqrt{\frac{\varepsilon}{\mu_c (\psi - \frac{1}{2})}}$. Consequently, the closed-loop system is UB.

Moreover, since

$$\begin{aligned}&\| u-u^{*}\| = \| -\frac{1}{2r} g\Phi_x^T(l)\tilde{W}_V - \varepsilon_u \| \\ &\le \frac{1}{2r} g(\|\phi_l\|_{\mathcal{L}_2} \|x\|_{\mathcal{L}_2} \|\tilde{W}_V\| + \|\varepsilon_{Vx}(l)\|) \\ &\le \frac{1}{2r} g\|x\|_{\mathcal{L}_2} (\|\phi_l\|_{\mathcal{L}_2}\|\tilde{W}_V\| + \varepsilon_{pM}),\end{aligned} \tag{88}$$

and $\|x\|_{\mathcal{L}_2}, \|\tilde{W}_V\|$ are proven to be bounded, the error between the actual and truly optimal control inputs is also UB where the bound can be reduced by increasing q_{min} as defined in Assumption 1, weight update parameters α_2, α_3 and c_1 and decreasing ε_{pM} by selecting large enough set of basis functions.

Proof of Lemma 2: $|\varepsilon_H|$ for Dirichlet boundary control condition can be derived as

$$|\varepsilon_H| \leqslant c_{H1}\|x\|_{\mathcal{L}_2}^2 + c_{H2}x(0)^2 + c_{H3}(\frac{\partial x}{\partial z}|_{z=l})^2, \tag{89}$$

where $\|\phi_{lz}\|_{\mathcal{L}_2}$ is abbreviation for $\|\frac{\partial \phi}{\partial z}|_{z=l}\|_{\mathcal{L}_2}$, $c_{H1} = l\varepsilon_{pM}\lambda_M + \frac{1}{4r}g^2 l\varepsilon_{pzM}(\varepsilon_{pzM} + \frac{2}{\sqrt{l}}\|\phi_{lz}\|_{\mathcal{L}_2}\|W_V\|) + l\varepsilon_{pzzM} + \frac{1}{2}l\varepsilon_{pzM}^2 + \frac{1}{2}l(|h|\varepsilon_{pM} + \varepsilon_{pzM})^2$ and $c_{H2} = c_{H3} = \frac{1}{2}$.

Moreover, recalling $\zeta_D = c_1\|x\|_{\mathcal{L}_2}^2 + c_2 x(0)^2 + c_3(\frac{\partial x}{\partial z}|_{z=l})^2 + c_4$, $|\varepsilon_H|$ can be expressed as

$$\begin{aligned}&|\varepsilon_H| \le (\frac{c_{H1}}{c_1})c_1\|x\|_{\mathcal{L}_2}^2 + (\frac{c_{H2}}{c_2})c_2 x(0)^2 \\ &+(\frac{c_{H3}}{c_3})c_3(\frac{\partial x}{\partial z}|_{z=l})^2 + c_4 \leqslant \xi_{DM}\zeta_D,\end{aligned} \tag{90}$$

where $\xi_{DM} = \max\{\frac{c_{H1}}{c_1}, \frac{c_{H2}}{c_2}, \frac{c_{H3}}{c_3}, 1\}$.

Proof of Theorem 3: Consider the same Lyapunov function as in Theorem 2. If update parameters satisfy $0 < \alpha_1 < 1$, $\alpha_2 > \alpha_3$ and $\alpha_3 > \theta_{\alpha_3}^D = \frac{\|\phi_{lz}\|_{\mathcal{L}_2}^4 g^4}{8c_1^2 r^2}$ define $\eta = \frac{1}{2}(\alpha_2 - \alpha_3)$. Similarly, taking the derivative of Lyapunov function and combining $\dot{L}_a$, $\dot{L}_b$ and $\dot{L}_c$ yield

$$\dot{L} \leq -\frac{q_{min}}{2} \| x \|_{\mathcal{L}_2}^2 - \mu_a \eta \| \tilde{W}_V \|^2 - \mu_c (\psi - \frac{1}{2}) \|\tilde{x}\|_{\mathcal{L}_2}^2$$
$$- \frac{1}{4} \mu_c \alpha_\lambda \left\| \tilde{W}_\lambda \right\|^2 + \varepsilon. \tag{91}$$

where $\varepsilon = \frac{3\mu_a}{4} \xi_{DM}^2 + \mu_a \frac{3W_{VM}^2 \varepsilon_\lambda^2 \| \phi \|_{\mathcal{L}_2}^2}{2c_1^2} + \frac{\mu_a \alpha_2}{2} W_{VM}^2 + \frac{81\mu_a \alpha_3}{4} W_{VM}^4 + \frac{\mu_a \alpha_3}{2} W_{VM}^6 + \frac{1}{2} \mu_c \alpha_\lambda W_{\lambda M}^2 + \frac{l}{2} \mu_c \varepsilon_{\lambda M}^2$.

Consequently, the closed-loop system is UB.

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III. OUTPUT FEEDBACK BOUNDARY CONTROL OF COUPLED SEMI-LINEAR PARABOLIC PDE USING NEURO DYNAMIC PROGRAMMING

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In this paper, neuro dynamic programming (NDP) based output feedback boundary control of distributed parameter systems (DPS) governed by uncertain coupled semi-linear parabolic partial differential equations (PDE) under Neumann or Dirichlet boundary control conditions is introduced. First, Hamilton-Jacobi-Bellman (HJB) equation is formulated in the original PDE domain for such systems and the optimal control policy is derived using the value functional as the solution of the HJB equation. Subsequently, a novel observer is developed to estimate the system states given the uncertain nonlinearity in PDE dynamics and measured outputs. The sub-optimal boundary control policy is obtained by forward-in-time estimation of the value functional using a neural network (NN) based online approximator and estimated state vector obtained from the NN observer. Novel adaptive tuning laws in continuous-time are proposed for learning the value functional online to satisfy the HJB equation along system trajectories while ensuring the closed-loop stability. Local ultimate boundedness (UB) of the closed-loop system is verified by using Lyapunov theory. The performance of proposed controller is verified via simulation on an unstable coupled diffusion reaction process.

1. INTRODUCTION

Significant number of industrial processes are inherently distributed in space so that their behavior depend upon spatial position and time [1],[2]. These systems are usually described by a set of partial differential equations (PDE) with homogenous or mixed boundary conditions. In particular, coupled semi-linear parabolic PDEs represent a wide range of industrial distributed parameter systems (DPS) such as reaction-diffusion processes [1] and fluid flow [2].

Control of DPS modeled as single PDE by using operator theory has been extensively studied in the literature [3]. The results were also extended to boundary control [4] which is mathematically more involved and physically more practical and relevant. However, in many DPS processes, there are interactions between different components and therefore it is desirable to simultaneously control a set of variables with dynamics that are modeled by coupled PDE equations.

In [5], boundary control of coupled PDE using operator theory has been studied in detail with particular attention to parabolic-hyperbolic coupling arising in acoustics. After the development of backstepping as a new approach for boundary control of PDE in the last decade [6], control of higher dimensional coupled PDE dynamics using this method was also studied in recent years [7],[1]. In [7], the linearized model of thermal-fluid convection has been treated by combining backstepping and Fourier series methods and in [1], the stabilization of n coupled linear diffusion reaction processes was studied.

Following the introduction [8] and development [9] of neuro dynamic programming (NDP) to solve optimal control problems in real-time for systems with large dimensional state spaces, NDP based control schemes were also developed for DPS. However, since dealing with PDE dynamics and infinite dimensional state spaces were difficult, the conventional method usually included extracting a finite dimensional space prior to the NDP-based controller design [10]-[11]. Moreover, existing NDP approaches [10]-[11] for control of DPS systems rely on availability of system states. However, spatially distributed nature of these systems makes output feedback control approaches significantly more useful and practical [12]-[14].

This paper addresses output feedback boundary control scheme using NDP for uncertain DPS expressed as coupled semi-linear parabolic PDE without using any model reduction. In contrast to previous NDP control designs for DPS [10]-[11], the optimal control problem is formulated in the original PDE domain and solved forward-in-time without using any finite dimensional model approximation prior to the control design. Moreover, a neural network (NN) observer is proposed for the online estimation of states of the coupled PDE when the system dynamics are partially uncertain so that the need for availability of system states and accurate dynamics are relaxed.

Traditionally, formulating the HJB equation for online control of DPS seemed impractical. The complexity of dynamics and large scale of system state space made the required time for finding a solution extremely lengthy, a problem commonly referred as curse of dimensionality. In order to avoid this difficulty, NDP approaches were proposed to find approximate but tractable solutions for optimal control problem [15]. Many recent NDP algorithms use policy or value iterations to find the approximate optimal control [16]. Although these algorithms attain the optimal control over time, they still have considerable computational cost which is inappropriate for DPS online control since the size of state space is large.

Therefore, the proposed boundary control policy in this paper is based on an iteration-free adaptive scheme with novel update laws that can achieve system stability and local optimality along trajectories whereas the knowledge for system states or dynamics is relaxed. Lyapunov analysis demonstrates that provided an initial admissible control policy, the error between the actual and truly optimal control will always remain bounded whereas all approximation reconstruction errors are considered. Simulation results confirm that the presented output feedback control approach has good convergence and control performance on an uncertain unstable coupled diffusion-reaction process.

In order to find the boundary control law, after defining an appropriate cost functional, the HJB equation is derived in the infinite dimensional space and the optimal control policy is obtained based on necessary conditions of optimality. Subsequently, motivated by the fact that the system states are not available in the whole of spatial domain and the nonlinearity in coupled PDE dynamics may be highly uncertain, a NN observer is proposed to estimate the states online when the nonlinearity in the coupled PDE dynamics is unknown. Consequently, by approximating the optimal value functional based on a novel adaptive framework, the actual control policy is derived.

The observer has a special design such that the boundedness of observer approximation error in original PDE domain is guaranteed. This is considerably different from ordinary differential equation (ODE)-based NN observers [17] where stability is solely assured for finite dimensional model. Combined with a novel NN adaptive boundary control method that minimizes the HJB approximation error forward-in-time in the PDE domain, the proposed observer-controller framework will provide the closed-loop stability in original infinite dimensional domain which is different from finite dimensional NDP approaches where stability is only guaranteed for reduced order model.

Notations: Throughout the paper, $\|\cdot\|$ stands for Euclidean norm for a vector or Frobenius norm for a matrix. We recall the inequality $\|\cdot\|_2 \le \|\cdot\|_F$ with $\|\cdot\|_2$ being the induced 2 norm. We also define for $\forall x \in [\mathcal{L}_2(0,l)]^n$, $\|x\|_{\mathcal{L}_2} = (\int_0^l \|x(z)\|_2^2 dz)^{\frac{1}{2}}$.

The rest of the paper is organized as follows. In Section 2, the class of DPS under consideration is described and the output feedback optimal control approaches are explained separately for different boundary conditions. Section 3 addresses the closed-loop stability of system under the proposed boundary control framework. Section 4 demonstrates the simulation results and Section 5 provides the concluding remarks.

2. NEURO DYNAMIC PROGRAMMING BOUNDARY CONTROL

The class of DPS considered in this paper is described by following semi-linear coupled parabolic PDE dynamics as

$$\begin{aligned}
&\frac{\partial x(z,t)}{\partial t} = A\frac{\partial^2 x(z,t)}{\partial z^2} + f(x,z) \\
&\left.\frac{\partial x}{\partial z}\right|_{z=l} = Gu(t), \qquad z \in [0,l] \\
&\left.\frac{\partial x}{\partial z}\right|_{z=0} + Px(0,t) = 0, \qquad x(z,0) = x_0(z), \\
&y(t) = \int_0^l C(z)x(z,t)dz,
\end{aligned} \tag{1}$$

where Neumann boundary control at $z = l$ is primarily chosen for convenience. Here, $x(z,t) = [x_1(z,t),...,x_n(z,t)]^T \in X$ is the state vector, $X = [\mathcal{H}_2(0,l)]^n$ is the solution space of PDE with $\mathcal{H}_2$ being the Sobolev space of second order, t represents time, $z \in [0,l]$ is the spatial variable, $l > 0$, $A \in \mathbb{R}^{n\times n}$ is a constant diagonal matrix with $A_{i,i} = a_i > 0$, $f(x,z) = [f_1(x,z),..., f_n(x,z)]^T \in [\mathcal{C}(0,l)]^n$ is an unknown Lipschitz continuous nonlinear vector function, $\frac{\partial x}{\partial t}$ and $\frac{\partial^2 x}{\partial z^2}$ denote the time and second spatial derivatives of state x. Here $t \in [0,\infty)$ and $z \in [0,l]$, the PDE domain is $D \subset [0,\infty)\times[0,l]^n$. Moreover $u(t) = [u_1(t),...,u_n(t)]^T \in \mathbb{R}^n$ is the vector of boundary control input signals, $G \in \mathbb{R}^{n\times n}$ is a known constant matrix, $P \in \mathbb{R}^{n\times n}$ is a diagonal negative definite matrix and $x_0(z) \in \mathcal{C}_2(0,l)$ represents the initial condition of the state with $\mathcal{C}_2$ being the space of second order differentiable functions. In addition, $y(z,t) = [y_1(z,t),..., y_n(z,t)]^T$ represents the measured output and $C(z) \in \mathcal{L}_2^{n\times n} : [\mathcal{H}_2(0,l)]^n \to [\mathcal{H}_2(0,l)]^n$ is a diagonal linear operator with $\|C_{ii}(z)\|_{\mathcal{L}_2} > 0$, $1 \le i \le n$.

Assumption 1: the function $f(.,z)$ is Lipschitz continuous with Lipschitz constant L_f described by

$$\| f(x_1,z) - f(x_2,z)\| \le L_f \|x_1 - x_2\|. \tag{2}$$

The goal is to provide a continuous control input that minimizes the cost functional over infinite horizon given by

$$\bar{V}(x,t_0,u)=\int_{t_0}^{\infty}(Q(x(z,t))+u^T(t)Ru(t))dt, \tag{3}$$

where $R\in\mathbb{R}^{n\times n}$ is a positive definite matrix and $Q(x)$ is an integral functional with a nonlinear integrand function $q:[\mathcal{H}_2]^n\times[0,l]\rightarrow C_1[0,l]$ expressed as

$$Q(x)=\int_0^l q(x(z,t),z)dz. \tag{4}$$

It is assumed that $Q(x)$ is also positive definite i.e. $Q(x)\geq q_{\min}\|x\|^2_{\mathcal{L}_2}$ with $q_{\min}>0$ and locally Lipschitz continuous. In contrast to finite dimensional systems where the state is denoted by a vector, there are infinitely many states $x_i(z,t)$ in DPS for $0\leq z\leq l$ at each instant of time t that are continuous on the domain $[0,l]$. Therefore, integral functional such as (4) is necessary [3] for definition of $Q(x)$ instead of finite dimensional summations in conventional optimal control [17].

If the optimal cost functional is represented by $\bar{V}^*$, it can be expressed as

$$\bar{V}^*(x,t_0)=\min_u\int_{t_0}^{\infty}Q(x)+u^TRudt, \tag{5}$$

Let H be the Hilbert space of square integrable vector functions $[L_2(0,l)]^n$ with inner product defined as

$$(x,y)_H=\int_0^l y^T(z)x(z)dz. \tag{6}$$

Also, let $W=[H_1(0,l)]^n$, the Hilbert space of L_2 functions with L_2 derivatives with inner product is defined as

$$(x,y)_W=\int_0^l y^T(z)x(z)+\frac{\partial y^T}{\partial z}(z)\frac{\partial x}{\partial z}(z)dz. \tag{7}$$

We assume that $\bar{V}^*:H\rightarrow R$ is Frechet differentiable everywhere with respect to x [18]. Denote the x-Frechet derivative of $\bar{V}^*$ at w evaluated at y by $[\bar{V}_x^*(w)]y$. Since $\bar{V}_x^*(w)$ is a bounded linear functional on H, the Riesz representation Theorem [19] guarantees that there is a unique $k\in H$ such that

$$[\bar{V}_x^*(w)]y=\int_0^l k^T(w,z)y(z)dz. \tag{8}$$

where $k\in H$. We also make an additional assumption that $k\in W$. By taking the time interval $[t,t+\delta t)$, $\bar{V}^*(x(t),t)$ in (3) can be expressed in the recursive form as

$$\bar{V}^*(x(t),t) = \min_u \{ \int_t^{t+\delta t} Q(x) + u^T R u dt$$
$$+ \bar{V}^*(x(t+\delta t), t+\delta t) \}, \tag{9}$$

Take $\delta t \to 0$, and use $x(t+\delta t) \approx x(t) + \delta t(\partial x / \partial t)$ and the definition of the Frechet derivative of $\bar{V}^*$ with respect to x to obtain

$$0 = \min_u \{ Q(x) + u^T R u + [\bar{V}_x^*(x(t))] \frac{\partial x}{\partial t} \} + \frac{\partial \bar{V}^*}{\partial t}(x(t),t). \tag{10}$$

Since the actual cost function does not depend on t, $\frac{\partial \bar{V}^*}{\partial t} = 0$. Next, use (8) and PDE dynamics (1) to obtain the HJB equation as

$$0 = \min_u \{ Q(x) + u^T R u + \int_0^l k^T(x,z)(A \frac{\partial^2 x}{\partial z^2} + f(x,z)) dz \}. \tag{11}$$

Integrating by parts in the second order term gives

$$0 = \min_u \{ Q(x) + u^T R u + k^T(x,l) G u + k^T(x,0) P x(t,0)$$
$$+ \int_0^l \frac{\partial k^T}{\partial z}(x,z)(f(x,z) - A \frac{\partial x}{\partial z}) dz \}. \tag{12}$$

By completing the square, it can be shown that the minimum is achieved for $u = u^*$ where

$$u^* = -\frac{1}{2} R^{-1} G^T k(x,l). \tag{13}$$

Let $\delta_l : H_1(0,l) \to \mathbb{R}$ be the Dirac delta operator at $x = l$. According to Riesz representation Theorem [19], there exists a unique function $d \in H_1$ such that for all $f \in H_1(0,l)$,

$$\delta_l(f) = f(l) = (f,d)_{H_1}. \tag{14}$$

This implies that the optimal control u^* can be represented as

$$u^* = -\frac{1}{2} R^{-1} G^T B k(x), \tag{15}$$

where the operator $Bk(x)$ is expressed as

$$Bk(x) = [(k_1(x),d)_{H_1}, \ldots, (k_n(x),d)_{H_1}]^T. \tag{16}$$

Defining $h(x) = GR^{-1}G^T Bk(x)$, one has

$$
\begin{aligned}
& u^{*T} R u^* = \frac{1}{4}[(k_1(x), d)_{H_1}, \ldots, (k_n(x), d)_{H_1}] h(x) \\
& = \frac{1}{4}\sum_{i=1}^{n} h_i(x)(k_i(x), d)_{H_1(0,l)} = \\
& \int_0^l (\frac{1}{4}\sum_{i=1}^{n} h_i(x)[\frac{\partial k_i}{\partial z}(x,z)\frac{\partial d}{\partial z}(z) + k_i(x,z)d(z)])dz \\
& = \int_0^l r(x,z)dz
\end{aligned} \tag{17}
$$

Substituting the above representation in (5) results

$$\bar{V}^*(x) = \int_0^\infty \int_0^l q(x,z) + r(x,z)dzdt, \tag{18}$$

which by switching the order of integration and using (8) gives

$$\bar{V}^*(x) = \int_0^l V^*(x,z)dz, \tag{19}$$

where $V_x^*(x,z) = k(x,z)$.

Note that since A is a diagonal matrix $A^T = A$. Taking $E^T = G^T A$ and substituting the optimal control (13) in (12), leads to the HJB equation representation given by

$$
\begin{aligned}
& H^* = 0 = Q(x) + \int_0^l V_x^{*T} f(x,z)dz - \int_0^l \frac{\partial V_x^{*T}}{\partial z} A \frac{\partial x}{\partial z} dz \\
& -\frac{1}{4} V_x^{*T}(l) E R^{-1} E^T V_x^*(l) + V_x^{*T}(0) P A x(0),
\end{aligned} \tag{20}
$$

where $V_x^*(x,z)$ is represented by $V_x^*(z)$ for brevity.

Since the system state x is necessary to implement the control policy (13), it is necessary to introduce an observer prior to control synthesis. Therefore, assuming that the system is observable [20], in the next section the design of a NN observer will be explained.

2.1. NEURAL NETWORK OBSERVER DESIGN

The objective in this section is to introduce an observer for the estimation of system states in the presence of unknown dynamics when states are available for measurement only in limited locations in spatial domain or at boundaries. The estimated states and approximated system dynamics can be subsequently used in the controller design. In order to proceed, the following assumption is introduced.

If the estimation of state x is denoted by $\hat{x}$, consider the observer with dynamics

$$\frac{\partial \hat{x}}{\partial t} = A\frac{\partial^2 \hat{x}}{\partial z^2} + \hat{f}(\hat{x}, z) + C^T(z)\Gamma(y - \int_0^l C(z)\hat{x}dz),$$
$$\frac{\partial \hat{x}}{\partial z}(l) = Gu, \qquad \frac{\partial \hat{x}}{\partial z}(0) = -P\hat{x}(0), \tag{21}$$
$$\hat{y}(t) = \int_0^l C(z)\hat{x}(z,t)dz,$$

where $\hat{f}$ is the NN approximation of f, $\hat{y}$ is the estimated output y, and $\Gamma \in \mathbb{R}^{n\times n}$ is a diagonal positive definite gain matrix.

Since f is a function over $X \times [0,l] \to [\mathcal{C}(0,l)]^n$, there exists a compact set $\Omega \subset X \times [0,l]$, such that for $(x,z) \in \Omega$ by choosing a set of smooth bounded and Lipschitz basis functions $\sigma_{fi} : [\mathcal{H}_2]^n \times [0,l] \to [\mathcal{C}(0,l)]^m$, the function $f(x,z)$ can be represented as [23]

$$f(x,z) = W_f^T \sigma_f(x,z) + \varepsilon_f(x,z), \tag{22}$$

where $W_f \in \mathbb{R}^{m\times n}$ represents NN identifier target weight matrix with bound $\|W_f\| \le W_{fM}$, the uniform bound of σ_f is denoted by σ_{fM} and $\varepsilon_f(x,z)$ is the approximation error which is also assumed to be bounded above such that $\|\varepsilon_f(x,z)\| \le \varepsilon_{fM}$ for all x and z. The estimation error bound ε_{fM} can be made arbitrarily small by increasing the number of basis functions [24].

Remark 1: Existence of compact set Ω can be deduced from the fact that as long as the domain D of PDE (1) is a bounded set satisfying the *cone condition*, any $\mathcal{L}_2$ bounded and closed set in the solution space X of PDE forms a compact set according to compact embedding theorem [25].

As may be expected, the approximated dynamics can be provided by

$$\hat{f}(\hat{x}, z) = \hat{W}_f^T \sigma_f(\hat{x}, z). \tag{23}$$

Subsequently, the NN tuning law for $\hat{W}_f$ is provided by

$$\dot{\hat{W}}_f = -\alpha_f \hat{W}_f + \int_0^l \sigma_f(\hat{x}, z)\tilde{y}^T dz, \tag{24}$$

where $\alpha_f > 0$ is the design parameter and $\tilde{y} = \int_0^l C(z)\tilde{x}dz$. The second term in (24) integrates the output estimation error $\tilde{y}$ with different basis functions σ_{fi} $(1 \le i \le m)$ as integral kernels. Therefore, if estimation $\hat{f}$ and consequently $\hat{x}$ in (21) is unacceptable, the second term in (24) will adjust the weights associated with σ_{fi} close to their desirable value.

Define the state estimation error $\tilde{x} = x - \hat{x}$ and its dynamics will be governed by

$$\begin{aligned}
&\frac{\partial \tilde{x}}{\partial t} = A\frac{\partial^2 \tilde{x}}{\partial z^2} + \tilde{W}_f \sigma_f(\hat{x}, z) + W_f^T(\sigma_f(x,z) - \sigma_f(\hat{x}, z)) \\
&+\varepsilon_f(x,z) - C^T(z)\Gamma\int_0^l C(z)\tilde{x}dz, \\
&\left.\frac{\partial \tilde{x}}{\partial z}\right|_{z=l} = 0, \qquad \left.\frac{\partial \tilde{x}}{\partial z}\right|_{z=0} = -P\tilde{x}(0),
\end{aligned} \tag{25}$$

where weight estimation error is given by $\tilde{W}_f = W_f - \hat{W}_f$. Noting $\dot{\tilde{W}}_f = -\dot{\hat{W}}_f$, the dynamics of NN identifier weight estimation error can be represented as

$$\dot{\tilde{W}}_f = \alpha_f \hat{W}_f - \int \sigma_f(x,z)\tilde{y}^T dz. \tag{26}$$

Assumption 2: There exists a diagonal positive definite matrix Γ such that for $\tilde{x}$ satisfying error dynamics (25), the quadratic term in terms of $\tilde{y}$ satisfies

$$\tilde{y}^T\Gamma\tilde{y} \geq 2\psi\|\tilde{x}\|_{\mathcal{L}_2}^2 . \tag{27}$$

where $\psi \geq L_f + \kappa$ with $\kappa > 0$.

The above assumption helps to prove a smaller less conservative bound for observation error. This assumption will not hold generally for all $x \in [\mathcal{H}_2(0,l)]^n$ but can be justified for estimation error of parabolic PDE as long as the nonlinear dynamics $f(x,z)$ and initial condition $x_0(z)$ are smooth enough. Since an estimate of system states is now available, in the next section the NDP approximate optimal control will be addressed.

2.2. NDP CONTROLLER DESIGN

By a simple study of equations (15) and (20), it would be clear that in order to find the optimal control policy, $V_x^*(x,z)$ should be found by solving equation (20) and then substituted in the control input. Equation (20) is generally a nonlinear quadratic partial integro-differential equation (PIDE) and therefore it has no closed form solution. Consequently, the objective is to find a suitable structure for estimation of $V^*(x,z)$. For this purpose, since $V^*(x,z)$ is an integral functional over $[\mathcal{H}_2]^n \times [0,l] \to \mathcal{C}_1[0,l]$, there exists a set of smooth bounded basis functions $\Phi_i : [\mathcal{H}_2]^n \times [0,l] \to \mathcal{C}_1[0,l]$, such that

$$V^*(x,z) = W_\Phi^T \Phi(x,z) + \varepsilon_V(x,z), \tag{28}$$

where $W_\Phi \in \mathbb{R}^r$ is the target NN weight vector with $\|W_\Phi\| \le W_{\Phi M}$ and the uniform bound of $\Phi(.,.)$ is denoted by Φ_M. It is assumed that the uniform norm of derivative of approximation error with respect to x, i.e. $\|\varepsilon_{Vx}\|$ and its second and third derivatives with respect to z, i.e. $\left\|\frac{\partial \varepsilon_{Vx}}{\partial z}\right\|$ and $\left\|\frac{\partial^2 \varepsilon_{Vx}}{\partial z^2}\right\|$ are bounded by ε_{VxM}, ε_{VxzM} and ε_{VxzzM} [24], respectively. The estimation error ε_V can be also made arbitrarily small by increasing the number of basis functions.

Assumption 3: The function $\Phi_x(.,z)$ is Lipschitz continuous with Lipschitz constant L_Φ i.e.

$$\|\Phi_x(x_1,z) - \Phi_x(x_2,z)\| \le L_\Phi \|x_1 - x_2\|_{\mathcal{L}_2}.$$

Therefore, the optimal control input can be represented as

$$u^* = -\frac{1}{2} R^{-1} E^T \Phi_x^T(x,l) W_\Phi + \varepsilon_u, \tag{29}$$

where $\varepsilon_u = \frac{-1}{2} R^{-1} E^T \varepsilon_{Vx}(x,l)$. Consequently, HJB equation can be represented in its approximated form as

$$\begin{aligned} H^* = 0 = Q(x) &+ \int_0^l W_\Phi^T \Phi_x(z) W_f^T \sigma_f(z) dz \\ &- \int_0^l W_\Phi^T \frac{\partial \Phi_x}{\partial z} A \frac{\partial x}{\partial z} dz - \frac{1}{4} W_\Phi^T \Phi_x(l) E R^{-1} E^T \Phi_x{}^T(l) W_\Phi \\ &+ W_\Phi^T \Phi_x(0) P A x(0) + \varepsilon_H, \end{aligned} \tag{30}$$

where $\Phi(z)$ and $\sigma_f(z)$ are the abbreviation for $\Phi(x,z)$ and $\sigma_f(x,z)$, respectively and ε_H can be derived as

$$\begin{aligned} \varepsilon_H = &\int_0^l \varepsilon_{Vx}^T(z) f(x,z) dz + \int_0^l W_\Phi \Phi_x^T(z) \varepsilon_f(x,z) dz \\ &- \int_0^l \frac{\partial \varepsilon_{Vx}^T}{\partial z} A \frac{\partial x}{\partial z} dz - \frac{1}{4} \varepsilon_{Vx}^T(l) E R^{-1} E^T \varepsilon_{Vx}(l) \\ &- \frac{1}{2} W_\Phi^T \Phi_x(l) E R^{-1} E^T \varepsilon_{Vx}(l) + \varepsilon_{Vx}^T(0) P A x(0). \end{aligned} \tag{31}$$

Since the state vector is unavailable for measurement, and if $V^*(x,z)$ is estimated by

$$\hat{V}^*(x,z) = \hat{W}_\Phi^T \Phi(\hat{x},z), \tag{32}$$

the approximated HJB can be represented as

$$\begin{aligned}\hat{H} &= Q(\hat{x}) + \int_0^l \hat{W}_\Phi^T \Phi_x(\hat{x},z)\hat{W}_f^T \sigma_f(z)dz \\ &- \int_0^l \hat{W}_\Phi^T \frac{\partial \Phi_x}{\partial z} A \frac{\partial \hat{x}}{\partial z} dz - \frac{1}{4}\hat{W}_\Phi^T \Phi_x(\hat{x},l) E R^{-1} \\ &E^T {\Phi_x}^T(\hat{x},l)\hat{W}_\Phi + \hat{W}_\Phi^T \Phi_x(\hat{x},0) P A \hat{x}(0).\end{aligned} \tag{33}$$

The control policy becomes

$$\hat{u} = -\frac{1}{2} R^{-1} E^T \Phi_x^T(\hat{x},l)\hat{W}_\Phi. \tag{34}$$

The value function NN weight tuning law is chosen as

$$\dot{\hat{W}}_\Phi = -\alpha_1 \frac{\omega_N}{\zeta_N^2}\hat{H} - \alpha_2 \hat{W}_\Phi - \alpha_3 \hat{W}_\Phi \left\|\hat{W}_\Phi\right\|^2, \tag{35}$$

where $\omega_N = \int_0^l \Phi_x(\hat{x},z)\hat{W}_f^T \sigma_f(z)dz - \int_0^l \frac{\partial \Phi_x}{\partial z}(\hat{x},z) A \frac{\partial \hat{x}}{\partial z} dz - \frac{1}{2}\Phi(\hat{x},l) E R^{-1} E^T \Phi^T(\hat{x},l)\hat{W}_\Phi + \Phi_x(\hat{x},0) P A \hat{x}(0)$,

$\zeta_N = c_1 \|\hat{x}\|_{\mathcal{L}_2} + c_2 \left\|\frac{\partial \hat{x}}{\partial z}\right\|_{\mathcal{L}_2} + c_3$ and α_1, α_2, α_3, c_1, c_2 and c_3 are suitable positive design constants. Here the first term in tuning law (35) reduces the approximated Hamiltonian while the other two terms are necessary for the stability of the system based on Lyapunov criterion that will be demonstrated in the proof. Under proposed control policy and adaptive tuning laws, it will be shown in Section 3 that closed-loop system will be ultimately bounded (UB).

The presented output feedback controller can be extended to the case of more involved Dirichlet boundary condition. This will be explained in the next section.

2.3. DIRICHLET BOUNDARY CONTROL

If the boundary conditions for PDE (1) is modified to

$$x(0,t) = Gu, \qquad \left.\frac{\partial x}{\partial z}\right|_{z=0} + Px(0,t) = 0, \tag{36}$$

where matrices G and P are defined in the same manner as Neumann boundary control, the control policy (15) and HJB equation (20) should also be changed accordingly. The following assumption is necessary before proceeding.

Assumption 4: $\frac{\partial x}{\partial z}|_{z=l}$ is available. Equivalently, the domain that output is measured includes the vicinity of boundary $z=l$.

In order to find the Hamiltonian, consider equation (11) and take integration by parts twice to obtain

$$\begin{aligned} H &= Q(x) + u^T Ru + \int_0^l V_x^{*T} f(x,z)dz + V_x^{*T}(x,l)A\frac{\partial x}{\partial z}|_{z=l} \\ &+ V_x^{*T}(x,0)APx(0) - \frac{\partial V_x^{*T}}{\partial z}|_{z=l} AGu + \frac{\partial V_x^{*T}}{\partial z}|_{z=0} Ax(0) \\ &- \int_0^l \frac{\partial^2 V_x^{*T}}{\partial z^2} Axdz, \end{aligned} \tag{37}$$

where $\frac{\partial V_x^*}{\partial z}$ and $\frac{\partial^2 V_x^*}{\partial z^2}$ are the first and second spatial derivatives of V_x^* with respect to z. Similar to Neumann boundary control, the Fréchet derivative of this equation with respect to control policy u should equal to zero. Therefore,

$$2Ru - G^T A\frac{\partial V_x^*}{\partial z}|_{z=l} = 0 \Rightarrow u^* = \frac{1}{2}R^{-1}G^T A\frac{\partial V_x^*}{\partial z}|_{z=l}. \tag{38}$$

Substituting the optimal control in (37), the HJB equation for DPS under Dirichlet boundary condition can be represented in the form

$$\begin{aligned} H^* = 0 &= Q(x) + \int_0^l V_x^{*T} f(x,z)dz - \int_0^l \frac{\partial^2 V_x^{*T}}{\partial z^2} Axdz \\ &- \frac{1}{4}V_{xz}^{*T}(l)ER^{-1}E^T V_{xz}^*(l) + V_x^{*T}(x,l)A\frac{\partial x}{\partial z}|_{z=l} \\ &+ V_x^{*T}(x,0)APx(0) + \frac{\partial V_x^{*T}}{\partial z}|_{z=0} Ax(0) \end{aligned} \tag{39}$$

Analogously, if V^* is represented in approximated form of (28), the optimal control would be

$$u^* = \frac{1}{2}R^{-1}E^T \frac{\partial \Phi_x^T}{\partial z}|_{z=l} W_\Phi + \varepsilon_u, \tag{40}$$

where $\varepsilon_u = \frac{1}{2}R^{-1}E^T \varepsilon_{Vxz}(x,l)$. The following assumption is equivalently to Assumption 2 for Neumann boundary control and necessary.

Assumption 5: The function $\frac{\partial \Phi_x}{\partial z}$ is Lipschitz continuous with Lipschitz constant $L_{\Phi z}$ i.e.

$$\left\|\frac{\partial \Phi_x(x_1)}{\partial z}-\frac{\partial \Phi_x(x_2)}{\partial z}\right\| \le L_{\Phi z}\left\|x_1-x_2\right\|_{\mathcal{L}_2} . \tag{41}$$

Accordingly, the HJB equation becomes

$$\begin{aligned} H^* = 0 = Q(x) + \int_0^l W_\Phi^T \Phi_x(z)(W_f^T \sigma_f(z))dz \\ -\int_0^l W_\Phi^T \frac{\partial^2 \Phi_x}{\partial z^2} Axdz - \frac{1}{4} W_\Phi^T \frac{\partial \Phi_x}{\partial z}\Big|_{z=l} ER^{-1}E^T \frac{\partial \Phi_x^T}{\partial z}\Big|_{z=l} W_\Phi \\ + W_\Phi^T \Phi_x(0) PAx(0) + W_\Phi^T \Phi_x(l) A \frac{\partial x}{\partial z}\Big|_{z=l} \\ + W_\Phi \frac{\partial \Phi_x^T}{\partial z}\Big|_{z=0} Ax(0) + \varepsilon_H , \end{aligned} \tag{42}$$

where ε_H is derived as

$$\begin{aligned} \varepsilon_H = \int_0^l \varepsilon_{Vx}^T(z) f(x,z) dz + \int_0^l W_\Phi \Phi_x^T(z) \varepsilon_f(x,z) dz \\ -\int_0^l \frac{\partial^2 \varepsilon_{Vx}^T}{\partial z^2} Axdz - \frac{1}{4} \frac{\partial \varepsilon_{Vx}^T}{\partial z}\Big|_{z=l} ER^{-1}E^T \frac{\partial \varepsilon_{Vx}^T}{\partial z}\Big|_{z=l} \\ -\frac{1}{2} W_\Phi^T \frac{\partial \Phi_x}{\partial z}\Big|_{z=l} ER^{-1}E^T \frac{\partial \varepsilon_{Vx}}{\partial z}\Big|_{z=l} + \varepsilon_{Vx}^T(0) PAx(0) \end{aligned} \tag{43}$$

$$+\varepsilon_{Vx}^T(l) A \frac{\partial x}{\partial z}\Big|_{z=l} + \frac{\partial \varepsilon_{Vx}^T}{\partial z}\Big|_{z=0} Ax(0).$$

If the value functional is estimated as

$$\hat{V}^*(x,z) = \hat{W}_\Phi^T \Phi(\hat{x}, z), \tag{44}$$

the approximated HJB would be

$$\begin{aligned} Q(\hat{x}) + \int_0^l \hat{W}_\Phi^T \Phi_x(\hat{x},z)(\hat{W}_f^T \sigma_f(z))dz \\ -\int_0^l \hat{W}_\Phi^T \frac{\partial^2 \Phi_x(\hat{x},z)}{\partial z^2} A\hat{x}dz - \frac{1}{4} \hat{W}_\Phi^T \frac{\partial \Phi_x(\hat{x},z)}{\partial z}\Big|_{z=l} ER^{-1}E^T \\ \frac{\partial \Phi_x^T(\hat{x},z)}{\partial z}\Big|_{z=l} \hat{W}_\Phi + \hat{W}_\Phi^T \Phi_x(\hat{x},0) PA\hat{x}(0) \\ + \hat{W}_\Phi^T \Phi_x(\hat{x},z) A \frac{\partial \hat{x}}{\partial z}\Big|_{z=l} + \hat{W}_\Phi \frac{\partial \Phi_x^T(\hat{x},z)}{\partial z}\Big|_{z=0} A\hat{x}(0), \end{aligned} \tag{45}$$

where $\hat{x}$ is the estimated state described in Section 2.1. Therefore, the control input can be represented by

$$\hat{u} = \frac{1}{2} R^{-1} E^T \frac{\partial \Phi_x^T(\hat{x},z)}{\partial z}\Big|_{z=l} \hat{W}_\Phi . \tag{46}$$

The value functional weight tuning law for Dirichlet boundary control is defined as

$$\dot{\hat{W}}_\Phi = -\alpha_1 \frac{\omega_D}{\zeta_D^2} \hat{H} - \alpha_2 \hat{W}_\Phi - \alpha_3 \hat{W}_\Phi \left\| \hat{W}_\Phi \right\|^2, \tag{47}$$

where

$$\begin{aligned} \omega_D = & \int_0^l \Phi_x(\hat{x}, z) \hat{W}_f^T \sigma_f(z) dz - \int_0^l \frac{\partial^2 \Phi_x(\hat{x}, z)}{\partial z^2} A\hat{x} dz \\ & - \frac{1}{2} \frac{\partial \Phi_x(\hat{x}, z)}{\partial z} \Big|_{z=l} E R^{-1} E^T \frac{\partial \Phi_x^T(\hat{x}, z)}{\partial z} \Big|_{z=l} \hat{W}_\Phi \\ & + \Phi_x(\hat{x}, 0) P A \hat{x}(0) + \Phi_x(\hat{x}, l) A \frac{\partial \hat{x}}{\partial z} \Big|_{z=l} \\ & + \frac{\partial \Phi_x^T(\hat{x}, z)}{\partial z} \Big|_{z=0} A x(0) \end{aligned} \tag{48}$$

and $\zeta_N = c_1 \left\| \hat{x} \right\|_{\mathcal{L}_2} + c_2 \left\| \frac{\partial \hat{x}}{\partial z} \right\|_{\mathcal{L}_2} + c_3 \left\| \frac{\partial x}{\partial z} \Big|_{z=l} \right\|$ with $c_1, ..., c_3$ and $\alpha_1, \alpha_2, \alpha_3$ being appropriate positive design parameters.

In the next section, the ultimate boundedness of the closed-loop system with the developed boundary control policies will be illustrated for Neumann and Dirichlet boundary conditions, respectively.

3. STABILITY ANALYSIS

System closed-loop stability will be examined by using Lyapunov criterion. First, it will be shown in Theorem 1 that in presence of bounded input, the observer dynamics and estimation error will be bounded. In Lemma 1, a bound will be found for Hamiltonian reconstruction error ε_H when Neumann boundary control policy is implemented. This bound will be used later in closed-loop stability proof. Consequently, Theorem 2 will address the ultimate boundedness (UB) of all closed-loop states for Neumann boundary control condition. Finally, Lemma 2 and Theorem 3 will state similar results when Dirichlet boundary control policy is pursued.

Theorem 1 *(Boundedness of NN observer)***:** Let the initial NN identifier weight estimation error $\tilde{W}_f$ and state estimation error $\tilde{x}$ be residing in a compact set $\mho_1$ and $\mho_2$, and the proposed NN observer and NN weight tuning law be provided by (21) and (24), respectively. In the presence of bounded inputs, there exists a positive definite observer gain matrix Γ and tuning parameter $\alpha_f > 0$ for the identifier weight update law such that the state estimation error $\tilde{x}$ and weight estimation error $\tilde{W}_f$ are all *UB*.

Proof: Refer to Appendix.

Lemma 1: For Neumann boundary control policy, the following inequality

$$\|\varepsilon_H\| \leq \xi_{NM}\zeta_N + c_{H1}\|\tilde{x}\|_{\mathcal{L}_2} + c_{H2}\left\|\frac{\partial \tilde{x}}{\partial z}\right\|_{\mathcal{L}_2}, \quad (49)$$

holds where c_{H1}, c_{H2} and ξ_{NM} are positive constants depending on approximation reconstruction errors.

Proof: Refer to Appendix.

Theorem 2 *(Performance of the output feedback NN controller for Neumann boundary condition)***:** Consider the DPS given by (1) and let the proposed NN observer and NN weight tuning law be provided by (21) and (24), respectively. Moreover, let control policy and tuning law for value function weights be provided by (34) and (35), respectively where $0 < \alpha_1 < 1$, $\alpha_2 > \alpha_3$ and $\alpha_3 > \theta_{\alpha_3}^N$ with $\theta_{\alpha_3}^N$ being a positive constant. Consequently, provided an initial admissible control, the system state x, state estimation error $\tilde{x}$ and weight estimation errors $\tilde{W}_f$ and $\tilde{W}_\Phi$ are all UB.

Proof: Refer to Appendix.

Lemma 2: For Dirichlet boundary control policy, the following inequality

$$|\varepsilon_H| \le \xi_{DM}\zeta_D + c_{H1}\|\tilde{x}\|_{\mathcal{L}_2} + c_{H2}\left\|\frac{\partial \tilde{x}}{\partial z}\right\|_{\mathcal{L}_2} \tag{50}$$

holds where c_{H1}, c_{H2} and ξ_{DM} are positive constants depending on approximation reconstruction errors.

Proof: Refer to the Appendix.

Theorem 2 *(Performance of the output feedback NN controller for Dirichlet boundary condition)***:** Consider the DPS given by (1) and let the proposed NN observer and NN weight tuning law be provided by (21) and (24), respectively. Moreover, let control policy and tuning law for value function weights be provided by (46) and (47), respectively where $0<\alpha_1<1$, $\alpha_2>\alpha_3$ and $\alpha_3>\theta_{\alpha_3}^D$ with $\theta_{\alpha_3}^D$ being a positive constant. Consequently, provided an initial admissible control, the system state x, state estimation error $\tilde{x}$ and weight estimation errors $\tilde{W}_f$ and $\tilde{W}_\Phi$ are all UB.

Proof: Refer to the Appendix.

In the next section, the performance of proposed NDP based NN controller is examined by simulation on a semi-linear diffusion reaction process.

4. SIMULATIONS

The controller developed in Section 0 was simulated using MATLAB on a diffusion reaction process to verify its performance. Next, Simulation results will be presented in separate parts for Neumann and Dirichlet boundary conditions.

4.1. NEUMANN BOUNDARY CONTROL

Consider the nonlinear coupled semilinear reaction-diffusion system given by

$$\begin{aligned}
&\frac{\partial x_1}{\partial t} = \frac{\partial^2 x_1}{\partial z^2} + \beta_{T_1} e^{-\gamma/(1+x_1)} x_2 + \frac{\beta_{\alpha 1}}{1+\vartheta(z-z_0)^2} x_2 \\
&\frac{\partial x_2}{\partial t} = \frac{\partial^2 x_2}{\partial z^2} + \beta_{T_2} e^{-\gamma/(1+x_1)} x_2 + \frac{\beta_{\alpha 2}}{1+\vartheta(z-z_0)^2} x_1 \\
&\left.\frac{\partial x}{\partial z}\right|_{z=0} = 5x(z,0), \ \left.\frac{\partial x}{\partial z}\right|_{z=1} = u \quad x(z,0) = \begin{bmatrix} sin(\pi z) \\ 2 \end{bmatrix},
\end{aligned} \tag{51}$$

where $u = [u_1, u_2]^T$ is the control input that exists only at the boundary $z = 1$. Comparing (51) to dynamics (1) it will be deduced that $A = G = I_{2\times 2}$ and $P = -5I_{2\times 2}$. The following values were given to process parameters: $\beta_{T_1} = 15, \ \beta_{T_2} = 20, \ \beta_{\alpha_1} = 8, \ \beta_{\alpha_2} = 12, \ \gamma = 5.0$, $\vartheta = 4$ and $z_0 = 0.5.$ It was verified that the open loop system was highly unstable when these values for system parameters were used.

The cost functional that should be minimized was chosen to be

$$V(x) = \int_0^\infty \{\int_0^1 x^T x dz + u^2\} dt. \tag{52}$$

The system output is defined as $y(t) = \int_0^1 C(z) x(z,t) dz$ where $C(z)$ is a diagonal 2×2 matrix with $C_{i,i}(z), \ i = 1,2$ defined as

$$C_{i,i}(z) = \begin{cases} \delta(z-z_0) & \text{if } z_0 \in \{0, 0.5, 1\} \\ 0 & \text{Otherwise} \end{cases}. \tag{53}$$

In order to estimate the states, the observer gain matrix is chosen as $\Gamma = 100 I_{2\times 2}$. The NN activation functions for identifying $f(x,z)$ were chosen as

$$\sigma_k(\hat{x}, z) = \frac{1}{1+\lambda(z-z_i)^2} \varphi_j(\hat{x}) \ , 1 \le k \le 18 \tag{54}$$

with $z_i \in \{0.2, 0.5, 0.8\}$, $1 \le i \le 3$, $\lambda = 4$, $\varphi_j(x) = \rho(\{1, x_1, x_1^2, x_2, x_2^2, x_1 x_2\})$, $1 \le j \le 6$, and $\rho(.) = \text{tansig}(.)$. The identifier update parameter was chosen as $\alpha_f = 10$. Fifteen basis functions were chosen to approximate V^* with following structure

$$\Phi_k(\hat{x}, z) = \frac{0.5}{1 + \upsilon(\overline{z} - \overline{z}_i)^2} \int_{\Delta_i} \psi_j(\hat{x}) dz \tag{55}$$

where $1 \le k \le 15$, $\psi_j(x) \in \rho(\{x_1^2, x_1^4, x_2^2, x_2^4, x_1^2 x_2^2\})$, $1 \le j \le 5$, $z_i \in \{0.2, 0.5, 0.8\}$, $\Delta_i \in \{(0, 0.35), (0.35, 0.65), (0.65, 1)\}$, $1 \le i \le 3$ and $\upsilon = 8$.

The value functional update parameters were chosen to be $\alpha_1 = 0.9$, $\alpha_2 = 0.2$, $\alpha_3 = 0.15$, $c_3 = 1$ and $c_1 = c_2 = 0.2$. The initial admissible control was chosen as $-45, -2^T$.

Note that design conditions are satisfied with these parameters, i.e. $\alpha_2 > \alpha_3$ and $\alpha_3 > \theta_{\alpha_3}^N \simeq 0.1$. The controller was subsequently implemented on the system using MATLAB with spatial step size of $dz = 0.05$. The NN weights were updated and the controller was applied in real-time setting with step size of 1 msec in order to have a convincing performance and computational rate.

As shown in Figure 4.1.a the proposed controller can stabilize the system to zero in less than 50 milliseconds. The control inputs which were the spatial derivatives of the states at the boundary $z = 1$ are shown along with HJB error in Figure 4.2. Since the system was unstable, its regulation required a considerable control effort at the beginning of simulation. Moreover, faster convergence of HJB error to zero compared to control input verifies that the adaptive law (35) is capable of reducing the HJB error. Finally, Figure 4.3 shows the performance of observer for different values of Γ. As shown, increasing the observer gain reduce the bound $\|e\|_{\mathcal{L}_2} = \|x - \hat{x}\|_{\mathcal{L}_2}$.

4.2. DIRICHLET BOUNDARY CONTROL

Consider the same reaction-diffusion system (51) but with boundary conditions:

$$\left.\frac{\partial x}{\partial z}\right|_{z=0} = 5x(z, 0), \qquad \left.\frac{\partial x}{\partial z}\right|_{z=1} = u \tag{56}$$

The process parameters were chosen as $\beta_{T_1} = 15$, $\beta_{T_2} = 20$, $\beta_{\alpha_1} = 12$, $\beta_{\alpha_2} = 15$, $\gamma = 5.0$, $\vartheta = 4$ and $z_0 = 0.5$. It was verified that with these parameter values the system is still highly unstable under Dirichlet boundary control condition.

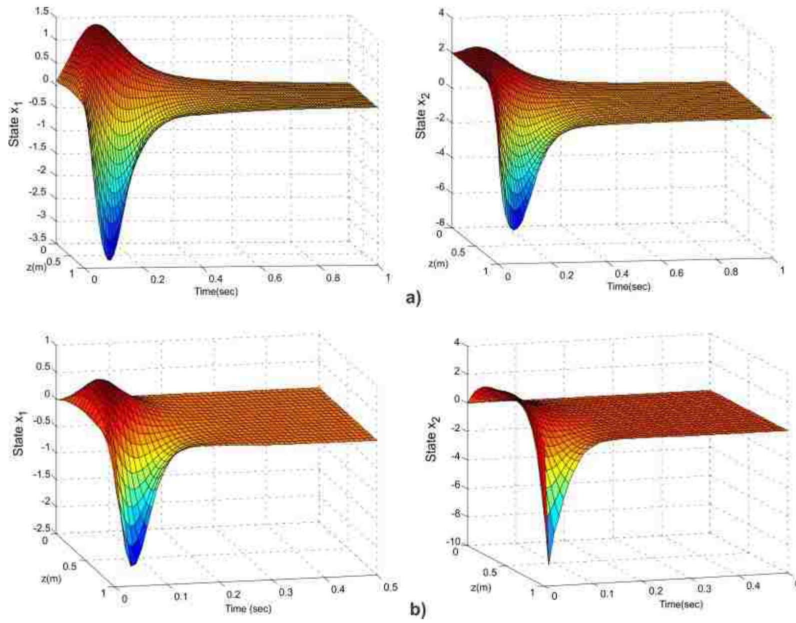

Figure 4.1. State profile history. a) Profile history of states x_1 and x_2 for Neumann boundary control condition. b) Profile history of states for Dirichlet boundary control condition.

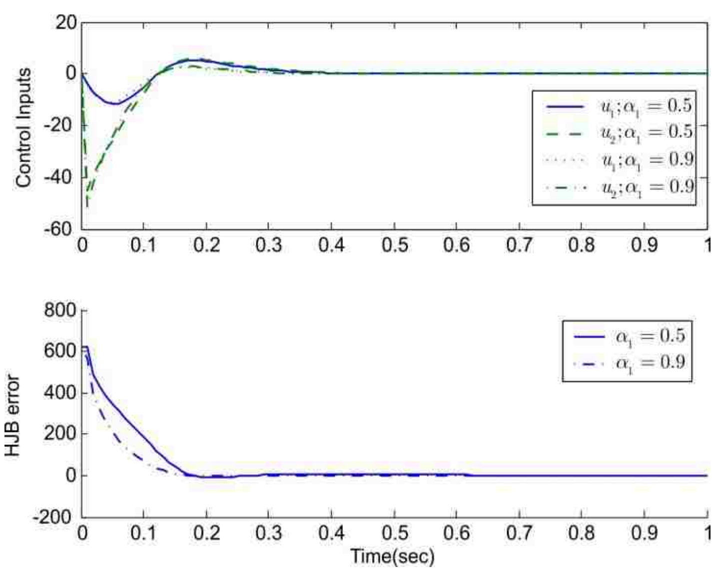

Figure 4.2. Convergence of control inputs and HJB error for Neumann boundary control.

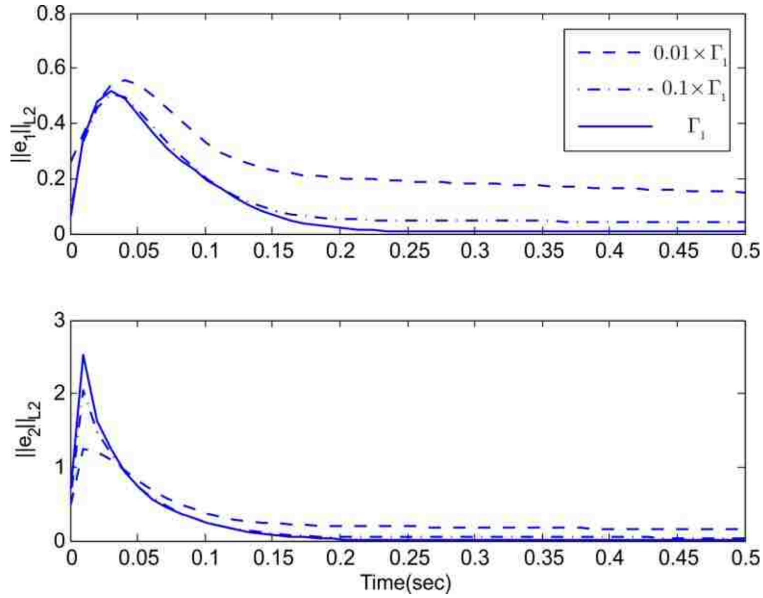

Figure 4.3. Performance of observer for different values of gain Γ .

The cost functional was expressed so as to compare with (52). The same update parameters were chosen for the observer as in Neumann boundary control. The basis functions for approximating V^* were chosen so as to compare with (55). The value functional update parameters were accordingly chosen to be $\alpha_1 = 0.3,\ \ \alpha_2 = 0.2,\ \ \alpha_3 = 0.1$, $c_3 = 1$, $c_1 = 0.2$, $c_2 = 0.02$. The initial admissible control was chosen as $-8.5,-0.2^{T}$. Note that design conditions are satisfied with these parameters, i.e. $\alpha_2 > \alpha_3$ and $\alpha_3 > \theta_{\alpha_3}^{D} \simeq 0.04$.

As shown in Figure 4.1.b, the proposed controller is capable of stabilizing the system to zero in less than 50 msecs. The time profiles of control inputs and HJB error are depicted in Figure 4.4. Similar to Figure 4.2 for Neumann boundary control, the HJB error converges more rapidly to zero when the update parameter α_1 is increased.

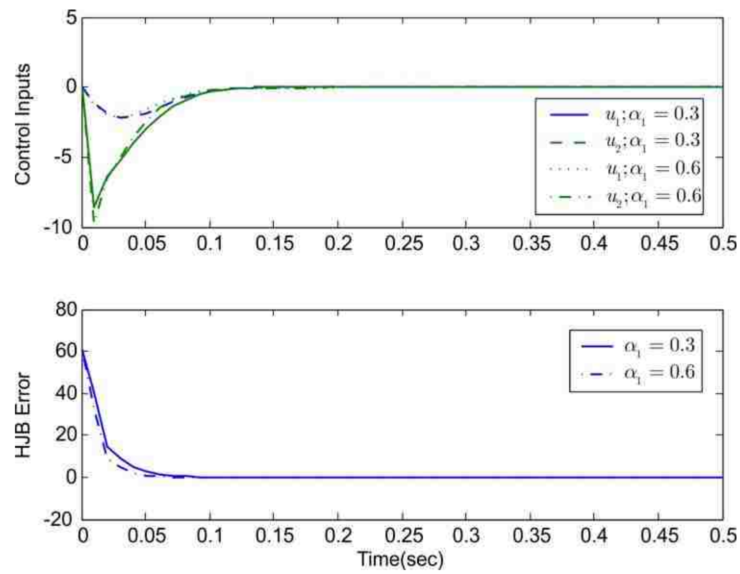

Figure 4.4. Convergence of control inputs and HJB error for Dirichlet control.

5. CONCLUSIONS

This paper developed a novel NN-based output feedback near optimal boundary control scheme for DPS governed by semi-linear coupled parabolic PDE under Neumann boundary control condition without any model reduction prior to control design. By defining an integral cost functional and formulating the HJB equation based on calculus in infinite dimensional state space, a closed-form of optimal control policy was derived. The proposed adaptive NN framework made online approximation of optimal control policy along system trajectories possible when the PDE dynamics were partially unknown. Lyapunov stability analysis indicated that the error between approximated and truly optimal weights and state trajectories would remain UB. The performance of proposed observer and controller in estimating states, stabilizing the system and reducing the HJB error was successfully verified on a coupled semi-linear diffusion process.

APPENDIX

In order to avoid long derivations in the proof, we first define $x_z = \frac{\partial x}{\partial z}$, $x_{zz} = \frac{\partial^2 x}{\partial z^2}$, $\Phi_x = \frac{\partial \Phi}{\partial x}$, $\Phi_{xz} = \frac{\partial \Phi_x}{\partial z}$, $\varepsilon_{Vxz} = \frac{\partial \varepsilon_{Vx}}{\partial z}$ and $\varepsilon_{Vxzz} = \frac{\partial^2 \varepsilon_{Vx}}{\partial z^2}$. First the proof for Theorem 1 will be provided.

Proof of Theorem 1: In order to prove stability of state feedback identifier, consider the Lyapunov function

$$L_c = \frac{1}{2}\mu_c \int_0^l \tilde{x}^T \tilde{x} dz + \frac{1}{2}\mu_c tr(\tilde{W}_f^T \tilde{W}_f), \tag{57}$$

where μ_c is a positive scalar. Taking its derivative $\dot{L}_c$ and substituting error dynamics (25) and (26) yields

$$\begin{aligned} &\dot{L}_c \le -\mu_c \tilde{y}^T \Gamma \tilde{y} + \mu_c \int_0^l \tilde{x}^T A \tilde{x}_{zz} dz + \mu_c \int_0^l \tilde{x}^T \varepsilon_f(x,z) dz \\ &+\mu_c \int_0^l \tilde{x}^T \tilde{W}_f^T \sigma_f(\hat{x},z) dz + \mu_c \int_0^l \tilde{x}^T W_f^T (\sigma_f(\overline{x},\overline{z}) \\ &-\sigma_f(\hat{\overline{x}},\overline{z}))dz + \mu_c tr(\tilde{W}_f^T \alpha_f (W_f - \tilde{W}_f)) \\ &-\mu_c tr(\int_0^l \tilde{W}_f \sigma_f(x,z) \tilde{y}^T dz). \end{aligned} \tag{58}$$

Taking integration by parts, substituting PDE dynamics (25) for state estimation error $\tilde{x}$ and using Poincare inequality ($\int_0^l w^2 dz \le 2lw^2(1) + 2l^2 \int_0^l w_z^2 dz$) yields

$$\begin{aligned} &\int_0^l \tilde{x}^T A \tilde{x}_{zz} dz = \tilde{x}^T(0) AP \tilde{x}(0) - \int_0^l \tilde{x}_z^T A \tilde{x}_z dz \le -\frac{1}{4l^2} \\ &\int_0^l \tilde{x}^T A \tilde{x} dz - \frac{1}{2}\lambda_{\min}(A) \left\| \tilde{x}_z \right\|_{\mathcal{L}_2} + \lambda_{\max}(AP) \left\| \tilde{x}(0) \right\|^2. \end{aligned} \tag{59}$$

Note that matrix AP is negative definite. Moreover, using Young's inequality ($\beta_1\beta_2 \le \frac{1}{2\gamma}\beta_1^2 + \frac{\gamma}{2}\beta_2^2$) yields

$$\begin{aligned} &\mu_c \int_0^l \tilde{x}^T \varepsilon_f(x,z) dz \le \mu_c \sqrt{l} \varepsilon_{fM} \left\| \tilde{x} \right\|_{\mathcal{L}_2} \le \mu_c \frac{\gamma_1}{2} \left\| \tilde{x} \right\|_{\mathcal{L}_2}^2 \\ &+\mu_c \frac{l}{2\gamma_1} \varepsilon_{fM}^2 \end{aligned} \tag{60}$$

and

$$\mu_c \int_0^l \tilde{x}^T W_f^T (\sigma_f(\bar{x},\bar{z}) - \sigma_f(\hat{\bar{x}},\bar{z}))dz \le \mu_c 2\sqrt{l}\,\|\tilde{x}\|_{\mathcal{L}_2} \tag{61}$$

$$W_{fM}\sigma_{fM} \le \mu_c \frac{\gamma_2}{2}\|\tilde{x}\|_{\mathcal{L}_2}^2 + \mu_c \frac{2l}{\gamma_2} W_{fM}^2 \sigma_{fM}^2 .$$

Moreover,

$$\mu_c tr(\int_0^l \tilde{W}_f \sigma_f(x,z)\tilde{y}^T dz) \le \mu_c \frac{1}{2}\|\tilde{W}_f\|^2 + \mu_c \frac{1}{2} l \sigma_{fM}^2 \|\tilde{y}\|_{\mathcal{L}_2}^2 .$$

Therefore,

$$\begin{aligned} \dot{L}_c \le & -\mu_c(\frac{1}{4l^2}\lambda_{\min}(A) - \frac{\gamma_1+\gamma_2}{2})\|\tilde{x}\|_{\mathcal{L}_2}^2 - \frac{1}{2}\lambda_{\min}(A)\|\tilde{x}_z\|_{\mathcal{L}_2} \\ & +\lambda_{\max}(AP)\|\tilde{x}(0)\|^2 - \frac{1}{2}\mu_c\alpha_f\|\tilde{W}_f\|^2 - \mu_c\lambda_{\min}(\Gamma)\|\tilde{y}\|_{\mathcal{L}_2}^2 \\ & +\frac{1}{2}\mu_c\alpha_f\|W_f\|^2 + \frac{l}{2\gamma_1}\mu_c\varepsilon_{fM}^2 + \mu_c\frac{2l}{\gamma_2}W_{fM}^2\sigma_{fM}^2 \\ & +\mu_c\frac{1}{2}\|\tilde{W}_f\|^2 + \mu_c\frac{1}{2}l\sigma_{fM}^2\|\tilde{y}\|_{\mathcal{L}_2}^2 , \end{aligned} \tag{62}$$

Therefore choosing $\lambda_{\min}(\Gamma) > \frac{1}{2} l\sigma_{fM}^2$, $\alpha_f > 1$, $\gamma_1 + \gamma_2 < \frac{1}{2l^2}\lambda_{\min}(A)$ and defining $\psi_1 = \frac{1}{8l^2}\lambda_{\min}(A) - \frac{\gamma_1+\gamma_2}{2}$, $\psi_2 = \alpha_f - 1$, $\psi_3 = \frac{1}{2}\lambda_{\min}(A)$, $\psi_4 = \lambda_{\max}(AP)$ one has

$$\begin{aligned} \dot{L}_c \le & -\mu_c\psi_1\|\tilde{x}\|_{\mathcal{L}_2}^2 - \frac{1}{2}\mu_c\psi_2\|\tilde{W}_f\|^2 - \frac{1}{2}\mu_c\psi_3\|\tilde{x}_z\|_{\mathcal{L}_2}^2 \\ & -\mu_c\psi_4\|\tilde{x}(0)\|^2 + \varepsilon_c , \end{aligned} \tag{63}$$

where $\varepsilon_c = \frac{1}{2}\mu_c\alpha_f W_{fM}^2 + \frac{l}{2\gamma_1}\mu_c\varepsilon_{fM}^2 + \mu_c\frac{2l}{\gamma_2}W_{fM}^2\sigma_{fM}^2$. Therefore the identifier dynamics are UB.

Remark 2: The aforementioned bound ε_c can be made smaller if Assumption 2 holds. Accordingly, instead of equations (60) and (61) we have:

$$\mu_c \int_0^l \tilde{x}^T \{W_f^T(\sigma_f(\bar{x},\bar{z}) - \sigma_f(\hat{\bar{x}},\bar{z})) + \varepsilon_f\}dz \le \mu_c L_f \|\tilde{x}\|_{\mathcal{L}_2}^2 . \tag{64}$$

Therefore, by combining the terms, instead of (62) we have

$$\dot{L}_c \le -\mu_c(\kappa + \frac{1}{8l^2}\lambda_{\min}(A))\|\tilde{x}\|_{\mathcal{L}_2}^2 - \frac{1}{2}\lambda_{\min}(A)\|\tilde{x}_z\|_{\mathcal{L}_2}$$
$$+\lambda_{\max}(AP)\|\tilde{x}(0)\|^2 - \frac{1}{2}\mu_c\alpha_f\|\tilde{W}_f\|^2 - \frac{1}{2}\mu_c\lambda_{\min}(\Gamma)\|\tilde{y}\|_{\mathcal{L}_2}^2$$
$$+\frac{1}{2}\mu_c\alpha_f\|W_f\|^2 + \mu_c\frac{1}{2}\|\tilde{W}_f\|^2 + \mu_c\frac{1}{2}l\sigma_{fM}^2\|\tilde{y}\|_{\mathcal{L}_2}^2, \tag{65}$$

Therefore, choosing $\lambda_{\min}(\Gamma) > l\sigma_{fM}^2$, $\alpha_f > 1$, and defining $\psi_1 = \frac{\pi^2}{8}\lambda_{\min}(A) + \kappa$, $\psi_2 = \alpha_f - 1$, $\psi_3 = \frac{1}{2}\lambda_{\min}(A)$, $\psi_4 = \lambda_{\max}(AP)$ one has

$$\dot{L}_c \le -\mu_c\psi_1\|\tilde{x}\|_{\mathcal{L}_2}^2 - \frac{1}{2}\mu_c\psi_2\|\tilde{W}_f\|^2 - \frac{1}{2}\mu_c\psi_3\|\tilde{x}_z\|_{\mathcal{L}_2}^2$$
$$-\mu_c\psi_4\|\tilde{x}(0)\|^2 + \varepsilon_c, \tag{66}$$

where $\varepsilon_c = \frac{1}{2}\mu_c\alpha_f W_{fM}^2$.

Proof of Lemma 1: Using integration by parts, one has

$$\varepsilon_H = \int_0^l \varepsilon_{Vx}^T(z)f(x,z)dz + \int_0^l W_\Phi^T\Phi_x(z)\varepsilon_f(x,z)dz +$$
$$\int_0^l \varepsilon_{Vxzz}^T Axdz - \varepsilon_{Vxz}^T\big|_{z=l} Ax(l) + \varepsilon_{Vxz}^T\big|_{z=0} Ax(0) \tag{67}$$
$$-\frac{1}{4}\varepsilon_{Vx}^T(l)ER^{-1}E^T\varepsilon_{Vx}(l) - \frac{1}{2}W_\Phi^T\Phi_x(l)ER^{-1}E^T\varepsilon_{Vx}(l)$$
$$+\varepsilon_{Vx}(0)PAx(0).$$

Therefore,

$$|\varepsilon_H| \le \left|\int_0^l \varepsilon_{Vx}^T(z)f(x,z)dz\right| + \left|\int_0^l W_\Phi\Phi_x^T(z)\varepsilon_f(x,z)dz\right|$$
$$+\left|\int_0^l \varepsilon_{Vxzz}^T Axdz\right| + \left|\varepsilon_{Vxz}^T\big|_{z=l} Ax(l)\right| + \left|\varepsilon_{Vxz}^T\big|_{z=0} Ax(0)\right| \tag{68}$$
$$+\left|\frac{1}{4}\varepsilon_{Vx}^T(l)ER^{-1}E^T\varepsilon_{Vx}(l)\right| + \left|\frac{1}{2}W_\Phi^T\Phi_x(l)ER^{-1}E^T\varepsilon_{Vx}(l)\right|$$
$$+\left|\varepsilon_{Vx}(0)PAx(0)\right|.$$

According to Hölder's inequality ($\int a(z)b(z)dz \le \|a\|_{\mathcal{L}_2}\|b\|_{\mathcal{L}_2}$)

$$\int_0^l \|x(z)\|dz \le \sqrt{l}\,\|x\|_{\mathcal{L}_2}, \tag{69}$$

where $\|.\|_{\mathcal{L}_2} = (\int \|.\|_2^2 dz)^{\frac{1}{2}}$. Therefore, using Cauchy's, Young's and Hölder's inequalities yield

$$
\begin{aligned}
&|\varepsilon_H| \le l\varepsilon_{VxM} W_{fM}\sigma_{fM} + l\varepsilon_{VxM}\varepsilon_{fM} + l\varepsilon_{fM} W_{\Phi M}\Phi_{xM} \\
&+\sqrt{l}\varepsilon_{VxzzM}\|A\|\|x\|_{\mathcal{L}_2} + \varepsilon_{VxzM}\|A\|\|x(l)\| + \varepsilon_{VxzM}\|A\| \\
&\|x(0)\| + \frac{1}{4}\delta\varepsilon_{VxM}^2 + \frac{1}{2}\delta W_{\Phi M}\Phi_{xM}\varepsilon_{VxM} + \|PA\|\|x(0)\|\varepsilon_{VxM} \\
&= \chi_{H1}\|x\|_{\mathcal{L}_2} + \chi_{H2}\|x(l)\| + \chi_{H3}\|x(0)\| + \chi_{H4},
\end{aligned} \tag{70}
$$

where $\delta = \|ER^{-1}E^T\|$, $\chi_{H1} = \sqrt{l}\varepsilon_{VxzzM}\|A\|$, $\chi_{H2} = \varepsilon_{VxzM}\|A\|$, $\chi_{H3} = \varepsilon_{VxzM}\|A\| + \|PA\|\varepsilon_{VxM}$ and

$$
\begin{aligned}
&\chi_{H4} = l\varepsilon_{VxM}(W_{fM}\sigma_{fM} + \varepsilon_{fM}) + l\varepsilon_{fM} W_{\Phi M}\Phi_{xM} \\
&+ \frac{1}{4}\delta\varepsilon_{VxM}^2 + \frac{1}{2}\delta W_{\Phi M}\Phi_{xM}\varepsilon_{VxM}
\end{aligned} \tag{71}
$$

Moreover, recalling $\|x(0)\|, \|x(l)\| \le k(\|x\|_{\mathcal{L}_2} + \|x_z\|_{\mathcal{L}_2})$, $k > 0$ $|\varepsilon_H|$ can be expressed as

$$
\begin{aligned}
&|\varepsilon_H| \le (\chi_{H1} + k\chi_{H2} + k\chi_{H3})\|x\|_{\mathcal{L}_2} \\
&(k\chi_{H2} + k\chi_{H3})\|x_z\|_{\mathcal{L}_2} + \chi_{H4} \\
&\le c_{H1}(\|\hat{x}\|_{\mathcal{L}_2} + \|\tilde{x}\|_{\mathcal{L}_2}) + c_{H2}(\|\hat{x}_z\|_{\mathcal{L}_2} + \|\tilde{x}_z\|_{\mathcal{L}_2}) + c_{H3} \\
&\le \xi_{NM}\zeta_N + c_{H1}\|\tilde{x}\|_{\mathcal{L}_2} + c_{H2}\|\tilde{x}_z\|_{\mathcal{L}_2}
\end{aligned} \tag{72}
$$

where $c_{H1} = \chi_{H1} + k\chi_{H2} + k\chi_{H3}$, $c_{H2} = k\chi_{H2} + k\chi_{H3}$, $c_{H3} = \chi_{H4}$ and $\xi_{NM} = \max\{\frac{c_{H1}}{c_1}, \frac{c_{H2}}{c_2}, \frac{c_{H3}}{c_3}\}$.

Proof of Theorem 2: First, Consider the Lyapunov function

$$
L_a = \frac{1}{2}\tilde{W}_\Phi^T\tilde{W}_\Phi, \tag{73}
$$

where $\tilde{W}_\Phi = \hat{W}_\Phi - W_\Phi$. Taking its derivative $\dot{L}_a$ yields

$$
\dot{L}_a = -\alpha_1\frac{\tilde{W}_\Phi^T\omega_N}{\zeta_N^2}\hat{H} + \tilde{W}_\Phi^T\{-\alpha_2\hat{W}_\Phi - \alpha_3\hat{W}_\Phi\|\hat{W}_\Phi\|^2\}. \tag{74}
$$

Since $H^* = 0$, $\hat{H} = \hat{H} - H^*$ and it can be represented in the following form

$$
\begin{aligned}
&\hat{H} = Q(\hat{x}) + \int_0^l \hat{W}_\Phi^T\Phi_x(\hat{x},z)\hat{W}_f^T\sigma_f(z)dz \\
&-\int_0^l \hat{W}_\Phi^T\Phi_{xz}A\hat{x}_z dz - \frac{1}{4}\hat{W}_\Phi^T\Phi_x(\hat{x},l)ER^{-1}E^T\Phi_x{}^T(\hat{x},l)\hat{W}_\Phi \\
&+\hat{W}_\Phi^T\Phi_x(\hat{x},0)PA\hat{x}(0) - Q(x) - \int_0^l W_\Phi^T\Phi_x(z) \\
&(W_f^T\sigma_f(z))dz + \int_0^l W_\Phi^T\Phi_{xz}Ax_z dz + \frac{1}{4}W_\Phi^T\Phi_x(l) \\
&ER^{-1}E^T\Phi_x{}^T(l)W_\Phi - W_\Phi^T\Phi_x(0)PAx(0) - \varepsilon_H.
\end{aligned} \tag{75}
$$

Since $|Q(\hat{x})-Q(x)|\le L_Q\|\tilde{x}\|_{\mathcal{L}_2}$, and

$$
\begin{aligned}
&-\int_0^l \hat{W}_\Phi^T\Phi_{xz}(\hat{x},z)A\hat{x}_z dz+\int_0^l W_\Phi^T\Phi_{xz}(x,z)Ax_z dz,\\
&=-\int_0^l \tilde{W}_\Phi^T\Phi_{xz}(\hat{x},z)A\hat{x}_z dz+\int_0^l W_\Phi^T\Phi_{xz}(x,z)A\tilde{x}_z dz\\
&+\int_0^l W_\Phi^T(\Phi_{xz}(x,z)-\Phi_{xz}(\hat{x},z))A\hat{x}_z dz,
\end{aligned}
\tag{76}
$$

and

$$
\begin{aligned}
&\underbrace{-\frac{1}{4}(W_\Phi+\tilde{W}_\Phi)\Phi_x(\hat{x},l)ER^{-1}E^T\Phi_x{}^T(\hat{x},l)(W_\Phi+\tilde{W}_\Phi)}_{\text{expand}}\\
&+\frac{1}{4}W_\Phi^T\Phi_x(x,l)ER^{-1}E^T\Phi_x{}^T(x,l)W_\Phi\\
&=-\frac{1}{2}\tilde{W}_\Phi^T\Phi_x(\hat{x},l)ER^{-1}E^T\Phi_x{}^T(\hat{x},l)\hat{W}_\Phi+\frac{1}{4}\tilde{W}_\Phi^T\Phi_x(\hat{x},l)\\
&-\Phi_x(\hat{x},l)ER^{-1}E^T\Phi_x{}^T(\hat{x},l)]W_\Phi\\
&ER^{-1}E^T\Phi_x{}^T(\hat{x},l)\tilde{W}_\Phi+\frac{1}{4}W_\Phi^T[\Phi_x(x,l)ER^{-1}E^T\Phi_x{}^T(x,l)
\end{aligned}
\tag{77}
$$

and

$$
\begin{aligned}
&\hat{W}_\Phi^T\Phi_x(\hat{x},0)PA\hat{x}(0)-W_\Phi^T\Phi_x(x,0)PAx(0)\\
&=\tilde{W}_\Phi^T\Phi_x(\hat{x},0)PA\hat{x}(0)-W_\Phi^T\Phi_x(x,0)PA\tilde{x}(0)\\
&W_\Phi^T(\Phi_x(\hat{x},0)-\Phi_x(x,0))PA\hat{x}(0)
\end{aligned}
\tag{78}
$$

Combining all the terms and using Cauchy Schwartz inequality yield

$$
\begin{aligned}
&\hat{H}\le\tilde{W}_\Phi^T\omega_N+b_1\|\tilde{x}\|_{\mathcal{L}_2}+b_2\|\tilde{x}_z\|_{\mathcal{L}_2}+b_3\|\tilde{x}(0)\|+b_4\\
&\|\hat{x}(0)\|\|\tilde{x}\|_{\mathcal{L}_2}+b_5\|\hat{x}_z\|_{\mathcal{L}_2}\|\tilde{x}\|_{\mathcal{L}_2}+b_6\|\tilde{W}_f\|+b_7\|\tilde{W}_\Phi\|^2+b_8
\end{aligned}
\tag{79}
$$

where $b_1=L_Q+W_{\Phi M}W_{fM}(l\Phi_{xM}L_{\sigma f}+\sqrt{l}\sigma_{fM}L_\Phi)+\frac{1}{2}L_\Phi\delta\Phi_{xM}W_{\Phi M}^2+c_{H1}$, $b_2=\sqrt{l}W_{\Phi M}\Phi_{xzM}\|A\|+c_{H2}$, $b_3=W_{\Phi M}\Phi_{xM}\|PA\|$, $b_4=W_{\Phi M}\Phi_{xM}\|PA\|$, $b_5=\sqrt{l}W_{\Phi M}\Phi_{xzM}$, $b_6=lW_{\Phi M}\Phi_{xM}\sigma_{fM}$, $b_7=\frac{1}{4}\Phi_{xM}^2\delta$ and $b_8=\xi_{NM}\zeta_N$.

Subsequently, expanding $-\tilde{W}_\Phi^T\hat{W}_\Phi\|\hat{W}_\Phi\|^2$ and using Young's and Cauchy-Schwarz inequalities yield

$$-\tilde{W}_\Phi^T \hat{W}_\Phi \left\|\hat{W}_\Phi\right\|^2 = -\tilde{W}_\Phi^T (W_\Phi + \tilde{W}_\Phi)(W_\Phi + \tilde{W}_\Phi)^T$$
$$(W_\Phi + \tilde{W}_\Phi) \le -\frac{1}{4}\left\|\tilde{W}_\Phi\right\|^4 + \frac{1}{2}\left\|\tilde{W}_\Phi\right\|^2 + \frac{81}{4}\left\|W_\Phi\right\|^4 \tag{80}$$
$$+\frac{1}{2}\left\|W_\Phi\right\|^6 .$$

Consequently $\dot{L}_a$ can be derived as

$$\dot{L}_a \le -\alpha_1 \frac{\tilde{W}_\Phi^T \omega_N}{\zeta_N^2}\{\tilde{W}_\Phi^T \omega_N + b_1\left\|\tilde{x}\right\|_{\mathcal{L}_2} + b_2\left\|\tilde{x}_z\right\|_{\mathcal{L}_2} + b_3\left\|\tilde{x}(0)\right\|$$
$$+b_4\left\|\hat{x}(0)\right\|\left\|\tilde{x}\right\|_{\mathcal{L}_2} + b_5\left\|\hat{x}_z\right\|_{\mathcal{L}_2}\left\|\tilde{x}\right\|_{\mathcal{L}_2} + b_6\left\|\tilde{W}_f\right\| \tag{81}$$
$$+b_7\left\|\tilde{W}_\Phi\right\|^2 + b_8\} - \alpha_2 \tilde{W}_\Phi^T (W_\Phi + \tilde{W}_\Phi) - \alpha_3 \tilde{W}_\Phi^T \hat{W}_\Phi \left\|\hat{W}_\Phi\right\|^2 .$$

Therefore, according to Young's inequality and since

$\beta_N \zeta_N \ge 1, \left\|\hat{x}\right\|_{\mathcal{L}_2}, \left\|\hat{x}_z\right\|_{\mathcal{L}_2}$ where $\beta_N = \max\{\frac{1}{c_1}, \frac{1}{c_2}, \frac{1}{c_3}\}$, we have

$$\dot{L}_a \le (-\alpha_1 + \alpha_1^2)\frac{\tilde{W}_\Phi^T \omega_N \omega_N^T \tilde{W}_\Phi}{9\zeta_N^2} + \frac{9b_1^2\beta_N^2\left\|\tilde{x}\right\|_{\mathcal{L}_2}^2}{4}$$
$$+\frac{9b_2^2\beta_N^2\left\|\tilde{x}_z\right\|_{\mathcal{L}_2}^2}{4} + \frac{9b_3^2\beta_N^2\left\|\tilde{x}(0)\right\|^2}{4} + \frac{9b_4^2 k\beta_N^2\left\|\tilde{x}\right\|_{\mathcal{L}_2}^2}{4} \tag{82}$$
$$+\frac{9b_5^2\beta_N^2\left\|\tilde{x}\right\|_{\mathcal{L}_2}^2}{4} + \frac{9\beta_N^2 b_6^2}{4}\left\|\tilde{W}_f\right\|^4 + \frac{9b_7^2}{4c_3^2}\left\|\tilde{W}_\Phi\right\|^4$$
$$+\frac{9}{4}\xi_{NM}^2 + (-\frac{\alpha_2}{2} + \frac{\alpha_3}{2})\left\|\tilde{W}_\Phi\right\|^2 - \frac{\alpha_3}{4}\left\|\tilde{W}_\Phi\right\|^4$$

$$+\frac{\alpha_2}{2}\left\|W_\Phi\right\|^2 + \frac{81\alpha_3}{4}\left\|W_\Phi\right\|^4 + \frac{\alpha_3}{2}\left\|W_\Phi\right\|^6 .$$

If $\alpha_2 \ge \alpha_3$, $\alpha_3 \ge \theta_{\alpha_3}^N = \frac{9b_7^2}{c_3^2}$ define $\eta = \frac{1}{2}(\alpha_2 - \alpha_3)$. If $0 < \alpha_1 < 1$ then

$$\dot{L}_a \le -\alpha_1(1-\alpha_1)\frac{\tilde{W}_\Phi^T \omega_N \omega_N^T \tilde{W}_\Phi}{\zeta_N^2} - \eta_1\left\|\tilde{W}_\Phi\right\|^2 - \eta_2\left\|\tilde{W}_\Phi\right\|^4$$
$$+\frac{9(b_1^2 + b_5^2 + kb_4^2)\beta_N^2\left\|\tilde{x}\right\|_{\mathcal{L}_2}^2}{4} + \frac{9b_2^2\beta_N^2\left\|\tilde{x}_z\right\|_{\mathcal{L}_2}^2}{4} + \frac{9b_3^2\left\|\tilde{x}(0)\right\|^2}{4\zeta_N^2} \tag{83}$$
$$+\frac{9\beta_N^2 b_6^2}{4}\left\|\tilde{W}_f\right\|^2 + \varepsilon_a ,$$

where $\varepsilon_a = \frac{9}{4}\xi_{NM}^2 + \frac{\alpha_2}{2}W_{\Phi M}^2 + \frac{81\alpha_3}{4}W_{\Phi M}^4 + \frac{\alpha_3}{2}W_{\Phi M}^6$.

Defining $L_b = \mu_b \int_0^l V^*(x)dz$ and taking its derivative $\dot{L}_b$ yield

$$\begin{aligned}\dot{L}_b &= \mu_b \int_0^l V_x^*(x_{zz} + f(x,z))dz = \mu_b \int_0^l V_x^* f(x,z) \\ &+ \mu_b \int_0^l V_x^* x_{zz} dz = \mu_b \int_0^l V_x^* f(x,z)dz - \mu_b \int_0^l V_{xz}^* x_z dz \\ &+ \mu_b V_x^*(l) GA(u^* + \tilde{u}) - \mu_b V_x^*(0) x_z(0) \\ &\leq -\mu_b q_{min} \|x\|_{\mathcal{L}_2}^2 - \frac{1}{2}\mu_b V_x^*(l) E R^{-1} E^T \Phi_x^T(\hat{x},l)\tilde{W}_\Phi \\ &- \frac{1}{2}\mu_b V_x^*(l) E R^{-1} E^T \{\Phi_x^T(\hat{x},l) - \Phi_x^T(x,l)\} W_\Phi \\ &+ \frac{1}{2}\mu_b (\Phi_x^T(l) W_\Phi + \varepsilon_{Vx}(l)) E R^{-1} E^T \varepsilon_{Vx}(l).\end{aligned} \tag{84}$$

By using Cauchy-Schwartz and Young inequalities one has

$$\begin{aligned}&\frac{1}{2}\left|\mu_b V_x^*(l) E R^{-1} E^T \{\Phi_x^T(\hat{x},l) - \Phi_x^T(x,l)\} W_\Phi\right| \\ &\leq \frac{1}{4}\mu_b W_{\Phi M}^2 + b_1' \|\tilde{x}\|_{\mathcal{L}_2}^2,\end{aligned} \tag{85}$$

where $b_1' = \frac{1}{4}\mu_b \delta^2 (\Phi_{xM} W_{\Phi M} + \varepsilon_{VxM})^2 L_\Phi^2$. And

$$\begin{aligned}&\frac{1}{2}\left|\mu_b V_x^*(l) E R^{-1} E^T \Phi_x^T(\hat{x},l)\tilde{W}_\Phi\right| \leq \frac{1}{8}\mu_b (\|W_\Phi\|^2 + \varepsilon_{VxM}^2) \\ &+ \frac{1}{2}\mu_b \delta^2 (\Phi_{xM}^4 + \Phi_{xM}^2)\|\tilde{W}_\Phi\|^2,\end{aligned} \tag{86}$$

where $b_2' = \frac{1}{2}\delta^2(\Phi_{xM}^4 + \Phi_{xM}^2)$. Moreover,

$$\begin{aligned}&\frac{1}{2}(\Phi_x^T(l) W_\Phi + \varepsilon_{Vx}(l)) E R^{-1} E^T \varepsilon_{Vx}(l) \\ &\leq \frac{1}{2}\delta(\Phi_{xM}\|W_\Phi\| + \varepsilon_{VxM})\varepsilon_{VxM}.\end{aligned} \tag{87}$$

Therefore

$$\begin{aligned}\dot{L}_b &\leq -\mu_b q_{min}\|x\|_{\mathcal{L}_2}^2 + \mu_b b_1' \|\tilde{x}\|_{\mathcal{L}_2}^2 \\ &+ \mu_b b_2' \|\tilde{W}_\Phi\|^2 + \varepsilon_b,\end{aligned} \tag{88}$$

where

$$\begin{aligned}\varepsilon_b &= \frac{1}{4}\mu_b W_{\Phi M}^2 + \frac{1}{8}\mu_b(\|W_\Phi\|^2 + \varepsilon_{VxM}^2) + \frac{1}{2}\mu_b \delta(\Phi_{xM} \\ &\|W_\Phi\| + \varepsilon_{VxM})\varepsilon_{VxM}.\end{aligned} \tag{89}$$

Closed loop stability proof: In order to prove the overall closed loop stability of the system, Consider the Lyapunov function L defined by $L = L_a + L_b + L_c$. Taking its derivative $\dot{L}$ yields

$$\begin{aligned}
&\dot{L} \le -\mu_c\psi_1 \|\tilde{x}\|_{\mathcal{L}_2}^2 - \frac{1}{2}\mu_c\psi_2 \|\tilde{W}_f\|^2 - \frac{1}{2}\mu_c\psi_3 \|\tilde{x}_z\|_{\mathcal{L}_2}^2 \\
&-\mu_c\psi_4 \|\tilde{x}(0)\|^2 + \frac{9(b_1^2 + b_5^2 + kb_4^2)\beta_N^2 \|\tilde{x}\|_{\mathcal{L}_2}^2}{4} \\
&+\frac{9b_2^2\beta_N^2 \|\tilde{x}_z\|_{\mathcal{L}_2}^2}{4} + \frac{9b_3^2\beta_N^2 \|\tilde{x}(0)\|^2}{4} + \frac{9\beta_N^2 b_6^2}{4}\|\tilde{W}_f\|^4 \\
&-\eta\| \tilde{W}_\Phi\|^2 - \mu_b q_{min}\|x\|_{\mathcal{L}_2}^2 + \mu_b b_2' \| \tilde{W}_\Phi\|^2 + \mu_b b_1' \|\tilde{x}\|_{\mathcal{L}_2}^2 + \varepsilon
\end{aligned} \tag{90}$$

where $\varepsilon = \varepsilon_a + \varepsilon_b + \varepsilon_c$. Therefore, choosing $\mu_b < \frac{\eta}{2b_2'}$ and $\mu_c > \max\{\frac{9(b_1^2 + b_5^2 + kb_4^2)\beta_N^2}{2\psi_1} + \frac{\mu_b b_1'}{2\psi_1}, \frac{9b_2^2\beta_N^2}{\psi_3}, \frac{9b_3^2\beta_N^2}{4\psi_4}, \frac{9\beta_N^2 b_6^2}{\psi_2}\}$ $\dot{L}$ is always less than zero if $\|x\|_{\mathcal{L}_2} > \sqrt{\frac{\varepsilon}{\mu_b q_{\min}}}$ or $\| \tilde{W}_\Phi\| > \sqrt{\frac{2\varepsilon}{\eta}}$ or $\|\tilde{x}\|_{\mathcal{L}_2} > \sqrt{\frac{2\varepsilon}{\mu_c\psi_1}}$ or $\|\tilde{W}_f\| > \sqrt{\frac{4\varepsilon}{\mu_c\psi_2}}$ and the closed-loop system is UB.

Aside from increasing number of neurons to reduce the bound ε, the bounds for $\|x\|$, $\|\tilde{x}\|$ and $\|\tilde{W}_f\|$ can be arbitrarily reduced by increasing $q_{\min}$, κ in ψ_1 and α_f in ψ_2.

Proof of Lemma 2: $\|\varepsilon_H\|$ for Dirichlet boundary control condition can be derived as $\|\varepsilon_H\| \le \xi_{DM}\zeta_D + c_{H1}\|\tilde{x}\|_{\mathcal{L}_2} + c_{H2}\|\tilde{x}_z\|_{\mathcal{L}_2}$ where $c_{H1} = \sqrt{l}\varepsilon_{VxzzM}\|A\| + k(\|PA\|\varepsilon_{VxM} + \varepsilon_{VxzM}\|A\|)$, $c_{H2} = k(\|PA\|\varepsilon_{VxM} + \varepsilon_{VxzM}\|A\|)$, $c_{H3} = \varepsilon_{VxM}\|A\|$ and $\xi_{DM} = \max\{\frac{c_{H1}}{c_1}, \frac{c_{H2}}{c_2}, \frac{c_{H3}}{c_3}\}$.

Proof of Theorem 3: Consider the same Lyapunov function as in Theorem 2. If update parameters satisfy $\alpha_2 \ge \alpha_3$, $\alpha_3 \ge \theta_{\alpha_3}^D = \frac{9b_8^2}{4c_4^2}$ and $0 < \alpha_1 < 1$. Define $\eta = \frac{1}{2}(\alpha_2 - \alpha_3)$ and $\beta_D = \max\{\frac{1}{c_1}, \frac{1}{c_2}, \frac{1}{c_3}\}$. Similarly, taking the derivative of Lyapunov function and combining $\dot{L}_a$, $\dot{L}_b$ and $\dot{L}_c$ yield,

$$\dot{L} \le -\mu_c \psi_1 \|\tilde{x}\|_{\mathcal{L}_2}^2 - \frac{1}{2}\mu_c \psi_2 \|\tilde{W}_f\|^2 - \frac{1}{2}\mu_c \psi_3 \|\tilde{x}_z\|_{\mathcal{L}_2}^2$$
$$-\mu_c \psi_4 \|\tilde{x}(0)\|^2 + \frac{9(b_1^2 + b_4^2 + b_5^2 + b_6^2)\beta_D^2 \|\tilde{x}\|_{\mathcal{L}_2}^2}{4} \tag{91}$$
$$+\frac{9b_2^2\beta_D^2 \|\tilde{x}_z\|_{\mathcal{L}_2}^2}{4} + \frac{9b_3^2\beta_D^2 \|\tilde{x}(0)\|^2}{4} + \frac{9\beta_D^2 b_7^2}{4}\|\tilde{W}_f\|^4$$
$$-\eta \| \tilde{W}_\Phi \|^2 - \mu_b q_{min} \|x\|_{\mathcal{L}_2}^2 + \mu_b b_2' \| \tilde{W}_\Phi \|^2 + \mu_b b_1' \|\tilde{x}\|_{\mathcal{L}_2}^2 + \varepsilon$$

where

$$\varepsilon = \varepsilon_c + \frac{1}{4}\mu_b W_{\Phi M}^2 + \frac{1}{8}\mu_b (W_{\Phi M}^2 + \varepsilon_{VxzM}^2) + \frac{1}{2}\mu_b \delta(\Phi_{xzM}$$
$$W_{\Phi M} + \varepsilon_{VxzM})\varepsilon_{VxzM} + \frac{9}{4}\xi_{DM}^2 + \frac{\alpha_2}{2}W_{\Phi M}^2 \tag{92}$$
$$+\frac{81\alpha_3}{4}W_{\Phi M}^4 + \frac{\alpha_3}{2}W_{\Phi M}^6,$$

$\psi_1, ..., \psi_4$ are defined as in proof of theorem 1 and $b_1 = L_Q + W_{\Phi M} W_{fM} (l\Phi_{xM} L_{\sigma f} + \sqrt{l}\sigma_{fM} L_\Phi) + \sqrt{l} W_{\Phi M} \Phi_{xzzM} \|A\| + \frac{1}{2} L_\Phi \delta \Phi_{xzM} W_{\Phi M}^2 + c_{H1}, b_2 = c_{H2}$,

$b_3 = W_{\Phi M}\Phi_{xM} \|PA\| + W_{\Phi M}\Phi_{xzM} \|A\|$, $b_4 = W_{\Phi M} L_{\Phi x} \|PA\| + W_{\Phi M} L_{\Phi xz} \|A\|$, $b_5 = W_{\Phi M} L_{\Phi x} \|A\|$,

$b_6 = \sqrt{l} W_{\Phi M} L_{\Phi xzz} \|A\|$, $b_7 = l W_{\Phi M} \Phi_{xM} \sigma_{fM}$, $b_1' = \frac{1}{4}\mu_b \delta^2 (\Phi_{xzM} W_{\Phi M} + \varepsilon_{VxzM})^2 L_{\Phi z}^2$, $b_2' = \frac{1}{2}\delta^2(\Phi_{xzM}^4 + \Phi_{xzM}^2)$.

Therefore, choosing $\mu_b < \frac{\eta_1}{2b_2'}$ and

$\mu_c > \max\{\frac{9(b_1^2 + b_4^2 + b_5^2 + b_6^2)\beta_D^2}{2\psi_1} + \frac{\mu_b b_1'}{2\psi_1}, \frac{9b_2^2\beta_D^2}{\psi_3}, \frac{9b_3^2\beta_D^2}{4\psi_4}, , \frac{9\beta_D^2 b_6^2}{\psi_2}\}$, $\dot{L}$ is always less than zero if

$\|x\|_{\mathcal{L}_2} > \sqrt{\frac{\varepsilon}{\mu_b q_{\min}}}$ or $\| \tilde{W}_\Phi \| > \sqrt{\frac{2\varepsilon}{\eta}}$ or $\|\tilde{x}\|_{\mathcal{L}_2}^2 > \sqrt{\frac{2\varepsilon}{\mu_c \psi_1}}$ or $\|\tilde{x}_z\|_{\mathcal{L}_2}^2 > \sqrt{\frac{4\varepsilon}{\mu_c \psi_3}}$ or $\| \tilde{W}_f \| > \sqrt{\frac{4\varepsilon}{\mu_c \psi_2}}$.

Consequently, the closed-loop system is UB.

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IV. BOUNDARY CONTROL OF TWO DIMENSIONAL BURGERS' PDE USING APPROXIMATE DYNAMIC PROGRAMMING

B. Talaei, S. Jagannathan, and J. Singler

An approximate dynamic programming (ADP) based near optimal boundary control of distributed parameter systems (DPS) governed by uncertain two dimensional (2D) Burgers equation under Neumann boundary condition is introduced. First, Hamilton-Jacobi-Bellman (HJB) equation is formulated without any model reduction and optimal boundary control policy is derived in terms of value functional which is obtained as the solution to the HJB equation. Subsequently, a novel neural network (NN) identifier is developed to estimate the unknown nonlinearity in the partial differential equation (PDE) dynamics. The suboptimal control policy is obtained by forward-in-time approximation of the value functional using a second NN-based online approximator and identified dynamics. Adaptive tuning laws are proposed for online learning the value functional and identifier. Local ultimate boundedness (UB) of the closed-loop system is verified by using Lyapunov theory. Simulation results confirm the good performance of proposed controller on an unstable 2D Burgers equation.

1. INTRODUCTION

Stabilization of multi-dimensional fluids is one of the most challenging problems in the control theory [1]. In particular, boundary control of fluid dynamics modeled either by Navier Stokes or Burgers equation expressed as partial difference equations (PDE) gained special attention [2] after the development of micro-electro-mechanical systems (MEMS) which made the distributed control of fluids at the boundaries possible.

Boundary control of one dimensional Burgers equation has been studied in the literature. In [3], nonlinear Galerkin model reduction is used for designing a controller for finite dimensional model. In [4], a combination of nonlinear transformation and backstepping method is utilized to design a locally asymptotic controller and an estimate of region of attraction is provided. Earlier effort in this area included “radiation boundary control” which proves local exponential stability by using scalar feedback gains in the boundary conditions [5].

Since the fluid flow or mixing applications are usually modeled by using multi-dimensional Navier Stokes or simplified Burgers equations, boundary control of these multi-dimensional PDEs have also been introduced in the literature. In [6], the stabilization problem of two dimensional (2D) Navier Stokes equation using backstepping method is considered whereas the authors in [3] have applied their nonlinear model reduction technique for 2D Navier Stokes equation. The Riccati equation for optimal boundary feedback stabilization of incompressible Navier Stokes flows have also been formulated in [7]. The control of 2D PDEs is more involved and difficult in comparison with a 1D PDE.

Approximate dynamic programming (ADP) has been extensively utilized for the optimal control of one dimensional PDE [8]-[10],. Here, the conventional method of solving the PDE [8]-[10] using ADP involves obtaining a reduced order finite dimensional representation and subsequently using the ADP on the reduced order ordinary differential equations (ODE). Recently, in [11] and [12], an ADP approach is introduced for boundary control of linear and nonlinear one dimensional parabolic PDE without model reduction, respectively. However, to the best knowledge of the authors, no known work on the optimal control of uncertain multi-dimensional PDE has been reported using ADP. Due to the importance of higher dimensional PDE in various applications such as fluid flow, in this paper, the authors intend to solve the boundary control of multi-dimensional nonlinear PDE like 2D Burgers equation with unknown uncertainty in the system dynamics.

Therefore, this paper addresses an ADP-based boundary control of 2D uncertain Burgers equation without using model reduction. The optimal control problem is formulated in the original PDE domain and solved forward-in-time without using any finite dimensional model approximation prior to the control design. After defining an appropriate value functional, the Hamilton-Jacobi-Bellman (HJB) equation is derived in the infinite dimensional space and the optimal control policy, which requires the solution to the HJB equation, is obtained based on necessary conditions of optimality.

Afterwards, a NN identifier is introduced for the online approximation of nonlinearity in the PDE so that the need for the system dynamics is relaxed. Subsequently, by approximating the optimal value functional based on a novel adaptive framework, and by using the NN identifier, the actual control policy is derived. No value or policy iterations [13] have been used in the ADP design. Instead, the value functional NN weights are tuned online based on conventional adaptive techniques. Eventually, Lyapunov analysis is utilized to demonstrate the ultimate boundedness (UB) of the closed-loop system.

Throughout the paper $\|.\|$ stands for Euclidean norm for a vector and Frobenius norm for a matrix. We recall the inequality $\|.\|_2 \le \|.\|_F$ with $\|.\|_2$ being the induced 2 norm. We also define for $\forall w \in [\mathcal{L}_2(0,l)]$, $\|w\|_{\mathcal{L}_2} = (\int_0^l \|w(z)\|_2^2 dz)^{\frac{1}{2}}$ and $\forall x \in [\mathcal{L}_2(0,l)^2]$, $\|x\|_{\mathcal{L}_2} = (\int_0^l \int_0^l x^2(z,y)dzdy)^{\frac{1}{2}}$.

The rest of the paper is organized as follows. In Section 2, the class of DPS under consideration is described and the state feedback optimal control approach is explained. The stability of developed approach is discussed in Section 3. Section 4 demonstrates the simulation results and Section 5 provides the conclusions of the paper.

2. APPROXIMATE OPTIMAL CONTROL

The class of DPS considered in this paper is described by following Burger's PDE dynamics in a square domain $[0,l]\times[0,l]$ with Neumann boundary control as

$$\begin{aligned}
&\frac{\partial x}{\partial t}=\frac{\partial^2 x}{\partial z^2}+\frac{\partial^2 x}{\partial y^2}-x\frac{\partial x}{\partial z}-x\frac{\partial x}{\partial y}+f(x,z,y),\\
&\frac{\partial x}{\partial z}|_{z=0,y}=g_1(y)u_1(y),\ \frac{\partial x}{\partial z}|_{z=l,y}=g_2(y)u_2(y),\\
&\frac{\partial x}{\partial y}|_{z,y=0}=h_1(z)v_1(z),\ \frac{\partial x}{\partial y}|_{z,y=l}=h_2(z)v_2(z),\\
&x(z,y,0)=x_0(z,y),
\end{aligned} \tag{1}$$

where $x(z,y,t)$ represents the system state belonging to the solution space of the PDE $X\subset\mathcal{H}_1[0,l]^2$, with $\mathcal{H}_1$ being the Sobolev space of first order, t denotes time $z,y\in[0,l]$, $l>0$, being the spatial variable, $f(x,z,y)$ is an unknown Lipschitz continuous nonlinear vector function, and $\frac{\partial x}{\partial t}$ and $\frac{\partial^2 x}{\partial z^2}$ denote the time and second spatial derivatives of state x. Moreover $u_1(y),u_2(y),v_1(z),v_2(z)$ denote boundary control input signals, and $g_1(y),g_2(y),h_1(z),h_2(z)$ are bounded functions. Next the following assumption on the nonlinearity is needed.

Assumption 1: The function $f(.,z,y)$ is Lipschitz continuous with Lipschitz constant L_f i.e.

$$\|f(x_1,z,y)-f(x_2,z,y)\|\le L_f\|x_1-x_2\|. \tag{2}$$

The objective is to provide a continuous control policy by minimizing the cost functional over infinite horizon as

$$\bar{V}(x,t_0,u,v)=\int_{t_0}^{\infty}V(x,u,v)dt, \tag{3}$$

where

$$\begin{aligned}
&V(x,u,v)=\int_0^l\int_0^l q(x,y,z)dydz\\
&+\sum_{i=1,2}\int_0^l r_i(y)u_i^2(y)dy+\sum_{i=1,2}\int_0^l s_i(z)v_i^2(z)dz,
\end{aligned} \tag{4}$$

with $q(x,z,y)$ being a nonlinear integrand function and functions $r_i(y),s_i(z)>0$ for $y,z\in[0,l]$, $i=1,2$.

By using Riesz representation Theorem [14], it can be proven that

$$\bar{V}^*(x,t) = \int_0^l \int_0^l V^*(x,z,y)dzdy. \tag{5}$$

where $\bar{V}^*$ is the optimal value functional and V^* is its optimal integrand. The steps are omitted here for the sake of brevity. By taking the current time interval $[t, t+\delta t)$, $\bar{V}(x,t,u)$ in (3) can be expressed in the recursive form as

$$\bar{V}(x,t,u,v) = \int_t^{t+\delta t} V(x,u,v)dt + \bar{V}(x,t+\delta t,u,v). \tag{6}$$

Therefore, the optimal $\bar{V}^*$ can be expressed as

$$\bar{V}^*(x,t) = \min_{u,v}\{\int_t^{t+\delta t} V(x,u,v)dt + \bar{V}(x,t+\delta t,u,v)\}. \tag{7}$$

It is assumed that $\bar{V}^*(x,t)$ is Frechet analytic in neighborhood of $(x(z,t),t)$. According to calculus of variations [15], $\bar{V}^*(x,t+\delta t)$ can be represented in its first order approximation form as

$$\bar{V}^*(x,t+\delta t) \approx \bar{V}^*(x,t) + \int_0^l \int_0^l V_x^* \delta x dzdy + \frac{\partial \bar{V}^*}{\partial t}\delta t, \tag{8}$$

Since the cost function $\bar{V}$ in (3) is infinite horizon, $\bar{V}^*$ does not explicitly depend on time and $\frac{\partial \bar{V}^*}{\partial t} = 0$. Consequently, substituting approximation (8) into (7), canceling $\bar{V}^*(x,t)$ on both sides, dividing throughout by δt, letting $\delta t \to 0$ and finally substituting PDE dynamics (1) for $\frac{\partial x}{\partial t}$ results in the HJB equation given by

$$\begin{aligned} H^* = 0 = \min_u\{&\int_0^l \int_0^l q(x,y,z)dydz + \sum_{i=1,2}\int_0^l r_i(y)u_i^2(y)dy \\ &+ \sum_{i=1,2}\int_0^l s_i(z)v_i^2(z)dz + \int_0^l \int_0^l V_x^*(\frac{\partial^2 x}{\partial z^2} + \frac{\partial^2 x}{\partial y^2} \\ &- x\frac{\partial x}{\partial z} - x\frac{\partial x}{\partial y} + f(x,z,y))dzdy\} \end{aligned} \tag{9}$$

Define the Hamiltonian as

$$H = \int_0^l \int_0^l q(x,y,z)dydz + \sum_{i=1,2} \int_0^l r_i(y)u_i^2(y)dy + \sum_{i=1,2} \int_0^l s_i(z)v_i^2(z)dz + \int_0^l \int_0^l V_x \left(\frac{\partial^2 x}{\partial z^2} + \frac{\partial^2 x}{\partial y^2} - x\frac{\partial x}{\partial z} - x\frac{\partial x}{\partial y} + f(x,z,y)\right)dzdy. \tag{10}$$

Using integration by parts, Hamiltonian can be rewritten as

$$H = \int_0^l \int_0^l q(x,y,z)dydz + \sum_{i=1,2} \int_0^l r_i(y)u_i^2(y)dy + \sum_{i=1,2} \int_0^l s_i(z)v_i^2(z)dz + \int_0^l V_x(l,y)g_2(y)u_2(y)dy - \int_0^l V_x(0,y)g_1(y)u_1(y)dy + \int_0^l V_x(z,l)h_2(z)v_2(z)dz - \int_0^l V_x(z,0)h_1(z)v_1(z)dz - \int_0^l \int_0^l \left(\frac{\partial V_x}{\partial z}\frac{\partial x}{\partial z} + \frac{\partial V_x}{\partial y}\frac{\partial x}{\partial y}\right)dzdy + \int_0^l \int_0^l V_x^*\left(-x\frac{\partial x}{\partial z} - x\frac{\partial x}{\partial y} + f(x,z,y)\right)dzdy. \tag{11}$$

The necessary conditions of optimality requires that for the control policy to be minimizing the Hamiltonian (11), $\frac{dH}{du} = 0$. Therefore

$$u_1(y) = \frac{1}{2r_1(y)} g_1(y)V_x^*(0,y), \quad u_2(y) = -\frac{1}{2r_2(y)} g_2(y)V_x^*(l,y) \tag{12}$$

$$v_1(z) = \frac{1}{2s_1(z)} h_1(z)V_x^*(z,0), \quad v_2(z) = -\frac{1}{2s_2(z)} h_2(z)V_x^*(z,l).$$

By substituting the optimal control policy (12) in (11), the HJB is given by

$$\begin{aligned} H^* &= \int_0^l \int_0^l q(x,y,z)dydz \\ &-\frac{1}{4}\int_0^l \frac{g_1^2(y)}{r_1(y)} V_x^{*2}(0,y)dy - \frac{1}{4}\int_0^l \frac{g_2^2(y)}{r_2(y)} V_x^{*2}(l,y)dy \\ &-\frac{1}{4}\int_0^l \frac{h_1^2(z)}{s_1(z)} V_x^{*2}(z,0)dz - \frac{1}{4}\int_0^l \frac{h_2^2(z)}{s_2(z)} V_x^{*2}(z,l)dz \\ &-\int_0^l \int_0^l \left(\frac{\partial V_x^*}{\partial z}\frac{\partial x}{\partial z} + \frac{\partial V_x^*}{\partial y}\frac{\partial x}{\partial y}\right)dzdy \\ &+\int_0^l \int_0^l V_x^*\left(-x\frac{\partial x}{\partial z} - x\frac{\partial x}{\partial y} + f(x,z,y)\right)dzdy. \end{aligned} \tag{13}$$

Remark: The control inputs (12) at all boundary edges depend on the value functional $\bar{V}^*$ which is the solution to the HJB for the Burgers equation expressed as 2D PDE. The HJB for multi-dimensional nonlinear PDE problem cannot be generally solved separately for each dimension and therefore two decoupled optimal controllers for y and z directions are not a substitute for the proposed control scheme in this paper.

Since the nonlinear function $f(x,z,y)$ in the system dynamics is unknown, it is necessary to introduce an identifier prior to control synthesis. Therefore, in the next section the design of NN identifier will be introduced.

2.1. IDENTIFIER DESIGN

Since f is a function over $X \times [0,l]^2 \to C(0,l)^2$, and the domain $X \times [0,l]^2$ is a Banach space, there exists a compact set $\Omega \subset X \times [0,l]^2$, such that for $(x,z,y) \in \Omega$ by choosing a smooth bounded basis function vector $\sigma_f : \mathcal{H}_1[0,l]^2 \times [0,l]^2 \to [C(0,l)^2]^m$ with the uniform norm bound σ_{fM} the function $f(x,z,y)$ can be written as [16]

$$f(x,z,y) = W_f^T \sigma_f(x,z,y) + \varepsilon_f(x,z,y), \tag{14}$$

where $W_f \in \mathbb{R}^m$ is the target weight vector with the Euclidean norm bound W_{fM} and ε_f is the approximation error with the bound ε_{fM} . Hence, the NN approximation of the nonlinearity is given by

$$\hat{f}(x,z,y) = \hat{W}_f^T \sigma_f(x,z,y). \tag{15}$$

In order to find the tuning law for the estimated weight vector $\hat{W}_f$, a state estimator with following PDE dynamics is initially considered as

$$\frac{\partial\hat{x}}{\partial t}=\frac{\partial^2\hat{x}}{\partial z^2}+\frac{\partial^2\hat{x}}{\partial y^2}-x\frac{\partial x}{\partial z}-x\frac{\partial x}{\partial y}+\hat{f}(x,z,y)+k\tilde{x},$$
$$\frac{\partial\hat{x}}{\partial z}|_{z=0,y}=g_1(y)u_1(y),\ \ \frac{\partial\hat{x}}{\partial z}|_{z=l,y}=g_2(y)u_2(y), \tag{16}$$
$$\frac{\partial\hat{x}}{\partial z}|_{z,y=0}=h_1(z)v_1(z),\ \ \frac{\partial\hat{x}}{\partial z}|_{z,y=l}=h_2(z)v_2(z).$$

The state estimation error $\tilde{x}$ is defined as $\tilde{x}=x-\hat{x}$.

Select NN tuning law for $\hat{W}_f$ as

$$\dot{\hat{W}}_f=-\alpha_f\hat{W}_f+\int_0^l\int_0^l\sigma_f(x,z,y)\tilde{x}dydz. \tag{17}$$

Accordingly, the state estimation error $\tilde{x}$ dynamics will be governed by

$$\frac{\partial\tilde{x}}{\partial t}=\frac{\partial^2\tilde{x}}{\partial z^2}+\frac{\partial^2\tilde{x}}{\partial y^2}+\tilde{W}_f\sigma_f(x,y,z)+\varepsilon_f(x,y,z)-k\tilde{x},$$
$$\frac{\partial\tilde{x}}{\partial z}|_{z=0,y}=\frac{\partial\tilde{x}}{\partial z}|_{z=l,y}=0,$$
$$\frac{\partial\tilde{x}}{\partial z}|_{z,y=0}==\frac{\partial\tilde{x}}{\partial z}|_{z,y=l}=0, \tag{18}$$

where the NN weight estimation error for the identifier is given by $\tilde{W}_f=W_f-\hat{W}_f$. Noting $\dot{\tilde{W}}_f=-\dot{\hat{W}}_f$, the dynamics of NN identifier weight estimation error can be represented as

$$\dot{\tilde{W}}_f=\alpha_f\hat{W}_f-\int_0^l\int_0^l\sigma_f(x,z,y)\tilde{x}dydz. \tag{19}$$

Since an estimate of system dynamics is now available, in the next section, the ADP approximate optimal control policy will be addressed.

2.2. APPROXIMATE CONTROLLER DESIGN

Equation (13) is generally a nonlinear quadratic partial integro-differential equation (PIDE) and therefore it has no closed-form solution. Therefore, the objective is to find a suitable structure for estimation of $V^*(x,z)$ in (13). For this purpose, it is assumed that $V^*(x,z)$ is an integral functional over $[\mathcal{H}_1]^n\times[0,l]^2\to C_1[0,l]^2$, and therefore there exists a smooth bounded basis function vector $\Phi:[\mathcal{H}_1]^n\times[0,l]^2\to[C_1[0,l]^2]^p$ with the uniform norm bound Φ_M such that [16]

$$V^*(x,z,y)=W_\Phi^T\Phi(x,z,y)+\varepsilon_V(x,z,y). \tag{20}$$

Assumption 2: It is assumed that the weight vector $W_\Phi\in\mathbb{R}^p$ is bounded with the bound $W_{\Phi M}$ and the derivative of approximation error with respect to x, i.e. ε_{Vx} and its second and third derivatives with respect to z and y are bounded by uniform norm bounds ε_{VxM} , ε_{VxzM} , ε_{VxyM} , ε_{VxzzM} and ε_{VxyyM} [16] .

Therefore the optimal control input can be written as

$$\begin{aligned} u_1^*(y) &= \frac{1}{2r_1(y)} g_1(y)\Phi_x^T(x,0,y)W_\Phi + \varepsilon_{u_1}, \\ u_2^*(y) &= -\frac{1}{2r_2(y)} g_2(y)\Phi_x^T(x,l,y)W_\Phi + \varepsilon_{u_2}, \\ v_1^*(z) &= \frac{1}{2s_1(z)} h_1(z)\Phi_x^T(x,z,0)W_\Phi + \varepsilon_{v_1}, \\ v_2^*(z) &= -\frac{1}{2s_2(z)} h_2(z)\Phi_x^T(x,z,l)W_\Phi + \varepsilon_{v_2}, \end{aligned} \tag{21}$$

where

$$\begin{cases} \varepsilon_{u_1} = \dfrac{1}{2r_1(y)} g_1(y)\varepsilon_{Vx}(x,0,y), \\ \varepsilon_{u_2} = -\dfrac{1}{2r_2(y)} g_2(y)\varepsilon_{Vx}(x,l,y), \\ \varepsilon_{v_1} = \dfrac{1}{2s_1(z)} h_1(z)\varepsilon_{Vx}(x,z,0), \\ \varepsilon_{v_2} = -\dfrac{1}{2s_2(z)} h_2(z)\varepsilon_{Vx}(x,z,l). \end{cases} \tag{22}$$

Consequently, the HJB equation can be represented in its approximated form as

$$\begin{aligned} & H^* = 0 = \int_0^l\int_0^l q(x,y,z)dydz - \frac{1}{4}\int_0^l W_\Phi^T\Phi_x(0,y) \\ & \frac{g_1^2(y)}{r_1(y)}\Phi_x{}^T(0,y)W_\Phi dy - \frac{1}{4}\int_0^l W_\Phi^T\Phi_x(l,y)\frac{g_2^2(y)}{r_2(y)}\Phi_x{}^T(l,y) \\ & W_\Phi dy - \frac{1}{4}\int_0^l W_\Phi^T\Phi_x(z,0)\frac{h_1^2(z)}{s_1(z)}W_\Phi^T\Phi_x(z,0)dz \\ & -\frac{1}{4}\int_0^l W_\Phi^T\Phi_x(z,l)\frac{h_2^2(z)}{s_2(z)}W_\Phi^T\Phi_x(z,l)dz \\ & -\int_0^l\int_0^l (W_\Phi^T\frac{\partial\Phi_x}{\partial z}\frac{\partial x}{\partial z} + W_\Phi^T\frac{\partial\Phi_x}{\partial y}\frac{\partial x}{\partial y})dzdy \\ & +\int_0^l\int_0^l W_\Phi^T\Phi_x(-x\frac{\partial x}{\partial z} - x\frac{\partial x}{\partial y} + W_f^T\sigma_f(z,y))dzdy + \varepsilon_H, \end{aligned} \tag{23}$$

where the function Φ is expressed as $\Phi(z,y)$ for brevity, using similar notation for functions ε_V and σ_f in the following equation, ε_H can be derived as

$$\begin{aligned} \varepsilon_H = & -\frac{1}{4}\int_0^l \frac{g_1^2(y)}{r_1(y)}\varepsilon_{Vx}^2(y,0)dy - \frac{1}{4}\int_0^l \frac{g_2^2(y)}{r_2(y)}\varepsilon_{Vx}^2(y,l)dy \\ & -\frac{1}{4}\int_0^l \frac{h_1^2(z)}{s_1(z)}\varepsilon_{Vx}^2(0,z)dz - \frac{1}{4}\int_0^l \frac{h_2^2(z)}{s_2(z)}\varepsilon_{Vx}^2(l,z)dz \end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}\int_0^l W_\Phi^T \Phi_x(0,y)\frac{g_1^2(y)}{r_1(y)}\varepsilon_{Vx}(0,y)W_\Phi dy-\frac{1}{2}\int_0^l W_\Phi^T \Phi_x(l,y)\\
&\frac{g_2^2(y)}{r_2(y)}\varepsilon_{Vx}(l,y)W_\Phi dy-\frac{1}{2}\int_0^l W_\Phi^T \Phi_x(z,0)\frac{h_1^2(z)}{s_1(z)}\varepsilon_{Vx}(z,0)dz\\
&-\frac{1}{2}\int_0^l W_\Phi^T \Phi_x(z,l)\frac{h_2^2(z)}{s_2(z)}\varepsilon_{Vx}(z,l)dz\\
&-\int_0^l\int_0^l\left(\frac{\partial \varepsilon_{Vx}}{\partial z}\frac{\partial x}{\partial z}+\frac{\partial \varepsilon_{Vx}}{\partial y}\frac{\partial x}{\partial y}\right)dzdy+\int_0^l\int_0^l \varepsilon_{Vx}\left(-x\frac{\partial x}{\partial z}\right.\\
&\left.-x\frac{\partial x}{\partial y}+W_f^T\sigma_f(z,y)\right)dzdy+\int_0^l\int_0^l W_\Phi^T\Phi_x\varepsilon_f(z,y)dzdy.
\end{aligned}
\tag{24}
$$

The approximated Hamiltonian can be represented as

$$
\begin{aligned}
&\hat{H}=\int_0^l\int_0^l q(x,y,z)dydz-\frac{1}{4}\int_0^l \hat{W}_\Phi^T\Phi_x(0,y)\frac{g_1^2(y)}{r_1(y)}\\
&\Phi_x{}^T(0,y)\hat{W}_\Phi dy-\frac{1}{4}\int_0^l \hat{W}_\Phi^T\Phi_x(l,y)\frac{g_2^2(y)}{r_2(y)}\Phi_x{}^T(l,y)\hat{W}_\Phi dy\\
&-\frac{1}{4}\int_0^l \hat{W}_\Phi^T\Phi_x(z,l)\frac{h_1^2(z)}{s_1(z)}\hat{W}_\Phi^T\Phi_x(z,l)dz-\frac{1}{4}\int_0^l \hat{W}_\Phi^T\Phi_x(z,l)\\
&\frac{h_2^2(z)}{s_2(z)}\hat{W}_\Phi^T\Phi_x(z,l)dz-\int_0^l\int_0^l\left(\hat{W}_\Phi^T\frac{\partial \Phi_x}{\partial z}\frac{\partial x}{\partial z}+\hat{W}_\Phi^T\frac{\partial \Phi_x}{\partial y}\frac{\partial x}{\partial y}\right)\\
&dzdy+\int_0^l\int_0^l \hat{W}_\Phi^T\Phi_x\left(-x\frac{\partial x}{\partial z}-x\frac{\partial x}{\partial y}+\hat{W}_f^T\sigma_f(z,y)\right)dzdy,
\end{aligned}
\tag{25}
$$

Next, the control policy in terms of estimated value functional weight vector becomes

$$
\begin{cases}
\hat{u}_1(y)=\dfrac{1}{2r_1(y)}g_1(y)\Phi_x^T(0,y)\hat{W}_\Phi,\\
\hat{u}_2(y)=-\dfrac{1}{2r_2(y)}g_2(y)\Phi_x^T(l,y)\hat{W}_\Phi\\
\hat{v}_1(z)=\dfrac{1}{2s_1(z)}h_1(z)\Phi_x^T(z,0)\hat{W}_\Phi,\\
\hat{v}_2(z)=-\dfrac{1}{2s_2(z)}h_2(z)\Phi_x^T(z,l)\hat{W}_\Phi
\end{cases}
\tag{26}
$$

Select the value function NN weight tuning law as

$$
\dot{\hat{W}}_\Phi=-\alpha_1\frac{\omega_N}{\zeta_N^2}\hat{H}-\alpha_2\hat{W}_\Phi-\alpha_3\hat{W}_\Phi\left\|\hat{W}_\Phi\right\|^2,
\tag{27}
$$

where

$$\omega_N = \int_0^l \int_0^l \Phi_x(-x\frac{\partial x}{\partial z} - x\frac{\partial x}{\partial y} + \hat{W}_f^T \sigma_f(z,y))dzdy$$
$$-\int_0^l \int_0^l (\frac{\partial \Phi_x}{\partial z}\frac{\partial x}{\partial z} + \frac{\partial \Phi_x}{\partial y}\frac{\partial x}{\partial y})dzdy - \frac{1}{2}\int_0^l \Phi_x(0,y)\frac{g_1^2(y)}{r_1(y)}$$
$$\Phi_x^T(0,y)\hat{W}_\Phi dy - \frac{1}{2}\int_0^l \Phi_x(l,y)\frac{g_2^2(y)}{r_2(y)}\Phi_x^T(l,y)\hat{W}_\Phi dy$$
$$-\frac{1}{2}\int_0^l \Phi_x(z,0)\frac{h_1^2(z)}{s_1(z)}\hat{W}_\Phi^T \Phi_x(z,0)dz$$
$$-\frac{1}{2}\int_0^l \Phi_x(z,l)\frac{h_2^2(z)}{s_2(z)}\hat{W}_\Phi^T \Phi_x(z,l)dz, \tag{28}$$

$\alpha_1, \alpha_2, \alpha_3$ represent positive constants and

$$\zeta_N = c_1\|x\|_{\mathcal{L}_2} + c_2\|x^2\|_{\mathcal{L}_2} + c_3\|x^2(z,l)\|_{\mathcal{L}_2} + c_4\|x^2(z,0)\|_{\mathcal{L}_2}$$
$$+c_5\|x^2(l,y)\|_{\mathcal{L}_2} + c_6\|x^2(l,y)\|_{\mathcal{L}_2} + c_7\|x(l,y)\|_{\mathcal{L}_2}$$
$$+c_8\|x(0,y)\|_{\mathcal{L}_2} + c_9\|x(z,l)\|_{\mathcal{L}_2} + c_{10}\|x(z,0)\|_{\mathcal{L}_2} + c_{11} \tag{29}$$

with $c_1,...,c_{11}$ being positive constants. Next the stability analysis of the closed-loop system is fully analyzed by using Lyapunov criterion. Specifically, ultimate boundedness (UB) of all closed-loop states for Neumann boundary control condition will be addressed.

3. STABILITY ANALYSIS

System closed-loop stability will be fully analyzed by using Lyapunov criterion. First, it will be shown in Theorem 1 that in presence of bounded input, the identifier dynamics and estimation error will be bounded. In Lemma 1, a bound will be found for Hamiltonian reconstruction error ε_H when Neumann boundary control policy is implemented. This bound will be used later in closed-loop stability proof. Consequently, Theorem 2 will address ultimately boundedness (UB) of all closed-loop states for Neumann boundary control condition.

Theorem 1 *(Boundedness of the NN identifier)*: Let the initial NN identifier weight estimation error $\tilde{W}_f$ and state estimation error $\tilde{x}$ be residing in a compact set $\mho_1$, and the proposed NN identifier, the state estimator and NN weight tuning law be provided by (15), (16) and (17), respectively. In the presence of bounded inputs, there exists a positive definite gain $k > 1/2$ and tuning parameter $\alpha_f > 0$ for the identifier weight update law such that the state estimation error $\tilde{x}$ and weight estimation error $\tilde{W}_f$ are all *UB*.

Proof: Refer to Appendix.

The boundedness of the control input is relaxed in the main closed-loop stability theorem. The following lemma is needed before we proceed.

Lemma 1: The following inequality

$$\begin{aligned}
&|\varepsilon_H| \le c_{H1}\|x\|_{\mathcal{L}_2} + c_{H2}\|x^2\|_{\mathcal{L}_2} + c_{H3}\|x^2(z,l)\|_{\mathcal{L}_2} \\
&+c_{H4}\|x^2(z,0)\|_{\mathcal{L}_2} + c_{H5}\|x^2(l,y)\|_{\mathcal{L}_2} + c_{H6}\|x^2(l,y)\|_{\mathcal{L}_2} \\
&+c_{H7}\|x(l,y)\|_{\mathcal{L}_2} + c_{H8}\|x(0,y)\|_{\mathcal{L}_2} + c_{H9}\|x(z,l)\|_{\mathcal{L}_2} \\
&+c_{H10}\|x(z,0)\|_{\mathcal{L}_2} + c_{H11}
\end{aligned} \tag{30}$$

holds where $c_{H1},...,c_{H11}$ are positive constants. Moreover note that $|\varepsilon_H| \le \xi_{NM}\zeta_N$ where ζ_N is defined in (29) and ξ_{NM} is a positive constant.

Proof: Refer to the Appendix.

Theorem 2 *(Performance of the state feedback NN controller)*: Consider the DPS given by (1) and let the NN identifier, the state estimator and NN weight update law be provided by (15), (16) and (17), respectively. Moreover, let control policy and tuning law for value function weights be provided by (26) and (27), respectively where $0 < \alpha_1 < 1$, $\alpha_2 > \alpha_3$ and $\alpha_3 > \theta_{\alpha_3}^N$ with $\theta_{\alpha_3}^N$ being a positive constant. Then the system state x, state estimation error $\tilde{x}$ and weight estimation errors $\tilde{W}_f$ and $\tilde{W}_\Phi$ are all UB.

Proof: Refer to the Appendix.

In the next section, the performance of proposed ADP based NN controller is examined on the 2D Burgers PDE via simulation.

4. SIMULATIONS

We consider the plant with the PDE dynamics

$$\begin{aligned}
&\frac{\partial x}{\partial t} = \frac{\partial^2 x}{\partial z^2} + \frac{\partial^2 x}{\partial y^2} - x\frac{\partial x}{\partial z} - x\frac{\partial x}{\partial y} + f(x,z,y),\\
&\frac{\partial x}{\partial z}|_{z=0,y} = u_1(y),\ \ \frac{\partial x}{\partial z}|_{z=l,y} = u_2(y),\\
&\frac{\partial x}{\partial y}|_{z,y=0} = v_1(z),\ \ \frac{\partial x}{\partial y}|_{z,y=l} = v_2(z),\\
&x(z,y,0) = x_0(z,y),
\end{aligned} \tag{31}$$

in the region $z, y \in (0,1)\times(0,1)$. The unknown nonlinearity $f(x,z,y)$ is given by

$$f(x,z,y) = \frac{100}{1+100((z-0.5)^2+(y-0.25)^2)}x^2, \tag{32}$$

and state initial condition is expressed as

$$x_0(z,y) = \frac{10}{1+100((z-0.75)^2+(y-0.25)^2)}. \tag{33}$$

The 4th order Runge-Kutta finite difference method was used for numerical simulation of Burgers PDE with $\Delta z = \Delta y = 0.05$ and $\Delta t = 5\times10^{-4}$. The function $q(x,z,y)$ in cost functional (4) was chosen to be

$$q(x,z,y) = \frac{1}{1+10((z-0.5)^2+(y-0.25)^2)}x^2, \tag{34}$$

and $r_1 = r_2 = s_1 = s_2 = 1$. The NN activation functions for identifying $f(x)$ were chosen as

$$\sigma(x,z,y) = \frac{1}{1+10((z-z_i)^2+(y-y_i)^2)}\lambda(x), \tag{35}$$

where $\lambda(x) \in \{x, x^2\}$. Forty eight basis functions were chosen to approximate $V^*(x,z,y)$ with following structure

$$\begin{aligned}
&\Phi_i(x,z,y) = \frac{1}{1+10((z-z_i)^2+(y-y_i)^2)}\\
&\iint_{\Delta_i} c_j\psi_j(x)dzdy,
\end{aligned} \tag{36}$$

where $\psi(x) \in \{x^2, x^4, x^6\}$, $c_j = 10^{-2}, 10^{-4}$ or 10^{-6} and spatial domain is divided into 16 identical square subregions Δ_i, $1 \le i \le 16$. The value functional update parameters were chosen to be $\alpha_1 = 0.9,\ \alpha_2 = 0.2,\ \alpha_3 = 0.1$, $c_{11} = 0.5$, $c_1 = c_{10} = 10^{-3}$ and identifier update parameters were selected as $\alpha_f = 2$ and $k = 50$.

Figure 4.1 verifies that the open loop system was highly unstable with these values for system parameters. Figure 4.2 confirms the instability of open loop PDE by displaying the divergent behavior of $\mathcal{L}_2$ norm of state for initial 100 msec.

Figure 4.3 depicts the closed-loop state evolution of system in space and time verifying that the boundary controller can stabilize the unstable states at zero in less than 0.5 second. Figure 4.4 shows the convergence of system state $x(z, y)$ to zero more clearly by displaying the norm of state x over time. As shown the boundary controller can stabilize the state norm at zero in less than 0.5 second. Stabilizing the norm of state is a very good indication that the states have converged to zero. Moreover, the norm history shows how elegantly the controller performs in presence of external disturbance.

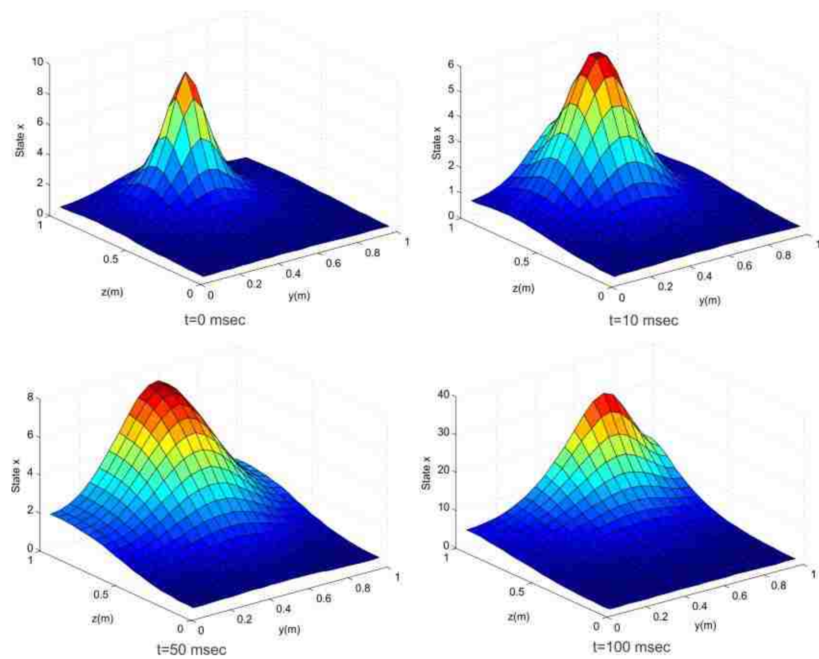

Figure 4.1. Snapshots of system open loop state behavior.

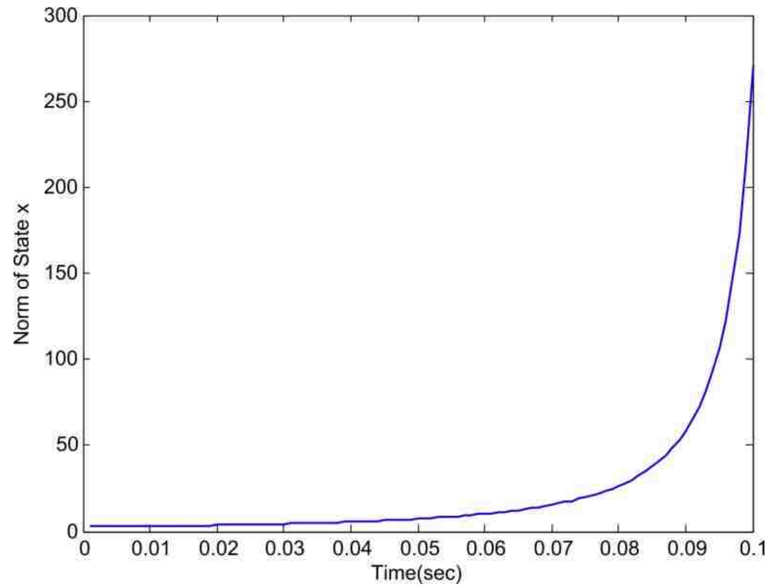

Figure 4.2. Open loop time history of norm of state x when there is no boundary. control.

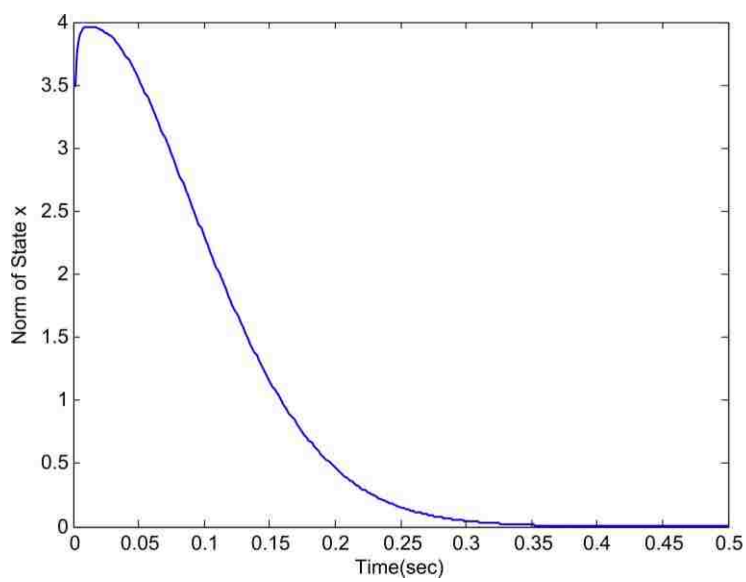

Figure 4.3. Time profile of norm of state using the proposed boundary control policy.

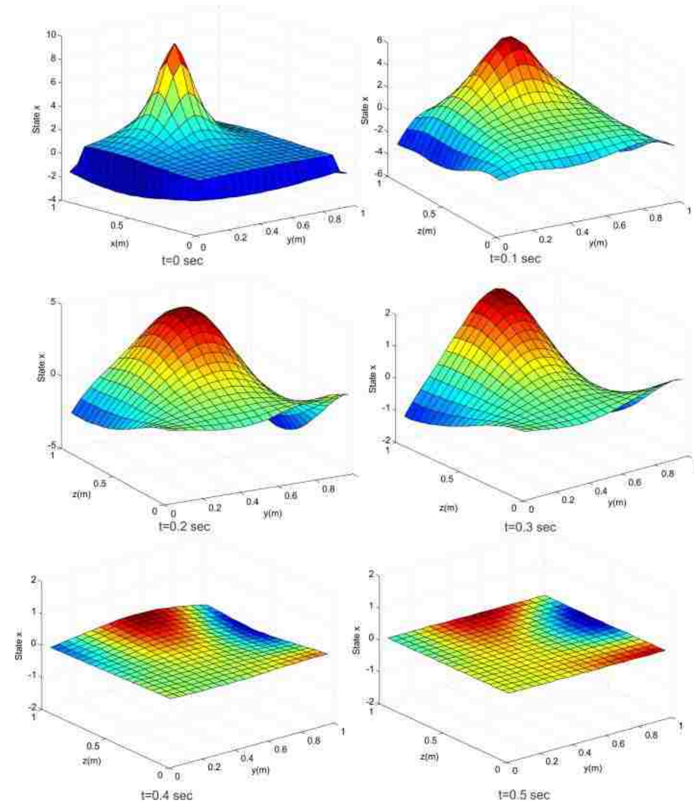

Figure 4.4. Snapshots of closed loop state behavior.

Figures 4.5 and 4.6 show the time profile of control policy and convergence of HJB error, respectively. The $\mathcal{L}_2$ norm of boundary control inputs are solely shown in Fig 5 for brevity. This figure shows that the initial admissible control effort is relatively high since the PDE dynamics are unstable. Subsequently, faster convergence of HJB error in this figure compared to state trajectory and control inputs verifies that the HJB error is reduced by the proposed adaptive law before convergence of the states to zero. Moreover, increasing the value of α_1 in the tuning law (27) results in a more rapid convergence of the HJB error.

Finally, Figure 4.7 verifies the performance of proposed identifier developed in Section 2.1. As shown, increasing the gain k in state estimator has a significant role in convergence of the identification error to zero.

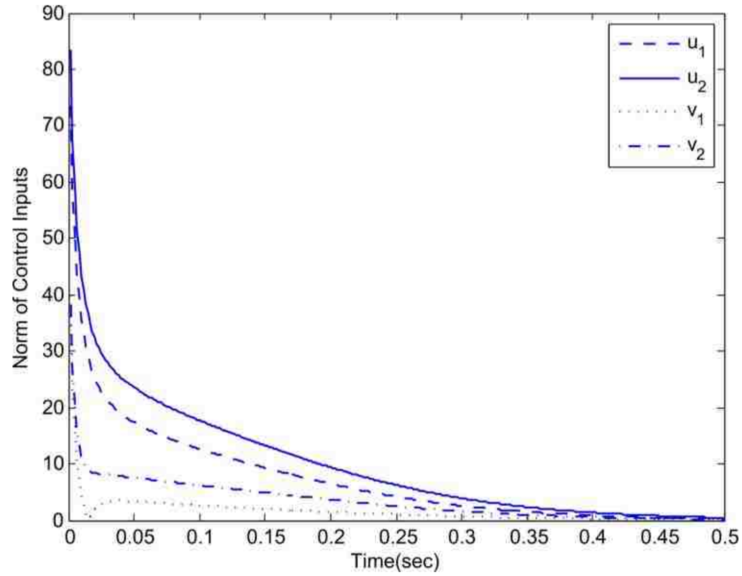

Figure 4.5. Time profile of norm of boundary control inputs.

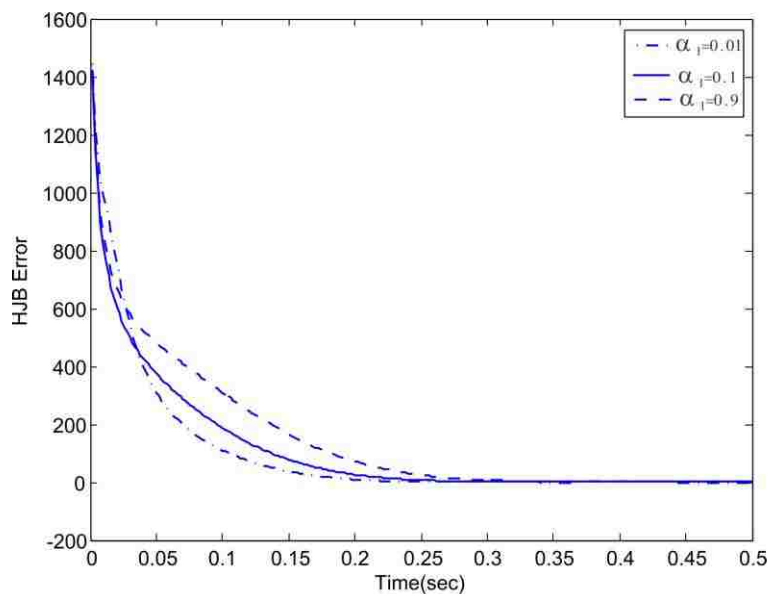

Figure 4.6. Time profile of HJB error for different values of design parameters.

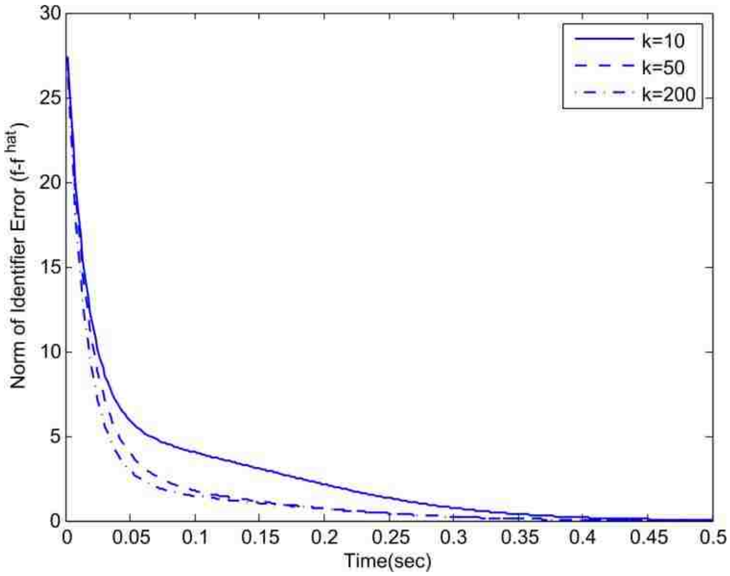

Figure 4.7. Time profile of identifier error for different values of design parameters.

5. CONCLUSIONS

This paper introduced a novel ADP optimal boundary control scheme for 2D Burgers equation under Neumann boundary condition without any model reduction prior to the control design. By formulating the HJB equation based on calculus in infinite dimensional state space, boundary optimal control policy was derived. The proposed adaptive NN framework made online approximation of optimal control policy along system trajectories possible whereas the PDE dynamics were partially unknown. Lyapunov stability analysis indicated the error between approximated and truly optimal weights and state trajectories would remain UB. Simulation results confirm the effectiveness of developed boundary control approach on an unstable 2D Burgers equation.

APPENDIX

Proof of Theorem 1: In order to prove stability of state feedback identifier, consider the Lyapunov function

$$L_c = \frac{1}{2}\mu_c \int_0^l \int_0^l \tilde{x}^T \tilde{x} dydz + \frac{1}{2}\mu_c \tilde{W}_f^T \tilde{W}_f , \quad (37)$$

where μ_c is a positive scalar. Taking its derivative $\dot{L}_c$ and substituting error dynamics (18) and (19) yields

$$\dot{L}_c = \mu_c \int_0^l \int_0^l \tilde{x}(\frac{\partial^2 \tilde{x}}{\partial z^2} + \frac{\partial^2 \tilde{x}}{\partial y^2} + \tilde{W}_f \sigma_f (x,y,z) + \varepsilon_f - k\tilde{x}) dydz \quad (38)$$

$$+\mu_c \tilde{W}_f^T (\alpha_f \hat{W}_f - \int_0^l \int_0^l \sigma_f (x,z,y)\tilde{x} dydz) = -\mu_c k \int_0^l \int_0^l \tilde{x}^2 dydz$$
$$+\mu_c \int_0^l \int_0^l \tilde{x}\varepsilon_f dydz + \mu_c \int_0^l \int_0^l \tilde{x}(\frac{\partial^2 \tilde{x}}{\partial z^2} + \frac{\partial^2 \tilde{x}}{\partial y^2}) dydz$$
$$+\mu_c \alpha_f \tilde{W}_f^T (W_f - \tilde{W}_f).$$

Taking integration by parts and substituting PDE dynamics (18) for state estimation error $\tilde{x}$ yield

$$\int_0^l \int_0^l \tilde{x}(\frac{\partial^2 \tilde{x}}{\partial z^2} + \frac{\partial^2 \tilde{x}}{\partial y^2}) dydz \le \int_0^l \tilde{x}(l,y)\frac{\partial \tilde{x}}{\partial z}|_{z=l,y}$$
$$-\tilde{x}(0,y)\frac{\partial \tilde{x}}{\partial z}|_{z=0,y}\, dy + \int_0^l \tilde{x}(z,l)\frac{\partial \tilde{x}}{\partial y}|_{z,y=l} - \tilde{x}(z,0)\frac{\partial \tilde{x}}{\partial y}|_{z,y=0} dz \quad (39)$$
$$-\int_0^l \int_0^l (\frac{\partial \tilde{x}}{\partial z})^2 + (\frac{\partial \tilde{x}}{\partial y})^2 dydz = -\int_0^l \int_0^l (\frac{\partial \tilde{x}}{\partial z})^2 + (\frac{\partial \tilde{x}}{\partial y})^2 dydz \le 0.$$

Therefore, using Young's inequality $(\beta_1\beta_2 \le \frac{1}{2}\beta_1^2 + \frac{1}{2}\beta_2^2)$ yields

$$\dot{L}_c \le -\mu_c k \int_0^l \int_0^l \tilde{x}^2 dydz + \frac{1}{2}\mu_c \int_0^l \int_0^l \tilde{x}^2 dydz + \frac{l^2}{2}\varepsilon_{fM}^2$$
$$-\frac{1}{2}\mu_c\alpha_f \left\|\tilde{W}_f\right\|^2 + \frac{1}{2}\mu_c\alpha_f \left\|W_f\right\|^2 = -\mu_c(k - \frac{1}{2})\left\|\tilde{x}\right\|_{\mathcal{L}_2}^2$$
$$-\frac{1}{2}\mu_c\alpha_f \left\|\tilde{W}_f\right\|^2 + \varepsilon_c, \quad (40)$$

where $\varepsilon_c = \mu_c \frac{l^2}{2}\varepsilon_{fM}^2 + \frac{1}{2}\mu_c \alpha_f \left\|W_f\right\|^2$. Therefore by selecting $k > \frac{1}{2}$, the identifier dynamics are UB with the bounds for $\left\|\tilde{x}\right\|_{\mathcal{L}_2}^2$ and $\left\|\tilde{W}_f\right\|^2$ can be arbitrarily reduced through the selection of design parameters.

Proof of Lemma 1: Define $\varepsilon_{Vxz} = \frac{\partial \varepsilon_{Vx}}{\partial z}$, $\varepsilon_{Vxy} = \frac{\partial \varepsilon_{Vx}}{\partial y}$, $\varepsilon_{Vxzz} = \frac{\partial^2 \varepsilon_{Vx}}{\partial z^2}$, $\varepsilon_{Vxyy} = \frac{\partial^2 \varepsilon_{Vx}}{\partial y^2}$, and ε_H can be derived using integration by parts as

$$
\begin{aligned}
\varepsilon_H &= -\frac{1}{4}\int_0^l \frac{g_1^2(y)}{r_1(y)}\varepsilon_{Vx}^2(y,0)dy - \frac{1}{4}\int_0^l \frac{g_2^2(y)}{r_2(y)}\varepsilon_{Vx}^2(y,l)dy \\
&-\frac{1}{4}\int_0^l \frac{h_1^2(z)}{s_1(z)}\varepsilon_{Vx}^2(0,z)dz - \frac{1}{4}\int_0^l \frac{h_2^2(z)}{s_2(z)}\varepsilon_{Vx}^2(l,z)dz \\
&-\frac{1}{2}\int_0^l W_\Phi^T \Phi_x(0,y)\frac{g_1^2(y)}{r_1(y)}\varepsilon_{Vx}(0,y)dy - \frac{1}{2}\int_0^l W_\Phi^T \Phi_x(l,y) \\
&\frac{g_2^2(y)}{r_2(y)}\varepsilon_{Vx}(l,y)dy - \frac{1}{2}\int_0^l W_\Phi^T \Phi_x(z,0)\frac{h_1^2(z)}{s_1(z)}\varepsilon_{Vx}(z,0)dz \\
&-\frac{1}{2}\int_0^l W_\Phi^T \Phi_x(z,l)\frac{h_2^2(z)}{s_2(z)}\varepsilon_{Vx}(z,l)dz + \int_0^l\int_0^l \varepsilon_{Vxzz} x dz dy \\
&-\int_0^l [\varepsilon_{Vxz}\,|_{z=l,y}\; x(l,y)dy - \varepsilon_{Vxz}\,|_{z=0,y}\; x(0,y)dy] \\
&+\int_0^l\int_0^l \varepsilon_{Vxyy} x dz dy - \int_0^l [\varepsilon_{Vxy}\,|_{y=l,z}\; x(z,l)dz \\
&-\varepsilon_{Vxy}\,|_{y=0,z}\; x(z,0)dz] + \int_0^l\int_0^l \varepsilon_{Vx} W_f^T \sigma_f(z,y)dz dy \\
&+\int_0^l\int_0^l \varepsilon_{Vxz} x^2 dz dy - \int_0^l [\varepsilon_{Vx}\,|_{z=l,y}\; x^2(l,y)dy \\
&-\varepsilon_{Vx}\,|_{z=0,y}\; x^2(0,y)dy] + \int_0^l\int_0^l \varepsilon_{Vxy} x^2 dz dy \\
&-\int_0^l [\varepsilon_{Vx}\,|_{y=l,z}\; x^2(z,l)dy - \varepsilon_{Vx}\,|_{y=0,z}\; x^2(z,l)dy].
\end{aligned}
\tag{41}
$$

Therefore

$$
\begin{aligned}
\left|\varepsilon_H\right| &\le c_{H1}\left\|x\right\|_{\mathcal{L}_2} + c_{H2}\left\|x^2\right\|_{\mathcal{L}_2} + c_{H3}\left\|x^2(z,l)\right\|_{\mathcal{L}_2} \\
&+c_{H4}\left\|x^2(z,0)\right\|_{\mathcal{L}_2} + c_{H5}\left\|x^2(l,y)\right\|_{\mathcal{L}_2} + c_{H6}\left\|x^2(l,0)\right\|_{\mathcal{L}_2} \\
&+c_{H7}\left\|x(l,y)\right\|_{\mathcal{L}_2} + c_{H8}\left\|x(0,y)\right\|_{\mathcal{L}_2} + c_{H9}\left\|x(z,l)\right\|_{\mathcal{L}_2} \\
&+c_{H10}\left\|x(z,0)\right\|_{\mathcal{L}_2} + c_{H11}.
\end{aligned}
\tag{42}
$$

where,

$c_{H1} = l(\varepsilon_{VxzzM} + \varepsilon_{VxyyM})$, $c_{H2} = l\varepsilon_{VxzM}$, $c_{H3} = c_{H4} = \sqrt{l}\varepsilon_{VxM}$, $c_{H5} = c_{H6} = \sqrt{l}\varepsilon_{VxM}$, $c_{H7} = c_{H8} = \sqrt{l}\varepsilon_{VxzM}$,

$c_{H9} = c_{H10} = \sqrt{l}\varepsilon_{VxyM}$ and $c_{H11} = \frac{1}{4}\sqrt{l}(\sum_{i=1}^{4}\gamma_i)\varepsilon_{VxM}^2 + \frac{1}{2}W_{\Phi M}\Phi_{xM}(\sum_{i=1}^{4}\gamma_i)\varepsilon_{VxM}$ with $\gamma_1 = \max_y \frac{g_1^2(y)}{r_1(y)}$, $\gamma_2 = \max_y \frac{g_2^2(y)}{r_2(y)}$, $\gamma_3 = \max_z \frac{h_1^2(z)}{s_1(z)}$ and $\gamma_4 = \max_z \frac{h_2^2(z)}{s_2(z)}$.

Moreover, recalling ζ_N in (29), $|\varepsilon_H|$ can be expressed as

$$
\begin{aligned}
|\varepsilon_H| = (\frac{c_{H1}}{c_1})c_1\|x\|_{\mathcal{L}_2} + (\frac{c_{H2}}{c_2})c_2\|x^2\|_{\mathcal{L}_2} + (\frac{c_{H3}}{c_3})c_3 \\
\|x^2(z,l)\|_{\mathcal{L}_2} + (\frac{c_{H4}}{c_4})c_4\|x^2(z,0)\|_{\mathcal{L}_2} + (\frac{c_{H5}}{c_5})c_5\|x^2(l,y)\|_{\mathcal{L}_2} \\
+(\frac{c_{H6}}{c_6})c_6\|x^2(l,0)\|_{\mathcal{L}_2} + (\frac{c_{H7}}{c_7})c_7\|x(l,y)\|_{\mathcal{L}_2} \\
+(\frac{c_{H8}}{c_8})c_8\|x(0,y)\|_{\mathcal{L}_2} + (\frac{c_{H9}}{c_9})c_9\|x(z,l)\|_{\mathcal{L}_2} \\
+(\frac{c_{H10}}{c_{10}})c_{10}\|x(z,0)\|_{\mathcal{L}_2} + (\frac{c_{H11}}{c_{11}})c_{11} \le \xi_{NM}\zeta_N,
\end{aligned}
\tag{43}
$$

where $\xi_{NM} = \max_{1\le i\le 11}\frac{c_{Hi}}{c_i}$.

Proof of Theorem 2: Since $H^* = 0$, $\hat{H} = \hat{H} - H^*$ and it can be represented in the following form:

$$
\begin{aligned}
\hat{H} = -\int_0^l\int_0^l((\hat{W}_\Phi^T - W_\Phi^T)\frac{\partial\Phi_x}{\partial z}\frac{\partial x}{\partial z} + (\hat{W}_\Phi^T - W_\Phi^T) \\
\frac{\partial\Phi_x}{\partial y}\frac{\partial x}{\partial y})dzdy + \int_0^l\int_0^l(\hat{W}_\Phi^T - W_\Phi^T)\Phi_x(-x\frac{\partial x}{\partial z} - x\frac{\partial x}{\partial y})dzdy \\
+\int_0^l\int_0^l\hat{W}_\Phi^T\Phi_x(\hat{W}_f^T\sigma_f(z,y))dzdy - \int_0^l\int_0^l W_\Phi^T\Phi_x(W_f^T\sigma_f(z,y)) \\
dzdy - \frac{1}{4}\int_0^l\hat{W}_\Phi^T\Phi_x(0,y)\frac{g_1^2(y)}{r_1(y)}\Phi_x{}^T(0,y)\hat{W}_\Phi dy - \frac{1}{4}\int_0^l\hat{W}_\Phi^T \\
\Phi_x(l,y)\frac{g_2^2(y)}{r_2(y)}\Phi_x{}^T(l,y)\hat{W}_\Phi dy - \frac{1}{4}\int_0^l\hat{W}_\Phi^T\Phi_x(z,0)\frac{h_1^2(z)}{s_1(z)}\hat{W}_\Phi^T
\end{aligned}
\tag{44}
$$

$$\Phi_x(z,0)dz-\frac{1}{4}\int_0^l \hat{W}_\Phi^T\Phi_x(z,l)\frac{h_2^2(z)}{s_2(z)}\hat{W}_\Phi^T\Phi_x(z,l)dz$$
$$-\frac{1}{4}\int_0^l W_\Phi^T\Phi_x(0,y)\frac{g_1^2(y)}{r_1(y)}\Phi_x{}^T(0,y)W_\Phi dy-\frac{1}{4}\int_0^l W_\Phi^T$$
$$\Phi_x(l,y)\frac{g_2^2(y)}{r_2(y)}\Phi_x{}^T(l,y)W_\Phi dy-\frac{1}{4}\int_0^l W_\Phi^T\Phi_x(z,0)\frac{h_1^2(z)}{s_1(z)}$$
$$W_\Phi^T\Phi_x(z,0)dz-\frac{1}{4}\int_0^l W_\Phi^T\Phi_x(z,l)\frac{h_2^2(z)}{s_2(z)}W_\Phi^T\Phi_x(z,l)dz-\varepsilon_H$$
$$=\tilde{W}_\Phi^T\omega_N-\int_0^l\int_0^l W_\Phi^T\Phi_x(\tilde{W}_f^T\sigma_f(z,y))dzdy+\frac{1}{4}\int_0^l\tilde{W}_\Phi^T$$
$$\Phi_x(0,y)\frac{g_1^2(y)}{r_1(y)}\Phi_x{}^T(0,y)\tilde{W}_\Phi dy+\frac{1}{4}\int_0^l\tilde{W}_\Phi^T\Phi_x(l,y)\frac{g_2^2(y)}{r_2(y)}$$
$$\Phi_x{}^T(l,y)\tilde{W}_\Phi dy+\frac{1}{4}\int_0^l\tilde{W}_\Phi^T\Phi_x(z,0)\frac{h_1^2(z)}{s_1(z)}\tilde{W}_\Phi^T\Phi_x(z,0)dz$$
$$+\frac{1}{4}\int_0^l\tilde{W}_\Phi^T\Phi_x(z,l)\frac{h_2^2(z)}{s_2(z)}\tilde{W}_\Phi^T\Phi_x(z,l)dz-\varepsilon_H.$$

Subsequently, expanding $-\tilde{W}_\Phi^T\hat{W}_\Phi\left\|\hat{W}_\Phi\right\|^2$ and using Young's and Cauchy-Schwarz inequalities to get

$$-\tilde{W}_\Phi^T\hat{W}_\Phi\left\|\hat{W}_\Phi\right\|^2=-\tilde{W}_\Phi^T(W_\Phi+\tilde{W}_\Phi)(W_\Phi+\tilde{W}_\Phi)^T$$
$$(W_\Phi+\tilde{W}_\Phi)=-\tilde{W}_\Phi^T W_\Phi\left\|W_\Phi\right\|^2-2(\tilde{W}_\Phi^T W_\Phi)^2-\left\|\tilde{W}_\Phi\right\|^2 \tag{45}$$
$$\left\|W_\Phi\right\|^2-3\tilde{W}_\Phi^T W_\Phi\left\|\tilde{W}_\Phi\right\|^2-\left\|\tilde{W}_\Phi\right\|^4\le\frac{1}{2}\left\|\tilde{W}_\Phi\right\|^2+\frac{1}{2}\left\|W_\Phi\right\|^6$$
$$+\frac{3}{4}\left\|\tilde{W}_\Phi\right\|^4+\frac{81}{4}\left\|W_\Phi\right\|^4-\left\|\tilde{W}_\Phi\right\|^4\le-\frac{1}{4}\left\|\tilde{W}_\Phi\right\|^4$$
$$+\frac{1}{2}\left\|\tilde{W}_\Phi\right\|^2+\frac{81}{4}\left\|W_\Phi\right\|^4+\frac{1}{2}\left\|W_\Phi\right\|^6.$$

Consequently $\dot{L}_a$ can be derived as

$$\dot{L}_a=-\alpha_1\frac{\tilde{W}_\Phi^T\omega_N}{\zeta_N^2}\{\tilde{W}_\Phi^T\omega_N-\int_0^l\int_0^l W_\Phi^T\Phi_x(\tilde{W}_f^T\sigma_f(z,y))dzdy$$
$$+\frac{1}{4}\int_0^l\tilde{W}_\Phi^T\Phi_x(0,y)\frac{g_1^2(y)}{r_1(y)}\Phi_x{}^T(0,y)\tilde{W}_\Phi dy+\frac{1}{4}\int_0^l\tilde{W}_\Phi^T$$
$$\Phi_x(l,y)\frac{g_2^2(y)}{r_2(y)}\Phi_x{}^T(l,y)\tilde{W}_\Phi dy+\frac{1}{4}\int_0^l\tilde{W}_\Phi^T\Phi_x(z,0)\frac{h_1^2(z)}{s_1(z)}$$
$$\tilde{W}_\Phi^T\Phi_x(z,0)dz+\frac{1}{4}\int_0^l\tilde{W}_\Phi^T\Phi_x(z,l)\frac{h_2^2(z)}{s_2(z)}\tilde{W}_\Phi^T\Phi_x(z,l)dz \tag{46}$$
$$-\varepsilon_H\}-\alpha_2\tilde{W}_\Phi^T(W_\Phi+\tilde{W}_\Phi)-\alpha_3\tilde{W}_\Phi^T\hat{W}_\Phi\left\|\hat{W}_\Phi\right\|^2.$$

According to Young's Inequality

$$\alpha_1 \frac{\tilde{W}_\Phi^T \omega_N}{\zeta_N^2}\left(\int_0^l\int_0^l W_\Phi^T \Phi_x (\tilde{W}_f^T \sigma_f(z,y))dzdy\right) \leq \alpha_1^2 \frac{\tilde{W}_\Phi^T \omega_N \omega_N^T \tilde{W}_\Phi}{6\zeta_N^2} + \frac{3}{2} l^4 W_{\Phi M}^2 \Phi_{xM}^2 \sigma_{fM}^2 \left\|\tilde{W}_f\right\|^2 . \tag{47}$$

Moreover,

$$\alpha_1 \frac{\tilde{W}_\Phi^T \omega_N}{\zeta_N^2}\left(\frac{1}{4}\int_0^l \tilde{W}_\Phi^T \Phi_x(0,y)\frac{g_1^2(y)}{r_1(y)}\Phi_x{}^T(0,y)\tilde{W}_\Phi dy\right) \leq \alpha_1^2 \frac{\tilde{W}_\Phi^T \omega_N \omega_N^T \tilde{W}_\Phi}{6\zeta_N^2} + \frac{3\gamma_1^2 \Phi_{xM}^4}{32\zeta_N^2}\left\|\tilde{W}_\Phi\right\|^4 , \tag{48}$$

Therefore,

$$\dot{L}_a \leq (-\alpha_1 + \alpha_1^2)\frac{\tilde{W}_\Phi^T \omega_N \omega_N^T \tilde{W}_\Phi}{\zeta_N^2} + \frac{3(\sum_{i=1}^4 \gamma_i^2)\Phi_{xM}^4}{32\zeta_N^2}\left\|\tilde{W}_\Phi\right\|^4 + \frac{3}{2} l^4 W_{\Phi M}^2 \Phi_{xM}^2 \sigma_{fM}^2 \left\|\tilde{W}_f\right\|^2 + \frac{3\varepsilon_H^2}{2\zeta_N^2}, \tag{49}$$

where γ_i , $1 \leq i \leq 4$ are defined in Lemma 1. By observing the fact that $\frac{1}{\zeta_N} \leq \frac{1}{c_{11}}$ for all $x \in [\mathcal{H}_2]^n$, if $\alpha_2 \geq \alpha_3$, $\alpha_3 \geq \theta_{\alpha_3}^N = \frac{3(\sum_{i=1}^4 \gamma_i^2)\Phi_{xM}^4}{32c_{11}^2}$ define $\eta_1 = \frac{1}{2}(\alpha_2 - \alpha_3)$ and $\eta_2 = \frac{1}{4}(\alpha_3 - \theta_{\alpha_3}^N)$. If $0 < \alpha_1 < 1$ then

$$\dot{L}_a \leq -\alpha_1(1-\alpha_1)\frac{\tilde{W}_\Phi^T \omega_N \omega_N^T \tilde{W}_\Phi}{\zeta_N^2} - \eta_1 \| \tilde{W}_\Phi \|^2 - \eta_2 \| \tilde{W}_\Phi \|^4 + \frac{3}{2} l^4 W_{\Phi M}^2 \Phi_{xM}^2 \sigma_{fM}^2 \left\|\tilde{W}_f\right\|^2 + \varepsilon_a , \tag{50}$$

where by using Lemma 2, $\varepsilon_a = \frac{3}{2}\xi_{NM}^2 + \frac{\alpha_2}{2} W_{\Phi M}^2 + \frac{81\alpha_3}{4} W_{\Phi M}^4 + \frac{\alpha_3}{2} W_{\Phi M}^6$.

Defining $L_b = \mu_b \int_0^l\int_0^l V^*(x,z,y)dzdy$ and taking its derivative $\dot{L}_b$ yields

$$\dot{L}_b = \mu_b \int_0^l\int_0^l V_x^*\left(\frac{\partial^2 x}{\partial z^2} + \frac{\partial^2 x}{\partial y^2} - x\frac{\partial x}{\partial z} - x\frac{\partial x}{\partial y} + f(x,z,y)\right)dzdy = \mu_b \int_0^l\int_0^l V_x^*\left(-x\frac{\partial x}{\partial z} - x\frac{\partial x}{\partial y} + f(x,z,y)\right)dzdy \tag{51}$$

$$+\mu_b\int_0^l\int_0^l V_x^*(\frac{\partial^2 x}{\partial z^2}+\frac{\partial^2 x}{\partial y^2})dzdy=\mu_b\int_0^l\int_0^l V_x^*(-x\frac{\partial x}{\partial z}-x\frac{\partial x}{\partial y}$$

$$+f(x,z,y))dzdy-\mu_b\int_0^l\int_0^l(\frac{\partial V_x^*}{\partial z}\frac{\partial x}{\partial z}+\frac{\partial V_x^*}{\partial y}\frac{\partial x}{\partial y})dzdy$$

$$-\mu_b\int_0^l V_x^*(0,y)g_1(y)(u_1^*(y)+\tilde{u}_1(y))dy+\mu_b\int_0^l V_x^*(l,y)$$

$$g_2(y)(u_2^*(y)+\tilde{u}_2(y))dy-\mu_b\int_0^l V_x^*(z,0)h_1(z)(v_1^*(z)$$

$$+\tilde{v}_1(z))dz+\mu_b\int_0^l V_x^*(z,l)h_2(z)(v_2^*(z)+\tilde{v}_2(z))dz$$

$$\le-\mu_b q_{\min}\|x\|^2+\mu_b\int_0^l V_x^*(0,y)g_1(y)(\frac{1}{2r_1(y)}g_1(y)$$

$$\Phi_x^T(0,y)\tilde{W}_\Phi-\varepsilon_{u_1})dy+\mu_b\int_0^l V_x^*(l,y)g_2(y)(-\frac{1}{2r_2(y)}$$

$$g_2(y)\Phi_x^T(l,y)\tilde{W}_\Phi-\varepsilon_{u_2})dy+\mu_b\int_0^l V_x^*(z,0)h_1(z)$$

$$(\frac{1}{2s_1(z)}h_1(z)\Phi_x^T(z,0)\tilde{W}_\Phi-\varepsilon_{v_1})dz+\mu_b\int_0^l V_x^*(z,l)h_2(z)$$

$$(-\frac{1}{2s_2(z)}h_2(z)\Phi_x^T(z,l)\tilde{W}_\Phi-\varepsilon_{v_2})dz$$

$$=-\mu_b q_{\min}\|x\|^2+\mu_b\int_0^l(W_\Phi^T\Phi_x(0,y)+\varepsilon_V)g_1(y)(\frac{1}{2r_1(y)}g_1(y)$$

$$\Phi_x^T(0,y)\tilde{W}_\Phi-\varepsilon_{u_1})dy+\mu_b\int_0^l(W_\Phi^T\Phi_x(l,y)+\varepsilon_V)g_2(y)$$

$$(-\frac{1}{2r_2(y)}g_2(y)\Phi_x^T(l,y)\tilde{W}_\Phi-\varepsilon_{u_2})dy+\mu_b\int_0^l(W_\Phi^T\Phi_x(z,0)$$

$$+\varepsilon_V)h_1(z)(\frac{1}{2s_1(z)}h_1(z)\Phi_x^T(z,0)\tilde{W}_\Phi-\varepsilon_{v_1})dz+\mu_b\int_0^l(W_\Phi^T$$

$$\Phi_x(z,l)+\varepsilon_V)h_2(z)(-\frac{1}{2s_2(z)}h_2(z)\Phi_x^T(z,l)\tilde{W}_\Phi-\varepsilon_{v_2})dz.$$

We have

$$\int_0^l(W_\Phi^T\Phi_x(0,y))g_1(y)(\frac{1}{2r_1(y)}g_1(y)\Phi_x^T(0,y)\tilde{W}_\Phi)dy$$
$$\le\frac{l}{2}W_{\Phi M}\Phi_{xM}^2 g_{1M}\gamma_1\tilde{W}_{\Phi M}\le\frac{1}{2}\tilde{W}_{\Phi M}^2+\frac{l^2}{8}W_{\Phi M}^2\Phi_{xM}^4 g_{1M}^2\gamma_1^2. \tag{52}$$

And

$$\int_0^l(W_\Phi^T\Phi_x(0,y))g_1(y)\varepsilon_{u_1}dy=\int_0^l(W_\Phi^T\Phi_x(0,y))g_1(y)$$
$$(\frac{1}{2r_1(y)}g_1(y)\varepsilon_{Vx}(x,0,y))dy\le\frac{l}{2}W_{\Phi M}\Phi_{xM}g_{1M}\gamma_1\varepsilon_{VxM}. \tag{53}$$

Moreover,

$$\int_0^l \varepsilon_V g_1(y)(-\frac{1}{2r_1(y)} g_1(y)\Phi_x^T(x,0,y)\tilde{W}_\Phi)dy \\ \le \frac{l}{2}\varepsilon_{VM}\Phi_{xM} g_{1M}\gamma_1\tilde{W}_{\Phi M} \le \frac{1}{2}\tilde{W}_{\Phi M}^2 + \frac{l^2}{8}\varepsilon_{VM}^2\Phi_{xM}^2 g_{1M}^2\gamma_1^2. \tag{54}$$

And

$$\int_0^l \varepsilon_V g_1(y)\varepsilon_{u_1} dy = \int_0^l \varepsilon_V g_1(y)(\frac{1}{2r_1(y)} g_1(y)\varepsilon_{Vx}(x,0,y))dy \\ \le \frac{l}{2}\varepsilon_V g_{1M}\gamma_1\varepsilon_{VxM}. \tag{55}$$

Therefore

$$\dot{L}_b \le -\mu_b q_{\min}\|x\|^2 + 2\mu_b\tilde{W}_{\Phi M}^2 + \varepsilon_b, \tag{56}$$

where

$$\varepsilon_b = \frac{l^2}{8}\mu_b W_{\Phi M}^2\Phi_{xM}^4(g_{1M}^2\gamma_1^2 + g_{2M}^2\gamma_2^2 + h_{1M}^2\gamma_3^2 + h_{2M}^2\gamma_3^2) \\ + \frac{l}{2}\mu_b W_{\Phi M}\Phi_{xM}(g_{1M}\gamma_1 + g_{2M}\gamma_2 + h_{1M}\gamma_3 + h_{2M}\gamma_4)\varepsilon_{VxM} \\ + \frac{l^2}{8}\mu_b\varepsilon_{VM}^2\Phi_{xM}^2(g_{1M}^2\gamma_1^2 + g_{2M}^2\gamma_2^2 + h_{1M}^2\gamma_3^2 + h_{2M}^2\gamma_4^2) \\ + \frac{l}{2}\mu_b\varepsilon_V(g_{1M}\gamma_1 + g_{2M}\gamma_2 + h_{1M}\gamma_3 + h_{2M}\gamma_4)\varepsilon_{VxM}. \tag{57}$$

Closed loop stability proof: In order to prove the overall closed loop stability of the system, Consider the Lyapunov function L defined by $L = L_a + L_b + L_c$. Taking its derivative $\dot{L}$ yields

$$\dot{L} \le -\mu_b q_{\min}\|x\|_{\mathcal{L}_2}^2 + [-\eta_1 + 2\mu_b]\|\tilde{W}_\Phi\|^2 \\ + l^4 W_{\Phi M}^2\Phi_{xM}^2\sigma_{fM}^2\|\tilde{W}_f\|^2 - \mu_c(\lambda_{\min}(K) - \frac{1}{2})\|\tilde{x}\|_{\mathcal{L}_2}^2 \\ -\frac{1}{2}\mu_c\alpha_f\|\tilde{W}_f\|^2 + \varepsilon, \tag{58}$$

where $\varepsilon = \varepsilon_a + \varepsilon_b + \varepsilon_c$. Therefore, choosing $\mu_b < \frac{1}{4}\eta_1$ and $\mu_c > \frac{4}{\alpha_f} l^4 W_{\Phi M}^2\Phi_{xM}^2\sigma_{fM}^2$, $\dot{L}$ is always less than zero when $\|x\|_{\mathcal{L}_2} > \sqrt{\frac{\varepsilon}{\mu_b q_{\min}}}$ or $\|\tilde{W}_\Phi\| > \sqrt{\frac{2\varepsilon}{\eta_1}}$ or $\|\tilde{x}\|_{\mathcal{L}_2}^2 > \sqrt{\frac{\varepsilon}{\mu_c(\lambda_{\min}(K) - \frac{1}{2})}}$ or $\|\tilde{W}_f\| > \sqrt{\frac{4\varepsilon}{\mu_c\alpha_f}}$.

Consequently, the closed-loop system is UB and the bounds for the state x, state estimation $\tilde{x}$ and weight estimation errors $\tilde{W}_\Phi$, $\tilde{W}_f$ can be reduced by using design parameters.

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V. OUTPUT FEEDBACK BOUNDARY CONTROL OF TWO DIMENSIONAL NONLINEAR PARABOLIC PDE: AN ADAPTIVE DYNAMIC PROGRAMMING APPROACH

B. Talaei, S. Jagannathan, and J. Singler

In this paper, adaptive dynamic programming (ADP) based output feedback boundary control of two dimensional nonlinear parabolic partial differential equations (PDE) under Neumann boundary control condition is introduced. The Hamilton-Jacobi-Bellman (HJB) equation is formulated in the original PDE domain and the optimal boundary control policy is derived using the value functional as the solution of the HJB equation. Subsequently, a novel PDE observer is developed to estimate the system states given the nonlinearity in PDE dynamics and measured outputs. Eventually, the sub-optimal boundary control policy is obtained by forward-in-time estimation of the value functional using a neural network (NN) online approximator and estimated state obtained from the NN observer. Novel adaptive tuning laws in continuous-time are proposed for learning the value functional online to satisfy the HJB equation along system trajectories while ensuring the closed-loop stability. Local ultimate boundedness (UB) of the closed-loop system is verified by using Lyapunov theory. The performance of proposed controller is verified via simulation on an unstable two dimensional nonlinear diffusion reaction process.

1. INTRODUCTION

Many physical phenomena are described by the interaction of convection, diffusion and reaction [1]. In fact, the convection–diffusion and the diffusion–reaction processes modeled by nonlinear partial differential equations (PDE) are basic in describing a wide variety of problems in physics, chemistry, biology, and engineering. It is well known that the Burgers equation is a simple nonlinear model equation representing phenomena described by convection and diffusion [2]. The Fisher equation is another nonlinear model equation which arises in a wide variety of problems involving diffusion and reaction such as propagation of an advantageous gene in a population [3]. The Fisher equation is a particular case of a general model equation, called the nonlinear reaction–diffusion equations. Other problems described by this kind of PDE include the propagation of chemical waves [4], the spread of animal or plant populations [5] and the evolution of neutron populations in a nuclear reactor [6].

The optimal control of nonlinear parabolic PDE was first studied in [7]. In the early stages of development, the existence and uniqueness of solutions to Hamilton-Jacobi-Bellman (HJB) equation were solely studied in the abstract form using operator theory. Several other researchers subsequently extended the results to boundary control and different kinds of settings [8]-[11]. However, a practical solution to the HJB equation could not be provided because of computational complexity of solving the HJB equation with huge number of states involved in the PDE system.

From the engineering aspect of view, approximate dynamic programming (ADP) [12], as part of optimal control, seeks computationally feasible solutions of HJB for the cases where the state space is considerably large and the dynamics are nonlinear. Therefore, the conventional method of controlling the PDE using ADP has been previously used to control different kinds of one dimensional parabolic PDE [13]-[17]. The conventional control method involves obtaining a reduced order finite dimensional representation and subsequently using the ADP algorithms on the reduced order ordinary differential equations (ODE).

The real-world applications of PDE and in particular nonlinear reaction diffusion systems are almost in multi-dimensional domains. Although the design and analysis is much more challenging in two or three dimensions, very few ADP approaches were designed in these settings. Moreover, since the system is spatially distributed, it is necessary that the controller only rely on availability of few measurable outputs rather than the state on whole spatial domain. However, most of developed ADP approaches in the literature, design and analyze the controller based on the assumption that state is available throughout the space.

This paper designs and analyzes an output feedback ADP control method for two dimensional (2D) nonlinear reaction diffusion PDE. The design extends recently developed approach [18],[19] for one dimensional setting which does not utilize model reduction. Hence, the HJB is derived and an adaptive algorithm is developed to approximate its solution in the PDE domain forward-in-time and without using policy or value iterations [20]. A PDE observer is designed to estimate the unavailable state. Moreover, the stability analysis is also carried in the original infinite dimensional domain using calculus. The boundary control problem which is more theoretically challenging and practically relevant is addressed. Since abstract operator theory is avoided, the paper is comprehensible for majority of engineers that are not quite familiar with functional analysis. Simulation results confirm that the presented output feedback control method has good convergence and control performance on an unstable 2D diffusion-reaction process.

Notations: Throughout the paper, $\|.\|$ stands for Euclidean norm for a vector and Frobenius norm for a matrix. We recall the inequality $\|.\|_2 \le \|.\|_F$ with $\|.\|_2$ being the induced 2 norm. We also define for $\forall w \in [\mathcal{L}^2(0,l)]$, $\|w\|_{\mathcal{L}^2} = (\int_0^l \|w(z)\|_2^2 dz)^{\frac{1}{2}}$ and $\forall x \in [\mathcal{L}^2(0,l)^2]$, $\|x\|_{\mathcal{L}^2} = (\int_0^l \int_0^l x^2(z,y)dzdy)^{\frac{1}{2}}$.

The rest of the paper is organized as follows. In Section 2, the class of PDE under consideration is described and the output feedback boundary optimal control approach is explained. Section 3 addresses the closed-loop stability of system under the proposed boundary control framework. Section 4 demonstrates the simulation results and Section 0 provides the conclusions.

2. ADAPTIVE DYNAMIC PROGRAMMING BOUNDARY CONTROL

The class of PDE considered in this paper is described by following 2D parabolic dynamics in a square domain $[0,l]\times[0,l]$ with Neumann boundary control as

$$\begin{aligned}
&\frac{\partial x}{\partial t}=\frac{\partial^2 x}{\partial z^2}+\frac{\partial^2 x}{\partial y^2}+f(x,z,y)+d(z,y,t),\\
&\frac{\partial x}{\partial z}|_{z=0,y}=g_1(y)u_1,\ \frac{\partial x}{\partial z}|_{z=l,y}=g_2(y)u_2,\\
&\frac{\partial x}{\partial y}|_{z,y=0}=h_1(z)v_1,\ \frac{\partial x}{\partial y}|_{z,y=l}=h_2(z)v_2,\\
&x(z,y,0)=x_0(z,y),\\
&w(t)=\int_0^l\int_0^l C(z,y)x(z,y)dzdy,
\end{aligned} \tag{1}$$

where $x(z,y,t)$ represents the system state belonging to the solution space of the PDE $X\subset\mathcal{H}^1[0,l]^2$, with $\mathcal{H}^1$ being the Sobolev space of first order, t denotes time, $z,y\in[0,l]$, $l>0$, being the spatial variable, $f(x,z,y)$ is an Lipschitz continuous nonlinear vector function, $\frac{\partial x}{\partial t}$, $\frac{\partial^2 x}{\partial z^2}$ and $\frac{\partial^2 x}{\partial y^2}$ denote the time and second spatial derivatives of state x and $d(z,y,t)$ is an exogenous disturbance. Moreover, u_1,u_2,v_1,v_2 denote boundary control input signals, and $g_1(.),g_2(.),h_1(.),h_2(.)$ are bounded functions with uniform bounds g_{1M}, g_{2M}, h_{1M}, h_{2M}. In addition, $w(z,y)$ represents the measured output and $C(z,y)\in\mathcal{L}^2:\mathcal{H}^1(0,l)^2\to\mathcal{H}^1(0,l)^2$ is a function with $\|C(z,y)\|_{\mathcal{L}^2}>0$. Next, the following assumptions on the nonlinearity are required.

Assumption 1: The function $f(.,z,y)$ is Lipschitz continuous with Lipschitz constant L_f i.e.

$$\|f(x_1,z,y)-f(x_2,z,y)\|\le L_f\|x_1-x_2\|. \tag{2}$$

Assumption 2: The exogenous disturbance $d(z,y,t)$ is bounded, i.e. $\forall z,y\in[0,l],t\in[0,\infty)$, $d(z,y,t)\le D$ where $D\in\mathbb{R}$.

The objective is to provide a continuous optimal boundary control policy by minimizing the cost functional over infinite horizon as

$$\bar{V}(x,t_0,u,v)=\int_{t_0}^{\infty}V(x,u,v)dt, \tag{3}$$

where

$$V(x,u,v)=\int_0^l\int_0^l q(x,y,z)dydz+\sum_{i=1,2} r_i u_i^2+\sum_{i=1,2} s_i v_i^2, \tag{4}$$

with $q(x,z,y)$ being a nonlinear integrand function and scalars $r_i, s_i > 0$ for $i = 1,2$.

The HJB equation for above optimal problem has been previously stated in [9] when the control input is distributed in space. The results were further extended to boundary control in [10] when there is a discount factor in the cost function. As mentioned in [9], existence of discount factor in the cost function is not necessary. Moreover, approximation of time independent value function is much more feasible using neural networks. In what follows, a formal derivation of HJB equation is provided using Bellman principle of optimality. It is assumed that the value functional is Frechet differentiable [21]. In [11], it has been shown that in case of smooth initial conditions and Lipschitz cost function, the differentiability assumption of value function is reasonable.

Let $\mathcal{L}^2(0,l)^2$ be the Hilbert space of square integrable functions with inner product defined as

$$(x_1,x_2)_{\mathcal{L}^2}=\int_0^l\int_0^l x_1(z,y)x_2(z,y)dzdy. \tag{5}$$

Also, let $\mathcal{H}^1(0,l)^2$ be the Hilbert space of $\mathcal{L}^2$ functions with $\mathcal{L}^2$ derivatives with inner product defined as

$$(x_1,x_2)_{\mathcal{H}^1}=\int_0^l\int_0^l (x_1x_2+\frac{\partial x_1}{\partial z}\frac{\partial x_2}{\partial z}+\frac{\partial x_1}{\partial y}\frac{\partial x_2}{\partial y})dzdy. \tag{6}$$

We assume that the optimal value functional $\bar{V}^*:\mathcal{L}^2\to\mathbb{R}$ is Frechet differentiable everywhere with respect to x [1]. Denote the x-Frechet derivative of $\bar{V}^*$ at ρ_1 evaluated at ρ_2 by $[\bar{V}_x^*(\rho_1)]\rho_2$. Since $\bar{V}_x^*(\rho)$ is a bounded linear functional on $\mathcal{L}^2$, the Riesz representation Theorem [22] guarantees that there is a unique $\phi\in\mathcal{L}^2$ such that

$$[\bar{V}_x^*(\rho_1)]\rho_2=\int_0^l\int_0^l \phi(\rho_1,z,y)\rho_2(z,y)dzdy. \tag{7}$$

We also make an additional assumption that $\phi\in\mathcal{H}^1$.

By taking the time interval $[t,t+\delta t)$, $\bar{V}^*(x(t),t)$ can be expressed in the recursive form as

$$\bar{V}^*(x,t,u,v) = \min_{u,v}\{\int_t^{t+\delta t} V(x,u,v)dt + \bar{V}(x,t+\delta t,u,v)\}. \tag{8}$$

Take $\delta t \to 0$, and use $x(t+\delta t) \approx x(t) + \delta t(\partial x / \partial t)$ and the definition of the Frechet derivative of $\bar{V}^*$ with respect to x to obtain

$$\begin{aligned} 0 = \min_{u,v}\{&\int_0^l \int_0^l q(x,y,z)dydz + \sum_{i=1,2} r_i u_i^2 \\ &+ \sum_{i=1,2} s_i v_i^2 + [\bar{V}_x^*(x(t))]\frac{\partial x}{\partial t}\} + \frac{\partial \bar{V}^*}{\partial t}(x(t),t). \end{aligned} \tag{9}$$

Since the actual infinite horizon cost function does not depend on t, $\frac{\partial \bar{V}^*}{\partial t} = 0$. Next, use (7) and PDE dynamics (1) to obtain the HJB equation as

$$\begin{aligned} 0 = \min_{u,v}\{&\int_0^l \int_0^l q(x,y,z)dydz + \sum_{i=1,2} r_i u_i^2 \\ &+ \sum_{i=1,2} s_i v_i^2 + \int_0^l \int_0^l \phi(x,z,y)(\frac{\partial^2 x}{\partial z^2} + \frac{\partial^2 x}{\partial y^2} \\ &+ f(x,z,y) + d(z,y,t))dzdy\}. \end{aligned} \tag{10}$$

Integrating by parts in the second order term gives

$$\begin{aligned} 0 = \min_{u,v}\{&\int_0^l \int_0^l q(x,y,z)dydz + \sum_{i=1,2} r_i u_i^2 \\ &+ \sum_{i=1,2} s_i v_i^2 + \int_0^l \phi(x,l,y)g_2(y)u_2 dy \\ &- \int_0^l \phi(x,0,y)g_1(y)u_1 dy + \int_0^l \phi(x,z,l)h_2(z)v_2 dz \\ &- \int_0^l \phi(x,z,0)h_1(z)v_1 dz - \int_0^l \int_0^l (\frac{\partial \phi}{\partial z}\frac{\partial x}{\partial z} + \frac{\partial \phi}{\partial y}\frac{\partial x}{\partial y})dzdy \\ &+ \int_0^l \int_0^l \phi(x,z,y)(f(x,z,y) + d(t,z,y))dzdy\}. \end{aligned} \tag{11}$$

By completing the square, it can be shown that the minimum is achieved when

$$\begin{aligned} u_1^* &= \frac{1}{2r_1}\int_0^l g_1(y)\phi(x,0,y)dy, \\ u_2^* &= -\frac{1}{2r_2}\int_0^l g_2(y)\phi(x,l,y)dy, \end{aligned} \tag{12}$$

$$v_1^* = \frac{1}{2s_1}\int_0^l h_1(z)\phi(x,z,0)dz,$$
$$v_2^* = -\frac{1}{2s_2}\int_0^l h_2(z)\phi(x,z,l)dz.$$

Moreover, define

$$I_1(\phi) = \int_0^l g_1(y)\phi(x,0,y)dy,$$
$$I_2(\phi) = \int_0^l g_2(y)\phi(x,l,y)dy, \tag{13}$$

$$I_3(\phi) = \int_0^l h_1(z)\phi(x,z,0)dy,$$
$$I_4(\phi) = \int_0^l h_2(z)\phi(x,z,l)dy.$$

According to Trace Theorem [23], since $\phi(x,.,.) \in \mathcal{H}^1$ for any fixed $x \in \mathcal{H}^1$, $I_i : \mathcal{H}^1 \to \mathbb{R}$ is a bounded linear functional. By the Riesz Representation Theorem, there exists $e_i \in \mathcal{H}^1$ such that

$$I_i(\phi) = (\phi, e_i)_{\mathcal{H}_1} = \int_0^l \int_0^l \phi(x,z,y)e_i(x,z,y) + \frac{\partial\phi}{\partial z}\frac{\partial e_i}{\partial z} + \frac{\partial\phi}{\partial y}\frac{\partial e_i}{\partial y}dzdy. \tag{14}$$

Therefore,

$$r_1u_1^{*2} = \frac{1}{4r_1}\left(\int_0^l g_1(y)\phi(x,0,y)dy\right)(\phi, e_1)_{\mathcal{H}_1} = \int_0^l \int_0^l p_1(x,y,x)dydz \tag{15}$$

The terms $r_2u_2^{*2}$, $s_1v_1^{*2}$ and $s_2v_2^{*2}$ can be represented in a similar fashion. Substituting the above representations in (3) results

$$\bar{V}^*(x) = \int_0^\infty \int_0^l \int_0^l q(x,z,y) + p(x,z,y)dzdydt, \tag{16}$$

where $p(.,.,.) = \sum_{i=1}^{4} p_i(.,.,.)$. By switching the order of integration one has

$$\bar{V}^*(x) = \int_0^l \int_0^l V^*(x,z,y)dzdy, \tag{17}$$

where $V^*(x,z,y)=\int_0^\infty q(x,z,y)+p(x,z,y)dt$ and $V_x^*(x,z,y)=\phi(x,z,y)$.

By substituting the optimal control policy (12) in (11), the HJB equation is expressed as

$$
\begin{aligned}
H^* = {} & \int_0^l\int_0^l q(x,z,y)dydz-\frac{1}{4r_1}(\int_0^l g_1(y)V_x^*(0,y)dy)^2 \\
& -\frac{1}{4r_2}(\int_0^l g_2(y)V_x^*(l,y)dy)^2-\frac{1}{4s_1}(\int_0^l h_1(z)V_x^*(z,0)dz)^2 \\
& -\frac{1}{4s_2}(\int_0^l h_2(z)V_x^*(z,l)dz)^2-\int_0^l\int_0^l(\frac{\partial V_x^*}{\partial z}\frac{\partial x}{\partial z} \\
& +\frac{\partial V_x^*}{\partial y}\frac{\partial x}{\partial y})dzdy+\int_0^l\int_0^l V_x^*(f(x,z,y)+d(z,y,t))dzdy.
\end{aligned}
\tag{18}
$$

Remark 1: The control inputs (12) at all boundary edges depend on the value functional $\bar{V}^*$ which is the solution to the HJB equation for the 2D parabolic PDE. The HJB for multi-dimensional nonlinear PDE problem cannot be generally solved separately for each dimension and therefore two decoupled optimal controllers for y and z directions are not a substitute for the proposed control scheme in this paper.

Since the system state x is necessary to implement the control policy (12), it is necessary to introduce an observer prior to control synthesis. Therefore, assuming that the system is observable [24], in the next section the design of a NN observer will be explained.

2.1. NEURAL NETWORK OBSERVER DESIGN

The objective in this section is to introduce an observer for the estimation of system state x in the presence of dynamics and disturbance when states are available for measurement only in limited locations in spatial domain or at boundaries. The estimated states can be subsequently used in the controller design.

If the estimation of state x is denoted by $\hat{x}$, consider the observer with dynamics

$$
\begin{aligned}
\frac{\partial \hat{x}}{\partial t} = {} & \frac{\partial^2 \hat{x}}{\partial z^2}+\frac{\partial^2 \hat{x}}{\partial y^2}+f(\hat{x},z,y) \\
& +\eta C(z)(y-\int_0^l\int_0^l C(z)\hat{x}dzdy),
\end{aligned}
$$

$$\frac{\partial \hat{x}}{\partial z}|_{z=0,y} = g_1(y)u_1, \quad \frac{\partial \hat{x}}{\partial z}|_{z=l,y} = g_2(y)u_2,$$
$$\frac{\partial \hat{x}}{\partial y}|_{z,y=0} = h_1(z)v_1, \quad \frac{\partial \hat{x}}{\partial y}|_{z,y=l} = h_2(z)v_2, \tag{19}$$
$$\hat{x}(z,y,0) = \hat{x}_0(z,y),$$
$$\hat{y}(t) = \int_0^l \int_0^l C(z)\hat{x}(z,t)dzdy,$$

where $\hat{y}$ is the estimation of output y, and $\eta \in \mathbb{R}$ is a positive definite gain constant.

Define the state estimation error $\tilde{x} = x - \hat{x}$. Therefore, its dynamics will be governed by

$$\begin{aligned} \frac{\partial \hat{x}}{\partial t} &= \frac{\partial^2 \tilde{x}}{\partial z^2} + \frac{\partial^2 \tilde{x}}{\partial y^2} + f(x,z,y) - f(\hat{x},z,y) \\ &\quad - \eta C(z,y)\int_0^l \int_0^l C(z,y)\tilde{x}dzdy, \\ \frac{\partial \tilde{x}}{\partial z}&|_{z=0,y} = 0, \ \frac{\partial \tilde{x}}{\partial z}|_{z=l,y} = 0, \\ \frac{\partial \tilde{x}}{\partial y}&|_{z,y=0} = 0, \ \frac{\partial \tilde{x}}{\partial y}|_{z,y=l} = 0, \end{aligned} \tag{20}$$

Assumption 2: There exists a positive definite constant η such that for $\tilde{x}$ satisfying error dynamics (20), the quadratic term in terms of $\tilde{y}$ satisfies

$$\eta \tilde{y}^2 \geq 2\beta \|\tilde{x}\|_{\mathcal{L}^2}^2 . \tag{21}$$

where $\beta \geq L_f + \kappa$ with $\kappa > 0$.

The above assumption helps to prove a smaller less conservative bound for observation error. This assumption will not hold generally for all $x \in \mathcal{H}^1(0,l)^2$ but can be justified for estimation error of parabolic PDE as long as the nonlinear dynamics $f(x,z,y)$ and initial condition $x_0(z,y)$ are smooth. Since an estimate of system state is now available, in the next section the ADP approximate optimal control will be addressed.

2.2. ADP CONTROLLER DESIGN

Equation (18) is generally a nonlinear quadratic 2D partial integro-differential equation (PIDE) and therefore it has no closed-form solution. Therefore, the objective is to find a suitable structure for estimation of $V^*(x,z,y)$ in (18). For this purpose, it is assumed that $V^*(x,z,y)$ is an integral functional over $[\mathcal{H}^1]^n \times [0,l]^2 \to C^1[0,l]^2$, and therefore there exists a smooth bounded basis function vector $\Phi : [\mathcal{H}^1]^n \times [0,l]^2 \to [C^1[0,l]^2]^p$ with the uniform norm bound Φ_M such that [25]

$$V^*(x,z,y) = W_\Phi^T \Phi(x,z,y) + \varepsilon_V(x,z,y). \tag{22}$$

Assumption 4: It is assumed that the optimal weight vector $W_\Phi \in \mathbb{R}^p$ is bounded with the bound $W_{\Phi M}$ and the derivative of approximation error with respect to x, i.e. ε_{Vx} and its second and third derivatives with respect to z and y are bounded by uniform norm bounds ε_{VxM}, ε_{VxzM}, ε_{VxyM}, ε_{VxzzM} and ε_{VxyyM} [26].

Therefore the optimal control policies can be represented as

$$\begin{cases} u_1^* = \dfrac{1}{2r_1}\displaystyle\int_0^l g_1(y)\Phi_x^T(x,0,y)W_\Phi\, dy + \varepsilon_{u_1}, \\ u_2^* = -\dfrac{1}{2r_2}\displaystyle\int_0^l g_2(y)\Phi_x^T(x,l,y)W_\Phi\, dy + \varepsilon_{u_2}, \\ v_1^* = \dfrac{1}{2s_1}\displaystyle\int_0^l h_1(z)\Phi_x^T(x,z,0)W_\Phi\, dy + \varepsilon_{v_1}, \\ v_2^* = -\dfrac{1}{2s_2}\displaystyle\int_0^l h_2(z)\Phi_x^T(x,z,l)W_\Phi\, dy + \varepsilon_{v_2}, \end{cases} \tag{23}$$

where

$$\begin{cases} \varepsilon_{u_1} = \dfrac{1}{2r_1}\displaystyle\int_0^l g_1(y)\varepsilon_{Vx}(x,0,y)dy, \\ \varepsilon_{u_2} = -\dfrac{1}{2r_2}\displaystyle\int_0^l g_2(y)\varepsilon_{Vx}(x,l,y)dy, \\ \varepsilon_{v_1} = \dfrac{1}{2s_1}\displaystyle\int_0^l h_1(z)\varepsilon_{Vx}(x,z,0)dy, \\ \varepsilon_{v_2} = -\dfrac{1}{2s_2}\displaystyle\int_0^l h_2(z)\varepsilon_{Vx}(x,z,l)dy. \end{cases} \tag{24}$$

Consequently, the HJB equation can be represented in its approximated form as

$$\begin{aligned} H^* = 0 = &\int_0^l\int_0^l q(x,z,y)dydz - \frac{1}{4r_1}\Big(\int_0^l g_1(y)W_\Phi^T\Phi_x(0,y) \\ &dy\Big)^2 - \frac{1}{4r_2}\Big(\int_0^l g_2(y)W_\Phi^T\Phi_x(l,y)dy\Big)^2 - \frac{1}{4s_1}\Big(\int_0^l h_1(z)W_\Phi^T \\ &\Phi_x(z,0)dz\Big)^2 - \frac{1}{4s_2}\Big(\int_0^l h_2(z)W_\Phi^T\Phi_x(z,l)dz\Big)^2 \\ &-\int_0^l\int_0^l \Big(W_\Phi^T\frac{\partial\Phi_x}{\partial z}\frac{\partial x}{\partial z} + W_\Phi^T\frac{\partial\Phi_x}{\partial y}\frac{\partial x}{\partial y}\Big)dzdy \\ &+\int_0^l\int_0^l W_\Phi^T\Phi_x(W_f^T\sigma_f(z,y))dzdy + \varepsilon_H, \end{aligned} \tag{25}$$

where the function Φ is expressed as $\Phi(z,y)$ for brevity. Using similar notation for functions ε_V and σ_f in the following equation, ε_H can be derived as

$$
\begin{aligned}
\varepsilon_H &= -\frac{1}{4r_1}\Big(\int_0^l g_1(y)\varepsilon_{Vx}(y,0)dy\Big)^2 - \frac{1}{4r_2}\Big(\int_0^l g_2(y)\varepsilon_{Vx}(y,l)dy\Big)^2 \\
&-\frac{1}{4s_1}\Big(\int_0^l h_1(y)\varepsilon_{Vx}(0,z)dz\Big)^2 - \frac{1}{4s_2}\Big(\int_0^l h_2(y)\varepsilon_{Vx}(l,z)dz\Big)^2 \\
&-\frac{1}{2r_1}\int_0^l g_1(y)W_\Phi^T\Phi_x(0,y)dy\int_0^l g_1(y)\varepsilon_{Vx}(0,y)dy \\
&-\frac{1}{2r_2}\int_0^l g_2(y)W_\Phi^T\Phi_x(l,y)dy\int_0^l g_2(y)\varepsilon_{Vx}(l,y)dy \\
&-\frac{1}{2s_1}\int_0^l h_1(z)W_\Phi^T\Phi_x(z,0)dz\int_0^l h_1(z)\varepsilon_{Vx}(z,0)dz \\
&-\frac{1}{2s_2}\int_0^l h_2(z)W_\Phi^T\Phi_x(z,l)dz\int_0^l h_2(z)\varepsilon_{Vx}(z,l)dz \\
&-\int_0^l\int_0^l\Big(\frac{\partial\varepsilon_{Vx}}{\partial z}\frac{\partial x}{\partial z}+\frac{\partial\varepsilon_{Vx}}{\partial y}\frac{\partial x}{\partial y}\Big)dzdy + \int_0^l\int_0^l \varepsilon_{Vx}(W_f^T\sigma_f(z,y))dzdy \\
&+\int_0^l\int_0^l W_\Phi^T\Phi_x(\varepsilon_f(z,y)+d(z,y,t))dzdy.
\end{aligned}
\tag{26}
$$

Since the state vector is unavailable for measurement $V^*(x,z,y)$ is estimated by

$$\hat{V}^*(x,z,y) = \hat{W}_\Phi^T\Phi(\hat{x},z,y). \tag{27}$$

Hence, the approximated HJB can be represented as

$$
\begin{aligned}
\hat{H} &= \int_0^l\int_0^l q(\hat{x},y,z)dydz - \frac{1}{4r_1}\Big(\int_0^l g_1(y)\hat{W}_\Phi^T\Phi_x(\hat{x},0,y) \\
&dy\Big)^2 - \frac{1}{4r_2}\Big(\int_0^l g_2(y)\hat{W}_\Phi^T\Phi_x(\hat{x},l,y)dy\Big)^2 - \frac{1}{4s_1}\Big(\int_0^l h_1(z)\hat{W}_\Phi^T \\
&\Phi_x(\hat{x},z,0)dz\Big)^2 - \frac{1}{4s_2}\Big(\int_0^l h_2(z)\hat{W}_\Phi^T\Phi_x(\hat{x},z,l)dz\Big)^2 \\
&-\int_0^l\int_0^l\Big(\hat{W}_\Phi^T\frac{\partial\Phi_x}{\partial z}\frac{\partial\hat{x}}{\partial z}+\hat{W}_\Phi^T\frac{\partial\Phi_x}{\partial y}\frac{\partial\hat{x}}{\partial y}\Big)dzdy \\
&+\int_0^l\int_0^l\hat{W}_\Phi^T\Phi_x(\hat{W}_f^T\sigma_f(\hat{x},z,y))dzdy.
\end{aligned}
\tag{28}
$$

Next, the control policy in terms of estimated value functional weight vector would be represented as

$$
\begin{cases}
\hat{u}_1 = \dfrac{1}{2r_1}\displaystyle\int_0^l g_1(y)\Phi_x^T(\hat{x},0,y)\hat{W}_\Phi\, dy, \\
\hat{u}_2 = -\dfrac{1}{2r_2}\displaystyle\int_0^l g_2(y)\Phi_x^T(\hat{x},l,y)\hat{W}_\Phi\, dy,
\end{cases}
\tag{29}
$$

$$\begin{cases} \hat{v}_1 = \dfrac{1}{2s_1}\int_0^l h_1(z)\Phi_x^T(\hat{x},z,0)\hat{W}_\Phi\, dy, \\ \hat{v}_2 = -\dfrac{1}{2s_2}\int_0^l h_2(z)\Phi_x^T(\hat{x},z,l)\hat{W}_\Phi\, dy. \end{cases} \tag{30}$$

Note that the designed boundary control inputs does not vary with spatial variables z, y and therefore applicable with finite number of actuators at boundary edges. Select the value function NN weight tuning law as

$$\dot{\hat{W}}_\Phi = -\alpha_1 \frac{\omega_N}{\zeta_N^2}\hat{H} - \alpha_2 \hat{W}_\Phi - \alpha_3 \hat{W}_\Phi \left\| \hat{W}_\Phi \right\|^2 , \tag{31}$$

where

$$\begin{aligned} \omega_N = {} & \int_0^l \int_0^l \Phi_x(\hat{W}_f^T \sigma_f(\hat{x},z,y))dzdy - \int_0^l\int_0^l (\frac{\partial \Phi_x}{\partial z}\frac{\partial \hat{x}}{\partial z} \\ & + \frac{\partial \Phi_x}{\partial y}\frac{\partial \hat{x}}{\partial y})dzdy - \frac{1}{2r_1}\int_0^l g_1(y)\Phi_x(\hat{x},0,y)dy \\ & \int_0^l g_1(y)\Phi_x{}^T(\hat{x},0,y)\hat{W}_\Phi dy - \frac{1}{2r_2}\int_0^l g_2(y)\Phi_x(\hat{x},l,y)dy \\ & \int_0^l g_2(y)\Phi_x{}^T(\hat{x},l,y)\hat{W}_\Phi dy - \frac{1}{2s_1}\int_0^l h_1(z)\Phi_x(\hat{x},z,0)dz \\ & \int_0^l h_1(z)\hat{W}_\Phi^T\Phi_x(\hat{x},z,0)dz - \frac{1}{2s_2}\int_0^l h_1(z)\Phi_x(\hat{x},z,l)dz \\ & \int_0^l h_1(z)\hat{W}_\Phi^T\Phi_x(\hat{x},z,l)dz, \end{aligned} \tag{32}$$

$\alpha_1, \alpha_2, \alpha_3$ represent positive constants and

$$\zeta_N = c_1\left\|\hat{x}\right\|_{\mathcal{L}_2} + c_2\left\|\frac{\partial \hat{x}}{\partial z}\right\|_{\mathcal{L}_2} + c_3\left\|\frac{\partial \hat{x}}{\partial y}\right\|_{\mathcal{L}_2} + c_4 \tag{33}$$

with $c_1,...,c_4$ being positive constants.

Remark 2: The first term in update law (31), minimizes the approximated Hamiltonian whereas the other terms are necessary to insure the boundedness of weights as will be explained in the proof. Under an initial admissible control policy, it will be shown in Section 3 that the update law (31) and control policies (29)-(30) along with developed observer in Section 2.1, cause the system state vector x ,state estimation error $\tilde{x}$ and the weights estimation errors $\tilde{W}_\Phi$, $\tilde{W}_f$ to be ultimately bounded (UB).

Remark 3: Observe from the definition (31) and the Hamiltonian approximation (28) that both the value function and the Hamiltonian become zero when $\| x\| = 0$. Hence, when the system state converges to zero, the value function approximation is no longer properly updated. This can be viewed as a persistency of excitation (PE) [27] requirement for the inputs to the NN. Therefore, the system states must be persistently exciting long enough for the NN to learn the optimal value function. The PE condition can be guaranteed to be satisfied by adding exploration noise or a perturbation term to the control input.

Next the stability analysis of the closed-loop system is addressed.

3. STABILITY ANALYSIS

System closed-loop stability will be examined by using Lyapunov criterion. First, it will be shown in Theorem 1 that in presence of bounded input, the observer dynamics will be bounded. In Lemma 1, a bound will be found for Hamiltonian reconstruction error ε_H when the output feedback boundary control policy is implemented. This bound will be used later in closed-loop stability proof. Consequently, Theorem 2 will address the ultimate boundedness (UB) of all closed-loop states.

Theorem 1 *(Local asymptotic stability of observer)***:** Let the state estimation error $\tilde{x}$ be residing in a compact set Ω_1, and the proposed NN observer be provided by (19). In the presence of bounded inputs, there exists a positive definite observer gain η such that the state estimation error $\tilde{x}$ is asymptotically stable.

Proof: Refer to Appendix.

Lemma 1: The following inequality

$$\left\|\varepsilon_H\right\| \le \xi_{NM}\zeta_N + \chi_{H1}\left\|\tilde{x}\right\|_{\mathcal{L}_2} + \chi_{H2}\left\|\frac{\partial\tilde{x}}{\partial z}\right\|_{\mathcal{L}_2} + \chi_{H3}\left\|\frac{\partial\tilde{x}}{\partial y}\right\|_{\mathcal{L}_2}, \tag{34}$$

holds where $\chi_{H1},\dots\chi_{H3}$, and ξ_{NM} are positive constants depending on approximation reconstruction errors.

Proof: Refer to Appendix.

Theorem 2 *(Performance of the output feedback NN controller)***:** Consider the DPS given by (1) and let the proposed NN observer be provided by (19). Moreover, let control policies and tuning law for value function weights be provided by (29)-(30) and (31), respectively where $0<\alpha_1<1$, $\alpha_2>\alpha_3$ and $\alpha_3>\theta_{\alpha_3}^N$ with $\theta_{\alpha_3}^N$ being a positive constant. Consequently, provided an initial admissible control, the system state x, state estimation error $\tilde{x}$ and weight estimation errors $\tilde{W}_f$ and $\tilde{W}_\Phi$ are all UB.

Proof: Refer to Appendix.

In the next section, the performance of proposed ADP based NN controller is examined by simulation on a two dimensional nonlinear diffusion reaction process.

4. SIMULATIONS

We consider the plant with the PDE dynamics (1) in the region $z, y \in (0,1)\times(0,1)$. The nonlinearity $f(x,z,y)$ is given by

$$f(x,z,y) = \frac{50}{1+100\sqrt{(z-0.5)^2+(y-0.25)^2}} e^{\frac{|x|}{1+|x|}} x^2, \tag{35}$$

the disturbance $d(t) = 5\sin(100t)$, $g_1(y) = g_2(y) = \dfrac{1}{1+(y-0.5)^2}$ and $h_1(z) = h_2(z) = \dfrac{1}{1+(z-0.5)^2}$.

The state initial condition is expressed as

$$x_0(z,y) = \frac{10}{1+100((z-0.75)^2+(y-0.5)^2)}. \tag{36}$$

The implicit-explicit finite difference method [28] was used for numerical simulation of nonlinear parabolic PDE with $\Delta z = \Delta y = 0.05$ and $\Delta t = 5\times 10^{-4}$. However, the mesh size for control design was set to $\Delta z = \Delta y = 0.1$. The diffusion term $\dfrac{\partial^2 x}{\partial z^2} + \dfrac{\partial^2 x}{\partial y^2}$ was handled implicitly while the nonlinearity $f(x,z,y)$ was modeled in the explicit form. The function $C(.,.)$ was chosen as $C(z,y) = \delta(z - z_0, y - y_0)$, where $z_0, y_0 \in \{0.25, 0.5, 0.75\}$. Moreover, The function $q(x,z,y)$ in cost functional (4) was chosen to be

$$q(x,z,y) = \frac{1}{1+10((z-0.5)^2+(y-0.25)^2)} x^2, \tag{37}$$

and $r_1 = r_2 = s_1 = s_2 = 1$.

Thirty two basis functions were chosen to approximate $V^*(x,z,y)$ with following structure

$$\Phi_i(x,z,y) = \frac{c_j}{1+10((z-z_i)^2+(y-y_i)^2)} \times \iint_{\Delta_i} \psi_j(x) dz dy, \tag{38}$$

where $\psi_j(x) \in \{x^2, x^4\}$, $c_j = 10^{-2}, 10^{-4}$ and spatial domain is divided into 16 identical square subregions Δ_i, $1 \le i \le 16$.

The value functional update parameters were chosen to be $\alpha_1 = 0.9,\ \alpha_2 = 0.2,\ \alpha_3 = 0.1$, $c_1 = c_3 = 10^{-3}$, $c_4 = 0.5$ and observer update parameters were selected as $\alpha_f = 1$ and $\eta = 1$.

Figure 4.1 verifies that the open loop system was highly unstable with these values for system parameters.

Figure 4.2 depicts the closed-loop state evolution of system in space and time verifying that the boundary controller can stabilize the unstable state at zero in less than 0.5 second.

Figure 4.3 shows the convergence of system state $x(z, y)$ to zero more clearly by displaying the snapshots of state x over time.

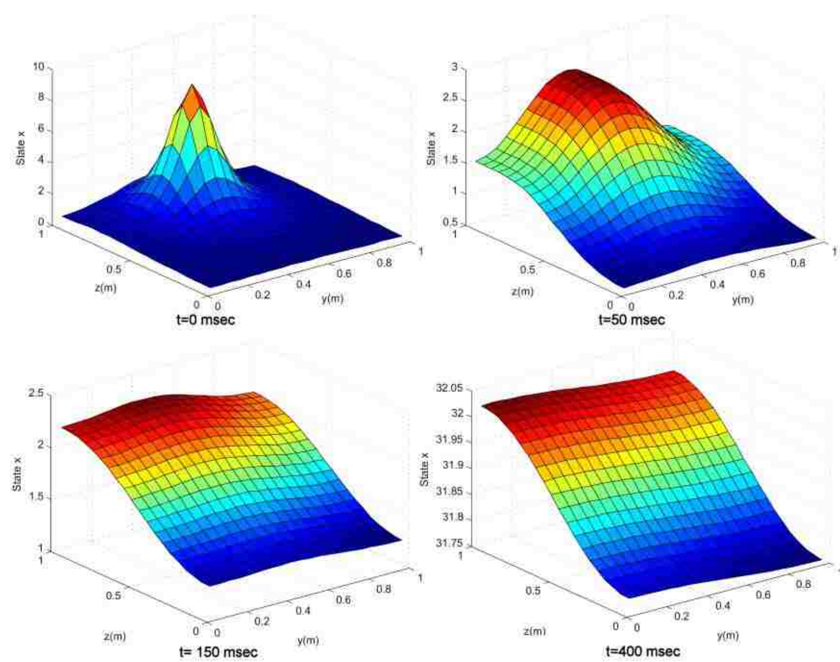

Figure 4.1. Snapshots of system open loop state behavior.

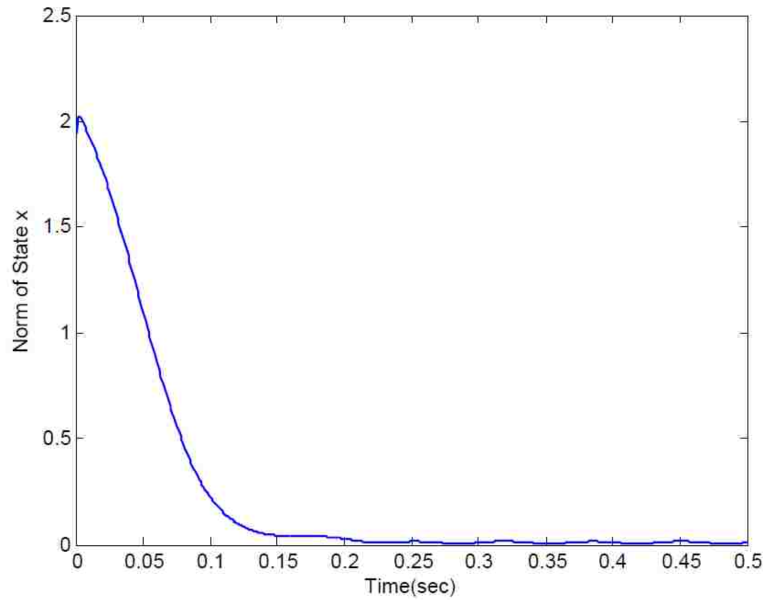

Figure 4.2. Time profile of norm of state using the proposed boundary control policy.

Figures 4.4 and 4.5 show the time profile of control policy and convergence of HJB error, respectively. Figure 4.4 basically shows that the initial admissible control effort is relatively high since the PDE dynamics are unstable. Subsequently, faster convergence of HJB error in this figure compared to state trajectory and control inputs verifies that the HJB error is reduced by the proposed adaptive law before convergence of the states to zero. Moreover, increasing the value of α_1 in the tuning law (31) results in a more rapid convergence of the HJB error.

Finally, Figure 4.6 verifies the performance of proposed observer developed in Section 2.1. As shown, increasing the gain η in the PDE observer has a significant role in convergence of estimation error to zero.

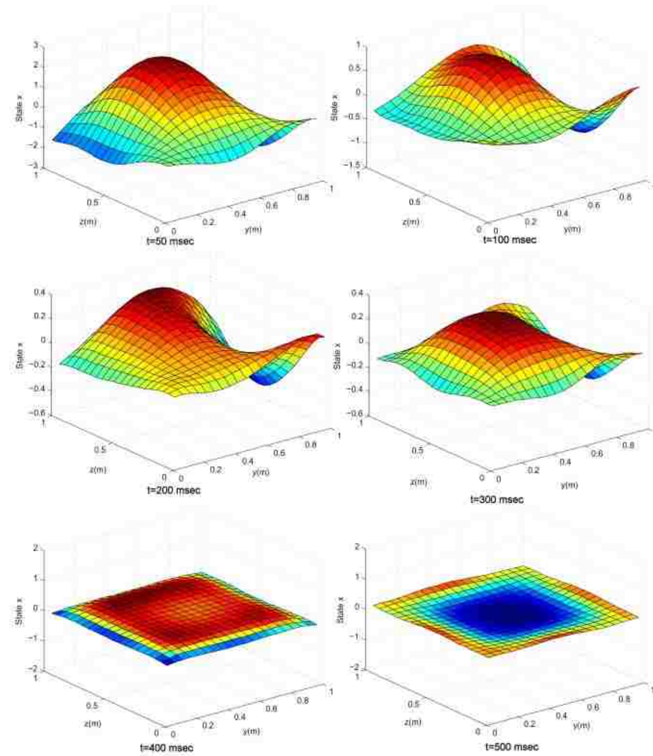

Figure 4.3. Snapshots of closed loop state behavior.

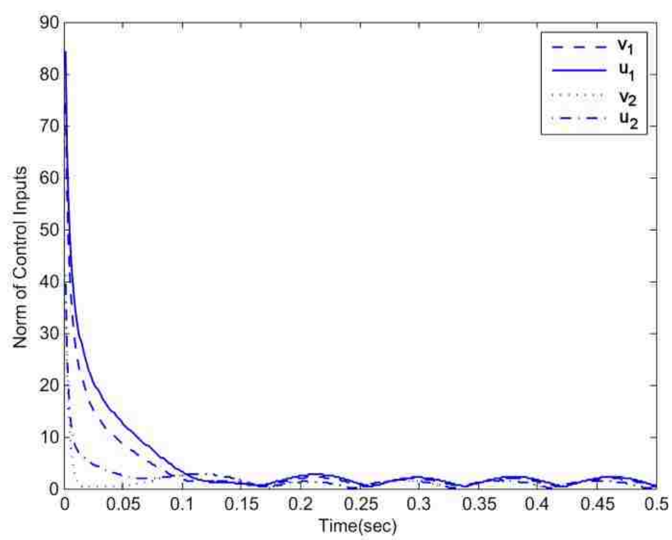

Figure 4.4. Time profile of norm of boundary control inputs.

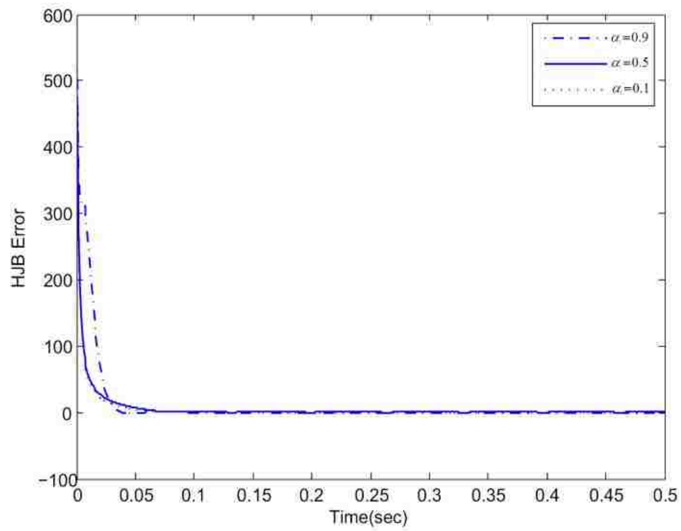

Figure 4.5. Time profile of HJB error for different values of design parameters.

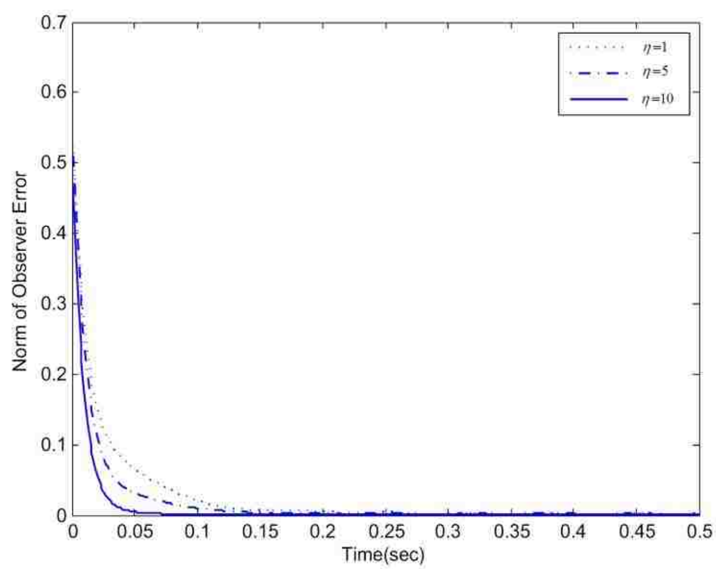

Figure 4.6. Time profile of observer error for different values of design parameters.

5. CONCLUSIONS

This paper developed a novel NN-based output feedback near optimal boundary control scheme for nonlinear 2D parabolic PDE under Neumann boundary control condition without any model reduction prior to control design. By defining an integral cost functional and formulating the HJB equation based on calculus in infinite dimensional state space, a closed-form of optimal control policy was derived. The proposed adaptive NN framework made online approximation of optimal control policy along system trajectories possible when the PDE state was not available and exogenous disturbance was present. Lyapunov stability analysis indicated that the error between approximated and truly optimal weights and state trajectories would remain UB. The performance of proposed observer and controller in estimating states, stabilizing the system and reducing the HJB error was successfully verified on an unstable 2D nonlinear diffusion reaction process.

APPENDIX

In order to avoid long derivations in the proof, we first define $x_z = \frac{\partial x}{\partial z}$, $x_y = \frac{\partial x}{\partial y}$, $x_{zz} = \frac{\partial^2 x}{\partial z^2}$, $x_{yy} = \frac{\partial^2 x}{\partial y^2}$, $\Phi_x = \frac{\partial \Phi}{\partial x}$, $\Phi_{xz} = \frac{\partial \Phi_x}{\partial z}$, $\varepsilon_{Vxz} = \frac{\partial \varepsilon_{Vx}}{\partial z}$ and $\varepsilon_{Vxzz} = \frac{\partial^2 \varepsilon_{Vx}}{\partial z^2}$. First, the proof of Theorem 1 will be provided.

Proof of Theorem 1: In order to prove stability of observer, consider the Lyapunov function

$$L_c = \frac{1}{2}\mu_c \int_0^l \int_0^l \tilde{x}^2(z,y)dzdy, \tag{39}$$

where μ_c is a positive scalar. Taking its derivative $\dot{L}_c$ and substituting error dynamics (20) yields

$$\begin{aligned} \dot{L}_c \le & -\mu_c \eta \tilde{w}^2 + \mu_c \int_0^l \int_0^l \tilde{x}(\tilde{x}_{zz} + \tilde{x}_{yy})dzdy \\ & + \mu_c \int_0^l \int_0^l \tilde{x}(f(x,z,y) - f(\hat{x},z,y))dzdy, \end{aligned} \tag{40}$$

where $\tilde{w} = w - \hat{w}$. Taking integration by parts, substituting PDE dynamics (20) for state estimation error $\tilde{x}$ and using Poincare inequality $(\int_0^l \int_0^l \omega^2 dzdy \le C_p \int_0^l \int_0^l \omega_z^2 dzdy)$ with constant $C_p > 0$, yields

$$\begin{aligned} & \int_0^l \int_0^l \tilde{x}(\tilde{x}_{zz} + \tilde{x}_{yy})dzdy = -\int_0^l \int_0^l \tilde{x}_z^2 + \tilde{x}_y^2 dzdy \le \\ & -\frac{1}{2C_p}\int_0^l \int_0^l \tilde{x}^2 dzdy - \frac{1}{2}\int_0^l \int_0^l \tilde{x}_z^2 + \tilde{x}_y^2 dzdy. \end{aligned} \tag{41}$$

Moreover,

$$\mu_c \int_0^l \int_0^l \tilde{x}(f(x,z,y) - f(\hat{x},z,y))dzdy \le \mu_c L_f \|\tilde{x}\|_{\mathcal{L}^2}^2 \tag{42}$$

Therefore using (21) yields,

$$\dot{L}_c \le -\mu_c(\kappa + \frac{1}{2C_p})\|\tilde{x}\|^2_{\mathcal{L}^2} - \frac{1}{2}\|\tilde{x}_z\|_{\mathcal{L}^2} - \frac{1}{2}\|\tilde{x}_y\|_{\mathcal{L}^2}, \tag{43}$$

Therefore, defining $\psi = \frac{1}{2C_p} + \kappa$, one has

$$\dot{L}_c \le -\mu_c\psi\|\tilde{x}\|^2_{\mathcal{L}^2} - \frac{1}{2}\mu_c(\|\tilde{x}_z\|^2_{\mathcal{L}^2} + \|\tilde{x}_y\|^2_{\mathcal{L}^2}), \tag{44}$$

Proof of Lemma 1: Using integration by parts, one has

$$\begin{aligned}
\varepsilon_H = &-\frac{1}{4r_1}(\int_0^l g_1(y)\varepsilon_{Vx}(y,0)dy)^2 - \frac{1}{4r_2}(\int_0^l g_2(y) \\
&\varepsilon_{Vx}(y,l)dy)^2 - \frac{1}{4s_1}(\int_0^l h_1(z)\varepsilon_{Vx}(0,z)dz)^2 - \frac{1}{4s_2} \\
&(\int_0^l h_2(z)\varepsilon_{Vx}(l,z)dz)^2 - \frac{1}{2r_1}\int_0^l g_1(y)W_\Phi^T\Phi_x(0,y)dy \\
&\int_0^l g_1(y)\varepsilon_{Vx}(0,y)dy - \frac{1}{2r_2}\int_0^l g_2(y)W_\Phi^T\Phi_x(l,y)dy \\
&\int_0^l g_2(y)\varepsilon_{Vx}(l,y)dy - \frac{1}{2s_1}\int_0^l h_1(z)W_\Phi^T\Phi_x(z,0)dz \\
&\int_0^l h_1(z)\varepsilon_{Vx}(z,0)dz - \frac{1}{2s_2}\int_0^l h_2(z)W_\Phi^T\Phi_x(z,l)dz \\
&\int_0^l h_2(z)\varepsilon_{Vx}(z,l)dz + \int_0^l\int_0^l \varepsilon_{Vxzz}xdzdy \\
&-\int_0^l[\varepsilon_{Vxz}|_{z=l,y}\ x(l,y)dy - \varepsilon_{Vxz}|_{z=0,y}\ x(0,y)dy] \\
&+\int_0^l\int_0^l \varepsilon_{Vxyy}xdzdy - \int_0^l[\varepsilon_{Vxy}|_{y=l,z}\ x(z,l)dz \\
&-\varepsilon_{Vxy}|_{y=0,z}\ x(z,0)dz] + \int_0^l\int_0^l \varepsilon_{Vx}f(x,z,y)dzdy \\
&+\int_0^l\int_0^l W_\Phi^T\Phi_x d(z,y,t)dzdy.
\end{aligned} \tag{45}$$

Therefore

$$\begin{aligned}
|\varepsilon_H| \le\ & c_{H1}\|x\|_{\mathcal{L}^2} + c_{H2}\|x(l,y)\|_{\mathcal{L}^2} + c_{H3}\|x(0,y)\|_{\mathcal{L}^2} \\
&+ c_{H4}\|x(z,l)\|_{\mathcal{L}^2} + c_{H5}\|x(z,0)\|_{\mathcal{L}^2} + c_{H6}.
\end{aligned} \tag{46}$$

where $c_{H1} = l(\varepsilon_{VxzzM} + \varepsilon_{VxyyM})$, $c_{H2} = c_{H3} = \sqrt{l}\varepsilon_{VxzM}$, $c_{H4} = c_{H5} = \sqrt{l}\varepsilon_{VxyM}$

and $c_{H6} = \frac{1}{4}l^2(\sum_{i=1}^4\gamma_i)\varepsilon^2_{VxM} + \frac{1}{2}l^2W_{\Phi M}\Phi_{xM}(\sum_{i=1}^4\gamma_i)\varepsilon_{VxM} + l^2\varepsilon_{VxM}f_M + l^2W_{\Phi M}\Phi_{xM}D$ with $\gamma_1 = \frac{g^2_{1M}}{r_1}$,

$\gamma_2 = \frac{g^2_{2M}}{r_2}$, $\gamma_3 = \frac{h^2_{1M}}{s_1}$ and $\gamma_4 = \frac{h^2_{2M}}{s_2}$.

Using the Trace Theorem [23], $\|x(l,y)\|_{\mathcal{L}^2}$, $\|x(0,y)\|_{\mathcal{L}^2}$, $\|x(z,l)\|_{\mathcal{L}^2}, \|x(z,0)\|_{\mathcal{L}^2} \le k(\|x\|_{\mathcal{L}^2} + \|x_z\|_{\mathcal{L}^2} + \|x_y\|_{\mathcal{L}^2})$, $k > 0$. Hence $|\varepsilon_H|$ can be expressed as

$$|\varepsilon_H| \le \chi_{H1}\|x\|_{\mathcal{L}^2} + \chi_{H2}\|x_z\|_{\mathcal{L}^2} + \chi_{H3}\|x_y\|_{\mathcal{L}^2} + \chi_{H4}.$$

where $\chi_{H1} = c_{H1} + k[c_{H2} + c_{H3} + c_{H4} + c_{H5}]$, $\chi_{H2} = \chi_{H3} = k(c_{H2} + c_{H3} + c_{H4} + c_{H5})$ and $\chi_{H4} = c_{H6}$. Therefore, using ($\|x\| \le \|\hat{x}\| + \|\tilde{x}\|$) yields

$$|\varepsilon_H| \le \xi_{NM}\zeta_N + \chi_{H1}\|\tilde{x}\|_{\mathcal{L}_2} + \chi_{H2}\|\tilde{x}_z\|_{\mathcal{L}_2} + \chi_{H3}\|\tilde{x}_y\|_{\mathcal{L}_2},$$

where $\xi_{NM} = \max\{\frac{\chi_{H1}}{c_1}, \frac{\chi_{H2}}{c_2}, \frac{\chi_{H3}}{c_3}, \frac{\chi_{H4}}{c_4}\}$.

Proof of Theorem 2: First, consider the quadratic function

$$L_a = \frac{1}{2}\tilde{W}_\Phi^T\tilde{W}_\Phi, \tag{47}$$

where $\tilde{W}_\Phi = \hat{W}_\Phi - W_\Phi$. Taking its derivative $\dot{L}_a$ yields

$$\dot{L}_a = -\alpha_1\frac{\tilde{W}_\Phi^T\omega_N}{\zeta_N^2}\hat{H} + \tilde{W}_\Phi^T\{-\alpha_2\hat{W}_\Phi - \alpha_3\hat{W}_\Phi\left\|\hat{W}_\Phi\right\|^2\}. \tag{48}$$

Since $H^* = 0$, $\hat{H} = \hat{H} - H^*$ and it can be represented in the following form

$$\begin{aligned}
\hat{H} &= Q(\hat{x}) - Q(x) - \int_0^l\int_0^l (\hat{W}_\Phi^T \frac{\partial\Phi_x(\hat{x},z,y)}{\partial z}\frac{\partial\hat{x}}{\partial z} + \hat{W}_\Phi^T \\
&\frac{\partial\Phi_x(\hat{x},z,y)}{\partial y}\frac{\partial\hat{x}}{\partial y})dzdy + \int_0^l\int_0^l (W_\Phi^T \frac{\partial\Phi_x(x,z,y)}{\partial z}\frac{\partial x}{\partial z} + W_\Phi^T \\
&\frac{\partial\Phi_x(x,z,y)}{\partial y}\frac{\partial x}{\partial y})dzdy + \int_0^l\int_0^l \hat{W}_\Phi^T\Phi_x(f(\hat{x},z,y))dzdy \\
&-\int_0^l\int_0^l W_\Phi^T\Phi_x(f(x,z,y))dzdy - \frac{1}{4r_1}\int_0^l g_1(y)\hat{W}_\Phi^T \\
&\Phi_x(\hat{x},0,y)dy\int_0^l g_1(y)\Phi_x{}^T(\hat{x},0,y)\hat{W}_\Phi dy \\
&-\frac{1}{4r_2}\int_0^l g_2(y)\hat{W}_\Phi^T\Phi_x(\hat{x},l,y)dy\int_0^l g_2(y)\Phi_x{}^T(\hat{x},l,y)\hat{W}_\Phi dy \\
&-\frac{1}{4s_1}\int_0^l h_1(z)\hat{W}_\Phi^T\Phi_x(\hat{x},z,0)dz\int_0^l h_1(z)\hat{W}_\Phi^T\Phi_x(\hat{x},z,0)dz
\end{aligned} \tag{49}$$

$$-\frac{1}{4s_2}\int_0^l h_2(z)\hat{W}_\Phi^T\Phi_x(\hat{x},z,l)\int_0^l h_2(z)\hat{W}_\Phi^T\Phi_x(\hat{x},z,l)dz$$
$$+\frac{1}{4r_1}\int_0^l g_1(y)W_\Phi^T\Phi_x(x,0,y)dy\int_0^l g_1(y)\Phi_x{}^T(x,0,y)W_\Phi dy$$
$$+\frac{1}{4r_2}\int_0^l g_2(y)W_\Phi^T\Phi_x(x,l,y)dy\int_0^l g_2(y)\Phi_x{}^T(x,l,y)W_\Phi dy$$
$$+\frac{1}{4s_1}\int_0^l h_1(z)W_\Phi^T\Phi_x(x,z,0)dz\int_0^l h_1(z)W_\Phi^T\Phi_x(x,z,0)dz$$
$$+\frac{1}{4s_2}\int_0^l h_2(z)W_\Phi^T\Phi_x(x,z,l)dz\int_0^l h_2(z)W_\Phi^T\Phi_x(x,z,l)dz-\varepsilon_H$$

Since $|Q(\hat{x})-Q(x)|\le L_Q\|\tilde{x}\|_{\mathcal{L}^2}$, and

$$-\int_0^l\int_0^l(\hat{W}_\Phi^T\frac{\partial\Phi_x(\hat{x})}{\partial z}\frac{\partial\hat{x}}{\partial z}+\hat{W}_\Phi^T\frac{\partial\Phi_x(\hat{x})}{\partial y}\frac{\partial\hat{x}}{\partial y})dzdy$$
$$+\int_0^l\int_0^l(W_\Phi^T\frac{\partial\Phi_x}{\partial z}\frac{\partial x}{\partial z}+W_\Phi^T\frac{\partial\Phi_x}{\partial y}\frac{\partial x}{\partial y})dzdy$$
$$\pm\int_0^l\int_0^l W_\Phi^T(\Phi_{xz}(\hat{x})\hat{x}_z+\Phi_{xy}(\hat{x})\hat{x}_y)dzdy$$
$$\pm\int_0^l\int_0^l W_\Phi^T(\Phi_{xz}(x)\hat{x}_z+\Phi_{xy}(x)\hat{x}_y)dzdy, \tag{50}$$
$$=-\int_0^l\int_0^l \tilde{W}_\Phi^T(\Phi_{xz}(\hat{x})\hat{x}_z+\Phi_{xy}(\hat{x})\hat{x}_y)dzdy$$
$$+\int_0^l\int_0^l W_\Phi^T(\Phi_{xz}(x)\tilde{x}_z+\Phi_{xy}(x)\tilde{x}_y)dzdy$$
$$+\int_0^l\int_0^l W_\Phi^T(\Phi_{xz}(x)-\Phi_{xz}(\hat{x}))\hat{x}_z dzdy$$
$$+\int_0^l\int_0^l W_\Phi^T(\Phi_{xy}(x)-\Phi_{xy}(\hat{x}))\hat{x}_y dzdy,$$

and

$$\int_0^l\int_0^l \hat{W}_\Phi^T\Phi_x(\hat{x})f(\hat{x})dzdy-\int_0^l\int_0^l W_\Phi^T\Phi_x(x)$$
$$f(x)dzdy\pm\int_0^l\int_0^l W_\Phi^T\Phi_x(\hat{x})f(\hat{x})dzdy$$

$$
\begin{aligned}
&= \int_0^l \int_0^l \tilde{W}_\Phi^T \Phi_x(\hat{x}) f(\hat{x}) dz dy + \int_0^l \int_0^l W_\Phi^T \Phi_x(\hat{x}) f(\hat{x}) dz dy \\
&- \int_0^l \int_0^l W_\Phi^T \Phi_x(x) f(x) dz dy \pm \int_0^l \int_0^l W_\Phi^T \Phi_x(\hat{x}) f(x) dz dy \\
&= \int_0^l \int_0^l \tilde{W}_\Phi^T \Phi_x(\hat{x}) f(\hat{x}) dz dy + \int_0^l \int_0^l W_\Phi^T \Phi_x(\hat{x})(f(\hat{x}) \\
&- f(x)) dz dy + \int_0^l \int_0^l W_\Phi^T (\Phi_x(\hat{x}) - \Phi_x(x)) f(x) dz dy,
\end{aligned} \tag{51}
$$

and

$$
\begin{aligned}
&-\frac{1}{4r_1}\int_0^l g_1(y)(W_\Phi + \tilde{W}_\Phi)^T \Phi_x(\hat{x},0,y)dy \int_0^l g_1(y) \\
&\Phi_x{}^T(\hat{x},0,y)(W_\Phi + \tilde{W}_\Phi)dy + \frac{1}{4r_1}\int_0^l g_1(y) W_\Phi^T \Phi_x(x,0,y)dy \\
&\int_0^l g_1(y)\Phi_x{}^T(x,0,y) W_\Phi dy = -\frac{1}{2r_1}\int_0^l g_1(y)\tilde{W}_\Phi{}^T \\
&\Phi_x(\hat{x},0,y)dy \int_0^l g_1(y)\Phi_x{}^T(\hat{x},0,y)\hat{W}_\Phi dy + \frac{1}{4r_1}\int_0^l g_1(y) \\
&\tilde{W}_\Phi{}^T \Phi_x(\hat{x},0,y)dy \int_0^l g_1(y)\Phi_x{}^T(\hat{x},0,y)\tilde{W}_\Phi dy - \frac{1}{4r_1}\int_0^l g_1(y) \\
&W_\Phi{}^T \Phi_x(\hat{x},0,y)dy \int_0^l g_1(y)(\Phi_x(\hat{x},0,y) - \Phi_x(x,0,y))^T W_\Phi dy \\
&+\frac{1}{4r_1}\int_0^l g_1(y) W_\Phi{}^T \Phi_x(x,0,y)dy \int_0^l g_1(y)(\Phi_x(\hat{x},0,y) \\
&-\Phi_x(x,0,y))^T W_\Phi dy,
\end{aligned}
$$

Combining all the terms and using Cauchy Schwartz and Hölder's inequalities yield

$$
\begin{aligned}
&\hat{H} \le \tilde{W}_\Phi^T \omega_N + b_1 \|\tilde{x}\|_{\mathcal{L}^2} + b_2 \|\tilde{x}_z\|_{\mathcal{L}^2} + b_3 \|\tilde{x}_y\|_{\mathcal{L}^2} + b_4 \|\hat{x}_y\|_{\mathcal{L}^2} \\
&\|\tilde{x}\|_{\mathcal{L}^2} + b_5 \|\hat{x}_z\|_{\mathcal{L}^2} \|\tilde{x}\|_{\mathcal{L}^2} + b_6 \|\tilde{W}_\Phi\|^2 + b_7
\end{aligned} \tag{52}
$$

where $b_1 = L_Q + W_{\Phi M} l(\Phi_{xM} L_f + f_M L_\Phi) + \frac{l^2}{2} L_\Phi W_{\Phi M}^2 \Phi_{xM}^2 (\sum_{i=1}^{4}\gamma_i) + \chi_{H1}$, $b_2 = l W_{\Phi M} \Phi_{xzM} + \chi_{H2}$,

$b_3 = l W_{\Phi M} \Phi_{xyM} + \chi_{H3}$, $b_4 = W_{\Phi M} L_{\Phi y}$, $b_5 = W_{\Phi M} L_{\Phi z}$, $b_6 = \frac{l^2}{4}\Phi_{xM}^2 (\sum_{i=1}^{4}\gamma_i)$ and $b_7 = \xi_{NM}\zeta_N$.

Subsequently, expanding $-\tilde{W}_\Phi^T \hat{W}_\Phi \|\hat{W}_\Phi\|^2$ and using Young's and Cauchy-Schwarz inequalities yield

$$-\tilde{W}_\Phi^T \hat{W}_\Phi \left\|\hat{W}_\Phi\right\|^2 = -\tilde{W}_\Phi^T (W_\Phi + \tilde{W}_\Phi)(W_\Phi + \tilde{W}_\Phi)^T$$
$$(W_\Phi + \tilde{W}_\Phi) \le -\frac{1}{4}\left\|\tilde{W}_\Phi\right\|^4 + \frac{1}{2}\left\|\tilde{W}_\Phi\right\|^2 + \frac{81}{4}\left\|W_\Phi\right\|^4 \tag{53}$$
$$+\frac{1}{2}\left\|W_\Phi\right\|^6.$$

Consequently $\dot{L}_a$ can be derived as

$$\dot{L}_a \le -\alpha_1 \frac{\tilde{W}_\Phi^T \omega_N}{\zeta_N^2}\{\tilde{W}_\Phi^T \omega_N + b_1\left\|\tilde{x}\right\|_{\mathcal{L}^2} + b_2\left\|\tilde{x}_z\right\|_{\mathcal{L}^2} + b_3\left\|\tilde{x}_y\right\|_{\mathcal{L}^2}$$
$$+b_4\left\|\hat{x}_y\right\|_{\mathcal{L}^2}\left\|\tilde{x}\right\|_{\mathcal{L}^2} + b_5\left\|\hat{x}_z\right\|_{\mathcal{L}^2}\left\|\tilde{x}\right\|_{\mathcal{L}^2} + b_6\left\|\tilde{W}_\Phi\right\|^2 \tag{54}$$
$$+b_7\} - \alpha_2 \tilde{W}_\Phi^T (W_\Phi + \tilde{W}_\Phi) - \alpha_3 \tilde{W}_\Phi^T \hat{W}_\Phi \left\|\hat{W}_\Phi\right\|^2.$$

Therefore, according to Young's inequality and since $\zeta_N \ge c_1\left\|\hat{x}\right\|_{\mathcal{L}^2}, c_2\left\|\hat{x}_z\right\|_{\mathcal{L}^2}, c_3\left\|\hat{x}_y\right\|_{\mathcal{L}^2}, c_4$, we have

$$\dot{L}_a \le (-\alpha_1 + \alpha_1^2)\frac{\tilde{W}_\Phi^T \omega_N \omega_N^T \tilde{W}_\Phi}{8\zeta_N^2} + 2b_1^2\beta_N^2\left\|\tilde{x}\right\|_{\mathcal{L}_2}^2$$
$$+2b_2^2\beta_N^2\left\|\tilde{x}_z\right\|_{\mathcal{L}_2}^2 + 2b_3^2\beta_N^2\left\|\tilde{x}_y\right\|_{\mathcal{L}_2}^2 + 2b_4^2\beta_N^2\left\|\tilde{x}\right\|_{\mathcal{L}_2}^2$$
$$+2b_5^2\beta_N^2\left\|\tilde{x}\right\|_{\mathcal{L}_2}^2 + \frac{2b_6^2}{c_4^2}\left\|\tilde{W}_\Phi\right\|^4 \tag{55}$$
$$+2\xi_{NM}^2 + (-\frac{\alpha_2}{2} + \frac{\alpha_3}{2})\left\|\tilde{W}_\Phi\right\|^2 - \frac{\alpha_3}{4}\left\|\tilde{W}_\Phi\right\|^4$$
$$+\frac{\alpha_2}{2}\left\|W_\Phi\right\|^2 + \frac{81\alpha_3}{4}\left\|W_\Phi\right\|^4 + \frac{\alpha_3}{2}\left\|W_\Phi\right\|^6.$$

where $\beta_N = \max\{\frac{1}{c_1},...,\frac{1}{c_4}\}$. If $\alpha_2 > \alpha_3$, $\alpha_3 \ge \theta_{\alpha_3}^N = \frac{9b_7^2}{c_3^2}$ define $\lambda = \frac{1}{2}(\alpha_2 - \alpha_3)$. Choosing $0 < \alpha_1 < 1$ yields

$$\dot{L}_a \le -\alpha_1(1-\alpha_1)\frac{\tilde{W}_\Phi^T \omega_N \omega_N^T \tilde{W}_\Phi}{\zeta_N^2} - \lambda\left\|\tilde{W}_\Phi\right\|^2$$
$$+2(b_1^2 + b_4^2 + b_5^2)\beta_N^2\left\|\tilde{x}\right\|_{\mathcal{L}_2}^2 + 2b_2^2\beta_N^2\left\|\tilde{x}_z\right\|_{\mathcal{L}_2}^2 \tag{56}$$
$$+2\beta_N^2 b_3^2\left\|\tilde{x}_y\right\|_{\mathcal{L}_2}^2 + \varepsilon_a,$$

where $\varepsilon_a = 2\xi_{NM}^2 + \frac{\alpha_2}{2}W_{\Phi M}^2 + \frac{81\alpha_3}{4}W_{\Phi M}^4 + \frac{\alpha_3}{2}W_{\Phi M}^6$.

Defining $L_b = \mu_b \int_0^l \int_0^l V^*(x,z,y)dzdy$ and taking its derivative $\dot{L}_b$ yields

$$\begin{aligned}\dot{L}_b &= \mu_b \int_0^l \int_0^l V_x^* (\frac{\partial^2 x}{\partial z^2} + \frac{\partial^2 x}{\partial y^2} + f(x,z,y) + d(z,y,t))dzdy \\ &= \mu_b \int_0^l \int_0^l V_x^* (f(x,z,y) + d(z,y,t))dzdy\end{aligned} \tag{57}$$

$$\begin{aligned}
&+\mu_b \int_0^l \int_0^l V_x^* (\frac{\partial^2 x}{\partial z^2} + \frac{\partial^2 x}{\partial y^2})dzdy = \mu_b \int_0^l \int_0^l V_x^* (f(x,z,y) \\
&+d(z,y,t))dzdy - \mu_b \int_0^l \int_0^l (\frac{\partial V_x^*}{\partial z}\frac{\partial x}{\partial z} + \frac{\partial V_x^*}{\partial y}\frac{\partial x}{\partial y})dzdy \\
&-\mu_b \int_0^l V_x^*(0,y) g_1(y)(u_1^* + \tilde{u}_1)dy + \mu_b \int_0^l V_x^*(l,y) \\
&g_2(y)(u_2^* + \tilde{u}_2)dy - \mu_b \int_0^l V_x^*(z,0) h_1(z)(v_1^* + \tilde{v}_1)dz \\
&+\mu_b \int_0^l V_x^*(z,l) h_2(z)(v_2^* + \tilde{v}_2)dz \\
&\le -\mu_b q_{\min} \|x\|^2 + \mu_b \int_0^l V_x^*(0,y) g_1(y) \frac{1}{2r_1}(\int_0^l g_1(y)\Phi_x^T(0,y) \\
&\tilde{W}_\Phi dy - \varepsilon_{u_1})dy + \mu_b \int_0^l V_x^*(l,y) g_2(y)(-\frac{1}{2r_2} \\
&\int_0^l g_2(y)\Phi_x^T(l,y)\tilde{W}_\Phi dy - \varepsilon_{u_2})dy + \mu_b \int_0^l V_x^*(z,0) h_1(z)
\end{aligned}$$

$$\begin{aligned}
&(\frac{1}{2s_1}\int_0^l h_1(z)\Phi_x^T(x,z,0)\tilde{W}_\Phi dz - \varepsilon_{v_1})dz + \mu_b \int_0^l V_x^*(x,z,l) h_2(z) \\
&(-\frac{1}{2s_2}\int_0^l h_2(z)\Phi_x^T(z,l)\tilde{W}_\Phi dz - \varepsilon_{v_2})dz \\
&+\mu_b \int_0^l V_x^*(x,0,y) g_1(y) \frac{1}{2r_1}(\int_0^l g_1(y)(\Phi_x^T(\hat{x},0,y) - \Phi_x^T(x,0,y)) \\
&W_\Phi dy)dy - \mu_b \int_0^l V_x^*(x,l,y) g_2(y) \frac{1}{2r_1}(\int_0^l g_2(y)(\Phi_x^T(\hat{x},l,y) \\
&-\Phi_x^T(x,l,y))W_\Phi dy)dy + \mu_b \int_0^l V_x^*(x,z,0) h_1(z) \\
&\frac{1}{2r_1}(\int_0^l h_1(z)(\Phi_x^T(\hat{x},z,0) - \Phi_x^T(x,z,0))W_\Phi dz)dz \\
&-\mu_b \int_0^l V_x^*(x,z,l) h_1(z) \frac{1}{2r_1}(\int_0^l h_2(z)(\Phi_x^T(\hat{x},z,l) \\
&-\Phi_x^T(x,z,l))W_\Phi dz)dz \\
&\le -\mu_b q_{\min} \|x\|^2 \\
&+\mu_b \int_0^l (W_\Phi^T \Phi_x(x,0,y) + \varepsilon_V) g_1(y)(\frac{1}{2r_1}\int_0^l g_1(y)\Phi_x^T(0,y)\tilde{W}_\Phi dy \\
&-\varepsilon_{u_1})dy + \mu_b \int_0^l (W_\Phi^T \Phi_x(x,l,y) + \varepsilon_V) g_2(y) \\
&(-\frac{1}{2r_2}\int_0^l g_2(y)\Phi_x^T(l,y)\tilde{W}_\Phi dy - \varepsilon_{u_2})dy + \mu_b \int_0^l (W_\Phi^T \Phi_x(z,0)
\end{aligned}$$

$$+\varepsilon_V)h_1(z)(\frac{1}{2s_1}\int_0^l h_1(z)\Phi_x^T(x,z,0)\tilde{W}_\Phi dz-\varepsilon_{v_1})dz+\mu_b\int_0^l(W_\Phi^T$$

$$\Phi_x(x,z,l)+\varepsilon_{Vx})h_2(z)(-\frac{1}{2s_2}\int_0^l h_2(z)\Phi_x^T(x,z,l)\tilde{W}_\Phi dz-\varepsilon_{v_2})dz$$

$$+\mu_b\int_0^l(W_\Phi^T\Phi_x(x,0,y)+\varepsilon_{Vx})g_1(y)\frac{1}{2r_1}(\int_0^l g_1(y)(\Phi_x^T(\hat{x},0,y)$$

$$-\Phi_x^T(x,0,y))W_\Phi dy)dy-\mu_b\int_0^l(W_\Phi^T\Phi_x(x,l,y)+\varepsilon_{Vx})g_2(y)\frac{1}{2r_1}$$

$$(\int_0^l g_2(y)(\Phi_x^T(\hat{x},l,y)-\Phi_x^T(x,l,y))W_\Phi dy)dy$$

$$+\mu_b\int_0^l(W_\Phi^T\Phi_x(x,z,0)+\varepsilon_{Vx})h_1(z)\frac{1}{2r_1}(\int_0^l h_1(z)(\Phi_x^T(\hat{x},z,0)$$

$$-\Phi_x^T(x,z,0))W_\Phi dz)dz-\mu_b\int_0^l(W_\Phi^T\Phi_x(x,z,l)+\varepsilon_{Vx})h_1(z)\frac{1}{2r_1}$$

$$(\int_0^l h_2(z)(\Phi_x^T(\hat{x},z,l)-\Phi_x^T(x,z,l))W_\Phi dz)dz.$$

We have

$$\int_0^l(W_\Phi^T\Phi_x(0,y))g_1(z)(\frac{1}{2r_1}\int_0^l g_1(z)\Phi_x^T(0,y)\tilde{W}_\Phi dy)dy$$
$$\leq\frac{l^2}{2}W_{\Phi M}\Phi_{xM}^2\gamma_1\tilde{W}_{\Phi M}\leq\frac{l^4}{2}\Phi_{xM}^4\gamma_1^2\tilde{W}_{\Phi M}^2+\frac{1}{8}W_{\Phi M}^2. \tag{58}$$

And,

$$\int_0^l(W_\Phi^T\Phi_x(x,0,y))g_1\varepsilon_{u_1}dy=\int_0^l(W_\Phi^T\Phi_x(x,0,y))g_1(z)$$
$$(\frac{1}{2r_1}\int_0^l g_1(z)\varepsilon_{Vx}(x,0,y)dy)dy\leq\frac{l^2}{2}W_{\Phi M}\Phi_{xM}g_{1M}\gamma_1\varepsilon_{VxM}. \tag{59}$$

Moreover,

$$\int_0^l\varepsilon_V g_1(z)(-\frac{1}{2r_1}\int_0^l g_1(z)\Phi_x^T(x,0,y)\tilde{W}_\Phi dy)dy$$
$$\leq\frac{l^2}{2}\varepsilon_{VM}\Phi_{xM}g_{1M}\gamma_1\tilde{W}_{\Phi M}\leq\frac{1}{2}\tilde{W}_{\Phi M}^2+\frac{l^4}{8}\varepsilon_{VM}^2\Phi_{xM}^2g_{1M}^2\gamma_1^2. \tag{60}$$

And,

$$\int_0^l\varepsilon_V g_1(z)\varepsilon_{u_1}dy=\int_0^l\varepsilon_V g_1(z)(\frac{1}{2r_1}\int_0^l g_1(z)$$
$$\varepsilon_{Vx}(x,0,y)dy)dy\leq\frac{l^2}{2}\varepsilon_V g_1\gamma_1\varepsilon_{VxM}. \tag{61}$$

Subsequently,

$$\int_0^l (W_\Phi^T \Phi_x(x,0,y) + \varepsilon_{Vx}) g_1(y) \frac{1}{2r_1} (\int_0^l g_1(y)(\Phi_x^T(\hat{x},0,y)$$
$$-\Phi_x^T(x,0,y))W_\Phi dy)dy \le l(W_{\Phi M}\Phi_{xM} + \varepsilon_{VxM})\gamma_1 W_{\Phi M} L_\Phi \tag{62}$$
$$\|\tilde{x}\|_{\mathcal{L}_2} \le \frac{1}{2} W_{\Phi M}^2 + \frac{1}{2} l^2 (W_{\Phi M}\Phi_{xM} + \varepsilon_{VxM})^2 \gamma_1^2 L_\Phi^2 \|\tilde{x}\|_{\mathcal{L}_2}^2$$

Therefore

$$\dot{L}_b \le -\mu_b q_{\min} \|x\|_{\mathcal{L}^2}^2 + \mu_b b_1' \tilde{W}_{\Phi M}^2 + \mu_b b_2' \|\tilde{x}\|_{\mathcal{L}^2}^2 + \varepsilon_b, \tag{63}$$

where $b_1' = 2 + \frac{l^4}{2} \Phi_{xM}^4 \sum_{i=1}^4 \gamma_i^2 \quad b_2' = \frac{1}{2} l^2 (W_{\Phi M}\Phi_{xM} + \varepsilon_{VxM})^2 \; L_\Phi^2 \sum_{i=1}^4 \gamma_i^2$ and

$$\varepsilon_b = \frac{l^2}{2} \mu_b W_{\Phi M} \Phi_{xM} (g_{1M}\gamma_1 + g_{2M}\gamma_2 + h_{1M}\gamma_3 + h_{2M}\gamma_4)\varepsilon_{VxM}$$
$$+\frac{l^4}{8} \mu_b \varepsilon_{VM}^2 \Phi_{xM}^2 (g_{1M}^2\gamma_1^2 + g_{2M}^2\gamma_2^2 + h_{1M}^2\gamma_3^2 + h_{2M}^2\gamma_4^2) + \frac{l^2}{2} \tag{64}$$
$$\mu_b \varepsilon_V (g_{1M}\gamma_1 + g_{2M}\gamma_2 + h_{1M}\gamma_3 + h_{2M}\gamma_4)\varepsilon_{VxM} + \frac{5}{2}\mu_b W_{\Phi M}^2.$$

Closed loop stability proof: In order to prove the overall closed loop stability of the system, Consider the Lyapunov function L defined by $L = L_a + L_b + L_c$. Taking its derivative $\dot{L}$ yields

$$\dot{L} \le -\mu_c \psi \|\tilde{x}\|_{\mathcal{L}_2}^2 - \frac{1}{2}\mu_c \|\tilde{x}_z\|_{\mathcal{L}_2}^2 - \frac{1}{2}\mu_c \|\tilde{x}_y\|_{\mathcal{L}^2}^2$$
$$-\lambda \|\tilde{W}_\Phi\|^2 + 2(b_1^2 + b_4^2 + b_5^2)\beta_N^2 \|\tilde{x}\|_{\mathcal{L}^2}^2 \tag{65}$$
$$+2b_2^2\beta_N^2 \|\tilde{x}_z\|_{\mathcal{L}^2}^2 + 2b_3^2\beta_N^2 \|\tilde{x}_y\|_{\mathcal{L}^2}^2$$
$$-\mu_b q_{\min} \|x\|_{\mathcal{L}^2}^2 + \mu_b b_1' \tilde{W}_{\Phi M}^2 + \mu_b b_2' \|\tilde{x}\|_{\mathcal{L}^2}^2 + \varepsilon$$

where $\varepsilon = \varepsilon_a + \varepsilon_b + \varepsilon_c$. Therefore, choosing $\mu_b < \frac{\lambda}{2b_1'}$ and

$\mu_c > \max\{\frac{4(b_1^2 + b_5^2 + b_4^2)\beta_N^2}{\psi_1} + \frac{2\mu_b b_2'}{\psi}, 8b_2^2\beta_N^2, 8b_3^2\beta_N^2\}$, $\dot{L}$ is always less than zero if $\|x\|_{\mathcal{L}^2} > \sqrt{\frac{\varepsilon}{\mu_b q_{\min}}}$

or $\|\tilde{W}_\Phi\| > \sqrt{\frac{2\varepsilon}{\lambda}}$ or $\|\tilde{x}\|_{\mathcal{L}^2} > \sqrt{\frac{2\varepsilon}{\mu_c \psi_1}}$ and the closed-loop system is UB.

Aside from increasing number of neurons to reduce the bound ε, the bounds for $\|x\|$, and $\|\tilde{x}\|$ can be arbitrarily reduced by increasing $q_{\min}$, and κ in ψ.

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SECTION

2. CONCLUSIONS AND FUTURE WORK

2.1. CONCLUSIONS

In this dissertation novel ADP control approaches based on state and output feedback were proposed for uncertain parabolic PDE in one and two dimensions. The controller design included an ADP-based approach in the early lumping category which was addressed in Paper I. Consequently, the other four papers focused on control design without model reduction for different kinds of parabolic PDE, namely: uncertain linear, coupled nonlinear and two dimensional nonlinear parabolic and Burgers PDE. The notable attributes of designed controllers can be classified as achieving the following contributions:

1. The infinite horizon optimal control problem was solved in the forward-in-time manner and applied in real time to the DPS without the requirement of solving an algebraic Riccati equation (ARE) or value and policy iterations.
2. The distributed nature and large scale of state space makes pure state feedback control design for DPS practically obsolete. Therefore, special attention was paid to output feedback designs using limited-point sensors in the domain or only at boundaries
3. Considerable effort has been made in this research, which were specifically featured Papers II–V to avoid model reduction before ADP controller design and, therefore, make system stability analysis possible in original infinite dimensional domain. Using Lyapunov criterion, the ultimate boundedness (UB) result that was common in most ADP control approaches was successfully verified for closed-loop PDEs regulated by developed controllers.
4. Finally, Papers IV and V shows that the developed ADP control method can be extended to multi-dimensional and more complicated PDE dynamics like Burgers and nonlinear parabolic equations.

In Paper I, an output feedback ADP near optimal NN boundary controller was designed for DPS which is described by a one-dimensional semi-linear parabolic PDE with control input constraints and unknown nonlinearity in dynamics. For controller synthesis, a discretized model of DPS was developed which uses finite difference approximation (FDM), and this model provided satisfactory results. The FDM approach led to an affine nonlinear finite dimensional dynamical representation of DPS where the boundary control input could be designed based on an optimal control method for finite dimensional systems. Then a novel ADP scheme was provided by using two NNs to estimate unavailable states under uncertain system dynamics and to approximate the optimal value function when only a few states were available for measurement in the spatial domain. Since the input constraints were incorporated as a priori in the design, the control input lay inherently within the actuator limits. Ultimate boundedness (UB) of the closed-loop system was successfully verified by using standard Lyapunov theory. Finally, simulation results confirmed the effectiveness of an output feedback controller on a diffusion reaction process.

An ADP-based near optimal boundary control scheme for DPS governed by uncertain linear one-dimensional parabolic PDE under both Neumann and Dirichlet actuation conditions without any finite dimensional model approximation prior to control design was introduced in Paper II. By defining the value functional as the extension of its definition from linear LPS optimal control design, the HJB equation was derived in original infinite dimensional state space. The proposed identifier was effective in estimating the unknown coefficient over the space in system dynamics. Based on defined structure for the value functional as a surface integral, a radial basis network (RBN) was proposed to estimate its unknown parameters as a continuous two-variable kernel function. The update law for RBN unknown weights was defined to reduce the HJB error effectively and insuring system stability whereas PDE dynamics was uncertain. Ultimate boundedness of the closed-loop system was verified by using the standard Lyapunov theory with consideration of all approximation reconstruction errors. Since model reduction was not utilized in control development, the design is more reliable and can be applied to achieve accurate control and closed loop stability of the original infinite dimensional system. The performance of the proposed control method was successfully verified on a diffusion reaction process.

Paper III presents a novel NN-based output feedback near optimal boundary control scheme for DPS governed by semi-linear coupled parabolic PDE under Neumann boundary control condition. No model reduction prior to control design is needed with the new control scheme. By defining an integral cost function and formulating the HJB equation based on calculus in infinite dimensional state space, a closed-form of optimal control policy was derived. The proposed adaptive NN framework made online approximation of optimal control policy along system trajectories possible when PDE dynamics were partially unknown. Lyapunov stability analysis indicated that the error between approximated and truly optimal weights and state trajectories would remain UB. The performance of proposed observer and controller in estimating states, stabilizing the system and reducing the HJB error was successfully verified on a coupled semi-linear diffusion process.

In Paper IV, the development of a novel ADP optimal boundary control scheme is introduced for 2D Burgers equation under Neumann boundary condition without any model reduction. By formulating the HJB equation based on calculus in infinite dimensional state space, boundary optimal control policy was derived. The proposed adaptive NN framework made online approximation of optimal control policy along system trajectories possible whereas the PDE dynamics were partially unknown. Lyapunov stability analysis indicated the error between approximated and truly optimal weights and state trajectories would remain UB. Simulation results confirm the effectiveness of the developed boundary control approach on an unstable 2D Burgers equation.

The requirement for infinite number of actuators or sensors in DPS control is a typical challenging problem and if a control design method can tackle these problems appropriately, it has an obvious advantage. The control method developed in the Paper III requires finite number actuators for implementation. Moreover, the effect of external disturbance on system stability has been considered.

The next stage of research as reported in Paper V was focusing on output feedback control design for parabolic PDE in the two dimensional domain. Until now, the ADP control approaches for multi-dimensional PDE have not been studied in detail. Paper IV presents a novel framework for this purpose without model reduction. However, the proposed approach relies on availability of system state. Measuring the state over the entire domain is a formidable task and impractical. Hence, the necessity for an output feedback control design is much more crucial in multi-dimensional PDE domains. Accordingly, the observer design and stability proof for two dimensional parabolic PDE was the primary task to be addressed in the final stage of this research.

2.2. FUTURE WORK

In contrast to ODEs, no general control methodology can be developed for PDEs, either for analysis or for control synthesis. Two of the most basic categories studied in literature are parabolic and hyperbolic PDEs, with standard examples being heat equations and wave equations. Therefore, as part of future work, other PDE problems like hyperbolic PDE can be considered. The oscillating nature of this type of PDEs makes their controller design and stability guarantee much more challenging.

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VITA

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His area of research lied in optimal control of distributed parameter systems based on adaptive dynamic programming. Applications included but not limited to the control of: reaction-diffusion processes, fluid dynamics, rechargeable batteries and flexible structures. His previous area of research was control and motion planning of robotic manipulators.