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Model based fault diagnosis and prognosis of class of linear and nonlinear distributed parameter systems modeled by partial differential equations

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MODEL BASED FAULT DIAGNOSIS AND PROGNOSIS OF CLASS OF LINEAR AND NONLINEAR DISTRIBUTED PARAMETER SYSTEMS MODELED BY PARTIAL DIFFERENTIAL EQUATIONS

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ABSTRACT

With the rapid development of modern control systems, a significant number of industrial systems may suffer from component failures. An accurate yet faster fault prognosis and resilience can improve system availability and reduce unscheduled downtime. Therefore, in this dissertation, model-based prognosis and resilience control schemes have been developed for online prediction and accommodation of faults for distributed parameter systems (DPS). First, a novel fault detection, estimation and prediction framework is introduced utilizing a novel observer for a class of linear DPS with bounded disturbance by modeling the DPS as a set of partial differential equations.

To relax the state measurability in DPS, filters are introduced to redesign the detection observer. Upon detecting a fault, an adaptive term is activated to estimate the multiplicative fault and a tuning law is derived to tune the fault parameter magnitude. Then based on this estimated fault parameter together with its failure limit, time-to-failure (TTF) is derived for prognosis. A novel fault accommodation scheme is developed to handle actuator and sensor faults with boundary measurements. Next, a fault isolation scheme is presented to differentiate actuator, sensor and state faults with a limited number of measurements for a class of linear and nonlinear DPS.

Subsequently, actuator and sensor fault detection and prediction for a class of nonlinear DPS are considered with bounded disturbance by using a Luenberger observer. Finally, a novel resilient control scheme is proposed for nonlinear DPS once an actuator fault is detected by using an additional boundary measurement. In all the above methods, Lyapunov analysis is utilized to show the boundedness of the closed-loop signals during fault detection, prediction and resilience under mild assumptions.

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SECTION

1. INTRODUCTION

In the past few decades, industry control systems have become more and more complicated, thereby increasing the possibility of faults and failures to occur. An ultimate objective of resilient control systems is state awareness which is an extensive sense of safety for critical infrastructures [1]. It is critical to design a fault detection and prediction scheme in order to improve system reliability. Therefore, fault diagnosis and prognosis, which is utilized to detect and predict unexpected faults and system failures, has drawn increasing attention [2].

In order to detect faults in physical systems, data-driven and model-based detection schemes are developed [3]. Data-driven approaches [4] are preferred when the mathematical model of the system is not available or cannot be derived. Usually data-driven methods require significant quantities of data based on both healthy and faulty systems. As a result, it is crucial to design a generic data-driven fault diagnosis and prognosis framework which is applicable to a variety of industrial systems. In addition, collecting faulty data is costly and is impossible under certain conditions.

Compared to data-driven based fault detection, model-based detection utilizes a mathematical representation of the overall scheme for detecting faults based on which a detection observer is designed to estimate system physical states and output. Figure 1.1 shows an overview of the model-based fault detection scheme. Detection residual is generated by comparing the measured output with estimated output given by the observer. Under healthy operating conditions, because the estimated system output provided by observer is close enough to the measured one and thus the residual maintains operation below a predefined threshold. In the presence of a fault, the system dynamics will change due to the fault even though the observer dynamics remain unchanged.

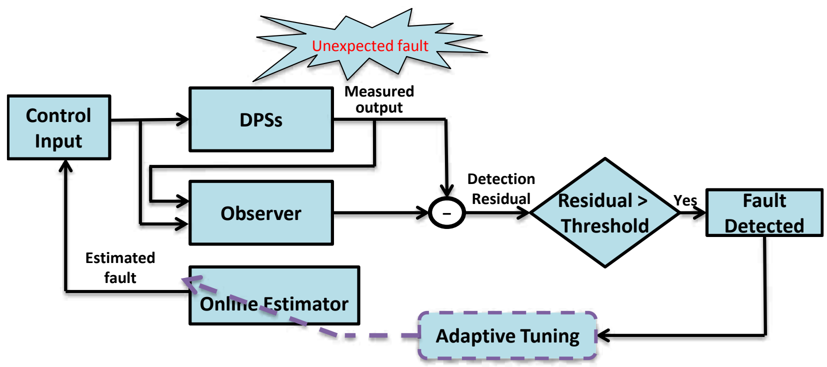


Figure 1.1. Model based fault detection and accommodaiton scheme.

Therefore, the measured output will deviate from the estimated output due to the presence of the fault, which causes the detection residual to increase. A fault is declared active once the residual exceeds the predefined threshold. The selection of the detection threshold depends upon the initial condition and upper bound of uncertainties and disturbances. After detection of a fault, if the fault type is unknown fault isolation techniques will be applied to determine the fault type and location. Once both of the fault type and location is identified, an online estimator with an appropriate tuning algorithm will be applied to estimate the fault dynamics based on which failure prediction can be obtained by estimating the remaining useful life of the system. The estimated fault dynamics can also be utilized to accommodate faults by reconfiguring the actuator input; thus, fault resilience is accomplished. In addition, a resilient pit can be generated based on the change of the output tracking error to evaluate the performance of the proposed resilient control scheme and the time to resilience (TTR) is able to be estimated to predict the time a system needs to go back to a normal level.

In recent years, significant efforts have been made as noted in the literature on model-based diagnosis and accommodation for lumped parameter systems (LPS) modeled by ordinary differential equations (ODEs) [5, 6]. However, many industrial systems including fluid flow, chemical reaction, and thermal convection systems are classified as distributed parameter systems (DPS) or infinite dimensional systems since the system states are a function of both time and space. The mathematical models of such systems cannot be represented by ODEs any more instead partial differential equations (PDEs) are utilized to describe the system dynamics.

Compared to LPS, fault diagnosis for DPS are more involved because of their complex distributed nature. A fault occurring at one point can cause the change of the system state over the entire space while the possible locations of faults are many. However, it is impossible to measure the system state at each point of the system. Therefore, fault diagnosis and prognosis have to be achieved by using a limited number of measurements.

Fault diagnosis plays a significant role in improving the reliability of modern industrial systems and reliable resilient control systems demand timely fault detection, real time fault analysis and advanced notice of system failures. Next, an overview of current fault diagnosis and prognosis methodologies for LPS and DPS will be discussed.

Subsequently, the organization of this dissertation as well as its contributions will be presented.

1.1. OVERVIEW OF FAULT DIAGNOSIS METHODOLOGIES

Recently, different methods of fault detection and prediction have been proposed in the literature for LPS. Authors in [7] develop a prognostic scheme to identify faults on a mechanical component, subsystem or system by using the mathematical representation of the practical system. A fault detection and isolation framework is introduced in [8] based on the system representation while a model-based fault detection and diagnosis scheme is introduced by generating symptoms in [9]. A fault distribution function is addressed in [10] by using an adaptive observer, which is complementary to the one developed for fault detection and isolation in dynamics systems [11].

In order to estimate fault detection residual, an adaptive threshold is given in [12]. A stochastic process model is presented in [13] to estimate the fault and the remaining useful life (RUL) of the system while a dynamic wavelet neural network (NN) is used to estimate the RUL in [14].

Another imperative issue in the aspect of fault diagnosis and prognosis is associated with the DPS. Industrial systems such as thermal convection, fluid flow, chemical reaction systems, etc. have complicated temporal as well as spatial dynamics. Fault detection and prediction for DPS are more involved when compared to LPS due to the distributed nature of the system state. The ODE representation does not apply when estimating the behavior of DPS.

Over the past few decades, researchers have dedicated to studying control and observer designs for both linear and nonlinear DPSs. In order to deal with the distributed system states, the PDEs representing DPS can be considered as a set of infinite bank of ODEs. Subsequently, the system model can be approximated with finite dimensional of dominating ODE by applying Galerkin' method [15]. On one hand, nonlinear finite dimensional output feedback controllers are presented in [16] for quasi-linear parabolic PDE by combing Galerkin's method with a design of approximated inertial manifolds to derive the applicable ODEs. A general scheme was proposed to control the parabolic PDE with input constraints [17], and an adaptive optimal controller was designed by using neuron-dynamic programming for highly dissipative nonlinear PDEs [18].

On the other hand, fault detection frameworks are introduced for mechanical and aerospace engineering systems by using PDE models [19]. In order to address the incipient actuator faults, an adaptive fault detection and accommodation scheme was developed in [20], and a geometric approach was introduced to detect and isolate dissipative parabolic PDE [21]. In spite of attractive results, all of these methods [15–21] address the problem by converting the original PDE representation to an approximated finite dimensional ODE resulting in inaccurate fault detection due to model reduction.

In addition, in the presence of a fault, the dynamics of the DPS will be changed and the reduced ODE may not be applicable. In order to avoid this problem, a few researchers [22, 23] pursued the controller design directly based on the original PDE. A state feedback boundary controller was designed for a class of linear parabolic PDE without discretizing the spatial variable [22] and adaptive controllers were presented in [23] to stabilize the parabolic PDE with unknown destabilizing parameters. For the sake of obtaining output feedback boundary control, exponentially convergent observers without disturbance and uncertainty were proposed by [24] for parabolic PDE with boundary measurements. The work in [25] extends the boundary control of the system with scalar states to a system with coupled parabolic linear PDE. An extended Luenberger observer is proposed for semilinear DPS in the absence of disturbance and uncertainty with observer gains determined by linearizing observer error dynamics [26].

In summary, fault diagnosis and prognosis frameworks for DPS have been investigated by converting the PDE representation into finite dimensional ODE; thus, fault detection can be obtained based on those ODE models. Although controller and observer designs have been studied recently, fault diagnosis for DPS using the original PDE has not been investigated. Therefore, in this dissertation, model based fault diagnosis and prognosis schemes are outlined for linear and nonlinear DPS by designing the detection observer directly on the basis of their PDE representation. The performance of the observer is evaluated with bounded disturbance or uncertainty. In addition, the Lyapunov stability analysis for the proposed frameworks is guaranteed in this dissertation.

1.2. ORGANIZATION OF THE DISSERTATION

This dissertation presents model-based fault diagnosis and prognosis schemes for a class of linear and nonlinear distributed parameter systems represented by parabolic PDE in the form of five papers as shown in Figure 1.2. All five papers deal with fault prognosis and resilience control of DPS. The first two papers address fault detection and accommodation of linear DPS. The third paper investigates fault isolation and location determination while the fourth and fifth papers address nonlinear DPS.

Paper I develops novel fault detection and estimation framework by designing a detection observer based on original system PDE representation. At first, a Luenberger-type observer is introduced by using the system model to estimate the system state as well as output. Detection residual is generated by comparing the measured output with estimated output; furthermore, its asymptotical stability can be guaranteed without disturbance and uncertainty under healthy conditions. A fault can be detected when the residual exceeds a predefined threshold, which is decided by the initial conditions. Once detecting a fault, an adaptive term is incorporated by the observer to estimate fault dynamics with a novel update law. However, the fault estimation demands systems states over the entire space which is a major disadvantage. Therefore, the detection observer is redesigned considering bounded uncertainty and disturbance by using an input filter along with two output filters based on the linear property of the PDE, and the adaptive term can be tuned by an update law with measured output alone. In addition, given the estimated fault parameter and its failure limit, an explicit formula is given to estimate the time to failure (TTF) or RUL of the system on the real time.

Subsequently, a fault accommodation scheme is proposed for multi-input and multi-output (MIMO) coupled linear DPS with actuator and sensor faults in Paper II. A filter-based observer is utilized to generate a residual for fault detection and the corresponding fault dynamics is approximated by using an adaptive term for an actuator or sensor fault. Next, based on the estimated fault dynamics, fault accommodation can be achieved for actuator and sensor faults. Moreover, given the limit values of the tracking errors and by using the dynamics of the tracking error, the time to accommodation (TTA) can be predicted, which can provide useful information for the maintenance schedule.

Paper III solves a critical problem of fault prognosis, which involves the isolation and location determination of faults in DPS. The proposed fault isolation scheme for linear DPS can differentiate actuator, sensor, and state faults using actuator and sensor fault isolation estimators. The location determination scheme presented in this paper is able to provide useful information of the state fault location which is critical for further fault estimation and prediction. In addition, a fault isolation framework for nonlinear DPS is introduced to isolate different types of faults with boundary measurements alone.

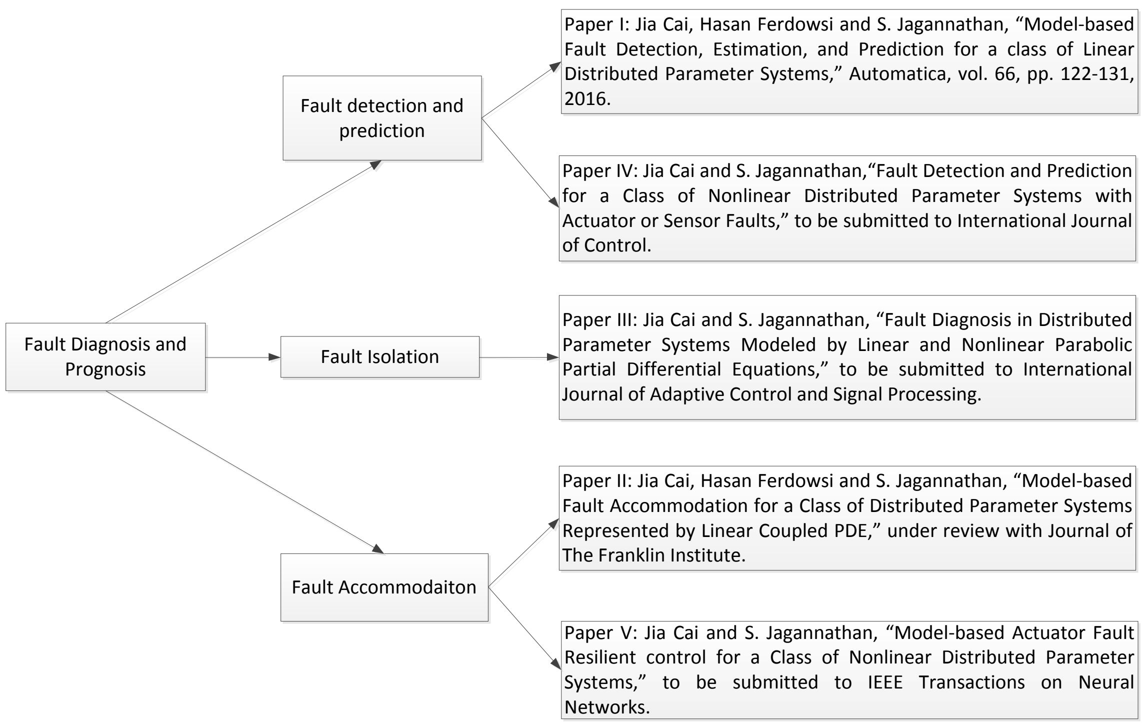


Figure 1.2. Dissertation overview.

By comparison, Paper IV introduces a novel fault detection and prediction scheme for MIMO nonlinear DPSs with bounded disturbance. Because the presence of the nonlinear term, the filter based observer presented in the previous work is not applicable to this type of system. Therefore, an extended Luenberger-type observer is utilized instead for both of output control design and fault detection. The stability of the observer error can be guaranteed with observer gains selecting by linearizing the observer error dynamics under healthy conditions. Detectability conditions for both of the actuator and sensor faults are discussed in this paper. The actuator/sensor fault dynamics can be estimated with a novel update law and the TTF is estimated by comparing the measured outputs with their failure limits. Paper V proposes a fault resilience scheme to mitigate the unexpected fault and obtain fault resilience through tracking error by reconfiguring the actuator input. Moreover, the estimated TTR can be used to predict the time when a system can recover from a faulty state.

In summary, a significant number of industrial systems are classified as DPS whereas limited work has been done in this area. The purpose of this dissertation is to investigate fault diagnosis and prognosis for linear and nonlinear DPS with fault types. The proposed schemes are generic to accommodate different practical systems and fault types; moreover, different examples are used to demonstrate the effectiveness of the proposed schemes.

1.3. CONTRIBUTIONS OF THE DISSERTATION

This dissertation covers model-based fault diagnosis, prognosis and accommodation framework for linear and nonlinear DPS with both multiplicative and additive faults. Unlike the literature [19–21] where the original DPS is converted to infinite dimensional ODEs and the system model can be approximated with finite dimensional dominating ODEs, the fault prognostic and accommodation is obtained directly based on the system PDEs representation. All the schemes proposed in this dissertation consider bounded disturbances.

The contributions of Paper I include the design of the fault detection and estimation scheme by utilizing a Luenberger-type observer based on the system PDE representation with detectability conditions for both actuator and sensor faults. Then in order to relax the requirements of all states available, a filter-based observer was redesigned with measured output alone in the presence of bounded disturbance and uncertainty for fault detection, estimation and prediction. Subsequently, an estimated TTF formula was developed to predict the remaining useful life of the system.

The contributions of Paper II include the development of an innovative model-based fault accommodation scheme for a class of MIMO DPS with actuator and sensor faults represented by coupled parabolic PDEs. The fault resilience is accomplished by reconfiguring the control input. In addition, time to accommodation is estimated based by using the dynamics of the tracking error.

As for Paper III, a fault isolation scheme is introduced to differentiate actuator, sensor and state faults and determine the location of a state fault for a class of linear DPSs and a detectability condition is proposed for state faults. In addition, a fault isolable condition for actuator, sensor and state faults are discussed herein. These schemes have not investigated in the literature [19-21]. Moreover, a fault isolation framework for a class of nonlinear DPSs is also included in this paper.

The contributions of Paper IV involve the design of an online detection observer with detectability conditions for a nonlinear MIMO DPS with bounded uncertainty and disturbance. An innovative update law is derived to tune an adaptive term in order to estimate fault dynamics. In the end, based on the failure limit of the output, an estimated TTF is given to predict the RUL of the system.

Finally, the fault accommodation and resilient control discussed in Paper V further explains its role in nonlinear DPS by providing an estimated time to resilience (TTR) to predict the recovery time of the system in the presence of an actuator fault.

PAPER

I. MODEL-BASED FAULT DETECTION, ESTIMATION, AND PREDICTION FOR A CLASS OF LINEAR DISTRIBUTED PARAMETER SYSTEMS

Jia Cai, Hasan Ferdowsi and S. Jagannathan

This paper addresses a new model-based fault detection, estimation, and prediction scheme for linear distributed parameter systems (DPSs) described by a class of partial differential equations (PDEs). An observer is proposed by using the PDE representation and the detection residual is generated by taking the difference between the observer and the physical system outputs. A fault is detected by comparing the residual to a predefined threshold. Subsequently, the fault function is estimated, and its parameters are tuned via a novel update law. Though state measurements are utilized initially in the parameter update law for the fault function estimation, the output and input filters in the modified observer subsequently relax this requirement. The actuator and sensor fault functions are estimated and the time to failure (TTF) is calculated with output measurements alone. Finally, the performance of detection, estimation and a prediction scheme is evaluated on a heat transfer reactor with sensor and actuator faults.

1. INTRODUCTION

The design of fault detection and prediction scheme is a critical part of improving system reliability [1]. Therefore several model-based detection and prognostics schemes have been introduced in the literature for industrial systems, which are traditionally described by ordinary differential equations (ODEs). By utilizing a mathematical model of the physical systems [2], a robust prognostic scheme was developed by authors in [3]. Authors of [4] proposed a detection and isolation scheme by using system representation. The work [5] introduced a model-based fault detection and diagnosis scheme by generating symptoms. Authors of [6] utilized an adaptive observer to handle a fault distribution function. Authors in [7] developed complementary approaches in fault detection and isolation in dynamic systems.

An adaptive threshold was generated in the research of [8] to evaluate the fault detection residual. Works of [9] and [10] developed a stochastic process model to approximate the fault and estimate the remaining useful life (RUL) or time to failure (TTF) of the system whereas the RUL was estimated in [11] by applying the dynamic wavelet neural network (NN).

A variety of industrial systems including fluid flows, thermal convection and chemical reaction processes are classified as distributed parameter systems (DPS) since the system state changes with both time and space. Therefore, the ODE models given by lumped parameter representation for DPS are unsuitable to mimic their behavior [12]. Instead, the state of a DPS is described by a partial differential equation (PDE).

Several fault detection and diagnosis schemes have been introduced in the literature for DPS. The author of [13] approximated DPS with finite dimensional ODEs;

then, the reduced order ODE model was utilized in the development of fault detection and diagnosis schemes. A detection observer based on the approximate finite dimensional slow subsystem was introduced to detect and isolate faults in [14]. Authors of [15] introduced a finite-dimensional geometric method for fault detection and isolation (FDI) of parabolic PDEs by constructing a set of residuals such that each one is only affected by a fault. Despite these interesting results, these detection and diagnosis schemes proposed in [14] and [15] used a finite dimensional ODE representation of DPS; consequently, they may suffer from false and missed alarms due to model reduction. In addition, the fault can change the dynamics of the overall system, thereby causing the reduced order model and resulting fault detection and diagnostics scheme to be inaccurate.

By contrast, this paper introduces a novel fault detection and estimation scheme by using a novel observer, which is designed directly based on PDE representation of DPS. Initially, a Luenberger-type observer was designed using healthy DPS dynamics to estimate system state and output. The estimated and measured system outputs are compared to generate the detection residual, which is shown to converge under healthy operating conditions in the absence of disturbance and uncertainty. An actuator/sensor fault on the DPS can act as an external input to the detection residual dynamics causing the residual to increase. The fault is detected when this residual exceeds a predefined threshold.

Upon detecting a fault, an adaptive term is added to the observer to learn the fault function. Although the fault detection observer only requires the system output, the parameter update law requires the system state to be available at all positions, which is a major drawback.

Therefore, by using the linear property of the PDE representation, an input filter along with two output filters are utilized to develop a new observer, which allows the determination of a parameter update law that tunes unknown fault parameter estimation with measured system output alone. Upon detecting a fault by using the filter-based observer, the detection and estimation scheme is revisited.

With state and output availability, the detection residual and parameter estimation errors are shown to be bounded in the presence of any bounded uncertainties or disturbances while asymptotic convergence is demonstrated in the absence of these terms. In addition, with output alone the detection residual and parameter estimation errors are shown to be bounded under faults with bounded uncertainties or disturbances. Moreover, by comparing the estimated fault parameters with their failure limits, an explicit formula for online estimation of TTF or RUL is proposed.

The contributions of this paper include: a) the development of a novel model-based detection and estimation scheme by using the PDE-based detection observer with detectability conditions, b) the design of the detection, estimation and prediction scheme by using a filter-based observer, which not only requires the system output alone but also allows the estimation of actuator and sensor faults, and c) TTF prediction with outputs alone.

This paper is organized as follows. A class of linear DPS described by a parabolic PDE is introduced in Section 2. Then the detection and estimation scheme is developed in Section 3, when the state is measurable and in Section 4 with output alone. Finally, Section 5 applies the proposed scheme to a heat transfer reactor in simulations.

2. BACKGROUND AND SYSTEM DESCRIPTION

The notations used in this paper are standard. A scalar function $v(x) \in L^2(0,1)$ is a square integrable on Hilbert space $L^2(0,1)$ with the norm defined as $\|v\|_2 = \sqrt{\int_0^1 v^2(x)dx}$. Throughout the paper the norm of a function $v(x,t)$ is denoted by $\|v(t)\|$ and the norm of $\partial v(x,t)/\partial x$ is expressed as $\|v_x(t)\|$.

Consider a class of linear DPS expressed by the following parabolic PDE with Dirichlet actuation given by

$$v_t(x,t) = \varepsilon v_{xx}(x,t) + \lambda v(x,t) + d(v(x,t),x,t) \tag{1}$$

where x is the space variable and $t \geq 0$ is the time variable $t \geq 0$ with boundary conditions defined by

$$v_x(0,t) = -qv(0,t), \;\; v(1,t) = U(t), \; y(t) = v(0,t) \tag{2}$$

where $v:[0,1] \times R^+ \to R$ represents the distributed state of the system; $d(v(x,t),x,t)$ stands for the system uncertainty or disturbance; $U(t)$ denotes control input, $\lambda > 0$ is a positive constant; ε and q are constant scalars; $v_t = \partial v/\partial t$, $v_x = \partial v/\partial x$ and $v_{xx} = \partial^2 v/\partial x^2$ are partial derivatives of v and $y(t)$ is the system output.

Assumption 1: The system uncertainty or disturbance is bounded above such that $\|d(v,x,t)\| \leq \bar{d}$ for all (v,x) and $t \geq 0$, where $\bar{d} > 0$ is a known constant. A more specific representation can be found in [15] and [16].

In this paper, an actuator and sensor fault type at the boundary condition are considered and will be described next.

2.1. ACTUATOR FAULT

Under a multiplicative actuator fault at the boundary condition of the DPS, the system in (1) and (2) can be described by

$$v_t(x,t) = \varepsilon v_{xx}(x,t) + \lambda v(x,t) + d(v(x,t),x,t)\,, \tag{3}$$

subject to the boundary conditions given by

$$v_x(0,t) = -qv(0,t),\;\; v(1,t) = \theta U(t),\;\; y(t) = v(0,t)\,, \tag{4}$$

where θ is the multiplicative fault parameter bounded by $|\theta| \le \theta_{\max}$. Alternatively, the boundary condition with the actuator fault can be expressed as $v(1,t) = U(t) + h(U(t),t)$, where $h(U(t),t) = \Theta U(t) = (\theta - 1)U(t)$ and $\Theta = \theta - 1$.

Moreover, the fault function can be written as

$$h(U(t),t) = \Omega(t - t_i)\Theta U(t)\;\;, \tag{5}$$

where t_i is the time of fault occurrence and $\Omega(t - t_i)$ is the time profile of the fault defined by $\Omega(\tau) = \begin{cases} 0 & ,if\;\; \tau < 0 \\ 1 - e^{-\kappa\tau} & ,if\;\; \tau \ge 0 \end{cases}$, where κ represents the fault growth rate, which should be a constant. This time profile allows both incipient and abrupt faults with different growth rates κ to be represented. However, for fault prediction, incipient faults are considered.

2.2. SENSOR FAULT

In the presence of a multiplicative sensor fault, the system measured output is modified as

$$y(t) = \theta_s v(0)\,, \tag{6}$$

where θ_s is a positive scalar representing a multiplicative sensor fault bounded by $\theta_{s\min} \le |\theta_s| \le \theta_{s\max}$. Under healthy conditions, the value of θ_s is taken as unity whereas it changes in the presence of a sensor fault. The following standard assumptions are required in order to proceed.

Assumption 2: There exists a stabilizing controller that guarantees the boundedness of the system state under healthy operating conditions.

Remark 1: This assumption separates a fault with instability of the system. For fault detection, the closed-loop DPS should be stable. Authors in [17] proposed a state and output feedback controller by using the backstepping approach to stabilize the parabolic PDE by using a control input which is a function of output $y(t)$.

Assumption 3: The fault type is known. Moreover, a single fault occurs on the system at any given time.

Remark 2: This assumption is used for fault estimation.

Before presenting the case where only the output is available, initially the system state and output are considered available over the entire range of space so that an actuator fault can be estimated. Next when the system output alone is available, the fault detection observer is redesigned using input and output filters. Fault estimation and prognosis are performed for both actuator and sensor faults. The next section investigates the former scenario and the latter is discussed in Section 4.

3. DETECTION AND ESTIMATION WITH STATE MEASUREMENTS

In this section, the system is considered initially without any disturbance and uncertainty, i.e., $d(v,x,t)=0$. An observer acting as a model is used to estimate the system state and output by utilizing DPS dynamics in healthy conditions. Figure 3.1 shows that under healthy conditions with no disturbances and uncertainties, and through the selection of observer gains, the estimated output will converge to measured value and thus the detection residual, which is defined as the difference between the estimated and the measured outputs, will converge to zero. During an actuator fault, the control input applied to the original system will be different than that of the observer. Thus, the measured output will deviate from the estimated output and lead to an increase in the residual [18].

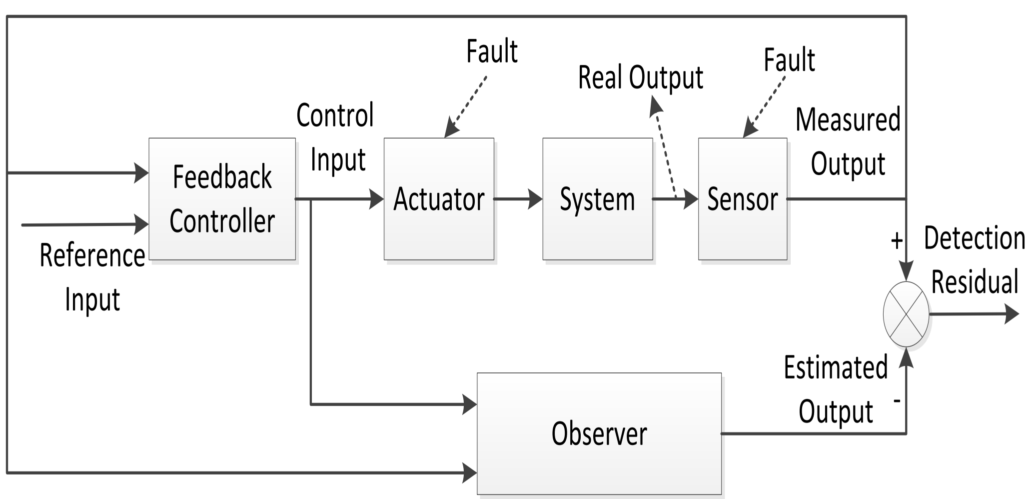


Figure 3.1. Architecture of fault detection scheme.

A sensor fault, on the other hand, will cause a change in the measured output, which will change the estimated output through feedback control input. Thus, a sensor fault can be detected as it leads to changes in the measured and estimated outputs differently causing the detection residual to increase over a threshold.

Remark 3: In this paper, the measurements are considered noise free. In addition, in the presence of bounded uncertainties and disturbances, the detection residual is shown to be bounded instead of converging to zero.

3.1. FAULT DETECTION OBSERVER

An observer generates the state of the DPS and is tuned by the output. By using the approach proposed by [19], define the fault detection observer as

$$\hat{v}_t(x,t) = \varepsilon \hat{v}_{xx}(x,t) + \lambda \hat{v}(x,t) + p_1(x)(y - \hat{y}), \tag{7}$$

$$\begin{aligned} &\hat{v}_x(0,t) = -qv(0,t) + p_{10}(y - \hat{y}), \quad \hat{v}(1,t) = U(t), \\ &\hat{y}(t) = \hat{v}(0,t), \quad e(t) = y(t) - \hat{y}(t), \end{aligned} \tag{8}$$

where $\hat{v}(x,t)$ is the estimated system state, $\hat{y}$ represents estimated output, $p_1(x)$ and p_{10} denote observer gains, and $e(t)$ is the detection residual.

Note that the fault detection observer is constructed using measured output. However, it will be observed later that when an adaptive term is incorporated into this observer to estimate fault function upon detection, the parameter tuning law will require the system state to be available.

To move on, define a state residual or state estimation error as $\tilde{v} = v - \hat{v}$ so as to analyze the performance of the observer. The state residual can also be used for fault detection in this section due to the availability of the system state; however, this is not necessary since the linear PDE in (3) transfers the effect of actuator fault to the output;

thus, fault can be detected by using the output residual e. In the absence of disturbances and uncertainties, $d(v,x,t)=0$, the state residual dynamics without a fault is represented as

$$\tilde{v}_t\left(x,t\right)=\varepsilon\tilde{v}_{xx}\left(x,t\right)+\lambda\tilde{v}\left(x,t\right)-p_1(x)\tilde{v}(0,t)\,, \tag{9}$$

$$\tilde{v}_x\left(0,t\right)=-p_{10}\tilde{v}(0,t),\;\;\tilde{v}\left(1,t\right)=0\;\;. \tag{10}$$

Next the following Lemma is needed in order to proceed.

Lemma 1 [20]: Consider the Volterra integral transformation utilized by the authors of [20]

$$\tilde{v}\left(x,t\right)=\xi\left(x,t\right)-\int_0^x L\left(x,\tau\right)\xi\left(\tau,t\right)d\tau\,, \tag{11}$$

where

$$L(x,\tau)=-\frac{(\lambda+b)(1-x)}{\varepsilon}\sum_{n=0}^{\infty}\frac{[\frac{(\lambda+b)}{\varepsilon}(2-x-\tau)(x-\tau)]^n}{2^{2n+1}n!(n+1)!}\,, \tag{12}$$

is the solution to the hyperbolic PDE given by

$$\varepsilon L_{xx}\left(x,\tau\right)-\varepsilon L_{\tau\tau}\left(x,\tau\right)=-(\lambda+b)L\left(x,\tau\right), \tag{13}$$

$$L\left(1,\tau\right)=0,\qquad L\left(x,x\right)=(\lambda+b)(x-1)/(2\varepsilon)\,. \tag{14}$$

Select observer gains $p_1(x)$ and p_{10} as

$$p_1(x)=\varepsilon L_\tau\left(x,0\right),\qquad p_{10}=L(0,0)\;\;. \tag{15}$$

For the sake of eliminating the unstable term $\lambda\tilde{v}(x,t)$ in (9), the transformation (11) with observer gains (15) converts the state residual dynamics in (9) and (10) to

$$\xi_t\left(x,t\right)=\varepsilon\xi_{xx}\left(x,t\right)-b\xi\left(x,t\right), \tag{16}$$

$$\xi_x\left(0,t\right)=0,\quad \xi\left(1,t\right)=0, \tag{17}$$

where $b \geq 0$ is an arbitrary constant that defines the convergence rate. Next, the following lemma will show that the transformation (11) is invertible.

Lemma 2: The inverse transformation of (11) is given by

$$\xi(x,t) = \tilde{v}(x,t) + \int_0^x M(x,\eta)\tilde{v}(\eta,t)d\eta \,, \tag{18}$$

where

$$M(x,\eta) = -\frac{(\lambda+b)(1-x)}{\varepsilon}\sum_{n=0}^{\infty}\frac{(-1)^n[\frac{(\lambda+b)}{\varepsilon}(2-x-\eta)(x-\eta)]^n}{2^{2n+1}n!(n+1)!} \,, \tag{19}$$

is the solution obtained through successive approximation [20] to the following hyperbolic PDE given by

$$M_{xx}(x,\eta) - M_{\eta\eta}(x,\eta) = (\lambda+b)M(x,\eta)/\varepsilon \,, \tag{20}$$

$$M(x,x) = (\lambda+b)(x-1)/2\varepsilon, \quad M(1,\eta) = 0 \,. \tag{21}$$

Proof: Follow steps in [20].

It will be shown that with the new observer presented herein, the detection residual converges to zero asymptotically under healthy conditions without any bounded disturbances or uncertainties and will remain ultimately bounded (UB) with them. A fault is detected by comparing the detection residual $e(t)$ with a predefined threshold ρ. The threshold is selected by using both the initial conditions and the bound on any system uncertainty or disturbances. The following theorem demonstrates the stability of detection residual under healthy conditions and provides fault detectability conditions.

Theorem 1 (Fault detection observer performance): Let the observer given in (7) and (8) be used to monitor the DPS defined by (3), (4) and (6). Then the state estimation error $\tilde{v}$ and the output detection residual $e(t)$ will converge to zero asymptotically under healthy operating conditions. In addition, a fault is declared active when the output detection residual $e(t)$ crosses the detection threshold ρ. A fault initiated at time t_i is detectable if there exists a time $T \geq t_i$ and a positive constant $\mathrm{H} > \rho$, such that

I. $\left|h\left(U\left(T\right),\mathrm{T}\right)\right| = H$ and $\left|h\left(U\left(t\right),t\right)\right| \geq H$ for $t > T$ in the case of an actuator fault, or

II. $\left|\left(\theta_s - 1\right)U(\mathrm{T})\right| = H$ and $\left|\left(\theta_s - 1\right)U(t)\right| \geq H$ for $t > T$ in the case of a sensor fault.

Proof: It is already known that the transformation (11) can map the state residual dynamics into the target system of (16) and (17) if $p_1(x)$ and p_{10} are defined by (15). The stability of the residual dynamics can be concluded from the stability of the target system given by (16) and (17) due to the transformation made possible by (11) [21]. To discuss the stability of the PDE described in (16) with boundary conditions given by (17), one must select a positive definite Lyapunov function candidate, which is half of the squared Sobolev norm of the spatial profile defined in a Hilbert space $H_1(0,1)$ as utilized in [20]

$$V(t) = \frac{\|\xi(t)\|^2}{2} + \frac{\|\xi_x(t)\|^2}{2} = \frac{\int_0^1 \xi^2(x,t)dx}{2} + \frac{\int_0^1 \xi_x^2(x,t)dx}{2} \quad . \tag{22}$$

The derivative of (22) is given by

$$\dot{V}(t) = \int_0^1 \xi(x,t)\frac{\partial \xi(x,t)}{\partial t}dx + \int_0^1 \frac{\partial \xi(x,t)}{\partial x}\frac{\partial^2 \xi(x,t)}{\partial t \partial x}dx \quad . \tag{23}$$

By using (16) and (17) in the equation above and applying both integration by parts and Poincare inequality [22]), $\dot{V}(t)$ becomes

$$\dot{V}(t) \leq -\frac{\pi^2 \varepsilon}{4}\int_0^1 \xi^2(x,t)dx + \int_0^1 \xi_x^2(x,t)dx = -\frac{\pi^2 \varepsilon V(t)}{2},$$

which is exponentially converging. It further yields that $\|\xi(t)\|^2 + \|\xi_x(t)\|^2 \le e^{-\frac{\varepsilon\pi^2}{2}t}(\|\xi(0)\|^2 + \|\xi_x(0)\|^2)$.

Therefore, the system defined by (16) and (17) is exponentially stable in $H_1(0,1)$. By using Agmon's inequality [19] we get

$$\begin{aligned} &\max_{x\in[0,1]}|\xi(x,t)|^2 \le 2\|\xi(t)\|\|\xi_x(t)\| \\ &\le \|\xi(t)\|^2 + \|\xi_x(t)\|^2 \le e^{-\frac{\varepsilon\pi^2}{2}t}(\|\xi(0)\|^2 + \|\xi_x(0)\|^2) \end{aligned},$$

which implies that $\xi(x,t)$ converges to zero asymptotically for all $x \in [0,1]$. By using the relationship between the detection residual $\tilde{v}(x,t)$ and $\xi(x,t)$ from (11), we can conclude that as $\xi(x,t) \to 0$ asymptotically, state residual $\tilde{v}(x,t) \to 0$ during healthy operating conditions.

To determine detectability conditions, an actuator fault is considered first. When an actuator fault occurs at t_i then the state estimation error dynamics described by (9) is subject to following boundary conditions given by

$$\tilde{v}_x(0,t) = -p_{10}\tilde{v}(0,t), \qquad \tilde{v}(1,t) = h(U(t),t) \quad . \tag{24}$$

Applying the transformation (11) on (9) and (24) leads to (16) with boundary conditions given by

$$\xi_x(0,t) = 0, \quad \xi(1,t) = h(U(t),t), \tag{25}$$

for $t \ge t_i$. Because $\tilde{v}(1,t) = \xi(1,t) - \int_0^1 L(1,\tau)\xi(\tau,t)\,d\tau$ and $L(1,\tau) = 0$ for all $0 \le \tau \le 1$, we can get $\tilde{v}(1,t) = \xi(1,t)$. Notice that $b = 0$ before fault detection and $b > 0$ after fault detection. Now a bounded state variable $\phi(x,t)$ is introduced which is defined by (16) with $b=0$ $\phi_t(x,t)=\varepsilon\phi_{xx}(x,t)$ subject to $\phi_x(0,t)=0$ and $\phi(1,t)=H$ and $\phi(x,T)=\xi(x,T)$. If $\left|h(U(t),t)\right|\geq H$ for $t\geq T$ then $\left|\xi(x,t)\right|\geq\phi(x,t)$ for $0\leq x\leq 1$ and $t\geq T$. Because $\bar{\phi}(x,t)=\phi(x,t)-H$ satisfies the following exponentially stable PDE given by $\bar{\phi}_t(x,t)=\varepsilon\bar{\phi}_{xx}(x,t),\bar{\phi}_x(0,t)=0,\ \ \bar{\phi}(1,t)=0$, the new state variable $\phi(\mathrm{x,t})$ converges to H. Hence, for any $\omega>0$, there exists a time $t_d\geq T$, such that $\phi(0,t_d)\geq H-\omega$. Therefore, for any $\omega>0$, there exists a time $t_d\geq T$ such that $\left|e(t_d)\right|=\left|\xi(0,t_d)\right|\geq\phi(0,t_d)\geq H-\omega$. Because $H>\rho$ and if ω is selected as $\omega<H-\rho$, then $\left|e(t_d)\right|>\rho$ for some $t_d\geq T$, so that the detection of an actuator fault is guaranteed when the detectability condition in Theorem 1 is satisfied.

In the case of a sensor fault, the detection residual can be written as $e(t)=y(t)-\hat{y}(t)=\theta_s v(0,t)-\hat{v}(0,t)$. If we define a new distributed variable $\Delta(x,t)=\theta_s v(x,t)-\hat{v}(x,t)$, then the detection residual can be expressed as $e(t)=\Delta(0,t)$. By using the definition of Δ and the observer dynamics in (7) and (8), it can be shown that Δ satisfies the PDE given by

$$\Delta_t(x,t)=\varepsilon\Delta_{xx}(x,t)+\lambda\Delta(x,t)-p_1(x)\Delta(0,t)\ ,$$

$$\Delta_x(0,t)=-p_{10}\Delta(0,t),\qquad \Delta(1,t)=(\theta_s-1)U(t)\,,$$

for $0\leq x\leq 1$ and $t\geq t_i$. If $\tilde{\mathrm{v}}$ in transformation (11) is replaced by Δ, applying this transformation to the above PDE will lead to

$$\xi_t(x,t)=\varepsilon\xi_{xx}(x,t)-b\xi(x,t)\,,$$

$$\xi_x(0,t)=0,\quad \xi(1,t)=(\theta_s-1)U(t)\,,$$

which is exactly the same as (16) and (25) except that the term $h\left(U\left(t\right),t\right)$ is replaced by $\left(\theta_s - 1\right)U(t)$. Thus, by using the same steps taken in the case of an actuator fault, it can be shown that if $\left|\left(\theta_s - 1\right)U(t)\right| \geq H$ for $t \geq T$, then for any $\omega > 0$, there exists a time $t_d \geq T$ such that $\left|e(t_d)\right| = \left|\Delta(0,t_d)\right| \geq H - \omega$. Selecting $\omega < H - \rho$, results in $\left|e(t_d)\right| > \rho$ which declares the presence of a fault.

Remark 4: In the presence of bounded uncertainties or disturbances, the dynamics of the observer error becomes $\tilde{v}_t\left(x,t\right) = \varepsilon\tilde{v}_{xx}\left(x,t\right) + \lambda\tilde{v}\left(x,t\right) - p_1(x)\tilde{v}(0,t) + d(v,x,t)$ with boundary condition (10). By applying inverse transformation (18) to these dynamics, you get $\xi_t\left(x,t\right) = \varepsilon\xi_{xx}\left(x,t\right) - b\xi\left(x,t\right) + d(v,x,t) + \int_0^x d(v,\eta,t)M(x,\eta)d\eta$ and (17). Now select (22), and it can be shown that with $b = 0$, $\left\|\xi(t)\right\| \leq \frac{2\sqrt{4+\pi^2}}{\varepsilon\pi^2} d_M$ and $\left\|\xi_x(t)\right\| \leq \frac{2\sqrt{4+\pi^2}}{\varepsilon\pi^2} d_M$ with $d_M = (1+\bar{m})\bar{d}$ and $\bar{m} = \max_{0\leq x,\eta\leq 1}\left|M(x,\eta)\right|$.

Remark 5: By using Agmon's inequality and the results of Remark 4, the detection residual can be expressed as $\left|e(t)\right| = \left|\xi(0,t)\right| \leq \frac{2\sqrt{2(4+\pi^2)}}{\varepsilon\pi^2} d_M (1+\bar{m})\bar{d}$. To detect the fault, a predefined threshold must be modified as $\rho = \rho_0 + k_m\bar{d}$ where ρ_0 is the threshold without uncertainties and disturbances, and $k_m = \sqrt{\frac{2\sqrt{2}(\varepsilon+2b)}{\sqrt{\varepsilon(\varepsilon+b)}\varepsilon b\pi^2}(1+\bar{m})}$. Note the newly defined threshold is greater than the one without uncertainties and disturbances.

3.2. FAULT ESTIMATION

Upon detection, the fault parameter has to be estimated. Although, both actuator and sensor faults are detectable by the proposed detection observer, the current method does not allow estimation of sensor fault function, since the dynamics of observer and residual change due to a sensor fault, makes the transformation (11) inadmissible. Therefore, in this section, the fault estimation is performed for actuator faults only.

An adaptive estimator, which is only activated upon detection, is added to the boundary condition of the observer (7) and (8) as (7) with boundary conditions

$$\hat{v}_x\left(0,t\right) = -qv(0,t) + p_{10}e\left(t\right) \quad , \tag{26}$$

$$\hat{v}(1,t) = U(t) + \hat{h}(U(t);\hat{\Theta}(t)),\ \hat{y}(t) = \hat{v}(0,t) \quad , \tag{27}$$

where $\hat{h}$ is the estimated fault dynamics given by the adaptive estimator

$$\hat{h}(U(t),\hat{\Theta}(t)) = \hat{\Theta}(t)U(t), \tag{28}$$

with $\hat{\Theta}(t)$ as the estimated fault parameter where $\hat{\Theta}(0) = 0$.

By taking the difference between the observer dynamics in (7), (26), and (27) and the actual system dynamics in (3) and (4) and applying Assumption 2, the state residual dynamics upon detecting an actuator fault can be expressed as (9) subject to

$$\tilde{v}_x\left(0,t\right) = -p_{10}e(t),\ \tilde{v}\left(1,t\right) = \tilde{\Theta}(t)U(t), \tag{29}$$

where $\tilde{\Theta}(t) = \Theta - \hat{\Theta}(t)$ is the fault parameter estimation error. Next the performance of the observer is discussed in the presence of an actuator fault.

Theorem 2 (Performance of an actuator fault estimation): Let the boundary condition of the observer in (8) be modified using (26) and (27) in order to estimate the state and output of the system defined in (3) and (4). In the presence of an actuator fault, consider the parameter tuning law

$$\begin{aligned}\dot{\hat{\Theta}}(t) &= \beta\varepsilon U(t)(\tilde{v}_x(1,t)+\int_0^1 M_x(1,\eta)\tilde{v}(\eta)d\eta) \\ &\quad -\hat{\Theta}(t)\gamma\left\|\tilde{v}(x,t)+\int_0^x M(x,\eta)\tilde{v}(\eta,t)d\eta\right\|^2,\end{aligned} \tag{30}$$

for fault estimation where $\beta > 0$ is the adaptation rate, $0 < \gamma < 2\beta b / \Theta_{\max}^2$ is the stabilizing term, and $M(x,\eta)$ is given by (19), then the state residual converges to zero and the parameter estimation error is bounded.

Proof: First apply transformation (11) on the residual dynamics (9) and (29) to get PDE (16) with boundary conditions given by

$$\xi_x(0,t)=0, \quad \xi(1,t)=\tilde{\Theta}(t)U(t) \ . \tag{31}$$

Now select the Lyapunov function candidate

$$V(t)=\int_0^1 \xi^2(x,t)dx/2+\tilde{\Theta}^2(t)/(2\beta) \ , \tag{32}$$

whose first derivative is given by $\dot{V}(t)=\int_0^1 \xi(x,t)\xi_t(x,t)dx+\tilde{\Theta}(t)\dot{\tilde{\Theta}}(t)/\beta$. By substituting (16) in the first derivative, we will arrive at

$$\dot{V}(t)=\varepsilon\int_0^1 \xi(x,t)\xi_{xx}(x,t)dx-b\int_0^1 \xi^2(x,t)dx+\tilde{\Theta}(t)\dot{\tilde{\Theta}}(t)/\beta$$

Applying integration by parts and using boundary conditions given by (31) will lead to

$$\dot{V}(t)=\varepsilon\xi_x(1,t)\tilde{\Theta}(t)U(t)-\varepsilon\int_0^1 \xi_x^2(x,t)dx-b\int_0^1 \xi^2(x,t)dx+\tilde{\Theta}(t)\dot{\tilde{\Theta}}(t)/\beta \ . \tag{33}$$

To represent this update law in terms of a transformed system state, instead of the actual system state, the inverse transformation (18) and its first derivative with respect to x given by $\xi_x(1,t)=\tilde{v}_x(1,t)+\int_0^1 M_x(1,\eta)\tilde{v}(\eta,t)d\eta$ will be utilized in (30) to get

$$\dot{\hat{\Theta}}(t) = \beta\varepsilon U(t)\xi_x(1,t) - \hat{\Theta}(t)\gamma\|\xi(t)\|^2 \ . \tag{34}$$

Equation (34) eliminates the positive term $\varepsilon\xi_x(1,t)\tilde{\Theta}(t)U(t)$. By applying the proposed parameter update law in the derivative of the Lyapunov function candidate, we get

$$\begin{aligned}\dot{V}(t) &= -\varepsilon\int_0^1 \xi_x^2(x,t)dx - b\|\xi(t)\|^2 + \tilde{\Theta}(t)\hat{\Theta}(t)\gamma\|\xi(t)\|^2 / \beta \\ &\le -b\|\xi(t)\|^2 - \gamma\|\xi(t)\|^2[\tilde{\Theta}^2(t) - \Theta^2{}_{\max}]/2\beta \\ &\le -(2\beta b - \Theta^2{}_{\max}\gamma)\|\xi(t)\|^2/(2\beta) - \gamma\tilde{\Theta}^2(t)\|\xi(t)\|^2/(2\beta)\end{aligned} .$$

Thus, when $\gamma < 2\beta b / \Theta^2_{\max}$, $\dot{V} \le 0$ and $\|\xi(t)\|$ and $|\tilde{\Theta}|$ are bounded. Now, define $S = \left\{\|\xi(t)\|, \tilde{\Theta}(t) \middle| \dot{V}(\|\xi(t)\|, \tilde{\Theta}(t)) = 0\right\}$, when $\dot{V} = 0$. Since the largest invariant set contained in S, is same as S, the asymptotic convergence of $\|\xi(t)\|$ to zero and boundedness of $|\tilde{\Theta}|$ can be shown by using LaSalle's theorem [23].

Remark 6: In the presence of $d(v,x,t)$, it can be shown that with (30) $\|\xi(t)\| \le \sqrt{\beta / [b(\beta b - \Theta^2{}_{\max}\gamma)]}(1+\bar{m})\bar{d}$. Therefore, $\xi(x,t)$ is bounded for $x \in [0,1]$. From (31), notice that $\xi(1,t) = \tilde{\Theta}(t)U(t)$, so that $\tilde{\Theta}(t)$ is also bounded.

4. ESTIMATION AND PREDICTION WITH OUTPUTS

In this case, only the system output is considered measurable without the system state being available. The detection observer had to be redesigned and its tuning law had to be carefully selected.

4.1. FILTER-BASED OBSERVER

In the case of the redesign of the filter-base observer, the boundary value $v(0,t)$ is available. The following steps have to be taken: (a) Convert the system dynamics to an observable form. (b) Design a filter-based observer based on known information, namely the control input and the measured output. (c) Prove the stability of the observer under healthy conditions. (d) Incorporate adaptive terms in the observer to estimate unknown fault parameters, upon detection of a fault.

In the first step, the system plant (3, (4) and (6) is converted to an observable form by utilizing the transformation [17]

$$z(x,t) = v(x,t) - \int_0^x l(x,\tau)v(\tau,t)\,d\tau \ , \tag{35}$$

where $z(x,t)$ is the new state variable of the system in the observable form and $l(x,\eta)$ is the solution of the hyperbolic PDE $l_{xx} - l_{\eta\eta} = \lambda l(x,\eta)/\varepsilon$, $l(1,\eta) = 0$ and $l(x,x) = \lambda(1-x)/2\varepsilon$. Therefore, this transformation can convert the system (3), (4) and (6) in the presence of an actuator and a sensor, both of which fault respectively into

$$\begin{aligned} z_t(x,t) &= \varepsilon z_{xx}(x,t) - \varepsilon[l_\eta(x,0) + ql(x,0)]z(0,t) \\ &\quad + d(v(x),x,t) - \int_0^x d(v(\tau),\tau,t)l(x,\tau)\,d\tau \end{aligned} \ , \tag{36}$$

$$z_x(0,t) = -(\lambda/2\varepsilon + q)z(0,t), \quad z(1,t) = \theta U(t) \ , \tag{37}$$

and

$$y(t) = \theta_s z(0,t) \ . \tag{38}$$

Note that $v(0,t) = z(0,t)$. Under healthy conditions where $\theta = \theta_s = 1$, consider linear DPS in (36) and (37) with $U(t)$ $-\varepsilon(l_\eta(x,0) + ql(x,0))v(0,t)$ and $-(\lambda / 2\varepsilon + q)v(0,t)$ as external inputs [20].

By using the superposition principle, its solution can be expressed by summing the response of the PDE due to each external input. Therefore, $z(x,t)$ can be expressed as a combination of the solutions defined by [20]

$$\Lambda_t(x,t) = \varepsilon\Lambda_{xx}(x,t) \quad , \ \Lambda_x(0,t) = 0, \quad \Lambda(1,t) = U(t),$$

where $\Lambda(x,t)$ is referred to as an input filter since it is derived from the input of the actual system $U(t)$ [20]. Next consider

$$A_t(x,t) = \varepsilon A_{xx}(x,t) \ , \quad A_x(0,t) = y(t), \quad A(1,t) = 0,$$

where $A(x,t)$ is an output filter since it is derived from output of the actual system $y(t)$. In addition, consider

$$\begin{gathered} B_t(x,\eta,t) = \varepsilon B_{xx}(x,\eta,t) + \delta(x-\eta)y(t) \\ B_x(0,\eta,t) = 0, \quad B(1,\eta,t) = 0 \end{gathered} \ ,$$

where $B(x,\eta,t)$ is another output filter.

Define the observer in terms of the new state variable as

$$\begin{aligned} &\hat{z}(x,t) = \Lambda(x,t) + [-\lambda/(2\varepsilon) - q]\mathrm{A}(x,t) \\ &+\int_0^1 \varepsilon[-l_\eta(s,0) - ql(s,0)]\mathrm{B}(x,s,t)ds, \quad \hat{y}(t) = \hat{z}(0,t) \end{aligned} \ , \tag{39}$$

where $\hat{z}(x,t)$ is an estimate of $z(x,t)$ and $\hat{y}(t)$ is an estimate of $y(t)$ under healthy operating conditions.

It is shown in the next theorem that the observer state estimation error $\tilde{z}(x,t) = z(x,t) - \hat{z}(x,t)$ is ultimately bounded under healthy conditions with $\theta = \theta_s = 1$. To perform fault detection, based on the observer error, one must define the detection residual $\bar{e}(t) = y(t) - \hat{y}(t)$ since the only available measurement is $y(t)$. A fault is detected when the detection residual $\bar{e}(t)$ exceeds the predefined detection threshold ρ. Next, the fault detectability condition is introduced in the following theorem.

Theorem 3 (Output-based fault detection observer performance): Let the observer in (39) be used to monitor the DPS defined by (36)–(38) with bounded disturbances. Then the state estimation error $\tilde{z}$ and detection residual $\bar{e}(t)$ are bounded under healthy operating conditions. Further, a fault initiated at time t_i is detectable if there exists a time $T \ge t_i$ and a positive constant H, such that

I. $\left|(\theta-1)U(t)\right| \ge H$ for $t \ge T$ with $H > \rho + k_l\bar{d}$ in the case of an actuator fault, or

II. $\left|(\theta_s-1)U(t)\right| \ge H$ for $t \ge T$ with $H > \rho + k_l\bar{d}\theta_{s\max}$ in the case of a sensor fault.

where $k_l = 2\sqrt{2(\pi^2+4)}(1+\bar{l})/(\varepsilon\pi^2)$ and $\bar{l} = \max\limits_{0\le x,\eta\le 1} l(x,\eta)$.

Proof: During healthy conditions with $\theta = \theta_s = 1$, the state residual satisfies the stable PDE given by

$$\tilde{z}_t(x,t) = \varepsilon\tilde{z}_{xx}(x,t) + d(v,x,t) - \int_0^x d(v,\tau,t)l(x,\tau)\,d\tau\,, \tag{40}$$

$$\tilde{z}_x(0,t) = 0,\ \ \tilde{z}(1,t) = 0\,. \tag{41}$$

Select a positive definite Lyapunov function candidate as $V(t) = \left\|\tilde{z}(t)\right\|^2/2 + \left\|\tilde{z}_x(t)\right\|^2/2$, whose first derivative of $V(t)$ becomes

$$\dot{V}(t) = \int_0^1 \tilde{z}(x,t)\tilde{z}_t(x,t)dx + \int_0^1 \tilde{z}_x(x,t)\tilde{z}_{tx}(x,t)dx\,.$$

By using (40) and (41) in the above equation and applying both integration by parts and Poincare inequality, we get

$$\begin{aligned}
&\dot{V}(t) = -\varepsilon\int_0^1 \tilde{z}_x^2(x,t)dx + \int_0^1 d(v,x,t)\tilde{z}(x,t)dx \\
&-\int_0^1 \tilde{z}(x,t)\int_0^x d(v,\eta,t)l(x,\eta)d\eta dx - \varepsilon\int_0^1 \tilde{z}_{xx}^2(x,t)dx \\
&-\int_0^1 d(v,x,t)\tilde{z}_{xx}(x,t)dx + \int_0^1 \tilde{z}_{xx}(x,t)\int_0^x d(v,\eta,t)l(x,\eta)d\eta dx\,, \\
&\le -\varepsilon\int_0^1 \tilde{z}_x^2(x,t)dx - \varepsilon\int_0^1 \tilde{z}_{xx}^2(x,t)dx \\
&\quad + d_l\int_0^1 |\tilde{z}(x,t)|dx + d_l\int_0^1 |\tilde{z}_{xx}(x,t)|dx
\end{aligned}$$

where $d_l = (1+\bar{l}\,)\bar{d}$. By using Cauchy-Schwarz and Poincare inequalities we get

$$\dot{V}(t) \le -\frac{\varepsilon\pi^2}{8}\int_0^1 \tilde{z}^2(x,t)dx - \frac{\varepsilon\pi^2}{8}\int_0^1 \tilde{z}_x^2(x,t)dx + \frac{(\pi^2+4)}{2\varepsilon\pi^2}d_l^2\ .$$

Thus, $\dot{V}(t)$ will be less than zero if

$$\|\tilde{z}(t)\| > 2d_l\sqrt{\pi^2+4}\,/\,\varepsilon\pi^2 \ \text{ or } \|\tilde{z}_x(t)\| > 2d_l\sqrt{\pi^2+4}\,/\,\varepsilon\pi^2\,.$$

By Agmon's inequality $\max_{x\in[0,1]}|\tilde{z}(x,t)|^2 \le 2\|\tilde{z}(t)\|\|\tilde{z}_x(t)\|$, we can get $\max_{x\in[0,1]}|\tilde{z}(x,t)| \le 2(1+\bar{l}\,)\bar{d}\sqrt{2(\pi^2+4)}\,/\,\varepsilon\pi^2$ and $|e(t)| = |\tilde{z}(0,t)| \le 2(1+\bar{l}\,)\bar{d}\sqrt{2(\pi^2+4)}\,/\,\varepsilon\pi^2$.

Therefore, the detection threshold must be selected as $\rho = \rho_0 + k_l\bar{d}$ where ρ_0 depends on the initial conditions and $k_l = 2(1+\bar{l}\,)\sqrt{2(\pi^2+4)}\,/\,\varepsilon\pi^2$.

If an actuator fault happens ($\theta \ne 1, \theta_s = 1$) at t_i, then the state estimation error dynamics are given by

$$\tilde{z}_t(x,t) = \varepsilon\tilde{z}_{xx}(x,t) + d(v,x,t) - \int_0^x d(v,\tau,t)l\left(x,\tau\right)d\tau\,,$$

$$\tilde{z}_x(0,t)=0,\ \tilde{z}(1,t)=(\theta-1)U(t) \quad ,$$

for $t \geq t_i$. Now, we can define a new PDE as

$$\psi_t(x,t)=\varepsilon\psi_{xx}(x,t)+d(v,x,t)-\int_0^x d(v,\tau,t)l\left(x,\tau\right)d\tau \ ,$$

$$\psi_x(0,t)=0,\ \psi(1,t)=H,$$

for $t \geq T$ and let $\psi(x,T)=\tilde{z}(x,T)$. If $\left|(\theta-1)U(t)\right| \geq H$ for $t \geq T \geq t_i$ then $\left|\tilde{z}(x,t)\right| \geq \psi(x,t)$ for $0 \leq x \leq 1$ and $t \geq T$. Note that the dynamics of $\left[\psi\left(x,t\right)-H\right]$ is the same as (40-41), thus $\left|\psi\left(0,t\right)-H\right| \leq k_l\bar{d}$ ultimately. Thus, for any $\omega > 0$, there exists a time $t_d \geq T$, such that $\psi\left(0,t_d\right) \geq H-k_l\bar{d}-\omega$. That means, for any $\omega > 0$, there exists a time $t_d \geq T$ such that $\left|\bar{e}(t_d)\right| = \left|\tilde{z}(0,t_d)\right| \geq \psi\left(0,t_d\right) \geq H-k_l\bar{d}-\omega$. Because $H > \rho + k_l\bar{d}$ and if ω is selected as $\omega < H-(\rho+k_l\bar{d})$, then $\left|\bar{e}(t_d)\right| > \rho$ for some $t_d \geq T$, and the detection of actuator fault is guaranteed.

Since the sensor fault will affect the system through the feedback control, the detectability condition for the sensor fault is going to be very similar to that of the actuator fault. In the presence of a sensor fault ($\theta=1, \theta_s \neq 1$), the detection residual is given by $\bar{e}\left(t\right)=y(t)-\hat{y}(t)=\theta_s z\left(0,t\right)-\hat{z}(0,t)$.

If we define a new distributed variable Δ such that $\Delta(x,t)=\theta_s z(x,t)-\hat{z}(x,t)$ for $0 \leq x \leq 1$, then by using (36)–(39), Δ can be described by

$$\Delta_t(x,t)=\varepsilon\Delta_{xx}(x,t)+\theta_s[d(v,x,t)-\int_0^x d(v,\tau,t)l\left(x,\tau\right)d\tau],$$

$$\Delta_x(0,t)=0,\ \Delta(1,t)=(\theta_s-1)U(t).$$

and the detection residual can be defined as $\bar{e}\left(t\right) = \Delta(0,t)$. Similar to the case of actuator fault, a new bounded state variable defined by the following PDE is introduced

$$\Psi_t(x,t) = \varepsilon\Psi_{xx}(x,t) + \theta_s[d(v,x,t) - \int_0^x d(v,\tau,t)l\left(x,\tau\right)d\tau],$$

$$\Psi_x(0,t) = 0,\ \Psi(1,t) = H\ ,$$

for $t \geq T$ and let $\Psi(x,T) = \Delta(x,T)$. If $\left|\left(\theta_s - 1\right)U(t)\right| \geq H$ for $t \geq T$ then $\left|\Delta(x,t)\right| \geq \Psi(x,t)$ for $0 \leq x \leq 1$ and $t \geq T$. Similarly, it can be obtained that $\left|\Psi\left(0,t\right) - H\right| \leq k_l\bar{d}\theta_{s\max}$ ultimately, which means that for any $\omega > 0$, there exists a time $t_d \geq T$ such that $\left|\Delta\left(0,t_d\right)\right| \geq \Psi\left(0,t_d\right) \geq H - k_l\bar{d}\theta_{s\max} - \omega$. Therefore, by selecting $\omega < H - (\rho + k_l\bar{d}\theta_{s\max})$, it is easy to see that there exists a time $t_d \geq T$ where $\left|\bar{e}(t_d)\right| = \left|\Delta\left(0,t_d\right)\right| > \rho$, which guarantees the detection of a fault.

Remark 7: When $d(v,x,t) = 0$, observer error dynamics $\tilde{z}(x,t)$ satisfies $\tilde{z}_t(x,t) = \varepsilon\tilde{z}_{xx}(x,t), \tilde{z}_x(0,t) = 0, \tilde{z}(1,t) = 0$ under healthy conditions. Therefore, the observer error will converge to zero asymptotically, and the detection threshold $\rho = \rho_0$ only depends upon initial conditions.

4.2. FAULT ESTIMATION

Upon detecting a fault and knowing the fault type, an adaptive term will be incorporated into the observer defined in (39). If an actuator fault is detected, the observer in this case is described by

$$\begin{aligned}\hat{z}(x,t) &= \hat{\theta}(t)\Lambda(x,t) + [-\lambda/(2\varepsilon) - q]\mathrm{A}(x,t) \\ &+ \int_0^1 \varepsilon[-l_\eta(s,0) - ql(s,0)]\mathrm{B}(x,s,t)ds, \quad \hat{y}(t) = \hat{z}(0,t)\end{aligned}, \tag{42}$$

where $\hat{\theta}(t)$ is the estimated parameter of an actuator fault and $\hat{y}(t)$ represents the estimated output.

For a sensor fault, the observer will be described by

$$\begin{aligned}\hat{z}(x,t) &= \Lambda(x,t) + [-\lambda/(2\varepsilon) - q]\mathrm{A}(x,t)/\hat{\theta}_s(t) \\ &+\{\int_0^1 \varepsilon[-l_\eta(s,0) - ql(s,0)]\mathrm{B}(x,s,t)ds\}/\hat{\theta}_s(t)\ , \\ \hat{y}(t) &= \hat{\theta}_s(t)\hat{z}(0,t)\end{aligned} \tag{43}$$

where $\hat{\theta}_s(t)$ is the estimated parameter of a sensor fault. To assure $\hat{\theta}_s(t) \neq 0$ since $\hat{\theta}_s(t)$ is in the denominator in (43), two cases are considered.

Upon detecting a sensor fault, the residual becomes $\bar{e}(t) = y(t) - \hat{y}(t) = \theta_s z(0,t) - \hat{\theta}_s \hat{z}(0,t)$ where $\hat{\theta}_s$ is initialized as $\hat{\theta}_s(0) = 1$ and will not be updated before the detection of a sensor fault. The next theorem demonstrates the boundedness of closed-loop system with faults.

Theorem 4 (Output based fault estimation): Let the observer in (42) be used to estimate the state and output of DPS (36)–(37) with $\hat{\theta}(0) = 1$. The tuning law

$$\dot{\hat{\theta}} = \beta\Lambda(0,t)\bar{e}(t) - \gamma\hat{\theta}, \tag{44}$$

is activated upon detection of an actuator fault. Similarly, allow the observer from (43) to estimate the system state and output when $\hat{\theta}_s(0) = 1$ with the tuning law

$$\dot{\hat{\theta}}_s(t) = \begin{cases} 0 \quad if \quad \beta\Lambda(0,t)\bar{e}(t) - \gamma\hat{\theta}_s(t) < 0 \ \&\ \hat{\theta}_s(t) = \theta_{s\min} \\ \beta\Lambda(0,t)\bar{e}(t) - \gamma\hat{\theta}_s(t) \qquad otherwise \end{cases}, \tag{45}$$

upon detection of a sensor fault, where $\theta_{smin} > 0$ is a known lower bound on the sensor fault parameter. Then the residual $\bar{e}$, parameter estimation errors $\tilde{\theta} = \theta - \hat{\theta}$

and $\tilde{\theta}_s = \theta_s - \hat{\theta}_s$ in the presence of an actuator and sensor faults respectively will be ultimately bounded (UB).

Proof: For an actuator fault, an error signal is defined as

$$\begin{aligned}\mu(x,t) &= z(x,t) - \theta\Lambda(x,t) - (-\lambda/2\varepsilon - q)\mathrm{A}(x,t)\\ &-\int_0^1 \varepsilon[-l_\eta(s,0) - ql(s,0)]\mathrm{B}(x,s,t)ds\end{aligned},$$

and in the case of a sensor fault, it is defined as

$$\begin{aligned}\mu(x,t) &= z(x,t) - \Lambda(x,t) - (-\lambda/2\varepsilon - q)\mathrm{A}(x,t)/\theta_s\\ &-\{\int_0^1 \varepsilon[-l_y(s,0) - ql(s,0)]\mathrm{B}(x,s,t)ds\}/\theta_s\end{aligned}.$$

This error signal in both cases clearly satisfies

$$\mu_t(x,t) = \varepsilon\mu_{xx}(x,t) + d(v,x,t) - \int_0^x d(v,\tau,t)l\left(x,\tau\right)d\tau,$$

$$\mu_x(0,t) = 0, \quad \mu(1,t) = 0.$$

(a) Actuator fault

Now, a Lyapunov function candidate is selected as

$$V = \|\mu\|^2/(2\varepsilon) + \tilde{\theta}^2/(2\beta).$$

By taking the derivative of the Lyapunov function with respect to time and applying integration by parts, we obtain

$$\begin{aligned}\dot{V} &= -\int_0^1 \mu_x^2(x,t)dx + \int_0^1 d(v(x,t),x,t)\mu(x,t)dx/\varepsilon\\ &-\int_0^1 \mu(x,t)\int_0^x d(v(\eta,t),\eta,t)l(x,\eta)d\eta dx/\varepsilon + \tilde{\theta}(t)\dot{\tilde{\theta}}(t)/\beta.\end{aligned}$$

Substituting (44) in the above equation yields

$$\begin{aligned}\dot{V} &= -\|\mu_x\|^2 + \int_0^1 d(v(x,t),x,t)\mu(x,t)dx/\varepsilon - \tilde{\theta}(t)\overline{e}(t)\Lambda(0,t)\\ &+\gamma\tilde{\theta}(t)\hat{\theta}(t)/\beta - \int_0^1 \mu(x,t)\int_0^x d(v(\eta,t),\eta,t)l(x,\eta)d\eta dx/\varepsilon.\end{aligned}$$

Since $\tilde{z}(0,t)-\mu(0,t)=\tilde{\theta}(\mathrm{t})\Lambda(0,\mathrm{t})$ and by using Cauchy-Schwarz inequality, $\dot{V}$ can be rewritten as

$$\begin{aligned}\dot{V} &= -\|\mu_x\|^2 + \int_0^1 d(v(x,t),x,t)\mu(x,t)dx/\varepsilon \\ &\quad - \int_0^1 \mu(x,t)\int_0^x d(v(\eta,t),\eta,t)l(x,\eta)d\eta dx/\varepsilon \\ &\quad - \bar{e}^2(t) + \bar{e}(t)\mu(0,t) + \gamma\tilde{\theta}(t)\hat{\theta}(t)/\beta \\ &\le -\|\mu_x\|^2 - \bar{e}^2(t)/2 + \mu^2(0,t)/2 \\ &\quad - \gamma[\tilde{\theta}^2(t) - \theta_{\max}^2]/(2\beta) + d_l\int_0^1 |\mu(x,t)| dx/\varepsilon\end{aligned}.$$

By using the Poincare $\|\mu\|^2 \le 4\|\mu_x\|^2/\pi^2$ and Agmon's inequalities $\mu^2(0,t) \le 2\|\mu\|\|\mu_x\| \le \|\mu\|^2 + \|\mu_x\|^2$, we get

$$\begin{aligned}\dot{V} &\le -(\pi^2-4)\|\mu\|^2/8 + d_l\int_0^1 |\mu(x,t)| dx/\varepsilon - \gamma[\tilde{\theta}^2(t)-\theta_{\max}^2]/(2\beta) \\ &\le -(\pi^2-8)\|\mu\|^2/8 - \gamma[\tilde{\theta}^2(t)-\theta_{\max}^2]/(2\beta) + d_l^2/2\varepsilon^2\end{aligned}$$

. Therefore, $\dot{V}<0$ when

$$\|\mu\| > \sqrt{\frac{4(\gamma\theta_{\max}^2 + \beta d_l^2/\varepsilon^2)}{(\pi^2-8)\beta}} \quad or \quad |\tilde{\theta}| > \sqrt{\frac{\gamma\theta_{\max}^2 + \beta d_l^2/\varepsilon^2}{\gamma}} .$$

Hence, μ and $\tilde{\theta}$ are UB with the bounds defined above. Since $\bar{e}(t)=\mu(0,t)+\tilde{\theta}(t)\Lambda(0,t)$ and Λ is bounded, $\bar{e}$ is also bounded.

Remark 8: In the ideal case, when $d(v,x,t)=0$, one can show that $\|\mu\| \le \sqrt{4\gamma/[(\pi^2-4)\beta]}\theta_{\max}$ and $|\tilde{\theta}| \le \theta_{\max}$.

(b) Sensor fault

Similarly, in this case, consider the Lyapunov function

$$V = \|\mu\|^2/(2\varepsilon) + \tilde{\theta}_s^{\,2}/(2\beta\theta_{s\max}^2) \tag{46}$$

where θ_{smax} is the upper bound on the sensor fault magnitude θ_s . Taking the derivative of (46) with respect to time and applying integration by parts leads to

$$\dot{V} = -\int_0^1 \mu_x^2(x,t)dx + \int_0^1 d(v(x,t),x,t)\mu(x,t)dx/\varepsilon$$
$$-\frac{1}{\varepsilon}\int_0^1 \mu(x,t)\int_0^x d(v(\eta,t),\eta,t)l(x,\eta)d\eta dx + \frac{\theta_s(t)\dot{\tilde{\theta}}_s(t)}{\beta\theta_{s\max}^2}.$$

The sensor fault parameter $\hat{\theta}_s$ is tuned using $\dot{\hat{\theta}}_s(t) = \beta\Lambda(0,t)\bar{e}(t) - \gamma\hat{\theta}_s(t)$. However, $\hat{\theta}_s$ appears in the denominator of (43), and the update law is modified as (45) to ensure it is nonzero. With this update law, if $\beta\Lambda(0,t)\bar{e}(t) - \gamma\hat{\theta}_s(t) < 0$ & $\hat{\theta}_s = \theta_{s\min}$, $\dot{V}$ is given by $\dot{V} \le -\|\mu_x\|^2 + d_l\int_0^1 |\mu(x,t)|dx/\varepsilon$; otherwise, it should satisfy $\dot{V} \le -\|\mu_x\|^2 - \frac{\tilde{\theta}_s\Lambda(0,t)\bar{e}(t)}{\theta_{s\max}^2} + \frac{\gamma\tilde{\theta}_s(t)\hat{\theta}_s(t)}{\beta\theta_{s\max}^2} + \frac{d_l}{\varepsilon}\int_0^1 |\mu(x,t)|dx$.

In the first case, by applying Poincare inequality $\|\mu\|^2 \le \frac{4\|\mu_x\|^2}{\pi^2}$, we can show $\dot{V} \le -\frac{(\pi^2-2)}{4}\|\mu\|^2 + \frac{d_l^2}{2\varepsilon^2}$ which means $\dot{V}$ will be less than zero if $\|\mu\| > \sqrt{2/[\varepsilon^2(\pi^2-2)]}d_l$. Therefore, $\|\mu\|$ and $|\tilde{\theta}_s|$ are bounded. Now for the second case, as $\bar{e}(t) - \theta_s\mu(0,t) = \tilde{\theta}_s(t)\Lambda(0,t)$, $\dot{V}$ can be written as

$$\begin{aligned}\dot{V} &\le -\|\mu_x\|^2 - [\bar{e}(t) - \theta_s\mu(0,t)]\bar{e}(t)/\theta_{s\max}^2 + \gamma\tilde{\theta}_s(t)\hat{\theta}_s(t)/(\beta\theta_{s\max}^2) + d_l\int_0^1 |\mu(x,t)|dx/\varepsilon \\ &\le -\|\mu_x\|^2 + d_l\int_0^1 |\mu(x,t)|dx/\varepsilon + \mu^2(0,t)/2 - \gamma\tilde{\theta}_s^{\,2}(t)/(2\beta\theta_{s\max}^2) + \gamma/(2\beta)\end{aligned},$$

Applying Poincare $\|\mu\|^2 \le 4\|\mu_x\|^2/\pi^2$ and Agmon's inequalities $\mu^2(0,t) \le 2\|\mu\|\|\mu_x\| \le \|\mu\|^2 + \|\mu_x\|^2$ will lead to

$$\dot{V} \le -(\pi^2 - 4)\|\mu\|^2 / 8 + d_l \int_0^1 |\mu(x,t)| dx / \varepsilon - \gamma \tilde{\theta}_s^{\,2}(t) / (2\beta\theta_{s\max}^2) + \gamma / (2\beta)$$
$$\le -(\pi^2 - 8)\|\mu\|^2 / 8 - \gamma \tilde{\theta}_s^{\,2}(t) / 2\beta\theta_{s\max}^2 + \gamma / 2\beta + d_l^2 / 2\varepsilon^2.$$

Thus, $\dot{V} < 0$ when

$$\|\mu\| > \sqrt{\frac{4(\gamma + \beta d_l^2 / \varepsilon^2)}{(\pi^2 - 8)\beta}} \quad or \quad |\tilde{\theta}_s| > \sqrt{\frac{\gamma + \beta d_l^2 / \varepsilon^2}{\gamma}} \theta_{s\max} ,$$

are satisfied, we can see how μ and $\tilde{\theta}_s$ are UB. Because $\bar{e}(t) = \theta_s \mu(0,t) + \tilde{\theta}_s(t)\Lambda(0,t)$ and θ_s , μ , $\tilde{\theta}_s$ and Λ are bounded, $\bar{e}$ is also bounded. Therefore, the closed-loop system is bounded for both cases.

Remark 9: Without disturbance one can show that $\|\mu\| \le \sqrt{4\gamma / [(\pi^2 - 4)\beta]}$ and $|\tilde{\theta}| \le \theta_{s\max}$.

4.3. FAILURE PREDICTION

A system may remain functional after an incipient fault; however, it cannot function after a failure. The TTF scheme can predict the RUL of the system upon detection by using the estimated parameter trajectories. The TTF can be defined as the difference between current time t and the time of failure t_f $TTF(t) = t_f - t$. The TTF can be predicted by using the parameter update laws (44) and (45) given the parameter failure limits as proposed next.

Remark 10 (TTF prediction for an actuator fault): Given the detection residual of an actuator fault, the input filter state and the upper bound of the actuator fault parameter, upon detecting an actuator fault by using the observer defined by (39), TTF can then be estimated as

$$TTF(t) = \frac{1}{\gamma} Ln\left(\frac{\gamma\hat{\theta}(t) - \beta\Lambda(0,t)\bar{e}(t)}{\gamma\theta_f - \beta\Lambda(0,t)\bar{e}(t)} \right), \tag{47}$$

where θ_f is the failure limit for θ. Considering $\beta\Lambda(0,t)\bar{e}(t)$ as an input to (44), the fault parameter estimation can be solved as

$\hat{\theta}(t_f) = e^{-\gamma(t_f - t)}\hat{\theta}(t) + \int_t^{t_f} e^{-\gamma(t_f - \tau)}\beta\Lambda(0,\tau)\bar{e}(\tau)d\tau \quad for\ t_f > t$ where t is the current time instant and t_f refers to future time. Now assume that the term $\beta\Lambda(0,t)\bar{e}(t)$ is held in the interval $[t, t_f]$ and let t_f be the first time when the value of $\hat{\theta}$ reaches its failure limit θ_f as

$\hat{\theta}(t_f) = \theta_f = e^{-\gamma(t_f - t)}\hat{\theta}(t) + \beta\Lambda(0,t)\bar{e}(t)[1 - e^{-\gamma(t_f - t)}]/\gamma.$

By substituting $TTF = t_f - t$ in the above equation and solving it, we will get (47).

Remark 11 (TTF prediction for a sensor fault): Given the detection residual of a sensor fault, input filter state and the upper bound of the sensor fault parameter, upon detecting a sensor fault by the observer (39), the TTF can be estimated as

$$TTF(t) = \frac{1}{\gamma} Ln\left(\frac{\gamma\hat{\theta}_s(t) - \beta\Lambda(0,t)\bar{e}(t)}{\gamma\theta_{s_f} - \beta\Lambda(0,t)\bar{e}(t)} \right), \tag{48}$$

where θ_{sf} is the failure limit for θ_s. Similarly, for the sensor fault, assume that in the interval $[t, t_f]$, the term $\beta\Lambda(0,t)\bar{e}(t)$ is held and suppose t_f is the first time when the value of $\hat{\theta}_s$ reaches its failure limit θ_{s_f} ; hence, the estimated TTF is given by (48).

5. SIMULATION RESULTS

Consider a thin rod whose heat conduction can be represented by parabolic PDEs. The heat equation with an actuator fault can be expressed as

$$v_t(x,t) = \varepsilon v_{xx}(x,t) + \lambda v(x,t) + d(v(x,t),x,t), \tag{49}$$

$$v_x(0,t) = -qv(0,t), \quad v(1,t) = \theta U(t), \tag{50}$$

where $d(x,t) = 0.05e^{-0.5(x-0.2)^2}\sin(t)$ is the disturbance. The system state $v(x,t)$ represents the temperature in this heated rod with $v_d(0,t) = 5(1-e^{0.5t})$ being the desired output temperature profile. In addition, $U(t) = v_d(1,t) + \int_0^1 k(x)[v(x,t) - v_d(x,t)]dx$ is the control input designed to regulate the temperature profile of the heated rod with $k(x) = -\frac{\lambda}{2}\sum_{n=0}^{50}\frac{[\lambda(1-x^2)]^n}{4^n n!(n+1)!}$ being the controller gain, and $v_d(x,t) = 5[\cos(\sqrt{\lambda}x) - e^{-0.5t}\cos(\sqrt{0.5+\lambda}x)]$ being the desired full state trajectory. Notice that in the output measurement case, $\hat{v}(x,t)$ will be utilized instead of $v(x,t)$ resulting in $U(t) = v_d(1,t) + \int_0^1 k(x)[\hat{v}(x,t) - v_d(x,t)]dx$.

The system parameters chosen for the simulation are given by $\varepsilon = 1$, and $q = 0$. Now $\theta = 1 - .5\Omega(t-5)$ is the actuator fault $\lambda = 10$ parameter seeded at $t = 5s$ where $\Omega(t-5) = \begin{cases} 0 & ,if\ t < 5 \\ 1-e^{-0.3(t-5)} & ,if\ t \ge 5 \end{cases}$.

For the numerical simulation, the closed-loop system and observer are discretized by using the finite difference method with 20 grid points with respect to the space. The total time for simulation is taken as 25 seconds. The number of discretization points should be selected carefully since it may result in inaccurate results for the PDE system. Simulation results for full state measurement case can be found in research reported by the work of [24]. Here, results are obtained with the output temperature at $x = 0$ as measured.

5.1. ACTUATOR FAULT

The initial condition of actual system (49)–(50) is selected as $v(x,0) = 0.2\cos(x)$, and the initial values for the filters are set at zero. The estimated fault parameters are initialized as $\hat{\theta}(0) = 1$, $\theta_s(0) = 1$, and the threshold is selected as $\rho = 0.5$. First, by applying the transformation (35), the DPSs (49) and (50) are converted to the observable form in (36) and (37) where $l_\eta(x,0) = 25(1-x)\sum_{n=0}^{50} \frac{(-1)^n [10x(2-x)]^n}{4^n n!(n+2)!}$ and $q = 0$. Then two output filters along with one input filter are employed to estimate the state and output of the transformed system. Prior to the fault occurrence, the detection residual is expected to decrease, whereas it will increase once a fault occurs. This is clearly observed in Figure 5.1. Fault is detected at approximately $t = 6.3s$ when the detection residual exceeds the threshold.

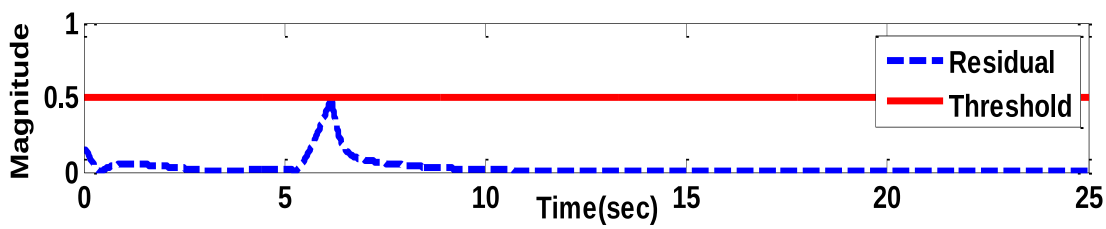


Figure 5.1. Detection residual of actuator fault.

Upon detection of the actuator fault, the adaptive estimator will be utilized to learn the fault dynamics. In this case, we just have one available measurement, so the update law (44) is utilized to estimate the actuator fault parameter where update parameters are chosen as $\beta = 0.2$ and $\gamma = 0.01$. The fault estimation results can be seen from Figure 5.2(a). According to the estimated fault parameter and the failure limit of fault parameter defined as $\theta_f = 0.7$, the estimated TTF is obtained by using the formula (47). Estimated TTF is plotted in Figure 5.2(b).

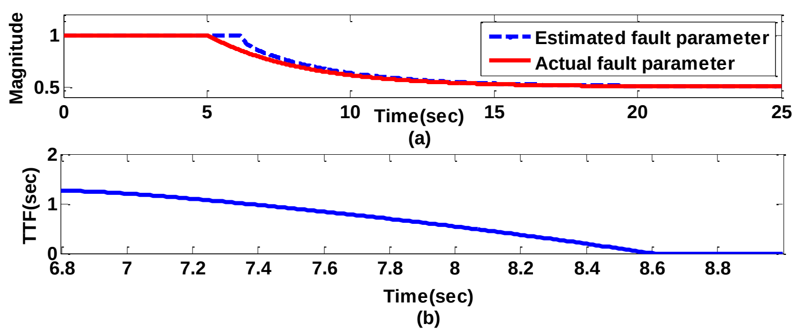


Figure 5.2. (a) Actual and estimated actuator fault; (b) estimated TTF.

5.2. SENSOR FAULT

As discussed in Section 4, this method is able to deal with a sensor fault as well. Thus, a sensor fault is expressed as $y = \theta_s v(0,t)$ with the fault function being described by $\theta_s(t) = 1 + \Omega(t-5)$ where $\Omega(t) = \begin{cases} 0 & ,if\ t < 5 \\ 1 - e^{-0.3(t-5)} & ,if\ t \geq 5 \end{cases}$ and the disturbance is selected as $d(x,t)=0.03e^{-0.5(x-0.3)^2}\sin(2t)$. The detection observer in (39) is again used to detect the fault. Figure 5.3 depicts the detection residual as exceeding the threshold around one and half seconds after the initiation of the fault. Upon detecting the sensor fault, the adaptive terms are activated in the observer as described in (43). By using the parameter tuning law given by Theorem 4 with the parameters selected as $\beta=0.35$ and $\gamma=0.01$, the fault parameter estimation can be performed, the TTF can be estimated by (48), and the failure limit utilized in the formula is $\theta_{sf}=1.5$. Fault estimation and TTF prediction results are shown in Figure 5.4.

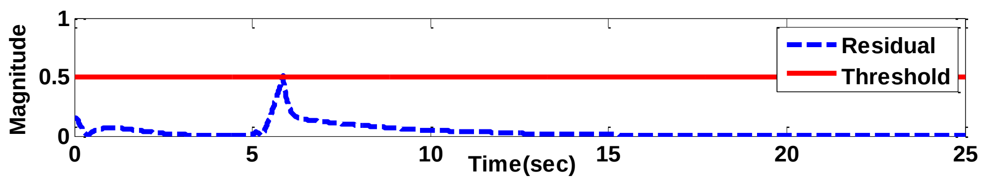


Figure 5.3. Detection residual of a sensor fault.

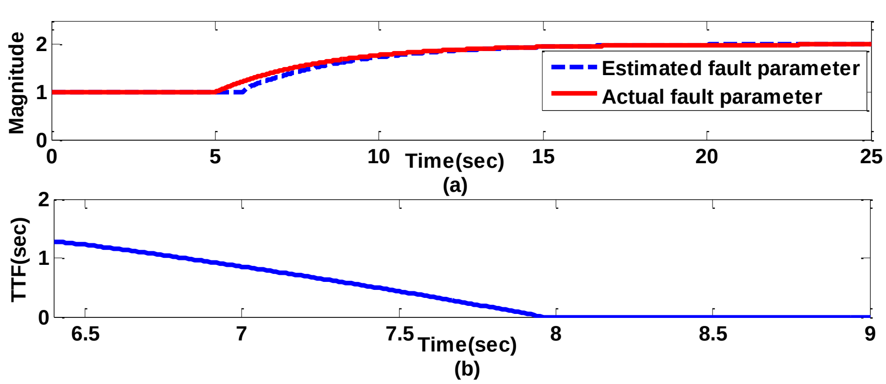


Figure 5.4. (a) Actual and estimated sensor fault dynamics and (b) TTF.

6. CONCLUSIONS

A novel observer based on the PDE representation of a DPS provides a more accurate estimation of the state which is beneficial to both fault detection and estimation. The adaptive term incorporated in the observer appears to estimate the fault function. The TTF can be predicted based on both estimated fault parameters and a failure threshold provided the fault type is known.

The filter-based approach is quite important when dealing with implementation of the scheme on real practical systems, and it also allows the estimation of actuator and sensor faults provided the fault type is given. Simulation results confirm the theoretical claims. Future research will involve fault isolation and extension to other PDEs.

APPENDIX

The dynamics of state residual in the absence of uncertainty and disturbances, i.e. $d(v,x,t)=0$ is expressed as

$$\tilde{v}_t(x,t)=\varepsilon\tilde{v}_{xx}(x,t)+\lambda\tilde{v}(x,t)-p_1(x)\tilde{v}(0,t), \tag{A.1}$$

$$\tilde{v}_x(0,t)=-p_{10}\tilde{v}(0,t),\ \ \tilde{v}(1,t)=0\ , \tag{A.2}$$

Lemma 1[20]: Consider the Volterra integral transformation utilized in the work of [20]

$$\tilde{v}(x,t)=\xi(x,t)-\int_0^x L(x,\tau)\xi(\tau,t)d\tau\ , \tag{A.3}$$

where $L(x,\tau)=-\frac{(\lambda+b)(1-x)}{\varepsilon}\sum_{n=0}^{\infty}\frac{[\frac{(\lambda+b)}{\varepsilon}(2-x-\tau)(x-\tau)]^n}{2^{2n+1}n!(n+1)!}$ is obtained as the solution to the hyperbolic PDE

$$\varepsilon L_{xx}(x,\tau)-\varepsilon L_{\tau\tau}(x,\tau)=-(\lambda+b)L(x,\tau), \tag{A.4}$$

$$L(1,\tau)=0, \tag{A.5}$$

$$L(x,x)=\frac{(\lambda+b)}{2\varepsilon}(x-1), \tag{A.6}$$

with $L_{xx}=\partial^2 L/\partial x^2$ and $L_{\tau\tau}=\partial^2 L/\partial\tau^2$. Select observer gains $p_1(x)$ and p_{10} as

$$p_1(x)=\varepsilon L_\tau(x,0),\qquad p_{10}=L(0,0), \tag{A.7}$$

for the sake of eliminating the unstable term $\lambda\tilde{v}(x,t)$ in (A.1), the transformation (A.3) with observer gains (A.7) converts the state residual dynamics (A.1)-(A.2) to

$$\xi_t(x,t)=\varepsilon\xi_{xx}(x,t)-b\xi(x,t)\ , \tag{A.8}$$

$$\xi_x(0,t)=0,\quad \xi(1,t)=0, \tag{A.9}$$

where $\xi_t = \partial\xi/\partial t$, $\xi_x = \partial\xi/\partial x$, $\xi_{xx} = \partial^2\xi/\partial x^2$ with $b > 0$ is a constant parameter that is adjusted to tune the convergence speed of the observer.

Proof: Part (a) It has to be shown that $L(x,\tau) = -\frac{(\lambda+b)(1-x)}{\varepsilon}\sum_{n=0}^{\infty}\frac{[\frac{(\lambda+b)}{\varepsilon}(2-x-\tau)(x-\tau)]^n}{2^{2n+1}n!(n+1)!}$ is the solution to (A.4)-(A.6).

In order to find a solution of the PDE (A.4)-(A.6), first it is converted to an integral equation. Introduce the change of variables as

$$\alpha = x+\tau\,, \omega = x-\tau\,, L(x,\tau) = G(\alpha,\omega)\,.$$

Then we can know that

$$L_x = G_\alpha + G_\omega \ , L_{xx} = G_{\alpha\alpha} + 2G_{\alpha\omega} + G_{\omega\omega} \ ,$$

$$L_\tau = G_\alpha - G_\omega \ , L_{\tau\tau} = G_{\alpha\alpha} - 2G_{\alpha\omega} + G_{\omega\omega} \ ,$$

where $L_x = \partial L/\partial x$, $L_\tau = \partial L/\partial\tau$, $G_\alpha = \partial L/\partial\alpha$, $G_\omega = \partial G/\partial w$, $G_{\alpha\alpha} = \partial^2 L/\partial\alpha^2$ and $G_{\omega\omega} = \partial^2 G/\partial w^2$. Substituting equations above to (A.4)-(A.6) we get

$$G_{\alpha\omega}(\alpha,\omega) = -\frac{\lambda+b}{4\varepsilon}G(\alpha,\omega) \ , \tag{A.10}$$

$$G(\alpha, 2-\alpha) = 0 \ , \tag{A.11}$$

$$G(\alpha,0) = \frac{\lambda+b}{4\varepsilon}\alpha - \frac{\lambda+b}{2\varepsilon} \ . \tag{A.12}$$

Integrating (A.10) with respect to ω from 0 to $2-\omega$ gives

$$G_\alpha(\alpha, 2-\omega) = G_\alpha(\alpha,0) - \frac{\lambda+b}{4\varepsilon}\int_0^{2-\omega}G(\alpha,s)ds = \frac{\lambda+b}{4\varepsilon} - \frac{\lambda+b}{4\varepsilon}\int_0^{2-\omega}G(\alpha,s)ds \ . \tag{A.13}$$

Next, we integrate both sides of the equation given by (A.13) with respect to α from ω to α get

$$G(\alpha,2-\omega)=G(\omega,2-\omega)+\frac{\lambda+b}{4\varepsilon}(\alpha-\omega)$$
$$-\frac{\lambda+b}{4\varepsilon}\int_{\omega}^{\alpha}\int_{0}^{2-\omega}G(z,s)dsdz \qquad .$$
$$=\frac{\lambda+b}{4\varepsilon}(\alpha-\omega)-\frac{\lambda+b}{4\varepsilon}\int_{\omega}^{\alpha}\int_{0}^{2-\omega}G(z,s)dsdz$$

Replace $2-\omega$ with ω

$$G(\alpha,\omega)=-\frac{\lambda+b}{4\varepsilon}(2-\alpha-\omega)-\frac{\lambda+b}{4\varepsilon}\int_{2-\omega}^{\alpha}\int_{0}^{\omega}G(z,s)dsdz \ . \qquad \text{(A.14)}$$

Starting with an initial guess

$$G^0(\alpha,\omega)=0, \qquad \text{(A.15)}$$

and setting up the recursive formula for (A.14) as follows

$$G^{n+1}(\alpha,\omega)=-\frac{\lambda+b}{4\varepsilon}(2-\alpha-\omega)-\frac{\lambda+b}{4\varepsilon}\int_{2-\omega}^{\alpha}\int_{0}^{\omega}G^n(z,s)dsdz \ . \qquad \text{(A.16)}$$

If this converges, we can write the solution $G(\alpha,\omega)$ as

$$G(\alpha,\omega)=\lim_{n\to\infty}G^n(\alpha,\omega) \ . \qquad \text{(A.17)}$$

The difference between two consecutive terms are expressed as

$$\Delta G^n(\alpha,\omega)=G^{n+1}(\alpha,\omega)-G^n(\alpha,\omega) \ . \qquad \text{(A.18)}$$

Then

$$\Delta G^{n+1}(\alpha,\omega)=\frac{\lambda+b}{4\varepsilon}\int_{\alpha}^{2-\omega}\int_{0}^{\omega}\Delta G^n(z,s)dsdz \ , \qquad \text{(A.19)}$$

with (A.17) rewritten as

$$G(\alpha,\omega)=\sum_{n=0}^{\infty}\Delta G^n(\alpha,\omega) \ . \qquad \text{(A.20)}$$

In order to get the solution to the equation given by (A.20), Computing $\Delta G^n(\alpha,\omega)$ from (A.19) starting with

$$\Delta G^0(\alpha,\omega) = G^1(\alpha,\omega) = -\frac{\lambda+b}{4\varepsilon}(2-\alpha-\omega)$$
$$\Delta G^1(\alpha,\omega) = -(\frac{\lambda+b}{4\varepsilon})^2 \frac{(2-\alpha-\omega)(2-\alpha)\omega}{2}. \quad (A.21)$$

We can see the pattern which results in the following formula

$$\Delta G^n(\alpha,\omega) = -(\frac{\lambda+b}{4\varepsilon})^{n+1} \frac{(2-\alpha-\omega)(2-\alpha)^n\omega^n}{n!(n+1)!} \ . \quad (A.22)$$

The formula (A.22) can be verified by induction. Then the solution to (A.20) is given by

$$G(\alpha,\omega) = -\sum_{n=0}^{\infty}(\frac{\lambda+b}{4\varepsilon})^{n+1} \frac{(2-\alpha-\omega)(2-\alpha)^n\omega^n}{n!(n+1)!} \ . \quad (A.23)$$

Returning to the original x , τ variables

$$L(x,\tau) = -2\sum_{n=0}^{\infty}(\frac{\lambda+b}{4\varepsilon})^{n+1} \frac{(1-x)(2-x-\tau)^n(x-\tau)^n}{n!(n+1)!}$$
$$= -\frac{(\lambda+b)(1-x)}{\varepsilon}\sum_{n=0}^{\infty}\frac{[\frac{(\lambda+b)}{\varepsilon}(2-x-\tau)(x-\tau)]^n}{2^{2n+1}n!(n+1)!} \ . \quad (A.24)$$

which completes the proof of part (a).

Part (b): It needs to be shown that with the observer gains $p_1(x)$ and p_{10} from (A.5), the transformation (A.3) where $L(x,\tau)$ satisfies (A.4)-(A.6) converts the state residual dynamics (A.1)-(A.2) to (A.8)-(A.9).

In order to move on we will show that

$$\tilde{v}_t(x,t) - \varepsilon\tilde{v}_{xx}(x,t) = \lambda\tilde{v}(x,t) - p_1(x)\tilde{v}(0,t),$$

by using (A.13), (A.18) with observer gains selected as (A.7). By differentiating the transformation (A.3) with respect to t and substituting $\xi_t(x,t) = \varepsilon\xi_{xx}(x,t) - b\xi(x,t)$ from (A.8) we can get

$$
\begin{aligned}
&\tilde{v}_t(x,t) = \xi_t(x,t) - \int_0^x L(x,\tau)\xi_t(\tau,t)\,d\tau \\
&= \xi_t(x,t) - \varepsilon\int_0^x L(x,\tau)\xi_{\tau\tau}(\tau,t)\,d\tau + b\int_0^x L(x,\tau)\xi(\tau,t)\,d\tau \\
&= \xi_t(x,t) + b\int_0^x L(x,\tau)\xi(\tau,t)\,d\tau - \varepsilon\int_0^x L(x,\tau)\,d\xi_\tau(\tau,t) \\
&= \xi_t(x,t) + b\int_0^x L(x,\tau)\xi(\tau,t)\,d\tau - \varepsilon L(x,x)\xi_x(x,t) \\
&+ \varepsilon L(x,0)\xi_x(0,t) + \varepsilon\int_0^x L_\tau(x,\tau)\xi_\tau(\tau,t)\,d\tau \\
&= \xi_t(x,t) + b\int_0^x L(x,\tau)\xi(\tau,t)\,d\tau - \varepsilon L(x,x)\xi_x(x,t) \\
&+ \varepsilon L(x,0)\xi_x(0,t) + \varepsilon\int_0^x L_\tau(x,\tau)\,d\xi(\tau,t) \\
&= \xi_t(x,t) + b\int_0^x L(x,\tau)\xi(\tau,t)\,d\tau - \varepsilon L(x,x)\xi_x(x,t) + \varepsilon L(x,0)\xi_x(0,t) \\
&+ \varepsilon L_\tau(x,x)\xi(x,t) - \varepsilon L_\tau(x,0)\xi(0,t) - \varepsilon\int_0^x L_{\tau\tau}(x,\tau)\xi(\tau,t)\,d\tau
\end{aligned}
\quad . \qquad \text{(A.25)}
$$

By differentiating the transformation (A.3) with respect to x we get

$$
\tilde{v}_x(x,t) = \xi_x(x,t) - L(x,x)\xi(x,t) - \int_0^x L_x(x,\tau)\xi(\tau,t)\,d\tau \ , \qquad \text{(A.26)}
$$

$$
\begin{aligned}
&\tilde{v}_{xx}(x,t) = \xi_{xx}(x,t) - \frac{dL(x,x)}{dx}\xi(x,t) \\
&- L(x,x)\xi_x(x,t) - L_x(x,x)\xi(x,t) - \int_0^x L_{xx}(x,\tau)\xi(\tau,t)\,d\tau
\end{aligned}
\quad . \qquad \text{(A.27)}
$$

According to (A.25), (A.27) and $\xi_t(x,t) = \varepsilon\xi_{xx}(x,t) - b\xi(x,t)$ from (A.8) we can get $\tilde{v}_t(x,t) - \varepsilon\tilde{v}_{xx}(x,t)$ as

$$
\begin{aligned}
&\tilde{v}_t(x,t) - \varepsilon\tilde{v}_{xx}(x,t) \\
&= -b\xi(x,t) + b\int_0^x L(x,\tau)\xi(\tau,t)\,d\tau + 2\varepsilon\frac{dL(x,x)}{dx}\xi(x,t), \\
&- \varepsilon L_\tau(x,0)\xi(0,t) + \varepsilon\int_0^x [L_{xx}(x,\tau) - L_{\tau\tau}(x,\tau)]\xi(\tau,t)\,d\tau
\end{aligned}
\qquad \text{(A.28)}
$$

Substitute transformation (A.3) in equation (A.1) to get

$$
\begin{aligned}
&\tilde{v}_t(x,t) - \varepsilon\tilde{v}_{xx}(x,t) = \lambda\tilde{v}(x,t) - p_1(x)\tilde{v}(0,t) \\
&= \lambda\xi(x,t) - \lambda\int_0^x L(x,\tau)\xi(\tau,t)\,d\tau - p_1(x)\xi(0,t)
\end{aligned}
\ . \qquad \text{(A.29)}
$$

In order to guarantee the equation (A.1) holds, the right hand side of (A.28) should be equal to right hand side of (A.29) which means

$$\begin{aligned}&\lambda\xi(x,t)-\int_0^x \lambda L(x,\tau)\xi(\tau,t)\,d\tau - p_1(x)\xi(0,t)\\&=\int_0^x \varepsilon\left[L_{xx}(x,\tau)-L_{\tau\tau}(x,\tau)\right]\xi(\tau,t)\,d\tau + b\int_0^x L(x,\tau)\xi(\tau,t)\,d\tau\\&+2\varepsilon\frac{dL(x,x)}{dx}\xi(x,t)-b\xi(x,t)-\varepsilon L_\tau(x,0)\xi(0,t)\end{aligned}\quad , \tag{A.30}$$

which results in the following conditions

$$\varepsilon L_{xx}(x,\tau)-\varepsilon L_{\tau\tau}(x,\tau)=-(\lambda+b)L(x,\tau), \tag{A.31}$$

$$\frac{dL(x,x)}{dx}=\frac{(\lambda+b)}{2\varepsilon}. \tag{A.32}$$

As for as the boundary conditions are concerned, differentiate (A.3) with respect to x, set $x=0$ and substitute (A.2) in the resulting equation, to get

$$\xi_x(0,t)=\tilde{v}_x(0,t)+L(0,0)\xi(0,t)=[L(0,0)-p_{10}]\xi(0,t). \tag{A.33}$$

Then setting $x=1$ in (A.3) and substituting (A.2) and $p_1(x)$ given by (A.7) in the resulting equation leads to

$$\xi(1)=\int_0^1 L(1,\tau)\xi(\tau,t)\,d\tau. \tag{A.34}$$

Therefore, the boundary condition (A.9) can be obtained from (A.33) and (A.34) with $L(1,\tau)=0$ from (A.5) and observer gain selected as $p_{10}=L(0,0)$ given by (A.7). Because $L(1,1)=0$ from (A.5) and $\frac{dL(x,x)}{dx}=\frac{(\lambda+b)}{2\varepsilon}$ from (A.32), we can get (A.6). This completes the proof.

Lemma 2: The inverse transformation of $\tilde{v}(x,t)=\xi(x,t)-\int_0^x L(x,\tau)\xi(\tau,t)\,d\tau$ is given by

$$\xi(x,t) = \tilde{v}(x,t) + \int_0^x M(x,\eta)\tilde{v}(\eta,t)d\eta \,, \tag{A.35}$$

where

$$M(x,\eta) = -\frac{(\lambda+b)(1-x)}{\varepsilon}\sum_{n=0}^{\infty}\frac{(-1)^n[\frac{(\lambda+b)}{\varepsilon}(2-x-\eta)(x-\eta)]^n}{2^{2n+1}n!(n+1)!} \,, \tag{A.36}$$

is the solution obtained through successive approximation [20] to the following hyperbolic PDE given by

$$\begin{aligned} &M_{xx}(x,\eta) - M_{\eta\eta}(x,\eta) = (\lambda+b)M(x,\eta)/\varepsilon \\ &M(x,x) = (\lambda+b)(x-1)/2\varepsilon, \quad M(1,\eta) = 0 \end{aligned} , \tag{A.37}$$

with $M_{xx} = \partial^2 M / \partial x^2$ and $M_{\eta\eta} = \partial^2 M / \partial \eta^2$.

Proof: Comparing the PDE given by (A.4)-(A.6) with the PDE given by (A.37), we can observe that

$$M(x,\eta;\frac{(\lambda+b)}{\varepsilon}) = -L(x,\eta,-\frac{(\lambda+b)}{\varepsilon}) \,. \tag{A.38}$$

From (A.24), we can derive the solution to (A.37) as (A.36) which completes the proof.

The system dynamics without any disturbance and uncertainty, i.e. $d(v,x,t) = 0$ is expressed as

$$v_t(x,t) = \varepsilon v_{xx}(x,t) + \lambda v(x,t) \,, \tag{A.39}$$

$$v_x(0,t) = -qv(0,t),\ v(1,t) = U(t) + h(U(t),t),\ y(t) = \theta_s v(0) \,. \tag{A.40}$$

And the dynamics of the detection observer is represented as

$$\hat{v}_t(x,t) = \varepsilon \hat{v}_{xx}(x,t) + \lambda \hat{v}(x,t) + p_1(x)(y-\hat{y}) \,, \tag{A.41}$$

$$\hat{v}_x(0,t) = -qv(0,t) + p_{10}(y-\hat{y}), \quad \hat{v}(1,t) = U(t),$$

$$\hat{y}(t) = \hat{v}(0,t), \quad e(t) = y(t) - \hat{y}(t). \tag{A.42}$$

Theorem 1 (Fault detection observer performance): Let the observer given in (A.41) - (A.42) be used to monitor the DPS defined by (A.39) and (A.40). Then the state estimation error $\tilde{v}$ and the output detection residual $e(t)$ will converge to zero asymptotically under healthy operating conditions. In addition, a fault is declared active when the output detection residual $e(t)$ crosses the detection threshold ρ . A fault initiated at time t_i is detectable if there exists a time $T \geq t_i$ and a positive constant $\mathrm{H} > \rho$, such that

I. $\left|h\left(U\left(T\right),\mathrm{T}\right)\right| = H$ and $\left|h\left(U\left(t\right),t\right)\right| \geq H$ for $t > T$ in the case of an actuator fault, or

II. $\left|\left(\theta_s - 1\right)U(\mathrm{T})\right| = H$ and $\left|\left(\theta_s - 1\right)U(t)\right| \geq H$ for $t > T$ in the case of a sensor fault.

Proof: It is already known that the transformation (A.3) can map the state residual dynamics into the target system of (A.8)-(A.9) if $p_1(x)$ and p_{10} are defined by (A.7). The stability of the residual dynamics can be concluded from the stability of the target system given by (A.8)-(A.9) due to the transformation (A.3) [21].

To discuss the stability of the PDE described in (A.8) with boundary conditions given by (A.9), one must select a positive definite Lyapunov function candidate which is half of the squared Sobolev norm of the spatial profile defined in Hilbert space $H_1(0,1)$ utilized in [20]

$$V(t) = \frac{\left\|\xi(t)\right\|^2}{2} + \frac{\left\|\xi_x(t)\right\|^2}{2} = \frac{\int_0^1 \xi^2(x,t)dx}{2} + \frac{\int_0^1 \xi_x^2(x,t)dx}{2} \quad . \tag{A.43}$$

The derivative of (A.43) is given by

$$\begin{aligned}\dot{V}(t) &= \int_0^1 \xi(x,t)\xi_t(x,t)dx + \int_0^1 \xi_x(x,t)\xi_{tx}(x,t)dx \\ &= \varepsilon\int_0^1 \xi(x,t)\xi_{xx}(x,t)dx - b\int_0^1 \xi^2\left(x,t\right)dx + \int_0^1 \xi_x(x,t)d\xi_t(x,t)\end{aligned}$$

$$= \varepsilon\xi(1,t)\xi_x(1,t) - \varepsilon\xi(0,t)\xi_x(0,t) - \varepsilon\int_0^1 \xi_x^2(x,t)dx - b\int_0^1 \xi^2\left(x,t\right)dx$$

$$+ \xi_x(x,t)\xi_t(x,t)\,|_0^1 - \varepsilon\int_0^1 \xi_{xx}^2(x,t)dx + b\int_0^1 \xi(x,t)\xi_x(x,t)dx$$

$$= -b\int_0^1 \xi^2\left(x,t\right)dx - \varepsilon\int_0^1 \xi_x^2(x,t)dx - \varepsilon\int_0^1 \xi_{xx}^2(x,t)dx.$$

Note that from (A.9) $\xi_x\left(0,t\right)=0,\ \ \xi\left(1,t\right)=0$. According to the Poincare inequality

$$\int_0^1 (\xi(x,t)-\xi(1,t))^2 dx \le \frac{4}{\pi^2}\int_0^1 \xi_x^2(x,t)dx$$

and

$$\int_0^1 (\xi_x(x,t)-\xi_x(0,t))^2 dx \le \frac{4}{\pi^2}\int_0^1 \xi_{xx}^2(x,t)dx\,,$$

we can get

$$\dot{V}(t) \le -\frac{\varepsilon\pi^2}{4}\int_0^1 \xi^2(x,t)dx - \frac{\varepsilon\pi^2}{4}\int_0^1 \xi_x^2(x,t)dx = -\frac{\varepsilon\pi^2}{2}V(t)$$

Then it gives us

$$\|\xi(t)\|^2 + \|\xi_x(t)\|^2 \le e^{-\frac{\varepsilon\pi^2}{2}t}(\|\xi(0)\|^2 + \|\xi_x(0)\|^2)$$

By Agmon's inequality

$$\max_{x\in[0,1]}|\xi(x,t)|^2 \le 2\|\xi(t)\|\|\xi_x(t)\| \le \|\xi(t)\|^2 + \|\xi_x(t)\|^2 \le e^{-\frac{\varepsilon\pi^2}{2}t}(\|\xi(0)\|^2 + \|\xi_x(0)\|^2)\,.$$

Therefore $\xi(x,t)$ converges to zero asymptotically for all $x\in[0,1]$. From the transformation $\tilde{v}\left(x,t\right)=\xi\left(x,t\right)-\int_0^x L\left(x,\tau\right)\xi\left(\tau,t\right)d\tau$ we can know that $e(t)=\tilde{v}(0,t)=\xi(0,t)$. Since the transformed system state $\xi(x,t)$ converges to zero asymptotically for any $0\le x\le 1$, we can get that $e(t)=\xi(0,t)\to 0$ asymptotically under healthy conditions. Then a constant threshold maybe selected as $\rho=(1+a)|e(0)|$ where a is a small positive.

To determine detectability conditions, an actuator fault is considered first. When an actuator fault occurs at t_i then the state estimation error dynamics are described by (A.1) subject to following boundary conditions given by

$$\tilde{v}_x(0,t)=-p_{10}\tilde{v}(0,t), \quad \tilde{v}(1,t)=h(U(t),t)\,. \tag{A.44}$$

Applying the transformation (A.3) on (A.1) and (A.44) leads to (A.8) with boundary conditions given by

$$\xi_x(0,t)=0, \quad \xi(1,t)=h(U(t),t)\,, \tag{A.45}$$

for $t\geq t_i$. Because $\tilde{v}(1,t)=\xi(1,t)-\int_0^1 L(1,\tau)\xi(\tau,t)\,d\tau$ and $L(1,\tau)=0$ for all $0\leq\tau\leq1$, we can get $\tilde{v}(1,t)=\xi(1,t)$. Now a bounded state variable $\phi(x,t)$ is introduced which is defined by (A.8) $\phi_t(x,t)=\varepsilon\phi_{xx}(x,t)-b\phi(x,t)$ subject to $\phi_x(0,t)=0$, $\phi(1,t)=H$ and $\phi(x,T)=\xi(x,T)$. If $\left|h(U(t),t)\right|\geq H$ for $t\geq T$ then $|\xi(x,t)|\geq\phi(x,t)$ for $0\leq x\leq1$ and $t\geq T$. Because $\bar{\phi}(x,t)=\phi(x,t)-H$ satisfies the following exponentially stable PDE given by $\bar{\phi}_t(x,t)=\varepsilon\bar{\phi}_{xx}(x,t)-b\bar{\phi}(x,t)$, $\bar{\phi}_x(0,t)=0$, $\bar{\phi}(1,t)=0$, the new state variable $\phi(\mathrm{x},\mathrm{t})$ converges to H. Hence, for any $\omega>0$, there exists a time $t_d\geq T$, such that $\phi(0,t_d)\geq H-\omega$. Therefore, for any $\omega>0$, there exists a time $t_d\geq T$ such that $|e(t_d)|=|\xi(0,t_d)|\geq\phi(0,t_d)\geq H-\omega$. Because $H>\rho$ and if ω is selected as $\omega<H-\rho$, then $|e(t_d)|>\rho$ for some $t_d\geq T$, so that the detection of an actuator fault is guaranteed when the detectability condition in Theorem 1 is satisfied.

In the case of a sensor fault, the detection residual can be written as $e(t) = y(t) - \hat{y}(t) = \theta_s v(0,t) - \hat{v}(0,t)$. If we define a new distributed variable $\Delta(x,t) = \theta_s v(x,t) - \hat{v}(x,t)$, then the detection residual can be expressed as $e(t) = \Delta(0,t)$. By using the definition of Δ and the observer dynamics in (A.41) and (A.42), it can be shown that Δ satisfies the PDE given by

$$\Delta_t(x,t) = \varepsilon\Delta_{xx}(x,t) + \lambda\Delta(x,t) - p_1(x)\Delta(0,t),$$

$$\Delta_x(0,t) = -p_{10}\Delta(0,t), \qquad \Delta(1,t) = (\theta_s - 1)U(t) \ ,$$

for $0 \le x \le 1$ and $t \ge t_i$. If $\tilde{v}$ in transformation (A.3) is replaced by Δ , applying this transformation on the above PDE will lead to

$$\xi_t(x,t) = \varepsilon\xi_{xx}(x,t) - b\xi(x,t) \ ,$$

$$\xi_x(0,t) = 0, \quad \xi(1,t) = (\theta_s - 1)U(t),$$

which is exactly same as (A.8) and (A.45) except that the term $h(U(t),t)$ is replaced by $(\theta_s - 1)U(t)$. Thus, by using the same steps taken in the case of actuator fault, it can be shown that if $|(\theta_s - 1)U(t)| \ge H$ for $t \ge T$, then for any $\omega > 0$, there exists a time $t_d \ge T$ such that $|e(t_d)| = |\Delta(0,t_d)| \ge H - \omega$. Selecting $\omega < H - \rho$, results in $|e(t_d)| > \rho$ which declares the presence of a fault.

An adaptive estimator, which is only activated upon detection, is added to the boundary condition of the observer (A.41) and (A.42) as (A.41) with boundary conditions

$$\hat{v}_x(0,t) = -qv(0,t) + p_{10}e(t) \ , \tag{A.46}$$

$$\hat{v}(1,t) = U(t) + \hat{h}(U(t);\hat{\Theta}(t)), \ \hat{y}(t) = \hat{v}(0,t) . \tag{A.47}$$

Then the state error dynamics with an adaptive estimator can be expressed as (A.1) subjecting to

$$\tilde{v}_x(0,t) = -p_{10}e(t),\ \tilde{v}(1,t) = \tilde{\Theta}(t)U(t). \tag{A.48}$$

Theorem 2 (Performance of an actuator fault estimation): Let the boundary condition of the observer in (A.42) be modified to (A.46)-(A.47) in order to estimate the state and output of the system defined in (A.39) and (A.40) without a sensor fault, i.e. $\theta_s = 1$. In the presence of an actuator fault, consider the parameter tuning law

$$\dot{\hat{\Theta}}(t) = \beta\varepsilon U(t)(\tilde{v}_x(1,t) + \int_0^1 M_x(1,\eta)\tilde{v}(\eta)d\eta) - \hat{\Theta}(t)\gamma\left\|\tilde{v}(x,t) + \int_0^x M(x,\eta)\tilde{v}(\eta,t)d\eta\right\|^2, \tag{A.49}$$

for fault estimation where $\beta > 0$ is the adaptation rate, $0 < \gamma < 2\beta b / \Theta_{\max}^2$ is the stabilizing term and $M(x,\eta)$ is given by (A.36), then the state residual converges to zero and the parameter estimation error is bounded.

Proof: First apply the transformation (A.3) on the residual dynamics (A.1) and (A.48) to get the PDE (A.8) with boundary conditions given by

$$\xi_x(0,t) = 0, \quad \xi(1,t) = \tilde{\Theta}(t)U(t)\ , \tag{A.50}$$

Now select the Lyapunov function candidate

$$V(t) = \int_0^1 \xi^2(x,t)dx/2 + \tilde{\Theta}^2(t)/(2\beta), \tag{A.51}$$

whose first derivative is given by $\dot{V}(t) = \int_0^1 \xi(x,t)\xi_t(x,t)dx + \tilde{\Theta}(t)\dot{\tilde{\Theta}}(t)/\beta$. By substituting (A.8) in the first derivative, we will arrive at

$$\dot{V}(t) = \varepsilon\int_0^1 \xi(x,t)\xi_{xx}(x,t)dx - b\int_0^1 \xi^2(x,t)dx + \tilde{\Theta}(t)\dot{\tilde{\Theta}}(t)/\beta\ . \tag{A.52}$$

Applying integration by parts and using boundary conditions given by (A.50) will lead to

$$\begin{aligned}
\dot{V}(t) &= \varepsilon\int_0^1 \xi(x,t)\xi_{xx}(x,t)dx - b\int_0^1 \xi^2\left(x,t\right)dx + \tilde{\Theta}(t)\dot{\tilde{\Theta}}(t)/\beta \\
&= \varepsilon\xi(x,t)\xi_x(x,t)|_0^1 - \varepsilon\int_0^1 \xi_x^2(x,t)dx - b\int_0^1 \xi^2\left(x,t\right)dx + \tilde{\Theta}(t)\dot{\tilde{\Theta}}(t)/\beta \\
&= \varepsilon\xi(1,t)\xi_x(1,t) - \varepsilon\int_0^1 \xi_x^2(x,t)dx - b\int_0^1 \xi^2\left(x,t\right)dx + \tilde{\Theta}(t)\dot{\tilde{\Theta}}(t)/\beta \\
&= \varepsilon\xi_x(1,t)\tilde{\Theta}(t)U(t) - \varepsilon\int_0^1 \xi_x^2(x,t)dx - b\int_0^1 \xi^2\left(x,t\right)dx + \tilde{\Theta}(t)\dot{\tilde{\Theta}}(t)/\beta
\end{aligned} \quad . \tag{A.53}$$

To represent this update law in terms of transformed system state, instead of the actual system state, the inverse transformation (A.35) and its first derivative with respect to x given by $\xi_x(1,t) = \tilde{v}_x(1,t) + \int_0^1 M_x(1,\eta)\tilde{v}(\eta,t)d\eta$ will be utilized in (A.49) to get

$$\dot{\hat{\Theta}}(t) = \beta\varepsilon U(t)\xi_x(1,t) - \hat{\Theta}(t)\gamma\left\|\xi(t)\right\|^2 \ . \tag{A.54}$$

Equation (A.54) eliminates the positive term $\varepsilon\xi_x(1,t)\tilde{\Theta}(t)U(t)$.

By applying the proposed parameter update law in the derivative of Lyapunov function candidate we get

$$\begin{aligned}
\dot{V}(t) &= \varepsilon\tilde{\Theta}(t)U(t)\xi_x(1,t) - \varepsilon\int_0^1 \xi_x^2(x,t)dx - b\int_0^1 \xi^2\left(x,t\right)dx \\
&\quad + \frac{1}{\beta}\tilde{\Theta}(t)[-\beta\varepsilon U(t)\xi_x(1,t) + \hat{\Theta}(t)\gamma\left\|\xi(t)\right\|^2] \\
&= \varepsilon\tilde{\Theta}(t)U(t)\xi_x(1,t) - \varepsilon\int_0^1 \xi_x^2(x,t)dx - b\int_0^1 \xi^2\left(x,t\right)dx \\
&\quad - \varepsilon\tilde{\Theta}(t)U(t)\xi_x(1,t) + \frac{\gamma}{\beta}\tilde{\Theta}(t)\hat{\Theta}(t)\left\|\xi(t)\right\|^2 \\
&= -\varepsilon\int_0^1 \xi_x^2(x,t)dx - \frac{\gamma}{\beta}\tilde{\Theta}^2(t)\left\|\xi(t)\right\|^2 + \frac{\gamma}{\beta}\tilde{\Theta}(t)\Theta\left\|\xi(t)\right\|^2 - b\int_0^1 \xi^2\left(x,t\right)dx \\
&\le -\varepsilon\int_0^1 \xi_x^2(x,t)dx - b\int_0^1 \xi^2\left(x,t\right)dx - \frac{\gamma}{\beta}\tilde{\Theta}^2(t)\left\|\xi(t)\right\|^2 + \frac{\gamma}{2\beta}\tilde{\Theta}^2(t)\left\|\xi(t)\right\|^2 + \frac{\gamma}{2\beta}\Theta^2\left\|\xi(t)\right\|^2 \\
&\le -\varepsilon\int_0^1 \xi_x^2(x,t)dx - b\int_0^1 \xi^2\left(x,t\right)dx - \frac{\gamma}{2\beta}\tilde{\Theta}^2(t)\left\|\xi(t)\right\|^2 + \frac{\gamma}{2\beta}\Theta_{\max}^2\left\|\xi(t)\right\|^2 \\
&\le -\frac{(2\beta b - \Theta^2{}_{\max}\gamma)}{2\beta}\left\|\xi(t)\right\|^2 - \frac{\gamma}{2\beta}\tilde{\Theta}^2(t)\left\|\xi(t)\right\|^2
\end{aligned} \quad .$$

Thus, when $\gamma < 2\beta b/\Theta_{\max}^2$, $\dot{V} \le 0$ and $\left\|\xi(t)\right\|$ and $\left|\tilde{\Theta}\right|$ are bounded.

Now, define

$$S = \left\{ \|\xi(t)\|, \tilde{\Theta}(t) \middle| \dot{V}(\|\xi(t)\|, \tilde{\Theta}(t)) = 0 \right\} ,$$

when $\dot{V} = 0$. Since the largest invariant set contained in S, is same as S, the convergence of $\|\xi(t)\|$ to zero asymptotically and boundedness of $|\tilde{\Theta}|$ can be shown by using LaSalle's theorem (LaSalle, J. and Lefschetz, S., 1961).

By using the following transformation

$$z(x,t) = v(x,t) - \int_0^x l(x,\tau) v(\tau,t) d\tau \ , \tag{A.55}$$

the original system with disturbance or uncertainty can be converted as

$$\begin{aligned} z_t(x,t) &= \varepsilon z_{xx}(x,t) - \varepsilon[l_\eta(x,0) + ql(x,0)]z(0,t) \\ &+d(v(x),x,t) - \int_0^x d(v(\tau),\tau,t) l(x,\tau) d\tau \end{aligned} \ , \tag{A.56}$$

$$z_x(0,t) = -(\lambda / 2\varepsilon + q) z(0,t), \quad z(1,t) = \theta U(t) \ , \tag{A.57}$$

and

$$y(t) = \theta_s z(0,t) . \tag{A.58}$$

The system state $z(x,t)$ can be expressed as a combination of the solutions defined by [20]

$$\Lambda_t(x,t) = \varepsilon \Lambda_{xx}(x,t), \quad \Lambda_x(0,t) = 0, \quad \Lambda(1,t) = U(t) ,$$

where $\Lambda(x,t)$ is referred to an input filter since it is derived from the input of the actual system $U(x)$ [20] Next consider

$$A_t(x,t) = \varepsilon A_{xx}(x,t), \ A_x(0,t) = y(t), \ A(1,t) = 0 ,$$

where $A(x,t)$ is an output filter since it is derived from output of the actual system $y(t)$. In addition, consider

$$B_t(x,\eta,t)=\varepsilon B_{xx}(x,\eta,t)+\delta(x-\eta)y(t), B_x(0,\eta,t)=0, B(1,\eta,t)=0 \ ,$$

where $B(x,\eta,t)$ is another output filter. Then the observer is defined as

$$\begin{aligned}&\hat{z}(x,t)=\Lambda(x,t)+[-\lambda/(2\varepsilon)-q]\mathrm{A}(x,t)\\&+\int_0^1 \varepsilon[-l_\eta(s,0)-ql(s,0)]\mathrm{B}(x,s,t)ds, \quad \hat{y}(t)=\hat{z}(0,t)\end{aligned}. \tag{A.59}$$

Theorem 3 (Output-based fault detection observer performance)**:** Let the observer in (A.59) be used to monitor the DPS defined by (A.56)-(A.58) with bounded disturbances. Then the state estimation error $\tilde{z}$ and detection residual $\bar{e}(t)$ are bounded under healthy operating conditions. Further, a fault initiated at time t_i is detectable if there exists a time $T \geq t_i$ and a positive constant H, such that

I. $\left|(\theta-1)U(t)\right| \geq H$ for $t \geq T$ where $H > \rho + k_l\bar{d}$ in the case of an actuator fault, or

II. $\left|(\theta_s-1)U(t)\right| \geq H$ for $t \geq T$ where $H > \rho + k_l\bar{d}\,\theta_{s\max}$ in the case of a sensor fault.

where $k_l = \dfrac{2\sqrt{2(\pi^2+4)}}{\varepsilon\pi^2}(1+\bar{l})$ and $\bar{l} = \max\limits_{0\leq x,\eta\leq 1} l(x,\eta)$.

Proof: During healthy conditions with $\theta=\theta_s=1$, the state residual satisfies the stable PDE given by

$$\tilde{z}_t(x,t)=\varepsilon\tilde{z}_{xx}(x,t)+d(v,x,t)-\int_0^x d(v,\tau,t)l\left(x,\tau\right)d\tau \ , \tag{A.60}$$

$$\tilde{z}_x(0,t)=0, \ \tilde{z}(1,t)=0 \,. \tag{A.61}$$

Select a positive definite Lyapunov function candidate as $V(t)=\left\|\tilde{z}(t)\right\|^2/2+\left\|\tilde{z}_x(t)\right\|^2/2$, whose first derivative of $V(t)$ becomes

$$\dot{V}(t) = \int_0^1 \tilde{z}(x,t)\tilde{z}_t(x,t)dx + \int_0^1 \tilde{z}_x(x,t)\tilde{z}_{tx}(x,t)dx \ .$$

By using (A.60) and (A.61) in the above equation and applying both integration by parts and Poincare inequality, to get

$$\begin{aligned}
\dot{V}(t) &= -\varepsilon\int_0^1 \tilde{z}_x^2(x,t)dx + \int_0^1 d(v,x,t)\tilde{z}(x,t)dx \\
&-\int_0^1 \tilde{z}(x,t)\int_0^x d(v,\eta,t)l(x,\eta)d\eta dx - \varepsilon\int_0^1 \tilde{z}_{xx}^2(x,t)dx \\
&-\int_0^1 d(v,x,t)\tilde{z}_{xx}(x,t)dx + \int_0^1 \tilde{z}_{xx}(x,t)\int_0^x d(v,\eta,t)l(x,\eta)d\eta dx \\
&\le -\varepsilon\int_0^1 \tilde{z}_x^2(x,t)dx - \varepsilon\int_0^1 \tilde{z}_{xx}^2(x,t)dx + d_l\int_0^1 |\tilde{z}(x,t)|dx + d_l\int_0^1 |\tilde{z}_{xx}(x,t)|dx,
\end{aligned}$$

where $d_l = (1+\bar{l})\bar{d}$. By using Cauchy-Schwarz and Poincare inequalities to get

$$\dot{V}(t) \le -\frac{\varepsilon\pi^2}{8}\int_0^1 \tilde{z}^2(x,t)dx - \frac{\varepsilon\pi^2}{8}\int_0^1 \tilde{z}_x^2(x,t)dx + \frac{(\pi^2+4)}{2\varepsilon\pi^2}d_l^2 \ .$$

Thus, $\dot{V}(t)$ will be less than zero if

$$\|\tilde{z}(t)\| > \frac{2\sqrt{\pi^2+4}}{\varepsilon\pi^2}d_l \ \text{or} \ \|\tilde{z}_x(t)\| > \frac{2\sqrt{\pi^2+4}}{\varepsilon\pi^2}d_l \, .$$

By Agmon's inequality $\max_{x\in[0,1]}|\tilde{z}(x,t)|^2 \le 2\|\tilde{z}(t)\|\|\tilde{z}_x(t)\|$ we can get $\max_{x\in[0,1]}|\tilde{z}(x,t)| \le \frac{2\sqrt{2(\pi^2+4)}}{\varepsilon\pi^2}(1+\bar{l})\bar{d}$ and $|e(t)| = |\tilde{z}(0,t)| \le \frac{2\sqrt{2(\pi^2+4)}}{\varepsilon\pi^2}(1+\bar{l})\bar{d}$. Therefore, the detection threshold must be selected as $\rho = \rho_0 + k_l\bar{d}$ where ρ_0 depends upon the initial conditions with $k_l = \frac{2\sqrt{2(\pi^2+4)}}{\varepsilon\pi^2}(1+\bar{l})$.

If an actuator fault happens ($\theta \ne 1, \theta_s = 1$) at t_i , then we can get the state estimation error dynamics by taking the difference between the system and observer dynamics which are given by

$$\tilde{z}_t(x,t) = \varepsilon \tilde{z}_{xx}(x,t) + d(v,x,t) - \int_0^x d(v,\tau,t) l\left(x,\tau\right) d\tau \ ,$$

$$\tilde{z}_x(0,t) = 0,\ \tilde{z}(1,t) = (\theta - 1)U(t)\,,$$

for $t \geq t_i$. Now define a new PDE as

$$\psi_t(x,t) = \varepsilon \psi_{xx}(x,t) + d(v,x,t) - \int_0^x d(v,\tau,t) l\left(x,\tau\right) d\tau,\ \psi_x(0,t) = 0,\ \psi(1,t) = H$$

for $t \geq T$ and let $\psi(x,T) = \tilde{z}(x,T)$. If $\left|(\theta - 1)U(t)\right| \geq H$ for $t \geq T \geq t_i$ then $\left|\tilde{z}(x,t)\right| \geq \psi(x,t)$ for $0 \leq x \leq 1$ and $t \geq T$. Note that the dynamics of $\left[\psi\left(x,t\right) - H\right]$ is the same as (A.60-61), thus $\left|\psi\left(0,t\right) - H\right| \leq k_l \bar{d}$ ultimately. Thus, for any $\omega > 0$, there exists a time $t_d \geq T$, such that $\psi\left(0,t_d\right) \geq H - k_l \bar{d} - \omega$. That means, for any $\omega > 0$, there exists a time $t_d \geq T$ such that $\left|\bar{e}(t_d)\right| = \left|\tilde{z}(0,t_d)\right| \geq \psi\left(0,t_d\right) \geq H - k_l \bar{d} - \omega$. Because $H > \rho + k_l \bar{d}$ and if ω is selected as $\omega < H - (\rho + k_l \bar{d})$, then $\left|\bar{e}(t_d)\right| > \rho$ for some $t_d \geq T$, and the detection of actuator fault is guaranteed.

Since the sensor fault will affect the system through the feedback control, the detectability condition for the sensor fault is going to be very similar to that of the actuator fault. In the presence of a sensor fault ($\theta = 1, \theta_s \neq 1$), the detection residual is given by $\bar{e}\left(t\right) = y(t) - \hat{y}(t) = \theta_s z\left(0,t\right) - \hat{z}(0,t)$.

If we define a new distributed variable Δ such that $\Delta(x,t) = \theta_s z(x,t) - \hat{z}(x,t)$ for $0 \leq x \leq 1$, then by using (A.56)-(A.59), Δ can be described by

$$\Delta_t(x,t) = \varepsilon \Delta_{xx}(x,t) + \theta_s [d(v,x,t) - \int_0^x d(v,\tau,t) l\left(x,\tau\right) d\tau] \ ,$$

$$\Delta_x(0,t) = 0,\ \Delta(1,t) = (\theta_s - 1)U(t)$$

and the detection residual can be defined as $\bar{e}(t) = \Delta(0,t)$. Similar to the case of actuator fault, a new bounded state variable defined by the following PDE is introduced

$$\Psi_t(x,t) = \varepsilon\Psi_{xx}(x,t) + \theta_s[d(v,x,t) - \int_0^x d(v,\tau,t)l(x,\tau)\,d\tau] \ ,$$

$$\Psi_x(0,t) = 0,\ \Psi(1,t) = H \ ,$$

for $t \geq T$ and let $\Psi(x,T) = \Delta(x,T)$. If $|(\theta_s - 1)U(t)| \geq H$ for $t \geq T$ then $|\Delta(x,t)| \geq \Psi(x,t)$ for $0 \leq x \leq 1$ and $t \geq T$. Similarly, it can be obtained that $|\Psi(0,t) - H| \leq k_l \bar{d}\theta_{s\max}$ ultimately, which means that for any $\omega > 0$, there exists a time $t_d \geq T$, such that $|\Delta(0,t_d)| \geq \Psi(0,t_d) \geq H - k_l\bar{d}\theta_{s\max} - \omega$. Therefore, by selecting $\omega < H - (\rho + k_l\bar{d}\theta_{s\max})$, it is easy to see that there exists a time $t_d \geq T$ such that $|\bar{e}(t_d)| = |\Delta(0,t_d)| > \rho$, which guarantees the detection of a fault.

If an actuator fault is detected, the observer in this case is described by

$$\begin{aligned} &\hat{z}(x,t) = \hat{\theta}(t)\Lambda(x,t) + [-\lambda/(2\varepsilon) - q]\mathrm{A}(x,t) \\ &+\int_0^1 \varepsilon[-l_\eta(s,0) - ql(s,0)]\mathrm{B}(x,s,t)ds, \quad \hat{y}(t) = \hat{z}(0,t) \end{aligned} . \tag{A.62}$$

For a sensor fault, the observer will be described by

$$\begin{aligned} &\hat{z}(x,t) = \Lambda(x,t) + [-\lambda/(2\varepsilon) - q]\mathrm{A}(x,t)/\hat{\theta}_s(t) \\ &+\{\int_0^1 \varepsilon[-l_\eta(s,0) - ql(s,0)]\mathrm{B}(x,s,t)ds\}/\hat{\theta}_s(t), \quad \hat{y}(t) = \hat{\theta}_s(t)\hat{z}(0,t) \end{aligned} . \tag{A.63}$$

Theorem 4 (Output based fault estimation): Let the observer in (A.62) be used to estimate the state and output of DPS (A.56)-(A.57) with $\hat{\theta}(0) = 1$. The tuning law

$$\dot{\hat{\theta}} = \beta\Lambda(0,t)\bar{e}(t) - \gamma\hat{\theta} \ , \tag{A.64}$$

is activated upon detection of an actuator fault. Similarly consider the observer from (A.63) to estimate the system state and output when $\hat{\theta}_s(0)=1$ with the tuning law

$$\dot{\hat{\theta}}_s(t)=\begin{cases}0 & if \quad \beta\Lambda(0,t)\bar{e}(t)-\gamma\hat{\theta}_s(t)<0 \ \& \ \hat{\theta}_s(t)=\theta_{s\min} \\ \beta\Lambda(0,t)\bar{e}(t)-\gamma\hat{\theta}_s(t) & otherwise\end{cases}, \qquad \text{(A.65)}$$

upon detection of a sensor fault, where $\theta_{smin}>0$ is a known lower bound on the sensor fault parameter. Then the residual $\bar{e}$, parameter estimation errors $\tilde{\theta}=\theta-\hat{\theta}$ and $\tilde{\theta}_s=\theta_s-\hat{\theta}_s$ in the presence of an actuator and sensor faults respectively will be ultimately bounded (UB).

Proof: For an actuator fault, an error signal is defined as

$$\mu(x,t)=z(x,t)-\theta\Lambda(x,t)-(-\lambda/2\varepsilon-q)\mathrm{A}(x,t)-\int_0^1\varepsilon[-l_\eta(s,0)-ql(s,0)]\mathrm{B}(x,s,t)ds$$

and in the case of a sensor fault, it is defined as

$$\begin{aligned}\mu(x,t)&=z(x,t)-\Lambda(x,t)-(-\lambda/2\varepsilon-q)\mathrm{A}(x,t)/\theta_s \\ &-\{\int_0^1\varepsilon[-l_y(s,0)-ql(s,0)]\mathrm{B}(x,s,t)ds\}/\theta_s\end{aligned}.$$

This error signal in both cases clearly satisfies

$$\mu_t(x,t)=\varepsilon\mu_{xx}(x,t)+d(v,x,t)-\int_0^x d(v,\tau,t)l(x,\tau)d\tau, \mu_x(0,t)=0, \mu(1,t)=0. \qquad \text{(A.66)}$$

(a) Actuator fault

Now, a Lyapunov function candidate is selected as

$$V=\frac{\|\mu\|^2}{2\varepsilon}+\frac{\tilde{\theta}^2}{2\beta}=\frac{\int_0^1\mu^2(x,t)dx}{2\varepsilon}+\frac{\tilde{\theta}^2}{2\beta}.$$

By taking the derivative of the Lyapunov function with respect to time and substituting (A.66) to the equation above to get

$$\dot{V} = \int_0^1 \mu(x,t)\mu_t(x,t)dx/\varepsilon + \tilde{\theta}(t)\dot{\tilde{\theta}}(t)/\beta$$
$$= \int_0^1 \mu(x,t)\mu_{xx}(x,t)dx + \frac{\tilde{\theta}(t)\dot{\tilde{\theta}}(t)}{\beta}$$
$$+ \frac{\int_0^1 d(v(x,t),x,t)\mu(x,t)dx}{\varepsilon} - \frac{\int_0^1 \mu(x,t)\int_0^x d(v(\eta,t),\eta,t)l(x,\eta)d\eta dx}{\varepsilon}.$$

Then applying integration by parts and Substituting (A.64) in the above equation we obtain

$$\dot{V} = -\int_0^1 \mu_x^2(x,t)dx + \frac{\tilde{\theta}(t)\dot{\tilde{\theta}}(t)}{\beta}$$
$$+ \frac{\int_0^1 d(v(x,t),x,t)\mu(x,t)dx}{\varepsilon} - \frac{\int_0^1 \mu(x,t)\int_0^x d(v(\eta,t),\eta,t)l(x,\eta)d\eta dx}{\varepsilon}.$$
$$= -\|\mu_x\|^2 + \int_0^1 d(v(x,t),x,t)\mu(x,t)dx/\varepsilon - \tilde{\theta}(t)\bar{e}(t)\Lambda(0,t)$$
$$+ \gamma\tilde{\theta}(t)\hat{\theta}(t)/\beta - \int_0^1 \mu(x,t)\int_0^x d(v(\eta,t),\eta,t)l(x,\eta)d\eta dx/\varepsilon$$

Since $\tilde{z}(0,t) - \mu(0,t) = \tilde{\theta}(\mathrm{t})\Lambda(0,\mathrm{t})$ and by using Cauchy-Schwarz inequality, $\dot{V}$ can be rewritten as

$$\dot{V} = -\|\mu_x\|^2 + \int_0^1 d(v(x,t),x,t)\mu(x,t)dx/\varepsilon - \int_0^1 \mu(x,t)\int_0^x d(v(\eta,t),\eta,t)l(x,\eta)d\eta dx/\varepsilon$$
$$-\bar{e}^2(t) + \bar{e}(t)\mu(0,t) + \gamma\tilde{\theta}(t)\hat{\theta}(t)/\beta \quad .$$
$$\le -\|\mu_x\|^2 - \bar{e}^2(t)/2 + \mu^2(0,t)/2 - \gamma[\tilde{\theta}^2(t) - \theta_{\max}^2]/(2\beta) + d_l\int_0^1 |\mu(x,t)|dx/\varepsilon$$

By using the Poincare $\|\mu\|^2 \le 4\|\mu_x\|^2/\pi^2$ and Agmon's inequalities $\mu^2(0,t) \le 2\|\mu\|\|\mu_x\| \le \|\mu\|^2 + \|\mu_x\|^2$, we get

$$\dot{V} \le -(\pi^2 - 4)\|\mu\|^2/8 + d_l\int_0^1 |\mu(x,t)|dx/\varepsilon - \gamma[\tilde{\theta}^2(t) - \theta_{\max}^2]/(2\beta)$$
$$= -\frac{(\pi^2 - 8)\|\mu\|^2}{8} - \frac{\|\mu\|^2}{2} + d_l\int_0^1 |\mu(x,t)|dx/\varepsilon - \frac{\gamma\tilde{\theta}^2(t)}{2\beta} + \frac{\gamma\theta_{\max}^2}{2\beta}$$
$$\le -(\pi^2 - 8)\|\mu\|^2/8 - \gamma[\tilde{\theta}^2(t) - \theta_{\max}^2]/(2\beta) + d_l^2/(2\varepsilon^2)$$

Therefore, $\dot{V} < 0$ when

$$\|\mu\| > \sqrt{\frac{4(\gamma\theta_{\max}^2 + \beta d_l^2 / \varepsilon^2)}{(\pi^2 - 8)\beta}} \ \ or \ \ |\tilde{\theta}| > \sqrt{\frac{\gamma\theta_{\max}^2 + \beta d_l^2 / \varepsilon^2}{\gamma}} \ .$$

Hence, μ and $\tilde{\theta}$ are UB with the bounds defined above. Since $\bar{e}(t) = \mu(0,t) + \tilde{\theta}(t)\Lambda(0,t)$ and Λ is bounded, $\bar{e}$ is also bounded.

(b) Sensor fault

Similarly, in this case, consider the Lyapunov function

$$V = \|\mu(t)\|^2 / (2\varepsilon) + \tilde{\theta}_s^{\,2} / (2\beta\theta_{s\max}^2) = \int_0^1 \mu^2(x,t)dx / (2\varepsilon) + \tilde{\theta}_s^{\,2} / (2\beta\theta_{s\max}^2) \,,$$

where θ_{smax} is the upper bound on the sensor fault magnitude θ_s . By taking the derivative of the equation above with respect to time and applying integration by parts leads to

$$\begin{aligned} \dot{V} &= \int_0^1 \mu(x,t)\mu_t(x,t)dx / \varepsilon + \theta_s(t)\dot{\tilde{\theta}}_s(t) / (\beta\theta_{s\max}^2) \\ &= \int_0^1 \mu(x,t)\mu_{xx}(x,t)dx + \frac{\int_0^1 d(v(x,t),x,t)\mu(x,t)dx}{\varepsilon} \\ &\quad - \frac{\int_0^1 \mu(x,t)\int_0^x d(v(\eta,t),\eta,t)l(x,\eta)d\eta dx}{\varepsilon} + \frac{\theta_s(t)\dot{\tilde{\theta}}_s(t)}{\beta\theta_{s\max}^2} \ . \\ &= -\int_0^1 \mu_x^2(x,t)dx + \frac{\int_0^1 d(v(x,t),x,t)\mu(x,t)dx}{\varepsilon} \\ &\quad - \frac{1}{\varepsilon}\int_0^1 \mu(x,t)\int_0^x d(v(\eta,t),\eta,t)l(x,\eta)d\eta dx + \frac{\theta_s(t)\dot{\tilde{\theta}}_s(t)}{\beta\theta_{s\max}^2} \end{aligned}$$

The sensor fault parameter $\hat{\theta}_s$ is tuned using $\dot{\hat{\theta}}_s(t) = \beta\Lambda(0,t)\bar{e}(t) - \gamma\hat{\theta}_s(t)$. However, $\hat{\theta}_s$ appears in the denominator of (A.63), and the update law is modified as (A.65) to ensure it is nonzero. With this update law, if $\beta\Lambda(0,t)\bar{e}(t) - \gamma\hat{\theta}_s(t) < 0$ & $\hat{\theta}_s = \theta_{s\min}$, $\dot{V}$ is given by

$$\dot{V} \le -\|\mu_x(t)\|^2 + d_l \int_0^1 |\mu(x,t)| dx / \varepsilon$$
$$\le -\frac{\pi^2}{4}\|\mu(t)\|^2 + \frac{\|\mu(t)\|^2}{2} + \frac{d_l^2}{2\varepsilon^2} \quad ,$$

otherwise it should satisfy

$$\dot{V} \le -\|\mu_x(t)\|^2 - \frac{\tilde{\theta}_s \Lambda(0,t)\bar{e}(t)}{\theta_{s\max}^2} + \frac{\gamma\tilde{\theta}_s(t)\hat{\theta}_s(t)}{\beta\theta_{s\max}^2} + \frac{d_l}{\varepsilon}\int_0^1 |\mu(x,t)| dx \, .$$

In the first case, by applying Poincare inequality $\|\mu(t)\|^2 \le \frac{4\|\mu_x(t)\|^2}{\pi^2}$, we can show $\dot{V} \le -\frac{(\pi^2-2)}{4}\|\mu(t)\|^2 + \frac{d_l^2}{2\varepsilon^2}$ which means $\dot{V}$ will be less than zero if $\|\mu\| > \sqrt{\frac{2}{\varepsilon^2(\pi^2-2)}} d_l$. Therefore, $\|\mu(t)\|$ and $|\tilde{\theta}_s|$ are bounded. Now for the second case, as $\bar{e}(t) - \theta_s \mu(0,t) = \tilde{\theta}_s(t)\Lambda(0,t)$, $\dot{V}$ can be written as

$$\dot{V} \le -\|\mu_x\|^2 - [\bar{e}(t) - \theta_s\mu(0,t)]\bar{e}(t) / \theta_{s\max}^2 + \gamma\tilde{\theta}_s(t)\hat{\theta}_s(t) / (\beta\theta_{s\max}^2) + d_l \int_0^1 |\mu(x,t)| dx$$
$$\le -\|\mu_x\|^2 + d_l \int_0^1 |\mu(x,t)| dx - \bar{e}^2(t) / 2\theta_{s\max}^2$$
$$+\theta_s^2 \mu^2(0,t) / (2\theta_{s\max}^2) - \gamma\tilde{\theta}_s^{\,2}(t) / (2\beta\theta_{s\max}^2) + \gamma\theta_s^2 / (2\beta\theta_{s\max}^2) \quad .$$
$$\le -\|\mu_x\|^2 + d_l \int_0^1 |\mu(x,t)| dx + \mu^2(0,t)/2 - \gamma\tilde{\theta}_s^{\,2}(t) / (2\beta\theta_{s\max}^2) + \gamma / (2\beta)$$

Applying Poincare $\|\mu(t)\|^2 \le 4\|\mu_x(t)\|^2 / \pi^2$ and Agmon's inequalities $\mu^2(0,t) \le 2\|\mu(t)\|\|\mu_x(t)\| \le \|\mu(t)\|^2 + \|\mu_x(t)\|^2$ will lead to

$$\dot{V} \le -\frac{\|\mu_x(t)\|^2}{2} + \frac{\|\mu(t)\|^2}{2} - \frac{\gamma\tilde{\theta}_s^{\,2}(t)}{2\beta\theta_{s\max}^2} + \frac{\gamma}{2\beta} + \frac{d_l^2}{2\varepsilon^2}$$
$$\le -\frac{(\pi^2-4)\|\mu(t)\|^2}{8} + \frac{d_l \int_0^1 |\mu(x,t)| dx}{\varepsilon} - \frac{\gamma\tilde{\theta}_s^{\,2}(t)}{2\beta\theta_{s\max}^2} + \frac{\gamma}{2\beta}$$

$$\leq -\frac{(\pi^2-4)\left\|\mu(t)\right\|^2}{8}+\frac{\left\|\mu(t)\right\|^2}{2}+\frac{d_l^2}{2\varepsilon^2}-\frac{\gamma\tilde{\theta}_s^{\,2}(t)}{2\beta\theta_{s\max}^2}+\frac{\gamma}{2\beta}.$$

$$\leq -(\pi^2-8)\left\|\mu(t)\right\|^2/8-\gamma\tilde{\theta}_s^{\,2}(t)/(2\beta\theta_{s\max}^2)+\gamma/(2\beta)+d_l^2/(2\varepsilon^2)$$

Thus, $\dot{V}<0$ when

$$\left\|\mu(t)\right\|>\sqrt{\frac{4(\gamma+\beta d_l^2/\varepsilon^2)}{(\pi^2-8)\beta}} \quad or \quad \left|\tilde{\theta}_s\right|>\sqrt{\frac{\gamma+\beta d_l^2/\varepsilon^2}{\gamma}}\theta_{s\max}$$

are satisfied implying that μ and $\tilde{\theta}_s$ are UB. Because $\bar{e}(t)=\theta_s\mu(0,t)+\tilde{\theta}_s(t)\Lambda(0,t)$ and θ_s , μ , $\tilde{\theta}_s$ and Λ are bounded, $\bar{e}$ is also bounded. Therefore, the closed-loop system is bounded for both cases.

REFERENCES

[1] Jiang, B., Marcel, S., and Vincent, C. (2006). Fault accommodation for nonlinear dynamic systems. IEEE Transaction on Automatic Control, 51(9), 1578-1583.

[2] Bartelmus, W. (2001). Mathematical modeling and computer simulations as an aid to gearbox diagnostics. Mechanical Systems and Signal Processing, 15(5), 855-871.

[3] Hansen, R. J., Hall, D. L., and Kurtz, S. K. (1995). A new approach to the challenge of machinery prognostics. Journal of Engineering for Gas Turbines and Power, 117(2), 320-325.

[4] Vania, A, and Pennacchi, P. (2004). Experimental and theoretical application of fault identification measures of accuracy in rotating machine diagnostics. Mechanical Systems and Signal Processing, 18(2), 329–352.

[5] Isermann, R. (2004). Model-based fault-detection and diagnosis-status and applications. Annual Reviews in Ctrl, 29(1), 71-85.

[6] Jiang, B., Chowdhury, F. N. (2005). Parameter fault detection and estimation of a class of nonlinear systems using observers. Journal of the Franklin Institute, 342(7), 725-736.

[7] Biswas, G., Koutsoukos, X., Bregon, A., and Pulido, B. (2009, June). Analytic redundancy, possible conflicts, and TCG-based fault signature diagnosis applied to nonlinear dynamic systems. In Proc. of the 7th IFAC Symp. on Fault Detection, Supervision and Safety of Tech. Processes (pp. 1486-1491).

[8] Meseguer, J., Puig, V., and Escobet, T. (2006, August). Observer gain effect in linear interval observer-based fault detection. In Fault Detection, Supervision and Safety of Technical Processes (Vol. 6, No. 1, pp. 540-545).

[9] Chinnam, R. B., and Baruah, P. (2003). Autonomous diagnos-tics and prognostics through competitive learning driven HMM-based cluster. Proc. of the International Joint Conference on Neural Networks, New York, US (pp. 2466–2471).

[10] Kwan, C., Zhang, X., Xu, R., and Haynes, L. (2003). A novel approach to fault diagnostics and prognostics. Proc. of the IEEE Int. Conf. on Robotics and Auto, NY (pp. 604–609).

[11] Wang, P., and Vachtsevanos, G. (2001). Fault prognostics us-ing dynamic wavelet neural networks. AI EDAM-Artificial In-telligence for Engineering Design Analysis and Manufacturing, 15(4), pp. 349–365.

[12] Patan, M, and Ucinski, D. (2005). Optimal activation strategy of discrete scanning sensors for fault detection in distributed pa-rameter systems. Proceedings of the 16th IFAC world congress, Prague, Czech Republic (pp. 1836-1836).

[13] Christofides, P. D. (2001). Control of nonlinear distributed process systems: recent developments and challenges. AIChE Journal, 47(3), pp. 514-518.

[14] Demetriiou, M. A., and Ito, K. (2002). On-line fault detection and diagnosis for a class of positive real infinite dimensional systems. Proceedings of the 41st IEEE Conference on Decision and Control, Las Vegas, US (pp. 4359-4364).

[15] Baniamerian, A., and Khorasani, K. (2012, June). Fault detection and isolation of dissipative parabolic PDEs: Finite-dimensional geometric approach. In American Control Conference (ACC), 2012 (pp. 5894-5899).

[16] Yao, Z., and El-Farra, N. H. (2011, December). Robust fault detection and reconfiguration in sampled-data uncertain distributed processes. In Decision and Control and European Control Conference (CDC-ECC), 50th IEEE Conference on (pp. 4925-4930).

[17] Smyshlyaev, A., and Krstic, M. (2004). Backstepping observer for a class of parabolic PDEs. System and Control Letters, 54(7), 613-625.

[18] Gertler, J. (2005). Residual generation from principal component models for fault diagnosis in linear systems. Proc. of the IEEE International Sym. on Intelligent Control, Cyprus (pp. 634-639).

[19] Smyshlyaev, A., and Krstic, M. (2010). Adaptive control of parabolic PDEs. New Jersey: Princeton University Press.

[20] Krstic, M., and Smyshlyaev, A. (2008). Boundary control of PDEs. Philadelphia: Society for Industrial and Applied Mathematics.

[21] Balogh, A., and Krstic, M. (2002). Infinite dimensional backstepping-style feedback transformations for a heat equation with an arbitrary level of instability. European journal of control, 8(2), 165-175.

[22] Hardy, G. H., Littlewood, J. E., and Polya, G. (1952). Inequalities. Cambridge: Cambridge university press.

[23] LaSalle, J. and Lefschetz, S. (1961). Stability by Lyapunov's Direct Method with Applications, Academic Press, New York.

[24] Cai, J., Ferdowsi, H., and Jagannathan, S. (2014, December). Model-based fault diagnosis and prediction for a class of distributed parameter systems. Decision and Control (CDC), IEEE 53rd Annual Conference *on* (pp. 5758-5763).

II. MODEL-BASED FAULT ACCOMMODATION FOR A CLASS OF DISTRIBUTED PARAMETER SYSTEMS REPRESENTED BY LINEAR COUPLED PDE

Jia Cai, Hasan Ferdowsi and S. Jagannathan

A novel model-based fault detection and accommodation scheme is introduced for a class of linear distributed parameter systems (DPS) represented by partial differential equations (PDEs) in the presence of both actuator and sensor faults. A filter-based observer on the basis of the linear PDE model of the DPS is designed with output measurements. The estimated output from the observer and the measured outputs are utilized to generate a residual for fault detection. Upon detection, the fault function is estimated by using an unknown parameter vector and a known basis function. Novel update laws are introduced to estimate the unknown fault parameter vector for actuator and sensor faults. Next, the controller from the healthy scenario is modified to accommodate the actuator and sensor faults respectively by using output measurements alone. Next, an explicit formula is presented to predict the time-to-accommodation (TTA). Finally, a typical linearized diffusion-process is shown to illustrate the effectiveness of the proposed scheme.

1. INTRODUCTION

In modern control systems, reliability is as important as maintaining performance. System failures due to unexpected faults or degradation of the system components may cause a change in the system dynamics leading to unreliable operation. Therefore, fault diagnosis and accommodation (FDA) research, which is introduced to detect, isolate, and mitigate the effect of unexpected incipient faults, has attracted attention [1].

In the past two decades, significant level of effort is introduced in the literature [2, 3] on model-based diagnosis and fault-tolerant control of lumped parameter systems (LPS) represented as ordinary differential equations (ODEs). However, industrial systems such as fluid flows, thermal convection and chemical reaction systems are categorized as distributed parameter systems (DPS) or infinite dimensional systems because the system state changes not only with time but also with space.

The FDA for DPS represented by partial differential equations (PDEs) is more involved and challenging when compared to LPS due to the need to estimate the system state at all locations [4, 5, 6,7]. It is not possible to measure the system state of a DPS at all these locations. Though under certain assumptions, the DPS are represented as LPS, the ODE models from LPS representation [8] are no longer suitable to mimic the behavior of DPS accurately.

Because of the distributed nature and complicated dynamics, limited effort is being reported for fault detection and diagnosis of DPS. Recently, fault detection of mechanical and aerospace engineering systems have been studied in [9] and fault tolerant controller was considered in [10, 11, 12, 13] with actuator faults. Besides, an adaptive fault detection and accommodation scheme is presented in [14] in order to deal with incipient actuator faults.

On the other hand, fault-tolerant control of DPS with control constraints and actuator faults is introduced in [15]. In spite of these exciting results, these detection and accommodation schemes in [10, 11, 12, 13, 14, 15] have been developed based on approximate finite dimensional representation of DPS which may lead to false and missed alarms because of the model reduction. Moreover, the system dynamics change in the presence of a fault and thus reduced order models may be inaccurate for fault detection and accommodation in DPS.

Driven by these model reduction considerations, we developed a novel FDA scheme on the basis of the PDE representation for linear DPS with incipient faults. A filter-based observer is introduced for generating a residual which is utilized for fault detection. Next the approximation of fault dynamics is carried out by using an adaptive term under the assumption that the fault function is expressed as linear in the unknown parameters. This adaptive term is added to the filter-based observer upon detecting the fault. Both actuator and sensor faults are considered and suitable parameter tuning scheme using the output measurements alone is derived. Next, the fault accommodation is introduced on the basis of estimated fault function. The system stability is demonstrated through Lyapunov analysis.

Moreover, upon detecting a fault and by using the tracking error dynamics, estimated time to accommodation (TTA), which is defined as the time needed by the accommodation scheme to recover back to the normal operating regime, can be assessed online. The TTA is particularly useful when compared to the remaining useful life, since it can predict whether or not the accommodation scheme will work before the system reaches failure.

The main contributions of this paper include the development of: a) an innovative model-based FDA scheme for both actuator and sensor faults by using filter-based observer and system output, and b) TTA scheme on the basis of tracking error dynamics. This paper is constructed as follows. First, a class of DPS represented by linear parabolic PDE with actuator and sensor faults is presented in Section 2. Second, the design of FDA scheme for both actuator and sensor faults with output alone is considered in Section 3. Finally, the application of the proposed scheme in simulation on the heat conduction of a thin rod with actuator and sensor faults is demonstrated in Section 4.

2. SYSTEM DESCRIPTION AND FAULT FUNCTION

Before presenting the system description, the notation and the norm used throughout this paper is given [19]. A scalar function $v_1(x) \in L_2(0,1)$ means it is square integrable on the Hilbert space $L_2(0,1)$ with the corresponding norm

$$\|v_1(\cdot)\|_2 = \sqrt{\int_0^1 v_1^2(x)dx}\,, \tag{1}$$

Now consider

$$\begin{aligned}\dot{V} &\le -\|\mu_x(x,t)\|_{2,n}^2 - e^T(t)e(t)/2 + \mu^T(0,t)\mu(0,t)/2 \\ &\quad -\gamma[\tilde{\theta}_a^T(t)\tilde{\theta}_a(t) - \theta_{a\max}^2]/2\beta + d_l\int_0^1 \sqrt{\mu^T(x,t)\mu(x,t)}dx/\varepsilon \\ &\le -(\pi^2-4)\|\mu(x,t)\|_{2,n}^2/8 - \gamma\tilde{\theta}_a^T(t)\tilde{\theta}_a(t)/(2\beta) + \gamma\theta_{a\max}^2/(2\beta) + d_l^2/(4\varepsilon^2)\end{aligned}\,, \tag{2}$$

with the corresponding norm of a vector function $v(x,t) = [v_1(x,t), v_2(x,t), ..., v_n(x,t)]^T \in [L_2(0,1)]^n$ defined as

$$\|v(\cdot)\|_{2,n} = \sqrt{\sum_{i=1}^n \|v_i(x)\|_2^2} = \sqrt{\int_0^1 v^T(x)v(x)dx}\,. \tag{3}$$

2.1. SYSTEM DESCRIPTION

A class of n-dimensional linear DPS, which can be expressed by the following parabolic partial differential equation (PDE), is described by

$$v_t(x,t) = \varepsilon v_{xx}(x,t) + \Lambda v(x,t) + d(v,x,t)\,, \tag{4}$$

with boundary conditions defined by

$$v_x(0,t) = 0, \quad v(1,t) = U(t), \quad y(t) = v(0,t)\,, \tag{5}$$

for $x \in (0,1)$ and $t \ge 0$, where $d(v,x,t) = [d_1(v,x,t), ...d_n(v,x,t)] \in \Re^n$ stands for disturbance or uncertainty, $v(x,t) = [v_1(x,t), ..., v_n(x,t)]^T \in [L_2(0,1)]^n$ represents the state of the DPS,

v_t and v_x denote partial derivatives of $v(x,t)$ or $\partial v(x,t)/\partial t$ and $\partial v(x,t)/\partial x$ respectively, $U(t)=[u_1(x,t),...,u_n(x,t)]^T \in \Re^n$ denotes the control input, ε is a positive constant, and $\Lambda \in \Re^{n \times n}$ is a real valued square matrix. In addition, $y(t) \in \Re^n$ is the system output vector measured at the opposite end of both the actuator and controller. For fault accommodation, a controller is required prior to the fault.

Assumption 1: The system uncertainty or disturbance is bounded above such that $\|d(v,x,t)\| \le \bar{d}$ for all (v,x) and $t \ge 0$, where $\bar{d} > 0$ is a known constant. It is written as a general form in this paper, a more specific model can be found in [16, 17].

2.2. STATE FEEDBACK CONTROLLER DESIGN UNDER HEALTHY CONDITIONS

Given a reference output, a full-state desired trajectory satisfying the system dynamics given by (4) and (5) is required in order to design the control input $U(t)$ which in turn allows the system state to follow the trajectory. Given a reference output $v_d(0,t) \in \Re^n$, a desired state trajectory for $0 < x \le 1$ can be represented as [18]

$$v_d(x,t) = \sum_{k=0}^{\infty} a_k(t) \frac{x^k}{k!} \quad , \tag{6}$$

where $a_k(t) = [a_{k1}(t), a_{k2}(t),..., a_{kn}(t)] \in \Re^n$ represents time-varying coefficients of Taylor series expansion in x. These coefficients are determined by using the reference output and the system description (4)-(5).

Next define the state tracking error as $r(x,t) = [r_1(x,t),...,r_n(x,t)]^T = v(x,t) - v_d(x,t)$. The state tracking error dynamics can be obtained as

$$r_t(x,t) = \varepsilon r_{xx}(x,t) + \Lambda r(x,t) + d(v,x,t)\,, \tag{7}$$

$$r_x(0,t) = 0\,, \tag{8}$$

where $r_t = \partial r / \partial t$, $r_x = \partial r / \partial x$ and $r_{xx} = \partial^2 r / \partial x^2$. The open-loop system (7) and (8) with $r(1,t) = 0$ is unstable when Λ is positive definite with sufficiently large eigenvalues. Since $\Lambda r(x,t)$ is the source of instability, our aim is to eliminate this term by using both the Volterra integral transformation [18, 19] and a suitable controller. Apply the Volterra integral transformation given by

$$w(x,t) = r(x,t) - \int_0^x K(x,\tau) r(\tau,t) d\tau\,, \tag{9}$$

with feedback control input $U(t) = U_h(t)$

$$U(t) = U_h(t) = v_d(1,t) + \int_0^1 K(1,\tau) r(\tau,t) d\tau\,, \tag{10}$$

along with the boundary condition

$$r(1,t) = \int_0^1 K(1,\tau) r(\tau,t) d\tau\,, \tag{11}$$

to convert the system (7)-(8) and (11) into a stable target system described by

$$w_t(x,t) = \varepsilon w_{xx}(x,t) - Cw(x,t) + d(v,x,t) - \int_0^x K(x,\tau) d(v,\tau,t) d\tau\ , \tag{12}$$

$$w_x(0,t) = 0, \quad w(1,t) = 0\,. \tag{13}$$

Here

$$K(x,\tau) = -\sum_{n=0}^{\infty} \frac{(x^2 - \tau^2)^n (2x)}{(4\varepsilon)^{n+1} n!(n+1)!} [\sum_{i=0}^{n} \binom{n}{i} C^i (\Lambda + C) \Lambda^{n-i}]\ , \tag{14}$$

is an $n \times n$ controller kernel matrix obtained by using a backstepping approach through the well-posed hyperbolic PDE given by (Baccoli, Orlov & Pisano, 2014)

$$\varepsilon K_{xx}(x,\tau) - \varepsilon K_{\tau\tau}(x,\tau) = K(x,\tau)\Lambda + CK(x,\tau)\ , \tag{15}$$

$$K_{\tau}\left(x,0\right)=0 \;\; K(x,x)=-(\Lambda+C)x/2\varepsilon\,, \tag{16}$$

where $C \in \Re^{n\times n}$ is an arbitrary symmetric positive definite square matrix. Due to the invertability of (9) [18], the boundedness of $w(x,t)$ can guarantee the boundedness of $r(x,t)$.

It is important to note that the controller given in (10) clearly requires the state vector $v(x,t)$ at all positions. Therefore, the output feedback controller will be introduced in Section 3. Next actuator and sensor fault function, $h_a(t)\in\Re^n$ and $h_s(t)\in\Re^n$, respectively are considered at the boundary of the DPS.

2.3. ACTUATOR AND SENSOR FAULT DESCRIPTION

In the presence of actuator and sensor faults, the system description from (4) and (5) can be described by (4) subjected to the new boundary conditions

$$v_x\left(0,t\right)=0,\; v(1,t)=U(t)+h_a(t)\,, \tag{17}$$

$$y(t)=v(0,t)+h_s(t)\,, \tag{18}$$

Moreover, the fault function can be written as

$$h_a(t)=\Omega(t-t_0)\bar{h}_a(t)\,,\; h_s(t)=\Omega(t-t_0)\bar{h}_s(t)\,, \tag{19}$$

where $\Omega_i(t-t_0)$ is the time profile of the fault defined by $\Omega_i(\tau)=\begin{cases}0 & ,if\;\tau<0\\ 1-e^{-\kappa_i\tau} & ,if\;\tau\geq 0\end{cases}$ for $i=1,...,n$ with the constant κ_i represents the growth rate of the incipient fault and $\bar{h}_a(t)$ and $\bar{h}_s(t)$ denote actuator and sensor fault function dynamics respectively. Abrupt faults can also be modeled when a large κ_i is chosen. The following assumption is needed in order to proceed.

Assumption 2: The fault function can be expressed as linear in the unknown parameters (LIP)[20]. In other words, the actuator fault function $\bar{h}_a(t) = \Phi_a(U(t),t)\theta_a$ and the sensor fault function $\bar{h}_s(t) = \Phi_s(t)\theta_s$ with $\theta_a \in \Re^n$ and $\theta_s \in \Re^n$ being the unknown fault parameter vector satisfies $\|\theta_a\| \le \theta_{a\max}$, $\|\theta_s\| \le \theta_{s\max}$, with $\Phi_a(U(t),t) = diag(\sigma_i^{(a)}(U(t),t)) \in \Re^{n\times n}$ for an actuator fault and $\Phi_s(y(t),t) = diag[\sigma_i^{(s)} t)] \in \Re^{n\times n}$ for a sensor fault being known where $\sigma_i^{(a)}(\cdot)$ and $\sigma_i^{(s)}(\cdot) \in \Re\ (i = 1, 2, \ldots n)$ are smooth bounded function.

3. FAULT ACCOMMODATION SCHEME WITH OUTPUT MEASUREMENTS

In this section, a detection observer, which provides the estimated state information, is designed using an input and a couple of output filters. In addition, an adaptive tuning law has to be carefully selected to detect and approximate both the sensor and actuator fault functions using estimated state measurements under the assumption that the type of fault is known. The isolation of fault type is outside the scope of this work. The controller structure from the previous section with state measurements can be utilized with modifications for fault accommodation. The controller for the healthy case is introduced first and it is modified for the purpose of fault accommodation later.

3.1. OUTPUT FEEDBACK CONTROLLER DESIGN UNDER HEALTHY CONDITIONS

Now assume that the only the boundary value $v(0,t)$ is measured. In order to design the observer and output feedback controller, the DPS from (4) and (5) is first converted to an observable form, by utilizing the following transformation [22] given by

$$z(x,t) = v(x,t) - \int_0^x l(x,\tau) v(\tau,t) d\tau \,, \tag{20}$$

where $l(x,\tau) = -2(x-1)\sum_{n=0}^{\infty} \frac{(-1)^n[(\tau-1)^2-(x-1)^2]^n \Lambda^{n+1}}{(4\varepsilon)^{n+1} n!(n+1)!}$ being the solution of the hyperbolic PDE given by $l_{xx} - l_{\tau\tau} = l(x,\tau)\Lambda / \varepsilon$, $l(1,\tau)=0$ and $l(x,x) = \Lambda(1-x)/(2\varepsilon)$. The transformation (20) can convert the original system (4)-(5) to the following PDE given by

$$z_t(x,t) = \varepsilon z_{xx}(x,t) + G(x) z(0,t) + d(v,x,t) + \int_0^x l(x,) d(v,\tau,t) d\tau \,, \tag{21}$$

$$z_x(0,t) = L_0 z(0,t),\ z(1,t) = U(t) \,, \tag{22}$$

$$y(t) = z(0,t) \,, \tag{23}$$

where $L_0 = -\Lambda/(2\varepsilon)$ and $G(x) = -\varepsilon l_\tau(x,0)$. Note $z(0,t)$ is available since $z(0,t) = v(0,t)$. This transformation helps to avoid the unstable term $\Lambda v(x,t)$ from appearing in the design of filters which are introduced next.

The DPS given by (21) and (22) have $U(t)$, $L_0 v(0,t)$ and $G(x)v(0,t)$ as external inputs. According to superposition principle [22] of linear DPS, its solution can be expressed as the sum of the response of the PDEs for each external input acting alone. Therefore, $z(x,t) \in \Re^n$ can be expressed as a combination of the solution to three individual PDEs defined by

$$\Xi_t(x,t) = \varepsilon \Xi_{xx}(x,t), \quad \Xi_x(0,t) = 0, \quad \Xi(1,t) = U(t), \tag{24}$$

where $\Xi(x,t)$ is referred to an input filter since it is derived from the input of the actual system [22]. Next consider

$$A_t(x,t) = \varepsilon A_{xx}(x,t), A_x(0,t) = y(t), A(1,t) = 0, \tag{25}$$

where $A(x,t)$ is an output filter since it is derived from output of the actual system, $y(t)$. Finally consider

$$\Pi_t(x,\eta,t) = \varepsilon \Pi_{xx}(x,\eta,t) + \delta(x-\eta)y(t), \ \Pi_x(0,\eta,t) = 0, \ \Pi(1,\eta,t) = 0 \ . \tag{26}$$

where $\Pi(x,\eta,t)$ is a second output filter.

Define the observer with its state, $\hat{z}(x,t) \in \Re^n$, given by

$$\hat{z}(x,t) = \Xi(x,t) + L_0 \mathrm{A}(x,t) + \int_0^1 G(s)\Pi(x,s,t)ds, \tag{27}$$

with $\hat{y}(t) = \hat{z}(0,t)$, and $e(t) = y(t) - \hat{y}(t)$.

The observer state estimation error is obtained as $\tilde{z}(x,t) \in \Re^n = z(x,t) - \hat{z}(x,t)$ with its dynamics satisfying

$$\tilde{z}_t(x,t) = \varepsilon \tilde{z}_{xx}(x,t) + d(v,x,t) + \int_0^x l(x,)d(v,\tau,t)d\tau,\ \tilde{z}_x(0,t) = 0,\ \tilde{z}(0,t) = 0 \ . \qquad (28)$$

Since (27) provides $\hat{z}$ instead of $\hat{v}$, for the controller design we need the inverse transformation of (20) given by

$$v(x,t) = z(x,t) + \int_0^x M(x,\tau) z(\tau,t) d\tau , \qquad (29)$$

to obtain the estimated state $\hat{v}(x,t) = \hat{z}(x,t) + \int_0^x M(x,\tau)\hat{z}(\tau,t)d\tau$ where $M(x,\tau) \in \Re^n = -2(x-1)\sum_{n=0}^{\infty} \frac{[(\tau-1)^2-(x-1)^2]^n \Lambda^{n+1}}{(4\varepsilon)^{n+1} n!(n+1)!}$ is a bounded solution to the following hyperbolic PDE $M_{xx} - M_{\tau\tau} = -\frac{M(x,\tau)\Lambda}{\varepsilon}, M(1,\tau) = 0, M(x,x) = \frac{\Lambda(1-x)}{2\varepsilon}$.

Then the state estimation error is defined in terms of $M(x,\tau)$ and $\tilde{z}(\tau,t)$ as

$$\tilde{v}(x,t) = v(x,t) - \hat{v}(x,t) = \tilde{z}(x,t) + \int_0^x M(x,\tau)\tilde{z}(\tau,t)d\tau . \qquad (30)$$

Note that the boundedness of $\tilde{v}(x,t)$ is guaranteed due to the boundedness of $\tilde{z}(x,t)$. With the observer defined in (27), the stability of the observable system (21) and (22) as well as the original system (4) and (5) can be demonstrated with the controller designed as

$$\begin{aligned} U(t) &= \hat{U}_h(t) = v_d(1,t) + \int_0^1 K(1,\tau)[\hat{v}(\tau,t) - v_d(\tau,t)]d\tau \\ &= v_d(1,t) + \int_0^1 K(1,\tau) r(\tau,t)d\tau - \int_0^1 K(1,\tau)\tilde{v}(\tau,t)d\tau \end{aligned} . \qquad (31)$$

where $\hat{U}_h(t)$ is the control input using estimated state vector during healthy conditions. It is important to observe the difference between this controller using the estimated state vector $\hat{v}(x,t)$ and the controller (10) designed by using the measured state vector $v(x,t)$. They will be equivalent when $\tilde{v}(x,t) \to 0$.

Next, apply the controller (31) to the system (4) and (5), the state tracking error dynamics can be obtained as (7) and (8) with the following boundary condition

$$r(1,t)=\int_0^1 K(1,\tau)r(\tau,t)d\tau-\int_0^1 K(1,\tau)\tilde{v}(\tau,t)d\tau\,. \tag{32}$$

Then by asserting the transformation (9) to the state tracking error dynamics (7)-(8) and (32), we get (12) subjecting to

$$w_x(0,t)=0,\quad w(1,t)=-\int_0^1 K(1,\tau)\tilde{v}(\tau,t)d\tau\,. \tag{33}$$

Therefore from (30) and (33) we know that

$$\begin{aligned} &w^T(1,t)w(1,t)=[\int_0^1 K(1,\tau)\tilde{v}(\tau,t)d\tau]^T[\int_0^1 K(1,\tau)\tilde{v}(\tau,t)d\tau] \\ &\le \int_0^1 [K(1,\tau)\tilde{v}(\tau,t)]^T[K(1,\tau)\tilde{v}(\tau,t)]d\tau \\ &=\int_0^1 \tilde{v}^T(\tau,t)K^T(1,\tau)K(1,\tau)\tilde{v}(\tau,t)d\tau \le \bar{k}^2\int_0^1 \tilde{v}^T(\tau,t)\tilde{v}(\tau,t)d\tau=\bar{k}^2\left\|\tilde{v}\right\|_{2,n}^2 \le k_h\left\|\tilde{z}\right\|_{2,n}^2 \end{aligned}\;, \tag{34}$$

where $\bar{k}=\max_{0\le x\le 1}\left\|K(x,\tau)\right\|_2$, $k_h=2\bar{k}^2(\bar{m}^2+1)$ and $\bar{m}=\max_{0\le x\le 1}\left\|M(x,\tau)\right\|_2$.

The performance of the controller under healthy condition is shown in the Appendix. Now the assumption that the type of the fault is known is asserted and the actuator fault detection and accommodation is introduced using estimated states.

3.2. ACTUATOR FAULT DETECTION AND ACCOMMODATION

Recall the dynamics of transformed system with an actuator fault represented as (21) subjecting to

$$z_x(0,t)=L_0 z(0,t),\, z(1,t)=U(t)+h_a(t),\; y(t)=z(0,t)\,. \tag{35}$$

In order to approximate the fault dynamics upon detection, the design of the fault filter will be performed based on the observable form (21) which is expressed as

$$D_t(x,t)=\varepsilon D_{xx}(x,t)\,,\, D_x(0,t)=0, D(1,t)=[\sigma_1^{(a)},\sigma_2^{(a)},...\sigma_n^{(a)}]^T\,,$$

where $D(x,t) \in \Re^n$. Then the observer (27) after incorporating the adaptive term becomes

$$\begin{aligned} &\hat{z}(x,t) = \Xi(x,t) + \Gamma(x,t)\hat{\theta}_a(t) + L_0 \mathrm{A}(x,t) + \int_0^1 G(s)\Pi(x,s,t)ds \\ &\hat{y}(t) = \hat{z}(0,t), \;\; e(t) = y(t) - \hat{y}(t) \end{aligned}, \quad (36)$$

where $\hat{\theta}_a(t)$ is the estimated fault parameter vector with $\hat{\theta}_a(0) = 0$ since the fault parameter vector under healthy conditions is $\theta_a = 0$ and $\Gamma(x,t) = diag(D(x,t)) \in \Re^{n\times n}$ with $\Gamma(1,t) = \Phi_a(U(t),t)$. Next, an ideal signal $\bar{z}(x,t) \in \Re^n$ is introduced with an initial condition same as that of $\hat{z}(x,t)$. This ideal signal is viewed as the ultimate target of $\hat{z}(x,t)$ as $\hat{z}(x,t)$ gets tuned along with $\hat{\theta}_a(t)$. It is designed as

$$\bar{z}(x,t) = \Xi(x,t) + \Gamma(x,t)\theta_a + L_0 \mathrm{A}(x,t) + \int_0^1 G(s)\Pi(x,s,t)ds \,.$$

Then it is easy to obtain the dynamics of $\bar{z}(x,t)$ as

$$\bar{z}_t(x,t) = \varepsilon \bar{z}_{xx}(x,t) + G(x)z(0,t)\,, \quad (37)$$

$$\bar{z}_x(0,t) = L_0 z(0,t),\; \bar{z}(1,t) = U(t) + \Phi_a(U(t),t)\theta_a\,. \quad (38)$$

Notice $\bar{z}(x,t)$ has the same initial condition as that of $\hat{z}(x,t)$ while it has different initial condition from $z(x,t)$. Because $\bar{z}(x,t)$ has the same dynamics as that of DPS given by the observable form (21), it will be utilized in the proof of fault approximation with filters. The next theorem discusses the performance of this observer as a fault detection observer and provides a suitable parameter tuning law.

Theorem 1 (Detection and Fault Approximation): Let the observer in (36) be used to monitor the system defined by (21) and (35). Then the magnitude of output detection residual $e(t)$ will increase in the presence of an actuator fault and when it reaches the threshold, a fault is considered detected. Upon detecting a fault, select the parameter tuning law as

$$\dot{\hat{\theta}}_a = \beta\Gamma(0,t)e(t) - \gamma\hat{\theta}_a \ , \tag{39}$$

where $0 < \beta < 2$ is the leaning rate and $\gamma > 0$ is a design parameter. Then the observer estimation error, $\tilde{z}$, and parameter estimation error, $\tilde{\theta}_a = \theta_a - \hat{\theta}_a$, are ultimately bounded (UB).

Proof: Refer to the Appendix.

It is shown in the Appendix that with the parameter tuning law (39), the parameter estimation error $\tilde{\theta}_a$ stays in a bounded region and the bound of the observer residual $\tilde{z}$ is guaranteed due to $\Gamma(x,t)$ being bounded.

The approximated fault function given by $\Phi_a(U(t),t)\hat{\theta}_a(t)$ is utilized in the control input for accommodation. The overall input is designed as

$$U(t) = U_{accom}(t) = \hat{U}_h(t) - \Phi_a(t)\hat{\theta}_a(t) \tag{40}$$

yielding

$$z(1,t) = \hat{U}_h(t) + \Phi_a(t)\tilde{\theta}_a(t) \ , \tag{41}$$

in order to mitigate the effect of the fault on the system where $\hat{U}_h(t)$ is the control input under healthy conditions using filter based approach as given by (31). Then the dynamics of the transformed tracking error becomes (12) subject to

$$w_x(0,t) = 0, \ \ w(1,t) = \Phi_a(U(t),t)\tilde{\theta}_a(t) - \int_0^1 K(1,\tau)\tilde{v}(\tau,t)d\tau \ . \tag{42}$$

Noting that $\Phi_a^T(U(t),t)\Phi_a(U(t),t) \le \sigma_{a\max}^2$ and $\tilde{z}(x,t) = \mu(x,t) + \Gamma(x,t)\tilde{\theta}_a$, it follows that

$$w^T(1,t)w(1,t) \le 2\sigma_{a\max}^2 \tilde{\theta}_a^T \tilde{\theta}_a + 2\bar{k}\,\|\tilde{v}\|_{2,n}^2$$
$$\le 2\sigma_{a\max}^2 \tilde{\theta}_a^T \tilde{\theta}_a + 2k_h\,\|\tilde{z}\|_{2,n}^2 \le k_c[\tilde{\theta}_a^T \tilde{\theta}_a + \|\mu\|_{2,n}^2] . \qquad (43)$$

where $k_c = \max\{4k_h, (4k_h\bar{D} + 2\sigma_{a\max}^2)\}$, $\sigma_{a\max} = \sup\{\sqrt{\sum_{i=1}^{n}[\sigma_i^{(a)}(U(t)]^2, t)}\}$ and $\bar{D} = \max_{0\le x\le 1}\|D(x,t)\|^2$.

The next theorem shows the boundedness of tracking error with the proposed accommodation scheme.

Theorem 2 (Fault Accommodation in the General Case): Let the control law in (40) be used upon detecting an actuator fault. Then the parameter estimation, observer estimation and tracking errors are UB.

Proof: See Appendix.

Corollary 1 (Fault Accommodation in the Ideal Case): In the absence of disturbance or uncertainty, i.e. $d(v,x,t)=0$, let the control law in (40) be used upon detecting an actuator fault. Then the parameter estimation, observer estimation, and tracking errors are all UB with smaller bounds.

The boundedness of parameter estimation, state estimation and the tracking errors are shown in the Appendix.

Remark 1: Those bounds can be adjusted by using the designed parameter $c = \lambda_{\min}(C)$.

3.3. SENSOR FAULT DETECTION AND ACCOMMODATION

Upon detection of a sensor fault, the following two fault filters will be applied to estimate the fault dynamics

$$F_{1t}(x,t)=\varepsilon F_{1xx}(x,t), F_{1x}(0)=[\sigma_1^{(s)},\sigma_2^{(s)},...\sigma_n^{(s)}]^T, F_1(1,t)=0,$$

$$F_{2t}(x,\eta,t)=\varepsilon F_{2xx}(x,\eta,t)+\delta(x-\eta)[\sigma_1^{(s)},\sigma_2^{(s)},...\sigma_n^{(s)}]^T,\ F_{2x}(0,\eta,t)=0,\ F_2(1,\eta,t)=0.$$

The two output filters become

$$\mathrm{A}_t(x,t)=\varepsilon \mathrm{A}_{xx}(x,t),\ \mathrm{A}_x(0,t)=z(0,t)+\Phi_s(t)\theta_s,\ \mathrm{A}(1,t)=0,$$

and

$$\Pi_t(x,\eta,t)=\varepsilon\Pi_{xx}(x,\eta,t)+\delta(x-\eta)[z(0,t)+\Phi_s(t)\theta_s],\ \Pi_x(0,\eta,t)=0,\ \Pi(1,\eta,t)=0\ .$$

Then the corresponding observer will be redefined as

$$\hat{z}(x,t)=\Xi(x,t)+L_0\mathrm{A}(x,t)+\int_0^1 G(s)\Pi(x,s,t)ds-[L_0\Delta(x,t)+\int_0^1 G(s)\Psi(x,s,t)ds]\hat{\theta}_s(t)\ .(44)$$

where $\Delta(x,t)=diag(F_1(x,t))$, $\Psi(x,s,t)=diag(F_2(x,s,t))$ and $\hat{\theta}_s(t)\in\Re^n$ is the estimated sensor fault parameter vector. In order to proceed, similar to the actuator fault case, we introduce a variable defined by

$$\bar{z}(x,t)=\Xi(x,t)+L_0\mathrm{A}(x,t)+\int_0^1 G(s)\Pi(x,s,t)ds-[L_0\Delta(x,t)+\int_0^1 G(s)\Psi(x,s,t)ds]\theta_s(t)\ .$$

Thus we can get that

$$\hat{z}(x,t)=\bar{z}(x,t)+[L_0\Delta(x,t)+\int_0^1 G(s)\Psi(x,s,t)ds]\tilde{\theta}_s(t),$$

where $\tilde{\theta}_s(t)=\theta_s-\hat{\theta}_s(t)$ is the parameter estimation error. Defining an error signal as $\mu(x,t)=z(x,t)-\bar{z}(x,t)$, it is clear that

$$\mu_t(x,t)=\varepsilon\mu_{xx}(x,t)+d(v,x,t)-\int_0^x l(x,\tau)d(v,\tau,t)d\tau\ . \tag{45}$$

$$\mu_x(0,t)=0,\ \mu(1,t)=0\ . \tag{46}$$

Then the estimated state error $\tilde{z}(x,t)=z(x,t)-\hat{z}(x,t)$ can be represented as

$$\tilde{z}(x,t)=\mu(x,t)-[L_0\Delta(x,t)+\int_0^1 G(s)\Psi(x,s,t)ds]\tilde{\theta}_s(t)\ . \tag{47}$$

The next theorem evaluates the detection observer and presents an appropriate tuning law to approximate the fault function upon detection of the sensor fault.

Theorem 3 (Detection and Fault Approximation): Let the observer in (44) be used to monitor the system defined by (21)-(22) and (18). The magnitude of detection residual $e(t)$ will increase in the presence of a sensor fault and when it reaches the detection threshold, a fault is considered detected. Upon detecting a sensor fault, select the parameter tuning law as

$$\dot{\hat{\theta}}_s = \beta[\Phi_s(t) - F(0,t)]^T e(t) - \gamma\hat{\theta}_s \,, \tag{48}$$

where $F(x,t) = L_0\Delta(x,t) + \int_0^1 G(s)\Psi(x,s,t)ds$, $0 < \beta < 2$ is the leaning rate, $\gamma > 0$ is a design parameter, and $e(t)$ is the detection residual defined as $e(t) = y(t) - \Phi_s(t)\hat{\theta}_s(t) - \hat{z}(0,t) = \tilde{z}(0,t) + \Phi_s(t)\tilde{\theta}_s(t)$. Then the observer residual, $\tilde{z}$, and parameter estimation error, $\tilde{\theta}_s$, are bounded.

Proof: See Appendix.

The details of the proof for the above theorem are given in the Appendix. Next, it will be shown that with the controller given by

$$\begin{aligned} U(t) &= v_d(1,t) + \int_0^1 K(1,\tau)[r(\tau,t) - \tilde{v}(\tau,t)]d\tau \\ &= v_d(1,t) + \int_0^1 K(1,\tau)[r(\tau,t) - \tilde{z}(\tau,t) - \int_0^\tau M(\tau,\eta)\tilde{z}(\eta,t)d\eta]d\tau \end{aligned} \,, \tag{49}$$

where $\tilde{v}(x,t)$ is given by (30) with observer state defined by (44) , the state tracking error $r(x,t)$ will remain bounded. With the controller defined by (49), the dynamics of state tracking error in the presence of a sensor fault at the measured output can be expressed as (7), (8) and (32).

Theorem 4 (Fault Accommodation in the General Case): Let the control law in (49) be used upon detecting the sensor fault. Then the parameter estimation, observer estimation, and tracking errors are UB.

Proof: See Appendix.

Corollary 2 (Fault Accommodation in the Ideal Case): In the absence of disturbance or uncertainty, i.e. $d(v,x,t)=0$, let the control law in (49) be used upon detecting the sensor fault. Then the parameter estimation, observer estimation, and tracking errors are UB with smaller bounds.

It can be proven in the Appendix that $\tilde{\theta}_s$, $\tilde{z}$ and $r(0,t)$ are bounded.

Remark 2: Those bounds change with designed parameter $c=\lambda_{\min}(C)$.

3.4. TIME TO ACCOMMODATION (TTA)

In the previous sections, it has been shown that the tracking error will increase and exceed a limit in the presence of faults at the boundary conditions. Then the fault accommodation scheme will be applied to reduce the effect of faults on the tracking error. Now the estimated time to accommodation is introduced next. The estimated TTA is defined as the time available before the tracking error decreases below a predefined limit with the fault accommodation scheme. TTA using full states can be found in [23]. The following remark gives an explicit formula to estimate the TTA with output alone.

Remark 2: Given an initial value of the output tracking and estimated state tracking errors, and the tracking error limit, upon detecting and activating the fault accommodation scheme, the TTA can be estimated as

$$TTA(t)=\max_{1\le i\le n} t_{a(i)}(t)\,, \tag{50}$$

where $t_{a(i)} = \frac{1}{\lambda_i} Ln\left(\frac{\lambda_i \bar{\vartheta}_i + \varepsilon p_i \frac{\hat{r}(2h,t) - 2\hat{r}(h,t) + r(0,t)}{h^2}}{\lambda_i p_i r(0,t) + \varepsilon p_i \frac{\hat{r}(2h,t) - 2\hat{r}(h,t) + r(0,t)}{h^2}} \right)$. The formula (50) is derived based on the tracking error dynamics (7). The following transformation

$$\bar{r}(x,t) = Pr(x,t) \tag{51}$$

will be utilized when Λ is not diagonal to convert the dynamics of the tracking error (7) to

$$\bar{r}_t(x,t) = \varepsilon \bar{r}_{xx}(x,t) + \bar{\Lambda}\bar{r}(x,t) \tag{52}$$

where $\bar{\Lambda} = P\Lambda P^{-1} = diag(\lambda_1, \lambda_2, ...\lambda_n)$ with $\lambda_i (i = 1, 2, ...n)$ being the eigenvalue of Λ and $P = [p_1^T, p_2^T, ..., p_n^T]^T \in \Re^{n\times n}$.

By using finite difference method, $\bar{r}_{xx}(x,t)$ can be derived as $\bar{r}_{xx}(x,t) = Pr_{xx}(x,t) = P \lim_{h\to 0^+} [r(x+2h,t) - 2r(x+h,t) + r(x,t)]/h^2$, thus $\bar{r}_t(0,t)$ can be obtained as

$$\bar{r}_t(0,t) = \varepsilon P[r(2h,t) - 2r(h,t) + r(0,t)]/h^2 + \bar{\Lambda}\bar{r}(0,t) \ , \tag{53}$$

where $h > 0$ is a sampling interval. The solution $\bar{r}(0,t) = [\bar{r}_1(0,t), ..., \bar{r}_n(0,t))]^T$ to (53) in the interval $\left[t, t_{a(i)}\right]$ is given by

$$\begin{aligned} \bar{r}_i(0,t_{a(i)}) = &\int_t^{t_{a(i)}} e^{\bar{\Lambda}(t_{a(i)}-\tau)} \varepsilon p_i^T [r(2h,\tau) - 2r(h,\tau) + r(0,\tau)]/h^2 d\tau \\ &+ e^{\bar{\Lambda}(t_{a(i)}-t)} p_i^T r(0,t), \forall t_{a(i)} > t \ and \ i = 1, 2, ..., n \end{aligned},$$

where t is the current time instant and $t_{a(i)}$ is the future time when the value of $\bar{r}_i(0,t)$ decrease to its corresponding limit $\bar{\vartheta}_i = p_i[\vartheta_1, \vartheta_2, ..., \vartheta_n]^T$ for the first time where $\vartheta_i (i = 1, 2, ..., n)$ is the limiting value of output tracking error $r_i(0,t)$. Assume that the term $r(2h,\tau) - 2r(h,\tau) + r(0,\tau)$ is held in the interval $\left[t, t_{a(i)}\right]$, we can show that

$$t_{a(i)} = \frac{1}{\lambda_i} Ln\left(\frac{\lambda_i \overline{\vartheta}_i + \varepsilon p_i^T [r(2h,t) - 2r(h,t) + r(0,t)] / h^2}{\lambda_i p_i^T r(0,t) + \varepsilon p_i^T [r(2h,t) - 2r(h,t) + r(0,t)] / h^2} \right) . \quad (54)$$

And in the formula given by (54), since $r(2h,t)$ and $r(h,t)$ are unknown, we need to use $\hat{r}(2h,t) = \hat{v}(2h,t) - v_d(2h,t)$ and $\hat{r}(h,t) = \hat{v}(h,t) - v_d(h,t)$ instead for $t_{a(i)}$ as given before. Because the output tracking error for all the states must be less than their limits, the TTA is obtained as the maximum among all the individual TTA given by (50).

4. SIMULATION RESULTS

In order to demonstrate the proposed scheme for fault accommodation, it has been implemented on a heated rod whose temperature distribution can be represented by a parabolic PDE. The dynamics of the heat equation with an actuator fault at boundary condition can be expressed as

$$\begin{aligned} \partial v_1(x,t)/\partial t &= \partial^2 v_1(x,t)/\partial x^2 + 8v_1(x,t) + v_2(x,t) + d_1(x,t) \\ \partial v_2(x,t)/\partial t &= \partial^2 v_2(x,t)/\partial x^2 + 2v_1(x,t) + 10v_2(x,t) + d_2(x,t) \end{aligned}, \quad (55)$$

subject to

$$\begin{aligned} \partial v_1(0,t)/\partial x = 0, \quad v_1(1,t) = u_1(t) + \theta_1\sigma_1(t) \\ \partial v_2(0,t)/\partial x = 0, \quad v_2(1,t) = u_2(t) + \theta_2\sigma_2(t) \end{aligned}, \quad (56)$$

where $v(x,t)$ is the system state representing the temperature of a heated rod at the position $x \in [0,1]$ with time $t \geq 0$, $d_1(x,t) = 0.05e^{-0.5(x-0.2)^2}\sin(t)$, $d_2(x,t) = 0.06e^{-0.3(x-0.4)^2}\sin(2t)$ are disturbances and $\sigma_i(t) = \begin{cases} u_i(t), \ if \ |u_i(t)| < \bar{u}_i \\ \bar{u}_i, \quad else \end{cases}$ with $\bar{u}_i = 100$ being the maximum value of the actuator output and $i = 1,2$. Given reference outputs as $v_{d1}(0,t) = 5(1-e^{-0.5t})$ and $v_{d2}(0,t) = 4(1-e^{-0.5t})$ the corresponding controller under healthy conditions can be obtained using (31). The actuator fault is seeded at $t = 5s$ with the fault parameters being defined by $\theta_1(t) = 1.1\Omega_1(t-5)$ and $\theta_2(t) = 0.8\Omega_2(t-5)$ where $\Omega_i(t-5)$ for $i = 1,2$ is defined as $\Omega_i(t-5) = \begin{cases} 0 & , if \ t < 5 \\ 1-e^{-\kappa_i(t-5)} & , if \ t \geq 5 \end{cases}$ with $\kappa_1 = 0.3$ and $\kappa_2 = 0.6$ representing fault growth rates.

For the simulation results using MATLAB, the closed-loop system and observer are discretized over the entire space $0 \leq x \leq 1$ by using the finite difference method with 20 point grid. Next the performance of the detection and accommodation scheme is evaluated on this example when only output is available.

4.1. ACTUATOR FAULT SCENARIO

The total time for simulation in MATLAB is 25 seconds and the time interval for solving system PDE and filters is 0.01 second. By combing the solution of input filter along with output filters, the estimated state under healthy conditions given by observer (27) can be obtained.

We assume that only the output temperature $v(0,t)$ is measured at $x = 0$. First, the DPS (55) and (56) should be converted to the observable form by applying the transformation (20). Then two output filters (25) and (26) along with one input filter (24) are employed to estimate states over the space and the output of the transformed system. Prior to the fault occurrence, the detection residual is expected to be decreasing, whereas it will increase once a fault occurs. It is clearly observed in Figure 4.1(a) that the residual between the output solution to system dynamics of (55)-(56) and the estimated output of (27) can reach a steady state in a short time, but once a fault is activated at $t = 5s$, the residual increase because of the behavior of the system state changes. Then the fault is detected about one second after initiation, when the detection residual exceeds the threshold.

Upon detecting the actuator fault, a fault filter is activated to learn the fault dynamics. In this case, we just have available measurement at $x = 0$, so the update law

(39) is utilized to estimate the fault parameter vector. The fault detection and estimation results can be seen from Figure 4.1.

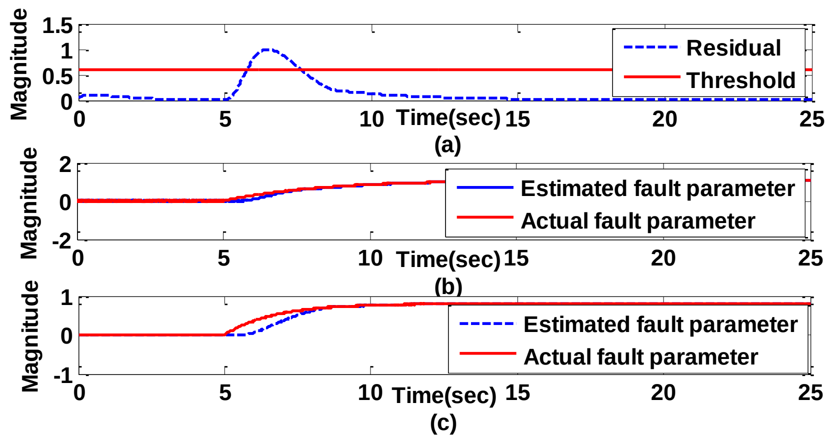
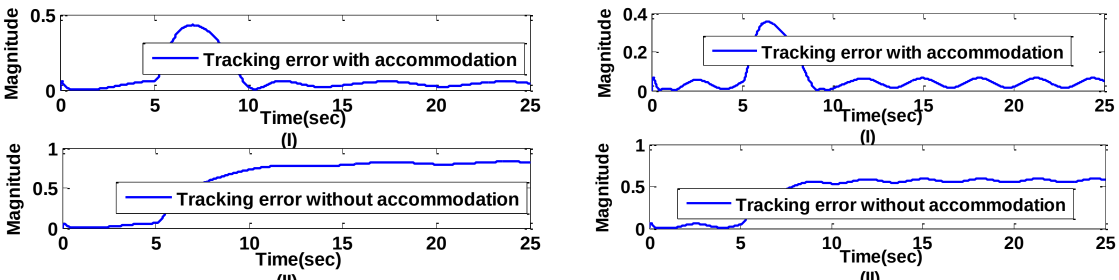


Figure 4.1. Output feedback of (a) residual; fault parameter of (b) θ_1 and (c) θ_2 .

Since the actuator fault will affect the controller of the actual system, the tracking error without accommodation will not decrease. However, if the fault tolerance controller is applied to the boundary condition, tracking error will first increase and then decrease once the adaptation is activated to estimate the fault dynamics. The comparison with and without accommodation results can be seen from Figure 4.2 (a) and (b).

By utilizing formula (50), and setting the limits as $\vartheta_1 = \vartheta_2 = 0.2$, the TTA can be estimated which can predict the time when the output tracking errors of the system shown in Figure 4.2 decrease below their limit values. The fault is considered being accommodated completely when both tracking errors approach below their limits as shown in Figure 4.3.



(a) (b)

Figure 4.2. Comparison with and without accommodation (a) $r_1(0,t)$; (b) $r_2(0,t)$.

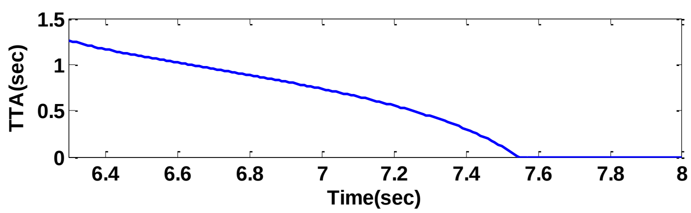


Figure 4.3. Estimated time-to-accommodation for an actuator fault.

4.2. SENSOR FAULT SCENARIO

As mentioned in Section 3.3, with output available a sensor fault can be dealt. The sensor fault is represented as

$$y_1(t) = v_1(0,t) + \theta_{s1}\sigma_1(t) \ , \ y_2(t) = v_2(0,t) + \theta_{s2}\sigma_2(t) \ ,$$

where $\sigma_1(t) = v_{d1}(0,t)$, $\sigma_2(t) = v_{d2}(0,t)$ and fault parameters are expressed as $\theta_{s1}(t) = 0.8\Omega_1(t-8)$ and $\theta_{s2}(t) = 1.2\Omega_2(t-8)$ where $\Omega_i(t-8)$ for $i = 1,2$ is defined as

$$\Omega_i(t-8) = \begin{cases} 0 & , if \ t < 8 \\ 1 - e^{-\kappa_i(t-5)} & , if \ t \geq 8 \end{cases}$$ with $\kappa_1 = 0.3$ and $\kappa_2 = 0.6$.

Here, the detection observer (27) is used to detect the fault. Figure 4.4(a) shows that the sensor fault can be detected in a short time. Once detecting of the fault, fault filters will be incorporated in the observer (44) with the update law (48) to approximate the fault parameter. Fault parameter estimation results are shown in Figure 4.4(b) and (c). In the presence of a sensor fault, outputs of the system will not track the reference output. Besides, the control input based on output feedback will increase the error between actual output and reference output. It can be observed from Figure 4.5 that in the occurrence of a sensor fault, the tracking error will increase immediately while if the control input (49) is utilized once detecting of the fault, the tracking error will be reduced.

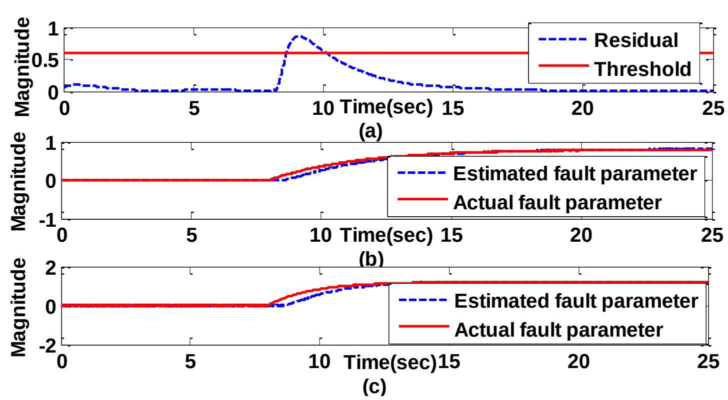
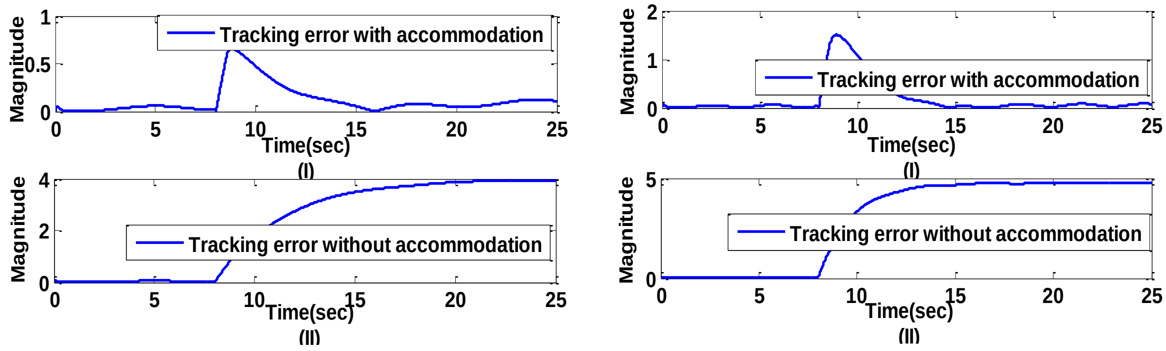


Figure 4.4. Output feedback of (a) residual; fault parameter of (b) θ_1 and (c) θ_2.



(a) (b)

Figure 4.5. Comparison with and without accommodation (a) $r_1(0,t)$; (b) $r_2(0,t)$.

Based on the dynamics of the tracking error and given the limit value of the tracking error as $\vartheta_1 = \vartheta_2 = -0.2$, we can estimate the TTA which can predict the time when those tracking errors are reduced below their limit values. Figure 4.6 plots the TTA by utilizing the proposed formula (50), it shows that fault can be accommodated completed within 4.5 seconds.

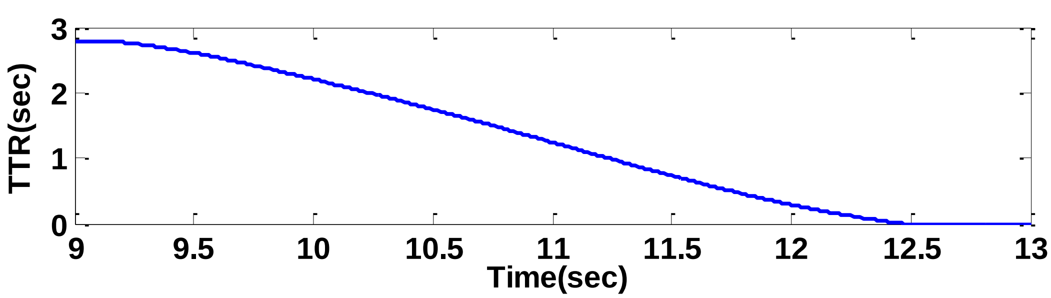


Figure 4.6. Estimated time-to-accommodation for a sensor fault.

5. CONCLUSIONS

In this paper, first we propose a filter-based detection observer using output measurement alone. Compared to ODE representation of DPS, our PDE-based observers provide a more accurate estimation of the state, which is beneficial to both fault detection and accommodation. Furthermore with the filter based observer, both actuator and sensor faults are accommodated provided they occur one at a time. Upon detection, the proposed adaptive estimator incorporated in the observer provides valuable information about the fault function in order to estimate the time-to-accommodation. The filter based approach is critical when dealing with the implementation on practical systems. The effectiveness of the fault accommodation is guaranteed by the Lyapunov analysis. Finally, the simulation results are included to verify the theoretical claims.

APPENDIX

Proof of the performance of the filter based controller (31) under healthy conditions: Select the Lyapunov candidate given by

$$V = \frac{1}{2\varepsilon}\left\|\tilde{z}(x,t)\right\|_{2,n}^{2} + \frac{1}{2\varepsilon k_h}\int_0^1\int_0^x w^T(\eta,t)w(\eta,t)d\eta dx \ , \tag{A.1}$$

and its derivative of V with respect to t can be obtained as

$$\dot{V} = \frac{1}{\varepsilon}\int_0^1 \tilde{z}^T(x,t)\tilde{z}_t(x,t)dx + \frac{1}{\varepsilon k_h}\int_0^1\int_0^x w^T(\eta,t)w_t(\eta,t)d\eta dx \ .$$

By substituting (28), (12), (33) and (34) to the equation above and applying the integration by parts, to get

$$\begin{aligned}
\dot{V} &= -\left\|\tilde{z}_x(x,t)\right\|_{2,n}^{2} + \frac{1}{\varepsilon}\int_0^1 \tilde{z}^T(x,t)\int_0^x l(x,\eta)d(v,\eta,t)d\eta dx + \frac{1}{\varepsilon}\int_0^1 \tilde{z}^T(x,t)d(v,x,t)dx \\
&+\int_0^1 w^T(x,t)w_x(x,t)dx/k_h + \int_0^1\int_0^x w^T(\eta,t)d(v,\eta,t)d\eta dx/(\varepsilon k_h) \\
&-\int_0^1\int_0^x w^T(\eta,t)\int_0^\eta K(\eta,\tau)d(v,\tau,t)d\tau d\eta dx/(\varepsilon k_h) \\
&-[\int_0^1\int_0^x w_\eta^T(\eta,t)w_\eta(\eta,t)d\eta dx + \int_0^1\int_0^x w^T(\eta,t)Cw(\eta,t)d\eta dx/\varepsilon]/k_h \\
&\le -(\pi^2-2)\left\|\tilde{z}(x,t)\right\|_{2,n}^{2}/4 - c\int_0^1\int_0^x w^T(\eta,t)w(\eta,t)d\eta dx/\varepsilon k_h \qquad , \\
&+\frac{d_l}{\varepsilon}\int_0^1\sqrt{\tilde{z}^T(x,t)\tilde{z}(x,t)}dx + \frac{d_k}{\varepsilon k_h}\int_0^1\int_0^x\sqrt{w^T(\eta,t)w(\eta,t)}d\eta dx \\
&\le -(\pi^2-4)\left\|\tilde{z}(x,t)\right\|_{2,n}^{2}/4 + d_l^2/2\varepsilon^2 + d_k^2/(4c\varepsilon k_h) - c\int_0^1\int_0^x w^T(\eta,t)w(\eta,t)d\eta dx/(2\varepsilon k_h)
\end{aligned}$$

where $d_l = (1+\bar{l})\bar{d}$, $d_k = (1+\bar{k})\bar{d}$ with $\bar{k} = \max_{0\le x\le 1}\left\|K(x,\tau)\right\|_2$ and $\bar{l} = \max_{0\le x\le 1}\left\|l(x,\eta)\right\|_2$.

Then $\dot{V} < 0$ when

$$\left\|\tilde{z}(x,t)\right\|_{2,n} > \sqrt{\frac{2d_l^2}{\varepsilon^2(\pi^2-4)} + \frac{d_k^2}{c\varepsilon k_h(\pi^2-4)}} \ or \int_0^1\int_0^x w^T(\eta,t)w(\eta,t)d\eta dx > \frac{k_h d_l^2}{\varepsilon c} + \frac{d_k^2}{2c^2} \ .$$

Therefore $\tilde{z}$ and $w(x,t)$ will be bounded. The boundedness of $\tilde{v}$ and r are also guaranteed because of (20) and the invertible of (9).

Proof of Theorem 1: This is an extension of the scalar case from [21]. To show the boundedness of observer and parameter estimation errors in the presence of fault, an error is first defined as $\mu(x,t) = z(x,t) - \bar{z}(x,t)$. It is clear that

$$\mu_t(x,t) = \varepsilon\mu_{xx}(x,t) + d(v,x,t) - \int_0^x l\left(x,\tau\right) d(v,\tau,t)d\tau,\ \mu_x(0,t) = 0,\ \mu(1,t) = 0. \tag{A.2}$$

Now select a Lyapunov function candidate in the form of

$$V = \left\|\mu(x,t)\right\|_{2,n}^2 / (2\varepsilon) + \tilde{\theta}_a^T(t)\tilde{\theta}_a(t) / (2\beta)\ , \tag{A.3}$$

which is positive definite. Then the derivative of the Lyapunov function with respect to time can be obtained as

$$\dot{V} = \int_0^1 \mu^T(x,t)\mu_t(x,t)dx / \varepsilon + \tilde{\theta}_a^T(t)\dot{\tilde{\theta}}_a(t) / \beta \quad .$$

Substituting the update law (39) in the above equation and notice that $e(t) = \tilde{z}(0,t) = \mu(0,t) + \Gamma(0,t)\tilde{\theta}_a\left(\mathrm{t}\right)$, results in

$$\begin{aligned}\dot{V} &\le -\left\|\mu_x(x,t)\right\|_{2,n}^2 - e^T(t)e(t)/2 + \mu^T(0,t)\mu(0,t)/2 \\ &\quad -\gamma[\tilde{\theta}_a^T(t)\tilde{\theta}_a(t) - \theta_{a\max}^2]/2\beta + d_l\int_0^1\sqrt{\mu^T(x,t)\mu(x,t)}dx/\varepsilon \\ &\le -(\pi^2-4)\left\|\mu(x,t)\right\|_{2,n}^2/8 - \gamma\tilde{\theta}_a^T(t)\tilde{\theta}_a(t)/(2\beta) + \gamma\theta_{a\max}^2/(2\beta) + d_l^2/(4\varepsilon^2)\end{aligned} \quad .$$

Therefore, $\dot{V}$ will be less than zero when

$$\left\|\mu\right\|_{2,n} > \sqrt{\frac{4\gamma\varepsilon^2\theta_{a\max}^2 + 2\beta d_l^2}{(\pi^2-4)\beta\varepsilon^2}}\ \text{ or } \left\|\tilde{\theta}_a(t)\right\| > \sqrt{\theta_{a\max}^2 + \frac{\beta d_l^2}{2\gamma\varepsilon^2}}\ . \tag{A.4}$$

It is shown that with the parameter tuning law (39), the derivative of this function will be less than zero if μ or $\tilde{\theta}_a$ stays in a bounded region. Note that since $\tilde{z}(x,t)=\mu(x,t)+\Gamma(x,t)\tilde{\theta}_a(\mathrm{t})$, the bound of the observer residual $\tilde{z}$ is guaranteed since $\Gamma(x,t)$ is bounded.

Proof of Theorem 2: Notice that with controller modified as (40) the boundary condition of $\bar{z}(x,t)$ stays the same as $z(x,t)$ satisfying $\bar{z}_x(0,t)=L_0 z(0,t),\ \bar{z}(1,t)=\hat{U}(t)+\Phi_a(U(t),t)\tilde{\theta}_a$, thus the dynamics of $\mu(x,t)$ is maintained as (A.2). Now select a Lyapunov function candidate as

$$V=\frac{R}{2\varepsilon}\|\mu(x,t)\|_{2,n}^2+\frac{\tilde{\theta}_a^T\tilde{\theta}_a}{2\beta}+\frac{\gamma}{4\varepsilon k_c}\int_0^1\int_0^x w^T(\eta,t)w(\eta,t)d\eta dx. \qquad \text{(A.5)}$$

By taking the derivative of the Lyapunov function with respect to time and applying integration by parts with (12) we can obtain

$$\begin{aligned}\dot{V}\le&-R\|\mu_x(x,t)\|_{2,n}^2+\frac{d_k}{2\varepsilon k_c}\int_0^1\int_0^x\sqrt{w^T(\eta,t)w(\eta,t)}d\eta dx+\frac{Rd_l}{\varepsilon}\int_0^1\sqrt{\mu^T(x,t)\mu(x,t)}dx\\&-\frac{\gamma c\int_0^1\int_0^x w^T(\eta,t)w(\eta,t)d\eta dx}{2\varepsilon k_c}+\frac{\gamma}{2k_c}\int_0^1\int_0^x w^T(\eta,t)w_{\eta\eta}(\eta,t)d\eta dx+\frac{1}{\beta}\tilde{\theta}_a^T(t)\dot{\tilde{\theta}}_a(t)\end{aligned}.$$

Substituting the parameter update law from (39) and applying Poincare inequality in the above equation to get

$$\begin{aligned}\dot{V}\le&-\frac{R\pi^2}{4}\|\mu(x,t)\|_{2,n}^2+\frac{Rd_l}{\varepsilon}\int_0^1\sqrt{\mu^T(x,t)\mu(x,t)}dx+\frac{d_k}{2\varepsilon k_c}\int_0^1\int_0^x\sqrt{w^T(\eta,t)w(\eta,t)}d\eta dx\\&-\tilde{\theta}_a^T(t)\Gamma(0,t)e(t)-\frac{\gamma c}{2\varepsilon k_c}\int_0^1\int_0^x w^T(\eta,t)w(\eta,t)d\eta dx\\&+\frac{\gamma\tilde{\theta}_a^T(t)\hat{\theta}_a(t)}{\beta}+\frac{\gamma}{2k_c}\int_0^1\int_0^x w^T(\eta,t)w_{\eta\eta}(\eta,t)d\eta dx.\end{aligned}$$

Because $\tilde{z}(0,t)-\mu(0,t)=\Gamma(0,t)\tilde{\theta}_a(\mathrm{t})$ and $\tilde{z}(0,t)=e(t)$, the derivative of Lyapunov function can be rewritten as

$$\dot{V} \le -\frac{R(\pi^2-2)}{4}\|\mu(x,t)\|_{2,n}^2 + \mu^T(0,t)e(t) + \frac{d_k^2}{4c\varepsilon k_h} + \frac{Rd_l^2}{2\varepsilon^2} - e^T(t)e(t)$$

$$-\frac{\gamma c}{4\varepsilon k_c}\int_0^1\int_0^x w^T(\eta,t)w(\eta,t)d\eta dx + \frac{\gamma}{2k_c}\int_0^1\int_0^x w^T(\eta,t)w_{\eta\eta}(\eta,t)d\eta dx + \frac{\gamma\tilde{\theta}_a^T(t)\hat{\theta}_a(t)}{\beta}$$

$$\le -\frac{R(\pi^2-2)}{4}\|\mu(x,t)\|_{2,n}^2 - \frac{\gamma c}{4\varepsilon k_c}\int_0^1\int_0^x w^T(\eta,t)w(\eta,t)d\eta dx - \frac{e^T(t)e(t)}{2} \quad .$$

$$+\frac{\mu^T(0,t)\mu(0,t)}{2} + \frac{\gamma\theta_{a\max}^2}{2\beta} + \frac{\gamma}{4k_c}\left[w^T(1,t)w(1,t) - w^T 0,t)w(0,t)\right]$$

$$+\frac{d_k^2}{4c\varepsilon k_h} + \frac{Rd_l^2}{2\varepsilon^2} - \frac{\gamma\tilde{\theta}_a^T(t)\tilde{\theta}_a(t)}{2\beta} - \frac{\gamma}{2k_c}\int_0^1\int_0^x w_\eta^T(\eta,t)w_\eta(\eta,t)d\eta dx$$

From (43) $w^T(1,t)w^T(1,t) \le k_c(\tilde{\theta}_a^T\tilde{\theta}_a + \|\mu\|_{2,n}^2)$, then

$$\dot{V} \le -\frac{R(\pi^2-2)-\gamma-2}{4}\|\mu(x,t)\|_{2,n}^2 - \frac{1}{2}e^T(t)e(t) - \frac{\gamma}{2\beta}\tilde{\theta}_a^T(t)\tilde{\theta}_a(t) + \frac{\gamma}{4}\tilde{\theta}_a^T(t)\tilde{\theta}_a(t)$$

$$-\frac{\gamma c}{4\varepsilon k_c}\int_0^1\int_0^x w^T(\eta,t)w(\eta,t)d\eta dx - \frac{\gamma}{2k_c}[\int_0^1\int_0^x w_\eta^T(\eta,t)w_\eta(\eta,t)d\eta dx + \frac{1}{2}w^T 0,t)w(0,t)] \quad .$$

$$+\frac{\gamma}{2\beta}\theta_{a\max}^2 + \frac{Rd_l^2}{2\varepsilon^2} + \frac{d_k^2}{4c\varepsilon k_h}$$

By applying Poincare inequality [24], we have $\int_0^1\int_0^x w^T(\eta,t)w(\eta,t)d\eta dx \le 2w^T 0,t)w(0,t) + 4\int_0^1\int_0^x w_\eta^T(\eta,t)w_\eta(\eta,t)d\eta dx$. Then the first derivative of the Lyapunov function becomes

$$\dot{V} \le -\frac{R(\pi^2-2)-\gamma-2}{4}\|\mu(x,t)\|_{2,n}^2 - \frac{\gamma(2-\beta)}{4\beta}\tilde{\theta}_a^T(t)\tilde{\theta}_a(t) + \frac{\gamma\theta_{a\max}^2}{2\beta} + \frac{Rd_l^2}{2\varepsilon^2} + \frac{d_k^2}{4c\varepsilon k_c}$$

$$-\gamma(\varepsilon+2c)\int_0^1\int_0^x w^T(\eta,t)w(\eta,t)d\eta dx/8\varepsilon k_c,$$

where $R > \frac{\gamma+2}{\pi^2-2}$ and $c = \lambda_{\min}(C)$. Therefore, the derivative of Lyapunov function will be less than zero when

$$\left\|\mu(x,t)\right\|_{2,n} > \sqrt{\frac{2\gamma\varepsilon^2 ck_c\theta_{a\max}^2 + 2R\beta ck_c d_l^2 + \beta\varepsilon}{\beta\varepsilon^2 ck_c[R(\pi^2-2)-\gamma-2]}}$$
$$or \left\|\tilde{\theta}_a(t)\right\| > \sqrt{\frac{2\theta_{a\max}^2}{2-\beta} + \frac{2R\beta d_l^2}{\gamma\varepsilon^2(2-\beta)} + \frac{\beta d_k^2}{\gamma\varepsilon ck_c(2-\beta)}} \quad .$$
$$or \int_0^1\int_0^x w^T(\eta,t)w(\eta,t)d\eta dx > \frac{4\varepsilon k_c\theta_{a\max}^2}{\beta(\varepsilon+2c)} + \frac{4Rk_c d_l^2}{\gamma\varepsilon^2(\varepsilon+2c)} + \frac{2d_k^2}{4\gamma(\varepsilon+2c)}$$

Hence, μ and $\tilde{\theta}_a$ are ultimately bounded with the bounds defined above. Since $\tilde{z}(x,t) = \mu(x,t) + \Gamma(x,t)\tilde{\theta}_a(t)$, $\tilde{z}$ is also bounded due to boundedness of $\Gamma(x,t)$.

So far we have shown the boundedness of $\int_0^1\int_0^x w^T(\eta,t)w(\eta,t)d\eta dx$ and because $w(x,t)$ is continuous in $x\in[0,1]$, the transformed tracking error $w(0,t)$ is also bounded. Now, given the transformation (9) we know that $w(0,t)=r(0,t)$, the boundedness of the tracking error $r(0,t)$ is ensured.

Proof of Theorem 3: This is an extension of [21] where only scalar actuator fault is considered. We have shown that under healthy condition the observer estimation error $\tilde{z}$ will converge. Note that $\hat{\theta}_s$ is initialized as $\hat{\theta}_s(0)=0$ and it will not be updated until the detection of a sensor fault. Now select a positive definite Lyapunov function candidate as

$$V = \left\|\mu(x,t)\right\|_{2,n}^2 / (2\varepsilon) + \tilde{\theta}_s^T\tilde{\theta}_s / (2\beta) \qquad \text{(A.6)}$$

With the update law (48) and the using fact that

$$e(t) = \tilde{z}(0,t) + \Phi_s(t)\tilde{\theta}_s(t) = \mu(0,t) - F(0,t)\tilde{\theta}_s(t) + \Phi_s(t)\tilde{\theta}_s(t),$$

the derivative of the Lyapunov function candidate is given by

$$\begin{aligned}\dot{V} &= -\left\|\mu_x(x,t)\right\|_{2,n}^2 - \tilde{\theta}_s^T(t)[\Phi_s(t) - F(0,t)]^T e(t) \\ &\quad + \gamma\tilde{\theta}_s^T\hat{\theta}_s/\beta + d_l\int_0^1\sqrt{\mu^T(x,t)\mu(x,t)}dx/\varepsilon \\ &\le -(\pi^2-4)\left\|\mu(x,t)\right\|_{2,n}^2/8 - \gamma\tilde{\theta}_s^T\tilde{\theta}_s/(2\beta) + \gamma\theta_{s\max}^2/(2\beta) + d_l^2/(4\varepsilon^2)\end{aligned} \quad .$$

Then, $\dot{V} < 0$ when

$$\|\mu(x,t)\|_{2,n} > \frac{4\gamma\varepsilon^2\theta_{s\max}^2 + 2\beta d_l^2}{(\pi^2-4)\beta\varepsilon^2} \; or \; \|\tilde{\theta}_s(t)\| > \sqrt{\theta_{s\max}^2 + \frac{\beta d_l^2}{2\gamma\varepsilon^2}} \,. \tag{A.7}$$

Therefore, μ and $\tilde{\theta}_s$ are ultimately bounded with the bounds defined above. Since $\tilde{z}(x,t) = \mu(x,t) - F(x,t)\tilde{\theta}_s(t)$, $\tilde{z}$ is also bounded because of the boundedness of $F(x,t)$ and thus $\tilde{v}$ is bounded due to (30).

Proof of Theorem 4: The dynamics of state tracking error $r(x,t)$ can be obtained as (7)-(8) and (32). Apply transformation (9) to (7)-(8) and (32), it leads to (12) and (33).

Now select a Lyapunov function candidate as

$$V = \frac{R}{2\varepsilon}\|\mu(x,t)\|_{2,n}^2 + \frac{\tilde{\theta}_s^T\tilde{\theta}_s}{2\beta} + \frac{\gamma}{4\varepsilon k_s}\int_0^1\int_0^x w^T(\eta,t)w(\eta,t)d\eta dx \,. \tag{A.8}$$

By taking the derivative of the above with respect to time we will arrive at

$$\dot{V} = \frac{R}{\varepsilon}\int_0^1 \mu^T(x,t)\mu_t(x,t)dx + \frac{\tilde{\theta}_s^T(t)\dot{\tilde{\theta}}_s(t)}{\beta} + \frac{\gamma}{4\varepsilon k_s}\int_0^1\int_0^x w^T(\eta,t)w_t(\eta,t)d\eta dx \,.$$

Substituting (45), (12) and the update law (48) in the equation above and applying integration by parts

$$\dot{V} \le -\frac{R(\pi^2-2)}{4}\|\mu(x,t)\|_{2,n}^2 + \mu^T(0,t)e(t) + \frac{d_k^2}{4c\varepsilon k_h} + \frac{Rd_l^2}{2\varepsilon^2} - e^T(t)e(t)$$
$$-\frac{\gamma c}{4\varepsilon k_c}\int_0^1\int_0^x w^T(\eta,t)w(\eta,t)d\eta dx + \frac{\gamma}{2k_c}\int_0^1\int_0^x w^T(\eta,t)w_{\eta\eta}(\eta,t)d\eta dx + \frac{\gamma\tilde{\theta}_s(t)\hat{\theta}_s(t)}{\beta}$$
$$\le -\frac{R(\pi^2-2)}{4}\|\mu(x,t)\|_{2,n}^2 - \frac{\gamma c}{4\varepsilon k_c}\int_0^1\int_0^x w^T(\eta,t)w(\eta,t)d\eta dx - \frac{e^T(t)e(t)}{2}$$
$$+\frac{\gamma}{4k_c}\left[w^T(1,t)w(1,t) - w^T 0,t)w(0,t)\right] - \frac{\gamma}{2k_c}\int_0^1\int_0^x w_\eta^T(\eta,t)w_\eta(\eta,t)d\eta dx$$
$$+\frac{\mu^T(0,t)\mu(0,t)}{2} + \frac{\gamma\theta_{a\max}^2}{2\beta} + \frac{d_k^2}{4c\varepsilon k_h} + \frac{Rd_l^2}{2\varepsilon^2} - \frac{\gamma\tilde{\theta}_s^T(t)\tilde{\theta}_s(t)}{2\beta}$$

From (34) and (47) we can see that $w^T(1,t)w(1,t) \le \bar{k}\,\|\tilde{v}\|_{2,n}^2 \le k_h \|\tilde{z}\|_{2,n}^2 \le k_s[\tilde{\theta}_s^T\tilde{\theta}_s + \|\mu\|_{2,n}^2]$,

where $k_s = \max\{2k_h, 2k_h\bar{f}\}$ and $\bar{f} = \max_{0\le x\le 1}\|F(x,t)\|_2$, it leads to

$$\begin{aligned}\dot{V} \le &-\frac{R(\pi^2-2)-\gamma-2}{4}\|\mu(x,t)\|_{2,n}^2 - \frac{1}{2}e^T(t)e(t) - \frac{\gamma}{2\beta}\tilde{\theta}_s^T(t)\tilde{\theta}_s(t) + \frac{\gamma}{4}\tilde{\theta}_s^T(t)\tilde{\theta}_s(t) + \frac{d_k^2}{4c\varepsilon k_h} \\ &-\frac{\gamma c}{4\varepsilon k_c}\int_0^1\int_0^x w^T(\eta,t)w(\eta,t)d\eta dx + \frac{\gamma}{2\beta}\theta_{a\max}^2 + \frac{Rd_l^2}{2\varepsilon^2} \\ &-\frac{\gamma}{2k_c}[\int_0^1\int_0^x w_\eta^T(\eta,t)w_\eta(\eta,t)d\eta dx + \frac{1}{2}w^T 0,t)w(0,t)]\end{aligned}.$$

Then, apply Poincare inequality [24] to arrive at

$$\begin{aligned}\dot{V} \le &-\frac{R(\pi^2-2)-\gamma-2}{4}\|\mu(x,t)\|_{2,n}^2 - \frac{\gamma(2-\beta)}{4\beta}\tilde{\theta}_s^T(t)\tilde{\theta}_s(t) \\ &-\frac{\gamma(\varepsilon+2c)}{8\varepsilon k_c}\int_0^1\int_0^x w^T(\eta,t)w(\eta,t)d\eta dx + \frac{\gamma\theta_{s\max}^2}{2\beta} + \frac{Rd_l^2}{2\varepsilon^2} + \frac{d_k^2}{4c\varepsilon k_c}\end{aligned}.$$

Therefore, $\dot{V} < 0$ if $R > \dfrac{\gamma+2}{\pi^2-2}$ and

$$\begin{aligned}&\|\mu(x,t)\|_{2,n} > \sqrt{\frac{2\gamma\varepsilon^2ck_c\theta_{s\max}^2 + 2R\beta ck_cd_l^2 + \beta\varepsilon}{\beta\varepsilon^2ck_c[R(\pi^2-2)-\gamma-2]}} \\ &or\ \|\tilde{\theta}_s(t)\| > \sqrt{\frac{2\theta_{s\max}^2}{2-\beta} + \frac{2R\beta d_l^2}{\gamma\varepsilon^2(2-\beta)} + \frac{\beta d_k^2}{\gamma\varepsilon ck_c(2-\beta)}} \\ &or\ \int_0^1\int_0^x w^T(\eta,t)w(\eta,t)d\eta dx > \frac{4\varepsilon k_c\theta_{s\max}^2}{\beta(\varepsilon+2c)} + \frac{4Rk_cd_l^2}{\gamma\varepsilon^2(\varepsilon+2c)} + \frac{2d_k^2}{4\gamma(\varepsilon+2c)}\end{aligned}. \qquad (A.9)$$

Thus, μ and $\tilde{\theta}_s$ are ultimately bounded. Next $\tilde{z}$ is also bounded since $\tilde{z}(x,t) = \mu(x,t) - \mathrm{F}(x,t)\tilde{\theta}_s(t)$ and $F(x,t)$ is bounded. It has been shown that $\int_0^1\int_0^x w^T(\eta,t)w(\eta,t)d\eta dx$ and $w(x,t)$ is continuous in $x\in[0,1]$, so $w(x,t)$ is bounded. Then we know that $r(0,t)$ is also bounded because $r(0,t) = w(0,t)$ from equation (9).

REFERENCES

[1] G. G. Yen, and L. W. Ho, Online multiple-model-based fault diagnosis and accommodation. Industrial Electronics, IEEE Transactions on 50, 2 (2003), pp. 296-312.

[2] P. M. Frank, Fault diagnosis in dynamic systems using analytical and knowledge-based redundancy: A survey and some new results. Automatica 26, 3 (1990), pp. 459-474.

[3] J. J. Gertler, Survey of model-based failure detection and isolation in complex plants. Control Systems Magazine, IEEE 8, 6 (1988), pp. 3-11.

[4] J. Gertler, Fault detection and diagnosis in engineering systems. CRC press, 1998.

[5] F. Woittennek and J. Rudolph, Controller canonical forms and flatness-based state feedback for 1D hyperbolic systems. In 7th Vienna International Conference on Mathematical Modelling, MATHMOD, 2012.

[6] C. Bertoglio, D. Chapelle, M. A. Fernández, J. F. Gerbeau and P. Moireau, State observers of a vascular fluid–structure interaction model through measurements in the solid. Computer Methods in Applied Mechanics and Engineering, 256 (2013), pp. 149-168.

[7] S. Sallberg, P. S. Maybeck, and M. E. Oxley, Infinite-dimensional sampled-data Kalman filtering and the stochastic heat equation. Proc. of the Decision and Control, 2010, December, pp. 5062-5067.

[8] D. Yu, and S. Chakravotry, A randomly perturbed iterative proper orthogonal decomposition technique for filtering distributed parameter systems. In American Control Conference, Montreal, Canada, 2012.

[9] M. Patan and D. Ucinski, Optimal activation strategy of discrete scanning sensors for fault detection in distributed-parameter systems. Proc. of the 16th IFAC world congress, Prague, Czech Republic, 2005, pp. 4-8.

[10] H. Baruh, Actuator failure detection in the control of distributed systems. Journal of Guidance, Control and Dynamics, 2 (1986), pp. 181-189.

[11] M. Demetriou, K. Ito and R. C. Smith, Adaptive monitoring and accommodation of nonlinear actuator faults in positive real infinite dimensional systems. Automatic Control, IEEE Transactions on 52, 12 (2007), 2332-2338.

[12] N. H. El- Farra and S. Ghantasala, Actuator fault isolation and reconfiguration in transport-reaction processes. AIChE Journal 53, 6 (2007), pp. 1518-1537.

[13] S. Ghantasala and N. H. El-Farra, Model-based fault isolation and reconfigurable control of transport-reaction processes with actuator faults. Proc. ff the American Control Conference, 2007, pp. 2066-2071.

[14] P. Mhaskar, A. Gani, N. H. El- Farra, C. McFall, P. D. Christofides and J. F., Davis, Integrated fault- detection and fault- tolerant control of process systems. AIChE Journal 52, 6 (2006), pp. 2129-2148.

[15] M. Demetriou, Ito, K. and R. C. Smith, On-line monitoring and accommodation of nonlinear actuator faults in positive real infinite dimensional systems. Proc. of the IEEE Conference on Decision and Control, 3 (2004), pp. 2871-2875.

[16] Z. Yao and N. H. El-Farra, Robust fault detection and reconfiguration in sampled-data uncertain distributed processes. Proc. CDC, Orlando, FL, 2011, pp. 4925-4930.

[17] A. Baniamerian and K. Khorasani, Fault detection and isolation of dissipative parabolic PDEs: Finite-dimensional geometric approach. Proc. of the American Control Conference, Montreal, QC, 2012, pp. 5894-5899.

[18] M. Krstic and A. Smyshlyaev, Boundary control of PDEs: A course on backstepping designs. SIAM, 2008.

[19] A. Baccoli, Y. Orlov and A. Pisano, On the boundary control of coupled reaction-diffusion equations having the same diffusivity parameters. Proc. of the IEEE Conference on Decision and Control, Los Angeles, US, 2014, pp. 5222-5228.

[20] J. Sarangapani, Neural network control of nonlinear discrete-time systems. CRC Press, 2006. .

[21] J. Cai, H. Ferdowsi and S. Jagannathan, Model-based fault diagnosis and prediction for a class of distributed parameter systems. Proc. of the IEEE Cobference on Decision and Control, 2014, pp. 5758-5763.

[22] A. Smyshlyaev and M. Krstic, Backstepping observers for a class of parabolic PDEs. Systems & Control Letters 54, 7 (2005), pp. 613-625.

[23] J. Cai, H. Ferdowsi and S. Jagannathan, Model-Based Actuator Fault Accommodation for Distributed Parameter Systems Represented by Coupled Linear PDEs. Proc. of the IEEE Multi-Conference on Systems and Control, to appear in 2015.

[24] G. H. Hardy, J. E. Littlewood and G. Pólya, Inequalities. Cambridge university press, 1952.

III. FAULT DIAGNOSIS IN DISTRIBUTED PARAMETER SYSTEMS MODELED BY LINEAR AND NONLINEAR PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS

Jia Cai and S. Jagannathan

This paper covers model-based fault detection and isolation for linear and nonlinear distributed parameter systems (DPS). The first part mainly deals with actuator, sensor and state fault detection and isolation for a class of DPS represented by a set of coupled linear partial differential equations (PDE). A filter based observer is designed based on the linear PDE representation using which a detection residual is generated. A fault is detected when the magnitude of the detection residual exceeds a detection threshold. Upon detection, several isolation estimators are designed whose output residuals are compared with predefined isolation thresholds. A fault is declared to be of certain type if the corresponding isolation estimator output residual is below its isolation threshold while the other fault isolation estimator output residual is above its threshold. Next, the fault location is determined when a state fault is identified. The second part of this paper revisits fault detection and isolation of nonlinear DPS by using a Luenberger type observer. Here fault isolation framework is introduced to isolate actuator, sensor and state faults with isolability condition by using additional boundary measurements. Finally, the effectiveness of the proposed fault detection and isolation schemes for both linear and nonlinear DPS are demonstrated through simulation. Keywords—Fault detection, isolation, linear and nonlinear partial differential equation systems

1. INTRODUCTION

Generally in order to increase system availability and reliability, fault diagnosis has drawn significant attention in the area of modern control systems. Usually fault diagnosis consists of [1] (a) detection- to indicate the presence of a fault; (b) isolation- to determine the root cause and location of a fault; and (c) identification- to estimate the magnitude of a fault function. Fault isolation is a crucial step in fault diagnosis.

A variety of fault diagnosis approaches have been studied in the past two decades and of them, model-based methods [2] have found appealing since significant amount of healthy and faulty data is no longer required. Model-based fault detection and isolation methods have been developed for lumped parameter systems (LPS) represented by ordinary differential equations (ODEs) by using adaptive observer [3] sliding mode [4] and fuzzy observers [5]. Despite the comprehensive effort, they [3-5] are only applicable for LPS.

However, many fluid flow systems, thermal convection and spatially distributed chemical reaction-based systems are characterized as distributed parameter systems (DPS) or infinite dimensional systems. Because of their distributed nature, the ODE representation cannot describe the DPS behavior [6] and they are usually modeled by partial differential equations (PDEs). Fault diagnosis of DPS is more complicated and challenging when compared to LPS since the system parameters are defined over a continuous range of both time and space [7].

In the early effort, the DPS is approximated by finite dimensional ODE using Gelenkin's method [8] by assuming that the DPS is dominated by finite dimensional system with slow eigenvalues [9]. Subsequently, several articles appeared in the literature including an actuator failure detection method for DPS by identifying the actuator input [10]. An adaptive observer is developed in [11] to monitor the distributed parameter system and to provide information for the diagnosis of actuator faults. A geometric fault diagnosis approach, on the other hand, is introduced in [12] by approximating the PDE representation with a finite dimensional ODE. Despite these attractive results [8-12], the fault detection and isolation of DPS based on approximated finite dimensional ODE can lead to an inaccurate model description and thus can result in false or missed alarms due to incorrect isolation.

Motivated by the model reduction concerns, authors derived a fault detection and isolation (FDI) scheme based on PDE representation of linear DPS in [13]. Unlike [8-12], authors [13] use an infinite dimensional adaptive observer to detect faults. In order to monitor system behavior, a detection residual signal, which is defined as the difference between the actual and estimated output of the observer, was generated. In the absence of a fault, this detection residual remains below a predefined detection threshold. A fault acts as an unwanted input to the detection residual dynamics and increasing it. A fault is declared active when this residual crosses the detection threshold. However, detectability condition for state faults and isolation is not covered [13].

Therefore, this paper extends the fault detection and prediction framework from [13] to fault isolation by utilizing the PDE representation of linear DPS. First, the detectability condition of state faults is introduced. Upon detection by using the detection observer from [13], actuator and sensor fault isolation estimators are developed to identify the fault type when the output residual of the corresponding fault isolation estimator is below a predefined isolation threshold while the output residual of the other fault isolation estimator is above its threshold. In the event that the fault type is not an actuator and sensor, several state fault estimators located over the space are introduced to help determine the location of the state fault by using a second output measurement-spatial average over the sensed region. Several state fault isolation estimator residuals at different locations are derived and the one that is the minimum among them will determine the location of a potential state fault. Next, the magnitude of the fault parameter vectors are estimated upon fault identification for actuator, sensor and state faults.

In the case of a nonlinear DPS, a Luenberger type observer from [14] is used for fault detection in the presence of bounded disturbances. For nonlinear DPS, due to lack of fault filters, isolation estimators cannot be derived and additional measurements are needed for fault isolation. By using additional measurements at the boundary conditions and estimated output of the detection observer, an actuator/sensor isolation residual is generated.

When the actuator/sensor isolation residual exceeds its isolation threshold, the corresponding fault is isolated and if neither of them does, a state fault is considered to have occurred. Next, the isolability conditions are introduced to define the class of faults which can be isolated using the proposed scheme.

In the analysis, it is shown that the proposed observer can estimate measured and unmeasured system parameters satisfactorily under healthy condition with limited output measurements. The main contribution of this paper includes the development of: (a) a novel model-based fault isolation and location determination scheme for linear DPS with actuator, sensor and state faults; (b) fault isolable condition for faults in linear DPS and (c) a fault detection and isolation framework for nonlinear DPS with actuator, sensor and state faults with isolable conditions.

The paper is arranged as follows. First of all, a class of DPS represented by linear parabolic PDE with actuator, sensor and state faults is presented in Section 2. A fault isolation scheme is introduced for linear DPS in Section 3. Then fault detection and isolation of nonlinear DPS is discussed in Section 4. Finally, the proposed schemes are demonstrated in simulation in Section 5.

2. NOTATION AND LINEAR SYSTEM DESCRIPTION

Before introducing the system description, the notation is briefly introduced [15]. A scalar function $v_1(x) \in L_2(0,1)$ implies it is square integrable on the Hilbert space $L_2(0,1)$ with its corresponding norm defined by

$$\|v_1\|_2 = \sqrt{\int_0^1 v_1^2(x)dx} \quad . \tag{1}$$

Now consider

$$[L_2(0,1)]^n = \underbrace{L_2(0,1) \times L_2(0,1) \times ... \times L_2(0,1)}_{n\ times} \; , \tag{2}$$

with $v(x,t) = [v_1(x,t), v_2(x,t), ..., v_n(x,t)]^T \in [L_2(0,1)]^n$ and the norm of a vector function is defined as

$$\|v\|_{2,n} = \sqrt{\sum_{i=1}^{n} \|v_i\|_2^2} = \sqrt{\int_0^1 v^T(x)v(x)dx} \quad . \tag{3}$$

In addition, $\|\cdot\|$ denotes a Frobenius norm for a matrix or Euclidean norm for a vector. For sake of saving space, a vector, $v(x,t)$ and its partial derivatives are represented as

$$v_t(x,t) = \partial v(x,t) / \partial t \text{ , } v_x(x,t) = \partial v(x,t) / \partial x \text{, and } v_{xx}(x,t) = \partial^2 v(x,t) / \partial x^2 \; .$$

2.1. LINEAR SYSTEM DESCRIPTION

Consider a class of linear DPS expressed by the following parabolic PDE with Dirichlet actuation expressed as

$$v_t(x,t) = \varepsilon v_{xx}(x,t) + \Lambda v(x,t) + d(x,t) \; , \tag{4}$$

where $x \in [0,1]$ is the space variable and $t \geq 0$ is the time variable with boundary conditions defined by

$$v_x(0,t)=0, \quad v(1,t)=U(t) \ , \tag{5}$$

for $x \in (0,1)$ and $t \geq 0$, where $v(x,t)=[v_1(x,t),...,v_n(x,t)]^T \in [L_2(0,1)]^n$ is the state vector of the DPS, $d(x,t)$ is a bounded disturbance vector, $U(t)=[u_1(x,t),...,u_n(x,t)]^T \in \Re^n$ denotes the control input vector, ε is a positive constant, $\Lambda \in \Re^{n\times n}$ is a real valued square matrix, and $y(t) \in \Re^n$ is the system output given by

$$y(t)=v(0,t) \ . \tag{6}$$

A second output will be utilized for location determination of a state fault and it is expressed as

$$y_i(t)=\int_0^1 C(x)v(x)dx \ , \tag{7}$$

where $C(x) \in \Re^{n\times n}$ with $\int_0^1 \|C(x)\|^2 dx \leq \bar{c}^2$ is a known function.

Remark 1: The output defined in (6) is an ideal point sensor and the output given by (7) represents a spatial weighting function of sensors which is a spatial average over the sensed region [18]. The output equation (7) is required only for location determination when a state fault is identified. Next, the fault description is defined.

2.2. FAULT DESCRIPTION FOR LINEAR DPS

The DPS (4) with a state fault is described as

$$v_t(x,t)=\varepsilon v_{xx}(x,t)+\Lambda v(x,t)+d(x,t)+h_c(y,x_f,x,t), \tag{8}$$

and the boundary conditions with actuator and sensor faults can be written as

$$v_x(0,t)=0, \ v(1,t)=U(t)+h_a(t), \tag{9}$$

and

$$y(t) = v(0,t) + h_s(t), \tag{10}$$

where x_f is the location of a state fault, h_c , h_a and h_s represent state, actuator and sensor fault functions respectively. The fault functions are described by

$$\begin{aligned} &h_a(t) = \Omega(t-t_f)\Phi_a\left(U(t),t\right)\theta_a, h_s(t) = \Omega(t-t_f)\Phi_s(t)\theta_s \\ &h_c\left(y, x_f, x, t\right) = \Omega(t-t_f)\Phi_c\left(y, x, t\right)\Delta(x-x_f)\theta_c \end{aligned}, \tag{11}$$

where t_f represents the time when a fault occurs, $\theta_a \in \Re^n$, $\theta_s \in \Re^n$ and $\theta_c \in \Re^n$ are the unknown actuator, sensor and state fault parameter magnitude vector, respectively, with $\Phi_a(U(t),t) = diag(\sigma_i^{(a)}(U(t),t)) \in \Re^{n\times n}$ is an actuator fault basis function, $\Phi_s(t) = diag[\sigma_i^{(s)}(t)] \in \Re^{n\times n}$ denotes a sensor fault basis function, $\Delta(x-x_f) = diag[\delta_i^{(c)}(x-x_f)] \in \Re^{n\times n}$ determines the location of the state fault, and $\Phi_c(t) = diag[\sigma_i^{(c)}(y,x,t)] \in \Re^{n\times n}$ is a state fault basis function.

The term $\Omega(t-t_f) = diag[\Omega_i(t-t_f)], i = 1, 2, ...n$ represents the time profile of the fault defined by $\Omega_i(\tau) = \begin{cases} 0 & , if\ \tau < 0 \\ 1-e^{-\kappa_i \tau} & , if\ \tau \geq 0 \end{cases}$ with constant κ_i denoting the growth rate of an incipient fault. Abrupt faults can be represented with large κ_i . The following standard assumptions are required in order to proceed.

Assumption 1: The disturbance vector is bounded above such that $\|d(x,t)\| \leq \bar{d}$ for all x and $t \geq 0$, where $\bar{d} > 0$ is a known constant. A general form is given in this paper and a more specific representation is found in [11].

Remark 2: The upper bound of the disturbance $\bar{d}$ is required to determine the fault detection threshold.

Assumption 2: The magnitude of the fault parameter vector is considered unknown but assumed to belong to a known compact set Θ_N (i.e. $\theta_N \in \Theta_N \subset \Re^n$, $N = a, s, c$ where a, s, and c denote actuator, sensor and state faults respectively), Θ_a represents an actuator fault, Θ_s represents a sensor fault and Θ_c stands for a state fault, $\sigma_i^{(N)}$ is a known smooth function with $\sigma_i^{(a)}$ representing an actuator fault, $\sigma_i^{(s)}$ represents a sensor fault and $\sigma_i^{(c)}$ stands for a state fault.

Remark 3: This assumption is needed to assist in selecting isolation thresholds.

Assumption 3: Sensor, actuator or state fault types are considered and only a single fault occurs at a given time.

Assumption 4: For the sake of isolating the actuator, sensor, and state faults, it is assumed that the DPS functions longer than the isolation time t_i.

Assumption 5: The fault functions are considered bounded.

Next a filter-based detection observer is revisited from [13] to monitor the linear DPS and generate the detection residual.

3. FAULT DETECTION AND ISOLATION FOR LINEAR DPS

A fault detection scheme for state fault and isolation framework will be introduced for linear DPS in this section. In order to detect unexpected faults, an observer acting as a model under healthy conditions is utilized to monitor system behavior. A fault causes the residual to increase beyond a detection threshold indicating the presence of a fault. Upon detection, a fault isolation scheme is subsequently applied to differentiate the actuator, sensor and state faults. The location will be determined if a fault is identified as a state fault.

3.1. DETECTION OBSERVER DESIGN

A filter-based observer was designed utilizing an input and a couple of output filters based on an observable form under healthy conditions. The filter-based observer relaxes the need for state vector measurements over the range of space. Next, the detection residual was generated by comparing the estimated outputs from the observer with measured outputs. Since only the output $y(t)=v(0,t)$ is available, the DPS from (4) and (5) is first converted into the observable form by using the transformation [16] given by

$$z(x,t)=v(x,t)-\int_0^x l(x,\tau)v(\tau,t)\,d\tau\ , \tag{12}$$

where $l(x,\tau)$ is the solution to the hyperbolic PDE satisfying $l_{xx}-l_{\tau\tau}=l(x,\tau)\Lambda/\varepsilon$, $l(1,\tau)=0$ and $l(x,x)=\Lambda(1-x)/(2\varepsilon)$. The following observable form

$$z_t(x,t)=\varepsilon z_{xx}(x,t)+G(x)z(0,t)+d_l(x,t)\ , \tag{13}$$

$$z_x(0,t)=L_0 z(0,t),\ z(1,t)=U(t)\ , \tag{14}$$

$$y(t) = z(0,t) \ , \tag{15}$$

is obtained where $L_0 = -\Lambda / (2\varepsilon)$, $G(x) = -\varepsilon l_\tau(x,0)$ and $d_l(x,t) = d(x,t) - \int_0^x d(\tau,t) l\left(x,\tau\right) d\tau$ is bounded since $d(x,t)$ and $l(x,\tau)$ are bounded. Notice $z(0,t)$ is available since $v(0,t) = z(0,t)$. This transformation prevents the unstable term, $\Lambda v(x,t)$, from appearing in the design of filters which are described next.

The system model given by (13) and (14) is a linear PDE with $G(x)z(0,t)$, $L_0 z(0,t)$ and $U(t)$ viewed as external inputs. According to superposition principle, its solution can be expressed by summing the response of the PDE due to each external input [16] considered individually. Therefore, $z(x,t) \in \Re^n$ can be represented by a combination of the solution defined by

$$\Xi_t(x,t) = \varepsilon \Xi_{xx}(x,t), \ \Xi_x(0,t) = 0, \ \Xi(1,t) = U(t), \tag{16}$$

where $\Xi(x,t)$ is denoted as an input filter, since it is derived from the input of the actual system $U(t)$ [16]

Then consider

$$A_t(x,t) = \varepsilon A_{xx}(x,t), A_x(0,t) = y(t), \ A(1,t) = 0 , \tag{17}$$

where $A(x,t)$ is an output filter since it is derived from output of the actual system $y(t)$. It is also important to consider

$$\Pi_t(x,\eta,t) = \varepsilon \Pi_{xx}(x,\eta,t) + \delta(x-\eta) y(t), \qquad \Pi_x(0,\eta,t) = 0, \qquad \Pi(1,\eta,t) = 0 \ , \tag{18}$$

where $\Pi(x,\eta,t)$ is a second output filter. Therefore, the observer with its state, $\hat{z}(x,t) \in \Re^n$, is defined as

$$\hat{z}(x,t) = \Xi(x,t) + L_0 \mathrm{A}(x,t) + \int_0^1 G(s) \Pi(x,s,t) ds \, . \tag{19}$$

The estimated output and detection residual are given by

$$\hat{y}(t) = \hat{z}(0,t), \text{ and } e(t) = y(t) - \hat{y}(t) .$$

The dynamics of the observer error $\tilde{z}(x,t) \in \Re^n = z(x,t) - \hat{z}(x,t)$ under healthy condition satisfies

$$\tilde{z}_t(x,t) = \varepsilon \tilde{z}_{xx}(x,t) + d_l(x,t), \tilde{z}_x(0,t) = 0, \tilde{z}(1,t) = 0 \ . \tag{20}$$

The detectability condition for the state fault is given next while the fault detection framework, and detectability condition for actuator and sensor faults are reported in [13]. In the presence of a state fault, the system dynamics are modified as (8) with boundary conditions given by (5). Take the partial derivative of the transformation (12) with respect to t as

$$z_t(x,t) = v_t(x,t) - \int_0^x l(x,\tau) v_t(\tau,t) d\tau ,$$

Substitute the dynamics given by (8) to the equation above and apply integration by parts to get

$$\begin{aligned}
&z_t(x,t) = \varepsilon v_{xx}(x,t) - \int_0^x l(x,\tau)[\varepsilon v_{\tau\tau}(\tau,t) + \Lambda v(\tau,t)]d\tau + \lambda v(x,t) \\
&+d(x,t) + \int_0^x l(x,\tau) d(\tau,t) d\tau + h_c(y,x,x_f,t) - \int_0^x l(x,\tau) h_c(y,\tau,x_f,t) d\tau \\
&= \varepsilon v_{xx}(x,t) + \Lambda v(x,t) + \varepsilon l(x,0) v_x(0,t) - \varepsilon l_x(x,x) v(x,t) \\
&+\varepsilon l_\tau(x,x) v(x,t) - \varepsilon l_\tau(x,0) v(0,t) - \varepsilon \int_0^x l_{\tau\tau}(x,\tau) v(\tau,t) d\tau \\
&-\int_0^x l(x,\tau) \Lambda v(\tau,t) d\tau + d_l(x,t) + h_c(y,x,x_f,t) - \int_0^x l(x,\tau) h_c(y,\tau,x_f,t) d\tau.
\end{aligned} \tag{21}$$

Differentiating the transformation given by (12) with respect to x we can get the derivative of $z_x(x,t)$ as

$$z_x(x,t) = v_x(x,t) - l(x,x) v(x,t) - \int_0^x l_x(x,\tau) v(\tau,t) d\tau, \tag{22}$$

$$z_{xx}(x,t) = v_{xx}\left(x,t\right) - \frac{dl(x,x)}{dx}v(x,t) - l(x,x)v_x(x,t)$$
$$-l_x(x,x)v(x,t) - \int_0^x l_{xx}\left(x,\tau\right)v\left(\tau,t\right)d\tau. \quad (23)$$

Subtracting $\varepsilon \times (23)$ from (21) and applying the dynamics (8) yields

$$z_t(x,t) - \varepsilon z_{xx}(x,t)$$
$$= \left[\Lambda - 2\varepsilon\frac{dl(x,x)}{dx}\right]v(x,t) - \varepsilon l_\tau(x,0)v(0,t) + d_l(x,t) + h_c(y,x,x_f,t)$$
$$-\int_0^x l(x,\tau)h_c(y,\tau,x_f,t)d\tau + \int_0^x \left[\varepsilon l_{xx}\left(x,\tau\right) - \varepsilon l_{\tau\tau}\left(x,\tau\right) - l\left(x,\tau\right)\Lambda\right]v\left(\tau,t\right)d\tau.$$

By using the fact that $l_{xx} - l_{\tau\tau} = l(x,\tau)\Lambda/\varepsilon$, $l(1,\tau) = 0$ and $l(x,x) = \Lambda(1-x)/(2\varepsilon)$ we get

$$z_t(x,t) = \varepsilon z_{xx}(x,t) + G(x)z(0,t) + d_l(x,t)$$
$$+h_c(y,x,x_f,t) - \int_0^x l(x,\tau)h_c(y,\tau,x_f,t)d\tau \quad , \quad (24)$$

with boundary conditions (14) and (15) where $G(x)$ is defined after equation (15). Next, the following theorem will introduce a detectability condition for a state fault by using (24).

Theorem 1 (State fault detectability condition): Consider the observer defined by (19) is utilized to monitor (24) and (14–15). A state fault initiated at the time instant, t_f , and location, x_f , is detectable if there exists a time $T \geq t_f$ such that for all $t > T$, the following condition

$$\left\|\sum_{n=0}^{\infty}\int_{t_f}^{t}\left\{2\int_0^1[h_c(y,x,x_f,\tau) - \int_0^x l(x,\eta)\,h_c(y,\eta,x_f,\tau)\right.\right.$$
$$\left.\left.\times d\eta]\cos[(n+0.5)\pi x]dx\right\}e^{-\varepsilon[(n+0.5)\pi]^2(t-\tau)}d\tau\right\| > 2\rho \quad , \quad (25)$$

is satisfied where $n = 0,1,2,...$ is an integer.

Proof: See Appendix.

Remark 4: The proof shown in the Appendix demonstrates that a state fault satisfying the detectability condition given by (25) can be detectable by using the observer given in (19).

The next step is to determine the type and location of the fault.

3.2. FAULT ISOLATION SCHEME

Upon detecting a fault, the fault type has to be identified followed by fault magnitude estimation. In the case of a state fault, the location has to be found.

To determine the fault type, first an additive actuator and sensor fault isolation estimators, to be presented next, are activated as shown in Figure 3.1 to generate the corresponding time-varying estimator output residuals, $e_a(t)$ and $e_s(t)$, for actuator and sensor respectively which are to be defined later. The actuator and sensor fault locations are trivial. The isolation scheme in Figure 3.1 shows that when one of the isolation residuals stays below its isolation threshold ρ_a or ρ_s for actuator or sensor respectively, the fault is considered to be of that type while the others are above their threshold. A fault is categorized a state fault when both the sensor and actuator isolation residuals exceed their thresholds. Next, the actuator and sensor isolation estimators will be introduced.

3.2.1. Actuator Fault Isolation Estimator. Upon detection of a fault, for an additive actuator fault, a fault filter given by

$$F_t(x,t) = \varepsilon F_{xx}(x,t), F_x(0,t) = 0, \tag{26}$$

$$F(1,t) = [\sigma_1^{(a)}(U(t),t),...,\sigma_n^{(a)}(U(t),t)]^T, \tag{27}$$

is incorporated into the observer (19) to construct an actuator fault isolation estimator where $F(x,t) \in \Re^n$ is utilized to estimate the fault function with initial condition

$F(x,t_d)=0$. In order to match the dimension of $\Phi_a(U(t),t)\in\Re^{n\times n}$, $F_a(x,t)=diag[F(x,t)]$ is used to estimate the fault function. The next theorem will cover the performance of an actuator isolation estimator.

Remark 5: By representing $\Phi_a(U(t),t)$ in (27) as a diagonal matrix to derive the actuator fault filter, the number of PDE equations can be reduced from $n\times n$ to n. In addition, if $[\sigma_1^{(a)}(U(t),t),...,\sigma_n^{(a)}(U(t),t)]^T=U(t)$, the fault filter given by (26) and (27) will be same as the input filter described by (16).

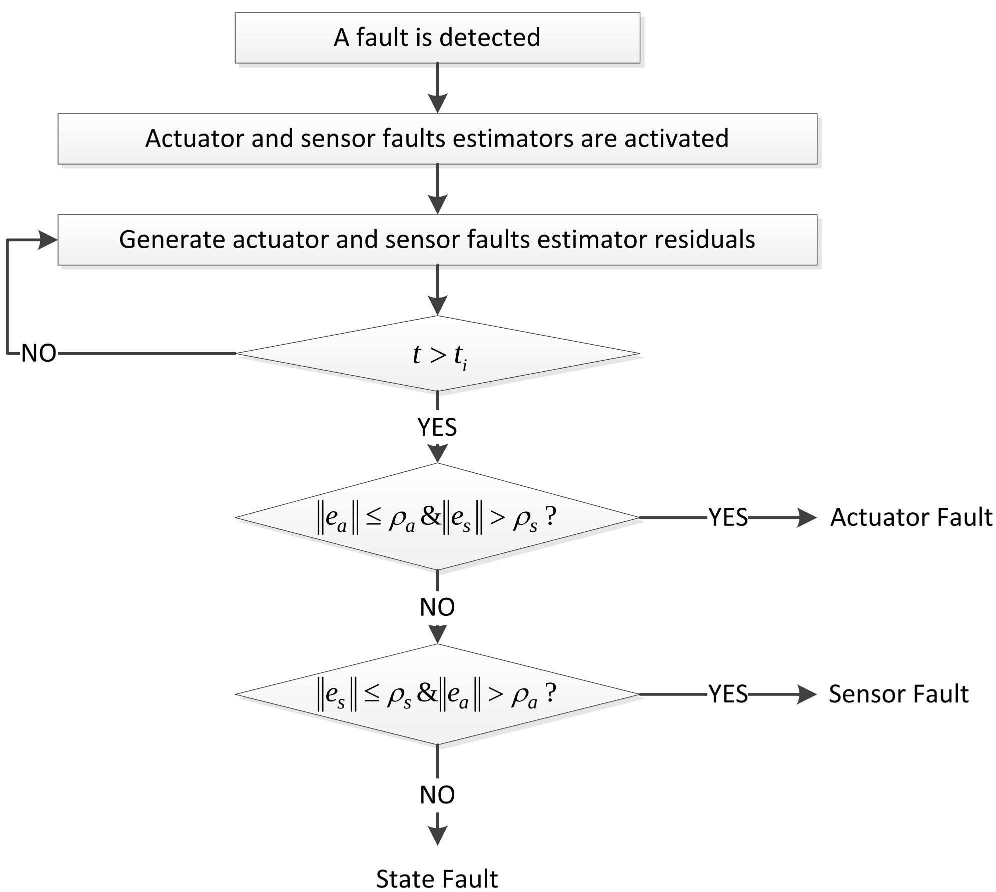


Figure 3.1. Fault isolation scheme.

Theorem 2 (Actuator fault isolation estimator performance): Once detecting a fault at time t_d, consider

$$\hat{z}_a(x,t) = \hat{z}(x,t) + F_a(x,t)\hat{\theta}_a(t),\ \hat{y}_a(t) = \hat{z}_a(0,t)\ , \tag{28}$$

as the estimator at $t \geq t_d$ for the state and output of the system in the presence of a bounded actuator fault, where $\hat{z}(x,t)$ is given by (19), and $\hat{\theta}_a(t) \in \Re^n$ is the estimated actuator fault parameter vector. Consider the projection algorithm given by

$$\dot{\hat{\theta}}_a(t) = \mathcal{P}_{\Theta_a}\{\beta F_a(0,t)e(t)\}, \tag{29}$$

to tune the parameter vector where $\beta > 0$ is the adaptation rate and $\mathcal{P}_{\Theta_a}\{\cdot\}$ is the projection operator. The actuator output isolation residual, $e_a(t) = y(t) - y_a(t)$, will remain bounded and stays within an fault isolation threshold ρ_a .

Proof: See Appendix.

Remark 6: By defining the actuator fault isolation threshold ρ_a as

$$\rho_a(t) = \rho + \kappa_a(t)\left\|F_a(0,t)\right\| + \bar{D}\ , \tag{30}$$

it can be shown in the Appendix that $\left\|e_a(t)\right\| \leq \rho_a(t)$ by using estimator defined by (28) with parameter vector tuned by (29). This ensures that an actuator fault can be isolated. Similarly, a sensor fault isolation estimator will be proposed next.

3.2.2. Sensor Fault Isolation Estimator. The presence of a sensor fault changes the value of $y(t)$ and thus causes the dynamics of two output filters given by (17) and (18) to provide inaccurate state estimates. Two fault filters are needed in order to mitigate the changes. Upon detecting the fault, consider

$$F_{1t}(x,t) = \varepsilon F_{1xx}(x,t), F_{1x}(0) = [\sigma_1^{(s)},...,\sigma_n^{(s)}]^T, F_1(1,t) = 0\ , \tag{31}$$

$$F_{2t}(x,\eta,t)=\varepsilon F_{2xx}(x,\eta,t)+\delta(x-\eta)[\sigma_1,...,\sigma_n(t)]^T\ , \tag{32}$$

$$F_{2x}(0,\eta,t)=0,\ F_2(1,\eta,t)=0\ , \tag{33}$$

where $F_1(x,t)$ and $F_2(x,t)\in\Re^n$ are states of fault filters. Then the following theorem will establish a sensor fault isolation estimator and define its performance based on these fault filters given by equations above.

Theorem 3 (Sensor fault isolation estimator performance): Upon detecting a fault, consider the sensor fault isolation estimator for $t\geq t_d$ given by

$$\hat{z}_s(x,t)=\hat{z}(x,t)-[L_0M(x,t)+\int_0^1 G(s)\Psi(x,s,t)ds]\hat{\theta}_s(t)\ , \tag{34}$$

with

$$\hat{y}_s(t)=\hat{z}_s(0,t)+\Phi_s(t)\hat{\theta}_s(t)\ , \tag{35}$$

to estimate the state and output of DPS, where $\mathrm{M}(x,t)=diag(F_1(x,t))$, $\Psi(x,s,t)=diag(F_2(x,s,t))$ and $\hat{\theta}_s(t)\in\Re^n$ represent the estimated sensor fault parameter vector. Consider the parameter tuning law given by

$$\dot{\hat{\theta}}_s(t)=\mathcal{P}_{\Theta_s}\{\beta F_s^T(0,t)e(t)\}\ , \tag{36}$$

where $F_s(0,t)=\Phi_s(t)-[L_0\mathrm{M}(0,t)+\int_0^1 G(s)\Psi(0,s,t)ds]$, L_0 is defined after the equation (15) and $\beta>0$ is the adaptation rate.

Then for $t>t_d$, the sensor fault estimator output isolation residual, $e_s(t)=y(t)-\hat{y}_s(t)$, will be bounded and remains below a predefined sensor fault isolation threshold ρ_s .

Proof: Refer to Appendix.

Remark 7: Define the sensor fault isolation threshold as

$$\rho_s(t) = \rho + \kappa_s(t)\left\|F_s(0,t)\right\| + \bar{D} \ . \tag{37}$$

By utilizing the sensor fault estimator given by (34) and output defined by (35) along with the parameter tuned by (36), we can show $\left\|e_s(t)\right\| \le \rho_s(t)$ in the Appendix.

Remark 8: It is shown that in the presence of an actuator or sensor fault, the corresponding isolation estimator output residual should be within its corresponding isolation threshold ρ_a or ρ_s, respectively while the other residual exceeds its isolation threshold. To the contrary, when both sensor and actuator fault isolation estimator output residuals exceed their corresponding isolation thresholds, a state fault is considered to have occurred.

Note the difference between the time-varying isolation thresholds ρ_a or ρ_s and the constant detection threshold ρ. The isolation thresholds (30) and (37) are generally higher than the detection threshold. For example, as shown in Figure 3.2 (a) the magnitude of the actuator estimator output residual $e_a(t)$ will cross the detection threshold ρ and yet always stay within the isolation threshold for the actuator fault estimator $\rho_a(t)$ in the presence of an actuator fault.

The identification of a state fault requires location determination, which is introduced next.

3.2.3. Location Determination of a State Fault. First, several state fault filters $i = 1, 2, 3, ..., p$, with p represents the number of filters, which divides the system space $x \in (0,1)$ into $p+1$ identical segments, will be designed next to construct the state fault estimator. By comparing the estimated isolation outputs given by estimators with the measured output, p isolation estimator errors will be generated. The estimator generating the minimum error magnitude is believed to be closest to the actual state fault position. Notice that placing more estimators, p, will result in the determination of accurate fault location but this will increase the computational cost. After introducing the state fault filters and the estimator, the performance of the estimator will be demonstrated and the isolability condition which defines the class of isolable faults will be given. Next, the state fault filters will be introduced.

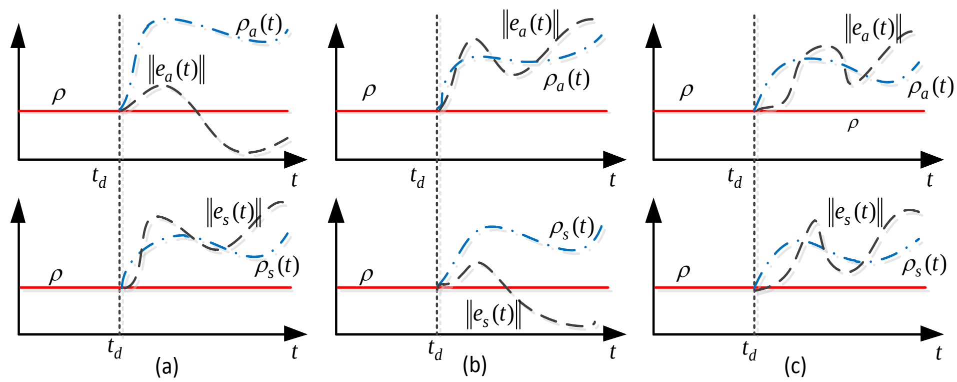


Figure 3.2. Isolation with (a) an actuator fault, (b) sensor fault, and (c) state fault.

The system dynamics with a state fault can be written as

$$z_t(x,t) = \varepsilon z_{xx}(x,t) + G(x)z(0,t) + \bar{\Phi}_c(y,x,x_f,t)\theta_c + d_l(v,x,t), \tag{38}$$

with boundary conditions given by (14) and (15) where

$$\bar{\Phi}_c(y,x,x_f,t) = \Phi_c(y,x,t)\Delta(x-x_f) - \int_0^x l\left(x,\tau\right)\Phi_c(y,\tau,t)\Delta(\tau - x_f)d\tau. \qquad (39)$$

In order to construct the state fault isolation estimators, fault filters are incorporated into the observer (19). The state of the estimator, $\hat{z}^{(i)}(x,t)$ at location $x = x_i$ with corresponding estimated output $\hat{y}^{(i)}(t)$ can be represented as

$$\hat{z}^{(i)}(x,t) = \hat{z}(x,t) + F_c^{(i)}(x,t)\hat{\theta}_c^{(i)}(t) \ , \qquad (40)$$

$$\hat{y}^{(i)}(t) = \hat{z}^{(i)}(0,t) \ . \qquad (41)$$

where $F_c^{(i)}(x,t)$ represents i^{th} fault filter at position $x = x_i$ for $x_i \in (0,1)$ with $i = 1,...p$. The fault filter is designed using

$$\partial F_c^{(i)}(x,t) / \partial t = \varepsilon \partial^2 F_c^{(i)}(x,t) / \partial x^2 + \bar{\Phi}_c(y,x,x_i,t) \ , \qquad (42)$$

$$\partial F_c^{(i)}(0,t) / \partial x = 0, \qquad F^{(i)}(1,t) = 0 \ , \qquad (43)$$

with

$$\bar{\Phi}_c(y,x,x_i,t) = \Phi_c(y,x,t)\Delta(x-x_i) - \int_0^x l\left(x,\tau\right)\Phi_c(y,\tau,t)\Delta(\tau - x_i)d\tau$$

where $F_c^{(i)}(x,t) \in \Re^{n\times n}$ is the i^{th} fault filter state, $\hat{\theta}^{(i)}(t)$ is the adaptive parameter vector of i^{th} state fault estimator. The state estimation error is defined as

$$\tilde{z}^{(i)}(x,t) = z(x,t) - \hat{z}^{(i)}(x,t) \ , \qquad (44)$$

whereas the output residual is given by $e^{(i)}(t) = y(t) - \hat{y}^{(i)}(t)$. In order to study the performance of the estimation error $\tilde{z}^{(i)}(x,t)$, define

$$\bar{z}^{(i)}(x,t) = \hat{z}(x,t) + F_c^{(i)}(x,t)\theta_c \ . \qquad (45)$$

It can be observed that as $\hat{\theta}_c^{(i)}(t) \to \theta_c$, the estimator state defined by (40) is the same as (45).

Define $\mu^{(i)}(x,t) = z(x,t) - \bar{z}^{(i)}(x,t)$ and its dynamics are given by

$$\mu_t^{(i)}(x,t) = \varepsilon \mu_{xx}^{(i)}(x,t) + d_l(x,t) + [\bar{\Phi}_c(y,x,x_f,t) - \bar{\Phi}_c(y,x,x_i,t)]\theta_c \ , \tag{46}$$

$$\mu_x^{(i)}(0,t) = 0, \quad \mu^{(i)}(1,t) = 0 \ . \tag{47}$$

From the definition of $\tilde{z}^{(i)}(x,t)$ and $\mu^{(i)}(x,t)$ we can get $\tilde{z}^{(i)}(x,t) = \mu^{(i)}(x,t) + F_c^{(i)}(x,t)\tilde{\theta}_c^{(i)}(t)$. If the estimator is located at the same position as the actual fault, i.e. $x_i = x_f$, $\mu^{(i)}(x,t)$ will have same dynamics as the one given by (20) which is bounded for all $x \in [0,1]$, $t \ge t_d$ and the bound only depends on the upper bound of the disturbance. An adaptive update law is proposed to tune the adaptive parameter and an identifiable condition, which defines the class of state faults whose location can be identified using the proposed estimators, is included in the next theorem.

Theorem 4 (State fault estimator performance): Let the state fault estimator be defined by (40) and (41) with parameter update law be presented as

$$\dot{\hat{\theta}}_c^{(i)}(t) = \beta [F_c^{(i)}(0,t)]^T e^{(i)}(t) - \gamma \hat{\theta}_c^{(i)}(t) \ , \tag{48}$$

where γ is a positive constant and $0 < \beta < (\pi^2 - 4)/2$ is the adaptation rate parameter to be used to estimate the system state described by (38) and (14) upon detecting a state fault. By comparing the actual isolation output defined in (7) with the estimated isolation output defined by

$$\hat{y}_i(t) = \int_0^1 C(x)\hat{v}^{(i)}(x)dx , \tag{49}$$

where $\hat{v}^{(i)}$ is the estimated system state given by $\hat{v}^{(i)}(x,t) = \hat{z}^{(i)}(x,t) + \int_0^x K(x,\tau)\hat{z}^{(i)}(\tau,t)d\tau$ with $\bar{k} = \underset{x\in[0,1]}{\|K(x,\tau)\|}$ and $K(x,\tau)$ being the kennel matrix of the inverse transformation

$$v^{(i)}\left(x,t\right)=z^{(i)}\left(x,t\right)+\int_0^x K\left(x,\tau\right)z^{(i)}\left(\tau,t\right)d\tau \ , \tag{50}$$

the location of a state fault occurred at position $x=x_f$ is identifiable when the state fault mismatch function $\eta_i(x)=\bar{\Phi}_c(y,x,x_f,t)-\bar{\Phi}_c(y,x,x_i,t)$ and fault filters defined by (42) and (43) satisfy

$\|\eta_s\|>\|\eta_r\|$ and $\chi_s\int_0^1\left\|F_c^{(s)}(x,t)\right\|dx>\chi_r\int_0^1\left\|F_c^{(r)}(x,t)\right\|dx$ when $\left|x_s-x_f\right|>\left|x_r-x_f\right|$

for s and $r\in 1,\ldots p$, (51)

where $\chi_i=\sqrt{(\|\eta_i\|^2+d_l^2)/\varepsilon\gamma+\theta_{c\max}^2}$, $i=r,s.$

Proof: See Appendix.

Remark 9: It is shown in the Appendix that the isolation output residual defined by $\tilde{y}_i^{(i)}(t)=y_i(t)-\hat{y}_i^{(i)}(t)$ is bounded by

$$\left\|\tilde{y}_i^{(i)}(t)\right\|\le\bar{c}\sqrt{(2+4\bar{k}^2)}[\sqrt{2\gamma\theta_{c\max}^2+2[\|\eta_i\|^2+d_l^2]/\varepsilon}+\chi_i\int_0^1\left\|F_c^{(i)}(x,t)\right\|dx], \tag{52}$$

when (51) holds and it is clear that the less the distance between the actual fault and filter location given by $S_i=\left|x_f-x_i\right|$, the smaller the bound given by (52). Therefore, the true fault location is determined as the one that is closest to the state fault estimator generating a residual that is minimum over others.

Remark 10: The identifiable condition defined by (51) has two parts because from the isolation output residual given by (52), we can obtained that the magnitude of the residual is determined by the value of both $\|\eta_i\|$ and $\chi_i\int_0^1\left\|F_c^{(i)}(x,t)\right\|$. In order to isolate an actuator, sensor and state fault, an isolable condition is required which will be introduced next.

3.2.4. Fault Isolability Condition. In this part, a fault isolability condition is derived on the basis of the proposed fault isolation scheme to define the class of faults that can be isolated. Faults which can produce enough difference on the measurements are simpler to isolate. For the sake of expressing this difference, define a fault mismatch function

$$h^{rm}(t) \triangleq F_r(0,t)\theta_r - F_m(0,t)\hat{\theta}_m(t), \tag{53}$$

where $r = a,s,c$ and $F_r(0,t)\theta_r$ represents the change of the measured output caused by an actuator fault, sensor fault or state fault respectively, $m = a,s$ and $F_m(0,t)$ denotes effect caused by an estimated actuator fault or sensor fault on the output and $r \neq m$. The fault mismatch function can be viewed as the difference between the actual change of the output $F_r(0,t)\theta_r$ due to the fault and estimated change of the output $F_m(0,t)\theta_m$ given by any other fault estimator m whose framework does not match with the actual fault r.

A fault r that has been detected is isolable if for each estimator $m \in \{a,s\} \setminus \{r\}$, there exists a time $t_i > t_d$ such that the fault mismatch function defined by (53) satisfies the following inequality

$$\left\| h^{rm}(t) \right\| > 2\rho + \kappa_m(t) \left\| F_m(0,t) \right\| + \bar{D}. \tag{54}$$

Proof: See Appendix.

Next the fault detection and isolation of nonlinear DPS is introduced.

4. NONLINEAR SYSTEM DESCRIPTION

A class of DPS represented by a bank of nonlinear PDEs will be introduced in this section. The system description under healthy conditions will be presented first and with actuator and sensor faults will be given in the second part.

4.1. SYSTEM DESCRIPTION WITHOUT FAULTS

The state representation of a class of nonlinear DPS is expressed as

$$\frac{\partial v(x,t)}{\partial t} = c\frac{\partial^2 v(x,t)}{x^2} + f(v,x) + d(v,x,t)\,, \tag{55}$$

with boundary conditions given by

$$v_x\left(0,t\right) = Qv(0,t),\ \ v(1,t) = u(t), \tag{56}$$

and

$$y(t) = v(0,t),\quad y_s(t) = v_x(0,t),\quad y_a(t) = v(1,t)\,, \tag{57}$$

where $x \in [0,1]$ is the space variable, $t \ge 0$ is the time variable, $v(x,t) \in \Re^n$ represents the state vector.

Notice that $y(t)$ is the measured output for observer design and fault detection, $y_s(t)$ is an additional required measurement for sensor fault isolation while $y_a(t)$ is the required measurement for an actuator fault isolation, $f(v,x) \in \Re^n$ is the nonlinear vector function, $d(x,t) \in \Re^n$ denotes the disturbance, $Q \in \Re^{n\times n}$ is a nonzero square matrix, and $c > 0$ is a constant.

Assumption 6: The nonlinear vector function $f(v,x)$ satisfies the following conditions

a. $f(v,x)$ is Lipschitz continuous in v, $\mathbb{C}^0$ in x, $\mathbb{C}^1$ in t and v for $x \in [0,1]$, $t \geq 0$ and $v(x) \in L_2(0,1)$.

b. $f(v,x)$ should satisfies $f(v+\Delta v,x) - f(v,x) = \dfrac{\partial f(v,x)}{\partial v}\Delta v + \varepsilon_f(\Delta v,x)$,where Δv represents a small change in v and $\varepsilon_f(\Delta v,x)$ is the approximation error satisfying $\left\|\varepsilon_f\right\|_{2,n} \leq \bar{\varepsilon}_f$.

Remark 11: Assumption 6 (a) indicates that $\dfrac{\partial f(v,x)}{\partial v}$ is bounded.

Remark 12: In order to meet the requirement $\left\|\varepsilon_f\right\|_{2,n} \leq \bar{\varepsilon}_f$ in Assumption 6 (b), Δv needs to be small enough implying that the initial condition of the observer which will be introduced in Subsection 4.2 should be close to the initial condition of the system described by (55) and (56).

In the presence of a state fault, the state representation given by (55) is modified as

$$\frac{\partial v(x,t)}{\partial t} = c\frac{\partial^2 v(x,t)}{x^2} + f(v,x) + d(x,t) + h_c(u,y) \quad . \tag{58}$$

Similarly, the boundary conditions are changed as

$$v_x(0,t) = 0,\ v(1,t) = u(t) + h_a(u) \ , \tag{59}$$

in the presence of an actuator fault and

$$y(t) = v(0,t) + h_s(t) \ , \tag{60}$$

in the presence of a sensor fault.

Assumption 7: The fault type considered in the nonlinear system is state, actuator or sensor faults and only one fault occurs at any time.

Next, a detection observer will be first presented and then a fault isolation scheme for differentiating state, actuator and sensor faults will be proposed.

4.2. OBSERVER DESIGN

First the design of the observer will be introduced. Next for the sake of selecting suitable gains of the observer, the observer error dynamics will be considered. It will be shown that by appropriately selecting observer gains, the error dynamics will be bounded. In order to monitor the system behavior described by (55), (56) and (57), a detection observer is proposed as

$$\frac{\partial \hat{v}(x,t)}{\partial t} = c\frac{\partial^2 \hat{v}(x,t)}{\partial x^2} + f(\hat{v},x) + P_1(x,t)(y-\hat{y}) \ , \tag{61}$$

$$\frac{\partial \hat{v}(0,t)}{\partial t} = Qy(t) + P_{10}(t)(y-\hat{y}),\ \hat{v}(1,t) = u(t) \ , \tag{62}$$

$$\hat{y}(t) = \hat{v}(0,t) \ , \tag{63}$$

where $\hat{v}(x,t) \in \Re^n$ represents the observer state, $P_1(x) \in \Re^{n\times n}$ and $P_{10} \in \Re^{n\times n}$ are observer gains and $\hat{y}(t) \in \Re^n$ is the estimated output.

Define detection residual as $e(t) \in \Re^n = y(t) - \hat{y}(t)$, and the observer error is given by

$$\tilde{v} \in \Re^n = v - \hat{v} \ .$$

Then, by applying Assumption 6, the dynamics of the observer error can be obtained as

$$\tilde{v}_t(x,t) = c\tilde{v}_{xx}(x,t) + A(t)\tilde{v}(x,t) + \varepsilon_f(\tilde{v},x) - P_1(x,t)e(t) + d(x,t) \ , \tag{64}$$

subject to the boundary conditions given by

$$\tilde{v}_x(0,t) = -P_{10}(t)e(t), \quad \tilde{v}(1,t) = 0 \ , \tag{65}$$

where $A(t)=\left.\frac{f(v,x,t)}{v^T}\right|_{v=\hat{v}}$. It can be shown that when the observer gains are selected as [14]

$$P_1(x,t)=c\frac{\partial L(x,0,t)}{\partial\tau},\quad P_{10}(t)=L(0,0,t)\ , \tag{66}$$

then by applying the transformation

$$\tilde{v}(x,t)=\Xi(x,t)-\int_0^x L(x,\tau,t)\Xi(\tau,t)d\tau\,, \tag{67}$$

to the observer error dynamics described by (64) and (65), it will be converted into a stable system given by

$$\frac{\partial\Xi(x,t)}{\partial t}=c\frac{\partial^2\Xi(x,t)}{\partial x^2}-b(t)\Xi(x,t)+\varepsilon_{fM}(\tilde{v},x)+d_M(x,t)\ , \tag{68}$$

$$\frac{\partial\Xi(0,t)}{\partial x}=0,\quad \Xi(1,t)=0\,. \tag{69}$$

where $L(x,\tau,t)$ is the unique solution to the well-posed PDE [14] given by

$$\frac{\partial L(x,\tau,t)}{\partial t}=A(t)L(x,\tau,t)+b(t)L(x,\tau,t)+c[\frac{\partial^2 L(x,\tau,t)}{\partial\tau^2}-\frac{\partial^2 L(x,\tau,t)}{\partial x^2}], \tag{70}$$

$$L(1,\tau,t)=0,\quad L(x,x,t)=\frac{(x-1)}{2c}[A(t)+b(t)I_{n\times n}]\ , \tag{71}$$

$\Xi(x,t)\in\Re^n$, $L(x,\tau,t)\in\Re^{n\times n}$, and $b(t)\geq 0$ is an arbitrary scalar, $d_M(x,t)=d(x,t)+\int_0^x M(x,\eta,t)d(\eta,t)d\eta$ and $\varepsilon_{fM}(\tilde{v},x)=\varepsilon_f(\tilde{v},x)+\int_0^x M(x,\eta,t)\varepsilon_f(\tilde{v},\eta)d\eta$ with $M(x,\eta,t)\in\Re^{n\times n}$ is the kennel matrix of the inverse transformation given by

$$\Xi(x,t)=\tilde{v}(x,t)+\int_0^x M(x,\eta,t)\tilde{v}(\eta,t)d\eta\,. \tag{72}$$

The following theorem shows the performance of the detection observer defined by (61), (62) and (63).

Theorem 5 (Detection observer performance): Let the observer defined by (61), (62), and (63) to estimate the unmeasured states and measured output of the DPS given by (55), (56) and (57). In the absence of a fault, detection residual $e(t)$ will be bounded and maintained below a detection threshold ρ. A fault can cause $e(t)$ to increase and exceed the threshold ρ.

Proof: Refer to Appendix.

Remark 13: It is shown in the Appendix that under healthy conditions the detection residual defined as $e(t) = v(0,t)$ is bounded by

$$\|e(t)\| \leq 2\sqrt{\frac{17c^3}{\sqrt{2[c+2b(t)][16b(t)+1]}}}(\bar{d}_M + \bar{\varepsilon}_{fM}) \ ,$$

and the bound depends upon the disturbance bound. Based on this bound, a predefined threshold ρ is selected, and in the absence of any fault, the magnitude of the detection residual should be below the threshold ρ. In the presence of any type of fault (Figure 4.1), the measured output will deviate from the estimated output and thereby cause the detection residual to increase and exceed the predefined threshold. In that case, a fault is declared to be active.

The fault isolation scheme will be introduced next.

4.3. FAULT ISOLATION SCHEME

Once a fault is detected by using the proposed observer as shown in Subsection 4.2, the fault type needs to be identified. In order to isolate the faults, it is assumed that the system operates longer than the isolation time t_i. The proposed isolation scheme can only identify the fault type and the location determination is out of the scope of this paper.

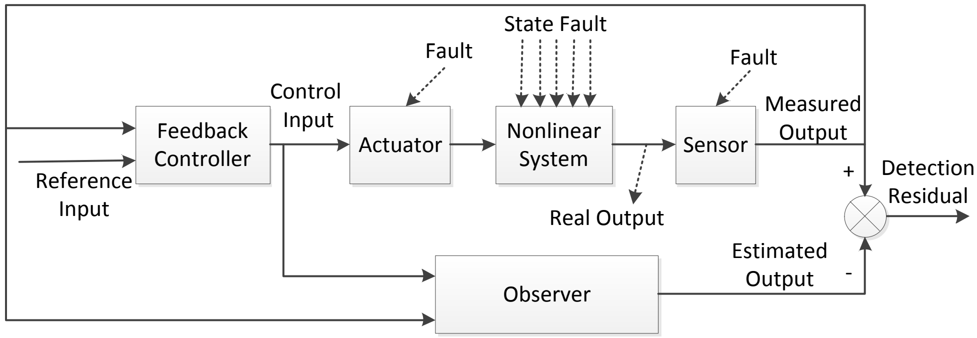


Figure 4.1. Fault detection scheme.

The isolation scheme given by Figure 4.2 indicates that after detecting a fault, by using the measurements defined by (57) and the estimated output given by the observer, the actuator and sensor fault isolation residuals (e_a and e_s) which will be defined next are generated. Because the presence of an actuator/sensor fault can only cause the corresponding fault isolation residual to increase, if one of the fault isolation residual (e_a / e_s) exceeds its isolation threshold, the corresponding fault will be declared; and if neither of them does, the fault is considered a state fault.

Theorem 6 (Fault isolability condition): Upon a fault is detected at $t = t_d$, let the additional measurements y_a and y_s defined by (57) be used to generate the actuator and sensor fault isolation residuals defined as $e_a(t) = y_a(t) - \hat{v}(1,t)$ and $e_s(t) = y_s(t) - Qy(t)$ respectively. Then

I. An actuator fault will be isolable if there exists a time $t_a > t_d$ such that the magnitude of the actuator fault satisfies $\left\| h_a(u; t_a) \right\| > \rho_a$;

II. A sensor fault will be isolable if there exists a time $t_s > t_d$ that the magnitude of the sensor fault satisfies $\|Qh_s(t_s)\| > \rho_s$;

III. A state fault will be identified if $\|e_a(t)\| < \rho_a$ and $\|e_s(t)\| < \rho_s$ for all $t_d < t \le t_i$.

Proof: See Appendix.

Remark 14: Based on the analysis in the Appendix, it is known that either the actuator fault or the sensor fault will cause the fault residual to exceed its corresponding isolation threshold. Therefore, if a fault is detected at t_d and $\|e_a(t)\| < \rho_a$, $\|e_s(t)\| < \rho_s$ for all $t_d < t \le t_i$, a state fault will be considered to occur. The selection of ρ_a and ρ_a can be based on the upper bound of the sensor noise.

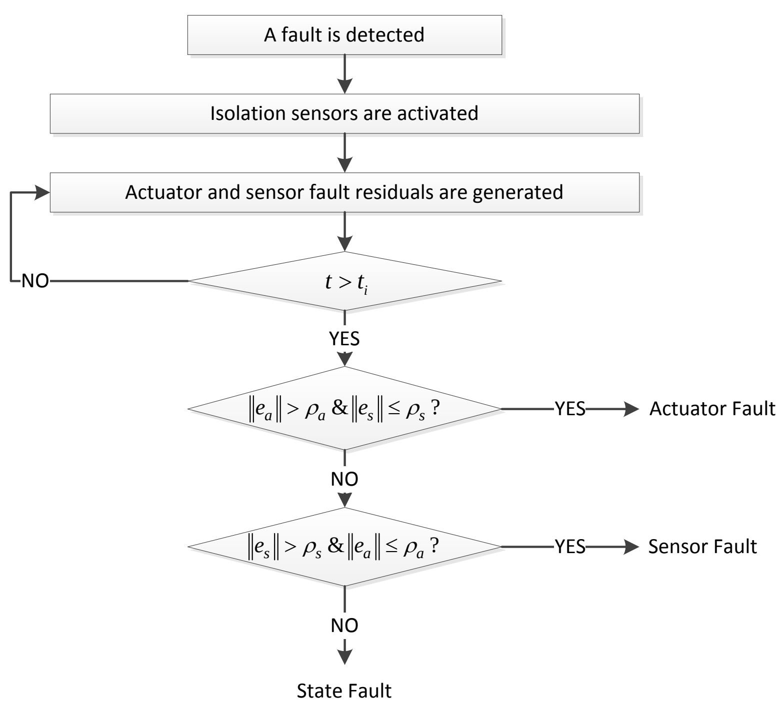


Figure 4.2. Fault isolation scheme.

5. SIMULATION RESULTS

The proposed fault detection and isolation scheme for linear DPS will be demonstrated in the first part of this section in the simulations by using MATLAB, and the verification of the scheme for nonlinear DPS will be introduced in the second part with a normalized heat equation.

5.1. FAULT ISOLATION OF A LINEAR SYSTEM

The linear DPS described by linear parabolic PDEs are given by

$$\frac{\partial v(x,t)}{\partial t} = \frac{\partial^2 v(x,t)}{\partial x^2} + \begin{bmatrix} 8 & 1 \\ 2 & 10 \end{bmatrix} v(x,t) + d(x,t) \ , \tag{73}$$

$$\frac{\partial v(0,t)}{\partial x} = [0;0], \quad v(1,t) = u(t) \, , \tag{74}$$

$$y(t) = [y_1(t), y_2(t)]^T = v(0,t) \, , \tag{75}$$

for $x \in [0,1]$ and $t > 0$ where $v(x,t) \in \Re^{2\times 1}$ represents the system state, $d(x,t) = \begin{bmatrix} 0.05e^{-.5(x-0.2)^2} \sin(t) \\ 0.06e^{-.3(x-0.4)^2} \sin(2t) \end{bmatrix}$ denotes the disturbance, $u(t)$ is the control input implemented at the position $x = 1$, and the output, $y(t)$, is measured at the opposite end.

In order to solve the system represented by PDE (73) - (74) and the detection observer using MATLAB, the space and time intervals are selected as $\Delta x = 0.05$ and $\Delta t = 0.01$. Upon detection of a fault, the actuator and sensor fault estimator with outputs given by (28), (34) and (35) are employed to isolate the sensor, actuator and sensor faults. Figure 5.1 shows that the sensor fault residual keeps within its isolation threshold all the time while an actuator fault residual exceeds its threshold. Combining the isolation results with the fault isolation scheme described in Figure 4.1 indicates a sensor fault.

Once a sensor fault is identified, the update law given by (36) will be utilized to estimate fault parameters. After an initial adaptation, as shown in Figure 5.2(b) and (c), the fault parameter vector can be estimated satisfactorily, which means the detection residual is reduced below the threshold again as shown in Figure 5.2(a).

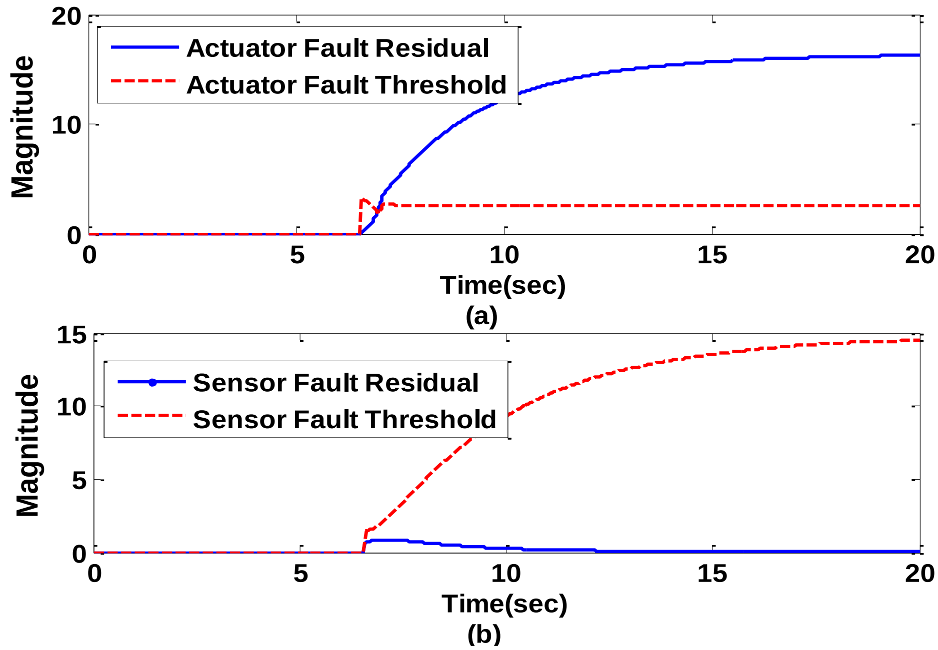


Figure 5.1. Fault isolation of a sensor fault.

Next, a state fault seeded at $x_f = 0.2$ is considered and the fault function is characterized as

$$h_c(y,x,t) = diag[y_1^2(t), y_2^2(t)]\theta_c(t)\delta(x-0.2) \,, \tag{76}$$

where $\theta_c(t) = \Omega(t-6)\begin{bmatrix} 1.2 \\ 2.3 \end{bmatrix}$ represents the state fault parameter vector and $\Omega(t-6) = diag(\Omega_i(t-6))$ for $i = 1,2$ which is the time profile of the state fault where $\Omega_i(t-6) = \begin{cases} 0, & if\ t < 6 \\ 1 - e^{-\kappa_i(t-6)}, & if\ t \geq 6 \end{cases}$ with $\kappa_1 = 0.3$ and $\kappa_2 = 0.6$.

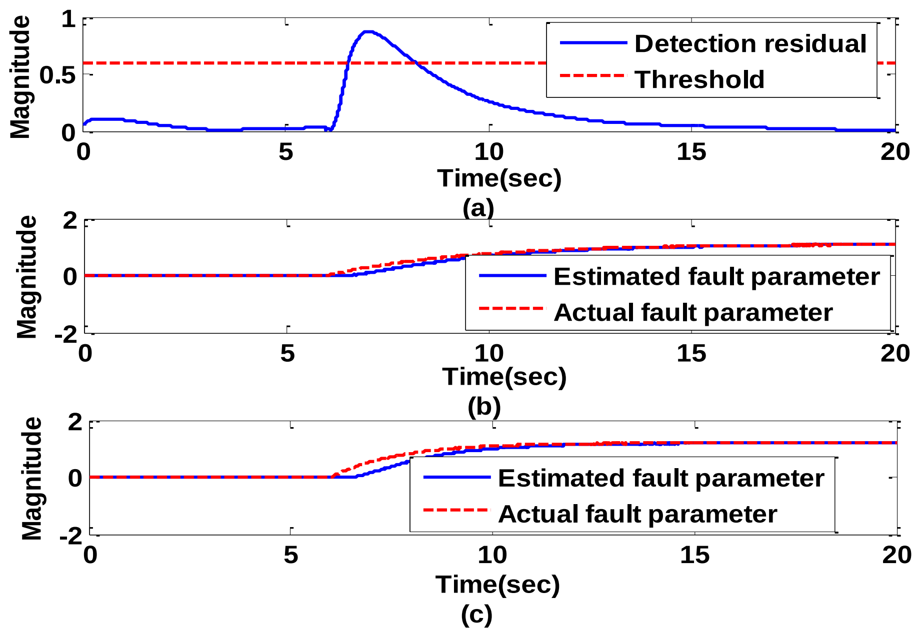


Figure 5.2. Fault detection and estimation results.

As noted previously, once a fault is detected, the actuator and sensor fault estimators are utilized to generate the corresponding fault residuals. It is obvious from Figure 5.3 that both the actuator and sensor fault residuals cross their thresholds implying a state fault.

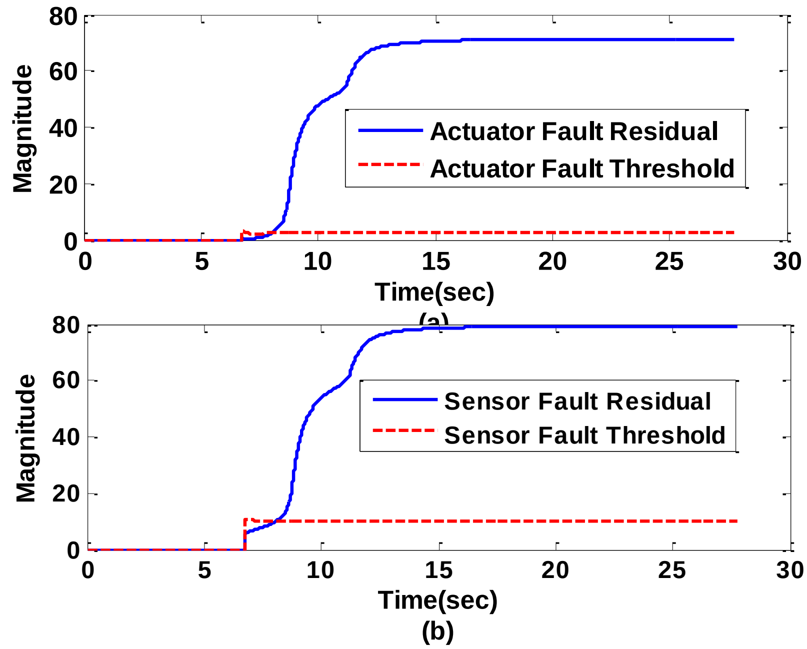


Figure 5.3. Fault isolation of a state fault.

After the identification of a state fault, the next step is to determine the fault location. In order to achieve this, four fault filters seeded at four different locations $x_i = 0.2, 0.4, 0.6, 0.8$ (see Figure 5.4) will be applied with isolation output selected as (notice that the isolation output is not limited to the one defined next)

$$y_i(t) = [v(0.1,t) + v(0.3,t) + v(0.5,t)]/3. \tag{77}$$

Each fault filter can generate an estimated isolation output, and using which four isolation residuals are generated by taking the difference between the actual and estimated isolation outputs. The state fault location is determined as $x_f = 0.2$ since Figure 5.4 shows that the magnitude of the isolation error generated by adding the fault filter at position $x_i = 0.2$ is the minimum.

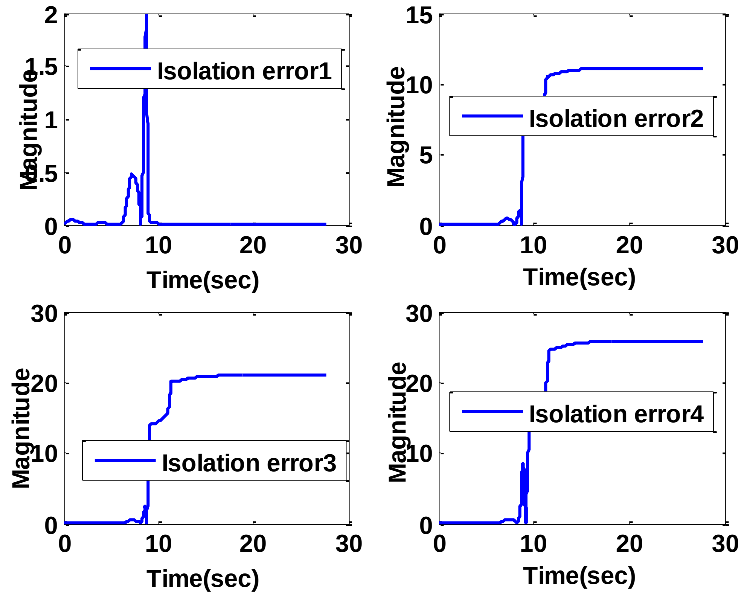


Figure 5.4. Location determination of a state fault.

5.2. FAULT ISOLATION OF A NONLINEAR SYSTEM

A heat equation with a nonlinear term is expressed as

$$\frac{\partial v(x,t)}{\partial t}=\frac{\partial^2 v(x,t)}{\partial x^2}+4v(x,t)+20e^{-\frac{5}{1+v(x,t)}}+d(x,t)\ , \tag{78}$$

subjecting to the boundary conditions

$$\frac{\partial v(0,t)}{\partial x}=0.5v(0,t),\ v(1,t)=u(t)\ , \tag{79}$$

where $v(x,t)$ is the system state, $u(t)$ represents the control input, and $d(x,t)=0.01\sin(t)e^{-100(x-0.5)^2}$ denotes the disturbance and the measured output for observer design defined as

$$y(0,t)=v(0,t)\ . \tag{80}$$

The observer is developed based on (61)–(63) to monitor system behavior. A fault is declared activated when the detection residual exceeds the detection threshold. Next, the actuator, sensor and state fault are incorporated into the system, respectively, and only one fault is considered at one specific time. The fault functions are expressed as

$$h_a = -0.5(1-e^{-0.8(t-0.6)})u(t)\ ,\ h_s(t) = 1.5(1-e^{-0.5(t-8)})y_d(t),$$

$$h_c(t) = 0.8(1-e^{-0.9(t-t_f)})(1+y(t))^2 e^{-15(x-0.3)^2}\ ,$$

where $y_d(t) = 0.3\sin(1.5t)+0.5$, which is the desired trajectory of the output. In order to differentiate these three types of faults, two measurements at different locations are utilized which are defined as $y_a(t) = v(1,t)$ and $y_s(t) = v_x(0,t)$.

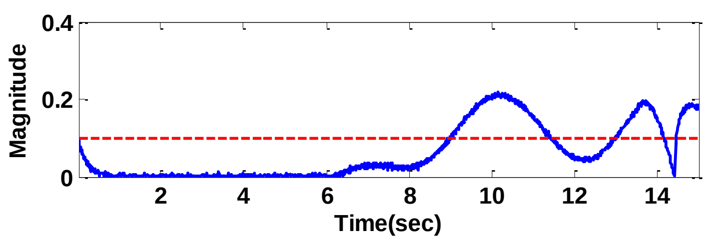


Figure 5.5. Fault detection of an actuator fault.

In the presence of an actuator fault seeded at $t_f = 6s$, it can be observed from Figure 5.5 that the fault can be detected within 2.5 s. Figure 5.6 shows that only the actuator fault residual exceeded its threshold; thus, an actuator fault is identified. In the case of a sensor fault, it can be seen from Figure 5.8 that just the sensor fault isolation residual goes across the threshold indicating a sensor fault. However, the isolation results as shown in the Figure 5.10 indicates that neither of the actuator and sensor fault isolation residuals exceed their isolation thresholds so according to the fault isolation scheme of nonlinear DPS, a state fault is identified. Above all, the actuator, sensor, and state faults can be isolated by checking the status of the actuator and sensor fault isolation residuals.

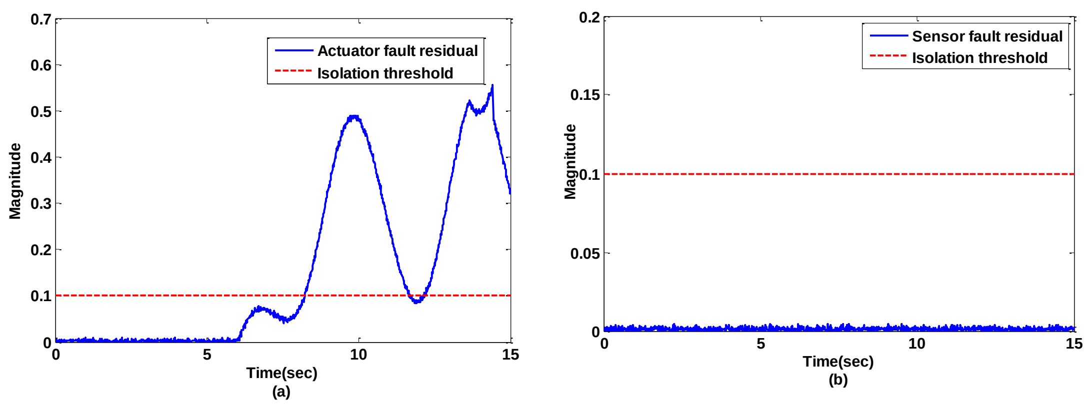


Figure 5.6. Fault isolation results of an actuator fault.

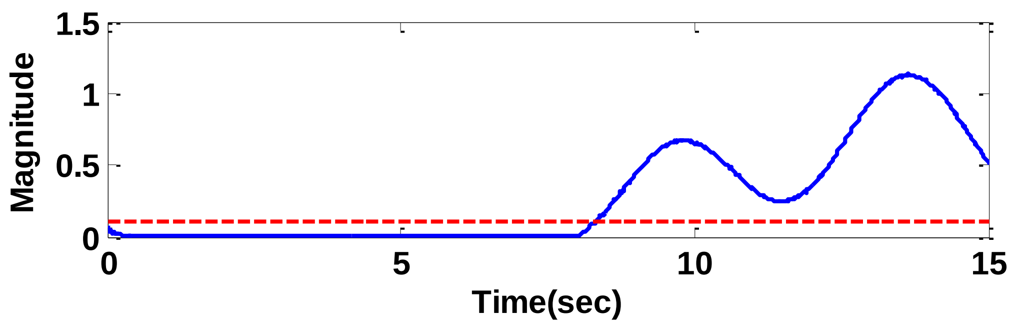


Figure 5.7. Fault detection result of a sensor fault.

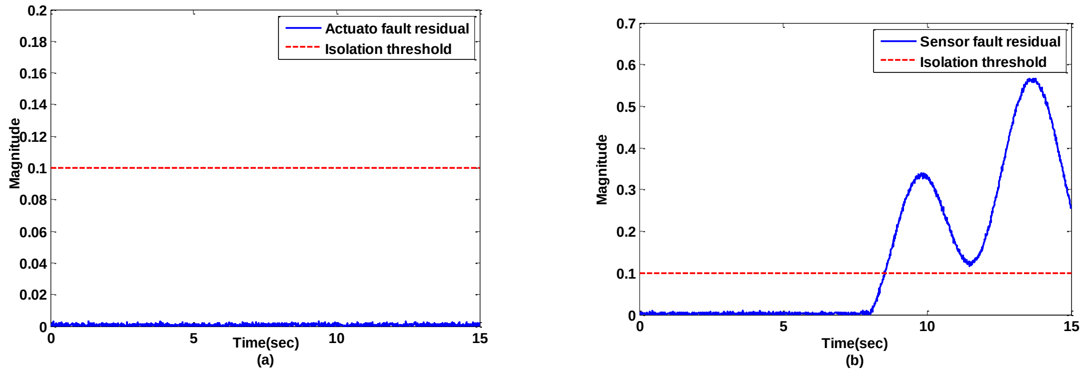


Figure 5.8. Fault isolation results of a sensor fault.

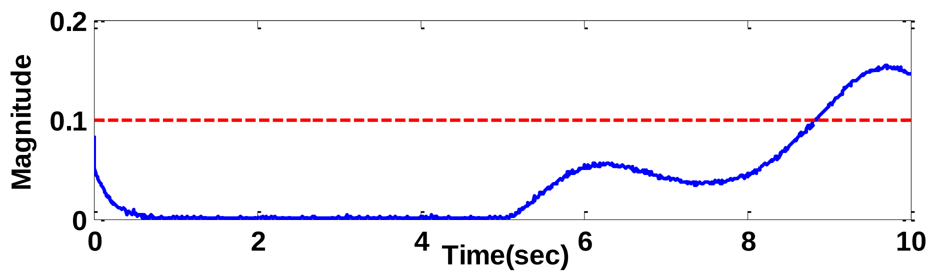


Figure 5.9. Fault detection result of a state fault.

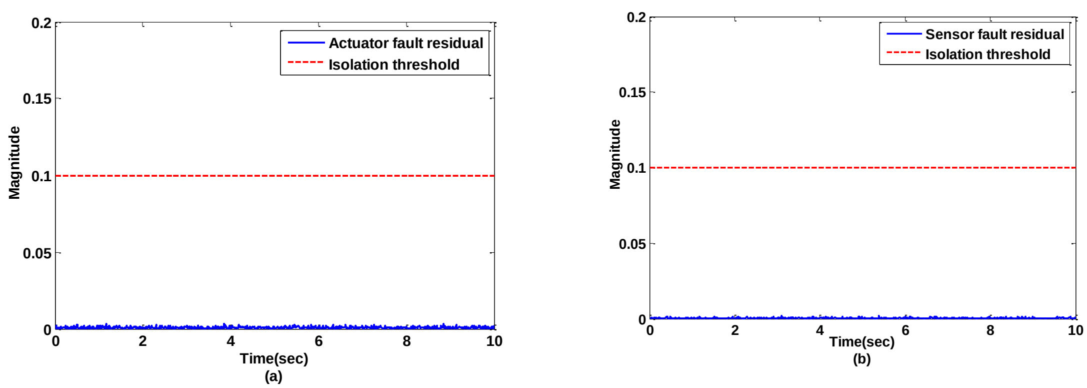


Figure 5.10. Fault isolation results of a state fault.

6. CONCLUSIONS

Fault isolation for DPS is more involved when compared to LPS because the system state in DPS is defined by spatial variations besides temporal variations. The developed actuator and sensor fault estimators for linear DPS with boundary measurement can be utilized to assist in differentiating actuator, sensor and state faults occurring on linear DPS. In addition, the proposed location determination scheme along with the isolation measurement is useful for identifying the location of a state fault. The fault detection framework using a Luenberger type observer can be applied to monitor the abnormal behavior of nonlinear DPS and the introduced fault isolation scheme is capable of isolating actuator, sensor and state faults with additional measurements at boundary conditions since fault filters are not available. The determined fault type and location developed in this research can provide useful information for fault estimation and accommodation.

APPENDIX

Proof of Theorem 1: In the presence of a state fault, the dynamics of the observer error becomes

$$\tilde{z}_t(x,t) = \varepsilon \tilde{z}_{xx}(x,t) + d_l(x,t) + h_c(y,x,x_f,t) - \int_0^x l(x,\tau) h_c(y,\tau,x_f,t) d\tau \ ,$$

$$\tilde{z}_x(0,t) = 0, \tilde{z}(0,t) = 0 \, .$$

Solving the PDE defined above yields [19]

$$\begin{aligned}\tilde{z}(x,t) &= \sum_{n=0}^{\infty} e^{-\varepsilon[(n+0.5)\pi]^2(t-t_f)} \tilde{z}_n(t_f) \cos[(n+0.5)\pi x] \\ &+ \sum_{n=0}^{\infty} \int_{t_f}^{t} d_m(\tau) e^{-\varepsilon[(n+0.5)\pi]^2(t-\tau)} d\tau \cos[(n+0.5)\pi x] \\ &+ \sum_{n=0}^{\infty} \int_{t_f}^{t} e^{-\varepsilon[(n+0.5)\pi]^2(t-\tau)} h_m(\tau) d\tau \cos[(n+0.5)\pi x],\end{aligned}$$

where $\tilde{z}_n(t_f) \in \Re^n$ depends upon the initial condition $e(t_f)$, $d_m(t) = 2\int_0^1 d_l(x,t) \cos[(n+0.5)\pi x] dx$ and $h_m(t) = 2\int_0^1 [h_c(y,x,x_f,t) - \int_0^x l(x,\eta) h_c(y,\eta,x_f,t) d\eta] \cos[(n+0.5)\pi x] dx$. The first term in the above equation is the response due to initial condition and the second one is the response due to the fault function and bounded disturbance. By noting detection residual being $e(t) = \tilde{z}(0,t)$, the solution to the detection residual is obtained by substituting $x = 0$ in the above equation as

$$e(t) = \sum_{n=0}^{\infty} \left\{ \tilde{z}_n(t_f) e^{-\varepsilon[(n+0.5)\pi]^2(t-t_f)} + \int_{t_f}^{t} d_m(\tau) e^{-\varepsilon[(n+0.5)\pi]^2(t-\tau)} d\tau \right\} + \sum_{n=0}^{\infty} \int_{t_f}^{t} h_m(\tau) e^{-\varepsilon[(n+0.5)\pi]^2(t-\tau)} d\tau.$$

According to triangle inequality ($\|a_1 + a_2\| \geq \|a_2\| - \|a_1\|$) and the equation above we can get

$$\|e(t)\| \geq \left\| \sum_{n=0}^{\infty} \int_{t_f}^{t} h_m(\tau) e^{-\varepsilon[(n+0.5)\pi]^2 (t-\tau)} d\tau \right\| -$$
$$\left\| \sum_{n=0}^{\infty} \left\{ \tilde{z}_n(t_f) e^{-\varepsilon[(n+0.5)\pi]^2 (t-t_f)} + \int_{t_f}^{t} d_m(\tau) e^{-\varepsilon[(n+0.5)\pi]^2 (t-\tau)} d\tau \right\} \right\|$$
$$> 2\rho - \kappa_\rho \left\{ \left\| \sum_{n=0}^{\infty} \left\{ e^{-\varepsilon[(n+0.5)\pi]^2 (t-t_f)} \tilde{z}_n(t_f) + \int_{t_f}^{t} d_m(\tau) e^{-\varepsilon[(n+0.5)\pi]^2 (t-\tau)} d\tau \right\} \right\| \right\} = 2\rho - \rho = \rho,$$

when (25) holds and the detection threshold is selected as

$$\rho = \kappa_\rho \left\{ \left\| \sum_{n=0}^{\infty} \left\{ \tilde{z}_n(t_f) e^{-\varepsilon[(n+0.5)\pi]^2 (t-t_f)} + \int_{t_f}^{t} d_m(\tau) e^{-\varepsilon[(n+0.5)\pi]^2 (t-\tau)} d\tau \right\} \right\| \right\},$$

where $\kappa_\rho > 1$ is a constant, thus assuring the detection of a state fault.

Proof of Theorem 2: The actuator isolation estimator state residual, $\tilde{z}_a(x,t) = z(x,t) - \hat{z}_a(x,t)$, can be written as $\tilde{z}_a(x,t) = \mu(x,t) + F_a(x,t)\tilde{\theta}_a(t)$. Then, the actuator fault estimator output isolation residual can be expressed as

$$e_a(t) = \tilde{z}_a(0,t) = \mu(0,t) + F_a(0,t)\tilde{\theta}_a(t) \,, \tag{A.1}$$

where $\mu(x,t) = z(x,t) - \bar{z}_a(x,t)$ with $\bar{z}_a(x,t)$ defined as

$$\bar{z}_a(x,t) = \hat{z}(x,t) + F_a(x,t)\theta_a.$$

Equation above is viewed as the ultimate target of $\hat{z}_a(x,t)$ when $\hat{\theta}_a$ is being tuned by (29) and it has the same initial condition as $\hat{z}_a(x,t)$ i.e. $\hat{z}_a(t_d) = \bar{z}_a(t_d)$. In the presence of an actuator fault, the system dynamics is described by (13) and (15) with modified boundary conditions given by

$$z_x(0,t) = L_0 z(0,t),\ z(1,t) = U(t) + \Phi_a(U(t),t)\theta_a\,. \tag{A.2}$$

By using the system dynamics given by (13), (15) and (A.2) and the observer defined by (19), we can obtain the dynamics of $\mu(x,t)$ as

$$\mu_t(x,t) = \varepsilon\mu_{xx}(x,t) + d_l(v,x,t),\ \mu_x(0,t) = 0,\ \mu(1,t) = 0, \tag{A.3}$$

where $d_l(v,x,t)$ is defined after (15). The error dynamics defined in (A.3) is same as the observer error dynamics given by (20) whose stability has been shown in [13]. Now to obtain the isolation residual, recall (A.1), when $t \geq t_d$, and take the norm on both sides to get

$$\|e_a(t)\| \leq \|\mu(0,t)\| + \|\tilde{\theta}_a(t)\|\|F_a(0,t)\|. \tag{A.4}$$

By solving the PDE given by (A.3) and substituting $x = 0$ to the solution we can get for $t \geq t_d$,

$$\mu(0,t) = \sum_{n=0}^{\infty} e^{-\varepsilon[(n+0.5)\pi]^2(t-t_d)}\mu_n(t_d) + \sum_{n=0}^{\infty}\int_{t_d}^{t} d_m(\tau)e^{-\varepsilon[(n+0.5)\pi]^2(t-\tau)}d\tau.$$

Substituting $\|\mu(0,t_d)\| = \|e(0,t_d)\| = \rho$ to the equation above to get

$$\|\mu(0,t)\| \leq \rho + \left\|\sum_{n=0}^{\infty}\int_{t_d}^{t} d_m(\tau)e^{-\varepsilon[(n+0.5)\pi]^2(t-\tau)}d\tau\right\| = \rho + \bar{D}.$$

where $\bar{D} = \left\|\sum_{n=0}^{\infty}\int_{t_d}^{t} d_m(\tau)e^{-\varepsilon[(n+0.5)\pi]^2(t-\tau)}d\tau\right\|$. Recalling the inequality given by (A.4) we can obtain

$$\|e_a(t)\| \leq \|\mu(0,t)\| + \|\tilde{\theta}_a(t)\|\|F_a(0,t)\| \leq \rho + \bar{D} + \kappa_a(t)\|F_a(0,t)\|,$$

where $\kappa_a(t) \geq \|\tilde{\theta}_a(t)\|$ depends upon the geometric properties of the compact set Θ_a. Recall the actuator fault isolation threshold ρ_a defined by (30) to get $\|e_a(t)\| \leq \rho_a(t)$, which completes the proof.

Proof of Theorem 3: The sensor fault estimator output error is expressed as

$$e_s(t) = y(t) - \hat{y}_s(t) = \tilde{z}_s(0,t) + \Phi_s(t)\tilde{\theta}_s(t) = \mu(0,t) + F_s(0,t)\tilde{\theta}_s(t)\ , \tag{A.5}$$

where $\tilde{z}_s(x,t) = z(x,t) - \hat{z}_s(x,t)$ is the sensor fault isolation estimator state residual, $\mu(x,t) = z(x,t) - \overline{z}_s(x,t)$ with $\overline{z}(x,t)$ is defined as

$$\overline{z}_s(x,t) = \hat{z}(x,t) - [L_0 \mathrm{M}(x,t) + \int_0^1 G(s)\Psi(x,s,t)ds]\theta_s \ ,$$

which is viewed as the ultimate target of $\hat{z}_s(x,t)$ when $\hat{\theta}_s$ is being tuned by (36) and has the same initial condition as $\hat{z}_s(x,t)$. In the presence of a sensor fault, the system dynamics becomes (13) and (14) with output expressed as

$$y(t) = z(0,t) + \Phi_s(t)\theta_s \ ,$$

By taking partial derivative of $\mu(x,t)$ with respective to t and x, we can get that the dynamics of $\mu(x,t)$ satisfying (A.3) indicating the stability of $\mu(x,t)$. Thus, for $t > t_d$ taking the norm on both sides of (A.5) we can obtain

$$\|e_s(t)\| \le \|\mu(0,t)\| + \kappa_s \|F_s(0,t)\| \le \rho + \kappa_s(t)\|F_s(0,t)\| + \bar{D} \ ,$$

where $\|\mu(0,t)\| \le \rho + \bar{D}$ for $t > t_d$ and $\kappa_s(t) \ge \left\|\theta_s - \hat{\theta}_s(t)\right\|$ relies on the geometric properties of the compact set Θ_s and $\bar{D}$ is decided by disturbance or uncertainty bound. Substitute the sensor fault isolation threshold defined by (37) to the inequality above yielding $\|e_s(t)\| \le \rho_s(t)$, which accomplishes the proof.

Proof of Theorem 4: Define a Lyapunov function candidate as

$$V = \int_0^1 [\mu^{(i)}(x,t)]^T \mu^{(i)}(x,t)dx/2 + (\tilde{\theta}_c^{(i)})^T \tilde{\theta}_c^{(i)}/2 \, ,$$

the derivative of this Lyapunov function with respect to time is given by $\dot{V} = \int_0^1 [\mu^{(i)}(x,t)]^T \dfrac{\partial \mu^{(i)}(x,t)}{\partial t} dx + [\tilde{\theta}_c^{(i)}]^T \dot{\tilde{\theta}}_c^{(c)}$. By substituting (48) to get

$$\dot{V} = \varepsilon\int_0^1 [\mu^{(i)}(x,t)]^T \mu_{xx}^{(i)}(x,t)dx + \int_0^1 [\mu^{(i)}(x,t)]^T d_l(x,t)dx + \int_0^1 [\mu^{(i)}(x,t)]^T [\bar{\Phi}_c(y,x,x_f,t) - \bar{\Phi}_c(y,x,x_i,t)]\theta_c dx + [\tilde{\theta}_c^{(i)}]^T \dot{\tilde{\theta}}_c^{(i)},$$

By integration by parts and using Poincare inequality [17] $\|\mu^{(i)}\|_{2,n}^2 \le \frac{4}{\pi^2}\|\mu_x^{(i)}\|_{2,n}^2$ and using the adaptive update law (48), we obtain

$$\begin{aligned}\dot{V} \le &-\frac{\varepsilon\pi^2}{4}\int_0^1 [\mu^{(i)}(x,t)]^T \mu^{(i)}(x,t)dx + \int_0^1 [\mu^{(i)}(x,t)]^T d_l(x,t)dx \\ &+ \int_0^1 [\mu^{(i)}(x,t)]^T [\bar{\Phi}_c(y,x,x_f,t) - \bar{\Phi}_c(y,x,x_i,t)]\theta_c dx \\ &-\beta[\tilde{\theta}_c^{(i)}]^T [F_c^{(i)}(0,t)]^T e^{(i)}(t) + \gamma[\tilde{\theta}_c^{(i)}]^T \hat{\theta}_c^{(i)}\end{aligned}.$$

Because $e^{(i)}(t) = \tilde{z}^{(i)}(0,t) = \mu^{(i)}(0,t) + F_c^{(i)}(0,t)\tilde{\theta}_c^{(i)}(t)$ the above inequality can be rewritten as

$$\begin{aligned}\dot{V} = &-\varepsilon\pi^2\int_0^1 [\mu^{(i)}(x,t)]^T \mu^{(i)}(x,t)dx/4 + \int_0^1 [\mu^{(i)}(x,t)]^T d_l(x,t)dx + \int_0^1 [\mu^{(i)}(x,t)]^T \eta_i(x)dx \\ &-\beta[e^{(i)}(t) - \mu^{(i)}(0,t)]^T e^{(i)}(t) + \gamma[\tilde{\theta}_c^{(i)}]^T \hat{\theta}_c^{(i)} \\ \le &-\varepsilon\pi^2\int_0^1 [\mu^{(i)}(x,t)]^T \mu^{(i)}(x,t)dx/4 \\ &+\int_0^1 [\mu^{(i)}(x,t)]^T d_l(x,t)dx + \int_0^1 [\mu^{(i)}(x,t)]^T \eta_i(x)dx \\ &-\beta[e^{(i)}(t)]^T e^{(i)}(t)/2 + \beta[\mu^{(i)}(0,t)]^T \mu^{(i)}(0,t)/2 - \gamma[\tilde{\theta}_c^{(i)}]^T \tilde{\theta}_c^{(i)}/2 + \gamma\theta_{c\max}^2/2 \\ \le &-[\frac{\varepsilon\pi^2}{4} - \frac{\varepsilon(\beta+2)}{2}]\int_0^1 [\mu^{(i)}(x,t)]^T \mu^{(i)}(x,t)dx \\ &-\gamma[\tilde{\theta}_c^{(i)}]^T \tilde{\theta}_c^{(i)}/2 + \gamma\theta_{c\max}^2/2 + (\bar{d}_l^2 + \|\eta_i\|^2)/2\varepsilon\end{aligned}$$

where $\theta_{c\max} \ge \|\theta_c\|$. Therefore, the derivative of Lyapunov function will be less than zero if

$$\|\mu^{(i)}\| > \sqrt{2\gamma\theta_{c\max}^2 + \frac{2[\|\eta_i\|^2 + d_l^2]}{\varepsilon}} \text{ or } \|\tilde{\theta}_c^{(i)}\| > \sqrt{\frac{\|\eta_i\|^2 + d_l^2}{\varepsilon\gamma} + \theta_{c\max}^2}. \qquad \text{(A.6)}$$

With the bounds given by (A.6), the bound of isolation output residual defined by $\tilde{y}_i^{(i)}(t) = y_i(t) - \hat{y}_i^{(i)}(t)$ can be obtained as

$$\begin{aligned}
\left\|\tilde{y}_i^{(i)}(t)\right\| &= \left\|y_i(t) - \hat{y}_i^{(i)}(t)\right\| = \left\|\int_0^1 C(x)[v(x) - \hat{v}^{(i)}(x)]dx\right\| \\
&= \left\|\int_0^1 C(x)\tilde{v}^{(i)}(x)dx\right\| \le \bar{c}\sqrt{(2+4\bar{k}^2)}\left\|\tilde{z}^{(i)}(x)\right\|_{2,n} \\
&\le \bar{c}\sqrt{(2+4\bar{k}^2)}[\left\|\mu^{(i)}(x)\right\|_{2,n} + \left\|\tilde{\theta}_c^{(i)}\right\|\int_0^1\left\|F_c^{(i)}(x,t)\right\|dx] \\
&\le \bar{c}\sqrt{(2+4\bar{k}^2)}[\sqrt{2\gamma\theta_{c\max}^2 + 2[\|\eta_i\|^2 + d_l^2]/\varepsilon} + \sqrt{(\|\eta_i\|^2 + d_l^2)/\varepsilon\gamma + \theta_{c\max}^2}\int_0^1\left\|F_c^{(i)}(x,t)\right\|dx]
\end{aligned},$$

where $\tilde{v}^{(i)}(x) = v(x) - \hat{v}^{(i)}(x)$ is the state error. The bound on the magnitude of the isolation output error of the state fault estimator $\tilde{y}_i^{(i)}$ depends upon the value of $\|\eta_i\|$ and $\chi_i\int_0^1\left\|F_c^{(i)}(x,t)\right\|dx$. Because the mismatch function $\|\eta_i\|$ and $\chi_i\int_0^1\left\|F_c^{(i)}(x,t)\right\|dx$ varies with the distance between the actual fault and filter location given by $S_i = |x_f - x_i|$ yielding the magnitude of $\tilde{y}_i^{(i)}$ changes with the distance S_i. When the condition (51) is satisfied, the location of the state fault will be identified by comparing the isolation output residual generated by state fault estimators at different locations. The true fault location is determined as the one that is closest to the state fault estimator generating a residual that is minimum over others.

Proof of isolability condition for linear PDS: Upon detecting a fault, recalling equations given by (A.1) and (A.5) the actuator/sensor fault estimator error satisfies

$$e_m(t) = \mu(0,t) + h^{rm}(t) \ .$$

According to triangle inequality $\|a_1 + a_2\| \ge \|a_2\| - \|a_1\|$ and the equation above we can get $\|e_m(t)\| \ge \|h^{rm}(t)\| - \|\mu(0,t)\|$. If the condition (55) is satisfied and recall that $\|\mu(0,t)\| \le \rho + \bar{D}$, it is clear that

$$\|e_m(t)\| > 2\rho + \kappa_m(t)\|F_m(0,t)\| + D - \|\mu(0,t)\| \geq \rho + \kappa_m(t)\|F_m(0,t)\| = \rho_m(t)$$

where ρ_m is the threshold used for fault isolation defined by (30) and (37).

Proof of Theorem 5: A Lyapunov function candidate is selected as

$$V(t) = \|\Xi(t)\|_{2,n}^2 / 2c + \|\Xi_x(t)\|_{2,n}^2 / 2c$$

whose derivative with respective to t is obtained as

$$\begin{aligned}\dot{V}(t) &= \int_0^1 \Xi^T(x,t)\Xi_t(x,t)dx / c + \int_0^1 \Xi_x^T(x,t)\Xi_{xt}(x,t)dx / c \\ &= \int_0^1 \Xi^T(x,t)\Xi_{xx}(x,t)dx - b(t)\int_0^1 \Xi^T(x,t)\Xi(x,t)dx / c \\ &+ \int_0^1 \Xi^T(x,t)[d_M(x,t) + \varepsilon_{fM}(\tilde{v},x)]dx / c + \int_0^1 \Xi_x^T(x,t)d\Xi_t(x,t) / c\end{aligned},$$

Substitute the dynamics described by (68) and (69) to the equation above to get

$$\begin{aligned}\dot{V}(t) &\leq -\|\Xi_x(t)\|_{2,n}^2 - b(t)\|\Xi(t)\|_{2,n}^2 / c + \int_0^1 \Xi^T(x,t)[d_M(x,t) + \varepsilon_{fM}(\tilde{v},x)]dx / c \\ &- \|\Xi_{xx}(t)\|_{2,n}^2 + b(t)\int_0^1 \Xi_{xx}^T(x,t)\Xi(x,t)dx / c - \int_0^1 \Xi_{xx}^T(x,t)[d_M(x,t) + \varepsilon_{fM}(\tilde{v},x)]dx / c \\ &\leq -b(t)\|\Xi(t)\|_{2,n}^2 / c - [c + b(t)]\|\Xi_x(t)\|_{2,n}^2 / c - \|\Xi_{xx}(t)\|_{2,n}^2 \\ &+ \frac{(\bar{d}_M + \bar{\varepsilon}_{fM})}{c}\int_0^1 \left[\sqrt{\Xi^T(x,t)\Xi(x,t)} + \sqrt{\Xi_{xx}^T(x,t)\Xi_{xx}(x,t)}\right]dx \\ &\leq -\frac{1 + 16b(t)}{16c}\|\Xi(t)\|_{2,n}^2 - \frac{c + 2b(t)}{2c}\|\Xi_x(t)\|_{2,n}^2 + \frac{17c^2(\bar{d}_M + \bar{\varepsilon}_{fM})^2}{4},\end{aligned}$$

where $\bar{d}_M \geq \|d_M\|$ and $\bar{\varepsilon}_{fM} \geq \|\varepsilon_{fM}\|$. Therefore, $\dot{V}(t) < 0$ if one of the following conditions is satisfied

$$\|\Xi(t)\|_{2,n} > 2(\bar{d}_M + \bar{\varepsilon}_{fM})\sqrt{\frac{17c^3}{16b(t)+1}}, \text{ or } \|\Xi_x(t)\|_{2,n} > \sqrt{\frac{17c^3}{2[c + 2b(t)]}}(\bar{d}_M + \bar{\varepsilon}_{fM}).$$

By Agmon's inequality

$$\max_{x\in[0,1]} \|\Xi(x,t)\|_2^2 \leq 2\|\Xi(t)\|_{2,n}\|\Xi_x(t)\|_{2,n},$$

we can get $\|e(t)\| \le 2\sqrt{\frac{17c^3}{\sqrt{2[c+2b(t)][16b(t)+1]}}}(\bar{d}_M + \bar{\varepsilon}_{fM})$. Therefore, the detection residual is bounded and based on the bound defined above, a detection threshold ρ can be selected to assure that in the absence of faults the magnitude of the detection residual is below the threshold ρ all the time while the presence of a fault can cause the magnitude of the detection residual to increase and exceed ρ .

Proof of Theorem 6: In the presence of an actuator fault, the boundary conditions are modified as (59), and we can get $e_a(t) = h_a(u;t)$ for $t \ge t_f$. If $\|h_a(u;t_a)\| > \rho_a$, then it can be guaranteed that $\|e_a(t_a)\| > \rho_a$ and thus, an actuator fault is isolated. On the hand, the presence of a sensor fault or state fault will not cause $\|e_a(t)\|$ to go across the isolation threshold ρ_a for all $t_d < t \le t_i$.

In the case of a sensor fault, the sensor fault residual will become as $e_s(t) = Qv(0,t) - Q[v(0,t) + h_s(t)] = -Qh_s(t)$ for $t > t_f$ due to the sensor fault. It is obvious that if $\|Qh_s(t_s)\| > \rho_s$ then $\|e_s(t_s)\| > \rho_s$ and thus a sensor fault is isolated. However, the occurrence of an actuator or state faults will not make the magnitude of the sensor fault residual to exceed its threshold for all $t_d < t \le t_i$.

REFERENCES

[1] X. Zhang, M. M. Polycarpou and T. Parisini, "A robust detection and isolation scheme for abrupt and incipient faults in nonlinear systems," Automatic Control, IEEE Transactions on, vol. 47(4), pp. 576-593, 2002.

[2] R. J. Patton, "Robustness in model-based fault diagnosis: the 1995 situation," Annual reviews in control, vol. 21, pp. 103-123, 1997.

[3] H. Ferdowsi and S. Jagannathan, "A unified model-based fault diagnosis scheme for non-linear discrete-time systems with additive and multiplicative faults," Transactions of the Institute of Measurement and Control, 2013.

[4] C. Edwards, S. K. Spurgeon and R. J. Patton, "Sliding mode observers for fault detection and isolation," Automatica, vol. 36(4), pp. 541-553, 2000.

[5] R. J. Patton, J. Chen and C. J. Lopez-Toribio, "Fuzzy observers for nonlinear dynamic systems fault diagnosis," In Decision and Control, Proc. of the 37th IEEE Conference on, vol. 1, pp. 84-89, 1998.

[6] M. Patan and D. Ucinski, "Optimal activation strategy of discrete scanning sensors for fault detection in distributed parameter systems," Proc. of the 16th IFAC World Congress, Prague, Czech Republic, pp. 4-8, 2005.

[7] S. Omatu and J. H. Seinfeld, "Distributed parameter systems: theory and applications," Clarendon Press, 1989.

[8] A. Friedman, "Partial differential equations of parabolic type," Courier Corporation, 2013.

[9] P. D. Christofides, "Nonlinear and robust control of PDE systems: Methods and applications to transport-reaction processes," Springer Science & Business Media, 2001.

[10] H. Baruh, "Actuator failure detection in the control of distributed systems," Journal of Guidance, Control, and Dynamics, vol. 9(2), 181-189, 1986.

[11] M. A. Demetriou, K. Ito and R. C. Smith, "Adaptive monitoring and accommodation of nonlinear actuator faults in positive real infinite dimensional systems," Automatic Control, IEEE Transactions on 52, vol. 12, pp. 2332-2338, 2007.

[12] A. Baniamerian and K. Khorasani, "Fault detection and isolation of dissipative parabolic PDEs: Finite-dimensional geometric approach," In American Control Conference (ACC), pp. 5894-5899, June 2012.

[13] J. Cai, H. Ferdowsi, and S. Jagannathan, “Model-based fault detection, estimation, and prediction for a class of linear distributed parameter systems,” Automatica, vol. 66, pp. 122-131, 2016.

[14] T. Meurer, “On the extended Luenberger-type observer for semilinear distributed-parameter systems,” Automatic Control, IEEE Transactions on, 58(7), pp. 1732-1743, 2013.

[15] A. Baccoli, Y. Orlov and A. Pisano, “On the boundary control of coupled reaction–diffusion equations having the same diffusivity parameters,” Proc. CDC, Los Angeles, pp. 5222-5228, Dec 2014.

[16] M. Krstic and A. Smyshlyaev, “Boundary control of PDEs: A course on backstepping designs,” vol. 16, SIAM, 2008.

[17] G. H. Hardy, J. E. Littlewood and G. Polya, “Inequalities,” Cambridge university press, 1952.

[18] Y. Sakawa, “Observability and related problems for partial differential equations of parabolic type,” SIAM Journal on Control, vol. 13(1), pp. 14-27, 1975.

[19] P. DuChateau and D. Zachmann, “Applied partial differential equations,” Courier Corporation, 2012.

IV. FAULT DETECTION AND PREDICTION FOR A CLASS OF NONLINEAR DISTRIBUTED PARAMETER SYSTEMS WITH ACTUATOR OR SENSOR FAULTS

Jia Cai and S. Jagannathan

This paper presents a new model-based fault detection and prediction framework for a class of multi-input and multi-output (MIMO) nonlinear distributed parameter systems (DPS) described by partial differential equations (PDE) with actuator and sensor faults. The fault functions cover both abrupt and incipient faults. A Luenberger type observer is used to monitor the health of the DPS as a detection observer on the basis of the nonlinear PDE representation of the system with measured output vector. By taking the difference between measured and estimated outputs from this observer, a residual signal is generated for fault detection. If the detection residual exceeds a predefined threshold, a fault will be claimed to be active. Once an actuator or a sensor fault is detected and the fault type is identified, an appropriate fault parameter update law is developed to learn the fault dynamics online with the help of an additional output measurement. Later, an explicit formula is introduced to estimate the time-to-failure in the presence of an actuator/sensor fault by utilizing the limiting values of the output vector along with the estimated fault parameter vector. Eventually, the proposed detection and prediction framework is demonstrated on a nonlinear process.

1. INTRODUCTION

In order to improve reliability and availability of complex dynamic systems, reliable fault detection and prediction framework is needed even in the presence of unknown system uncertainties. Because of the high risk of component failures, faults occur before system failures and when a forthcoming system failure can be predicted through early fault diagnosis, maintenance can be scheduled in advance thus preventing unscheduled downtime.

Normally, fault diagnosis methods are categorized as either data-driven or model-based [1]. Usually, data-driven fault diagnosis methods require significant quantities of both healthy and faulty data which is costly and time consuming. In addition, online estimation of fault dynamics for the purpose of prognosis is not straightforward. In contrast, model-based fault diagnosis methods can minimize the requirements of a priori data [2] and can estimate fault function online.

Research has been in place to develop model-based fault detection and prediction schemes for lumped parameter systems (LPS) based on their system representation described by ordinary partial equations (ODE). An observer, which can provide an estimate of measured and unmeasured states, is utilized to detect faults in [3]. On the other hand, a robust fault diagnosis scheme was introduced in [4] whose detection residual was insensitive to uncertainties. A supplementary observer was utilized together with an observer in [5] to reconstruct the fault function and estimate the linear system state vector in the presence of unknown disturbances and system uncertainties. Authors of [6] presented a sliding-model observer by using an online fault-detection framework to estimate the fault signal.

In spite of attractive results, these research efforts [3-6] dealt with fault diagnosis for LPS represented by ODEs. However, many industrial processes like transport-reaction processes are classified as distributed parameter systems (DPS) modeled by partial differential equations [7] . Fault detection and prediction for DPS is more complicated when compared to LPS because of their distributed nature.

Controller design for DPS has drawn a lot of attention recently [8] and [9], but limited work has been done for fault detection and prediction of DPS. The existing fault diagnosis approaches [10-11] are based on the fact that the PDE model of DPS can be represented by an infinite set of ODEs. Subsequently, by applying Gelerkin's method an approximated finite dimensional ODE representation is obtained for fault diagnosis [12]. Based on the reduced order ODE representation, fault detection and accommodation schemes were developed in [10] and [11].

However, the fault detection and accommodation schemes [10] and [11] utilizing an approximated finite dimensional ODE to represent PDE model of a DPS may result in false and missed alarms due to model reduction. Instead, authors in [13] and [14] presented a model-based fault detection and prediction scheme for linear DPS with actuator faults directly using its PDE representation.

The filter based observer utilized in [13] is not applicable to the nonlinear DPS considered herein because superposition principle cannot be applied to nonlinear systems thus needing a novel detection observer. Detectability condition derived in [13] to define the class of detectable faults for a class of linear DPS with actuator or sensor faults need to be developed for nonlinear DPS. In addition, fault filters proposed by [14] are not implementable for nonlinear DPS to estimate the actuator or sensor fault signal because of the presence of system nonlinearity necessitating a new way to estimate the fault dynamics online.

In order to mitigate the problems mentioned above, a fault detection and prediction scheme is presented in this paper for multi-input and multi-output (MIMO) nonlinear DPS by utilizing a Luenberger type observer proposed in [15] which is introduced for controller design on the basis of original nonlinear PDE representation.

This Luenberger observer is extended to the MIMO case with actuator/sensor faults and unknown disturbances for the purpose of detection. Appropriate observer gains are selected to guarantee the stability of the linearized observer error dynamics around the estimated state, with which it can be shown that the observer error is bounded under healthy conditions [15]. In order to stabilize the dynamics of the observer error, time varying observer gains are needed because of the presence of the nonlinear term in the PDE model. Next, by taking the difference between the measured output vector and the estimated value from the observer, a detection residual is generated for nonlinear DPS.

In the presence of an actuator/sensor fault, the dynamics of nonlinear DPS will change causing the system state/output to deviate from the estimated one given by observer leading to an increase in magnitude of the detection residual. A fault is believed to have occurred when the magnitude of the detection residual exceeds a predefined detection threshold.

Once a fault is detected, an online estimator will be subsequently added to the observer to estimate the nonlinear fault dynamics by utilizing both the measured output and a secondary measured output which will be introduced later in the paper. Update law is developed to estimate the magnitude of the fault parameter vector.

The detection residual as well as the fault parameter estimation error is shown to be bounded in the presence of an actuator fault. The class of faults that can be detected by using this approach is derived as part of detectability conditions. Since it is not clear that the unknown parameter vector has a failure limit, and therefore by comparing the measured output with its failure limit, an explicit formula for online estimation of time to failure (TTF) or remaining useful life (RUL) is proposed.

Therefore, the contributions of this paper involve: a) the design of an online fault detection scheme including detectability conditions for nonlinear DPS with an actuator or sensor faults b) estimation of nonlinear fault dynamics with a novel parameter tuning law guaranteeing boundedness of estimation errors by using a second output, and c) TTF prediction provided a limited output measurements.

This paper is established as follows. Section 2 introduces the nonlinear DPS under research while Section 3 develops a fault detection and estimation scheme with performance and stability analysis in detail and derives an explicit formula to predict TTF. Subsequently, an example is used to illustrate its effectiveness of the proposed scheme.

2. SYSTEM DESCRIPTION AND FAULT FUNCTION

Before presenting the system description, the notation and the norm used throughout this paper is given [16]. A scalar function $v_1(x) \in L_2(0,l)$ means it is square integrable on the Hilbert space $L_2(0,l)$ with the corresponding norm

$$\|v_1\|_2 = \sqrt{\int_0^l v_1^2(x)dx} \ . \quad (1)$$

Now consider

$$[L_2(0,l)]^n = \underbrace{L_2(0,l) \times L_2(0,l) \times ... \times L_2(0,l)}_{n\ times}, \quad (2)$$

with the corresponding norm of a vector function $v(x,t) = [v_1(x,t), v_2(x,t), ..., v_n(x,t)]^T \in [L_2(0,l)]^n$ defined as

$$\|v\|_{2,n} = \sqrt{\sum_{i=1}^{n} \|v_i(x)\|_2^2} = \sqrt{\int_0^l v^T(x)v(x)dx} \ . \quad (3)$$

In addition, $\|\cdot\|$ denotes a Frobenius norm for a matrix or Euclidean norm for a vector. In order to save space, a vector, $v(x,t)$ and its partial derivatives are represented as

$$v_t(x,t) = \partial v(x,t) / \partial t\ ,\ v_x(x,t) = \partial v(x,t) / \partial x,\ \text{and}\ v_{xx}(x,t) = \partial^2 v(x,t) / \partial x^2 \ .$$

Next, the system under consideration is discussed.

2.1. SYSTEM DESCRIPTION

A class of n-dimensional nonlinear DPS, which can be expressed by the following parabolic partial differential equation (PDE), is described by

$$\frac{\partial v(x,t)}{\partial t} = c\frac{\partial^2 v(x,t)}{x^2} + f(v,x) + d(x,t), \quad (4)$$

with boundary conditions defined by

$$v_x\left(0,t\right)=Qv(0,t),\quad v(l,t)=u(t)\ , \tag{5}$$

$$y(t)=v(0,t)\,, \tag{6}$$

for $x\in(0,l)$ and $t\geq 0$, where $v(x,t)=[v_1(x,t),...,v_n(x,t)]^T\in[L_2(0,l)]^n$ represents the state vector of the DPS, $d(x,t)=[d_1(x,t),...d_n(x,t)]\in\Re^n$ stands for disturbance vector, $u(t)=[u_1(x,t),...,u_n(x,t)]^T\in\Re^n$ denotes the control input vector, $c>0$ is a positive constant, $Q\in\Re^{n\times n}$ is a full rank square matrix and $f(v,x)\in\Re^n$ is a Lipschitz continuous nonlinear vector function. In addition, $y(t)\in\Re^n$ is the system output vector measured at the opposite end of the actuator which is utilized for observer design and generating the detection residual.

In addition, for the sake of estimating an actuator and sensor fault, additional measurements defined by

$$y_a(t)=v(l,t)\,,\ y_s(t)=v_x(0,t)=Qv(0,t)\,, \tag{7}$$

are needed. The measurement $y_a(t)$ is required only when estimating the actuator fault parameter vector while $y_s(t)$ is needed for the sensor fault estimation.

Remark 1: The measurement $y_a(t)$ defined by (7) is used for actuator fault since the presence of an actuator fault will cause a change in $v(l,t)$. On the other hand, the measurement $y_s(t)$ defined by (7) is utilized to construct the correct value of the output using $v(0,t)=Q^{-1}y_s(t)$.

The output $y_s(t)$ is a derivative type measurement which means flux in the fluid flow systems. In addition, measurement defined by (7) is helpful for fault isolation. Next the following assumptions are required.

Assumption 1: The disturbance is bounded above such that $\|d(x,t)\| \leq \bar{d}$ for all x and $t \geq 0$, where $\bar{d} > 0$ is the upper bound of the disturbance which is a known constant. It is written as a general form in this paper whereas a more specific model is given in [17] and [18].

Remark 2: The disturbance bound given above is used to determine detection residual.

Assumption 2: The nonlinear vector function $f(v,x)$ satisfies the following conditions:

a. $f(v,x)$ is Lipschitz continuous in v, $\mathbb{C}^0$ in x, $\mathbb{C}^1$ in t and v for $x \in [0,l]$, $t \geq 0$ and $v(x) \in L_2(0,l)$.

b. $f(v,x)$ should satisfies

$$f(v+\Delta v, x) - f(v,x) = \frac{\partial f(v,x)}{\partial v}\Delta v + \varepsilon_f(\Delta v, x), \tag{8}$$

where Δv represents a small change in v and $\varepsilon_f(\Delta v, x)$ is the approximation error satisfying $\|\varepsilon_f\|_{2,n} \leq \bar{\varepsilon}_f$

Remark 3: Assumption (a) assures that $\frac{\partial f(v,x,t)}{\partial v}$ is bounded implying the linearization coefficient $A(t)$ of observer error dynamics which will be presented in Subsection 3.1 is bounded. In order to meet the requirement $\|\varepsilon_f\|_{2,n} \leq \bar{\varepsilon}_f$ in Assumption 1 (b), Δv is viewed small implying that the initial conditions of the observer which will be introduced in Section 3 should be close to the initial condition of the system described by (4), (5) and (6).

Assumption 3: The system is controllable and there exists a controller, $u(t)$, that can guarantee the stability of the system before and after the presence of an actuator/sensor fault.

Next the actuator and sensor fault function are considered.

2.2. ACTUATOR FAULT DESCRIPTION

In the presence of an actuator fault, the system described by (4) and (5) becomes (4) subject to the new boundary conditions given by

$$v_x\left(0,t\right)=0,\ v(l,t)=u(t)+h_a(u,t),\ y(t)=v(0,t)\,. \tag{9}$$

The system output defined by (5) with a sensor fault will be given by

$$y(t)=v(0,t)+h_s(t)\,, \tag{10}$$

whereas the actuator fault function $h_a(u,t)\in\Re^n$ and sensor fault function, $h_s(t)$, can be written as

$$\begin{aligned} h_a\left(u,t\right)&=\left[h_1^{(a)}\left(u,t\right),...,h_n^{(a)}\left(u,t\right)\right]^T=diag[\Omega_i(t-t_a)]\bar{h}_a(t)\\ h_s\left(t\right)&=\left[h_1^{(s)}\left(t\right),...,h_n^{(s)}\left(t\right)\right]^T=diag[\Omega_i(t-t_s)]\bar{h}_s(t)\end{aligned}\ , \tag{11}$$

with $\Omega_i(t-t_a)$ and $\Omega_i(t-t_s)$ represent the time profile of the actuator and actuator faults respectively defined by $\Omega_i(\tau)=\begin{cases}0 & ,if\ \tau<0\\ 1-e^{-\kappa_i\tau} & ,if\ \tau\geq 0\end{cases}$ for $i=1,...,n$, the constant κ_i represents the growth rate of the incipient fault, t_a and t_s denote fault occurrence time, $\bar{h}_a(t)$ and $\bar{h}_s(t)$ are fault magnitude.

Abrupt faults can also be modeled when a large κ_i is chosen. However, for the purpose of detection, only incipient faults are considered. The following assumption is needed on the fault function.

Assumption 4: The fault functions, $\bar{h}_a(u,t)$ and $\bar{h}_s(t)$, can be expressed as linear in the unknown parameters (LIP)[19] i.e. the fault function $\bar{h}_a\left(u,t\right)=W_a^T\phi_a(u(t),t)$ and $\bar{h}_s\left(t\right)=W_s^T\phi_s(t)$ where $W_a\in\Re^{n\times n}$ and $W_s\in\Re^{n\times n}$ are unknown actuator and sensor fault parameter matrix respectively satisfying $\|W_a\|\le\bar{W}_a$ and $\|W_s\|\le\bar{W}_s$, $\phi_a(u(t),t)=[\phi_1^{(a)}(u(t),t),...\phi_n^{(a)}(u(t),t)]^T\in\Re^n$ and $\phi_s(t)=[\phi_1^{(s)}(t),...\phi_n^{(s)}(t)]^T\in\Re^n$ are known nonlinear regression function.

In the next section, fault detection and estimation due to actuator and sensor faults are introduced.

3. ACTUATOR FAULT DETECTION AND ESTIMATION

In this section, an adaptive observer is designed in order to generate the estimated state and output of the DPS. It can be shown that under healthy conditions, the detection residual is ultimately bounded (UB). In the presence of an actuator or a sensor fault, the residual will increase and exceeds a predefined threshold since the fault acts as an unwanted input to the residual dynamics. After a fault is detected, an adaptive term to estimate the fault function is included in the observer. An update law tuned with an additional measurement, $y_a(t)$, will be utilized to estimate the actuator fault vector function. If a sensor fault is detected, the detection observer will be modified to estimate the fault function with an adaptive term and by using a second measurement $y_s(t)$.

3.1. DETECTION OBSERVER DESIGN

Instead of converting the DPS (4) and (5) into an infinite set of ODEs, define the fault detection observer along with boundary conditions from [15] given by

$$\frac{\partial \hat{v}(x,t)}{\partial t} = c\frac{\partial^2 \hat{v}(x,t)}{\partial x^2} + f(\hat{v},x) + P_1(x,t)(y-\hat{y}) \ , \tag{12}$$

$$\hat{v}_x\left(0,t\right) = Qv(0,t) + P_{10}(t)(y-\hat{y}),\ \hat{v}\left(l,t\right) = u(t) \ , \tag{13}$$

$$\hat{y}(t) = \hat{v}(0,t)\,, \tag{14}$$

where $\hat{v}(x,t) \in \Re^n$ is the observer state, $\hat{v}_t(x,t) = \frac{\partial \hat{v}(x,t)}{\partial t}$ and $\hat{v}_x(x,t) = \frac{\partial \hat{v}(x,t)}{\partial x}$ represents the first order of partial derivatives of $\hat{v}(x,t)$ with respect to the time t and to the space x , $P_1(x) \in \Re^{n\times n}$ and $P_{10} \in \Re^{n\times n}$ denote observer gain matrices which will be defined later.

Define $e(t) \in \Re^n = y(t) - \hat{y}(t)$ being the output estimation error or detection residual, to correct the state estimation error given by $\tilde{v} \in \Re^n = v - \hat{v}$, resulting from initial conditions. The observer gains of the Luenberger observer can be found in [15]and will be introduced briefly next.

By taking the difference between the observer dynamics in (12) and (13) and the actual system dynamics from (4) and (5), and by applying Assumption 2, the state estimation error dynamics under healthy conditions are given by

$$\tilde{v}_t(x,t) = c\tilde{v}_{xx}(x,t) + A(x,t)\tilde{v}(x,t) + \varepsilon_f(\tilde{v},x) - P_1(x,t)e(t) + d(x,t), \tag{15}$$

with boundary conditions

$$\tilde{v}_x(0,t) = -P_{10}(t)e(t), \quad \tilde{v}(l,t) = 0, \tag{16}$$

where $A(x,t) = \left.\dfrac{\partial f(v,x)}{\partial v}\right|_{v=\hat{v}} \in \Re^{n\times n}$ since $f(v,x) - f(\hat{v},x) = \left.\dfrac{\partial f(v,x)}{\partial v}\right|_{v=\hat{v}} \tilde{v}(x,t) + \varepsilon_f(\tilde{v},x)$.

Note that the observer error dynamics described by (15) has a term $A(t)\tilde{v}(x,t)$ which can make the system be unstable when $A(t)$ become positively large.

For the sake of eliminating this term, appropriate observer gains have to be selected through a transformation. Apply the Volterra integral transformation [15]

$$\tilde{v}(x,t) = \Xi(x,t) - \int_0^x L(x,\tau,t)\Xi(\tau,t)\,d\tau \ , \tag{17}$$

and select the observer gains $P_1(x,t)$ and $P_{10}(t)$ as [15]

$$P_1(x,t) = c\frac{\partial L(x,0,t)}{\partial \tau}, \quad P_{10}(t) = L(0,0,t), \tag{18}$$

to convert the observer error dynamics (15) and (16) into a stable system given by

$$\Xi_t(x,t) = c\Xi_{xx}(x,t) - b\Xi(x,t) + \varepsilon_{fM}(\tilde{v},x) + d_M(x,t), \tag{19}$$

$$\frac{\partial \Xi(0,t)}{\partial x} = 0, \quad \Xi\left(l,t\right) = 0 \ , \tag{20}$$

where $L(x,\tau,t)$ is the unique solution to the well-posed PDE [15]given by

$$\frac{\partial L(x,\tau,t)}{\partial t} = A(t)L(x,\tau,t) + bL(x,\tau,t) + c[\frac{\partial^2 L(x,\tau,t)}{\partial \tau^2} - \frac{\partial^2 L(x,\tau,t)}{\partial x^2}] \ , \tag{21}$$

$$L\left(l,\tau,t\right) = 0, \quad L\left(x,x,t\right) = \frac{(x-l)}{2c}[A(t) + bI_{n\times n}] \ , \tag{22}$$

$\Xi(x,t) \in \Re^n$, $L(x,\tau,t) \in \Re^{n\times n}$, $b > 0$ is an arbitrary scalar, $d_M(x,t) = d(x,t) + \int_0^x M(x,\eta,t)d(\eta,t)d\eta$, $\varepsilon_{fM}(\tilde{v},x) = \varepsilon_f(\tilde{v},x) + \int_0^x M(x,\eta,t)\varepsilon_f(\tilde{v},\eta)d\eta$ and $M(x,\eta,t) \in \Re^{n\times n}$ is the kennel matrix of the inverse transformation $\Xi\left(x,t\right) = \tilde{v}\left(x,t\right) + \int_0^x M\left(x,\eta,t\right)\tilde{v}\left(\eta,t\right)d\eta$.

The observer performance in the healthy conditions without any disturbances is discussed in [15]whereas in this paper, the observer is extended to the MIMO case in the presence of a fault and disturbances for the purpose of detection.

It will be shown that with the observer presented herein, the detection residual remains ultimately bounded (UB) during healthy conditions in the presence of bounded disturbances. A fault on the nonlinear DPS will drive the system state or output off the desired trajectory and thus cause the detection residual to increase. A fault is detected by comparing the detection residual $e(t)$ with a predefined threshold ρ . The threshold is selected by using both the initial conditions and the disturbances.

In addition it will be shown in the following theorem that a fault can be detectable provided it satisfies certain conditions given in the theorem. The following theorem demonstrates the stability of detection residual under healthy conditions and provides fault detectability conditions.

Theorem 1 (Fault detection observer performance): Let the observer given in (12) and (13) be used to monitor the DPS defined by (4) and (5). Then

a. The state estimation error $\tilde{v}$ and the output detection residual $e(t)$ will remain bounded under healthy operating conditions.

b. A fault is declared active when the output detection residual $e(t)$ crosses the detection threshold ρ.

c. An actuator and sensor faults can be detectable when the following are satisfied:

   i. An actuator fault $h_a(u,t)$ initiated at time t_a is detectable if there exists a time $T \geq t_a$ such that for all $t > T$

$$\left\| h_a(u,t) - \sum_{n=0}^{\infty} \int_{t_a}^{t} \left\{ \frac{2}{l} \int_0^l [\dot{h}_a(u,\tau) + b h_a(u,\tau)] \times \cos(\frac{2n+1}{2l}\pi x)\, dx \right\} e^{-[\varepsilon(\frac{2n+1}{2l}\pi)^2 + b](t-\tau)} d\tau \right\| > 2\rho \; ; \tag{23}$$

   ii. A sensor fault $h_s(t)$ initiated at time t_s is detectable if there exists a time $T > t_s$ such that for all $t > T$

$$\left\| \sum_{n=0}^{\infty} \int_{t_s}^{t} \left\{ \frac{2}{l} \int_0^l [h_{s1}(x,\tau) + b h_{s2}(\tau)(x-l) - \dot{h}_{s2}(\tau)(x-l)] \times \cos(\frac{2n+1}{2l}\pi x) dx \right\} e^{-\varepsilon[(\frac{2n+1}{2l}\pi)^2 + b](t-\tau)} d\tau - l h_{s2}(t) + h_s(t) \right\| > 2\rho, \tag{24}$$

where $h_{s1}(x,t) = -P_1(x,t)h_s(t) - \int_0^x M(x,\eta,t)P_1(\eta,t)h_s(t)d\eta$ and $h_{s2}(t) = -P_{10}(t)h_s(t)$.

Proof: See Appendix.

Remark 3: It is shown in the proof that under healthy conditions the detection residual is bounded by $\|e(t)\| \le k_l(\bar{\varepsilon}_{fM} + \bar{d}_M)$ where $k_l = 2l\sqrt{2(1+4l^2)}/c$, $\bar{\varepsilon}_{fM} \ge \|\varepsilon_{fM}\|$ and $\bar{d}_M \ge \|d_M\|$, and stays within the detection threshold ρ which is determined by the bound above. A fault acts as an unwanted input to the detection residual dynamics. The presence of a fault causes an increase in the detection residual beyond the threshold. The detectability condition given by the theorem 1 (c) defines the class of faults which can be detected by the proposed observer.

Before estimating the fault dynamics, one needs to determine the fault type first. By utilizing the measurements defined by (6) and (7), actuator and sensor fault isolation residuals are generated as $e_a = y_a(t) - \hat{v}(l,t)$ and $e_s(t) = y_s(t) - Qy(t)$ respectively. It is important to note that both these isolation residuals must be kept close to zero and should remain below their isolation thresholds ρ_a and ρ_s under healthy conditions. An actuator fault makes the magnitude of the actuator fault isolation residual $e_a(t)$ to increase and cross its isolation threshold ρ_a while it cannot change the magnitude of the sensor fault isolation residual $e_s(t)$. On the contrary, the presence of a sensor fault causes the magnitude of the sensor fault isolation residual $e_s(t)$ to increase and exceed its isolation threshold $\rho_s(t)$ while it will not have an effect the magnitude of the actuator fault residual $e_a(t)$. Based on the analysis above, the fault type is identified as the one exceeding its isolation threshold. More details on isolating faults will be studied in our future work.

The next step is to estimate the actuator and sensor fault functions.

3.2. ACTUATOR FAULT ESTIMATION

Upon detecting an actuator fault, an online estimator will be added to the observer defined by (12) and (13) to estimate the fault function. Then the boundary condition becomes

$$\frac{\partial \hat{v}(0,t)}{\partial x} = Qv(0,t) + P_{10}(y - \hat{y}), \tag{25}$$

$$\hat{v}(l,t) = u(t) + \hat{W}_a^T(t)\phi_a(u,t), \quad \hat{y}(t) = v(0,t), \tag{26}$$

where $\hat{W}_a(t)$ represents estimated parameter matrix and $\hat{W}_a^T(t)\phi_a(u,t)$ is the fault estimation. Thus the observer error dynamics is expressed as (15) with following boundary conditions

$$\tilde{v}_x(0,t) = -P_{10}(t)e(t), \quad \tilde{v}(l,t) = \tilde{W}_a^T(t)\phi_a(u,t), \tag{27}$$

where $\tilde{W}_a(t) = W_a - \hat{W}_a(t)$ is the parameter estimation error. Then with the transformation (17), the dynamics of the observer error can be converted to (19) subject to

$$\Xi_x(0,t) = 0, \quad \Xi(l,t) = \tilde{W}_a^T(t)\phi_a(u,t). \tag{28}$$

The performance of the adaptive approximation will be discussed in the next theorem.

Theorem 2 (Actuator fault function estimation): Let the online estimator be incorporated into the boundary conditions of the observer given by (12) and (13) to estimate the state and output of the system given by (4) and (9). Upon detecting an actuator fault, consider the boundary condition of the observer given by (13) is modified as (25) and (26) in order to estimate the state and output of the system defined in (4) and (5). In the presence of an actuator fault, let the parameter tuning law given by

$$\dot{\hat{W}}_a(t) = c\beta\phi_a(u,t)[y^T{}_a(t) - \hat{v}^T(l,t)] - \gamma\hat{W}_a(t), \tag{29}$$

is utilized to estimate the fault function with $\beta > 0$ being the tuning rate and $\gamma > 0$ is the stabilizing term. Then the observer and the parameter estimation errors are ultimately bounded (UB).

Proof: Refer to the Appendix.

Remark 4: The first term of the update law given by (29) is used to eliminate the extra term of the observer error dynamics caused by the actuator fault; the second term is added to relax the PE condition and to assure the boundedness of fault parameter estimation error. Here, the initial condition of the estimated fault parameter $\hat{W}_a(t_d)$ can set to zero if an incipient fault is considered since $W_a(t_a) = 0$ and t_a is the actuator fault occurrence time. However, in the case of an abrupt fault the initial condition of the estimated fault parameter should be close to the actual fault parameter in order to meet the requirement given by (8) to be satisfied. The proof shows that once an actuator fault is detected, if the boundary condition of the observer is modified as (25) and (26) with estimated fault parameter vector tuned by (29) the observer error and parameter estimation error will be UB.

It can be shown in the Appendix that $\Xi(x,t)$ and $\tilde{W}_a(t)$ are UB and thus it implies that the observer error $\tilde{v}(x,t)$ is also bounded. Next, the performance of the observer for detecting the sensor faults is described.

3.3. SENSOR FAULT ESTIMATION

When a sensor fault is detected, the observer defined by (10) is modified as

$$\hat{v}_t(x,t) = c\hat{v}_{xx}(x,t) + f(\hat{v},x,t) + P_1(x,t)[Q^{-1}y_s(t) - \hat{v}(0,t)], \tag{30}$$

subject to

$$\hat{v}_x\left(0,t\right)=y_s(t)+P_{10}(t)(y-\hat{y}),\ \hat{v}\left(l,t\right)=u(t)\ , \tag{31}$$

$$\hat{y}(t)=\hat{v}(0,t)+\hat{W}_s^T(t)\phi_s(t)\,, \tag{32}$$

where $\hat{W}_s^T(t)\phi_s(t)$ is the adaptive term to estimate the sensor fault. Thus, the observer error dynamics is expressed as (15) with boundary conditions

$$\tilde{v}_x\left(0,t\right)=-P_{10}(t)\tilde{v}(0,t)-P_{10}(t)\tilde{W}_s^T(t)\phi_s(t),\ \tilde{v}\left(l,t\right)=0\,, \tag{33}$$

where $\tilde{W}_s^T(t)\phi_s(t)$ is the sensor fault estimation error and $\tilde{W}_s(t)=W_s-\hat{W}_s(t)$ is the parameter estimation error.

Apply the transformation (17) to the observer error dynamics given by (15) and (33) to get (19) subject to

$$\frac{\partial\Xi(0,t)}{\partial x}=-P_{10}(t)h_s(t),\quad \Xi\left(l,t\right)=0\,. \tag{34}$$

Notice that the sensor fault effecting the measurements defined by (7) cannot cause the detection residual $e(t)$ to increase but will make the error defined by $e_s(t)=Qy(t)-y_s(t)$ to grow, so it can be easily isolated from the sensor fault defined by (10). The performance of the adaptive estimation will be shown in the next theorem.

Theorem 3 (Sensor fault function estimation): Let the online approximator be added to the estimated output of the observer as (30), (31) and (32) to estimate the state and output of the system given by (4), (5) and (6). Upon detecting a sensor fault, consider the parameter tuning law

$$\dot{\hat{W}}_s(t)=c\beta\phi_s(t)[y_s^T(t)Q^{-T}-\hat{v}^T(0,t)]P_{10}(t)-\gamma\hat{W}_s(t)\,, \tag{35}$$

to estimate the sensor fault function where $\beta > 0$ is the tuning rate and $\gamma > 0$ is the robust term. Then the observer error described by (30), (31) and (32) and the sensor fault estimation error are UB.

Proof: See Appendix.

Remark 5: The first term of the updated law proposed by (35) is utilized to eliminate the extra term of the observer error dynamics induced by the sensor fault; the second term is used to overcome the PE condition and to guarantee the boundedness of the fault parameter estimation error at the same time. It will be shown in the Appendix that in the presence of a sensor fault, by incorporating the adaptive term $\hat{W}_s^T(t)\phi_s(u,t)$ into the observer, with update law defined by (35), the observer as well as the parameter vector estimation errors will be UB.

3.4. FAILURE PREDICTION

It has been shown in the previous sections that in the presence of an actuator fault, the output will deviate from the desired trajectory. Unlike the TTF prediction scheme using the failure limit of the fault parameter presented in [10] the estimated TTF proposed next is based on the deviation of the output to predict the remaining useful life of the system since sometimes the limit of the fault parameter is not available while the output limit is more reasonable. The estimated TTF is defined as the time available before the output reaches its limit value. The following derivation gives an explicit formula to estimate the TTF.

3.4.1. Actuator Fault. Given an initial value of the output, estimated states, and the limiting value for the output, upon detecting an actuator fault and activating the fault estimation scheme, the TTF can be obtained as

$$TTF(t) = \min_{1 \le i \le n} t_{f(i)}^{(a)}(t)\,, \tag{36}$$

where

$$t_{f(i)}^{(a)} = \frac{\bar{y}_i(t) - v_i(0,t)}{\frac{c[\hat{v}_i(2\delta,t) - 2\hat{v}_i(\delta,t) + \hat{v}_i(0,t)]}{\delta^2} + f(v_i(0,\mathrm{t}),0)}\,. \tag{37}$$

The formula (36) is derived based on the system dynamics given by (4). By applying the finite difference method, $v_{xx}(x,t)$ can be obtained as

$$v_{xx}(x,t) = \lim_{\delta \to 0^+} \frac{v(x+2\delta,t) - 2v(x+\delta,t) + v(x,t)}{\delta^2}\,.$$

Therefore we can get $\dot{y}(t) = v_t(0,t)$ expressed as

$$\dot{y}(t) = [v(2\delta,t) - 2v(\delta,t) + v(0,t)]/\delta^2 + f(v(0,t),0) + d(0,t)\;, \tag{38}$$

where $\delta > 0$ is a sampling interval. We assume that the term $[v(2\delta,t) - 2v(\delta,t) + v(0,t)]/\delta^2 + f(v(0,t),0)$ is held within a small time interval $[t, t_{r(i)}]$ for the purpose of prediction and let $t_{r(i)}$ be the first time when the value of $y_i(t)$ reaches its limit value $\bar{y}_i$. The solution $y(t) = [y_1(t), ..., y_n(t))]^T$ to (38) at $t_{r(i)}$ is approximated as $\bar{y}_i - v_i(0,t) = \left[c\frac{v_i(2\delta,t) - 2v_i(\delta,t) + v_i(0,t)}{\delta^2} + f(v_i(0,t),0)\right]\left(t_{r(i)} - t\right)$, for $i = 1, 2, ..., n$. By substituting $t_{f(i)} = t_{r(i)} - t$ in the equation above, we can get

$$t_{f(i)=} \frac{\bar{y}_i(t) - v_i(0,t)}{c\frac{v_i(2\delta,t) - 2v_i(\delta,t) + v_i(0,t)}{\delta^2} + f(v_i(0,\mathrm{t}),0)}\;. \tag{39}$$

In (39), because $v_i(2\delta,t)$ and $v_i(\delta,t)$ are not known in advance, we need to use $\hat{v}_i(2\delta,t)$ and $\hat{v}_i(\delta,t)$ instead for $t_{f(i)}$ as given before. Each element of the output vector must be less than its limit, so the overall TTF is defined as the minimum among all the individual TTF given by (37).

3.4.2. Sensor Fault. The TTF prediction in the presence of a sensor is estimated as

$$TTF(t) = \min_{1 \le i \le n} t_{f(i)}^{(s)}(t)\,, \tag{40}$$

where

$$t_{f(i)}^{(s)} = \frac{\bar{y}_i\left(t\right) - \hat{w}_{s(i)}^{T}(t)\phi_s(t) - v_i(0,t)}{c\dfrac{\hat{v}_i(2\delta,t) - 2\hat{v}_i(\delta,t) + \hat{v}_i(0,t)}{\delta^2} + f\left(v_i(0,\mathrm{t}),0\right)}. \tag{41}$$

Similar as the actuator fault, we can approximate $v_t(0,t)$ as

$$v_t(0,t) = [v(2\delta,t) - 2v(\delta,t) + v(0,t)]/\delta^2 + f(v(0,t),0) + d(x,t).$$

Solve the equation above and assume that $[v(2\delta,t) - 2v(\delta,t) + v(0,t)]/\delta^2 + f(v(0,t),0)$ is held in the interval $\left[t,\bar{t}\,\right]$ to approximate $v(0,t)$ as

$$v(0,\bar{t}\,) - v(0,t) = \left[c\frac{v(2\delta,t) - 2v(\delta,t) + v(0,t)}{\delta^2} + f(v_i(0,t),0)\right]\left(\bar{t} - t\right), \tag{42}$$

where t represents the current time and $\bar{t}$ denotes the future time. When $\hat{w}_s^T(i)\phi_s(t)$ is held in the interval $\left[t,\bar{t}\,\right]$ we can approximate $y(\bar{t}\,)$ as

$$y_i(\bar{t}\,) = v_i(0,\bar{t}\,) + \hat{w}_{s(i)}^{T}(t)\phi_s(t).$$

where $\hat{W}_s(t) = [\hat{w}_{s(1)},\ldots,\hat{w}_{s(n)}]$ and $i = 1,2,\ldots n$. Substituting $\bar{t} = t_{r(i)}^{(s)}$, $y_i(t_{r(i)}^{(s)}) = \bar{y}_i$, $t_{f(i)}^{(s)} = t_{r(i)}^{(s)} - t$ and the equation above to the equation given by (42) we can obtain

$$t_{f(i)}^{(s)} = \frac{\bar{y}_i\left(t\right) - \hat{w}_{s(i)}^{T}(t)\phi_s(t) - v_i(0,t)}{c\dfrac{v_i(2\delta,t) - 2v_i(\delta,t) + v_i(0,t)}{\delta^2} + f\left(v_i(0,\mathrm{t}),0\right)},$$

where $t_{f(i)}^{(s)}$ represents the first time i^{th} output reaching its limit value. Since $v_i(2\delta,t)$ and $v_i(\delta,t)$ are not known, we will use $\hat{v}_i(2\delta,t)$ and $\hat{v}_i(\delta,t)$ instead in the formula which has been given before. The overall TTF in the presence of the sensor fault is defined as the minimum among all the individual TTF described by (41).

4. SIMULATION RESULTS

In order to evaluate the effectiveness of the presented fault detection and prediction scheme, a nonlinear DPS whose system state can be represented by a parabolic PDE is considered next. The system dynamics in the presence of an actuator fault can be described as

$$\frac{\partial v_1(x,t)}{\partial t} = \frac{\partial^2 v_1(x,t)}{\partial x^2} + 1.2v_1^2(x,t) + 10v_1(x,t) + 2v_2(x,t) + 0.05e^{-(x-0.5)^2}\sin(2t),$$

$$\frac{\partial v_2(x,t)}{\partial t} = \frac{\partial^2 v_2(x,t)}{\partial x^2} + 1.5v_2^2(x,t) + 1.5v_1(x,t) + 10v_2(x,t) + 0.03e^{-(x-0.3)^2}\cos(3t),$$

subject to

$$\frac{\partial v_1(0,t)}{\partial x} = 2v_1(0,t), \quad v_1(1,t) = u_1(t) + \theta_1\sigma_1(t),$$

$$\frac{\partial v_2(0,t)}{\partial x} = 2v_2(0,t), \quad v_2(1,t) = u_2(t) + \theta_2\sigma_2(t),$$

where $v(x,t) = \left[v_1(x,t), v_2(x,t)\right]^T$ represents the system state at the position $x \in [0,1]$ with time $t \geq 0$ and $\sigma_i(u_i,t) = \begin{cases} |u_i(t)|, \; if \;\; |u_i(t)| < \bar{u}_i \\ \bar{u}_i, \quad else \end{cases}$, $i = 1,2$ where $\bar{u}_i$ is the maximum value of the actuator output with $\bar{u}_i = 3$. The desired output trajectory is chosen as $y_d(t) = [0.8(1-e^{-0.6t}), 0.7(1-e^{-0.5t})]^T$. Fault parameters of the actuator fault initiating at $t = 6s$ are given by $\theta_1(t) = 1.5\Omega_1(t-4)$ and $\theta_2(t) = 1.8\Omega_2(t-4)$ where $\Omega_i(t-4)$ is defined as $\Omega_i(t-4) = \begin{cases} 0 & , if \;\; t < 4 \\ 1 - e^{-\kappa_i(t-4)}, & if \;\; t \geq 4 \end{cases}$ with $\kappa_1 = 0.3$ and $\kappa_2 = 0.6$.

Simulation results are obtained by using MATLAB, in order to solve the PDEs of closed-loop system and detection observer, finite difference method with 20 points grid are applied to discretize the entire space $0 \le x \le 1$. Next this example is utilized to demonstrate the performance of the proposed detection and prediction scheme.

The detection observer defined as (12) and (13) with observer gain selected as (18) is applied to generate the detection residual which is defined as the difference between the actual output and estimated output given by the detection observer. Figure 4.1 shows the detection residual with different initial conditions under healthy conditions, it can be observed that detection residual is bounded in all the cases. In this example, the initial conditions of the DPS are selected as $v_1(x,0) = v_2(x,0) = 0.2$ and the one for observer is set as zero. It is shown in Figure 4.2 that observer errors are maintained bounded in the absence of any faults.

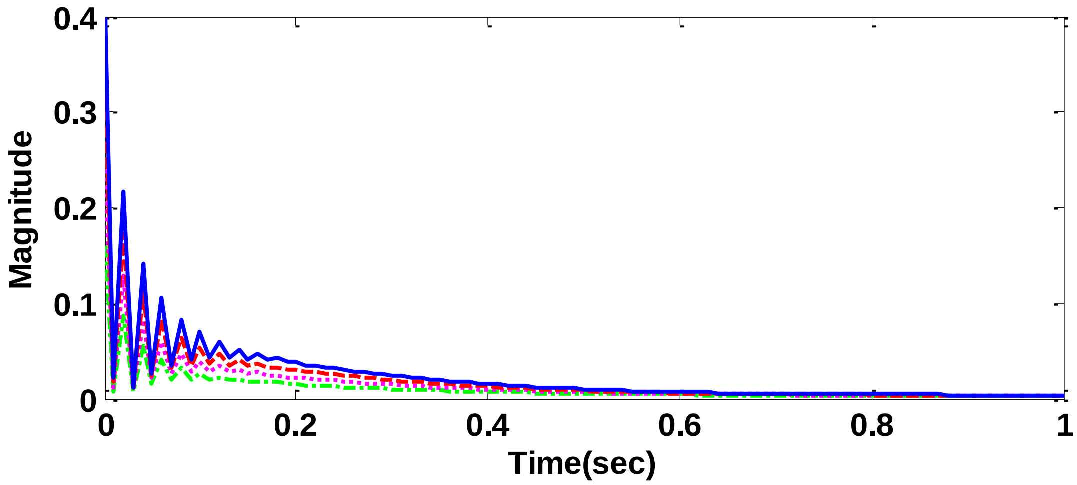
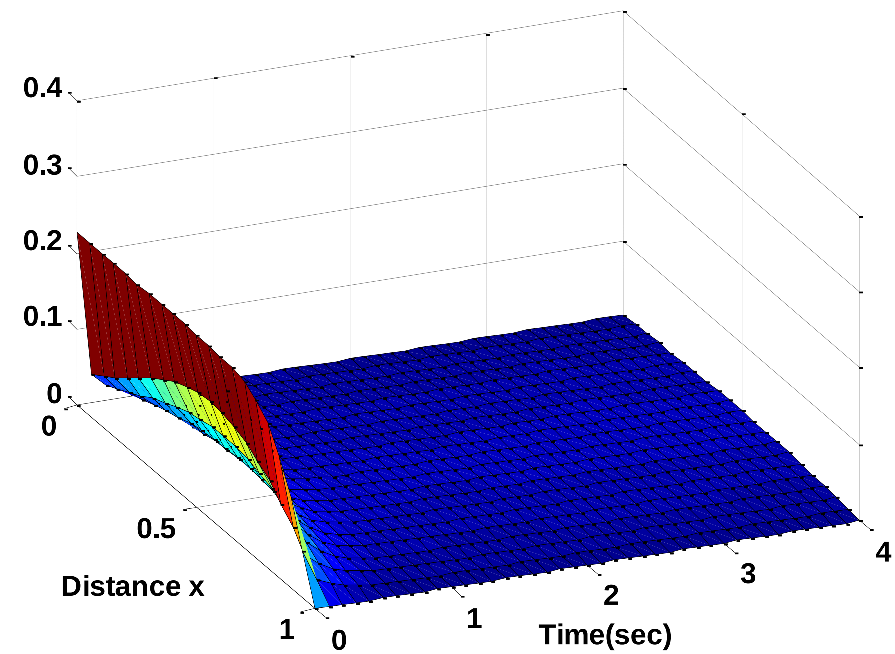
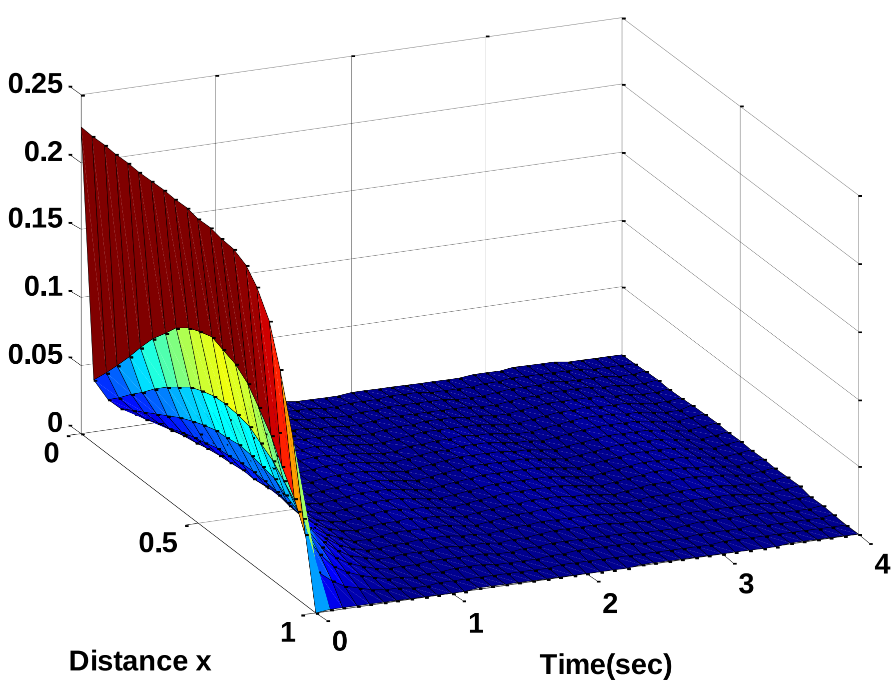


Figure 4.1. Detection Residual with different initial conditions.



(a)



(b)

Figure 4.2. Observer error under healthy conditions of (a) v_1 and (b) v_2.

Notice that the detection residual can be maintained below a predefined threshold (the solid line shown in Figure 4.3) under healthy condition when the time is before t = 4s. As observed from Figure 4.3, once an actuator fault occurs, and due to the fact that the output of the system diverges from the estimated output, the detection residual starts to increase and exceed the predefined threshold implying the occurrence of a fault. Upon detecting an actuator fault, an adaptive term will be incorporated into the detection observer to estimate the actuator fault parameters θ_1 and θ_2 .

From Figure 4.3, we can see that the fault can be detected at about t = 5s, then the fault parameter is estimated and after some quick adaptation it can match with the actual fault parameter vector which is shown in Figure 4.4.

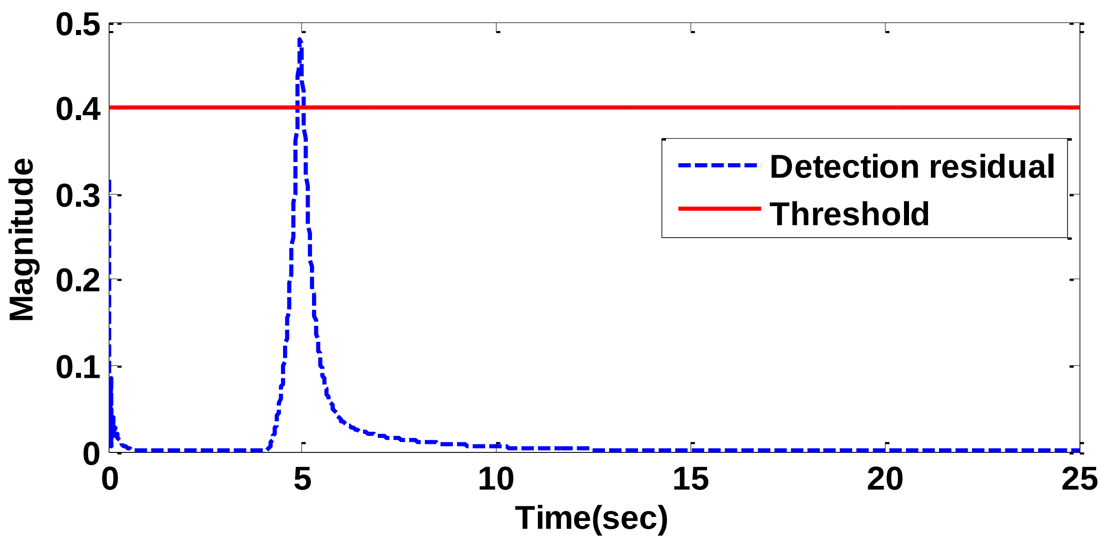


Figure 4.3. Actuator fault detection result.

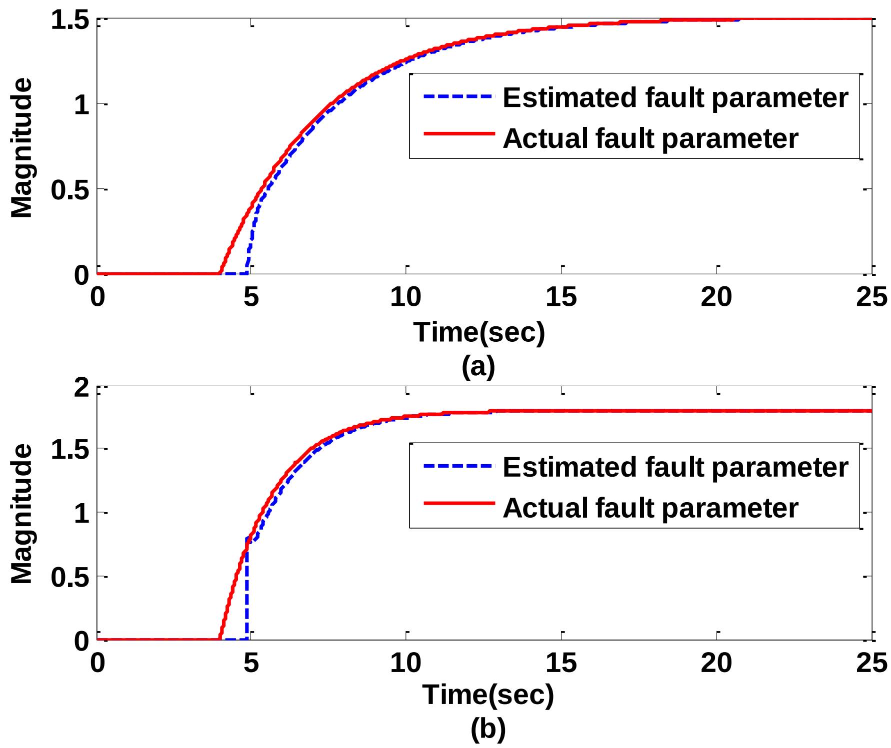


Figure 4.4. Actuator fault parameters estimation of (a) θ_1 and (b) θ_2.

Based on the dynamics of the output and given the limit values of the outputs as $\bar{y}_1 = 2$ and $\bar{y}_2 = 1.8$, we can estimate the overall TTF which can predict the time when the first output reaches its limit value. Figure 4.5 plots the TTF by utilizing the proposed formula (36).

As discussed in Section 3.3, the proposed fault detection and estimation framework can deal with sensor fault as well. The outputs with a sensor fault are expressed as

$$y_1(t) = v_1(0,t) + \theta_1^{(s)}(t)\left|y_{d1}(t)\right|,$$

$$y_2(t) = v_2(0,t) + \theta_2^{(s)}(t)\left|y_{d2}(t)\right|.$$

where $\theta_1^{(s)} = 1.5\Omega_1(t-5)$, $\theta_2^{(s)} = 1.5\Omega_2(t-5)$ and the fault time profile is defined as $\Omega_i(t-5) = \begin{cases} 0 & , if\ t < 5 \\ 1-e^{-\kappa_5(t-5)} & , if\ t \geq 5 \end{cases}$, $i = 1,2$ with $\kappa_1 = 0.3$ and $\kappa_2 = 0.6$. Once a sensor fault is detected as shown in the Figure 4.6, the measurement defined in (7) will be used to reconstruct the observer given by (30), (31) and (32) with an adaptive term.

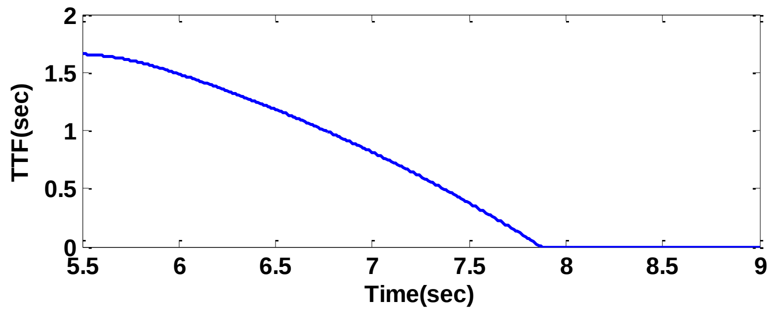


Figure 4.5. Estimated time-to-failure for an actuator fault.

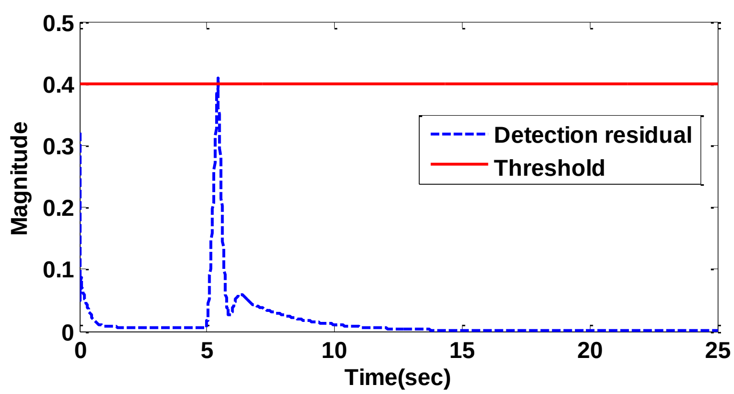


Figure 4.6. Sensor fault detection result.

Figure 4.7 shows that with the update law proposed in (35) the sensor fault parameters are tuned correctly within 2 seconds. Similar to the case of actuator fault, given the limiting value of the output, the time to failure can be estimated by using the formula given by (40) which can be observed in Figure 4.8.

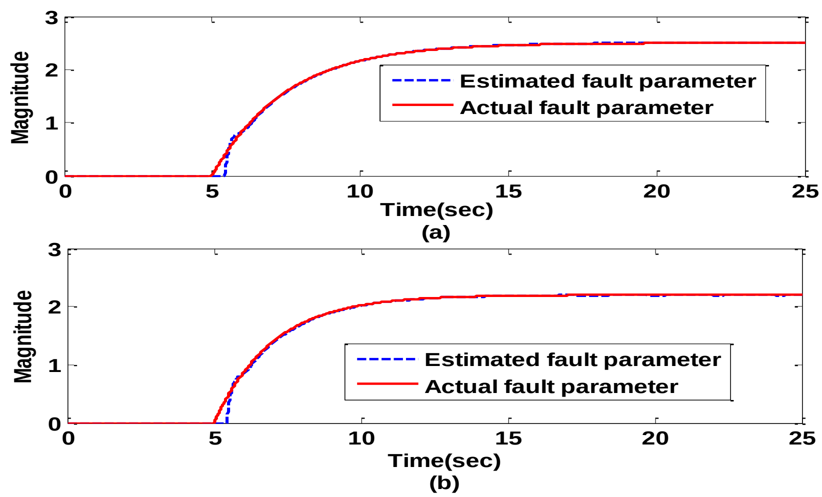


Figure 4.7. Sensor fault parameters estimation of (a) $\theta_1^{(s)}$ and $\theta_2^{(s)}$.

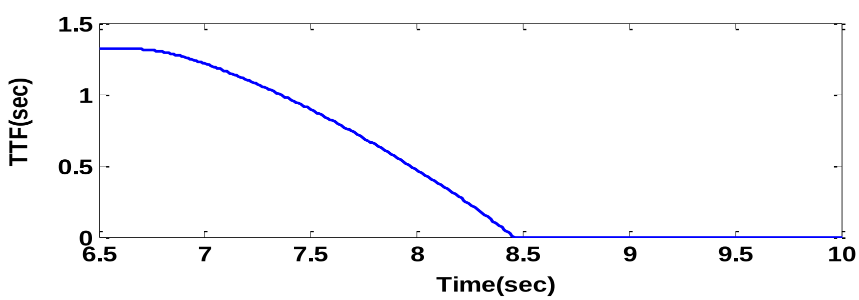


Figure 4.8. Estimated time-to-failure for a sensor fault.

5. CONCLUSIONS

This paper investigates the fault detection and prediction for a class of nonlinear DPS. The fault detection scheme which is developed based on a Luenberger type observer is capable of monitoring the behavior of nonlinear DPS with only boundary measurement. The proposed detection observer using nonlinear PDE representation provides accurate estimation of the measured and unmeasured state vector of the DPS provided measured output is available. Estimated fault dynamics given by the online estimator can assist in failure prediction and root cause analysis which is useful for maintenance schedule. Though this approach is generic, however, the proposed fault detection and prediction framework is limited to the class of nonlinear DPS represented by the parabolic PDE specified in this paper.

APPENDIX

Proof of Theorem 1: It is known that the transformation (17) can map the state residual dynamics into the target system given by (19) and (20) if the observer gains $P_1(x)$ and P_{10} are defined by (18). The stability of the residual dynamics can be concluded from the stability of the target system given by (19) and (20) due to the transformation made possible by (15) [20].

To discuss the stability of the PDE described in (19) with boundary conditions given by (20), one must select a positive definite Lyapunov function candidate, which is half of the squared Sobolev norm of the spatial profile defined in a Hilbert space $H_1(0,l)$ as per [20] and it is given by

$$V(t) = \|\Xi(t)\|_{2,n}^2 / 2 + \|\Xi_x(t)\|_{2,n}^2 / 2 = \int_0^l \Xi^T(x,t)\Xi(x,t)dx / 2 + \int_0^l \Xi_x^2(x,t)dx / 2,$$

The derivative of $V(t)$ with respect to t is obtained as

$$\dot{V}(t) = \int_0^l \Xi^T(x,t)\Xi_t(x,t)dx + \int_0^l \Xi_x^T(x,t)\Xi_{tx}(x,t)dx \; .$$

Substituting the equation (19) and (20) into the equation above and applying integration by parts, we will arrive at

$$\begin{aligned}
\dot{V}(t) &= c\int_0^l \Xi^T(x,t)\Xi_{xx}(x,t)dx - b\int_0^l \Xi^T(x,t)\Xi(x,t)dx \\
&\quad + \int_0^l \Xi^T(x,t)[\varepsilon_{fM}(\tilde{v},x,t) + d_M(v,x,t)]dx + \int_0^l \Xi_x^T(x,t)d\Xi_t(x,t) \\
&= -c\int_0^1 \Xi_x^T(x,t)\Xi_x(x,t)dx - b\int_0^l \Xi^T(x,t)\Xi(x,t)dx \\
&+\int_0^l \Xi^T(x,t)[\varepsilon_{fM}(\tilde{v},x,t) + d_M(v,x,t)]dx - \int_0^l \Xi_{xx}^T(x,t)\Xi_t(x,t)dx \\
&= -c\int_0^l \Xi_x^T(x,t)\Xi_x(x,t)dx - b\int_0^l \Xi^T(x,t)\Xi(x,t)dx + \int_0^l \Xi^T(x,t)[\varepsilon_{fM}(\tilde{v},x,t) + d_M(v,x,t)]dx \\
&-c\int_0^l \Xi_{xx}^T(x,t)\Xi_{xx}(x,t)dx - b\int_0^l \Xi_x^T(x,t)\Xi_x(x,t)dx - \int_0^l \Xi_{xx}^T(x,t)[\varepsilon_{fM}(\tilde{v},x,t) + d_M(v,x,t)]dx
\end{aligned}$$

$$\leq -c\|\Xi_x\|_{2,n} - c\|\Xi_{xx}\|_{2,n} + (\bar{\varepsilon}_{fM} + \bar{d}_M)[\int_0^l \sqrt{\Xi^T(x,t)\Xi(x,t)}dx$$
$$-b\|\Xi\|_{2,n} - b\|\Xi_x\|_{2,n} + \int_0^l \sqrt{\Xi_{xx}^T(x,t)\Xi_{xx}(x,t)}dx].$$

Then we apply Poincare inequality [22] $\|\Xi\|_{2,n} \leq 4l^2\|\Xi_x\|_{2,n}$ and $\|\Xi_x\|_{2,n} \leq 4l^2\|\Xi_{xx}\|_{2,n}$ to the equation above to get

$$\dot{V}(t) \leq -\left(\frac{c}{8l^2} + b\right)\|\Xi\|_{2,n}^2 - \left(\frac{c}{8l^2} + b\right)\|\Xi_x\|_{2,n}^2 + \frac{(1+4l^2)(\bar{\varepsilon}_{fM} + \bar{d}_M)^2}{2c}.$$

Therefore $\dot{V}(t) < 0$ if

$$\|\Xi\|_{2,n} > \frac{2l\sqrt{1+4l^2}}{\sqrt{c(c+8bl^2)}}(\bar{\varepsilon}_{fM} + \bar{d}_M) \text{ or } \|\Xi_x\|_{2,n} > \frac{2l\sqrt{1+4l^2}}{\sqrt{c(c+8bl^2)}}(\bar{\varepsilon}_{fM} + \bar{d}_M).$$

By using Agmon's inequality [21]we get

$$\|e(t)\| \leq \max_{x\in[0,l]}\|\Xi(x,t)\| \leq 2\|\Xi\|_{2,n}\|\Xi\|_{2,n} \leq k_l(\bar{\varepsilon}_{fM} + \bar{d}_M),$$

where $k_l = 2l\sqrt{2(1+4l^2)}/c$, $\bar{\varepsilon}_{fM} \geq \|\varepsilon_{fM}\|$ and $\bar{d}_M \geq \|d_M\|$, which means the detection error $e(t)$ will remain bounded under healthy conditions. The selection of the detection threshold ρ depends on the initial condition as well as the bound given above.

In the presence of an actuator fault at t_a, the boundary condition of the observer error dynamics from (15) will be modified as

$$\tilde{v}_x(0,t) = -P_{10}(t)e(t), \quad \tilde{v}(l,t) = h_a(u,t),$$

for $t \geq t_a$. Apply the transformation (17) to (15) and the equation above to get (19) with following boundary conditions

$$\Xi_x(0,t) = 0, \quad \Xi(l,t) = h_a(u,t). \tag{A.1}$$

Solving the PDE described by (19) and (A.1) we can obtain

$$\Xi(x,t)=\sum_{n=0}^{\infty}\tilde{v}_n(t_a)e^{-\beta_n(t-t_a)}\cos(\frac{2n+1}{2l}\pi x)d\tau+h_a(u,t)$$
$$+\sum_{n=0}^{\infty}\int_{t_a}^{t}\left[\varepsilon_m(\tau)+d_m(\tau)-h_{am}(\tau)\right]e^{-\beta_n(t-\tau)}\cos(\frac{2n+1}{2l}\pi x)d\tau, \tag{A.2}$$

where $\varepsilon_m(t)=\frac{2}{l}\int_0^l \varepsilon_{fM}(\tilde{v},x,t)\cos(\frac{2n+1}{2l}\pi x)dx$, $d_m(t)=\frac{2}{l}\int_0^l d_M(v,x,t)\cos(\frac{2n+1}{2l}\pi x)dx$

and $h_{am}(x,t)=\frac{2}{l}\int_0^l[\dot{h}_a(u,x)+bh_a(u,t)]\cos(\frac{2n+1}{2l}\pi x)dx$. Set $x=0$ in the solution given by (A.2) to get

$$e(t)=\Xi(0,t)=\sum_{n=0}^{\infty}\tilde{v}_n(t_a)e^{-\beta_n(t-t_a)}d\tau+h_a(u,t)+\sum_{n=0}^{\infty}\int_{t_a}^{t}\left[\varepsilon_m(\tau)+d_m(\tau)-h_{am}(\tau)\right]e^{-\beta_n(t-\tau)}d\tau,$$

where $\beta_n=\varepsilon(\frac{2n+1}{2l}\pi)^2+b$. Take norm on the both sides of the equation above and apply triangle inequality to get

$$\|e(t)\|\geq\left\|h_a(u,t)-\sum_{n=0}^{\infty}\int_{t_a}^{t}h_{am}(\tau)e^{-\beta_n(t-\tau)}d\tau\right\|-$$
$$\left\|\sum_{n=0}^{\infty}\tilde{v}_n(t_a)e^{-\beta_n(t-t_a)}d\tau+\sum_{n=0}^{\infty}\int_{t_a}^{t}\left[\varepsilon_m(\tau)+d_m(\tau)\right]e^{-\beta_n(t-\tau)}d\tau\right\|>2\rho-\rho=\rho,$$

when (23) holds and the detection threshold is chosen as

$$\rho=\kappa_a\left\|\sum_{n=0}^{\infty}e^{-\beta_n(t-t_s)}\tilde{v}_n(t_a)+\sum_{n=0}^{\infty}\int_{t_a}^{t}e^{-\beta_n(t-\tau)}[\varepsilon_m(\tau)+d_m(\tau)]d\tau\right\|,$$

where $\kappa_a>1$, which complete the proof.

When a sensor fault occurs, the dynamics of the observer error are changed to

$$\begin{aligned}\tilde{v}_t(x,t)&=c\tilde{v}_{xx}(x,t)+A(t)\tilde{v}(x,t)+\varepsilon_f(\tilde{v},x,t)\\&\quad-P_1(x,t)v(0,t)-P_1(x,t)h_s(t)+d(v,x,t)\end{aligned},$$

subject to

$$\tilde{v}_x(0,t)=-P_{10}(t)\tilde{v}(0,t)-Qh_s(t)-L_0(t)h_s(t),\tilde{v}(l,t)=0,$$

for $t \geq t_s$ By applying the transformation (17) to the dynamics above we can get

$$\frac{\partial \Xi(x,t)}{\partial t} = c\frac{\partial^2 \Xi(x,t)}{\partial t^2} - b\Xi(x,t) + \varepsilon_{fM}(\tilde{v},x,t) + d_M(v,x,t) + h_{s1}(x,t),$$

$$\frac{\partial \Xi(0,t)}{\partial x} = h_{s1}(t), \quad \Xi(l,t) = 0.$$

Solving the PDE represented by the equations above yields

$$\tilde{v}(0,t) = \Xi(0,t) = \sum_{n=0}^{\infty} e^{-\beta_n(t-t_s)}\tilde{v}_n(0) - lh_{s2}(t) + \sum_{n=0}^{\infty}\int_{t_s}^{t} e^{-\beta_n(t-\tau)}[\varepsilon_m(\tau) + d_m(\tau) + h_{sm}(\tau)]d\tau.$$

Then the detection residual is obtained as

$$e(t) = \tilde{v}(0,t) + h_s(t) = \sum_{n=0}^{\infty} e^{-\beta_n(t-t_s)}\tilde{v}_n(t_s) - lh_{s2}(t)$$
$$+ \sum_{n=0}^{\infty}\int_{t_s}^{t} e^{-\beta_n(t-\tau)}[\varepsilon_m(\tau) + d_m(\tau) + h_{sm}(\tau)]d\tau + h_s(t),$$

wher $d_m(t) = \frac{2}{l}\int_0^l d_M(v,x,t)\cos(\frac{2n+1}{2l}\pi x)dx$, $\varepsilon_m(\tilde{v},x,t) = \frac{2}{l}\int_0^l \varepsilon_{fM}(\tilde{v},x,t)\cos(\frac{2n+1}{2l}\pi x)dx$

and $h_{sm}(x,t) = \frac{2}{l}\int_0^l \left\{h_{s1}(x,t) + \left[bh_{s2}(t) - \dot{h}_{s2}(t)\right](x-l)\right\} \times \cos(\frac{2n+1}{2l}\pi x)dx$. When (24) holds and the detection threshold is selected as

$$\rho = \kappa_s \left\| \sum_{n=0}^{\infty} e^{-\beta_n(t-t_s)}\tilde{v}_n(t_s) + \sum_{n=0}^{\infty}\int_{t_s}^{t} e^{-\beta_n(t-\tau)}[\varepsilon_m(\tau) + d_m(\tau)]d\tau \right\|,$$

where $\kappa_s > 1$, it will lead to

$$\|e(t)\| \geq \left\| \sum_{n=0}^{\infty}\int_{t_s}^{t} e^{-\beta_n(t-\tau)}h_{sm}(\tau)d\tau - lh_{s2}(t) + h_s(t) \right\|$$
$$- \left\| \sum_{n=0}^{\infty}\int_{t_s}^{t} e^{-\beta_n(t-\tau)}[\varepsilon_m(\tau) + d_m(\tau)]d\tau + \sum_{n=0}^{\infty} e^{-\beta_n(t-t_s)}\tilde{v}_n(t_s) \right\|$$
$$> 2\rho - \rho = \rho,$$

Therefore, the detection of a sensor fault is guaranteed.

Proof of Theorem 2: A Lyapunov function candidate is chosen as

$$V(t)=\frac{1}{2}\int_0^l\int_0^x \Xi^T(\eta,t)\Xi(\eta,t)\,d\eta dx+\frac{1}{4\beta}\left\|\tilde{W}_a(t)\right\|^2 .$$

By applying the dynamics given by (19) and (28) and using the integration by parts, we can obtain the derivative of $V(t)$ with respect to t as

$$\begin{aligned}
\dot{V}(t)&=\int_0^l\int_0^x \Xi^T(\eta,t)\Xi_t(\eta,t)\,d\eta dx+\frac{1}{2\beta}tr\left[\tilde{W}_a^T(t)\dot{\tilde{W}}_a(t)\right]\\
&=c\int_0^l\int_0^x \Xi^T(\eta,t)\Xi_{\eta\eta}(\eta,t)\,d\eta dx+\frac{1}{2\beta}tr\left[\tilde{W}_a^T(t)\dot{\tilde{W}}_a(t)\right]-b(\mathrm{t})\int_0^l\int_0^x \Xi^T(\eta,t)\Xi(\eta,t)\,d\eta dx\\
&+\int_0^l\int_0^x \Xi^T(\eta,t)\left[\varepsilon_{fM}(\tilde{v},\eta,t)+d_M(v,\eta,t)\right]d\eta dx\\
&=c\int_0^l \Xi^T(x,t)\Xi_x(x,t)\,dx-c\int_0^l\int_0^x \Xi_\eta^T(\eta,t)\Xi_\eta(\eta,t)\,d\eta dx\\
&-b(\mathrm{t})\int_0^l\int_0^x \Xi^T(\eta,t)\Xi(\eta,t)\,d\eta dx+\frac{1}{2\beta}tr\left[\tilde{W}_a^T(t)\dot{\tilde{W}}_a(t)\right]\\
&+\int_0^l\int_0^x \Xi^T(\eta,t)\left[\varepsilon_{fM}(\tilde{v},\eta,t)+d_M(v,\eta,t)\right]d\eta dx\\
&=\frac{c}{2}\Xi^T(l,t)\Xi(l,t)-\frac{c}{2}\Xi^T(0,t)\Xi(0,t)-c\int_0^l\int_0^x \Xi_\eta^T(\eta,t)\Xi_\eta(\eta,t)\,d\eta dx\\
&-b(\mathrm{t})\int_0^l\int_0^x \Xi^T(\eta,t)\Xi(\eta,t)\,d\eta dx+\frac{1}{2\beta}tr\left[\tilde{W}_a^T(t)\dot{\tilde{W}}_a(t)\right]\\
&+\int_0^l\int_0^x \Xi^T(\eta,t)\left[\varepsilon_{fM}(\tilde{v},\eta,t)+d_M(v,\eta,t)\right]d\eta dx.
\end{aligned}$$

Substitute the update law (29) into the equation above to get

$$\begin{aligned}
&\dot{V}(t)=c\Xi^T(l,t)\Xi(l,t)/2-c\Xi^T(0,t)\Xi(0,t)/2-\\
&c\int_0^l\int_0^x \Xi_\eta^T(\eta,t)\Xi_\eta(\eta,t)\,d\eta dx-\frac{c}{2}tr\left[\tilde{W}_a^T(t)\phi(u,t)v^T(l,t)\right]\\
&-b(\mathrm{t})\int_0^l\int_0^x \Xi^T(\eta,t)\Xi(\eta,t)\,d\eta dx+\frac{\gamma}{2\beta}tr\left[\tilde{W}_a^T(t)\hat{W}_a(t)\right]\\
&+\int_0^l\int_0^x \Xi^T(\eta,t)\left[\varepsilon_{fM}(\tilde{v},\eta,t)+d_M(v,\eta,t)\right]d\eta dx.
\end{aligned}$$

Notice that $\tilde{v}(l,t)=\Xi(l,t)$ due to $L(1,\tau,t)=0$ from (20), then we get

$$\dot{V}(t) = -c\Xi^T(0,t)\Xi(0,t)/2 - c\int_0^l\int_0^x \Xi_\eta^T(\eta,t)\Xi_\eta(\eta,t)\,d\eta dx + \frac{\gamma}{2\beta}tr\left[\tilde{W}_a^T(t)W_a(t)\right]$$
$$-b(\mathrm{t})\int_0^l\int_0^x \Xi^T(\eta,t)\Xi(\eta,t)\,d\eta dx - \frac{\gamma}{2\beta}tr\left[\tilde{W}_a^T(t)\tilde{W}_a(t)\right]$$

$$+\int_0^l\int_0^x \Xi^T(\eta,t)\left[\varepsilon_{fM}(\tilde{v},\eta,t) + d_M(v,\eta,t)\right]d\eta dx$$
$$\leq -b(t)\int_0^l\int_0^x \Xi^T(\eta,t)\Xi(\eta,t)\,d\eta dx - \gamma\left\|\tilde{W}_a(t)\right\|^2/(4\beta)$$
$$+(\bar{\varepsilon}_{fM} + \bar{d}_M)\int_0^l\int_0^x \sqrt{\Xi^T(\eta,t)\Xi(\eta,t)}\,d\eta dx + \gamma\bar{W}_a^2/(4\beta)$$

where $\left\|\varepsilon_{fM}\right\| \leq \bar{\varepsilon}_{fM}$, $\left\|d_M\right\| \leq \bar{d}_M$ and $\left\|W_a\right\| \leq \bar{W}_a$. Then $\dot{V}(t) < 0$ if one of the following conditions is satisfied

$$\sqrt{\int_0^l\int_0^x \Xi^T(\eta,t)\Xi(\eta,t)\,d\eta dx} > \sqrt{2/b(t)}\bar{\varepsilon} \text{ or}$$

$$\left\|\tilde{W}_a(t)\right\| > \sqrt{4\beta/\gamma}\bar{\varepsilon} \;,$$

where $\bar{\varepsilon} = \sqrt{\dfrac{\gamma\bar{W}_a^2}{4\beta} + \dfrac{l\left(\bar{\varepsilon}_{fM} + \bar{d}_M\right)^2}{2b}}$. Therefore $\Xi(x,t)$ and $\tilde{W}_a(t)$ are UB. This implies that the observer error $\tilde{v}(x,t)$ is also bounded.

Proof of Theorem 3: Select a Lyapunov function candidate as

$$V(t) = \frac{1}{2}\|\Xi\|_{2,n} + \frac{1}{2\beta}\left\|\tilde{W}_s(t)\right\|^2 = \frac{1}{2}\int_0^l \Xi^T(x,t)\Xi(x,t)dx + \frac{1}{2\beta}tr\left[\tilde{W}_s^T(t)\tilde{W}_s(t)\right] .$$

By using the equations (19) and (34) and applying the integration by parts, we get the derivative of $V(t)$ with respect to time as

$$\dot{V}(t) = \int_0^l \Xi^T(x,t)\Xi_t(x,t)dx + \frac{1}{\beta}tr\left[\tilde{W}_s^T(t)\dot{\tilde{W}}_s(t)\right]$$
$$= c\int_0^l \Xi^T(x,t)\Xi_{xx}(x,t)dx - b(t)\int_0^l \Xi^T(x,t)\Xi(x,t)dx$$
$$+\int_0^l \Xi^T(x,t)\left[d_M(x,t) + \varepsilon_{fM}\right]dx + \frac{1}{\beta}tr\left[\tilde{W}_s^T(t)\dot{\tilde{W}}_s(t)\right]$$

$$= -c\int_0^l \Xi_x^T(x,t)\Xi_x(x,t)dx - c\Xi^T(0,t)\Xi_x(0,t) - b(t)\int_0^l \Xi^T(x,t)\Xi(x,t)dx$$
$$+\int_0^l \Xi^T(x,t)\left[d_M(x,t)+\varepsilon_{fM}\right]dx + \frac{1}{\beta}tr\left[\tilde{W}_s^T(t)\dot{\tilde{W}}_s(t)\right].$$

Next, substitute the update law (35) to the equation above and apply Poincare inequality [22] $\|\Xi\|_{2,n}^2 \le \frac{4l^2\|\Xi_x\|_{2,n}^2}{\pi^2}$ to obtain

$$\dot{V}(t) = -c\|\Xi_x\|_{2,n}^2 + c\tilde{v}^T(0,t)P_{10}(t)\tilde{W}_s^T(t)\phi_s(t) - b(t)\|\Xi\|_{2,n}^2 + \frac{\gamma}{\beta}tr\left[\tilde{W}_s^T(t)\hat{W}_s(t)\right]$$
$$+\int_0^l \Xi^T(x,t)\left[d_M(x,t)+\varepsilon_{fM}\right]dx - c\left[y_s^T(t)Q^{-T} - \hat{v}^T(0,t)\right]P_{10}(t)\tilde{W}_s^T(t)\phi_s(t)$$
$$\le -c\|\Xi_x\|_{2,n}^2 - b(t)\|\Xi\|_{2,n}^2 + \int_0^l \Xi^T(x,t)\left[d_M(x,t)+\varepsilon_{fM}\right]dx - \frac{\gamma}{2\beta}\left\|\tilde{W}_s(t)\right\|^2 + \frac{\gamma}{2\beta}\bar{W}_s^2$$
$$\le -\left(\frac{\pi^2 c}{4l^2} + b(t)\right)\|\Xi\|_{2,n}^2 - \frac{\gamma}{2\beta}\left\|\tilde{W}_s(t)\right\|^2 + \frac{\gamma}{2\beta}\bar{W}_s^2 + \int_0^l \Xi^T(x,t)\left[d_M(x,t)+\varepsilon_{fM}(x,t)\right]dx$$
$$\le -\left(\frac{\pi^2 c}{8l^2} + b(t)\right)\|\Xi\|_{2,n}^2 + \frac{2l\left[\bar{d}_M(x,t)+\bar{\varepsilon}_{fM}(x,t)\right]^2}{\pi^2 c} - \frac{\gamma}{2\beta}\left\|\tilde{W}_s(t)\right\|^2 + \frac{\gamma}{2\beta}\bar{W}_s^2.$$

It can be observed that $\dot{V}(t) < 0$ if one of the following conditions is satisfied

$$\|\Xi\|_{2,n} > \sqrt{\frac{16l^3\beta\left[\bar{d}_M(x,t)+\bar{\varepsilon}_{fM}(x,t)\right]^2 + 4l^2\pi^2 c\gamma\bar{W}_s^2}{\beta\pi^2 c\left[\pi^2 c + 8l^2 b(t)\right]}} \text{ or}$$

$$\left\|\tilde{W}_s(t)\right\| > \sqrt{\frac{4l^3\beta\left[\bar{d}_M(x,t)+\bar{\varepsilon}_{fM}(x,t)\right]^2 + \pi^2 c\gamma\bar{W}_s^2}{\gamma\pi^2 c}}.$$

Therefore, $\Xi(x,t)$ and $\tilde{W}_s(t)$ are bounded for all $x \in [0,l]$ and $t > t_s$. Moreover, the boundedness of observer error $v(x,t)$ and the detection residual $e(t) = \Xi(0,t) + \tilde{W}_s^T(t)\phi_s(t)$ can be guaranteed.

REFERENCES

[1] D. Miljković, "Fault detection methods: A literature survey," In MIPRO, 2011.

[2] R. Isermann, "Model-based fault-detection and diagnosis–status and applications," Annual Reviews in control, 29(1), pp. 71-85, 2005.

[3] J. J. Gertler, "Survey of model-based failure detection and isolation in complex plants," IEEE Control Systems Magazine, vol. 8(6), pp. 3-11, 1988.

[4] J. Chen and R. J. Patton, "Robust Model-Based Fault Diagnosis for Dynamic Systems," Kluwer Academic publishers, MA, USA, 1999.

[5] C. P. Tan, F. Crusca and M. Aldeen, "Extended results on robust state estimation and fault detection," Automatica, vol. 44(8), pp. 2027-2033, 2008.

[6] X. G. Yan and C. Edwards, "Adaptive sliding-mode-observer-based fault reconstruction for nonlinear systems with parametric uncertainties," IEEE Trans. On Industrial Electronics, vol. 55(11), pp. 4029-4036, 2008.

[7] S. Ghantasala and N. H. El-Farra, "Robust actuator fault isolation and management in constrained uncertain parabolic PDE systems," Automatica, vol. 45(10), pp. 2368-2373, 2009.

[8] A. Alonso and B. E. Ydstie, "Stabilization of distributed systems using irreversible thermodynamics," Automatica, vol. 37, pp. 1739–1755, 2001.

[9] M. Demetriou and N. Kazantzis, "A new actuator activation policy for performance enhancement of controlled diffusion processes," Automatica, vol. 40, pp. 415–421, 2004.

[10] A. Armaou and M. Demetriou, "Robust detection and accommodation of incipient component and actuator faults in nonlinear distributed processes," AIChE Journal, vol. 54, pp. 2651–2662, 2008.

[11] M. Demetriou, A. S. Ackleh and S. Reich, "Detection and accommodation of second order distributed parameter systems with abrupt changes in the input term: Existence and approximation," Kybernetika, vol. 36(1), pp. 117–132, 2000.

[12] A. Friedman, "Partial differential equations of parabolic type," Courier Corporation, 2013.

[13] J. Cai, H. Ferdowsi and S. Jagannathan, "Model-based fault detection, estimation, and prediction for a class of linear distributed parameter systems," Automatica, vol. 66, pp.122-131, 2016.

[14] J. Cai, H. Ferdowsi and S. Jagannathan, “Model-based actuator fault accommodation for distributed parameter systems represented by coupled linear PDEs,” Proc. of Control Applications (CCA), IEEE Conference on, pp. 978-983, 2015.

[15] T. Meurer, “On the extended Luenberger-type observer for semilinear distributed-parameter systems,” Automatic Control, IEEE Transactions on, vol. 58(7), pp. 1732-1743, 2013.

[16] A. Baccoli Y. Orlov and A. Pisano, “On the boundary control of coupled reaction-diffusion equations having the same diffusivity parameters,” Proc. of IEEE Conference on Decision and Control (CDC), pp. 5222-5228, 2014.

[17] Z. Yao and N. H. El-Farra, “Robust fault detection and reconfiguration in sampled-data uncertain distributed processes,” Proc. of IEEE Conference on Decision and Control (CDC), pp. 4925-4930, 2011.

[18] A. Baniamerian and K. Khorasani, “Fault detection and isolation of dissipative parabolic PDEs: Finite-dimensional geometric approach,” Proc. of the American Control Conference, pp. 5894-5899, 2012.

[19] J. Sarangapani, “Neural network control of nonlinear discrete-time systems,” vol. 21, CRC Press, 2006.

[20] M. Krstic and A. Smyshlyaev, “Boundary control of PDEs: A course on backstepping designs,” vol. 16, Siam, 2008.

[21] A. Smyshlyaev and M. Krstic, “Adaptive control of parabolic PDEs,” Princeton University Press, 2010.

[22] G. H. Hardy, J. E. Littlewood and G. Pólya, “Inequalities,” Cambridge university press, 1952.

V. MODEL-BASED ACTUATOR FAULT RESILIENT CONTROL FOR A CLASS OF NONLINEAR DISTRIBUTED PARAMETER SYSTEMS

Jia Cai and S. Jagannathan

This paper presents a new model-based fault resilient control scheme for a class of nonlinear distributed parameter systems (DPS) represented by parabolic partial differential equations (PDE) in the presence of actuator faults. A Luenberger-like observer on the basis of nonlinear PDE representation of DPS is developed with boundary measurements. A detection residual is generated by taking the difference between the measured output of the DPS and the estimated one given by the observer. Once a fault is detected, an unknown actuator fault parameter vector together with a known basis function is utilized to estimate the fault dynamics. A novel tuning algorithm is derived to estimate the unknown actuator fault parameter vector. Next, in order to achieve resilient, the controller from the healthy scenario is adjusted to mitigate the faults by using both estimated fault dynamics and a secondary measurement. Subsequently, an explicit formula is developed to estimate the time-to-resilience (TTR). Finally, a nonlinear example is utilized to illustrate the effectiveness of the proposed scheme.

1. INTRODUCTION

Recently, modern control systems have become increasingly complex which can lead to a significant number of component faults and system failures. A resilient control system is defined as the one that can maintain state awareness, an extensive sense of security and safety [1], and normal operational behavior in the presence of unexpected faults or threats [2]. As mentioned in [3], resilience is not a generic characteristic of a system but is associated with a specific type of fault or threat. It means that a system is resilient to a class of faults [4].

Over the past two decades, resilient control systems have been investigated in various industrial arena spanning different applications [4-6]. In particular, a resilient control framework is designed [4] for cyber-physical systems. Authors in [5] present a model based resilient control strategy to resist disturbances or component faults. A resilient control scheme is proposed in [6] for wireless networked control systems to mitigate interference incidents. Despite attractive results, the past literature [4-6] covers the control systems whose mathematical models are represented by ordinary differential equations (ODEs). However, a significant number of industrial systems including fluid flows, thermal convection and chemical reaction systems are classified as distributed parameter systems (DPS) or infinite dimensional systems because the system variables are defined over a continuous range of both space and time [7].

Compared to the systems discussed in [4-6], the fault detection and resilience (FDR) for DPS modeled by partial differential equations (PDEs) is more complicated and challenging due to their distributed nature. It is not practical to measure all the state vector over a continuous range in order to detect abnormal system behavior.

Limited work has been done for fault detection of distributed parameter systems due to its complex dynamics. More recently, a model-based actuator failure detection method is presented for DPS in [8]. Fault detection and accommodation is introduced [9] for nonlinear DPS with actuator faults by using an adaptive detection observer and fault-tolerant control is developed [10] for nonlinear distributed parameter processes with actuator failures.

Though interesting results have been obtained, the fault tolerant control schemes proposed in [8-10] are developed by approximating the PDE representation of the DPS with a set of ODEs which may result in missed or false alarms due to the reduced model. In addition, the presence of faults can cause the system dynamics to change and further lead to inaccurate fault detection.

Motivated by the reduced model considerations, a novel fault detection and resilience scheme directly based on the PDE representation of nonlinear DPS with actuator faults is developed. A Luenberger like observer from [11] is utilized for both fault detection and output feedback control design. The fault dynamics are estimated by utilizing a tuning term assuming that the fault function can be written as linear in the unknown parameters. The tuning term is activated to estimate the unknown fault parameter vector once an actuator fault is detected with boundary measurements alone. Next, a fault resilient scheme is introduced to mitigate the actuator fault by using the estimated fault function. The closed-loop system stability is guaranteed through Lyapunov analysis.

Upon detecting a fault and by using the output tracking error dynamics, estimated time to resilience (TTR), which is defined as the time needed by the resilience scheme to recover back to the normal operating regime, can be assessed online by using a resilient pit. The TTR is particularly useful when compared to the remaining useful life, since it can predict whether or not the resilient scheme will work before the system reaches failure.

The main contributions of this paper include the development of: a) an innovative model-based FDR scheme for actuator faults by using a Luenberger observer and system output, and b) TTR scheme by using a resilient pit on the basis of output tracking error dynamics, and 3) Lyapunov analysis of the closed-loop system by using the FDR scheme. This paper is constructed as follows. First, a class of nonlinear DPS represented by parabolic PDE with actuator faults is introduced in Section 2. Second, the development of FDR scheme for actuator faults with boundary measurements is considered in Section 3. Finally, the application of the proposed scheme in simulation on a nonlinear DPS with an actuator fault is demonstrated in Section 4.

2. SYSTEM DESCRIPTION AND FAULT FUNCTION

Before introducing the system description, the notation and the norm used throughout this paper is given [12]. A scalar function $v_1(x) \in L_2(0,l)$ indicates it is square integrable on the Hilbert space $L_2(0,l)$ with the corresponding norm

$$\|v_1(\cdot)\|_2 = \sqrt{\int_0^l v_1^2(x)dx}. \tag{1}$$

Now take

$$[L_2(0,l)]^n = \underbrace{L_2(0,l) \times L_2(0,l) \times ... \times L_2(0,l)}_{n\ times}, \tag{2}$$

into account with the corresponding norm of a vector

function $v(x,t) = [v_1(x,t),...,v_n(x,t)]^T \in [L_2(0,l)]^n$ defined as

$$\|v\|_{2,n} = \sqrt{\sum_{i=1}^{n} \|v_i\|_2^2} = \sqrt{\int_0^l v^T(x)v(x)dx}. \tag{3}$$

In addition, $\|\cdot\|$ stands for a Frobenius norm for a matrix or Euclidean norm for a vector. In order to save space, any vector defined in $L_2(0,l)$, $v(x,t)$ and its partial derivatives are written as

$$v_t(x,t) = \partial v(x,t)/\partial t,\ v_x(x,t) = \partial v(x,t)/\partial x, \text{ and } v_{xx}(x,t) = \partial^2 v(x,t)/\partial x^2.$$

2.1. SYSTEM DESCRIPTION

A class of *n*-dimensional nonlinear DPS, which can be represented by the following parabolic PDE, is expressed as

$$\frac{\partial v(x,t)}{\partial t} = c\frac{\partial^2 v(x,t)}{x^2} + f(v,x) + d(x,t), \tag{4}$$

subject to boundary conditions given by

$$v_x\left(0,t\right)=Qv(0,t), \quad v(l,t)=u(t), \tag{5}$$

$$y(t)=v(0,t),\ y_a(t)=v(l,t)\,, \tag{6}$$

for $x\in(0,l)$ and $t\geq 0$, where $v(x,t)=[v_1(x,t),...,v_n(x,t)]^T\in[L_2(0,l)]^n$ is the state vector, $d(x,t)=[d_1(x,t),...d_n(x,t)]\in\Re^n$ represents disturbance vector, $u(t)=[u_1(t),...,u_n(t)]^T\in\Re^n$ stands for the control input vector, $c>0$ is a positive constant, $Q\in\Re^{n\times n}$ is a full rank square matrix and $f(v,x)\in\Re^n$ is a Lipschitz continuous nonlinear vector function. In addition, the measured output vector $y(t)=[y_1(t),...,y_n(t)]^T\in\Re^n$ is located at the opposite end of the actuator. For fault resilience, a controller is required under healthy conditions prior to the fault.

Remark 1: The measurement $y(t)$ defined by (6) is used for observer design and to generate detection residual. The secondary measurement $y_a(t)$ given by (6) is required to estimate unknown fault parameter after fault occurrence.

Assumption 1: The system uncertainty or disturbance is bounded above such that $\left\|d(x,t)\right\|\leq\bar{d}$ for all $x\in[0,l]$ and $t\geq 0$, where $\bar{d}>0$ is a known constant. It is written as a general form in this paper, whereas a more specific model can be found in [13].

Remark 2: The upper bound of the disturbance $\bar{d}$ is needed to determine the detection threshold.

Assumption 2: The nonlinear vector function $f(v,x)$ satisfies the following conditions:

a. $f(v,x)$ is Lipschitz continuous in v , $\mathbb{C}^0$ in x , $\mathbb{C}^1$ in t and v for $x\in[0,l]$, $t\geq 0$ and $v(x,t)\in L_2(0,l)$.

b. $f(v,x)$ should satisfies $f(v+\Delta v,x)-f(v,x)=\dfrac{\partial f(v,x)}{\partial v}\Delta v+\varepsilon_f(\Delta v,x),$ where Δv represents a small change in v and $\varepsilon_f(\Delta v,x)$ is the approximation error satisfying $\left\|\varepsilon_f\right\|_{2,n}\le\bar{\varepsilon}_f$.

Remark 3: Assumption (a) can guarantee that $\dfrac{\partial f(v,x)}{\partial v}$ is bounded.

Remark 4: In order to meet the requirement $\left\|\varepsilon_f\right\|_{2,n}\le\bar{\varepsilon}_f$ in Assumption 2 (b), Δv is small implying that the initial conditions of the observer which will be introduced in Section 3 should be close to the system described by (4), (5) and (6). In addition, it also indicates that the initial value of the system state is close to its desired value.

2.2. STATE FEEDBACK CONTROL DESIGN UNDER HEALTHY CONDITIONS

Given a reference output, a full-state desired trajectory satisfying the system dynamics described by (4) and (5) in the absence of disturbance can be obtained by using flatness-based methods [14] to design the control input $u(t)$ which in turn allows the system state to follow the trajectory.

Given a reference output $y_d(t)=v_d(0,t)\in\Re^n$, a full-state desired trajectory for $0<x\le l$ is obtained as [14]

$$v_d(x,t)=\sum_{k=0}^{\infty}a_k(t)x^k, \tag{7}$$

where $a_k(t)=[a_{k1}(t),a_{k2}(t),...,a_{kn}(t)]\in\Re^n$ denotes time-varying coefficients of formal power series. These coefficients are decided by utilizing the given reference output and the system dynamics given by (4) and (5).

Next a state tracking error is introduced as $r(x,t)=[r_1(x,t),...,r_n(x,t)]^T \in \Re^n = v(x,t)-v_d(x,t)$. By applying the Assumption 2, the state tracking error dynamics is obtained as

$$r_t(x,t)=\varepsilon r_{xx}(x,t)+\Lambda(x,t)r(x,t)+\varepsilon_r(r,x)+d(x,t), \qquad (8)$$

$$r_x(0,t)=0 \ , \qquad (9)$$

where $\Lambda(x,t)=\left.\frac{\partial f(v,x)}{\partial v}\right|_{v=v_d}$ is considered bounded and $\varepsilon_r(r,x)$ represents the approximation error . The open-loop system (8) and (9) with $r(l,t)=0$ is unstable when $\Lambda(x,t)$ is positive definite with sufficiently large eigenvalues. Because $\Lambda(x,t)r(x,t)$ is the cause of instability, our objective is to eliminate this term by using both the Volterra integral transformation [15] and an appropriate controller.

Apply the Volterra integral transformation given by

$$w(x,t)=r(x,t)-\int_0^x K(x,\tau,t)r(\tau,t)d\tau, \qquad (10)$$

with state feedback control input $U(t)=U_h(t)$ defined by

$$u(t)=u_h(t)=v_d(l,t)+\int_0^l K(1,\tau,t)r(\tau,t)d\tau \ , \qquad (11)$$

along with the boundary condition

$$r(l,t)=\int_0^l K(l,\tau,t)r(\tau,t)d\tau \ , \qquad (12)$$

to convert the system (7)-(8) and (11) into a target system described by

$$w_t(x,t)=cw_{xx}(x,t)-aw(x,t)+\varepsilon_{rK}(r,x)+d_K(x,t), \qquad (13)$$

$$w_x(0,t)=0, \quad w(l,t)=0\,. \qquad (14)$$

where $\varepsilon_{rK}(r,x)=\varepsilon_r(r,x)-\int_0^x K(x,\eta,t)\varepsilon_f(\tilde{v},\eta)d\eta$, $d_K(x,t)=d(x,t)-\int_0^x K(x,\eta,t)d(\eta,t)d\eta$ and $K(x,\tau,t)$ is an $n\times n$ controller kernel matrix satisfying the following hyperbolic PDE given by

$$K_t\left(x,\tau,t\right)=cK_{xx}\left(x,\tau,t\right)-cK_{\tau\tau}\left(x,\tau,t\right)-K\left(x,\tau,t\right)\Lambda(x,t)-aK\left(x,\tau,t\right), \tag{15}$$

$$K_\tau\left(x,0,t\right)=K\left(x,0,t\right)Q, \tag{16}$$

$$K(x,x,t)=-\left[\Lambda(x,t)+a\right]x/2c+Q, \tag{17}$$

where $a>0$ is an arbitrary positive scalar. By considering $\varepsilon_{rK}(r,x)$ and $d_K(x,t)$ bounded, the target system described by (13) and (14) is stable. Due to the invertability of (10) [15], the stability of $w(x,t)$ can assure the stability of $r(x,t)$.

Note that the controller given by (11) requires the state vector $v(x,t)$ to be measurable at all positions which is impractical. Therefore, an output feedback controller will be introduced in Section 3. Next an actuator fault, $h_a(t)\in\Re^n$, is considered at the boundary of the nonlinear DPS.

2.3. ACTUATOR FAULT DESCRIPTION

The system description from (4) and (5) with an actuator fault can be described by (4) subject to the new boundary conditions

$$v_x\left(0,t\right)=Qv(0,t),\ v(l,t)=u(t)+h_a(y,t). \tag{18}$$

Moreover, the fault function can be written as

$$h_a(t)=\Omega(t-t_0)\bar{h}_a(t)\,, \tag{19}$$

where $\Omega_i(t-t_a)$ is the time profile of the fault defined by $\Omega_i(\tau)=\begin{cases}0 & ,if\ \tau<0\\ 1-e^{-\kappa_i\tau} & ,if\ \tau\geq 0\end{cases}$ for $i=1,...,n$, t_a denotes the fault occurrence time, the constant κ_i represents the growth rate of an incipient fault and $\bar{h}_a(t)$ describes the actuator fault function dynamics. Abrupt faults can be represented as well when a large κ_i is selected. Nevertheless, for the purpose of resilience, only incipient faults are considered. The following assumption is required in order to proceed.

Assumption 3: The fault function can be written as linear in the unknown parameters (LIP) [16]. In other words, the actuator fault function, $\bar{h}_a(t)=\Phi_a(y,t)\theta_a$, where $\theta_a\in\Re^n$ is the unknown fault parameter vector satisfies $\|\theta_a\|\leq\bar{\theta}_a$, $\Phi_a(y,t)=diag\left[\sigma_i(y_i,t)\right]\in\Re^{n\times}$ is known and $\sigma_i(\cdot)$ with $i=1,2,...,n$ is a smooth function.

3. FAULT RESILIENCE SCHEME WITH OUTPUT MEASUREMENTS

In this section, a Luenberger observer, which can provide the estimated state information, is introduced based on the representation of nonlinear DPS for both fault detection and output feedback resilient controller design. Moreover, an adaptive algorithm is derived to tune the unknown fault parameter with the help of a secondary measurement at the boundary condition. Then, by using the estimated fault parameter, the controller structure under healthy conditions can be reconfigured to accommodate the fault. In addition, a resilient pit is introduced to asset the performance of the fault resilient scheme through the output tracking errors and an explicit formula of time to resilient (TTR) is proposed. Next, the output feedback controller in the absence of faults is introduced first and it is modified for the sake of fault resilience later.

3.1. OUTPUT FEEDBACK CONTROLLER DESIGN UNDER HEALTHY CONDITIONS

Now consider that only the boundary value $y(t) = v(0,t)$ is available. An observer with its state, $\hat{v}(x,t) \in \Re^n$, based on the system representation given by (4) and (5) is introduced as

$$\frac{\partial \hat{v}(x,t)}{\partial t} = c\frac{\partial^2 \hat{v}(x,t)}{\partial x^2} + f(\hat{v},x) + P_1(x,t)(y-\hat{y}) \ , \tag{20}$$

with the following boundary conditions

$$\hat{v}_x(0,t) = Qv(0,t) + P_{10}(t)(y-\hat{y}),\ \hat{v}(l,t) = u(t)\,, \tag{21}$$

$$\hat{y}(t) = \hat{z}(0,t),\ e(t) = y(t) - \hat{y}(t)\,, \tag{22}$$

where $\hat{y}(t)$ is the estimated output and $e(t)$ denotes the detection residual. The observer estimation error is defined as $\tilde{v}(x,t) \in \Re^n = v(x,t) - \hat{v}(x,t)$ whose dynamics are

$$\tilde{v}_t\left(x,t\right)=c\tilde{v}_{xx}\left(x,t\right)+A(x,t)\tilde{v}\left(x,t\right)+\varepsilon_f(\tilde{v},x)-P_1(x,t)e(t)+d(x,t), \tag{23}$$

subject to

$$\tilde{v}_x\left(0,t\right)=-P_{10}(t)e(t),\quad \tilde{v}\left(l,t\right)=0, \tag{24}$$

where $A(x,t)=\left.\frac{\partial f(v,x)}{\partial v}\right|_{v=\hat{v}}\in\Re^{n\times n}$ is obtained by applying Assumption 2 to get $f(v,x)-f(\hat{v},x)=\left.\frac{\partial f(v,x)}{\partial v}\right|_{v=\hat{v}}\tilde{v}\left(x,t\right)+\varepsilon_f(\tilde{v},x)$.

Next, apply the Volterra integral transformation [11]

$$\tilde{v}\left(x,t\right)=\Xi\left(x,t\right)-\int_0^x L\left(x,\tau,t\right)\Xi\left(\tau,t\right)d\tau\ , \tag{25}$$

with the observer gains $P_1(x,t)$ and $P_{10}(t)$ selected as

$$P_1(x,t)=c\frac{\partial L\left(x,0,t\right)}{\partial\tau},\quad P_{10}(t)=L(0,0,t), \tag{26}$$

to convert the observer error dynamics (23) and (24) into a stable system given by

$$\Xi_t\left(x,t\right)=c\Xi_{xx}\left(x,t\right)-b\Xi\left(x,t\right)+\varepsilon_{fM}(\tilde{v},x)+d_M(x,t), \tag{27}$$

$$\frac{\partial\Xi(0,t)}{\partial x}=0,\quad \Xi\left(l,t\right)=0\ , \tag{28}$$

where $L(x,\tau,t)\in\Re^{n\times n}$ is the unique solution to the following well-posed PDE [11] given by

$$\frac{\partial L(x,\tau,t)}{\partial t}=A(x,t)L(x,\tau,t)+bL(x,\tau,t)+c[\frac{\partial^2 L(x,\tau,t)}{\partial\tau^2}-\frac{\partial^2 L(x,\tau,t)}{\partial x^2}]\ , \tag{29}$$

$$L\left(l,\tau,t\right)=0,\quad L\left(x,x,t\right)=(x-l)[A(t)+bI_{n\times n}]/(2c)\ , \tag{30}$$

$\Xi(x,t)\in\Re^n$, $L(x,\tau)\in\Re^{n\times n}$, $b>0$ is an arbitrary scalar, $d_M(x,t)=d(x,t)+\int_0^x M(x,\eta,t)d(\eta,t)d\eta$ and

$\varepsilon_{fM}(\tilde{v},x)=\varepsilon_f(\tilde{v},x)+\int_0^x M(x,\eta,t)\varepsilon_f(\tilde{v},\eta)d\eta$ with $M(x,\eta,t)\in\Re^{n\times n}$ is the kennel matrix of the inverse transformation $\Xi\left(x,t\right)=\tilde{v}\left(x,t\right)+\int_0^x M\left(x,\eta,t\right)\tilde{v}\left(\eta,t\right)d\eta$.

It is important to notice that the stability of $\tilde{v}(x,t)$ is guaranteed because of the stability of $\Xi(x,t)$. By using the observer defined by (20) and (21), the stability of the state tracking error dynamics can be demonstrated with the controller designed as

$$\begin{aligned} u(t)=\hat{u}_h(t)&=v_d(l,t)+\int_0^1 K(l,\tau)[\hat{v}(\tau,t)-v_d(\tau,t)]d\tau \\ &=v_d(l,t)+\int_0^1 K(l,\tau,t)r(\tau,t)d\tau-\int_0^1 K(1,\tau,t)\tilde{v}(\tau,t)d\tau \end{aligned} \tag{31}$$

where $\hat{U}_h(t)$ denotes the control input using estimated state vector given by the observer during healthy conditions.

It is worthy to point out that when compared with the controller using real state vector $v(x,t)$ given by the equation (11), the extra term $-\int_0^1 K(1,\tau,t)\tilde{v}(\tau,t)d\tau$ presented in the controller defined by (31) utilizing the estimated state vector $\hat{v}(x,t)$ is the result of the observer error $\tilde{v}(x,t)$. Both of them will be equivalent when the observer error $\tilde{v}(x,t)\to 0$.

Next, substitute the controller (31) into the DPS (4) and (5), the dynamics of the state tracking error are obtained as (8) and (9) with the boundary condition given by

$$r(l,t)=\int_0^l K(l,\tau,t)r(\tau,t)d\tau-\int_0^l K(l,\tau,t)\tilde{v}(\tau,t)d\tau . \tag{32}$$

Then by applying the transformation (10) to the state tracking error dynamics described by (8), (9) and (32), we can obtain (13) subject to

$$w_x(0,t)=0, \quad w(l,t)=-\int_0^l K(l,\tau,t)\tilde{v}(\tau,t)d\tau . \tag{33}$$

Therefore, by using (25), (33) and applying Cauchy Schwarz inequality and Young's inequality we get that

$$w^T(l,t)w(l,t) = [\int_0^l K(l,\eta,t)\tilde{v}(\eta,t)d\eta]^T[\int_0^l K(l,\eta,t)\tilde{v}(\eta,t)d\eta]$$

$$= \left\{\int_0^l K(l,\eta,t)\left[\Xi(\eta,t) - \int_0^\eta L(\eta,s,t)\Xi(s,t)\,ds\right]d\eta\right\}^T$$

$$\times\left\{\int_0^l K(l,\eta,t)\left[\Xi(\eta,t) - \int_0^\eta L(\eta,s,t)\Xi(s,t)\,ds\right]d\eta\right\} \tag{34}$$

$$\le 2\bar{k}\int_0^l \Xi^T(\eta,t)\Xi(\eta,t)\,d\eta + 2\bar{k}\bar{L}\int_0^l\int_0^\eta \Xi^T(s,t)\Xi(s,t)\,dsd\eta$$

$$\le (1+2\bar{L})\bar{k}\int_0^l\int_0^\eta \Xi^T(s,t)\Xi(s,t)dsd\eta$$

$$+4\bar{k}\int_0^l\int_0^\eta \Xi_s^T(s,t)\Xi_s(s,t)dsd\eta + 2l\Xi^T(0,t)\Xi(0,t).$$

where $\bar{k} \ge \int_0^l \|K(l,\eta,t)\|^2 d\eta$ and $\bar{L} = \max_{0\le x\le l}\int_0^l \|L(x,\eta,t)\|^2 d\eta$. The inequality given by (34) implies that the boundedness of $w(x,t)$ can be assured if $\Xi(x,t)$ is bounded.

The following theorem discusses the performance of the output feedback controller given by (31) using estimated state vector provided by the observer.

Theorem 1 (Controller Performance under Healthy Condition): Let the controller defined by (31) be applied to stabilize the DPS defined by (4) and (5). Then the tracking error described by (8), (9) and (32) will be ultimately bounded in the absence of faults.

Proof: See Appendix.

Remark 5: It can be shown in the Appendix that $\sqrt{\int_0^l\int_0^x \Xi^T(\eta,t)\Xi(\eta,t)\,d\eta dx} \le \sqrt{\dfrac{\bar{\varepsilon}}{bR-(1+2\bar{L})\bar{k}c}}$ and $\sqrt{\int_0^l\int_0^x w^T(\eta,t)w(\eta,t)\,d\eta dx} \le \sqrt{\dfrac{\bar{\varepsilon}}{a}}$,

where $\bar{\varepsilon} = \dfrac{2(\bar{\varepsilon}_{rK}+\bar{d}_K)}{a} + \dfrac{2R(\bar{\varepsilon}_{fM}+\bar{d}_M)}{b}$, $\|\varepsilon_{fM}\| \le \bar{\varepsilon}_{fM}$, $\|d_M\| \le \bar{d}_M$, $\|\varepsilon_{rK}\| \le \bar{\varepsilon}_{rK}$, $\|d_K\| \le \bar{d}_K$ and

$R > 0$ is a positive constant to construct the Lyapunov function satisfying $R \geq \max\left\{4l, 4\bar{k}, \frac{2\bar{k}c(1+2\bar{L})}{b}\right\}$. Therefore $\Xi(x,t)$ and $w(x,t)$ are UB under healthy condition which implies that the observer error $\tilde{v}(x,t)$ as well as the tracking error $r(x,t)$ are bounded. It should be noted that the bound given above can be reduced by appropriately selecting the values of a, b and R.

Next a resilient pit is introduced to show the control system performance in the presence of an actuator fault.

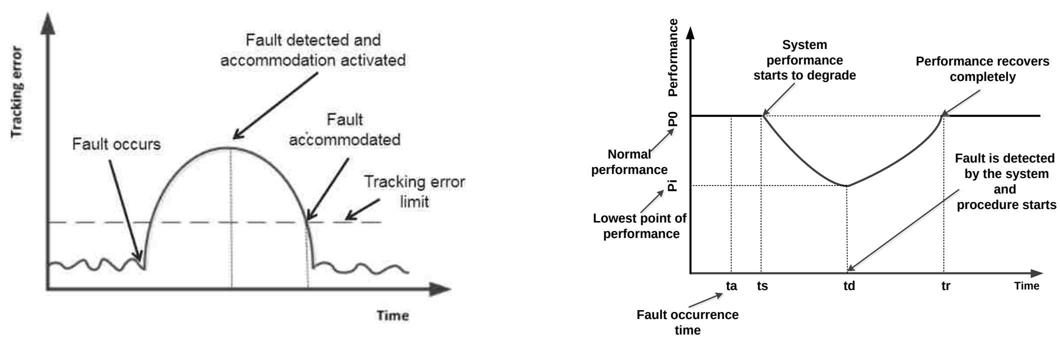
3.2. RESILIENT PIT

In order to evaluate the control system performance in the presence of faults, a resilient pit is introduced by using the output tracking error. Define the system performance as

$$P_s = \begin{cases} P_0 & if \quad \|r(0,t)\| \leq \bar{r} \\ P_0 \dfrac{\bar{r}}{\|r(0,t)\|} & if \quad \|r(0,t)\| > \bar{r} \end{cases}, \tag{35}$$

where P_0 represents normal behavior value, $\bar{r}$ denotes the limiting value of output tracking error.

The change of output tracking error in the presence of a fault is plotted in Figure 3.1 (a). When a fault occurs, the magnitude of the tracking error increases and exceeds its limit since the output is not able to follow the desired trajectory. After a fault is detected, if a resilient control is subsequently applied to mitigate the fault, then the output tracking error will be reduced below the limited value again, otherwise the output tracking error will not decrease. Based on the change of the output tracking error and by using the formula defined by (35), we can obtain a resilient pit which is shown in Figure 3.1 (b) where t_d is the detection time and t_r represents the time when the system performance goes back to normal, i.e. the time when the magnitude of the output tracking error is reduced below its limited value. If the time interval $\Delta t = t_r - t_d$ is small, it is believed that the system performance can recover from a fault using the fault resilient control scheme.



(a) (b)

Figure 3.1. (a) Output tracking error; (b) resilient pit.

Next the actuator fault detection and resilient control is introduced using estimated states.

3.3. ACTUATOR FAULT DETECTION AND RESILIENCE

It has been shown in the Subsection 3.1 that the proposed observer is able to assist in the output feedback controller design. Next the observer performance of fault detection and resilience is illustrated. In the presence of an actuator fault, the boundary condition of the observer error dynamics defined by (23) will be changed to

$$\tilde{v}_x(0,t) = -P_{10}(t)e(t), \quad \tilde{v}(l,t) = h_a(y,t), \tag{36}$$

By applying the transformation given by (25) to the modified observer error dynamics given by (23) and (36) we can get (27) subject to

$$\Xi_x(0,t) = 0, \quad \Xi(l,t) = h_a(y,t), \tag{37}$$

The presence of the actuator fault will change the boundary condition of the state tracking error dynamics given by (8) and (9) as

$$r(l,t) = \int_0^l K(l,\tau,t)r(\tau,t)d\tau - \int_0^l K(l,\tau,t)\tilde{v}(\tau,t)d\tau + h_a(y,t) \ , \tag{38}$$

Apply the transformation (10) to (8), (9) and the equation above to get (13) subject to

$$w_x(0,t) = 0, \tag{39}$$

$$w(l,t) = -\int_0^l K(1,\tau,t)\tilde{v}(\tau,t)d\tau + h_a(y,t). \tag{40}$$

The following theorem demonstrates the performance of the detection observer defined by (20), (21) and (22).

Theorem 2 (Detection observer performance): Let the observer given by (20), (21) and (22) to estimate the unmeasured states and measured output of the DPS described by (4), (5) and (6). During healthy conditions, detection residual $e(t)$ will be bounded and remained below a detection threshold ρ. An actuator fault can cause $e(t)$ to increase and exceed the threshold ρ indicating the presence of a fault. In addition, the occurrence of the fault will change the tracking error dynamics and cause the magnitude of the tracking error to increase.

Proof: Refer to Appendix.

Remark 6: It is shown in the Appendix that in the absence of faults, the detection residual is ultimately bounded by

$$\|e(t)\| \leq 2\sqrt{\frac{17c^3}{\sqrt{2(c+2b)(16b+1)}}}(\bar{d}_M + \bar{\varepsilon}_{fM}).$$

By using the bounded above, a predefined threshold ρ can be determined so that during healthy conditions the magnitude of the detection residual is below ρ for all the time.

Remark 7: In contrast, in the presence of an actuator fault and by using the same Lyapunov function candidate as in Theorem 1, it will be shown in the Appendix that

$$\sqrt{\int_0^l \int_0^x \Xi^T(\eta,t)\Xi(\eta,t)\,d\eta dx} \leq \sqrt{\frac{\bar{\varepsilon}_a}{bR-2(1+2\bar{L})\bar{k}c}} \text{ and } \sqrt{\int_0^l \int_0^x w^T(\eta,t)w(\eta,t)\,d\eta dx} \leq \sqrt{\frac{\bar{\varepsilon}_a}{a}},$$

where $\bar{\varepsilon}_a = \bar{\varepsilon} + (R+2)c\|h_a(y,t)\|^2$. It is clear that the bound given here is greater than the one presented in the Remark 5 due to the presence of the fault.

Once a fault is detected, an estimated fault function given by $\Phi_a(y,t)\hat{\theta}_a(t)$ is utilized to reconfigure the control input for resilience with $\hat{\theta}_a(t)$ is tuned by the following update law

$$\dot{\hat{\theta}}_a(t) = c\beta\Phi_a(y,t)[y_a(t) - \hat{v}(l,t)] - \gamma\hat{\theta}_a(t). \tag{41}$$

where $\beta > 0$ is the tuning rate, $\gamma > 0$ is the stabilizing term. The overall input is given by

$$U(t) = U_{accom}(t) = \hat{U}_h(t) - \Phi_a(y,t)\hat{\theta}_a(t), \tag{42}$$

yielding

$$v(1,t) = \hat{U}_h(t) + \Phi_a(y,t)\tilde{\theta}_a(t), \tag{43}$$

for the purpose of eliminating the effect on the DPS due to the presence of an actuator fault where $\hat{U}_h(t)$ is the control input designed during healthy conditions given by (31).

Then the dynamics of the transformed tracking error becomes (13) subject to the following boundary conditions

$$w_x(0,t) = 0, \tag{44}$$

and

$$w(l,t) = \Phi_a(y,t)\tilde{\theta}_a(t) - \int_0^1 K(1,\tau,t)\tilde{v}(\tau,t)d\tau. \tag{45}$$

The next theorem shows the boundedness of the tracking error with the proposed resilient scheme.

Theorem 3 (Actuator Fault Resilience): Let the resilient control law in (42) be applied after detecting an actuator fault. Then the parameter estimation, observer estimation and state tracking errors are UB.

Proof: See Appendix.

Remark 8: By using the modified controller given by (42), new bounds can be obtained as

$$\sqrt{\int_0^l \int_0^x \Xi^T(\eta,t)\Xi(\eta,t)\,d\eta dx} \le \sqrt{\frac{\bar{\varepsilon}_m}{bR - 2(1+2\bar{L})\bar{k}c}} \ , \ \sqrt{\int_0^l \int_0^x w^T(\eta,t)w(\eta,t)\,d\eta dx} \le \sqrt{\bar{\varepsilon}_m / a}$$

and $\left\|\tilde{\theta}_a(t)\right\| \le \sqrt{\dfrac{2\beta\bar{\varepsilon}_m}{\gamma(R+2)}}$, where $\bar{\varepsilon}_m = \sqrt{\bar{\varepsilon} + \dfrac{\gamma(R+2)\bar{\theta}_a^2}{2\beta}}$. (46)

By comparing the bounds given above with those given by Remark 7 without fault resilient control scheme, bounds defined by (46) can be significantly reduced by appropriately choosing γ and β .

3.4. TIME TO RESILIENCE (TTR)

It has been mentioned in the Subsection 3.2 that the magnitude of the output tracking error will increase in the presence of an actuator fault since the output cannot follow the desired trajectory with a faulty control input. Once a fault is detected, a fault resilience scheme introduced in Subsection 3.3 will be activated to force the output to follow its desired trajectory again and thus the magnitude of the output tracking can be reduced. Next the estimated time to resilience (TTR) is introduced which is defined as the time available before the magnitude of the tracking error is reduced below a given limit by using the proposed fault resilience scheme. The following remark presents an explicit formula to predict the TTR with output alone.

Remark 9: Given an initial value of the output tracking and estimated state tracking errors, and the limit values of each output tracking error, once a fault is detected and the fault resilient scheme is activated, the TTR can be estimated as

$$TTR(t) = \max_{1\le i\le n} t_{a(i)}(t) \ , \tag{47}$$

where

$$t_{a(i)} = \frac{\overline{r}_i - r_i(0,t)}{\left\{c[\hat{r}_i(2h,t) - 2\hat{r}_i(h,t) + r_i(0,t)]/h^2 + f_i(v(0,t),0) - f_i(v_d(0,t),0)\right\}} \, . \tag{48}$$

The formula given by (47) is developed by using the tracking error dynamics (8). Let $r(0,t) = [r_1(0,t),...,r_n(0,t)]^T$, $f(v(0,t),0) = [f_1(v(0,t),0),..., f_n(v(0,t),0)]^T$, then by substituting $x = 0$ to (8) for each $i = 1,...,n$ we can get

$$\frac{\partial r_i(0,t)}{\partial t} = c\frac{\partial^2 r_i(0,t)}{\partial x^2} + f_i(v(0,t),0) - f_i(v_d(0,t),0) + d_i(x,t). \tag{49}$$

By applying finite difference method, we can obtain

$$\frac{\partial^2 r_i(0,t)}{\partial x^2} = \lim_{h\to 0^+} \frac{[r_i(2h,t) - 2r_i(h,t) + r_i(0,t)]}{h^2} \, , \tag{50}$$

thus the output tracking error dynamics given by (49) can be rewritten as

$$\frac{\partial r_i(0,t)}{\partial t} = c\frac{[r_i(2h,t) - 2r_i(h,t) + r_i(0,t)]}{h^2} + f_i(v(0,t),0) - f_i(v_d(0,t),0) + d_i(x,t). \, , \quad (51)$$

where $h > 0$ is a sampling interval. Assume that $r_i(2h,t) - 2r_i(h,t) + r_i(0,t)$ and $f_i(v(0,t),0) - f_i(v_d(0,t),0)$ are held in the interval $\left[t_c, t_{r(i)}\right]$ to approximate $r_i(0,t_{r(i)})$ as

$$\begin{aligned} r_i(0,t_{r(i)}) &= \left\{c[r_i(2h,t) - 2r_i(h,t) + r_i(0,t)]/h^2 \right. \\ &\left. + f_i(v(0,t),0) - f_i(v_d(0,t),0)\right\}\left[t - t_{r(i)}\right] + r_i(0,t) \end{aligned}, \quad (52)$$

where t_c is the current time instant and $t_{r(i)}$ represents the future time when the value of $r_i(0,t)$ is reduced below its corresponding limit $\overline{r}_i$ for all $t \geq t_{r(i)}$ where $\overline{r}_i > 0 \, (i = 1,2,...,n)$ denotes the limited value of each output tracking error $r_i(0,t)$. Substitute $t_{a(i)} = t - t_{r(i)}$ and $r_i(0,t_{r(i)}) = \overline{r}_i$ to (50) to get

$$t_{a(i)} = \frac{\overline{r}_i - r_i(0,t)}{\left\{c[r_i(2h,t) - 2r_i(h,t) + r_i(0,t)]/h^2 + f_i(v(0,t),0) - f_i(v_d(0,t),0)\right\}} \, . \quad (53)$$

In the formula (53), because $r(2h,t)$ and $r(h,t)$ are not available, we need to use $\hat{r}(2h,t) = \hat{v}(2h,t) - v_d(2h,t)$ and $\hat{r}(h,t) = \hat{v}(h,t) - v_d(h,t)$ instead for $t_{a(i)}$ as given by (48). Because the output tracking error for all the states must be less than their limits, the TTR is obtained as the maximum among all the individual TTR given by (47).

4. SIMULATION RESULTS

In order to demonstrate the proposed fault resilient scheme, a nonlinear DPS is considered whose dynamics with an actuator fault at boundary condition can be expressed as

$$\frac{\partial v(x,t)}{\partial t}=\frac{\partial^2 v(x,t)}{\partial x^2}+2v(x,t)+\frac{2v^2(x,t)}{1+4(x-0.5)^2}+d(x,t)\ , \tag{54}$$

subject to

$$\frac{\partial v(0,t)}{\partial x}=0.5v(0,t),\quad v(1,t)=u(t)+\theta\sigma(y,t)\ , \tag{55}$$

where $v(x,t)$ represents the system for state $x\in[0,1]$ and time $t\geq 0$, $d(x,t)=0.02e^{-0.5(x-0.3)^2}\sin(2t)$ is the disturbance and $\sigma(y,t)=y^2(t)$. Given reference output as $v_d(0,t)=1.1(1-e^{-1.2t})$, the corresponding controller under healthy conditions can be obtained using (31). The actuator fault is seeded at $t=6s$ with the fault parameters being defined by $\theta(t)=0.25\Omega(t-6)$ where $\Omega(t-6)$ is defined as $\Omega(t-6)=\begin{cases}0 & ,if\ t<6\\ 1-e^{-0.8(t-6)} & ,if\ t\geq 6\end{cases}$.

For the simulation results using MATLAB, the closed-loop system and observer are discretized over the entire space $0\leq x\leq 1$ by using the finite difference method with 20 point grid. Next the performance of the detection and resilient scheme is evaluated on this example when only output is available.

The total time for simulation in MATLAB is taken as 15 seconds and the time interval for solving system PDE and observer is considered as 0.01 seconds. The estimated state and output under healthy conditions given by observer (20), (21) and (22) are obtained. Prior to the fault occurrence, the detection residual is expected to be decreasing, whereas it will increase once a fault occurs.

It is clearly observed in Figure 4.1 that the residual between the output from the system dynamics of (54)-(55) and the estimated output given by the observer can reach a steady state in a short time, but once a fault is activated at $t = 6s$, the residual increases because of the behavior of the system state changes. Then the fault is detected about 1.5 seconds after its inception, when the detection residual exceeds the threshold.

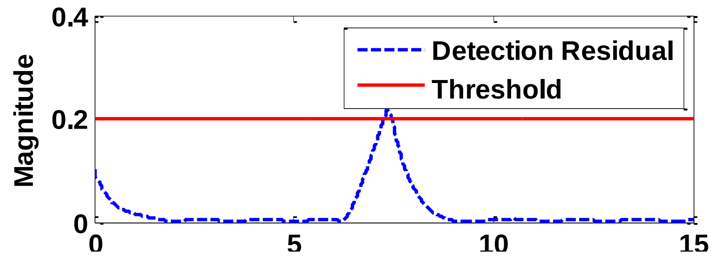


Figure 4.1. Fault detection.

Upon detecting the actuator fault, an online estimator is activated to learn the fault dynamics. The update law (41) is utilized to estimate the fault parameter. The fault detection estimation result can be seen from Figure 4.2.

Since the actuator fault will affect the controller of the actual system, the tracking error without mitigation will not decrease. However, if the fault resilient controller is applied to the boundary condition, tracking error will first increase and then decrease once the adaptation is activated to estimate the fault dynamics. The comparison with and without mitigation results can be seen from Figure 4.3 (a) and (b).

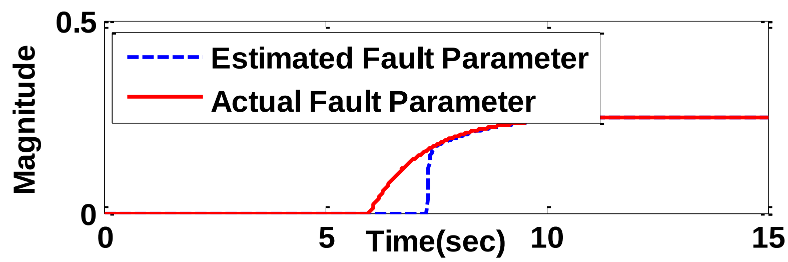


Figure 4.2. Fault parameter estimation.

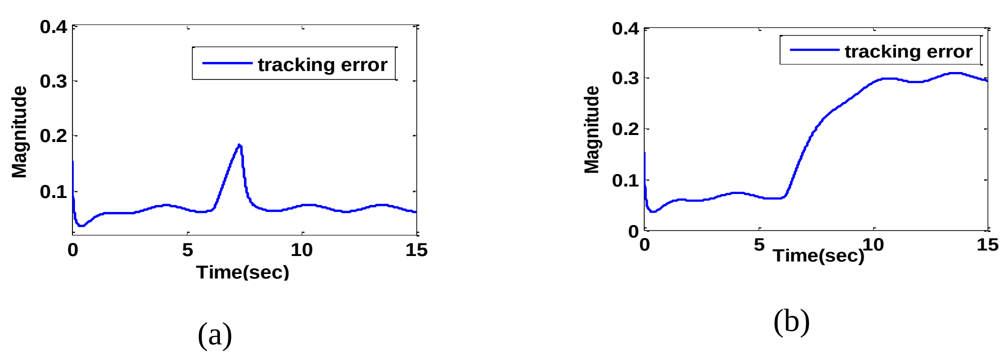


Figure 4.3. (a) with fault mitigation; (b) without fault mitigation.

By utilizing formula (47), and setting the limits as $\bar{r} = 0.16$, the TTR can be estimated which can predict the time when the tracking errors of the system shown in

Figure 4.4 decrease to their limit values. The fault is considered being accommodated completely when the output tracking error approach below its limit as shown in Figure 4.3 (a). By using the output tracking error and the system performance defined by (35) with $P_0 = 1$, a resilient pit is generated as shown in Figure 4.5. It can be observed from Figure 4.5 that the difference between the fault detection and recovery time is within 0.5 seconds implying that the system performance can recovery from the actuator fault very quickly by applying the proposed fault resilient scheme provided the actuator fault is bounded.

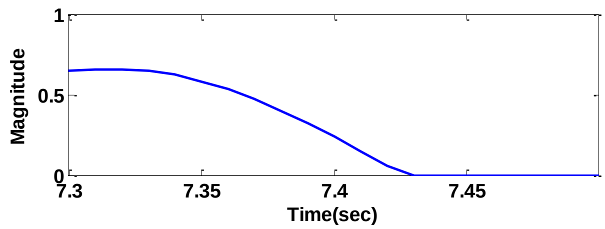


Figure 4.4. Estimated time-to-resilience for an actuator fault.

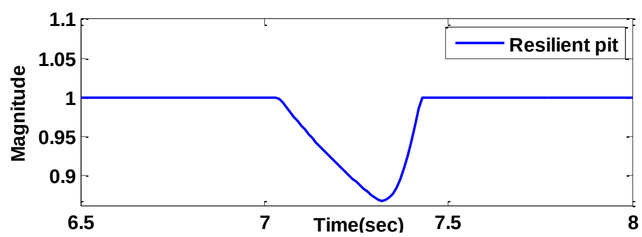


Figure 4.5. Resilient pit in the presence of an actuator fault.

5. CONCLUSIONS

In this paper, first a Luenberger observer is utilized for both fault detection and output feedback control design. Compared to ODE representation of DPS, the PDE-based observer provides a more accurate estimation of the state, which is beneficial to both fault detection and resilient control. Furthermore with a novel adaptive scheme to tune the fault parameter, the magnitude of the output tracking error can be reduced by reconfiguring the control input. The proposed adaptive estimator provides valuable information about the fault function for predicting the time-to-resilience. The proposed scheme with boundary measurements alone is critical when dealing with the implementation on practical systems. The effectiveness of the fault resilience is guaranteed by the Lyapunov analysis.

APPENDIX

Proof of Theorem 1: Select the Lyapunov candidate given by

$$V(t)=\frac{R}{2}\int_0^l\int_0^x \Xi^T(\eta,t)\Xi(\eta,t)\,d\eta dx+\frac{1}{2}\int_0^l\int_0^x w^T(\eta,t)w(\eta,t)\,d\eta dx, \quad , \tag{A.1}$$

and its derivative of V with respect to t can be obtained as

$$\dot{V}(t)=R\int_0^l\int_0^x \Xi^T(\eta,t)\Xi_t(\eta,t)\,d\eta dx+\int_0^l\int_0^x w^T(\eta,t)w_t(\eta,t)\,d\eta dx\,.$$

By applying the dynamics given by (13), (27) and using the integration by parts we can get

$$\begin{aligned}
\dot{V}&=cR\int_0^l\int_0^x \Xi^T(\eta,t)\Xi_{\eta\eta}(\eta,t)\,d\eta dx-bR\int_0^l\int_0^x \Xi^T(\eta,t)\Xi(\eta,t)\,d\eta dx\\
&+R\int_0^l\int_0^x \Xi^T(\eta,t)\left[\varepsilon_{fM}(\tilde{v},\eta,t)+d_M(v,\eta,t)\right]d\eta dx+c\int_0^l\int_0^x w^T(\eta,t)w_{\eta\eta}(\eta,t)\,d\eta dx\\
&-a\int_0^l\int_0^x w^T(\eta,t)w(\eta,t)\,d\eta dx+\int_0^l\int_0^x w^T(\eta,t)\left[\varepsilon_{rK}(\tilde{v},\eta,t)+d_K(v,\eta,t)\right]d\eta dx\\
&=-\frac{cR}{2}\Xi^T(0,t)\Xi(0,t)-cR\int_0^l\int_0^x \Xi_\eta^T(\eta,t)\Xi_\eta(\eta,t)\,d\eta dx\\
&-bR\int_0^l\int_0^x \Xi^T(\eta,t)\Xi(\eta,t)\,d\eta dx+R\int_0^l\int_0^x \Xi^T(\eta,t)\left[\varepsilon_{fM}(\tilde{v},\eta,t)+d_M(v,\eta,t)\right]d\eta dx\\
&+\frac{c}{2}w^T(l,t)w(l,t)-\frac{c}{2}w^T(0,t)w(0,t)+\int_0^l\int_0^x w^T(\eta,t)\left[\varepsilon_{rK}(\tilde{v},\eta,t)+d_K(v,\eta,t)\right]d\eta dx\\
&-c\int_0^l\int_0^x w_\eta^T(\eta,t)w_\eta(\eta,t)\,d\eta dx-a\int_0^l\int_0^x w^T(\eta,t)w(\eta,t)\,d\eta dx.
\end{aligned}$$

Substitute the inequality given by (34) to the equation above to get

$$\begin{aligned}
\dot{V}&\le-(\frac{R}{2}-l)c\Xi^T(0,t)\Xi(0,t)-(R-2\bar{k})c\int_0^l\int_0^x \Xi_\eta^T(\eta,t)\Xi_\eta(\eta,t)\,d\eta dx\\
&-[bR-(1+2\bar{L})\bar{k}c/2]\int_0^l\int_0^x \Xi^T(\eta,t)\Xi(\eta,t)\,d\eta dx\\
&+R\int_0^l\int_0^x \Xi^T(\eta,t)\left[\varepsilon_{fM}(\tilde{v},\eta,t)+d_M(v,\eta,t)\right]d\eta dx\\
&-\frac{c}{2}w^T(0,t)w(0,t)-c\int_0^l\int_0^x w_\eta^T(\eta,t)w_\eta(\eta,t)\,d\eta dx\\
&-a\int_0^l\int_0^x w^T(\eta,t)w(\eta,t)\,d\eta dx+\int_0^l\int_0^x w^T(\eta,t)\left[\varepsilon_{rK}(\tilde{v},\eta,t)+d_K(v,\eta,t)\right]d\eta dx
\end{aligned}$$

$$\leq -[\frac{bR}{2}-\frac{(1+2\bar{L})\bar{k}c}{2}]\int_0^l\int_0^x \Xi^T(\eta,t)\Xi(\eta,t)\,d\eta dx$$
$$-\frac{a}{2}\int_0^l\int_0^x w^T(\eta,t)w(\eta,t)\,d\eta dx+\frac{R}{b}(\bar{\varepsilon}_{fM}+\bar{d}_M)+\frac{1}{a}(\bar{\varepsilon}_{rK}+\bar{d}_K),$$

Then $\dot{V}<0$ when

$$\sqrt{\int_0^l\int_0^x \Xi^T(\eta,t)\Xi(\eta,t)\,d\eta dx}>\sqrt{\frac{\bar{\varepsilon}}{bR-(1+2\bar{L})\bar{k}c}}\ or\ \sqrt{\int_0^l\int_0^x w^T(\eta,t)w(\eta,t)\,d\eta dx}>\sqrt{\frac{\bar{\varepsilon}}{a}},$$

where $\bar{\varepsilon}=\frac{2(\bar{\varepsilon}_{rK}+\bar{d}_K)}{a}+\frac{2R(\bar{\varepsilon}_{fM}+\bar{d}_M)}{b}$. Therefore $\Xi(x,t)$ and $w(x,t)$ will be bounded considering $\Xi(x,t)$ and $w(x,t)$ are continuous on $x\in[0,l]$. The boundedness of $\tilde{v}$ and r are also guaranteed because of (25) and the invertible of (10).

Proof of Theorem 2: In order to discuss the performance of the detection observer under healthy conditions, a Lyaponov candidate is selected as

$$V(t)=\|\Xi(t)\|_{2,n}^2/2+\|\Xi_x(t)\|_{2,n}^2/2=\int_0^l \Xi^T(x,t)\Xi(x,t)dx/2+\int_0^l \Xi_x^2(x,t)dx/2, \quad \text{(A.2)}$$

which is positive definite. Then the derivative of the Lyapunov function with respect to time can be obtained as

$$\dot{V}(t)=\int_0^l \Xi^T(x,t)\Xi_t(x,t)dx+\int_0^l \Xi_x^T(x,t)\Xi_{tx}(x,t)dx\ .$$

Substituting the equation (27) and (28) into the equation above and applying integration by parts, we will arrive at

$$\begin{aligned}
\dot{V}(t)&=c\int_0^l \Xi^T(x,t)\Xi_{xx}(x,t)dx-b\int_0^l \Xi^T(x,t)\Xi(x,t)dx\\
&+\int_0^l \Xi^T(x,t)[\varepsilon_{fM}(\tilde{v},x,t)+d_M(v,x,t)]dx+\int_0^l \Xi_x^T(x,t)d\Xi_t(x,t)\\
&\leq -c\|\Xi_x\|_{2,n}-c\|\Xi_{xx}\|_{2,n}+(\bar{\varepsilon}_{fM}+\bar{d}_M)[\int_0^l\sqrt{\Xi^T(x,t)\Xi(x,t)}dx\\
&-b\|\Xi\|_{2,n}-b\|\Xi_x\|_{2,n}+\int_0^l\sqrt{\Xi_{xx}^T(x,t)\Xi_{xx}(x,t)}dx].
\end{aligned}$$

Then we apply Poincare inequality [17] $\|\Xi\|_{2,n} \le 4l^2 \|\Xi_x\|_{2,n}$ and $\|\Xi_x\|_{2,n} \le 4l^2 \|\Xi_{xx}\|_{2,n}$ to the equation above to get

$$\dot{V}(t) \le -\left(\frac{c}{8l^2}+b\right)\|\Xi\|_{2,n}^2 - \left(\frac{c}{8l^2}+b\right)\|\Xi_x\|_{2,n}^2 + \frac{(1+4l^2)(\bar{\varepsilon}_{fM}+\bar{d}_M)^2}{2c}.$$

Therefore, $\dot{V}$ will be less than zero when

$$\|\Xi\|_{2,n} > \frac{2l\sqrt{1+4l^2}}{\sqrt{c(c+8bl^2)}}(\bar{\varepsilon}_{fM}+\bar{d}_M) \text{ or} \|\Xi_x\|_{2,n} > \frac{2l\sqrt{1+4l^2}}{\sqrt{c(c+8bl^2)}}(\bar{\varepsilon}_{fM}+\bar{d}_M) \ .$$

By using Agmon's inequality [18] we get

$$\|e(t)\| \le \max_{x\in[0,l]}\|\Xi(x,t)\| \le 2\|\Xi\|_{2,n}\|\Xi\|_{2,n} \le k_l(\bar{\varepsilon}_{fM}+\bar{d}_M) \ ,$$

where $k_l = 2l\sqrt{2(1+4l^2)}/c$, which means the detection error $e(t)$ will remain bounded under healthy conditions.

In the presence of an actuator fault, if the same Lyapunov function candidate given by (A.1) is used here, the derivative of $V(t)$ with respect to t is obtained as

$$\begin{aligned}
&\dot{V}(t) = R\int_0^l\int_0^x \Xi^T(\eta,t)\Xi_t(\eta,t)\,d\eta dx + \int_0^l\int_0^x w^T(\eta,t)w_t(\eta,t)\,d\eta dx \\
&= \frac{cR}{2}\Xi^T(l,t)\Xi(l,t) - \frac{cR}{2}\Xi^T(0,t)\Xi(0,t) - cR\int_0^l\int_0^x \Xi_\eta^T(\eta,t)\Xi_\eta(\eta,t)\,d\eta dx \\
&-bR\int_0^l\int_0^x \Xi^T(\eta,t)\Xi(\eta,t)\,d\eta dx + R\int_0^l\int_0^x \Xi^T(\eta,t)\left[\varepsilon_{fM}(\tilde{v},\eta,t)+d_M(v,\eta,t)\right]d\eta dx \\
&+\frac{c}{2}w^T(l,t)w(l,t) - \frac{c}{2}w^T(0,t)w(0,t) + \int_0^l\int_0^x w^T(\eta,t)\left[\varepsilon_{rK}(\tilde{v},\eta,t)+d_K(v,\eta,t)\right]d\eta dx \\
&-c\int_0^l\int_0^x w_\eta^T(\eta,t)w_\eta(\eta,t)\,d\eta dx - a\int_0^l\int_0^x w^T(\eta,t)w(\eta,t)\,d\eta dx \\
&\le -[\frac{bR}{2} - (1+2\bar{L})\bar{k}c]\int_0^l\int_0^x \Xi^T(\eta,t)\Xi(\eta,t)\,d\eta dx \\
&-\frac{a}{2}\int_0^l\int_0^x w^T(\eta,t)w(\eta,t)\,d\eta dx + \frac{R}{b}(\bar{\varepsilon}_{fM}+\bar{d}_M) + \frac{1}{a}(\bar{\varepsilon}_{rK}+\bar{d}_K) + \frac{(R+2)c}{2}\|h_a(y,t)\|^2 .
\end{aligned}$$

Then $\dot{V}(t) < 0$ if one of the following conditions is satisfied

$$\sqrt{\int_0^l \int_0^x \Xi^T(\eta,t)\Xi(\eta,t)\,d\eta dx} > \sqrt{\frac{\bar{\varepsilon}_a}{bR-2(1+2\bar{L})\bar{k}c}}$$

or $\sqrt{\int_0^l \int_0^x w^T(\eta,t)w(\eta,t)\,d\eta dx} > \sqrt{\frac{\bar{\varepsilon}_a}{a}}$, where $\bar{\varepsilon}_a = \bar{\varepsilon} + \frac{(R+2)c}{2}\left\|h_a(y,t)\right\|^2$.

Proof of Theorem 3: Select a Lyapunov function candidate as

$$V(t) = \frac{R}{2}\int_0^l \int_0^x \Xi^T(\eta,t)\Xi(\eta,t)\,d\eta dx + \frac{1}{2}\int_0^l \int_0^x w^T(\eta,t)w(\eta,t)\,d\eta dx + \frac{R+2}{4\beta}\left\|\tilde{\theta}_a(t)\right\|^2,$$

By applying the dynamics given by (13), (27) and using the integration by parts, we can obtain the derivative of $V(t)$ with respect to t as

$$\dot{V}(t) = R\int_0^l \int_0^x \Xi^T(\eta,t)\Xi_t(\eta,t)\,d\eta dx + \int_0^l \int_0^x w^T(\eta,t)w_t(\eta,t)\,d\eta dx + \frac{R+2}{2\beta}\tilde{\theta}_a^T(t)\dot{\tilde{\theta}}_a(t)$$

$$= \frac{cR}{2}\Xi^T(l,t)\Xi(l,t) - cR\int_0^l \int_0^x \Xi_\eta^T(\eta,t)\Xi_\eta(\eta,t)\,d\eta dx$$

$$-bR\int_0^l \int_0^x \Xi^T(\eta,t)\Xi(\eta,t)\,d\eta dx + \frac{R+2}{2\beta}\tilde{\theta}_a^T(t)\dot{\tilde{\theta}}_a(t)$$

$$+R\int_0^l \int_0^x \Xi^T(\eta,t)\left[\varepsilon_{fM}(\tilde{v},\eta,t) + d_M(v,\eta,t)\right]d\eta dx$$

$$+\frac{c}{2}w^T(l,t)w(l,t) - c\int_0^l \int_0^x w_\eta^T(\eta,t)w_\eta(\eta,t)\,d\eta dx$$

$$-\frac{c}{2}w^T(0,t)w(0,t) - a\int_0^l \int_0^x w^T(\eta,t)w(\eta,t)\,d\eta dx$$

$$+\int_0^l \int_0^x w^T(\eta,t)[\varepsilon_{rK}(\tilde{v},\eta,t) + d_K(v,\eta,t)]\,d\eta dx - \frac{cR}{2}\Xi^T(0,t)\Xi(0,t)$$

$$= \frac{cR}{2}\tilde{\theta}_a^T(t)\Phi_a^T(y,t)\tilde{v}(l,t) - (\frac{cR}{2} - 2lc)\Xi^T(0,t)\Xi(0,t)$$

$$-(R-4\bar{k})c\int_0^l \int_0^x \Xi_\eta^T(\eta,t)\Xi_\eta(\eta,t)\,d\eta dx$$

$$-[bR-(1+2\bar{L})\bar{k}c]\int_0^l \int_0^x \Xi^T(\eta,t)\Xi(\eta,t)\,d\eta dx$$

$$+R\int_0^l\int_0^x \Xi^T(\eta,t)\left[\varepsilon_{fM}(\tilde{v},\eta,t)+d_M(v,\eta,t)\right]d\eta dx$$

$$-\frac{R+2}{2\beta}\tilde{\theta}_a^T(t)\left[c\beta\Phi_a^T(y,t)\tilde{v}(l,t)-\gamma\hat{\theta}_a(t)\right]$$

$$+c\tilde{\theta}_a^T(t)\Phi_a^T(y,t)\tilde{v}(l,t)-c\int_0^l\int_0^x w_\eta^T(\eta,t)w_\eta(\eta,t)d\eta dx$$

$$-a\int_0^l\int_0^x w^T(\eta,t)w(\eta,t)d\eta dx-\frac{c}{2}w^T(0,t)w(0,t)$$

$$+\int_0^l\int_0^x w^T(\eta,t)[\varepsilon_{rK}(\tilde{v},\eta,t)+d_K(v,\eta,t)]d\eta dx$$

$$\le -[\frac{bR}{2}-(1+2\bar{L})\bar{k}c]\int_0^l\int_0^x \Xi^T(\eta,t)\Xi(\eta,t)d\eta dx$$

$$-\frac{a}{2}\int_0^l\int_0^x w^T(\eta,t)w(\eta,t)d\eta dx-\frac{\gamma(R+2)}{4\beta}\left\|\tilde{\theta}_a(t)\right\|^2$$

$$+\frac{R}{b}(\bar{\varepsilon}_{fM}+\bar{d}_M)+\frac{1}{a}(\bar{\varepsilon}_{rK}+\bar{d}_K)+\frac{\gamma(R+2)}{4\beta}\bar{\theta}_a^2.$$

Therefore, the derivative of Lyapunov function will be less than zero when

$$\sqrt{\int_0^l\int_0^x \Xi^T(\eta,t)\Xi(\eta,t)d\eta dx}>\sqrt{\frac{\bar{\varepsilon}_m}{bR-2(1+2\bar{L})\bar{K}c}} \text{ or}$$

$$\sqrt{\int_0^l\int_0^x w^T(\eta,t)w(\eta,t)d\eta dx}>\sqrt{\bar{\varepsilon}_m/a} \text{ or}$$

$$\left\|\tilde{\theta}_a(t)\right\|>\sqrt{\frac{2\beta\bar{\varepsilon}_m}{\gamma(R+2)}} \text{ , where } \bar{\varepsilon}_m=\sqrt{\bar{\varepsilon}+\frac{\gamma(R+2)\bar{\theta}_a^2}{2\beta}}.$$

So far we have shown the boundedness of $\int_0^1\int_0^x w^T(\eta,t)w(\eta,t)d\eta dx$ and $\int_0^1\int_0^x \Xi^T(\eta,t)\Xi(\eta,t)d\eta dx$, because $w(x,t)$ and $\Xi(x,t)$ are continuous on $x\in[0,1]$, the transformed tracking error $w(0,t)$ is also bounded. Now, given the transformation (25) and the invertability of the transformation (10), the boundedness of the observer estimation error $\tilde{v}(x,t)$ state tracking error $r(x,t)$ are ensured.

REFERENCES

[1] C. I. Protection, “Department of Homeland Security Faces Challenges in Fulfilling Cyber security Responsibilities,” GAO-05-434, 2005.

[2] C. G. Rieger, D. I. Gertman, and M. A. McQueen, “Resilient control systems: next generation design research,” In Human System Interactions, 2009. HSI'09. 2nd Conference on, pp. 632-636, 2009.

[3] L. Mili and N. V. Center, “Taxonomy of the characteristics of power system operating states,” In 2nd NSF-VT Resilient and Sustainable Critical Infrastructures (RESIN) Workshop, Tucson, AZ, pp. 13-15, 2011.

[4] Q. Zhu and T. Başar, “Robust and resilient control design for cyber-physical systems with an application to power systems,” Proc. of Decision and Control and European Control Conference (CDC-ECC), 2011 50th IEEE Conference on, pp. 4066-4071, 2011.

[5] K. Ji, Y. Lu, L. Liao, Z. Song and D. Wei, “Prognostics enabled resilient control for model-based building automation systems,” Proc. of International Building Performance Simulation Association, pp. 286-293, 2011.

[6] K. Ji, and D. Wei, “Resilient control for wireless networked control systems,” Proc. of International Journal of Control, Automation and Systems, vol. 9(2), pp. 285-293, 2011.

[7] S. Omatu and J. H. Seinfeld, “Distributed parameter systems: theory and applications,” Clarendon Press, 1989.

[8] H. Baruh, “Actuator failure detection in the control of distributed systems,” Journal of Guidance, Control, and Dynamics, vol. 9(2), pp. 181-189, 1986,

[9] M. Demetriou, K. Ito and R. C. Smith, “Adaptive monitoring and accommodation of nonlinear actuator faults in positive real infinite dimensional systems,” Automatic Control, IEEE Transactions on, vol. 52(12), pp. 2332-2338, 2007.

[10] P. Mhaskar, A. Gani, N. H. El-Farra, C. McFall, P. D. Christofides and J. F. Davis, “Integrated fault-detection and fault-tolerant control of process systems,” AIChE Journal, vol. 52(6), pp. 2129-2148, 2006.

[11] T. Meurer, “On the extended Luenberger-type observer for semilinear distributed-parameter systems”, Automatic Control, IEEE Transactions on, vol. 58(7), pp. 1732-1743, 2013.

[12] A. Baccoli, Y. Orlov and A. Pisano, “On the boundary control of coupled reaction-diffusion equations having the same diffusivity parameters,” Proc. of the IEEE Conference on Decision and Control, Los Angeles, US, pp. 5222-5228, 2014.

[13] Z. Yao and N. H. El-Farra, “Robust fault detection and reconfiguration in sampled-data uncertain distributed processes,” Proc. CDC, Orlando, FL, pp. 4925-4930, 2011.

[14] T. Meurer, “Feedforward and feedback tracking control of diffusion-convection-reaction systems using summability methods,” Automatisierungstechnik, vol. 54(1/2006), pp.47-48, 2006.

[15] M. Krstic and A. Smyshlyaev, “Boundary control of PDEs: A course on backstepping designs,” vol. 16, Siam, 2008.

[16] J. Sarangapani, “Neural network control of nonlinear discrete-time systems,” vol. 21, CRC Press, 2006.

[17] G. H. Hardy, J. E. Littlewood and G. Pólya, “Inequalities,” Cambridge university press, 1952.

[18] A. Smyshlyaev and M. Krstic, “Backstepping observers for a class of parabolic PDEs,” Systems and Control Letters, vol. 54(7), pp. 613-625, 2005.

SECTION

2. CONCLUSIONS AND FUTURE WORK

In this dissertation, an online adaptive approach was utilized to design model-based fault diagnosis and prognosis schemes for a class of linear and nonlinear DPS described by parabolic PDEs. A Luenberger observer is used to detect and estimate actuator faults using state availability for linear parabolic PDEs. Subsequently, the requirement of measured full state availability is relaxed by redesigning the detection observer based on input and output filters. Then the prediction scheme is introduced to estimate TTF by using estimated fault parameters.

Based on the estimated fault dynamics, fault accommodation can be generated to mitigate faults by reconfiguring the controller. In addition, a fault isolation scheme is developed to identify actuator, sensor and state faults by using actuator and sensor fault isolation estimators and a location determination scheme is developed to provide information of the state fault location for linear DPS. A fault isolation framework differentiating different types of faults is introduced for nonlinear DPS as well. The proposed fault diagnosis and prognosis scheme is applicable to both incipient and abrupt faults. Stability is guaranteed with bounded uncertainty and disturbance. Moreover, an extended Luenberger-type observer is utilized to detect faults and predict system failures for nonlinear MIMO distributed parameter systems. Fault parameters are estimated using a novel tuning algorithm which is applied to reconfigure control input in order to achieve resilient control.

2.1. CONCLUSIONS

In Paper I, an innovative observer acting on the basis of the system PDE representation gives a better estimation of the system and thus provides more reliable fault detection and estimation. The adaptive term tuned by a novel update law online is incorporated into the observer to approximate the fault function. The predicted TTF is obtained by using the estimated fault parameter and its failure limit. The filter-based observer introduced in the paper only requires boundary measurement for fault detection and estimation. It is critical when implementing the proposed scheme on a practical system that both actuator and sensor faults are detected if the fault type is known. The proposed fault diagnosis framework is applicable to systems with software modification and can minimize the cost of sensor placement.

Paper II presents the model-based fault detection and accommodation scheme for linear MIMO distributed parameter systems with bounded uncertainty and disturbance. Instead of accommodating the faults based on a reduced ODE-based model, the detection observer is developed directly based on the original PDE representation which can estimate system states more accurately for the sake of reducing false or missed alarms. Several fault filters are applied to approximate additive actuator and sensor faults with boundary measurements; furthermore, the control input will be modified to mitigate actuator and sensor faults once detecting a fault if the type is known. In addition, time to accommodation is introduced which can be compared to the TTF presented in Paper I to determine if the system needs to be shut down for maintenance.

The first two papers address fault prognosis and accommodation under the assumption that the fault type is known a priori. The unsolved problem is how to isolate different faults. The main challenge is to determine the infinite possible locations of faults with a limited number of measurements.

Therefore, the third paper presents a fault isolation scheme for DPSs to identify different actuators, sensors and state faults by incorporating estimated actuator and sensor fault dynamics to the detection observer and thereby generate actuator and sensor fault residuals. Then by comparing those residuals to their isolation thresholds, the fault type is determined. If a state fault is identified, several filter-based estimators at different locations are then applied to identify the fault location by comparing the errors among estimators.

The fourth paper presents a model-based fault detection and prediction scheme for a class of MIMO nonlinear distributed parameter systems. The detection observer is developed based on nonlinear PDE representation, and the observer gains are selected by linearizing the observer error dynamics. A detectability conditions for actuator and sensor faults are provided and an online estimator is utilized to estimate the actuator/sensor fault dynamics with boundary measurements. In the end, a TTF prediction scheme is presented to estimate the remaining useful life of the system by using the failure limit of the output.

In the last paper, the fault resilient control of nonlinear distributed parameter systems is investigated. A Luenberger observer is utilized for output feedback controller design and fault detection. Once an actuator fault is detected, an online estimator with a tuning term is incorporated to learn the fault dynamics using which the control input is reconfigured to obtain fault resilient control. Based on the change of the output tracking error in the presence of the fault, a resilient pit is introduced to asset the system performance with the proposed resilient control.

2.2. FUTURE WORK

The proposed fault location determination scheme can be extended to nonlinear DSP to identify the location of a state fault. Because of the system nonlinearity and limited number of sensors, it may become extremely complicated to determine the location of a state fault. In addition, deriving state fault detectability condition for nonlinear DPS is necessary, and the state fault estimators need to be developed in order to identify the fault location.

A fault accommodation scheme will be proposed for nonlinear MIMO DPS to mitigate state faults. A new update law needs to be derived to estimate the fault parameter for reconfiguring the control input. The stability of the fault accommodation has to be guaranteed by using an adaptive term for nonlinear DPS. Finally, the fault resilience performance has to be evaluated through the tracking error in the presence of state faults.

Another part of the future work involves the implementation of the proposed fault diagnosis and prognosis to a practical system. Although the proposed schemes have been demonstrated by using simulation examples, it is necessary to implement the proposed scheme on a real system. As a next step, hardware implementation has to be pursued to resolve any issues that cannot be found in simulation studies.

REFERENCES

[1] C. I. Protection, “Department of Homeland Security Faces Challenges in Fulfilling Cyber security Responsibilities,” GAO-05-434, May, 2005.

[2] G. G. Yen and L. W. Ho, “Online multiple-model-based fault diagnosis and accommodation,” Industrial Electronics, IEEE Transactions on 50, 2 (2003), pp. 296-312.

[3] D. Miljković, “Fault detection methods: A literature survey,” In MIPRO 2011. Hrvatska znanstvena bibliografija i MZOS-Svibor, 2011.

[4] P. M. Frank, “Fault diagnosis in dynamic systems using analytical and knowledge-based redundancy: A survey and some new results,” Automatica 26, 3 (1990), pp. 459-474.

[5] J. Gertler, “Fault detection and diagnosis in engineering systems,” CRC press, 1998.

[6] S. M. Namburu, W. Mark, C. Shunsuke, Q. Liu, C. Kihoon, and P. Krishna, “Systematic data-driven approach to real-time fault detection and diagnosis in automotive engines,” In *Autotestcon, 2006 IEEE*, pp. 59-65. IEEE, 2006.

[7] R. J. Hansen, D. L. Hall and S. K Kurtz, “A new approach to the challenge of machinery prognostics,” Journal of Engineering for Gas Turbines and Power, 117(2), pp. 320-325, 1995.

[8] A., Vania, and P. Pennacchi, “Experimental and theoretical application of fault identification measures of accuracy in rotating machine diagnostics,” Mechanical Systems and Signal Processing 18, no. 2 (2004): pp. 329-352.

[9] R. Isermann, “Model-based fault-detection and diagnosis–status and applications,” *Annual Reviews in control* 29, no. 1 (2005): pp. 71-85.

[10] B. Jiang and F. N. Chowdhury, “Parameter fault detection and estimation of a class of nonlinear systems using observers,” Journal of the Franklin Institute, 342(7), pp. 725-736, 2005.

[11] G. Biswas, X. Koutsoukos, A. Bregon and B. Pulido, “Analytic redundancy, possible conflicts, and TCG-based fault signature diagnosis applied to nonlinear dynamic systems,” In Proc. of the 7th IFAC Symp. on Fault Detection, Supervision and Safety of Tech. Processes, pp. 1486-1491, 2009.

[12] J., Meseguer, V. Puig and T. Escobet, “Observer gain effect in linear interval observer-based fault detection,” In Fault Detection, Supervision and Safety of Technical Processes Vol. 6, No. 1, pp. 540-545, 2006.

[13] C. Kwan, X. Zhang, R. Xu, and L. Haynes, “A novel approach to fault diagnostics and prognostics,” Proc. of the IEEE Int. Conf. on Robotics and Auto, NY pp. 604–609, 2003.

[14] P. Wang and G. Vachtsevanos, “Fault prognostics using dynamic wavelet neural networks,” AI EDAM-Artificial Intelligence for Engineering Design Analysis and Manufacturing, 15(4), pp. 349–365, 2001.

[15] P. D. Christofides, “Nonlinear and robust control of PDE systems: Methods and applications to transport-reaction processes,” Springer Science & Business Media, 2001.

[16] P. D. Christofides and P Daoutidis, “Finite-dimensional control of parabolic PDE systems using approximate inertial manifolds,” In *Decision and Control, 1997., Proceedings of the 36th IEEE Conference on*, vol. 2, pp. 1068-1073. IEEE, 1997.

[17] N. H. El-Farra, A. Armaou and P. D. Christofides, “Analysis and control of parabolic PDE systems with input constraints,” *Automatica* 39, no. 4 (2003): pp. 715-725.

[18] B. Luo, H. N. Wu and H. X. Li, “Adaptive Optimal Control of Highly Dissipative Nonlinear Spatially Distributed Processes with Neuro-Dynamic Programming,” *Neural Networks and Learning Systems, IEEE Transactions on* 26, no. 4 (2015): pp. 684-696.

[19] H. Baruh, “Actuator failure detection in the control of distributed systems,” Journal of Guidance, Control, and Dynamics, vol. 9(2), pp. 181-189, 1986.

[20] M. A. Demetriou, K. Ito and R. C. Smith, “On-line monitoring and accommodation of nonlinear actuator faults in positive real infinite dimensional systems,” In Decision and Control, 2004. CDC. 43rd IEEE Conference on, vol. 3, pp. 2871-2875, Dec 2004.

[21] A. Baniamerian and K. Khorasani, “Fault detection and isolation of dissipative parabolic PDEs: Finite-dimensional geometric approach,” In American Control Conference (ACC), pp. 5894-5899, June 2012.

[22] A. Smyshlyaev and M. Krstic, “Closed-form boundary state feedbacks for a class of 1-D partial integro-differential equations,” *Automatic Control, IEEE Transactions on* 49, no. 12 (2004): pp. 2185-2202.

[23] M. Krstic and A. Smyshlyaev, “Adaptive boundary control for unstable parabolic PDEs—Part I: Lyapunov design,” *Automatic Control, IEEE Transactions on* 53, no. 7 (2008): pp. 1575-1591.

[24] A. Smyshlyaev and M. Krstic, “Backstepping observers for a class of parabolic PDEs,” *Systems & Control Letters* 54, no. 7 (2005): pp. 613-625.

[25] A. Baccoli, Y. Orlov and A. Pisano, “On the boundary control of coupled reaction-diffusion equations having the same diffusivity parameters,” Proc. of the IEEE Conference on Decision and Control, Los Angeles, US, 2014, pp. 5222-5228.

[26] T. Meurer, “On the extended Luenberger-type observer for semilinear distributed-parameter systems,” *Automatic Control, IEEE Transactions on* 58, no. 7 (2013): pp. 1732-1743.

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