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# UNDERGRADUATE MATHEMATICS STUDENTS' CONNECTIONS BETWEEN THEIR GROUP HOMOMORPHISM AND LINEAR TRANSFORMATION CONCEPT IMAGES

Jeffrey Slye
*University of Kentucky*, jeffrey.slye@uky.edu
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Jeffrey Slye, Student

Dr. David C. Royster, Major Professor

Dr. Peter D. Hislop, Director of Graduate Studies

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DISSERTATION

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A dissertation submitted in partial
fulfillment of the requirements for
the degree of Doctor of Philosophy
in the College of Arts and Sciences
at the University of Kentucky

By
Jeffrey Slye
Lexington, Kentucky

Director: Dr. David C. Royster, Professor of Mathematics
Lexington, Kentucky
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ABSTRACT OF DISSERTATION


# UNDERGRADUATE MATHEMATICS STUDENTS' CONNECTIONS BETWEEN THEIR GROUP HOMOMORPHISM AND LINEAR TRANSFORMATION CONCEPT IMAGES

It is well documented that undergraduate students struggle with the more formal and abstract concepts of vector space theory in a first course on linear algebra. Some of these students continue on to classes in abstract algebra, where they learn about algebraic structures such as groups. It is clear to the seasoned mathematician that vector spaces are in fact groups, and so linear transformations are group homomorphisms with extra restrictions. This study explores the question of whether or not students see this connection as well. In addition, I probe the ways in which students' stated understandings are the same or different across contexts, and how these differences may help or hinder connection making across domains. Students' understandings are also briefly compared to those of mathematics professors in order to highlight similarities and discrepancies between reality and idealistic expectations.

The data for this study primarily comes from clinical interviews with ten undergraduates and three professors. The clinical interviews contained multiple card sorts in which students expressed the connections they saw within and across the domains of linear algebra and abstract algebra, with an emphasis specifically on linear transformations and group homomorphisms. Qualitative data was analyzed using abductive reasoning through multiple rounds of coding and generating themes.

Overall, I found that students ranged from having very few connections, to beginning to form connections once placed in the interview setting, to already having a well-integrated morphism schema across domains. A considerable portion of this paper explores the many and varied ways in which students succeeded and failed in making mathematically correct connections, using the language of research on analogical reasoning to frame the discussion. Of particular interest were the ways in which isomorphisms did or did not play a role in understanding both morphisms, how students did not regularly connect the concepts of matrices and linear transformations, and how vector spaces were not fully aligned with groups as algebraic structures.

KEYWORDS: group homomorphism, linear transformation, undergraduate mathematics education, analogical reasoning, concept image

Author's signature: Jeffrey Slye

Date: July 15, 2019

# UNDERGRADUATE MATHEMATICS STUDENTS' CONNECTIONS BETWEEN THEIR GROUP HOMOMORPHISM AND LINEAR TRANSFORMATION CONCEPT IMAGES

By
Jeffrey Slye

Director of Dissertation: David C. Royster

Director of Graduate Studies: Peter D. Hislop

Date: July 15, 2019

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# Chapter 1 Introduction

## 1.1 Impetus for Research and Research Questions

Multiple factors contributed to the genesis of this project. First, the tools that students learn in a linear algebra class have immediate applications to applied problems in the sciences. Vector space theory also serves as a bedrock tool for most all of the subfields of pure mathematics. As linear algebra lies at such an important nexus in mathematics education, I desired to contribute to the young and burgeoning body of research on its teaching and learning.

Another factor was my reading of Tall's *How Humans Learn to Think Mathematically* (Tall, 2013). Among many other things, Tall introduces in this book the concept of a *met-before*. Inspired by the research of Lakoff and his colleagues on the role of metaphor in learning (e.g. Lakoff & Johnson, 1980; Lakoff & Núñez, 1997), Tall notes that the term is a play on the word metaphor itself. While Lakoff and others are focused on a grander scale of human cognition, Tall focuses on the individual learner, asking a student what they have met before that influenced their thinking on the mathematics at hand. Sometimes, a concept can be a helpful met-before, such as the addition of natural numbers naturally extending to the addition of negative integers. Other times, such as when familiarity with multiplication of natural numbers leads a particular student to believe that multiplication of fractions will always make a number larger, met-befores can be more problematic. Of course, the idea that prior knowledge can have beneficial and negative impacts on learning and understanding is not new. But, it was this individual take that influenced my present research. I began thinking about what met-befores students bring to bear during and after a semester in linear algebra. Following the line of reasoning given below, I then transitioned to thinking about concepts in linear algebra as met-befores for concepts in abstract algebra.

Over the last few decades, the number of studies focusing on the teaching and learning of both undergraduate linear algebra and abstract algebra has increased. With this has come preliminary research into how students understand the core concepts of each subject. However, there is little regarding the connections students make between similar concepts from both classes.

Since the 1970's, a strong emphasis has been placed on beginning linear algebra classes in the United States with matrices and computations. At institutions where students usually only take one semester of linear algebra, abstract theory is placed at a later point in the semester, and students' understanding of real vector spaces is leveraged to teach abstract vector spaces (Uhlig, 2003). According to Stewart (2008), the issue with this approach is that "while students seem to learn matrix algebra rather efficiently, very few students ever learn the more advanced concepts, which require understanding the more abstract material" (p. 3). Consequently, many studies have given attention to solutions for crossing the divide into the formal realm of abstract linear algebra.

Of course, building up students' abstract algebraic reasoning is a central goal of any undergraduate abstract algebra course. Thus, it seems reasonable to ask if students in such a course are able to reason more abstractly about linear algebra upon revisiting the subject. While most undergraduate abstract algebra courses may only tangentially mention linear algebra, there are many hopeful points of crossover from a higher point of view. For example, linear transformations and group homomorphisms are both morphisms in their respective categories which must satisfy more requirements than being simply set functions (Awodey, 2010). They are uniquely tied to the structure of vector spaces or groups.

In thinking of a starting point for the teaching of linear algebra which permeates the entirety of the subject, Uhlig (2003) states that "we only have one candidate for this role: ‘Linear Transformations’. This notion satisfies our basic requirement of being fundamental to the whole field” (p. 152). Likewise in abstract algebra, Hausberger (2013) states that “the (homo)morphism and isomorphism concepts [are] central in abstract algebra” (p. 2348). The centrality of these concepts, and their similarities, led me to study how students who have taken abstract algebra understand both morphisms. By centering the research around group homomorphisms and linear transformations, many other related concepts fit naturally into the conversation. Once a student creates a mental link between morphisms, it seems reasonable to believe the next natural link would be between the objects: vector spaces and groups. One could also talk about the structure preserved by these morphisms, bringing binary operations, sub-objects, or generators into the conceptual landscape. It is thus important not only to study students’ understandings of group homomorphisms and linear transformations in the abstract, but to discuss how they are conceived by students in relation to other concepts.

What should be the expectations for the connections that students make? There are three possibilities regarding the possible helpful or problematic links students make between group homomorphisms and linear transformations. The first possibility is that students taking abstract algebra and learning about group homomorphisms for the first time will recognize their similarity to linear transformations, and use their prior knowledge to inform their understanding of group homomorphisms. The second is that students, upon learning of group homomorphisms, reflect upon linear transformations and use their newly learned concept to recall and inform their understanding of linear transformations. Both of these possibilities could occur with the same student. The last possibility is that students see no connections between the two morphisms. These same three possibilities likewise apply to those concepts closely related to morphisms.

Seeing as each of these possibilities seem plausible, initial exploratory work is needed to determine what connections students are making without intervention, and if students regularly make these connections at all. In either case, it is important to understand what mental images students have regarding group homomorphisms and linear transformations individually. It could then be possible to find insight into problematic conceptions preventing connections, and helpful mental images leading to connections. Finally, I do not wish to pit the connections students make against my own mental connections. I could cite various undergraduate and graduate mathematics texts about the ways in which structures from vector space theory are also structures in group theory. However, this carefully curated "ideal" would be unfair as a comparison to students, who would be asked to generate or recite connections on the spot in an interview setting. It would be fairer to compare students' insights to those of professors who have taken both graduate linear algebra and graduate abstract algebra.

All of these considerations led me to form the following three guiding research questions:

(1) How are students' understandings of group homomorphisms and linear transformations the same or different?

(2) What connections do students make between the concepts related to group homomorphisms and linear transformations?

(3) How do students' understandings of these morphisms compare to those of professors?

## 1.2 Frameworks for Students' Conceptualizations of Mathematics

In mathematics, the mental conjuring of a particular term can immediately give rise to a host of related concepts. Simply mentioning the term "function" to a Calculus student will trigger perhaps the written notation "$f(x)$," a mathematical definition requiring $x_1 = x_2$ implying $f(x_1) = f(x_2)$, a vague connection to something called a "vertical line test," related terms such as domain and range, examples such as

$f(x) = \sqrt{x}$ or $g(x) = \sin(x)$, or even misconceptions such as $h(x) = 5$ not being a function. The collection of relations, notations, images, examples, and representations conjured by an expert would of course be even larger and richer than this, and would vary from person to person. To give a name to this personalized collection of immediate knowledge, Tall and Vinner (1981) introduced the terms *concept image* and *concept definition* to the lexicon of mathematics education. The concept image is used to "describe the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes" (Tall & Vinner, 1981, p. 152). This definition attempts to capture not only the logical connections and imagery held by learners, but also the illogical and incoherent parts as well. While concept image is used to describe the ideal whole, the term *evoked concept image* is used to describe the subset of mental constructs conveyed by a student at a particular time. Different stimuli, social contexts, emotional states, and so forth can trigger a specific portion of a learner's understanding. It is thus technically the evoked concept images which comprise the data for analysis in a study such as my own. By piecing together these evoked concept images, one can abstractly speak of a theoretical overall concept image. On the utility of the concept image framework, Tall (2003) ruminates, "Speaking of concept image can sometimes be vague, but... this is precisely what makes it so useful. It helps us to grasp that there are subtleties in mathematical thinking that cannot be precisely conveyed by the apparent precision of mathematics." The term concept image highlights the various, and potentially conflicting, aspects of mathematics central to a person's understanding.

A part of one's concept image is one's concept definition. A concept definition, which is unique to each individual, is not always the same as a formal mathematical definition. Rather, a concept definition is the learner's "form of words used to specify that concept" (Tall & Vinner, 1981, p. 152). This likewise can change between conveyances, as a concept definition is drawn from an evoked concept image. A concept definition can have its own surrounding concept images, or it can be a simple rote utterance of a formal definition.

The theoretical framework of concept images is very much related to that of the ubiquitous *schema* theory in cognitive psychology. Rumelhart and Ortony (1977) defined schemata as "data structures for representing the generic concepts stored in memory" (p. 101). Both concept images and schemata are collections of conceptual networks available to a student when encountering a relevant situation. The framework of concept image/definition simply places more emphasis on the fact that the human brain is hardly a consistent input-output machine. While a student may be able to recite a particular mathematical definition, they may instead apply understanding from their concept image in order to solve a problem (Edwards & Ward, 2004; Vinner & Dreyfus, 1989; Wawro, Sweeney, & Rabin, 2011). This incongruence is potentially not fully captured if one attempts to describe a student schema with either their concept definition or evoked concept image. By being aware of the image/definition distinction, the researcher can design a study in a way that gauges both aspects of students' thinking.

## 1.3 Research on the Teaching and Learning of Linear Transformations and Group Homomorphisms

Unlike abstract algebra, which by nature strongly focuses on teaching students abstract, formal mathematics, linear algebra asks students to work within three distinct, yet intertwined, modes of thinking. Hillel (2000) refers to the modes of description as *abstract*, *algebraic*, and *geometric*. The abstract mode, and its associated language, are at the heart of this study; it concerns the notions of vector spaces, linear transformations, and vector space theory in general. The algebraic mode, perhaps a term more misleading when placed in the context of my research, is the mode used when speaking of $\mathbb{R}^n$ and its vectors, matrices, and associated matrix computations.

Finally, the geometric mode of description is when students describe the concepts of linear algebra in terms of points, drawn vectors, lines, planes, and visual transformations. In a very similar description of the nature of linear algebra, Sierpinska (2000) first splits students' modes of thinking into two domains: *synthetic* and *analytic*. The synthetic mode is a practical way of thinking, where students can directly describe something such as a line or plane, without concern for its axiomatic definition. The analytic mode of thinking, according to Sierpinska, is theoretical in nature, and is the realm of intangible concepts given birth through definitions. This difference between practical and theoretical thinking fits nicely with Tall's and Vinner's rationale for introducing the distinction between mathematical definitions, concept definitions, and concept images. According to Sierpinska, the synthetic mode of thinking in linear algebra is by-and-large geometric in nature, as discussed with Hillel's modes of description. Thus, Sierpinska's first mode of thinking is *synthetic-geometric*. Sierpinska breaks the analytic mode into *analytic-arithmetic* and *analytic-structural*. The analytic-arithmetic mode is characterized by computing values through algorithms with matrices and vectors, while the in the analytic-structural mode students work directly with concepts in the theory of vector spaces, using their characteristic properties to make deductions. Clearly, Sierpinska's analytic-arithmetic and analytic-structural modes are similar to Hillel's algebraic and abstract modes, respectively. The main emphasis here is that students' concept images of linear algebra are often woven together with an intermingling of the above three modes. For the rest of this paper, I will refer to these three modes as simply geometric, arithmetic, and structural. Though I have not encountered any research comparing these modes of thinking and description to those in abstract algebra classes, I would venture to say that the structural and arithmetic modes have similar counterparts for students in abstract algebra (with the emphasis on matrix calculations being replaced with calculations of integers, rational numbers, or real numbers for the arithmetic mode).

There is research to show that students learning group theory, in encountering and attempting to reduce multiple levels of abstraction, sometimes reduce abstraction by instead thinking of number systems they are familiar with, such as the integers (Hazzan, 1999). Thus, concepts from abstract algebra also are conceived on multiple planes similar to the structural and arithmetic modes mentioned above, but it is not quite the same as the threefold ways of thinking found in linear algebra. Geometry is present in abstract algebra for certain groups, such as permutation groups, but it does not permeate every fiber of the subject as it does in linear algebra.

In my review of the literature, I found a handful of studies explicitly discussing how students conceive of linear transformations. In a study of two students' responses to interview questions concerning their understanding of linear transformations, de Oliveira and Lins (2002) found that their participants did not understand linear transformations as morphisms between structures. Instead, the students seemed to think about vector spaces functionally as a "naturalized space" such as the space we live in, and linear transformations as moving or stretching within that space. Vector spaces were not part of their understanding, but simply the insignificant quasi-physical place where vectors exist. The students in this study had a strong desire to see vectors, even those not in $\mathbb{R}^n$, visually in order to understand a linear transformation.

Zandieh, Ellis, and Rasmussen (2017) studied the similarities and differences between students' concept images of function and linear transformation. In their analysis of ten students' responses to an in-class questionnaire and interview questions, they found that students' evoked concept images centered on certain properties, computations, and "clusters of metaphorical expressions." Students often discussed functions in terms of input and output or equations, while they discussed linear transformations in terms of morphing one thing into another or as a machine that produces something based on an action. Some students had particularly well-aligned concept images, while others struggled to align across contexts or were still actively working to reconcile the two. In specifically looking at how students viewed one-to-one functions and linear transformations, they found that certain procedures, such as the horizontal line test or linear independence of columns of a matrix representation, presented barriers to making connections. The authors suggest continued research into unified concept images.

A few other studies have discussed students' reasoning on linear transformations as part of a larger conversation on geometric transformations. Dreyfus, Hillel, and Sierpinska (1999) found that with their particular CABRI learning environment, students confused the notation $T(\mathbf{v})$ for the image of a single vector $\mathbf{v}$ shown on the screen, rather than being the transformation $T$ of an arbitrary vector $\mathbf{v}$. In a study on geometric transformations, primarily with students in mathematics education, Portnoy, Grundmeier, and Graham (2006) found that students primarily view transformations as processes rather than objects; they are actions which are performed on other things, but not concepts which in their own right can be the objects of other processes.

Meanwhile, my review of the literature turned up very little on students' conceptions of group homomorphisms. Homomorphisms appear incidentally in studies focused on students' proof writing techniques in the abstract algebra classroom (e.g. Selden & Selden, 1987; Weber, 2001, 2002), or in exploring how students grapple with other concepts such as group, subgroup, isomorphisms, quotient groups, and so forth (e.g. Dubinsky, Dautermann, Leron, & Zazkis, 1994; Larsen, 2009). Concerning the research on students' use and understanding of isomorphisms, Weber and Alcock (2004) and Leron, Hazzan, and Zazkis (1995) both found that undergraduate students may have knowledge of the definition of isomorphism, but not an intuition regarding the sameness conveyed by isomorphisms. Such an intuition was used instead by graduate students and professors interviewed by Weber and Alcock (2004). Leron et al. (1995) also go on to point out how isomorphisms have their own conceptual pitfalls, such as isomorphisms as objects to be used in proof having a directionality arising from their nature as a function. This directionality is not present in the simpler idea of sameness.

## 1.4 Analogical Reasoning

A common way of comparing two concepts, such as those at the heart of this study, is to place them in an analogy. Outside of its use in bygone national testing, the intention of an analogy is to bring understanding of a more well-understood domain into the realm of a less well-grasped domain. Again using the concept of function as an example, teachers often make analogies when teaching students mathematical functions for the first time. Often, the instructor will ask students to think of a function like a "machine," perhaps a machine that one might see on a conveyor belt in a factory. Students already have the prior knowledge that machines take an input, perform a pre-programmed process on that input, and produce an output. The instructor hopes that this prior knowledge will serve as a springboard in the learning process, and identifies the inputs of the machine with the concept of domain, the machine itself with the function, and the outputs of the machine with the range. The analogy can then lend itself easily to certain extensions, such as the composition of functions. However, the same analogy can also lead to misunderstandings, such as a student picturing a machine that always changes its input then leading to confusion surrounding an identity function.

The key feature of an analogy, such as the example above, is the juxtaposition of parallel relations in different domains (Holyoak, 2012; Gentner & Maravilla, 2018). According to Gentner's (1983) structure-mapping theory, an analogy is established via a mapping of objects and relations from a *base* domain into a *target* domain. Both the target and the base are placed in *structural alignment*, a one-to-one correspondence between some objects of the two structures which preserves the relations by ensuring that parallel relations are also mapped to one another (Gentner & Markman, 1997). The surface attributes of the base and target objects are less important than the relations when creating a sound structural alignment. What constitutes a surface attribute varies from researcher to researcher (Holyoak, 2012), and what is considered a surface attribute to an expert may be an important attribute or relation to a novice (Lobato, 2008a). Relations between the objects not only dictate the mapping, but also determine the scope and effectiveness of the analogy. A guiding aspect of structure mapping theory is that of the *systematicity principle*, the idea that having a high level of connectedness – higher-order relations between lower-order relations, increases the likelihood that a relation will be mapped across domains (Gentner, 1983), a principle that has also played out in practice (Wharton et al., 1994). If there is a relationship between two objects that has little bearing on the other relationships in the base, it will be ignored in the structural alignment and not carried over into the target of the analogy. The strength of the analogy is also determined by systematicity. The more higher-order relations that can be mapped to corresponding relations in the target, the more likely it is that the person making the analogy rates it as a good analogy (Gentner, Ratterman, & Forbus, 1993). Gentner and Maravilla (2018) posit that "this desire for systematicity reflects an implicit preference for coherence and inferential power" (p. 188).

Once an analogy has been established via structure mapping, objects and relations in the source are used to infer new relationships between objects in the target (Holyoak, 2012). In other words, the next natural step after aligning clear structural parallels is to begin aligning potential new parallels. It is thus by this process of inference that learners can use prior knowledge of an understood source and apply it to new targets. A series of analogical mappings can also lead to the creation of a new schema. By focusing on the relational similarities of domains, a learner may develop a schema which abstracts the structure of the base and target (Gick & Holyoak, 1983).

This schema can then be used to more easily identify future analogical mappings (Gick & Holyoak, 1983).

The initiation of structure mapping depends on the presence of both the base and target in working memory (Holyoak, 2012). This means that one potential barrier to constructing and using analogies in this way is the retrieval of the domains. For example, if a learner is provided with a target by an instructor, and asked what structure(s) in their education "look similar" to the target, the learner is given the extra burden of sifting through long-term memory for a potential source. Should not enough cues be provided to the learner, no source will be found, and the learner will be unable to produce a mapping. Failure to retrieve a source with relational similarity (at least a source intended for by a researcher) has been shown to be quite common, even in a setting where the source was encountered not too long before the target (Gentner & Maravilla, 2018). Interestingly, sometimes all that is required to greatly increase the likelihood of a successful analogy is simply a hint that a recent topic is a good source candidate (Gick & Holyoak, 1983).

There are also certain caveats to the findings on retrieval. Some studies have shown that retrieval is much more common in the real world than the narrow definition found in laboratory settings would imply (Dunbar, 1995, 1997). In fact, giving a person the ability to choose their own sources (rather than hoping for a specific source to be chosen) increases retrieval and still leads to analogies with deep structural alignment (Blanchette & Dunbar, 2000). Additionally, if a target has a corresponding source with a high degree of surface similarity (so that their objects have shared characteristics), the rate of retrieval greatly increases (Gentner et al., 1993).

Unfortunately, surface similarity is a double-edged sword for recall. Despite the systematicity principle, and the evidence that analogies with a high degree of relational similarity are highly rated, surface similarities are in fact the strongest predictors of retrieval (Gentner et al., 1993). Surface similarities can be particularly problematic in a *cross-mapping*, where similar objects appear in both domains, but play different roles. In this case, the surface similarity has been shown to interfere with a proper mapping of the relations in the analogy (Gentner & Toupin, 1986).

Once a structural mapping and its subsequent inferences are made, the analogy is evaluated for its soundness (Gentner & Maravilla, 2018). In the event that the analogy leads to clearly problematic inferences, the analogy may be discarded or ignored by its creator. Or as mentioned earlier, in some clinical studies, the analogy is reported, but rated by its creator as having poor soundness to the researchers. According to Gentner and Maravilla (2018), the evaluation of an analogy is determined by the adaptability of its inferences to the target, the relevance of the analogy to the goals of the subject, and the power of the analogy to generate novel inferences.

The use of analogical structure mapping to generate inferences based on a source and a target is part of a much wider body of research on *transfer*. Specifically, the above description aligns with what some may call a traditional conceptualization of transfer. Lobato (2008b) characterizes this more classical view of transfer as "The application of knowledge learned in one situation to a new situation" (p. 291). In recent decades, this definition and the academic culture surrounding it have come into question. One of the first main critics was Lave, who noted that the traditional model measures successful transfer in terms of pre-defined researcher outcomes, and attempts to separate the cognitive processes of the learner from their experience situated in the real world (Lobato, 2006). In response to criticisms such as these, Lobato and her colleagues developed a framework called *Actor-Oriented Transfer* (AOT). Under this view, transfer is more broadly defined as any "generalization of learning" made by the learner or, as named in the theory, the actor (Lobato, 2008b). In contrast to the traditional experimental approaches, AOT asks researchers to look for any relational similarities found by learners in a more naturalistic setting, and encourages the use of ethnographic methods (Lobato, 2008b). Transfer is also not determined by correct or incorrect mathematical notions. It is possible that a learner finds similarities and makes inferences that are mathematically incorrect. From the point of view of AOT, a study should take note of these generalizations of knowledge, and look for what aspects of the environment encouraged this transfer (Lobato, 2008b). Finally, transfer is not seen as a static application of knowledge from one setting to another, but more so as a dynamic creation of new relationships and the restructuring of knowledge (Lobato, 2008b).

In this thesis, I use a synthesis of the mainstream and actor-oriented transfer frameworks. The emphasis on what prior knowledge the learner has transferred, rather than just if transfer has occurred for a very specific researcher-driven outcome, aligns well with my research question concerning what concepts students align across algebraic domains. It also aligns with Tall's idea of met-before, asking individual students what concepts they have met before that cause them to think about a process or object in a certain manner. However, the level of detail in the theoretical frameworks of mainstream analogical reasoning researchers such as Gentner and Holyoak provides a language for describing and theorizing about the ways that students use analogies to make sense of mathematics. Thus, I approach the data with an open-minded, student-focused lens, while also looking for schema alignment and comparisons of relational structures. To place these two frameworks as more synergistic than discordant is not new, as researchers such as Reed (2012) see alternative transfer frameworks as natural extensions of the classical frameworks. In his article, Reed (2012) proposes a taxonomy of analogical mapping, placing the various viewpoints of transfer on a two-dimensional grid indexed by the type of mapping (one to one, one to many, or partial) and type of situation (problems, representations, solutions, or contexts). In the taxonomy, Reed places classical studies such as Gick and Holyoak (1983) or Gentner (1983) under the "problems" situation, where a one-to-one or partial mapping is used in solving a specific problem of interest to the researcher. Meanwhile, a piece such as Lobato (2008b) is categorized as analyzing partial mappings of “representations.” Rather than specific problems, the research is focused on mapping “information from diagrams, formal concepts, mathematical symbols, and formulas” (Reed, 2012, p. 19).

Of course, Reed (2012) is specifically choosing to focus on the commonalities of the theories in order to draw the field together and highlight the advantages of Gentner’s mapping perspective in new settings. There are still important differences between classical and AOT transfer frameworks. Classical analogical mapping theory deals explicitly with abstract objects, relations, and schemata as if they are actual structures within the mind of the learner. These are aligned and then used to make inferences in one direction. AOT focuses on the social, physical, and cultural aspects of the situation of knowledge, in addition to the mental (Lobato, 2008b). Various pieces of these aspects gain the attention of the learner, and the result of their focus encourages the construction of relations that actively change and restructure knowledge of all planes and domains simultaneously. I recognize that by using clinical one-on-one interviews, I am unable to capture the social, physical, and cultural aspects of students’ learning processes. I also recognize that, because of their fundamental differences in the way they view the restructuring of knowledge, analogical transfer and AOT are not fully compatible. Nevertheless, I find the emphasis on actors’ point of view indispensable in an exploratory endeavor such as my own.

For this analysis, anytime I make use of classical cognitive language and theory, I use it with an understanding that the knowledge structures and mappings are only a tool, and that the mental plane is only one small part of the overall context of learning and understanding mathematics. The theoretical discussions of ethereal mental constructs are meant to reveal potential barriers to learning mathematics, and not a one-to-one account of the cognitive processes of a student. A single analogical mapping could be just one small fragment of a much larger simultaneous cognitive restructuring. Additionally, when a student makes mention of various other planes of knowledge, and their role in the process, I will also be sure to include these as important factors.

# Chapter 2 Research Design and Methodology

## 2.1 Participants

In order to identify departments within a reasonable proximity with potential student research subjects, I conducted a review of the course offerings for all four year undergraduate institutions in the state of Kentucky. Once I compiled a list of colleges and universities offering both a linear algebra and at least one abstract algebra class, I contacted department heads of potential mathematics programs in Kentucky. Interested departments forwarded an advertisement email to potential student subjects. I also posted fliers at my home campus. Interested students were then contacted directly by email. The main bottlenecks of recruitment of subjects were the timing of abstract algebra courses (often offered once every two years, with potential students graduating before the time of this study), and lack of responses from either departments or individual students. The final sample of students was eleven undergraduates (one being part of a pilot version of the study) across three institutions in the state of Kentucky. All participating students had either completed or were near completing both classes. In the event that a student was still taking the course, the interview date was set late in the semester, and students were asked about having seen certain terms in their course. Linear algebra and abstract algebra grades were self-reported. Every student interviewed was either a mathematics or mathematics education major, though many students were pursuing multiple majors. All students were still undergradutes at the time of interview. However, both Chaz and Kyle had both taken multiple graduate classes in mathematics, including graduate abstract algebra. See Table 2.1 for details on the demographics of these students.

A convenience sampling of three mathematics professors was used for comparison to students' responses. Two of these professors had taught undergraduate linear algebra, and the other had taught a section of undergraduate abstract algebra (second semester). Their research areas included algebra and topology.

## 2.2 Data Collection

### Format and Instruments

All participants individually completed two semi-structured clinical interviews. Because of the time frame for this research and the logistics of being able to follow the particular type of students needed for this research, the clinical interview format was chosen over a naturalistic, longitudinal study. The clinical interview format is very appropriate for research concerning students' cognitive structures. This is expressed by Ginsburg (1981), who said, "We see then that at least for the identification and description of complex cognitive structure, it is desirable and usually necessary to employ a method other than naturalistic observation or standard tests. For Piaget, and for researchers concerned with mathematical thinking, the method of choice is the clinical interview" (p. 7). A semi-structured format was chosen to adapt to the emergent lines of thought in student thinking, while maintaining a consistent question set for all participants. Protocols for both interviews were created, piloted, and revised before collecting the main data for this study.

The first interview centered around two main activities: definition writing and card sorts. The cards sorts in this study were primarily open, single-criterion sorts. What this means is that participants were presented with a set of cards with related terms, and then asked to create groupings based on a single criterion, which is open to their choosing (Rugg & McGeorge, 2005). While card sorts often require participants to partition the complete set of cards into groups (e.g. Whaley & Longoria, 2009; Rugg & McGeorge, 2005), this particular study was less restrictive. Instead, participants were asked only to group at least two cards together and explain how they are related. These cards were then placed back into the grid and the procedure was repeated. The

Table 2.1. Student Participants and Demographics

| **Pseudonym** | **Interview Sem.** | **Campus** | **Status** | **Majors** | **LA Sem.** | **AA I,II Sems.** | **LA Grade** | **AA Grade** |
|---|---|---|---|---|---|---|---|---|
| Arthur | Fall 2017 | B | Soph. | Math, Comp. Sci. | Fall 2017 | I Fall 2017 | A | A |
| Chaz | Fall 2017 | B | Junior | Math, Economics | Unknown | Unknown | A | A, A |
| Emilie | Fall 2017 | B | Senior | Math, Education | ∼Spring 2015 | I Fall 2017 | C | B |
| Flint | Fall 2017 | C | Senior | Math Education | Fall 2016 | I Fall 2017 | A | B |
| Jake | Fall 2017 | A | Junior | Math | Fall 2016 | I Fall 2017 | A | A |
| Kyle | Fall 2017 | B | Senior | Math, Economics | Fall 2014 | I Spring 2014, II Spring 2015 | A | A, A |
| Lucas | Fall 2017 | B | Senior | Classics, Math | Fall 2016 | I Fall 2016, II Spring 2017 | A | A, A |
| Maureen | Fall 2017 | B | Senior | Math | Fall 2017 | I Fall 2017 | C | A |
| Robyn | Fall 2017 | B | Soph. | Math, Education | Spring 2017 | I Spring 2017, II Fall 2017 | A | B, A |
| Sander | Fall 2017 | B | Senior | Math, Bio., Chem. | Spring 2017 | I Fall 2017 | B | B |
| Tamara | Spring 2018 | C | Senior | Math Education | ∼Spring 2015 | II Spring 2017, I Fall 2017 | A | C |

AA = abstract algebra

LA = linear algebra

I, II = first/second semester in abstract algebra

∼ = student could not recall exact semester inspiration for this style of card sort came from Eli, Mohr-Schroeder, and Lee (2013). In every open card sort, participants were allowed to create their own cards. This option was included in order to allow participants to feel free to express their concept images and connections without feeling restrained by the given terms. Card sorts can also be closed, where part of the above procedure is limited in some way (Fincher & Tenenberg, 2005). The closed card sort in this study was limited in the sense that the cards were given to participants, and they were asked to determine if they were related or not. All cards were reviewed by three other mathematics educators for their appropriateness in relation to group homomorphisms or linear transformations. According to Rugg and McGeorge (2005), all cards should be on "the same semantic level as each other," so only individual concepts, rather than equations, definitions, and theorems, were included. The purpose of the card sorts was twofold: 1) to serve as a framing device for students to expand on their concept images of the included terms, and 2) to create a quantitative record of connections within and between domains.

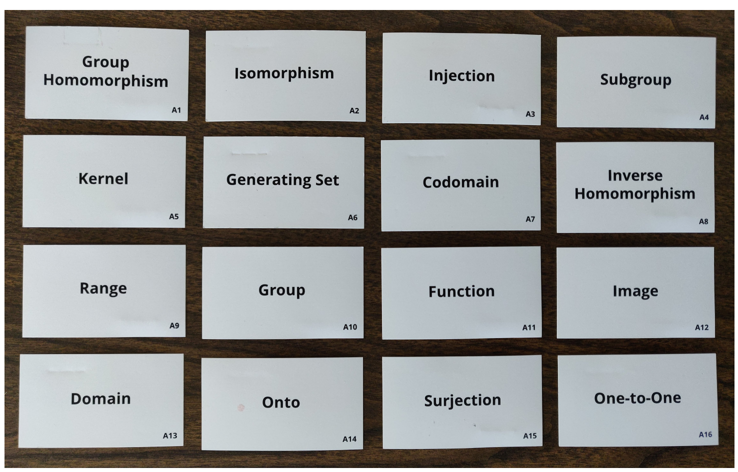


Figure 2.1. Layout of Group Homomorphism Card Sort

These card sorts were interspersed throughout the duration of the first interview. After collecting demographic data, participants were asked to define group homomorphisms. If the participant was unable recall this definition, they were asked to write whatever came to mind, and then were shown a definition which was prepared and printed beforehand (see Appendix B). This was followed by a series of questions asking students to describe their understanding of the nature and purpose of a group homomorphism. Participants were then shown an example of the open card sort format, which involved cards labeled with ten concepts related to derivatives as they normally appear in a first year calculus course. Once a participant expressed their comfort with the format, they were asked to complete an open card sort using 16 terms related to group homomorphisms. This process of definitions, questions, and card sort was then repeated with linear transformations. The section for linear transformations included additional questions asking participants about the relatedness of linear transformations and group homomorphism, as well as the impact of recalling group homomorphisms on linear transformation recall. The linear transformation card sort included 19 cards. Before starting this card sort, participants were shown any groupings from the previous sort that could still be created in the new grid. (For example, the cards for domain, range, and function were present in both of the first two card sort grids.) Each participant was asked if they would still create such a grouping in the new grid, and if their description of such connections would still remain the same. If a participant indicated that such a grouping still would have the same connections in linear algebra (e.g. "Functions, regardless of what class you're in, still have a domain and a range."), then the grouping was included in the records for the second card sort, as well. A third card sort activity followed, using all cards from the previous two sets. Because cards with the exact same concept name were only included once, this final card sort involved 24 pre-made cards. Any participant created cards from the prior two sorts were included as well. Participants were then asked to complete a closed card sort involving seven pairs of words in succession, in which they were asked if the two cards were related and, if so, how. Some pairs were across domains (e.g. subspace and subgroup), while others were within a single domain (e.g. matrix and linear transformation). The protocol in Appendix A contains lists of all cards included in each card sort.

The order of group homomorphisms and related concepts first, followed by linear transformations and related concepts second, was intentional. Because students typically study linear algebra before (or at least concurrently with) abstract algebra, it was presumed that concepts from the latter would be recalled more clearly than the former. Placing abstract algebra concept recall at the beginning of the interview would thus provide the more ideal setting for students to make connections between domains. It was also assumed that placing the more easily recalled domain first would build students' confidence, and avoid at least some of the impact of anxiety on students' ability to make analogies (Tohill & Holyoak, 2000).

All interviews were videotaped, and all written artifacts, such as definitions and spontaneous written explanations, were saved and scanned. The entirety of students' interviews were transcribed by myself. Card groupings were recorded during the interview.

### Pilot Study

The format of the study was piloted informally with two graduate students before their first semester in graduate school. In the informal pilot, the first interview originally used concept maps as a framing mechanism, instead of card sorts. Students were asked to create a concept map centered around group homomorphisms, and then a separate concept map focusing on linear transformations. At the end, they were asked to talk about the similarities of the two. Students had difficulty recalling the needed definitions to create the necessary concept maps. However, it was hypothesized that the increased length of time since taking the classes involved was a strongly influential factor. The informal pilot also revealed that the second problem solving interview contained too many problems, and would often likely take longer than the targeted two hour time. Thus, questions were cut to reduce the length of the second interview.

In the formal pilot for the study, I interviewed the undergraduate by the pseudonym of Chaz using an updated concept map protocol format. At this time, another version of the interview protocol using the card sort format had already been drafted. The intention was to pilot both versions and choose the version best suited for the study. After Chaz's first interview, it was clear that the card sort interview format would be chosen. The concept map format asked students to connect concepts using markers, sticky notes, and a large whiteboard. It was evident that Chaz had a good grasp of both linear transformations and group homomorphisms. However, the format left him to sit and ponder all possible connections at one time. Despite being prompted to explore his thoughts aloud, the number of possibilities for both the number of concepts and how they should be connected led to long periods of silence as he thought deeply about his concept maps. Chaz's reticence to draw an arrow spreading across the diagram also revealed the difficulty that students would face in this format to create a digestible two-dimensional picture of their connections within a subject. In retrospect, this probably would have caused certain connections between concepts not to be drawn due to the pictorial aspect of the format. Additionally, Chaz created groupings of sticky notes as he brainstormed his concept maps, indicating a natural fit for the card sort format. After drawing the two concept maps focusing on individual algebraic domains, Chaz was asked to look for and discuss aspects of the two concept maps that "play similar roles" in their respective domains. Despite the depth of knowledge shown by Chaz in each subject, he only connected two items. The format made it difficult for students to seek out related terms, and it was also clear that a portion of the interview needed to ask directly about terms that students may have simply forgotten to mention. Overall, the concept map format was too lengthy, yielded sparse data on student thinking due to periods of silence, and placed too much cognitive burden on students for the wrong reasons. The card sort format more adequately addressed these concerns, and was chosen for the main format for the first interview.

## 2.3 Analysis of the Data

### Development of Themes and Assertions from Qualitative Data

Each transcribed student card sort interview was initially coded with a combination of descriptive coding, structural coding, and concept coding methods (Saldaña, 2016). I performed all coding using Nvivo 12. For those unfamiliar with qualitative analysis, coding refers to the process of using words and phrases to summarize and assign meaning to portions of qualitative data (Saldaña, 2016). Coffey and Atkinson (1996) describe how this labeling of the data plays a different role than in quantitative analysis, saying,

> In this sense, coding qualitative data differs from quantitative analysis, for we are not merely counting. Rather, we are attaching codes as a way of identifying and reordering data, allowing the data to be thought about in new and different ways. Coding is the mechanics of a more subtle process of having ideas and using concepts about the data. (p. 29)

According to Saldaña (2016), codes are not necessarily the themes that appear in the final assertions or theory at the end of a research project. Rather, themes arise from the analysis and categorization of the codes.

Due to the exploratory nature of this study, I did not have an *a priori* coding scheme; the initial codes were formed inductively from the data. However, this does not mean that I entered into coding as a blank slate. Instead, my research questions guided the coding process, and my prior understanding of the preexisting theories, such as concept images, Tall's met-befores, transfer, and others not mentioned in this text, shaped the lens through which descriptive codes and concept codes were formed. This means that during the process of coding, I was simultaneously looking for emergent patterns in student thinking, evidence of connections between mathematical concepts, and indications of transfer or problematic concept images. As I coded more passages from the transcripts, the number of codes increased to reflect the different thought processes and perspectives of students. The presence of these new codes then warranted new passes through previous transcripts. This process of continual inductive coding passes continued until all transcripts were coded and a saturation point was reached regarding the number of new codes coming from a new pass through the data.

Because interviews were semi-structured, students answered a set of common interview questions between card sorts. These common prompts made it easier for me to compare and contrast students' responses for this portion of the data. Thus, I constructed structural codes marking the beginning and ending of answers to interview questions to quickly find each student's response in Nvivo. These responses were then summarized and placed into a spreadsheet organized by students on one axis, and interview questions on the other. This allowed me to analyze this portion of the data both within individual cases and across cases. This occurred concurrently with the descriptive coding and concept coding process above. Insights gained from this spreadsheet were integrated into the above coding process by finding the original passages and coding them accordingly.

After these first rounds of coding, the list of codes was pared down through two processes. The first of these was the coalescence of related codes either through merging codes or making one a subcode of the other. The other process was the pruning of codes from full consideration in the theoretical analysis. Codes were pruned either due to their presence in only a single passage, or their irrelevance to the research questions.

Through these processes, I found that most of the connections formed by students were either stated directly as analogies, or could be interpreted as such. While the theoretical framework of concept image and concept definition proved useful for discussing students' understandings of group homomorphisms and linear transformations, it did not provide a sufficient framework for analyzing students' reasoning through analogies. Rather than build an entirely new framework from induction alone, I continued research into preexisting theories on transfer through analogies. This is what led to the use of Gentner's and Holyoak's writings on analogical reasoning as a lens for viewing student connections. Similar to my use of both student-driven and theory-driven codes, my later analysis was thus neither purely inductive nor deductive. Instead, I engaged in what Coffey and Atkinson (1996) refer to as abductive reasoning. According to Coffey and Atkinson, most qualitative researchers do not engage in purely inductive or deductive logic to analyze their data. In pursuit of the generation of ideas and theories from the data, researchers oscillate back and forth between the two, looking to "go beyond the data themselves, to locate them in explanatory or interpretive frameworks," but also to use data to "come up with new configurations of ideas" when these existing frameworks do not fit (p. 156).

With the theoretical framework of analogical reasoning in mind, I constructed another spreadsheet, this time listing all coded analogies with a summary of a student's wording and the concepts aligned in the analogy. This, together with other overlapping codes, led to the summaries found in Section 3.2. This also gave rise to the use of mapping diagrams to convey visually the important analogies at work in students' reasoning. In each diagram, the abstract algebra domain is given on the left, while the linear algebra domain is on the right. Parallel structures are displayed in similar layouts across domains, with one structure being either a translation or reflection of the other. Concepts are shown as ellipses, and are linked by arrows representing relations expressed or implied by the student being discussed. Key concepts aligned across domains are also connected by arrows either labeled as "is analogous to" or using a more precise wording given by the student. Connections and relations shown with a dashed line were inferences generated by students based on the existing relations and connections indicated by arrows with solid lines. Dotted lines with "×" represent an important concept which has a mathematical parallel in the diagram, but is not present in the student's structural alignment.

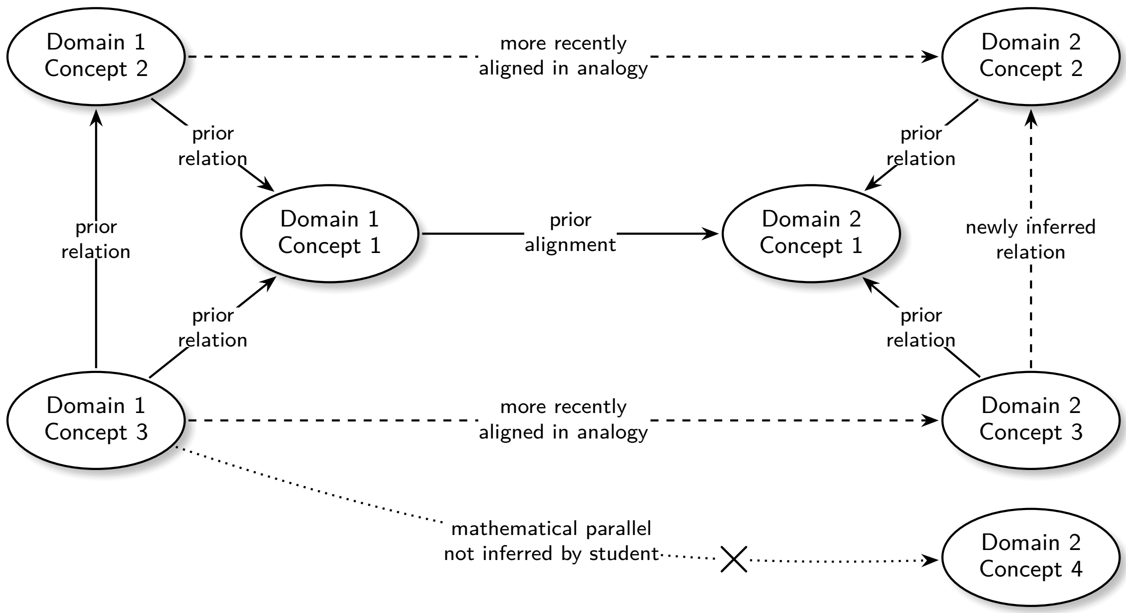


Figure 2.2. Example of Diagram for Displaying Analogical Reasoning

I regularly make assertions about participants' concept images, their analogical reasoning, and general themes across cases. As a form of triangulation, during the process of constructing assertions, I searched the data for other times a student described a relevant concept or connection. As a form of transparency, when the data is sparse or contains a discrepant case counter to the general assertion, I have reported as much in my analysis.

### Construction of Adjacency Matrices

Transfer across domains requires an understanding of both the base and target domains in order for an individual to begin engaging in structural alignment. Thus, it was important to find a way to communicate the overall structure of participants' understandings of each individual domain. Analogies between domains constructed by participants, either verbally or with explicit cards in the combined card sort, lent themselves nicely to qualitative coding and analysis. The one-to-one nature of structural alignment, and common linguistic indicators such as *similarly*, *parallel*, *like*, and so forth, made finding and reporting between-domain connections easy with the tools of qualitative analysis. However, within-domain connections are made by participants for a variety of reasons often not including analogical reasoning, and are particularly much more numerous. A qualitative discussion of these other connections, when they are not subservient to the analogical reasoning at the heart of this study, is beyond my current scope.

The first step in reducing the potential data is the card sort itself. Participants were asked to create groupings based on a thematic criterion of their choosing. This allowed me to immediately gain a record of a within-domain connection, and the reason for making that connection. However, lists of card sort groupings from each participant are still difficult to compare between students and between card sorts. If one student makes three large seven-card groupings in a card sort, and another makes twelve small two-card groupings, how are we to compare the connectedness of their schema within that particular domain? This hypothetical example shows that simply reporting the number of groupings made by individuals could greatly misrepresent the interconnected nature of their schema. For this study, the solution to this problem comes in the form of adjacency matrices.

In this paper, an adjacency matrix is an integer-valued matrix which encodes the edges connecting the vertices of a graph. While these can be used to describe directed graphs, all adjacency matrices herein will represent undirected graphs. An $n \times n$ adjacency matrix $A = [a_{ij}]$ encodes the edge data for a graph with $n$ vertices. The entry $a_{ij}$ represents the number of edges joining vertex $i$ with vertex $j$. For undirected graphs, this means that $a_{ij} = a_{ji}$, i.e. all adjacency matrices throughout this writing will be symmetric.

Two adjacency matrices were constructed for each participant: one for the group homomorphism card sort and one for the linear transformation card sort. All matrices can be found in Appendix E and Appendix F. Each concept written on a card (including participant-contributed cards) is a vertex of a participant's adjacency matrix. Two vertices are connected with one edge for each time their respective concept cards were grouped together by the participant during the card sort. Each card is counted as grouped with itself, so that the diagonal of the adjacency matrix gives the number of times each card was used by a participant. While this practice is helpful for displaying information about the card sorts when reading through individual adjacency matrices, creating loops does not have a meaningful purpose in calculations involving the numeration of edges, and so in the calculations for Table 3.2 these loops were removed and the diagonal set to all zeros. For the presentation of the adjacency matrices, a different order was chosen than the random order given to participants. In the group homomorphism matrices, the given 16 cards are listed, then student-contributed cards, then professor-only contributed cards — each alphabetically. Similar contributed cards, such as *normal* and *normal subgroup*, were combined and counted as a single concept. For the linear transformation matrices, the 16 cards having the same name or parallel role to their counterparts in the previous card sort were listed in the same order as the group homomorphism matrices. The remaining three given cards of column space, matrix, and null space come next. This is followed by student-contributed cards and then professor-contributed cards, each again alphabetically.

At the individual level, the graphs represented by the adjacency matrices were compared quantitatively using two measures: number of components and edge-to-vertex ratios. The inspiration for this second metric comes from Ferrari and Munarini (2014), wherein the authors cite a variety of other studies displaying the usefulness of such a metric in pure and applied graph theory. Together, these provide a quick picture of the connectedness and density of students' schema surrounding group homomorphisms and linear transformations.

Because participants were able to create their own cards, choices had to be made concerning their impact on the above two quantities. The main objective is to compare participants' connections between the given 16 group homomorphism cards, the parallel 16 linear transformation cards, and the full 19 linear transformation cards. The most straightforward way to do this is to use the submatrix of the desired cards. However, performing this action gives a subgraph where some vertices that were once connected in the parent graph become disconnected in the subgraph. For example, a student could connect matrix and determinant in one grouping and then determinant and isomorphism in another. Matrix and isomorphism would be connected in the parent graph, but not in a subgraph which does not contain determinant. Failing to report such a connection could be seen as a misrepresentation of participants' understandings. Thus, Table 3.2 contains columns for both scenarios. One column gives the number of components in the subgraph with no modifications. The other accounts for these "missing" connections. This was done by performing a slight modification to the parent graph before taking a desired subgraph. Given a vertex that would be dropped in the subgraph, every other vertex which was an immediate neighbor to that particular vertex was identified. Edges were then added to the graph to form the complete graph on all identified neighbors, thus connecting concepts which were previously linked through the soon-to-be-discarded concept. This technique was only used for the secondary measure of graph components. Because the number of edges on a complete graph $K_n$ is $n!$, using this version of the graph would only be a detriment to the reliability of the edge-to-vertex ratio.

Almost all groupings presented by participants were recorded as displayed on the table during the interview. The only exception was surface-level connections devoid of any mathematical content. Such groupings included timing of content in the curriculum (e.g. "I've seen these together in class...") or "gut" feelings without any backing (e.g. "I feel like there's a connection that could be made here..."). These were excluded from the adjacency matrices due to the groupings representing connections clearly tangential to the purpose of the study. The inclusion of such reasoning in an argument was not enough to bar a grouping from being present in an adjacency matrix. It was only when such statements were presented in isolation from any mathematical backing that they were excluded.

# Chapter 3 Results

## 3.1 On the Definitions of the Morphisms

### Concept Images of Group Homomorphisms

Of the ten students interviewed, four (Arthur, Jake, Kyle, and Robyn) were able to recall the definition of a group homomorphism with no errors or one minor error. Minor errors included the presence of the extraneous condition $f(a^{-1}) = (f(a))^{-1}$, and restriction of the domain to only certain elements. Two other students (Maureen and Sander) provided definitions containing the group homomorphism property somewhere in a related definition. When asked to define a group homomorphism, Sander requested to give the definition of group isomorphism, as he could only recall learning about isomorphisms. The definition of isomorphism which he produced was fully correct. Maureen also attempted to recall the definition of a group isomorphism, rather than group homomorphism. However, the definition Maureen produced was actually a ring homomorphism (without the identity preservation property).

Of the four remaining students, three (Emilie, Flint, and Lucas) proposed that a group homomorphism is a function or mapping. Interestingly, both Lucas and Flint initially assumed that the morphism maps a group to itself, with Lucas later realizing that this would instead be a requirement for a group automorphism. Tallying up all of the cases above, nine out of the ten students interviewed recalled group homomorphisms were related to either functions or mappings. Only Tamara was unable to recall anything at all about group homomorphisms on her own. At a bare minimum, this means that for a majority of the students, the core idea of a group homomorphism being a function or mapping is present in their evoked concept images before being reminded of the mathematical definition.

Once students had either produced or seen a correct mathematical definition of group homomorphism, they were asked in various ways to explain the concept in their own words. Many students at some point described the characteristic property of a group homomorphism in a very literal fashion. For example, Jake, in his initial description of group homomorphism, stated, “A group homomorphism is a function that holds the property that if you take an element from a group — or, two elements from a group, and you have $f$ of two — the multiplication of those two elements equals $f$ of the multip— $f$ of the first element times $f$ of the second element of that group.” For a few students, the interpretation ended there or shortly after. Emilie attempted to make sense of the group homomorphism property by describing it as a type of “commutativity” or “smushing.” For her, the symbols and letters were being “pulled together” in a similar manner to the commutative property. The imagery of her evoked concept image at that time was limited to the literal symbols on the paper. At the farthest end of the spectrum, Tamara was hesitant to make any claims about group homomorphisms beyond rereading the given definition.

However, for the majority of students, some extra meaning was ascribed to group homomorphisms beyond their own concept definition. In these cases, discussion on the larger concept image centered around three main themes: preservation of the group structure, relating or comparing groups, and sameness under isomorphism. Upon reflection, it is easy to see that all three of these themes are intertwined with one another. This is reflected by the fact that some students drew on multiple core tenants within the same breath. However, the subtle differences in these themes — and the fact that multiple students made use of only one or two of them instead of all three — warrant a look at each individually. It is also important to note that these themes inevitably mirror or mimic those used in both students’ textbooks and classrooms. The discussion below is not to assert that students’ concept images exist in vacuum, but rather to identify those aspects of morphisms that stick with students and remain at the core of their concept images.

Table 3.1. Themes of Students' Concept Images of Morphisms

| **Pseudonym** | **GH Compares Structures** | **GH Preserves Structure** | **I Denotes Sameness (AA)** | **LT Compares Structure** | **LT Preserves Structure** | **Geometric Reasoning (LA)** |
|---|---|---|---|---|---|---|
| Arthur | × | × | × | | × | × |
| Emilie | | | | | | |
| Flint | | | | | | × |
| Jake | | × | × | | | ×* |
| Kyle | × | × | × | | × | × |
| Lucas | × | × | | × | × | |
| Maureen | | × | | | × | ×* |
| Robyn | × | | | × | | |
| Sander | | | × | | | × |
| Tamara | | | | | | |

AA = abstract algebra
LA = linear algebra
GH = group homomorphism
LT = linear transformation
I = isomorphism

*Preserving Group Structure*

The core purpose of a group homomorphism is to preserve group structure. With a set-function being too relaxed to prove meaningful results in group theory, and an isomorphism being too rigid, the group homomorphism serves as the desired function to map between groups while maintaining structure. For those that can interpret the symbols in the definition, it is clear that group homomorphisms preserve the group operation. However, the morphism's minimalist definition hides the fact that it preserves the identity and inverses so that the image of a group is a subgroup (though this is often the first theorem to be proved after the introduction of the definition of group homomorphism). It is then important to observe whether or not students are able to communicate this key feature of group homomorphisms.

Half of the students interviewed (Arthur, Jake, Kyle, Lucas, and Maureen) showed some evidence of perceiving group homomorphisms as preserving group structure. With the exception of Jake, this evidence was given without the context of isomorphism. Specific examples of group homomorphism structure preservation were given by Arthur, Kyle, and Maureen.

Perhaps due to his continued experience with structure-preserving morphisms in graduate classes, Kyle in particular saw group homomorphisms as structure preserving at their core. He described a group homomorphism in his own words as "a map... or... a way of comparing two groups, where there's some structure in the first group that is carried over into the structure of the second group." In line with this concept image, his initial concept definition included the preservation of identity and inverses directly in its formulation. When asked if all three conditions were necessary and sufficient, he concluded that the identity property was indeed already given by the core group homomorphism property, but he maintained the preservation of inverses as necessary. Though Kyle's definition is not minimal, his work in constructing the definition and his subsequent description show a clear awareness of the structure preserving nature of this morphism. Kyle would later go on to describe how group homomorphisms send subgroups to subgroups and generating sets to generating sets of images.

Despite claiming that she had a stronger understanding of abstract algebra than linear algebra, Maureen was the only student to transfer consistently her intuition from the base domain of linear transformations into the target domain of group homomorphisms. Similar to Kyle, she began by describing a group homomorphism as "a relationship between the groups that preserves the structure of the groups." And, later in the first interview, Maureen would make note of the fact that group homomorphisms preserve cyclic groups. But interestingly, when asked to describe a group homomorphism to a theoretical other person, Maureen said, "I would describe it as a linear transformation." Maureen then relied on her linear transformation concept image, describing the preservation of the operations (as her group homomorphism definition contained requirements for a ring homomorphism) as preserving distances. She later described seeing groups as being geometrically similar to vector spaces, being comprised of vectors in $\mathbb{R}^3$. This geometric notion of preserving distances would be repeated by others concerning linear transformations, as will be discussed later. Maureen indeed drew a coordinate grid ($\mathbb{R}^2$) when solving a group homomorphism problem involving $\mathbb{Z} \times \mathbb{Z}$, but it is unknown how helpful or hurtful this particular concept image of preserving distances would be for group homomorphisms outside of this very particular context.

The notion of structure preservation was less prevalent for Arthur, Jake, and Lucas, but still present in some capacity. Arthur made mention that group homomorphisms preserve the identity element, and that one must be aware of group structures in general when working with group homomorphisms. Jake noted that isomorphisms allow mathematicians to classify abelian groups, implying the preservation of the abelian nature of a group. Finally, Lucas expressed that with group homomorphisms (and linear transformations), the "work" that is done in the domain has a clear relationship with its image; the addition "holds up" (i.e. is preserved) after taking the morphism. Overall, there is good evidence that a subset of students recognized the role of group homomorphism as preserving group structure.

*Relating or Comparing Groups*

Lucas's previously mentioned phrasing leads nicely into the second theme of students' group homomorphism concept images. A few students (Arthur, Kyle, Lucas, and Robyn) described the idea that a group homomorphism *relates* or *compares* two groups. On its own, the phrase "relates two groups" can be ambiguous when trying to interpret what a particular student means by the phrase. On the one hand, all functions are themselves relations, in both a colloquial sense and an exact mathematical sense. So, when a student explains that a group homomorphism relates two groups without further elaborating, that student could simply be appealing to the fact that a group homomorphism is a function between two groups. Indeed, sometimes the language chosen by students leans this way, such as when Arthur stated, "they're functions that relate one group to another group" when speaking of group homomorphisms, inverse homomorphisms, and isomorphisms. On the other hand, because group homomorphisms preserve group structure, this relationship can be used to leverage knowledge about one group and understand properties of another. Thus, a student observing that a group homomorphism "relates" two groups could in fact be making a deeper statement about the nature of group homomorphisms than their status as a function. This is evident when Robyn noted that the "comparison [given by a group homomorphism] is able to give you more properties about that specific group and what's happening from... G to G prime." Robyn's and Kyle's specific use of the word "comparison" (or "compare") indicates a purpose for the relation that makes use of group properties. Recall that this is also reflected in Kyle's initial description of group homomorphism, which blends the ideas of comparison and preservation. Sources such as Vinner and Dreyfus (1989) and Zandieh et al. (2017) make mention of students' noticing of the relational aspect of a function, labelling it as "correspondence," "rule," or "mapping." However the notions mentioned in these sources (which deal with functions in high school, calculus, and linear algebra classes) possess the more intentional nature of the comparison notion expressed here, indicating that this particular notion of morphism is more regularly expressed in abstract algebra.

Like the preservation theme, the typical example for comparing and relating groups was most often the idea of isomorphism. All three of the students who mentioned this theme were sure to point out that an isomorphism allows one to understand one of the two related groups in terms of the other in the most direct sense. Beyond this, only Kyle attempted to provide details in the case where an isomorphism is not present. He recalled that a surjection in the absence of an injection will imply that one group is larger than the other, and made a similar observation regarding an injection in the absence of a surjection. This particular example does not convey how the comparison or relation leverages the group homomorphism property. Thus, it is interesting to note that, even for students that mentioned the relational power of the group homomorphism, there was a lack of immediate example for what such relational power is outside of isomorphisms. Arthur's reflections on as much provide a very fitting firsthand summary of the above findings.

> A: I mean, we haven't necessarily gone over any, like, super great examples of the use of a homomorphism, but I feel like it is something that's really interesting, and something that you would want to take a note of, because it could be very helpful later on.
>
> I: Do you know why it would be interesting or helpful?
>
> A: It allows you to figure out how the two group structures can relate to each other, which if they end up being like an isomorphism, for example, then it's really nice to work with because practically the same.

*Isomorphism as Sameness*

As just seen with both the structural preservation and relational sections, isomorphisms play a role in students' (Arthur, Jake, Kyle, and Sander) understandings of homomorphisms. Arthur's and Jake's use of isomorphism was already mentioned above. Both responses were given in the context of being asked why mathematicians would define group homomorphisms the way they do. While Arthur's description of categorizing abelian groups is more concrete, both responses highlight how the definition of homomorphism ties into isomorphism, and that isomorphisms are "nice" for mathematicians to work with. Kyle's response to mathematicians' use of homomorphism is also met with a succinct "you can use them to show that two groups are the same — that they're isomorphic." This notion of sameness is then described as the ability to align elements side-by-side (as in a comparison of group operation tables) in a way that the multiplication of elements is exactly the same, with only a change in the names of the elements. Kyle later referred to this method when attempting to prove that a particular map in the problem solving interview is a group homomorphism. In this problem, Kyle put forth that he would draw a table of the domain elements, a table of the elements in the image, and compare the two. Thus, for Kyle, the notion of sameness in isomorphism is useful when carried over to understanding group homomorphisms in an enacted way.

Compare this with the interesting case of Sander. Recall that Sander was unable to produce a definition of group homomorphism — only group isomorphism. Even after being shown the definition of homomorphism, Sander's understanding remained very couched in the notion of isomorphism. Asked to describe the homomorphism property, Sander said he would think of it like a "group of apples" and a "group of oranges," and that "they're the same group, they're just being represented by different pieces. So there's some way to imply that even though this [motions to left with hands] is a group of apples [motions to right with hands] and this is a group of oranges, they're the same group overall. If you... got rid of the individual elements and just simplified it." This is similar to the name change notion brought up by Kyle. However, unlike Kyle, this was never contrasted against situations in which a homomorphism is not one-to-one, and there was no notion of preserving or relating only a portion of the structure. Sander's well-developed concept image of isomorphism was not yet transferred to an understanding of homomorphisms as a broader concept.

Emily, Flint, and Maureen also discussed isomorphisms in relation to homomorphisms. However, none of these three used isomorphisms to discuss sameness, or even group structure at all. Emilie could only recall that an isomorphism is a group homomorphism, but not how the two were different. Flint consistently referred to isomorphism as "the function that actually maps the group onto the other group" whenever a group homomorphism is used. It is possible that Flint only recalled seeing the homomorphism property as an aspect of isomorphisms, and not its own concept. This would account for Flint's assertion. Finally, like many others, Maureen described an isomorphism as a bijective homomorphism. It was also in attempting to recall isomorphisms that Maureen in fact produced a ring homomorphism. Thus, as will also be clear later in Table 3.1, students linked the concepts of isomorphism and homomorphism a sizeable number of times. That said, only the previous four students above used this link to more fully understand homomorphisms and how they relate algebraic structures.

**Concept Images of Linear Transformations**

Three of the ten students (Arthur, Kyle, and Maureen) were able to produce a definition of a linear transformation close to the standard definition. For Arthur, his original definition was defined in terms of $\mathbb{R}^n$ and $\mathbb{R}^m$, as is standard in many textbooks for the first introduction of linear transformations. In the case of Kyle, who had taken various graduate classes in mathematics, he attempted to give the definition more generally as a module homomorphism. This was mostly correct, and contained the critical condition preserving addition and scalar multiplication. However, this definition indirectly defined modules as a group with a group action, rather than an abelian group with a ring action. Finally, Maureen's definition contained the minor error that both the domain and codomain are given as $V$. Maureen never referred to the domain and codomain as being the same vector space for the rest of the interview.

In contrast to how nine of the ten students initially linked group homomorphisms to functions, only six did the same with linear transformations. Additionally, while almost every student recalled that a group homomorphism is a function between groups, only four students indicated the involvement of some type of vector space when attempting to recall linear transformations (before being shown the mathematical definition in the interview protocol). Three of these students were discussed above, with the remaining fourth being Lucas, who knew that a linear transformation was a function from $\mathbb{R}^n$ and $\mathbb{R}^m$, but could not recall more. For those students that conjectured that linear transformations relate to functions, but made no mention of vector spaces, their guesses were based on surface level features. Sander stated that he guessed a linear transformation was a function (between groups) because of the discussion on group homomorphisms earlier. Tamara remarked that she picked up on the "linear" aspect, and so guessed that a linear transformation must be a function involving lines, such as $y = 2x + 5$.

This influence of the appearance of the words used to describe mathematical concepts was a common theme across various terms, but was especially notable here. In a manner similar to the above, Emilie, Flint, and Jake all initially proposed that a linear transformation must involve one of the two eponymous terms. In the case of Emilie, she wrote that a linear transformation must involve "taking a line and transferring it to another line." Flint and Jake, meanwhile, both picked up on the transformation portion of the term, triggering recall of transformations of graphs in college algebra. While Jake quickly discarded the notion that this prior knowledge would be helpful in a linear algebra context upon seeing the correct definition, Flint continued to make use of such imagery throughout his interview. For Flint, a linear transformation must be like the transformation of a graph. However, instead of translating and scaling graphs, in this context the transformation must be doing the same to vectors, as seen in the quotation below.

> I: Mmhmm. Okay. Uh, how would you describe a linear transformation to someone else? How would you help them think about it intuitively?
>
> F: I would probably start out with the s— again, something simple, maybe drawing a picture of moving a graph and like asking how you got from one place to the other, and like that's your transformation. Um, and then probably bringing that back to vectors and how you can do the same thing with vectors. You can just add them together, or you can — like if you had this vector and you were trying to make it, you know, bigger, this much [points] bigger, you know, how would you do that? Something like that.

It would stand to reason that if the appearance and structure of the term "linear transformation" influenced recall of problematic prior knowledge, then appearance and structure of "group homomorphism" may have triggered potential helpful prior knowledge. The presence of "group" in the term immediately ties it to its respective object, while the presence of "morphism" may have allowed students to orient their concept images to isomorphisms and connected notions and properties. Note Sander's response when he was first asked to define group homomorphism, and recall that he eventually was only able to define isomorphism. His process started out as in the quote below.

> I guess isomorphism is what first comes to my mind, and connection between all the $x$'s and $y$'s, $x$'s would be in the set of the — or in the first group, and then $y$'s would be in the... group of the second one. And then there's some correspondence between them. I don't understand what homomorphism is.

For students attempting to recall terminology from months or even years ago, the wording of linear transformation places it at a disadvantage as compared to group homomorphism.

Before discussing the themes present in students' linear transformation concept images, it is important to note the themes which were *not* prevalent in the context of linear transformations. Mentions of comparing or relating two structures dropped significantly, with only Robyn attempting to make an analogy connecting each morphism's comparative nature (discussed in Section 3.2), and Lucas making parallel statements to his group homomorphism statement concerning the morphism allowing one to transfer work from one structure to the other.

Additionally, the theme of isomorphism as sameness is completely absent in the context of linear transformations. Isomorphisms are mentioned in terms of bijective functions, as an idea that spans across multiple subjects. However, the preservation of structure, which for homomorphisms was often immediately tied to the idea of isomorphism as an example, is no longer tied to isomorphisms in this context. Likely due to its presence in the abstract algebra curriculum before group homomorphisms, isomorphisms were discussed by half of all students when they were asked what concepts were most critical to understanding group homomorphisms. For the theme of "isomorphisms as sameness" to be missing in the context of linear transformations marks a large distinction between students' understanding of group homomorphisms and their understanding of linear transformations. This stands out as one of the largest differences between student thinking on morphisms between classes, alongside the later geometric theme of linear transformations. It also likely marks a difference in how these two subjects are both taught and utilized by mathematicians in teaching. For some students, they directly expressed that they had not encountered isomorphisms in linear algebra. It is also greatly possible that in many linear algebra classes, the emphasis on $\mathbb{R}^n$ for a majority of the course leads to a much lesser need for a discussion on isomorphisms of vector spaces.

There were then two main themes in students' linear transformation concept images: geometric reasoning and preservation of structure. As with the themes present with group homomorphisms, these overlapped and intertwined in some students' explanations. However, the geometric situations described by students varied, and sometimes tied into aspects of linear algebra which are tangential to linear transformations.

*Geometric Reasoning*

For six of the ten students, geometric reasoning was either displayed or referenced with regards to linear algebra. In the cases of Jake and Maureen, these were general references to "seeing" linear algebra more geometrically, or referring to vectors as existing in different planes. They made no attempt to describe linear transformations geometrically.

For Flint and Sander, both students attempted to reason through the properties of a linear transformation using prior knowledge of geometric representations of vector addition and scalar multiplication. Using the aforementioned knowledge of transfor-

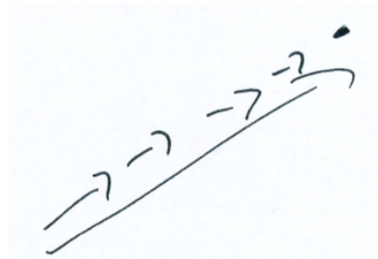

Figure 3.1. Flint's Geometric Reasoning on the Scalar Multiple Property

mations from college algebra, Flint first reasoned that the scalar multiple property of a linear transformation is scaling a vector. When drawing the picture in Figure 3.1, Flint stated of the second property,

> I would say that that's, like, your scalar multiple or whatever... just like that's what I was talking about when you're trying to make something bigger. Like if you're trying to get to this point up here [draws dot], this is the only vector you've got [drawing in direction of point but stopping short], you know, what would be your multiple to make this, like how many of these, you know, are you going to need, basically, to get up there?

Likewise, when Flint reasoned about the first linear transformation property, he drew out a triangle representative of vector addition. (Note that this particular passage is from a portion of the interview where Flint was asked to compare group homomorphisms and linear transformations, which led to the notation of $T(\mathbf{x} * \mathbf{y})$ in Figure 3.2 instead of the standard $T(\mathbf{x} + \mathbf{y})$.) Explaining this property, he stated, "like these two added together [points at $T(\mathbf{x})$, $T(\mathbf{y})$, and then left side of equation containing $T(\mathbf{x} * \mathbf{y})$] is going to give you this one thing [points at longest vector in triangle]." It is important to note that throughout these examples, and in all of Flint's other similar explanations, there is a lack of recognition of the function $T$ in the geometric reasoning. Both are explanations of scalar multiplication and vector addition, with $T$ existing in the algebraic symbols but having no impact on the actual geometric imagery. This is an extension of the above remark on students' lesser understanding of linear transformation as a function. A similar pattern of thinking occurred in Sander's interview. In Sander's case, the picture drawn was that of a

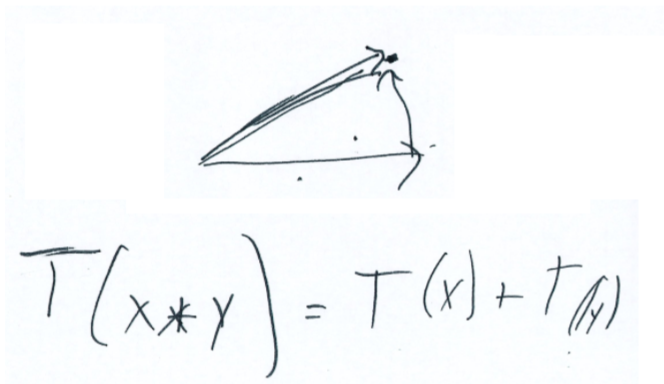


Figure 3.2. Flint's Geometric Reasoning on the Vector Addition Property

parallelogram, usually drawn to represent the commutativity of vector addition in a visual manner. Scalar multiplication was represented in the same manner as Flint, in the same diagram. In this case, the properties were verbally described as "closure"

Figure 3.3. Sander's Geometric Reasoning on Both Properties

of the operations of addition and scalar multiplication. Similar to Tamara, Sander seemingly latched onto the idea of $x$ and $y$ to the first power as being related to lines from college algebra. At one point Sander referred to the sum $\mathbf{x} + \mathbf{y}$ as a line, and posited that in a different vector space, the contents of $T(\square)$ could instead be a different "equation" (non-linear function), so that the transformation isn't linear. Thus, linearity is not a property of the function, but of the space and the lines (or line segments / vectors) drawn in that space. In both Sander and Flint's interviews, their attempts to draw on geometric reasoning from class ignored the algebraic statements in the definition. This then lead to misconceptions about linear transformations, and the oversight of the functional nature of linear transformations.

Finally, it was Arthur and Kyle who described their concept images of linear transformation geometrically. The core of Kyle's concept image was matrices, which he stated as much when he said, "When I think about linear transformations, I think — I see a matrix." However, Kyle was not limited to this arithmetic mode of thinking. When asked why mathematicians define linear transformations as they do, Kyle's initial response was "It keeps the "shape" [uses air quotes] of the spaces the same. So if you start with two vectors that are on the same line originally, you will end — the transformation will send them to vectors that are on the same line. They're multiples of each other, up to scalars." This indicates that while geometric reasoning was not Kyle's primary mode in his concept image, he was able to synthesize the two modes successfully in order to construct a unified schema. Arthur instead consistently emphasized the importance of geometry in understanding linear transformations, saying "That's just the main thing I think, for linear transformations, is, very visually, what does it look like?" Arthur's geometric explanation of a linear transformation is based more off of concrete examples than on a synthesis of algebra and geometric as Kyle's was. This is illustrated when Arthur described:

> [A linear transformation] kind of preserves like, the term I use is like the rigidity of the space. Um, not necessarily, not necessarily preserving distances, but, preserving relative distances, like everything — say I'm just going like a two-dimensional to a two-dimensional, you could have like just your regular grid plane, and then you could shear it in like a shear transformation, um, everything is preserved there, granted your distances changed, but they still have the same relative distance.

Arthur's use of the phrase "relative distance" is similar to Kyle's use of "shape" in his geometric explanation of a linear transformation. However, there is no direct reference to the properties of a linear transformation, and Arthur later directly says that he hasn't given the second property of linear transformations much thought beyond simply memorizing it. While this more intuition-based concept image proved problematic in some ways, such as when Arthur offhandedly seems to imply that a map from a two-dimensional space to a one-dimensional space does not preserve enough structure, it did not hamper his ability to produce and work with linear transformations in an arithmetically-heavy problem-solving context later on. Finally, it is interesting to see that Arthur's use of the word "distance" for what is being preserved echos Maureen's description of group homomorphisms (through the lens of the linear transformation concept image).

*Preserving Structure*

As seen above, much of Arthur's and Kyle's geometric reasoning pertains to the preservation of the structure of a vector space. Because the entirety of Arthur's evoked concept image pertaining to preservation of structure is geometric in nature, it stands in contrast to his algebraic intuition concerning isomorphic sameness and preservation of identity in describing group homomorphisms. Meanwhile, Kyle's statements concerning preservation of structure greatly mirror those from his description of group homomorphisms, again likely due to his continued exposure to morphisms in graduate classes. In addition to his geometric statements above, Kyle makes mention of the algebraic ideas of the translation of addition through linear transformation, image of basis as basis for the image, and subspaces as being thought of as images of injective linear transformations. Each of these notions was described using very similar language and with the corresponding group equivalents. The only missing parallel statement was already addressed earlier — there is no use of isomorphism in the linear algebra context to describe preservation of structure or sameness. Isomorphism is only described as a bijection. Even with this exception, there is a great deal of mirroring of language between contexts, and the overall body of Kyle's interview gives good evidence of Kyle's integrated morphism schema. Because group homomorphisms and linear transformations are part of a greater concept image, activation of this integrated schema results in similar statements regarding the shared properties and intuitions of these morphisms. More about this will be discussed later.

Lucas and Maureen likewise discussed preservation of structure by linear transformations in a similar manner to how they presented group homomorphisms. In the case of Lucas, this is because he described preservation of structure at a point in the interview when he was discussing a common feature to both morphisms. As noted earlier, for Lucas this preservation is the idea that "work" (such as addition) is maintained by the morphism from one space to the next. This is not elaborated upon in either context besides slight variations of the theme. As with group homomorphisms, Maureen directly describes linear transformations as preserving structure. Like Lucas, these references are in tandem with statements about group homomorphisms. Specifically, Maureen makes reference to the preservation of the addition operation. For both Lucas and Maureen, there is clearly a connection being made between the two morphisms and their preservation of structure. The lack of elaboration beyond the connection of the first condition of linear transformations and the group homomorphism property makes it unclear how robust this connection is compared to that of Kyle's integrated morphism schema.

### More on the Connections Between Morphisms

Though the levels of detail varied, it is evident from the above that some students are actively making some connections between these morphisms, without directly being told of said connections. Not only were Arthur, Kyle, Lucas, and Maureen using similar notions to describe both morphisms, but they made direct statements about how these notions manifested in both settings. Whether geometric or algebraic in nature, most of these links between overall intuition in concept images hinged on the common theme just discussed — that of the preservation of structure in a category by its respective morphism. However, for most of these students, the connections forged between concepts were not just similarities or intuitions. Arthur, Kyle, and Lucas all successfully recognized that the presence of the group homomorphism condition in the definition of linear transformation mathematically implied that a linear transformation is in fact a group homomorphism. The arc of this realization was different for each student.

Taking linear algebra and abstract algebra concurrently, Arthur saw the similarities between isomorphism, homomorphism, and linear transformation at different points during the semester. As he was ending this semester at the time of interview, he was able to recall the time line of these connections in detail, saying,

> I was looking up some definitions... to make sure I was understanding it, and it explained isomorphism as a specific type of homomorphism, and I was like, 'Well we didn't learn about homomorphisms.' So I kind of saw through there, I was like, 'Oh, okay, it's just a broadened isomorphism.' And then, I remembered seeing that and thought, 'Huh, that looks similar to a linear transformation,' but I didn't think much of it then.

When Arthur's linear algebra class progressed further into abstract vector spaces and linear transformations, this connection remained with Arthur, and he decided to ask his professor, as he recounts,

> When we were learning about it — not necessarily linear transformations, but I think this was once we got further into vector spaces... we talked about transformations with respect to vector spaces, specifically. I asked our professor after class, like, I explained I was in [abstract] algebra. I was like, 'Is this essentially the same thing?' And she was like, 'Yeah, it's practically the same thing. It is the same thing.' ... So it is a connection that I made, but then I did confirm it, I guess, is how I'd put that.

Thus, for Arthur, seeing the definitions in one class (concurrent with the other) was enough to trigger a sense of similarity and curiosity. By the time of interview, Arthur was able to clarify that the linear transformations and group homomorphisms were not the same, per se, but that "a linear transformation is a specific group homomorphism where it has that extra condition." Arthur was now able to categorize the morphisms into a clear hierarchy.

Kyle makes a similar, but more extended, statement regarding the two morphisms, when he says a linear transformation is "the group homomorphism that you would want for vector spaces. So, it has more conditions that make sense for the additional structure that a vector space has, in the sense that there are scalars and there are vectors, as opposed to just one type of element in a group." When Arthur came to the scalar condition, he remarked that it was just a condition that needed to be memorized, and that it probably helped with the "linearity" in some sense. Kyle, having taken abstract algebra at both the undergraduate and graduate levels, sees that the second condition is tied to the fact that vector spaces have the additional action of a field on the abelian group. Because this action (scalar multiplication) is an integral part of the structure of vector spaces, it must be included in the morphism for that space as well. Kyle does not give a full account of how his integrated schema of morphisms came to be, but does recall that taking abstract algebra before linear algebra helped to contextualize the latter subject as a specific case of the former.

Lucas's timeline for connecting concept images across morphisms was rather different than the two accounts above, as the progression happened in the course of the first interview. Lucas was not able to recall either definition when initially prompted. However, upon seeing both definitions, Lucas immediately saw the similarities. When asked how he would explain linear transformations to someone else, he responded that he would in fact use the "very clear" connection to group homomorphisms. This is when he first used the preservation of "work" language discussed above. Then, when grouping elements in the combined card sort, Lucas reformulates the connection between the two morphisms as being "essentially equivalent." By the end of this grouping, this is then reformulated again, as he says, "Perhaps equivalent wouldn't be the best word, but a more specific example... Linear transformations with a vector space is a more specific example of group homomorphisms." This quick evolution from noticing similar conditions, to equivalency, to a more nuanced statement of equivalency was perhaps due to Lucas's experience in two semesters of abstract algebra — though this wasn't true for the other students in a similar situation. It is also interesting to see that Lucas learned about both morphisms in the same semester, like Arthur, but had not made these connections during the semester, unlike Arthur.

The remaining question one might have is if students found such common intuitions helpful in the production of their concept definitions. Specifically, in this interview, did thinking about group homomorphisms help students recall linear transformations? Recall that only Arthur, Kyle, and Maureen were able to produce linear transformation definitions. Each of the other three students was someone who had already produced a definition containing the group homomorphism property without being shown the definition. Maureen had already thought of group homomorphisms in terms of linear transformations, and was the only person to do so during the group homomorphism portion of the interview. Thus, there is little surprise that Maureen believed that the first half of the interview was helpful in recalling the linear transformation definition — she had already begun drawing parallels in the concept images and concept definitions before being prompted. Arthur noted that the similar "form" of the morphisms' properties indeed helped him (in some part) to recall the definition of linear transformation correctly. Finally, Kyle was unique in saying that while the format of the interview did not help recall on that particular day, he did find group homomorphisms helpful in learning about linear transformations originally while taking linear algebra.

For the remaining students, seeing the definition of group homomorphism before being asked for the definition of linear transformation was not very helpful. The only impact that this may have had on recall was for Emilie, Lucas, and Sander. These students were able to recall (or guess) that a linear transformation must be a function, as with the first term they were asked to define. Sander took this one step further in guessing that a linear transformation must also be a function between groups. However, this was as far as the priming activity was able to facilitate recall of the definition of linear transformation. Thus, for a majority of the students interviewed, despite the presence of the group homomorphism property in the linear transformation requirements, thinking about group homomorphisms was not sufficient to trigger recall on its own. Instead, those students that did find such priming helpful were those that had already taken time to forge a connection between morphisms before the time of interview.

While only a few students entered into the interview with unified schema from which to draw, it is important to note that other students also were able to make connections during the course of the interview. In fact, nine of the ten students interviewed pointed out a similarity between the definitions of group homomorphism and of linear transformation. Already discussed in detail above is the case of Lucas, who went on to describe a linear transformation as a specific type of group homomorphism. Jake and Robyn both discovered that the group homomorphism condition exists within the linear transformation conditions. Jake pointed to each part of the notation in the first property, described how the $T$ is just $f$, and that the name of the operation, multiplication or addition, doesn't matter with regards to the group homomorphism definition. Robyn phrased this connection as "you see with a group homomorphism if you have two groups that are both, um, groups with addition, then it's... the same definition as linear transformation. Um, you see [they?] replace [it?] with a T, you've got a phi of x plus y is equal to phi of x plus phi of y, so that's very similar." In the remaining cases of Emilie, Flint, and Tamara, this similarity was remarked upon without explicitly stating that the first property was the same as a group homomorphism condition.

## 3.2 Greater Connections Across Domains

In addition to gathering data on how students think about linear transformations and group homomorphisms themselves, one of the expressed purposes of this project was to study how students make connections between the greater networks of concepts related to these morphisms. The delimitations of which terms were explicitly provided to students in the card sorts are detailed in the protocol in Appendix A. Recall that students were also given space to create their own concept cards for the card sort, as well.

The students in this study ranged across a spectrum of understandings regarding the connections that can (and cannot) be made across mathematical domains. Students also varied on the amount of connections made prior to the time of interview. Both axes of quality of connections and timing of connections would be valid ways to present the data. For the purpose of this thesis, I will choose the latter, and will group students into three roughly bounded categories: students with substantial prior analogical reasoning, students primarily engaged in new analogical reasoning, and students engaged in little analogical reasoning. After a brief discussion of the results of the adjacency matrices in the first two card sorts, each category will be exemplified by one to two case studies following students through their respective card sorts and highlighting the analogies that they used to demonstrate prior connections or gain understanding across domains. This will be followed by a briefer discussion of the similarities and differences of other students' responses in the same category. Using this format will allow for the analysis to stay close to the data and provide a look at the unique ways in which students displayed their reasoning, while also condensing these varied stories into a more digestible format.

Recall that one potential barrier to analogy formation is the failure to retrieve information from a particular domain. Thus, it is worth considering students' preparedness in both abstract and linear algebra. Table 3.2 shows the number of components

Table 3.2. Adjacency Matrix Components and Connections

| **Pseudonym** | **GH Card Sort** | | | | | **LT Card Sort, Parallel** | | | | | **LT Card Sort, All Cards** | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | C | (R) | E | V | E/V | C | (R) | E | V | E/V | C | (R) | E | V | E/V |
| *Prior* | | | | | | | | | | | | | | | |
| Arthur | 1 | 1 | 127 | 16 | 7.94 | 1 | 1 | 62 | 16 | 3.88 | 1 | 1 | 77 | 19 | 4.05 |
| Kyle | 1 | 1 | 70 | 16 | 4.38 | 1 | 1 | 32 | 16 | 2.00 | 1 | 1 | 49 | 19 | 2.58 |
| Maureen | 4 | 4 | 39 | 16 | 2.44 | 4 | 2 | 31 | 16 | 1.94 | 3 | 2 | 40 | 19 | 2.11 |
| *Current* | | | | | | | | | | | | | | | |
| Emilie | 8 | 8 | 9 | 16 | 0.56 | 8 | 8 | 11 | 16 | 0.69 | 10 | 10 | 14 | 19 | 0.74 |
| Jake | 3 | 3 | 41 | 16 | 2.56 | 4 | 4 | 51 | 16 | 3.19 | 6 | 6 | 52 | 19 | 2.74 |
| Lucas | 3 | 3 | 29 | 16 | 1.81 | 4 | 4 | 30 | 16 | 1.88 | 5 | 5 | 35 | 19 | 1.84 |
| Robyn | 4 | 4 | 25 | 16 | 1.56 | 4 | 2 | 26 | 16 | 1.63 | 2 | 2 | 37 | 19 | 1.95 |
| *Little* | | | | | | | | | | | | | | | |
| Flint | 4 | 4 | 40 | 16 | 2.50 | 9 | 9 | 25 | 16 | 1.56 | 11 | 11 | 29 | 19 | 1.53 |
| Sander | 5 | 5 | 21 | 16 | 1.31 | 6 | 3 | 16 | 16 | 1.00 | 3 | 3 | 44 | 19 | 2.32 |
| Tamara | 5 | 5 | 48 | 16 | 3.00 | 8 | 8 | 24 | 16 | 1.50 | 11 | 11 | 24 | 19 | 1.26 |
| *Professors* | | | | | | | | | | | | | | | |
| Brady | 1 | 1 | 87 | 16 | 5.44 | 1 | 1 | 80 | 16 | 5.00 | 1 | 1 | 97 | 19 | 5.11 |
| Greer | 1 | 1 | 70 | 16 | 4.38 | 1 | 1 | 107 | 16 | 6.69 | 1 | 1 | 125 | 19 | 6.58 |
| Powell | 2 | 2 | 64 | 16 | 4.00 | 2 | 2 | 76 | 16 | 4.75 | 2 | 2 | 90 | 19 | 4.74 |
| Stud. Mean | 3.7 | 3.7 | 45.0 | - | 2.81 | 4.9 | 4.2 | 30.8 | - | 1.93 | 5.3 | 5.2 | 40.1 | - | 2.11 |
| Prof. Mean | 1.3 | 1.3 | 73.7 | - | 4.60 | 1.3 | 1.3 | 87.7 | - | 5.48 | 1.3 | 1.3 | 104.0 | - | 5.47 |

C = number of components in graph of adjacency matrix truncated to only given cards
(R) = number of components of graph of adjacency matrix obtained through vertex replacement
E = number of edges in graph of adjacency matrix truncated to only given cards
V = number of vertices in graph of adjacency matrix truncated to only given cards
E/V = edges divided by vertices

and edges in participants' adjacency matrices, as well as their edge-to-vertex ratios. As discussed in Section 2.3, these two numbers together are used to give a rough overview of the quality of students' within-domain connections. Each section of the table describes an adjacency matrix obtained by truncating the full matrices shown in Appendix E and Appendix F down to a submatrix containing certain common concepts. The results under the first two headings, *GH Card Sort* and *LT Card Sort, Parallel*, were each truncated to the first 16 terms — the initial given grid of terms in the group homomorphism card sort and their closest respective parallels in the linear transformation card sort. The final heading refers to the first 19 terms of the linear transformation adjacency matrices, which include all of the previous 16 terms plus matrix, null space, and column space. Additionally, diagonal entries, which would produce loops for each included term, are not included in the calculations in this table.

On average, students had more connected adjacency matrices in the group homomorphism card sort than the linear transformation card sort, in terms of both number of components and edge-vertex ratios. Statistically, however, the difference in edge-to-vertex ratios is not significant. Comparing the means using a paired-differences $t$-test (Agresti, 2009), I obtained $\alpha = 0.083$ for the comparison between *GH Card Sort* and *LT Card Sort, Parallel* edge-to-vertex ratio means, and $\alpha = 0.164$ for the *GH Card Sort* and *LT Card Sort, All Cards* means. Neither of these is significant at the 0.05 level. However, this difference would make sense, because for many students, abstract algebra was the more recent class, and presumably slightly easier to recall. Students were grouped into categories based on the qualitative data analysis, and before consulting this quantitative component. Thus, it is a bit odd that the second category of students are almost exclusively the only students to run (ever so slightly) counter to this trend in terms of edge-to-vertex ratios. In terms of comparing groups of participants, the professors and those in the *Prior* category have, as expected, higher edge-to-vertex ratios and a lower number of components on the whole. However, there appears to be no trend between those in the *Current* and *Little* categories. Thus, the reason for some of these students' willingness or ability to create analogies between subjects is either unknown or requires a deeper look at the qualitative data.

### Students with Substantial Prior Analogical Reasoning

The breakdown of which students were able to successfully make connections between classes before the time of interview can be partially surmised from the prior section on the definitions of the morphisms alone. The students who successfully made connections between linear transformations and group homomorphisms were the same students who continued on to connect other terms across domains. Thus, in this section I discuss the more highly connected schema of Arthur, Kyle, and Maureen.

*Arthur*

For both the group homomorphisms card sort and the linear transformation card sort, Arthur had among the most connected adjacency graphs (1 component for both), and among the highest edge-to-vertex ratios. Thus, Arthur entered into the study with a strong understanding of both domains. If any analogies were to be made, Arthur seemed an ideal candidate for placing the worlds of linear and abstract algebra into structural alignment to prepare for such analogies.

In an earlier quote from Arthur about the analogy between group homomorphisms and linear transformations, Arthur recalled that he did not think much of the similarities between the two until his class began studying vector spaces as an abstract structure. To continue his above account, Arthur continued:

> So I guess that made it — once... we talked about linear transformation of just a general vector space, not just of vectors, I think that definitely resonated, because it — I was able to — at that point I had made the parallel between a group and a vector space, so that made it very easy to parallel between that transformation and a homomorphism.

The commonalities of groups and vector spaces, specifically that vector spaces are groups with respect to addition of vectors, was the initial "parallel" (analogy) which, as Arthur put it, resonated with him. This analogy is then what allowed Arthur to infer that linear transformations are group homomorphisms, and further extend the analogy. This analogy based off of the respective objects is exactly what Arthur used to explain how thinking about group homomorphisms helped him to recall the definition of linear transformation. Thus, there is some evidence that once students make these connections, they remain useful when recalling information.

Arthur continued to align terms during the linear transformation card sort — before most students would draw such parallels in the combined card sort. The very

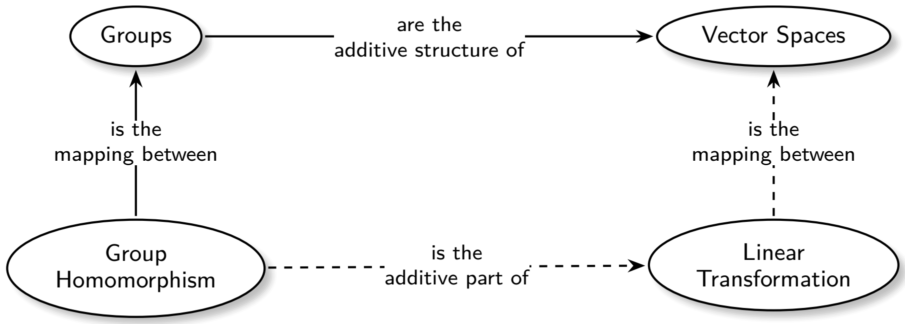


Figure 3.4. Arthur's Use of Object Structure to Align Morphisms

first grouping in Arthur's linear transformation card sort was of kernel and null space. Arthur was able to identify that the identity element concerned in the null space is specifically the zero vector in the setting of vector spaces. From there, the next alignment was of column space and image, or range. For both this and the previous grouping, Arthur correctly described the role of the matrix, the multiplication by which yields the linear transformation.

Another interesting use of analogical reasoning came when Arthur described subspaces. Similar to other students, Arthur provided the analogy that subspaces of vector spaces are similar to subgroups of groups. At the same time, Arthur could not recall the exact properties of or requirements for subspaces. Thus, Arthur drew upon his concept definition of subgroups. First, Arthur presumed, because "a homomorphism in a subgroup still applies to the subgroup," then a "linear transformation within the subspace is still going to be within the subspace." This could be interpreted as "the image of a subgroup/subspace element is contained within the image of the subgroup/subspace," though Arthur's wording is a bit unclear. More clearly, however, Arthur recalled that a subgroup contains the identity element, has inverses, is associative, and is closed under the operation. Thus, he said, a subspace must have similar properties, with the zero vector, additive inverses, associativity of addition, and closure under addition. Of course, this particular analogy failed to trigger the subspace requirement for closure under scalar multiplication. However, this indeed gives another instance of well-developed schema leading to helpful recall through analogical reasoning.

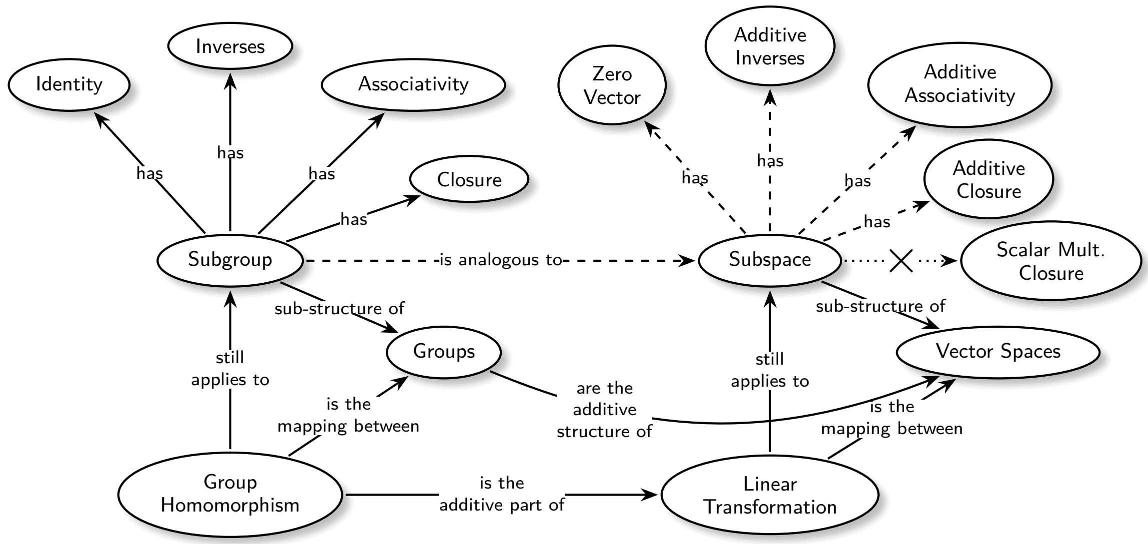


Figure 3.5. Arthur's Reasoning on Subgroups and Subspaces

After this, Arthur described the similarity of isomorphisms across subjects, where bijectivity is still a requirement, but linear transformation takes the place of group homomorphism. Next, Arthur made another inference, this time because he had not seen inverse transformations. Based on his experience with inverse homomorphisms (and, most likely, inverse functions in general), Arthur easily surmised that these transformations must be "the transformation that undoes a standard linear transformation" and swaps the codomain and domain.

One of the final analogies constructed by Arthur was one of his most interesting. His previous analogies were all between structures that Arthur had already mentally connected at some level before the interview. Mid-interview, though, Arthur realized that there is a connection between the generating set of a group and the basis of a vector space. According to Arthur, both "generate the entire space," with a difference being in exactly how the elements are combined in order to generate either a group or a vector space. By progressing through the interview, constantly placing algebraic structures in structural alignment, Arthur was able to extend the analogy between structures and generate a new (correct) connection. At the end of the interview, Arthur remarked on as much, saying, “the activity actually helped me make connections between them [concepts from both subjects], like for basis and generating set, for example.” This extension of his structural analogy will then become a part of the larger schema storing these common features, and will likely strengthen Arthur’s understanding of both algebraic structures.

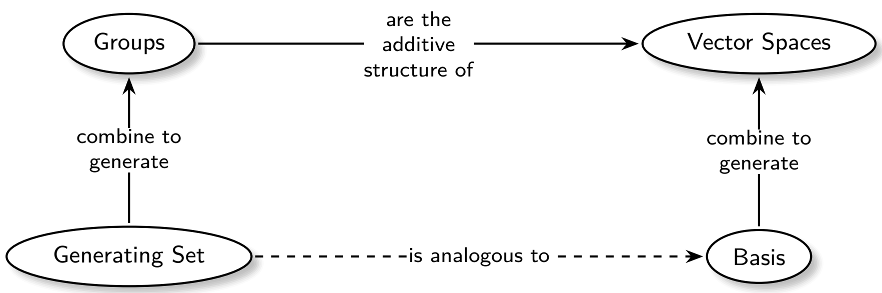


Figure 3.6. Arthur’s New Connection Between Generating Set and Basis

The ability to make such a connection in the moment is dependent upon a student’s understanding of at least one of the two algebraic structures considered. Arthur’s strong understanding of both subjects, reflected by the high density and connectivity in his adjacency matrices, allowed him to create a useful overarching analogy. This abundance of relations leading to productive analogies shows that the systematicity principle of analogical mapping is at play in Arthur’s process of connecting between abstract and linear algebra.

*Comments on Remaining Students*

Recall that Kyle had already taken some graduate classes in mathematics. Additionally, he had taken some abstract algebra classes before linear algebra. Kyle’s time spent thinking about these concepts across many mathematics classes corresponded to a great deal of connections within and across domains. Kyle’s interview paralleled Arthur’s in various fashions. Both of Kyle’s adjacency matrices yielded a graph with a single component, and the ratios of edges-to-vertices for both his graphs were some of the highest among students. At the beginning of his linear transformation card sort, Kyle remarked, “A lot of these [groupings] will be the same... up to a change of one card,” referencing the idea that linear transformation would take the place of group homomorphism. After that, Kyle referenced the connections between column space and image/range, basis and generating set, group and vector space, subgroup and subspace, and null space and kernel. There were also similar mentions about how isomorphisms are the bijective morphisms in each case. Also similar to Arthur, Kyle was unsure of what inverse homomorphisms/transformations were, but put forth that they would be functions in the opposite directions for their respective structures. In fact, this was true of almost all students, regardless of their categorization here. Both terms seemed unfamiliar to students, but they were able to surmise that because group homomorphisms and linear transformations were similar, then “inverse” should be taken at face value, and the inverse of each must be related in the sense that they go in a backwards direction. Thus, this will be the last comment on inverse morphism connections.

Maureen grouped many of these same cards, but was much more unsure of how to verbalize the connections between them. Some examples of this are when Maureen said, “I know that the null space and the kernel of a matrix are really similar, but I don’t remember the difference,” or that her grouping of basis and generating set was just “a reflex association.” Statements like this show that at some point, Maureen saw (at least partially) the connections between these concepts. Maureen made enough of these intuition-based statements to indicate that she had heard or considered many of the related concepts in terms of one another before the time of interview. Unfortunately, similar to the students in the little connections category, Maureen was sometimes a bit reluctant to expand her thinking beyond initial intuition, usually due to being unable to remember mathematical definitions. Her difficulty in recalling concepts from each individual content domain is reflected in her open card sorts. While her overall numbers were higher than a number of students in other categories, Maureen's adjacency matrix graphs are more disconnected and less dense than Arthur's and Kyle's. Part of this is due to the fact that a large number of Marueen's later groupings focused on her own added concepts such as determinant, eigenvalues, and eigenvectors.

An example of Maureen's attempts to align algebraic structures based on tenuous associations, despite her difficulty in recalling some key mathematical definitions, came when she spoke about the relationship between subgroup and subspace. This can be seen in the exchange below.

> M: I associate these concepts — I associate groups and vector spaces and subgroups and subspaces.
>
> I: Okay, would you be able to say how so?
>
> M: I see them as having the same underlying structure. Like I see a subgroup of a group as similar to a subspace of a vector space.
>
> I: Okay. Is there any sense in which they're similar at all? Like would you be able to say how you see — yeah, just how you see them as similar?
>
> M: I see them as subsets of these larger groups or spaces that fulfill certain attributes. For subgroups, you know, a subgroup must include an identity element, um... the inverse — identity, inverse, associativity, and it must be closed. So I think of that similarly as a subspace of vector spaces, although I'm not sure of the formal definition of a subspace.

Maureen's analogy went beyond the idea of subsets. She recalled that subgroups must have certain attributes, and used this fact and her running analogy to bolster her claim that subspaces must have certain attributes as well. However, Maureen stopped the analogy short of identifying the additive structure of subspaces with that of subgroups.

*General Comments*

Common among all the students in this category is their interconnected view of the subjects of abstract algebra and linear algebra. Already noted above is how Kyle, who took abstract algebra first, came to view linear algebra as a special case of the other subject, and how Arthur actively looked for and asked his professor about the connections between the two. Arthur also discussed how drawing parallels can be more time consuming, but that trying to think of one in terms of the other will help him to remember it better in the future. Maureen described how taking abstract algebra moved her away from a calculation mentality, and toward a more connected understanding. As she put it,

> I think for a long time I was very engrossed with just the calculation of math... but I don't think I had ever considered that it was actually doing something, like, that math doesn't happen in a void. And I think that has changed a lot after having [abstract] algebra.... And this is all very much closer related than I thought it was....

All three of these students show evidence that some aspect of abstract algebra, whether taking it before, concurrently with, or after linear algebra led to connections which helped some of these students construct more robust concept images. Sometimes, these connections were leveraged to learn one subject at the time it was taken. Other times, they were used to help with recall. At one point, these connections led to a new, undiscovered connection during the interview. In Maureen's case, though her analogies were more tenuous in nature, she gained an appreciation for mathematics as an interconnected field of study. Given more experience, as in the case of Kyle, it is not far-fetched to conjecture that Maureen's initial connections would blossom into a more robust larger concept image for objects and morphisms.

### Students Primarily Engaged in New Analogical Reasoning

A plurality of students attempted to make connections across domains at the time of interview, but had not considered such connections prior to it. For some, such as Jake and Lucas, multiple pieces fell into place during the interview, and connections were made more clear. For others, namely Emilie and Robyn, they engaged in inference making based on analogies, but were unsure of the conclusions they drew. In either case, there was a clear effort to align the two algebraic structures, and understand one in terms of the other. All of these students did not initially recall the definition of linear transformation. However, upon seeing both definitions, they expressed some form of connection between the two.

*Robyn*

Robyn outright verbalized the key difference between students in this category and the last when she said,

> I've never thought about math, these connections, this way before. It's really interesting.... I think in the classes I've taken between abstract [algebra] and [linear algebra], I very much just tried my best to understand like one concept at a time. And there have been times where it's like, 'Oh yeah, I recognize that from [linear algebra]' or from abstract [algebra], but I've never seen all of the connections that exist, like I've never seen that... the definition of linear transformation and group homomorphism can be the same thing. And, honestly, this helps me understand both of those things: better understand [linear algebra] and remember [linear algebra], and understand how abstract [algebra] relates to [linear algebra] and all of those things. Because I understood [linear algebra] so much more, and enjoyed that class so much more than I have my abstract [algebra] classes because I don't feel the concreteness in abstract [algebra] that I did in [linear algebra].

Like Arthur, Robyn mentioned that certain terms that showed up in one class reminded her of the other. However, Robyn hadn't taken the time to return to such surface-level connections and form stronger analogies based on relational structures. Now that Robyn had seen both the definitions of the morphisms and the similar grid items of the card sorts, she immediately saw the power of using one class to understand another.

Before delving into Robyn's analogies across domains, it is necessary to look at her concept image of matrix, and how it was linked to other concepts in certain ways. Because her linear algebra class contained a large focus on computations with matrices (as many linear algebra classes do), Robyn linked many concepts to the idea of matrix. Originally, Robyn grouped function, matrix, domain, and codomain together. However, eventually Robyn asked for function to be dropped from the grouping, and asserted that domain and codomain mainly related to matrix. This came from the prominence of augmented matrices in Robyn's concept image. By Robyn's account, a matrix is composed of individual functions, such as $2x+3y+4z=7$. The left-hand portion of the (augmented) matrix, also known as the coefficient matrix, is the domain, while the right-hand portion is the range. This stands in

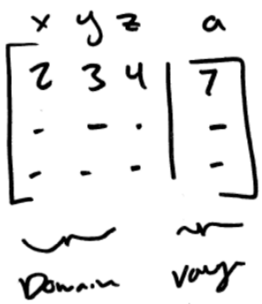


Figure 3.7. Robyn's Account of Domain and Range of a Matrix

contrast to both the previous group's and instructors' more standard usage of domain and codomain with linear transformation. Here, a matrix was imagined as a composite of individual actions, rather than a single object to be multiplied times vectors in order to yield a linear transformation. An emphasis on computations with matrices means that Robyn's concept image was firmly focused on processes from class, rather than theory. Matrix computations continued to take the spotlight throughout Robyn's second card sort. She described how, in her linear algebra class, they found the null space, column space, vector spaces, and basis of a matrix. For null space and column space, these indeed usually appear as aspects of a matrix in an undergraduate course. However, this led Robyn to speculate that a vector space is also an aspect of a matrix. In the case of basis, the repeated exposure to routine problems asking for a basis, using a matrix composed of a spanning set, had caused Robyn to view basis as an aspect of a matrix. The role of the vector space had been downplayed to the point where Robyn said, "So I definitely — I remember that the basis related — or, was a vector space? I think. They were very related in some way, that if you could find the basis of a matrix, or if you could find the basis of a vector space, or if you could find the vector space of a matrix, then you could find the other one.” Vector space was almost certainly included in the directions along side basis and vectors/matrix, but because the matrix was the main tool in finding the basis for the vector space, the other terms only have some vague link to matrix, rather than to each other.

This role of matrix as the main object or tool of linear algebra then resulted in a sort of cross mapping error in Robyn’s combined card sort. Robyn herself stated that the grouping of group, group homomorphism, linear transformation, and matrix is composed of “the two different... types of learning.... A linear transformation... is a way... to classify what is happening to a matrix. And then group homomorphism explains what a map is doing to a group.” The surface-level feature of being the main object of study provided interference as Robyn attempted to recall the relationship between matrices and linear transformations. The language of “classify” to describe the two morphisms is unclear, though it could stem from directions asking students whether a function is a group homomorphism/linear transformation, so that the students are classifying functions.

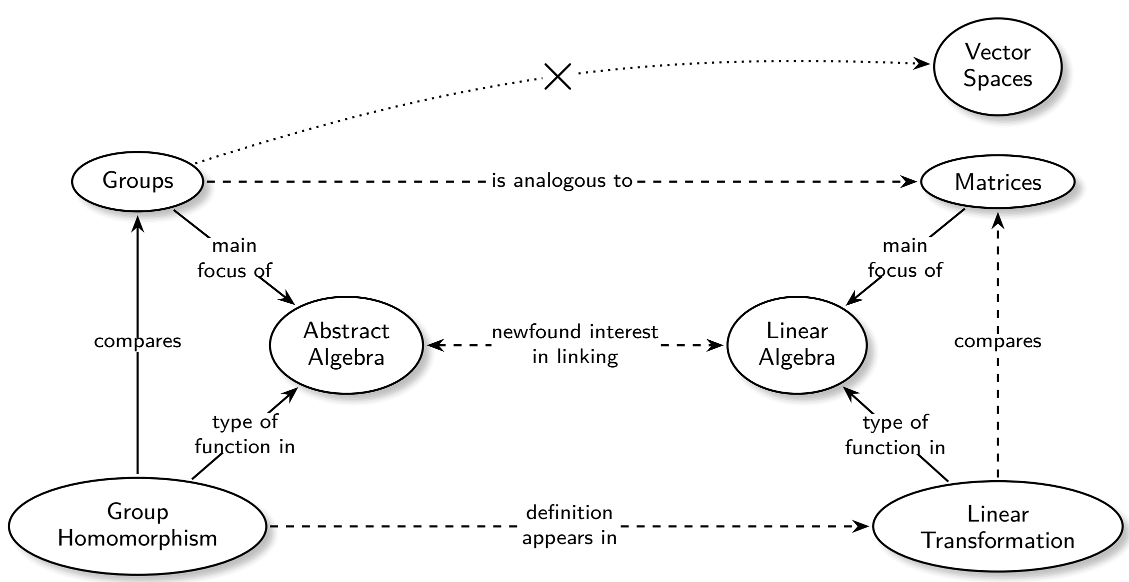


Figure 3.8. Robyn’s Alignment of Groups and Matrices

The next connection between subgroup and subspace, while correct, has both tenuous and problematic reasoning. The analogy began similarly to Arthur's, as Robyn said, "the subspace is basically the [linear algebra] version of the subgroup for groups." Robyn then recounted that subgroups must have the identity element, must be well defined under the group operation (i.e., closure), and that "the inverse must exist in the subgroup." However, the well-defined stipulation was then connected by Robyn to a subspace being a linear transformation, possibly due to the similar requirements on vector addition and scalar multiplication. Then, the inverse property was linked to to a subspace being an invertible matrix. The prominence of this term was mostly likely due to the much stronger emphasis on inverses of matrices in a linear algebra class, rather than additive inverses of vectors. Note that in both this and the last analogy, vector spaces are not present.

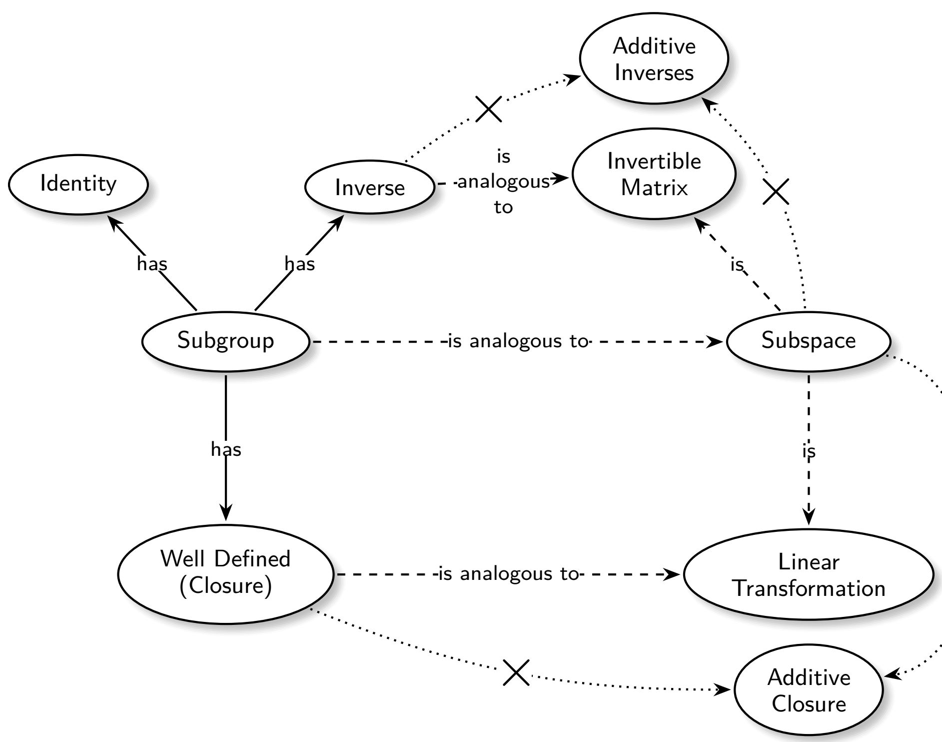


Figure 3.9. Robyn's Attempt to Align Sub-Objects

The first time that vector space was used across domains was in a pairing with generating set. Misremembering the term for basis, Robyn remarked that the two are similar in that a generating set determines the elements of a group, while a vector space defines all matrices in the "space." This was finally rectified near the end of the interview, when Robyn realized that the term she was looking for was basis, and that a basis creates all matrices (note that she still did not use the term "vectors") in the vector space. This example, combined with the previous two, shows both Robyn's difficulty with remembering and aligning vector spaces and ever-changing conception of the role of matrices.

One other important example of Robyn's interesting placement of the term matrix came when describing the similarities of the kernel across contexts. Kernel was mistaken for the order of the element of a group. Thus, because Robyn's concept definition for the kernel of a group was the number of times an element must be raised to a certain power or multiplied by an integer in order to get the identity element, she attempted to make the analogy that a kernel in linear algebra must be the number of times a matrix is added to itself in order to get the identity matrix. This, for Robyn, was strengthened by the fact that the kernel of a matrix yields the null space of a matrix (or possibly the column space, she also proposed). A matrix has a null space, which is like a kernel, and so a matrix must be the correct parallel to group element.

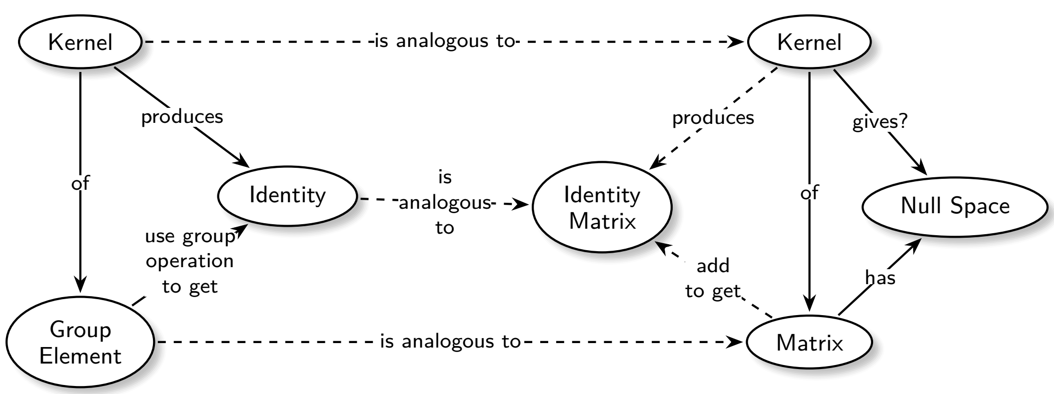


Figure 3.10. Robyn's Attempt to Align Group Element and Matrix

The difficulties presented in aligning matrices and vector spaces persisted into Robyn's closed card sort. Asked directly about the connection between group and vector space, Robyn decided that there was no relation to be expressed. Concerning vector space and linear transformation, Robyn at first proposed that a vector space could be a linear transformation, but ultimately decided this was incorrect. Her final statement on this pairing was that there was a definite connection, but she remained unsure of what that connection would be. Finally, of linear transformation and matrix, Robyn repeated her original analogy at the beginning of this section asserting how linear transformations compare matrices.

*Jake*

Jake began his linear transformation card sort by saying, "So... some of these things I don't really remember talking about, but... if this is similar to a... group homomorphism, I'm willing to assume it's similar for in vector space as well." This trend continued throughout Jake's interview; there were many terms that Jake had forgotten from linear algebra, but he was able to align both structures enough to infer properties of linear transformations and vector spaces from his knowledge in abstract algebra. This was evidenced by his constant use of "I imagine" before constructing an analogy between domains. Thus, for Jake, the discussion aligning linear transformations and group homomorphisms triggered a willingness to extend the analogy and transfer his knowledge from abstract algebra into linear algebra.

The first of Jake's connections in the linear transformation card sort stemmed from the terms closest to functions in general. As with most all students in the study, Jake was confident that injections/one-to-one functions and surjections/onto functions were "basically the same thing" as how he described them in the context of group homomorphisms. Kernels, according to Jake, must still be the set of elements mapped to the identity, but there was no indication of whether or not Jake was aware of what the identity element of a vector space is. Jake also concluded isomorphism must have the same requirements as in the previous context. He correctly stated that linear transformations are between vector spaces, in the same way that group homomorphisms are between groups. Later in the closed card sort, however, Jake was unwilling to say that groups and vector spaces were closely related, beyond both being sets of elements. Thus, while Jake was indeed beginning to form connections across domains, the analogy of similar requirements in morphisms had not yet been extended to objects and their similarities as algebraic structures. Part of this confusion may have stemmed from Jake's understanding of vector spaces in linear algebra. At the end of the closed card sort, Jake was unsure of how to define the relationship between matrix and linear transformation. Unsure of whether or not vector spaces include matrices, Jake concluded that linear transformations must map matrices as they do vectors. This, for Jake, was supported by the fact that in abstract algebra, the group operation could be addition or multiplication. Matrices can be multiplied, so the linear transformation must map multiplication of matrices as well. Jake later noticed that the definition given referred to addition of $\mathbf{x}$ and $\mathbf{y}$. Thus, after Jake stated that in order to understand linear transformations, "You would have to understand the multiplication of elements in that group, so multiplication of matrices or vectors and stuff like that," he then followed up with "...or addition of vectors, actually it looks like." It is unclear if this was meant to finally exclude matrices from vector spaces, or exclude multiplication as an operation of vectors. Regardless, this shows how Jake's concept definition of vector space interferes with his ability to make analogies.

The remaining connections also occurred during the closed card sort. Jake recalled a generating set as the set of elements generated by an element coprime to the order of the group (most likely thinking of cyclic groups), and a basis as a linearly independent set of vectors that generates all of a vector space. Based on this, Jake (correctly) believed the two to be similar in the sense of generating a larger set.

In contrast to this, Jake did not recall subspaces from his linear algebra class. However, he used his knowledge of subgroups and his analogy-in-progress between

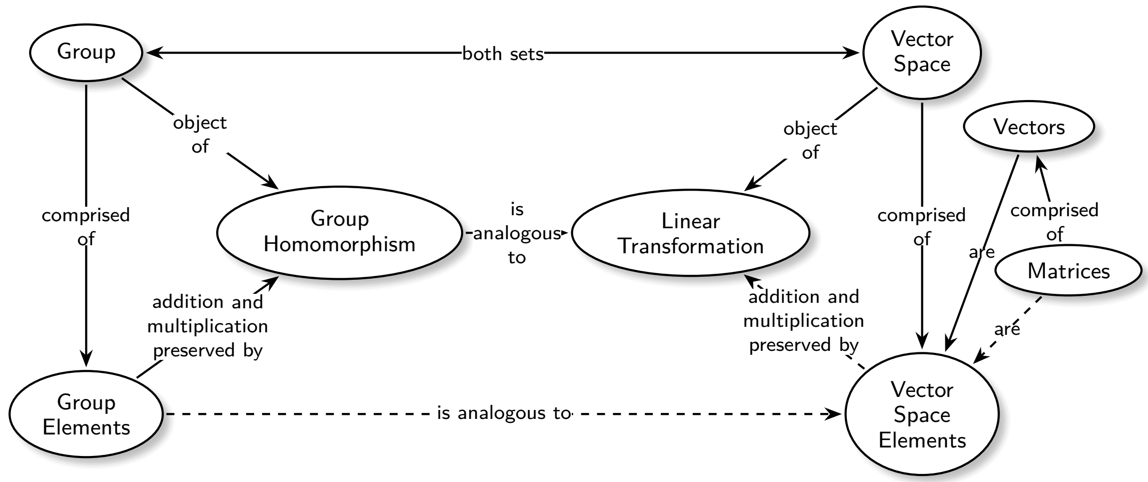


Figure 3.11. Jake's Inclusion of Matrices as Elements of Vector Spaces

domains to infer that a subspace of a vector space is probably like a subgroup of a group. As a subgroup has all the properties of a group, a subspace must have all the properties of a vector space, with each being "portions of their bigger brother." So, while Jake had still not connected the actual structural similarities between groups and vector spaces, he had started the process of extending his concept image of substructure gained from studying abstract algebra to include structures from linear algebra.

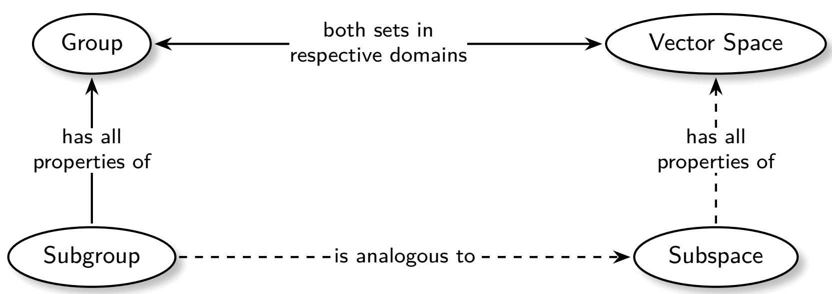


Figure 3.12. Jake's Analogy Between Sub-Objects

*Comments on Remaining Students*

The two accounts above are representative of the breakdown in this category of students still forming substantial connections. All of these students made multiple attempts to extend the analogy between morphisms to include other concepts from the card sort. In some cases these attempts proved fruitful, and allowed for a path to recall or new insight. Other times, the attempt to integrate new parallels led to problematic connections based on surface features or incorrect concept definitions.

Emilie's reasoning throughout her interview was similar to Robyn's in a few important ways. The first of these is Emilie's concept image of matrix, and the most dominant features within that concept image. Emilie noted that matrices are composed of vectors, but, like Robyn, saw a matrix as a function (or a collection of functions) rather than producing a function by way of matrix multiplication. The left-hand side represents multiples of "x's," and the "domain is all the x's." This description is similar to Robyn's picture in Figure 3.7, where the domain was labeled as the left hand side of an augmented matrix.

The other large similarity between these two interviews is each student's difficulty in recalling and aligning vector spaces and linear transformations to their respective structures in abstract algebra. Part of this difficulty lay in the fact that Emilie's concept images of vector spaces, basis, and linear transformations also proved to be problematic. Like Robyn, Emilie noticed the similarities between generating set and basis, but had trouble with the structural alignment of terms related to these two ideas. Instead of aligning group and vector space, Emilie claimed that the parallel to group in such an analogy would be a linear transformation. By her account, "The generating set can kind of create the whole group. It's like an element that can create the whole group, whereas the basis can create the whole... linear transformation." At an earlier point, Emilie also described the basis as being a "function" that makes other functions in a linear transformation go to the identity. These statements can be interpreted in a variety of manners. For example, Emilie could have been thinking of the linear independence condition on vectors, and calling the vectors functions (as she did in describing the rows of a matrix). Or, Emilie could have also been using the term linear transformation while in fact attempting to describe a vector space.

Or, Emilie could have been describing part of a process of Gaussian elimination. Regardless, it is still remarkable that Emilie was not able to align groups and vector spaces after the alignment of generating set and basis. This confusion persisted when Emilie was asked about the relationship between linear transformations and vector spaces, and she conjectured, “Since linear transformations are basically vectors just kind of dealing with each other, I would guess that there’s a vector space within linear transformations?” Though their particular analogies were different, both Robyn and Emilie made non-standard, problematic alignments of linear transformation, remained unsure of the role of vector spaces, and did not place vector spaces in analogy to groups in crucial alignments.

It is clear from Jake’s and Robyn’s interviews that recall of concepts from both domains was a substantial barrier to analogy construction. This was true for most all students in both this and the following category. However, Emilie stands out especially in this regard. Having taken linear algebra about two years before the time of interview, it would stand to reason that this domain would prove difficult to recall. However, Emilie’s numbers in Table 3.2 display an equally telling story for her recall of abstract algebra concepts — a class which she was taking at the time. This struggle to recall definitions from both classes was certainly a large factor in Emilie’s construction of non-standard and problematic analogies later on.

As a final point of discussion on Emilie’s analogical reasoning, despite the above difficulties, Emilie also made the parallel between subgroups and subspaces. Unlike Robyn, Emilie used this analogy to infer that vector spaces must be similar to groups. Asked at the end of the closed card sort whether there was any relationship between groups and vector spaces, Emilie said, “Well I did say a subgroup and a subspace were similar, so I guess it would be fitting that group and vector space would also be related — since a group can be broken down into subgroups and vector space can be broken down into subspaces. They’re both just kind of the big pictures.” This was as far as the elaboration of the analogy extended, but it provides an interesting look at the potential of the sub-object analogy as a seed for working against problematic concept images and toward an understanding of vector spaces.

Lucas's analogical reasoning was similar to Jake's, in the sense that many other correct analogies fell into place after the initial alignment of group homomorphism and linear transformation. That being said, Lucas's analogies were stated as facts, whereas Jake's analogies were stated as conjectures. Part of this may have been that Lucas recalled more of linear algebra than Jake, so his analogies were not being used to transfer knowledge from abstract algebra in order to recall linear algebra. Nevertheless, Lucas was placed in this category due to his surprise in seeing the similarity of the morphisms' conditions. The realization that the two functions had such parallels seemed to be the catalyst for Lucas's confidence in future groupings. For example, it was immediate to Lucas that a group and a vector space must also be parallels because of the fact that they are the structures their respective functions are "transforming." Lucas stated that "we can look at the elements of some vector space as a group of elements," but this was the extent of his elaboration at the time. When pressed in the closed card sort about how vector spaces were a special case of groups, Lucas just stated, "Elements of a vector space are linear. Group elements are not necessarily." Thus, it is not clear if Lucas was ready to view vector spaces as containing a group structure or not. Whatever the case, the original analogy of objects ended with the justification falling back to morphisms, with the exchange:

> L: Linear transformations with vector space is a more specific example of group homomorphisms.
>
> I: Okay, and function is there because?
>
> L: Oh, because there's some f— in both cases the idea is that you have some function going from one group to another or from one vector space to another.
>
> I: Okay.
>
> L: Or sending elements from one into the other.

This was immediately followed up by, “I think there are actually a lot of them that are sort of like that, where you get things that are similar,” again indicating a newfound propensity for seeing connections across domains. During the remainder of the card sort interview, Lucas went on to describe basis, inverse homomorphism, and subspace as special cases or specific examples of their analogous abstract algebra counterparts. For sub-objects, Lucas also aligned the identity element of a group and the “zero element” of a vector space as being similar requirements for subgroup and subspace, a slight addition to Jake’s analogy.

#### *General Comments*

The first category of students brought to light the important fact that there are indeed undergraduates who display a great deal of analogical reasoning without intervention. This second category category showed the equally important ways in which students first attempt such analogical reasoning. While students’ ability or inability to recall concepts from individual domains impacted their ability to create mathematically correct analogies, it did not impact their willingness to explore potential connections.

### Students Engaged in Little Analogical Reasoning

The three remaining students — Flint, Sander, and Tamara — made fewer connections between linear and abstract algebra than the others in this study. This is not to say that they did not see similarities here or there. Recall that Flint and Tamara were able to recognize the group homomorphism property as present in the definition of linear transformation. Yet, brief glimpses of similarity didn’t seem to spark many attempts to align algebraic structures as in the previous set of students. Students in this final category were less likely to form analogies in the final open card sort, or were more likely to make surface level guesses in the closed card sort. Sometimes, this was due in part to an inability to remember one or both classes. It may have also been possible that these students' concept images of group homomorphism and linear transformation were too different to inspire deeper connections.

*Sander*

Sander was the only student to consistently question that there was even a relationship between group homomorphism and linear transformation. Asked at the end of the card sorts if he had any more comments about the relationship between group homomorphism and linear transformation, Sander replied, "Are they analogous? Are they supposed to be seen as the same thing?" Sander clearly had the feeling that the study was implying with its constant questioning that the two must be similar, but he did not see how this was the case. One of the barriers to Sander's willingness to group these terms was perhaps brought to light when he was asked about the relationship between linear transformation and matrix. Asked about his phrasing of "linear transformation of a matrix" when discussing null space, Sander responded,

> That's what I was going back [to the definition paper] — so I didn't really understand [why the definitions were given the way they were in the study]... I think this is how we did — it was like A— A x, something to b, or some — it was some way of doing a... linear transformation, and if the null space is where b equals zero. So if you did a linear transformation of a matrix, where you applied some matrix to some conditions you decided to — and then set the null space equals zero — or, set the set the goal equal to zero, which is the null space, and you would get different... I think... it would be x values? You'd have a set of x values that map the null space. And these [pointing around the $\mathbf{x}$ part of $A\mathbf{x}$] would be, I don't know if these would be — but this [underlines $A$] would be the matrix. This [pointing around $\mathbf{x}$] would be vectors, I believe. And then the null space would just be all [pointing at $\mathbf{b}$] zeros.

Sander correctly recalled, at least partially, that a linear transformation is given by $T(\mathbf{x}) = A\mathbf{x}$. Additionally, it seems that this was the most prominent feature of Sander's concept image: a working concept definition of linear transformation as involving $A\mathbf{x}$ somehow. Similar to how some of the previously mentioned students maintained a focus on equations of an augmented matrix that may have caused difficulty in seeing vector spaces as objects connected by linear transformations, Sander's view may have allowed the computational aspect of matrix multiplication to obscure the role of vector spaces. Similarly, Sander's understanding of group homomorphism as one aspect of isomorphism, which is "sameness" rather than a function, further prevented an object-morphism understanding to blossom across domains.

However, the notion that matrix multiplication was the primary aspect of Sander's linear transformation concept image is complicated by his grouping of function, linear transformation, and matrix. In this grouping, Sander seemed much less confident about the role of a matrix and its relationship with linear transformation, similar to the confusion of the role of matrix found in earlier discussions above. This time, Sander said, "I think a linear transformation is like a function, where a matrix is the variable?" This is similar to Jake's internal debate about whether to include matrices alongside vectors as the input of a linear transformation. Regardless, this shows Sander's difficulty to place linear transformation as a function between vector spaces.

Sander's difficulty to recall vector space was further highlighted when he attempted to make a parallel between subgroup and subspace. Sander paired group and subgroup correctly, but when faced with subspace, Sander said, "Is that similar to subgroup? I guess if there'd been a card for 'space' and subspace, I would have put them together in the same way I did group and subgroup." The concept of vector space was unfamiliar to Sander. Instead, in a manner quite similar to Robyn above, null space, column space, and subspace were described by Sander as "subspaces of a matrix," and later, "a way to represent the matrix." According to Sander, a vector space must be a "space" similar to these, with its own requirements, but these requirements were unclear. When asked about the relationship between groups and vector spaces, Sander did not believe that they were related, saying "I guess maybe if it said 'set,' it would have been okay," since vectors are a part of a set. The only recognition of vector spaces as the objects of linear transformations came when Sander recalled that the definition given in the study, where he said "I think linear transformations are applied over vector spaces, based on the definition that we saw." However, Sander had not yet integrated this new information from the definition into his concept image for linear transformation, based on his remaining responses in the interview.

*Comments on Remaining Students*

In contrast to Sander, Flint attempted to align group and vector space. His comparison of the two was similar to Emilie's final conclusion that the two objects are both "big pictures." Flint described groups and vector spaces as being "all the possibilit[ies]" or the "containment." Flint elaborated on this later when talking about the relationship between vector spaces and linear transformations. Recall that Flint (as well as Sander) ignored the function aspect of the definition of linear transformation and focused on the addition and scalar multiplication of vectors. Flint also heavily drew upon his understanding of transformations of graphs. Thus, Flint explained, a vector space is all possible linear transformations, all multiples of the vectors you are working with, similar to all integers when discussing the group of integers. Basis and generating set, as well as subgroup and subspace, were not seen as connected by Flint. Additionally, Flint saw inverse morphisms as related in the sense that one takes the inverse of a homomorphism and one can also take the inverse transformation of a graph or vector. Interestingly, Flint described matrices in much the same way as Robyn and Emilie: a matrix is composed of functions (each "function" is a line of the system of equations represented by an augmented matrix), and a function has a domain and range. This, combined with Flint's concept image of linear transformation, may have been a barrier to allowing Flint to capitalize on the connections between morphisms and infer more connections.

Tamara's difficulties in aligning structures and making analogies seemed to stem from her difficulty in recalling the prerequisite concepts from each domain. This was more pronounced than other students in the study. This also shows a particular shortcoming in using the edge-to-vertex ratio of the adjacency matrices to broadly summarize student performance. Tamara's first grouping links together group, group homomorphism, inverse homomorphism, isomorphism, and subgroup. Her reasoning for this grouping was that all terms involved are groups. This is incorrect, but still creates a large number of edges in her adjacency matrix graph. Tamara followed this up with a large single grouping of terms related to functions of all settings (domain, range, onto, etc.), once again adding many connections. By the end of the first card sort, Tamara's adjacency matrix graph still has five components, but a high edge-to-vertex ratio.

Tamara's other card sort groupings beyond function-related concepts showed that she was unsure of many of the details of the group theory. One of the abstract algebra concepts that Tamara was able to recall correctly was that isomorphism. She knew that an isomorphism must be injective, bijective, and surjective, though she couldn't remember which of these three terms had which definition. Across domains, she guessed that inverse transformations and inverse homomorphisms must be alike, because she had just seen a similarity between group homomorphisms and linear transformations. There was no mention of the inverse aspect, so this is entirely based on the presence of homomorphism and transformation in the names of the terms. Tamara also guessed that subspace and subgroup must be related, possibly due to the presence of "sub" in their monikers, since she gave no justification of this guess. Similar to Emilie, though, this caused Tamara to change her answer for groups and vector spaces and guess that the two must be related. Again, there was no justification for this relationship, giving some evidence toward the idea that Tamara's inability to recall algebraic structures meant that most parallels were based on features of the given words instead of their conceptual meanings.

*General Comments*

For all of these students, there was less of a willingness to form analogies across domains. Sander and Tamara both confirmed that concepts like domain, codomain, onto, and one-to-one continue to hold their meanings across domains. Flint confirmed domain and codomain. Like most all students (and professors) these were seen as properties of functions, true across all domains of math. Particular emphasis has not been placed on these particular terms in this analysis, as they do not involve analogical reasoning when discussed in this manner. But, it is important to highlight this to display that these students maintained at least some concepts across domains.

The difficulties these students experienced in widening these connections through analogy could partially be explained due to difficulty in recalling or understanding prerequisite concepts in either domain. However, this difficulty existed in both categories above to some extent in each student. This was especially true for Maureen, Jake, Robyn, and Emilie. Between this category and the previous, there also does not appear to be a pattern or clear correlation between the number of components in students' adjacency matrices, their edge to vertex ratios, the substance of their groupings, or semesters of instruction and their use of analogy across domains. The difference remains unclear in this study.

## 3.3 General Findings Across All Cases

### Summary Tables

As a supplement to the qualitative themes explored both above and later below, in this section I will present tables which summarize all connections made by students in both the open and closed card sorts. Table 3.2 in Section 3.2 already displays the number of components and edge-to-vertex ratios from students' and professors' adjacency matrices, which were created from the first two open card sorts. The discussion below focuses on exactly which connections were and were not made over the course of the interview, from a vantage point higher than the case analyses in the previous section.

Table 3.3. Students' Levels of Connections Between Concepts

| **Pseudonym** | **Gen. Set/ Basis** | **Group/ Vec. Sp.** | **Inv. Hom./ Inv. Tran.** | **Subgroup/ Subspace** | **Lin. Trans./ Matrix** | **Kernel/ Null Sp.** | **Image/ Col. Sp.** | **Group/ Group Hom.** | **Vec. Sp./ Lin. Trans.** |
|---|---|---|---|---|---|---|---|---|---|
| *Prior* | | | | | | | | | |
| Arthur | Y | Y | Y | Y | Y* | Y | Y | Y | Y |
| Kyle | Y | Y | Y* | Y | Y | Y | Y | Y | Y |
| Maureen | S | Y | S* | Y | P* | S | - | Y | Y |
| *Current* | | | | | | | | | |
| Emilie | P | Y* | Y* | Y | P* | - | - | Y* | P* |
| Jake | Y* | Y* | Y | Y* | P* | - | - | Y* | Y* |
| Lucas | Y | Y | Y | Y* | Y | P | - | Y | Y |
| Robyn | P* | N* | S | P | P* | P | - | Y | P* |
| *Little* | | | | | | | | | |
| Flint | N* | S | P* | N* | P* | - | - | P | P* |
| Sander | N* | N* | S* | P* | P | S | - | Y | Y |
| Tamara | N* | S* | S | S* | S* | - | - | P | Y |

Y = correct connection
S = surface-level connection or guess with no rationale
P = problematic concept image or connection
N = verbal confirmation of no relationship
- = connection never discussed
* indicates connection was not discussed until the closed card sort

Table 3.3 shows students' connections between key concepts from the study. All of these pairs were explicitly addressed in the closed card sort, with the exception of kernel/null space and image/column space. Entries marked with "P" are connections which students identified, but which they discussed in a way that contained mathematical errors or other mathematically problematic analogies. Many of these particular entries were discussed in more detail in Section 3.2. Entries marked with "S" represent connections which students identified, but defended only on the basis of guesses and feelings, with no mathematical backing. This label also includes connections which students directly identify as not being present except for a minor surface level feature. All other student connections are represented with "Y" as being correct connections. These entries do not imply that a student fully understands the connection between the two parallel concepts, but that they successfully identified them as connected and provided at least some mathematically correct reasoning which they deemed as sufficient. To demonstrate the timing of the connection, an asterisk is used to denote that the connection was confirmed or denied by the student only after being asked directly in the closed card sort. Thus, the absence of an asterisk indicates that a student identified the connection without being prompted.

Students in the *Prior* category identified most of the connections correctly. Though, as discussed earlier, Maureen was sometimes operating off of intuition. All three of these students made most of their connections without having to be prompted. With the *Current* category, there is a shift towards more items being left undiscussed until the closed card sort. This pattern is likely connected to the fact that in both this and the *Little* categories, the parallels to kernel and image with null space and column space arose much less often. Genearlly, students in the *Current* and *Little* categories also made more problematic statements. The reason that Lucas, the main outlier to this observation, was still included in the *Current* category was discussed above in Section 3.2. Finally, students in the *Little* category understandably denied more connections than students in the other two categories.

The standout column of Table 3.3 is that of linear transformation/matrix. As will be discussed later, professors agreed that the relationship of linear transformations and matrices is of paramount importance. This theme of students failing to connect linear transformations and matrices appropriately will be discussed in detail shortly in the subsection on general themes.

It is also clear from Table 3.3 that a greater deal of students had difficulty connecting generating set/basis, kernel/null space, and image/column space. These are significant, but do not require a greater discussion in the subsection on general themes for a few reasons. The first is that there is not much elaboration from students on these topics. The few items marked as problematic connections have mostly been explored earlier in Section 3.2. Beyond that, students either expressed that there was no connection, or did not address the connection at all. The second is that these three sets of connections share the distinction, in some sense, of being one step away from the analogy between morphisms and objects. Inverse morphisms and subobjects, while also not directly concerning the main objects and morphisms in their categories, have a similar naming scheme across domains, causing some students to initiate structural alignment based on this surface feature. The other remaining connections strictly concern the morphisms and objects themselves. However, generating set/basis, kernel/null space, and image/column space lack a common naming scheme. They also have slight quirks that make the parallels more opaque. Generating set would have a more direct parallel to the concept of spanning set in linear algebra. Linking basis to generating set must be nuanced in that a basis is a specific type of spanning set with the extra requirement that its vectors are linearly independent. Kernel and image are terms used not only in abstract algebra, but in linear algebra as well, and could be considered their own parallels across domains. While the null space and column space are subspaces, just as the kernel and image are subgroups, the nuance here comes from the fact that null space and column space are terms almost always associated to the matrix representation of a linear transformation. Given that students in this study had difficulty making the connection between matrices and linear transformations, it is understandable that these last two connections would prove difficult as well.

The final comment on Table 3.3 concerns the way that the group/vector space column does not convey the entire story to be told concerning this connection. As entries in this table were only marked as surface level for a very narrow criteria, a few students have been marked as identifying the connection correctly for reasoning that was technically correct, but lacking in terms of identifying algebraic parallels. For this reason, this connection is discussed in greater detail in the section on general themes.

Table 3.4 and Table 3.5 show the totals found by taking the sum of all student

Table 3.4. Student Group Homomorphism Card Sort Adjacency Matrix Totals

| | Codomain | Domain | Function | Generatin... | Group | Group Ho... | Image | Injection | Inverse Ho... | Isomorphism | Kernel | One-to-One | Onto | Range | Subgroup | Surjection | Bijection | Cyclic | Ideal | Identity | Normal | Relations | Trivial Su... |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Codomain | 15 | 9 | 9 | 0 | 3 | 4 | 5 | 3 | 2 | 2 | 1 | 3 | 5 | 8 | 1 | 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Domain | 9 | 16 | 10 | 1 | 4 | 4 | 3 | 4 | 1 | 3 | 2 | 6 | 4 | 12 | 0 | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Function | 9 | 10 | 21 | 0 | 8 | 6 | 3 | 6 | 3 | 7 | 3 | 8 | 7 | 7 | 1 | 7 | 1 | 1 | 0 | 1 | 0 | 0 | 0 |
| Generating Set | 0 | 1 | 0 | 11 | 8 | 1 | 2 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 2 | 1 | 0 | 0 | 0 | 0 |
| Group | 3 | 4 | 8 | 8 | 31 | 10 | 1 | 2 | 3 | 10 | 3 | 3 | 3 | 2 | 13 | 2 | 1 | 3 | 1 | 1 | 2 | 0 | 0 |
| Group Homomorphism | 4 | 4 | 6 | 1 | 10 | 21 | 3 | 6 | 4 | 14 | 4 | 5 | 5 | 2 | 7 | 6 | 1 | 1 | 0 | 1 | 2 | 0 | 0 |
| Image | 5 | 3 | 3 | 2 | 1 | 3 | 9 | 2 | 1 | 0 | 1 | 1 | 3 | 3 | 1 | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Injection | 3 | 4 | 6 | 0 | 2 | 6 | 2 | 16 | 0 | 7 | 2 | 13 | 10 | 1 | 0 | 13 | 2 | 0 | 0 | 0 | 0 | 0 | 0 |
| Inverse Transformation | 2 | 1 | 3 | 0 | 3 | 4 | 1 | 0 | 7 | 5 | 0 | 1 | 1 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| Isomorphism | 2 | 3 | 7 | 0 | 10 | 14 | 0 | 7 | 5 | 19 | 2 | 7 | 6 | 2 | 3 | 7 | 1 | 1 | 0 | 1 | 0 | 0 | 0 |
| Kernel | 1 | 2 | 3 | 1 | 3 | 4 | 1 | 2 | 0 | 2 | 7 | 1 | 0 | 1 | 2 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 |
| One-to-One | 3 | 6 | 8 | 0 | 3 | 5 | 1 | 13 | 1 | 7 | 1 | 17 | 12 | 4 | 0 | 10 | 2 | 0 | 0 | 0 | 0 | 0 | 0 |
| Onto | 5 | 4 | 7 | 0 | 3 | 5 | 3 | 10 | 1 | 6 | 0 | 12 | 16 | 3 | 0 | 13 | 2 | 0 | 0 | 0 | 0 | 0 | 0 |
| Range | 8 | 12 | 7 | 1 | 2 | 2 | 3 | 1 | 1 | 2 | 1 | 4 | 3 | 16 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Subgroup | 1 | 0 | 1 | 1 | 13 | 7 | 1 | 0 | 1 | 3 | 2 | 0 | 0 | 0 | 16 | 0 | 0 | 1 | 2 | 1 | 3 | 0 | 0 |
| Surjection | 4 | 3 | 7 | 0 | 2 | 6 | 3 | 13 | 0 | 7 | 1 | 10 | 13 | 1 | 0 | 16 | 2 | 0 | 0 | 0 | 0 | 0 | 0 |
| Bijection | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 2 | 1 | 1 | 0 | 2 | 2 | 0 | 0 | 2 | 3 | 0 | 0 | 0 | 0 | 0 | 0 |
| Cyclic | 0 | 0 | 1 | 2 | 3 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 4 | 0 | 0 | 0 | 0 | 0 |
| Ideal | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 0 |
| Identity | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| Normal | 0 | 0 | 0 | 0 | 2 | 2 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 3 | 0 | 0 | 0 | 0 | 0 | 3 | 0 | 0 |
| Relations | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Trivial Subgroup | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table 3.5. Student Linear Transformation Card Sort Adjacency Matrix Totals

| | Codomain | Domain | Function | Basis | Vector Space | Linear Tr... | Image | Injection | Inverse Tr... | Isomorphism | Kernel | One-to-One | Onto | Range | Subspace | Surjection | Column S... | Matrix | Null Space | Bijection | Determinant | Dimension | Eigenvalues | Eigenvectors | Gaussian... | Linear Ind... | Row Space | Span | System of... | Vector | Coordinate... | Spanning S... | Trivial Hom... |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Codomain | 12 | 9 | 5 | 0 | 1 | 3 | 3 | 1 | 1 | 0 | 1 | 1 | 1 | 8 | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Domain | 9 | 15 | 9 | 2 | 2 | 3 | 4 | 1 | 1 | 0 | 1 | 1 | 1 | 10 | 0 | 1 | 0 | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| Function | 5 | 9 | 19 | 1 | 3 | 7 | 4 | 3 | 1 | 3 | 1 | 4 | 3 | 8 | 0 | 3 | 0 | 4 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 2 | 0 | 0 | 0 |
| Basis | 0 | 2 | 1 | 14 | 6 | 4 | 3 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 6 | 2 | 0 | 0 | 2 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 3 | 0 | 0 | 0 |
| Vector Space | 1 | 2 | 3 | 6 | 18 | 5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 5 | 0 | 1 | 5 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 4 | 0 | 0 | 0 |
| Linear Transformation | 3 | 3 | 7 | 4 | 5 | 35 | 5 | 5 | 8 | 6 | 3 | 5 | 5 | 2 | 4 | 4 | 3 | 5 | 4 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 0 | 0 | 0 |
| Image | 3 | 4 | 4 | 3 | 0 | 5 | 17 | 2 | 1 | 0 | 5 | 1 | 1 | 5 | 2 | 2 | 3 | 3 | 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| Injection | 1 | 1 | 3 | 0 | 0 | 5 | 2 | 16 | 1 | 6 | 1 | 13 | 10 | 0 | 0 | 12 | 0 | 2 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Inverse Transformation | 1 | 1 | 1 | 0 | 0 | 8 | 1 | 1 | 9 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Isomorphism | 0 | 0 | 3 | 0 | 0 | 6 | 0 | 6 | 2 | 10 | 1 | 7 | 6 | 0 | 0 | 6 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Kernel | 1 | 1 | 1 | 1 | 0 | 3 | 5 | 1 | 0 | 1 | 11 | 1 | 1 | 2 | 1 | 1 | 2 | 3 | 6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| One-to-One | 1 | 1 | 4 | 0 | 0 | 5 | 1 | 13 | 0 | 7 | 1 | 16 | 12 | 0 | 0 | 10 | 0 | 2 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Onto | 1 | 1 | 3 | 0 | 0 | 5 | 1 | 10 | 0 | 6 | 1 | 12 | 14 | 0 | 0 | 12 | 0 | 1 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Range | 8 | 10 | 8 | 1 | 0 | 2 | 5 | 0 | 0 | 0 | 2 | 0 | 0 | 17 | 0 | 0 | 3 | 4 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| Subspace | 0 | 0 | 0 | 0 | 5 | 4 | 2 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 10 | 0 | 1 | 1 | 2 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Surjection | 1 | 1 | 3 | 0 | 0 | 4 | 2 | 12 | 0 | 6 | 1 | 10 | 12 | 0 | 0 | 14 | 0 | 1 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Column Space | 1 | 0 | 0 | 0 | 1 | 3 | 3 | 0 | 0 | 0 | 2 | 0 | 0 | 3 | 1 | 0 | 10 | 6 | 5 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 |
| Matrix | 0 | 3 | 4 | 6 | 5 | 5 | 3 | 2 | 0 | 0 | 3 | 2 | 1 | 4 | 1 | 1 | 6 | 26 | 7 | 0 | 4 | 1 | 2 | 1 | 1 | 3 | 1 | 0 | 1 | 5 | 0 | 0 | 0 |
| Null Space | 1 | 0 | 0 | 2 | 1 | 4 | 4 | 0 | 0 | 0 | 6 | 0 | 0 | 1 | 2 | 0 | 5 | 7 | 12 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Bijection | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 2 | 0 | 1 | 0 | 2 | 2 | 0 | 0 | 2 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Determinant | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 4 | 0 | 0 | 4 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| Dimension | 1 | 0 | 0 | 2 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| Eigenvalues | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 0 | 0 | 1 | 0 | 2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Eigenvectors | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Gaussian Elimination | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| Linear Independence | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 3 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 3 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| Row Space | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| Span | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| System of Equations | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| Vector | 0 | 0 | 2 | 3 | 4 | 2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 5 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 8 | 0 | 0 | 0 |
| Coordinate Vector | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Spanning Set | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Trivial Homomorphism | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

group homomorphism adjacency matrices and linear transformation adjacency matrices, respectively. Adjacency matrices for individuals, including those of the three interviewed professors, can be found in Appendix E and Appendix F. Recall that the diagonal of the adjacency matrices lists the number of groupings in which a card was used. The off-diagonal entries give the number of times the concept in the heading of a column was placed in a grouping with the concept given in the heading of that row. Looking at the first two card sorts through the lens of this quantitative data immediately shows the regularity with which students grouped together "function" concepts, such as domain, codomain, range, function, injection, and surjection. Indeed, these were also often the first cards for students to group together, due to their familiarity with them from classes before linear and abstract algebra. Range was more often chosen over range. "Matrix" was the second most used card in the linear transformation card sort, with its pairings being spread out over almost every other term.

As I am also concerned with how students' concept images are different between domains, one of the most useful tables to explore is Table 3.6. Each of the terms in the first sixteen rows and columns of the group homomorphism and linear transformation adjacency matrices either was present in both sorts or had a mathematically parallel concept in the other card sort. These parallel concepts were placed in correspondingly numbered rows/columns. Subtracting the top left $16 \times 16$ submatrices, in the order of (*linear transformation submatrix*) − (*group homomorphism submatrix*), yields Table 3.6. As is evident from the order of subtraction, a negative entry represents a card (on the diagonal) or grouping (in the off-diagonal) which was used more frequently in the group homomorphism card sort, while a positive entry represents a card or grouping that was used more frequently in the linear transformation card sort. As a rough measure of which items show the most significant difference between domains, I used Tukey's test for outliers; data points greater than 1.5 times the interquartile range (IQR) from the third quartile or less than 1.5 times the IQR from the first quartile were marked as outliers (Gonick & Smith, 1993). As the diagonal and off-diagonal measure different types of frequencies, I split the data sets into diagonal and off-diagonal sets and calculated separate IQRs. Let $Q_n$ be the $n$-th quartile. On the diagonal data set, $Q_1 = -2.25$, $Q_2 = -1$, and $Q_3 = 2.25$, so that the lower cutoff is $-6.75$ and the upper cutoff is 9. On the off-diagonal data set (specifically using only the upper triangular entries so as not to duplicate every entry, as the matrix is symmetric) $Q_1 = -2$, $Q_2 = -1$, and $Q_3 = 0$, so that the lower cutoff is $-5$ and an upper cutoff is 3.

Table 3.6. Difference in Core Cards Between Adjacency Matrix Totals

| | Codomain | Domain | Function | Gen. Set/B... | Object | Morphism | Image | Injection | Inverse Mor... | Isomorphism | Kernel | One-to-One | Onto | Range | Sub-object | Surjection |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Codomain | -3 | 0 | -4 | 0 | -2 | -1 | -2 | -2 | -1 | -2 | 0 | -2 | -4 | 0 | -1 | -3 |
| Domain | 0 | -1 | -1 | 1 | -2 | -1 | 1 | -3 | 0 | -3 | -1 | -5 | -3 | -2 | 0 | -2 |
| Function | -4 | -1 | -2 | 1 | -5 | 1 | 1 | -3 | -2 | -4 | -2 | -4 | -4 | 1 | -1 | -4 |
| Gen. Set/Basis | 0 | 1 | 1 | 3 | -2 | 3 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -1 | 0 |
| Object | -2 | -2 | -5 | -2 | -13* | -5 | -1 | -2 | -3 | -10† | -3 | -3 | -3 | -2 | -8† | -2 |
| Morphism | -1 | -1 | 1 | 3 | -5 | 14* | 2 | -1 | 4† | -8† | -1 | 0 | 0 | 0 | -3 | -2 |
| Image | -2 | 1 | 1 | 1 | -1 | 2 | 8 | 0 | 0 | 0 | 4† | 0 | -2 | 2 | 1 | -1 |
| Injection | -2 | -3 | -3 | 0 | -2 | -1 | 0 | 0 | 1 | -1 | -1 | 0 | 0 | -1 | 0 | -1 |
| Inverse Morphism | -1 | 0 | -2 | 0 | -3 | 4† | 0 | 1 | 2 | -3 | 0 | -1 | -1 | -1 | -1 | 0 |
| Isomorphism | -2 | -3 | -4 | 0 | -10† | -8† | 0 | -1 | -3 | -9* | -1 | 0 | 0 | -2 | -3 | -1 |
| Kernel | 0 | -1 | -2 | 0 | -3 | -1 | 4† | -1 | 0 | -1 | 4 | 0 | 1 | 1 | -1 | 0 |
| One-to-One | -2 | -5 | -4 | 0 | -3 | 0 | 0 | 0 | -1 | 0 | 0 | -1 | 0 | -4 | 0 | 0 |
| Onto | -4 | -3 | -4 | 0 | -3 | 0 | -2 | 0 | -1 | 0 | 1 | 0 | -2 | -3 | 0 | -1 |
| Range | 0 | -2 | 1 | 0 | -2 | 0 | 2 | -1 | -1 | -2 | 1 | -4 | -3 | 1 | 0 | -1 |
| Sub-object | -1 | 0 | -1 | -1 | -8† | -3 | 1 | 0 | -1 | -3 | -1 | 0 | 0 | 0 | -6 | 0 |
| Surjection | -3 | -2 | -4 | 0 | -2 | -2 | -1 | -1 | 0 | -1 | 0 | 0 | -1 | -1 | 0 | -2 |

∗ = outlier among diagonal entries

† = outlier among non-diagonal entries

Using the cutoffs above shows some interesting patterns. Groups were discussed much more frequently than vector spaces, while group homomorphisms and linear transformations showed the exact opposite pattern. It was already clear from the qualitative data that students were less sure about vector spaces than groups as far as their respective algebraic structures are concerned. The difficulty that students had in connecting these two objects is discussed later in the section on general themes. Linear transformations received a small boost in frequency from being discussed with column space, null space, and matrices, which are not seen in this matrix. However, this does not account for the full discrepancy between domains. From the data collected, it is not completely clear why this difference is present. The final item along the diagonal, isomorphism, has a difference more clearly backed by the qualitative data. As mentioned earlier in Section 3.1, students did not defer to isomorphisms as sameness as one of their main examples for their concept images for linear transformations, which stood in contrast to their more regular use of isomorphism imagery when speaking of group homomorphisms. This lack of prominence of isomorphisms in the linear transformation setting is also clear from Table 3.6. The most likely explanation for this is that isomorphisms, while often being an important piece of scaffolding in learning group homomorphisms, are almost an afterthought in linear algebra. While there is not much more to say in terms of the data that has not already been said in Section 3.1, I will return to the implications of this difference in Section 4.2.

Among the non-diagonal entries, students' use of isomorphism with both objects and morphisms was heavily skewed towards groups and group homomorphisms. This is expected given my remarks above. Sub-objects barely missed the cutoff for card usage, but the lower usage of subspaces, combined with the already discussed low usage of vector spaces, also correlated to significantly more students discussing the connection between subgroups and groups than subspaces and vector spaces. This reflects the focus of study on subgroups as identifiers of group structure in abstract algebra, which is in contrast to the goals of teaching subspaces in linear algebra. In linear algebra, subspaces are unnecessary for discerning the structure of a given vector space from other vector spaces, due to all real vector spaces of a given dimension being isomorphic. The last remaining significant entries, kernel/image and morphism/inverse morphism are mysterious even upon reviewing the transcripts or reflecting on the nature of abstract and linear algebra instruction. It would be reasonable to assume that the latter occurs more frequently due to students having experience with inverse matrices, but this hypothesis does not play out upon inspection of the data. A similar hypothesis could be made about kernels and images being grouped together more frequently in reference to null spaces and column spaces, but this too is not reflected in the data.

### General Themes

When reviewing the above qualitative findings and summaries of the open card sorts, a few overarching themes become evident. I have attempted to comment on similarities shared by students across cases. While some of these are shared by just two or three students, others extend across the study to an extent that they deserve their own mention here. These findings suggest that some of the largest barriers to forming mathematically correct analogies across domains are students' understandings of matrix and vector space.

#### *The Role of Matrix*

It is clear from Table 3.3 and the above accounts that the relationship between matrices and linear transformations remains unclear to many students. Matrices were not mentioned in the definition of linear transformation given to students. It is also the only term in the card sort that (for these students) does not have a shared meaning across domains or a parallel in abstract algebra. Because matrices are at the heart of undergraduate linear algebra, students remember many of their uses and properties. However, it is this same overabundance of concepts tied to matrices that presents students difficulty in remembering the relationship between matrices and linear transformations.

For a few students, it was unclear what elements serve as the input of a linear transformation. The symbols in the definition of linear transformation, $\mathbf{x}$ and $\mathbf{y}$, are said to be from a vector space $V$. As will be discussed later, the concept of vector space also proved problematic for students. Thus, students had to determine what sort of mathematical objects $\mathbf{x}$ and $\mathbf{y}$ must be. Sander mentioned that a linear transformation is a function with a matrix as the variable, though this statement conflicted with some of Sander's other notions of matrix and linear transformation. Jake debated whether or not to include matrices in addition to vectors as the inputs of linear transformations. In Jake's case it was clear that the confusion arose because a matrix is composed of vectors, and because of matrix multiplication serving as a potential parallel to group element multiplication. Robyn's analogies placed matrix in alignment with group, possibly due to matrices and groups both being the main foci of their respective classes. Later, it was seen that Robyn also paralleled matrix with group element.

Some of these parallels are of course not without their merits. Rather, they can form the building blocks for understanding matrix groups and matrix rings. Or, other parts of these students' statements could be interpreted as correct when discussing a vector space of $n \times m$ matrices. However, in the context of these interviews, where students were most often recalling properties of vector spaces with vectors of real-numbered entries, these aspects of matrices only served as problematic met-befores leading to troublesome cross mappings.

Another less troublesome, but still prevalent, evoked concept image of matrix was that of the augmented matrix. Though the term was never stated outright by students, it was alluded to in both words and pictures, with Robyn's drawing in Figure 3.7 being the clearest example. Emilie, Flint, and Robyn all referred to the horizontal lines of an augmented matrix as functions. Each horizontal line represents one equation in a system of equations. Since each has inputs such as $x$, $y$, and $z$, as well as an equals sign followed by another number, each is a function. All three grouped matrix with domain and range not because of linear transformations and vector spaces, but because of the idea of input and output of systems of equations (where a problem-solver is finding the input, given an output). This most likely arises because systems of equations are the foundational starting topic for many undergraduate linear algebra classes. The purpose of this common starting point is that the familiar process of solving systems can serve as motivation for the matrix equation $A\mathbf{v} = \mathbf{b}$. Such a notation, used regularly throughout a linear algebra course, should suggest thinking of "input" and "output" as vectors, each a singular gestalt rather than three individual real numbers. Yet, these students relied more heavily on the notation and imagery of systems of equation, probably due to the abundance of Gaussian elimination students continue to use to solve linear algebra problems, even when new notation and concepts are introduced. Thus, the concept of an input vector from a vector space, such as $\mathbf{v} = (x, y, z)^T$ in $\mathbb{R}^3$, is obscured when these students reason through analogies using augmented matrices as their primary evoked concept image. Function, domain, and range are tied to systems of equations, rather than linear transformations. This in turn makes the analogy between linear transformations/vector spaces and group homomorphisms/groups as a morphism/object pair less immediate.

Arthur, Kyle, and Lucas were the only students to correctly describe the main relationship between linear transformations and vector spaces. Kyle was the only student whose primary evoked concept image of linear transformation was entwined with matrix; he described how he "sees" a matrix when he thinks of linear transformation. This was obviously very prevalent in Lucas's concept image as well, since his first linear transformation grouping was with matrix. Arthur was the only of these three to not describe the relationship until the closed card sort.

*Difficulty in Aligning Vector Space*

Continuing the above discussion of morphism/object pairs, one surprising aspect of this study was number of students who did not extend their analogy between morphisms to include objects. Both groups and vector spaces were identified as the domain and range on the definitions given to students unable to recall a particular morphism. Yet, few students made reference to this parallel in their interviews. Robyn and Sander were unable to describe any similarities between the two. Emilie, Flint, and Jake described both as sets of elements, containments, or the big picture. Tamara and Emilie revised their answers concerning similarity after being asked about subgroup and subspace. The common "sub-" naming scheme for both was a surface feature that led to more students grouping sub-objects rather than the objects themselves. Sander also made this grouping, but unlike Emilie and Tamara, he was unable to capitalize on it and infer a group/vector space parallel. This is in part due to Sander referring to "space" as the container for a subspace, a term he saw as different from vector space.

The above shows that most students forming new connections or struggling to make analogies had difficulties recalling the definition of vector space. The subset of these students who were able to successfully make mathematically correct analogies did so either based on the newly established link of morphisms, or of sub-objects. In either case, the connection drawn was mainly that of their common identity as sets. No mention was made of a vector space in fact being a type of group — a statement that would display an understanding of how to align algebraic structures based on their theoretical, mathematical definitions.

Compare this with some of the students with prior analogical reasoning. Arthur directly described a vector space as a specific example of a group, and identified the operation of addition as the group operation. Kyle identified this relationship as well. Also recall that Kyle's definition of linear transformation was in fact a morphism of modules. For Kyle, a vector space is an example of a module, and modules are examples of groups. Further classes in abstract algebra correlated to Kyle developing a hierarchical schema for algebraic structures and their morphisms. Maureen knew that a vector space could be described in terms of a group, and made various parallels between the structures, but always remained unsure of the exact details. Also recall that Lucas described a vector space as a specific example of a group, but like Maureen did not align the additive structures in the way that Arthur and Kyle did above.

Thus, simply seeing the definitions of linear transformation and group homomorphism, with their inclusion of the terms vector space and group, is often not enough for students to align algebraic structures as a whole. Students need to have already considered the similarities of the algebraic structures over a longer period of time. Part of this struggle to align algebraic objects may stem from the way in which vector spaces must be defined in a first undergraduate course. In a more typical sequence, students of linear algebra have not seen groups, so it would not make pedagogical sense to define a vector space as an abelian group with the action of a field. However, this results in a deluge of addition and scalar multiplication properties. Given that students had difficulty recalling the two properties of a linear transformation, it is unlikely that students would be able to list all the properties of a vector space. This, in turn, makes it difficult for students to place the additive structures in structural alignment.

### 3.4 Comparison with Professors

Preservation of structure was by far the most used theme for both linear transformations and group homomorphisms, being used by all three professors for both morphisms. Dr. Brady, Dr. Greer, and Dr. Powell all used language similar Kyle and his idea of what kind of function one would "want" for the given setting. The rationale for the conditions of the morphism was the structure of a group or a vector space. Asked how she would describe a linear transformation to a student, Dr. Powell said,

> [I would explain it] almost the same way I would think of group homomorphisms. I would assume that if the conversation came up, then I'd have reason to believe they were already familiar with the objects themselves, either the vector spaces or the groups. And, if not, that would be the first step is to understand what the ingredients are. But once you understand what the ob— what the domain and codomain spaces are, then I would use the same intuition in relating it back to what they know about functions and saying that instead of just having the name for any function, we give this name to those that preserve the operations that we care about in the objects.

Less prevalent was the notion of comparing groups or vector spaces. Only Dr. Brady explicitly mentioned this idea. However, Dr. Brady heavily relied on this language, and gave specific examples of permutations of roots of polynomials or changing basis to an eigenbasis as times when morphisms are used to transfer structure in a way that helps to understand a mathematical phenomenon. Isomorphism as sameness and geometric reasoning were also not as prevalent, with only Dr. Powell specifically utilizing such language. These differences are almost definitely an example of evoked concept image over concept image as an abstract ideal. Given a sample of ten professors, it is likely more than one would exhibit the same themes in the context of the study. However, it remains that the themes expressed by students in this study are indeed present in the language of even a few professors.

Also interesting is these professors' comments on their explanations of linear transformation and group homomorphism. All three professors consistently expressed that the way they would describe each concept to a student would depend on the students themselves. A student with substantial prior knowledge and experience hypothetically could be engaged more directly in a conversation about preservation of structure. Meanwhile, professors noted that whether they were teaching group homomorphisms or linear transformations for the first time, or explaining them to an individual student early on in the undergraduate curriculum, they were much more likely to use examples. For group homomorphisms, professors saw the definition as too “abstract” or “not illuminating” enough on its own. Rather, students should get a better understanding of homomorphisms through repeated examples. For linear transformations, students should have repeated exposure to matrices. For both concepts, they should consider familiar examples from the real numbers and integers, such as the squaring function, or multiplication by a constant. Thus, for these professors, the concept image of preservation of structure should be initially born out of repeatedly seeing structure successfully preserved or destroyed in tangible situations, rather than abstract symbolic contemplation.

For the card sorts and connecting concepts related to the given morphisms, a few key themes and differences presented themselves. First, all three professors had a markedly different reaction to the card sorts than most students. Whereas some students were not sure what connections to make beyond the first few groupings, all three professors expressed a sense of being overwhelmed by the potential number of connections between the cards. This reaction, and the data from their adjacency matrices, show the deeply interconnected schema of individuals who have spent years making sense of the concepts at the heart of the study.

A small, but interesting, difference between students and professors was their descriptions of the role of inverse morphisms. Many students never recalled seeing inverse homomorphism or inverse transformation. Only Kyle and Lucas mentioned the relationship between an isomorphism and the presence of an inverse morphism. Yet at the same time, every professor was sure to mention either this relationship (Brady and Greer), or an inverse's relationship with injective morphisms (Powell). Dr. Brady went as far as to state that the presence of an inverse homomorphism is one of the most important things about a group isomorphism. For a few students, this was missed perhaps due to their thinking of isomorphisms as sameness as being a literal relabeling of a group (e.g. Sander's concept image of apples and oranges).

The idea of an isomorphism as a function is downplayed when thinking of one group being relabeled. Many students (and professors) in this study referred to functions "sending" elements from one thing to another. But, if one imagines an isomorphism as taking one group and changing the labels, there is no aspect of "sending" and less focus on two structures with elements. It would then make sense that inverse homomorphism would not play a prominent role in students' evoked isomorphism concept images. Of course, this difference between students and professors could also just as likely be due to a mismatch in what these professors identify as important and what was emphasized in the students' curricula.

Another similar and even more important difference was the role of matrix for professors. Of course, it was expected that the professors would discuss the connection between linear transformation and matrix. Yet even with this expectation, the difference is still striking. For all three, the idea that a matrix can represent any given linear transformation, and that every matrix yields a linear transformation, was one of the defining highlights of undergraduate linear algebra. Essentially, with a choice of basis, matrices and linear transformations are the same entity. Part of the reason for the study of matrices, and the emphasis on their various related computations, is to be able to understand linear transformations. For example the null space, column space, and determinant of a matrix all inform one about the nature of its related linear transformation. That this relationship between matrix and linear transformation would be described by professors as a keystone of the course, but be so overlooked by students, displays a substantial mismatch in expectations versus the reality of what students remember at the end of a linear algebra course. This disconnect was not lost on the professors. Dr. Brady described the transition from systems of equations, to augmented matrices, to matrices and vectors as steps of understanding which progressively cause more students to have difficulty in linear algebra. Dr. Powell expressed how students can complete matrix computations without actually knowing what they are computing. However, it is hard to conceive of these professors imagining the unique ways in which students attempted to integrate the concept of matrix into a budding analogy between linear transformations and group homomorphisms.

# Chapter 4 Discussion

## 4.1 Summary and Conclusions

Having examined the many and varied responses of both students and professors, I now return to my original research questions. In this section, I will summarize the results of the study as they pertain to these guiding questions.

*(1) How are students' understandings of group homomorphisms and linear transformations the same or different?*

As expected, the span of time between students' linear algebra class and the interviews for this study had an impact on students' ability to recall linear transformations. However, this was not the only disadvantage students faced when attempting to recall linear transformations. The term "group homomorphism" contains both the name of the objects (groups) and a word very similar to "isomorphism." This gives two potential routes to recalling the concept of group homomorphism. Meanwhile, the presence of "linear" and "transformation" each led to recall of problematic prior knowledge. "Linear" caused two students to guess that a linear transformation concerns lines, while "transformation" triggered thoughts of transformations of graphs.

Whether it be time passed or the physical structures of the terms, it remains that students had a slightly harder time recalling linear transformations than they did recalling group homomorphisms. Six students recalled some function containing the homomorphism property, versus three of those same students that recalled a definition relating to linear transformations — though two of the six did not recall group homomorphisms proper. Before the reveal of the linear transformation definition, significantly fewer students recalled that linear transformations were functions, or that they were related to vector spaces. Even after the reveal of the linear transformation definition, some students continued to talk about linear transformations as if there were no function involved. The presence of vector addition and scalar multiplication triggered certain geometric or algebraic images, but these notions of addition and scalar multiplication were sometimes described in a way devoid of the actual linear transformation, $T$.

The theme of preserving structure remained constant across domains, largely with the same subset of students. As structure preservation is not a property often discussed in classes before linear algebra and abstract algebra, it is significant to see this theme appear in both settings. On the whole, students who made use of such language or notions did not provide many examples of what structure is preserved. In the linear transformation setting, this structure preservation was sometimes intertwined with the geometric notion of preserving distances. The language of comparing respective structures was used more heavily in the group homomorphism setting. This can probably partially be attributed to the fact that the emphasis of most linear algebra classes is on linear transformations of $\mathbb{R}^m$ to $\mathbb{R}^n$ for some $m$ and $n$. Without more foreign vector spaces such as polynomial spaces or matrix spaces for students to consider over a long period of time, the notion of using linear transformations to compare vector spaces would seem rather silly. A student would not need linear transformations to compare the usual examples of $\mathbb{R}^2$ and $\mathbb{R}^3$; they already have an understanding of these spaces without linear transformations.

Isomorphisms played a larger role in students' understandings of group homomorphisms than their understandings of linear transformations. As is clear from Table 3.6, isomorphisms were discussed in tandem with groups and group homomorphisms much more than they were with their respective parallels in linear algebra. Additionally, isomorphisms provided an immediate example for students to talk about how a group homomorphism preserves or compares structures: isomorphisms are a type of homomorphism in which the two groups being related are in fact "the same."

This theme was completely absent from students' evoked concept images of linear transformations.

*(2) What connections do students make between the concepts related to group homomorphisms and linear transformations?*

The quantity and quality of connections varied greatly from student to student. Whether or not students made connections before or during the interviews also varied. Even the mathematical correctness and descriptive detail of connections varied within the categorizations of *Students with Substantial Prior Analogical Reasoning* and *Students Primarily Engaged in New Analogical Reasoning*. Students' propensity for engaging in structural alignment and analogy construction could sometimes be explained in terms of the strength of their connections within target and base domains, but this was not universally true.

Some students had already reached the conclusion that linear transformations are special group homomorphisms with an extra condition. One of these students reached this conclusion because the interview promoted their structural alignment. However, most students did not state the connection so directly. While only a few students found the connection between the two morphisms helpful for initial recall, most students saw the similarities between the group homomorphism condition and the first condition for linear transformations almost immediately after seeing the linear transformation definition.

Most students, regardless of their categorization in this study, saw domain, range/image, onto/surjection, and one-to-one/onto as properties of functions which they would have recognized before taking linear algebra or abstract algebra. They attested to these properties as still holding in these later settings for linear transformations and group homomorphisms. However, some students could not recall which definition of surjection and injection belonged to which term. The vast majority of students did not connect image/range to column space, and most were not able to correctly describe the relationship between kernel and null space. Inverse homomorphisms and inverse transformations seemed foreign to most students, though they were able to piece together their meaning through the presence of the word “inverse” and correctly conjecture on their relatedness.

One important finding of this study is that there are students who make connections between various morphism-related concepts across linear algebra and abstract algebra without an explicit component for this instruction in their curriculum. Two of these students were taking the classes simultaneously, while the other had revisited the concepts multiple times through both undergraduate and graduate classes. The strengths of these connections varied, from vague notions of relatedness, to a fully developed singular morphism schema covering an array of algebraic structures. Students in this category found their connections useful in different ways, including influencing the formation of a new connection during the interview, helping a student to contextualize their linear algebra class, and helping another student to see mathematics as a connected subject.

Students who began engaging in analogical reasoning as a result of the design of the study made both mathematically correct and mathematically incorrect connections across domains. Vector spaces proved problematic in the analogy-making process for a certain subset of students. There is some evidence that the structures of the words “subspace” and “subgroup,” both containing the “sub-” prefix, served as surface-level features which led to students placing the sub-objects into in structural alignment. A few students attempted to use this analogy to infer that vector spaces and groups are parallel concepts, but, on the whole, this was met with varying levels of success. Even among those for whom groups and vector spaces were placed in structural alignment, there remained students that only related the two as sets. The recognition of the group homomorphism property in the linear transformation definition was not enough for some students to see the group structure as present within vector spaces.

Matrices also presented a challenge for some students' structural alignments. Many students in this study did not recall the relationship between matrices and linear transformations once they had passed linear algebra, or even while they were enrolled. Some students placed matrices as parallel to group elements. Another student connected matrices to groups. Instead of yielding a function through multiplication by a matrix, a subset of students saw matrices as containing functions in each row of the matrix.

*(3) How do students' understandings of these morphisms compare to those of professors?*

Professors' connections and understandings were similar to those of students with substantial prior analogical reasoning. Overall, both of these two groups had very interconnected morphism schema, and connected the items listed in Table 3.3, showing that it was reasonable to expect each of the connections in Table 3.3 to appear in the context of a clinical interview, supposing a certain level of experience with the material. This of course includes the connection between groups and vector spaces, with which some students had particular trouble.

The themes derived from students' responses also appeared in the professors' responses. This supports the notion that these aspects of students' concept images are fairly normal in the larger mathematical community. Students' language is likely a reflection of that of their professors' and books' language. However, every professor did not express every theme.

Inverse homomorphisms and inverse transformations were almost universally called out by students as concepts which they had not heard of specifically. In contrast, professors universally identified the relationship between isomorphisms (or injective morphisms) and inverse morphisms.

The relationship between matrices and linear transformations was much more prominent for professors. Professors saw this connection as one of the key aspects of an undergraduate linear algebra class. Such an emphasis on its importance serves to highlight the significance of the fact that many students in this study did not or could not verbalize the correct link between linear transformations and their matrices.

## 4.2 Implications

The students in this study used isomorphisms as a reference point for understanding group homomorphisms more heavily than for understanding linear transformations. In retrospect, such a finding is expected, as many textbooks use isomorphisms as a gateway to group homomorphisms, and isomorphisms are greatly downplayed in linear algebra. Yet it is still important to discuss this issue from the viewpoint of helping students make connections across the curriculum. Presenting group homomorphisms as group isomorphisms with relaxed requirements shapes students' concept images in a different way than how presenting linear transformations as heavily geometric entities does. A stronger emphasis on isomorphisms as sameness in linear algebra could encourage structural alignment and strengthen the link between students' understandings of morphisms. The presumable appeal of isomorphisms as sameness is that it is easier to grasp this intuitive notion of isomorphism than to dive into the nuts and bolts of a functional understanding. A similar appeal to intuition could be made geometrically in linear algebra. Lines in $\mathbb{R}^2$ and $\mathbb{R}^3$ "look like" a copy of $\mathbb{R}^1$. Planes in $\mathbb{R}^3$ "look like" $\mathbb{R}^2$. Using this intuition as a stepping stone to introducing the formal definition of isomorphism in linear algebra would serve as a contextual parallel to its introduction in abstract algebra. Alternatively, these same examples could be used or revisited early in an abstract algebra class as examples of isomorphisms of groups.

Change of notation across domains was not an issue for students regarding making the connection between morphisms. It was not difficult for students to see that the group homomorphism property is contained within the requirements for a linear transformation. However, the fact that many of these same students either did not continue to see parallels across domains or constructed mathematically incorrect parallels means that educators should not consider the alignment of properties to be evidence for alignment of algebraic structures. Educators who wish to encourage connection making will need to devote instructional time to more than a simple example or exercise highlighting this definitional similarity.

One crucial connection which students need assistance discovering and articulating is the connection between groups and vector spaces. What could be considered the most "obvious" parallel to make after connecting linear transformations and group homomorphisms is in fact not so obvious. This is probably partially due to a much heavier emphasis on $\mathbb{R}^n$ than other real vector spaces in the undergraduate linear algebra curriculum. One of the hopes of this study was to find that students in abstract algebra would retroactively gain an understanding of abstract concepts such as vector spaces. This was not the case for the majority of the students in this study. Thus, students of abstract algebra would benefit from a more direct discussion of the relationship between vector spaces and groups. As the axiomatic definition for a vector space in undergraduate linear algebra is rather lengthy, students' ability to recall the definition would likely be improved by condensing some of the requirements into a single requirement that a vector space be an abelian group under vector addition. Students more readily connected subgroup and subspace in terms of intuition formed from their concept images. Thus, following a direct instruction of the objects' connections with a more student-driven exercise in aligning the sub-objects' mathematical requirements hopefully would prove fruitful.

Finally, many students in this study did not or could not articulate the relationship between matrices and linear transformations. If one were to hope for students to align linear transformations and group homomorphisms without the aid of directly being reminded of their definitions, this presents a large problem. Students of linear algebra spend a considerable amount of time working with matrices. In a typical linear algebra classroom, null spaces/kernels, column spaces/images, surjectivity, and injectivity are often discussed or computed using matrices. If a student cannot recall the connection between matrices and linear transformations, a significant piece of their linear algebra morphism concept image is excluded. Based on the findings of this study, care should be taken to emphasize emphasize linear transformations in a way that prevents students from glossing over the term and proceeding directly to matrix calculations. In the setting of abstract algebra, routing students' recall of linear transformations first through matrices would potentially help to mitigate some of the difficulties found in this study.

The subset of students who visualize matrices as being composed of functions, each stemming from a corresponding row of the matrix, also highlight an important conceptual shift that must occur when learning of linear transformations. Functions in undergraduate calculus most regularly return a single real number, i.e., their codomain is $\mathbb{R}$. When encountering matrices representing systems of equations, it seems that some students continue to focus on individual outputs, so that an augmented matrix (and eventually a linear transformation derived from a matrix) is primarily coordinating functions with single real number outputs. Such a view sidesteps the need for using the language of vectors, vector spaces, and linear transformations. This is another reason that revisiting these structures in abstract algebra, comparing and contrasting them to groups and group homomorphisms, could prove beneficial to students' understanding of linear transformations. The current state of matters shows that for some of these students, matrix computations and systems of equations are potential barriers to allowing such retroactive benefits to take hold.

This study shows that it is not unreasonable to expect students in abstract algebra to make analogies to structures from linear algebra. A few students at various points in their mathematical journeys were able to express these connections quite well. However, the majority of students' responses in this study suggest that leaving the responsibility of seeing connections completely up to students would be a mistake. It also suggests that simply helping students to align one set of concepts, such as linear transformations and group homomorphisms, is not always enough to trigger further connections between other, closely related concepts. If students are not readily using concept images and definitions from linear algebra in abstract algebra, this would lead one to question why linear algebra is required as a prerequisite for abstract algebra at some institutions, as is recommended by the 2015 CUPM Curriculum Guide's course report on abstract algebra (Isaacs, Bahls, Judson, Pollatsek, & White, 2015).

Yet just because many students do not make these connections on their own does not mean that I am implying we should abandon the effort to have students engage in their creation. Instead, I encourage educators to continue looking into ways to facilitate students' deeper understanding across the undergraduate curriculum. We need to continue making efforts to improve students' understandings of how matrices are intricately tied to linear transformations of vector spaces. This will lay the groundwork for future structural understanding. In abstract algebra, students are ready to begin linking concepts such as linear transformations and group homomorphisms, despite surface level notational differences. However, care should be taken to scaffold such connection making. In the end, there is evidence in this study suggesting that taking time to encourage this analogical reasoning can lead not only to new individual insights, but a deeper appreciation for the overall connectedness of mathematics.

## 4.3 Limitations and Recommendations for Future Research

Due to logistical limitations, I only was able to interview ten students in a clinical interview setting for the main results of this thesis. I did not have access to students' lectures, notes, quizzes, conversations with other students or professors, etc. It is important to realize that students' understanding of mathematics is constructed at both the individual and social levels (e.g. Cobb & Yackel, 1996; Rasmussen, Wawro, & Zandieh, 2014). Certain evoked concept images of morphisms, which could have surfaced in a more naturalistic classroom environment, were not captured by this restricted form of data collection. The students (or even professors) in this study may have been able to recall or connect more concepts with the support of a peer. And, despite my best attempts to place interviewees at ease, a few students I interviewed were visibly nervous during the interview process. Some converted this nervous energy into words and actions that kept their thoughts flowing throughout the interview. Others became a bit more reticent. It was beyond the scope of this particular study to extend beyond the level of individual and add a social component, but I hope that future studies integrate a social learning component in order to create a richer data set and mitigate students' anxieties. Observing students in a classroom setting or using a paired interview setting could provide data on concept images and analogies beyond what I found with clinical interviews with individual students.

Following students through their linear algebra and algebra classes in addition to interviews would provide a richer data set. This would also give insight into differences in instruction in abstract and linear algebra that lead to some of the difficulties students faced during this study. However, such a study would require specific circumstances. The linear algebra classroom(s) observed would need to contain a respectable number of students whom the researcher knows will be taking abstract algebra together in a later semester. Such a study would likely yield new themes and insights into the driving questions of this research.

An alternative route for future research would be examining the efficacy of mini-lessons in an abstract algebra class which directly ask students to make connections between related concepts. As discussed above, this study indicates that some students need more reminders and scaffolding than the minimal amount provided in the interviews. The results of this study also suggest that care should be taken to more explicitly outline the connection between vector spaces and groups. Providing students with access to definitions of the other terms in this study would help with recall. Given this scaffolding, it would be interesting to know if having students engage in analogy creation in a collaborative setting would yield much richer connections for a greater proportion of students. This could be extended to observing the impact of such an intervention on students' long term understanding of linear algebra concepts. Data could be collected at the start of the semester, at the end of the semester, and in a following semester to gauge how students who are actively asked to link these concepts recall said concepts over time.

### Appendix A Card Sort Interview Protocol

# Card Sort Interview Protocol (Semi-Structured)

## Before Starting

- *Camera check*
- *Mic check*
- *Regular marker*
- *Card sets E,A,B,C*
- *Blank cards*
- *Blank paper*
- *Consent form*

## Basic Information

Interviewee:

Time:

Place:

Status in Program:

Majors/Minors/Field of Study:

Names of Classes: /

Self-Reports of Linear Algebra/Modern Algebra Grades: /

## Introduction

- *Introduce self*
- *Purpose of overall study*
- *Consent form*
- *Ask if any questions before start*

## Purpose of Today's Interview

- Today we will talk about some concepts from abstract algebra and linear algebra. The purpose of today's interview is to see how you think about these concepts. We will not be solving any problems today.

## Definition of Group Homomorphism

- *Place a marker and paper in front of student.*

1. First, I would like you to write out the definition of group homomorphism, as best you can remember it.
    - *Clarify any unclear notation.*
    - *If student cannot remember definition at all or partially after 3 or 4 minutes, provide definition from book.*
2. How would you describe a group homomorphism to someone else? How would you help them think about it intuitively?
3. Why would mathematicians define group homomorphisms as they do?

## Card Sort Activity Introduction

### Purpose of Card Sort

- Today I will ask you to do something called a "card sort."
- (If student: This is not an exam. This will not have an impact on any of your grades.)
- There are no right or wrong answers in this activity, as it is meant to convey what you believe.
- This will give me some insight into how you understand and connect certain mathematical concepts.

### Card Sort Example

- *Place example (E) cards in front of interviewee. These cards are (in $2 \times 5$ grid):*

    1. *Derivative*
    2. *Limit*
    3. *Difference Quotient*
    4. *Rate of Change*
    5. *Function*
    6. *Slope*

7. *Variable*
8. *Continuous*
9. *Secant Line*
10. *Tangent Line*

- In front of you is a small example of a card sort. It contains cards with concepts relating to derivatives in Calculus I.
- In this activity, I will ask you to select groups of cards that you believe are related. Say I was asked to select a group from this array. Perhaps I would select (*pick up cards*) Derivative, Rate of Change, and Variable. When asked how these are related, I might say that the derivative calculates the instantaneous rate of change between an independent variable $x$ and a dependent variable $y$.
- When asked to select another group, I may pick Slope, Rate of Change, Function, Secant Line, and Tangent Line. Notice that I am reusing Rate of Change, which is fine. When asked how these are related, I might say that the slope of the secant and tangent lines to a function are measures of the rate of change of that function. I could clarify that the slope of a secant line gives an average rate of change, while the slope of a tangent line gives instantaneous rate of change.
- So there was an example of two rounds of a card sort. Do you have any questions?
- I'll get out our first set of cards and you can feel free to ask questions as we go along.

## Group Homomorphisms Card Sort

- The grid of cards I am about to place in front of you contains topics from abstract algebra.
- *Place group homomorphism (A) cards, blank cards, and marker in front of interviewee. These cards are (in* $4 \times 4$ *grid):*

1. *Group Homomorphism*
2. *Isomorphism*
3. *Injection*
4. *Subgroup*
5. *Kernel*
6. *Generating set*
7. *Codomain*
8. *Inverse homomorphism*
9. *Range*
10. *Group*

11. *Function*
12. *Image*
13. *Domain*
14. *Onto*
15. *Surjection*
16. *One-to-one*

- Please select a group of two or more cards that you believe are related.

  Additionally, if there is a concept you believe would be useful to include, you are welcome to create your own cards, add them to the grid, and use them in groupings.

- *Follow-up questions once group is selected:*
  - How do you believe these concepts are connected?
  - Would you like to add any other cards from the grid to this group?

- *When student is done explaining answers to all questions:* Please return the cards to the grid and select another group of cards you believe are related.

- *Repeat group selection and follow-ups until it appears interviewee cannot make any more groups.*

- Are you unable to create any more groupings? Would you like to move on to the next activity?

## Definition of Linear Transformation

- *Place a marker and blank paper in front of student.*

4. First, I would like you to write out the definition of linear transformation, as best you can remember it.
   - *Clarify any unclear notation.*
   - *If student cannot remember definition at all or partially after 3 or 4 minutes, provide definition from book.*

5. Did thinking about group homomorphisms help you recall the definition of linear transformation?

6. How would you describe a linear transformation to someone else? How would you help them think about it intuitively?

7. Why would mathematicians define linear transformations as they do?

8. How is the definition of linear transformation related to the definition of group homomorphism, if at all?

## Linear Transformations Card Sort

- The grid of cards I am about to place in front of you contains topics from linear algebra.
- *Place linear transformation (B) cards in front of interviewee. These cards are (in* $4 \times 5$ *grid):*

  1. *Linear transformation*
  2. *Kernel*
  3. *Surjection*
  4. *Onto*
  5. *Inverse transformation*
  6. *Null space*
  7. *Matrix*
  8. *Codomain*
  9. *Image*
  10. *One-to-one*
  11. *Domain*
  12. *Range*
  13. *Basis*
  14. *Isomorphism*
  15. *Vector space*
  16. *Function*
  17. *Subspace*
  18. *Injection*
  19. *Column space*

- You'll notice many of the same cards, but remember that now we are thinking of these concepts in terms of a linear algebra class. I will present you with any groupings from the previous set which can be repeated in this set and ask you if this grouping still applies in the context of linear algebra. If a previous grouping contains at least one card which is not on the current grid, then I will not present it to you.

  *Present any previous overlapping groupings one at a time and ask:* Do you believe this grouping still applies in the context of linear algebra?

- Now, once again, please select a group of two or more cards that you believe are related. These may be similar to groupings from the previous activity, or completely different.

  Additionally, if there is a concept you believe would be useful to include, you are welcome to create your own cards, add them to the grid, and use them in groupings.

- *Follow-up questions once group is selected:*
  - How do you believe these concepts are connected?
  - Would you like to add any other cards from the grid to this group?
- *When student is done explaining answers to all questions:* Please return the cards to the grid and select another group of cards you believe are related.
- *Repeat group selection and follow-ups until it appears interviewee cannot make any more groups.*
- Are you unable to create any more groupings? Would you like to move on to the next activity?

## Combined Card Sort

### Open Card Sort

- The grid of cards I am about to place in front of you contains all terms from the previous two grids.
- *Place combined (C) cards set, including interviewee-made cards, in front of interviewee. The pre-made cards are (in grid with 6 columns):*

  1. *Vector space*
  2. *Isomorphism*
  3. *Image*
  4. *Group*
  5. *Linear transformation*
  6. *Basis*
  7. *Subspace*
  8. *Surjection*
  9. *Function*
  10. *One-to-one*
  11. *Codomain*
  12. *Group Homomorphism*
  13. *Matrix*
  14. *Column space*
  15. *Kernel*
  16. *Null space*
  17. *Subgroup*

18. *Generating set*
19. *Range*
20. *Onto*
21. *Domain*
22. *Inverse homomorphism*
23. *Injection*
24. *Inverse transformation*

- I will present you with any groupings from the previous sets so that you do not need to repeat them.
- Now that all cards are together, please select a group of two or more cards that you believe are related. These may be similar to groupings from the previous activity, or completely different.
- Additionally, if there is a concept you believe would be useful to include, you are welcome to create your own cards, add them to the grid, and use them in groupings.
- *Follow-up questions once group is selected:*
    - How do you believe these concepts are connected?
    - Would you like to add any other cards from the grid to this group?
- *When student is done explaining answers to all questions:* Please return the cards to the grid and select another group of cards you believe are related.
- *Repeat group selection and follow-ups until it appears interviewee cannot make any more groups.*
- Are you unable to create any more groupings?

### Closed Card Sort

- Now I will select some cards and ask if you believe the cards are related. Don't assume that the cards are necessarily related. If you do not believe they are related, you may state so.
- *Present the following pairs to students:*
    - Group/vector space
    - Basis/generating set
    - Group/group homomorphism
    - Inverse homomorphism/inverse transformation
    - Subgroup/subspace
    - Vector space/linear transformation

- Linear transformation/matrix

## Combined Follow-Up Questions

- Finally, I will ask you a few questions about group homomorphisms and linear transformations.

9. What concepts would you say are most critical to understanding group homomorphisms?

10. What concepts would you say are most critical to understanding linear transformations?

(For students:)

11. Reflecting on the activities today, are there any concepts surrounding linear transformations that were not clear the first time, but are now clarified after learning about group homomorphisms?

    - Could you explain how learning about group homomorphism concepts clarified that concept for you?

12. Are there any concepts surrounding linear transformations that you find less clear after learning about group homomorphisms?

    - Could you explain how learning about group homomorphism concepts affected your understanding of that concept?

13. Are there any concepts surrounding linear transformations that helped you understand any group homomorphism concepts?

    - Could you explain how those linear transformation concepts helped you to understand the group homomorphism concepts?

14. Are there any concepts surrounding linear transformations that got in the way of your understanding of any group homomorphism concepts?

    - Could you explain how those linear transformation concepts got in the way of understanding the group homomorphism concepts?

(For experts:)

11. Reflecting on the concepts on your maps, are there any concepts surrounding linear transformations that may not be clear the first time for students, but are clarified after learning about group homomorphisms?

    - Could you explain how?

12. Are there any concepts surrounding linear transformations that students may find less clear after learning about group homomorphisms?
    - Could you explain how?
13. Are there any concepts surrounding linear transformations that may help students understand any group homomorphism concepts?
    - Could you explain how?
14. Are there any concepts surrounding linear transformations that may get in the way of students understanding of any group homomorphism concepts?
    - Could you explain how?

(For all:)

15. Any other comments about the connections between concept of group homomorphism and the concept of linear transformation?
16. How well do you believe the activities today reflect your knowledge of concepts related to group homomorphisms (if expert: as it pertains to an undergraduate abstract algebra course)?
17. What part of your knowledge of group homomorphisms (if expert: as it pertains to an undergraduate abstract algebra course) is not reflected in the activities today?
18. How well do you believe the activities today reflect your knowledge of concepts related to linear transformations (if expert: as it pertains to an undergraduate linear algebra course)?
19. What part of your knowledge of linear transformations (if expert: as it pertains to an undergraduate linear algebra course) is not reflected in the activities today?

- Anything else you would like to add today?

## Before Leaving

- *Review date and time for next meeting.*
- *Thank them for their time!*

## Appendix B Provided Definitions of Morphisms

### Group Homomorphism

Definition: A map $f$ of a group $G$ into a group $H$ is a homomorphism if the property

$$f(ab) = f(a)f(b)$$

holds for all $a$, $b$ in $G$.

## Linear Transformation

Definition: A transformation (or map) $T$ of a vector space $V$ into a vector space $W$ is a linear transformation if the properties

$$T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$$

and

$$T(c\mathbf{x}) = cT(\mathbf{x})$$

hold for all $\mathbf{x}$, $\mathbf{y}$ in $V$ and scalars $c$.

## Appendix C Institutional Review Board Exemption Approval Letter

University of Kentucky | Office of Research Integrity
IRB, RDRC

EXEMPTION CERTIFICATION

IRB Number: 42449

| | |
|---|---|
| TO: | Jeffrey Slye, PhD Student<br>Mathematics<br>PI phone #: 7174602100<br>PI email: jeffrey.slye@uky.edu |
| FROM: | Chairperson/Vice Chairperson<br>Non Medical Institutional Review Board (IRB) |
| SUBJECT: | Approval for Exemption Certification |
| DATE: | 8/2/2017 |

On 8/1/2017, it was determined that your project entitled "*Student Transfer Between Homomorphisms in Abstract Algebra and Linear Transformations in Linear Algebra*" meets federal criteria to qualify as an exempt study.

Because the study has been certified as exempt, you will not be required to complete continuation or final review reports. However, it is your responsibility to notify the IRB prior to making any changes to the study. Please note that changes made to an exempt protocol may disqualify it from exempt status and may require an expedited or full review.

The Office of Research Integrity will hold your exemption application for six years. Before the end of the sixth year, you will be notified that your file will be closed and the application destroyed. If your project is still ongoing, you will need to contact the Office of Research Integrity upon receipt of that letter and follow the instructions for completing a new exemption application. It is, therefore, important that you keep your address current with the Office of Research Integrity.

For information describing investigator responsibilities after obtaining IRB approval, download and read the document "PI Guidance to Responsibilities, Qualifications, Records and Documentation of Human Subjects Research" from the Office of Research Integrity's Guidance and Policy Documents web page. Additional information regarding IRB review, federal regulations, and institutional policies may be found through ORI's web site. If you have questions, need additional information, or would like a paper copy of the above mentioned document, contact the Office of Research Integrity at 859-257-9428.

# Appendix D Consent Form

Consent to Participate in a Research Study

**STUDENT TRANSFER BETWEEN HOMOMORPHISMS IN ABSTRACT ALGEBRA AND LINEAR TRANSFORMATIONS IN LINEAR ALGEBRA**

**WHY ARE YOU BEING INVITED TO TAKE PART IN THIS RESEARCH?**

You are being invited to take part in a research study about student connections between maps in linear algebra and abstract algebra. You are being invited to take part in this research study because (1) you were previously or are currently enrolled in a linear algebra course and (2) you were previously or are currently enrolled in an abstract algebra course. If you volunteer to take part in this study, you will be one of about fifteen students and four professors to do so.

**WHO IS DOING THE STUDY?**

The person in charge of this study is Jeffrey Slye, a Ph.D. candidate of the University of Kentucky Department of Mathematics. He is being guided in this research by David Royster, Ph.D., of the University of Kentucky Department of Mathematics.

**WHAT IS THE PURPOSE OF THIS STUDY?**

By doing this study, we hope to learn about what connections, if any, students make between certain concepts in linear algebra and in abstract algebra.

**ARE THERE REASONS WHY YOU SHOULD NOT TAKE PART IN THIS STUDY?**

Participants should be at least 18 years in age. Otherwise, there are no known reasons for you to not take part in this study.

**WHERE IS THE STUDY GOING TO TAKE PLACE AND HOW LONG WILL IT LAST?**

The research procedures will be conducted on your campus, in an interview room provided by your college or university. You will need to come to be interviewed twice during the study. The first visit should take about 1.5 hours. The second visit should take about 2 hours.

**WHAT WILL YOU BE ASKED TO DO?**

At a time after you have completed your class studies on group theory, you will be asked to schedule two interview times within about one week of one-another. These times will be chosen by you to fit into your schedule. Your interviews will be video- and audio-recorded, and may be transcribed for use in the study. One interview will involve talking about concepts from classes using markers, paper, and/or cards. The other interview will involve the solving of math problems by hand while explaining aloud.

**WHAT ARE THE POSSIBLE RISKS AND DISCOMFORTS?**

To the best of our knowledge, the things you will be doing have no more risk of harm than you would experience in everyday life.

**WILL YOU BENEFIT FROM TAKING PART IN THIS STUDY?**

You will not get any personal benefit from taking part in this study.

**DO YOU HAVE TO TAKE PART IN THE STUDY?**

If you decide to take part in the study, it should be because you really want to volunteer. You will not lose any benefits or rights you would normally have if you choose not to volunteer. You can stop at any time during the study and still keep the benefits and rights you had before volunteering. As a student, if you decide not to take part in this study, your choice will have no effect on your academic status or grade in the class.

**IF YOU DON'T WANT TO TAKE PART IN THE STUDY, ARE THERE OTHER CHOICES?**

If you do not want to be in the study, there are no other choices except not to take part in the study.

**WHAT WILL IT COST YOU TO PARTICIPATE?**

There are no costs associated with taking part in the study.

**WILL YOU RECEIVE ANY REWARDS FOR TAKING PART IN THIS STUDY?**

You will receive a $10 gift card for taking part in this study. This will be provided upon completion of the second interview session.

**WHO WILL SEE THE INFORMATION THAT YOU GIVE?**

We will make every effort to keep confidential all research records that identify you to the extent allowed by law.

Your information will be combined with information from other people taking part in the study. When we write about the study to share it with other researchers, we will write about the combined information we have gathered. You will not be personally identified in these written materials. We may publish the results of this study; however, we will keep your name and other identifying information private.

We will make every effort to prevent anyone who is not on the research team from knowing that you gave us information, or what that information is. All physical documents and data-storage devices will be kept in a locked filing cabinet. Electronic files on computing devices or cloud services will be password protected. These electronic files will have any personally-identifiable information removed or censored.

We will keep private all research records that identify you to the extent allowed by law. However, there are some circumstances in which we may have to show your information to other people. For example, the law may require us to show your information to a court. Also, we may be required to show information which identifies you to people who need to be sure we have done the research correctly; these would be people from such organizations as the University of Kentucky. This does not include your linear algebra or abstract algebra professors.

**CAN YOUR TAKING PART IN THE STUDY END EARLY?**

If you decide to take part in the study you still have the right to decide at any time that you no longer want to continue. You will not be treated differently if you decide to stop taking part in the study.

The researcher may need to withdraw you from the study. This may occur if you are not able to follow the directions they give you.

**WHAT IF NEW INFORMATION IS LEARNED DURING THE STUDY THAT MIGHT AFFECT YOUR DECISION TO PARTICIPATE?**

If the researcher learns of new information in regards to this study, and it might change your willingness to stay in this study, the information will be provided to you. You may be asked to sign a new informed consent form if the information is provided to you after you have joined the study.

**WHAT ELSE DO YOU NEED TO KNOW?**

There is a possibility that the data collected from you may be shared with other investigators in the future. If that is the case the data will not contain information that can identify you unless you give your consent or the UK Institutional Review Board (IRB) approves the research. The IRB is a committee that reviews ethical issues, according to federal, state and local regulations on research with human subjects, to make sure the study complies with these before approval of a research study is issued.

**WHAT IF YOU HAVE QUESTIONS, SUGGESTIONS, CONCERNS, OR COMPLAINTS?**

Before you decide whether to accept this invitation to take part in the study, please ask any questions that might come to mind now. Later, if you have questions, suggestions, concerns, or complaints about the study, you can contact the investigator, Jeffrey Slye, at 717-638-8890. If you have any questions about your rights as a volunteer in this research, contact the staff in the Office of Research Integrity at the University of Kentucky between the business hours of 8am and 5pm EST, Mon-Fri. at 859-257-9428 or toll free at 1-866-400-9428. We will give you a signed copy of this consent form to take with you.

_______________________________________ ____________
Signature of person agreeing to take part in the study Date

_______________________________________
Printed name of person agreeing to take part in the study

_______________________________________ ____________
Name of (authorized) person obtaining informed consent Date

# Appendix E Group Homomorphism Card Sort Adjacency Matrices

Table E.1. Arthur's Group Homomorphism Card Sort Adjacency Matrix

| | Codomain | Domain | Function | Generating... | Group | Group Ho... | Image | Injection | Inverse Ho... | Isomorphism | Kernel | One-to-One | Onto | Range | Subgroup | Surjection | Bijection | Cyclic | Ideal | Identity | Normal | Relations | Trivial Su... |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Codomain | 3 | 2 | 3 | 0 | 2 | 2 | 1 | 1 | 1 | 2 | 1 | 1 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Domain | 2 | 2 | 2 | 0 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Function | 3 | 2 | 6 | 0 | 3 | 4 | 1 | 2 | 2 | 4 | 2 | 2 | 2 | 1 | 1 | 2 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| Generating Set | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Group | 2 | 1 | 3 | 1 | 6 | 4 | 0 | 1 | 1 | 3 | 1 | 1 | 1 | 0 | 3 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| Group Homomorphism | 2 | 1 | 4 | 0 | 4 | 6 | 0 | 2 | 2 | 5 | 1 | 2 | 2 | 0 | 2 | 2 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| Image | 1 | 1 | 1 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 1 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Injection | 1 | 1 | 2 | 0 | 1 | 2 | 0 | 3 | 0 | 2 | 0 | 3 | 3 | 0 | 0 | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Inverse Homomorphism | 1 | 0 | 2 | 0 | 1 | 2 | 0 | 0 | 2 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Isomorphism | 2 | 1 | 4 | 0 | 3 | 5 | 0 | 2 | 2 | 5 | 1 | 2 | 2 | 0 | 1 | 2 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| Kernel | 1 | 1 | 2 | 0 | 1 | 1 | 1 | 0 | 0 | 1 | 2 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| One-to-One | 1 | 1 | 2 | 0 | 1 | 2 | 0 | 3 | 0 | 2 | 0 | 3 | 3 | 0 | 0 | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Onto | 1 | 1 | 2 | 0 | 1 | 2 | 0 | 3 | 0 | 2 | 0 | 3 | 3 | 0 | 0 | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Range | 1 | 1 | 1 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 1 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Subgroup | 0 | 0 | 1 | 0 | 3 | 2 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 3 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| Surjection | 1 | 1 | 2 | 0 | 1 | 2 | 0 | 3 | 0 | 2 | 0 | 3 | 3 | 0 | 0 | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Bijection | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Cyclic | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Ideal | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Identity | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| Normal | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Relations | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Trivial Subgroup | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table E.2. Emilie's Group Homomorphism Card Sort Adjacency Matrix

| | Codomain | Domain | Function | Generatin... | Group | Group Ho... | Image | Injection | Inverse Ho... | Isomorphism | Kernel | One-to-One | Onto | Range | Subgroup | Surjection | Bijection | Cyclic | Ideal | Identity | Normal | Relations | Trivial Su... |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Codomain | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Domain | 0 | 0 | 0 | 1 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Function | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Generating Set | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Group | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Group Homomorphism | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Image | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Injection | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Inverse Homomorphism | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Isomorphism | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Kernel | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| One-to-One | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Onto | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Range | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Subgroup | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Surjection | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Bijection | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Cyclic | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Ideal | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Identity | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Normal | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Relations | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Trivial Subgroup | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table E.3. Flint's Group Homomorphism Card Sort Adjacency Matrix

| | Codomain | Domain | Function | Generatin... | Group | Group Ho... | Image | Injection | Inverse Ho... | Isomorphism | Kernel | One-to-One | Onto | Range | Subgroup | Surjection | Bijection | Cyclic | Ideal | Identity | Normal | Relations | Trivial Su... |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Codomain | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Domain | 0 | 1 | 0 | 1 | 3 | 2 | 0 | 0 | 0 | 2 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Function | 0 | 1 | 0 | 0 | 2 | 2 | 0 | 0 | 0 | 2 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Generating Set | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Group | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Group Homomorphism | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Image | 0 | 1 | 0 | 0 | 2 | 2 | 0 | 0 | 0 | 2 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Injection | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Inverse Homomorphism | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Isomorphism | 1 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Kernel | 1 | 2 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| One-to-One | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Onto | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Range | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Subgroup | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Surjection | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Bijection | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Cyclic | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Ideal | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Identity | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Normal | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Relations | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Trivial Subgroup | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table E.4. Jake's Group Homomorphism Card Sort Adjacency Matrix

| | Codomain | Domain | Function | Generatin... | Group | Group Ho... | Image | Injection | Inverse Ho... | Isomorphism | Kernel | One-to-One | Onto | Range | Subgroup | Surjection | Bijection | Cyclic | Ideal | Identity | Normal | Relations | Trivial Su... |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Codomain | 2 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Domain | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Function | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Generating Set | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Group | 0 | 0 | 0 | 1 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Group Homomorphism | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Image | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Injection | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 2 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Inverse Homomorphism | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Isomorphism | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Kernel | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| One-to-One | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Onto | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Range | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Subgroup | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Surjection | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 2 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Bijection | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Cyclic | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Ideal | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Identity | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Normal | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Relations | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Trivial Subgroup | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table E.5. Kyle's Group Homomorphism Card Sort Adjacency Matrix

| | Codomain | Domain | Function | Generatin... | Group | Group Ho... | Image | Injection | Inverse Ho... | Isomorphism | Kernel | One-to-One | Onto | Range | Subgroup | Surjection | Bijection | Cyclic | Ideal | Identity | Normal | Relations | Trivial Su... |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Codomain | 2 | 0 | 1 | 0 | 0 | 2 | 2 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Domain | 0 | 4 | 1 | 1 | 1 | 2 | 1 | 1 | 1 | 1 | 1 | 2 | 0 | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Function | 1 | 1 | 2 | 0 | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Generating Set | 0 | 1 | 0 | 2 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Group | 0 | 1 | 0 | 1 | 2 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Group Homomorphism | 2 | 2 | 1 | 1 | 0 | 6 | 3 | 2 | 0 | 1 | 2 | 1 | 1 | 1 | 2 | 2 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| Image | 2 | 1 | 1 | 1 | 0 | 3 | 3 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Injection | 0 | 1 | 0 | 0 | 0 | 2 | 0 | 3 | 0 | 1 | 1 | 2 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Inverse Homomorphism | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 2 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Isomorphism | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 1 | 1 | 3 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Kernel | 0 | 1 | 0 | 0 | 0 | 2 | 0 | 1 | 0 | 0 | 2 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| One-to-One | 0 | 2 | 1 | 0 | 0 | 1 | 0 | 2 | 1 | 0 | 1 | 3 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Onto | 1 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 2 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Range | 0 | 3 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 1 | 0 | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Subgroup | 1 | 0 | 0 | 0 | 0 | 2 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| Surjection | 1 | 0 | 1 | 0 | 0 | 2 | 1 | 1 | 0 | 1 | 0 | 0 | 2 | 0 | 0 | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Bijection | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Cyclic | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Ideal | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Identity | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Normal | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| Relations | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Trivial Subgroup | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table E.6. Lucas's Group Homomorphism Card Sort Adjacency Matrix

| | Codomain | Domain | Function | Generatin... | Group | Group Ho... | Image | Injection | Inverse Ho... | Isomorphism | Kernel | One-to-One | Onto | Range | Subgroup | Surjection | Bijection | Cyclic | Ideal | Identity | Normal | Relations | Trivial Su... |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Codomain | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Domain | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Function | 0 | 1 | 3 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Generating Set | 0 | 0 | 0 | 2 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| Group | 0 | 0 | 1 | 1 | 3 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| Group Homomorphism | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Image | 0 | 0 | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Injection | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Inverse Homomorphism | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Isomorphism | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 2 | 0 | 2 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Kernel | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| One-to-One | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 2 | 0 | 2 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Onto | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Range | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Subgroup | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 0 |
| Surjection | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Bijection | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Cyclic | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Ideal | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 0 |
| Identity | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Normal | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Relations | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Trivial Subgroup | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table E.7. Maureen's Group Homomorphism Card Sort Adjacency Matrix

| | Codomain | Domain | Function | Generatin... | Group | Group Ho... | Image | Injection | Inverse Ho... | Isomorphism | Kernel | One-to-One | Onto | Range | Subgroup | Surjection | Bijection | Cyclic | Ideal | Identity | Normal | Relations | Trivial Su... |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Codomain | 2 | 2 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Domain | 2 | 2 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Function | 1 | 1 | 3 | 0 | 3 | 1 | 0 | 1 | 0 | 2 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| Generating Set | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| Group | 1 | 1 | 3 | 1 | 6 | 1 | 0 | 1 | 0 | 2 | 0 | 1 | 1 | 0 | 2 | 1 | 0 | 3 | 0 | 0 | 1 | 0 | 0 |
| Group Homomorphism | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| Image | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Injection | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Inverse Homomorphism | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Isomorphism | 0 | 0 | 2 | 0 | 2 | 1 | 0 | 1 | 0 | 2 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| Kernel | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| One-to-One | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Onto | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Range | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Subgroup | 0 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 |
| Surjection | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Bijection | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Cyclic | 0 | 0 | 1 | 1 | 3 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 3 | 0 | 0 | 0 | 0 | 0 |
| Ideal | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Identity | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Normal | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| Relations | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Trivial Subgroup | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table E.8. Robyn's Group Homomorphism Card Sort Adjacency Matrix

| | Codomain | Domain | Function | Generatin... | Group | Group Ho... | Image | Injection | Inverse Ho... | Isomorphism | Kernel | One-to-One | Onto | Range | Subgroup | Surjection | Bijection | Cyclic | Ideal | Identity | Normal | Relations | Trivial Su... |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Codomain | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Domain | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Function | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Generating Set | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Group | 0 | 0 | 0 | 1 | 2 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| Group Homomorphism | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| Image | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Injection | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 2 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Inverse Homomorphism | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Isomorphism | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Kernel | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| One-to-One | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Onto | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Range | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Subgroup | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| Surjection | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 2 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Bijection | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Cyclic | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Ideal | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Identity | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Normal | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| Relations | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Trivial Subgroup | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table E.9. Sander's Group Homomorphism Card Sort Adjacency Matrix

| | Codomain | Domain | Function | Generatin... | Group | Group Ho... | Image | Injection | Inverse Ho... | Isomorphism | Kernel | One-to-One | Onto | Range | Subgroup | Surjection | Bijection | Cyclic | Ideal | Identity | Normal | Relations | Trivial Su... |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Codomain | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Domain | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Function | 0 | 0 | 2 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| Generating Set | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Group | 0 | 0 | 1 | 0 | 3 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| Group Homomorphism | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| Image | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Injection | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| Inverse Homomorphism | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| Isomorphism | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| Kernel | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| One-to-One | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| Onto | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| Range | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Subgroup | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Surjection | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| Bijection | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 2 | 0 | 0 | 0 | 0 | 0 | 0 |
| Cyclic | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Ideal | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Identity | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Normal | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Relations | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Trivial Subgroup | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table E.10. Tamara's Group Homomorphism Card Sort Adjacency Matrix

| | Codomain | Domain | Function | Generatin... | Group | Group Ho... | Image | Injection | Inverse Ho... | Isomorphism | Kernel | One-to-One | Onto | Range | Subgroup | Surjection | Bijection | Cyclic | Ideal | Identity | Normal | Relations | Trivial Su... |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Codomain | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Domain | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Function | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Generating Set | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| Group | 0 | 0 | 0 | 0 | 2 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Group Homomorphism | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Image | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Injection | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 2 | 2 | 1 | 0 | 2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| Inverse Homomorphism | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Isomorphism | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Kernel | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| One-to-One | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 3 | 3 | 2 | 0 | 2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| Onto | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 3 | 3 | 2 | 0 | 2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| Range | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 2 | 2 | 2 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Subgroup | 0 | 0 | 0 | 0 | 2 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Surjection | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 2 | 2 | 1 | 0 | 2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| Bijection | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| Cyclic | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| Ideal | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Identity | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Normal | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Relations | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Trivial Subgroup | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table E.11. Dr. Brady's Group Homomorphism Card Sort Adjacency Matrix

| | Codomain | Domain | Function | Generatin... | Group | Group Ho... | Image | Injection | Inverse Ho... | Isomorphism | Kernel | One-to-One | Onto | Range | Subgroup | Surjection | Bijection | Cyclic | Ideal | Identity | Normal | Relations | Trivial Su... |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Codomain | 6 | 4 | 1 | 0 | 1 | 5 | 2 | 1 | 1 | 2 | 0 | 1 | 2 | 1 | 2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Domain | 4 | 5 | 1 | 0 | 1 | 5 | 0 | 1 | 1 | 2 | 1 | 2 | 1 | 0 | 2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Function | 1 | 1 | 2 | 1 | 0 | 2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| Generating Set | 0 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| Group | 1 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Group Homomorphism | 5 | 5 | 2 | 1 | 1 | 8 | 3 | 1 | 1 | 2 | 2 | 2 | 2 | 0 | 3 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| Image | 2 | 0 | 1 | 1 | 0 | 3 | 4 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| Injection | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Inverse Homomorphism | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Isomorphism | 2 | 2 | 0 | 0 | 0 | 2 | 0 | 1 | 1 | 2 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Kernel | 0 | 1 | 0 | 0 | 0 | 2 | 1 | 0 | 0 | 0 | 3 | 1 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| One-to-One | 1 | 2 | 0 | 0 | 0 | 2 | 0 | 1 | 0 | 1 | 1 | 2 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Onto | 2 | 1 | 0 | 0 | 0 | 2 | 1 | 1 | 0 | 1 | 0 | 1 | 2 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Range | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Subgroup | 2 | 2 | 1 | 0 | 0 | 3 | 1 | 0 | 0 | 0 | 2 | 1 | 1 | 0 | 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Surjection | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Bijection | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Cyclic | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Ideal | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Identity | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Normal | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Relations | 0 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| Trivial Subgroup | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table E.12. Dr. Greer's Group Homomorphism Card Sort Adjacency Matrix

| | Codomain | Domain | Function | Generatin... | Group | Group Ho... | Image | Injection | Inverse Ho... | Isomorphism | Kernel | One-to-One | Onto | Range | Subgroup | Surjection | Bijection | Cyclic | Ideal | Identity | Normal | Relations | Trivial Su... |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Codomain | 4 | 2 | 1 | 0 | 0 | 2 | 4 | 0 | 0 | 0 | 2 | 1 | 2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Domain | 2 | 2 | 1 | 0 | 0 | 1 | 2 | 0 | 0 | 0 | 2 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Function | 1 | 1 | 2 | 0 | 0 | 2 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Generating Set | 0 | 0 | 0 | 2 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Group | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| Group Homomorphism | 2 | 1 | 2 | 0 | 0 | 7 | 2 | 1 | 1 | 2 | 2 | 3 | 3 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| Image | 4 | 2 | 1 | 1 | 0 | 2 | 5 | 0 | 0 | 0 | 3 | 1 | 2 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Injection | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Inverse Homomorphism | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 2 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Isomorphism | 0 | 0 | 1 | 0 | 0 | 2 | 0 | 0 | 1 | 2 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Kernel | 2 | 2 | 1 | 1 | 0 | 2 | 3 | 0 | 0 | 0 | 5 | 2 | 1 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| One-to-One | 1 | 1 | 0 | 0 | 0 | 3 | 1 | 1 | 1 | 1 | 2 | 4 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| Onto | 2 | 1 | 0 | 0 | 0 | 3 | 2 | 0 | 1 | 1 | 1 | 2 | 4 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Range | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Subgroup | 0 | 0 | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| Surjection | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Bijection | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Cyclic | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Ideal | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Identity | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Normal | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Relations | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Trivial Subgroup | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 |

Table E.13. Dr. Powell's Group Homomorphism Card Sort Adjacency Matrix

| | Codomain | Domain | Function | Generatin... | Group | Group Ho... | Image | Injection | Inverse Ho... | Isomorphism | Kernel | One-to-One | Onto | Range | Subgroup | Surjection | Bijection | Cyclic | Ideal | Identity | Normal | Relations | Trivial Su... |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Codomain | 4 | 2 | 1 | 0 | 0 | 2 | 4 | 0 | 0 | 0 | 2 | 1 | 2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Domain | 2 | 2 | 1 | 0 | 0 | 1 | 2 | 0 | 0 | 0 | 2 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Function | 1 | 1 | 2 | 0 | 0 | 2 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Generating Set | 0 | 0 | 0 | 2 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Group | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| Group Homomorphism | 2 | 1 | 2 | 0 | 0 | 7 | 2 | 1 | 1 | 2 | 2 | 3 | 3 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| Image | 4 | 2 | 1 | 1 | 0 | 2 | 5 | 0 | 0 | 0 | 3 | 1 | 2 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Injection | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Inverse Homomorphism | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 2 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Isomorphism | 0 | 0 | 1 | 0 | 0 | 2 | 0 | 0 | 1 | 2 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Kernel | 2 | 2 | 1 | 1 | 0 | 2 | 3 | 0 | 0 | 0 | 5 | 2 | 1 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| One-to-One | 1 | 1 | 0 | 0 | 0 | 3 | 1 | 1 | 1 | 1 | 2 | 4 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| Onto | 2 | 1 | 0 | 0 | 0 | 3 | 2 | 0 | 1 | 1 | 1 | 2 | 4 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Range | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Subgroup | 0 | 0 | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| Surjection | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Bijection | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Cyclic | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Ideal | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Identity | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Normal | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Relations | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Trivial Subgroup | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 |

## Appendix F Linear Transformation Card Sort Adjacency Matrices

Table F.1. Arthur's Linear Transformation Card Sort Adjacency Matrix

| | Codomain | Domain | Function | Basis | Vector Space | Linear Tr... | Image | Injection | Inverse Tr... | Isomorphism | Kernel | One-to-One | Onto | Range | Subspace | Surjection | Column S... | Matrix | Null Space | Bijection | Determinant | Dimension | Eigenvalues | Eigenvectors | Gaussian... | Linear Ind... | Row Space | Span | System of... | Vector | Coordinate... | Spanning S... | Trivial Hom... |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Codomain | 2 | 2 | 1 | 0 | 0 | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Domain | 2 | 2 | 1 | 0 | 0 | 1 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Function | 1 | 1 | 3 | 0 | 0 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Basis | 0 | 0 | 0 | 2 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| Vector Space | 0 | 0 | 0 | 1 | 2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| Linear Transformation | 1 | 1 | 1 | 1 | 1 | 5 | 0 | 1 | 2 | 2 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Image | 1 | 1 | 1 | 0 | 0 | 0 | 4 | 0 | 0 | 0 | 2 | 0 | 0 | 4 | 0 | 0 | 2 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Injection | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 2 | 0 | 1 | 0 | 2 | 2 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Inverse Transformation | 1 | 1 | 1 | 0 | 0 | 2 | 0 | 0 | 2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Isomorphism | 0 | 0 | 1 | 0 | 0 | 2 | 0 | 1 | 1 | 2 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Kernel | 1 | 1 | 1 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 3 | 0 | 0 | 2 | 0 | 0 | 1 | 1 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| One-to-One | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 2 | 0 | 1 | 0 | 2 | 2 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Onto | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 2 | 0 | 1 | 0 | 2 | 2 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Range | 1 | 1 | 1 | 0 | 0 | 0 | 4 | 0 | 0 | 0 | 2 | 0 | 0 | 4 | 0 | 0 | 2 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Subspace | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Surjection | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 2 | 0 | 1 | 0 | 2 | 2 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Column Space | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 1 | 0 | 0 | 2 | 0 | 0 | 2 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Matrix | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Null Space | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 2 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Bijection | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Determinant | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Dimension | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Eigenvalues | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Eigenvectors | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Gaussian Elimination | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Linear Independence | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Row Space | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Span | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 |
| System of Equations | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Vector | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Coordinate Vector | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Spanning Set | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Trivial Homomorphism | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table F.2. Emilie's Linear Transformation Card Sort Adjacency Matrix

| | Codomain | Domain | Function | Basis | Vector Space | Linear Tr... | Image | Injection | Inverse Tr... | Isomorphism | Kernel | One-to-One | Onto | Range | Subspace | Surjection | Column S... | Matrix | Null Space | Bijection | Determinant | Dimension | Eigenvalues | Eigenvectors | Gaussian... | Linear Ind... | Row Space | Span | System of... | Vector | Coordinate... | Spanning S... | Trivial Hom... |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Codomain | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Domain | 0 | 2 | 2 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Function | 0 | 2 | 2 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Basis | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Vector Space | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Linear Transformation | 0 | 0 | 0 | 1 | 0 | 2 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Image | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Injection | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Inverse Transformation | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Isomorphism | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Kernel | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| One-to-One | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Onto | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Range | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Subspace | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Surjection | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Column Space | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Matrix | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Null Space | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Bijection | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Determinant | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Dimension | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Eigenvalues | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Eigenvectors | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Gaussian Elimination | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Linear Independence | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Row Space | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Span | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| System of Equations | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Vector | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Coordinate Vector | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Spanning Set | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Trivial Homomorphism | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table F.3. Flint's Linear Transformation Card Sort Adjacency Matrix

| | Codomain | Domain | Function | Basis | Vector Space | Linear Tr... | Image | Injection | Inverse Tr... | Isomorphism | Kernel | One-to-One | Onto | Range | Subspace | Surjection | Column S... | Matrix | Null Space | Bijection | Determinant | Dimension | Eigenvalues | Eigenvectors | Gaussian... | Linear Ind... | Row Space | Span | System of... | Vector | Coordinate... | Spanning S... | Trivial Hom... |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Codomain | 2 | 2 | 2 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Domain | 2 | 3 | 3 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 3 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| Function | 2 | 3 | 3 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 3 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| Basis | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| Vector Space | 0 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Linear Transformation | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Image | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Injection | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Inverse Transformation | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Isomorphism | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Kernel | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| One-to-One | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Onto | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Range | 2 | 3 | 3 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 3 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| Subspace | 0 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Surjection | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Column Space | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Matrix | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| Null Space | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Bijection | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Determinant | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Dimension | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Eigenvalues | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Eigenvectors | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Gaussian Elimination | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| Linear Independence | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Row Space | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Span | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| System of Equations | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 |
| Vector | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Coordinate Vector | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Spanning Set | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Trivial Homomorphism | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table F.4. Jake's Linear Transformation Card Sort Adjacency Matrix

| | Codomain | Domain | Function | Basis | Vector Space | Linear Tr... | Image | Injection | Inverse Tr... | Isomorphism | Kernel | One-to-One | Onto | Range | Subspace | Surjection | Column S... | Matrix | Null Space | Bijection | Determinant | Dimension | Eigenvalues | Eigenvectors | Gaussian... | Linear Ind... | Row Space | Span | System of... | Vector | Coordinate... | Spanning S... | Trivial Hom... |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Codomain | 2 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Domain | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Function | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Basis | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| Vector Space | 0 | 0 | 0 | 1 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 0 | 0 | 0 |
| Linear Transformation | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Image | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Injection | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 2 | 0 | 1 | 1 | 2 | 2 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Inverse Transformation | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Isomorphism | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Kernel | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| One-to-One | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 2 | 0 | 1 | 1 | 2 | 2 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Onto | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 2 | 0 | 1 | 1 | 2 | 2 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Range | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Subspace | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Surjection | 1 | 1 | 1 | 0 | 0 | 1 | 1 | 2 | 0 | 1 | 1 | 2 | 2 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Column Space | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Matrix | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 |
| Null Space | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Bijection | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Determinant | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Dimension | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Eigenvalues | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Eigenvectors | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Gaussian Elimination | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Linear Independence | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Row Space | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Span | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| System of Equations | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Vector | 0 | 0 | 0 | 1 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 0 | 0 | 0 |
| Coordinate Vector | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Spanning Set | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Trivial Homomorphism | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table F.5. Kyle's Linear Transformation Card Sort Adjacency Matrix

| | Codomain | Domain | Function | Basis | Vector Space | Linear Tr... | Image | Injection | Inverse Tr... | Isomorphism | Kernel | One-to-One | Onto | Range | Subspace | Surjection | Column S... | Matrix | Null Space | Bijection | Determinant | Dimension | Eigenvalues | Eigenvectors | Gaussian... | Linear Ind... | Row Space | Span | System of... | Vector | Coordinate... | Spanning S... | Trivial Hom... |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Codomain | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Domain | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Function | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Basis | 0 | 1 | 0 | 3 | 1 | 2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Vector Space | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Linear Transformation | 1 | 1 | 1 | 2 | 0 | 11 | 4 | 2 | 1 | 1 | 1 | 0 | 0 | 1 | 3 | 1 | 3 | 2 | 2 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Image | 0 | 1 | 0 | 1 | 0 | 4 | 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Injection | 0 | 0 | 0 | 0 | 0 | 2 | 0 | 3 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Inverse Transformation | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Isomorphism | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Kernel | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| One-to-One | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Onto | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Range | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Subspace | 0 | 0 | 0 | 0 | 0 | 3 | 2 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 3 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Surjection | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Column Space | 1 | 0 | 0 | 0 | 0 | 3 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 3 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Matrix | 0 | 0 | 0 | 1 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Null Space | 1 | 0 | 0 | 0 | 0 | 2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 2 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Bijection | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Determinant | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Dimension | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Eigenvalues | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Eigenvectors | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Gaussian Elimination | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Linear Independence | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Row Space | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Span | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| System of Equations | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Vector | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Coordinate Vector | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Spanning Set | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Trivial Homomorphism | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table F.6. Lucas's Linear Transformation Card Sort Adjacency Matrix

| | Codomain | Domain | Function | Basis | Vector Space | Linear Tr... | Image | Injection | Inverse Tr... | Isomorphism | Kernel | One-to-One | Onto | Range | Subspace | Surjection | Column S... | Matrix | Null Space | Bijection | Determinant | Dimension | Eigenvalues | Eigenvectors | Gaussian... | Linear Ind... | Row Space | Span | System of... | Vector | Coordinate... | Spanning S... | Trivial Hom... |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Codomain | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Domain | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Function | 0 | 1 | 3 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Basis | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Vector Space | 0 | 0 | 1 | 1 | 2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Linear Transformation | 0 | 0 | 1 | 0 | 1 | 5 | 0 | 1 | 1 | 2 | 1 | 2 | 2 | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Image | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Injection | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Inverse Transformation | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Isomorphism | 0 | 0 | 1 | 0 | 0 | 2 | 0 | 0 | 1 | 3 | 0 | 2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Kernel | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| One-to-One | 0 | 0 | 1 | 0 | 0 | 2 | 0 | 1 | 0 | 2 | 0 | 3 | 2 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Onto | 0 | 0 | 0 | 0 | 0 | 2 | 0 | 1 | 0 | 1 | 0 | 2 | 2 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Range | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Subspace | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Surjection | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Column Space | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Matrix | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Null Space | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Bijection | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Determinant | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Dimension | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Eigenvalues | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Eigenvectors | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Gaussian Elimination | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Linear Independence | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Row Space | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Span | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| System of Equations | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Vector | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Coordinate Vector | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Spanning Set | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Trivial Homomorphism | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table F.7. Maureen's Linear Transformation Card Sort Adjacency Matrix

| | Codomain | Domain | Function | Basis | Vector Space | Linear Tr... | Image | Injection | Inverse Tr... | Isomorphism | Kernel | One-to-One | Onto | Range | Subspace | Surjection | Column S... | Matrix | Null Space | Bijection | Determinant | Dimension | Eigenvalues | Eigenvectors | Gaussian... | Linear Ind... | Row Space | Span | System of... | Vector | Coordinate... | Spanning S... | Trivial Hom... |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Codomain | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Domain | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Function | 0 | 1 | 3 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Basis | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Vector Space | 0 | 0 | 1 | 1 | 2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Linear Transformation | 0 | 0 | 1 | 0 | 1 | 5 | 0 | 1 | 1 | 2 | 1 | 2 | 2 | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Image | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Injection | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Inverse Transformation | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Isomorphism | 0 | 0 | 1 | 0 | 0 | 2 | 0 | 0 | 1 | 3 | 0 | 2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Kernel | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| One-to-One | 0 | 0 | 1 | 0 | 0 | 2 | 0 | 1 | 0 | 2 | 0 | 3 | 2 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Onto | 0 | 0 | 0 | 0 | 0 | 2 | 0 | 1 | 0 | 1 | 0 | 2 | 2 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Range | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Subspace | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Surjection | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Column Space | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Matrix | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Null Space | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Bijection | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Determinant | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Dimension | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Eigenvalues | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Eigenvectors | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Gaussian Elimination | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Linear Independence | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Row Space | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Span | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| System of Equations | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Vector | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Coordinate Vector | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Spanning Set | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Trivial Homomorphism | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table F.8. Robyn's Linear Transformation Card Sort Adjacency Matrix

| | Codomain | Domain | Function | Basis | Vector Space | Linear Tr... | Image | Injection | Inverse Tr... | Isomorphism | Kernel | One-to-One | Onto | Range | Subspace | Surjection | Column S... | Matrix | Null Space | Bijection | Determinant | Dimension | Eigenvalues | Eigenvectors | Gaussian... | Linear Ind... | Row Space | Span | System of... | Vector | Coordinate... | Spanning S... | Trivial Hom... |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Codomain | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Domain | 1 | 2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Function | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Basis | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Vector Space | 0 | 0 | 0 | 1 | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Linear Transformation | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Image | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Injection | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 2 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Inverse Transformation | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Isomorphism | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Kernel | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| One-to-One | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 2 | 2 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Onto | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 2 | 2 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Range | 1 | 2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Subspace | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Surjection | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 2 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Column Space | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 2 | 1 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Matrix | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Null Space | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 2 | 1 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Bijection | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Determinant | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Dimension | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Eigenvalues | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Eigenvectors | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Gaussian Elimination | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Linear Independence | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Row Space | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Span | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| System of Equations | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Vector | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Coordinate Vector | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Spanning Set | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Trivial Homomorphism | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table F.9. Sander's Linear Transformation Card Sort Adjacency Matrix

| | Codomain | Domain | Function | Basis | Vector Space | Linear Tr... | Image | Injection | Inverse Tr... | Isomorphism | Kernel | One-to-One | Onto | Range | Subspace | Surjection | Column S... | Matrix | Null Space | Bijection | Determinant | Dimension | Eigenvalues | Eigenvectors | Gaussian... | Linear Ind... | Row Space | Span | System of... | Vector | Coordinate... | Spanning S... | Trivial Hom... |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Codomain | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Domain | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Function | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Basis | 0 | 0 | 0 | 2 | 0 | 0 | 2 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Vector Space | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Linear Transformation | 0 | 0 | 1 | 0 | 0 | 3 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Image | 0 | 0 | 0 | 2 | 0 | 0 | 2 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Injection | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Inverse Transformation | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Isomorphism | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Kernel | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| One-to-One | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Onto | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Range | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Subspace | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Surjection | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Column Space | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 2 | 2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| Matrix | 0 | 0 | 1 | 2 | 0 | 2 | 2 | 1 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 2 | 8 | 4 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| Null Space | 0 | 0 | 0 | 2 | 0 | 1 | 2 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 4 | 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Bijection | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Determinant | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Dimension | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Eigenvalues | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Eigenvectors | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Gaussian Elimination | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Linear Independence | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Row Space | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 |
| Span | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| System of Equations | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Vector | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Coordinate Vector | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Spanning Set | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Trivial Homomorphism | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table F.10. Tamara's Linear Transformation Card Sort Adjacency Matrix

| | Codomain | Domain | Function | Basis | Vector Space | Linear Tr... | Image | Injection | Inverse Tr... | Isomorphism | Kernel | One-to-One | Onto | Range | Subspace | Surjection | Column S... | Matrix | Null Space | Bijection | Determinant | Dimension | Eigenvalues | Eigenvectors | Gaussian... | Linear Ind... | Row Space | Span | System of... | Vector | Coordinate... | Spanning S... | Trivial Hom... |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Codomain | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Domain | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Function | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Basis | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Vector Space | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Linear Transformation | 0 | 0 | 0 | 0 | 1 | 2 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Image | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Injection | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 2 | 0 | 1 | 0 | 2 | 2 | 0 | 0 | 2 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Inverse Transformation | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Isomorphism | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Kernel | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| One-to-One | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 2 | 0 | 1 | 0 | 2 | 2 | 0 | 0 | 2 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Onto | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 2 | 0 | 1 | 0 | 2 | 2 | 0 | 0 | 2 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Range | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Subspace | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Surjection | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 2 | 0 | 1 | 0 | 2 | 2 | 0 | 0 | 2 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Column Space | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Matrix | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Null Space | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Bijection | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 2 | 0 | 1 | 0 | 2 | 2 | 0 | 0 | 2 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Determinant | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Dimension | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Eigenvalues | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Eigenvectors | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Gaussian Elimination | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Linear Independence | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Row Space | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Span | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| System of Equations | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Vector | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Coordinate Vector | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Spanning Set | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Trivial Homomorphism | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table F.11. Dr. Brady's Linear Transformation Card Sort Adjacency Matrix

| | Codomain | Domain | Function | Basis | Vector Space | Linear Tr... | Image | Injection | Inverse Tr... | Isomorphism | Kernel | One-to-One | Onto | Range | Subspace | Surjection | Column S... | Matrix | Null Space | Bijection | Determinant | Dimension | Eigenvalues | Eigenvectors | Gaussian... | Linear Ind... | Row Space | Span | System of... | Vector | Coordinate... | Spanning S... | Trivial Hom... |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Codomain | 5 | 3 | 2 | 0 | 1 | 4 | 2 | 1 | 0 | 1 | 0 | 1 | 2 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Domain | 3 | 4 | 2 | 0 | 1 | 4 | 0 | 1 | 0 | 1 | 1 | 2 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Function | 2 | 2 | 3 | 1 | 1 | 3 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| Basis | 0 | 0 | 1 | 2 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 2 | 0 |
| Vector Space | 1 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Linear Transformation | 4 | 4 | 3 | 1 | 1 | 6 | 2 | 1 | 0 | 1 | 1 | 2 | 2 | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| Image | 2 | 0 | 1 | 1 | 0 | 2 | 4 | 0 | 0 | 0 | 0 | 0 | 2 | 2 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| Injection | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Inverse Transformation | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Isomorphism | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 2 | 0 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Kernel | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 3 | 2 | 0 | 0 | 1 | 0 | 1 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| One-to-One | 1 | 2 | 0 | 0 | 0 | 2 | 0 | 1 | 0 | 1 | 2 | 3 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Onto | 2 | 1 | 0 | 0 | 0 | 2 | 2 | 1 | 0 | 1 | 0 | 1 | 3 | 1 | 0 | 2 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Range | 1 | 0 | 0 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 1 | 2 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Subspace | 1 | 1 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 2 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| Surjection | 1 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 0 | 1 | 2 | 1 | 0 | 3 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Column Space | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 1 | 2 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| Matrix | 0 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| Null Space | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| Bijection | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Determinant | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Dimension | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Eigenvalues | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Eigenvectors | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Gaussian Elimination | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Linear Independence | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 |
| Row Space | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Span | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| System of Equations | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Vector | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Coordinate Vector | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Spanning Set | 0 | 0 | 1 | 2 | 0 | 1 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 2 | 0 |
| Trivial Homomorphism | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table F.12. Dr. Greer's Linear Transformation Card Sort Adjacency Matrix

| | Codomain | Domain | Function | Basis | Vector Space | Linear Tr... | Image | Injection | Inverse Tr... | Isomorphism | Kernel | One-to-One | Onto | Range | Subspace | Surjection | Column S... | Matrix | Null Space | Bijection | Determinant | Dimension | Eigenvalues | Eigenvectors | Gaussian... | Linear Ind... | Row Space | Span | System of... | Vector | Coordinate... | Spanning S... | Trivial Hom... |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Codomain | 5 | 3 | 0 | 0 | 1 | 2 | 5 | 0 | 0 | 2 | 4 | 2 | 3 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 3 |
| Domain | 3 | 3 | 0 | 0 | 1 | 1 | 3 | 0 | 0 | 2 | 3 | 2 | 3 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 3 |
| Function | 0 | 0 | 4 | 0 | 2 | 4 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Basis | 0 | 0 | 0 | 2 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Vector Space | 1 | 1 | 2 | 0 | 4 | 2 | 1 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 3 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| Linear Transformation | 2 | 1 | 4 | 1 | 2 | 7 | 2 | 0 | 0 | 1 | 2 | 0 | 1 | 0 | 1 | 0 | 0 | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| Image | 5 | 3 | 0 | 0 | 1 | 2 | 5 | 0 | 0 | 2 | 4 | 2 | 3 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 3 |
| Injection | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Inverse Transformation | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Isomorphism | 2 | 2 | 1 | 0 | 1 | 1 | 2 | 0 | 1 | 5 | 2 | 4 | 4 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 |
| Kernel | 4 | 3 | 0 | 0 | 1 | 2 | 4 | 0 | 0 | 2 | 5 | 2 | 3 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 3 |
| One-to-One | 2 | 2 | 0 | 0 | 1 | 0 | 2 | 1 | 1 | 4 | 2 | 5 | 4 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 |
| Onto | 3 | 3 | 0 | 0 | 1 | 1 | 3 | 0 | 1 | 4 | 3 | 4 | 6 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 3 |
| Range | 1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Subspace | 1 | 1 | 1 | 0 | 3 | 1 | 1 | 0 | 0 | 1 | 1 | 1 | 1 | 0 | 5 | 0 | 1 | 1 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 |
| Surjection | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Column Space | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 3 | 0 | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Matrix | 0 | 0 | 2 | 1 | 1 | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Null Space | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 2 | 0 | 3 | 0 | 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Bijection | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Determinant | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Dimension | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Eigenvalues | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Eigenvectors | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Gaussian Elimination | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Linear Independence | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Row Space | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Span | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| System of Equations | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Vector | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Coordinate Vector | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Spanning Set | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Trivial Homomorphism | 3 | 3 | 0 | 0 | 1 | 1 | 3 | 0 | 0 | 2 | 3 | 2 | 3 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 3 |

Table F.13. Dr. Powell's Linear Transformation Card Sort Adjacency Matrix

| | Codomain | Domain | Function | Basis | Vector Space | Linear Tr... | Image | Injection | Inverse Tr... | Isomorphism | Kernel | One-to-One | Onto | Range | Subspace | Surjection | Column S... | Matrix | Null Space | Bijection | Determinant | Dimension | Eigenvalues | Eigenvectors | Gaussian... | Linear Ind... | Row Space | Span | System of... | Vector | Coordinate... | Spanning S... | Trivial Hom... |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Codomain | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Domain | 0 | 3 | 0 | 0 | 1 | 3 | 0 | 0 | 0 | 1 | 2 | 0 | 0 | 1 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Function | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Basis | 0 | 0 | 0 | 2 | 1 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| Vector Space | 0 | 1 | 0 | 1 | 2 | 2 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| Linear Transformation | 1 | 3 | 1 | 2 | 2 | 14 | 0 | 3 | 1 | 2 | 5 | 2 | 3 | 3 | 3 | 4 | 1 | 4 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| Image | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Injection | 0 | 0 | 0 | 0 | 0 | 3 | 0 | 3 | 1 | 1 | 2 | 2 | 1 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Inverse Homomorphism | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Isomorphism | 0 | 1 | 0 | 0 | 0 | 2 | 0 | 1 | 0 | 2 | 0 | 1 | 1 | 1 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Kernel | 0 | 2 | 0 | 0 | 1 | 5 | 0 | 2 | 1 | 0 | 5 | 1 | 0 | 0 | 3 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| One-to-One | 0 | 0 | 0 | 0 | 0 | 2 | 0 | 2 | 0 | 1 | 1 | 2 | 1 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Onto | 1 | 0 | 0 | 0 | 0 | 3 | 0 | 1 | 0 | 1 | 0 | 1 | 3 | 1 | 0 | 4 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Range | 1 | 1 | 0 | 0 | 0 | 3 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 3 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Subspace | 0 | 2 | 0 | 0 | 1 | 3 | 0 | 0 | 0 | 0 | 3 | 0 | 0 | 0 | 3 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Surjection | 1 | 0 | 0 | 0 | 0 | 4 | 0 | 2 | 0 | 2 | 0 | 2 | 4 | 1 | 0 | 6 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Column Space | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Matrix | 0 | 0 | 0 | 1 | 0 | 4 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 1 | 1 | 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Null Space | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Bijection | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Determinant | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Dimension | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Eigenvalues | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Eigenvectors | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Gaussian Elimination | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Linear Independence | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Row Space | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Span | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| System of Equations | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Vector | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Coordinate Vector | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| Spanning Set | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| Trivial Homomorphism | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

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**Vita**

**Jeffrey Ernest Slye**

## EDUCATION

- M.A. in Mathematics, University of Kentucky, December 2015
- B.A. in Mathematics with Teaching Certification, *Summa Cum Laude*, Messiah College, May 2010

## PROFESSIONAL POSITIONS

- Assistant Professor of Mathematics, SUNY Oswego, Starting August 2019
- Graduate Teaching Assistant, University of Kentucky, 2013-2019
- Mathematics Teacher, West Perry High School, 2010-2013