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MULTIFACTOR DIMENSIONALITY REDUCTION WITH P RISK SCORES PER PERSON

Ye Li
University of Kentucky, yeleah@hotmail.com
Author ORCID Identifier:
https://orcid.org/0000-0003-2305-8734
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Ye Li, Student

Dr. Richard Charnigo, Major Professor

Dr. Constance Wood, Director of Graduate Studies

MULTIFACTOR DIMENSIONALITY REDUCTION WITH P RISK SCORES PER PERSON

DISSERTATION

A dissertation submitted in partial
fulfillment of the requirements for
the degree of Doctor of Philosophy
in the College of Arts and Sciences
at the University of Kentucky

By
Ye Li
Lexington, Kentucky

Director: Dr. Richard Charnigo, Professor of Statistics
Lexington, Kentucky
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ABSTRACT OF DISSERTATION

MULTIFACTOR DIMENSIONALITY REDUCTION WITH P RISK SCORES PER PERSON

After reviewing Multifactor Dimensionality Reduction(MDR) and its extensions, an approach to obtain P(larger than 1) risk scores is proposed to predict the continuous outcome for each subject. We study the mean square error(MSE) of dimensionality reduced models fitted with sets of 2 risk scores and investigate the MSE for several special cases of the covariance matrix. A methodology is proposed to select a best set of P risk scores when P is specified a priori. Simulation studies based on true models of different dimensions(larger than 3) demonstrate that the selected set of P(larger than 1) risk scores outperforms the single aggregated risk score generated in AQMDR and illustrate that our methodology can determine a best set of P risk scores effectively. With different assumptions on the dimension of the true model, we considered the preferable set of risk scores from the best set of two risk scores and the best set of three risk scores. Further, we present a methodology to access a set of P risk scores when P is not given a priori. The expressions of asymptotic estimated mean square error of prediction(MSPE) are derived for a 1-dimensional model and 2-dimensional model. In the last main chapter, we apply the methodology of selecting a best set of risk scores where P has been specified a priori to Alzheimer's Disease data and achieve a set of 2 risk scores and a set of three risk scores for each subject to predict measurements on biomarkers that are crucially involved in Alzheimer's Disease.

KEYWORDS: Multifactor Dimensionality Reduction, Risk Score, Continuous outcome, Gene-gene Interaction

Author's signature: Ye Li

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By

Ye Li

Director of Dissertation: Richard Charnigo

Director of Graduate Studies: Constance Wood

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1.1 Introduction

Currently, many common diseases in human could be explained by the theory in genes, including genetypes' interaction with genotypes[1, 2] or with environment[3, 4]. For example, type 2 diabetes[5], Alzheimer's[6] and breast cancer[7] could be results from multivariate genetic and environmental interactions. Usually, phenotypes or occurrence of disease is decided by interactions among numerous gene factors simultaneously and thus, carriers of those genotypes have higher risks of disease than non-carriers. The following is a brief example of using a 2×2 contingency table to detect interaction between two factors.

Suppose there are two factors, smoking and genes relating to cancer. Further assume there are four groups of individuals of size 100 persons randomly selected. Each group is formed by individuals who smoke and carry genes which might cause cancer, individuals who smoke but do not carry genes which might cause cancer, individuals who do not smoke but carry genes which might cause cancer, and individuals who don't smoke and don't carry genes which might cause cancer. The numbers of cancer patients are displayed in the following table.

Table 1.1: Table of Cancer Patients Number Example

individuals with cancer	smoker(A)	non-smoker(a)
genes carrier(B)	70(AB)	35(aB)
genes non-carrier(b)	35(Ab)	40(ab)

From this table, the number of individuals with cancer in group consisting of smoking gene carriers is larger than the number of individuals with cancer in the rest three groups; the number of individuals with cancer in group consisting of nonsmoking gene non-carriers is larger than the number of individuals with cancer in the group consisting of smoking gene non-carriers and the number of individuals with cancer in the group consisting of nonsmoking gene carriers, i.e., $AB > ab > aB\&Ab$. The number of cancer patients in this table could be written as $n = 40 + I_A \times (-5) + I_B \times (-5) + I_{A\ and\ B} \times (30)$. Thus, there is an interaction between smoking and genes.

To identify the probability to obtain some phenotypes caused by genotypes or the relationship between some phenotypes and genotypes, parametric methods such as logistic regression has been adopted to predict discrete outcomes by genotypes[8]. However, when the problems are of large dimension including large amount of possible interactions within genotypes or between genotypes and environment, the method of regression will be limited because that empty cells present in contingency tables [9]. Besides, in situations involving large dimensional data, huge financial expenses will also limits experiments which intends to find interactions of all possible orders[10]. Further, "traditional approaches such as Fisher's exact test and Chi-square test which assess only marginal main effects of risk factors" will also be limited in their ability to find the relationship between genotypes and phenotypes in complex network with numerous genes[10].

In order to address these issues, Ritchie et al developed a novel statistical technique to compute and analyse [9], multifactor dimensionality reduction (MDR), [9] in 2001. MDR makes it possible to consider a one-dimensional variable which is transfered from data of high dimension and with gene-gene interactions. Upon development, MDR has been widely studies[11] and many extensions were developed to deal with problems of various types of phenotypes[12, 13], sparse cells[14, 15] and mis-classification into high and low risk groups[16], etc.

1.2 Multifactor-Dimensionality Reduction

Ritchie et al. first developed the MDR method to identify the possible interactions within genes or with environmental factors in high dimensional data[9]. This non-parametric statistical method increases the power to identify high order interactions when there are sparse cells in the contingency tables[9]. The outline of MDR method is presented as follows:

MDR uses a 10-fold cross-validation. In each CV fold,

1. Use 9/10 of the data to be the training set and the rest 1/10 to be the test set.

2. Suppose there are M single nucleotide polymorphisms(SNPs) in a case-control dataset, each SNP is carrying a unique gene allele pair. If the highest order interaction of interest is m ($m \leq M$) , select m SNPs.

3. Use those m SNPs to construct contingency tables for all possible multiloci interactions of 2-way, 3-way,..., m-way, based on training data.

4. In each contingency table of i-way interaction ($i = 2, ..., m$), label each multifactor cell with “high risk” if the case-control ratio exceeds some predetermined threshold T (e.g., 0.5 or 1.0); otherwise, label the cell with “low risk”.

5. Separate the data into two groups using the binary variable “high risk” or “low risk”. And make predictions on test set using this binary variable.

In each fold of the 10 folds CV, the prediction error is calculated as the proportion of individuals in test data with incorrect prediction. The classification error is the proportion of individuals misclassified into “high risk” or “low risk” groups in the training data. The best k-way interaction model is determined by the least average classification error across the 10 folds, $2 \leq k \leq m$. The best overall model is chosen by average prediction error across the 10 folds.

Thus, by MDR, we could either find the best model of each k-way gene-gene interaction or do an exhaustive search to find the overall best model among all possible order of interactions. This best overall model is evaluated by a hypothesis test on the CV consistency, i.e., how many times this model will be chosen as the best model by the least classification error rate in the training data across 10 folds. The right tail Monte Carlo P-value from a null distribution of CV consistency generated by 1000 permutations is used to conclude on the significance of this model's effect on outcome.

After testing MDR on a simulated data with predetermined threshold T=1 to evaluate its ability to decide the best multifactor model and it is reasonable to use CV in analysis, MDR with threshold 1 is applied to the Sporadic Breast Cancer data, which is a balanced case-control data with 10 polymorphisms isolated from 5 genes as the genotypic factors. The best model is chosen by two criteria, CV consistency and prediction error. A four way multilocus interaction model is determined to be the best model among 2-way to 9-way interactions with the largest CV consistency=9.80 at significance level 0.001, which is obtained by the empirical distribution from permutation test, and the smallest prediction error 46.73 out of 200 cases and 200 controls. Thus, there is a significant interaction between the four SNPs: COMT, CYP1A1m1, CBP1B1 codon 48, and CYP1B1 codon 432, on phenotype of sporadic breast cancer.

1.3 Quantitative Multifactor-Dimensionality Reduction (QMDR)

Developed by Gui et al. in 2013, QMDR is an extension of MDR aiming to "detect gene-gene interaction effect on quantitative outcomes such as BMI, tumor size and survival time"[17], with an advantage in identifying gene-gene interactions for high dimensional data with quantitative phenotypes and large variability[17]. The outline of QMDR method is similar to that of MDR, only differing in the following two aspects:

1. In the procedure to define High/Low risk level, one mean value for each entry in a contingency table associated with m selected SNP's is compared with the overall mean and it is labeled of high risk if the mean value exceeds the overall mean, otherwise, it is labeled with low risk.

2. The best multifactor model in each interaction order and the best overall model are not decided by BA because the outcomes are quantitative, but by a t-statistic whose distribution under null hypothesis, which is there is not any gene-gene interaction, has an approximately normal distribution with mean 0.

Two simulations are performed before applying QMDR to the PREVENT study(Prevention of Renal and Vascular End-Stage Disease)[18]. The first simulation is to study the null distribution of t-statistic and find the critical value for the t-test from the empirical null distribution. The second simulation is to compare the effectiveness of QMDR with MDR and GMDR[19] in finding the best overall model for predicting quantitative traits. The simulation results show QMDR and GMDR are more effective than MDR in performance and QMDR is the more efficient than GMDR in computation.

In the real data analysis, an unbalanced gender dataset is chosen to "explore genetic predictors of tissue plasminogen activator(t-PA) and plasminogen activator inhibitor-1(PAI-1), which are gender specific" with 7 reported SNPs. Thus the best overall interaction models are found for females with t-PA, females with PAI-1, males with t-PA and males with PAI-1, respectively. QMDR with a 10 fold CV is applied to the residual generated by linear regression on factors with significant main effects to find the best overall interaction model. Then the p-value of each of the above best overall models selected by QMDR,found by carrying out a 10000 fold permutation test, is compared with the corresponding p-value from previous simulated empirical null distribution.

By the permuted p-value, only the interaction model for females with PAI is significant, a 4-way interaction model among SNPs. And it is also found the empirical p-values are very close to the permutation p-values, thus empirical p-values could be applied to estimate the significance of QMDR models instead of using permutation test.

1.4 Aggregated Multifactor-Dimensionality Reduction (AMDR)

Proposed by Dai & Charnigo et al. in 2013[10], AMDR addresses aggregated gene-gene interaction effect in predicting the outcome using a cumulative risk score. The risk score is based on all significant gene-gene interactions where the significance of the interactions is decided by three measures: predisposing odds ratio(pOR), predisposing relative risk(pRR) and predisposing chi-square test statistic($pChi$).

The reason to use these three measures is that, in MDR method, after labeling each cell as "High risk" or "Low risk" in one contingency table, every subject in the table is contributing to either the cases in "High risk" group or "Low risk" group, which is a Bernoulli trial. However, one subject's assignment to "High risk" group or "Low risk" group is not independent of all the other cases' assignments. Thus, with the aid of F and F_0, pOR_i, pRR_i and $pChi_i$, we could know whether there is any significant interaction in each i-way interaction(F is the cdf of $\frac{n_{11}n_{22}}{n_{12}n_{21}}$,$F_0$ is the cdf of $\dfrac{n_{11}n_{22}}{n_{12}n_{21}}$ if there is no interaction, n_{11}=number of cases in "High risk" cells, n_{12}=number of controls in "High risk" cells, n_{21}=number of cases in "Low risk" cells, n_{22}=number of controls in "Low risk" cells). The critical value for the above statistics could be judged by permutation test.

For the nth selected individual, his aggregated k-way risk score $R(k,n)$ is the number of significant gene-gene interactions labeled as "high risk" for that individual, out of the number of his all possible gene-gene interactions.

In order to find the best cutoff for significance in interactions, ROC curves are built with aggregated risk scores using interactions identified at different significance levels $\alpha_1, \alpha_2, ...$ And the final cutoff for significance is chosen to be $\hat{\alpha}$ which maximizes the area under ROC curve(AUC).

AMDR with measure pOR, AMDR with measure pRR and AMDR with measure $pChi$ are compared with MDR on the performance of four models with three scenarios of gene-gene interactions. Each model has a unique penetrance function, but Model1 and Model2 are with MAF(minor allele frequency) 0.5 and Model3 and Model4 are with MAF 0.25. There are 5 SNPs from five loci, $l_1, ..., l_5$ into consideration and the three scenarios are: (I)only one 2-way interaction of $l_1 \times l_2$, (II)genetic heterogeneity where some of the cases are related to interaction of $l_1 \times l_2$ and the rest of the cases are related to interaction of $l_4 \times l_5$, (III)additive interactions from $l_1 \times l_2$ and $l_4 \times l_5$. 100 balanced datasets are randomly generated to perform the assessment and the assessment is evaluated by power and Type I error rate. For scenario I, AMDR and MDR both have strong power to find the interaction; for scenario II, MDR has less power to find the interaction with weaker effect on phenotypes; for scenario III, AMDR has more power to find both interactions.

AMDR is applied to a dataset including 34 SNPs collected from 104 individuals in order to investigate the gene-gene interaction effect on the effectiveness of using MTX(methotrexate) to treat JIA(Juvenile Idiopathic Arthritis). AMDR detected 15 significant 2-way interactions among 7 SNPs. $\hat{\alpha} = 0.0167$ is selected to be the threshold maximizing the AUC. The average or median aggregated risk scores of patients without weaker MTX effect are significantly higher than those with stronger

MTX effect($p < 0.0001$). AMDR has an accuracy of 82%in prediction while MDR only has a 75% prediction accuracy.

1.5 Other Extension of MDR

Pair-wise MDR(PW-MDR)

Introduced by He et al. in 2009[20], PW-MDR aims to solve the problem caused by sparseness in cells of contingency table. Supposing the highest order of interaction among SNPs is M, for specified m SNPs, $1 \leq m \leq M$, consider $\binom{m}{2}$ contingency tables for each individual and classify each individual into "High risk" group or "Low risk" group by case-control ratios in each contingency table. The risk score for one subject is how many times he is assigned to "High risk" group minus how many times he is assigned "Low risk" group among the $\binom{m}{2}$ contingency tables. Each subject is classified to be a case if his risk score is not smaller than 0, otherwise, this subject is classified to be a control. The theoretical support for this classification is that if there is no interaction among m SNPs, their joint distribution should be the same in case group and control group[20]. The best model among all plausible m-way interactions, where m is fixed, is determined by CV consistency or prediction error. An exhaustive search could also be performed for any m-way interaction where m is allowed to change from 1 to M and the best model is determined by largest CV consistency or smallest prediction error.

Robust MDR(RMDR)

Gui et. al introduced RMDR in 2011 to address the issue when the case-control ratio is close or equal to the predetermined threshold T as well as sparse or empty cells in contingency tables in MDR[12]. RMDR differs from MDR in labeling contingency table cells as "High risk" or "Low risk". For each contingency table with a cases and b controls, a Fisher's Exact test is performed on a symmetric 2×2 contingency table with a first row entry a and a second row entry b. If the critical value is smaller than the threshold α for Fisher's Exact test, the interaction is labeled as "High risk" or

"Low risk" by case-control ratio, as presented in MDR, otherwise, it is assigned to an "unknown" group. The authors also used an adjusted BA(balanced accuracy) score, $(BA - 0.5)\sqrt{coverage + 0.5}$, where coverage is the proportion of contingency tables labeled not as "unknown group", to measure accuracy of RMDR.

Chapter 2 MDR with P Risk Scores per Person

2.1 Motivation

Assuming there are n subjects and M genes with possible two-way interactions between different genes, it is of interest to consider the risk score of ith individual where $1 \leq i \leq n$.

Let $\boldsymbol{S}$ be a $k \times 1$ vector where $k = \binom{M}{2}$. $\boldsymbol{S}' = (s_1, ..., s_k)$ where s_j is the indicator of whether the jth interaction is significant or not, $1 \leq j \leq k$. If the jth interaction is significant, $s_j = 1$, if the jth interaction is not significant, $s_j = 0$.

Let $\boldsymbol{G_i}$ be a $1 \times k$ vector where $k = \binom{M}{2}$. $\boldsymbol{G_i} = (g_{i1}, ..., g_{ik})$ where g_{ij} is the indicator of whether the ith subject with the jth gene-gene interaction is of "High" risk level to be with the outcome or not, $1 \leq j \leq k$. If the ith subject with the jth gene-gene interaction is of "High" risk level , $g_{ij} = 1$, if the ith subject with the jth gene-gene interaction is not of "High" risk level, $g_{ij} = 0$.

Risk score of ith subject can be obtained by $\boldsymbol{G_i S}$[21]. A 1-dimensional linear model can then be fitted to predict a continuous outcome , such as blood pressure, using data collected from the n subjects to form a fitted risk score by $\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 \boldsymbol{G_i S}$ and the true model is denoted as $Y_i = \beta_0 + \beta_1 \boldsymbol{G_i S} + \varepsilon_i$.

If $\boldsymbol{G_i}$ is a $\binom{M}{2} \times \binom{M}{2}$ diagonal matrix, a $\binom{M}{2}$-dimensional linear model can be fitted to predict a continuous outcome using data collected from the n subjects to form $\binom{M}{2}$ risk scores per person. The fitted model is denoted by $\hat{Y}_i = \hat{\beta}_0 + \hat{\boldsymbol{\beta}}_{\boldsymbol{1}}' \boldsymbol{G_i S}$ and the true model is denoted as $Y_i = \beta_0 + \boldsymbol{\beta_1}' \boldsymbol{G_i S} + \varepsilon_i$ where $\hat{\boldsymbol{\beta}}_{\boldsymbol{1}}$ and $\boldsymbol{\beta_1}$ are $\binom{M}{2} \times 1$ vectors.

But if there is an agreement on a determination to just use p risk scores per person, i.e., to use a p-dimensional prediction and $p << \binom{M}{2}$, then $\boldsymbol{\beta_1}$ is a $p \times 1$ vector, $\boldsymbol{G_i}$ is a $p \times \binom{M}{2}$ matrix, $\boldsymbol{S}$ is still a $\binom{M}{2} \times 1$ vector. Then the question becomes how to choose $\boldsymbol{G_i}$ such that MSE of prediction is minimized?

For example, suppose $M = 3$ and $p = 3$, there are three genes A, B, C and all the possible two-way interactions, AB, AC, BC are significant. Suppose the true model is $Y_i = \beta_0 + \beta_1 I_{(AB)i} + \beta_2 I_{(BC)i} + \beta_3 I_{(AC)i} + \varepsilon_i$ where $I_{(AB)i}, I_{(BC)i}, I_{(AC)i}$ are indicators of whether the ith subject with genotype AB, BC, AC, respectively, is of "High" risk. Then $\boldsymbol{G_i} = \begin{pmatrix} g_{i1} & 0 & 0 \\ 0 & g_{i2} & 0 \\ 0 & 0 & g_{i3} \end{pmatrix}$ is a 3×3 diagonal matrix with each element on diagnal to be 0 or 1, indicating whether the ith subject is with "High" risk genotype AB, BC, AC, receptively. $\boldsymbol{S} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ is a vector of length 3 with every element to be 1 because the assumption above is that all the possible two-way interactions, AB, AC, BC are significant. The fitted 3-dimensional model is $\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}_1 I_{(AB)i} + \hat{\beta}_2 I_{(BC)i} + \hat{\beta}_3 I_{(AC)i}$. In this situation, the risk score $\boldsymbol{G_i} S$ for each subject is a vector of length 3, or to say 3 risk scores.

Further, suppose there is an agreement that it is sufficient to use just 1 risk score(or 1-dimensional model) and still assumes all the possible two-way interactions, AB, AC, BC are significant.

The fitted 1-dimensional model is $\hat{Y}_i = \hat{\beta}_0 + \hat{\beta}(I_{(AB)i} + I_{(BC)i} + I_{(AC)i})$ where $\hat{\beta}_1 = \hat{\beta}_2 = \hat{\beta}_3 = \hat{\beta}$, even though it is not necessarily true that $\beta_1 = \beta_2 = \beta_3$.

In this 1-dimensional situation, $\boldsymbol{G_i} = \begin{pmatrix} g_{i1} & g_{i2} & g_{i3} \end{pmatrix}$, with each element to be 0 or 1 and g_{i1} indicating whether ith subject has high risk genotype AB, g_{i2} indicating whether ith subject has high risk genotype BC, g_{i3} indicating whether ith subject has high risk genotype AC, respectively. $\boldsymbol{S} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ because the assumption is still that all the possible two-way interactions, AB, AC, BC are significant.

Similarly, suppose there is an agreement that it is sufficient to use just 2 risk score(or 2-dimensional model) and still assumes all the possible two-way interactions, AB, AC, BC are significant, i.e., $M = 3$ and $p = 2$. There are three 2-dimensional models in selection:

$Y = \beta_0 + \beta_1 X_1 + \beta_2 (X_2 + X_3) + \varepsilon$

$Y = \beta_0 + \beta_1 X_2 + \beta_2(X_1 + X_3) + \varepsilon$

$Y = \beta_0 + \beta_1 X_3 + \beta_2(X_1 + X_2) + \varepsilon$

where $Y = Y_i,\ X_1 = I_{(AB)i},\ X_2 = I_{(AC)i},\ X_3 = I_{(BC)i}$.

For the above three models, $\boldsymbol{S} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

If the ith person is of "High" risk on all the three gene-gene interactions, $\boldsymbol{G_i}$ is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$, respectively for each model. $\boldsymbol{G_i S}$ is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ for all the three models.

If for a more general situation, $\boldsymbol{G_i}$ is $\begin{pmatrix} g_{i1} & 0 & 0 \\ 0 & g_{i2} & g_{i3} \end{pmatrix}$, $\begin{pmatrix} 0 & g_{i2} & 0 \\ g_{i1} & 0 & g_{i3} \end{pmatrix}$, $\begin{pmatrix} 0 & 0 & g_{i3} \\ g_{i1} & g_{i2} & 0 \end{pmatrix}$, respectively for each model.

The goal becomes to choose the best model from the above three models in order to minimize the MSPE(Mean Square Prediction Error).

2.2 MSE of MDR with P Risk Scores per Person

Theorem 2.2.1 *(taken from [22] with editing and changes in notation)*

Suppose $Y \in R^1,\ \boldsymbol{X} \in R^n$ *and vector* $\begin{pmatrix} Y \\ \boldsymbol{X} \end{pmatrix} \in R^{n+1}$ *with finite expectation and covariance matrix* $\boldsymbol{\Sigma}$, *where* $E\begin{pmatrix} Y \\ \boldsymbol{X} \end{pmatrix} = \begin{pmatrix} \mu_Y \\ \boldsymbol{\mu_X} \end{pmatrix}$ *and* $cov\begin{pmatrix} Y \\ \boldsymbol{X} \end{pmatrix} = \boldsymbol{\Sigma} = \begin{pmatrix} \sigma_{11} & \boldsymbol{\Sigma_{12}} \\ \boldsymbol{\Sigma_{21}} & \boldsymbol{\Sigma_{22}} \end{pmatrix}$. *If* $\boldsymbol{\Sigma_{22}}$ *is positive definite, then* $min_{\boldsymbol{\beta_1} \in R^n, \beta_0 \in R^1} E(Y - \boldsymbol{\beta_1}^T \boldsymbol{X} - \beta_0)^2 = \sigma_{11} - \boldsymbol{\Sigma_{12}} \boldsymbol{\Sigma_{22}}^{-1} \boldsymbol{\Sigma_{21}}$ *when* $\hat{\boldsymbol{\beta}}_{\boldsymbol{1}} = \boldsymbol{\Sigma_{22}^{-1}} \boldsymbol{\Sigma_{21}},\ \hat{\beta}_0 = \mu_Y - \hat{\boldsymbol{\beta}}_{\boldsymbol{1}}^T \boldsymbol{\mu_X}$.

Proof 1 $E(Y - \boldsymbol{\beta_1}^T \boldsymbol{X} - \beta_0)^2 = var(Y - \boldsymbol{\beta_1}^T \boldsymbol{X}) + (\mu_Y - \boldsymbol{\beta_1}^T \boldsymbol{\mu_X} - \beta_0)^2$.

Given $\boldsymbol{\beta_1}$, *the above is minimized at* $\hat{\beta}_0 = \mu_Y - \boldsymbol{\beta_1}^T \boldsymbol{\mu_X}$. *Next find* $\hat{\boldsymbol{\beta}}_{\boldsymbol{1}}$ *to minimize* $var(Y - \boldsymbol{\beta_1}^T \boldsymbol{X})$.

Since $Y - \boldsymbol{\beta_1}^T\boldsymbol{X} = (1, -\boldsymbol{\beta_1}^T)\begin{pmatrix} Y \\ \boldsymbol{X} \end{pmatrix}$,

$var(Y - \boldsymbol{\beta_1}^T\boldsymbol{X}) = \sigma_{11} - \boldsymbol{\beta_1}^T\boldsymbol{\Sigma_{21}} - \boldsymbol{\Sigma_{12}}\boldsymbol{\beta_1} + \boldsymbol{\beta_1}^T\boldsymbol{\Sigma_{22}}\boldsymbol{\beta_1} = \sigma_{11} - 2\boldsymbol{\beta_1}^T\boldsymbol{\Sigma_{21}} + \boldsymbol{\beta_1}^T\boldsymbol{\Sigma_{22}}\boldsymbol{\beta_1}$.

Let $\boldsymbol{b} = \boldsymbol{\Sigma_{22}^{1/2}}\boldsymbol{\beta_1}$, *then*$\boldsymbol{\beta_1}^T\boldsymbol{\Sigma_{22}}\boldsymbol{\beta_1} = \boldsymbol{b}^T\boldsymbol{b}$. *Let* $\boldsymbol{c} = \boldsymbol{b} - \boldsymbol{\Sigma_{22}^{-1/2}}\boldsymbol{\Sigma_{21}}$, *then* $inf_{\boldsymbol{\beta_1}}(\sigma_{11} - 2\boldsymbol{\beta_1}^T\boldsymbol{\Sigma_{21}} + \boldsymbol{\beta_1}^T\boldsymbol{\Sigma_{22}}\boldsymbol{\beta_1}) = inf_{\boldsymbol{c}}(\sigma_{11} - \boldsymbol{\Sigma_{12}}\boldsymbol{\Sigma_{22}^{-1}}\boldsymbol{\Sigma_{21}} + \boldsymbol{c}^T\boldsymbol{c})$.

This is minimized when $\boldsymbol{c} = \boldsymbol{0}$, *then* $\hat{\boldsymbol{\beta}}_{\boldsymbol{1}} = \boldsymbol{\Sigma_{22}^{-1}}\boldsymbol{\Sigma_{21}}$. $\square$

By theorem 2.2.1, let $Y = Y$, $\boldsymbol{X} = \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$, $\boldsymbol{\mu_X} = \begin{pmatrix} \mu_{X_1} \\ \mu_{X_2} \\ \mu_{X_3} \end{pmatrix}$.

And suppose $\begin{pmatrix} Y \\ \boldsymbol{X} \end{pmatrix}$ has finite expectation $\begin{pmatrix} \mu_Y \\ \boldsymbol{\mu_X} \end{pmatrix}$ and covariance matrix $\boldsymbol{\Sigma}$,

where $\begin{pmatrix} \mu_Y \\ \boldsymbol{\mu_X} \end{pmatrix} = \begin{pmatrix} \mu_Y \\ \mu_{X_1} \\ \mu_{X_2} \\ \mu_{X_3} \end{pmatrix}$ and $\boldsymbol{\Sigma} = cov\begin{pmatrix} Y \\ \boldsymbol{X} \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \boldsymbol{\Sigma_{12}} \\ \boldsymbol{\Sigma_{21}} & \boldsymbol{\Sigma_{22}} \end{pmatrix}$

then $\begin{pmatrix} Y \\ X_1 \\ X_2 + X_3 \end{pmatrix}$ has mean $\boldsymbol{M}\begin{pmatrix} \mu_Y \\ \boldsymbol{\mu_X} \end{pmatrix}$ and covariance matrix $\boldsymbol{M\Sigma M}^T$,

since $\begin{pmatrix} Y \\ X_1 \\ X_2 + X_3 \end{pmatrix} = \boldsymbol{M}\begin{pmatrix} Y \\ X_1 \\ X_2 \\ X_3 \end{pmatrix}$, where $\boldsymbol{M} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$.

Partition matrix $\boldsymbol{M}$ as $\left(\begin{array}{c:ccc} 1 & 0 & 0 & 0 \\ \hdashline 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{array}\right)$, and denote this partition as

$\begin{pmatrix} m_{11} & \boldsymbol{M_{12}} \\ \boldsymbol{M_{21}} & \boldsymbol{M_{22}} \end{pmatrix}$.

Then $cov\begin{pmatrix} Y \\ X_1 \\ X_2+X_3 \end{pmatrix} = \begin{pmatrix} m_{11} & \boldsymbol{M_{12}} \\ \boldsymbol{M_{21}} & \boldsymbol{M_{22}} \end{pmatrix} \begin{pmatrix} \sigma_{11} & \boldsymbol{\Sigma_{12}} \\ \boldsymbol{\Sigma_{12}}^T & \boldsymbol{\Sigma_{22}} \end{pmatrix} \begin{pmatrix} m_{11} & \boldsymbol{M_{12}} \\ \boldsymbol{M_{21}} & \boldsymbol{M_{22}} \end{pmatrix}^T$

$= \begin{pmatrix} m_{11}\sigma_{11} + \boldsymbol{M_{12}}\boldsymbol{\Sigma_{12}}^T & m_{11}\boldsymbol{\Sigma_{12}} + \boldsymbol{M_{12}}\boldsymbol{\Sigma_{22}} \\ \boldsymbol{M_{21}}\sigma_{11} + \boldsymbol{M_{22}}\boldsymbol{\Sigma_{12}}^T & \boldsymbol{M_{21}}\boldsymbol{\Sigma_{12}} + \boldsymbol{M_{22}}\boldsymbol{\Sigma_{22}} \end{pmatrix} \begin{pmatrix} m_{11} & \boldsymbol{M_{21}}^T \\ \boldsymbol{M_{12}}^T & \boldsymbol{M_{22}}^T \end{pmatrix}$

$= \begin{pmatrix} (m_{11}\sigma_{11} + \boldsymbol{M_{12}}\boldsymbol{\Sigma_{12}}^T)m_{11} & (m_{11}\sigma_{11} + \boldsymbol{M_{12}}\boldsymbol{\Sigma_{12}}^T)\boldsymbol{M_{21}}^T \\ +(m_{11}\boldsymbol{\Sigma_{12}} + \boldsymbol{M_{12}}\boldsymbol{\Sigma_{22}})\boldsymbol{M_{12}}^T & +(m_{11}\boldsymbol{\Sigma_{12}} + \boldsymbol{M_{12}}\boldsymbol{\Sigma_{22}})\boldsymbol{M_{22}}^T \\ (\boldsymbol{M_{21}}\sigma_{11} + \boldsymbol{M_{22}}\boldsymbol{\Sigma_{12}}^T)m_{11} & (\boldsymbol{M_{21}}\sigma_{11} + \boldsymbol{M_{22}}\boldsymbol{\Sigma_{12}}^T)\boldsymbol{M_{21}}^T \\ +(\boldsymbol{M_{21}}\boldsymbol{\Sigma_{12}} + \boldsymbol{M_{22}}\boldsymbol{\Sigma_{22}})\boldsymbol{M_{12}}^T & +(\boldsymbol{M_{21}}\boldsymbol{\Sigma_{12}} + \boldsymbol{M_{22}}\boldsymbol{\Sigma_{22}})\boldsymbol{M_{22}}^T \end{pmatrix}$

Since $m_{11} = 1, \boldsymbol{M_{12}} = (0,0,0)$ and $\boldsymbol{M_{21}}^T = (0,0)$,

$cov\begin{pmatrix} Y \\ X_1 \\ X_2+X_3 \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \boldsymbol{\Sigma_{12}}\boldsymbol{M_{22}}^T \\ \boldsymbol{M_{22}}\boldsymbol{\Sigma_{12}}^T & \boldsymbol{M_{22}}\boldsymbol{\Sigma_{22}}\boldsymbol{M_{22}}^T \end{pmatrix}.$

Further, without loss of generality,

write $\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_{11} & \boldsymbol{\Sigma_{12}} \\ \boldsymbol{\Sigma_{12}^T} & \boldsymbol{\Sigma_{22}} \end{pmatrix}$ as $\begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} & \sigma_{24} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & \sigma_{34} \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_{44} \end{pmatrix}$,

where $\boldsymbol{\Sigma_{12}} = \begin{pmatrix} \sigma_{12} & \sigma_{13} & \sigma_{14} \end{pmatrix}$ and $\boldsymbol{\Sigma_{22}} = \begin{pmatrix} \sigma_{22} & \sigma_{23} & \sigma_{24} \\ \sigma_{32} & \sigma_{33} & \sigma_{34} \\ \sigma_{42} & \sigma_{43} & \sigma_{44} \end{pmatrix}$.

Then $\boldsymbol{\Sigma_{12}}\boldsymbol{M_{22}}^T = \begin{pmatrix} \sigma_{12} & \sigma_{13} & \sigma_{14} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \sigma_{12} & \sigma_{13}+\sigma_{14} \end{pmatrix}$

$\boldsymbol{M_{22}}\boldsymbol{\Sigma_{22}}\boldsymbol{M_{22}}^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \sigma_{22} & \sigma_{23} & \sigma_{24} \\ \sigma_{32} & \sigma_{33} & \sigma_{34} \\ \sigma_{42} & \sigma_{43} & \sigma_{44} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}^T$

$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} \sigma_{22} & \sigma_{23} & \sigma_{24} \\ \sigma_{23} & \sigma_{33} & \sigma_{34} \\ \sigma_{24} & \sigma_{34} & \sigma_{44} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} \sigma_{22} & \sigma_{23}+\sigma_{24} \\ \sigma_{23}+\sigma_{24} & \sigma_{33}+2\sigma_{34}+\sigma_{44} \end{pmatrix}$$

Thus, $cov \begin{pmatrix} Y \\ X_1 \\ X_2+X_3 \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13}+\sigma_{14} \\ \sigma_{12} & \sigma_{22} & \sigma_{23}+\sigma_{24} \\ \sigma_{13}+\sigma_{14} & \sigma_{23}+\sigma_{24} & \sigma_{33}+2\sigma_{34}+\sigma_{44} \end{pmatrix}$

Denote $cov \begin{pmatrix} Y \\ X_1 \\ X_2+X_3 \end{pmatrix}$ as $\mathbf{\Sigma^{(1)}} = \begin{pmatrix} \sigma_{11}^{(1)} & \mathbf{\Sigma_{12}^{(1)}} \\ \mathbf{\Sigma_{21}^{(1)}} & \mathbf{\Sigma_{22}^{(1)}} \end{pmatrix}$

where $\sigma_{11}^{(1)} = \sigma_{11}$, $\mathbf{\Sigma_{12}^{(1)}} = \mathbf{\Sigma_{21}^{(1)}}^T = \begin{pmatrix} \sigma_{12} & \sigma_{13}+\sigma_{14} \end{pmatrix}$,

$\mathbf{\Sigma_{22}^{(1)}} = \begin{pmatrix} \sigma_{22} & \sigma_{23}+\sigma_{24} \\ \sigma_{23}+\sigma_{24} & \sigma_{33}+2\sigma_{34}+\sigma_{44} \end{pmatrix}$.

The minimal MSE for the first 2-dimensional model is

$$\sigma_{11}^{(1)} - \mathbf{\Sigma_{12}^{(1)}}\mathbf{\Sigma_{22}^{(1)}}^{-1}\mathbf{\Sigma_{21}^{(1)}}$$

$$= \sigma_{11} - \begin{pmatrix} \sigma_{12} & \sigma_{13}+\sigma_{14} \end{pmatrix} \begin{pmatrix} \sigma_{22} & \sigma_{23}+\sigma_{24} \\ \sigma_{23}+\sigma_{24} & \sigma_{33}+2\sigma_{34}+\sigma_{44} \end{pmatrix}^{-1} \begin{pmatrix} \sigma_{12} \\ \sigma_{13}+\sigma_{14} \end{pmatrix}$$

$$= \sigma_{11} - \begin{pmatrix} \sigma_{12} & \sigma_{13}+\sigma_{14} \end{pmatrix} \frac{\begin{pmatrix} \sigma_{33}+2\sigma_{34}+\sigma_{44} & -(\sigma_{23}+\sigma_{24}) \\ -(\sigma_{23}+\sigma_{24}) & \sigma_{22} \end{pmatrix}}{\sigma_{22}(\sigma_{33}+2\sigma_{34}+\sigma_{44}) - (\sigma_{23}+\sigma_{24})^2} \begin{pmatrix} \sigma_{12} \\ \sigma_{13}+\sigma_{14} \end{pmatrix}$$

$$= \sigma_{11} - \frac{\sigma_{12}^2(\sigma_{33}+2\sigma_{34}+\sigma_{44}) - 2\sigma_{12}(\sigma_{13}+\sigma_{14})(\sigma_{23}+\sigma_{24}) + (\sigma_{13}+\sigma_{14})^2\sigma_{22}}{\sigma_{22}(\sigma_{33}+2\sigma_{34}+\sigma_{44}) - (\sigma_{23}+\sigma_{24})^2}.$$

Similarly, $\begin{pmatrix} Y \\ X_2 \\ X_1+X_3 \end{pmatrix}$ is with mean $\boldsymbol{P} \begin{pmatrix} \mu_Y \\ \boldsymbol{\mu_X} \end{pmatrix}$ and covariance matrix $\boldsymbol{P\Sigma P}^T$,

since $\begin{pmatrix} Y \\ X_2 \\ X_1+X_3 \end{pmatrix} = \boldsymbol{P} \begin{pmatrix} Y \\ X_1 \\ X_2 \\ X_3 \end{pmatrix}$, where $\boldsymbol{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$.

Partition matrix $\boldsymbol{P}$ as $\left(\begin{array}{c:ccc} 1 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{array}\right)$, which is denoted as $\begin{pmatrix} p_{11} & \boldsymbol{P_{12}} \\ \boldsymbol{P_{21}} & \boldsymbol{P_{22}} \end{pmatrix}$.

Then $cov\begin{pmatrix} Y \\ X_2 \\ X_1+X_3 \end{pmatrix} = \begin{pmatrix} p_{11} & \boldsymbol{P_{12}} \\ \boldsymbol{P_{21}} & \boldsymbol{P_{22}} \end{pmatrix} \begin{pmatrix} \sigma_{11} & \boldsymbol{\Sigma_{12}} \\ \boldsymbol{\Sigma_{12}}^T & \boldsymbol{\Sigma_{22}} \end{pmatrix} \begin{pmatrix} p_{11} & \boldsymbol{P_{12}} \\ \boldsymbol{P_{21}} & \boldsymbol{P_{22}} \end{pmatrix}^T$

$= \begin{pmatrix} p_{11}\sigma_{11} + \boldsymbol{P_{12}\Sigma_{12}}^T & p_{11}\boldsymbol{\Sigma_{12}} + \boldsymbol{P_{12}\Sigma_{22}} \\ \boldsymbol{P_{21}}\sigma_{11} + \boldsymbol{P_{22}\Sigma_{12}}^T & \boldsymbol{P_{21}\Sigma_{12}} + \boldsymbol{P_{22}\Sigma_{22}} \end{pmatrix} \begin{pmatrix} p_{11} & \boldsymbol{P_{21}}^T \\ \boldsymbol{P_{12}}^T & \boldsymbol{P_{22}}^T \end{pmatrix}$

$= \begin{pmatrix} (p_{11}\sigma_{11} + \boldsymbol{P_{12}\Sigma_{12}}^T)p_{11} + (p_{11}\boldsymbol{\Sigma_{12}} + \boldsymbol{P_{12}\Sigma_{22}})\boldsymbol{P_{12}}^T & (p_{11}\sigma_{11} + \boldsymbol{P_{12}\Sigma_{12}}^T)\boldsymbol{P_{21}}^T + (p_{11}\boldsymbol{\Sigma_{12}} + \boldsymbol{P_{12}\Sigma_{22}})\boldsymbol{P_{22}}^T \\ (\boldsymbol{P_{21}}\sigma_{11} + \boldsymbol{P_{22}\Sigma_{12}}^T)p_{11} + (\boldsymbol{P_{21}\Sigma_{12}} + \boldsymbol{P_{22}\Sigma_{22}})\boldsymbol{P_{12}}^T & (\boldsymbol{P_{21}}\sigma_{11} + \boldsymbol{P_{22}\Sigma_{12}}^T)\boldsymbol{P_{21}}^T + (\boldsymbol{P_{21}\Sigma_{12}} + \boldsymbol{P_{22}\Sigma_{22}})\boldsymbol{P_{22}}^T \end{pmatrix}$

Since $p_{11} = 1, \boldsymbol{P_{12}} = (0,0,0)$ and $\boldsymbol{P_{21}}^T = (0,0)$,

$cov\begin{pmatrix} Y \\ X_2 \\ X_1+X_3 \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \boldsymbol{\Sigma_{12}P_{22}}^T \\ \boldsymbol{P_{22}\Sigma_{12}}^T & \boldsymbol{P_{22}\Sigma_{22}P_{22}}^T \end{pmatrix}.$

Further, without loss of generality,

write $\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_{11} & \boldsymbol{\Sigma_{12}} \\ \boldsymbol{\Sigma_{12}^T} & \boldsymbol{\Sigma_{22}} \end{pmatrix}$ as $\begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} & \sigma_{24} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & \sigma_{34} \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_{44} \end{pmatrix}$,

where $\boldsymbol{\Sigma_{12}} = \begin{pmatrix} \sigma_{12} & \sigma_{13} & \sigma_{14} \end{pmatrix}$ and $\boldsymbol{\Sigma_{22}} = \begin{pmatrix} \sigma_{22} & \sigma_{23} & \sigma_{24} \\ \sigma_{32} & \sigma_{33} & \sigma_{34} \\ \sigma_{42} & \sigma_{43} & \sigma_{44} \end{pmatrix}$.

Then $\boldsymbol{\Sigma_{12}P_{22}}^T = \begin{pmatrix} \sigma_{12} & \sigma_{13} & \sigma_{14} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \sigma_{13} & \sigma_{12} + \sigma_{14} \end{pmatrix}$,

$\boldsymbol{P_{22}\Sigma_{22}P_{22}}^T = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma_{22} & \sigma_{23} & \sigma_{24} \\ \sigma_{32} & \sigma_{33} & \sigma_{34} \\ \sigma_{42} & \sigma_{43} & \sigma_{44} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}^T$

$= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma_{22} & \sigma_{23} & \sigma_{24} \\ \sigma_{23} & \sigma_{33} & \sigma_{34} \\ \sigma_{24} & \sigma_{34} & \sigma_{44} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$= \begin{pmatrix} \sigma_{33} & \sigma_{32} + \sigma_{34} \\ \sigma_{23} + \sigma_{43} & \sigma_{22} + 2\sigma_{24} + \sigma_{44} \end{pmatrix}.$$

Thus, $cov \begin{pmatrix} Y \\ X_2 \\ X_1 + X_3 \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{13} & \sigma_{12} + \sigma_{14} \\ \sigma_{13} & \sigma_{33} & \sigma_{32} + \sigma_{34} \\ \sigma_{12} + \sigma_{14} & \sigma_{23} + \sigma_{43} & \sigma_{22} + 2\sigma_{24} + \sigma_{44} \end{pmatrix}$.

Denote $cov \begin{pmatrix} Y \\ X_2 \\ X_1 + X_3 \end{pmatrix}$ as $\mathbf{\Sigma^{(2)}} = \begin{pmatrix} \sigma_{11}^{(2)} & \mathbf{\Sigma_{12}^{(2)}} \\ \mathbf{\Sigma_{21}^{(2)}} & \mathbf{\Sigma_{22}^{(2)}} \end{pmatrix}$,

where $\sigma_{11}^{(2)} = \sigma_{11}$, $\mathbf{\Sigma_{12}^{(2)}} = \mathbf{\Sigma_{21}^{(2)}}^T = \begin{pmatrix} \sigma_{13} & \sigma_{12} + \sigma_{14} \end{pmatrix}$,

$\mathbf{\Sigma_{22}^{(2)}} = \begin{pmatrix} \sigma_{33} & \sigma_{32} + \sigma_{34} \\ \sigma_{23} + \sigma_{43} & \sigma_{22} + 2\sigma_{24} + \sigma_{44} \end{pmatrix}$.

The minimal MSE for the second 2-dimensional model is

$$\sigma_{11}^{(2)} - \mathbf{\Sigma_{12}^{(2)}} \mathbf{\Sigma_{22}^{(2)}}^{-1} \mathbf{\Sigma_{21}^{(2)}}$$

$$= \sigma_{11} - \begin{pmatrix} \sigma_{13} & \sigma_{12} + \sigma_{14} \end{pmatrix} \begin{pmatrix} \sigma_{33} & \sigma_{32} + \sigma_{34} \\ \sigma_{23} + \sigma_{43} & \sigma_{22} + 2\sigma_{24} + \sigma_{44} \end{pmatrix}^{-1} \begin{pmatrix} \sigma_{13} \\ \sigma_{12} + \sigma_{14} \end{pmatrix}$$

$$= \sigma_{11} - \begin{pmatrix} \sigma_{13} & \sigma_{12} + \sigma_{14} \end{pmatrix} \frac{\begin{pmatrix} \sigma_{22} + 2\sigma_{24} + \sigma_{44} & -(\sigma_{32} + \sigma_{34}) \\ -(\sigma_{23} + \sigma_{43}) & \sigma_{33} \end{pmatrix}}{\sigma_{33}(\sigma_{22} + 2\sigma_{24} + \sigma_{44}) - (\sigma_{23} + \sigma_{34})^2} \begin{pmatrix} \sigma_{13} \\ \sigma_{12} + \sigma_{14} \end{pmatrix}$$

$$= \sigma_{11} - \frac{\sigma_{13}^2(\sigma_{22} + 2\sigma_{24} + \sigma_{44}) - 2\sigma_{13}(\sigma_{12} + \sigma_{14})(\sigma_{23} + \sigma_{34}) + (\sigma_{12} + \sigma_{14})^2 \sigma_{33}}{\sigma_{33}(\sigma_{22} + 2\sigma_{24} + \sigma_{44}) - (\sigma_{23} + \sigma_{34})^2}.$$

For the last model, $\begin{pmatrix} Y \\ X_3 \\ X_1 + X_2 \end{pmatrix}$ has mean $\boldsymbol{Q} \begin{pmatrix} \mu_Y \\ \boldsymbol{\mu_X} \end{pmatrix}$ and covariance matrix $\boldsymbol{Q}\mathbf{\Sigma}\boldsymbol{Q}^T$,

since $\begin{pmatrix} Y \\ X_3 \\ X_1 + X_2 \end{pmatrix} = \boldsymbol{Q} \begin{pmatrix} Y \\ X_1 \\ X_2 \\ X_3 \end{pmatrix}$, where $\boldsymbol{Q} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$.

Partition matrix $\boldsymbol{Q}$ as $\left(\begin{array}{c:ccc} 1 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right)$, which is denoted as $\begin{pmatrix} q_{11} & \boldsymbol{Q_{12}} \\ \boldsymbol{Q_{21}} & \boldsymbol{Q_{22}} \end{pmatrix}$.

Then $cov \begin{pmatrix} Y \\ X_3 \\ X_1 + X_2 \end{pmatrix} = \begin{pmatrix} q_{11} & \boldsymbol{Q_{12}} \\ \boldsymbol{Q_{21}} & \boldsymbol{Q_{22}} \end{pmatrix} \begin{pmatrix} \sigma_{11} & \boldsymbol{\Sigma_{12}} \\ \boldsymbol{\Sigma_{12}}^T & \boldsymbol{\Sigma_{22}} \end{pmatrix} \begin{pmatrix} q_{11} & \boldsymbol{Q_{12}} \\ \boldsymbol{Q_{21}} & \boldsymbol{Q_{22}} \end{pmatrix}^T$

$= \begin{pmatrix} q_{11}\sigma_{11} + \boldsymbol{Q_{12}\Sigma_{12}}^T & q_{11}\boldsymbol{\Sigma_{12}} + \boldsymbol{Q_{12}\Sigma_{22}} \\ \boldsymbol{Q_{21}}\sigma_{11} + \boldsymbol{Q_{22}\Sigma_{12}}^T & \boldsymbol{Q_{21}\Sigma_{12}} + \boldsymbol{Q_{22}\Sigma_{22}} \end{pmatrix} \begin{pmatrix} q_{11} & \boldsymbol{Q_{21}}^T \\ \boldsymbol{Q_{12}}^T & \boldsymbol{Q_{22}}^T \end{pmatrix}$

$= \begin{pmatrix} (q_{11}\sigma_{11} + \boldsymbol{Q_{12}\Sigma_{12}}^T)q_{11} & (q_{11}\sigma_{11} + \boldsymbol{Q_{12}\Sigma_{12}}^T)\boldsymbol{Q_{21}}^T \\ +(q_{11}\boldsymbol{\Sigma_{12}} + \boldsymbol{Q_{12}\Sigma_{22}})\boldsymbol{Q_{12}}^T & +(q_{11}\boldsymbol{\Sigma_{12}} + \boldsymbol{Q_{12}\Sigma_{22}})\boldsymbol{Q_{22}}^T \\ (\boldsymbol{Q_{21}}\sigma_{11} + \boldsymbol{Q_{22}\Sigma_{12}}^T)q_{11} & (\boldsymbol{Q_{21}}\sigma_{11} + \boldsymbol{Q_{22}\Sigma_{12}}^T)\boldsymbol{Q_{21}}^T \\ +(\boldsymbol{Q_{21}\Sigma_{12}} + \boldsymbol{Q_{22}\Sigma_{22}})\boldsymbol{Q_{12}}^T & +(\boldsymbol{Q_{21}\Sigma_{12}} + \boldsymbol{Q_{22}\Sigma_{22}})\boldsymbol{Q_{22}}^T \end{pmatrix}$

Since $q_{11} = 1, \boldsymbol{Q_{12}} = (0,0,0)$ and $\boldsymbol{Q_{21}}^T = (0,0)$,

$cov \begin{pmatrix} Y \\ X_3 \\ X_1 + X_2 \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \boldsymbol{\Sigma_{12}Q_{22}}^T \\ \boldsymbol{Q_{22}\Sigma_{12}}^T & \boldsymbol{Q_{22}\Sigma_{22}Q_{22}}^T \end{pmatrix}$.

Further, without loss of generality,

write $\boldsymbol{\Sigma} = \begin{pmatrix} \sigma_{11} & \boldsymbol{\Sigma_{12}} \\ \boldsymbol{\Sigma_{12}^T} & \boldsymbol{\Sigma_{22}} \end{pmatrix}$ as $\begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} & \sigma_{24} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} & \sigma_{34} \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_{44} \end{pmatrix}$,

where $\boldsymbol{\Sigma_{12}} = \begin{pmatrix} \sigma_{12} & \sigma_{13} & \sigma_{14} \end{pmatrix}$ and $\boldsymbol{\Sigma_{22}} = \begin{pmatrix} \sigma_{22} & \sigma_{23} & \sigma_{24} \\ \sigma_{32} & \sigma_{33} & \sigma_{34} \\ \sigma_{42} & \sigma_{43} & \sigma_{44} \end{pmatrix}$.

Then $\boldsymbol{\Sigma_{12}Q_{22}}^T = \begin{pmatrix} \sigma_{12} & \sigma_{13} & \sigma_{14} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} \sigma_{14} & \sigma_{12} + \sigma_{13} \end{pmatrix}$

$\boldsymbol{Q_{22}\Sigma_{22}Q_{22}}^T = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \sigma_{22} & \sigma_{23} & \sigma_{24} \\ \sigma_{32} & \sigma_{33} & \sigma_{34} \\ \sigma_{42} & \sigma_{43} & \sigma_{44} \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}^T$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} \sigma_{22} & \sigma_{23} & \sigma_{24} \\ \sigma_{23} & \sigma_{33} & \sigma_{34} \\ \sigma_{24} & \sigma_{34} & \sigma_{44} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \sigma_{44} & \sigma_{42} + \sigma_{43} \\ \sigma_{24} + \sigma_{34} & \sigma_{22} + 2\sigma_{23} + \sigma_{33} \end{pmatrix}.$$

Thus, $cov \begin{pmatrix} Y \\ X_3 \\ X_1 + X_2 \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{14} & \sigma_{12} + \sigma_{13} \\ \sigma_{14} & \sigma_{44} & \sigma_{42} + \sigma_{43} \\ \sigma_{12} + \sigma_{13} & \sigma_{24} + \sigma_{34} & \sigma_{22} + 2\sigma_{23} + \sigma_{33} \end{pmatrix}$.

Denote $cov \begin{pmatrix} Y \\ X_3 \\ X_1 + X_2 \end{pmatrix}$ as $\mathbf{\Sigma^{(3)}} = \begin{pmatrix} \sigma_{11}^{(3)} & \mathbf{\Sigma_{12}^{(3)}} \\ \mathbf{\Sigma_{21}^{(3)}} & \mathbf{\Sigma_{22}^{(3)}} \end{pmatrix}$,

where $\sigma_{11}^{(3)} = \sigma_{11}$, $\mathbf{\Sigma_{12}^{(3)}} = \mathbf{\Sigma_{21}^{(3)}}^T = \begin{pmatrix} \sigma_{14} & \sigma_{12} + \sigma_{13} \end{pmatrix}$,

$\mathbf{\Sigma_{22}^{(3)}} = \begin{pmatrix} \sigma_{44} & \sigma_{42} + \sigma_{43} \\ \sigma_{24} + \sigma_{34} & \sigma_{22} + 2\sigma_{23} + \sigma_{33} \end{pmatrix}$.

The minimal MSE for the third 2-dimensional model is

$$\sigma_{11}^{(3)} - \mathbf{\Sigma_{12}^{(3)}} \mathbf{\Sigma_{22}^{(3)}}^{-1} \mathbf{\Sigma_{21}^{(3)}}$$

$$= \sigma_{11} - \begin{pmatrix} \sigma_{14} & \sigma_{12} + \sigma_{13} \end{pmatrix} \begin{pmatrix} \sigma_{44} & \sigma_{42} + \sigma_{43} \\ \sigma_{24} + \sigma_{34} & \sigma_{22} + 2\sigma_{23} + \sigma_{33} \end{pmatrix}^{-1} \begin{pmatrix} \sigma_{14} \\ \sigma_{12} + \sigma_{13} \end{pmatrix}$$

$$= \sigma_{11} - \begin{pmatrix} \sigma_{14} & \sigma_{12} + \sigma_{13} \end{pmatrix} \frac{\begin{pmatrix} \sigma_{22} + 2\sigma_{23} + \sigma_{33} & -(\sigma_{42} + \sigma_{43}) \\ -(\sigma_{24} + \sigma_{34}) & \sigma_{44} \end{pmatrix}}{\sigma_{44}(\sigma_{22} + 2\sigma_{23} + \sigma_{33}) - (\sigma_{42} + \sigma_{43})^2} \begin{pmatrix} \sigma_{14} \\ \sigma_{12} + \sigma_{13} \end{pmatrix}$$

$$= \sigma_{11} - \frac{\sigma_{14}^2(\sigma_{22} + 2\sigma_{23} + \sigma_{33}) - 2\sigma_{14}(\sigma_{24} + \sigma_{34})(\sigma_{12} + \sigma_{13}) + (\sigma_{12} + \sigma_{13})^2\sigma_{44}}{\sigma_{44}(\sigma_{22} + 2\sigma_{23} + \sigma_{33}) - (\sigma_{42} + \sigma_{43})^2}.$$

2.3 Special Cases

All of the following cases are considering the simplified model $Y = \beta_0 + \beta_1 X_1 + \beta_2(X_2 + X_3) + \varepsilon$ vs the true model $Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \varepsilon$ with respect to transformation matrix $M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}$.

Case1

Assume $\mathbf{\Sigma} = cov\begin{pmatrix} Y \\ \boldsymbol{X} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, which is a case that $Y,\ X_1,\ X_2,\ X_3$ are pairwise uncorrelated.

$$\mathbf{\Sigma^{(1)}} = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13}+\sigma_{14} \\ \sigma_{12} & \sigma_{22} & \sigma_{23}+\sigma_{24} \\ \sigma_{13}+\sigma_{14} & \sigma_{23}+\sigma_{24} & \sigma_{33}+2\sigma_{34}+\sigma_{44} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$\text{MSE}_{True} = \sigma_{11} - \mathbf{\Sigma_{12}}\mathbf{\Sigma_{22}}^{-1}\mathbf{\Sigma_{21}} = 1 - 0 = 1.$

$\text{MSE}_{Simplified} = \sigma_{11}^{(1)} - \mathbf{\Sigma_{12}^{(1)}}\mathbf{\Sigma_{22}^{(1)}}^{-1}\mathbf{\Sigma_{21}^{(1)}} = 1.$

In Case 1, MSE_{True}=$\text{MSE}_{Simplified}$ because the covariance of Y and X_1, covariance of Y and X_2, covariance of Y and X_3 are the same and $X_1,\ X_2,\ X_3$ are pairwise uncorrelated.

Case2

Assume $\mathbf{\Sigma} = cov\begin{pmatrix} Y \\ \boldsymbol{X} \end{pmatrix} = \begin{pmatrix} 1 & \sigma_{12} & \sigma_{13} & \sigma_{14} \\ \sigma_{21} & 1 & 0 & 0 \\ \sigma_{31} & 0 & 1 & 0 \\ \sigma_{41} & 0 & 0 & 1 \end{pmatrix}$.

This is a case where Y is correlated with $X_1,\ X_2,\ X_3$ and correlation coefficient $\sigma_{12} = \sigma_{21},\ \sigma_{13} = \sigma_{31},\ \sigma_{14} = \sigma_{41}$, respectively and $X_1,\ X_2,\ X_3$ are pairwise uncorrelated.

$$\mathbf{\Sigma^{(1)}} = \begin{pmatrix} 1 & \sigma_{12} & \sigma_{13}+\sigma_{14} \\ \sigma_{12} & 1 & 0 \\ \sigma_{13}+\sigma_{14} & 0 & 2 \end{pmatrix}$$

$$\text{MSE}_{True} = \sigma_{11} - \mathbf{\Sigma_{12}}\mathbf{\Sigma_{22}}^{-1}\mathbf{\Sigma_{21}} = 1 - \begin{pmatrix} \sigma_{12} & \sigma_{13} & \sigma_{14} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \sigma_{12} \\ \sigma_{13} \\ \sigma_{14} \end{pmatrix}$$

$= 1 - (\sigma_{12}^2 + \sigma_{13}^2 + \sigma_{14}^2).$

$\text{MSE}_{Simplified} = \sigma_{11}^{(1)} - \mathbf{\Sigma_{12}^{(1)}} \mathbf{\Sigma_{22}^{(1)}}^{-1} \mathbf{\Sigma_{21}^{(1)}}$

$= 1 - \begin{pmatrix} \sigma_{12} & \sigma_{13} + \sigma_{14} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} \sigma_{12} \\ \sigma_{13} + \sigma_{14} \end{pmatrix}$

$= 1 - [\sigma_{12}^2 + \frac{1}{2}(\sigma_{13} + \sigma_{14})^2].$

In case 2, MSE_{True} is less than or equal to $\text{MSE}_{Simplified}$ because

$\text{MSE}_{True} - \text{MSE}_{Simplified} = 1 - (\sigma_{12}^2 + \sigma_{13}^2 + \sigma_{14}^2) - \{1 - [\sigma_{12}^2 + \frac{1}{2}(\sigma_{13} + \sigma_{14})^2]\}$

$= -\dfrac{1}{2}(\sigma_{13} - \sigma_{14})^2 \leq 0$

and $\text{MSE}_{True} = \text{MSE}_{Simplified}$ when $\sigma_{13} = \sigma_{14}$.

Case3

Assume $\mathbf{\Sigma} = cov \begin{pmatrix} Y \\ \boldsymbol{X} \end{pmatrix} = \begin{pmatrix} 1 & \sigma_{12} & \sigma_{13} & \sigma_{14} \\ \sigma_{21} & 1 & 0 & 0 \\ \sigma_{31} & 0 & 1 & \sigma_{34} \\ \sigma_{41} & 0 & \sigma_{43} & 1 \end{pmatrix}$.

This is a case where Y is correlated with $X_1,\ X_2,\ X_3$ and correlation coefficient $\sigma_{12} = \sigma_{21},\ \sigma_{13} = \sigma_{31},\ \sigma_{14} = \sigma_{41}$, respectively, X_1 is uncorrelated with X_2 or X_3 and X_2 is correlated with X_3 with correlation coefficient $\sigma_{34} = \sigma_{43}$.

$\mathbf{\Sigma^{(1)}} = \begin{pmatrix} 1 & \sigma_{12} & \sigma_{13} + \sigma_{14} \\ \sigma_{12} & 1 & 0 \\ \sigma_{13} + \sigma_{14} & 0 & 2 + 2\sigma_{34} \end{pmatrix}$

$\text{MSE}_{True} = \sigma_{11} - \mathbf{\Sigma_{12}} \mathbf{\Sigma_{22}}^{-1} \mathbf{\Sigma_{21}}$

$= 1 - \begin{pmatrix} \sigma_{12} & \sigma_{13} & \sigma_{14} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \sigma_{34} \\ 0 & \sigma_{34} & 1 \end{pmatrix}^{-1} \begin{pmatrix} \sigma_{12} \\ \sigma_{13} \\ \sigma_{14} \end{pmatrix}$

$= 1 - [\sigma_{12}^2 + \dfrac{\sigma_{13}^2 + \sigma_{14}^2 - 2\sigma_{13}\sigma_{14}\sigma_{34}}{1 - \sigma_{34}^2}].$

$\text{MSE}_{Simplified} = \sigma_{11}^{(1)} - \mathbf{\Sigma_{12}^{(1)}} \mathbf{\Sigma_{22}^{(1)}}^{-1} \mathbf{\Sigma_{21}^{(1)}}$

$= 1 - \begin{pmatrix} \sigma_{12} & \sigma_{13} + \sigma_{14} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2+2\sigma_{34}} \end{pmatrix} \begin{pmatrix} \sigma_{12} \\ \sigma_{13} + \sigma_{14} \end{pmatrix}$

$= 1 - [\sigma_{12}^2 + \frac{(\sigma_{13}+\sigma_{14})^2}{2+2\sigma_{34}}].$

In case 3, $\text{MSE}_{True} - \text{MSE}_{Simplified}$

$= 1 - [\sigma_{12}^2 + \dfrac{\sigma_{13}^2 + \sigma_{14}^2 - 2\sigma_{13}\sigma_{14}\sigma_{34}}{1-\sigma_{34}^2}] - \{1 - [\sigma_{12}^2 + \frac{(\sigma_{13}+\sigma_{14})^2}{2+2\sigma_{34}}]\}$

$= \dfrac{-2[\sigma_{13}^2 + \sigma_{14}^2 - 2\sigma_{13}\sigma_{14}\sigma_{34}] + (\sigma_{13}+\sigma_{14})^2(1-\sigma_{34})}{2(1-\sigma_{34})(1+\sigma_{34})}$

$= \dfrac{-\sigma_{13}^2 - \sigma_{14}^2 + 2\sigma_{13}\sigma_{14}\sigma_{34} + 2\sigma_{13}\sigma_{14} - \sigma_{13}^2\sigma_{34} - \sigma_{14}^2\sigma_{34}}{2(1-\sigma_{34})(1+\sigma_{34})}$

$= \dfrac{(\sigma_{13}-\sigma_{14})^2(-1-\sigma_{34})}{2(1-\sigma_{34})(1+\sigma_{34})}$

$= \dfrac{-(\sigma_{13}-\sigma_{14})^2}{2(1-\sigma_{34})}$

Thus, $\text{MSE}_{True} \le \text{MSE}_{Simplified}$ and $\text{MSE}_{True} = \text{MSE}_{Simplified}$ when $\sigma_{13} = \sigma_{14}$.

Case4

Assume $\mathbf{\Sigma} = cov\begin{pmatrix} Y \\ \boldsymbol{X} \end{pmatrix} = \begin{pmatrix} 1 & \sigma_{12} & \sigma_{13} & \sigma_{14} \\ \sigma_{21} & 1 & \sigma_{23} & 0 \\ \sigma_{31} & \sigma_{32} & 1 & 0 \\ \sigma_{41} & 0 & 0 & 1 \end{pmatrix}$.

This is a case where Y is correlated with $X_1,\ X_2,\ X_3$ and correlation coefficient $\sigma_{12} = \sigma_{21},\ \sigma_{13} = \sigma_{31},\ \sigma_{14} = \sigma_{41}$, respectively, X_3 is uncorrelated with X_1 or X_2 and X_1 is correlated with X_2 with correlation coefficient $\sigma_{23} = \sigma_{32}$.

$\mathbf{\Sigma^{(1)}} = \begin{pmatrix} 1 & \sigma_{12} & \sigma_{13}+\sigma_{14} \\ \sigma_{12} & 1 & \sigma_{23} \\ \sigma_{13}+\sigma_{14} & \sigma_{23} & 2 \end{pmatrix}$

$\text{MSE}_{True} = \sigma_{11} - \mathbf{\Sigma_{12}}\mathbf{\Sigma_{22}}^{-1}\mathbf{\Sigma_{21}}$

$= 1 - \begin{pmatrix} \sigma_{12} & \sigma_{13} & \sigma_{14} \end{pmatrix} \begin{pmatrix} 1 & \sigma_{23} & 0 \\ \sigma_{23} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} \sigma_{12} \\ \sigma_{13} \\ \sigma_{14} \end{pmatrix}$

$= 1 - \begin{pmatrix} \sigma_{12} & \sigma_{13} & \sigma_{14} \end{pmatrix} \begin{pmatrix} \dfrac{1}{1-\sigma_{23}^2} & \dfrac{-\sigma_{23}}{1-\sigma_{23}^2} & 0 \\ \dfrac{-\sigma_{23}}{1-\sigma_{23}^2} & \dfrac{1}{1-\sigma_{23}^2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sigma_{12} \\ \sigma_{13} \\ \sigma_{14} \end{pmatrix}$

$= 1 - [\frac{\sigma_{12}^2 - 2\sigma_{12}\sigma_{13}\sigma_{23} + \sigma_{13}^2}{1-\sigma_{23}^2} + \sigma_{14}^2]$.

$\text{MSE}_{Simplified} = \sigma_{11}^{(1)} - \mathbf{\Sigma_{12}^{(1)}}\mathbf{\Sigma_{22}^{(1)}}^{-1}\mathbf{\Sigma_{21}^{(1)}}$

$$= 1 - \begin{pmatrix} \sigma_{12} & \sigma_{13}+\sigma_{14} \end{pmatrix} \begin{pmatrix} 1 & \sigma_{23} \\ \sigma_{23} & 2 \end{pmatrix}^{-1} \begin{pmatrix} \sigma_{12} \\ \sigma_{13}+\sigma_{14} \end{pmatrix}$$

$$= 1 - \begin{pmatrix} \sigma_{12} & \sigma_{13}+\sigma_{14} \end{pmatrix} \frac{\begin{pmatrix} 2 & -\sigma_{23} \\ -\sigma_{23} & 1 \end{pmatrix}}{2-\sigma_{23}^2} \begin{pmatrix} \sigma_{12} \\ \sigma_{13}+\sigma_{14} \end{pmatrix}$$

$$= 1 - \frac{2\sigma_{12}^2+(\sigma_{13}+\sigma_{14})^2-2\sigma_{12}\sigma_{13}\sigma_{23}-2\sigma_{12}\sigma_{14}\sigma_{23}}{2-\sigma_{23}^2}.$$

Case5

Assume $\mathbf{\Sigma} = cov \begin{pmatrix} Y \\ \boldsymbol{X} \end{pmatrix} = \begin{pmatrix} 1 & \sigma_{12} & \sigma_{13} & \sigma_{14} \\ \sigma_{21} & 1 & \sigma_{23} & \sigma_{24} \\ \sigma_{31} & \sigma_{32} & 1 & \sigma_{34} \\ \sigma_{41} & \sigma_{42} & \sigma_{43} & 1 \end{pmatrix}.$

This is a case where Y is correlated with $X_1,\ X_2,\ X_3$ and correlation coefficient $\sigma_{12} = \sigma_{21},\ \sigma_{13} = \sigma_{31},\ \sigma_{14} = \sigma_{41}$, respectively; X_1 is correlated with X_2 and X_3 with correlation coefficient $\sigma_{23} = \sigma_{32},\ \sigma_{24} = \sigma_{42}$, respectively and X_2 is correlated with X_3 with correlation coefficient $\sigma_{34} = \sigma_{43}$.

$$\mathbf{\Sigma^{(1)}} = \begin{pmatrix} 1 & \sigma_{12} & \sigma_{13}+\sigma_{14} \\ \sigma_{12} & 1 & \sigma_{23}+\sigma_{24} \\ \sigma_{13}+\sigma_{14} & \sigma_{23}+\sigma_{24} & 2+2\sigma_{34} \end{pmatrix}$$

$\text{MSE}_{True} = \sigma_{11} - \mathbf{\Sigma_{12}\Sigma_{22}}^{-1}\mathbf{\Sigma_{21}}$

$$= 1 - \begin{pmatrix} \sigma_{12} & \sigma_{13} & \sigma_{14} \end{pmatrix} \begin{pmatrix} 1 & \sigma_{23} & \sigma_{24} \\ \sigma_{23} & 1 & \sigma_{34} \\ \sigma_{24} & \sigma_{34} & 1 \end{pmatrix}^{-1} \begin{pmatrix} \sigma_{12} \\ \sigma_{13} \\ \sigma_{14} \end{pmatrix}$$

$$= 1 - \begin{pmatrix} \sigma_{12} & \sigma_{13} & \sigma_{14} \end{pmatrix} \frac{\begin{pmatrix} 1-\sigma_{34}^2 & \sigma_{24}\sigma_{34}-\sigma_{23} & \sigma_{23}\sigma_{34}-\sigma_{24} \\ \sigma_{34}\sigma_{24}-\sigma_{23} & 1-\sigma_{24}^2 & \sigma_{23}\sigma_{24}-\sigma_{34} \\ \sigma_{23}\sigma_{34}-\sigma_{24} & \sigma_{23}\sigma_{24}-\sigma_{34} & 1-\sigma_{23}^2 \end{pmatrix}}{1-\sigma_{23}^2-\sigma_{24}^2-\sigma_{34}^2+2\sigma_{23}\sigma_{34}\sigma_{24}} \begin{pmatrix} \sigma_{12} \\ \sigma_{13} \\ \sigma_{14} \end{pmatrix}$$

$$= 1 - \frac{\sigma_{12}^2+\sigma_{13}^2+\sigma_{14}^2-\sigma_{12}^2\sigma_{34}^2-\sigma_{13}^2\sigma_{24}^2-\sigma_{14}^2\sigma_{23}^2-2(\sigma_{12}\sigma_{23}\sigma_{13}+\sigma_{13}\sigma_{14}\sigma_{34}+\sigma_{12}\sigma_{14}\sigma_{24})}{1-\sigma_{23}^2-\sigma_{24}^2-\sigma_{34}^2+2\sigma_{23}\sigma_{34}\sigma_{24}}$$

$$- \frac{2(\sigma_{12}\sigma_{13}\sigma_{24}\sigma_{34}+\sigma_{13}\sigma_{14}\sigma_{23}\sigma_{24}+\sigma_{12}\sigma_{14}\sigma_{23}\sigma_{34})}{1-\sigma_{23}^2-\sigma_{24}^2-\sigma_{34}^2+2\sigma_{23}\sigma_{34}\sigma_{24}}.$$

$$
\begin{aligned}
\text{MSE}_{Simplified} &= \sigma_{11}^{(1)} - \mathbf{\Sigma_{12}^{(1)}}\mathbf{\Sigma_{22}^{(1)}}^{-1}\mathbf{\Sigma_{21}^{(1)}} \\
&= 1 - \begin{pmatrix} \sigma_{12} & \sigma_{13}+\sigma_{14} \end{pmatrix} \begin{pmatrix} 1 & \sigma_{23}+\sigma_{24} \\ \sigma_{23}+\sigma_{24} & 2+2\sigma_{34} \end{pmatrix}^{-1} \begin{pmatrix} \sigma_{12} \\ \sigma_{13}+\sigma_{14} \end{pmatrix} \\
&= 1 - \begin{pmatrix} \sigma_{12} & \sigma_{13}+\sigma_{14} \end{pmatrix} \frac{\begin{pmatrix} 2+2\sigma_{34} & -(\sigma_{23}+\sigma_{24}) \\ -(\sigma_{23}+\sigma_{24}) & 1 \end{pmatrix}}{(2+2\sigma_{34})-(\sigma_{23}+\sigma_{24})^2} \begin{pmatrix} \sigma_{12} \\ \sigma_{13}+\sigma_{14} \end{pmatrix} \\
&= 1 - \frac{\sigma_{12}^2(2+2\sigma_{34})-2(\sigma_{13}+\sigma_{14})(\sigma_{23}+\sigma_{24})+(\sigma_{13}+\sigma_{14})^2}{(2+2\sigma_{34})-(\sigma_{23}+\sigma_{24})^2}.
\end{aligned}
$$

2.4 Numerical Evaluation of Special Cases

The following simulation is on comparison between simplified model $Y = \beta_0+\beta_1X_1+\beta_2(X_2+X_3)+\varepsilon$ and the true model $Y = \beta_0+\beta_1X_1+\beta_2X_2+\beta_3X_3+\varepsilon$.

Since in case 1 above, MSE_{True}=$\text{MSE}_{Simplified}$, the simulation is performed for cases 2-5 only.

In case 2, let $\sigma_{12}, \sigma_{13}, \sigma_{14}$ equal 0.3 or 0.7, respectively. Thus, there would be 8 possible combinations of $(\sigma_{12}, \sigma_{13}, \sigma_{14})$. For each combination, if the covariance matrix is positive definite, then calculate the corresponding MSE_{True}, $\text{MSE}_{Simplified}$, MSE_{True}-$\text{MSE}_{Simplified}$ and $\text{MSE}_{True}/\text{MSE}_{Simplified}$; otherwise, take this combination out of consideration.

In case 3, let $\sigma_{12}, \sigma_{13}, \sigma_{14}, \sigma_{34}$ equal 0.3 or 0.7, respectively. Thus, there would be 16 possible combinations of $(\sigma_{12}, \sigma_{13}, \sigma_{14}, \sigma_{34})$. For each combination, if the covariance matrix is positive definite, then calculate the corresponding MSE_{True}, $\text{MSE}_{Simplified}$, MSE_{True}-$\text{MSE}_{Simplified}$ and $\text{MSE}_{True}/\text{MSE}_{Simplified}$; otherwise, take this combination out of consideration.

In case 4, let $\sigma_{12}, \sigma_{13}, \sigma_{14}, \sigma_{23}$ equal 0.3 or 0.7, respectively. Thus, there would be 16 possible combinations of $(\sigma_{12}, \sigma_{13}, \sigma_{14}, \sigma_{23})$. For each combination, if the covariance matrix is positive definite, then calculate the corresponding MSE_{True}, $\text{MSE}_{Simplified}$, MSE_{True}-$\text{MSE}_{Simplified}$ and $\text{MSE}_{True}/\text{MSE}_{Simplified}$; otherwise, take this combination out of consideration.

In case 5, let $\sigma_{12}, \sigma_{13}, \sigma_{14}, \sigma_{23}, \sigma_{24}, \sigma_{34}$ equal 0.3 or 0.7, respectively. Thus, there would be 64 possible combinations of $(\sigma_{12}, \sigma_{13}, \sigma_{14}, , \sigma_{23}, \sigma_{24}, \sigma_{34})$. For each combination, if the covariance matrix is positive definite, then calculate the corresponding MSE_{True}, $\text{MSE}_{Simplified}$, MSE_{True}-$\text{MSE}_{Simplified}$ and $\text{MSE}_{True}/\text{MSE}_{Simplified}$; otherwise, take this combination out of consideration and the corresponding rows are left blank.

The simulation results are listed on the following Table 2.1, Table 2.2, Table 2.3, Table 2.4, respectively.

Conclusions on patterns found in tables are:

(i) In case 2, when X_i's are independent, the simplified model performs relatively good, especially when σ_{13} and σ_{14} are relatively close.

Table 2.1: Simulation results of MSE comparison between true model and simplified model in Case 2

σ_{12}	σ_{13}	σ_{14}	MSE_{True}	$MSE_{Simplified}$	$MSE_{True} - MSE_{Simplified}$	$\frac{MSE_{True}}{MSE_{Simplified}}$
0.3	0.3	0.3	0.73	0.73	0	1
0.7	0.3	0.3	0.33	0.33	0	1
0.3	0.7	0.3	0.33	0.41	-0.08	0.805
0.7	0.7	0.3				
0.3	0.3	0.7	0.33	0.41	-0.08	0.805
0.7	0.3	0.7				
0.3	0.7	0.7				
0.7	0.7	0.7				

(ii) In case 3, if σ_{13} and σ_{14} are relatively close, the simplified model performs relatively good. But if σ_{13} and σ_{14} are of relatively big difference, the simplified model performs bad if σ_{12} is small and σ_{34} is large, the simplified model performs worse when σ_{34} is small and σ_{12} is large and the simplified model performs relatively good when σ_{34} and σ_{12} are close.

Table 2.2: Simulation results of MSE comparison between true model and simplified model in Case 3

σ_{12}	σ_{13}	σ_{14}	σ_{34}	MSE_{True}	$MSE_{Simplified}$	$MSE_{True} - MSE_{Simplified}$	$\frac{MSE_{True}}{MSE_{Simplified}}$
0.3	0.3	0.3	0.3	0.772	0.772	0	1
0.7	0.3	0.3	0.3	0.372	0.372	0	1
0.3	0.7	0.3	0.3	0.411	0.525	-0.114	0.782
0.7	0.7	0.3	0.3	0.011	0.125	-0.114	0.089
0.3	0.3	0.7	0.3	0.411	0.525	-0.114	0.782
0.7	0.3	0.7	0.3	0.011	0.125	-0.114	0.089
0.3	0.7	0.7	0.3	0.156	0.156	0	1
0.7	0.7	0.7	0.3				
0.3	0.3	0.3	0.7	0.804	0.804	0	1
0.7	0.3	0.3	0.7	0.404	0.404	0	1
0.3	0.7	0.3	0.7	0.349	0.616	-0.267	0.567
0.7	0.7	0.3	0.7				
0.3	0.3	0.7	0.7	0.349	0.616	-0.267	0.567
0.7	0.3	0.7	0.7				
0.3	0.7	0.7	0.7	0.334	0.334	0	1
0.7	0.7	0.7	0.7				

(iii) In case 4, if σ_{13} and σ_{14} are relatively close, the simplified model performs relatively good, except when σ_{12} and σ_{23} are large. But when σ_{13} and σ_{14} are of relatively big difference, the simplified model performs bad if σ_{13}, σ_{23} are small and σ_{14} is large and if σ_{14}, σ_{12} are small and σ_{13}, σ_{23} are large. The simplified model performs relatively good for the rest values of σ_{12} and σ_{23}.

Table 2.3: Simulation results of MSE comparison between true model and simplified model in Case 4

σ_{12}	σ_{13}	σ_{14}	σ_{23}	MSE_{True}	$MSE_{Simplified}$	$MSE_{True} - MSE_{Simplified}$	$\frac{MSE_{True}}{MSE_{Simplified}}$
0.3	0.3	0.3	0.3	0.772	0.774	-0.002	0.997
0.7	0.3	0.3	0.3	0.411	0.43	-0.019	0.955
0.3	0.7	0.3	0.3	0.411	0.476	-0.065	0.863
0.7	0.7	0.3	0.3	0.156	0.183	-0.027	0.852
0.3	0.3	0.7	0.3	0.372	0.476	-0.105	0.78
0.7	0.3	0.7	0.3	0.011	0.183	-0.172	0.061
0.3	0.7	0.7	0.3	0.011	0.012	0	0.964
0.7	0.7	0.7	0.3				
0.3	0.3	0.3	0.7	0.804	0.809	-0.005	0.994
0.7	0.3	0.3	0.7	0.349	0.502	-0.153	0.696
0.3	0.7	0.3	0.7	0.349	0.497	-0.147	0.703
0.7	0.7	0.3	0.7	0.334	0.338	-0.004	0.988
0.3	0.3	0.7	0.7	0.404	0.497	-0.093	0.814
0.7	0.3	0.7	0.7				
0.3	0.7	0.7	0.7				
0.7	0.7	0.7	0.7				

(iv) In case 5, if $MSE_{True} - MSE_{Simplified} < -0.2$ or $\frac{MSE_{True}}{MSE_{Simplified}} < 0.6$. The simplified model is considered to perform relatively bad. Thus, simplified models with index numbers 14, 20, 35, 36, 37, 38, 42, 43, 45, 50, 51, 53, 59, 60, 61, 62 in Table 2.5 performs relatively bad.

Table 2.4: Simulation results of MSE comparison between true model and simplified model in Case 5

σ_{12}	σ_{13}	σ_{14}	σ_{23}	σ_{24}	σ_{34}	Index	MSE_{True}	$MSE_{Simplified}$	$MSE_{True} - MSE_{Simplified}$	$\frac{MSE_{True}}{MSE_{Simplified}}$
0.3	0.3	0.3	0.3	0.3	0.3	1	0.831	0.831	0	1
0.7	0.3	0.3	0.3	0.3	0.3	2	0.496	0.496	0	1
0.3	0.7	0.3	0.3	0.3	0.3	3	0.496	0.61	-0.114	0.813
0.7	0.7	0.3	0.3	0.3	0.3	4	0.246	0.36	-0.114	0.682
0.3	0.3	0.7	0.3	0.3	0.3	5	0.496	0.61	-0.114	0.813
0.7	0.3	0.7	0.3	0.3	0.3	6	0.246	0.36	-0.114	0.682
0.3	0.7	0.7	0.3	0.3	0.3	7	0.246	0.246	0	1
0.7	0.7	0.7	0.3	0.3	0.3	8	0.081	0.081	0	1
0.3	0.3	0.3	0.7	0.3	0.3	9	0.852	0.854	-0.002	0.998
0.7	0.3	0.3	0.7	0.3	0.3	10	0.422	0.504	-0.082	0.838
0.3	0.7	0.3	0.7	0.3	0.3	11	0.422	0.604	-0.182	0.699
0.7	0.7	0.3	0.7	0.3	0.3	12	0.42	0.454	-0.033	0.926
0.3	0.3	0.7	0.7	0.3	0.3	13	0.499	0.604	-0.104	0.827
0.7	0.3	0.7	0.7	0.3	0.3	14	0.133	0.454	-0.321	0.292
0.3	0.7	0.7	0.7	0.3	0.3	15	0.133	0.154	-0.021	0.863
0.7	0.7	0.7	0.7	0.3	0.3	16	0.194	0.204	-0.01	0.953
0.3	0.3	0.3	0.3	0.7	0.3	17	0.852	0.854	-0.002	0.998
0.7	0.3	0.3	0.3	0.7	0.3	18	0.422	0.504	-0.082	0.838
0.3	0.7	0.3	0.3	0.7	0.3	19	0.499	0.604	-0.104	0.827
0.7	0.7	0.3	0.3	0.7	0.3	20	0.133	0.454	-0.321	0.292
0.3	0.3	0.7	0.3	0.7	0.3	21	0.422	0.604	-0.182	0.699
0.7	0.3	0.7	0.3	0.7	0.3	22	0.42	0.454	-0.033	0.926
0.3	0.7	0.7	0.3	0.7	0.3	23	0.133	0.154	-0.021	0.863
0.7	0.7	0.7	0.3	0.7	0.3	24	0.194	0.204	-0.01	0.953
0.3	0.3	0.3	0.7	0.7	0.3	25	0.859	0.859	0	1
0.7	0.3	0.3	0.7	0.7	0.3	26	0.284	0.284	0	1
0.3	0.7	0.3	0.7	0.7	0.3	27	0.27	0.384	-0.114	0.703
0.7	0.7	0.3	0.7	0.7	0.3	28	0.395	0.509	-0.114	0.776
0.3	0.3	0.7	0.7	0.7	0.3	29	0.27	0.384	-0.114	0.703
0.7	0.3	0.7	0.7	0.7	0.3	30	0.395	0.509	-0.114	0.776
0.3	0.7	0.7	0.7	0.7	0.3	31				
0.7	0.7	0.7	0.7	0.7	0.3	32	0.234	0.234	0	1
0.3	0.3	0.3	0.3	0.3	0.7	33	0.852	0.852	0	1
0.7	0.3	0.3	0.3	0.3	0.7	34	0.499	0.499	0	1
0.3	0.7	0.3	0.3	0.3	0.7	35	0.422	0.689	-0.267	0.613
0.7	0.7	0.3	0.3	0.3	0.7	36	0.133	0.399	-0.267	0.332

Table 2.4(continued)

σ_{12}	σ_{13}	σ_{14}	σ_{23}	σ_{24}	σ_{34}	Index	MSE_{True}	$MSE_{Simplified}$	$MSE_{True} - MSE_{Simplified}$	$\frac{MSE_{True}}{MSE_{Simplified}}$
0.3	0.3	0.7	0.3	0.3	0.7	37	0.422	0.689	-0.267	0.613
0.7	0.3	0.7	0.3	0.3	0.7	38	0.133	0.399	-0.267	0.332
0.3	0.7	0.7	0.3	0.3	0.7	39	0.42	0.42	0	1
0.7	0.7	0.7	0.3	0.3	0.7	40	0.194	0.194	0	1
0.3	0.3	0.3	0.7	0.3	0.7	41	0.859	0.872	-0.013	0.985
0.7	0.3	0.3	0.7	0.3	0.7	42	0.27	0.506	-0.236	0.534
0.3	0.7	0.3	0.7	0.3	0.7	43	0.284	0.706	-0.421	0.403
0.7	0.7	0.3	0.7	0.3	0.7	44	0.395	0.473	-0.077	0.836
0.3	0.3	0.7	0.7	0.3	0.7	45	0.27	0.706	-0.436	0.383
0.7	0.3	0.7	0.7	0.3	0.7	46				
0.3	0.7	0.7	0.7	0.3	0.7	47	0.395	0.406	-0.011	0.974
0.7	0.7	0.7	0.7	0.3	0.7	48	0.234	0.306	-0.071	0.766
0.3	0.3	0.3	0.3	0.7	0.7	49	0.859	0.872	-0.013	0.985
0.7	0.3	0.3	0.3	0.7	0.7	50	0.27	0.506	-0.236	0.534
0.3	0.7	0.3	0.3	0.7	0.7	51	0.27	0.706	-0.436	0.383
0.7	0.7	0.3	0.3	0.7	0.7	52				
0.3	0.3	0.7	0.3	0.7	0.7	53	0.284	0.706	-0.421	0.403
0.7	0.3	0.7	0.3	0.7	0.7	54	0.395	0.473	-0.077	0.836
0.3	0.7	0.7	0.3	0.7	0.7	55	0.395	0.406	-0.011	0.974
0.7	0.7	0.7	0.3	0.7	0.7	56	0.234	0.306	-0.071	0.766
0.3	0.3	0.3	0.7	0.7	0.7	57	0.888	0.888	0	1
0.7	0.3	0.3	0.7	0.7	0.7	58	0.41	0.41	0	1
0.3	0.7	0.3	0.7	0.7	0.7	59	0.41	0.676	-0.267	0.606
0.7	0.7	0.3	0.7	0.7	0.7	60	0.243	0.51	-0.267	0.477
0.3	0.3	0.7	0.7	0.7	0.7	61	0.41	0.676	-0.267	0.606
0.7	0.3	0.7	0.7	0.7	0.7	62	0.243	0.51	-0.267	0.477
0.3	0.7	0.7	0.7	0.7	0.7	63	0.243	0.243	0	1
0.7	0.7	0.7	0.7	0.7	0.7	64	0.388	0.387	0	1

2.5 Simulation of Model Selection

Suppose there are $M = 3$ SNP alleles A(a), B(b), C(c) with possible two-way interactions A(a)B(b), A(a)C(c), B(b)C(c). The interaction results from A(a)B(b) producing elevated outcomes with either AA or BB, i.e., AABB, AABb, AAbb, aaBB, AaBB. The interaction results from A(a)C(c) producing elevated outcomes with either AA or CC, i.e., AACC, AACc, AAcc, aaCC, AaCC. The interaction results from B(b)C(c) producing elevated outcomes with either BB or CC, i.e., BBCC, BBCc, BBcc, bbCC, BbCC. Then build contingency tables for A(a)B(b), A(a)C(c), B(b)C(c), respectively. In each contingency table, if any of the cell mean exceeds the overall table mean then the individuals in the corresponding cells are labeled as "High" risk to the outcome, otherwise, those individuals are of "Low"risk of the outcome.[17]

Let X_1 be the indicator function of whether one individual is labeled as"High" risk in the contingency table for A(a)B(b), X_2 be the indicator function of whether one individual is labeled as"High" risk in the contingency table for A(a)C(c), X_3 be the indicator function of whether one individual is labeled as"High" risk in the contingency table for B(b)C(c).

The aim for this simulation is to compare the following 2-dimensional models and 1-dimensional models with the 3-dimensional model.

The 3-dimensional model is

$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \varepsilon$

The three 2-dimensional models are

$Y = \beta_0 + \beta_1 X_1 + \beta_2 (X_2 + X_3) + \varepsilon,$

$Y = \beta_0 + \beta_1 X_2 + \beta_2 (X_1 + X_3) + \varepsilon,$

$Y = \beta_0 + \beta_1 X_3 + \beta_2 (X_1 + X_2) + \varepsilon.$

The three 1-dimensional models regressing on one single variable are

$Y = \beta_0 + \beta_1 X_1 + \varepsilon,$

$Y = \beta_0 + \beta_1 X_2 + \varepsilon,$

$Y = \beta_0 + \beta_1 X_3 + \varepsilon,$

The 1-dimensional model regressing on the sum of three variables is

$Y = \beta_0 + \beta_1 (X_1 + X_2 + X_3) + \varepsilon.$

The true 3-dimensional model is $Y = 100 + 30 \times I_{AAorBB} + 15 \times I_{AAorCC} + 10 \times I_{BBorCC} + \varepsilon,\ \varepsilon \sim N(0, 20^2)$. And 100 iterations are performed with a training set consisting of 1000 individuals and a test set consisting of 500 individuals.

In each iteration, after fitting 2-dimensional models and 1-dimensional models based on training data, the following models are taken to make comparisons in prediction on the test data, the fitted 3-dimensional model, the best fitted 2-dimensional model which is with least MSPE(mean square prediction error) among all the 2-dimensional models, the best fitted 1-dimensional model which is with least MSPE among the first three 1-dimensional models, the fitted fourth 1-dimensional model.

Then, the testing data is used to calculated MSPE(Mean Square Prediction Error) of the above four models, respectively.

The average MSPEs with standard errors of 100 iterations for the above four models are given in Table 2.5.

Table 2.5: Average MSPEs from 100 iterations for simulated dimension reduced models and the 3-dimensional model

2-dim model(SE)	3-dim model(SE)	1-dim model(SE) (regress on single variable)	1-dim model(SE) (regress on sum of 3 variables)
401.3685(2.6339)	400.1580(2.6399)	486.0222 (3.0842)	429.2796(2.8158)

Table 2.5 shows that the 3-dimensional model and 2-dimensional model outperform the 1-dimensional models. But the MSE of prediction of 3-dimensional model is almost the same as the 2-dimensional model because the MSPE ratio is $400.1580 \div 401.3685 = 0.9970$.

Next, in order to show the 2-dimensional model is significant in its ability in prediction, the following statistic is used

$$t = \frac{r}{\sqrt{(1-r^2)/(N-2)}}[23],$$

where r is the correlation coefficient of observed response variable and predicted value of response variable. N is the size of dataset. This t statistic follows a t-distribution of degree freedom $N-2$ approximately under the null hypothesis that there is no significant linear correlation between observed response variable and predicted response variable. Average r and t statistic with corresponding standard error in 100 iterations from test data is calculated for the above 2-dimensional model, 3-dimensional model, 1-dimensional model regressed on single variable and 1-dimensional model regressed on the sum of 3 variables, as in Table 2.6. Applying the t-statistic on testing data can avoid the problem caused by over-fitting the model on training data. By introducing a testing data, we could study the performance of predicted response variables fitted by models of different dimensions on an independent dataset.

Table 2.6: Average r statistics and t statistics from 100 iterations for dimension reduced models and the 3-dimensional model

	2-dim model(SE)	3-dim model(SE)	1-dim model(regress on single variable)(SE)	1-dim model(regress on sum of 3 variables)(SE)
r	0.7442(0.0017)	0.7451(0.0017)	0.6782(0.0020)	0.7229(0.0018)
t	24.9188(0.1283)	24.9871(0.1293)	20.6343(0.1117)	23.4008(0.1253)

Since each test data's size is 500, $t \sim t_{498}$ under $H_0 : \rho \leq 0$ approximately. The one-sided critical value for significance level 0.05 of t_{498} is 1.64. Thus, according to the average t-values, all four models' abilities in prediction are statistically significant at 0.05 level.

Further, for comparison, table 2.7 gives the average difference in r-statistic and t-statistic between the best 2-dimensional model and the rest 3 models from table 2.6 along with their standard error from 100 iterations. Table 2.7 exhibits that the best 2-dimensional model does not perform as good as the 3-dimensional model in prediction on average as the average value of 2-dimensional models' r-statistic and t-statistic minus those of 3-dimensional model's are negative, respectively. But the best 2-dimensional model does outperform the 1-dimensional models in prediction on average in view of the fact that the average value of 2-dimensional models' r-statistic and t-statistic minus those of 1-dimensional models' are positive, respectively. Similarly, we could compare and give conclusions on the average performance for models from different dimensions by the rest tables on models' difference in average r-statistic and t-statistic.

Table 2.7: Average differences in r statistic and t statistic over 100 iterations between the 2-dimensional model and the rest models from table 2.6

	2-dim model - 3-dim model (SE)	2-dim model - 1-dim model(regress on single variable)(SE)	2-dim model -1-dim model(regress on sum of 3 variables)(SE)
r	-9e-04 (2e-04)	0.0661(0.0012)	0.0213(0.0008)
t	-0.0684(0.0172)	4.2845(0.0804)	1.5179(0.0564)

Then use "test of difference between two dependent correlations with one variable in common"[24] to test the difference between 2-dimensional model and 3-dimensional model, difference between 2-dimensional model and 1-dimensional model regressed on single variable, difference between 2-dimensional model and 1-dimensional model regressed on the sum of 3 variables. Table 2.8 gives the average p-value with standard error on test data of these three test in 100 iterations, how many p-values out of the 100 iterations are significant at 5% level,how many p-values out of the 100 iterations are significant at 10% level, respectively.

Table 2.8: Average p-values from 100 iterations comparing difference between the 2-dimensional model and the rest models from table 2.6

	2-dim vs 3-dim(SE)	2-dim vs 1-dim(single var)(SE)	2-dim vs 1-dim(sum of var's) (SE)
p-value	0.3918(0.0312)	2e-04(1e-04)	0.0391(0.0084)
5% sig.	16	100	80
10% sig.	24	100	90

The average p-value for difference between 2-dimensional model and 3-dimensional model is larger than 0.05, thus the difference between 2-dimensional model and 3-dimensional model is not significant at 0.05 significance level. 16 out of 100 iterations are with a p-value that is significant at a 0.05 significance level, 24 out of 100 iterations are with a p-value that is significant at a 0.10 significance level.

The average p-value for difference between 2-dimensional model and 1-dimensional model regressed on single variable is smaller than 0.05, thus the difference between 2-dimensional model and 1-dimensional model regressed on single variable is significant at 0.05 significance level. 100 out of 100 iterations are with a p-value that is significant at a 0.05 significance level.

The average p-value for difference between 2-dimensional model and 1-dimensional model regressed on the sum of 3 variables is smaller than 0.05, thus the difference between 2-dimensional model and 1-dimensional model regressed on the sum of 3 variables is significant at 0.05 significance level.80 out of 100 iterations are with a p-value that is significant at a 0.05 significance level, 90 out of 100 iterations are with a p-value that is significant at a 0.10 significance level.

The second model selection simulation is to suppose that there are $M = 4$ SNP alleles A(a), B(b), C(c), D(d) with 6 possible two-way interactions A(a)B(b), A(a)C(c), A(a)D(d), B(b)C(c), B(b)D(d), C(c)D(d). The interaction results from A(a)B(b) produce elevated outcomes with either AA or BB, i.e., AABB, AABb, AAbb, aaBB, AaBB. The interaction results from A(a)C(c) produce elevated outcomes with either AA or CC, i.e., AACC, AACc, AAcc, aaCC, AaCC. The interaction results from A(a)D(d) produce elevated outcomes with either AA or DD, i.e., AADD, AADd, AAdd, aaDD, AaDD. The interaction results from B(b)C(c) produce elevated outcomes with either BB or CC, i.e., BBCC, BBCc, BBcc, bbCC, BbCC. The interaction results from B(b)D(d) produce elevated outcomes with either BB or DD, i.e., BBDD, BBDd, BBdd, bbDD, BbDD. The interaction results from C(c)D(d) produce elevated outcomes with either CC or DD, i.e., CCDD, CCDd, CCdd, ccDD, CcDD.

Then build one contingency table for A(a)B(b), A(a)C(c), A(a)D(d), B(b)C(c), B(b)D(d), C(c)D(d), respectively. In each contingency table, if any of the cell means exceeds the overall table mean then the individuals in the corresponding cells are labeled as "High" risk to the outcome, otherwise, those individuals are of "Low" risk of the outcome.

Let X_1 be the indicator function of whether one individual is labeled as "High" risk in the contingency table for A(a)B(b), X_2 be the indicator function of whether one individual is labeled as "High" risk in the contingency table for A(a)C(c), X_3 be the indicator function of whether one individual is labeled as "High" risk in the contingency table for A(a)D(d), X_4 be the indicator function of whether one individual is labeled as "High" risk in the contingency table for B(b)C(c),X_5 be the indicator function of whether one individual is labeled as "High" risk in the contingency table for B(b)D(d), X_6 be the indicator function of whether one individual is labeled as "High" risk in the contingency table for C(c)D(d).

The aim for this simulation is to compare the following 2-dimensional models and 1-dimensional models with the 6-dimensional model.

The 6-dimensional model is

$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \beta_4 X_4 + \beta_5 X_5 + \beta_6 X_6 + \varepsilon$

The 31 2-dimensional models are

$Y = \beta_0 + \beta_1(X_1 + X_2 + X_3 + X_4 + X_5) + \beta_2 X_6 + \varepsilon,$

$Y = \beta_0 + \beta_1(X_1 + X_2 + X_3 + X_4 + X_6) + \beta_2 X_5 + \varepsilon,$

$Y = \beta_0 + \beta_1(X_1 + X_2 + X_3 + X_5 + X_6) + \beta_2 X_4 + \varepsilon,$

$Y = \beta_0 + \beta_1(X_1 + X_2 + X_4 + X_5 + X_6) + \beta_2 X_3 + \varepsilon,$

$Y = \beta_0 + \beta_1(X_1 + X_3 + X_4 + X_5 + X_6) + \beta_2 X_2 + \varepsilon,$

$Y = \beta_0 + \beta_1(X_2 + X_3 + X_4 + X_5 + X_6) + \beta_2 X_1 + \varepsilon,$

$Y = \beta_0 + \beta_1(X_3 + X_4 + X_5 + X_6) + \beta_2(X_1 + X_2) + \varepsilon,$

$Y = \beta_0 + \beta_1(X_2 + X_4 + X_5 + X_6) + \beta_2(X_1 + X_3) + \varepsilon,$

$$Y = \beta_0 + \beta_1(X_2 + X_3 + X_5 + X_6) + \beta_2(X_1 + X_4) + \varepsilon,$$

$$Y = \beta_0 + \beta_1(X_2 + X_3 + X_4 + X_6) + \beta_2(X_1 + X_5) + \varepsilon,$$

$$Y = \beta_0 + \beta_1(X_2 + X_3 + X_4 + X_5) + \beta_2(X_1 + X_6) + \varepsilon,$$

$$Y = \beta_0 + \beta_1(X_1 + X_4 + X_5 + X_6) + \beta_2(X_2 + X_3) + \varepsilon,$$

$$Y = \beta_0 + \beta_1(X_1 + X_3 + X_5 + X_6) + \beta_2(X_2 + X_4) + \varepsilon,$$

$$Y = \beta_0 + \beta_1(X_1 + X_2 + X_3 + X_6) + \beta_2(X_2 + X_5) + \varepsilon,$$

$$Y = \beta_0 + \beta_1(X_1 + X_3 + X_4 + X_5) + \beta_2(X_2 + X_6) + \varepsilon,$$

$$Y = \beta_0 + \beta_1(X_1 + X_2 + X_5 + X_6) + \beta_2(X_3 + X_4) + \varepsilon,$$

$$Y = \beta_0 + \beta_1(X_1 + X_2 + X_4 + X_6) + \beta_2(X_3 + X_5) + \varepsilon,$$

$$Y = \beta_0 + \beta_1(X_1 + X_2 + X_4 + X_5) + \beta_2(X_3 + X_6) + \varepsilon,$$

$$Y = \beta_0 + \beta_1(X_1 + X_2 + X_3 + X_6) + \beta_2(X_4 + X_5) + \varepsilon,$$

$$Y = \beta_0 + \beta_1(X_1 + X_2 + X_3 + X_5) + \beta_2(X_4 + X_6) + \varepsilon,$$

$$Y = \beta_0 + \beta_1(X_1 + X_2 + X_3 + X_4) + \beta_2(X_5 + X_6) + \varepsilon,$$

$$Y = \beta_0 + \beta_1(X_4 + X_5 + X_6) + \beta_2(X_1 + X_2 + X_3) + \varepsilon,$$

$$Y = \beta_0 + \beta_1(X_3 + X_5 + X_6) + \beta_2(X_1 + X_2 + X_4) + \varepsilon,$$

$$Y = \beta_0 + \beta_1(X_3 + X_4 + X_6) + \beta_2(X_1 + X_2 + X_5) + \varepsilon,$$

$$Y = \beta_0 + \beta_1(X_3 + X_4 + X_5) + \beta_2(X_1 + X_2 + X_6) + \varepsilon,$$

$$Y = \beta_0 + \beta_1(X_2 + X_5 + X_6) + \beta_2(X_1 + X_3 + X_4) + \varepsilon,$$

$$Y = \beta_0 + \beta_1(X_2 + X_4 + X_6) + \beta_2(X_1 + X_3 + X_5) + \varepsilon,$$

$$Y = \beta_0 + \beta_1(X_2 + X_4 + X_5) + \beta_2(X_1 + X_3 + X_6) + \varepsilon,$$

$$Y = \beta_0 + \beta_1(X_2 + X_3 + X_6) + \beta_2(X_1 + X_4 + X_5) + \varepsilon,$$

$$Y = \beta_0 + \beta_1(X_2 + X_3 + X_5) + \beta_2(X_1 + X_4 + X_6) + \varepsilon,$$

$$Y = \beta_0 + \beta_1(X_2 + X_3 + X_4) + \beta_2(X_1 + X_5 + X_6) + \varepsilon,$$

The six 1-dimensional models regressing on one single variable are

$$Y = \beta_0 + \beta_1 X_1 + \varepsilon,$$

$$Y = \beta_0 + \beta_1 X_2 + \varepsilon,$$

$$Y = \beta_0 + \beta_1 X_3 + \varepsilon,$$

$$Y = \beta_0 + \beta_1 X_4 + \varepsilon,$$

$$Y = \beta_0 + \beta_1 X_5 + \varepsilon,$$

$$Y = \beta_0 + \beta_1 X_6 + \varepsilon,$$

The 1-dimensional model regressing on the sum of six variables is

$Y = \beta_0 + \beta_1(X_1 + X_2 + X_3 + X_4 + X_5 + X_6) + \varepsilon.$

The true 6-dimensional model is $Y = 100 + 30 \times I_{AAorBB} + 55 \times I_{AAorCC} + 10 \times I_{AAorDD} + 70 \times I_{BBorCC} + 15 \times I_{BBorDD} + 40 \times I_{CCorDD} + \varepsilon,\ \varepsilon \sim N(0, 20^2)$. And 100 iterations with a training set consisting of 1000 individuals and a test set consisting of 500 individuals are performed.

In each iteration, after fitting 2-dimensional models and 1-dimensional models based on training data, the following models are taken to make comparisons in prediction on the testing data, the fitted 6-dimensional model, the best fitted 2-dimensional model which is with the largest R-squared among all 2-dimensional models, the best fitted 1-dimensional model which is with the largest R-squared among the first six 1-dimensional models, the fitted seventh 1-dimensional model.

Then, the testing data is used to calculate MSPE(Mean Square Prediction Error) of the above four models, respectively.

The average MSPE's of 100 iterations with corresponding standard error for the above four models are given in Table 2.9.

Table 2.9: Average MSPEs from 100 iterations for simulated dimension reduced models and the 6-dimensional model using the first 6-dimensional true model

2-dim model(SE)	6-dim model(SE)	1-dim model(SE) (regress on single variable)	1-dim model(SE) (regress on sum of 6 variables)
463.7085(3.0789)	404.1943(3.1983)	2288.9661(14.9826)	1044.7537 (6.6488)

Table 2.9 shows that the 6-dimensional model and 2-dimensional model outperform the 1-dimensional models. And the MSPE ratio of 6-dimensional model and the 2-dimensional model is $404.1943 \div 463.7085 = 0.8717$.

Next, in order to show the 2-dimensional model is significant in its ability in prediction, the following statistic is used

$t = \dfrac{r}{\sqrt{(1-r^2)/(N-2)}}$[23],

where r, N and the distribution of t are same as before. Average r and t statistic in 100 iterations with corresponding standard error from test data is calculated for the above 2-dimensional model, 6-dimensional model, 1-dimensional model regressed on single variable and 1-dimensional model regressed on the sum of 6 variables, as in Table 2.10.

Table 2.10: Average r statistic and t statistic from 100 iterations for dimension reduced models and the 6-dimensional model using the first 6-dimensional true model

	2-dim model(SE)	6-dim model(SE)	1-dim model(regress on single variable)(SE)	1-dim model(regress on sum of 6 variables)(SE)
r	0.9634 (0.0003)	0.9682 (0.0003)	0.8034(0.0016)	0.9153 (0.0007)
t	80.3798 (0.3301)	86.5693(0.3861)	30.2014(0.1723)	50.8539(0.2445)

Since each test data's size is 500, $t \sim t_{498}$ under $H_0 : \rho \leq 0$ approximately. The one-sided critical value for significance level 0.05 of t_{498} is 1.64. Thus, according to average t scores, all four models' abilities in prediction are statistically significant at 0.05 level.

Further, for comparison, table 2.11 gives the average difference in r-statistic and t-statistic between the best 2-dimensional model and the rest 3 models from table 2.10 along with their standard error from 100 iterations.

Table 2.11: Average differences in r statistic and t statistic over 100 iterations between the 2-dimension model and the rest models in table 2.9

	2-dim model - 6-dim model (SE)	2-dim model - 1-dim model(regress on single variable)(SE)	2-dim model -1-dim model(regress on sum of 3 variables)(SE)
r	-0.0048 (0.0001)	0.1600(0.0015)	0.0481(0.0007)
t	-6.1895(0.1835)	50.1784(0.3220)	29.5259(0.3375)

Then use "test of difference between two dependent correlations with one variable in common"[24]to test the difference between 2-dimensional model and 6-dimensional model, difference between 2-dimensional model and 1-dimensional model regressed on single variable,difference between 2-dimensional model and 1-dimensional model regressed on the sum of 6 variables. Table 2.12 gives the average p-value with corresponding standard error on test data of these three test in 100 iterations, how many p-values out of the 100 iterations are significant at 5% level,how many p-values out of the 100 iterations are significant at 10% level, respectively.

Table 2.12: Average p-values from 100 iterations in comparing difference between the 2-dimension model and the rest models in table 2.9

	2-dim vs 6-dim(SE)	2-dim vs 1-dim(single var)(SE)	2-dim vs 1-dim(sum of var's) (SE)
p-value	0.0041(0.0012)	1.0547e-53(6.5998e-54)	2.5058e-14(2.5059e-14)
5% sig.	97	100	100
10% sig.	100	100	100

The average p-value for difference between 2-dimensional model and 6-dimensional model is less than 0.05, thus the difference between 2-dimensional model and 6-dimensional model is significant at 0.05 significance level.97 out of 100 iterations are with a p-value that is significant at a 0.05 significance level, 100 out of 100 iterations are with a p-value that is significant at a 0.10 significance level.

The average p-value for difference between 2-dimensional model and 1-dimensional model regressed on single variable is smaller than 0.05, thus the difference between 2-dimensional model and 1-dimensional model regressed on single variable is significant at 0.05 significance level. 100 out of 100 iterations are with a p-value that is significant at a 0.05 significance level.

The average p-value for difference between 2-dimensional model and 1-dimensional model regressed on the sum of 6 variables is smaller than 0.05, thus on average the difference between 2-dimensional model and 1-dimensional model regressed on the sum of 6 variables is significant at 0.05 significance level. 100 out of 100 iterations are with a p-value that is significant at a 0.05 significance level.

Thus, even though the 2-dimensional model is not as good as the 6-dimensional model, it outperforms the 1-dimensional models.

Next, the true 6-dimensional model is set to be $Y = 100 + 30 \times I_{AAorBB} + 15 \times I_{AAorCC} + 10 \times I_{AAorDD} + 15 \times I_{BBorCC} + 10 \times I_{BBorDD} + 10 \times I_{CCorDD} + \varepsilon$, $\varepsilon \sim N(0, 20^2)$. And 100 iterations with a training set consisting of 1000 individuals and a test set consisting of 500 individuals are performed.

Same as previous simulations in this section, in each iteration, after fitting 2-dimensional models and 1-dimensional models based on training data, the following models are taken to make comparisons in prediction on the testing data, the fitted 6-dimensional model, the best fitted 2-dimensional model which is with the largest R-squared among all 2-dimensional models, the best fitted 1-dimensional model which is with the largest R-squared among the first six 1-dimensional models, the fitted seventh 1-dimensional model. Then, the testing data is used to calculate MSPE(Mean Square Prediction Error) of the above four models, respectively.

The average MSPE's of 100 iterations with corresponding standard error for the above four models fitted with data generated from the above second true 6-dimensional model are given in Table 2.13, average r and t statistic in 100 iterations with corresponding standard error from test data are listed as in Table 2.14 and table 2.15 gives the average difference in r-statistic and t-statistic between the best 2 dimensional model and the rest 3 models from table 2.14 along with their standard error from 100 iteration. Listed in Table 2.16 are the average p-value with corresponding standard error on test data to perform "test of difference between two dependent correlations with one variable in common"[24] to test the difference between 2-dimensional model and 6-dimensional model, difference between 2-dimensional model and 1-dimensional model regressed on single variable,difference between 2-dimensional model and 1-dimensional model regressed on the sum of 6 variables, how many p-values out of the 100 iterations are significant at 5% level,how many p-values out of the 100 iterations are significant at 10% level, respectively.

Table 2.13: Average MSPEs from 100 iterations for simulated dimension reduced models and the 6-dimensional model using the second 6-dimensional true model

2-dim model(SE)	6-dim model(SE)	1-dim model(SE) (regress on single variable)	1-dim model(SE) (regress on sum of 6 variables)
402.7036 (2.6789)	402.4459(2.6522)	707.1826 (4.3715)	453.8246(2.9847)

Table 2.13 shows that the 6-dimensional model and 2-dimensional model outperform the 1-dimensional models. And the MSPE ratio of 6-dimensional model and the 2-dimensional model is $402.4459 \div 402.7036 = 0.9993$.

Table 2.14: Average r statistics and t statistics from 100 iterations for dimension reduced models and the 6-dimensional model using the second 6-dimensional true model

	2-dim model(SE)	6-dim model(SE)	1-dim model(regress on single variable)(SE)	1-dim model(regress on sum of 6 variables)(SE)
r	0.8398(0.0011)	0.8399(0.0011)	0.6948(0.0019)	0.8172 (0.0014)
t	34.6306(0.1809)	34.6395(0.1825)	21.5923(0.1339)	31.7939(0.1692)

Since each test data's size is 500, $t \sim t_{498}$ under $H_0 : \rho \leq 0$ approximately. The one-sided critical value for significance level 0.05 of t_{498} is 1.64. Thus, according to average t statistic, all four models' abilities in prediction are statistically significant at 0.05 level.

Table 2.15: Average differences in r statistic and t statistic over 100 iterations between the 2-dimension model and the rest models in table 2.13

	2-dim model - 6-dim model (SE)	2-dim model - 1-dim model(regress on single variable)(SE)	2-dim model -1-dim model(regress on sum of 3 variables)(SE)
r	-0.0001(0.0002)	0.1450(0.0016)	0.0226(0.0006)
t	-0.0163(0.0214)	12.9959(0.1450)	2.8691(0.0715)

Table 2.16: Average p-values from 100 iterations in comparing difference between the 2-dimension model and the rest models in table 2.13

	2-dim vs 6-dim(SE)	2-dim vs 1-dim(single var)(SE)	2-dim vs 1-dim(sum of var's) (SE)
p-value	0.5034(0.0324)	6.9737e-13(5.1026e-13)	0.0062 (0.0028)
5% sig.	10	100	96
10% sig.	15	100	99

The average p-value for difference between 2-dimensional model and 6-dimensional model is not less than 0.05, thus on average the difference between 2-dimensional model and 6-dimensional model is not significant at 0.05 significance level. 10 out of 100 iterations are with a p-value that is significant at a 0.05 significance level, 15 out of 100 iterations are with a p-value that is significant at a 0.10 significance level.

The average p-value for difference between 2-dimensional model and 1-dimensional model regressed on single variable is smaller than 0.05, thus on average the difference between 2-dimensional model and 1-dimensional model regressed on single variable is significant at 0.05 significance level. 100 out of 100 iterations are with a p-value that is significant at a 0.05 significance level.

The average p-value for difference between 2-dimensional model and 1-dimensional model regressed on the sum of 6 variables is smaller than 0.05, thus on average the difference between 2-dimensional model and 1-dimensional model regressed on the sum of 6 variables is significant at 0.05 significance level. 96 out of 100 iterations are with a p-value that is significant at a 0.05 significance level, 99 out of 100 iterations are with a p-value that is significant at a 0.10 significance level.

Thus, for the dataset generated upon the true 6-dimensional model $Y = 100+30\times I_{AAorBB}+15\times I_{AAorCC}+10\times I_{AAorDD}+15\times I_{BBorCC}+10\times I_{BBorDD}+10\times I_{CCorDD}+\varepsilon,\ \varepsilon \sim N(0, 20^2)$, the best 2-dimensional model outperforms the 1-dimensional model. Further, from the p-value of test for difference between models, there is no significant difference between the performance in prediction of the best 2-dimensional and the 6-dimensional model, which is a different result when the true 6-dimensional model is set to be $Y = 100 + 30 \times I_{AAorBB} + 55 \times I_{AAorCC} + 10 \times I_{AAorDD} + 70 \times I_{BBorCC} + 15 \times I_{BBorDD} + 40 \times I_{CCorDD} + \varepsilon,\ \varepsilon \sim N(0, 20^2)$.

This is because

$Y = 100 + 30 \times I_{AAorBB} + 15 \times I_{AAorCC} + 10 \times I_{AAorDD} + 15 \times I_{BBorCC} + 10 \times I_{BBorDD} + 10 \times I_{CCorDD} + \varepsilon$

could be written as

$Y = 100 + 30 \times I_{AAorBB} + 15 \times (I_{AAorCC} + I_{BBorCC}) + 10v(I_{AAorDD} + I_{BBorDD} + I_{CCorDD}) + \varepsilon$, which could be viewed as a 3-dimensional model.

So the performance of best 2-dimensional model works better when the model could be viewed as a lower dimensional model than a 6-dimensional model.

2.6 A Method to Obtain a Relatively Good P-dimensional Model, Given P

This subsection gives a method, as shown below, to answer the following question: when M is given, how to get a relatively good p-dimensional model? It is necessary to answer the above question when $\binom{M}{k}$ is large and p is relatively small, where M is number of SNPs and k is order of multi-locus interaction. As an example, assume there are $M = 6$ SNPs and the multi-locus interaction order of our interest is $k = 2$, then $\binom{6}{2} = 15$ is the number of explanatory variables assuming that every 2-way multi-locus interaction is significant. Then, there will be $\binom{15}{1} + \binom{15}{2} + \binom{15}{3} + \binom{15}{4} + \binom{15}{5} + \binom{15}{6} + \binom{15}{7}$ possible 2-dimensional models. Thus, it becomes very complicated when $\binom{M}{k} >> p$.

***Step*1** Determine the indicator variables of risk of every significant SNP-SNP interactions for all subjects, denoted as $X_1, X_2, ..., X_n$, where $n = \binom{M}{k}$

***Step*2** Randomly assign the variables into p groups, and use the sum of variables in each group to define p variables, noted as $Y_1, Y_2, ..., Y_p$. Fit a linear regression of response variable Y on $Y_1, Y_2, ..., Y_p$. Denote its R-squared to be R1.

***Step*3** Randomly select one variable from any of the p groups, if the group contains no less than two variables, and randomly put this variable into another group of variables, otherwise repeat step3. Denote the sum of variables in each group as $Y_1', Y_2', ..., Y_p'$. Fit a linear regression of response variable Y on $Y_1', Y_2', ..., Y_p'$. Denote its R-squared to be R2.

***Step*4** If R2 is not less than R1, keep the movement in step 3, otherwise, keep the group assignment same as those in step2.

The number of permutations is how many times step3 are performed. The above permutation strategy is performed 1000 times on training data to find a relatively good p-dimensional model and evaluate it on the test data. Even though this above methodology is referred as "permutation method" in this thesis, it is performed as a "rearrangement" process.

There are two simulations performed regarding the above method.

First, use the same true 3-dimensional model as in section2.5.

$Y = 100 + 30 \times I_{AAorBB} + 15 \times I_{AAorCC} + 10 \times I_{BBorCC} + \varepsilon,\ \varepsilon \sim N(0, 20^2)$. 100 iterations are performed with a training set consisting of 1000 individuals and a test set consisting of 500 individuals.

After choosing a relatively good 2-dimensional model, evaluate it with the 3-dimensional model, the best 1-dimensional model fitted on single variable and the 1-dimensional model fitted on the sum of all variables with average MSPE from 100 iterations, average r statistic and t statistic mentioned in Section 2.5 from 100 iterations, along with their standard error, as shown in Table 2.17. Further, table 2.18 gives the average differences in r-statistic and t-statistic between the relatively good 2-dimensional model and the rest 3 models from table 2.17 along with their standard error from 100 iterations. Table 2.19 displays average p-value of testing for difference in significance in making predictions from 100 iterations, how many p-values out of the 100 iterations are significant at 5% level,how many p-values out of the 100 iterations are significant at 10% level, respectively.

Table 2.17: Average MSPEs, r statistics, t statistics from 100 iterations for relatively good 2-dimensional model, 1-dimensional models and the 3-dimensional model

	relatively good 2-dim(SE)	3-dim(SE)	1-dim(SE) (regress on single variable)	1-dim(SE) (regress on sum of 3 variables)
MSPE	401.3684(2.6339)	400.1580(2.6399)	486.0222 (3.0842)	429.2796(2.8158)
r	0.7442(0.0016)	0.7451(0.0017)	0.6782(0.0020)	0.7229(0.0018)
t	24.9187(0.1282)	24.9871(0.1293)	20.6343(0.1117)	23.4008(0.1253)

Table 2.18: Average differences in r statistic and t statistic over 100 iterations between relatively good 2-dimensional model and 1-dimensional models, and the 3-dimensional model

	2-dim model - 3-dim model (SE)	2-dim model - 1-dim model(regress on single variable)(SE)	2-dim model -1-dim model(regress on sum of 3 variables)(SE)
r	-0.0009 (0.0002)	0.0661(0.0012)	0.0213 (0.0008)
t	-0.0684(0.0172)	4.2845(0.0804)	1.5179 (0.0564)

Table 2.19: Average p-values from 100 iterations comparing difference between relatively good 2-dimensional model and 1-dimensional models, and the 3-dimensional model

	2-dim vs 3-dim(SE)	2-dim vs 1-dim(single var)(SE)	2-dim vs 1-dim(sum of var's) (SE)
p-value	0.3918(0.0312)	2e-04(1e-04)	0.0391(0.0084)
5% sig.	16	100	80
10% sig.	24	100	90

The average p-value for difference between relatively good 2-dimensional model and 3-dimensional model is larger than 0.05, thus on average the difference between 2-dimensional model and 3-dimensional model is not significant at 0.05 significance level. 16 out of 100 iterations are with a p-value that is significant at a 0.05 significance level, 24 out of 100 iterations are with a p-value that is significant at a 0.10 significance level.

The average p-value for difference between relatively good 2-dimensional model and 1-dimensional model regressed on single variable is smaller than 0.05, thus on average the difference between 2-dimensional model and 1-dimensional model regressed on single variable is significant at 0.05 significance level. 100 out of 100 iterations are with a p-value that is significant at a 0.05 significance level.

The average p-value for difference between relatively good 2-dimensional model and 1-dimensional model regressed on the sum of 3 variables is smaller than 0.05, thus on average the difference between 2-dimensional model and 1-dimensional model regressed on the sum of 3 variables is significant at 0.05 significance level.80 out of 100 iterations are with a p-value that is significant at a 0.05 significance level, 90 out of 100 iterations are with a p-value that is significant at a 0.10 significance level.

Second, use the first true 6-dimensional model as in Section 2.5.

$Y = 100 + 30 \times I_{AAorBB} + 55 \times I_{AAorCC} + 10 \times I_{AAorDD} + 70 \times I_{BBorCC} + 15 \times I_{BBorDD} + 40 \times I_{CCorDD} + \varepsilon,\ \varepsilon \sim N(0, 20^2)$. 100 iterations are performed with a training set consisting of 1000 individuals and a test set consisting of 500 individuals.

After choosing a relatively good 2-dimensional model, evaluate it with the 6-dimensional model, the best 1-dimensional model fitted on single variable and the 1-dimensional model fitted on the sum of all variables with average MSPE from 100 iterations, average r statistic and t statistic mentioned in Section 2.5 from 100 iterations, along with their standard error, as shown in Table 2.20. Further, table 2.21 gives the average differences in r-statistic and t-statistic between the relatively good 2-dimensional model and the rest 3 models from table 2.20 along with their standard error from 100 iterations. Table 2.22 displays average p-value of testing for difference in significance in making predictions from 100 iterations, how many p-values out of the 100 iterations are significant at 5% level,how many p-values out of the 100 iterations are significant at 10% level, respectively.

Table 2.20: Average MSPEs, r statistics, t statistics from 100 iterations for relatively good 2-dimensional model, 1-dimensional models and the 6-dimensional model using the first 6-dimensional true model

	relatively good 2-dim (SE)	6-dim(SE)	1-dim(SE) (regress on single variable)	1-dim(SE) (regress on sum of 6 variables)
MSPE	465.1182(3.1956)	404.1943(3.1983)	2288.9661(14.9826)	1044.7537 (6.6488)
r	0.9633(0.0003)	0.9682 (0.0003)	0.8034(0.0016)	0.9153 (0.0007)
t	80.2577 (0.3296)	86.5693(0.3861)	30.2014(0.1723)	50.8539(0.2445)

Table 2.21: Average differences in r statistic, t statistic from 100 iterations between relatively good 2-dimensional model and 1-dimensional models, and the 6-dimensional model using the first 6-dimensional true model

	2-dim model - 6-dim model (SE)	2-dim model - 1-dim model(regress on single variable)(SE)	2-dim model -1-dim model(regress on sum of 6 variables)(SE)
r	-0.0049 (0.0001)	0.1599 (0.0015)	0.0480 (0.0007)
t	-6.3115(0.1781)	50.0563(0.3197)	29.4039 (0.3489)

Table 2.22: Average p-values from 100 iterations in comparing the difference between relatively good 2-dimensional model and 1-dimensional models, and the 6-dimensional model using the first 6-dimensional true model

	2-dim vs 6-dim(SE)	2-dim vs 1-dim(single var)(SE)	2-dim vs 1-dim(sum of var's) (SE)
p-value	0.0053(0.0019)	7.5806e-53(7.3677e-53)	2.5058e-14(2.5058e-14)
5% sig.	96	100	100
10% sig.	99	100	100

The average p-value for difference between relatively good 2-dimensional model and 6-dimensional model is less than 0.05, thus on average the difference between 2-dimensional model and 6-dimensional model is significant at 0.05 significance level. 96 out of 100 iterations are with a p-value that is significant at a 0.05 significance level, 99 out of 100 iterations are with a p-value that is significant at a 0.10 significance level.

The average p-value for difference between relatively good 2-dimensional model and 1-dimensional model regressed on single variable is smaller than 0.05, thus on average the difference between 2-dimensional model and 1-dimensional model regressed on single variable is significant at 0.05 significance level. 100 out of 100 iterations are with a p-value that is significant at a 0.05 significance level.

The average p-value for difference between relatively good 2-dimensional model and 1-dimensional model regressed on the sum of 6 variables is smaller than 0.05, thus on average the difference between 2-dimensional model and 1-dimensional model regressed on the sum of 6 variables is significant at 0.05 significance level. 100 out of 100 iterations are with a p-value that is significant at a 0.05 significance level.

Thus, even though the relatively good 2-dimensional model is not as good as the 6-dimensional model, it outperforms the 1-dimensional models.

Further, the true 6-dimensional model to generated dataset is set to be

$Y = 100 + 30 \times I_{AAorBB} + 15 \times I_{AAorCC} + 10 \times I_{AAorDD} + 15 \times I_{BBorCC} + 10 \times I_{BBorDD} + 10 \times I_{CCorDD} + \varepsilon,\ \varepsilon \sim N(0, 20^2)$.

The relatively good 2-dimensional model is evaluated with the 6-dimensional model, the best 1-dimensional model fitted on single variable and the 1-dimensional model fitted on the sum of all variables with average MSPE from 100 iterations, average r statistic and t statistic mentioned in Section 2.5 from 100 iterations, along with their standard error, as shown in Table 2.23. Further, table 2.24 gives the average differences in r-statistic and t-statistic between the relatively good 2-dimensional model and the rest 3 models from table 2.23 along with their standard error from 100 iterations. Table 2.25 displays average p-value of testing for difference in significance in making predictions from 100 iterations, how many p-values out of the 100 iterations are significant at 5% level,how many p-values out of the 100 iterations are significant at 10% level, respectively.

Table 2.23: Average MSPEs, r statistics, t statistics from 100 iterations for relatively good 2-dimensional model, 1-dimensional models and the 6-dimensional model using the second 6-dimensional true model

	relatively good 2-dim (SE)	6-dim(SE)	1-dim(SE) (regress on single variable)	1-dim(SE) (regress on sum of 6 variables)
MSPE	408.3885(2.7239)	402.4459(2.6522)	707.1826 (4.3715)	453.8246(2.9847)
r	0.8373(0.0012)	0.8399(0.0011)	0.6948(0.0019)	0.8172 (0.0014)
t	34.2619 (0.1695)	34.6395(0.1825)	21.5923(0.1339)	31.7939(0.1692)

Table 2.24: Average differences in r statistic and t statistic over 100 iterations between relatively good 2-dimensional model and 1-dimensional models, and 6-dimensional model using the second 6-dimensional true model

	2-dim model - 6-dim model (SE)	2-dim model - 1-dim model(regress on single variable)(SE)	2-dim model -1-dim model(regress on sum of 6 variables)(SE)
r	-0.0026 (0.0004)	0.1425(0.0017)	0.0201 (0.0006)
t	-0.3543(0.0478)	12.6580(0.1486)	2.5311 (0.0774)

Table 2.25: Average p-value from 100 iterations in comparing difference between relatively good 2-dimensional model and 1-dimensional models, and the 6-dimensional model using the second 6-dimensional true model

	2-dim vs 6-dim(SE)	2-dim vs 1-dim(single var)(SE)	2-dim vs 1-dim(sum of var's) (SE)
p-value	0.3148(0.0292)	9.2503e-10(9.2439e-10)	0.0092 (0.0031)
5% sig.	20	100	95
10% sig.	33	100	99

The average p-value for difference between relatively good 2-dimensional model and 6-dimensional model is not less than 0.05, thus on average the difference between 2-dimensional model and 6-dimensional model is insignificant at 0.05 significance level. 20 out of 100 iterations are with a p-value that is significant at a 0.05 significance level, 33 out of 100 iterations are with a p-value that is significant at a 0.10 significance level.

The average p-value for difference between relatively good 2-dimensional model and 1-dimensional model regressed on single variable is smaller than 0.05, thus on average the difference between 2-dimensional model and 1-dimensional model regressed on single variable is significant at 0.05 significance level.100 out of 100 iterations are with a p-value that is significant at a 0.05 significance level.

The average p-value for difference between relatively good 2-dimensional model and 1-dimensional model regressed on the sum of 6 variables is smaller than 0.05, thus on average the difference between 2-dimensional model and 1-dimensional model regressed on the sum of 6 variables is significant at 0.05 significance level. 95 out of 100 iterations are with a p-value that is significant at a 0.05 significance level, 99 out of 100 iterations are with a p-value that is significant at a 0.10 significance level.

Thus, even though the relatively good 2-dimensional model is not significantly different from the 6-dimensional model, it outperforms the 1-dimensional models.

With permutation numbers to be 1000 in the permutation method introduced in this secion, the “relatively good” 2-dimensional model found by permutation could reach the best 2-dimensional model in 100 iterations out of 100 iterations when the true model is

$Y = 100 + 30 \times I_{AAorBB} + 15 \times I_{AAorCC} + 10 \times I_{BBorCC} + \varepsilon,\ \varepsilon \sim N(0, 20^2)$;

the “relatively good” 2-dimensional model found by permutation could reach the best 2-dimensional model in 52 iterations out of 100 iterations when the true model is the 6-dimensional model

$Y = 100 + 30 \times I_{AAorBB} + 55 \times I_{AAorCC} + 10 \times I_{AAorDD} + 70 \times I_{BBorCC} + 15 \times I_{BBorDD} + 40 \times I_{CCorDD} + \varepsilon,\ \varepsilon \sim N(0, 20^2)$;

the “relatively good” 2-dimensional model found by permutation could reach the best 2-dimensional model in 67 iterations out of 100 iterations when the true model is the 6-dimensional model

$Y = 100 + 30 \times I_{AAorBB} + 15 \times I_{AAorCC} + 10 \times I_{AAorDD} + 15 \times I_{BBorCC} + 10 \times I_{BBorDD} + 10 \times I_{CCorDD} + \varepsilon,\ \varepsilon \sim N(0, 20^2)$.

Last, we consider the true model $Y = 100 + 30 \times I_{AAorBB} + 15 \times I_{AAorCC} + 10 \times I_{AAorDD} + 15 \times I_{BBorCC} + 10 \times I_{BBorDD} + 10 \times I_{CCorDD} + \varepsilon,\ \varepsilon \sim N(0, 50^2)$. The variance of the random error is increased from 20^2 to 50^2 because the response variables of the Alzheimer’s disease data in Chapter 5 are with variances about 50^2. Table 2.26 lists the average MSPEs, average r statistics and average t statistics, along with their standard errors from 100 iterations, from 100 iterations for the best 2-dimensional model, relatively good 2-dimensional model, 6-dimensional model and 1-dimensional model regressing on single variable and 1-dimensional model regressing on the sum of six variables.

Table 2.26: Average MSPE, r statistic and t statistic from 100 iterations for the 2-dimensional models, 6-dimensional model and 1-dimensional models using the true model with larger variance

Model	best 2-dim	rela. good 2-dim	6-dim	1-dim(single var.)	1-dim(sum of var.)
MSPE(SE)	2569.6046(15.1543)	2571.6297(15.7251)	2512.3330(15.1976)	4417.1766(26.4874)	3132.2641(18.6894)
r(SE)	0.8355(0.0012)	0.8354(0.0012)	0.8396(0.0012)	0.6935(0.0021)	0.7948(0.0015)
t(SE)	34.0210(0.1663)	34.0091(0.1695)	34.5715(0.1683)	21.5318(0.1240)	29.3019(0.1509)

Table 2.26 illustrates that with a larger variance in response variable, the relatively

good 2-dimensional model selected by our method on average performs close to the 6-dimensional model in MSPE, r statistic and t statistic and it well represents the performance of best 2-dimensional model on average. The 2-dimensional models in table 2.26 also perform much better on average than the 1-dimensional models in MSPE, r statistic and t statistic.

Table 2.27 exhibits the average p-value for comparing performance difference between 2-dimensional models and the models of other dimensions mentioned in table 2.26, along with the standard errors on p-values from 100 iterations.

Table 2.27: Average p-value from 100 iterations in comparing difference between 2-dimensional models and 1-dimensional models, and the 6-dimensional model using the true model with larger variance

Comparison	p-value(SE)	5% sig.	10% sig.
best 2-dim vs. 6-dim	0.210(0.024)	36	45
best 2-dim vs. 1-dim(single var.)	4.7113e-11(4.5121e-11)	100	100
best 2-dim vs. 1-dim(sum of var.)	2.3447e-04(9.21234e-05)	100	100
rela. good 2-dim vs. 6-dim	0.2363(0.0286)	37	52
rela. good 2-dim vs. 1-dim(single var.)	5.1820e-11(4.5266e-11)	100	100
rela. good 2-dim vs. 1-dim(sum of var.)	3.3966e-04(1.6707e-04)	100	100

Results from table 2.27 shows that the 2-dimensional models are not significantly different in prediction performance with the 6-dimensional model on average. Among 100 iterations in comparing the difference in prediction between best 2-dimensional model and 6-dimensional model, 36 are significant at 5% significant level, 45 are significant at 10% significant level. Among 100 iterations in comparing the difference in prediction between relatively good 2-dimensional model and 6-dimensional model, 37 are significant at 5% significant level, 52 are significant at 10% significant level. In the tests comparing difference in prediction performance between 2-dimensional models and 1-dimensional models, every iteration is with a significant p-value at 5% significance level, thus on average the difference in prediction performance between 2-dimensional models and 1-dimensional models are significant. We also noticed that, considering table 2.12 and table 2.22, when the variance of response variable is increased from 20^2 to 50^2, the number of iterations with significant p-values in tests comparing difference in prediction performance between 2-dimensional models and 6-dimensional models decreased. However, every iteration is with a significant p-value at 5% significance level in testing difference in prediction performance between 2-dimensional models and 1-dimensional models when the variance increases.

Chapter 3 Methodology for Determining Value of P

In this chapter, cases of MDR with 3 risk scores is first discussed in order to analyze the effectiveness of MDR with p risk scores when $p = 3$. Then the function of MDR with 3 risk scores and MDR with 2 risk scores are compared of true models of different dimensions. At last, a methodology to determine the dimension of model to use given data is discussed.

3.1 Multifactor Dimensionality Reduction with Cases of 3-dimensional Models

This model selection simulation is to suppose that there are $M = 4$ SNP alleles A(a), B(b), C(c), D(d) with 6 possible two-way interactions A(a)B(b), A(a)C(c), A(a)D(d), B(b)C(c), B(b)D(d), C(c)D(d). The interaction results from A(a)B(b) produce elevated outcomes with either AA or BB, i.e., AABB, AABb, AAbb, aaBB, AaBB. The interaction results from A(a)C(c) produce elevated outcomes with either AA or CC, i.e., AACC, AACc, AAcc, aaCC, AaCC. The interaction results from A(a)D(d) produce elevated outcomes with either AA or DD, i.e., AADD, AADd, AAdd, aaDD, AaDD. The interaction results from B(b)C(c) produce elevated outcomes with either BB or CC, i.e., BBCC, BBCc, BBcc, bbCC, BbCC. The interaction results from B(b)D(d) produce elevated outcomes with either BB or DD, i.e., BBDD, BBDd, BBdd, bbDD, BbDD. The interaction results from C(c)D(d) produce elevated outcomes with either CC or DD, i.e., CCDD, CCDd, CCdd, ccDD, CcDD.

Then build one contingency table for A(a)B(b), A(a)C(c), A(a)D(d), B(b)C(c), B(b)D(d), C(c)D(d), respectively. In each contingency table, if any of the cell means exceeds the overall table mean then the individuals in the corresponding cells are labeled as “High” risk to the outcome, otherwise, those individuals are of “Low” risk of the outcome.

Let X_1 be the indicator function of whether one individual is labeled as “High” risk in the contingency table for A(a)B(b), X_2 be the indicator function of whether one individual is labeled as "High" risk in the contingency table for A(a)C(c), X_3 be the indicator function of whether one individual is labeled as "High" risk in the contingency table for A(a)D(d), X_4 be the indicator function of whether one individual is labeled as "High" risk in the contingency table for B(b)C(c), X_5 be the indicator function of whether one individual is labeled as "High" risk in the contingency table for B(b)D(d), X_6 be the indicator function of whether one individual is labeled as "High" risk in the contingency table for C(c)D(d).

The aim for this simulation is to compare the following 3-dimensional models and 1-dimensional models with the 6-dimensional model.

The 6-dimensional model is

$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \beta_4 X_4 + \beta_5 X_5 + \beta_6 X_6 + \varepsilon$

The 90 3-dimensional models are

$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 (X_3 + X_4 + X_5 + X_6) + \varepsilon,$

$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_3 + \beta_3 (X_2 + X_4 + X_5 + X_6) + \varepsilon,$

$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_4 + \beta_3 (X_2 + X_3 + X_5 + X_6) + \varepsilon,$

$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_5 + \beta_3 (X_2 + X_3 + X_4 + X_6) + \varepsilon,$

$Y = \beta_0 + \beta_1 X_1 + \beta_2 X_6 + \beta_3 (X_2 + X_3 + X_4 + X_5) + \varepsilon,$

$Y = \beta_0 + \beta_1 X_2 + \beta_2 X_3 + \beta_3 (X_1 + X_4 + X_5 + X_6) + \varepsilon,$

$Y = \beta_0 + \beta_1 X_2 + \beta_2 X_4 + \beta_3 (X_1 + X_3 + X_5 + X_6) + \varepsilon,$

$Y = \beta_0 + \beta_1 X_2 + \beta_2 X_5 + \beta_3 (X_1 + X_3 + X_4 + X_6) + \varepsilon,$

$Y = \beta_0 + \beta_1 X_2 + \beta_2 X_6 + \beta_3 (X_1 + X_3 + X_4 + X_5) + \varepsilon,$

$Y = \beta_0 + \beta_1 X_3 + \beta_2 X_4 + \beta_3 (X_1 + X_2 + X_5 + X_6) + \varepsilon,$

$Y = \beta_0 + \beta_1 X_3 + \beta_2 X_5 + \beta_3 (X_1 + X_2 + X_4 + X_6) + \varepsilon,$

$Y = \beta_0 + \beta_1 X_3 + \beta_2 X_6 + \beta_3 (X_1 + X_2 + X_4 + X_5) + \varepsilon,$

$Y = \beta_0 + \beta_1 X_4 + \beta_2 X_5 + \beta_3 (X_1 + X_2 + X_3 + X_6) + \varepsilon,$

$Y = \beta_0 + \beta_1 X_4 + \beta_2 X_6 + \beta_3 (X_1 + X_2 + X_3 + X_5) + \varepsilon,$

$Y = \beta_0 + \beta_1 X_5 + \beta_2 X_6 + \beta_3 (X_1 + X_2 + X_3 + X_4) + \varepsilon,$

$Y = \beta_0 + \beta_1 X_1 + \beta_2 (X_2 + X3) + \beta_3 (X_4 + X_5 + X_6) + \varepsilon,$

$Y = \beta_0 + \beta_1 X_1 + \beta_2 (X_2 + X4) + \beta_3 (X_3 + X_5 + X_6) + \varepsilon,$

$$Y = \beta_0 + \beta_1 X_1 + \beta_2(X_2 + X5) + \beta_3(X_3 + X_4 + X_6) + \varepsilon,$$
$$Y = \beta_0 + \beta_1 X_1 + \beta_2(X_2 + X6) + \beta_3(X_3 + X_4 + X_5) + \varepsilon,$$
$$Y = \beta_0 + \beta_1 X_1 + \beta_2(X_3 + X4) + \beta_3(X_2 + X_5 + X_6) + \varepsilon,$$
$$Y = \beta_0 + \beta_1 X_1 + \beta_2(X_3 + X5) + \beta_3(X_2 + X_4 + X_6) + \varepsilon,$$
$$Y = \beta_0 + \beta_1 X_1 + \beta_2(X_3 + X6) + \beta_3(X_2 + X_4 + X_5) + \varepsilon,$$
$$Y = \beta_0 + \beta_1 X_1 + \beta_2(X_4 + X5) + \beta_3(X_2 + X_3 + X_6) + \varepsilon,$$
$$Y = \beta_0 + \beta_1 X_1 + \beta_2(X_4 + X6) + \beta_3(X_2 + X_3 + X_5) + \varepsilon,$$
$$Y = \beta_0 + \beta_1 X_1 + \beta_2(X_5 + X6) + \beta_3(X_2 + X_3 + X_4) + \varepsilon,$$
$$Y = \beta_0 + \beta_1 X_2 + \beta_2(X_1 + X3) + \beta_3(X_4 + X_5 + X_6) + \varepsilon,$$
$$Y = \beta_0 + \beta_1 X_2 + \beta_2(X_1 + X4) + \beta_3(X_3 + X_5 + X_6) + \varepsilon,$$
$$Y = \beta_0 + \beta_1 X_2 + \beta_2(X_1 + X5) + \beta_3(X_3 + X_4 + X_6) + \varepsilon,$$
$$Y = \beta_0 + \beta_1 X_2 + \beta_2(X_1 + X6) + \beta_3(X_3 + X_4 + X_5) + \varepsilon,$$
$$Y = \beta_0 + \beta_1 X_2 + \beta_2(X_3 + X4) + \beta_3(X_1 + X_5 + X_6) + \varepsilon,$$
$$Y = \beta_0 + \beta_1 X_2 + \beta_2(X_3 + X5) + \beta_3(X_1 + X_4 + X_6) + \varepsilon,$$
$$Y = \beta_0 + \beta_1 X_2 + \beta_2(X_3 + X6) + \beta_3(X_1 + X_4 + X_5) + \varepsilon,$$
$$Y = \beta_0 + \beta_1 X_2 + \beta_2(X_4 + X5) + \beta_3(X_1 + X_3 + X_6) + \varepsilon,$$
$$Y = \beta_0 + \beta_1 X_2 + \beta_2(X_4 + X6) + \beta_3(X_1 + X_3 + X_5) + \varepsilon,$$
$$Y = \beta_0 + \beta_1 X_2 + \beta_2(X_5 + X6) + \beta_3(X_1 + X_3 + X_4) + \varepsilon,$$
$$Y = \beta_0 + \beta_1 X_3 + \beta_2(X_1 + X2) + \beta_3(X_4 + X_5 + X_6) + \varepsilon,$$
$$Y = \beta_0 + \beta_1 X_3 + \beta_2(X_1 + X4) + \beta_3(X_2 + X_5 + X_6) + \varepsilon,$$
$$Y = \beta_0 + \beta_1 X_3 + \beta_2(X_1 + X5) + \beta_3(X_2 + X_4 + X_6) + \varepsilon,$$
$$Y = \beta_0 + \beta_1 X_3 + \beta_2(X_1 + X6) + \beta_3(X_2 + X_4 + X_5) + \varepsilon,$$
$$Y = \beta_0 + \beta_1 X_3 + \beta_2(X_2 + X4) + \beta_3(X_1 + X_5 + X_6) + \varepsilon,$$
$$Y = \beta_0 + \beta_1 X_3 + \beta_2(X_2 + X5) + \beta_3(X_1 + X_4 + X_6) + \varepsilon,$$
$$Y = \beta_0 + \beta_1 X_3 + \beta_2(X_2 + X6) + \beta_3(X_1 + X_4 + X_5) + \varepsilon,$$
$$Y = \beta_0 + \beta_1 X_3 + \beta_2(X_4 + X5) + \beta_3(X_1 + X_2 + X_6) + \varepsilon,$$
$$Y = \beta_0 + \beta_1 X_3 + \beta_2(X_4 + X6) + \beta_3(X_1 + X_2 + X_5) + \varepsilon,$$
$$Y = \beta_0 + \beta_1 X_3 + \beta_2(X_5 + X6) + \beta_3(X_1 + X_2 + X_4) + \varepsilon,$$
$$Y = \beta_0 + \beta_1 X_4 + \beta_2(X_1 + X2) + \beta_3(X_3 + X_5 + X_6) + \varepsilon,$$
$$Y = \beta_0 + \beta_1 X_4 + \beta_2(X_1 + X3) + \beta_3(X_2 + X_5 + X_6) + \varepsilon,$$

$$Y = \beta_0 + \beta_1 X_4 + \beta_2(X_1 + X5) + \beta_3(X_2 + X_3 + X_6) + \varepsilon,$$
$$Y = \beta_0 + \beta_1 X_4 + \beta_2(X_1 + X6) + \beta_3(X_2 + X_3 + X_5) + \varepsilon,$$
$$Y = \beta_0 + \beta_1 X_4 + \beta_2(X_2 + X3) + \beta_3(X_1 + X_5 + X_6) + \varepsilon,$$
$$Y = \beta_0 + \beta_1 X_4 + \beta_2(X_2 + X5) + \beta_3(X_1 + X_3 + X_6) + \varepsilon,$$
$$Y = \beta_0 + \beta_1 X_4 + \beta_2(X_2 + X6) + \beta_3(X_1 + X_3 + X_5) + \varepsilon,$$
$$Y = \beta_0 + \beta_1 X_4 + \beta_2(X_3 + X5) + \beta_3(X_1 + X_2 + X_6) + \varepsilon,$$
$$Y = \beta_0 + \beta_1 X_4 + \beta_2(X_3 + X6) + \beta_3(X_1 + X_2 + X_5) + \varepsilon,$$
$$Y = \beta_0 + \beta_1 X_4 + \beta_2(X_5 + X6) + \beta_3(X_1 + X_2 + X_3) + \varepsilon,$$
$$Y = \beta_0 + \beta_1 X_5 + \beta_2(X_1 + X2) + \beta_3(X_3 + X_4 + X_6) + \varepsilon,$$
$$Y = \beta_0 + \beta_1 X_5 + \beta_2(X_1 + X3) + \beta_3(X_2 + X_4 + X_6) + \varepsilon,$$
$$Y = \beta_0 + \beta_1 X_5 + \beta_2(X_1 + X4) + \beta_3(X_2 + X_3 + X_6) + \varepsilon,$$
$$Y = \beta_0 + \beta_1 X_5 + \beta_2(X_1 + X6) + \beta_3(X_2 + X_3 + X_4) + \varepsilon,$$
$$Y = \beta_0 + \beta_1 X_5 + \beta_2(X_2 + X3) + \beta_3(X_1 + X_4 + X_6) + \varepsilon,$$
$$Y = \beta_0 + \beta_1 X_5 + \beta_2(X_2 + X4) + \beta_3(X_1 + X_3 + X_6) + \varepsilon,$$
$$Y = \beta_0 + \beta_1 X_5 + \beta_2(X_2 + X6) + \beta_3(X_1 + X_3 + X_4) + \varepsilon,$$
$$Y = \beta_0 + \beta_1 X_5 + \beta_2(X_3 + X4) + \beta_3(X_1 + X_2 + X_6) + \varepsilon,$$
$$Y = \beta_0 + \beta_1 X_5 + \beta_2(X_3 + X6) + \beta_3(X_1 + X_2 + X_4) + \varepsilon,$$
$$Y = \beta_0 + \beta_1 X_5 + \beta_2(X_4 + X6) + \beta_3(X_1 + X_2 + X_3) + \varepsilon,$$
$$Y = \beta_0 + \beta_1 X_6 + \beta_2(X_1 + X2) + \beta_3(X_3 + X_4 + X_5) + \varepsilon,$$
$$Y = \beta_0 + \beta_1 X_6 + \beta_2(X_1 + X3) + \beta_3(X_2 + X_4 + X_5) + \varepsilon,$$
$$Y = \beta_0 + \beta_1 X_6 + \beta_2(X_1 + X4) + \beta_3(X_2 + X_3 + X_5) + \varepsilon,$$
$$Y = \beta_0 + \beta_1 X_6 + \beta_2(X_1 + X5) + \beta_3(X_2 + X_3 + X_4) + \varepsilon,$$
$$Y = \beta_0 + \beta_1 X_6 + \beta_2(X_2 + X3) + \beta_3(X_1 + X_4 + X_5) + \varepsilon,$$
$$Y = \beta_0 + \beta_1 X_6 + \beta_2(X_2 + X4) + \beta_3(X_1 + X_3 + X_5) + \varepsilon,$$
$$Y = \beta_0 + \beta_1 X_6 + \beta_2(X_2 + X5) + \beta_3(X_1 + X_3 + X_4) + \varepsilon,$$
$$Y = \beta_0 + \beta_1 X_6 + \beta_2(X_3 + X4) + \beta_3(X_1 + X_2 + X_5) + \varepsilon,$$
$$Y = \beta_0 + \beta_1 X_6 + \beta_2(X_3 + X5) + \beta_3(X_1 + X_2 + X_4) + \varepsilon,$$
$$Y = \beta_0 + \beta_1 X_6 + \beta_2(X_4 + X5) + \beta_3(X_1 + X_2 + X_3) + \varepsilon,$$
$$Y = \beta_0 + \beta_1(X_1 + X_2) + \beta_2(X_3 + X4) + \beta_3(X_5 + X_6) + \varepsilon,$$
$$Y = \beta_0 + \beta_1(X_1 + X_2) + \beta_2(X_3 + X5) + \beta_3(X_4 + X_6) + \varepsilon,$$

$Y = \beta_0 + \beta_1(X_1 + X_2) + \beta_2(X_3 + X6) + \beta_3(X_4 + X_5) + \varepsilon,$

$Y = \beta_0 + \beta_1(X_1 + X_3) + \beta_2(X_2 + X4) + \beta_3(X_5 + X_6) + \varepsilon,$

$Y = \beta_0 + \beta_1(X_1 + X_3) + \beta_2(X_2 + X5) + \beta_3(X_4 + X_6) + \varepsilon,$

$Y = \beta_0 + \beta_1(X_1 + X_3) + \beta_2(X_2 + X6) + \beta_3(X_4 + X_5) + \varepsilon,$

$Y = \beta_0 + \beta_1(X_1 + X_4) + \beta_2(X_2 + X3) + \beta_3(X_5 + X_6) + \varepsilon,$

$Y = \beta_0 + \beta_1(X_1 + X_4) + \beta_2(X_2 + X5) + \beta_3(X_3 + X_6) + \varepsilon,$

$Y = \beta_0 + \beta_1(X_1 + X_4) + \beta_2(X_2 + X6) + \beta_3(X_3 + X_5) + \varepsilon,$

$Y = \beta_0 + \beta_1(X_1 + X_5) + \beta_2(X_2 + X3) + \beta_3(X_4 + X_6) + \varepsilon,$

$Y = \beta_0 + \beta_1(X_1 + X_5) + \beta_2(X_2 + X4) + \beta_3(X_3 + X_6) + \varepsilon,$

$Y = \beta_0 + \beta_1(X_1 + X_5) + \beta_2(X_2 + X6) + \beta_3(X_3 + X_4) + \varepsilon,$

$Y = \beta_0 + \beta_1(X_1 + X_6) + \beta_2(X_2 + X3) + \beta_3(X_4 + X_5) + \varepsilon,$

$Y = \beta_0 + \beta_1(X_1 + X_6) + \beta_2(X_2 + X4) + \beta_3(X_3 + X_5) + \varepsilon,$

$Y = \beta_0 + \beta_1(X_1 + X_6) + \beta_2(X_2 + X5) + \beta_3(X_3 + X_4) + \varepsilon,$

The six 1-dimensional models regressing on one single variable are

$Y = \beta_0 + \beta_1 X_1 + \varepsilon,$

$Y = \beta_0 + \beta_1 X_2 + \varepsilon,$

$Y = \beta_0 + \beta_1 X_3 + \varepsilon,$

$Y = \beta_0 + \beta_1 X_4 + \varepsilon,$

$Y = \beta_0 + \beta_1 X_5 + \varepsilon,$

$Y = \beta_0 + \beta_1 X_6 + \varepsilon,$

The 1-dimensional model regressing on the sum of six variables is

$Y = \beta_0 + \beta_1(X_1 + X_2 + X_3 + X_4 + X_5 + X_6) + \varepsilon.$

The true 6-dimensional model is $Y = 100 + 30 * I_{AAorBB} + 55 * I_{AAorCC} + 10 * I_{AAorDD} + 70 * I_{BBorCC} + 15 * I_{BBorDD} + 40 * I_{CCorDD} + \varepsilon,\ \varepsilon \sim N(0, 20^2)$. And 100 iterations with a training set consisting of 1000 individuals and a test set consisting of 500 individuals are performed.

In each iteration, after fitting 3-dimensional models and 1-dimensional models based on training data, the following models are taken to make comparisons in prediction on the testing data, the fitted 6-dimensional model from training data, the best fitted 3-dimensional model which is with the largest R-squared among all 3-

dimensional models from training data, the best fitted 1-dimensional model which is with the largest R-squared among the first six 1-dimensional models from training data, the fitted seventh 1-dimensional model from training data.

Then, the testing data is used to calculate MSPE(Mean Square Prediction Error) of the above four models, respectively.

The average MSPE's of 100 iterations with their standard error (SE) for the above four models are given in Table 3.1. Average r statistic and t statistic and their standard error in 100 iterations are given in Table 3.2. Table 3.3 displays the average difference in r-statistic and t-statistic between the best 3 dimensional model rest 3 model from table 3.2 along with their standard error from 100 iterations. Table 3.4 gives the average p-value with standard error on test data of these three test in 100 iterations, how many p-values out of the 100 iterations are significant at 5% level,how many p-values out of the 100 iterations are significant at 10% level, respectively.

Table 3.1: Average MSPEs from 100 iterations for best 3-dimensional model, 1-dimensional models and the 6-dimensional model

3-dim model(SE)	6-dim model(SE)	1-dim model(SE) (regress on single variable)	1-dim model(SE) (regress on sum of 6 variables)
429.5159(3.1562)	404.7234(3.1081)	2293.6515(11.9864)	1057.0427 (7.5094)

Table 3.2: Average r statistic, t statistic from 100 iterations for best 3-dimensional model, 1-dimensional models and the 6-dimensional model

	3-dim model(SE)	6-dim model(SE)	1-dim model(regress on single variable)(SE)	1-dim model(regress on sum of 6 variables)(SE)
r	0.9662 (0.0003)	0.9682(0.0003)	0.8032(0.0014)	0.9145(0.0007)
t	83.8884(0.3651)	86.6079(0.3945)	30.1588(0.1431)	50.5961(0.2465)

Table 3.3: Average r statistic and t statistic differences over 100 iterations between best 3-dimensional model and 1-dimensional models, and the 6-dimensional model

	3-dim model - 6-dim model (SE)	3-dim model - 1-dim model(regress on single variable)(SE)	3-dim model -1-dim model(regress on sum of 6 variables)(SE)
r	-1.9717e-03 (7.6088e-05)	0.1629(0.0012)	0.0517(0.0006)
t	-2.7196(0.1082)	53.7296(0.3269)	33.2923(0.3234)

Table 3.4: Average p-values from 100 iterations in comparing difference between best 3-dimensional model and 1-dimensional models, and the 6-dimensional model

	3-dim vs 6-dim(SE)	3-dim vs 1-dim(single var)(SE)	3-dim vs 1-dim(sum of var's)(SE)
p-value	0.0566 (0.0145)	1.0665e-62 (1.0408e-62)	2.1479e-22(1.5794e-22)
5% sig.	77	100	100
10% sig.	85	100	100

Table 3.1 shows that the 6-dimensional model and best 3-dimensional model outperform the 1-dimensional models. The performance of best 3-dimensional is close to the 6-dimensional model since the MSPE ratio of best 3-dimensional model to 6-dimensional model is $404.7234 \div 429.5159 = 0.9423$.

In Table 3.2, r statistic and t statistic are the same that was introduced in Chapter 2.5. Since the sample size of testing data to calculate r statistic and t statistic is 500, $t \sim t_{498}$ under $H_0 : \rho \leq 0$ approximately . The one-sided critical value for significance level 0.05 of t_{498} is 1.64. Thus, according to average t scores, all four models' abilities in prediction are statistically significant at 0.05 level.

Table 3.3 displays that the 6-dimensional model perform a little better than the best 3-dimensional model in prediction on average and the best 3-dimensional model has a slightly higher ability in prediction than the 1-dimensional models.

However, table 3.4 indicates that all 100 out of 100 iterations are with a p-value that is significant at a 0.05 significance level for testing on the difference in prediction between 3-dimensional model and 1-dimensional models. However, in the tests of prediction difference between 3-dimensional model and 6-dimensional models, 77 out of 100 iterations are with a p-value that is significant at a 0.05 significance level, 85 out of 100 iterations are with a p-value that is significant at a 0.10 significance level.

By the methodology to find a relatively good p-dimensional model proposed in chapter 2, a relatively good 3 dimensional model is found using testing data and compared with the relatively good 1-dimensional model regressed on single variable, the 1-dimensional model regressed on sum of all variables and the six dimensional model in MSPE, r statistic and t statistic, as listed in Table 3.5, and in significance of difference, as listed in Table 3.7, along with how many p-values out of the 100 iterations are significant at 5% level,how many p-values out of the 100 iterations are significant at 10% level. Table 3.6 exhibits the average differences in r-statistic and t-statistic between the relatively good 3-dimensional model and the rest 3 models from table 3.5 along with their standard errors from 100 iterations.

Table 3.5: Average MSPEs, r statistics, t statistics from 100 iterations for relatively good 3-dimensional model, 1-dimensional models and the 6-dimensional model

	relatively good 3-dim (SE)	6-dim(SE)	1-dim(SE) (regress on single variable)	1-dim(SE) (regress on sum of 6 variables)
MSPE	425.3672(3.3451)	404.7234(3.1081)	2293.6515(11.9864)	1057.0427(7.5094)
r	0.9665(0.0003)	0.9682(0.0003)	0.8032(0.0014)	0.9145(0.0007)
t	84.3462(0.4059)	86.6079(0.3945)	30.1588(0.1431)	50.5961(0.2465)

Table 3.6: Average r statistic and t statistic differences over 100 iterations between relatively good 3-dimensional model and 1-dimensional models, and the 6-dimensional model

	3-dim model - 6-dim model (SE)	3-dim model - 1-dim model(regress on single variable)(SE)	3-dim model -1-dim model(regress on sum of 6 variables)(SE)
r	-0.0017 (0.0001)	0.1633(0.0013)	0.0520 (0.0006)
t	-2.2618(0.1621)	54.1874(0.3709)	33.7501 (0.3582)

Table 3.7: Average p-values from 100 iterations in comparing difference between relatively good 3-dimensional model and 1-dimensional models, and the 6-dimensional model

	3-dim vs 6-dim(SE)	3-dim vs 1-dim(single var)(SE)	3-dim vs 1-dim(sum of var's) (SE)
p-value	0.1376(0.0250)	1.0662e-62(1.0408e-62)	4.4738e-21 (4.4735e-21)
5% sig.	65	100	100
10% sig.	72	100	100

3.2 Selecting from 2-dim and 3-dim Models

In this section, there are six true models in study, with dimensions of 1, 2, 3, 4, 5, 6, respectively. The performance of best two-dimensional model and best three-dimensional model are compared for each true model on simulated testing data, then the performance of relatively good two-dimensional model and relatively good three-dimensional model obtained by methodology in Chapter 2 are compared on simulated testing data.

In each case below, it gives the true model to generate data in 100 iterations. In each iteration, there are 1000 individuals in the training data and 500 individuals in the testing data.

The training data is first used to fit the best 2-dim model and the best 3-dim model whose average MSPE, average r statistic, average t statistic, average p-value for test of difference from 100 iterations are compared in testing data. Then the training data is used to find the relatively good 2-dim model and the relatively good 3-dim model by the permutation methodology introduced in Chapter 2.6 whose average MSPE, average r statistic, average t statistic, average p-value for test of difference from 100 iterations are compared in testing data.

We make the same assumption for SNPs numbers and SNP-SNP interactions as in Section 3.1, $X_1 = I_{AAorBB},\ X_2 = I_{AAorCC},\ X_3 = I_{AAorDD},\ X_4 = I_{BBorCC},\ X_5 = I_{BBorDD},\ X_6 = I_{CCorDD}$.

Case 1

True Model: $Y = 100 + 100 \times X_1 + 0 \times X_2 + 0 \times X_3 + 0 \times X_4 + 0 \times X_5 + 0 \times X_6 + \varepsilon$, $\varepsilon \sim N(0, 20^2)$				
	best 2-dim	best 3-dim	rela. good 2-dim	rela. good 3-dim
MSPE(SE)	401.3794(2.6598)	402.5526(2.6716)	498.8267(8.7246)	403.158896(2.7260)
r(SE)	0.9277(0.0005)	0.9274(0.0005)	0.9092(0.0017)	0.9273(0.0005)
t(SE)	55.5553(0.2230)	55.4596(0.2213)	49.4672(0.5556)	55.4160(0.2296)
p-value for diff test(SE)	0.4340(0.0303)		0.1457(0.0275)	
5% sig.	10		69	
10% sig.	20		74	
	best 2-dim - best 3-dim		rela 2-dim - rela 3-dim	
MSPE(SE)	-1.1732(0.2032)		95.6678(7.7940)	
r(SE)	2.2028e-04(3.7941e-05)		-0.0181(0.0015)	
t(SE)	0.0956(0.0165)		-5.9488(0.4726)	

Case 2

True Model: $Y = 100 + 70 \times X_1 + 30 \times X_2 + 0 \times X_3 + 0 \times X_4 + 0 \times X_5 + 0 \times X_6 + \varepsilon$, $\varepsilon \sim N(0, 20^2)$				
	best 2-dim	best 3-dim	rela. good 2-dim	rela. good 3-dim
MSPE(SE)	465.2058(3.0922)	415.3998(3.1395)	471.4914(3.5272)	423.8685(3.322)
r(SE)	0.8913(8e-04)	0.9035 (8e-04)	0.8897(9e-04)	0.9014(8e-04)
t(SE)	43.9715(0.1877)	47.1735(0.223)	43.6114(0.2095)	46.601(0.2326)
p-value for diff test(SE)	0.0125 (0.0054)		0.0454 (0.0155)	
5% sig.	93		86	
10% sig.	97		93	
	best 2-dim - best 3-dim		rela 2-dim - rela 3-dim	
MSPE(SE)	49.806 (1.8765)		47.6229 (2.4072)	
r(SE)	-0.0122 (5e-04)		-0.0117 (6e-04)	
t(SE)	-3.202 (0.1204)		-2.9897 (0.1503)	

Case 3

True Model: $Y = 100 + 50 \times X_1 + 30 \times X_2 + 20 \times X_3 + 0 \times X_4 + 0 \times X_5 + 0 \times X_6 + \varepsilon$, $\varepsilon \sim N(0, 20^2)$				
	best 2-dim	best 3-dim	rela. good 2-dim	rela. good 3-dim
MSPE(SE)	474.711(3.5697)	450.8205(3.2387)	474.9487(3.5448)	454.4775(3.4849)
r(SE)	0.8727(0.0010)	0.8794(9e-04)	0.8726(0.0010)	0.8784(0.0010)
t(SE)	39.9766(0.1870)	41.3308(0.1893)	39.9626(0.1876)	41.1338(0.2025)
p-value for diff test(SE)	0.1812 (0.0262)		0.2194 (0.0294)	
5% sig.	51		51	
10% sig.	60		56	
	best 2-dim - best 3-dim		rela 2-dim - rela 3-dim	
MSPE(SE)	23.8905 (2.0405)		20.4712 (1.9887)	
r(SE)	-0.0068 (6e-04)		-0.0058 (6e-04)	
t(SE)	-1.3541(0.1166)		-1.1712(0.1160)	

Case 4

True Model: $Y = 100 + 40 \times X_1 + 20 \times X_2 + 5 \times X_3 + 15 \times X_4 + 0 \times X_5 + 0 \times X_6 + \varepsilon$, $\varepsilon \sim N(0, 20^2)$				
	best 2-dim	best 3-dim	rela. good 2-dim	rela. good 3-dim
MSPE(SE)	418.3904(2.7692)	409.5602(2.6311)	422.1417(2.7566)	410.8684(2.7198)
r(SE)	0.8331(0.0013)	0.8370(0.0012)	0.8315(0.0012)	0.8364(0.0012)
t(SE)	33.6944(0.1681)	34.2108(0.1647)	33.4771(0.1611)	34.1374(0.1684)
p-value for diff test(SE)	0.2028 (0.0272)		0.1741 (0.0246)	
5% sig.	49		47	
10% sig.	57		60	
	best 2-dim - best 3-dim		rela 2-dim - rela 3-dim	
MSPE(SE)	8.8302 (0.6503)		11.2734 (0.9141)	
r(SE)	-0.0039 (3e-04)		-0.0049 (4e-04)	
t(SE)	-0.5163 (0.038)		-0.6603 (0.0544)	

Case 5

True Model: $Y = 100 + 20 \times X_1 + 30 \times X_2 + 10 \times X_3 + 5 \times X_4 + 35 \times X_5 + 0 \times X_6 + \varepsilon$, $\varepsilon \sim N(0, 20^2)$				
	best 2-dim	best 3-dim	rela. good 2-dim	rela. good 3-dim
MSPE(SE)	412.2716(2.8760)	409.4685(2.8269)	417.0436(2.8454)	410.1979(2.9674)
r(SE)	0.8613(0.0010)	0.8623(0.0010)	0.8596(0.0010)	0.8621(0.001)
t(SE)	37.9148(0.1756)	38.0897(0.1732)	37.6235(0.1728)	38.0598(0.1764)
p-value for diff test(SE)	0.4149 (0.0279)		0.3474 (0.0308)	
5% sig.	6		26	
10% sig.	16		34	
	best 2-dim - best 3-dim		rela 2-dim - rela 3-dim	
MSPE(SE)	2.8031 (0.5529)		6.8457 (1.1581)	
r(SE)	-0.001 (2e-04)		-0.0026 (4e-04)	
t(SE)	-0.175 (0.0344)		-0.4363 (0.0706)	

Case 6

True Model: $Y = 100 + 30 \times X_1 + 55 \times X_2 + 10 \times X_3 + 70 \times X_4 + 15 \times X_5 + 40 \times X_6 + \varepsilon$, $\varepsilon \sim N(0, 20^2)$				
	best 2-dim	best 3-dim	rela. good 2-dim	rela. good 3-dim
MSPE(SE)	465.7856(3.0297)	429.2765(2.7712)	469.3885(3.2331)	425.0592(3.0735)
r(SE)	0.9634(3e-04)	0.9663(2e-04)	0.9631(3e-04)	0.9667(2e-04)
t(SE)	80.3682(0.2977)	83.9571(0.3015)	80.0703(0.3171)	84.4411(0.3227)
p-value for diff test(SE)	0.0642 (0.0127)		0.0996 (0.0219)	
5% sig.	74		79	
10% sig.	81		82	
	best 2-dim - best 3-dim		rela 2-dim - rela 3-dim	
MSPE(SE)	36.5092 (1.6347)		44.3293 (2.4278)	
r(SE)	-0.0029 (1e-04)		-0.0035 (2e-04)	
t(SE)	-3.589 (0.1607)		-4.3709 (0.2369)	

In Case 1, the best 2-dimensional model has a smaller MSPE on the testing data than the best 3-dimensional model on average in 100 iterations. This is because the true model is just 1-dimension and 3-dimensional model has more error in prediction due to over-fitting. The relatively good 2-dimensional model has larger MSPE than the relatively good 3-dimensional model obtained through permutation methodology(Chapter 2.6) with permutation number 1000 on average in 100 iterations. This because the relatively good 2-dimensional model has more variation which can be reflected by its MSPE's standard error. The p-values of testing on difference in prediction that are significant at 5% and 10% displays that there is almost no significant difference between the best 2-dimensional model's and 3-dimensional model's ability in prediction, but due to the large variation of relatively good 2-dimensional model, a relatively good 3-dimensional model is more likely to have better performance in prediction than than a relatively good 2-dimensional model. The last few columns of the summary table displays the average differences between MSPE, r-statistic and t-statistic of best models of 2-dimension and 3-dimension, relatively good models of

2-dimension and 3-dimension, which illustrates a best 2-dimensional model would perform a slightly better in prediction than a best 3-dimensional model on average since the difference in average MSPE is negative and differences in average r-statistic and average t-statistic are positive, a relatively good 3-dimensional model would perform much better in prediction than a relatively good 2-dimensional model on average since the difference in average MSPE is positive and differences in average r-statistic and average t-statistic are negative. Conclusions about prediction performance between models can be drawn for the rest 5 cases in a similar way by average differences between MSPE, r-statistic and t-statistic.

In Case 2, the best 2-dimensional model has a larger MSPE on testing data than the best 3-dimensional model on average in 100 iterations. Even though the true model is 2-dimensional and the best 3-dimensional model might be over-fitting, the 3-dimensional models has a larger chance to reach the expression of the true model than the 2-dimensional models. For example, given the true model in Case 2, the pattern for the 2-dimensional models is $Y = \beta_0 + \beta_1 Z_1 + \beta_2 Z_2 + \varepsilon$, where Z_1 is the sum of some of the X_i's, $i = 1, 2, ..., 6$, and Z_2 is the sum of the rest X_i's, $i = 1, 2, ..., 6$. it will reach the true model only when Z_1 contains X_1 and Z_2 contains X_2 or Z_1 contains X_2 and Z_2 contains X_1. 15 out of the 31 2-dimensional models will have this format. However, the pattern for the 3-dimensional models is $Y = \beta_0 + \beta_1 Z_1 + \beta_2 Z_2 + \beta_3 Z_3 + \varepsilon$, where $Z_1 + Z_2 + Z_3 = X_1 + X_2 + X_3 + X_4 + X_5 + X_6$. It will reach the true model when Z_i contains X_1 and Z_j contains X_2 where $i, j = 1, 2, 3, i \neq j$. 65 out of the 90 3-dimensional models will have this format. Since the best models are chosen by R^2 and variability exists in the data, the selected best models can't be certain to reach the true model in every iteration, thus the 3-dimensional model has a smaller MSPE on average. The p-values of testing on difference in prediction shows that there exists significant difference between the best 2-dimensional model's and 3-dimensional model's ability in prediction, between the relatively good 2-dimensional model's and relatively good 3-dimensional model's ability in prediction, respectively. Besides, the last three columns shows the best 3 dimensional model outperforms the best 2 dimensional model on average, the relatively good 3 dimensional model outperforms the relatively good 2 dimensional model on average.

In Case 3-6, best 3-dimensional model has a smaller MSPE on the testing data than the best 2-dimensional model on average in 100 iterations. The relatively good 3-dimensional model has smaller MSPE than the relatively good 2-dimensional model obtained through permutation methodology(Chapter 2.6) with permutation number 1000 on average in 100 iterations. And their MSPE's differences are relatively small, respectively. But from the information provided from p-value to test difference in prediction, for Case 3 and Case 4, with around 50% opportunity, a 3-dimensional model will perform significantly better than a 2-dimensional model at 5% significance level. For Case 5, with around 60% opportunity, a best 3-dimensional model will perform significantly better than a best 2-dimensional model at 5% significance level, with around 30% opportunity, a relatively good 3-dimensional model will perform significantly better than a relatively good 2-dimensional model at 5% significance level. For Case 6, with around 80% opportunity, a 3-dimensional model will perform significantly better than a 2-dimensional model at 5% significance level.

In Case 1-6, the average r statistic of best 2-dimensional model is close to that of best 3-dimensional model, the average r statistic of relatively good 2-dimensional model is close to that of the relatively good 3-dimensional model, and the average t statistics where $t \sim t_{498}$ under $H_0 : \rho \leq 0$ approximately shows the four models ability in prediction are statistically significant at 0.05 level.

For all six cases, relatively good 3-dimensional model outperforms the relatively good 2-dimensional model in MSPE on average. The difference of average MSPE between relatively good 2-dimensional model and relatively good 3-dimensional model selected by permutation decreases from Case 1 to Case 4, then begins to increase from Case 5 to Case 6. Same pattern is observed for difference of average MSPE between best 2-dimensional model and best 3-dimensional model from Case 2 to Case 6, where best 3-dimensional model outperforms the best 2-dimensional model in MSPE on average, except for Case 1, where best 3-dimensional model does not outperform the best 2-dimensional model in MSPE on average. The average r statistic for best 2-dimensional model, best 3-dimensional model, relatively good 2-dimensional model, relatively good 3-dimensional model also preserves the same pattern that it decreases from Case 1 to Case 4 and then increases from Case 5 to Case 6, respectively.

The above trend is presumably caused by the variance of response variable. The variance of Y is listed below for Case 1 - Case 6.

Table 3.8: Average variance of response variable over 100 iterations

Case	1	2	3	4	5	6
Var(Y)	2868.226	2276.492	1992.234	1369.719	1583.819	2101.865

If the response variable has larger variability, in those cases where 3-dimensional model outperforms 2-dimensional model on average, it will have larger MSPE difference.

If the response variable has larger variability, in all six cases r statistic will have larger value because if the responses move in a larger range, then the predicted responses are also anticipated to move in a larger range. Thus, they will have a larger correlation than those moving in a smaller range.

3.3 Methodology to Select Relatively Good P Dimensional Model, P Unknown

Supposing there are N observations, m predictive variables and the dataset is available, this section gives a method to find the relatively good p-dimensional model to apply to the dataset. A 5-fold cross validation strategy is adopted.

***Step*1** Separate the dataset into 5 pieces, use the first piece as testing data and the rest as training data.

***Step*2** Use the length of l where l is the previous integer of $\sqrt{m}$ to cut m into $n+1$ pieces, approximately $\frac{m}{l}+1$ pieces, namely, m_1=1, m_2,..., $m_{n+1} = m$ with $m_j \approx (j-1)\sqrt{m}$ where $j = 2, ..., n$

***Step*3** Find relatively good m_1-dimensional model, m_2-dimensional model,..., m_n-dimensional model with the permutation methodology introduced in Chapter 2.5 by training data. Obtain their MSPE using testing data, noted as $MSPE_1$, $MSPE_2$,..., $MSPE_{n+1}$.

***Step*4** Repeat Step3 five times using different fold, respectively. Calculate the average MSPE, noted as $\overline{MSPE_1}$,$\overline{MSPE_2}$,..., $\overline{MSPE_{n+1}}$.

***Step*5** Find the least $\overline{MSPE_i}$, $i = 1, 2, ..., n$, noted as $\overline{MSPE_{i_0}}$. If there does not exist $\overline{MSPE_j}$ with a larger or smaller dimension than $\overline{MSPE_{i_0}}$, keep $\overline{MSPE_{i_0}}$ and the next largest $\overline{MSPE_i}$, noted as $\overline{MSPE_{i_1}}$(example shown as Figure 3.1), reset $m_{left} = min(i_0, i_1)$, $m_{right} = max(i_0, i_1)$; If there exist $\overline{MSPE_j}$'s with a larger and a smaller dimension than $\overline{MSPE_{i_0}}$, keep both of them, noted as $\overline{MSPE_{i_1}}$ and $\overline{MSPE_{i_2}}$(example shown as Figure 3.2), reset $m_{left} = min(i_1, i_2)$, $m_{right} = max(i_1, i_2)$.

***Step*6** Set the range of predictive variables from $[1, m]$ to $[m_{left}, m_{right}]$, which means $m = m_{right} - m_{left} + 1$ in Step2, repeat Step1-Step5 until $m < 8$, then move to Step 7.

***Step*7** Use the permutation method to find relatively good p-dimension models, $p = m_{left},\ m_{left} + 1,\ m_{left} + 2,\ ...,\ m_{right}$, respectively by training data in each CV fold and choose the model with the least average MSPE in 5-fold CV to be "relatively good" p-dimensional mdoel. Thus, the dimension is determined.

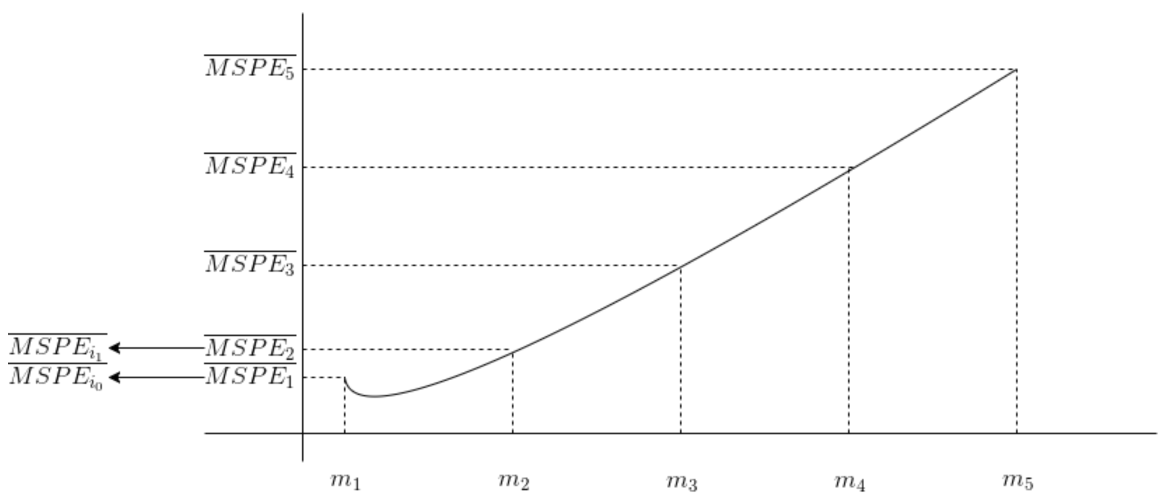

Figure 3.1: First example for Step 5

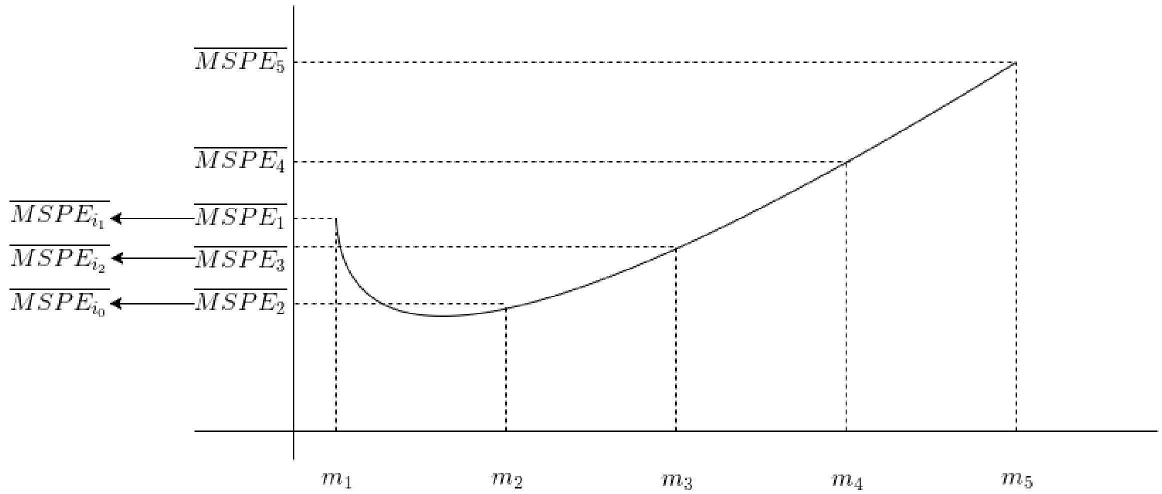

Figure 3.2: Second example for Step 5

Next, a hypothetical example is listed below to demonstrate how to use the above methodology.

- The assumption of this example to start is that the curve of number of prediction variables and MSPE is of the shape in Figure 3.2 where the lowest point of MSPE curve is not at either of the endpoints.

- Second assumption is that there are 98 explanatory variables and one continuous outcome, which means $m = 98$, $\sqrt{m} = 9.89 \approx 10$. Thus, $l = 10$, $m_1 = 1$, $m_2 = 1 \times 10$, $m_3 = 2 \times 10$, $m_4 = 3 \times 10$, $m_5 = 4 \times 10$, $m_6 = 5 \times 10$, $m_7 = 6 \times 10$, $m_8 = 7 \times 10$, $m_9 = 8 \times 10$, $m_{10} = 9 \times 10$, $m_{11} = 98$.

- Separate the data set into 5 parts randomly, use the ith part as testing data and the rest as training data in ith fold of 5-fold cross-validation, $i = 1, 2, 3, 4, 5$. On training data in ith fold, apply the permutation method in Chapter 2.5 to find "relatively good" m_j-dimensional model, $j = 1, ..., 11$. respectively. Then apply the eleven "relatively good" m_j-dimensional models, $j = 1, ..., 11$ on testing data to obtain $MSPE_{ij}$, $i = 1, 2, 3, 4, 5$, $j = 1, ..., 11$, respectively. Calculate $\overline{MSPE_j} = \frac{1}{5}(MSPE_{1j}+MSPE_{2j}+MSPE_{3j}+MSPE_{4j}+MSPE_{5j})$, $j = 1, ..., 11$, respectively.

- For illustration, assume $\overline{MSPE_1} = 500$, $\overline{MSPE_2} = 200$, $\overline{MSPE_3} = 100$, $\overline{MSPE_4} = 110$, $\overline{MSPE_5} = 115$, $\overline{MSPE_6} = 120$, $\overline{MSPE_7} = 130$, $\overline{MSPE_8} = 145$, $\overline{MSPE_9} = 170$, $\overline{MSPE_{10}} = 200$, $\overline{MSPE_{11}} = 240$.

- The smallest average MSPE from cross-validation is $\overline{MSPE_3}$, noted as $\overline{MSPE_{i_0}}$. There does exist $\overline{MSPE_2} = 200$ based on m_2 that is smaller than m_3 in dimension and $\overline{MSPE_4} = 110$ based on m_4 that is larger than m_3 in dimension. Denote $\overline{MSPE_2} = 200$ as $\overline{MSPE_{i_1}}$, Denote $\overline{MSPE_4} = 500$ as $\overline{MSPE_{i_2}}$, set $m_{left} = 10$, $m_{right} = 30$.

- Reset the range of prediction variables' number from $[1, 98]$ to $[10, 30]$, reset $m = 30 - 10 + 1 = 21$, $\sqrt{m} = 4.58 \approx 5$. Thus, $l = 5$, reset $m_1 = 10, m_2 = 15, m_3 = 20, m_4 = 25, m_5 = 30$.

- Separate the data set into 5 parts randomly, use the ith part as testing data and the rest as training data in ith fold of 5-fold cross-validation, $i = 1, 2, 3, 4, 5$. On training data in ith fold, apply the permutation method in Chapter 2.5 to find "relatively good" m_j-dimensional model, $j = 1, ..., 5$. respectively. Then apply the five "relatively good" m_j-dimensional models, $j = 1, ..., 5$ on testing data to obtain $MSPE_{ij}$, $i = 1, 2, 3, 4, 5$, $j = 1, ..., 5$, respectively. Calculate $\overline{MSPE_j} = \frac{1}{5}(MSPE_{1j} + MSPE_{2j} + MSPE_{3j} + MSPE_{4j} + MSPE_{5j})$, $j = 1, ..., 5$, respectively.

- For illustration, assume

  $\overline{MSPE_1} = 200$, $\overline{MSPE_2} = 150$, $\overline{MSPE_3} = 100$, $\overline{MSPE_4} = 95$, $\overline{MSPE_5} = 110$.

- The smallest average MSPE from cross-validation is $\overline{MSPE_4}$, noted as $\overline{MSPE_{i_0}}$. There does exist $\overline{MSPE_3} = 100$ based on m_3 that is smaller than m_4 in dimension and $\overline{MSPE_5} = 110$ based on m_5 that is larger than m_4 in dimension. Denote $\overline{MSPE_3} = 100$ as $\overline{MSPE_{i_1}}$, Denote $\overline{MSPE_5} = 110$ as $\overline{MSPE_{i_2}}$, set $m_{left} = 20$, $m_{right} = 30$.

- Reset the range of prediction variables' number from $[10, 30]$ to $[20, 30]$, reset $m = 30 - 20 + 1 = 11$, $\sqrt{m} = 3.32 \approx 3$. Thus, $l = 3$, reset $m_1 = 20, m_2 = 23, m_3 = 26, m_4 = 29, m_5 = 30$.

- Separate the data set into 5 parts randomly, use the ith part as testing data and the rest as training data in ith fold of 5-fold cross-validation, $i = 1, 2, 3, 4, 5$. On training data in ith fold, apply the permutation method in Chapter 2.5 to find "relatively good" m_j-dimensional model, $j = 1, ..., 5$. respectively. Then apply the five "relatively good" m_j-dimensional models, $j = 1, ..., 5$ on testing data to obtain $MSPE_{ij}$, $i = 1, 2, 3, 4, 5$, $j = 1, ..., 5$, respectively. Calculate $\overline{MSPE_j} = \frac{1}{5}(MSPE_{1j} + MSPE_{2j} + MSPE_{3j} + MSPE_{4j} + MSPE_{5j})$, $j = 1, ..., 5$, respectively.

- For illustration, assume

  $\overline{MSPE_1} = 100$, $\overline{MSPE_2} = 93$, $\overline{MSPE_3} = 97$, $\overline{MSPE_4} = 105$, $\overline{MSPE_5} = 110$.

- The smallest average MSPE from cross-validation is $\overline{MSPE_2} = 93$, noted as $\overline{MSPE_{i_0}}$. There does exist $\overline{MSPE_1} = 100$ based on m_1 that is smaller than m_2 in dimension and $\overline{MSPE_3} = 97$ based on m_3 that is larger than m_2 in dimension. Denote $\overline{MSPE_3} = 100$ as $\overline{MSPE_{i_1}}$, Denote $\overline{MSPE_1} = 97$ as $\overline{MSPE_{i_2}}$, set $m_{left} = 20$, $m_{right} = 26$.

- Now $m = 7 < 8$,separate the data set into 5 parts randomly, use the ith part as testing data and the rest as training data in ith fold of 5-fold cross-validation, $i = 1, 2, 3, 4, 5$. On training data in ith fold, apply the permutation method in Chapter 2.5 to find "relatively good" 20-dimensional model,21-dimensional model,22-dimensional model,23-dimensional model,24-dimensional model,25-dimensional model,26-dimensional model, respectively. Then apply the seven "relatively good" models on testing data to obtain $MSPE_{ij}$, $i = 1, ..., 7$, $j = 1, ..., 5$, respectively. Calculate $\overline{MSPE_j} = \frac{1}{5}(MSPE_{1j} + MSPE_{2j} + MSPE_{3j} + MSPE_{4j} + MSPE_{5j})$, $j = 1, ..., 5$, respectively.

- For illustration, assume

  $\overline{MSPE_1} = 100$, $\overline{MSPE_2} = 99$, $\overline{MSPE_3} = 91$, $\overline{MSPE_4} = 93$, $\overline{MSPE_5} = 93.5$, $\overline{MSPE_6} = 95$, $\overline{MSPE_7} = 97$. Thus, the relatively good model is determined to be 22-dimensional model because $\overline{MSPE_3} = 91$ is the smallest among $\overline{MSPE_i},\ i = 1, ..., 7$.

However, in real data, there might be multiple lowest average MSPEs from 5-fold cross-validation in Step 3 which are close to each other in value due to data's varibility, shown in Figure 3.3. In those cases, we could flexibly apply the above method, such as applying Step 5 respectively on each lowest average MSPE to narrow down the range of prediction variables by segments to find the relatively good p-dimensional model in each segment, then comparing the relatively good segment p-dimensional models to find the relatively good overall p-dimensional model.

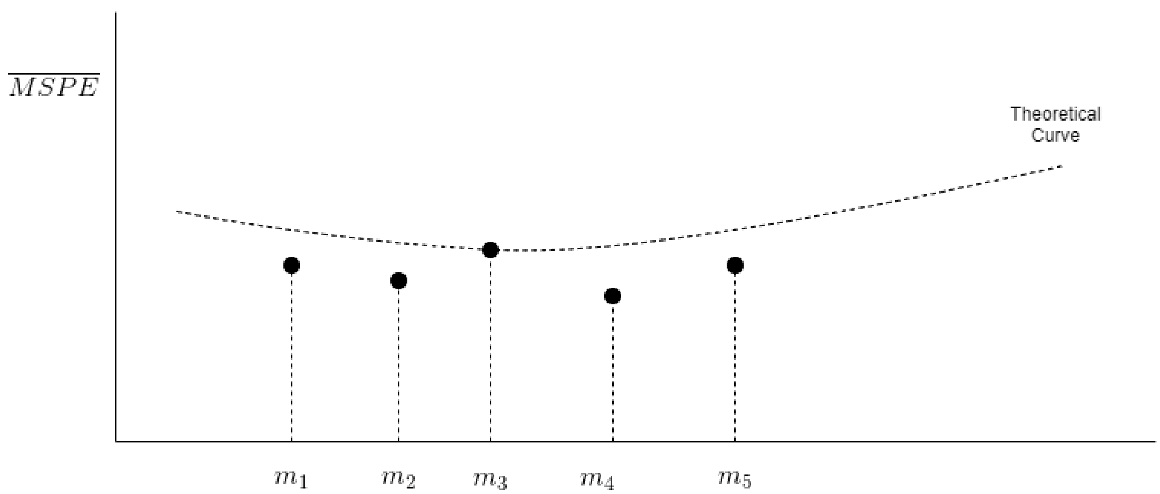

Figure 3.3: Example for multiple lowest average MSPEs

Chapter 4 Theoretical Support

In this Chapter, the response variable $Y = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_3 + \varepsilon$, where X_1, X_2, X_3, ε are mutually independent, $\varepsilon \sim N(0, \sigma^2)$.

Without loss of generality, we assume each observation of X_1 has a Bernoulli $Ber(p_1)$ distribution, each observation of X_2 has a Bernoulli $Ber(p_2)$ distribution, each observation of X_3 has a Bernoulli $Ber(p_3)$ distribution. The square error of prediction on a testing data of size 500 is expressed as below.

$$\begin{aligned} SPE &= \sum_{i=1}^{500} [\widehat{y}_{i,test} - y_{i,test}]^2 \\ &= \sum_{i=1}^{500} [X_{i,test}\widehat{\beta}_{train} - y_{i,test}]^2 \\ &= [X_{test}\widehat{\beta}_{train} - Y_{test}]^T [X_{test}\widehat{\beta}_{train} - Y_{test}] \end{aligned}$$

where $\widehat{\beta}_{train} = (X_{train}^T X_{train})^{-1}(X_{train}^T Y_{train})$.

Thus, $SPE = [X_{test}(X_{train}^T X_{train})^{-1}(X_{train}^T Y_{train}) - Y_{test}]^T [X_{test}(X_{train}^T X_{train})^{-1}$ $(X_{train}^T Y_{train}) - Y_{test}]$.

4.1 Asymptotic Estimation on MSPE of 1-Dimensional Model Fitted by Single Variable

Now, assume $X_{test} = \begin{pmatrix} 1 & x_{1,1,test} \\ 1 & x_{1,2,test} \\ \vdots & \vdots \\ 1 & x_{1,500,test} \end{pmatrix}$, $X_{train} = \begin{pmatrix} 1 & x_{1,1,train} \\ 1 & x_{1,2,train} \\ \vdots & \vdots \\ 1 & x_{1,1000,train} \end{pmatrix}$.

Then, $X_{train}^T X_{train} = \begin{pmatrix} 1000 & 1000\overline{X}_{1,train} \\ 1000\overline{X}_{1,train} & 1000\overline{X^2_1}_{,train} \end{pmatrix}$,

where $\overline{X}_{1,train} = \dfrac{1}{1000}(x_{1,1,train} + x_{1,2,train} + ... + x_{1,1000,train})$,

$$\overline{X_{1}^{2}}_{,train} = \frac{1}{1000}(x_{1,1,train}^2 + x_{1,2,train}^2 + ... + x_{1,1000,train}^2).$$

$$(X_{train}^T X_{train})^{-1} = \frac{\begin{pmatrix} 1000\overline{X_{1}^{2}}_{,train} & -1000\overline{X}_{1,train} \\ -1000\overline{X}_{1,train} & 1000 \end{pmatrix}}{1000^2\overline{X_{1}^{2}}_{,train} - 1000^2\overline{X}_{1,train}^2}$$

$$= \frac{\begin{pmatrix} \overline{X_{1}^{2}}_{,train} & -\overline{X}_{1,train} \\ -\overline{X}_{1,train} & 1 \end{pmatrix}}{1000(\overline{X_{1}^{2}}_{,train} - \overline{X}_{1,train}^2)}.$$

$$X_{test}(X_{train}^T X_{train})^{-1} = \frac{\begin{pmatrix} 1 & x_{1,1,test} \\ \vdots & \vdots \\ 1 & x_{1,500,test} \end{pmatrix}\begin{pmatrix} \overline{X_{1}^{2}}_{,train} & -\overline{X}_{1,train} \\ -\overline{X}_{1,train} & 1 \end{pmatrix}}{1000(\overline{X_{1}^{2}}_{,train} - \overline{X}_{1,train}^2)}$$

$$= \frac{\begin{pmatrix} \overline{X_{1}^{2}}_{,train} - x_{1,1,test}\overline{X}_{1,train} & -\overline{X}_{1,train} + x_{1,1,test} \\ \overline{X_{1}^{2}}_{,train} - x_{1,2,test}\overline{X}_{1,train} & -\overline{X}_{1,train} + x_{1,2,test} \\ \vdots & \vdots \\ \overline{X_{1}^{2}}_{,train} - x_{1,500,test}\overline{X}_{1,train} & -\overline{X}_{1,train} + x_{1,500,test} \end{pmatrix}}{1000(\overline{X_{1}^{2}}_{,train} - \overline{X}_{1,train}^2)}.$$

$$X_{train}^T Y_{train} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_{1,1,train} & x_{1,2,train} & \cdots & x_{1,1000,train} \end{pmatrix}\begin{pmatrix} y_{1,train} \\ y_{2,train} \\ \vdots \\ y_{1000,train} \end{pmatrix}$$

$$= \begin{pmatrix} 1000\overline{Y}_{train} \\ 1000\overline{X_{1,train}Y_{train}} \end{pmatrix}.$$

$$X_{test}\widehat{\beta}_{train} = X_{test}(X_{train}^T X_{train})^{-1} X_{train}^T Y_{train}$$

$$= \begin{pmatrix} \frac{(\overline{X^2_{1}}_{,train} - x_{1,1,test}\overline{X}_{1,train})\overline{Y}_{train} + (-\overline{X}_{1,train} + x_{1,1,test})\overline{X_{1,train}Y_{train}}}{(\overline{X^2_{1}}_{,train} - \overline{X}^2_{1,train})} \\ \frac{(\overline{X^2_{1}}_{,train} - x_{1,2,test}\overline{X}_{1,train})\overline{Y}_{train} + (-\overline{X}_{1,train} + x_{1,2,test})\overline{X_{1,train}Y_{train}}}{(\overline{X^2_{1}}_{,train} - \overline{X}^2_{1,train})} \\ \vdots \\ \frac{(\overline{X^2_{1}}_{,train} - x_{1,500,test}\overline{X}_{1,train})\overline{Y}_{train} + (-\overline{X}_{1,train} + x_{1,500,test})\overline{X_{1,train}Y_{train}}}{(\overline{X^2_{1}}_{,train} - \overline{X}^2_{1,train})} \end{pmatrix}.$$

$$X_{test}\widehat{\beta}_{train} - Y_{test} =$$

$$\begin{pmatrix} \frac{(\overline{X^2_{1}}_{,train} - x_{1,1,test}\overline{X}_{1,train})\overline{Y}_{train} + (-\overline{X}_{1,train} + x_{1,1,test})\overline{X_{1,train}Y_{train}}}{(\overline{X^2_{1}}_{,train} - \overline{X}^2_{1,train})} - y_{1,test} \\ \frac{(\overline{X^2_{1}}_{,train} - x_{1,2,test}\overline{X}_{1,train})\overline{Y}_{train} + (-\overline{X}_{1,train} + x_{1,2,test})\overline{X_{1,train}Y_{train}}}{(\overline{X^2_{1}}_{,train} - \overline{X}^2_{1,train})} - y_{2,test} \\ \vdots \\ \frac{(\overline{X^2_{1}}_{,train} - x_{1,500,test}\overline{X}_{1,train})\overline{Y}_{train} + (-\overline{X}_{1,train} + x_{1,500,test})\overline{X_{1,train}Y_{train}}}{(\overline{X^2_{1}}_{,train} - \overline{X}^2_{1,train})} - y_{500,test} \end{pmatrix}.$$

$$SPE = [X_{test}\widehat{\beta}_{train} - Y_{test}]^T [X_{test}\widehat{\beta}_{train} - Y_{test}]$$

$$= \sum_{j=1}^{500} [\frac{(\overline{X^2_{1}}_{,train} - x_{1,j,test}\overline{X}_{1,train})\overline{Y}_{train} + (-\overline{X}_{1,train} + x_{1,j,test})\overline{X_{1,train}Y_{train}}]}{(\overline{X^2_{1}}_{,train} - \overline{X}^2_{1,train})} - y_{j,test}]^2,$$

$$MSPE = E(SPE)$$

$$= E(\sum_{j=1}^{500} \frac{[(\overline{X^2_{1}}_{,train} - x_{1,j,test}\overline{X}_{1,train})\overline{Y}_{train} + (-\overline{X}_{1,train} + x_{1,j,test})\overline{X_{1,train}Y_{train}}]^2}{(\overline{X^2_{1}}_{,train} - \overline{X}^2_{1,train})^2})$$

$$+ E(\sum_{j=1}^{500} y^2_{j,test})$$

$$+E(\sum_{j=1}^{500}(-2)y_{j,test}\frac{[(\overline{X^2_{1}}_{,train} - x_{1,j,test}\overline{X}_{1,train})\overline{Y}_{train} + (-\overline{X}_{1,train} + x_{1,j,test})\overline{X_{1,train}Y_{train}}]}{(\overline{X^2_{1}}_{,train} - \overline{X}^2_{1,train})}).$$

Next, denote

$$A = \sum_{j=1}^{500} \frac{[(\overline{X^2_{1}}_{,train} - x_{1,j,test}\overline{X}_{1,train})\overline{Y}_{train} + (-\overline{X}_{1,train} + x_{1,j,test})\overline{X_{1,train}Y_{train}}]^2}{(\overline{X^2_{1}}_{,train} - \overline{X}^2_{1,train})^2},$$

$$B = \sum_{j=1}^{500} y^2_{j,test},$$

$$C = \sum_{j=1}^{500}(-2)y_{j,test}\frac{[(\overline{X^2_{1}}_{,train} - x_{1,j,test}\overline{X}_{1,train})\overline{Y}_{train} + (-\overline{X}_{1,train} + x_{1,j,test})\overline{X_{1,train}Y_{train}}]}{(\overline{X^2_{1}}_{,train} - \overline{X}^2_{1,train})},$$

and calculate the asymptotic estimation of $E(A)$, $E(B)$, $E(C)$, respectively.

$$
\begin{aligned}
A = &\sum_{j=1}^{500} \frac{1}{(\overline{X^2_1}_{,train}-\overline{X}^2_{1,train})^2}[(\overline{X^2_1}_{,train}-x_{1,j,test}\overline{X}_{1,train})^2\overline{Y}^2_{train} \\
&+(-\overline{X}_{1,train}+x_{1,j,test})^2\overline{X_{1,train}Y_{train}}^2 \\
&+2(\overline{X^2_1}_{,train}-x_{1,j,test}\overline{X}_{1,train})\overline{Y}_{train}(-\overline{X}_{1,train}+x_{1,j,test})\overline{X_{1,train}Y_{train}}] \\
&=\sum_{j=1}^{500} \frac{1}{(\overline{X^2_1}_{,train}-\overline{X}^2_{1,train})^2}(\overline{X^2_1}^2_{,train}\overline{Y}^2_{train}) \\
&+\sum_{j=1}^{500} \frac{1}{(\overline{X^2_1}_{,train}-\overline{X}^2_{1,train})^2}(x^2_{1,j,test}\overline{X}^2_{1,train}\overline{Y}^2_{train}) \\
&-2\sum_{j=1}^{500} \frac{1}{(\overline{X^2_1}_{,train}-\overline{X}^2_{1,train})^2}(\overline{X^2_1}_{,train}x_{1,j,test}\overline{X}_{1,train}\overline{Y}^2_{train}) \\
&+\sum_{j=1}^{500} \frac{1}{(\overline{X^2_1}_{,train}-\overline{X}^2_{1,train})^2}(\overline{X}^2_{1,train}\overline{X_{1,train}Y_{train}}^2) \\
&+\sum_{j=1}^{500} \frac{1}{(\overline{X^2_1}_{,train}-\overline{X}^2_{1,train})^2}(x^2_{1,j,test}\overline{X_{1,train}Y_{train}}^2) \\
&-2\sum_{j=1}^{500} \frac{1}{(\overline{X^2_1}_{,train}-\overline{X}^2_{1,train})^2}(\overline{X}_{1,train}x_{1,j,test}\overline{X_{1,train}Y_{train}}^2) \\
&-2\sum_{j=1}^{500} \frac{1}{(\overline{X^2_1}_{,train}-\overline{X}^2_{1,train})^2}(\overline{X^2_1}_{,train}\overline{X}_{1,train}\overline{Y}_{train}\overline{X_{1,train}Y_{train}}) \\
&+2\sum_{j=1}^{500} \frac{1}{(\overline{X^2_1}_{,train}-\overline{X}^2_{1,train})^2}(x_{1,j,test}\overline{X}^2_{1,train}\overline{Y}_{train}\overline{X_{1,train}Y_{train}}) \\
&+2\sum_{j=1}^{500} \frac{1}{(\overline{X^2_1}_{,train}-\overline{X}^2_{1,train})^2}(\overline{X^2_1}_{,train}x_{1,j,test}\overline{Y}_{train}\overline{X_{1,train}Y_{train}}) \\
&-2\sum_{j=1}^{500} \frac{1}{(\overline{X^2_1}_{,train}-\overline{X}^2_{1,train})^2}(x^2_{1,j,test}\overline{X}_{1,train}\overline{Y}_{train}\overline{X_{1,train}Y_{train}})
\end{aligned}
$$

$$
\begin{aligned}
C = &\textstyle\sum_{j=1}^{500}(-2)y_{j,test}\frac{[(\overline{X^2_1}_{,train}-x_{1,j,test}\overline{X}_{1,train})\overline{Y}_{train}+(-\overline{X}_{1,train}+x_{1,j,test})\overline{X_{1,train}Y_{train}}]}{(\overline{X^2_1}_{,train}-\overline{X}^2_{1,train})} \\
&=\textstyle\sum_{j=1}^{500}(-2)y_{j,test}\frac{\overline{X^2_1}_{,train}\overline{Y}_{train}}{(\overline{X^2_1}_{,train}-\overline{X}^2_{1,train})}+\sum_{j=1}^{500}2y_{j,test}\frac{x_{1,j,test}\overline{X}_{1,train}\overline{Y}_{train}}{(\overline{X^2_1}_{,train}-\overline{X}^2_{1,train})} \\
&+\textstyle\sum_{j=1}^{500}2y_{j,test}\frac{\overline{X}_{1,train}\overline{X_{1,train}Y_{train}}}{(\overline{X^2_1}_{,train}-\overline{X}^2_{1,train})}+\sum_{j=1}^{500}(-2)y_{j,test}\frac{x_{1,j,test}\overline{X_{1,train}Y_{train}}}{(\overline{X^2_1}_{,train}-\overline{X}^2_{1,train})}.
\end{aligned}
$$

To estimate the asymptotic approximations, the following expansions for $\mathbf{X}$ in different dimensions are used.

- **Case 1**

  Assume $\sqrt{n}(\begin{pmatrix} X_{n1} \\ X_{n2} \end{pmatrix} - \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}) \xrightarrow{D} N(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \Sigma)$, $\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} \\ \sigma_{12} & \sigma_2^2 \end{pmatrix}$ and $g(x_1, x_2)$ has continuous second order partial derivatives with

  $\frac{\partial^2 g(\mu_1,\mu_2)}{\partial x_1^2} \neq \frac{\partial^2 g(\mu_1,\mu_2)}{\partial x_1 \partial x_2}$, $\frac{\partial^2 g(\mu_1,\mu_2)}{\partial x_2^2} \neq \frac{\partial^2 g(\mu_1,\mu_2)}{\partial x_1 \partial x_2}$, where$\frac{\partial^2 g(\mu_1,\mu_2)}{\partial x_1^2}$ is used for denoting $\frac{\partial^2 g(x_1,x_2)}{\partial x_1^2}|_{(\mu_1,\mu_2)}$.

  Define the Hessian matrix at point (μ_1, μ_2) to be

  $H(\mu_1, \mu_2) = \begin{pmatrix} \frac{\partial^2 g(\mu_1,\mu_2)}{\partial x_1^2} & \frac{\partial^2 g(\mu_1,\mu_2)}{\partial x_1 \partial x_2} \\ \frac{\partial^2 g(\mu_1,\mu_2)}{\partial x_1 \partial x_2} & \frac{\partial^2 g(\mu_1,\mu_2)}{\partial x_2^2} \end{pmatrix}$,

  by second order Taylor expansion of $g(x_1, x_2)$,

$$g(x_1, x_2) - g(\mu_1, \mu_2) \sim$$

$$\begin{pmatrix} \frac{\partial g(\mu_1,\mu_2)}{\partial x_1} \\ \frac{\partial g(\mu_1,\mu_2)}{\partial x_2} \end{pmatrix}^T \begin{pmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{pmatrix}^T H(\mu_1, \mu_2) \begin{pmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \end{pmatrix}$$

$$= \frac{\partial g(\mu_1, \mu_2)}{\partial x_1}(x_1 - \mu_1) + \frac{\partial g(\mu_1, \mu_2)}{\partial x_2}(x_2 - \mu_2) + \frac{1}{2}\frac{\partial^2 g(\mu_1, \mu_2)}{\partial x_1^2}(x_1 - \mu_1)^2$$

$$+\frac{1}{2}\frac{\partial^2 g(\mu_1, \mu_2)}{\partial x_2^2}(x_2 - \mu_2)^2 + \frac{\partial^2 g(\mu_1, \mu_2)}{\partial x_1 \partial x_2}(x_1 - \mu_1)(x_2 - \mu_2).$$

  For convenience, denote $a = \frac{\partial^2 g(\mu_1,\mu_2)}{\partial x_1^2}$, $b = \frac{\partial^2 g(\mu_1,\mu_2)}{\partial x_1 \partial x_2}$, $c = \frac{\partial^2 g(\mu_1,\mu_2)}{\partial x_2^2}$, $d = \frac{\partial g(\mu_1,\mu_2)}{\partial x_1}$, $e = \frac{\partial g(\mu_1,\mu_2)}{\partial x_2}$.

  Then, $g(x_1, x_2) - g(\mu_1, \mu_2) \sim \frac{1}{2}b(x_1 - \mu_1 + x_2 - \mu_2)^2 + \frac{1}{2}(a-b)[(x_1 - \mu_1 + \dfrac{d}{a-b})^2 - (\dfrac{d}{a-b})^2] + \frac{1}{2}(c-b)[(x_2 - \mu_2 + \dfrac{e}{c-b})^2 - (\dfrac{e}{c-b})^2]$.

  Substitute x_1, x_2 by X_{n1}, X_{n2}[25], similar to what is on page 47 of that book,

  $n(g(X_{n1}, X_{n2}) - g(\mu_1, \mu_2)) \sim (\sigma_1^2 + \sigma_2^2 + 2\sigma_{12})\frac{b}{2}(\sqrt{n}\frac{(X_{n1}-\mu_1)}{\sqrt{\sigma_1^2+\sigma_2^2+2\sigma_{12}}} + \sqrt{n}\frac{(X_{n2}-\mu_2)}{\sqrt{\sigma_1^2+\sigma_2^2+2\sigma_{12}}})^2$

$$+\sigma_1^2\frac{(a-b)}{2}[(\frac{\sqrt{n}(X_{n1} - \mu_1)}{\sigma_1} + \frac{\sqrt{n}d}{\sigma_1(a-b)})^2 - (\frac{\sqrt{n}d}{\sigma_1(a-b)})^2]$$

$$+\sigma_2^2\frac{(c-b)}{2}[(\sqrt{n}\frac{(X_{n2}-\mu_2)}{\sigma_2}+\frac{\sqrt{n}e}{\sigma_2(c-b)})^2-(\frac{\sqrt{n}e}{\sigma_2(c-b)})^2].$$

By Central Limit Theorem,

$$\sqrt{n}\frac{(X_{n1}-\mu_1)}{\sqrt{\sigma_1^2+\sigma_2^2+2\sigma_{12}}}+\sqrt{n}\frac{(X_{n2}-\mu_2)}{\sqrt{\sigma_1^2+\sigma_2^2+2\sigma_{12}}}\xrightarrow{D}N(0,1),$$

$$\sqrt{n}\frac{(X_{n1}-\mu_1)}{\sigma_1}+\frac{\sqrt{n}d}{\sigma_1(a-b)}\overset{D}{\approx}N(\frac{\sqrt{n}d}{\sigma_1(a-b)},1),$$

$$\sqrt{n}\frac{(X_{n2}-\mu_2)}{\sigma_2}+\frac{\sqrt{n}e}{\sigma_2(c-b)}\overset{D}{\approx}N(\frac{\sqrt{n}e}{\sigma_2(c-b)},1).$$

Therefore,

$$[\sqrt{n}\frac{(X_{n1}-\mu_1)}{\sqrt{\sigma_1^2+\sigma_2^2+2\sigma_{12}}}+\sqrt{n}\frac{(X_{n2}-\mu_2)}{\sqrt{\sigma_1^2+\sigma_2^2+2\sigma_{12}}}]^2\xrightarrow{D}\chi_1^2,$$

$$[\sqrt{n}\frac{(X_{n1}-\mu_1)}{\sigma_1}+\frac{\sqrt{n}d}{\sigma_1(a-b)}]^2\overset{D}{\approx}\chi_1^2((\frac{\sqrt{n}d}{\sigma_1(a-b)})^2),$$

$$[\sqrt{n}\frac{(X_{n2}-\mu_2)}{\sigma_2}+\frac{\sqrt{n}e}{\sigma_2(c-b)}]^2\overset{D}{\approx}\chi_1^2((\frac{\sqrt{n}e}{\sigma_2(c-b)})^2).$$

Thus,

$$\begin{aligned}E[n(g(X_{n1},X_{n2})-g(\mu_1,\mu_2))] &\sim (\sigma_1^2+\sigma_2^2+2\sigma_{12})\frac{b}{2}\\ &+\sigma_1^2\frac{(a-b)}{2}[1+(\frac{\sqrt{n}d}{\sigma_1(a-b)})^2-(\frac{\sqrt{n}d}{\sigma_1(a-b)})^2]\\ &+\sigma_2^2\frac{(c-b)}{2}[1+(\frac{\sqrt{n}e}{\sigma_2(c-b)})^2-(\frac{\sqrt{n}e}{\sigma_2(c-b)})^2]\\ &=(\sigma_1^2+\sigma_2^2+2\sigma_{12})\frac{b}{2}+\sigma_1^2\frac{(a-b)}{2}+\sigma_2^2\frac{(c-b)}{2}\\ &=\sigma_1^2\frac{a}{2}+\sigma_2^2\frac{c}{2}+\sigma_{12}b.\end{aligned}$$

- **Case 2**

Assume $\sqrt{n}(\begin{pmatrix} X_{n1} \\ X_{n2} \\ X_{n3} \end{pmatrix} - \begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \end{pmatrix}) \xrightarrow{D} N(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \Sigma),\ \Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} \\ \sigma_{12} & \sigma_2^2 & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_3^2 \end{pmatrix}$,
and $g(x_1, x_2, x_3)$ has continuous second order partial derivatives with

$\frac{\partial^2 g(\mu_1,\mu_2,\mu_3)}{\partial x_1^2} \neq \frac{\partial^2 g(\mu_1,\mu_2,\mu_3)}{\partial x_1 \partial x_2} + \frac{\partial^2 g(\mu_1,\mu_2,\mu_3)}{\partial x_1 \partial x_3}$,

$\frac{\partial^2 g(\mu_1,\mu_2,\mu_3)}{\partial x_2^2} \neq \frac{\partial^2 g(\mu_1,\mu_2,\mu_3)}{\partial x_1 \partial x_2} + \frac{\partial^2 g(\mu_1,\mu_2,\mu_3)}{\partial x_2 \partial x_3}$,

$\frac{\partial^2 g(\mu_1,\mu_2,\mu_3)}{\partial x_3^2} \neq \frac{\partial^2 g(\mu_1,\mu_2,\mu_3)}{\partial x_1 \partial x_3} + \frac{\partial^2 g(\mu_1,\mu_2,\mu_3)}{\partial x_2 \partial x_3}$,

where $\frac{\partial^2 g(\mu_1,\mu_2,\mu_3)}{\partial x_1^2}$ is used to denote $\frac{\partial^2 g(x_1,x_2,x_3)}{\partial x_1^2}|_{(\mu_1,\mu_2,\mu_3)}$.

Define the Hessian matrix at point (μ_1, μ_2, μ_3) to be

$$H(\mu_1, \mu_2, \mu_3) = \begin{pmatrix} \frac{\partial^2 g(\mu_1,\mu_2,\mu_3)}{\partial x_1^2} & \frac{\partial^2 g(\mu_1,\mu_2,\mu_3)}{\partial x_1 \partial x_2} & \frac{\partial^2 g(\mu_1,\mu_2,\mu_3)}{\partial x_1 \partial x_3} \\ \frac{\partial^2 g(\mu_1,\mu_2,\mu_3)}{\partial x_1 \partial x_2} & \frac{\partial^2 g(\mu_1,\mu_2,\mu_3)}{\partial x_2^2} & \frac{\partial^2 g(\mu_1,\mu_2,\mu_3)}{\partial x_2 \partial x_3} \\ \frac{\partial^2 g(\mu_1,\mu_2,\mu_3)}{\partial x_1 \partial x_3} & \frac{\partial^2 g(\mu_1,\mu_2,\mu_3)}{\partial x_2 \partial x_3} & \frac{\partial^2 g(\mu_1,\mu_2,\mu_3)}{\partial x_3^2} \end{pmatrix}.$$

By second order Taylor expansion of $g(x_1, x_2, x_3)$,

$$g(x_1, x_2, x_3) - g(\mu_1, \mu_2, \mu_3) \sim \begin{pmatrix} \frac{\partial g(\mu_1,\mu_2,\mu_3)}{\partial x_1} \\ \frac{\partial g(\mu_1,\mu_2,,\mu_3)}{\partial x_2} \\ \frac{\partial g(\mu_1,\mu_2,\mu_3)}{\partial x_3} \end{pmatrix}^T \begin{pmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \\ x_3 - \mu_3 \end{pmatrix}$$
$$+ \frac{1}{2} \begin{pmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \\ x_3 - \mu_3 \end{pmatrix}^T H(\mu_1, \mu_2, \mu_3) \begin{pmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \\ x_3 - \mu_3 \end{pmatrix}$$

$$= \frac{\partial g(\mu_1, \mu_2, \mu_3)}{\partial x_1}(x_1 - \mu_1) + \frac{\partial g(\mu_1, \mu_2, \mu_3)}{\partial x_2}(x_2 - \mu_2)$$
$$+ \frac{\partial g(\mu_1, \mu_2, \mu_3)}{\partial x_3}(x_3 - \mu_3) + \frac{1}{2}\frac{\partial^2 g(\mu_1, \mu_2, \mu_3)}{\partial x_1^2}(x_1 - \mu_1)^2$$
$$+ \frac{\partial^2 g(\mu_1, \mu_2, \mu_3)}{\partial x_1 \partial x_2}(x_1 - \mu_1)(x_2 - \mu_2) + \frac{1}{2}\frac{\partial^2 g(\mu_1, \mu_2, \mu_3)}{\partial x_2^2}(x_2 - \mu_2)^2$$
$$+ \frac{\partial^2 g(\mu_1, \mu_2, \mu_3)}{\partial x_1 \partial x_3}(x_1 - \mu_1)(x_3 - \mu_3)$$

$$+\frac{\partial^2 g(\mu_1,\mu_2,\mu_3)}{\partial x_2 \partial x_3}(x_2-\mu_2)(x_3-\mu_3)+\frac{1}{2}\frac{\partial^2 g(\mu_1,\mu_2,\mu_3)}{\partial x_3^2}(x_3-\mu_3)^2.$$

For convenience, denote $a=\frac{\partial^2 g(\mu_1,\mu_2,\mu_3)}{\partial x_1^2}, b=\frac{\partial^2 g(\mu_1,\mu_2,\mu_3)}{\partial x_1 \partial x_2}, c=\frac{\partial^2 g(\mu_1,\mu_2,\mu_3)}{\partial x_2^2}, d=\frac{\partial^2 g(\mu_1,\mu_2,\mu_3)}{\partial x_1 \partial x_3}, e=\frac{\partial^2 g(\mu_1,\mu_2,\mu_3)}{\partial x_2 \partial x_3}, f=\frac{\partial^2 g(\mu_1,\mu_2,\mu_3)}{\partial x_3^2}, h=\frac{\partial g(\mu_1,\mu_2,\mu_3)}{\partial x_1}, i=\frac{\partial g(\mu_1,\mu_2,\mu_3)}{\partial x_2}, j=\frac{\partial g(\mu_1,\mu_2,\mu_3)}{\partial x_3}$.

Then, $g(x_1,x_2,x_3)-g(\mu_1,\mu_2,\mu_3)\sim\frac{1}{2}b(x_1-\mu_1+x_2-\mu_2)^2+\frac{1}{2}d(x_1-\mu_1+x_3-\mu_3)^2+\frac{1}{2}e(x_2-\mu_2+x_3-\mu_3)^2+\frac{1}{2}(a-b-d)[(x_1-\mu_1+\frac{h}{a-b-d})^2-(\frac{h}{a-b-d})^2]+\frac{1}{2}(c-b-e)[(x_2-\mu_2+\frac{i}{c-b-e})^2-(\frac{i}{c-b-e})^2]+\frac{1}{2}(f-d-e)[(x_3-\mu_3+\frac{j}{f-d-e})^2-(\frac{j}{f-d-e})^2]$.

Substitute x_1,x_2,x_3 by X_{n1},X_{n2},X_{n3}[25], similar to what is on page 47 of that book,

$$n(g(X_{n1},X_{n2},X_{n3})-g(\mu_1,\mu_2,\mu_3))\sim$$

$$(\sigma_1^2+\sigma_2^2+2\sigma_{12})\frac{b}{2}(\sqrt{n}\frac{(X_{n1}-\mu_1)}{\sqrt{\sigma_1^2+\sigma_2^2+2\sigma_{12}}}+\sqrt{n}\frac{(X_{n2}-\mu_2)}{\sqrt{\sigma_1^2+\sigma_2^2+2\sigma_{12}}})^2$$
$$+(\sigma_1^2+\sigma_3^2+2\sigma_{13})\frac{d}{2}(\sqrt{n}\frac{(X_{n1}-\mu_1)}{\sqrt{\sigma_1^2+\sigma_3^2+2\sigma_{13}}}+\sqrt{n}\frac{(X_{n3}-\mu_3)}{\sqrt{\sigma_1^2+\sigma_3^2+2\sigma_{13}}})^2$$
$$+(\sigma_2^2+\sigma_3^2+2\sigma_{23})\frac{e}{2}(\sqrt{n}\frac{(X_{n2}-\mu_2)}{\sqrt{\sigma_2^2+\sigma_3^2+2\sigma_{23}}}+\sqrt{n}\frac{(X_{n3}-\mu_3)}{\sqrt{\sigma_2^2+\sigma_3^2+2\sigma_{23}}})^2$$
$$+\sigma_1^2\frac{(a-b-d)}{2}[(\frac{\sqrt{n}(X_{n1}-\mu_1)}{\sigma_1}+\frac{\sqrt{n}h}{\sigma_1(a-b-d)})^2-(\frac{\sqrt{n}h}{\sigma_1(a-b-d)})^2]$$
$$+\sigma_2^2\frac{(c-b-e)}{2}[(\sqrt{n}\frac{(X_{n2}-\mu_2)}{\sigma_2}+\frac{\sqrt{n}i}{\sigma_2(c-b-e)})^2-(\frac{\sqrt{n}i}{\sigma_2(c-b-e)})^2]$$
$$+\sigma_3^2\frac{(f-d-e)}{2}[(\sqrt{n}\frac{(X_{n3}-\mu_3)}{\sigma_3}+\frac{\sqrt{n}j}{\sigma_3(f-d-e)})^2-(\frac{\sqrt{n}j}{\sigma_3(f-d-e)})^2].$$

By Central Limit Theorem,

$$\sqrt{n}\frac{(X_{n1}-\mu_1)}{\sqrt{\sigma_1^2+\sigma_2^2+2\sigma_{12}}}+\sqrt{n}\frac{(X_{n2}-\mu_2)}{\sqrt{\sigma_1^2+\sigma_2^2+2\sigma_{12}}}\xrightarrow{D}N(0,1),$$
$$\sqrt{n}\frac{(X_{n1}-\mu_1)}{\sqrt{\sigma_1^2+\sigma_3^2+2\sigma_{13}}}+\sqrt{n}\frac{(X_{n3}-\mu_3)}{\sqrt{\sigma_1^2+\sigma_3^2+2\sigma_{13}}}\xrightarrow{D}N(0,1),$$
$$\sqrt{n}\frac{(X_{n2}-\mu_2)}{\sqrt{\sigma_2^2+\sigma_3^2+2\sigma_{23}}}+\sqrt{n}\frac{(X_{n3}-\mu_3)}{\sqrt{\sigma_2^2+\sigma_3^2+2\sigma_{23}}}\xrightarrow{D}N(0,1),$$

$$\sqrt{n}\frac{(X_{n1}-\mu_1)}{\sigma_1}+\frac{\sqrt{n}h}{\sigma_1(a-b-d)} \overset{D}{\approx} N(\frac{\sqrt{n}h}{\sigma_1(a-b-d)},1),$$
$$\sqrt{n}\frac{(X_{n2}-\mu_2)}{\sigma_2}+\frac{\sqrt{n}i}{\sigma_2(c-b-e)} \overset{D}{\approx} N(\frac{\sqrt{n}i}{\sigma_2(c-b-e)},1),$$
$$\sqrt{n}\frac{(X_{n3}-\mu_3)}{\sigma_3}+\frac{\sqrt{n}j}{\sigma_3(f-d-e)} \overset{D}{\approx} N(\frac{\sqrt{n}j}{\sigma_3(f-d-e)},1).$$

Therefore,

$$[\sqrt{n}\frac{(X_{n1}-\mu_1)}{\sqrt{\sigma_1^2+\sigma_2^2+2\sigma_{12}}}+\sqrt{n}\frac{(X_{n2}-\mu_2)}{\sqrt{\sigma_1^2+\sigma_2^2+2\sigma_{12}}}]^2 \overset{D}{\to} \chi_1^2,$$
$$[\sqrt{n}\frac{(X_{n1}-\mu_1)}{\sqrt{\sigma_1^2+\sigma_3^2+2\sigma_{13}}}+\sqrt{n}\frac{(X_{n3}-\mu_3)}{\sqrt{\sigma_1^2+\sigma_3^2+2\sigma_{13}}}]^2 \overset{D}{\to} \chi_1^2,$$
$$[\sqrt{n}\frac{(X_{n2}-\mu_2)}{\sqrt{\sigma_2^2+\sigma_3^2+2\sigma_{23}}}+\sqrt{n}\frac{(X_{n3}-\mu_3)}{\sqrt{\sigma_2^2+\sigma_3^2+2\sigma_{23}}}]^2 \overset{D}{\to} \chi_1^2,$$
$$[\sqrt{n}\frac{(X_{n1}-\mu_1)}{\sigma_1}+\frac{\sqrt{n}h}{\sigma_1(a-b-d)}]^2 \overset{D}{\approx} \chi_1^2((\frac{\sqrt{n}h}{\sigma_1(a-b-d)})^2),$$
$$[\sqrt{n}\frac{(X_{n2}-\mu_2)}{\sigma_2}+\frac{\sqrt{n}i}{\sigma_2(c-b-e)}]^2 \overset{D}{\approx} \chi_1^2((\frac{\sqrt{n}i}{\sigma_2(c-b-e)})^2),$$
$$[\sqrt{n}\frac{(X_{n3}-\mu_3)}{\sigma_3}+\frac{\sqrt{n}j}{\sigma_3(f-d-e)}]^2 \overset{D}{\approx} \chi_1^2((\frac{\sqrt{n}j}{\sigma_3(f-d-e)})^2).$$

Thus, $E[n(g(X_{n1},X_{n2},X_{n3})-g(\mu_1,\mu_2,\mu_3))] \sim (\sigma_1^2+\sigma_2^2+2\sigma_{12})\frac{b}{2}$
$+(\sigma_1^2+\sigma_3^2+2\sigma_{13})\frac{d}{2}+(\sigma_2^2+\sigma_3^2+2\sigma_{23})\frac{e}{2}+\sigma_1^2\frac{(a-b-d)}{2}[1+(\frac{\sqrt{n}h}{\sigma_1(a-b-d)})^2-$
$(\frac{\sqrt{n}h}{\sigma_1(a-b-d)})^2]+\sigma_2^2\frac{(c-b-e)}{2}[1+(\frac{\sqrt{n}i}{\sigma_2(c-b-e)})^2-(\frac{\sqrt{n}i}{\sigma_2(c-b-e)})^2]$
$+\sigma_3^2\frac{(f-d-e)}{2}[1+(\frac{\sqrt{n}j}{\sigma_3(f-d-e)})^2-(\frac{\sqrt{n}j}{\sigma_3(f-d-e)})^2]$
$=(\sigma_1^2+\sigma_2^2+2\sigma_{12})\frac{b}{2}+(\sigma_1^2+\sigma_3^2+2\sigma_{13})\frac{d}{2}+(\sigma_2^2+\sigma_3^2+2\sigma_{23})\frac{e}{2}+\sigma_1^2\frac{(a-b-d)}{2}+$
$\sigma_2^2\frac{(c-b-e)}{2}+\sigma_3^2\frac{(f-d-e)}{2}$
$=\sigma_1^2\frac{a}{2}+\sigma_2^2\frac{c}{2}+\sigma_3^2\frac{f}{2}+\sigma_{12}b+\sigma_{13}d+\sigma_{23}e.$

From the above cases, there is a generalization to the following theorem with its proof.

Theorem 4.1.1 *Assume that*

$$\sqrt{n}(\begin{pmatrix} X_{n1} \\ X_{n2} \\ \vdots \\ X_{nk} \end{pmatrix} - \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_k \end{pmatrix}) \xrightarrow{D} N(\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \Sigma), \ \Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \cdots & \sigma_{1k} \\ \sigma_{12} & \sigma_2^2 & \ldots & \sigma_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{1k} & \sigma_{2k} & \cdots & \sigma_k^2 \end{pmatrix}.$$

Define the Hessian matrix at point $(\mu_1, \mu_2, \cdots, \mu_k)$ *to be*

$$H(\mu_1, \mu_2, \cdots, \mu_k) = \begin{pmatrix} \frac{\partial^2 g(\mu_1,\mu_2,\cdots,\mu_k)}{\partial x_1^2} & \frac{\partial^2 g(\mu_1,\mu_2,\cdots,\mu_k)}{\partial x_1 \partial x_2} & \ldots & \frac{\partial^2 g(\mu_1,\mu_2,\cdots,\mu_k)}{\partial x_1 \partial x_k} \\ \frac{\partial^2 g(\mu_1,\mu_2,\cdots,\mu_k)}{\partial x_1 \partial x_2} & \frac{\partial^2 g(\mu_1,\mu_2,\cdots,\mu_k)}{\partial x_2^2} & \ldots & \frac{\partial^2 g(\mu_1,\mu_2,\cdots,\mu_k)}{\partial x_2 \partial x_k} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial^2 g(\mu_1,\mu_2,\cdots,\mu_k)}{\partial x_1 \partial x_k} & \frac{\partial^2 g(\mu_1,\mu_2,\cdots,\mu_k)}{\partial x_2 \partial x_k} & \ldots & \frac{\partial^2 g(\mu_1,\mu_2,\cdots,\mu_k)}{\partial x_k^2} \end{pmatrix}$$

where $\frac{\partial^2 g(\mu_1,\mu_2,\cdots,\mu_k)}{\partial x_i \partial x_j}$ *is to denote* $\frac{\partial^2 g(x_1,x_2,\cdots,x_k)}{\partial x_i \partial x_j}|_{(\mu_1,\mu_2,\cdots,\mu_k)}, \ i,j = 1,...,k$

For convenience, denote $l_i = \frac{\partial g(\mu_1,\mu_2,\cdots,\mu_k)}{\partial x_i}, i = 1,...,k,$

$$H(\mu_1, \mu_2, \cdots, \mu_k) = \begin{pmatrix} h_{11} & h_{12} & \cdots & h_{1k} \\ h_{21} & h_{22} & \cdots & h_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ h_{1k} & h_{2k} & \cdots & h_{kk} \end{pmatrix}$$ *where* $h_{ij} = h_{ji}, \ i = 1,...,k, \ j = 1,...,k, \ i \neq j.$

Further, assume that $g(x_1, x_2, \cdots, x_k)$ *has continuous second order partial derivatives with* $h_{ii} \neq \sum_{j=1, i\neq j}^k h_{ij}, \ i = 1,...,k.$ *Then,*

$$E[n(g(X_{n1}, X_{n2}, \cdots, X_{nk}) - g(\mu_1, \mu_2, \cdots, \mu_k))] \sim \sum_{i=1}^k \sigma_i^2 \frac{h_{ii}}{2} + \sum_{i,j=1,i>j}^k \sigma_{ij} h_{ij}.$$

Proof By second order Taylor expansion of $g(x_1, x_2, \cdots, x_k)$,

$$g(x_1, x_2, \cdots, x_k) - g(\mu_1, \mu_2, \cdots, \mu_k) \sim \begin{pmatrix} l_1 \\ l_2 \\ \vdots \\ l_k \end{pmatrix}^T \begin{pmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \\ \vdots \\ x_k - \mu_k \end{pmatrix}$$

$$+\frac{1}{2} \begin{pmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \\ \vdots \\ x_k - \mu_k \end{pmatrix}^T H(\mu_1, \mu_2, \cdots, \mu_k) \begin{pmatrix} x_1 - \mu_1 \\ x_2 - \mu_2 \\ \vdots \\ x_{-}\mu_k \end{pmatrix}.$$

Then,

$$g(x_1, x_2, \cdots, x_k) - g(\mu_1, \mu_2, \cdots, \mu_k) \sim \sum_{i=1}^{k} l_i(x_i - \mu_i) + \sum_{i=1}^{k} \frac{h_{ii}}{2}(x_i - \mu_i)^2$$

$$+ \sum_{i,j=1,j>i}^{k} h_{ij}(x_i - \mu_i)(x_j - \mu_j)$$

$$= \sum_{i,j=1,j>i}^{k} \frac{1}{2} h_{ij}(x_i - \mu_i + x_j - \mu_j)^2$$

$$+ \sum_{i=1}^{k} \frac{1}{2}(h_{ii} - \sum_{j=1,j\neq i}^{k} h_{ij})[(x_i - \mu_i + \frac{l_i}{h_{ii} - \sum_{j=1,j\neq i}^{k} h_{ij}})^2 - (\frac{l_i}{h_{ii} - \sum_{j=1,j\neq i}^{k} h_{ij}})^2].$$

Substitute $x_1, x_2, \cdots, x_k$ by $X_{n1}, X_{n2}, \cdots, X_{nk}$[25], similar to what is on page 47 of that book, then

$$n(g(X_{n1}, X_{n2}, \cdots, X_{nk}) - g(\mu_1, \mu_2, \cdots, \mu_k)) \sim$$

$$\sum_{i,j=1,j>i}^{k} (\sigma_i^2 + \sigma_j^2 + 2\sigma_{ij}) \frac{h_{ij}}{2} (\sqrt{n} \frac{(X_{ni} - \mu_j)}{\sqrt{\sigma_i^2 + \sigma_j^2 + 2\sigma_{ij}}} + \sqrt{n} \frac{(X_{nj} - \mu_j)}{\sqrt{\sigma_i^2 + \sigma_j^2 + 2\sigma_{ij}}})^2$$

$$+ \sum_{i=1}^{k} \sigma_i^2 \frac{(h_{ii} - \sum_{j=1,j\neq i}^{k} h_{ij})}{2} [(\frac{\sqrt{n}(X_{ni} - \mu_i)}{\sigma_i} + \frac{\sqrt{n} l_i}{\sigma_i(h_{ii} - \sum_{j=1,j\neq i}^{k} h_{ij})})^2$$

$$-(\frac{\sqrt{n} l_i}{\sigma_i(h_{ii} - \sum_{j=1,j\neq i}^{k} h_{ij})})^2].$$

By Central Limit Theorem,

$$\sqrt{n} \frac{(X_{ni} - \mu_i)}{\sqrt{\sigma_i^2 + \sigma_j^2 + 2\sigma_{ij}}} + \sqrt{n} \frac{(X_{nj} - \mu_j)}{\sqrt{\sigma_i^2 + \sigma_j^2 + 2\sigma_{ij}}} \xrightarrow{D} N(0, 1), \quad i, j = 1, ...k, \ j > i,$$

$$\frac{\sqrt{n}(X_{ni} - \mu_i)}{\sigma_i} + \frac{\sqrt{n} l_i}{\sigma_i(h_{ii} - \sum_{j=1,j\neq i}^{k} h_{ij})} \overset{D}{\approx} N(\frac{\sqrt{n} l_i}{\sigma_i(h_{ii} - \sum_{j=1,j\neq i}^{k} h_{ij})}, 1), \quad i = 1, ..., k.$$

Therefore,

$$[\sqrt{n} \frac{(X_{ni} - \mu_i)}{\sqrt{\sigma_i^2 + \sigma_j^2 + 2\sigma_{ij}}} + \sqrt{n} \frac{(X_{nj} - \mu_j)}{\sqrt{\sigma_i^2 + \sigma_j^2 + 2\sigma_{ij}}}]^2 \xrightarrow{D} \chi_1^2, \quad i, j = 1, ...k, \ j > i,$$

$$[\frac{\sqrt{n}(X_{ni} - \mu_i)}{\sigma_i} + \frac{\sqrt{n} l_i}{\sigma_i(h_{ii} - \sum_{j=1,j\neq i}^{k} h_{ij})}]^2 \overset{D}{\approx} \chi_1^2((\frac{\sqrt{n} l_i}{\sigma_i(h_{ii} - \sum_{j=1,j\neq i}^{k} h_{ij})})^2), \quad i = 1, ..., k.$$

Thus,

$E[n(g(X_{n1}, X_{n2}, \cdots, X_{nk}) - g(\mu_1, \mu_2, \cdots, \mu_k))] \sim \sum_{i,j=1,j>i}^{k} (\sigma_i^2 + \sigma_j^2 + 2\sigma_{ij}) \frac{h_{ij}}{2}$
$+ \sum_{i=1}^{k} \sigma_i^2 \frac{(h_{ii} - \sum_{j=1,j\neq i}^{k} h_{ij})}{2} [1 + (\frac{\sqrt{n} l_i}{\sigma_i (h_{ii} - \sum_{j=1,j\neq i}^{k} h_{ij})})^2 - (\frac{\sqrt{n} l_i}{\sigma_i (h_{ii} - \sum_{j=1,j\neq i}^{k} h_{ij})})^2]$
$= \sum_{i,j=1,j>i}^{k} (\sigma_i^2 + \sigma_j^2) \frac{h_{ij}}{2} + \sum_{i=1}^{k} \sigma_i^2 \frac{(h_{ii} - \sum_{j=1,j\neq i}^{k} h_{ij})}{2}$
$= \sum_{i=1}^{k} \sigma_i^2 \frac{h_{ii}}{2} + \sum_{i,j=1,i>j}^{k} \sigma_{ij} h_{ij}. \square$

The derivatives in this Chapter are calculated from an on-line derivative calculator 'https://www.derivative-calculator.net/'[26].

Now

$\sqrt{n}(\overline{X}_{1,train} - \mu_1) \xrightarrow{D} N(0, \sigma_1^2),$
$\sqrt{n}(\overline{X^2_1}_{,train} - \mu_2) \xrightarrow{D} N(0, \sigma_2^2),$
$\sqrt{n}(\overline{Y}_{train} - \mu_3) \xrightarrow{D} N(0, \sigma_3^2),$
$\sqrt{n}(\overline{X_{1,train} Y_{train}} - \mu_4) \xrightarrow{D} N(0, \sigma_4^2),$

where

$\mu_1 = p_1,$

$\mu_2 = p_1,$

$\mu_3 = \beta_0 + \beta_1 p_1 + \beta_2 p_2 + \beta_3 p_3,$

$\mu_4 = \beta_0 p_1 + \beta_1 p_1 + \beta_2 p_1 p_2 + \beta_3 p_1 p_3,$

$\sigma_1^2 = var(X_{1,train}) = p_1(1 - p_1),$

$\sigma_2^2 = var(X^2_{1,train}) = E(X^4_{1,train}) - E^2(X^2_{1,train}) = \frac{\partial^4 (1 - p_1 + p_1 e^t)}{\partial t^4}|_{t=0} - p_1^2$
$= p_1(1 - p_1)$

$\sigma_3^2 = var(Y_{train}) = \beta_1^2 p_1(1 - p_1) + \beta_2^2 p_2(1 - p_2) + \beta_3^2 p_3(1 - p_3) + \sigma^2$

$\sigma_4^2 = var(X_{1,train} Y_{train})$
$= \beta_0^2 var(X_{1,train}) + \beta_1^2 var(X^2_{1,train}) + \beta_2^2 var(X_{1,train} X_{2,train}) + \beta_3^2 var(X_{1,train} X_{3,train}) + var(X_{1,train} \epsilon)$
$= \beta_0^2 p_1(1 - p_1) + \beta_1^2 p_1(1 - p_1) + \beta_2^2 p_1 p_2(1 - p_1 p_2) + \beta_3^2 p_1 p_3(1 - p_1 p_3) + \sigma^2 p_1$

$\sigma_{12} = cov(X_{1,train}, X^2_{1,train}) = E(X^3_{1,train}) - E(X_{1,train})E(X^2_{1,train})$
$= \frac{\partial^3 (1 - p_1 + p_1 e^t)}{\partial t^3}|_{t=0} - E(X_{1,train})E(X^2_{1,train}) = p_1 - p_1^2$
$= p_1(1 - p_1)$

$\sigma_{13} = cov(X_{1,train}, Y_{train}) = cov(X_{1,train}, \beta_1 X_{1,train}) = \beta_1 p_1(1 - p_1)$

$\sigma_{14} = cov(X_{1,train}, X_{1,train}Y_{train})$

$= cov(X_{1,train}, X_{1,train}\beta_0) + cov(X_{1,train}, X^2_{1,train}\beta_1) + cov(X_{1,train}, X_{1,train}X_{2,train}\beta_2) +$
$cov(X_{1,train}, X_{1,train}X_{3,train}\beta_3) + cov(X_{1,train}, X_{1,train}\epsilon_{train})$

$= \beta_0\sigma_1^2 + \beta_1\sigma_{12} + \beta_2[E(X^2_{1,train}X_{2,train}) - E(X_{1,train})E(X_{1,train}X_{2,train})]$

$+ \beta_3[E(X^2_{1,train}X_{3,train}) - E(X_{1,train})E(X_{1,train}X_{3,train})]$

$+ E(X^2_{1,train}\epsilon_{train}) - E(X_{1,train})E(X_{1,train}\epsilon_{train})$

$= \beta_0\sigma_1^2 + \beta_1\sigma_{12} + \beta_2(p_1p_2 - p_1^2p_2) + \beta_3(p_1p_3 - p_1^2p_3)$

$= p_1(1-p_1)[\beta_0 + \beta_1 + \beta_2p_2 + \beta_3p_3]$

$\sigma_{23} = cov(X^2_{1,train}, Y_{train})$

$= cov(X^2_{1,train}, \beta_0) + cov(X^2_{1,train}, X_{1,train}\beta_1) + cov(X^2_{1,train}, X_{2,train}\beta_2)$

$+ cov(X^2_{1,train}, X_{3,train}\beta_3) + cov(X^2_{1,train}, \epsilon_{train})$

$= \beta_1\sigma_1^2 = \beta_1p_1(1-p_1)$

$\sigma_{24} = cov(X^2_{1,train}, X_{1,train}Y_{train})$

$= cov(X^2_{1,train}, X_{1,train}\beta_0) + cov(X^2_{1,train}, X^2_{1,train}\beta_1) + cov(X^2_{1,train}, X_{1,train}X_{2,train}\beta_2) +$
$cov(X^2_{1,train}, X_{1,train}X_{3,train}\beta_3) + cov(X^2_{1,train}, X_{1,train}\epsilon_{train})$

$= \beta_0\sigma_{12} + \beta_1\sigma_2^2 + \beta_2[E(X^3_{1,train}X_{2,train}) - E(X^2_{1,train})E(X_{1,train}X_{2,train})]$

$+ \beta_3[E(X^3_{1,train}X_{3,train}) - E(X^2_{1,train})E(X_{1,train}X_{3,train})]$

$+ E(X^3_{1,train}\epsilon_{train}) - E(X^2_{1,train})E(X_{1,train}\epsilon_{train})$

$= \beta_0p_1(1-p_1) + \beta_1p_1(1-p_1) + \beta_2[p_1p_2 - p_1^2p_2] + \beta_3[p_1p_3 - p_1^2p_3]$

$= p_1(1-p_1)[\beta_0 + \beta_1 + \beta_2p_2 + \beta_3p_3]$

$\sigma_{34} = cov(Y_{train}, X_{1,train}Y_{train}) = E(Y^2_{train}X_{1,train}) - E(Y_{train})E(X_{1,train}Y_{train})$

$= E[\beta_0^2X_{1,train} + \beta_1^2X^3_{1,train} + \beta_2^2X^2_{2,train}X_{1,train} + \beta_3^2X^2_{3,train}X_{1,train} + \epsilon^2_{train}X_{1,train}$

$+ 2\beta_0\beta_1X^2_{1,train} + 2\beta_0\beta_2X_{1,train}X_{2,train} + 2\beta_0\beta_3X_{1,train}X_{3,train} + 2\beta_0X_{1,train}\epsilon_{train}$

$+ 2\beta_1\beta_2X^2_{1,train}X_{2,train} + 2\beta_1\beta_3X^2_{1,train}X_{3,train} + 2\beta_1X^2_{1,train}\epsilon_{train}$

$+2\beta_2\beta_3X_{1,train}X_{2,train}X_{3,train} + 2\beta_2X_{1,train}X_{2,train}\epsilon_{train} + 2\beta_3X_{1,train}X_{3,train}\epsilon_{train}] - \mu_3\mu_4$

$= \beta_0^2p_1 + \beta_1^2p_1 + \beta_2^2p_2p_1 + \beta_3^2p_3p_1 + \sigma^2p_1 + 2\beta_0\beta_1p_1 + 2\beta_0\beta_2p_1p_2 + 2\beta_0\beta_3p_1p_3 + 2\beta_1\beta_2p_1p_2 +$
$2\beta_1\beta_3p_1p_3 + 2\beta_2\beta_3p_1p_2p_3 - \mu_3\mu_4.$

Recall from the expressions of A, B, C on page 74 in this thesis,

$E(B) = E(\sum_{j=1}^{500} y^2_{j,test}) = 500 \times E(y^2_{1,test}) = 500 \times [\beta_0^2 + \beta_1^2p_1 + \beta_2^2p_2 + \beta_3^2p_3 + \sigma^2 +$
$2\beta_0\beta_1p_1 + 2\beta_0\beta_2p_2 + 2\beta_0\beta_3p_3 + 2\beta_1\beta_2p_1p_2 + 2\beta_1\beta_3p_1p_3 + 2\beta_2\beta_3p_2p_3].$

For convenience, in the following parts calculating $E(A)$ and $E(C)$, denote

$x = \overline{X}_{1,train},\ y = \overline{X^2_{1}}_{,train},\ z = \overline{Y}_{train},\ t = \overline{X_{1,train}Y_{train}}$

Let $A_1 = \sum_{j=1}^{500} \frac{1}{(\overline{X^2_{1}}_{,train} - \overline{X}^2_{1,train})^2}(\overline{X^2_{1}}^2_{,train}\overline{Y}^2_{train})$

$E(A_1) = \frac{1}{2} \times E(1000\frac{1}{(\overline{X^2_{1}}_{,train} - \overline{X}^2_{1,train})^2}(\overline{X^2_{1}}^2_{,train}\overline{Y}^2_{train}))$

$= \frac{1}{2}E[1000 \times \frac{y^2z^2}{(y-x^2)^2}]$

Let $A_2 = \sum_{j=1}^{500} \frac{1}{(\overline{X^2_{1}}_{,train} - \overline{X}^2_{1,train})^2}(x^2_{1,j,test}\overline{X}^2_{1,train}\overline{Y}^2_{train})$

$E(A_2) = E(x^2_{1,1,test})E(A_1) = p_1E(A_1)$

$= \frac{1}{2}E[1000 \times \frac{p_1y^2z^2}{(y-x^2)^2}]$

Let $A_3 = -2\sum_{j=1}^{500} \frac{1}{(\overline{X^2_{1}}_{,train} - \overline{X}^2_{1,train})^2}(\overline{X^2_{1}}_{,train}x_{1,j,test}\overline{X}_{1,train}\overline{Y}^2_{train})$

$E(A_3) = -E(x_{1,j,test})E(1000 \times \frac{1}{(\overline{X^2_{1}}_{,train} - \overline{X}^2_{1,train})^2}(\overline{X^2_{1}}_{,train}\overline{X}_{1,train}\overline{Y}^2_{train}))$

$= -p_1E(1000 \times \frac{1}{(\overline{X^2_{1}}_{,train} - \overline{X}^2_{1,train})^2}(\overline{X^2_{1}}_{,train}\overline{X}_{1,train}\overline{Y}^2_{train}))$

$= \frac{1}{2}E[1000 \times \frac{-2p_1yxz^2}{(y-x^2)^2}]$

Let $A_4 = \sum_{j=1}^{500} \frac{1}{(\overline{X^2_{1}}_{,train} - \overline{X}^2_{1,train})^2}(\overline{X}^2_{1,train}\overline{X_{1,train}Y_{train}}^2)$

$E(A_4) = \frac{1}{2}E(1000 \times \frac{1}{(\overline{X^2_{1}}_{,train} - \overline{X}^2_{1,train})^2}(\overline{X}^2_{1,train}\overline{X_{1,train}Y_{train}}^2))$

$= \frac{1}{2}E[1000 \times \frac{x^2t^2}{(y-x^2)^2}]$

Let $A_5 = \sum_{j=1}^{500} \frac{1}{(\overline{X^2_{1}}_{,train} - \overline{X}^2_{1,train})^2}(x^2_{1,j,test}\overline{X_{1,train}Y_{train}}^2)$

$E(A_5) = \frac{1}{2}E(x^2_{1,1,test})E(1000 \times \frac{1}{(\overline{X^2_{1}}_{,train} - \overline{X}^2_{1,train})^2}\overline{X_{1,train}Y_{train}}^2)$

$= \frac{1}{2}p_1E(1000 \times \frac{1}{(\overline{X^2_{1}}_{,train} - \overline{X}^2_{1,train})^2}\overline{X_{1,train}Y_{train}}^2)$

$= \frac{1}{2}E[1000 \times \frac{p_1t^2}{(y-x^2)^2}]$

Let $A_6 = -2\sum_{j=1}^{500} \frac{1}{(\overline{X^2_{1}}_{,train} - \overline{X}^2_{1,train})^2}(\overline{X}_{1,train}x_{1,j,test}\overline{X_{1,train}Y_{train}}^2)$

$E(A_6) = -E(x_{1,1,test})E(1000 \times \frac{1}{(\overline{X^2_{1}}_{,train} - \overline{X}^2_{1,train})^2}(\overline{X}_{1,train}\overline{X_{1,train}Y_{train}}^2))$

$= -p_1 E(1000 \times \frac{1}{(\overline{X^2_{1}}_{,train} - \overline{X}^2_{1,train})^2}(\overline{X}_{1,train}\overline{X_{1,train}Y_{train}}^2))$

$= \frac{1}{2}E[1000 \times \frac{-2p_1xt^2}{(y-x^2)^2}]$

Let $A_7 = -2\sum_{j=1}^{500} \frac{1}{(\overline{X^2_{1}}_{,train} - \overline{X}^2_{1,train})^2}(\overline{X^2_{1}}_{,train}\overline{X}_{1,train}\overline{Y}_{train}\overline{X_{1,train}Y_{train}})$

$E(A_7) = -E(1000 \times \frac{1}{(\overline{X^2_{1}}_{,train} - \overline{X}^2_{1,train})^2}(\overline{X^2_{1}}_{,train}\overline{X}_{1,train}\overline{Y}_{train}\overline{X_{1,train}Y_{train}}))$

$= \frac{1}{2}E[1000 \times \frac{-2xyzt}{(y-x^2)^2}]$

Let $A_8 = 2\sum_{j=1}^{500} \frac{1}{(\overline{X^2_{1}}_{,train} - \overline{X}^2_{1,train})^2}(x_{1,j,test}\overline{X}^2_{1,train}\overline{Y}_{train}\overline{X_{1,train}Y_{train}})$

$E(A_8) = E(x_{1,1,test})E[1000 \times \frac{1}{(\overline{X^2_{1}}_{,train} - \overline{X}^2_{1,train})^2}(\overline{X}^2_{1,train}\overline{Y}_{train}\overline{X_{1,train}Y_{train}})]$

$= p_1 E[1000 \times \frac{1}{(\overline{X^2_{1}}_{,train} - \overline{X}^2_{1,train})^2}(\overline{X}^2_{1,train}\overline{Y}_{train}\overline{X_{1,train}Y_{train}})]$

$= \frac{1}{2}E[1000 \times \frac{2p_1x^2zt}{(y-x^2)^2}]$

Let $A_9 = 2\sum_{j=1}^{500} \frac{1}{(\overline{X^2_{1}}_{,train} - \overline{X}^2_{1,train})^2}(\overline{X^2_{1}}_{,train}x_{1,j,test}\overline{Y}_{train}\overline{X_{1,train}Y_{train}})$

$E(A_9) = E(x_{1,1,test})E(1000 \times \frac{1}{(\overline{X^2_{1}}_{,train} - \overline{X}^2_{1,train})^2}(\overline{X^2_{1}}_{,train}\overline{Y}_{train}\overline{X_{1,train}Y_{train}}))$

$= p_1 E(1000 \times \frac{1}{(\overline{X^2_{1}}_{,train} - \overline{X}^2_{1,train})^2}(\overline{X^2_{1}}_{,train}\overline{Y}_{train}\overline{X_{1,train}Y_{train}}))$

$= \frac{1}{2}E[1000 \times \frac{2p_1yzt}{(y-x^2)^2}]$

Let $A_{10} = -2\sum_{j=1}^{500} \frac{1}{(\overline{X^2_{1}}_{,train} - \overline{X}^2_{1,train})^2}(x^2_{1,j,test}\overline{X}_{1,train}\overline{Y}_{train}\overline{X_{1,train}Y_{train}})$

$E(A_{10}) = -E(x^2_{1,1,test})E(1000 \times \frac{1}{(\overline{X^2_{1}}_{,train} - \overline{X}^2_{1,train})^2}(\overline{X}_{1,train}\overline{Y}_{train}\overline{X_{1,train}Y_{train}}))$

$= -p_1 E(1000 \times \frac{1}{(\overline{X^2_{1}}_{,train} - \overline{X}^2_{1,train})^2}(\overline{X}_{1,train}\overline{Y}_{train}\overline{X_{1,train}Y_{train}}))$

$= \frac{1}{2}E[1000 \times \frac{-2p_1xzt}{(y-x^2)^2}]$

$E(A) = \Sigma_{i=1}^{10} A_i$

$= \frac{1}{2}E[1000 \times (\frac{y^2z^2+x^2t^2-2xyzt+p_1(y^2z^2-2yxz^2+t^2-2xt^2+2x^2zt+2yzt-2xzt)}{(y-x^2)^2})]$

Now suppose

$$g_1 = \frac{y^2z^2+x^2t^2-2xyzt+p_1(y^2z^2-2yxz^2+t^2-2xt^2+2x^2zt+2yzt-2xzt)}{(y-x^2)^2}$$

and let $x = \mu_1, y = \mu_2, z = \mu_3, t = \mu_4, a = p_1$ in the following expressions of $h_{ij}^{(A)},\ i,j = 1,...,4, j \geqslant i$, where $\mu_i,\ i = 1,...,4$ are calculated on page 83 of this thesis.

Further, assume $\mu_1,\ \mu_2,\ \mu_3,\ \mu_4$ satisfies that the denominators in $h_{ij}^{(A)},\ i,j = 1,...,4, j \geqslant i$ are not equal to zero as well as $h_{ii}^{(A)} \neq \sum_{j=1,i\neq j}^{4} h_{ij}^{(A)},\ i = 1,...,4.$

Then

$h_{11}^{(A)} = [2((6atz+3t^2)x^4+(-12ayz^2+(-12ty-12at)z-12at^2)x^3+((10a+10)y^2z^2+36atyz+8t^2y+10at^2)x^2+(-12ay^2z^2+(-12ty^2-12aty)z-12at^2y)x+(2a+2)y^3z^2+6aty^2z+t^2y^2+2at^2y)]/(x^2-y)^4$

$h_{12}^{(A)} = -[2((((2a+2)x-a)z^2-tz)y^2+(((4a+4)x^3-8ax^2)z^2+(-8tx^2+12atx-2at)z+2t^2x-2at^2)y-3ax^4z^2+(-3tx^4+12atx^3-10atx^2)z+4t^2x^3-10at^2x^2+6at^2x)]/[(y-x^2)^4]$

$h_{13}^{(A)} = [2((((4a+4)x-2a)y^2-6ax^2y)z-ty^2+(-3tx^2+6atx-at)y+2atx^3-3atx^2)]/[(y-x^2)^3]$

$h_{14}^{(A)} = [2(((2x-2a)y+2x^3-6ax^2+4ax)t+(-y^2+(-3x^2+6ax-a)y+2ax^3-3ax^2)z)]/[(y-x^2)^3]$

$h_{22}^{(A)} = [2((((2a+2)x^2-2ax)z^2+(2at-2tx)z)y+((a+1)x^4-4ax^3)z^2+(-4tx^3+10atx^2-6atx)z+3t^2x^2-6at^2x+3at^2)]/[(y-x^2)^4]$

$$h_{23}^{(A)} = -\frac{2(y(2((a+1)x^2-ax)z-tx+at)-2ax^3z-tx^3+3atx^2-2atx)}{(y-x^2)^3}$$

$$h_{24}^{(A)} = -\frac{2((2x^2-4ax+2a)t+((a-x)y-x^3+3ax^2-2ax)z)}{(y-x^2)^3}$$

$$h_{33}^{(A)} = \frac{2y((a+1)y-2ax)}{(y-x^2)^2}$$

$$h_{34}^{(A)} = \frac{2((a-x)y+ax^2-ax)}{(y-x^2)^2}$$

$$h_{44}^{(A)} = \frac{2(x^2-2ax+a)}{(y-x^2)^2}$$

Thus, $E(A) \sim \dfrac{1}{2}[\sum_{i=1}^{4}\sigma_i^2 \dfrac{h_{ii}^{(A)}}{2} + \sum_{i,j=1,i>j}^{4}\sigma_{ij}h_{ij}^{(A)}]$

where $\sigma_{ij},\ i,j = 1,...,4,\ i \leq j$ are calculated on page 83 and page 84 of this thesis.

In the following part to calculate $E(C_i),\ i = 1,...,4,\ \mu_i,\ i = 1,...,4$, are calculated on page 83 of this thesis.

Let $C_1 = \sum_{j=1}^{500}(-2)y_{j,test}\frac{\overline{X^2_1}_{,train}\overline{Y}_{train}}{(\overline{X^2_1}_{,train} - \overline{X}^2_{1,train})}$

$E(C_1) = -E(y_{j,test})E(1000 \times \frac{\overline{X^2_1}_{,train}\overline{Y}_{train}}{(\overline{X^2_1}_{,train} - \overline{X}^2_{1,train})})$

$= -\mu_3 E(1000 \times \frac{\overline{X^2_1}_{,train}\overline{Y}_{train}}{(\overline{X^2_1}_{,train} - \overline{X}^2_{1,train})})$

$= E[1000 \times \frac{-\mu_3 yz}{y - x^2}]$

Let $C_2 = \sum_{j=1}^{500} 2y_{j,test}\frac{x_{1,j,test}\overline{X}_{1,train}\overline{Y}_{train}}{(\overline{X^2_1}_{,train} - \overline{X}^2_{1,train})}$

$E(C_2) = E(y_{1,test}x_{1,1,test})E(1000 \times \frac{\overline{X}_{1,train}\overline{Y}_{train}}{(\overline{X^2_1}_{,train} - \overline{X}^2_{1,train})})$

$= \mu_4 E(1000 \times \frac{\overline{X}_{1,train}\overline{Y}_{train}}{(\overline{X^2_1}_{,train} - \overline{X}^2_{1,train})})$

$= E[1000 \times \frac{\mu_4 xz}{y - x^2}]$

Let $C_3 = \sum_{j=1}^{500} 2y_{j,test}\frac{\overline{X}_{1,train}\overline{X_{1,train}Y_{train}}}{(\overline{X^2_1}_{,train} - \overline{X}^2_{1,train})}$

$E(C_3) = E(y_{j,test})E(1000 \times \frac{\overline{X}_{1,train}\overline{X_{1,train}Y_{train}}}{(\overline{X^2_1}_{,train} - \overline{X}^2_{1,train})})$

$= \mu_3 E(1000 \times \frac{\overline{X}_{1,train}\overline{X_{1,train}Y_{train}}}{(\overline{X^2_1}_{,train} - \overline{X}^2_{1,train})})$

$= E(1000 \times \frac{\mu_3 xt}{y - x^2}]$

Let $C_4 = \sum_{j=1}^{500}(-2)y_{j,test}\frac{x_{1,j,test}\overline{X_{1,train}Y_{train}}}{(\overline{X^2_1}_{,train} - \overline{X}^2_{1,train})}$

$E(C_4) = -E(y_{j,test}x_{1,j,test})E(1000 \times \frac{\overline{X_{1,train}Y_{train}}}{(\overline{X^2_1}_{,train} - \overline{X}^2_{1,train})})$

$= -\mu_4 E(1000 \times \frac{\overline{X_{1,train}Y_{train}}}{(\overline{X^2_1}_{,train} - \overline{X}^2_{1,train})})$

$= E[1000 \times \frac{-\mu_4 t}{y - x^2}]$

$E(C) = \Sigma_{i=1}^4 E(C_i)$

$= E[1000 \times (\frac{-\mu_3 yz + \mu_4 xz + \mu_3 xt - \mu_4 t}{y - x^2})]$

$= E[1000 \times \frac{\mu_3(xt - yz) + \mu_4(xz - t)}{y - x^2}]$

Now suppose $g_2 = \frac{\mu_3(xt - yz) + \mu_4(xz - t)}{y - x^2}$, let $x=\mu_1, y=\mu_2, z=\mu_3, t=\mu_4$ and $a=\mu_3,\ b=\mu_4$ in the following expressions of $h_{ij}^{(C)},\ i,j=1,...,4, j\geqslant i$, where $\mu_i,\ i=1,...,4$ are calculated on page 83 of this thesis.

Further, assume $\mu_1,\ \mu_2,\ \mu_3,\ \mu_4$ satisfies that the denominators in $h_{ij}^{(C)},\ i,j=1,...,4, j\geqslant i$ are not equal to zero as well as $h_{ii}^{(C)} \neq \sum_{j=1,i\neq j}^{4} h_{ij}^{(C)},\ i=1,...,4.$

then

$$h_{11}^{(C)} = -\frac{2\left((bz+at)\,x^3+(-3ayz-3bt)\,x^2+(3byz+3aty)\,x-ay^2z-bty\right)}{\left(x^2-y\right)^3},$$
$$h_{12}^{(C)} = \frac{((2ax-b)\,z-at)\,y+\left(2ax^3-3bx^2\right)z-3atx^2+4btx}{\left(y-x^2\right)^3},$$
$$h_{13}^{(C)} = -\frac{(2ax-b)\,y-bx^2}{\left(y-x^2\right)^2},$$
$$h_{14}^{(C)} = \frac{ay+ax^2-2bx}{\left(y-x^2\right)^2},$$
$$h_{22}^{(C)} = -\frac{2\,(ax-b)\,(xz-t)}{\left(y-x^2\right)^3},$$
$$h_{23}^{(C)} = \frac{x\,(ax-b)}{\left(y-x^2\right)^2},$$
$$h_{24}^{(C)} = -\frac{ax-b}{\left(y-x^2\right)^2},$$
$$h_{33}^{(C)} = 0,$$
$$h_{34}^{(C)} = 0,$$
$$h_{44}^{(C)} = 0.$$

Thus, $E(C) \sim \sum_{i=1}^{4} \sigma_i^2 \frac{h_{ii}^{(C)}}{2} + \sum_{i,j=1,i>j}^{4} \sigma_{ij} h_{ij}^{(C)}$,

where $\sigma_{ij},\ i,j=1,...,4,\ i\leq j$ are calculated on page 83 and page 84 of this thesis.

Therefore, asymptotic estimation of $MSPE$ of 1-dimensional model fitted by single variable X_1 is

$$\frac{1}{2}[\sum_{i=1}^{4} \sigma_i^2 \frac{h_{ii}^{(A)}}{2} + \sum_{i,j=1,i>j}^{4} \sigma_{ij} h_{ij}^{(A)}] + E(B) + \sum_{i=1}^{4} \sigma_i^2 \frac{h_{ii}^{(C)}}{2} + \sum_{i,j=1,i>j}^{4} \sigma_{ij} h_{ij}^{(C)},$$

where $\mu_1,\ \mu_2,\ \mu_3,\ \mu_4$ satisfies that the denominators in $h_{ii}^{(A)},\ h_{ij}^{(C)},\ i,j=1,...,4,$ $j\geqslant i$ are not equal to zero and $h_{ii}^{(A)} \neq \sum_{j=1,i\neq j}^{4} h_{ij}^{(A)},\ h_{ii}^{(C)} \neq \sum_{j=1,i\neq j}^{4} h_{ij}^{(C)},\ i=1,...,4,$ and $\mu_i,\ \sigma_{ij},\ i,j=1,...,4,\ i\leq j$ are calculated on page 83 and page 84 of this thesis.

4.2 Asymptotic Estimation on MSPE of 2-Dimensional Model

Now, assume $X_{test} = \begin{pmatrix} 1 & x_{1,1,test} & x_{2,1,test}+x_{3,1,test} \\ 1 & x_{1,2,test} & x_{2,2,test}+x_{3,2,test} \\ \vdots & \vdots & \vdots \\ 1 & x_{1,500,test} & x_{2,500,test}+x_{3,500,test} \end{pmatrix}$,

$X_{train} = \begin{pmatrix} 1 & x_{1,1,train} & x_{2,1,train}+x_{3,1,train} \\ 1 & x_{1,2,train} & x_{2,1,train}+x_{3,1,train} \\ \vdots & \vdots & \vdots \\ 1 & x_{1,1000,train} & x_{2,1,train}+x_{3,1,train} \end{pmatrix}$.

$X_{train}^T X_{train} =$

$1000 \times \begin{pmatrix} 1 & \overline{X}_{1,train} & (\overline{X}_{2,train}+\overline{X}_{3,train}) \\ \overline{X}_{1,train} & \overline{X^2_1}_{,train} & (\overline{X_1X_2}_{,train}+\overline{X_1X_3}_{,train}) \\ (\overline{X}_{2,train}+\overline{X}_{3,train}) & (\overline{X_1X_2}_{,train}+\overline{X_1X_3}_{,train}) & \overline{(X_2+X_3)^2}_{,train} \end{pmatrix}$

where

$\overline{X}_{i,train} = \dfrac{1}{1000}(x_{i,1,train}+x_{i,2,train}+...+x_{i,1000,train}), i=1,2,3$

$\overline{X^2_i}_{,train} = \dfrac{1}{1000}(x^2_{i,1,train}+x^2_{i,2,train}+...+x^2_{i,1000,train}),\ i=1,2,3$

$\overline{X_iX_j}_{,train} = \dfrac{1}{1000}[x_{i,1,train}x_{j,1,train}+x_{i,2,train}x_{j,2,train}+...+x_{i,1000,train}x_{j,1000,train}]$, $i,j=1,2,3$.

$\overline{(X_i+X_j)^2}_{,train} = \dfrac{1}{1000}[(x_{i,1,train}+x_{j,1,train})^2+(x_{i,2,train}x_{j,2,train})^2+...$ $+(x_{i,1000,train}x_{j,1000,train})^2]$, $i,j=1,2,3$.

$(X_{train}^T X_{train})^{-1} = \dfrac{\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}}{1000 \times D}$

where $a_{11} = \overline{X^2_1}_{,train}\overline{(X_2+X_3)^2}_{,train} - (\overline{X_1X_2}_{,train}+\overline{X_1X_3}_{,train})^2$

$a_{12} = (\overline{X}_{2,train}+\overline{X}_{3,train})(\overline{X_1X_2}_{,train}+\overline{X_1X_3}_{,train}) - \overline{X}_{1,train}\overline{(X_2+X_3)^2}_{,train}$

$a_{13} = \overline{X}_{1,train}(\overline{X_1X_2}_{,train}+\overline{X_1X_3}_{,train}) - (\overline{X}_{2,train}+\overline{X}_{3,train})\overline{X^2_1}_{,train}$

$a_{22} = \overline{(X_2+X_3)^2}_{,train} - (\overline{X}_{2,train}+\overline{X}_{3,train})^2$

$a_{23} = (\overline{X}_{2,train}+\overline{X}_{3,train})\overline{X}_{1,train} - (\overline{X_1X_2}_{,train}+\overline{X_1X_3}_{,train})$

$a_{33} = \overline{X^2_1}_{,train} - (\overline{X}_{1,train})^2$

where

$$D=\overline{X^2_1}_{,train}\overline{(X_2+X_3)^2}_{,train}+2\overline{X}_{1,train}(\overline{X_1X_2}_{,train}+\overline{X_1X_3}_{,train})(\overline{X}_{2,train}+\overline{X}_{3,train})-(\overline{X_1X_2}_{,train}+\overline{X_1X_3}_{,train})^2-(\overline{X}_{2,train}+\overline{X}_{3,train})^2\overline{X^2_1}_{,train}-(\overline{X}_{1,train})^2\overline{(X_2+X_3)^2}_{,train}$$

$$X_{test}(X^T_{train}X_{train})^{-1}=\frac{\begin{pmatrix} 1 & x_{1,1,test} & x_{2,1,test}+x_{3,1,test} \\ 1 & x_{1,2,test} & x_{2,2,test}+x_{3,2,test} \\ \vdots & \vdots & \vdots \\ 1 & x_{1,500,test} & x_{2,500,test}+x_{3,500,test} \end{pmatrix}\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}}{1000\times D}$$

$$=\frac{\begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} \\ b_{2,1} & b_{2,2} & b_{2,3} \\ \vdots & \vdots & \vdots \\ b_{500,1} & b_{500,2} & b_{500,3} \end{pmatrix}}{1000\times D},$$

where for $j=1,...,500$,

$b_{j,1}=a_{11}+a_{12}x_{1,j,test}+a_{13}(x_{2,j,test}+x_{3,j,test})$

$b_{j,2}=a_{12}+a_{22}x_{1,j,test}+a_{23}(x_{2,j,test}+x_{3,j,test})$

$b_{j,3}=a_{13}+a_{23}x_{1,j,test}+a_{33}(x_{2,j,test}+x_{3,j,test})$

$$X^T_{train}Y_{train}=1000\times\begin{pmatrix} \overline{Y}_{train} \\ \overline{X_{1,train}Y_{train}} \\ \overline{(X_{2,train}+X_{3,train})Y_{train}} \end{pmatrix}$$

$$X_{test}(X^T_{train}X_{train})^{-1}X^T_{train}Y_{train}-Y_{test}$$

$$=\begin{pmatrix} \frac{b_{1,1}\overline{Y}_{train}+b_{1,2}\overline{X_{1,train}Y_{train}}+b_{1,3}\overline{(X_{2,train}+X_{3,train})Y_{train}}}{D}-y_{1,test} \\ \frac{b_{2,1}\overline{Y}_{train}+b_{2,2}\overline{X_{1,train}Y_{train}}+b_{2,3}\overline{(X_{2,train}+X_{3,train})Y_{train}}}{D}-y_{2,test} \\ \vdots \\ \frac{b_{500,1}\overline{Y}_{train}+b_{500,2}\overline{X_{1,train}Y_{train}}+b_{500,3}\overline{(X_{2,train}+X_{3,train})Y_{train}}}{D}-y_{500,test} \end{pmatrix}$$

$$SPE=(X_{test}(X^T_{train}X_{train})^{-1}X^T_{train}Y_{train}-Y_{test})^T(X_{test}(X^T_{train}X_{train})^{-1}X^T_{train}Y_{train}-Y_{test})$$

$$=\sum_{j=1}^{500}[\frac{b_{j,1}\overline{Y}_{train}+b_{j,2}\overline{X_{1,train}Y_{train}}+b_{j,3}\overline{(X_{2,train}+X_{3,train})Y_{train}}}{D}-y_{j,test}]^2$$

$$MSPE=E(SPE)$$

$= E(\sum_{j=1}^{500}[\frac{b_{j,1}\overline{Y}_{train} + b_{j,2}\overline{X_{1,train}Y_{train}} + b_{j,3}\overline{(X_{2,train} + X_{3,train})Y_{train}}}{D}]^2)$
$+E(\sum_{j=1}^{500} y_{j,test}^2)$
$+E(\sum_{j=1}^{500}(-2)y_{j,test}\frac{b_{j,1}\overline{Y}_{train} + b_{j,2}\overline{X_{1,train}Y_{train}} + b_{j,3}\overline{(X_{2,train} + X_{3,train})Y_{train}}}{D})$

Denote the above three parts in MSPE as
$E(U) = E(\sum_{j=1}^{500}[\frac{b_{j,1}\overline{Y}_{train}+b_{j,2}\overline{X_{1,train}Y_{train}}+b_{j,3}\overline{(X_{2,train}+X_{3,train})Y_{train}}}{D}]^2)$,
$E(V) = E(\sum_{j=1}^{500} y_{j,test}^2)$,
$E(W) = E(\sum_{j=1}^{500}(-2)y_{j,test}\frac{b_{j,1}\overline{Y}_{train}+b_{j,2}\overline{X_{1,train}Y_{train}}+b_{j,3}\overline{(X_{2,train}+X_{3,train})Y_{train}}}{D})$,
where $E(V)$ is calculated as $E(B)$ in Chapter 4.1.

Now,
$\sqrt{n}(\overline{X}_{1,train} - \mu_1) \xrightarrow{D} N(0, \sigma_1^2)$,
$\sqrt{n}(\overline{X_1^2}_{,train} - \mu_2) \xrightarrow{D} N(0, \sigma_2^2)$,
$\sqrt{n}(\overline{Y}_{train} - \mu_3) \xrightarrow{D} N(0, \sigma_3^2)$,
$\sqrt{n}(\overline{X_{1,train}Y_{train}} - \mu_4) \xrightarrow{D} N(0, \sigma_4^2)$,
$\sqrt{n}(\overline{(X_2 + X_3)^2}_{,train} - \mu_5) \xrightarrow{D} N(0, \sigma_5^2)$,
$\sqrt{n}(\overline{X_1X_2}_{,train} - \mu_6) \xrightarrow{D} N(0, \sigma_6^2)$,
$\sqrt{n}(\overline{X_1X_3}_{,train} - \mu_7) \xrightarrow{D} N(0, \sigma_7^2)$,
$\sqrt{n}(\overline{X}_{2,train} - \mu_8) \xrightarrow{D} N(0, \sigma_8^2)$,
$\sqrt{n}(\overline{X}_{3,train} - \mu_9) \xrightarrow{D} N(0, \sigma_9^2)$,
$\sqrt{n}(\overline{(X_{2,train} + X_{3,train})Y_{train}} - \mu_{10}) \xrightarrow{D} N(0, \sigma_{10}^2)$,

where $\mu_1,\ \mu_2,\ \mu_3,\ \mu_4,\ \sigma_1^2,\ \sigma_2^2,\ \sigma_3^2,\ \sigma_4^2$ are calculated in Chapter 4.1 besides the pairwise covariance among $\overline{X}_{1,train}$, $\overline{X_1^2}_{,train}$, $\overline{Y}_{train}$ and $\overline{X_{1,train}Y_{train}}$.

Further,

$\mu_5 = E((X_2 + X_3)_{train}^2) = var(X_{2,train} + X_{3,train}) + E^2(X_{2,train} + X_{3,train})$
$= p_2(1 - p_2) + p_3(1 - p_3) + (p_2 + p_3)^2$
$= p_2 + p_3 + 2p_2p_3$

$\mu_6 = E(X_{1,train}X_{2,train}) = p_1p_2$

$\mu_7 = E(X_{1,train}X_{3,train}) = p_1p_3$

$\mu_8 = E(X_{2,train}) = p_2$

$\mu_9 = E(X_{3,train}) = p_3$

$\mu_{10} = E((X_{2,train} + X_{3,train})Y_{train})$

$= E[(\beta_0 + \beta_1 X_{1,train} + \beta_2 X_{2,train} + \beta_3 X_{3,train} + \varepsilon_{train})X_{2,train}] + E[(\beta_0 + \beta_1 X_{1,train} + \beta_2 X_{2,train} + \beta_3 X_{3,train} + \varepsilon_{train})X_{3,train}]$

$= \beta_0 p_2 + \beta_1 p_1 p_2 + \beta_2 p_2 + \beta_3 p_2 p_3 + \beta_0 p_3 + \beta_1 p_1 p_3 + \beta_2 p_2 p_3 + \beta_3 p_3$

$= \beta_0(p_2 + p_3) + \beta_1 p_1(p_2 + p_3) + \beta_2(1 + p_3)p_2 + \beta_3(1 + p_2)p_3$

$\sigma_5^2 = var((X_{2,train}+X_{3,train})^2) = E((X_{2,train}+X_{3,train})^4) - E^2((X_{2,train}+X_{3,train})^2)$

$= E[X_{2,train}^4 + X_{3,train}^4 + 6X_{2,train}^2 X_{3,train}^2 + 4X_{2,train}^3 X_{3,train} + 4X_{2,train} X_{3,train}^3]$

$- [E(X_{2,train}^2) + E(X_{3,train}^2) + 2E(X_{2,train} X_{3,train})]^2$

$= p_2 + p_3 + 14p_2 p_3 - (p_2 + p_3 + 2p_2 p_3)^2$

$\sigma_6^2 = var(X_{1,train} X_{2,train}) = p_1 p_2(1 - p_1 p_2)$

$\sigma_7^2 = var(X_{1,train} X_{3,train}) = p_1 p_3(1 - p_1 p_3)$

$\sigma_8^2 = var(X_{2,train}) = p_2(1 - p_2)$

$\sigma_9^2 = var(X_{3,train}) = p_3(1 - p_3)$

$\sigma_{10}^2 = var((X_{2,train} + X_{3,train})Y_{train})$

$= var[(\beta_0 + \beta_1 X_{1,train} + \beta_2 X_{2,train} + \beta_3 X_{3,train} + \varepsilon_{train})(X_{2,train} + X_{3,train})]$

$= var[\beta_0(X_{2,train} + X_{3,train})] + var[\beta_1 X_{1,train}(X_{2,train} + X_{3,train})]$

$+ var[\beta_2 X_{2,train}(X_{2,train} + X_{3,train})] + var[\beta_3 X_{3,train}(X_{2,train} + X_{3,train})]$

$+ var[\varepsilon_{train}(X_{2,train}+X_{3,train})] + cov(\beta_0(X_{2,train}+X_{3,train}), \beta_1 X_{1,train}(X_{2,train}+X_{3,train}))$

$+ cov(\beta_0(X_{2,train} + X_{3,train}), \beta_2 X_{2,train}(X_{2,train} + X_{3,train}))$

$+ cov(\beta_0(X_{2,train} + X_{3,train}, \beta_3 X_{3,train}(X_{2,train} + X_{3,train}))$

$+ cov(\beta_0(X_{2,train} + X_{3,train}), \varepsilon_{train}(X_{2,train} + X_{3,train}))$

$+ cov(\beta_1 X_{1,train}(X_{2,train} + X_{3,train}), \beta_2 X_{2,train}(X_{2,train} + X_{3,train}))$

$+ cov(\beta_1 X_{1,train}(X_{2,train} + X_{3,train}), \beta_3 X_{3,train}(X_{2,train} + X_{3,train}))$

$+ cov(\beta_1 X_{1,train}(X_{2,train} + X_{3,train}), \varepsilon_{train}(X_{2,train} + X_{3,train}))$

$+ cov(\beta_2 X_{2,train}(X_{2,train} + X_{3,train}), \beta_3 X_{3,train}(X_{2,train} + X_{3,train}))$

$+ cov(\beta_2 X_{2,train}(X_{2,train} + X_{3,train}), \varepsilon_{train}(X_{2,train} + X_{3,train}))$

$+ cov(\beta_3 X_{3,train}(X_{2,train} + X_{3,train}), \varepsilon_{train}(X_{2,train} + X_{3,train}))$

$= \beta_0^2(p_2(1-p_2) + p_3(1-p_3)) + \beta_1^2(p_1(p_2 + p_3 + 2p_2 p_3) - p_1^2(p_2 + p_3)^2) + \beta_2^2(p_2 + 3p_2 p_3 - (p_2 + p_2 p_3)^2) + \beta_3^2(p_3 + 3p_2 p_3 - (p_3 + p_2 p_3)^2) + \sigma^2(p_2 + p_3 + 2p_2 p_3) + \beta_0\beta_1 p_1(p_2 + p_3 + 2p_2 p_3 - (p_2 + p_3)^2) + \beta_0\beta_2(p_2 + 3p_2 p_3 - (p_2 + p_3)(p_2 + p_2 p_3)) + \beta_0\beta_3(p_3 + 3p_2 p_3 - (p_2 +$

$p_3)(p_3+p_2p_3))+\beta_1\beta_2p_1(p_2+3p_2p_3-(p_2+p_3)(p_2+p_2p_3))+\beta_1\beta_3p_1(p_3+3p_2p_3-(p_2+p_3)(p_3+p_2p_3))+\beta_2\beta_3(4p_2p_3-(p_2+p_2p_3)(p_3+p_2p_3))$

$\sigma_{15}=cov(X_{1,train},(X_{2,train}+X_{3,train})^2)=0$

$\sigma_{16}=cov(X_{1,train},X_{1,train}X_{2,train})=p_1p_2-p_1^2p_2$

$\sigma_{17}=cov(X_{1,train},X_{1,train}X_{3,train})=p_1p_3-p_1^2p_3$

$\sigma_{18}=cov(X_{1,train},X_{2,train})=0$

$\sigma_{19}=cov(X_{1,train},X_{3,train})=0$

$\sigma_{1\ 10}=cov(X_{1,train},(X_{2,train}+X_{3,train})Y_{train})=\beta_1p_1(1-p_1)(p_2+p_3)$

$\sigma_{25}=cov(X_{1,train}^2,(X_{2,train}+X_{3,train})^2)=0$

$\sigma_{26}=cov(X_{1,train}^2,X_{1,train}X_{2,train})=p_1p_2-p_1^2p_2$

$\sigma_{27}=cov(X_{1,train}^2,X_{1,train}X_{3,train})=p_1p_3-p_1^2p_3$

$\sigma_{28}=cov(X_{1,train}^2,X_{2,train})=0$

$\sigma_{29}=cov(X_{1,train}^2,X_{3,train})=0$

$\sigma_{2\ 10}=cov(X_{1,train}^2,(X_{2,train}+X_{3,train})Y_{train})=\beta_1p_1(1-p_1)(p_2+p_3)$

$\sigma_{35}=cov(Y_{train},(X_{2,train}+X_{3,train})^2)$

$=E[(\beta_0+\beta_1X_{1,train}+\beta_2X_{2,train}+\beta_3X_{3,train}+\varepsilon_{train})(X_{2,train}+X_{3,train})^2]-\mu_3\mu_5$

$=\beta_0\mu_5+\beta_1p_1\mu_5+\beta_2(p_2+3p_2p_3)+\beta_3(p_3+3p_2p_3)-\mu_3\mu_5$

$\sigma_{36}=cov(Y_{train},X_{1,train}X_{2,train})=\beta_0\mu_6+\beta_1\mu_6+\beta_2\mu_6+\beta_3p_1p_2p_3-\mu_3\mu_6$

$\sigma_{37}=cov(Y_{train},X_{1,train}X_{3,train})=\beta_0\mu_7+\beta_1\mu_7+\beta_2p_1p_2p_3+\beta_3\mu_7-\mu_3\mu_7$

$\sigma_{38}=cov(Y_{train},X_{2,train})=\beta_0p_2+\beta_1p_1p_2+\beta_2p_2+\beta_3p_2p_3-\mu_3p_2$

$\sigma_{39}=cov(Y_{train},X_{3,train})=\beta_0p_3+\beta_1p_1p_3+\beta_2p_2p_3+\beta_3p_3-\mu_3p_3$

$\sigma_{3\ 10}=cov(Y_{train},(X_{2,train}+X_{3,train})Y_{train})=E((\beta_0+\beta_1X_{1,train}+\beta_2X_{2,train}+\beta_3X_{3,train}+\varepsilon_{train})^2(X_{2,train}+X_{3,train}))-\mu_3\mu_{10}$

$=E[\beta_0\beta_1X_{1,train}(X_{2,train}+X_{3,train})+\beta_0\beta_2X_{2,train}(X_{2,train}+X_{3,train})$

$+\beta_0\beta_3X_{3,train}(X_{2,train}+X_{3,train})+\beta_0\varepsilon_{train}(X_{2,train}+X_{3,train})$

$+\beta_1X_{1,train}\beta_2X_{2,train}(X_{2,train}+X_{3,train})$

$+\beta_1X_{1,train}\beta_3X_{3,train}(X_{2,train}+X_{3,train})+\beta_1X_{1,train}\varepsilon_{train}(X_{2,train}+X_{3,train})$

$+\beta_2X_{2,train}\beta_3X_{3,train}(X_{2,train}+X_{3,train})+\beta_2X_{2,train}\varepsilon_{train}(X_{2,train}+X_{3,train})$

$+\beta_3X_{3,train}\varepsilon_{train}(X_{2,train}+X_{3,train})]-\mu_3\mu_{10}$

$= \beta_0\beta_1 p_1(p_2+p_3)+\beta_0\beta_2(p_2+p_2p_3)+\beta_0\beta_3(p_3+p_2p_3)+\beta_1\beta_2 p_1(p_2+p_2p_3)+\beta_1\beta_3 p_1(p_3+p_2p_3)+\beta_2\beta_3 2p_2p_3-\mu_3\mu_{10}$

$\sigma_{45} = cov(X_{1,train}Y_{train},(X_{2,train}+X_{3,train})^2) = E[(\beta_0+\beta_1 X_{1,train}+\beta_2 X_{2,train}+\beta_3 X_{3,train}+\varepsilon_{train})X_{1,train}(X_{2,train}+X_{3,train})^2]-\mu_4\mu_5$

$= E[\beta_0 X_{1,train}(X_{2,train}+X_{3,train})^2+\beta_1 X_{1,train}^2(X_{2,train}+X_{3,train})^2$

$+\beta_2 X_{2,train}X_{1,train}(X_{2,train}+X_{3,train})^2+\beta_3 X_{3,train}X_{1,train}(X_{2,train}+X_{3,train})^2$

$+\varepsilon_{train}X_{1,train}(X_{2,train}+X_{3,train})^2]-\mu_4\mu_5$

$= \beta_0 p_1\mu_5+\beta_1 p_1\mu_5+\beta_2 p_1(p_2+3p_2p_3)+\beta_3 p_1(p_3+3p_2p_3)-\mu_4\mu_5$

$\sigma_{46} = cov(X_{1,train}Y_{train},X_{1,train}X_{2,train})$

$= E[(\beta_0+\beta_1 X_{1,train}+\beta_2 X_{2,train}+\beta_3 X_{3,train}+\varepsilon_{train})X_{1,train}^2 X_{2,train}]-\mu_4\mu_6$

$= \beta_0 p_1p_2+\beta_1 p_1p_2+\beta_2 p_1p_2+\beta_3 p_1p_2p_3-\mu_4\mu_6$

$\sigma_{47} = cov(X_{1,train}Y_{train},X_{1,train}X_{3,train})$

$= E[(\beta_0+\beta_1 X_{1,train}+\beta_2 X_{2,train}+\beta_3 X_{3,train}+\varepsilon_{train})X_{1,train}^2 X_{3,train}]-\mu_4\mu_7$

$= \beta_0 p_1p_3+\beta_1 p_1p_3+\beta_2 p_1p_2p_3+\beta_3 p_1p_3-\mu_4\mu_7$

$\sigma_{48} = cov(X_{1,train}Y_{train},X_{2,train})$

$= E[(\beta_0+\beta_1 X_{1,train}+\beta_2 X_{2,train}+\beta_3 X_{3,train}+\varepsilon_{train})X_{1,train}X_{2,train}]-\mu_4\mu_8$

$= \beta_0 p_1p_2+\beta_1 p_1p_2+\beta_2 p_1p_2+\beta_3 p_1p_2p_3-\mu_4\mu_8$

$\sigma_{49} = cov(X_{1,train}Y_{train},X_{3,train})$

$= E[(\beta_0+\beta_1 X_{1,train}+\beta_2 X_{2,train}+\beta_3 X_{3,train}+\varepsilon_{train})X_{1,train}X_{3,train}]-\mu_4\mu_9$

$= \beta_0 p_1p_3+\beta_1 p_1p_3+\beta_2 p_1p_2p_3+\beta_3 p_1p_3-\mu_4\mu_9$

$\sigma_{4\ 10} = cov(X_{1,train}Y_{train},(X_{2,train}+X_{3,train})Y_{train})$

$= E((\beta_0+\beta_1 X_{1,train}+\beta_2 X_{2,train}+\beta_3 X_{3,train}+\varepsilon_{train})^2 X_{1,train}(X_{2,train}+X_{3,train}))-\mu_4\mu_{10}$

$= E[\beta_0\beta_1 X_{1,train}^2(X_{2,train}+X_{3,train})+\beta_0\beta_2 X_{2,train}X_{1,train}(X_{2,train}+X_{3,train})$

$+\beta_0\beta_3 X_{3,train}X_{1,train}(X_{2,train}+X_{3,train})+\beta_0\varepsilon_{train}X_{1,train}(X_{2,train}+X_{3,train})$

$+\beta_1 X_{1,train}\beta_2 X_{2,train}X_{1,train}(X_{2,train}+X_{3,train})$

$+\beta_1 X_{1,train}\beta_3 X_{3,train}X_{1,train}(X_{2,train}+X_{3,train})$

$+\beta_1 X_{1,train}\varepsilon_{train}X_{1,train}(X_{2,train}+X_{3,train})$

$+\beta_2 X_{2,train}\beta_3 X_{3,train}X_{1,train}(X_{2,train}+X_{3,train})$

$+\beta_2 X_{2,train}\varepsilon_{train}X_{1,train}(X_{2,train}+X_{3,train})$

$+\beta_3 X_{3,train}\varepsilon_{train}X_{1,train}(X_{2,train}+X_{3,train})]-\mu_4\mu_{10}$

$= \beta_0\beta_1 p_1(p_2 + p_3) + \beta_0\beta_2 p_1(p_2 + p_2p_3) + \beta_0\beta_3 p_1(p_3 + p_2p_3) + \beta_1\beta_2 p_1(p_2 + p_2p_3) + \beta_1\beta_3 p_1(p_3 + p_2p_3) + \beta_2\beta_3 2p_1p_2p_3 - \mu_4\mu_{10}$

$\sigma_{56} = cov((X_{2,train} + X_{3,train})^2, X_{1,train}X_{2,train}) = p_1(p_2 + 3p_2p_3) - \mu_5\mu_6$

$\sigma_{57} = cov((X_{2,train} + X_{3,train})^2, X_{1,train}X_{3,train}) = p_1(p_3 + 3p_2p_3) - \mu_5\mu_7$

$\sigma_{58} = cov((X_{2,train} + X_{3,train})^2, X_{2,train}) = (p_2 + 3p_2p_3) - \mu_5\mu_8$

$\sigma_{59} = cov((X_{2,train} + X_{3,train})^2, X_{3,train}) = (p_3 + 3p_2p_3) - \mu_5\mu_9$

$\sigma_{5\ 10} = cov((X_{2,train} + X_{3,train})^2, (X_{2,train} + X_{3,train})Y_{train})$

$= E[(\beta_0 + \beta_1 X_{1,train} + \beta_2 X_{2,train} + \beta_3 X_{3,train} + \varepsilon_{train})(X_{2,train} + X_{3,train})^3] - \mu_5\mu_{10}$

$= \beta_0(p_2 + p_3 + 6p_2p_3) + \beta_1 p_1(p_2 + p_3 + 6p_2p_3) + \beta_2(p_2 + 7p_2p_3) + \beta_3(p_3 + 7p_2p_3) - \mu_5\mu_{10}$

$\sigma_{67} = cov(X_{1,train}X_{2,train}, X_{1,train}X_{3,train}) = p_1p_2p_3 - \mu_6\mu_7$

$\sigma_{68} = cov(X_{1,train}X_{2,train}, X_{2,train}) = p_1p_2 - \mu_6\mu_8$

$\sigma_{69} = cov(X_{1,train}X_{2,train}, X_{3,train}) = 0$

$\sigma_{6\ 10} = cov(X_{1,train}X_{2,train}, (X_{2,train} + X_{3,train})Y_{train})$

$= E[(\beta_0 + \beta_1 X_{1,train} + \beta_2 X_{2,train} + \beta_3 X_{3,train} + \varepsilon_{train})X_{1,train}X_{2,train}(X_{2,train} + X_{3,train})] - \mu_6\mu_{10}$

$= \beta_0 p_1(p_2 + p_2p_3) + \beta_1 p_1(p_2 + p_2p_3) + \beta_2 p_1(p_2 + p_2p_3) + \beta_3 p_1 2p_2p_3 - \mu_6\mu_{10}$

$\sigma_{78} = cov(X_{1,train}X_{3,train}, X_{2,train}) = 0$

$\sigma_{79} = cov(X_{1,train}X_{3,train}, X_{3,train}) = p_1p_3 - \mu_7\mu_9$

$\sigma_{7\ 10} = cov(X_{1,train}X_{3,train}, (X_{2,train} + X_{3,train})Y_{train})$

$= E[(\beta_0 + \beta_1 X_{1,train} + \beta_2 X_{2,train} + \beta_3 X_{3,train} + \varepsilon_{train})X_{1,train}X_{3,train}(X_{2,train} + X_{3,train})] - \mu_7\mu_{10}$

$= \beta_0 p_1(p_3 + p_2p_3) + \beta_1 p_1(p_3 + p_2p_3) + \beta_2 p_1 2p_2p_3 + \beta_3 p_1(p_3 + p_2p_3) - \mu_7\mu_{10}$

$\sigma_{89} = cov(X_{2,train}, X_{3,train}) = 0$

$\sigma_{8\ 10} = cov(X_{2,train}, (X_{2,train} + X_{3,train})Y_{train}) = E[(\beta_0 + \beta_1 X_{1,train} + \beta_2 X_{2,train} + \beta_3 X_{3,train} + \varepsilon_{train})X_{2,train}(X_{2,train} + X_{3,train})] - \mu_8\mu_{10}$

$= \beta_0(p_2 + p_2p_3) + \beta_1 p_1(p_2 + p_2p_3) + \beta_2(p_2 + p_2p_3) + \beta_3 2p_2p_3 - \mu_8\mu_{10}$

$\sigma_{9\ 10} = cov(X_{3,train}, (X_{2,train} + X_{3,train})Y_{train})$

$= E[(\beta_0 + \beta_1 X_{1,train} + \beta_2 X_{2,train} + \beta_3 X_{3,train} + \varepsilon_{train})X_{3,train}(X_{2,train} + X_{3,train})] - \mu_9\mu_{10}$

$= \beta_0(p_3 + p_2p_3) + \beta_1 p_1(p_3 + p_2p_3) + \beta_2 2p_2p_3 + \beta_3(p_3 + p_2p_3) - \mu_9\mu_{10}$

For convenience, use the following notations to simplify the expression of $E(U)$ and $E(W)$.

$x = \overline{X}_{1,train}$,

$y = \overline{X_1^2}_{,train}$

$z = \overline{Y}_{train}$

$t = \overline{X_{1,train} Y_{train}}$

$l = \overline{(X_2 + X_3)^2}_{,train}$

$m = \overline{X_1 X_2}_{,train}$

$n = \overline{X_1 X_3}_{,train}$

$p = \overline{X}_{2,train}$

$q = \overline{X}_{3,train}$

$s = \overline{(X_{2,train} + X_{3,train}) Y_{train}}$

Therefore,

$a_{11} = yl - (m+n)^2$

$a_{12} = (p+q)(m+n) - xl$

$a_{13} = x(m+n) - (p+q)y$

$a_{22} = l - (p+q)^2$

$a_{23} = (p+q)x - (m+n)$

$a_{33} = y - x^2$

$D = yl + 2x(m+n)(p+q) - (m+n)^2 - (p+q)^2 y - x^2 l$

Thus,

$$E(U) = E(\textstyle\sum_{j=1}^{500} [\frac{b_{j,1}z + b_{j,2}t + b_{j,3}s}{D}]^2)$$

$$= E(\textstyle\sum_{j=1}^{500} [\frac{(b_{j,1}z)^2}{D^2} + \frac{(b_{j,2}t)^2}{D^2} + \frac{(b_{j,3}s)^2}{D^2} + \frac{2b_{j,1}zb_{j,2}t}{D^2} + \frac{2b_{j,1}zb_{j,3}s}{D^2} + \frac{2b_{j,2}tb_{j,3}s}{D^2}])$$

Let $U_1 = \sum_{j=1}^{500} \frac{(b_{j,1}z)^2}{D^2}$,

then $E(U_1) = \sum_{j=1}^{500} E(\frac{(b_{j,1}z)^2}{D^2})$

$$= \textstyle\sum_{j=1}^{500} E(\frac{(a_{11} + a_{12}x_{1,j,test} + a_{13}(x_{2,j,test} + x_{3,j,test}))^2 z^2}{D^2})$$

$$= \textstyle\sum_{j=1}^{500} E(\frac{a_{11}^2}{D^2 z^{-2}} + \frac{a_{12}^2 x_{1,j,test}^2}{D^2 z^{-2}} + \frac{a_{13}^2 (x_{2,j,test} + x_{3,j,test})^2}{D^2 z^{-2}} + \frac{2a_{11}a_{12}x_{1,j,test}}{D^2 z^{-2}} + \frac{2a_{11}a_{13}(x_{2,j,test} + x_{3,j,test})}{D^2 z^{-2}} + \frac{2a_{12}x_{1,j,test}a_{13}(x_{2,j,test} + x_{3,j,test})}{D^2 z^{-2}})$$

$$= \frac{1}{2}E(1000 \times \frac{a_{11}^2}{D^2 z^{-2}})$$
$$+\frac{1}{2}E(x_{1,1,test}^2)E(1000 \times \frac{a_{12}^2}{D^2 z^{-2}})$$
$$+\frac{1}{2}E((x_{2,j,test} + x_{3,j,test})^2)E(1000 \times \frac{a_{13}^2}{D^2 z^{-2}})$$
$$+E(x_{1,j,test})E(1000 \times \frac{a_{11}a_{12}}{D^2 z^{-2}})$$
$$+E((x_{2,j,test} + x_{3,j,test}))E(1000 \times \frac{a_{11}a_{13}}{D^2 z^{-2}})$$
$$+E(x_{1,j,test}(x_{2,j,test} + x_{3,j,test}))E(1000 \times \frac{a_{12}a_{13}}{D^2 z^{-2}})$$
$$= E(1000 \times \frac{a_{11}^2}{2D^2 z^{-2}})$$
$$+p_1 E(1000 \times \frac{a_{12}^2}{2D^2 z^{-2}})$$
$$+(p_2 + p_3 + 2p_2 p_3)E(1000 \times \frac{a_{13}^2}{2D^2 z^{-2}})$$
$$+p_1 E(1000 \times \frac{2a_{11}a_{12}}{2D^2 z^{-2}})$$
$$+(p_2 + p_3)E(1000 \times \frac{2a_{11}a_{13}}{2D^2 z^{-2}})$$
$$+(p_1 p_2 + p_1 p_3)E(1000 \times \frac{2a_{12}a_{13}}{2D^2 z^{-2}})$$
$$= E[1000 \times \frac{a_{11}^2 + p_1 a_{12}^2 + (p_2+p_3+2p_2p_3)a_{13}^2 + 2p_1 a_{11}a_{12} + 2(p_2+p_3)a_{11}a_{13} + 2(p_1p_2+p_1p_3)a_{12}a_{13}}{2D^2 z^{-2}}]$$

Let $U_2 = \sum_{j=1}^{500} \frac{(b_{j,2}t)^2}{D^2}$

$$E(U_2) = \sum_{j=1}^{500} E(\frac{(a_{12} + a_{22}x_{1,j,test} + a_{23}(x_{2,j,test} + x_{3,j,test}))^2 t^2}{D^2})$$
$$= \sum_{j=1}^{500} E(\frac{a_{12}^2}{D^2 t^{-2}} + \frac{a_{22}^2 x_{1,j,test}^2}{D^2 t^{-2}} + \frac{a_{23}^2 (x_{2,j,test} + x_{3,j,test})^2}{D^2 t^{-2}} + \frac{2a_{12}a_{22}x_{1,j,test}}{D^2 t^{-2}}$$
$$+ \frac{2a_{12}a_{23}(x_{2,j,test} + x_{3,j,test})}{D^2 t^{-2}} + \frac{2a_{22}x_{1,j,test}a_{23}(x_{2,j,test} + x_{3,j,test})}{D^2 t^{-2}})$$
$$= \frac{1}{2}E(1000 \times \frac{a_{12}^2}{D^2 t^{-2}}) + \frac{1}{2}E(1000 \times \frac{a_{22}^2 x_{1,j,test}^2}{D^2 t^{-2}}) + \frac{1}{2}E(1000 \times \frac{a_{23}^2 (x_{2,j,test} + x_{3,j,test})^2}{D^2 t^{-2}})$$
$$+E(1000 \times \frac{a_{12}a_{22}x_{1,j,test}}{D^2 t^{-2}}) + E(1000 \times \frac{a_{12}a_{23}(x_{2,j,test} + x_{3,j,test})}{D^2 t^{-2}})$$
$$+E(1000 \times \frac{a_{22}x_{1,j,test}a_{23}(x_{2,j,test} + x_{3,j,test})}{D^2 t^{-2}})$$
$$= \frac{1}{2}E(1000 \times \frac{a_{12}^2}{D^2 t^{-2}}) + \frac{1}{2}p_1 E(1000 \times \frac{a_{22}^2}{D^2 t^{-2}}) + \frac{1}{2}(p_2 + p_3 + 2p_2p_3)E(1000 \times \frac{a_{23}^2}{D^2 t^{-2}})$$
$$+p_1 E(1000 \times \frac{a_{12}a_{22}}{D^2 t^{-2}}) + (p_2 + p_3)E(1000 \times \frac{a_{12}a_{23}}{D^2 t^{-2}}) + (p_1p_2 + p_1p_3)E(1000 \times \frac{a_{22}a_{23}}{D^2 t^{-2}})$$
$$= E[1000 \times \frac{a_{12}^2 + p_1 a_{22}^2 + (p_2+p_3+2p_2p_3)a_{23}^2 + 2p_1 a_{12}a_{22} + 2(p_2+p_3)a_{12}a_{23} + 2(p_1p_2+p_1p_3)a_{22}a_{23}}{2D^2 t^{-2}}]$$

Let $U_3 = \sum_{j=1}^{500} \frac{(b_{j,3}s)^2}{D^2}$, then

$$E(U_3) = \sum_{j=1}^{500} E(\frac{(b_{j,3}s)^2}{D^2})$$

$= \sum_{j=1}^{500} E(\frac{(a_{13} + a_{23}x_{1,j,test} + a_{33}(x_{2,j,test} + x_{3,j,test}))^2}{D^2 s^{-2}})$

$= \sum_{j=1}^{500} E(\frac{a_{13}^2}{D^2 s^{-2}}) + \sum_{j=1}^{500} E(\frac{a_{23}^2 x_{1,j,test}^2}{D^2 s^{-2}}) + \sum_{j=1}^{500} E(\frac{a_{33}^2 (x_{2,j,test} + x_{3,j,test})^2}{D^2 s^{-2}})$

$+ \sum_{j=1}^{500} E(\frac{2a_{13}a_{23}x_{1,j,test}}{D^2 s^{-2}}) + \sum_{j=1}^{500} E(\frac{2a_{13}a_{33}(x_{2,j,test} + x_{3,j,test})}{D^2 s^{-2}})$

$+ \sum_{j=1}^{500} E(\frac{2a_{23}x_{1,j,test}a_{33}(x_{2,j,test} + x_{3,j,test})}{D^2 s^{-2}})$

$= \frac{1}{2}E(1000 \times \frac{a_{13}^2}{D^2 s^{-2}}) + \frac{1}{2}p_1 E(1000 \times \frac{a_{23}^2}{D^2 s^{-2}})$

$+\frac{1}{2}(p_2 + p_3 + 2p_2p_3)E(1000 \times \frac{a_{33}^2}{D^2 s^{-2}}) + \frac{1}{2}p_1 E(1000 \times \frac{2a_{13}a_{23}}{D^2 s^{-2}})$

$+\frac{1}{2}(p_2 + p_3)E(1000 \times \frac{2a_{13}a_{33}}{D^2 s^{-2}}) + \frac{1}{2}(p_1p_2 + p_1p_3)E(1000 \times \frac{2a_{23}a_{33}}{D^2 s^{-2}})$

$= E[1000 \times \frac{a_{13}^2 + p_1 a_{23}^2 + (p_2+p_3+2p_2p_3)a_{33}^2 + 2p_1 a_{13}a_{23} + 2(p_2+p_3)a_{13}a_{33} + 2p_1(p_2+p_3)a_{23}a_{33}}{2D^2 s^{-2}}]$

Let $U_4 = \sum_{j=1}^{500} \frac{2b_{j,1}zb_{j,2}t}{D^2}$,

then $E(U_4)$

$= 2\sum_{j=1}^{500} E[\frac{(a_{11}+a_{12}x_{1,j,test}+a_{13}(x_{2,j,test}+x_{3,j,test}))(a_{12}+a_{22}x_{1,j,test}+a_{23}(x_{2,j,test}+x_{3,j,test}))}{D^2(zt)^{-1}}]$

$= 2\sum_{j=1}^{500} E[\frac{a_{11}a_{12}}{D^2(zt)^{-1}}] + 2\sum_{j=1}^{500} E[\frac{a_{11}a_{22}x_{1,j,test}}{D^2(zt)^{-1}}] + 2\sum_{j=1}^{500} E[\frac{a_{11}a_{23}(x_{2,j,test}+x_{3,j,test})}{D^2(zt)^{-1}}]$

$+2\sum_{j=1}^{500} E[\frac{a_{12}^2 x_{1,j,test}}{D^2(zt)^{-1}}] + 2\sum_{j=1}^{500} E[\frac{a_{12}a_{22}x_{1,j,test}^2}{D^2(zt)^{-1}}]$

$+2\sum_{j=1}^{500} E[\frac{a_{12}x_{1,j,test}a_{23}(x_{2,j,test}+x_{3,j,test})}{D^2(zt)^{-1}}] + 2\sum_{j=1}^{500} E[\frac{a_{13}(x_{2,j,test}+x_{3,j,test})a_{12}}{D^2(zt)^{-1}}]$

$+2\sum_{j=1}^{500} E[\frac{a_{13}(x_{2,j,test}+x_{3,j,test})a_{22}x_{1,j,test}}{D^2(zt)^{-1}}] + 2\sum_{j=1}^{500} E[\frac{a_{13}a_{23}(x_{2,j,test}+x_{3,j,test})^2}{D^2(zt)^{-1}}]$

$= E[1000 \times \frac{a_{11}a_{12}}{D^2(zt)^{-1}}] + E(x_{1,j,test})E[1000 \times \frac{a_{11}a_{22}}{D^2(zt)^{-1}}]$

$+E(x_{2,j,test} + x_{3,j,test})E[1000 \times \frac{a_{11}a_{23}}{D^2(zt)^{-1}}] + E(x_{1,j,test})E[1000 \times \frac{a_{12}^2}{D^2(zt)^{-1}}]$

$+E(x_{1,j,test}^2)E[1000 \times \frac{a_{12}a_{22}}{D^2(zt)^{-1}}] + E(x_{1,j,test}(x_{2,j,test} + x_{3,j,test}))E[1000 \times \frac{a_{12}a_{23}}{D^2(zt)^{-1}}]$

$+E((x_{2,j,test} + x_{3,j,test}))E[1000 \times \frac{a_{13}a_{12}}{D^2(zt)^{-1}}] + E((x_{2,j,test} + x_{3,j,test})x_{1,j,test})E[1000 \times \frac{a_{13}a_{22}}{D^2(zt)^{-1}}] + E((x_{2,j,test} + x_{3,j,test})^2)E[1000 \times \frac{a_{13}a_{23}}{D^2(zt)^{-1}}]$

$= E[1000 \times \frac{a_{11}a_{12}}{D^2(zt)^{-1}}] + p_1 E[1000 \times \frac{a_{11}a_{22}}{D^2(zt)^{-1}}]$

$+(p_2 + p_3)E[1000 \times \frac{a_{11}a_{23}}{D^2(zt)^{-1}}] + p_1 E[1000 \times \frac{a_{12}^2}{D^2(zt)^{-1}}]$

$+p_1 E[1000 \times \frac{a_{12}a_{22}}{D^2(zt)^{-1}}] + (p_1p_2 + p_1p_3)E[1000 \times \frac{a_{12}a_{23}}{D^2(zt)^{-1}}]$

$+(p_2 + p_3)E[1000 \times \frac{a_{13}a_{12}}{D^2(zt)^{-1}}] + (p_1p_2 + p_1p_3)E[1000 \times \frac{a_{13}a_{22}}{D^2(zt)^{-1}}] + (p_2 + p_3 + 2p_2p_3)E[1000 \times \frac{a_{13}a_{23}}{D^2(zt)^{-1}}]$

$= E[1000 \times \frac{1}{D^2(zt)^{-1}}[a_{11}a_{12} + p_1(a_{11}a_{22} + a_{12}a_{22} + a_{12}^2) + (p_2 + p_3)(a_{11}a_{23} + a_{13}a_{12}) + (p_1p_2 + p_1p_3)(a_{12}a_{23} + a_{13}a_{22}) + (p_2 + p_3 + 2p_2p_3)a_{13}a_{23}]]$

Let $U_5 = \sum_{j=1}^{500} \frac{2b_{j,1}zb_{j,3}s}{D^2}$

Then $E(U_5)$

$= 2\sum_{j=1}^{500} E[\frac{(a_{11}+a_{12}x_{1,j,test}+a_{13}(x_{2,j,test}+x_{3,j,test}))(a_{13}+a_{23}x_{1,j,test}+a_{33}(x_{2,j,test}+x_{3,j,test}))}{D^2(zs)^{-1}}]$

$= 2\sum_{j=1}^{500} E[\frac{a_{11}a_{13}}{D^2(zs)^{-1}}]$

$+2\sum_{j=1}^{500} E[\frac{a_{11}a_{23}x_{1,j,test}}{D^2(zs)^{-1}}]$

$+2\sum_{j=1}^{500} E[\frac{a_{11}a_{33}(x_{2,j,test} + x_{3,j,test})}{D^2(zs)^{-1}}]$

$+2\sum_{j=1}^{500} E[\frac{a_{12}x_{1,j,test}a_{13}}{D^2(zs)^{-1}}]$

$+2\sum_{j=1}^{500} E[\frac{a_{12}x_{1,j,test}a_{23}x_{1,j,test}}{D^2(zs)^{-1}}]$

$+2\sum_{j=1}^{500} E[\frac{a_{12}x_{1,j,test}a_{33}(x_{2,j,test} + x_{3,j,test})}{D^2(zs)^{-1}}]$

$+2\sum_{j=1}^{500} E[\frac{a_{13}(x_{2,j,test} + x_{3,j,test})a_{13}}{D^2(zs)^{-1}}]$

$+2\sum_{j=1}^{500} E[\frac{a_{13}(x_{2,j,test} + x_{3,j,test})a_{23}x_{1,j,test}}{D^2(zs)^{-1}}]$

$+2\sum_{j=1}^{500} E[\frac{a_{13}(x_{2,j,test} + x_{3,j,test})a_{33}(x_{2,j,test} + x_{3,j,test})}{D^2(zs)^{-1}}]$

$= E[1000 \times \frac{a_{11}a_{13}}{D^2(zs)^{-1}}]$

$+E(x_{1,1,test})E[1000 \times \frac{a_{11}a_{23}}{D^2(zs)^{-1}}]$

$+E(x_{2,1,test} + x_{3,1,test})E[1000 \times \frac{a_{11}a_{33}}{D^2(zs)^{-1}}]$

$+E(x_{1,1,test})E[1000 \times \frac{a_{12}a_{13}}{D^2(zs)^{-1}}]$

$+E(x_{1,1,test}^2)E[1000 \times \frac{a_{12}a_{23}}{D^2(zs)^{-1}}]$

$+E(x_{1,1,test}(x_{2,1,test} + x_{3,1,test}))E[1000 \times \frac{a_{12}a_{33}}{D^2(zs)^{-1}}]$

$+E(x_{2,1,test} + x_{3,1,test})E[1000 \times \frac{a_{13}^2}{D^2(zs)^{-1}}]$

$+E((x_{2,1,test} + x_{3,1,test})x_{1,1,test})E[1000 \times \frac{a_{13}a_{23}}{D^2(zs)^{-1}}]$

$+E((x_{2,1,test} + x_{3,1,test})^2)E[1000 \times \frac{a_{13}a_{33}}{D^2(zs)^{-1}}]$

$= E[1000 \times \frac{1}{D^2(zs)^{-1}}[a_{11}a_{13} + p_1a_{11}a_{23} + (p_2 + p_3)a_{11}a_{33} + p_1a_{12}a_{13} + p_1a_{12}a_{23} + (p_1p_2 + p_1p_3)a_{12}a_{33} + (p_2 + p_3)a_{13}^2 + (p_1p_2 + p_1p_3)a_{13}a_{23} + (p_2 + p_3 + 2p_2p_3)a_{13}a_{33}]$

$= E[1000 \times \frac{1}{D^2(zs)^{-1}}[a_{11}a_{13} + p_1(a_{11}a_{23} + a_{12}a_{13} + a_{12}a_{23}) + (p_2 + p_3)(a_{11}a_{33} + a_{13}^2) + (p_1p_2 + p_1p_3)(a_{12}a_{33} + a_{13}a_{23}) + (p_2 + p_3 + 2p_2p_3)a_{13}a_{33}]$

Let $U_6 = \sum_{j=1}^{500} \frac{2b_{j,2}tb_{j,3}s}{D^2}$

Then $E(U_6)$

$= 2\sum_{j=1}^{500} E[\frac{(a_{12}+a_{22}x_{1,j,test}+a_{23}(x_{2,j,test}+x_{3,j,test}))(a_{13}+a_{23}x_{1,j,test}+a_{33}(x_{2,j,test}+x_{3,j,test}))}{D^2(ts)^{-1}}]$

$= 2\sum_{j=1}^{500} E[\frac{a_{12}a_{13}}{D^2(ts)^{-1}}]$

$+2\sum_{j=1}^{500} E[\frac{a_{12}a_{23}x_{1,j,test}}{D^2(ts)^{-1}}]$

$+2\sum_{j=1}^{500} E[\frac{a_{12}a_{33}(x_{2,j,test} + x_{3,j,test})}{D^2(ts)^{-1}}]$

$+2\sum_{j=1}^{500} E[\frac{a_{22}x_{1,j,test}a_{13}}{D^2(ts)^{-1}}]$

$+2\sum_{j=1}^{500} E[\frac{a_{22}x_{1,j,test}a_{23}x_{1,j,test}}{D^2(ts)^{-1}}]$

$+2\sum_{j=1}^{500} E[\frac{a_{22}x_{1,j,test}a_{33}(x_{2,j,test} + x_{3,j,test})}{D^2(ts)^{-1}}]$

$+2\sum_{j=1}^{500} E[\frac{a_{23}(x_{2,j,test} + x_{3,j,test})a_{13}}{D^2(ts)^{-1}}]$

$+2\sum_{j=1}^{500} E[\frac{a_{23}(x_{2,j,test} + x_{3,j,test})a_{23}x_{1,j,test}}{D^2(ts)^{-1}}]$

$+2\sum_{j=1}^{500} E[\frac{a_{23}(x_{2,j,test} + x_{3,j,test})a_{33}(x_{2,j,test} + x_{3,j,test})}{D^2(ts)^{-1}}]$

$= E[1000 \times \frac{a_{12}a_{13}}{D^2(ts)^{-1}}]$

$+E(x_{1,1,test})E[1000 \times \frac{a_{12}a_{23}}{D^2(ts)^{-1}}]$

$+E((x_{2,1,test} + x_{3,1,test}))E[1000 \times \frac{a_{12}a_{33}}{D^2(ts)^{-1}}]$

$+E(x_{1,1,test})E[1000 \times \frac{a_{22}a_{13}}{D^2(ts)^{-1}}]$

$+E(x_{1,1,test}^2)E[1000 \times \frac{a_{22}a_{23}}{D^2(ts)^{-1}}]$

$+E(x_{1,1,test}(x_{2,1,test} + x_{3,1,test}))E[1000 \times \frac{a_{22}a_{33}}{D^2(ts)^{-1}}]$

$+E(x_{2,1,test} + x_{3,1,test})E[1000 \times \frac{a_{23}a_{13}}{D^2(ts)^{-1}}]$

$+E((x_{2,1,test} + x_{3,1,test})x_{1,1,test})E[1000 \times \frac{a_{23}^2}{D^2(ts)^{-1}}]$

$+E((x_{2,1,test} + x_{3,1,test})^2)E[1000 \times \frac{a_{23}a_{33}}{D^2(ts)^{-1}}]$

$= E[1000 \times \frac{1}{D^2(ts)^{-1}}[a_{12}a_{13} + p_1a_{12}a_{23} + (p_2 + p_3)a_{12}a_{33} + p_1a_{22}a_{13} + p_1a_{22}a_{23} + (p_1p_2 + p_1p_3)a_{22}a_{33} + (p_2 + p_3)a_{23}a_{13} + (p_1p_2 + p_1p_3)a_{23}^2 + (p_2 + p_3 + 2p_2p_3)a_{23}a_{33}]$

$= E[1000 \times \frac{1}{D^2(ts)^{-1}}[a_{12}a_{13} + p_1(a_{12}a_{23} + a_{22}a_{13} + a_{22}a_{23}) + (p_2 + p_3)(a_{12}a_{33} + a_{23}a_{13}) + (p_1p_2 + p_1p_3)(a_{22}a_{33} + a_{23}^2) + (p_2 + p_3 + 2p_2p_3)a_{23}a_{33}]$

Now suppose $g_3 = \frac{1}{2D^2z^{-2}}[a_{11}^2 + p_1a_{12}^2 + (p_2 + p_3 + 2p_2p_3)a_{13}^2 + 2p_1a_{11}a_{12} + 2(p_2 + p_3)a_{11}a_{13} + 2(p_1p_2 + p_1p_3)a_{12}a_{13}] + \frac{1}{2D^2t^{-2}}[a_{12}^2 + p_1a_{22}^2 + (p_2 + p_3 + 2p_2p_3)a_{23}^2 + 2p_1a_{12}a_{22} + 2(p_2 + p_3)a_{12}a_{23} + 2(p_1p_2 + p_1p_3)a_{22}a_{23}] + \frac{1}{2D^2s^{-2}}[a_{13}^2 + p_1a_{23}^2 + (p_2 + p_3 + 2p_2p_3)a_{33}^2 + 2p_1a_{13}a_{23} + 2(p_2 + p_3)a_{13}a_{33} + 2p_1(p_2 + p_3)a_{23}a_{33}] + \frac{1}{D^2(zt)^{-1}}[a_{11}a_{12} + p_1(a_{11}a_{22} + a_{12}a_{22} + a_{12}^2) + (p_2 + p_3)(a_{11}a_{23} + a_{13}a_{12}) + (p_1p_2 + p_1p_3)(a_{12}a_{23} + a_{13}a_{22}) + (p_2 + p_3 + 2p_2p_3)a_{13}a_{23}] + \frac{1}{D^2(zs)^{-1}}[a_{11}a_{13} + p_1(a_{11}a_{23} + a_{12}a_{13} + a_{12}a_{23}) + (p_2 + p_3)(a_{11}a_{33} + a_{13}^2) + (p_1p_2 + p_1p_3)(a_{12}a_{33} + a_{13}a_{23}) + (p_2 + p_3 + 2p_2p_3)a_{13}a_{33}] + \frac{1}{D^2(ts)^{-1}}[a_{12}a_{13} + p_1(a_{12}a_{23} + a_{22}a_{13} + a_{22}a_{23}) + (p_2 + p_3)(a_{12}a_{33} + a_{23}a_{13}) + (p_1p_2 + p_1p_3)(a_{22}a_{33} + a_{23}^2) + (p_2 + p_3 + 2p_2p_3)a_{23}a_{33}]$.

Now, for convenience in calculation, express g_3 as the sum of the following six functions $g_{3k},\ k = 1, ..., 6$, where $a = p_1,\ b = p_2,\ c = p_3$ in the expressions of $g_{3k},\ k = 1, ..., 6$ and $h_{ij}^{(U_k)},\ i, j = 1, ..., 10, j \geqslant i,\ k = 1, ..., 6$.

$g_{31} = \frac{1}{2D^2z^{-2}}[a_{11}^2 + p_1a_{12}^2 + (p_2 + p_3 + 2p_2p_3)a_{13}^2 + 2p_1a_{11}a_{12} + 2(p_2 + p_3)a_{11}a_{13} + 2(p_1p_2 + p_1p_3)a_{12}a_{13}]$

$= \frac{1}{2[yl + 2x(m+n)(p+q) - (m+n)^2 - (p+q)^2y - x^2l]^2z^{-2}}[(yl - (m+n)^2)^2 + a((p+q)(m+n) - xl)^2 + (b+c+2bc)(x(m+n) - (p+q)y)^2 + 2a(yl - (m+n)^2)((p+q)(m+n) - xl) + 2(b+c)(yl - (m+n)^2)(x(m+n) - (p+q)y) + 2(ab+ac)((p+q)(m+n) - xl)(x(m+n) - (p+q)y)]$

$g_{32} = \frac{1}{2D^2t^{-2}}[a_{12}^2 + p_1a_{22}^2 + (p_2 + p_3 + 2p_2p_3)a_{23}^2 + 2p_1a_{12}a_{22} + 2(p_2 + p_3)a_{12}a_{23} + 2(p_1p_2 + p_1p_3)a_{22}a_{23}]$

$= \frac{1}{2[yl + 2x(m+n)(p+q) - (m+n)^2 - (p+q)^2y - x^2l]^2t^{-2}}[((p+q)(m+n) - xl)^2 + a(l - (p+q)^2)^2 + (b+c+2bc)((p+q)x - (m+n))^2 + 2a((p+q)(m+n) - xl)(l - (p+q)^2) + 2(b+c)((p+q)(m+n) - xl)((p+q)x - (m+n)) + 2(ab+ac)(l - (p+q)^2)((p+q)x - (m+n))]$

$g_{33} = \frac{1}{2D^2s^{-2}}[a_{13}^2 + p_1a_{23}^2 + (p_2 + p_3 + 2p_2p_3)a_{33}^2 + 2p_1a_{13}a_{23} + 2(p_2 + p_3)a_{13}a_{33} + 2p_1(p_2 + p_3)a_{23}a_{33}]$

$= \frac{1}{2[yl + 2x(m+n)(p+q) - (m+n)^2 - (p+q)^2y - x^2l]^2s^{-2}}[(x(m+n) - (p+q)y)^2 + a((p+q)x - (m+n))^2 + (b+c+2bc)(y - x^2)^2 + 2a(x(m+n) - (p+q)y)((p+q)x - (m+n)) + 2(b+c)(x(m+n) - (p+q)y)(y - x^2) + 2a(b+c)((p+q)x - (m+n))(y - x^2)]$

$g_{34} = \frac{1}{D^2(zt)^{-1}}[a_{11}a_{12} + p_1(a_{11}a_{22} + a_{12}a_{22} + a_{12}^2) + (p_2 + p_3)(a_{11}a_{23} + a_{13}a_{12}) + (p_1p_2 + p_1p_3)(a_{12}a_{23} + a_{13}a_{22}) + (p_2 + p_3 + 2p_2p_3)a_{13}a_{23}]$

$= \frac{1}{[yl + 2x(m+n)(p+q) - (m+n)^2 - (p+q)^2y - x^2l]^2(zt)^{-1}}[(yl - (m+n)^2)((p+q)(m+n) - xl) + a((yl - (m+n)^2)(l - (p+q)^2) + ((p+q)(m+n) - xl)(l - (p+q)^2) + ((p+q)(m+n) - xl)^2) + (b+c)((yl - (m+n)^2)((p+q)x - (m+n)) + (x(m+n) - (p+q)y)((p+q)(m+n) - xl)) + (ab+ac)(((p+q)(m+n) - xl)((p+q)x - (m+n)) + (x(m+n) - (p+q)y)(l - (p+q)^2)) + (b + c + 2bc)(x(m+n) - (p+q)y)((p+q)x - (m+n))]$

$g_{35} = \frac{1}{D^2(zs)^{-1}}[a_{11}a_{13} + p_1(a_{11}a_{23} + a_{12}a_{13} + a_{12}a_{23}) + (p_2 + p_3)(a_{11}a_{33} + a_{13}^2) + (p_1p_2 + p_1p_3)(a_{12}a_{33} + a_{13}a_{23}) + (p_2 + p_3 + 2p_2p_3)a_{13}a_{33}]$

$= \frac{1}{[yl + 2x(m+n)(p+q) - (m+n)^2 - (p+q)^2y - x^2l]^2(zs)^{-1}}[(yl-(m+n)^2)(x(m+n) - (p+q)y) + a((yl - (m+n)^2)((p+q)x - (m+n)) + ((p+q)(m+n) - xl)(x(m+n) - (p+q)y) + ((p+q)(m+n) - xl)((p+q)x - (m+n))) + (b+c)((yl - (m+n)^2)(y - x^2) + (x(m+n) - (p+q)y)^2) + (ab+ac)(((p+q)(m+n) - xl)(y - x^2) + (x(m+n) - (p+q)y)((p+q)x - (m+n))) + (b + c + 2bc)(x(m+n) - (p+q)y)(y - x^2)]$

$g_{36} = \frac{1}{D^2(ts)^{-1}}[a_{12}a_{13} + p_1(a_{12}a_{23} + a_{22}a_{13} + a_{22}a_{23}) + (p_2 + p_3)(a_{12}a_{33} + a_{23}a_{13}) + (p_1p_2 + p_1p_3)(a_{22}a_{33} + a_{23}^2) + (p_2 + p_3 + 2p_2p_3)a_{23}a_{33}]$

$= \frac{1}{[yl + 2x(m+n)(p+q) - (m+n)^2 - (p+q)^2y - x^2l]^2(ts)^{-1}}[((p+q)(m+n) - xl)(x(m+n) - (p+q)y) + a(((p+q)(m+n) - xl)((p+q)x - (m+n)) + (l - (p+q)^2)(x(m+n) - (p+q)y) + (l - (p+q)^2)((p+q)x - (m+n))) + (b+c)(((p+q)(m+n) - xl)(y - x^2) + ((p+q)x - (m+n))(x(m+n) - (p+q)y)) + (ab+ac)((l - (p+q)^2)(y - x^2) + ((p+q)x - (m+n))^2) + (b + c + 2bc)((p+q)x - (m+n))(y - x^2)]$

Then $h_{ij}^{(U)} = \frac{\partial^2 g_3(\mu_1,...,\mu_{10})}{\partial x_i \partial x_j} = \Sigma_{k=1}^6 h_{ij}^{(U_k)} = \Sigma_{k=1}^6 \frac{\partial^2 g_{3k}(\mu_1,...,\mu_{10})}{\partial x_i \partial x_j},\ i \leq j,\ i,j = 1,...,10$

$h_{ij}^{(U_k)}, i \leq j,\ i,j = 1,...,10,\ k = 1,...,6$ are given in appendix with $x = \mu_1,\ y = \mu_2,\ z = \mu_3,\ t = \mu_4,\ l = \mu_5,\ m = \mu_6,\ n = \mu_7,\ p = \mu_8,\ q = \mu_9,\ s = \mu_{10}$.

Further, assume $\mu_i,\ i = 1,...,10$ satisfies that the denominators in $h_{ij}^{(U)},\ i,j = 1,...,10, j \geqslant i$ are not equal to zero as well as $h_{ii}^{(U)} \neq \sum_{j=1,i\neq j}^{10} h_{ij}^{(U)},\ i \leq j,\ i = 1,...,10$.

Thus, $E(U) \sim \sum_{i=1}^{10} \sigma_i^2 \frac{h_{ii}^{(U)}}{2} + \sum_{i,j=1,i>j}^{10} \sigma_{ij} h_{ij}^{(U)}$

$E(W) = E(\sum_{j=1}^{500}(-2)y_{j,test}\frac{b_{j,1}z + b_{j,2}t + b_{j,3}s}{D})$

$= (-2)\sum_{j=1}^{500} E[\frac{y_{j,test}}{Dz^{-1}}(a_{11} + a_{12}x_{1,j,test} + a_{13}(x_{2,j,test} + x_{3,j,test}))$

$+ \dfrac{y_{j,test}}{Dt^{-1}}(a_{12} + a_{22}x_{1,j,test} + a_{23}(x_{2,j,test} + x_{3,j,test}))$
$+ \dfrac{y_{j,test}}{Ds^{-1}}(a_{13} + a_{23}x_{1,j,test} + a_{33}(x_{2,j,test} + x_{3,j,test}))]$

$= (-2)\sum_{j=1}^{500} E(y_{j,test})E[\dfrac{a_{11}}{Dz^{-1}}]$
$+(-2)\sum_{j=1}^{500} E(y_{j,test}x_{1,j,test})E[\dfrac{a_{12}}{Dz^{-1}}]$
$+(-2)\sum_{j=1}^{500} E(y_{j,test}(x_{2,j,test} + x_{3,j,test}))E[\dfrac{a_{13}}{Dz^{-1}}]$
$+(-2)\sum_{j=1}^{500} E(y_{j,test})E[\dfrac{a_{12}}{Dt^{-1}}]$
$+(-2)\sum_{j=1}^{500} E(y_{j,test}x_{1,j,test})E[\dfrac{a_{22}}{Dt^{-1}}]$
$+(-2)\sum_{j=1}^{500} E(y_{j,test}(x_{2,j,test} + x_{3,j,test}))E[\dfrac{a_{23}}{Dt^{-1}}]$
$+(-2)\sum_{j=1}^{500} E(y_{j,test})E[\dfrac{a_{13}}{Ds^{-1}}]$
$+(-2)\sum_{j=1}^{500} E(y_{j,test}x_{1,j,test})E[\dfrac{a_{23}}{Ds^{-1}}]$
$+(-2)\sum_{j=1}^{500} E(y_{j,test}(x_{2,j,test} + x_{3,j,test}))E[\dfrac{a_{33}}{Ds^{-1}}]$

$= -E(y_{1,test})E[1000 \times \dfrac{a_{11}}{Dz^{-1}}]$
$-E(y_{1,test}x_{1,1,test})E[1000 \times \dfrac{a_{12}}{Dz^{-1}}]$
$-E(y_{1,test}(x_{2,1,test} + x_{3,1,test}))E[1000 \times \dfrac{a_{13}}{Dz^{-1}}]$
$-E(y_{1,test})E[1000 \times \dfrac{a_{12}}{Dt^{-1}}]$
$-E(y_{1,test}x_{1,1,test})E[1000 \times \dfrac{a_{22}}{Dt^{-1}}]$
$-E(y_{1,test}(x_{2,1,test} + x_{3,1,test}))E[1000 \times \dfrac{a_{23}}{Dt^{-1}}]$
$-E(y_{1,test})E[1000 \times \dfrac{a_{13}}{Ds^{-1}}]$
$-E(y_{1,test}x_{1,1,test})E[1000 \times \dfrac{a_{23}}{Ds^{-1}}]$
$-E(y_{1,test}(x_{2,1,test} + x_{3,1,test}))E[1000 \times \dfrac{a_{33}}{Ds^{-1}}]$

$= -\mu_3 E[1000 \times \dfrac{a_{11}}{Dz^{-1}}] - \mu_4 E[1000 \times \dfrac{a_{12}}{Dz^{-1}}] - \mu_{10} E[1000 \times \dfrac{a_{13}}{Dz^{-1}}]$
$-\mu_3 E[1000 \times \dfrac{a_{12}}{Dt^{-1}}] - \mu_4 E[1000 \times \dfrac{a_{22}}{Dt^{-1}}] - \mu_{10} E[1000 \times \dfrac{a_{23}}{Dt^{-1}}]$
$-\mu_3 E[1000 \times \dfrac{a_{13}}{Ds^{-1}}] - \mu_4 E[1000 \times \dfrac{a_{23}}{Ds^{-1}}] - \mu_{10} E[1000 \times \dfrac{a_{33}}{Ds^{-1}}]$
$= E[-1000 \times \frac{\mu_3(a_{11}z+a_{12}t+a_{13}s)+\mu_4(a_{12}z+a_{22}t+a_{23}s)+\mu_{10}(a_{13}z+a_{23}t+a_{33}s)}{D}]$

Now suppose $g_4 = -\frac{\mu_3(a_{11}z+a_{12}t+a_{13}s)+\mu_4(a_{12}z+a_{22}t+a_{23}s)+\mu_{10}(a_{13}z+a_{23}t+a_{33}s)}{D}$.

Further, assume $\mu_i,\ i = 1, ..., 10$, satisfies that the denominators in $h_{ij}^{(W)},\ i, j = 1, ..., 10, j \geqslant i$, are not equal to zero as well as $h_{ii}^{(W)} \neq \sum_{j=1, i\neq j}^{10} h_{ij}^{(W)},\ i = 1, ..., 10$.

For notation convenience, let $a = \mu_3,\ b = \mu_4,\ c = \mu_{10}$ in the expressions of $h_{ij}^{(W)},\ i, j = 1, ..., 10, j \geqslant i$. Then

$g_4 = -\frac{1}{yl + 2x(m+n)(p+q) - (m+n)^2 - (p+q)^2 y - x^2 l}[a((yl - (m+n)^2)z + ((p+q)(m+n) - xl)t + (x(m+n) - (p+q)y)s) + b(((p+q)(m+n) - xl)z + (l - (p+q)^2)t + ((p+q)x - (m+n))s) + c((x(m+n) - (p+q)y)z + ((p+q)x - (m+n))t + (y - x^2)s)]$,

$h_{ij}^{(W)}, i \leq j,\ i,j = 1, ..., 10,\ k = 1, ..., 6$ are given in appendix with $x = \mu_1,\ y = \mu_2,\ z = \mu_3,\ t = \mu_4,\ l = \mu_5,\ m = \mu_6,\ n = \mu_7,\ p = \mu_8,\ q = \mu_9,\ s = \mu_{10}$.

Thus, $E(W) \sim \sum_{i=1}^{10} \sigma_i^2 \frac{h_{ii}^{(W)}}{2} + \sum_{i,j=1,i>j}^{10} \sigma_{ij} h_{ij}^{(W)}$.

Therefore, asymptotic estimation of $MSPE$ of 2-dimensional model fitted by single variable X_1 and $X_2 + X_3$ is

$\sum_{i=1}^{10} \sigma_i^2 \frac{h_{ii}^{(U)}}{2} + \sum_{i,j=1,i>j}^{10} \sigma_{ij} h_{ij}^{(U)} + E(V) + \sum_{i=1}^{10} \sigma_i^2 \frac{h_{ii}^{(W)}}{2} + \sum_{i,j=1,i>j}^{10} \sigma_{ij} h_{ij}^{(W)}$,

where $\mu_1,\ \mu_2,\ ...,\ \mu_{10}$ satisfies that the denominators in $h_{ii}^{(U)},\ h_{ij}^{(W)},\ i,j = 1, ..., 10,\ j \geqslant i$ are not equal to zero and $h_{ii}^{(U)} \neq \sum_{j=1,i\neq j}^{10} h_{ij}^{(U)},\ h_{ii}^{(W)} \neq \sum_{j=1,i\neq j}^{10} h_{ij}^{(W)},\ i = 1, ..., 10$, and $\mu_i,\ \sigma_{ij},\ i,j = 1, ..., 4,\ i \leq j$ are calculated on page 83 and page 84 of this thesis. $\mu_i,\ \sigma_{ij},\ i,j = 5, ..., 10,\ i \leq j$ and $\sigma_{ij},\ i,j = 5, ..., 10,\ 1 \leq i \leq 4,\ 5 \leq j \leq 10$ are calculated on page 92-96 of this thesis.

Last, we calculate the asymptotically estimated MSPE of the 1-dimensional model fitted by X_1 and the asymptotically estimated MSPE of the 2-dimensional model fitted by $X_1,\ X_2 + X_3$.

Assume the true model to generate the training data with sample points 1000 and testing data with sample points 500 is $Y = X_1 + X_2 + X_3 + \epsilon,\ \epsilon \sim N(0, 1.3^2)$, $X_1 \sim Ber(0.71)$, $X_2 \sim Ber(0.39)$, $X_3 \sim Ber(0.23)$ and $\epsilon,\ X_1,\ X_2,\ X_3$ are mutually independent. Thus, $\beta_0 = 0,\ \beta_1 = 1,\ \beta_2 = 1,\ \beta_3 = 1,\ \sigma = 1.3,\ p_1 = 0.71,\ p_2 = 0.39,\ p_3 = 0.23$,

Then $\mu_i,\ \sigma_{ij},\ i,j = 1, ..., 10,\ i \leq j$ on page 83-84 and 92-96 are with the following value.

$$\begin{pmatrix} \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \mu_5 \\ \mu_6 \\ \mu_7 \\ \mu_8 \\ \mu_9 \\ \mu_{10} \end{pmatrix} = \begin{pmatrix} 0.7100 \\ 0.7100 \\ 1.3300 \\ 1.1502 \\ 0.7994 \\ 0.2769 \\ 0.1633 \\ 0.3900 \\ 0.2300 \\ 1.2396 \end{pmatrix}$$

,

$$\begin{pmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} & \sigma_{14} & \sigma_{15} & \sigma_{16} & \sigma_{17} & \sigma_{18} & \sigma_{19} & \sigma_{1\ 10} \\ \sigma_{12} & \sigma_2^2 & \sigma_{23} & \sigma_{24} & \sigma_{25} & \sigma_{26} & \sigma_{27} & \sigma_{28} & \sigma_{29} & \sigma_{2\ 10} \\ \sigma_{13} & \sigma_{23} & \sigma_3^2 & \sigma_{34} & \sigma_{35} & \sigma_{36} & \sigma_{37} & \sigma_{38} & \sigma_{39} & \sigma_{3\ 10} \\ \sigma_{14} & \sigma_{24} & \sigma_{34} & \sigma_4^2 & \sigma_{45} & \sigma_{46} & \sigma_{47} & \sigma_{48} & \sigma_{49} & \sigma_{4\ 10} \\ \sigma_{15} & \sigma_{25} & \sigma_{35} & \sigma_{45} & \sigma_5^2 & \sigma_{56} & \sigma_{57} & \sigma_{58} & \sigma_{59} & \sigma_{5\ 10} \\ \sigma_{16} & \sigma_{26} & \sigma_{36} & \sigma_{46} & \sigma_{56} & \sigma_6^2 & \sigma_{67} & \sigma_{68} & \sigma_{69} & \sigma_{6\ 10} \\ \sigma_{17} & \sigma_{27} & \sigma_{37} & \sigma_{47} & \sigma_{57} & \sigma_{67} & \sigma_7^2 & \sigma_{78} & \sigma_{79} & \sigma_{7\ 10} \\ \sigma_{18} & \sigma_{28} & \sigma_{38} & \sigma_{48} & \sigma_{58} & \sigma_{68} & \sigma_{78} & \sigma_8^2 & \sigma_{89} & \sigma_{8\ 10} \\ \sigma_{19} & \sigma_{29} & \sigma_{39} & \sigma_{49} & \sigma_{59} & \sigma_{69} & \sigma_{79} & \sigma_{89} & \sigma_9^2 & \sigma_{9\ 10} \\ \sigma_{1\ 10} & \sigma_{2\ 10} & \sigma_{3\ 10} & \sigma_{4\ 10} & \sigma_{5\ 10} & \sigma_{6\ 10} & \sigma_{7\ 10} & \sigma_{8\ 10} & \sigma_{9\ 10} & \sigma_{10}^2 \end{pmatrix} =$$

$$\begin{pmatrix} 0.2059 & 0.2059 & 0.2059 & 0.3336 & 0.0000 & 0.08037 & 0.0474 & 0.0000 & 0.0000 & 0.1277 \\ 0.2059 & 0.2059 & 0.2059 & 0.33367 & 0.0000 & 70.0803 & 0.0474 & 0.0000 & 0.0000 & 0.1277 \\ 0.2059 & 0.2059 & 2.3109 & 1.8281 & 0.6626 & 0.24927 & 0.1731 & 0.2379 & 0.1771 & -0.9017 \\ 0.3336 & 0.3336 & 1.8281 & 1.7427 & 70.4704 & 70.2990 & 0.2025 & 0.1689 & 0.1257 & -0.7308 \\ 0.0000 & 0.0000 & 0.6626 & 0.4704 & 1.2368 & 0.24667 & 0.2238 & 0.3473 & 0.3152 & 1.7072 \\ 0.0803 & 0.0803 & 0.2492 & 0.2990 & 0.2466 & 0.2002 & 0.0185 & 0.1689 & 0.0000 & 3.2241 \\ 0.0474 & 0.0474 & 0.1731 & 70.2025 & 0.2238 & 0.0185 & 0.1366 & 0.0000 & 0.1257 & 0.3789 \\ 0.0000 & 0.0000 & 0.2379 & 0.1689 & 0.3473 & 0.1689 & 0.0000 & 0.2379 & 0.0000 & 0.5162 \\ 0.0000 & 0.0000 & 0.1771 & 0.1257 & 0.3152 & 0.0000 & 0.1257 & 0.0000 & 0.1771 & 0.4410 \\ 0.1277 & 0.1277 & -0.9017 & -0.7308 & 1.7072 & 3.2241 & 0.3789 & 0.5162 & 0.4410 & 0.8028 \end{pmatrix}$$

Then $h_{ij}^{(A)},\ i,j=1,...,4$ on page 87, $h_{ij}^{(C)},\ i,j=1,...,4$ on page 89, $h_{ij}^{(U)},\ i,j=1,...,10$ on page 103, $h_{ij}^{(W)},\ i,j=1,...,10$ on page 105 are the following,

$$H^{(A)}=\left(h_{ij}^{(A)})\right)=\begin{pmatrix} -1213.5 & 986.22 & -68.499 & 1.7484 \\ 986.22 & -728.83 & 77.051 & -19.427 \\ -68.499 & 77.051 & 6.8966 & -6.8966 \\ 1.7484 & -19.427 & -6.8966 & 9.7135 \end{pmatrix},$$

$$H^{(C)}=\left(h_{ij}^{(C)})\right)=\begin{pmatrix} -2.0787 & 0.8742 & 1.3103 & -0.4371 \\ 0.8742 & -9.7135 & -3.4483 & 4.8567 \\ 1.3103 & -3.4483 & 0 & 0 \\ -0.4371 & 4.8567 & 0 & 0 \end{pmatrix},$$

$$H^{(U)}=\left(h_{ij}^{(U)})\right)=$$

$$\begin{pmatrix} 35.7 & -27.081 & -19.443 & 23.632 & -17.711 & -60.13 & -60.13 & 63.298 & 63.298 & 4.4924 \\ -27.081 & 14.57 & 6.8966 & -9.7135 & 0 & 52.687 & 52.687 & -37.408 & -37.408 & 0 \\ -19.443 & 6.8966 & 3.1859 & -3.4483 & 16.217 & 38.452 & 38.452 & -47.041 & -47.041 & -2.9984 \\ 23.632 & -9.7135 & -3.4483 & 4.8567 & 0 & -47.83 & -47.83 & 33.959 & 33.959 & 0 \\ -17.711 & 0 & 16.217 & 0 & 52.997 & 34.086 & 34.086 & -37.74 & -37.74 & -31.676 \\ -60.13 & 52.687 & 38.452 & -47.83 & 34.086 & 105.98 & 105.98 & -90.485 & -90.485 & -12.765 \\ -60.13 & 52.687 & 38.452 & -47.83 & 34.086 & 105.98 & 105.98 & -90.485 & -90.485 & -12.765 \\ 63.298 & -37.408 & -47.041 & 33.959 & -37.74 & -90.485 & -90.485 & 90.896 & 90.896 & 24.521 \\ 63.298 & -37.408 & -47.041 & 33.959 & -37.74 & -90.485 & -90.485 & 90.896 & 90.896 & 24.521 \\ 4.4924 & 0 & -2.9984 & 0 & -31.676 & -12.765 & -12.765 & 24.521 & 24.521 & 10.355 \end{pmatrix},$$

$$H^{(W)}=\left(h_{ij}^{(W)})\right)=$$

$$\begin{pmatrix} -8.7491 & 6.8966 & 4.3745 & -3.4483 & 2.988 & 9.8845 & 9.8845 & -8.7491 & -8.7491 & -1.494 \\ 6.8966 & -9.7135 & -3.4483 & 4.8567 & 0 & -9.7135 & -9.7135 & 6.8966 & 6.8966 & 0 \\ 4.3745 & -3.4483 & 0 & 0 & -1.494 & -4.9423 & -4.9423 & 4.3745 & 4.3745 & 0 \\ -3.4483 & 4.8567 & 0 & 0 & 0 & 4.8567 & 4.8567 & -3.4483 & -3.4483 & 0 \\ 2.988 & 0 & -1.494 & 0 & -4.8193 & -4.8193 & -4.8193 & 2.988 & 2.988 & 2.4096 \\ 9.8845 & -9.7135 & -4.9423 & 4.8567 & -4.8193 & -14.533 & -14.533 & 9.8845 & 9.8845 & 2.4096 \\ 9.8845 & -9.7135 & -4.9423 & 4.8567 & -4.8193 & -14.533 & -14.533 & 9.8845 & 9.8845 & 2.4096 \\ -8.7491 & 6.8966 & 4.3745 & -3.4483 & 2.988 & 9.8845 & 9.8845 & -8.7491 & -8.7491 & -1.494 \\ -8.7491 & 6.8966 & 4.3745 & -3.4483 & 2.988 & 9.8845 & 9.8845 & -8.7491 & -8.7491 & -1.494 \\ -1.494 & 0 & 0 & 0 & 2.4096 & 2.4096 & 2.4096 & -1.494 & -1.494 & 0 \end{pmatrix}.$$

Next we obtain the asymptotic estimation of $E(A)$, $E(C)$, $E(U)$, $E(W)$, by page 87, $E(A)\sim 2.7906$, by page 89, $E(C)\sim 9.345621e-17\approx 0$, by page 103, $E(U)\sim -37.5539$, by page 105, $E(W)\sim 6.6478$.

By page 84 and page 92, $E(B) = E(V) = 2039.9$.

Thus, the asymptotic estimation of 1-dimensional model fitted by X_1 is $E(A) + E(B) + E(C) = 2042.691$, which is larger than the asymptotic estimation of 2-dimensional model fitted by X_1, $X_2 + X_3$ because $E(U) + E(V) + E(W) = 2008.994$.

Chapter 5 Application to Alzheimer's Disease Data

5.1 Introduction to Alzheimer's Disease and Research

Being a major type of dementia, Alzheimer's Disease(AD) is a neurodegenerative disease caused by brain cell death[27], whose patients at the beginning will experience cognitive impairment such as vision issues, defects in reasoning and judgment, then gradually suffer symptoms including memory loss, inability to recognize families and even difficulties to communicate or swallow[28]. The progress of AD is irreversible and no effective medical treatment has been found to prevent the worsening of AD.[28] In 2017, about 5.5 million Americans are with AD, which is the sixth dominant cause of death in US.[29]

AD types are classified as early-onset and late-onset, in both of which genetic components are discovered by scientists. While early-onset AD is a rare AD type that usually develops between a patient's 30s and 60s, late-onset is the most common AD type occurring around a patient's 60s and possibly involving apolipoprotein E4 (APOE4) gene on chromosome 19 as a risk factor. The APOE4 allele number for each person could be 0, 1, and 2, but a larger APOE4 allele number increases the person's risk to get AD.[30]

Since AD always presents with brain abnormalities, biological markers including cerebrospinal fluid (CSF) tau protein and β-amyloid (Aβ) peptide are measured to reflect the brain's pathologic change in order to diagnose AD for patients at mild cognitive impairment status, where high CSF tau concentration and low CSF Aβ are regarded as specific and sensitive biomarkers in predicting AD's progressing, respectively.[31]

Depending on previous analysis on examining susceptibility loci for late-onset AD[32], as well as genes that might have association with CSF Aβ[33] or CSF tau[34, 35, 36], Crouch considered 23 SNPs and examined the significance of all the two-way and three-way SNP-SNP interactions involving APOE4 for CSF tau and CSF

Aβ with AQMDR(Aggregated Quantitative Multifactor-Dimensionality Reduction) and QMDR(Quantitative Multifactor-Dimensionality Reduction), for AD status normal control(NL), mild cognitive impairment(MCI) and mild Alzheimer's disease(AD), respectively, using those subjects whose baseline measurements of CSF tau and CSF Aβ are available in Alzheimer's Disease Neuroimaging Initiative (ADNI) data[21][37]. The results about significant SNP-SNP interactions for CSF Aβ and CSF tau, which are from Crouch 2016[21], are listed below in Table 5.1 and Table 5.2 where the interactions chosen by AQMDR are significant at 0.1 significance level. The significant "High Risk" genotypes in Table 5.3, Table 5.5, Table 5.7, Table 5.9, are also from Crouch 2016[21].

Table 5.1: 2-way & 3-way gene-gene interaction candidates for CSF Aβ measurement

2-way & 3-way gene-gene interaction candidates for CSF Aβ measurement		
Stratification	AQMDR Significant Interaction	QMDR Optimal Interaction
NL	(APOE4, CR1, SLC24A4) (APOE4, BIN1, SLC24A4)	(APOE4, BIN1, PICALM)
MCI	(APOE4, PTK2B) (APOE4, PTK2B, MAPT) (APOE4, PTK2B, TREM2) (APOE4, PTK2B, NME8) (APOE4, PTK2B, ZCWPW1) (APOE4, PTK2B, CELF1) (APOE4, PTK2B, SORL1) (APOE4, PTK2B, FERMT2) (APOE4, PTK2B, PSEN1) (APOE4, PTK2B, DSG2) (APOE4, PTK2B, ABCA7) (APOE4, PTK2B, CD33) (APOE4, PTK2B, CASS4)	(APOE4, PTK2B)
AD	–	(APOE4, TREM2)

Table 5.2: 2-way & 3-way gene-gene interaction candidates for CSF tau measurement

2-way & 3-way gene-gene interaction candidates for CSF tau measurement		
Stratification	AQMDR Significant Interaction	QMDR Optimal Interaction
NL	(APOE4, ABCA7) (APOE4, CD2AP, PICALM) (APOE4 , ABCA7, NME8)	(APOE4, ABCA7)
MCI	(APOE4, CR1, MS4A6A)	(APOE4, CR1, MS4A6A)
AD	(APOE4, MEF2C, CD2AP) (APOE4, TREM2, ZCWPW1) (APOE4, MAPT, CELF1)	(APOE4, MAPT, CELF1)

In the above two tables, (Gene1, Gene2, Gene3) represents the three-way interaction among gene1, gene2 and gene3, (Gene1, Gene2) represents the two-way interaction between gene1 and gene2. The table on gene-gene interaction candidate of CSF Aβ measurement displays that, combining results from AQMDR and QMDR, there are 3 interaction candidates for normal control status, 13 interaction candidate for mild cognitive impairment status and only 1 interaction candidates for mild Alzheimer's disease status. The table on gene-gene interaction candidate on CSF tau measurement displays that, combining results from AQMDR and QMDR, there are 3 interaction candidates for normal control status, 1 interaction candidate for mild cognitive impairment status and 3 interaction candidates for mild Alzheimer's disease status.

Depending on results shown in these above two tables, in next section, we will generate a set of three risk scores to predict measurements of CSF Aβ for MCI status, a set of two risk scores to predict measurements of CSF Aβ for N status, and achieve a set of two risk scores to predict CSF tau level for NL status and AD status, respectively. The reason to conduct analysis by stratification on AD status is that the population differs from each other in different AD status[21]. The residuals after regressing CSF tau and CSF Aβ on variables age, gender and education are taken as the response variable, noted as Y_{tau}, $Y_{A\beta}$, respectively[21]. NL, MCI or AD can be added to the subscripts to indicate which status the response variable is from. For illustration, $Y_{tau,NL}$ represents those response variables Y_{tau} that belong to normal control status.

5.2 Study Method and Results

The sample in this study involves 626 subjects whose disease status has been diagnosed to be stable in visits with 199 subjects in normal control status, 327 subjects in mild cognitive impairment status and 100 subjects in mild AD status. The risk score generated from each SNP-SNP interaction is built independently in every disease status stratification with the subjects whose all interaction involved SNPs are observed without any missing value. The risk scores are used to predict the new response variables Y_{tau}^{new}, $Y_{A\beta}^{new}$, which are the residuals after regressing Y_{tau}, $Y_{A\beta}$ on all the SNP factors as covariates in the corresponding stratification[21]. For example, when building the risk score of interaction (APOE4, CR1, SLC24A4) to predict CSF Aβ for subjects in NL status, only subjects' records with no missing value in APOE4, CR1, SLC24A4, BIN1, PICALM will be used because there are two other SNP-SNP interactions, (APOE4, BIN1, SLC24A4) and (APOE4, BIN1, PICALM), that are being considered as significant interactions in this situation as shown in Table 5.1. Similarly, binary risk scores of (APOE4, BIN1, SLC24A4) and binary risk scores of (APOE4, BIN1, PICALM) can be generated for each subject. Then, $Y_{A\beta,NL}^{new}$ is created by regressing $Y_{A\beta,NL}$ on categorical covariates APOE4, CR1, SLC24A4, BIN1, PICALM with those subjects in stratification NL which do not contain any missed observation on APOE4, CR1, SLC24A4, BIN1, PICALM.

5.2.1 A Set of P Risk Scores for CSF tau

In this subsection, a set of 2 aggregated risk scores are built for CSF tau measurement in NL status from the three original risk scores where one risk score is formed from the two-way interaction (APOE4, ABCA7) and the other two original risk scores are formed from three-way interactions, (APOE4, CD2AP, PICALM) and (APOE4 , ABCA7, NME8), respectively by methodology introduced in chapter 2.6. The response variable $Y_{tau,NL}^{new}$ is the residual after regressing $Y_{tau,NL}$ on all the six SNP factors that are contained in the interactions. The three risk scores are listed in table 5.3. In the column of " High Risk" genotype, (a,b) or (a,b,c) represents the genotype combinations that will be assigned to "High" risk group where a, b, c equal 0 or 1 or 2. For illustration, in the first row of Table 5.3, a subject whose genotype (APOE4, ABCA7) is APOE4=0, ABCA7=1 or APOE4=1, ABCA7=0 or APOE4=2, ABCA7=2 will be defined of "high risk" with risk score $X_1 = 1$. There are three risk scores generated as displayed in Table 5.3. It should be noticed that there are only two significant three-way genetic interaction risk scores. Thus, the cumulative risk score of three-way genetic interaction from our view of MDR with P risk scores can only be $X_2 + X_3$. In order to fully leverage the data, we apply our methodology to X_1, X_2, X_3, neglecting their difference in interaction orders, in order to generate a set of two risk scores.

Then this "relatively good" 2-dimensional model is compared with the model selected by AQMDR, the model selected by QMDR, the model fitted by sum of all three risk scores and the model fitted by all three risk scores in Table 5.4. 167 subjects' records out of 199 sample points in Normal Control Status are involved in this analysis for CSF tau measurement.

Table 5.3: Risk Scores for CSF tau in Normal Control Status

Risk Scores for CSF tau in Normal Control Status		
Interactions	"High Risk" Genotype	Notation
(APOE4, ABCA7)	(0,1) (1,0) (2,2)	X_1
(APOE4, CD2AP,PICALM)	(0,0,0) (0,0,1) (0,1,0) (0,1,2) (0,2,0) (1,0,1) (1,1,1) (1,1,2) (1,2,0) (1,2,1) (2,0,0) (2,0,1) (2,1,0)	X_2
(APOE4 , ABCA7, NME8)	(0,0,2) (0,1,0) (0,1,1) (1,0,0) (1,0,1) (2,0,0) (2,0,2) (2,2,0)	X_3

Table 5.4: Models to Predict CSF tau in Normal Control Status

Models to Predict CSF tau in Normal Control Status			
Models	risk score	R^2	$R^2_{adjusted}$
MDR with 2 risk scores(chosen by "permutation" methodology)	X_1, $X_2 + X_3$	0.1309	0.1202
AQMDR	X_1, $X_2 + X_3$	0.1309	0.1202
QMDR	X_1	0.04857	0.04277
all variable	X_1, X_2, X_3	0.1322	0.1161
sum of variables	$X_1 + X_2 + X_3$	0.1225	0.1172

Table 5.4 indicates that the two-dimensional model chosen by this methodology is the same as the model defined by AQMDR. Its R^2 is larger than the model chosen by QMDR and the model fitted by sum of all three risk scores, as well as very close to the model fitted by three risk scores. But the model has the largest $R^2_{adjusted}$ among all the models compared in Table 5.4.

Similarly, three risk scores from significant SNP interactions are generated to predict CSF tau measurement $Y^{new}_{tau,AD}$, the residual after regressing $Y_{tau,AD}$ on all the seven distinct SNP factors in table 5.5, by genetic information from subjects in Mild Alzheimer's Disease status. Table 5.6 presents the results of MDR with 2 risk scores for each subject, compared in R^2 and $R^2_{adjusted}$ from AQMDR, QMDR, the 3-dimensional model and the 1-dimensional model.

Table 5.5: Risk Scores for CSF tau in Mild Alzheimer's Disease Status

Risk Scores for CSF tau in Mild Alzheimer's Disease Status		
Interactions	"High Risk" Genotype	Notation
(APOE4, MEF2C, CD2AP)	(0,0,1) (0,1,2) (0,2,0) (1,0,0) (1,1,1) (1,2,1) (2,1,0) (2,1,1)	X_4
(APOE4,TREM2, ZCWPW1)	(0,0,0) (0,0,1) (1,0,2) (2,0,1)	X_5
(APOE4, MAPT, CELF1)	(0,0,1) (0,0,2) (0,1,1) (0,1,2) (1,0,0) (1,2,0) (2,1,0) (2,1,1)	X_6

In Table 5.6, the "relatively good" two-dimensional model chosen by methodology in chapter 2.6 has the largest $R^2_{adjusted}$. The R^2 statistic of this model is quite close to the three-dimensional model and larger than the rest of the models considered in Table 5.6. 77 subjects' records out of 100 sample points in Mild AD Status are involved in the CSF tau measurement analysis.

Table 5.6: Models to Predict CSF tau in Mild Alzheimer's Disease Status

Models to Predict CSF tau in Mild Alzheimer's Disease Status			
Models	risk score	R^2	$R^2_{adjusted}$
MDR with 2 risk scores(chosen by "permutation" methodology)	X_4, $X_5 + X_6$	0.1133	0.08897
AQMDR	$X_4 + X_5 + X_6$	0.09463	0.0824
QMDR	X_6	0.04186	0.02891
all variable	X_4, X_5, X_6	0.1138	0.07686
sum of variables	$X_4 + X_5 + X_6$	0.09463	0.0824

5.2.2 A Set of P Risk Scores for CSF Abeta

In this subsection, we study models built with risk scores form significant SNP interactions to predict CSF Aβ measurements in Mild Cognitive status and Normal Control status for each subject.

By data in Mild Cognitive Impairment status where there is no missing value in SNP factors shown in table 5.7, thirteen risk scores X_7,..., X_{19} are generated to predict CSF Aβ measurement $Y^{new}_{A\beta,MCL}$, the residual after regressing $Y_{A\beta,MCL}$ on all the 14 SNP factors in table 5.7.

Table 5.7: Risk Scores for CSF Aβ in Mild Cognitive Impairment Status

Risk Scores for CSF Aβ in Mild Cognitive Impairment Status		
Interactions	"High Risk" Genotype	Notation
(APOE4, PTK2B)	(0,0) (0,2) (1,1) (2,0)	X_7
(APOE4, PTK2B,MAPT)	(0,0,0) (0,0,1) (0,2,0) (0,2,2) (1,1,0) (1,1,1) (1,1,2) (2,0,0) (2,0,1) (2,1,1)	X_8
(APOE4, PTK2B, TREM2)	(0,0,0) (0,0,1) (0,2,0) (1,1,0) (2,0,0)	X_9
(APOE4, PTK2B, NME8)	(0,0,0) (0,0,1) (0,0,2) (0,2,1) (0,2,2) (1,1,0) (1,1,1) (1,1,2) (2,0,0) (2,0,1) (2,0,2) (2,1,1) (2,1,2)	X_{10}
(APOE4, PTK2B, ZCWPW1)	(0,0,0) (0,2,0) (0,2,1) (1,1,0) (1,1,1) (1,1,2) (1,2,0) (2,0,0) (2,0,1) (2,0,2)	X_{11}
(APOE4, PTK2B, CELF1)	(0,0,0) (0,0,1) (0,0,2) (0,1,2) (0,2,0) (0,2,1) (0,2,2) (1,1,0) (1,1,1) (1,1,2) (2,0,0) (2,0,1) (2,1,1)	X_{12}
(APOE4, PTK2B, SORL1)	(0,0,0) (0,1,1) (0,2,0) (1,1,0) (1,1,1) (2,0,0) (2,1,0)	X_{13}
(APOE4, PTK2B, FERMT2)	(0,0,0) (0,0,1) (0,1,1) (0,2,0) (0,2,1) (1,1,0) (1,1,1) (2,0,0) (2,0,1) (2,1,1)	X_{14}
(APOE4, PTK2B, PSEN1)	(0,0,0) (0,1,1) (0,2,0) (1,1,0) (2,0,0)	X_{15}
(APOE4, PTK2B, DSG2)	(0,0,0) (0,2,0) (0,2,1) (1,1,0) (2,0,0) (2,0,1)	X_{16}
(APOE4, PTK2B, ABCA7)	(0,0,0) (0,0,1) (0,2,0) (0,2,1) (0,2,2) (1,0,2) (1,1,0) (1,1,1) (1,1,2) (2,0,0) (2,1,1)	X_{17}
(APOE4, PTK2B, CD33)	(0,0,0) (0,0,1) (0,0,2) (0,2,0) (0,2,1) (0,2,2) (1,0,2) (1,1,0) (1,1,1) (1,2,1) (2,0,0) (2,0,1) (2,1,1)	X_{18}
(APOE4, PTK2B, CASS4)	(0,0,0) (0,0,1) (0,0,2) (0,2,0) (1,1,0) (2,0,0) (2,0,1)	X_{19}

We have observed that risk score X_7 equals to X_{16} for each subject in this data with no missing observations in genotypes and other demographic variables. Thus, including the two-way interaction risk score X_7 with the rest of the three-way interaction risk scores in Table 5.7 will not impact the performance of MDR with P risk scores.

The “relatively good” two-dimensional model and “relatively good” 3-dimensional model chosen by methodology in chapter 2.6 are evaluated in Table 5.8 with the model chosen by AQMDR, the model chosen by QMDR, the 1-dimensional model fitted by sum of all 13 risk scores and the 13-dimensional model. 141 subjects’ records out of 327 sample points in Mild Cognitive Impairment Status are involved in the CSF Aβ measurement analysis.

Table 5.8: Models to Predict CSF Aβ in Mild Cognitive Impairment Status

Models to Predict CSF Aβ in Mild Cognitive Impairment Status			
Models	risk score	R^2	$R^2_{adjusted}$
MDR with 2 risk scores(chosen by “permutation” methodology)	$X_7 + X_{16}$, $X_8 + X_9 + X_{10} + X_{11} + X_{12}+$ $X_{13} + X_{14} + X_{15} + X_{17}+$ $X_{18} + X_{19}$	0.1589	0.1466
MDR with 3 risk scores(chosen by “permutation” methodology)	X_{16}, $X_{12} + X_{13} + X_{14} + X_{10}+$ $X_8 + X_{17} + X_{19} + X_7$, $X_{15} + X_{18} + X_9 + X_{11}$	0.1816	0.1636
AQMDR	X_7, $X_8 + X_9 + X_{10} + X_{11} + X_{12}+$ $X_{13} + X_{14} + X_{15} + X_{16}+$ $X_{17} + X_{18} + X_{19}$	0.1589	0.1466
QMDR	X_{18}	0.1119	0.1055
all variable	X_7, X_8, X_9, X_{10}, X_{11}, X_{12}, X_{13}, X_{14}, X_{15}, X_{16}, X_{17}, X_{18}, X_{19}	0.1978	0.122
sum of variables	$X_7 + X_8 + X_9 + X_{10} + X_{11} + X_{12}+$ $X_{13} + X_{14} + X_{15} + X_{16}+$ $X_{17} + X_{18} + X_{19}$	0.1005	0.09401

Table 5.8 exhibits the “relatively good” three-dimensional model has the largest $R^2_{adjusted}$ and is with a R^2 which is quite close to the thirteen-dimensional model and larger than the rest of the models considered in table 5.8. The “relatively good” two-dimensional model has the same R^2 and $R^2_{adjusted}$ with QMDR because X_{16} equals X_7

for each subject used to fit the model. Besides, we do not seek a higher dimensional model than 3 dimensions since the R^2 of three-dimensional model is already quite close to that of the thirteen-dimensional model. In Table 5.8, the "relatively good" three-dimensional model chosen by methodology in chapter 2.6 has the largest $R^2_{adjusted}$.

By data in Normal Control status where there is no missing value in SNP factors shown in table 5.9, three risk scores X_{20}, X_{21}, X_{22} are generated to predict CSF Aβ measurement $Y^{new}_{A\beta,NL}$, the residual after regressing $Y_{A\beta,NL}$ on all the 5 SNP factors in table 5.9. In Table 5.10, the "relatively good" two-dimensional model chosen by methodology in chapter 2.6 has the largest $R^2_{adjusted}$. The R^2 statistic of this model is quite close to the three-dimensional model and larger than the rest of the models considered in Table 5.10. 68 subjects' records out of 199 sample points in Normal Control Status are involved in the CSF Aβ measurement analysis.

Table 5.9: Risk Scores for CSF Aβ in Normal Control Status

Risk Scores for CSF Aβ in Normal Control Status		
Interactions	"High Risk" Genotype	Notation
(APOE4, CR1, SLC24A4)	(0,0,1) (0,1,0) (1,0,1) (1,1,1) (2,0,0)	X_{20}
(APOE4,BIN1, SLC24A4)	(0,0,0) (0,1,1) (0,2,0) (1,1,1) (1,2,0) (2,0,0) (2,2,0)	X_{21}
(APOE4, BIN1, PICALM)	(0,0,1) (0,0,2) (0,1,0) (0,1,1) (0,2,0) (1,1,0) (1,1,1) (1,1,2) (1,2,0) (2,2,0)	X_{22}

Table 5.10: Models to Predict CSF Aβ in Normal Control Status

Models to Predict CSF Aβ in Normal Control Status			
Models	risk score	R^2	$R^2_{adjusted}$
MDR with 2 risk scores(chosen by "permutation" methodology)	X_{21}, $X_{20}+X_{22}$	0.1501	0.1235
AQMDR	$X_{20}+X_{21}$	0.1035	0.08969
QMDR	X_{22}	0.0617	0.04727
all variable	X_{20}, X_{21}, X_{22}	0.1508	0.1104
sum of variables	$X_{20}+X_{21}+X_{22}$	0.1468	0.1337

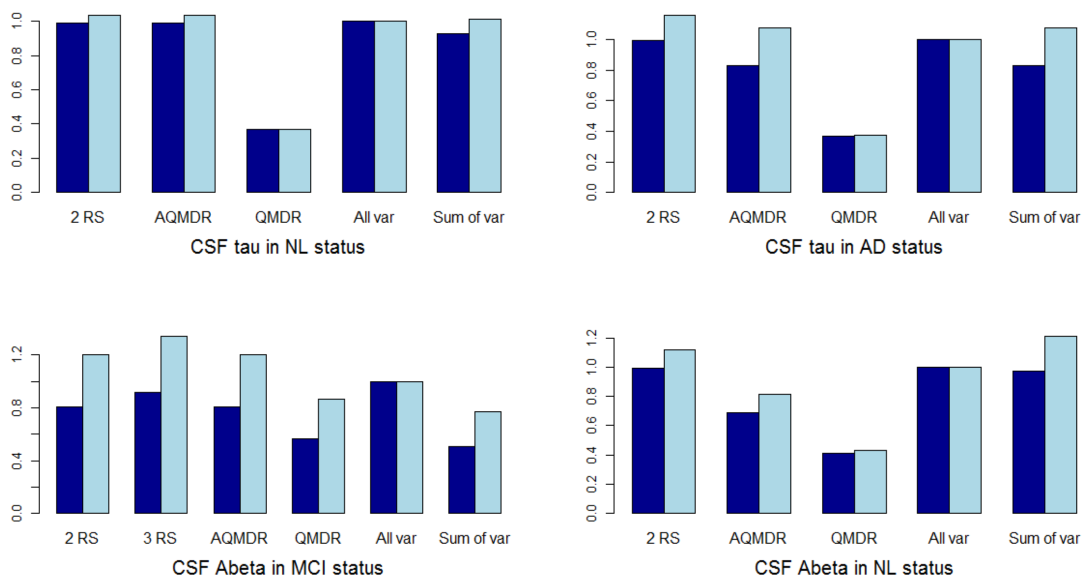

Figure 5.1: Ratio in Rsquared and adjusted Rsquared between dimension reduced model and full dimension model

To visually display the performance of our methodology comparing with QMDR and AQMDR, barplots in figure 5.1 are utilized to represent the R^2 and $R^2_{adjusted}$ ratio between dimension reduced models and the full dimension model in each stratum. The dark blue bars are the ratios on R^2 between model obtained by MDR with P risk score and model fitted with all variables, between AQMDR model and model fitted with all variables, between QMDR model and model fitted with all variables, between model regressed on sum of all variables and model fitted with all variables, respectively in each stratum. The light blue bar is the ratio on $R^2_{adjusted}$ in corresponding model comparison. The model fitted with all variables is "compared" to itself in R^2 and $R^2_{adjusted}$, thus the bars heights of models fitted on all variables are always 1. "2RS" represents the results from model obtained by MDR with 2 risk scores, "3RS" represents the results from model obtained by MDR with 3 risk scores.

The R^2 ratio for models with P risk score are almost in the same height of the full dimensional model except the model with 2 risk scores for CSF Aβ measurements in MCI status. Further, the bar representing adjusted R^2 ratio for models with P risk score is always higher than the bar of the corresponding full dimensional model and the difference is especially apparent in barplots for CSF tau measurements in AD status, CSF Aβ measurements in MCI status and CSF Aβ measurements in NL status.

In each barplot, QMDR model is with the least R^2 ratio and adjusted R^2 ratio. Meanwhile, even though the difference in R^2 ratio or adjusted R^2 ratio among MDR with 2 risk scores model, AQMDR model and model fitted on sum of all variables are not intuitively of much difference in the barplot for CSF tau measurements in NL status, the bars for R^2 ratio and adjusted R^2 ratio of MDR with 2 risk scores model are higher than those of AQMDR model and model fitted on sum of all variables in the barplot for CSF tau measurements in AD status, the bars for R^2 ratio and adjusted R^2 ratio of model obtained from MDR with 3 risk scores are higher than those of AQMDR model and model fitted on sum of all variables in the barplot for CSF Aβ measurements in MCI status.

These demonstrate that it is appropriate to apply the MDR with P risk scores method in each stratum to predict the measurements of CSF tau and CSF Aβ. Thus, with risk factors generated from subjects' genotype, the level of CSF tau or CSF Aβ, can be estimated in order to predict a marker of AD's progressing.

5.3 Conclusions

Table 5.4, Table 5.6, Table 5.8 and Table 5.10 exhibit the R^2 and $R^2_{adjusted}$ for each model considered in them and illustrate the performance of MDR with P risk score as well. However, all the R^2 and $R^2_{adjusted}$ statistics are numerically small, around 0.1 or 0.2. This is due to the fact that the response variables used are the residuals after regressing CSF *tau* or CSF Aβ measurements on the demographic variables and the corresponding SNP factors. The variance in biomarker measurements can be separated into environmental variance, including demography, and genetic variance.[38] Typically only a small portion of the phenotype(or outcome) variability is attributed to SNPs(so called missing heritability)[39], the remaining variability explained by genetic interaction effects is also small.

The reason that response variables are the residuals after regressing original biomarker measurements on demographic variables and SNP factors is that our method focuses on constructing risk scores with significant SNP-SNP interactions for continuous response variables, where the interactions are binary variables and demographic variables contribute to the variances of biomarker measurements. Through leveraging the residuals as response variables, we can study how many additional variabilities can be explained by risk scores built with significant SNP-SNP interactions after eliminating effects from demography and SNP factors.

Missing data on SNPs due to genotype array and quality control is a limitation of the analysis.[40] In each AD stratification on ADNI data, a large portion of the subjects' data can not be leveraged because the corresponding SNPs related to identified significant interactions are missing. This leads to the decrease in power to explain variance in the response variables. Moreover, patterns exist in missing SNPs and can cause bias in analysis. The missing SNP patterns occur due to the difficulty in SNP calling for some common genomic structures and individually specific genomic patterns.[40]

Chapter 6 Other Genetic Interaction Cases and Further Work

6.1 Other Significant Gene-Gene Interaction Cases

In the assumed significance pattern of two-way gene-gene interactions in previous examples, they all have the following format exhibited with contingency tables with SNPs A(a), B(b) as an illustration, where the cells with blue genotypes in bold italic type convey high-risk and the cells with black genotypes represent the genotype is of low risk. And all the blue cells in bold italic type happen to represent all the genotypes that contain at least one homozygous dominant gene, i.e, all genotypes that contain AA or BB.

	AA	Aa	aa
BB	***AABB***	***AaBB***	***aaBB***
Bb	***AABb***	AaBb	aaBb
bb	***AAbb***	Aabb	aabb

To generalize the method to select P risk scores to other significant gene-gene interaction patterns, the following 2 cases are taken as examples to evaluate the performance of methodology to select relatively good P risk scores in Chapter 2.6. Similar with previous simulations, the outputs of each case below contain the average MSPE, average r-statistic and average t-statistic obtained from 100 iterations with 1000 training data and 500 testing data, and $t \sim t_{498}$ under $H_0 : \rho = 0$ as introduced in Chapter 2.6 in each iteration. Comments on the results listed as Table 6.1 and Table 6.2, in Case 1 and Case 2 are given after Table 6.2.

Case 1

In case 1, the assumption is that there are three SNPs A(a), B(b), C(c) with the following two-way interaction significance pattern, where the blue genotype in bold italic type represents high risk, otherwise, the genotype is of low risk.

Besides, the true model for Case 1 is $Y = 100 + 30 \times X_1 + 15 \times X_2 + 10 \times X_3 + \epsilon, \epsilon \sim N(0, 20^2)$, where $X_1 = I_{AABB}$, $X_2 = I_{AaCC,aaCC,AaCc,aaCc}$, $X_3 = I_{BbCc,Bbcc}$.

Case 2

	AA	Aa	aa
BB	***AABB***	AaBB	aaBB
Bb	AABb	AaBb	aaBb
bb	AAbb	Aabb	aabb

	AA	Aa	aa
CC	AACC	***AaCC***	***aaCC***
Cc	AACc	***AaCc***	***aaCc***
cc	AAcc	Aacc	aacc

	BB	Bb	bb
CC	BBCC	BbCC	bbCC
Cc	BBCc	***BbCc***	bbCc
cc	BBcc	***Bbcc***	bbcc

Table 6.1: Outputs for comparison between reduced dimension models and true 3-dimensional model for genetic interaction case 1

Model	MSPE(SE)	r-statistic(SE)	t-statistic(SE)
best 2-dim	427.5132 (2.4426)	0.3363 (0.0039)	7.9935 (0.1053)
rela. good 2-dim	427.5132 (2.4426)	0.3363 (0.0039)	7.9935 (0.1053)
best 3-dim	427.3565 (2.452)	0.3368 (0.0039)	8.0051 (0.1058)
best 1-dim(regress on single var.)	456.0167 (2.5973)	0.2331 (0.0041)	5.364 (0.099)
best 1-dim(regress on sum of var.)	428.3714 (2.422)	0.3333 (0.004)	7.9131 (0.1061)

In case 2, the assumption is that there are four SNPs A(a), B(b), C(c), D(d) with the following two-way interaction significance pattern, where the blue genotype in bold italic type represents high risk , otherwise, the genotype is of low risk.

	AA	Aa	aa
BB	AABB	AaBB	aaBB
Bb	AABb	***AaBb***	***aaBb***
bb	AAbb	***Aabb***	***aabb***

	AA	Aa	aa
CC	AACC	***AaCC***	aaCC
Cc	AACc	AaCc	aaCc
cc	AAcc	Aacc	aacc

	AA	Aa	aa
DD	***AADD***	***AaDD***	***aaDD***
Dd	***AADd***	AaDd	aaDd
dd	***AAdd***	Aadd	aadd

	BB	Bb	bb
CC	BBCC	BbCC	bbCC
Cc	***BBCc***	***BbCc***	bbCc
cc	BBcc	Bbcc	bbcc

	BB	Bb	bb
DD	BBDD	BbDD	***bbDD***
Dd	BBDd	BbDd	bbDd
dd	***BBdd***	Bbdd	bbdd

	CC	Cc	cc
DD	***CCDD***	CcDD	ccDD
Dd	CCDd	CcDd	ccDd
dd	CCdd	Ccdd	***ccdd***

The true model for Case 2 is $Y = 100 + 30 \times X_1 + 55 \times X_2 + 10 \times X_3 + 70 \times X_4 + 15 \times X_5 + 40 \times X_6 + \epsilon, \epsilon \sim N(0, 20^2)$, where $X_1 = I_{AaBb,aaBb,Aabb,aabb}$, $X_2 = I_{AaCC}$,$X_3 = I_{AA\ or\ DD}$, $X_4 = I_{BBCc,BbCc}$, $X_5 = I_{bbDD,BBdd}$, $X_6 = I_{CCDD,ccdd}$.

Table 6.2: Outputs for comparison between reduced dimension models and true 3-dimensional model for genetic interaction case 2

Model	MSPE(SE)	r-statistic(SE)	t-statistic(SE)
best 3-dim	717.0344 (5.821)	0.7537 (0.002)	25.6759 (0.1597)
rela. good 3-dim	716.37 (5.8329)	0.754 (0.002)	25.6965 (0.1607)
best 6-dim	714.2592 (5.8286)	0.7548 (0.002)	25.7645 (0.1598)
best 1-dim(regress on single var.)	1102.1041 (7.941)	0.5798 (0.0032)	15.9359 (0.133)
best 1-dim(regress on sum of var.)	758.4711 (6.3694)	0.7368 (0.0023)	24.4152 (0.1705)

The output tables of above two cases exhibit that the performance of the method to select P risk scores when P is given a priori(P=2 and P=3 in above examples) works stably for other patterns of significant gene-gene interactions since the average MSPE, average r-statistic and average t-statistic of the relatively good models are quite close to these of the corresponding best models. But it is noticeable that the average r-statistic in Case 1 is only around 0.3, this is due to variation in the original linear model that is deployed to generate the data. If the variation is reduced to 2^2, the average r-statistic will increase to about 0.8.

6.2 Generalization of P Risk Scores to Higher Order Interactions

In previous chapters, we introduced how to select 2 or 3 risk scores when the total gene-gene significant interaction numbers are larger than 2 or 3 only for 2-way genetic interactions. In this section, we will illustrate how to select P risk scores when the order of genetic interactions is 3 or higher by an example of four-way genetic interaction.

Supposing the four-way gene-gene interaction between SNPs A(a), B(b), C(c), D(d) is significant, a four-way contingency table, as shown in next page, is built for this interaction with blue cells in bold italic type representing high risk interactions and the low risk interactions are labels by black color, where the high risk genotype is determined by the fact that the cell mean exceeds the overall mean of the four-way contingency table.

Then a risk score can be created as $X = I_{AabbCcDd \text{ or } AabbCcdd \text{ or } aaBbccdd}$ for each subject to predict his continuous response variable.

Similarly, for higher order significant gene-gene interactions, risk scores could be created in the same method as the above example.

		CC			Cc			cc		
		BB	Bb	bb	BB	Bb	bb	BB	Bb	bb
DD	AA	AABBCCDD	AABbCCDD	AAbbCCDD	AABBCcDD	AABbCcDD	AAbbCcDD	AABBccDD	AABbccDD	AAbbccDD
DD	Aa	AaBBCCDD	AaBbCCDD	AabbCCDD	AaBBCcDD	AaBbCcDD	AabbCcDD	AaBBccDD	AaBbccDD	AabbccDD
DD	aa	aaBBCCDD	aaBbCCDD	aabbCCDD	aaBBCcDD	aaBbCcDD	aabbCcDD	aaBBccDD	aaBbccDD	aabbccDD
Dd	AA	AABBCCDd	AABbCCDd	AAbbCCDd	AABBCcDd	AABbCcDd	AAbbCcDd	AABBccDd	AABbccDd	AAbbccDd
Dd	Aa	AaBBCCDd	AaBbCCDd	AabbCCDd	AaBBCcDd	AaBbCcDd	***AabbCcDd***	AaBBccDd	AaBbccDd	AabbccDd
Dd	aa	aaBBCCDd	aaBbCCDd	aabbCCDd	aaBBCcDd	aaBbCcDd	aabbCcDd	aaBBccDd	aaBbccDd	aabbccDd
dd	AA	AABBCCdd	AABbCCdd	AAbbCCdd	AABBCcdd	AABbCcdd	AAbbCcdd	AABBccdd	AABbccdd	AAbbccdd
dd	Aa	AaBBCCdd	AaBbCCdd	AabbCCdd	AaBBCcdd	AaBbCcdd	***AabbCcdd***	AaBBccdd	AaBbccdd	Aabbccdd
dd	aa	aaBBCCdd	aaBbCCdd	aabbCCdd	aaBBCcdd	aaBbCcdd	aabbCcdd	aaBBccdd	***aaBbccdd***	aabbccdd

If there are only four SNPs A(a), B(b), C(c), D(d) in the above four-way gene-gene interaction example, there is only one risk score generated. However, if more SNPs with significant four-way gene-gene interactions are added into this example, such as E(e) and F(f), there would be $\binom{6}{4} = 15$ risk scores generated given all the four way gene-gene interactions within the six SNPs are significant. Therefore, the methodologies proposed in Chapter 2.6 and Chapter 3.3 could be applied to select a set of relatively good p risk scores to predict each subject's continuous outcome when p is given a priori or not, respectively.

6.3 Further Work

Further work on MDR with p risk scores per person could include extension to other linear regression models. As an example, the methodologies proposed in Chapter 2.6 and Chapter 3.3 could be combined with logistic regression, weighted regression or ridge regression.

A weighted regression is a linear model with following expression,

$Y_i = \beta_0 + \beta_1 X_{i1} + ... + \beta_{m-1} X_{i,m-1} + \epsilon_i$, $\epsilon_i \sim N(0, \sigma_i^2)$, $i = 1, ..., n$,[41]

where Y_i is the continuous outcome for ith subject, X_{ij}, $j = 1, ..., m-1$ are the predictive variables, ϵ_i, $i = 1, ..., n$ are independent error terms following normal distribution that are all centered at 0 but plausibly with different standard deviations.

There are two situations regarding variances of $\epsilon_i, i = 1, .., n$. One is the error variances are known, then the diagonal weight matrix W can be expressed as $(w_{ii})_n, i = 1, ..., n$ where w_{ii} is defined as $\frac{1}{\sigma_i^2}$. And $\hat{\boldsymbol{\beta}}$ can be estimated by $(X'WX)^{-1}X'WY$[41] where

$$\hat{\boldsymbol{\beta}} = \begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \vdots \\ \hat{\beta}_{m-1} \end{pmatrix}, X = \begin{pmatrix} 1 & x_{11} & \cdots & x_{1,m-1} \\ 1 & x_{21} & \cdots & x_{2,m-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_{n1} & \cdots & x_{n,m-1} \end{pmatrix}, Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}.$$

If each variable is a risk score generated from significant gene-gene interactions, then we could develop similar methodologies as in Chapter 2.6 to select a set of relative good aggregated p risk scores for weighted linear models when p is assigned a priori.

The other situation of a weighted linear model is that the error variance is unknown. One method to obtain the estimates of variances σ_i^2, $i = 1, ..., n$, is to perform a residual plot against the absolute of kth predictive variable $|X_k|$ or the square of kth predictive variable X_k^2, k in $(1, ..., m-1)$, to fit for σ_i^2. Then w_{ii} is estimated by the inverse of fitted σ_i^2.[41] Thus, for this condition, a new methodology could be proposed with aim to select a set of p_1 relatively good aggregated risk scores to fit σ_i^2 first and then select a set of relatively good aggregated p_2 risk scores to predict the continuous outcome.

In ridge regression, even though a bias constant is added into the model after transforming all the predictive variables by a standardized transformation[41], the bias constant could not reduce the dimension of the ridge regression model but introduces a penalty to control the estimated coefficients' variances. Methodologies could be studied to reduce the dimension of a ridge regression model fitted with risk scores by generating a set of p aggregated risk scores to fit a ridge regression model, where p is much smaller than the total number of predictive risk scores.

The methodologies to select a set of p risk scores could be applied beyond studies on genetic interactions. One example is compliance management on patients receiving prescribed therapies. Non-compliant patients in therapy have behaviors such as taking medicine at an incorrect dose or at an incorrect time, lag in presenting on doctor appointments and terminating treatment too early.[42] Non-compliance can impact the clinical treatment consequences, introduce social financial loss and mislead physician prescription.[42] A collection of factors are listed in [42], smoking, compliance history, medicine side effect, financial support, clinic experience and many other factors. A large portion of the factors listed can be treated as binary predictive variables in a study to predict a continuous risk score, the response variable, measuring the risk of non-compliance for each patient. With our method, the dimension of the model predicting this risk score can be reduced to a relatively small number rather than fitting a linear regression model with all the binary factors and interactions that are related to non-compliance.

Another example of the application is in predicting the demand of a new product when the price is fixed. A demand curve is a plot reflecting the relationship between quantity(horizontal axis) and price(vertical axis).[43] The demand curve is affected to shift left or right by factors such as income level, product type, existence of complements, etc. [43] We can apply our method to select a set of p risk scores leveraging all the binary predictive variables that are related to demand to concisely reflect the product demand at a targeted price, and then combine the selected p risk scores with other continuous predictive variables to predict the expected demand with the price given.

Appendix: Expression of Second Order Partial Derivatives in Chapter 4.2

Expression of second order partial derivatives in Chapter 4.2

$h_{ij}^{(U)} = \frac{\partial^2 g_3(\mu_1,...,\mu_{10})}{\partial x_i \partial x_j} = \Sigma_{k=1}^{6} h_{ij}^{(U_k)} = \Sigma_{k=1}^{6} \frac{\partial^2 g_{3k}(\mu_1,...,\mu_{10})}{\partial x_i \partial x_j},\ i,j = 1,...,10$

Further, assume $\mu_i,\ i = 1,...,10$ satisfies that the denominators in $h_{ij}^{(U)},\ i,j = 1,...,10, j \geqslant i$ are not equal to zero as well as $h_{ii}^{(U)} \neq \sum_{j=1,i\neq j}^{10} h_{ij}^{(U)},\ i = 1,...,10.$

$$h_{11}^{(U_1)} = \frac{\left(2\left(2bc+c+b\right)\left(n+m\right)^2 - 4\left(ac+ab\right)l\left(n+m\right) + 2al^2\right)z^2}{2\left(-lx^2 + 2\left(n+m\right)\left(q+p\right)x - \left(q+p\right)^2 y + ly - \left(n+m\right)^2\right)^2}$$

$-[2z^2(2(n+m)(q+p)-2lx)(2(2bc+c+b)(n+m)((n+m)x-(q+p)y)-2(ac+ab)l((n+m)x-(q+p)y)+2(ac+ab)(n+m)((n+m)(q+p)-lx)-2al((n+m)(q+p)-lx)+2(c+b)(n+m)(ly-(n+m)^2)-2al(ly-(n+m)^2))]/[(-lx^2+2(n+m)(q+p)x-(q+p)^2y+ly-(n+m)^2)^3]+[2lz^2((2bc+c+b)((n+m)x-(q+p)y)^2+a((n+m)(q+p)-lx)^2+2(ac+ab)((n+m)(q+p)-lx)((n+m)x-(q+p)y)+2(c+b)(ly-(n+m)^2)((n+m)x-(q+p)y)+2a(ly-(n+m)^2)((n+m)(q+p)-lx)+(ly-(n+m)^2)^2)]/[(-lx^2+2(n+m)(q+p)x-(q+p)^2y+ly-(n+m)^2)^3]+[3z^2(2(n+m)(q+p)-2lx)^2((2bc+c+b)((n+m)x-(q+p)y)^2+a((n+m)(q+p)-lx)^2+2(ac+ab)((n+m)(q+p)-lx)((n+m)x-(q+p)y)+2(c+b)(ly-(n+m)^2)((n+m)x-(q+p)y)+2a(ly-(n+m)^2)((n+m)(q+p)-lx)+(ly-(n+m)^2)^2)]/[(-lx^2+2(n+m)(q+p)x-(q+p)^2y+ly-(n+m)^2)^4]$

$h_{12}^{(U_1)} = [(2(2bc+c+b)(n+m)(-q-p)-2(ac+ab)l(-q-p)+2(c+b)l(n+m)-2al^2)z^2]/[2(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^2]+[3(l-(q+p)^2)(2(n+m)(q+p)-2lx)z^2((2bc+c+b)((n+m)x-(q+p)y)^2+(ly-(n+m)^2)^2+2(c+b)(ly-(n+m)^2)((n+m)x-(q+p)y)+2(ac+ab)((n+m)(q+p)-lx)((n+m)x-(q+p)y)+2a((n+m)(q+p)-lx)(ly-(n+m)^2)+a((n+m)(q+p)-lx)^2)]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^4]-[(2(n+m)(q+p)-2lx)z^2(2(2bc+c+b)(-q-p)((n+m)x-(q+p)y)+2(c+b)l((n+m)x-(q+p)y)+2(c+b)(-q-p)(ly-(n+m)^2)+2l(ly-(n+m)^2)+2(ac+ab)(-q-p)((n+m)(q+p)-lx)+2al((n+m)(q+p)-lx))]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^3]-[(l-(q+p)^2)z^2(2(2bc+c+b)(n+m)((n+m)x-(q+p)y)-2(ac+ab)l((n+m)x-(q+p)y)+2(c+b)(n+m)(ly-(n+m)^2)-2al(ly-(n+m)^2)+2(ac+ab)(n+m)((n+m)(q+p)-lx)-$

$2al((n+m)(q+p)-lx))]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^3]$

$h_{13}^{(U_1)} = [(2(2bc+c+b)(n+m)((n+m)x-(q+p)y)-2(ac+ab)l((n+m)x-(q+p)y)+2(c+b)(n+m)(ly-(n+m)^2)-2al(ly-(n+m)^2)+2(ac+ab)(n+m)((n+m)(q+p)-lx)-2al((n+m)(q+p)-lx))z]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^2]-[2(2(n+m)(q+p)-2lx)((2bc+c+b)((n+m)x-(q+p)y)^2+(ly-(n+m)^2)^2+2(c+b)(ly-(n+m)^2)((n+m)x-(q+p)y)+2(ac+ab)((n+m)(q+p)-lx)((n+m)x-(q+p)y)+2a((n+m)(q+p)-lx)(ly-(n+m)^2)+a((n+m)(q+p)-lx)^2)z]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^3]$

$h_{14}^{(U_1)} = 0$

$h_{15}^{(U_1)} = [2xz^2((yl-(n+m)^2)^2+a((n+m)(q+p)-xl)^2+2a((n+m)(q+p)-xl)(yl-(n+m)^2)+2(c+b)((n+m)x-(q+p)y)(yl-(n+m)^2)+2(ac+ab)((n+m)x-(q+p)y)((n+m)(q+p)-xl)+(2bc+c+b)((n+m)x-(q+p)y)^2)]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^3]+[3(y-x^2)z^2(2(n+m)(q+p)-2xl)((yl-(n+m)^2)^2+a((n+m)(q+p)-xl)^2+2a((n+m)(q+p)-xl)(yl-(n+m)^2)+2(c+b)((n+m)x-(q+p)y)(yl-(n+m)^2)+2(ac+ab)((n+m)x-(q+p)y)((n+m)(q+p)-xl)+(2bc+c+b)((n+m)x-(q+p)y)^2)]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^4]-[(y-x^2)z^2(-2al(yl-(n+m)^2)+2(c+b)(n+m)(yl-(n+m)^2)-2al((n+m)(q+p)-xl)+2(ac+ab)(n+m)((n+m)(q+p)-xl)-2(ac+ab)((n+m)x-(q+p)y)l+2(2bc+c+b)(n+m)((n+m)x-(q+p)y))]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^3]-[z^2(2(n+m)(q+p)-2xl)(2y(yl-(n+m)^2)-2ax(yl-(n+m)^2)+2ay((n+m)(q+p)-xl)-2ax((n+m)(q+p)-xl)+2(c+b)y((n+m)x-(q+p)y)-2(ac+ab)x((n+m)x-(q+p)y))]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^3]+[z^2(-2a(yl-(n+m)^2)-2a((n+m)(q+p)-xl)-2ayl+2axl-2(ac+ab)((n+m)x-(q+p)y)+2(c+b)(n+m)y-2(ac+ab)(n+m)x)]/[2(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^2]$

$h_{16}^{(U_1)} = -[z^2(2(q+p)x-2(m+n))(2(c+b)(m+n)(ly-(m+n)^2)-2al(ly-(m+n)^2)+2(2bc+c+b)(m+n)(x(m+n)-(q+p)y)-2(ac+ab)l(x(m+n)-(q+p)y)+2(ac+ab)(m+n)((q+p)(m+n)-lx)-2al((q+p)(m+n)-lx))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^3]-[z^2(2(q+p)(m+n)-2lx)(-4(m+n)(ly-(m+n)^2)+2(c+b)x(ly-(m+n)^2)+2a(q+p)(ly-(m+n)^2)-4(c+b)(m+n)(x(m+n)-(q+p)y)+2(2bc+c+b)x(x(m+n)-(q+p)y)+2(ac+ab)(q+p)(x(m+n)-(q+p)y)-$

$4a(m+n)((q+p)(m+n)-lx)+2(ac+ab)x((q+p)(m+n)-lx)+2a(q+p)((q+p)(m+n)-lx))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^3]-[2(q+p)z^2((ly-(m+n)^2)^2+(2bc+c+b)(x(m+n)-(q+p)y)^2+a((q+p)(m+n)-lx)^2+2(c+b)(x(m+n)-(q+p)y)(ly-(m+n)^2)+2a((q+p)(m+n)-lx)(ly-(m+n)^2)+2(ac+ab)((q+p)(m+n)-lx)(x(m+n)-(q+p)y))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^3]+[3z^2(2(q+p)x-2(m+n))(2(q+p)(m+n)-2lx)((ly-(m+n)^2)^2+(2bc+c+b)(x(m+n)-(q+p)y)^2+a((q+p)(m+n)-lx)^2+2(c+b)(x(m+n)-(q+p)y)(ly-(m+n)^2)+2a((q+p)(m+n)-lx)(ly-(m+n)^2)+2(ac+ab)((q+p)(m+n)-lx)(x(m+n)-(q+p)y))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^4]+[z^2(-4(c+b)(m+n)^2+2(c+b)(ly-(m+n)^2)+2(2bc+c+b)(x(m+n)-(q+p)y)+2(ac+ab)((q+p)(m+n)-lx)+2(2bc+c+b)x(m+n)+2(ac+ab)(q+p)(m+n)+4al(m+n)-2(ac+ab)lx-2al(q+p))]/[2(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^2]$

$h_{17}^{(U_1)}=-[z^2(2(q+p)x-2(n+m))(2(c+b)(n+m)(ly-(n+m)^2)-2al(ly-(n+m)^2)+2(2bc+c+b)(n+m)(x(n+m)-(q+p)y)-2(ac+ab)l(x(n+m)-(q+p)y)+2(ac+ab)(n+m)((q+p)(n+m)-lx)-2al((q+p)(n+m)-lx))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]-[z^2(2(q+p)(n+m)-2lx)(-4(n+m)(ly-(n+m)^2)+2(c+b)x(ly-(n+m)^2)+2a(q+p)(ly-(n+m)^2)-4(c+b)(n+m)(x(n+m)-(q+p)y)+2(2bc+c+b)x(x(n+m)-(q+p)y)+2(ac+ab)(q+p)(x(n+m)-(q+p)y)-4a(n+m)((q+p)(n+m)-lx)+2(ac+ab)x((q+p)(n+m)-lx)+2a(q+p)((q+p)(n+m)-lx))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]-[2(q+p)z^2((ly-(n+m)^2)^2+(2bc+c+b)(x(n+m)-(q+p)y)^2+a((q+p)(n+m)-lx)^2+2(c+b)(x(n+m)-(q+p)y)(ly-(n+m)^2)+2a((q+p)(n+m)-lx)(ly-(n+m)^2)+2(ac+ab)((q+p)(n+m)-lx)(x(n+m)-(q+p)y))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]+[3z^2(2(q+p)x-2(n+m))(2(q+p)(n+m)-2lx)((ly-(n+m)^2)^2+(2bc+c+b)(x(n+m)-(q+p)y)^2+a((q+p)(n+m)-lx)^2+2(c+b)(x(n+m)-(q+p)y)(ly-(n+m)^2)+2a((q+p)(n+m)-lx)(ly-(n+m)^2)+2(ac+ab)((q+p)(n+m)-lx)(x(n+m)-(q+p)y))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^4]+[z^2(-4(c+b)(n+m)^2+2(c+b)(ly-(n+m)^2)+2(2bc+c+b)(x(n+m)-(q+p)y)+2(ac+ab)((q+p)(n+m)-lx)+2(2bc+c+b)x(n+m)+2(ac+ab)(q+p)(n+m)+4al(n+m)-2(ac+ab)lx-2al(q+p))]/[2(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^2]$

$$h_{18}^{(U_1)} = [(-2(2bc+c+b)(n+m)y+2(ac+ab)ly+2(ac+ab)(n+m)^2-2al(n+m))z^2]/[2(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^2]-[z^2(2(n+m)(p+q)-2lx)(-2(2bc+c+b)y((n+m)x-y(p+q))+2(ac+ab)(n+m)((n+m)x-y(p+q))-2(ac+ab)y((n+m)(p+q)-lx)+2a(n+m)((n+m)(p+q)-lx)-2(c+b)y(ly-(n+m)^2)+2a(n+m)(ly-(n+m)^2))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]-[z^2(2(n+m)x-2y(p+q))(2(2bc+c+b)(n+m)((n+m)x-y(p+q))-2(ac+ab)l((n+m)x-y(p+q))+2(ac+ab)(n+m)((n+m)(p+q)-lx)-2al((n+m)(p+q)-lx)+2(c+b)(n+m)(ly-(n+m)^2)-2al(ly-(n+m)^2))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]-[2(n+m)z^2((2bc+c+b)((n+m)x-y(p+q))^2+a((n+m)(p+q)-lx)^2+2(ac+ab)((n+m)(p+q)-lx)((n+m)x-y(p+q))+2(c+b)(ly-(n+m)^2)((n+m)x-y(p+q))+2a(ly-(n+m)^2)((n+m)(p+q)-lx)+(ly-(n+m)^2)^2)]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]+[3z^2(2(n+m)(p+q)-2lx)(2(n+m)x-2y(p+q))((2bc+c+b)((n+m)x-y(p+q))^2+a((n+m)(p+q)-lx)^2+2(ac+ab)((n+m)(p+q)-lx)((n+m)x-y(p+q))+2(c+b)(ly-(n+m)^2)((n+m)x-y(p+q))+2a(ly-(n+m)^2)((n+m)(p+q)-lx)+(ly-(n+m)^2)^2)]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^4]$$

$$h_{19}^{(U_1)} = [(-2(2bc+c+b)(n+m)y+2(ac+ab)ly+2(ac+ab)(n+m)^2-2al(n+m))z^2]/[2(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^2]-[z^2(2(n+m)(q+p)-2lx)(-2(2bc+c+b)y((n+m)x-y(q+p))+2(ac+ab)(n+m)((n+m)x-y(q+p))-2(ac+ab)y((n+m)(q+p)-lx)+2a(n+m)((n+m)(q+p)-lx)-2(c+b)y(ly-(n+m)^2)+2a(n+m)(ly-(n+m)^2))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]-[z^2(2(n+m)x-2y(q+p))(2(2bc+c+b)(n+m)((n+m)x-y(q+p))-2(ac+ab)l((n+m)x-y(q+p))+2(ac+ab)(n+m)((n+m)(q+p)-lx)-2al((n+m)(q+p)-lx)+2(c+b)(n+m)(ly-(n+m)^2)-2al(ly-(n+m)^2))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]-[2(n+m)z^2((2bc+c+b)((n+m)x-y(q+p))^2+a((n+m)(q+p)-lx)^2+2(ac+ab)((n+m)(q+p)-lx)((n+m)x-y(q+p))+2(c+b)(ly-(n+m)^2)((n+m)x-y(q+p))+2a(ly-(n+m)^2)((n+m)(q+p)-lx)+(ly-(n+m)^2)^2)]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]+[3z^2(2(n+m)(q+p)-2lx)(2(n+m)x-2y(q+p))((2bc+c+b)((n+m)x-y(q+p))^2+a((n+m)(q+p)-lx)^2+2(ac+ab)((n+m)(q+p)-lx)((n+m)x-y(q+p))+2(c+b)(ly-(n+m)^2)((n+m)x-y(q+p))+2a(ly-(n+m)^2)((n+m)(q+p)-lx)+(ly-(n+m)^2)^2)]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^4]$$

$$h_{1\ 10}^{(U_1)} = 0$$

$$h_{22}^{(U_1)} = [(4(c+b)l(-q-p)+2(2bc+c+b)(-q-p)^2+2l^2)z^2]/[2(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^2]+[3(l-(q+p)^2)^2z^2((2bc+c+b)((n+m)x-(q+p)y)^2+(ly-(n+m)^2)^2+2(c+b)(ly-(n+m)^2)((n+m)x-(q+p)y)+2(ac+ab)((n+m)(q+p)-lx)((n+m)x-(q+p)y)+2a((n+m)(q+p)-lx)(ly-(n+m)^2)+a((n+m)(q+p)-lx)^2)]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^4]-[2(l-(q+p)^2)z^2(2(2bc+c+b)(-q-p)((n+m)x-(q+p)y)+2(c+b)l((n+m)x-(q+p)y)+2(c+b)(-q-p)(ly-(n+m)^2)+2l(ly-(n+m)^2)+2(ac+ab)(-q-p)((n+m)(q+p)-lx)+2al((n+m)(q+p)-lx))]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^3]$$

$$h_{23}^{(U_1)} = [(2(2bc+c+b)(-q-p)((n+m)x-(q+p)y)+2(c+b)l((n+m)x-(q+p)y)+2(c+b)(-q-p)(ly-(n+m)^2)+2l(ly-(n+m)^2)+2(ac+ab)(-q-p)((n+m)(q+p)-lx)+2al((n+m)(q+p)-lx))z]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^2]-[2(l-(q+p)^2)((2bc+c+b)((n+m)x-(q+p)y)^2+(ly-(n+m)^2)^2+2(c+b)(ly-(n+m)^2)((n+m)x-(q+p)y)+2(ac+ab)((n+m)(q+p)-lx)((n+m)x-(q+p)y)+2a((n+m)(q+p)-lx)(ly-(n+m)^2)+a((n+m)(q+p)-lx)^2)z]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^3]$$

$$h_{24}^{(U_1)} = 0$$

$$h_{25}^{(U_1)} = -[z^2((yl-(n+m)^2)^2+a((n+m)(q+p)-xl)^2+2a((n+m)(q+p)-xl)(yl-(n+m)^2)+2(c+b)((n+m)x-(q+p)y)(yl-(n+m)^2)+2(ac+ab)((n+m)x-(q+p)y)((n+m)(q+p)-xl)+(2bc+c+b)((n+m)x-(q+p)y)^2)]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^3]+[3(y-x^2)z^2(l-(q+p)^2)((yl-(n+m)^2)^2+a((n+m)(q+p)-xl)^2+2a((n+m)(q+p)-xl)(yl-(n+m)^2)+2(c+b)((n+m)x-(q+p)y)(yl-(n+m)^2)+2(ac+ab)((n+m)x-(q+p)y)((n+m)(q+p)-xl)+(2bc+c+b)((n+m)x-(q+p)y)^2)]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^4]-[(y-x^2)z^2(2l(yl-(n+m)^2)+2(c+b)(-q-p)(yl-(n+m)^2)+2al((n+m)(q+p)-xl)+2(ac+ab)(-q-p)((n+m)(q+p)-xl)+2(c+b)((n+m)x-(q+p)y)l+2(2bc+c+b)(-q-p)((n+m)x-(q+p)y))]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^3]-[z^2(l-(q+p)^2)(2y(yl-(n+m)^2)-2ax(yl-(n+m)^2)+2ay((n+m)(q+p)-xl)-2ax((n+m)(q+p)-xl)+2(c+b)y((n+m)x-(q+p)y)-2(ac+ab)x((n+m)x-(q+p)y))]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^3]+[z^2(2(yl-(n+m)^2)+2a((n+m)(q+p)-xl)+2yl-2axl+2(c+b)((n+m)x-(q+p)y)+2(c+$$

$b)(-q-p)y-2(ac+ab)(-q-p)x)]/[2(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^2]$

$h_{26}^{(U_1)} = [z^2(-4(c+b)(-q-p)(m+n)-4l(m+n)+2(2bc+c+b)(-q-p)x+2(c+b)lx+2(ac+ab)(-q-p)(q+p)+2al(q+p))]/[2(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^2]-[(l-(q+p)^2)z^2(-4(m+n)(ly-(m+n)^2)+2(c+b)x(ly-(m+n)^2)+2a(q+p)(ly-(m+n)^2)-4(c+b)(m+n)(x(m+n)-(q+p)y)+2(2bc+c+b)x(x(m+n)-(q+p)y)+2(ac+ab)(q+p)(x(m+n)-(q+p)y)-4a(m+n)((q+p)(m+n)-lx)+2(ac+ab)x((q+p)(m+n)-lx)+2a(q+p)((q+p)(m+n)-lx))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^3]-[z^2(2(q+p)x-2(m+n))(2(c+b)(-q-p)(ly-(m+n)^2)+2l(ly-(m+n)^2)+2(2bc+c+b)(-q-p)(x(m+n)-(q+p)y)+2(c+b)l(x(m+n)-(q+p)y)+2(ac+ab)(-q-p)((q+p)(m+n)-lx)+2al((q+p)(m+n)-lx))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^3]+[3(l-(q+p)^2)z^2(2(q+p)x-2(m+n))((ly-(m+n)^2)^2+(2bc+c+b)(x(m+n)-(q+p)y)^2+a((q+p)(m+n)-lx)^2+2(c+b)(x(m+n)-(q+p)y)(ly-(m+n)^2)+2a((q+p)(m+n)-lx)(ly-(m+n)^2)+2(ac+ab)((q+p)(m+n)-lx)(x(m+n)-(q+p)y))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^4]$

$h_{27}^{(U_1)} = [z^2(-4(c+b)(-q-p)(n+m)-4l(n+m)+2(2bc+c+b)(-q-p)x+2(c+b)lx+2(ac+ab)(-q-p)(q+p)+2al(q+p))]/[2(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^2]-[(l-(q+p)^2)z^2(-4(n+m)(ly-(n+m)^2)+2(c+b)x(ly-(n+m)^2)+2a(q+p)(ly-(n+m)^2)-4(c+b)(n+m)(x(n+m)-(q+p)y)+2(2bc+c+b)x(x(n+m)-(q+p)y)+2(ac+ab)(q+p)(x(n+m)-(q+p)y)-4a(n+m)((q+p)(n+m)-lx)+2(ac+ab)x((q+p)(n+m)-lx)+2a(q+p)((q+p)(n+m)-lx))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]-[z^2(2(q+p)x-2(n+m))(2(c+b)(-q-p)(ly-(n+m)^2)+2l(ly-(n+m)^2)+2(2bc+c+b)(-q-p)(x(n+m)-(q+p)y)+2(c+b)l(x(n+m)-(q+p)y)+2(ac+ab)(-q-p)((q+p)(n+m)-lx)+2al((q+p)(n+m)-lx))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]+[3(l-(q+p)^2)z^2(2(q+p)x-2(n+m))((ly-(n+m)^2)^2+(2bc+c+b)(x(n+m)-(q+p)y)^2+a((q+p)(n+m)-lx)^2+2(c+b)(x(n+m)-(q+p)y)(ly-(n+m)^2)+2a((q+p)(n+m)-lx)(ly-(n+m)^2)+2(ac+ab)((q+p)(n+m)-lx)(x(n+m)-(q+p)y))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^4]$

$h_{28}^{(U_1)} = [z^2(-2(2bc+c+b)((n+m)x-y(p+q))-2(ac+ab)((n+m)(p+q)-lx)-2(2bc+c+b)y(-p-q)+2(ac+ab)(n+m)(-p-q)-2(c+b)(ly-(n+m)^2)-2(c+b)ly+2al(n+m))]/[2(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^2]-[z^2(-2(2bc+c+$

$b)y((n+m)x-y(p+q))+2(ac+ab)(n+m)((n+m)x-y(p+q))-2(ac+ab)y((n+m)(p+q)-lx)+2a(n+m)((n+m)(p+q)-lx)-2(c+b)y(ly-(n+m)^2)+2a(n+m)(ly-(n+m)^2))(l-(p+q)^2)]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]-[z^2(2(n+m)x-2y(p+q))(2(2bc+c+b)(-p-q)((n+m)x-y(p+q))+2(c+b)l((n+m)x-y(p+q))+2(ac+ab)(-p-q)((n+m)(p+q)-lx)+2al((n+m)(p+q)-lx)+2(c+b)(ly-(n+m)^2)(-p-q)+2l(ly-(n+m)^2))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]+[2z^2(p+q)((2bc+c+b)((n+m)x-y(p+q))^2+a((n+m)(p+q)-lx)^2+2(ac+ab)((n+m)(p+q)-lx)((n+m)x-y(p+q))+2(c+b)(ly-(n+m)^2)((n+m)x-y(p+q))+2a(ly-(n+m)^2)((n+m)(p+q)-lx)+(ly-(n+m)^2)^2)]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]+[3z^2(2(n+m)x-2y(p+q))(l-(p+q)^2)((2bc+c+b)((n+m)x-y(p+q))^2+a((n+m)(p+q)-lx)^2+2(ac+ab)((n+m)(p+q)-lx)((n+m)x-y(p+q))+2(c+b)(ly-(n+m)^2)((n+m)x-y(p+q))+2a(ly-(n+m)^2)((n+m)(p+q)-lx)+(ly-(n+m)^2)^2)]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^4]$

$h_{29}^{(U_1)}=[z^2(-2(2bc+c+b)((n+m)x-y(q+p))-2(ac+ab)((n+m)(q+p)-lx)-2(2bc+c+b)y(-q-p)+2(ac+ab)(n+m)(-q-p)-2(c+b)(ly-(n+m)^2)-2(c+b)ly+2al(n+m))]/[2(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^2]-[z^2(-2(2bc+c+b)y((n+m)x-y(q+p))+2(ac+ab)(n+m)((n+m)x-y(q+p))-2(ac+ab)y((n+m)(q+p)-lx)+2a(n+m)((n+m)(q+p)-lx)-2(c+b)y(ly-(n+m)^2)+2a(n+m)(ly-(n+m)^2))(l-(q+p)^2)]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]-[z^2(2(n+m)x-2y(q+p))(2(2bc+c+b)(-q-p)((n+m)x-y(q+p))+2(c+b)l((n+m)x-y(q+p))+2(ac+ab)(-q-p)((n+m)(q+p)-lx)+2al((n+m)(q+p)-lx)+2(c+b)(ly-(n+m)^2)(-q-p)+2l(ly-(n+m)^2))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]+[2z^2(q+p)((2bc+c+b)((n+m)x-y(q+p))^2+a((n+m)(q+p)-lx)^2+2(ac+ab)((n+m)(q+p)-lx)((n+m)x-y(q+p))+2(c+b)(ly-(n+m)^2)((n+m)x-y(q+p))+2a(ly-(n+m)^2)((n+m)(q+p)-lx)+(ly-(n+m)^2)^2)]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]+[3z^2(2(n+m)x-2y(q+p))(l-(q+p)^2)((2bc+c+b)((n+m)x-y(q+p))^2+a((n+m)(q+p)-lx)^2+2(ac+ab)((n+m)(q+p)-lx)((n+m)x-y(q+p))+2(c+b)(ly-(n+m)^2)((n+m)x-y(q+p))+2a(ly-(n+m)^2)((n+m)(q+p)-lx)+(ly-(n+m)^2)^2)]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^4]$

$h_{2\ 10}^{(U_1)}=0$

$h_{33}^{(U_1)} = [(2bc+c+b)((n+m)x-(q+p)y)^2+(ly-(n+m)^2)^2+2(c+b)(ly-(n+m)^2)((n+m)x-(q+p)y)+2(ac+ab)((n+m)(q+p)-lx)((n+m)x-(q+p)y)+2a((n+m)(q+p)-lx)(ly-(n+m)^2)+a((n+m)(q+p)-lx)^2]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^2]$

$h_{34}^{(U_1)} = 0$

$h_{35}^{(U_1)} = [z(2y(yl-(n+m)^2)-2ax(yl-(n+m)^2)+2ay((n+m)(q+p)-xl)-2ax((n+m)(q+p)-xl)+2(c+b)y((n+m)x-(q+p)y)-2(ac+ab)x((n+m)x-(q+p)y))]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^2]-[2(y-x^2)z((yl-(n+m)^2)^2+a((n+m)(q+p)-xl)^2+2a((n+m)(q+p)-xl)(yl-(n+m)^2)+2(c+b)((n+m)x-(q+p)y)(yl-(n+m)^2)+2(ac+ab)((n+m)x-(q+p)y)((n+m)(q+p)-xl)+(2bc+c+b)((n+m)x-(q+p)y)^2)]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^3]$

$h_{36}^{(U_1)} = [z(-4(m+n)(ly-(m+n)^2)+2(c+b)x(ly-(m+n)^2)+2a(q+p)(ly-(m+n)^2)-4(c+b)(m+n)(x(m+n)-(q+p)y)+2(2bc+c+b)x(x(m+n)-(q+p)y)+2(ac+ab)(q+p)(x(m+n)-(q+p)y)-4a(m+n)((q+p)(m+n)-lx)+2(ac+ab)x((q+p)(m+n)-lx)+2a(q+p)((q+p)(m+n)-lx))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^2]-[2z(2(q+p)x-2(m+n))((ly-(m+n)^2)^2+(2bc+c+b)(x(m+n)-(q+p)y)^2+a((q+p)(m+n)-lx)^2+2(c+b)(x(m+n)-(q+p)y)(ly-(m+n)^2)+2a((q+p)(m+n)-lx)(ly-(m+n)^2)+2(ac+ab)((q+p)(m+n)-lx)(x(m+n)-(q+p)y))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^3]$

$h_{37}^{(U_1)} = [z(-4(n+m)(ly-(n+m)^2)+2(c+b)x(ly-(n+m)^2)+2a(q+p)(ly-(n+m)^2)-4(c+b)(n+m)(x(n+m)-(q+p)y)+2(2bc+c+b)x(x(n+m)-(q+p)y)+2(ac+ab)(q+p)(x(n+m)-(q+p)y)-4a(n+m)((q+p)(n+m)-lx)+2(ac+ab)x((q+p)(n+m)-lx)+2a(q+p)((q+p)(n+m)-lx))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^2]-[2z(2(q+p)x-2(n+m))((ly-(n+m)^2)^2+(2bc+c+b)(x(n+m)-(q+p)y)^2+a((q+p)(n+m)-lx)^2+2(c+b)(x(n+m)-(q+p)y)(ly-(n+m)^2)+2a((q+p)(n+m)-lx)(ly-(n+m)^2)+2(ac+ab)((q+p)(n+m)-lx)(x(n+m)-(q+p)y))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]$

$h_{38}^{(U_1)} = [z(-2(2bc+c+b)y((n+m)x-y(p+q))+2(ac+ab)(n+m)((n+m)x-y(p+q))-2(ac+ab)y((n+m)(p+q)-lx)+2a(n+m)((n+m)(p+q)-lx)-2(c+b)y(ly-(n+m)^2)+2a(n+m)(ly-(n+m)^2))]/[(-y(p+q)^2+2(n+m)x(p+q)+$

$ly - lx^2 - (n+m)^2)^2] - [2z(2(n+m)x - 2y(p+q))((2bc+c+b)((n+m)x - y(p+q))^2 + a((n+m)(p+q) - lx)^2 + 2(ac+ab)((n+m)(p+q) - lx)((n+m)x - y(p+q)) + 2(c+b)(ly - (n+m)^2)((n+m)x - y(p+q)) + 2a(ly - (n+m)^2)((n+m)(p+q) - lx) + (ly - (n+m)^2)^2)]/[(-y(p+q)^2 + 2(n+m)x(p+q) + ly - lx^2 - (n+m)^2)^3]$

$h_{39}^{(U_1)} = [z(-2(2bc+c+b)y((n+m)x - y(q+p)) + 2(ac+ab)(n+m)((n+m)x - y(q+p)) - 2(ac+ab)y((n+m)(q+p) - lx) + 2a(n+m)((n+m)(q+p) - lx) - 2(c+b)y(ly - (n+m)^2) + 2a(n+m)(ly - (n+m)^2))]/[(-y(q+p)^2 + 2(n+m)x(q+p) + ly - lx^2 - (n+m)^2)^2] - [2z(2(n+m)x - 2y(q+p))((2bc+c+b)((n+m)x - y(q+p))^2 + a((n+m)(q+p) - lx)^2 + 2(ac+ab)((n+m)(q+p) - lx)((n+m)x - y(q+p)) + 2(c+b)(ly - (n+m)^2)((n+m)x - y(q+p)) + 2a(ly - (n+m)^2)((n+m)(q+p) - lx) + (ly - (n+m)^2)^2)]/[(-y(q+p)^2 + 2(n+m)x(q+p) + ly - lx^2 - (n+m)^2)^3]$

$h_{3\ 10}^{(U_1)} = 0$

$h_{44}^{(U_1)} = 0$

$h_{45}^{(U_1)} = 0$

$h_{46}^{(U_1)} = 0$

$h_{47}^{(U_1)} = 0$

$h_{48}^{(U_1)} = 0$

$h_{49}^{(U_1)} = 0$

$h_{4\ 10}^{(U_1)} = 0$

$h_{55}^{(U_1)} = [(2y^2 - 4axy + 2ax^2)z^2]/[2(yl - x^2l - (q+p)^2y + 2(n+m)(q+p)x - (n+m)^2)^2] + [3(y-x^2)^2z^2((yl - (n+m)^2)^2 + a((n+m)(q+p) - xl)^2 + 2a((n+m)(q+p) - xl)(yl - (n+m)^2) + 2(c+b)((n+m)x - (q+p)y)(yl - (n+m)^2) + 2(ac+ab)((n+m)x - (q+p)y)((n+m)(q+p) - xl) + (2bc+c+b)((n+m)x - (q+p)y)^2)]/[(yl - x^2l - (q+p)^2y + 2(n+m)(q+p)x - (n+m)^2)^4] - [2(y-x^2)z^2(2y(yl - (n+m)^2) - 2ax(yl - (n+m)^2) + 2ay((n+m)(q+p) - xl) - 2ax((n+m)(q+p) - xl) + 2(c+b)y((n+m)x - (q+p)y) - 2(ac+ab)x((n+m)x - (q+p)y))]/[(yl - x^2l - (q+p)^2y + 2(n+m)(q+p)x - (n+m)^2)^3]$

$h_{56}^{(U_1)} = [z^2(-4y(m+n) + 4ax(m+n) + 2(c+b)xy + 2a(q+p)y - 2(ac+ab)x^2 - 2a(q+p)x)]/[2(-(m+n)^2 + 2(q+p)x(m+n) - (q+p)^2y + ly - lx^2)^2] - [(y-x^2)z^2(-4(m+n)(ly - (m+n)^2) + 2(c+b)x(ly - (m+n)^2) + 2a(q+p)(ly - (m+n)^2) - 4(c+b)(m+n)(x(m+n) - (q+p)y) + 2(2bc+c+b)x(x(m+n) - (q+p)y) + 2(ac+ab)(q+p)(x(m+n) - (q+$

$p)y)-4a(m+n)((q+p)(m+n)-lx)+2(ac+ab)x((q+p)(m+n)-lx)+2a(q+p)((q+p)(m+n)-lx))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^3]-[z^2(2(q+p)x-2(m+n))(2y(ly-(m+n)^2)-2ax(ly-(m+n)^2)+2(c+b)y(x(m+n)-(q+p)y)-2(ac+ab)x(x(m+n)-(q+p)y)+2ay((q+p)(m+n)-lx)-2ax((q+p)(m+n)-lx))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^3]+[3(y-x^2)z^2(2(q+p)x-2(m+n))((ly-(m+n)^2)^2+(2bc+c+b)(x(m+n)-(q+p)y)^2+a((q+p)(m+n)-lx)^2+2(c+b)(x(m+n)-(q+p)y)(ly-(m+n)^2)+2a((q+p)(m+n)-lx)(ly-(m+n)^2)+2(ac+ab)((q+p)(m+n)-lx)(x(m+n)-(q+p)y))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^4]$

$h_{57}^{(U_1)}=[z^2(-4y(n+m)+4ax(n+m)+2(c+b)xy+2a(q+p)y-2(ac+ab)x^2-2a(q+p)x)]/[2(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^2]-[(y-x^2)z^2(-4(n+m)(ly-(n+m)^2)+2(c+b)x(ly-(n+m)^2)+2a(q+p)(ly-(n+m)^2)-4(c+b)(n+m)(x(n+m)-(q+p)y)+2(2bc+c+b)x(x(n+m)-(q+p)y)+2(ac+ab)(q+p)(x(n+m)-(q+p)y)-4a(n+m)((q+p)(n+m)-lx)+2(ac+ab)x((q+p)(n+m)-lx)+2a(q+p)((q+p)(n+m)-lx))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]-[z^2(2(q+p)x-2(n+m))(2y(ly-(n+m)^2)-2ax(ly-(n+m)^2)+2(c+b)y(x(n+m)-(q+p)y)-2(ac+ab)x(x(n+m)-(q+p)y)+2ay((q+p)(n+m)-lx)-2ax((q+p)(n+m)-lx))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]+[3(y-x^2)z^2(2(q+p)x-2(n+m))((ly-(n+m)^2)^2+(2bc+c+b)(x(n+m)-(q+p)y)^2+a((q+p)(n+m)-lx)^2+2(c+b)(x(n+m)-(q+p)y)(ly-(n+m)^2)+2a((q+p)(n+m)-lx)(ly-(n+m)^2)+2(ac+ab)((q+p)(n+m)-lx)(x(n+m)-(q+p)y))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^4]$

$h_{58}^{(U_1)}=[(-2(c+b)y^2+2(ac+ab)xy+2a(n+m)y-2a(n+m)x)z^2]/[2(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^2]-[(y-x^2)z^2(-2(2bc+c+b)y((n+m)x-y(p+q))+2(ac+ab)(n+m)((n+m)x-y(p+q))-2(ac+ab)y((n+m)(p+q)-lx)+2a(n+m)((n+m)(p+q)-lx)-2(c+b)y(ly-(n+m)^2)+2a(n+m)(ly-(n+m)^2))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]-[z^2(2(n+m)x-2y(p+q))(2(c+b)y((n+m)x-y(p+q))-2(ac+ab)x((n+m)x-y(p+q))+2ay((n+m)(p+q)-lx)-2ax((n+m)(p+q)-lx)+2y(ly-(n+m)^2)-2ax(ly-(n+m)^2))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]+[3(y-x^2)z^2(2(n+m)x-2y(p+q))((2bc+c+b)((n+m)x-y(p+q))^2+a((n+m)(p+q)-lx)^2+2(ac+ab)((n+m)(p+q)-lx)((n+m)x-y(p+q))+2(c+b)(ly-(n+m)^2)((n+m)x-y(p+q))+2a(ly-(n+m)^2)(((n+m)(p+$

$q)-lx)+(ly-(n+m)^2)^2)]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^4]$

$h_{59}^{(U_1)}=[(-2(c+b)y^2+2(ac+ab)xy+2a(n+m)y-2a(n+m)x)z^2]/[2(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^2]-[(y-x^2)z^2(-2(2bc+c+b)y((n+m)x-y(q+p))+2(ac+ab)(n+m)((n+m)x-y(q+p))-2(ac+ab)y((n+m)(q+p)-lx)+2a(n+m)((n+m)(q+p)-lx)-2(c+b)y(ly-(n+m)^2)+2a(n+m)(ly-(n+m)^2))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]-[z^2(2(n+m)x-2y(q+p))(2(c+b)y((n+m)x-y(q+p))-2(ac+ab)x((n+m)x-y(q+p))+2ay((n+m)(q+p)-lx)-2ax((n+m)(q+p)-lx)+2y(ly-(n+m)^2)-2ax(ly-(n+m)^2))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]+[3(y-x^2)z^2(2(n+m)x-2y(q+p))((2bc+c+b)((n+m)x-y(q+p))^2+a((n+m)(q+p)-lx)^2+2(ac+ab)((n+m)(q+p)-lx)((n+m)x-y(q+p))+2(c+b)(ly-(n+m)^2)((n+m)x-y(q+p))+2a(ly-(n+m)^2)((n+m)(q+p)-lx)+(ly-(n+m)^2)^2)]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^4]$

$h_{5\ 10}^{(U_1)}=0$

$h_{66}^{(U_1)}=-[2z^2(2(q+p)x-2(m+n))(-4(m+n)(ly-(m+n)^2)+2(c+b)x(ly-(m+n)^2)+2a(q+p)(ly-(m+n)^2)-4(c+b)(m+n)(x(m+n)-(q+p)y)+2(2bc+c+b)x(x(m+n)-(q+p)y)+2(ac+ab)(q+p)(x(m+n)-(q+p)y)-4a(m+n)((q+p)(m+n)-lx)+2(ac+ab)x((q+p)(m+n)-lx)+2a(q+p)((q+p)(m+n)-lx))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^3]+[2z^2((ly-(m+n)^2)^2+(2bc+c+b)(x(m+n)-(q+p)y)^2+a((q+p)(m+n)-lx)^2+2(c+b)(x(m+n)-(q+p)y)(ly-(m+n)^2)+2a((q+p)(m+n)-lx)(ly-(m+n)^2)+2(ac+ab)((q+p)(m+n)-lx)(x(m+n)-(q+p)y))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^3]+[3z^2(2(q+p)x-2(m+n))^2((ly-(m+n)^2)^2+(2bc+c+b)(x(m+n)-(q+p)y)^2+a((q+p)(m+n)-lx)^2+2(c+b)(x(m+n)-(q+p)y)(ly-(m+n)^2)+2a((q+p)(m+n)-lx)(ly-(m+n)^2)+2(ac+ab)((q+p)(m+n)-lx)(x(m+n)-(q+p)y))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^4]+[z^2(8(m+n)^2-4(ly-(m+n)^2)-4(c+b)(x(m+n)-(q+p)y)-4a((q+p)(m+n)-lx)-8(c+b)x(m+n)-8a(q+p)(m+n)+2(2bc+c+b)x^2+4(ac+ab)(q+p)x+2a(q+p)^2)]/[2(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^2]$

$h_{67}^{(U_1)}=-[2z^2(2(q+p)x-2(n+m))(-4(n+m)(ly-(n+m)^2)+2(c+b)x(ly-(n+m)^2)+2a(q+p)(ly-(n+m)^2)-4(c+b)(n+m)(x(n+m)-(q+p)y)+2(2bc+c+b)x(x(n+m)-(q+p)y)+2(ac+ab)(q+p)(x(n+m)-(q+p)y)-4a(n+m)((q+p)(n+$

$$m)-lx)+2(ac+ab)x((q+p)(n+m)-lx)+2a(q+p)((q+p)(n+m)-lx))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]+[2z^2((ly-(n+m)^2)^2+(2bc+c+b)(x(n+m)-(q+p)y)^2+a((q+p)(n+m)-lx)^2+2(c+b)(x(n+m)-(q+p)y)(ly-(n+m)^2)+2a((q+p)(n+m)-lx)(ly-(n+m)^2)+2(ac+ab)((q+p)(n+m)-lx)(x(n+m)-(q+p)y))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]+[3z^2(2(q+p)x-2(n+m))^2((ly-(n+m)^2)^2+(2bc+c+b)(x(n+m)-(q+p)y)^2+a((q+p)(n+m)-lx)^2+2(c+b)(x(n+m)-(q+p)y)(ly-(n+m)^2)+2a((q+p)(n+m)-lx)(ly-(n+m)^2)+2(ac+ab)((q+p)(n+m)-lx)(x(n+m)-(q+p)y))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^4]+[z^2(8(n+m)^2-4(ly-(n+m)^2)-4(c+b)(x(n+m)-(q+p)y)-4a((q+p)(n+m)-lx)-8(c+b)x(n+m)-8a(q+p)(n+m)+2(2bc+c+b)x^2+4(ac+ab)(q+p)x+2a(q+p)^2)]/[2(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^2]$$

$$h_{68}^{(U_1)}=[z^2(2(ac+ab)((n+m)x-y(p+q))+2a((n+m)(p+q)-lx)-2(ac+ab)y(p+q)+2a(n+m)(p+q)+2a(ly-(n+m)^2)-2(2bc+c+b)xy+4(c+b)(n+m)y+2(ac+ab)(n+m)x-4a(n+m)^2)]/[2(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^2]-[z^2(2(n+m)x-2y(p+q))(2(ac+ab)(p+q)((n+m)x-y(p+q))+2(2bc+c+b)x((n+m)x-y(p+q))-4(c+b)(n+m)((n+m)x-y(p+q))+2a(p+q)((n+m)(p+q)-lx)+2(ac+ab)x((n+m)(p+q)-lx)-4a(n+m)((n+m)(p+q)-lx)+2a(ly-(n+m)^2)(p+q)+2(c+b)x(ly-(n+m)^2)-4(n+m)(ly-(n+m)^2))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]-[z^2(2x(p+q)-2(n+m))(-2(2bc+c+b)y((n+m)x-y(p+q))+2(ac+ab)(n+m)((n+m)x-y(p+q))-2(ac+ab)y((n+m)(p+q)-lx)+2a(n+m)((n+m)(p+q)-lx)-2(c+b)y(ly-(n+m)^2)+2a(n+m)(ly-(n+m)^2))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]-[2xz^2((2bc+c+b)((n+m)x-y(p+q))^2+a((n+m)(p+q)-lx)^2+2(ac+ab)((n+m)(p+q)-lx)((n+m)x-y(p+q))+2(c+b)(ly-(n+m)^2)((n+m)x-y(p+q))+2a(ly-(n+m)^2)((n+m)(p+q)-lx)+(ly-(n+m)^2)^2)]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]+[3z^2(2x(p+q)-2(n+m))(2(n+m)x-2y(p+q))((2bc+c+b)((n+m)x-y(p+q))^2+a((n+m)(p+q)-lx)^2+2(ac+ab)((n+m)(p+q)-lx)((n+m)x-y(p+q))+2(c+b)(ly-(n+m)^2)((n+m)x-y(p+q))+2a(ly-(n+m)^2)((n+m)(p+q)-lx)+(ly-(n+m)^2)^2)]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^4]$$

$$h_{69}^{(U_1)}=[z^2(2(ac+ab)((n+m)x-y(q+p))+2a((n+m)(q+p)-lx)-2(ac+ab)y(q+p)+2a(n+m)(q+p)+2a(ly-(n+m)^2)-2(2bc+c+b)xy+4(c+b)(n+m)y+2(ac+ab)(n+$$

$m)x-4a(n+m)^2)]/[2(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^2]-[z^2(2(n+m)x-2y(q+p))(2(ac+ab)(q+p)((n+m)x-y(q+p))+2(2bc+c+b)x((n+m)x-y(q+p))-4(c+b)(n+m)((n+m)x-y(q+p))+2a(q+p)((n+m)(q+p)-lx)+2(ac+ab)x((n+m)(q+p)-lx)-4a(n+m)((n+m)(q+p)-lx)+2a(ly-(n+m)^2)(q+p)+2(c+b)x(ly-(n+m)^2)-4(n+m)(ly-(n+m)^2))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]-[z^2(2x(q+p)-2(n+m))(-2(2bc+c+b)y((n+m)x-y(q+p))+2(ac+ab)(n+m)((n+m)x-y(q+p))-2(ac+ab)y((n+m)(q+p)-lx)+2a(n+m)((n+m)(q+p)-lx)-2(c+b)y(ly-(n+m)^2)+2a(n+m)(ly-(n+m)^2))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]-[2xz^2((2bc+c+b)((n+m)x-y(q+p))^2+a((n+m)(q+p)-lx)^2+2(ac+ab)((n+m)(q+p)-lx)((n+m)x-y(q+p))+2(c+b)(ly-(n+m)^2)((n+m)x-y(q+p))+2a(ly-(n+m)^2)((n+m)(q+p)-lx)+(ly-(n+m)^2)^2)]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]+[3z^2(2x(q+p)-2(n+m))(2(n+m)x-2y(q+p))((2bc+c+b)((n+m)x-y(q+p))^2+a((n+m)(q+p)-lx)^2+2(ac+ab)((n+m)(q+p)-lx)((n+m)x-y(q+p))+2(c+b)(ly-(n+m)^2)((n+m)x-y(q+p))+2a(ly-(n+m)^2)((n+m)(q+p)-lx)+(ly-(n+m)^2)^2)]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^4]$

$h^{(U_1)}_{6\ 10} = 0$

$h^{(U_1)}_{77} = -[2z^2(2(q+p)x-2(n+m))(-4(n+m)(ly-(n+m)^2)+2(c+b)x(ly-(n+m)^2)+2a(q+p)(ly-(n+m)^2)-4(c+b)(n+m)(x(n+m)-(q+p)y)+2(2bc+c+b)x(x(n+m)-(q+p)y)+2(ac+ab)(q+p)(x(n+m)-(q+p)y)-4a(n+m)((q+p)(n+m)-lx)+2(ac+ab)x((q+p)(n+m)-lx)+2a(q+p)((q+p)(n+m)-lx))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]+[2z^2((ly-(n+m)^2)^2+(2bc+c+b)(x(n+m)-(q+p)y)^2+a((q+p)(n+m)-lx)^2+2(c+b)(x(n+m)-(q+p)y)(ly-(n+m)^2)+2a((q+p)(n+m)-lx)(ly-(n+m)^2)+2(ac+ab)((q+p)(n+m)-lx)(x(n+m)-(q+p)y))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]+[3z^2(2(q+p)x-2(n+m))^2((ly-(n+m)^2)^2+(2bc+c+b)(x(n+m)-(q+p)y)^2+a((q+p)(n+m)-lx)^2+2(c+b)(x(n+m)-(q+p)y)(ly-(n+m)^2)+2a((q+p)(n+m)-lx)(ly-(n+m)^2)+2(ac+ab)((q+p)(n+m)-lx)(x(n+m)-(q+p)y))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^4]+[z^2(8(n+m)^2-4(ly-(n+m)^2)-4(c+b)(x(n+m)-(q+p)y)-4a((q+p)(n+m)-lx)-8(c+b)x(n+m)-8a(q+p)(n+m)+2(2bc+c+b)x^2+4(ac+ab)(q+p)x+2a(q+p)^2)]/[2(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^2]$

$h_{78}^{(U_1)} = [z^2(2(ac+ab)((n+m)x-y(p+q))+2a((n+m)(p+q)-lx)-2(ac+ab)y(p+q)+2a(n+m)(p+q)+2a(ly-(n+m)^2)-2(2bc+c+b)xy+4(c+b)(n+m)y+2(ac+ab)(n+m)x-4a(n+m)^2)]/[2(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^2]-[z^2(2(n+m)x-2y(p+q))(2(ac+ab)(p+q)((n+m)x-y(p+q))+2(2bc+c+b)x((n+m)x-y(p+q))-4(c+b)(n+m)((n+m)x-y(p+q))+2a(p+q)((n+m)(p+q)-lx)+2(ac+ab)x((n+m)(p+q)-lx)-4a(n+m)((n+m)(p+q)-lx)+2a(ly-(n+m)^2)(p+q)+2(c+b)x(ly-(n+m)^2)-4(n+m)(ly-(n+m)^2))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]-[z^2(2x(p+q)-2(n+m))(-2(2bc+c+b)y((n+m)x-y(p+q))+2(ac+ab)(n+m)((n+m)x-y(p+q))-2(ac+ab)y((n+m)(p+q)-lx)+2a(n+m)((n+m)(p+q)-lx)-2(c+b)y(ly-(n+m)^2)+2a(n+m)(ly-(n+m)^2))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]-[2xz^2((2bc+c+b)((n+m)x-y(p+q))^2+a((n+m)(p+q)-lx)^2+2(ac+ab)((n+m)(p+q)-lx)((n+m)x-y(p+q))+2(c+b)(ly-(n+m)^2)((n+m)x-y(p+q))+2a(ly-(n+m)^2)((n+m)(p+q)-lx)+(ly-(n+m)^2)^2)]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]+[3z^2(2x(p+q)-2(n+m))(2(n+m)x-2y(p+q))((2bc+c+b)((n+m)x-y(p+q))^2+a((n+m)(p+q)-lx)^2+2(ac+ab)((n+m)(p+q)-lx)((n+m)x-y(p+q))+2(c+b)(ly-(n+m)^2)((n+m)x-y(p+q))+2a(ly-(n+m)^2)((n+m)(p+q)-lx)+(ly-(n+m)^2)^2)]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^4]$

$h_{79}^{(U_1)} = [z^2(2(ac+ab)((n+m)x-y(q+p))+2a((n+m)(q+p)-lx)-2(ac+ab)y(q+p)+2a(n+m)(q+p)+2a(ly-(n+m)^2)-2(2bc+c+b)xy+4(c+b)(n+m)y+2(ac+ab)(n+m)x-4a(n+m)^2)]/[2(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^2]-[z^2(2(n+m)x-2y(q+p))(2(ac+ab)(q+p)((n+m)x-y(q+p))+2(2bc+c+b)x((n+m)x-y(q+p))-4(c+b)(n+m)((n+m)x-y(q+p))+2a(q+p)((n+m)(q+p)-lx)+2(ac+ab)x((n+m)(q+p)-lx)-4a(n+m)((n+m)(q+p)-lx)+2a(ly-(n+m)^2)(q+p)+2(c+b)x(ly-(n+m)^2)-4(n+m)(ly-(n+m)^2))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]-[z^2(2x(q+p)-2(n+m))(-2(2bc+c+b)y((n+m)x-y(q+p))+2(ac+ab)(n+m)((n+m)x-y(q+p))-2(ac+ab)y((n+m)(q+p)-lx)+2a(n+m)((n+m)(q+p)-lx)-2(c+b)y(ly-(n+m)^2)+2a(n+m)(ly-(n+m)^2))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]-[2xz^2((2bc+c+b)((n+m)x-y(q+p))^2+a((n+m)(q+p)-lx)^2+2(ac+ab)((n+m)(q+p)-lx)((n+m)x-y(q+p))+2(c+b)(ly-(n+m)^2)((n+m)x-y(q+p))+2a(ly-(n+m)^2)((n+m)(q+p)-lx)+(ly-(n+m)^2)^2)]/[(-y(q+p)^2+2(n+$

$m)x(q+p)+ly-lx^2-(n+m)^2)^3]+[3z^2(2x(q+p)-2(n+m))(2(n+m)x-2y(q+p))((2bc+c+b)((n+m)x-y(q+p))^2+a((n+m)(q+p)-lx)^2+2(ac+ab)((n+m)(q+p)-lx)((n+m)x-y(q+p))+2(c+b)(ly-(n+m)^2)((n+m)x-y(q+p))+2a(ly-(n+m)^2)((n+m)(q+p)-lx)+(ly-(n+m)^2)^2)]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^4]$

$h_{7\ 10}^{(U_1)} = 0$

$h_{88}^{(U_1)} = [(2(2bc+c+b)y^2-4(ac+ab)(n+m)y+2a(n+m)^2)z^2]/[2(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^2]-[2z^2(2(n+m)x-2y(p+q))(-2(2bc+c+b)y((n+m)x-y(p+q))+2(ac+ab)(n+m)((n+m)x-y(p+q))-2(ac+ab)y((n+m)(p+q)-lx)+2a(n+m)((n+m)(p+q)-lx)-2(c+b)y(ly-(n+m)^2)+2a(n+m)(ly-(n+m)^2))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]+[2yz^2((2bc+c+b)((n+m)x-y(p+q))^2+a((n+m)(p+q)-lx)^2+2(ac+ab)((n+m)(p+q)-lx)((n+m)x-y(p+q))+2(c+b)(ly-(n+m)^2)((n+m)x-y(p+q))+2a(ly-(n+m)^2)((n+m)(p+q)-lx)+(ly-(n+m)^2)^2)]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]+[3z^2(2(n+m)x-2y(p+q))^2((2bc+c+b)((n+m)x-y(p+q))^2+a((n+m)(p+q)-lx)^2+2(ac+ab)((n+m)(p+q)-lx)((n+m)x-y(p+q))+2(c+b)(ly-(n+m)^2)((n+m)x-y(p+q))+2a(ly-(n+m)^2)((n+m)(p+q)-lx)+(ly-(n+m)^2)^2)]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^4]$

$h_{89}^{(U_1)} = [(2(2bc+c+b)y^2-4(ac+ab)(n+m)y+2a(n+m)^2)z^2]/[2(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^2]-[2z^2(2(n+m)x-2y(q+p))(-2(2bc+c+b)y((n+m)x-y(q+p))+2(ac+ab)(n+m)((n+m)x-y(q+p))-2(ac+ab)y((n+m)(q+p)-lx)+2a(n+m)((n+m)(q+p)-lx)-2(c+b)y(ly-(n+m)^2)+2a(n+m)(ly-(n+m)^2))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]+[2yz^2((2bc+c+b)((n+m)x-y(q+p))^2+a((n+m)(q+p)-lx)^2+2(ac+ab)((n+m)(q+p)-lx)((n+m)x-y(q+p))+2(c+b)(ly-(n+m)^2)((n+m)x-y(q+p))+2a(ly-(n+m)^2)((n+m)(q+p)-lx)+(ly-(n+m)^2)^2)]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]+[3z^2(2(n+m)x-2y(q+p))^2((2bc+c+b)((n+m)x-y(q+p))^2+a((n+m)(q+p)-lx)^2+2(ac+ab)((n+m)(q+p)-lx)((n+m)x-y(q+p))+2(c+b)(ly-(n+m)^2)((n+m)x-y(q+p))+2a(ly-(n+m)^2)((n+m)(q+p)-lx)+(ly-(n+m)^2)^2)]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^4]$

$h_{8\ 10}^{(U_1)} = 0$

$h_{99}^{(U_1)} = [(2(2bc+c+b)y^2 - 4(ac+ab)(n+m)y + 2a(n+m)^2)z^2]/[2(-y(q+p)^2 + 2(n+m)x(q+p) + ly - lx^2 - (n+m)^2)^2] - [2z^2(2(n+m)x - 2y(q+p))(-2(2bc+c+b)y((n+m)x - y(q+p)) + 2(ac+ab)(n+m)((n+m)x - y(q+p)) - 2(ac+ab)y((n+m)(q+p) - lx) + 2a(n+m)((n+m)(q+p) - lx) - 2(c+b)y(ly - (n+m)^2) + 2a(n+m)(ly - (n+m)^2))]/[(-y(q+p)^2 + 2(n+m)x(q+p) + ly - lx^2 - (n+m)^2)^3] + [2yz^2((2bc+c+b)((n+m)x - y(q+p))^2 + a((n+m)(q+p) - lx)^2 + 2(ac+ab)((n+m)(q+p) - lx)((n+m)x - y(q+p)) + 2(c+b)(ly - (n+m)^2)((n+m)x - y(q+p)) + 2a(ly - (n+m)^2)((n+m)(q+p) - lx) + (ly - (n+m)^2)^2)]/[(-y(q+p)^2 + 2(n+m)x(q+p) + ly - lx^2 - (n+m)^2)^3] + [3z^2(2(n+m)x - 2y(q+p))^2((2bc+c+b)((n+m)x - y(q+p))^2 + a((n+m)(q+p) - lx)^2 + 2(ac+ab)((n+m)(q+p) - lx)((n+m)x - y(q+p)) + 2(c+b)(ly - (n+m)^2)((n+m)x - y(q+p)) + 2a(ly - (n+m)^2)((n+m)(q+p) - lx) + (ly - (n+m)^2)^2)]/[(-y(q+p)^2 + 2(n+m)x(q+p) + ly - lx^2 - (n+m)^2)^4]$

$h_{9\ 10}^{(U_1)} = 0$

$h_{10\ 10}^{(U_1)} = 0$

$h_{11}^{(U_2)} = [(2(2bc+c+b)(q+p)^2 - 4(c+b)l(q+p) + 2l^2)t^2]/[2(-lx^2 + 2(n+m)(q+p)x - (q+p)^2y + ly - (n+m)^2)^2] - [2t^2(2(n+m)(q+p) - 2lx)(2(2bc+c+b)(q+p)((q+p)x - n - m) - 2(c+b)l((q+p)x - n - m) + 2(c+b)(q+p)((n+m)(q+p) - lx) - 2l((n+m)(q+p) - lx) + 2(ac+ab)(q+p)(l - (q+p)^2) - 2al(l - (q+p)^2))]/[(-lx^2 + 2(n+m)(q+p)x - (q+p)^2y + ly - (n+m)^2)^3] + [2lt^2((2bc+c+b)((q+p)x - n - m)^2 + ((n+m)(q+p) - lx)^2 + 2(c+b)((n+m)(q+p) - lx)((q+p)x - n - m) + 2(ac+ab)(l - (q+p)^2)((q+p)x - n - m) + 2a(l - (q+p)^2)((n+m)(q+p) - lx) + a(l - (q+p)^2)^2)]/[(-lx^2 + 2(n+m)(q+p)x - (q+p)^2y + ly - (n+m)^2)^3] + [3t^2(2(n+m)(q+p) - 2lx)^2((2bc+c+b)((q+p)x - n - m)^2 + ((n+m)(q+p) - lx)^2 + 2(c+b)((n+m)(q+p) - lx)((q+p)x - n - m) + 2(ac+ab)(l - (q+p)^2)((q+p)x - n - m) + 2a(l - (q+p)^2)((n+m)(q+p) - lx) + a(l - (q+p)^2)^2)]/[(-lx^2 + 2(n+m)(q+p)x - (q+p)^2y + ly - (n+m)^2)^4]$

$h_{12}^{(U_2)} = [3(l - (q+p)^2)t^2(2(n+m)(q+p) - 2lx)((2bc+c+b)((q+p)x - n - m)^2 + ((n+m)(q+p) - lx)^2 + 2(c+b)((n+m)(q+p) - lx)((q+p)x - n - m) + 2(ac+ab)(l - (q+p)^2)((q+p)x - n - m) + 2a(l - (q+p)^2)((n+m)(q+p) - lx) + a(l - (q+p)^2)^2)]/[(-(q+p)^2y + ly - lx^2 + 2(n+m)(q+p)x - (n+m)^2)^4] - [(l - (q+p)^2)t^2(2(2bc+c+b)(q+p)((q+p)x - n - m) - 2(c+b)l((q+p)x - n - m) + 2(c+$

$b)(q+p)((n+m)(q+p)-lx)-2l((n+m)(q+p)-lx)+2(ac+ab)(q+p)(l-(q+p)^2)-2al(l-(q+p)^2))]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^3]$

$h_{13}^{(U_2)}=0$

$h_{14}^{(U_2)}=[(2(2bc+c+b)(q+p)((q+p)x-n-m)-2(c+b)l((q+p)x-n-m)+2(c+b)(q+p)((n+m)(q+p)-lx)-2l((n+m)(q+p)-lx)+2(ac+ab)(q+p)(l-(q+p)^2)-2al(l-(q+p)^2))t]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^2]-[2(2(n+m)(q+p)-2lx)((2bc+c+b)((q+p)x-n-m)^2+((n+m)(q+p)-lx)^2+2(c+b)((n+m)(q+p)-lx)((q+p)x-n-m)+2(ac+ab)(l-(q+p)^2)((q+p)x-n-m)+2a(l-(q+p)^2)((n+m)(q+p)-lx)+a(l-(q+p)^2)^2)t]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^3]$

$h_{15}^{(U_2)}=[2t^2x(((n+m)(q+p)-xl)^2+a(l-(q+p)^2)^2+2a(l-(q+p)^2)((n+m)(q+p)-xl)+2(c+b)((q+p)x-n-m)((n+m)(q+p)-xl)+2(ac+ab)((q+p)x-n-m)(l-(q+p)^2)+(2bc+c+b)((q+p)x-n-m)^2)]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^3]+[3t^2(y-x^2)(2(n+m)(q+p)-2xl)(((n+m)(q+p)-xl)^2+a(l-(q+p)^2)^2+2a(l-(q+p)^2)((n+m)(q+p)-xl)+2(c+b)((q+p)x-n-m)((n+m)(q+p)-xl)+2(ac+ab)((q+p)x-n-m)(l-(q+p)^2)+(2bc+c+b)((q+p)x-n-m)^2)]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^4]-[t^2(y-x^2)(-2l((n+m)(q+p)-xl)+2(c+b)(q+p)((n+m)(q+p)-xl)-2al(l-(q+p)^2)+2(ac+ab)(q+p)(l-(q+p)^2)-2(c+b)((q+p)x-n-m)l+2(2bc+c+b)(q+p)((q+p)x-n-m))]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^3]-[t^2(2(n+m)(q+p)-2xl)(-2x((n+m)(q+p)-xl)+2a((n+m)(q+p)-xl)-2ax(l-(q+p)^2)+2a(l-(q+p)^2)-2(c+b)x((q+p)x-n-m)+2(ac+ab)((q+p)x-n-m))]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^3]+[t^2(-2((n+m)(q+p)-xl)-2a(l-(q+p)^2)+2xl-2al-2(c+b)((q+p)x-n-m)-2(c+b)(q+p)x+2(ac+ab)(q+p))]/[2(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^2]$

$h_{16}^{(U_2)}=[(2(c+b)(q+p)^2-2l(q+p)-2(2bc+c+b)(q+p)+2(c+b)l)t^2]/[2(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^2]-[t^2(2(q+p)x-2(m+n))(2(c+b)(q+p)((q+p)(m+n)-lx)-2l((q+p)(m+n)-lx)+2(2bc+c+b)(q+p)(-m+(q+p)x-n)-2(c+b)l(-m+(q+p)x-n)+2(ac+ab)(q+p)(l-(q+p)^2)-2al(l-(q+p)^2))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^3]-[t^2(2(q+p)(m+n)-2lx)(2(q+p)((q+p)(m+n)-lx)-2(c+b)((q+p)(m+n)-lx)+2(c+b)(q+p)(-m+(q+p)x-n)-2(2bc+c+b)(-m+(q+p)x-n)+2a(q+p)(l-(q+p)^2)-2(ac+ab)(l-(q+p)^2))]/[(-(m+n)^2+$

$2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^3]-[2(q+p)t^2(((q+p)(m+n)-lx)^2+2(c+b)(-m+(q+p)x-n)((q+p)(m+n)-lx)+2a(l-(q+p)^2)((q+p)(m+n)-lx)+2(ac+ab)(l-(q+p)^2)(-m+(q+p)x-n)+(2bc+c+b)(-m+(q+p)x-n)^2+a(l-(q+p)^2)^2)]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^3]+[3t^2(2(q+p)x-2(m+n))(2(q+p)(m+n)-2lx)(((q+p)(m+n)-lx)^2+2(c+b)(-m+(q+p)x-n)((q+p)(m+n)-lx)+2a(l-(q+p)^2)((q+p)(m+n)-lx)+2(ac+ab)(l-(q+p)^2)(-m+(q+p)x-n)+(2bc+c+b)(-m+(q+p)x-n)^2+a(l-(q+p)^2)^2)]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^4]$

$h_{17}^{(U_2)} = [(2(c+b)(q+p)^2-2l(q+p)-2(2bc+c+b)(q+p)+2(c+b)l)t^2]/[2(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^2]-[t^2(2(q+p)x-2(n+m))(2(c+b)(q+p)((q+p)(n+m)-lx)-2l((q+p)(n+m)-lx)+2(2bc+c+b)(q+p)(-n+(q+p)x-m)-2(c+b)l(-n+(q+p)x-m)+2(ac+ab)(q+p)(l-(q+p)^2)-2al(l-(q+p)^2))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]-[t^2(2(q+p)(n+m)-2lx)(2(q+p)((q+p)(n+m)-lx)-2(c+b)((q+p)(n+m)-lx)+2(c+b)(q+p)(-n+(q+p)x-m)-2(2bc+c+b)(-n+(q+p)x-m)+2a(q+p)(l-(q+p)^2)-2(ac+ab)(l-(q+p)^2))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]-[2(q+p)t^2(((q+p)(n+m)-lx)^2+2(c+b)(-n+(q+p)x-m)((q+p)(n+m)-lx)+2a(l-(q+p)^2)((q+p)(n+m)-lx)+2(ac+ab)(l-(q+p)^2)(-n+(q+p)x-m)+(2bc+c+b)(-n+(q+p)x-m)^2+a(l-(q+p)^2)^2)]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]+[3t^2(2(q+p)x-2(n+m))(2(q+p)(n+m)-2lx)(((q+p)(n+m)-lx)^2+2(c+b)(-n+(q+p)x-m)((q+p)(n+m)-lx)+2a(l-(q+p)^2)((q+p)(n+m)-lx)+2(ac+ab)(l-(q+p)^2)(-n+(q+p)x-m)+(2bc+c+b)(-n+(q+p)x-m)^2+a(l-(q+p)^2)^2)]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^4]$

$h_{18}^{(U_2)} = [t^2(-4(ac+ab)(p+q)^2+2(ac+ab)(l-(p+q)^2)+2(2bc+c+b)(x(p+q)-n-m)+2(c+b)((n+m)(p+q)-lx)+2(2bc+c+b)x(p+q)+2(c+b)(n+m)(p+q)+4al(p+q)-2(c+b)lx-2l(n+m))]/[2(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^2]-[t^2(2(n+m)x-2y(p+q))(2(ac+ab)(p+q)(l-(p+q)^2)-2al(l-(p+q)^2)+2(2bc+c+b)(p+q)(x(p+q)-n-m)-2(c+b)l(x(p+q)-n-m)+2(c+b)(p+q)((n+m)(p+q)-lx)-2l((n+m)(p+q)-lx))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]-[t^2(2(n+m)(p+q)-2lx)(-4a(p+q)(l-(p+q)^2)+2(ac+ab)x(l-(p+q)^2)+2a(n+m)(l-(p+q)^2)-4(ac+ab)(p+q)(x(p+q)-n-m)+2(2bc+c+b)x(x(p+q)-n-m)+2(c+b)(n+m)(x(p+q)-n-m)-4a(p+q)((n+m)(p+q)-lx)+2(c+$

$b)x((n+m)(p+q)-lx)+2(n+m)((n+m)(p+q)-lx))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]-[2(n+m)t^2(a(l-(p+q)^2)^2+(2bc+c+b)(x(p+q)-n-m)^2+((n+m)(p+q)-lx)^2+2(ac+ab)(x(p+q)-n-m)(l-(p+q)^2)+2a((n+m)(p+q)-lx)(l-(p+q)^2)+2(c+b)((n+m)(p+q)-lx)(x(p+q)-n-m))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]+[3t^2(2(n+m)(p+q)-2lx)(2(n+m)x-2y(p+q))(a(l-(p+q)^2)^2+(2bc+c+b)(x(p+q)-n-m)^2+((n+m)(p+q)-lx)^2+2(ac+ab)(x(p+q)-n-m)(l-(p+q)^2)+2a((n+m)(p+q)-lx)(l-(p+q)^2)+2(c+b)((n+m)(p+q)-lx)(x(p+q)-n-m))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^4]$

$h_{19}^{(U_2)}=[t^2(-4(ac+ab)(q+p)^2+2(ac+ab)(l-(q+p)^2)+2(2bc+c+b)(x(q+p)-n-m)+2(c+b)((n+m)(q+p)-lx)+2(2bc+c+b)x(q+p)+2(c+b)(n+m)(q+p)+4al(q+p)-2(c+b)lx-2l(n+m))]/[2(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^2]-[t^2(2(n+m)x-2y(q+p))(2(ac+ab)(q+p)(l-(q+p)^2)-2al(l-(q+p)^2)+2(2bc+c+b)(q+p)(x(q+p)-n-m)-2(c+b)l(x(q+p)-n-m)+2(c+b)(q+p)((n+m)(q+p)-lx)-2l((n+m)(q+p)-lx))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]-[t^2(2(n+m)(q+p)-2lx)(-4a(q+p)(l-(q+p)^2)+2(ac+ab)x(l-(q+p)^2)+2a(n+m)(l-(q+p)^2)-4(ac+ab)(q+p)(x(q+p)-n-m)+2(2bc+c+b)x(x(q+p)-n-m)+2(c+b)(n+m)(x(q+p)-n-m)-4a(q+p)((n+m)(q+p)-lx)+2(c+b)x((n+m)(q+p)-lx)+2(n+m)((n+m)(q+p)-lx))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]-[2(n+m)t^2(a(l-(q+p)^2)^2+(2bc+c+b)(x(q+p)-n-m)^2+((n+m)(q+p)-lx)^2+2(ac+ab)(x(q+p)-n-m)(l-(q+p)^2)+2a((n+m)(q+p)-lx)(l-(q+p)^2)+2(c+b)((n+m)(q+p)-lx)(x(q+p)-n-m))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]+[3t^2(2(n+m)(q+p)-2lx)(2(n+m)x-2y(q+p))(a(l-(q+p)^2)^2+(2bc+c+b)(x(q+p)-n-m)^2+((n+m)(q+p)-lx)^2+2(ac+ab)(x(q+p)-n-m)(l-(q+p)^2)+2a((n+m)(q+p)-lx)(l-(q+p)^2)+2(c+b)((n+m)(q+p)-lx)(x(q+p)-n-m))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^4]$

$h_{1\ 10}^{(U_2)}=0$

$h_{22}^{(U_2)}=[3(l-(q+p)^2)^2t^2((2bc+c+b)((q+p)x-n-m)^2+((n+m)(q+p)-lx)^2+2(c+b)((n+m)(q+p)-lx)((q+p)x-n-m)+2(ac+ab)(l-(q+p)^2)((q+p)x-n-m)+2a(l-(q+p)^2)((n+m)(q+p)-lx)+a(l-(q+p)^2)^2)]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^4]$

$$h_{23}^{(U_2)} = 0$$

$$h_{24}^{(U_2)} = -[2(l-(q+p)^2)((2bc+c+b)((q+p)x-n-m)^2+((n+m)(q+p)-lx)^2+2(c+b)((n+m)(q+p)-lx)((q+p)x-n-m)+2(ac+ab)(l-(q+p)^2)((q+p)x-n-m)+2a(l-(q+p)^2)((n+m)(q+p)-lx)+a(l-(q+p)^2)^2)t]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^3]$$

$$h_{25}^{(U_2)} = -[t^2(((n+m)(q+p)-xl)^2+a(l-(q+p)^2)^2+2a(l-(q+p)^2)((n+m)(q+p)-xl)+2(c+b)((q+p)x-n-m)((n+m)(q+p)-xl)+2(ac+ab)((q+p)x-n-m)(l-(q+p)^2)+(2bc+c+b)((q+p)x-n-m)^2)]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^3]+[3t^2(y-x^2)(l-(q+p)^2)(((n+m)(q+p)-xl)^2+a(l-(q+p)^2)^2+2a(l-(q+p)^2)((n+m)(q+p)-xl)+2(c+b)((q+p)x-n-m)((n+m)(q+p)-xl)+2(ac+ab)((q+p)x-n-m)(l-(q+p)^2)+(2bc+c+b)((q+p)x-n-m)^2)]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^4]-[t^2(l-(q+p)^2)(-2x((n+m)(q+p)-xl)+2a((n+m)(q+p)-xl)-2ax(l-(q+p)^2)+2a(l-(q+p)^2)-2(c+b)x((q+p)x-n-m)+2(ac+ab)((q+p)x-n-m))]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^3]Maxima$$

$$h_{26}^{(U_2)} = [3(l-(q+p)^2)t^2(2(q+p)x-2(m+n))(((q+p)(m+n)-lx)^2+2(c+b)(-m+(q+p)x-n)((q+p)(m+n)-lx)+2a(l-(q+p)^2)((q+p)(m+n)-lx)+2(ac+ab)(l-(q+p)^2)(-m+(q+p)x-n)+(2bc+c+b)(-m+(q+p)x-n)^2+a(l-(q+p)^2)^2)]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^4]-[(l-(q+p)^2)t^2(2(q+p)((q+p)(m+n)-lx)-2(c+b)((q+p)(m+n)-lx)+2(c+b)(q+p)(-m+(q+p)x-n)-2(2bc+c+b)(-m+(q+p)x-n)+2a(q+p)(l-(q+p)^2)-2(ac+ab)(l-(q+p)^2))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^3]$$

$$h_{27}^{(U_2)} = [3(l-(q+p)^2)t^2(2(q+p)x-2(n+m))(((q+p)(n+m)-lx)^2+2(c+b)(-n+(q+p)x-m)((q+p)(n+m)-lx)+2a(l-(q+p)^2)((q+p)(n+m)-lx)+2(ac+ab)(l-(q+p)^2)(-n+(q+p)x-m)+(2bc+c+b)(-n+(q+p)x-m)^2+a(l-(q+p)^2)^2)]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^4]-[(l-(q+p)^2)t^2(2(q+p)((q+p)(n+m)-lx)-2(c+b)((q+p)(n+m)-lx)+2(c+b)(q+p)(-n+(q+p)x-m)-2(2bc+c+b)(-n+(q+p)x-m)+2a(q+p)(l-(q+p)^2)-2(ac+ab)(l-(q+p)^2))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]$$

$$h_{28}^{(U_2)} = -[t^2(-4a(p+q)(l-(p+q)^2)+2(ac+ab)x(l-(p+q)^2)+2a(n+m)(l-(p+q)^2)-4(ac+ab)(p+q)(x(p+q)-n-m)+2(2bc+c+b)x(x(p+q)-n-m)+2(c+$$

$b)(n+m)(x(p+q)-n-m)-4a(p+q)((n+m)(p+q)-lx)+2(c+b)x((n+m)(p+q)-lx)+2(n+m)((n+m)(p+q)-lx))(l-(p+q)^2)]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]+[2t^2(p+q)(a(l-(p+q)^2)^2+(2bc+c+b)(x(p+q)-n-m)^2+((n+m)(p+q)-lx)^2+2(ac+ab)(x(p+q)-n-m)(l-(p+q)^2)+2a((n+m)(p+q)-lx)(l-(p+q)^2)+2(c+b)((n+m)(p+q)-lx)(x(p+q)-n-m))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]+[3t^2(2(n+m)x-2y(p+q))(l-(p+q)^2)(a(l-(p+q)^2)^2+(2bc+c+b)(x(p+q)-n-m)^2+((n+m)(p+q)-lx)^2+2(ac+ab)(x(p+q)-n-m)(l-(p+q)^2)+2a((n+m)(p+q)-lx)(l-(p+q)^2)+2(c+b)((n+m)(p+q)-lx)(x(p+q)-n-m))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^4]$

$h_{29}^{(U_2)} = -[t^2(-4a(q+p)(l-(q+p)^2)+2(ac+ab)x(l-(q+p)^2)+2a(n+m)(l-(q+p)^2)-4(ac+ab)(q+p)(x(q+p)-n-m)+2(2bc+c+b)x(x(q+p)-n-m)+2(c+b)(n+m)(x(q+p)-n-m)-4a(q+p)((n+m)(q+p)-lx)+2(c+b)x((n+m)(q+p)-lx)+2(n+m)((n+m)(q+p)-lx))(l-(q+p)^2)]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]+[2t^2(q+p)(a(l-(q+p)^2)^2+(2bc+c+b)(x(q+p)-n-m)^2+((n+m)(q+p)-lx)^2+2(ac+ab)(x(q+p)-n-m)(l-(q+p)^2)+2a((n+m)(q+p)-lx)(l-(q+p)^2)+2(c+b)((n+m)(q+p)-lx)(x(q+p)-n-m))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]+[3t^2(2(n+m)x-2y(q+p))(l-(q+p)^2)(a(l-(q+p)^2)^2+(2bc+c+b)(x(q+p)-n-m)^2+((n+m)(q+p)-lx)^2+2(ac+ab)(x(q+p)-n-m)(l-(q+p)^2)+2a((n+m)(q+p)-lx)(l-(q+p)^2)+2(c+b)((n+m)(q+p)-lx)(x(q+p)-n-m))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^4]$

$h_{2\ 10}^{(U_2)} = 0$

$h_{33}^{(U_2)} = 0$

$h_{34}^{(U_2)} = 0$

$h_{35}^{(U_2)} = 0$

$h_{36}^{(U_2)} = 0$

$h_{37}^{(U_2)} = 0$

$h_{38}^{(U_2)} = 0$

$h_{39}^{(U_2)} = 0$

$h_{3\ 10}^{(U_2)} = 0$

$h_{44}^{(U_2)} = [(2bc+c+b)((q+p)x-n-m)^2+((n+m)(q+p)-lx)^2+2(c+b)((n+m)(q+$

$p)-lx)((q+p)x-n-m)+2(ac+ab)(l-(q+p)^2)((q+p)x-n-m)+2a(l-(q+p)^2)((n+m)(q+p)-lx)+a(l-(q+p)^2)^2]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^2]$

$h_{45}^{(U_2)} = [t(-2x((n+m)(q+p)-xl)+2a((n+m)(q+p)-xl)-2ax(l-(q+p)^2)+2a(l-(q+p)^2)-2(c+b)x((q+p)x-n-m)+2(ac+ab)((q+p)x-n-m))]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^2]-[2t(y-x^2)(((n+m)(q+p)-xl)^2+a(l-(q+p)^2)^2+2a(l-(q+p)^2)((n+m)(q+p)-xl)+2(c+b)((q+p)x-n-m)((n+m)(q+p)-xl)+2(ac+ab)((q+p)x-n-m)(l-(q+p)^2)+(2bc+c+b)((q+p)x-n-m)^2)]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^3]$

$h_{46}^{(U_2)} = [t(2(q+p)((q+p)(m+n)-lx)-2(c+b)((q+p)(m+n)-lx)+2(c+b)(q+p)(-m+(q+p)x-n)-2(2bc+c+b)(-m+(q+p)x-n)+2a(q+p)(l-(q+p)^2)-2(ac+ab)(l-(q+p)^2))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^2]-[2t(2(q+p)x-2(m+n))(((q+p)(m+n)-lx)^2+2(c+b)(-m+(q+p)x-n)((q+p)(m+n)-lx)+2a(l-(q+p)^2)((q+p)(m+n)-lx)+2(ac+ab)(l-(q+p)^2)(-m+(q+p)x-n)+(2bc+c+b)(-m+(q+p)x-n)^2+a(l-(q+p)^2)^2)]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^3]$

$h_{47}^{(U_2)} = [t(2(q+p)((q+p)(n+m)-lx)-2(c+b)((q+p)(n+m)-lx)+2(c+b)(q+p)(-n+(q+p)x-m)-2(2bc+c+b)(-n+(q+p)x-m)+2a(q+p)(l-(q+p)^2)-2(ac+ab)(l-(q+p)^2))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^2]-[2t(2(q+p)x-2(n+m))(((q+p)(n+m)-lx)^2+2(c+b)(-n+(q+p)x-m)((q+p)(n+m)-lx)+2a(l-(q+p)^2)((q+p)(n+m)-lx)+2(ac+ab)(l-(q+p)^2)(-n+(q+p)x-m)+(2bc+c+b)(-n+(q+p)x-m)^2+a(l-(q+p)^2)^2)]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]$

$h_{48}^{(U_2)} = [t(-4a(p+q)(l-(p+q)^2)+2(ac+ab)x(l-(p+q)^2)+2a(n+m)(l-(p+q)^2)-4(ac+ab)(p+q)(x(p+q)-n-m)+2(2bc+c+b)x(x(p+q)-n-m)+2(c+b)(n+m)(x(p+q)-n-m)-4a(p+q)((n+m)(p+q)-lx)+2(c+b)x((n+m)(p+q)-lx)+2(n+m)((n+m)(p+q)-lx))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^2]-[2t(2(n+m)x-2y(p+q))(a(l-(p+q)^2)^2+(2bc+c+b)(x(p+q)-n-m)^2+((n+m)(p+q)-lx)^2+2(ac+ab)(x(p+q)-n-m)(l-(p+q)^2)+2a((n+m)(p+q)-lx)(l-(p+q)^2)+2(c+b)((n+m)(p+q)-lx)(x(p+q)-n-m))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]$

$h_{49}^{(U_2)} = [t(-4a(q+p)(l-(q+p)^2)+2(ac+ab)x(l-(q+p)^2)+2a(n+m)(l-(q+p)^2)-4(ac+ab)(q+p)(x(q+p)-n-m)+2(2bc+c+b)x(x(q+p)-n-m)+2(c+b)(n+m)(x(q+p)-n-m)-4a(q+p)((n+m)(q+p)-lx)+2(c+b)x((n+m)(q+p)-lx)+2(n+m)((n+$

$m)(q+p)-lx))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^2]-[2t(2(n+m)x-2y(q+p))(a(l-(q+p)^2)^2+(2bc+c+b)(x(q+p)-n-m)^2+((n+m)(q+p)-lx)^2+2(ac+ab)(x(q+p)-n-m)(l-(q+p)^2)+2a((n+m)(q+p)-lx)(l-(q+p)^2)+2(c+b)((n+m)(q+p)-lx)(x(q+p)-n-m))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]$

$h_{4\ 10}^{(U_2)} = 0$

$h_{55}^{(U_2)} = [t^2(2x^2-4ax+2a)]/[2(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^2]+[3t^2(y-x^2)^2(((n+m)(q+p)-xl)^2+a(l-(q+p)^2)^2+2a(l-(q+p)^2)((n+m)(q+p)-xl)+2(c+b)((q+p)x-n-m)((n+m)(q+p)-xl)+2(ac+ab)((q+p)x-n-m)(l-(q+p)^2)+(2bc+c+b)((q+p)x-n-m)^2)]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^4]-[2t^2(y-x^2)(-2x((n+m)(q+p)-xl)+2a((n+m)(q+p)-xl)-2ax(l-(q+p)^2)+2a(l-(q+p)^2)-2(c+b)x((q+p)x-n-m)+2(ac+ab)((q+p)x-n-m))]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^3]$

$h_{56}^{(U_2)} = [t^2(-2(q+p)x+2(c+b)x+2a(q+p)-2(ac+ab))]/[2(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^2]-[t^2(2(q+p)x-2(m+n))(-2x((q+p)(m+n)-lx)+2a((q+p)(m+n)-lx)-2(c+b)x(-m+(q+p)x-n)+2(ac+ab)(-m+(q+p)x-n)-2a(l-(q+p)^2)x+2a(l-(q+p)^2))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^3]-[t^2(y-x^2)(2(q+p)((q+p)(m+n)-lx)-2(c+b)((q+p)(m+n)-lx)+2(c+b)(q+p)(-m+(q+p)x-n)-2(2bc+c+b)(-m+(q+p)x-n)+2a(q+p)(l-(q+p)^2)-2(ac+ab)(l-(q+p)^2))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^3]+[3t^2(y-x^2)(2(q+p)x-2(m+n))(((q+p)(m+n)-lx)^2+2(c+b)(-m+(q+p)x-n)((q+p)(m+n)-lx)+2a(l-(q+p)^2)((q+p)(m+n)-lx)+2(ac+ab)(l-(q+p)^2)(-m+(q+p)x-n)+(2bc+c+b)(-m+(q+p)x-n)^2+a(l-(q+p)^2)^2)]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^4]$

$h_{57}^{(U_2)} = [t^2(-2(q+p)x+2(c+b)x+2a(q+p)-2(ac+ab))]/[2(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^2]-[t^2(2(q+p)x-2(n+m))(-2x((q+p)(n+m)-lx)+2a((q+p)(n+m)-lx)-2(c+b)x(-n+(q+p)x-m)+2(ac+ab)(-n+(q+p)x-m)-2a(l-(q+p)^2)x+2a(l-(q+p)^2))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]-[t^2(y-x^2)(2(q+p)((q+p)(n+m)-lx)-2(c+b)((q+p)(n+m)-lx)+2(c+b)(q+p)(-n+(q+p)x-m)-2(2bc+c+b)(-n+(q+p)x-m)+2a(q+p)(l-(q+p)^2)-2(ac+ab)(l-(q+p)^2))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]+[3t^2(y-x^2)(2(q+p)x-2(n+m))(((q+p)(n+m)-lx)^2+2(c+b)(-n+(q+p)x-m)((q+p)(n+m)-lx)+2a(l-$

$(q+p)^2)((q+p)(n+m)-lx)+2(ac+ab)(l-(q+p)^2)(-n+(q+p)x-m)+(2bc+c+b)(-n+(q+p)x-m)^2+a(l-(q+p)^2)^2)]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^4]$

$h_{58}^{(U_2)} = [t^2(4ax(p+q)-4a(p+q)-2(c+b)x^2-2(n+m)x+2(ac+ab)x+2a(n+m))]/[2(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^2]-[t^2(y-x^2)(-4a(p+q)(l-(p+q)^2)+2(ac+ab)x(l-(p+q)^2)+2a(n+m)(l-(p+q)^2)-4(ac+ab)(p+q)(x(p+q)-n-m)+2(2bc+c+b)x(x(p+q)-n-m)+2(c+b)(n+m)(x(p+q)-n-m)-4a(p+q)((n+m)(p+q)-lx)+2(c+b)x((n+m)(p+q)-lx)+2(n+m)((n+m)(p+q)-lx))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]-[t^2(2(n+m)x-2y(p+q))(-2ax(l-(p+q)^2)+2a(l-(p+q)^2)-2(c+b)x(x(p+q)-n-m)+2(ac+ab)(x(p+q)-n-m)-2x((n+m)(p+q)-lx)+2a((n+m)(p+q)-lx))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]+[3t^2(y-x^2)(2(n+m)x-2y(p+q))(a(l-(p+q)^2)^2+(2bc+c+b)(x(p+q)-n-m)^2+((n+m)(p+q)-lx)^2+2(ac+ab)(x(p+q)-n-m)(l-(p+q)^2)+2a((n+m)(p+q)-lx)(l-(p+q)^2)+2(c+b)((n+m)(p+q)-lx)(x(p+q)-n-m))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^4]$

$h_{59}^{(U_2)} = [t^2(4ax(q+p)-4a(q+p)-2(c+b)x^2-2(n+m)x+2(ac+ab)x+2a(n+m))]/[2(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^2]-[t^2(y-x^2)(-4a(q+p)(l-(q+p)^2)+2(ac+ab)x(l-(q+p)^2)+2a(n+m)(l-(q+p)^2)-4(ac+ab)(q+p)(x(q+p)-n-m)+2(2bc+c+b)x(x(q+p)-n-m)+2(c+b)(n+m)(x(q+p)-n-m)-4a(q+p)((n+m)(q+p)-lx)+2(c+b)x((n+m)(q+p)-lx)+2(n+m)((n+m)(q+p)-lx))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]-[t^2(2(n+m)x-2y(q+p))(-2ax(l-(q+p)^2)+2a(l-(q+p)^2)-2(c+b)x(x(q+p)-n-m)+2(ac+ab)(x(q+p)-n-m)-2x((n+m)(q+p)-lx)+2a((n+m)(q+p)-lx))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]+[3t^2(y-x^2)(2(n+m)x-2y(q+p))(a(l-(q+p)^2)^2+(2bc+c+b)(x(q+p)-n-m)^2+((n+m)(q+p)-lx)^2+2(ac+ab)(x(q+p)-n-m)(l-(q+p)^2)+2a((n+m)(q+p)-lx)(l-(q+p)^2)+2(c+b)((n+m)(q+p)-lx)(x(q+p)-n-m))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^4]$

$h_{5\ 10}^{(U_2)} = 0$

$h_{66}^{(U_2)} = [(2(q+p)^2-4(c+b)(q+p)+2(2bc+c+b))t^2]/[2(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^2]-[2t^2(2(q+p)x-2(m+n))(2(q+p)((q+p)(m+n)-lx)-2(c+b)((q+p)(m+n)-lx)+2(c+b)(q+p)(-m+(q+p)x-n)-2(2bc+c+b)(-m+$

$(q+p)x-n)+2a(q+p)(l-(q+p)^2)-2(ac+ab)(l-(q+p)^2))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^3]+[2t^2(((q+p)(m+n)-lx)^2+2(c+b)(-m+(q+p)x-n)((q+p)(m+n)-lx)+2a(l-(q+p)^2)((q+p)(m+n)-lx)+2(ac+ab)(l-(q+p)^2)(-m+(q+p)x-n)+(2bc+c+b)(-m+(q+p)x-n)^2+a(l-(q+p)^2)^2)]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^3]+[3t^2(2(q+p)x-2(m+n))^2(((q+p)(m+n)-lx)^2+2(c+b)(-m+(q+p)x-n)((q+p)(m+n)-lx)+2a(l-(q+p)^2)((q+p)(m+n)-lx)+2(ac+ab)(l-(q+p)^2)(-m+(q+p)x-n)+(2bc+c+b)(-m+(q+p)x-n)^2+a(l-(q+p)^2)^2)]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^4]$

$h_{67}^{(U_2)}=[(2(q+p)^2-4(c+b)(q+p)+2(2bc+c+b))t^2]/[2(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^2]-[2t^2(2(q+p)x-2(n+m))(2(q+p)((q+p)(n+m)-lx)-2(c+b)((q+p)(n+m)-lx)+2(c+b)(q+p)(-n+(q+p)x-m)-2(2bc+c+b)(-n+(q+p)x-m)+2a(q+p)(l-(q+p)^2)-2(ac+ab)(l-(q+p)^2))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]+[2t^2(((q+p)(n+m)-lx)^2+2(c+b)(-n+(q+p)x-m)((q+p)(n+m)-lx)+2a(l-(q+p)^2)((q+p)(n+m)-lx)+2(ac+ab)(l-(q+p)^2)(-n+(q+p)x-m)+(2bc+c+b)(-n+(q+p)x-m)^2+a(l-(q+p)^2)^2)]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]+[3t^2(2(q+p)x-2(n+m))^2(((q+p)(n+m)-lx)^2+2(c+b)(-n+(q+p)x-m)((q+p)(n+m)-lx)+2a(l-(q+p)^2)((q+p)(n+m)-lx)+2(ac+ab)(l-(q+p)^2)(-n+(q+p)x-m)+(2bc+c+b)(-n+(q+p)x-m)^2+a(l-(q+p)^2)^2)]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^4]$

$h_{68}^{(U_2)}=[t^2(-4a(p+q)^2+2a(l-(p+q)^2)+2(c+b)(x(p+q)-n-m)+2((n+m)(p+q)-lx)+2(c+b)x(p+q)+2(n+m)(p+q)+4(ac+ab)(p+q)-2(2bc+c+b)x-2(c+b)(n+m))]/[2(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^2]-[t^2(2(n+m)x-2y(p+q))(2a(p+q)(l-(p+q)^2)-2(ac+ab)(l-(p+q)^2)+2(c+b)(p+q)(x(p+q)-n-m)-2(2bc+c+b)(x(p+q)-n-m)+2(p+q)((n+m)(p+q)-lx)-2(c+b)((n+m)(p+q)-lx))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]-[t^2(2x(p+q)-2(n+m))(-4a(p+q)(l-(p+q)^2)+2(ac+ab)x(l-(p+q)^2)+2a(n+m)(l-(p+q)^2)-4(ac+ab)(p+q)(x(p+q)-n-m)+2(2bc+c+b)x(x(p+q)-n-m)+2(c+b)(n+m)(x(p+q)-n-m)-4a(p+q)((n+m)(p+q)-lx)+2(c+b)x((n+m)(p+q)-lx)+2(n+m)((n+m)(p+q)-lx))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]-[2t^2x(a(l-(p+q)^2)^2+(2bc+c+b)(x(p+q)-n-m)^2+((n+m)(p+$

$q)-lx)^2+2(ac+ab)(x(p+q)-n-m)(l-(p+q)^2)+2a((n+m)(p+q)-lx)(l-(p+q)^2)+2(c+b)((n+m)(p+q)-lx)(x(p+q)-n-m))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]+[3t^2(2x(p+q)-2(n+m))(2(n+m)x-2y(p+q))(a(l-(p+q)^2)^2+(2bc+c+b)(x(p+q)-n-m)^2+((n+m)(p+q)-lx)^2+2(ac+ab)(x(p+q)-n-m)(l-(p+q)^2)+2a((n+m)(p+q)-lx)(l-(p+q)^2)+2(c+b)((n+m)(p+q)-lx)(x(p+q)-n-m))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^4]$

$h_{69}^{(U_2)}=[t^2(-4a(q+p)^2+2a(l-(q+p)^2)+2(c+b)(x(q+p)-n-m)+2((n+m)(q+p)-lx)+2(c+b)x(q+p)+2(n+m)(q+p)+4(ac+ab)(q+p)-2(2bc+c+b)x-2(c+b)(n+m))]/[2(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^2]-[t^2(2(n+m)x-2y(q+p))(2a(q+p)(l-(q+p)^2)-2(ac+ab)(l-(q+p)^2)+2(c+b)(q+p)(x(q+p)-n-m)-2(2bc+c+b)(x(q+p)-n-m)+2(q+p)((n+m)(q+p)-lx)-2(c+b)((n+m)(q+p)-lx))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]-[t^2(2x(q+p)-2(n+m))(-4a(q+p)(l-(q+p)^2)+2(ac+ab)x(l-(q+p)^2)+2a(n+m)(l-(q+p)^2)-4(ac+ab)(q+p)(x(q+p)-n-m)+2(2bc+c+b)x(x(q+p)-n-m)+2(c+b)(n+m)(x(q+p)-n-m)-4a(q+p)((n+m)(q+p)-lx)+2(c+b)x((n+m)(q+p)-lx)+2(n+m)((n+m)(q+p)-lx))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]-[2t^2x(a(l-(q+p)^2)^2+(2bc+c+b)(x(q+p)-n-m)^2+((n+m)(q+p)-lx)^2+2(ac+ab)(x(q+p)-n-m)(l-(q+p)^2)+2a((n+m)(q+p)-lx)(l-(q+p)^2)+2(c+b)((n+m)(q+p)-lx)(x(q+p)-n-m))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]+[3t^2(2x(q+p)-2(n+m))(2(n+m)x-2y(q+p))(a(l-(q+p)^2)^2+(2bc+c+b)(x(q+p)-n-m)^2+((n+m)(q+p)-lx)^2+2(ac+ab)(x(q+p)-n-m)(l-(q+p)^2)+2a((n+m)(q+p)-lx)(l-(q+p)^2)+2(c+b)((n+m)(q+p)-lx)(x(q+p)-n-m))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^4]$

$h_{6\ 10}^{(U_2)}=0$

$h_{77}^{(U_2)}=[(2(q+p)^2-4(c+b)(q+p)+2(2bc+c+b))t^2]/[2(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^2]-[2t^2(2(q+p)x-2(n+m))(2(q+p)((q+p)(n+m)-lx)-2(c+b)((q+p)(n+m)-lx)+2(c+b)(q+p)(-n+(q+p)x-m)-2(2bc+c+b)(-n+(q+p)x-m)+2a(q+p)(l-(q+p)^2)-2(ac+ab)(l-(q+p)^2))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]+[2t^2(((q+p)(n+m)-lx)^2+2(c+b)(-n+(q+p)x-m)((q+p)(n+m)-lx)+2a(l-(q+p)^2)((q+p)(n+m)-lx)+2(ac+ab)(l-(q+$

$p)^2)(-n+(q+p)x-m)+(2bc+c+b)(-n+(q+p)x-m)^2+a(l-(q+p)^2)^2)]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]+[3t^2(2(q+p)x-2(n+m))^2(((q+p)(n+m)-lx)^2+2(c+b)(-n+(q+p)x-m)((q+p)(n+m)-lx)+2a(l-(q+p)^2)((q+p)(n+m)-lx)+2(ac+ab)(l-(q+p)^2)(-n+(q+p)x-m)+(2bc+c+b)(-n+(q+p)x-m)^2+a(l-(q+p)^2)^2)]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^4]$

$h_{78}^{(U_2)} = [t^2(-4a(p+q)^2+2a(l-(p+q)^2)+2(c+b)(x(p+q)-n-m)+2((n+m)(p+q)-lx)+2(c+b)x(p+q)+2(n+m)(p+q)+4(ac+ab)(p+q)-2(2bc+c+b)x-2(c+b)(n+m))]/[2(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^2]-[t^2(2(n+m)x-2y(p+q))(2a(p+q)(l-(p+q)^2)-2(ac+ab)(l-(p+q)^2)+2(c+b)(p+q)(x(p+q)-n-m)-2(2bc+c+b)(x(p+q)-n-m)+2(p+q)((n+m)(p+q)-lx)-2(c+b)((n+m)(p+q)-lx))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]-[t^2(2x(p+q)-2(n+m))(-4a(p+q)(l-(p+q)^2)+2(ac+ab)x(l-(p+q)^2)+2a(n+m)(l-(p+q)^2)-4(ac+ab)(p+q)(x(p+q)-n-m)+2(2bc+c+b)x(x(p+q)-n-m)+2(c+b)(n+m)(x(p+q)-n-m)-4a(p+q)((n+m)(p+q)-lx)+2(c+b)x((n+m)(p+q)-lx)+2(n+m)((n+m)(p+q)-lx))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]-[2t^2x(a(l-(p+q)^2)^2+(2bc+c+b)(x(p+q)-n-m)^2+((n+m)(p+q)-lx)^2+2(ac+ab)(x(p+q)-n-m)(l-(p+q)^2)+2a((n+m)(p+q)-lx)(l-(p+q)^2)+2(c+b)((n+m)(p+q)-lx)(x(p+q)-n-m))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]+[3t^2(2x(p+q)-2(n+m))(2(n+m)x-2y(p+q))(a(l-(p+q)^2)^2+(2bc+c+b)(x(p+q)-n-m)^2+((n+m)(p+q)-lx)^2+2(ac+ab)(x(p+q)-n-m)(l-(p+q)^2)+2a((n+m)(p+q)-lx)(l-(p+q)^2)+2(c+b)((n+m)(p+q)-lx)(x(p+q)-n-m))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^4]$

$h_{79}^{(U_2)} = [t^2(-4a(q+p)^2+2a(l-(q+p)^2)+2(c+b)(x(q+p)-n-m)+2((n+m)(q+p)-lx)+2(c+b)x(q+p)+2(n+m)(q+p)+4(ac+ab)(q+p)-2(2bc+c+b)x-2(c+b)(n+m))]/[2(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^2]-[t^2(2(n+m)x-2y(q+p))(2a(q+p)(l-(q+p)^2)-2(ac+ab)(l-(q+p)^2)+2(c+b)(q+p)(x(q+p)-n-m)-2(2bc+c+b)(x(q+p)-n-m)+2(q+p)((n+m)(q+p)-lx)-2(c+b)((n+m)(q+p)-lx))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]-[t^2(2x(q+p)-2(n+m))(-4a(q+p)(l-(q+p)^2)+2(ac+ab)x(l-(q+p)^2)+2a(n+m)(l-(q+p)^2)-4(ac+ab)(q+p)(x(q+p)-n-m)+2(2bc+c+b)x(x(q+p)-n-m)+2(c+$

$b)(n+m)(x(q+p)-n-m)-4a(q+p)((n+m)(q+p)-lx)+2(c+b)x((n+m)(q+p)-lx)+2(n+m)((n+m)(q+p)-lx))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]-[2t^2x(a(l-(q+p)^2)^2+(2bc+c+b)(x(q+p)-n-m)^2+((n+m)(q+p)-lx)^2+2(ac+ab)(x(q+p)-n-m)(l-(q+p)^2)+2a((n+m)(q+p)-lx)(l-(q+p)^2)+2(c+b)((n+m)(q+p)-lx)(x(q+p)-n-m))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]+[3t^2(2x(q+p)-2(n+m))(2(n+m)x-2y(q+p))(a(l-(q+p)^2)^2+(2bc+c+b)(x(q+p)-n-m)^2+((n+m)(q+p)-lx)^2+2(ac+ab)(x(q+p)-n-m)(l-(q+p)^2)+2a((n+m)(q+p)-lx)(l-(q+p)^2)+2(c+b)((n+m)(q+p)-lx)(x(q+p)-n-m))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^4]$

$h_{7\ 10}^{(U_2)} = 0$

$h_{88}^{(U_2)} = [t^2(8a(p+q)^2-4a(l-(p+q)^2)-4(ac+ab)(x(p+q)-n-m)-4a((n+m)(p+q)-lx)-8(ac+ab)x(p+q)-8a(n+m)(p+q)+2(2bc+c+b)x^2+4(c+b)(n+m)x+2(n+m)^2)]/[2(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^2]-[2t^2(2(n+m)x-2y(p+q))(-4a(p+q)(l-(p+q)^2)+2(ac+ab)x(l-(p+q)^2)+2a(n+m)(l-(p+q)^2)-4(ac+ab)(p+q)(x(p+q)-n-m)+2(2bc+c+b)x(x(p+q)-n-m)+2(c+b)(n+m)(x(p+q)-n-m)-4a(p+q)((n+m)(p+q)-lx)+2(c+b)x((n+m)(p+q)-lx)+2(n+m)((n+m)(p+q)-lx))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]+[2t^2y(a(l-(p+q)^2)^2+(2bc+c+b)(x(p+q)-n-m)^2+((n+m)(p+q)-lx)^2+2(ac+ab)(x(p+q)-n-m)(l-(p+q)^2)+2a((n+m)(p+q)-lx)(l-(p+q)^2)+2(c+b)((n+m)(p+q)-lx)(x(p+q)-n-m))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]+[3t^2(2(n+m)x-2y(p+q))^2(a(l-(p+q)^2)^2+(2bc+c+b)(x(p+q)-n-m)^2+((n+m)(p+q)-lx)^2+2(ac+ab)(x(p+q)-n-m)(l-(p+q)^2)+2a((n+m)(p+q)-lx)(l-(p+q)^2)+2(c+b)((n+m)(p+q)-lx)(x(p+q)-n-m))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^4]$

$h_{89}^{(U_2)} = [t^2(8a(q+p)^2-4a(l-(q+p)^2)-4(ac+ab)(x(q+p)-n-m)-4a((n+m)(q+p)-lx)-8(ac+ab)x(q+p)-8a(n+m)(q+p)+2(2bc+c+b)x^2+4(c+b)(n+m)x+2(n+m)^2)]/[2(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^2]-[2t^2(2(n+m)x-2y(q+p))(-4a(q+p)(l-(q+p)^2)+2(ac+ab)x(l-(q+p)^2)+2a(n+m)(l-(q+p)^2)-4(ac+ab)(q+p)(x(q+p)-n-m)+2(2bc+c+b)x(x(q+p)-n-m)+2(c+b)(n+m)(x(q+p)-n-m)-4a(q+p)((n+m)(q+p)-lx)+2(c+b)x((n+m)(q+p)-lx)+2(n+m)((n+m)(q+p)-lx))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]+[2t^2y(a(l-(q+$

$p)^2)^2+(2bc+c+b)(x(q+p)-n-m)^2+((n+m)(q+p)-lx)^2+2(ac+ab)(x(q+p)-n-m)(l-(q+p)^2)+2a((n+m)(q+p)-lx)(l-(q+p)^2)+2(c+b)((n+m)(q+p)-lx)(x(q+p)-n-m))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]+[3t^2(2(n+m)x-2y(q+p))^2(a(l-(q+p)^2)^2+(2bc+c+b)(x(q+p)-n-m)^2+((n+m)(q+p)-lx)^2+2(ac+ab)(x(q+p)-n-m)(l-(q+p)^2)+2a((n+m)(q+p)-lx)(l-(q+p)^2)+2(c+b)((n+m)(q+p)-lx)(x(q+p)-n-m))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^4]$

$h^{(U_2)}_{8\ 10}=0$

$h^{(U_2)}_{99}=[t^2(8a(q+p)^2-4a(l-(q+p)^2)-4(ac+ab)(x(q+p)-n-m)-4a((n+m)(q+p)-lx)-8(ac+ab)x(q+p)-8a(n+m)(q+p)+2(2bc+c+b)x^2+4(c+b)(n+m)x+2(n+m)^2)]/[2(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^2]-[2t^2(2(n+m)x-2y(q+p))(-4a(q+p)(l-(q+p)^2)+2(ac+ab)x(l-(q+p)^2)+2a(n+m)(l-(q+p)^2)-4(ac+ab)(q+p)(x(q+p)-n-m)+2(2bc+c+b)x(x(q+p)-n-m)+2(c+b)(n+m)(x(q+p)-n-m)-4a(q+p)((n+m)(q+p)-lx)+2(c+b)x((n+m)(q+p)-lx)+2(n+m)((n+m)(q+p)-lx))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]+[2t^2y(a(l-(q+p)^2)^2+(2bc+c+b)(x(q+p)-n-m)^2+((n+m)(q+p)-lx)^2+2(ac+ab)(x(q+p)-n-m)(l-(q+p)^2)+2a((n+m)(q+p)-lx)(l-(q+p)^2)+2(c+b)((n+m)(q+p)-lx)(x(q+p)-n-m))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]+[3t^2(2(n+m)x-2y(q+p))^2(a(l-(q+p)^2)^2+(2bc+c+b)(x(q+p)-n-m)^2+((n+m)(q+p)-lx)^2+2(ac+ab)(x(q+p)-n-m)(l-(q+p)^2)+2a((n+m)(q+p)-lx)(l-(q+p)^2)+2(c+b)((n+m)(q+p)-lx)(x(q+p)-n-m))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^4]$

$h^{(U_2)}_{9\ 10}=0$

$h^{(U_2)}_{10\ 10}=0$

$h^{(U_3)}_{11}=[s^2(8(2bc+c+b)x^2-4(2bc+c+b)(y-x^2)-4a(c+b)((q+p)x-n-m)-4(c+b)((n+m)x-(q+p)y)-8a(c+b)(q+p)x-8(c+b)(n+m)x+2a(q+p)^2+4a(n+m)(q+p)+2(n+m)^2)]/[2(-lx^2+2(n+m)(q+p)x-(q+p)^2y+ly-(n+m)^2)^2]-[2s^2(2(n+m)(q+p)-2lx)(-4(2bc+c+b)x(y-x^2)+2a(c+b)(q+p)(y-x^2)+2(c+b)(n+m)(y-x^2)-4a(c+b)x((q+p)x-n-m)+2a(q+p)((q+p)x-n-m)+2a(n+m)((q+p)x-n-m)-4(c+b)x((n+m)x-(q+p)y)+2a(q+p)((n+m)x-(q+p)y)+2(n+m)((n+m)x-(q+p)y))]/[(-lx^2+2(n+m)(q+p)x-(q+p)^2y+ly-(n+m)^2)^3]+[2ls^2((2bc+c+b)(y-x^2)^2+a((q+p)x-n-m)^2+((n+m)x-(q+p)y)^2+2a(c+b)((q+p)x-n-$

$m)(y-x^2)+2(c+b)((n+m)x-(q+p)y)(y-x^2)+2a((n+m)x-(q+p)y)((q+p)x-n-m))]/[(-lx^2+2(n+m)(q+p)x-(q+p)^2y+ly-(n+m)^2)^3]+[3s^2(2(n+m)(q+p)-2lx)^2((2bc+c+b)(y-x^2)^2+a((q+p)x-n-m)^2+((n+m)x-(q+p)y)^2+2a(c+b)((q+p)x-n-m)(y-x^2)+2(c+b)((n+m)x-(q+p)y)(y-x^2)+2a((n+m)x-(q+p)y)((q+p)x-n-m))]/[(-lx^2+2(n+m)(q+p)x-(q+p)^2y+ly-(n+m)^2)^4]$

$h_{12}^{(U_3)}=[s^2(-4(c+b)(-q-p)x-4(2bc+c+b)x+2a(-q-p)(q+p)+2a(c+b)(q+p)+2(n+m)(-q-p)+2(c+b)(n+m))]/[2(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^2]+[3(l-(q+p)^2)s^2(2(n+m)(q+p)-2lx)((((n+m)x-(q+p)y)^2+(2bc+c+b)(y-x^2)^2+2(c+b)(y-x^2)((n+m)x-(q+p)y)+2a((q+p)x-n-m)((n+m)x-(q+p)y)+2a(c+b)((q+p)x-n-m)(y-x^2)+a((q+p)x-n-m)^2)]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^4]-[(l-(q+p)^2)s^2(-4(c+b)x((n+m)x-(q+p)y)+2a(q+p)((n+m)x-(q+p)y)+2(n+m)((n+m)x-(q+p)y)-4(2bc+c+b)x(y-x^2)+2a(c+b)(q+p)(y-x^2)+2(c+b)(n+m)(y-x^2)-4a(c+b)x((q+p)x-n-m)+2a(q+p)((q+p)x-n-m)+2a(n+m)((q+p)x-n-m))]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^3]-[s^2(2(n+m)(q+p)-2lx)(2(-q-p)((n+m)x-(q+p)y)+2(c+b)((n+m)x-(q+p)y)+2(c+b)(-q-p)(y-x^2)+2(2bc+c+b)(y-x^2)+2a(-q-p)((q+p)x-n-m)+2a(c+b)((q+p)x-n-m))]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^3]$

$h_{13}^{(U_3)}=0$

$h_{14}^{(U_3)}=0$

$h_{15}^{(U_3)}=[2s^2x(((n+m)x-(q+p)y)^2+(2bc+c+b)(y-x^2)^2+2(c+b)(y-x^2)((n+m)x-(q+p)y)+2a((q+p)x-n-m)((n+m)x-(q+p)y)+2a(c+b)((q+p)x-n-m)(y-x^2)+a((q+p)x-n-m)^2)]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^3]-[s^2(y-x^2)(-4(c+b)x((n+m)x-(q+p)y)+2a(q+p)((n+m)x-(q+p)y)+2(n+m)((n+m)x-(q+p)y)-4(2bc+c+b)x(y-x^2)+2a(c+b)(q+p)(y-x^2)+2(c+b)(n+m)(y-x^2)-4a(c+b)x((q+p)x-n-m)+2a(q+p)((q+p)x-n-m)+2a(n+m)((q+p)x-n-m))]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^3]+[3s^2(y-x^2)(((n+m)x-(q+p)y)^2+(2bc+c+b)(y-x^2)^2+2(c+b)(y-x^2)((n+m)x-(q+p)y)+2a((q+p)x-n-m)((n+m)x-(q+p)y)+2a(c+b)((q+p)x-n-m)(y-x^2)+a((q+p)x-n-m)^2)(2(n+m)(q+p)-2xl)]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^4]$

$h_{16}^{(U_3)}=[s^2(2(x(m+n)-(q+p)y)+2x(m+n)-2a(m+n)+2a(-m+(q+p)x-n)+$

$2(c+b)(y-x^2)-4(c+b)x^2+2a(q+p)x+4a(c+b)x-2a(q+p))]/[2(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^2]-[s^2(2(q+p)x-2(m+n))(2(m+n)(x(m+n)-(q+p)y)-4(c+b)x(x(m+n)-(q+p)y)+2a(q+p)(x(m+n)-(q+p)y)+2a(-m+(q+p)x-n)(m+n)+2(c+b)(y-x^2)(m+n)-4a(c+b)x(-m+(q+p)x-n)+2a(q+p)(-m+(q+p)x-n)-4(2bc+c+b)x(y-x^2)+2a(c+b)(q+p)(y-x^2))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^3]-[s^2(2(q+p)(m+n)-2lx)(2x(x(m+n)-(q+p)y)-2a(x(m+n)-(q+p)y)+2ax(-m+(q+p)x-n)-2a(-m+(q+p)x-n)+2(c+b)x(y-x^2)-2a(c+b)(y-x^2))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^3]-[2(q+p)s^2((x(m+n)-(q+p)y)^2+2a(-m+(q+p)x-n)(x(m+n)-(q+p)y)+2(c+b)(y-x^2)(x(m+n)-(q+p)y)+2a(c+b)(y-x^2)(-m+(q+p)x-n)+a(-m+(q+p)x-n)^2+(2bc+c+b)(y-x^2)^2)]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^3]+[3s^2(2(q+p)x-2(m+n))(2(q+p)(m+n)-2lx)((x(m+n)-(q+p)y)^2+2a(-m+(q+p)x-n)(x(m+n)-(q+p)y)+2(c+b)(y-x^2)(x(m+n)-(q+p)y)+2a(c+b)(y-x^2)(-m+(q+p)x-n)+a(-m+(q+p)x-n)^2+(2bc+c+b)(y-x^2)^2)]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^4]$

$h_{17}^{(U_3)} = [s^2(2(x(n+m)-(q+p)y)+2x(n+m)-2a(n+m)+2a(-n+(q+p)x-m)+2(c+b)(y-x^2)-4(c+b)x^2+2a(q+p)x+4a(c+b)x-2a(q+p))]/[2(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^2]-[s^2(2(q+p)x-2(n+m))(2(n+m)(x(n+m)-(q+p)y)-4(c+b)x(x(n+m)-(q+p)y)+2a(q+p)(x(n+m)-(q+p)y)+2a(-n+(q+p)x-m)(n+m)+2(c+b)(y-x^2)(n+m)-4a(c+b)x(-n+(q+p)x-m)+2a(q+p)(-n+(q+p)x-m)-4(2bc+c+b)x(y-x^2)+2a(c+b)(q+p)(y-x^2))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]-[s^2(2(q+p)(n+m)-2lx)(2x(x(n+m)-(q+p)y)-2a(x(n+m)-(q+p)y)+2ax(-n+(q+p)x-m)-2a(-n+(q+p)x-m)+2(c+b)x(y-x^2)-2a(c+b)(y-x^2))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]-[2(q+p)s^2((x(n+m)-(q+p)y)^2+2a(-n+(q+p)x-m)(x(n+m)-(q+p)y)+2(c+b)(y-x^2)(x(n+m)-(q+p)y)+2a(c+b)(y-x^2)(-n+(q+p)x-m)+a(-n+(q+p)x-m)^2+(2bc+c+b)(y-x^2)^2)]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]+[3s^2(2(q+p)x-2(n+m))(2(q+p)(n+m)-2lx)((x(n+m)-(q+p)y)^2+2a(-n+(q+p)x-m)(x(n+m)-(q+p)y)+2(c+b)(y-x^2)(x(n+m)-(q+p)y)+2a(c+b)(y-x^2)(-n+(q+p)x-m)+a(-n+(q+p)x-m)^2+(2bc+c+b)(y-x^2)^2)]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^4]$

$$h_{18}^{(U_3)} = [s^2(2a((n+m)x-y(p+q))+2a(x(p+q)-n-m)-2ay(p+q)+2ax(p+q)+2a(c+b)(y-x^2)+4(c+b)xy-2(n+m)y-4a(c+b)x^2+2a(n+m)x)]/[2(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^2]-[s^2(2(n+m)x-2y(p+q))(2a(p+q)((n+m)x-y(p+q))-4(c+b)x((n+m)x-y(p+q))+2(n+m)((n+m)x-y(p+q))+2a(p+q)(x(p+q)-n-m)-4a(c+b)x(x(p+q)-n-m)+2a(n+m)(x(p+q)-n-m)+2a(c+b)(y-x^2)(p+q)-4(2bc+c+b)x(y-x^2)+2(c+b)(n+m)(y-x^2))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]-[s^2(2(n+m)(p+q)-2lx)(-2y((n+m)x-y(p+q))+2ax((n+m)x-y(p+q))-2ay(x(p+q)-n-m)+2ax(x(p+q)-n-m)-2(c+b)y(y-x^2)+2a(c+b)x(y-x^2))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]-[2(n+m)s^2(((n+m)x-y(p+q))^2+a(x(p+q)-n-m)^2+2a(x(p+q)-n-m)((n+m)x-y(p+q))+2(c+b)(y-x^2)((n+m)x-y(p+q))+2a(c+b)(y-x^2)(x(p+q)-n-m)+(2bc+c+b)(y-x^2)^2)]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]+[3s^2(2(n+m)(p+q)-2lx)(2(n+m)x-2y(p+q))(((n+m)x-y(p+q))^2+a(x(p+q)-n-m)^2+2a(x(p+q)-n-m)((n+m)x-y(p+q))+2(c+b)(y-x^2)((n+m)x-y(p+q))+2a(c+b)(y-x^2)(x(p+q)-n-m)+(2bc+c+b)(y-x^2)^2)]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^4]$$

$$h_{19}^{(U_3)} = [s^2(2a((n+m)x-y(q+p))+2a(x(q+p)-n-m)-2ay(q+p)+2ax(q+p)+2a(c+b)(y-x^2)+4(c+b)xy-2(n+m)y-4a(c+b)x^2+2a(n+m)x)]/[2(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^2]-[s^2(2(n+m)x-2y(q+p))(2a(q+p)((n+m)x-y(q+p))-4(c+b)x((n+m)x-y(q+p))+2(n+m)((n+m)x-y(q+p))+2a(q+p)(x(q+p)-n-m)-4a(c+b)x(x(q+p)-n-m)+2a(n+m)(x(q+p)-n-m)+2a(c+b)(y-x^2)(q+p)-4(2bc+c+b)x(y-x^2)+2(c+b)(n+m)(y-x^2))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]-[s^2(2(n+m)(q+p)-2lx)(-2y((n+m)x-y(q+p))+2ax((n+m)x-y(q+p))-2ay(x(q+p)-n-m)+2ax(x(q+p)-n-m)-2(c+b)y(y-x^2)+2a(c+b)x(y-x^2))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]-[2(n+m)s^2(((n+m)x-y(q+p))^2+a(x(q+p)-n-m)^2+2a(x(q+p)-n-m)((n+m)x-y(q+p))+2(c+b)(y-x^2)((n+m)x-y(q+p))+2a(c+b)(y-x^2)(x(q+p)-n-m)+(2bc+c+b)(y-x^2)^2)]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]+[3s^2(2(n+m)(q+p)-2lx)(2(n+m)x-2y(q+p))(((n+m)x-y(q+p))^2+a(x(q+p)-n-m)^2+2a(x(q+p)-n-m)((n+m)x-y(q+p))+2(c+b)(y-x^2)((n+m)x-y(q+p))+2a(c+b)(y-x^2)(x(q+p)-n-m)+(2bc+c+b)(y-x^2)^2)]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^4]$$

$h_{1\ 10}^{(U_3)} = [(-4(c+b)x((n+m)x-(q+p)y)+2a(q+p)((n+m)x-(q+p)y)+2(n+m)((n+m)x-(q+p)y)-4(2bc+c+b)x(y-x^2)+2a(c+b)(q+p)(y-x^2)+2(c+b)(n+m)(y-x^2)-4a(c+b)x((q+p)x-n-m)+2a(q+p)((q+p)x-n-m)+2a(n+m)((q+p)x-n-m))s]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^2]-[2(2(n+m)(q+p)-2lx)(((n+m)x-(q+p)y)^2+(2bc+c+b)(y-x^2)^2+2(c+b)(y-x^2)((n+m)x-(q+p)y)+2a((q+p)x-n-m)((n+m)x-(q+p)y)+2a(c+b)((q+p)x-n-m)(y-x^2)+a((q+p)x-n-m)^2)s]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^3]$

$h_{22}^{(U_3)} = [(4(c+b)(-q-p)+2(-q-p)^2+2(2bc+c+b))s^2]/[2(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^2]+[3(l-(q+p)^2)^2s^2(((n+m)x-(q+p)y)^2+(2bc+c+b)(y-x^2)^2+2(c+b)(y-x^2)((n+m)x-(q+p)y)+2a((q+p)x-n-m)((n+m)x-(q+p)y)+2a(c+b)((q+p)x-n-m)(y-x^2)+a((q+p)x-n-m)^2)]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^4]-[2(l-(q+p)^2)s^2(2(-q-p)((n+m)x-(q+p)y)+2(c+b)((n+m)x-(q+p)y)+2(c+b)(-q-p)(y-x^2)+2(2bc+c+b)(y-x^2)+2a(-q-p)((q+p)x-n-m)+2a(c+b)((q+p)x-n-m))]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^3]$

$h_{23}^{(U_3)} = 0$

$h_{24}^{(U_3)} = 0$

$h_{25}^{(U_3)} = -[s^2(((n+m)x-(q+p)y)^2+(2bc+c+b)(y-x^2)^2+2(c+b)(y-x^2)((n+m)x-(q+p)y)+2a((q+p)x-n-m)((n+m)x-(q+p)y)+2a(c+b)((q+p)x-n-m)(y-x^2)+a((q+p)x-n-m)^2)]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^3]-[s^2(y-x^2)(2(-q-p)((n+m)x-(q+p)y)+2(c+b)((n+m)x-(q+p)y)+2(c+b)(-q-p)(y-x^2)+2(2bc+c+b)(y-x^2)+2a(-q-p)((q+p)x-n-m)+2a(c+b)((q+p)x-n-m))]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^3]+[3s^2(y-x^2)(((n+m)x-(q+p)y)^2+(2bc+c+b)(y-x^2)^2+2(c+b)(y-x^2)((n+m)x-(q+p)y)+2a((q+p)x-n-m)((n+m)x-(q+p)y)+2a(c+b)((q+p)x-n-m)(y-x^2)+a((q+p)x-n-m)^2)(l-(q+p)^2)]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^4]$

$h_{26}^{(U_3)} = [s^2(2(-q-p)x+2(c+b)x-2a(-q-p)-2a(c+b))]/[2(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^2]-[(l-(q+p)^2)s^2(2x(x(m+n)-(q+p)y)-2a(x(m+n)-(q+p)y)+2ax(-m+(q+p)x-n)-2a(-m+(q+p)x-n)+2(c+b)x(y-x^2)-2a(c+b)(y-x^2))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^3]-[s^2(2(q+p)x-2(m+n))(2(-q-p)(x(m+n)-(q+p)y)+2(c+b)(x(m+n)-(q+p)y)+2a(-q-p)(-m+$

$(q+p)x-n)+2a(c+b)(-m+(q+p)x-n)+2(c+b)(-q-p)(y-x^2)+2(2bc+c+b)(y-x^2))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^3]+[3(l-(q+p)^2)s^2(2(q+p)x-2(m+n))((x(m+n)-(q+p)y)^2+2a(-m+(q+p)x-n)(x(m+n)-(q+p)y)+2(c+b)(y-x^2)(x(m+n)-(q+p)y)+2a(c+b)(y-x^2)(-m+(q+p)x-n)+a(-m+(q+p)x-n)^2+(2bc+c+b)(y-x^2)^2)]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^4]$

$h_{27}^{(U_3)} = [s^2(2(-q-p)x+2(c+b)x-2a(-q-p)-2a(c+b))]/[2(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^2]-[(l-(q+p)^2)s^2(2x(x(n+m)-(q+p)y)-2a(x(n+m)-(q+p)y)+2ax(-n+(q+p)x-m)-2a(-n+(q+p)x-m)+2(c+b)x(y-x^2)-2a(c+b)(y-x^2))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]-[s^2(2(q+p)x-2(n+m))(2(-q-p)(x(n+m)-(q+p)y)+2(c+b)(x(n+m)-(q+p)y)+2a(-q-p)(-n+(q+p)x-m)+2a(c+b)(-n+(q+p)x-m)+2(c+b)(-q-p)(y-x^2)+2(2bc+c+b)(y-x^2))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]+[3(l-(q+p)^2)s^2(2(q+p)x-2(n+m))((x(n+m)-(q+p)y)^2+2a(-n+(q+p)x-m)(x(n+m)-(q+p)y)+2(c+b)(y-x^2)(x(n+m)-(q+p)y)+2a(c+b)(y-x^2)(-n+(q+p)x-m)+a(-n+(q+p)x-m)^2+(2bc+c+b)(y-x^2)^2)]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^4]$

$h_{28}^{(U_3)} = [s^2(-2((n+m)x-y(p+q))-2a(x(p+q)-n-m)-2y(-p-q)+2ax(-p-q)-2(c+b)(y-x^2)-2(c+b)y+2a(c+b)x)]/[2(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^2]-[s^2(-2y((n+m)x-y(p+q))+2ax((n+m)x-y(p+q))-2ay(x(p+q)-n-m)+2ax(x(p+q)-n-m)-2(c+b)y(y-x^2)+2a(c+b)x(y-x^2))(l-(p+q)^2)]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]-[s^2(2(n+m)x-2y(p+q))(2(-p-q)((n+m)x-y(p+q))+2(c+b)((n+m)x-y(p+q))+2a(-p-q)(x(p+q)-n-m)+2a(c+b)(x(p+q)-n-m)+2(c+b)(y-x^2)(-p-q)+2(2bc+c+b)(y-x^2))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]+[2s^2(p+q)(((n+m)x-y(p+q))^2+a(x(p+q)-n-m)^2+2a(x(p+q)-n-m)((n+m)x-y(p+q))+2(c+b)(y-x^2)((n+m)x-y(p+q))+2a(c+b)(y-x^2)(x(p+q)-n-m)+(2bc+c+b)(y-x^2)^2)]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]+[3s^2(2(n+m)x-2y(p+q))(l-(p+q)^2)(((n+m)x-y(p+q))^2+a(x(p+q)-n-m)^2+2a(x(p+q)-n-m)((n+m)x-y(p+q))+2(c+b)(y-x^2)((n+m)x-y(p+q))+2a(c+b)(y-x^2)(x(p+q)-n-m)+(2bc+c+b)(y-x^2)^2)]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^4]$

$h_{29}^{(U_3)} = [s^2(-2((n+m)x-y(q+p))-2a(x(q+p)-n-m)-2y(-q-p)+2ax(-q-$

$p)-2(c+b)(y-x^2)-2(c+b)y+2a(c+b)x)]/[2(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^2]-[s^2(-2y((n+m)x-y(q+p))+2ax((n+m)x-y(q+p))-2ay(x(q+p)-n-m)+2ax(x(q+p)-n-m)-2(c+b)y(y-x^2)+2a(c+b)x(y-x^2))(l-(q+p)^2)]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]-[s^2(2(n+m)x-2y(q+p))(2(-q-p)((n+m)x-y(q+p))+2(c+b)((n+m)x-y(q+p))+2a(-q-p)(x(q+p)-n-m)+2a(c+b)(x(q+p)-n-m)+2(c+b)(y-x^2)(-q-p)+2(2bc+c+b)(y-x^2))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]+[2s^2(q+p)(((n+m)x-y(q+p))^2+a(x(q+p)-n-m)^2+2a(x(q+p)-n-m)((n+m)x-y(q+p))+2(c+b)(y-x^2)((n+m)x-y(q+p))+2a(c+b)(y-x^2)(x(q+p)-n-m)+(2bc+c+b)(y-x^2)^2)]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]+[3s^2(2(n+m)x-2y(q+p))(l-(q+p)^2)(((n+m)x-y(q+p))^2+a(x(q+p)-n-m)^2+2a(x(q+p)-n-m)((n+m)x-y(q+p))+2(c+b)(y-x^2)((n+m)x-y(q+p))+2a(c+b)(y-x^2)(x(q+p)-n-m)+(2bc+c+b)(y-x^2)^2)]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^4]$

$h^{(U_3)}_{2\ 10}=[(2(-q-p)((n+m)x-(q+p)y)+2(c+b)((n+m)x-(q+p)y)+2(c+b)(-q-p)(y-x^2)+2(2bc+c+b)(y-x^2)+2a(-q-p)((q+p)x-n-m)+2a(c+b)((q+p)x-n-m))s]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^2]-[2(l-(q+p)^2)(((n+m)x-(q+p)y)^2+(2bc+c+b)(y-x^2)^2+2(c+b)(y-x^2)((n+m)x-(q+p)y)+2a((q+p)x-n-m)((n+m)x-(q+p)y)+2a(c+b)((q+p)x-n-m)(y-x^2)+a((q+p)x-n-m)^2)s]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^3]$

$h^{(U_3)}_{33}=0$

$h^{(U_3)}_{34}=0$

$h^{(U_3)}_{35}=0$

$h^{(U_3)}_{36}=0$

$h^{(U_3)}_{37}=0$

$h^{(U_3)}_{38}=0$

$h^{(U_3)}_{39}=0$

$h^{(U_3)}_{3\ 10}=0$

$h^{(U_3)}_{44}=0$

$h^{(U_3)}_{45}=0$

$h^{(U_3)}_{46}=0$

$h_{47}^{(U_3)} = 0$

$h_{48}^{(U_3)} = 0$

$h_{49}^{(U_3)} = 0$

$h_{4\ 10}^{(U_3)} = 0$

$h_{55}^{(U_3)} = [3s^2(y-x^2)^2(((n+m)x-(q+p)y)^2+(2bc+c+b)(y-x^2)^2+2(c+b)(y-x^2)((n+m)x-(q+p)y)+2a((q+p)x-n-m)((n+m)x-(q+p)y)+2a(c+b)((q+p)x-n-m)(y-x^2)+a((q+p)x-n-m)^2)]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^4]$

$h_{56}^{(U_3)} = [3s^2(y-x^2)(2(q+p)x-2(m+n))((x(m+n)-(q+p)y)^2+2a(-m+(q+p)x-n)(x(m+n)-(q+p)y)+2(c+b)(y-x^2)(x(m+n)-(q+p)y)+2a(c+b)(y-x^2)(-m+(q+p)x-n)+a(-m+(q+p)x-n)^2+(2bc+c+b)(y-x^2)^2)]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^4]-[s^2(y-x^2)(2x(x(m+n)-(q+p)y)-2a(x(m+n)-(q+p)y)+2ax(-m+(q+p)x-n)-2a(-m+(q+p)x-n)+2(c+b)x(y-x^2)-2a(c+b)(y-x^2))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^3]$

$h_{57}^{(U_3)} = [3s^2(y-x^2)(2(q+p)x-2(n+m))((x(n+m)-(q+p)y)^2+2a(-n+(q+p)x-m)(x(n+m)-(q+p)y)+2(c+b)(y-x^2)(x(n+m)-(q+p)y)+2a(c+b)(y-x^2)(-n+(q+p)x-m)+a(-n+(q+p)x-m)^2+(2bc+c+b)(y-x^2)^2)]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^4]-[s^2(y-x^2)(2x(x(n+m)-(q+p)y)-2a(x(n+m)-(q+p)y)+2ax(-n+(q+p)x-m)-2a(-n+(q+p)x-m)+2(c+b)x(y-x^2)-2a(c+b)(y-x^2))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]$

$h_{58}^{(U_3)} = [3s^2(y-x^2)(2(n+m)x-2y(p+q))(((n+m)x-y(p+q))^2+a(x(p+q)-n-m)^2+2a(x(p+q)-n-m)((n+m)x-y(p+q))+2(c+b)(y-x^2)((n+m)x-y(p+q))+2a(c+b)(y-x^2)(x(p+q)-n-m)+(2bc+c+b)(y-x^2)^2)]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^4]-[s^2(y-x^2)(-2y((n+m)x-y(p+q))+2ax((n+m)x-y(p+q))-2ay(x(p+q)-n-m)+2ax(x(p+q)-n-m)-2(c+b)y(y-x^2)+2a(c+b)x(y-x^2))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]$

$h_{59}^{(U_3)} = [3s^2(y-x^2)(2(n+m)x-2y(q+p))(((n+m)x-y(q+p))^2+a(x(q+p)-n-m)^2+2a(x(q+p)-n-m)((n+m)x-y(q+p))+2(c+b)(y-x^2)((n+m)x-y(q+p))+2a(c+b)(y-x^2)(x(q+p)-n-m)+(2bc+c+b)(y-x^2)^2)]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^4]-[s^2(y-x^2)(-2y((n+m)x-y(q+p))+2ax((n+m)x-y(q+p))-2ay(x(q+p)-n-m)+2ax(x(q+p)-n-m)-2(c+b)y(y-$

$x^2) + 2a(c+b)x(y-x^2))]/[(-y(q+p)^2 + 2(n+m)x(q+p) + ly - lx^2 - (n+m)^2)^3]$

$h_{5\ 10}^{(U_3)} = -[2(y-x^2)(((n+m)x-(q+p)y)^2+(2bc+c+b)(y-x^2)^2+2(c+b)(y-x^2)((n+m)x-(q+p)y)+2a((q+p)x-n-m)((n+m)x-(q+p)y)+2a(c+b)((q+p)x-n-m)(y-x^2)+a((q+p)x-n-m)^2)s]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^3]$

$h_{66}^{(U_3)} = [s^2(2x^2-4ax+2a)]/[2(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^2]-[2s^2(2(q+p)x-2(m+n))(2x(x(m+n)-(q+p)y)-2a(x(m+n)-(q+p)y)+2ax(-m+(q+p)x-n)-2a(-m+(q+p)x-n)+2(c+b)x(y-x^2)-2a(c+b)(y-x^2))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^3]+[2s^2((x(m+n)-(q+p)y)^2+2a(-m+(q+p)x-n)(x(m+n)-(q+p)y)+2(c+b)(y-x^2)(x(m+n)-(q+p)y)+2a(c+b)(y-x^2)(-m+(q+p)x-n)+a(-m+(q+p)x-n)^2+(2bc+c+b)(y-x^2)^2)]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^3]+[3s^2(2(q+p)x-2(m+n))^2((x(m+n)-(q+p)y)^2+2a(-m+(q+p)x-n)(x(m+n)-(q+p)y)+2(c+b)(y-x^2)(x(m+n)-(q+p)y)+2a(c+b)(y-x^2)(-m+(q+p)x-n)+a(-m+(q+p)x-n)^2+(2bc+c+b)(y-x^2)^2)]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^4]$

$h_{67}^{(U_3)} = [s^2(2x^2-4ax+2a)]/[2(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^2]-[2s^2(2(q+p)x-2(n+m))(2x(x(n+m)-(q+p)y)-2a(x(n+m)-(q+p)y)+2ax(-n+(q+p)x-m)-2a(-n+(q+p)x-m)+2(c+b)x(y-x^2)-2a(c+b)(y-x^2))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]+[2s^2((x(n+m)-(q+p)y)^2+2a(-n+(q+p)x-m)(x(n+m)-(q+p)y)+2(c+b)(y-x^2)(x(n+m)-(q+p)y)+2a(c+b)(y-x^2)(-n+(q+p)x-m)+a(-n+(q+p)x-m)^2+(2bc+c+b)(y-x^2)^2)]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]+[3s^2(2(q+p)x-2(n+m))^2((x(n+m)-(q+p)y)^2+2a(-n+(q+p)x-m)(x(n+m)-(q+p)y)+2(c+b)(y-x^2)(x(n+m)-(q+p)y)+2a(c+b)(y-x^2)(-n+(q+p)x-m)+a(-n+(q+p)x-m)^2+(2bc+c+b)(y-x^2)^2)]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^4]$

$h_{68}^{(U_3)} = [s^2(-2xy+2ay+2ax^2-2ax)]/[2(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^2]-[s^2(2x(p+q)-2(n+m))(-2y((n+m)x-y(p+q))+2ax((n+m)x-y(p+q))-2ay(x(p+q)-n-m)+2ax(x(p+q)-n-m)-2(c+b)y(y-x^2)+2a(c+b)x(y-x^2))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]-[s^2(2(n+m)x-2y(p+q))(2x((n+m)x-y(p+q))-2a((n+m)x-y(p+q))+2ax(x(p+q)-n-m)-2a(x(p+q)-n-m)+2(c+b)x(y-x^2)-2a(c+b)(y-x^2))]/[(-y(p+q)^2+$

$2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]-[2s^2x(((n+m)x-y(p+q))^2+a(x(p+q)-n-m)^2+2a(x(p+q)-n-m)((n+m)x-y(p+q))+2(c+b)(y-x^2)((n+m)x-y(p+q))+2a(c+b)(y-x^2)(x(p+q)-n-m)+(2bc+c+b)(y-x^2)^2)]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]+[3s^2(2x(p+q)-2(n+m))(2(n+m)x-2y(p+q))(((n+m)x-y(p+q))^2+a(x(p+q)-n-m)^2+2a(x(p+q)-n-m)((n+m)x-y(p+q))+2(c+b)(y-x^2)((n+m)x-y(p+q))+2a(c+b)(y-x^2)(x(p+q)-n-m)+(2bc+c+b)(y-x^2)^2)]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^4]$

$h_{69}^{(U_3)}=[s^2(-2xy+2ay+2ax^2-2ax)]/[2(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^2]-[s^2(2x(q+p)-2(n+m))(-2y((n+m)x-y(q+p))+2ax((n+m)x-y(q+p))-2ay(x(q+p)-n-m)+2ax(x(q+p)-n-m)-2(c+b)y(y-x^2)+2a(c+b)x(y-x^2))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]-[s^2(2(n+m)x-2y(q+p))(2x((n+m)x-y(q+p))-2a((n+m)x-y(q+p))+2ax(x(q+p)-n-m)-2a(x(q+p)-n-m)+2(c+b)x(y-x^2)-2a(c+b)(y-x^2))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]-[2s^2x(((n+m)x-y(q+p))^2+a(x(q+p)-n-m)^2+2a(x(q+p)-n-m)((n+m)x-y(q+p))+2(c+b)(y-x^2)((n+m)x-y(q+p))+2a(c+b)(y-x^2)(x(q+p)-n-m)+(2bc+c+b)(y-x^2)^2)]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]+[3s^2(2x(q+p)-2(n+m))(2(n+m)x-2y(q+p))(((n+m)x-y(q+p))^2+a(x(q+p)-n-m)^2+2a(x(q+p)-n-m)((n+m)x-y(q+p))+2(c+b)(y-x^2)((n+m)x-y(q+p))+2a(c+b)(y-x^2)(x(q+p)-n-m)+(2bc+c+b)(y-x^2)^2)]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^4]$

$h_{6\ 10}^{(U_3)}=[(2x((n+m)x-(q+p)y)-2a((n+m)x-(q+p)y)+2(c+b)x(y-x^2)-2a(c+b)(y-x^2)+2ax((q+p)x-n-m)-2a((q+p)x-n-m))s]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^2]-[2(2(q+p)x-2(n+m))(((n+m)x-(q+p)y)^2+(2bc+c+b)(y-x^2)^2+2(c+b)(y-x^2)((n+m)x-(q+p)y)+2a((q+p)x-n-m)((n+m)x-(q+p)y)+2a(c+b)((q+p)x-n-m)(y-x^2)+a((q+p)x-n-m)^2)s]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^3]$

$h_{77}^{(U_3)}=[s^2(2x^2-4ax+2a)]/[2(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^2]-[2s^2(2(q+p)x-2(n+m))(2x(x(n+m)-(q+p)y)-2a(x(n+m)-(q+p)y)+2ax(-n+(q+p)x-m)-2a(-n+(q+p)x-m)+2(c+b)x(y-x^2)-2a(c+b)(y-x^2))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]+[2s^2((x(n+m)-(q+$

$p)y)^2+2a(-n+(q+p)x-m)(x(n+m)-(q+p)y)+2(c+b)(y-x^2)(x(n+m)-(q+p)y)+2a(c+b)(y-x^2)(-n+(q+p)x-m)+a(-n+(q+p)x-m)^2+(2bc+c+b)(y-x^2)^2)]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]+[3s^2(2(q+p)x-2(n+m))^2((x(n+m)-(q+p)y)^2+2a(-n+(q+p)x-m)(x(n+m)-(q+p)y)+2(c+b)(y-x^2)(x(n+m)-(q+p)y)+2a(c+b)(y-x^2)(-n+(q+p)x-m)+a(-n+(q+p)x-m)^2+(2bc+c+b)(y-x^2)^2)]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^4]$

$h_{78}^{(U_3)}=[s^2(-2xy+2ay+2ax^2-2ax)]/[2(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^2]-[s^2(2x(p+q)-2(n+m))(-2y((n+m)x-y(p+q))+2ax((n+m)x-y(p+q))-2ay(x(p+q)-n-m)+2ax(x(p+q)-n-m)-2(c+b)y(y-x^2)+2a(c+b)x(y-x^2))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]-[s^2(2(n+m)x-2y(p+q))(2x((n+m)x-y(p+q))-2a((n+m)x-y(p+q))+2ax(x(p+q)-n-m)-2a(x(p+q)-n-m)+2(c+b)x(y-x^2)-2a(c+b)(y-x^2))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]-[2s^2x(((n+m)x-y(p+q))^2+a(x(p+q)-n-m)^2+2a(x(p+q)-n-m)((n+m)x-y(p+q))+2(c+b)(y-x^2)((n+m)x-y(p+q))+2a(c+b)(y-x^2)(x(p+q)-n-m)+(2bc+c+b)(y-x^2)^2)]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]+[3s^2(2x(p+q)-2(n+m))(2(n+m)x-2y(p+q))(((n+m)x-y(p+q))^2+a(x(p+q)-n-m)^2+2a(x(p+q)-n-m)((n+m)x-y(p+q))+2(c+b)(y-x^2)((n+m)x-y(p+q))+2a(c+b)(y-x^2)(x(p+q)-n-m)+(2bc+c+b)(y-x^2)^2)]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^4]$

$h_{79}^{(U_3)}=[s^2(-2xy+2ay+2ax^2-2ax)]/[2(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^2]-[s^2(2x(q+p)-2(n+m))(-2y((n+m)x-y(q+p))+2ax((n+m)x-y(q+p))-2ay(x(q+p)-n-m)+2ax(x(q+p)-n-m)-2(c+b)y(y-x^2)+2a(c+b)x(y-x^2))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]-[s^2(2(n+m)x-2y(q+p))(2x((n+m)x-y(q+p))-2a((n+m)x-y(q+p))+2ax(x(q+p)-n-m)-2a(x(q+p)-n-m)+2(c+b)x(y-x^2)-2a(c+b)(y-x^2))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]-[2s^2x(((n+m)x-y(q+p))^2+a(x(q+p)-n-m)^2+2a(x(q+p)-n-m)((n+m)x-y(q+p))+2(c+b)(y-x^2)((n+m)x-y(q+p))+2a(c+b)(y-x^2)(x(q+p)-n-m)+(2bc+c+b)(y-x^2)^2)]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]+[3s^2(2x(q+p)-2(n+m))(2(n+m)x-2y(q+p))(((n+m)x-y(q+p))^2+a(x(q+p)-n-m)^2+2a(x(q+p)-n-m)((n+m)x-$

$y(q+p)) + 2(c+b)(y-x^2)((n+m)x - y(q+p)) + 2a(c+b)(y-x^2)(x(q+p) - n - m) + (2bc+c+b)(y-x^2)^2)]/[(-y(q+p)^2 + 2(n+m)x(q+p) + ly - lx^2 - (n+m)^2)^4]$

$h_{7\ 10}^{(U_3)} = [(2x((n+m)x - (q+p)y) - 2a((n+m)x - (q+p)y) + 2(c+b)x(y-x^2) - 2a(c+b)(y-x^2) + 2ax((q+p)x - n - m) - 2a((q+p)x - n - m))s]/[(-(q+p)^2y + ly - lx^2 + 2(n+m)(q+p)x - (n+m)^2)^2] - [2(2(q+p)x - 2(n+m))(((n+m)x - (q+p)y)^2 + (2bc+c+b)(y-x^2)^2 + 2(c+b)(y-x^2)((n+m)x - (q+p)y) + 2a((q+p)x - n - m)((n+m)x - (q+p)y) + 2a(c+b)((q+p)x - n - m)(y-x^2) + a((q+p)x - n - m)^2)s]/[(-(q+p)^2y + ly - lx^2 + 2(n+m)(q+p)x - (n+m)^2)^3]$

$h_{88}^{(U_3)} = [s^2(2y^2 - 4axy + 2ax^2)]/[2(-y(p+q)^2 + 2(n+m)x(p+q) + ly - lx^2 - (n+m)^2)^2] - [2s^2(2(n+m)x - 2y(p+q))(-2y((n+m)x - y(p+q)) + 2ax((n+m)x - y(p+q)) - 2ay(x(p+q) - n - m) + 2ax(x(p+q) - n - m) - 2(c+b)y(y-x^2) + 2a(c+b)x(y-x^2))]/[(-y(p+q)^2 + 2(n+m)x(p+q) + ly - lx^2 - (n+m)^2)^3] + [2s^2y(((n+m)x - y(p+q))^2 + a(x(p+q) - n - m)^2 + 2a(x(p+q) - n - m)((n+m)x - y(p+q)) + 2(c+b)(y-x^2)((n+m)x - y(p+q)) + 2a(c+b)(y-x^2)(x(p+q) - n - m) + (2bc+c+b)(y-x^2)^2)]/[(-y(p+q)^2 + 2(n+m)x(p+q) + ly - lx^2 - (n+m)^2)^3] + [3s^2(2(n+m)x - 2y(p+q))^2(((n+m)x - y(p+q))^2 + a(x(p+q) - n - m)^2 + 2a(x(p+q) - n - m)((n+m)x - y(p+q)) + 2(c+b)(y-x^2)((n+m)x - y(p+q)) + 2a(c+b)(y-x^2)(x(p+q) - n - m) + (2bc+c+b)(y-x^2)^2)]/[(-y(p+q)^2 + 2(n+m)x(p+q) + ly - lx^2 - (n+m)^2)^4]$

$h_{89}^{(U_3)} = [s^2(2y^2 - 4axy + 2ax^2)]/[2(-y(q+p)^2 + 2(n+m)x(q+p) + ly - lx^2 - (n+m)^2)^2] - [2s^2(2(n+m)x - 2y(q+p))(-2y((n+m)x - y(q+p)) + 2ax((n+m)x - y(q+p)) - 2ay(x(q+p) - n - m) + 2ax(x(q+p) - n - m) - 2(c+b)y(y-x^2) + 2a(c+b)x(y-x^2))]/[(-y(q+p)^2 + 2(n+m)x(q+p) + ly - lx^2 - (n+m)^2)^3] + [2s^2y(((n+m)x - y(q+p))^2 + a(x(q+p) - n - m)^2 + 2a(x(q+p) - n - m)((n+m)x - y(q+p)) + 2(c+b)(y-x^2)((n+m)x - y(q+p)) + 2a(c+b)(y-x^2)(x(q+p) - n - m) + (2bc+c+b)(y-x^2)^2)]/[(-y(q+p)^2 + 2(n+m)x(q+p) + ly - lx^2 - (n+m)^2)^3] + [3s^2(2(n+m)x - 2y(q+p))^2(((n+m)x - y(q+p))^2 + a(x(q+p) - n - m)^2 + 2a(x(q+p) - n - m)((n+m)x - y(q+p)) + 2(c+b)(y-x^2)((n+m)x - y(q+p)) + 2a(c+b)(y-x^2)(x(q+p) - n - m) + (2bc+c+b)(y-x^2)^2)]/[(-y(q+p)^2 + 2(n+m)x(q+p) + ly - lx^2 - (n+m)^2)^4]$

$h_{8\ 10}^{(U_3)} = [(-2y((n+m)x - (q+p)y) + 2ax((n+m)x - (q+p)y) - 2(c+b)y(y-x^2) + 2a(c+b)x(y-x^2) - 2a((q+p)x - n - m)y + 2ax((q+p)x - n - m))s]/[(-(q+$

$p)^2y + ly - lx^2 + 2(n+m)(q+p)x - (n+m)^2)^2] - [2(2(n+m)x - 2(q+p)y)(((n+m)x - (q+p)y)^2 + (2bc+c+b)(y-x^2)^2 + 2(c+b)(y-x^2)((n+m)x - (q+p)y) + 2a((q+p)x - n - m)((n+m)x - (q+p)y) + 2a(c+b)((q+p)x - n - m)(y - x^2) + a((q+p)x - n - m)^2)s]/[(-(q+p)^2y + ly - lx^2 + 2(n+m)(q+p)x - (n+m)^2)^3]$

$h_{99}^{(U_3)} = [s^2(2y^2 - 4axy + 2ax^2)]/[2(-y(q+p)^2 + 2(n+m)x(q+p) + ly - lx^2 - (n+m)^2)^2] - [2s^2(2(n+m)x - 2y(q+p))(-2y((n+m)x - y(q+p)) + 2ax((n+m)x - y(q+p)) - 2ay(x(q+p) - n - m) + 2ax(x(q+p) - n - m) - 2(c+b)y(y-x^2) + 2a(c+b)x(y-x^2))]/[(-y(q+p)^2 + 2(n+m)x(q+p) + ly - lx^2 - (n+m)^2)^3] + [2s^2y(((n+m)x - y(q+p))^2 + a(x(q+p) - n - m)^2 + 2a(x(q+p) - n - m)((n+m)x - y(q+p)) + 2(c+b)(y-x^2)((n+m)x - y(q+p)) + 2a(c+b)(y-x^2)(x(q+p) - n - m) + (2bc+c+b)(y-x^2)^2)]/[(-y(q+p)^2 + 2(n+m)x(q+p) + ly - lx^2 - (n+m)^2)^3] + [3s^2(2(n+m)x - 2y(q+p))^2(((n+m)x - y(q+p))^2 + a(x(q+p) - n - m)^2 + 2a(x(q+p) - n - m)((n+m)x - y(q+p)) + 2(c+b)(y-x^2)((n+m)x - y(q+p)) + 2a(c+b)(y-x^2)(x(q+p) - n - m) + (2bc+c+b)(y-x^2)^2)]/[(-y(q+p)^2 + 2(n+m)x(q+p) + ly - lx^2 - (n+m)^2)^4]$

$h_{9\ 10}^{(U_3)} = [(-2y((n+m)x - (q+p)y) + 2ax((n+m)x - (q+p)y) - 2(c+b)y(y-x^2) + 2a(c+b)x(y-x^2) - 2a((q+p)x - n - m)y + 2ax((q+p)x - n - m))s]/[(-(q+p)^2y + ly - lx^2 + 2(n+m)(q+p)x - (n+m)^2)^2] - [2(2(n+m)x - 2(q+p)y)(((n+m)x - (q+p)y)^2 + (2bc+c+b)(y-x^2)^2 + 2(c+b)(y-x^2)((n+m)x - (q+p)y) + 2a((q+p)x - n - m)((n+m)x - (q+p)y) + 2a(c+b)((q+p)x - n - m)(y-x^2) + a((q+p)x - n - m)^2)s]/[(-(q+p)^2y + ly - lx^2 + 2(n+m)(q+p)x - (n+m)^2)^3]$

$h_{10\ 10}^{(U_3)} = [((n+m)x - (q+p)y)^2 + (2bc+c+b)(y-x^2)^2 + 2(c+b)(y-x^2)((n+m)x - (q+p)y) + 2a((q+p)x - n - m)((n+m)x - (q+p)y) + 2a(c+b)((q+p)x - n - m)(y-x^2) + a((q+p)x - n - m)^2]/[(-(q+p)^2y + ly - lx^2 + 2(n+m)(q+p)x - (n+m)^2)^2]$

$h_{11}^{(U_4)} = [(2(2bc+c+b)(n+m)(q+p) - 2(ac+ab)l(q+p) - 2(c+b)l(n+m) + 2al^2)tz]/[(-lx^2 + 2(n+m)(q+p)x - (q+p)^2y + ly - (n+m)^2)^2] - [4tz(2(n+m)(q+p) - 2lx)((ac+ab)(-l((q+p)x - n - m) + (q+p)((n+m)(q+p) - lx) + (n+m)(l - (q+p)^2)) + (c+b)(-l((n+m)x - (q+p)y) + (n+m)((n+m)(q+p) - lx) + (q+p)(ly - (n+m)^2)) + a(-2l((n+m)(q+p) - lx) - l(l - (q+p)^2)) + (2bc+c+b)(n+m)((q+p)x - n - m) + (2bc+c+b)(q+p)((n+m)x - (q+p)y) - l(ly - (n+m)^2))]/[(-lx^2 + 2(n+m)(q+p)x - (q+p)^2y + ly - (n+m)^2)^3] + [4ltz(a(((n+m)(q+p) - lx)^2 + (l - (q+p)^2)((n+m)(q+p) -$

$lx)+(l-(q+p)^2)(ly-(n+m)^2))+(ac+ab)(((n+m)(q+p)-lx)((q+p)x-n-m)+(l-(q+p)^2)((n+m)x-(q+p)y))+(c+b)((ly-(n+m)^2)((q+p)x-n-m)+((n+m)(q+p)-lx)((n+m)x-(q+p)y))+(2bc+c+b)((n+m)x-(q+p)y)((q+p)x-n-m)+(ly-(n+m)^2)((n+m)(q+p)-lx))]/[(-lx^2+2(n+m)(q+p)x-(q+p)^2y+ly-(n+m)^2)^3]+[6tz(2(n+m)(q+p)-2lx)^2(a(((n+m)(q+p)-lx)^2+(l-(q+p)^2)((n+m)(q+p)-lx)+(l-(q+p)^2)(ly-(n+m)^2))+(ac+ab)(((n+m)(q+p)-lx)((q+p)x-n-m)+(l-(q+p)^2)((n+m)x-(q+p)y))+(c+b)((ly-(n+m)^2)((q+p)x-n-m)+((n+m)(q+p)-lx)((n+m)x-(q+p)y))+(2bc+c+b)((n+m)x-(q+p)y)((q+p)x-n-m)+(ly-(n+m)^2)((n+m)(q+p)-lx))]/[(-lx^2+2(n+m)(q+p)x-(q+p)^2y+ly-(n+m)^2)^4]$

$h_{12}^{(U_4)} = [((c+b)(l(q+p)-l(-q-p))+(2bc+c+b)(-q-p)(q+p)-l^2)tz]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^2]-[2t(2(n+m)(q+p)-2lx)((c+b)(l((q+p)x-n-m)+(-q-p)((n+m)(q+p)-lx))+(2bc+c+b)(-q-p)((q+p)x-n-m)+l((n+m)(q+p)-lx)+(ac+ab)(-q-p)(l-(q+p)^2)+al(l-(q+p)^2))z]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^3]+[6(l-(q+p)^2)t(2(n+m)(q+p)-2lx)z((c+b)(((n+m)(q+p)-lx)((n+m)x-(q+p)y)+((q+p)x-n-m)(ly-(n+m)^2))+(ac+ab)((l-(q+p)^2)((n+m)x-(q+p)y)+((n+m)(q+p)-lx)((q+p)x-n-m))+a((l-(q+p)^2)(ly-(n+m)^2)+((n+m)(q+p)-lx)^2+(l-(q+p)^2)((n+m)(q+p)-lx))+(2bc+c+b)((q+p)x-n-m)((n+m)x-(q+p)y)+((n+m)(q+p)-lx)(ly-(n+m)^2))]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^4]-[2(l-(q+p)^2)tz((c+b)(-l((n+m)x-(q+p)y)+(q+p)(ly-(n+m)^2)+(n+m)((n+m)(q+p)-lx))+(2bc+c+b)(q+p)((n+m)x-(q+p)y)-l(ly-(n+m)^2)+(ac+ab)(-l((q+p)x-n-m)+(q+p)((n+m)(q+p)-lx)+(n+m)(l-(q+p)^2))+a(-2l((n+m)(q+p)-lx)-l(l-(q+p)^2))+(2bc+c+b)(n+m)((q+p)x-n-m))]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^3]$

$h_{13}^{(U_4)} = [t((c+b)(-l((n+m)x-(q+p)y)+(q+p)(ly-(n+m)^2)+(n+m)((n+m)(q+p)-lx))+(2bc+c+b)(q+p)((n+m)x-(q+p)y)-l(ly-(n+m)^2)+(ac+ab)(-l((q+p)x-n-m)+(q+p)((n+m)(q+p)-lx)+(n+m)(l-(q+p)^2))+a(-2l((n+m)(q+p)-lx)-l(l-(q+p)^2))+(2bc+c+b)(n+m)((q+p)x-n-m))]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^2]-[2t(2(n+m)(q+p)-2lx)((c+b)(((n+m)(q+p)-lx)((n+m)x-(q+p)y)+((q+p)x-n-m)(ly-(n+m)^2))+(ac+ab)((l-(q+p)^2)((n+m)x-(q+p)y)+((n+m)(q+p)-lx)((q+p)x-n-m))+a((l-(q+p)^2)(ly-(n+m)^2)+((n+m)(q+p)-lx)^2+$

$(l-(q+p)^2)((n+m)(q+p)-lx))+(2bc+c+b)((q+p)x-n-m)((n+m)x-(q+p)y)+((n+m)(q+p)-lx)(ly-(n+m)^2))]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^3]$

$h_{14}^{(U_4)} = [((c+b)(-l((n+m)x-(q+p)y)+(q+p)(ly-(n+m)^2)+(n+m)((n+m)(q+p)-lx))+(2bc+c+b)(q+p)((n+m)x-(q+p)y)-l(ly-(n+m)^2)+(ac+ab)(-l((q+p)x-n-m)+(q+p)((n+m)(q+p)-lx)+(n+m)(l-(q+p)^2))+a(-2l((n+m)(q+p)-lx)-l(l-(q+p)^2))+(2bc+c+b)(n+m)((q+p)x-n-m))z]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^2]-[2(2(n+m)(q+p)-2lx)((c+b)(((n+m)(q+p)-lx)((n+m)x-(q+p)y)+((q+p)x-n-m)(ly-(n+m)^2))+(ac+ab)((l-(q+p)^2)((n+m)x-(q+p)y)+((n+m)(q+p)-lx)((q+p)x-n-m))+a((l-(q+p)^2)(ly-(n+m)^2)+((n+m)(q+p)-lx)^2+(l-(q+p)^2)((n+m)(q+p)-lx))+(2bc+c+b)((q+p)x-n-m)((n+m)x-(q+p)y)+((n+m)(q+p)-lx)(ly-(n+m)^2))z]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^3]$

$h_{15}^{(U_4)} = [4txz(a(((n+m)(q+p)-xl)^2+(l-(q+p)^2)(yl-(n+m)^2)+(l-(q+p)^2)((n+m)(q+p)-xl))+(c+b)(((q+p)x-n-m)(yl-(n+m)^2)+((n+m)x-(q+p)y)((n+m)(q+p)-xl))+(ac+ab)(((q+p)x-n-m)((n+m)(q+p)-xl)+((n+m)x-(q+p)y)(l-(q+p)^2))+((n+m)(q+p)-xl)(yl-(n+m)^2)+(2bc+c+b)((q+p)x-n-m)((n+m)x-(q+p)y)))]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^3]+[6t(y-x^2)z(2(n+m)(q+p)-2xl)(a(((n+m)(q+p)-xl)^2+(l-(q+p)^2)(yl-(n+m)^2)+(l-(q+p)^2)((n+m)(q+p)-xl))+(c+b)(((q+p)x-n-m)(yl-(n+m)^2)+((n+m)x-(q+p)y)((n+m)(q+p)-xl))+(ac+ab)(((q+p)x-n-m)((n+m)(q+p)-xl)+((n+m)x-(q+p)y)(l-(q+p)^2))+((n+m)(q+p)-xl)(yl-(n+m)^2)+(2bc+c+b)((q+p)x-n-m)((n+m)x-(q+p)y)))]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^4]-[2t(y-x^2)z((c+b)((q+p)(yl-(n+m)^2)+(n+m)((n+m)(q+p)-xl)-((n+m)x-(q+p)y)l)+a(-2l((n+m)(q+p)-xl)-l(l-(q+p)^2))+(ac+ab)((q+p)((n+m)(q+p)-xl)+(n+m)(l-(q+p)^2)-((q+p)x-n-m)l)-l(yl-(n+m)^2)+(2bc+c+b)(q+p)((n+m)x-(q+p)y)+(2bc+c+b)(n+m)((q+p)x-n-m))]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^3]-[2tz(2(n+m)(q+p)-2xl)(a(-2x((n+m)(q+p)-xl)+y(l-(q+p)^2)-x(l-(q+p)^2)+yl-xl+(n+m)(q+p)-(n+m)^2)-x(yl-(n+m)^2)+y((n+m)(q+p)-xl)+(c+b)(((q+p)x-n-m)y-x((n+m)x-(q+p)y))+(ac+ab)(-(q+p)y-x((q+p)x-n-m)+(n+m)x))]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^3]+[tz(a(-2((n+$

$m)(q+p)-xl)+2xl-2l+(q+p)^2)-2yl+(c+b)(2(q+p)y-2(n+m)x)+(ac+ab)(-2(q+p)x+2n+2m)+(n+m)^2)]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^2]$

$h_{16}^{(U_4)} = [tz(2l(m+n)-(2bc+c+b)(m+n)+(2bc+c+b)(-m+(q+p)x-n)+(2bc+c+b)(q+p)x-2(c+b)lx-2al(q+p)+2(ac+ab)l)]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^2]-[2tz(2(q+p)x-2(m+n))(-l(ly-(m+n)^2)+(c+b)((q+p)(ly-(m+n)^2)-l(x(m+n)-(q+p)y)+(m+n)((q+p)(m+n)-lx))+(ac+ab)((q+p)((q+p)(m+n)-lx)+(l-(q+p)^2)(m+n)-l(-m+(q+p)x-n))+a(-2l((q+p)(m+n)-lx)-l(l-(q+p)^2))+(2bc+c+b)(q+p)(x(m+n)-(q+p)y)+(2bc+c+b)(-m+(q+p)x-n)(m+n))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^3]-[4(q+p)tz(a(((q+p)(m+n)-lx)^2+(l-(q+p)^2)(ly-(m+n)^2)+(l-(q+p)^2)((q+p)(m+n)-lx))+((q+p)(m+n)-lx)(ly-(m+n)^2)+(c+b)((-m+(q+p)x-n)(ly-(m+n)^2)+((q+p)(m+n)-lx)(x(m+n)-(q+p)y))+(ac+ab)((l-(q+p)^2)(x(m+n)-(q+p)y)+(-m+(q+p)x-n)((q+p)(m+n)-lx))+(2bc+c+b)(-m+(q+p)x-n)(x(m+n)-(q+p)y))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^3]+[6tz(2(q+p)x-2(m+n))(2(q+p)(m+n)-2lx)(a(((q+p)(m+n)-lx)^2+(l-(q+p)^2)(ly-(m+n)^2)+(l-(q+p)^2)((q+p)(m+n)-lx))+((q+p)(m+n)-lx)(ly-(m+n)^2)+(c+b)((-m+(q+p)x-n)(ly-(m+n)^2)+((q+p)(m+n)-lx)(x(m+n)-(q+p)y))+(ac+ab)((l-(q+p)^2)(x(m+n)-(q+p)y)+(-m+(q+p)x-n)((q+p)(m+n)-lx))+(2bc+c+b)(-m+(q+p)x-n)(x(m+n)-(q+p)y))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^4]-[2tz(2(q+p)(m+n)-2lx)((c+b)((m+n)^2+(q+p)(x(m+n)-(q+p)y)+x((q+p)(m+n)-lx)-2(-m+(q+p)x-n)(m+n)-ly)+(q+p)(ly-(m+n)^2)+a(2(q+p)((q+p)(m+n)-lx)-2(l-(q+p)^2)(m+n)+(q+p)(l-(q+p)^2))-(2bc+c+b)(x(m+n)-(q+p)y)-2(m+n)((q+p)(m+n)-lx)+(ac+ab)(-(q+p)(m+n)+(q+p)(-m+(q+p)x-n)+(l-(q+p)^2)x+lx)+(2bc+c+b)x(-m+(q+p)x-n))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^3]$

$h_{17}^{(U_4)} = [tz(2l(n+m)-(2bc+c+b)(n+m)+(2bc+c+b)(-n+(q+p)x-m)+(2bc+c+b)(q+p)x-2(c+b)lx-2al(q+p)+2(ac+ab)l)]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^2]-[2tz(2(q+p)x-2(n+m))(-l(ly-(n+m)^2)+(c+b)((q+p)(ly-(n+m)^2)-l(x(n+m)-(q+p)y)+(n+m)((q+p)(n+m)-lx))+(ac+ab)((q+p)((q+p)(n+m)-lx)+(l-(q+p)^2)(n+m)-l(-n+(q+p)x-m))+a(-2l((q+p)(n+m)-lx)-l(l-(q+p)^2))+(2bc+c+b)(q+p)(x(n+m)-(q+p)y)+(2bc+c+b)(-n+(q+p)x-m)(n+$

$$m))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]-[4(q+p)tz(a(((q+p)(n+$$
$$m)-lx)^2+(l-(q+p)^2)(ly-(n+m)^2)+(l-(q+p)^2)((q+p)(n+m)-lx))+((q+p)(n+$$
$$m)-lx)(ly-(n+m)^2)+(c+b)((-n+(q+p)x-m)(ly-(n+m)^2)+((q+p)(n+m)-$$
$$lx)(x(n+m)-(q+p)y))+(ac+ab)((l-(q+p)^2)(x(n+m)-(q+p)y)+(-n+(q+p)x-$$
$$m)((q+p)(n+m)-lx))+(2bc+c+b)(-n+(q+p)x-m)(x(n+m)-(q+p)y))]/[(-(n+$$
$$m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]+[6tz(2(q+p)x-2(n+m))(2(q+p)(n+$$
$$m)-2lx)(a(((q+p)(n+m)-lx)^2+(l-(q+p)^2)(ly-(n+m)^2)+(l-(q+p)^2)((q+p)(n+$$
$$m)-lx))+((q+p)(n+m)-lx)(ly-(n+m)^2)+(c+b)((-n+(q+p)x-m)(ly-(n+m)^2)+$$
$$((q+p)(n+m)-lx)(x(n+m)-(q+p)y))+(ac+ab)((l-(q+p)^2)(x(n+m)-(q+p)y)+$$
$$(-n+(q+p)x-m)((q+p)(n+m)-lx))+(2bc+c+b)(-n+(q+p)x-m)(x(n+m)-$$
$$(q+p)y))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^4]-[2tz(2(q+p)(n+m)-$$
$$2lx)((c+b)((n+m)^2+(q+p)(x(n+m)-(q+p)y)+x((q+p)(n+m)-lx)-2(-n+(q+$$
$$p)x-m)(n+m)-ly)+(q+p)(ly-(n+m)^2)+a(2(q+p)((q+p)(n+m)-lx)-2(l-(q+$$
$$p)^2)(n+m)+(q+p)(l-(q+p)^2))-(2bc+c+b)(x(n+m)-(q+p)y)-2(n+m)((q+p)(n+$$
$$m)-lx)+(ac+ab)(-(q+p)(n+m)+(q+p)(-n+(q+p)x-m)+(l-(q+p)^2)x+lx)+$$
$$(2bc+c+b)x(-n+(q+p)x-m))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]$$

$$h_{18}^{(U_4)}=[tz((2bc+c+b)((n+m)x-y(p+q))+a(2l(p+q)-2l(n+m))-(2bc+c+$$
$$b)y(p+q)+2(c+b)ly+(2bc+c+b)(n+m)x-2(ac+ab)lx)]/[(-y(p+q)^2+2(n+m)x(p+$$
$$q)+ly-lx^2-(n+m)^2)^2]-[2tz(2(n+m)(p+q)-2lx)((ac+ab)(-y(l-(p+q)^2)-2(p+$$
$$q)((n+m)x-y(p+q))+(n+m)(x(p+q)-n-m)+x((n+m)(p+q)-lx))+a((n+$$
$$m)(l-(p+q)^2)-2(p+q)((n+m)(p+q)-lx)+2(n+m)((n+m)(p+q)-lx)-2(ly-(n+$$
$$m)^2)(p+q))+(c+b)((n+m)((n+m)x-y(p+q))-y((n+m)(p+q)-lx)+x(ly-(n+$$
$$m)^2))+(2bc+c+b)x((n+m)x-y(p+q))-(2bc+c+b)y(x(p+q)-n-m)+(n+m)(ly-$$
$$(n+m)^2))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]-[2tz(2(n+m)x-$$
$$2y(p+q))((ac+ab)((n+m)(l-(p+q)^2)-l(x(p+q)-n-m)+(p+q)((n+m)(p+q)-$$
$$lx))+a(-l(l-(p+q)^2)-2l((n+m)(p+q)-lx))+(c+b)(-l((n+m)x-y(p+q))+(n+$$
$$m)((n+m)(p+q)-lx)+(ly-(n+m)^2)(p+q))+(2bc+c+b)(p+q)((n+m)x-y(p+q))+$$
$$(2bc+c+b)(n+m)(x(p+q)-n-m)-l(ly-(n+m)^2))]/[(-y(p+q)^2+2(n+m)x(p+q)+$$
$$ly-lx^2-(n+m)^2)^3]-[4(n+m)tz(a(((n+m)(p+q)-lx)^2+((n+m)(p+q)-lx)(l-(p+$$
$$q)^2)+(ly-(n+m)^2)(l-(p+q)^2))+(ac+ab)(((n+m)x-y(p+q))(l-(p+q)^2)+((n+$$

$m)(p+q)-lx)(x(p+q)-n-m))+(c+b)(((n+m)(p+q)-lx)((n+m)x-y(p+q))+(ly-(n+m)^2)(x(p+q)-n-m))+(2bc+c+b)(x(p+q)-n-m)((n+m)x-y(p+q))+(ly-(n+m)^2)((n+m)(p+q)-lx))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]+[6tz(2(n+m)(p+q)-2lx)(2(n+m)x-2y(p+q))(a(((n+m)(p+q)-lx)^2+((n+m)(p+q)-lx)(l-(p+q)^2)+(ly-(n+m)^2)(l-(p+q)^2))+(ac+ab)(((n+m)x-y(p+q))(l-(p+q)^2)+((n+m)(p+q)-lx)(x(p+q)-n-m))+(c+b)(((n+m)(p+q)-lx)((n+m)x-y(p+q))+(ly-(n+m)^2)(x(p+q)-n-m))+(2bc+c+b)(x(p+q)-n-m)((n+m)x-y(p+q))+(ly-(n+m)^2)((n+m)(p+q)-lx))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^4]$

$h_{19}^{(U_4)} = [tz((2bc+c+b)((n+m)x-y(q+p))+a(2l(q+p)-2l(n+m))-(2bc+c+b)y(q+p)+2(c+b)ly+(2bc+c+b)(n+m)x-2(ac+ab)lx)]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^2]-[2tz(2(n+m)(q+p)-2lx)((ac+ab)(-y(l-(q+p)^2)-2(q+p)((n+m)x-y(q+p))+(n+m)(x(q+p)-n-m)+x((n+m)(q+p)-lx))+a((n+m)(l-(q+p)^2)-2(q+p)((n+m)(q+p)-lx)+2(n+m)((n+m)(q+p)-lx)-2(ly-(n+m)^2)(q+p))+(c+b)((n+m)((n+m)x-y(q+p))-y((n+m)(q+p)-lx)+x(ly-(n+m)^2))+(2bc+c+b)x((n+m)x-y(q+p))-(2bc+c+b)y(x(q+p)-n-m)+(n+m)(ly-(n+m)^2))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]-[2tz(2(n+m)x-2y(q+p))((ac+ab)((n+m)(l-(q+p)^2)-l(x(q+p)-n-m)+(q+p)((n+m)(q+p)-lx))+a(-l(l-(q+p)^2)-2l((n+m)(q+p)-lx))+(c+b)(-l((n+m)x-y(q+p))+(n+m)((n+m)(q+p)-lx)+(ly-(n+m)^2)(q+p))+(2bc+c+b)(q+p)((n+m)x-y(q+p))+(2bc+c+b)(n+m)(x(q+p)-n-m)-l(ly-(n+m)^2))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]-[4(n+m)tz(a(((n+m)(q+p)-lx)^2+((n+m)(q+p)-lx)(l-(q+p)^2)+(ly-(n+m)^2)(l-(q+p)^2))+(ac+ab)(((n+m)x-y(q+p))(l-(q+p)^2)+((n+m)(q+p)-lx)(x(q+p)-n-m))+(c+b)(((n+m)(q+p)-lx)((n+m)x-y(q+p))+(ly-(n+m)^2)(x(q+p)-n-m))+(2bc+c+b)(x(q+p)-n-m)((n+m)x-y(q+p))+(ly-(n+m)^2)((n+m)(q+p)-lx))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]+[6tz(2(n+m)(q+p)-2lx)(2(n+m)x-2y(q+p))(a(((n+m)(q+p)-lx)^2+((n+m)(q+p)-lx)(l-(q+p)^2)+(ly-(n+m)^2)(l-(q+p)^2))+(ac+ab)(((n+m)x-y(q+p))(l-(q+p)^2)+((n+m)(q+p)-lx)(x(q+p)-n-m))+(c+b)(((n+m)(q+p)-lx)((n+m)x-y(q+p))+(ly-(n+m)^2)(x(q+p)-n-m))+(2bc+c+b)(x(q+p)-n-m)((n+m)x-y(q+p))+(ly-(n+m)^2)((n+m)(q+p)-lx))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^4]$

$h_{1\ 10}^{(U_4)} = 0$

$h_{22}^{(U_4)} = [6(l-(q+p)^2)^2tz((c+b)(((n+m)(q+p)-lx)((n+m)x-(q+p)y)+((q+p)x-n-m)(ly-(n+m)^2))+(ac+ab)((l-(q+p)^2)((n+m)x-(q+p)y)+((n+m)(q+p)-lx)((q+p)x-n-m))+a((l-(q+p)^2)(ly-(n+m)^2)+((n+m)(q+p)-lx)^2+(l-(q+p)^2)((n+m)(q+p)-lx))+(2bc+c+b)((q+p)x-n-m)((n+m)x-(q+p)y)+((n+m)(q+p)-lx)(ly-(n+m)^2))]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^4]-[4(l-(q+p)^2)t((c+b)(l((q+p)x-n-m)+(-q-p)((n+m)(q+p)-lx))+(2bc+c+b)(-q-p)((q+p)x-n-m)+l((n+m)(q+p)-lx)+(ac+ab)(-q-p)(l-(q+p)^2)+al(l-(q+p)^2))z]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^3]$

$h_{23}^{(U_4)} = [t((c+b)(l((q+p)x-n-m)+(-q-p)((n+m)(q+p)-lx))+(2bc+c+b)(-q-p)((q+p)x-n-m)+l((n+m)(q+p)-lx)+(ac+ab)(-q-p)(l-(q+p)^2)+al(l-(q+p)^2))]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^2]-[2(l-(q+p)^2)t((c+b)(((n+m)(q+p)-lx)((n+m)x-(q+p)y)+((q+p)x-n-m)(ly-(n+m)^2))+(ac+ab)((l-(q+p)^2)((n+m)x-(q+p)y)+((n+m)(q+p)-lx)((q+p)x-n-m))+a((l-(q+p)^2)(ly-(n+m)^2)+((n+m)(q+p)-lx)^2+(l-(q+p)^2)((n+m)(q+p)-lx))+(2bc+c+b)((q+p)x-n-m)((n+m)x-(q+p)y)+((n+m)(q+p)-lx)(ly-(n+m)^2))]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^3]$

$h_{24}^{(U_4)} = [((c+b)(l((q+p)x-n-m)+(-q-p)((n+m)(q+p)-lx))+(2bc+c+b)(-q-p)((q+p)x-n-m)+l((n+m)(q+p)-lx)+(ac+ab)(-q-p)(l-(q+p)^2)+al(l-(q+p)^2))z]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^2]-[2(l-(q+p)^2)((c+b)(((n+m)(q+p)-lx)((n+m)x-(q+p)y)+((q+p)x-n-m)(ly-(n+m)^2))+(ac+ab)((l-(q+p)^2)((n+m)x-(q+p)y)+((n+m)(q+p)-lx)((q+p)x-n-m))+a((l-(q+p)^2)(ly-(n+m)^2)+((n+m)(q+p)-lx)^2+(l-(q+p)^2)((n+m)(q+p)-lx))+(2bc+c+b)((q+p)x-n-m)((n+m)x-(q+p)y)+((n+m)(q+p)-lx)(ly-(n+m)^2))z]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^3]$

$h_{25}^{(U_4)} = -[2tz(a(((n+m)(q+p)-xl)^2+(l-(q+p)^2)(yl-(n+m)^2)+(l-(q+p)^2)((n+m)(q+p)-xl))+(c+b)(((q+p)x-n-m)(yl-(n+m)^2)+((n+m)x-(q+p)y)((n+m)(q+p)-xl))+(ac+ab)(((q+p)x-n-m)((n+m)(q+p)-xl)+((n+m)x-(q+p)y)(l-(q+p)^2))+((n+m)(q+p)-xl)(yl-(n+m)^2)+(2bc+c+b)((q+p)x-n-m)((n+m)x-(q+p)y))]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^3]+$

$[6t(y-x^2)z(l-(q+p)^2)(a(((n+m)(q+p)-xl)^2+(l-(q+p)^2)(yl-(n+m)^2)+(l-(q+p)^2)((n+m)(q+p)-xl))+(c+b)(((q+p)x-n-m)(yl-(n+m)^2)+((n+m)x-(q+p)y)((n+m)(q+p)-xl))+(ac+ab)(((q+p)x-n-m)((n+m)(q+p)-xl)+((n+m)x-(q+p)y)(l-(q+p)^2))+((n+m)(q+p)-xl)(yl-(n+m)^2)+(2bc+c+b)((q+p)x-n-m)((n+m)x-(q+p)y))]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^4]-[2tz(l-(q+p)^2)(a(-2x((n+m)(q+p)-xl)+y(l-(q+p)^2)-x(l-(q+p)^2)+yl-xl+(n+m)(q+p)-(n+m)^2)-x(yl-(n+m)^2)+y((n+m)(q+p)-xl)+(c+b)(((q+p)x-n-m)y-x((n+m)x-(q+p)y))+(ac+ab)(-(q+p)y-x((q+p)x-n-m)+(n+m)x))]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^3]-[2t(y-x^2)z((c+b)((-q-p)((n+m)(q+p)-xl)+((q+p)x-n-m)l)+l((n+m)(q+p)-xl)+al(l-(q+p)^2)+(ac+ab)(-q-p)(l-(q+p)^2)+(2bc+c+b)(-q-p)((q+p)x-n-m))]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^3]+[tz(a(l-(q+p)^2)-2xl+al+(c+b)((q+p)x-(-q-p)x-n-m)+(n+m)(q+p)+(ac+ab)(-q-p))]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^2]$

$h_{26}^{(U_4)}=[((c+b)((-q-p)(q+p)-l)+l(q+p)-(2bc+c+b)(-q-p))tz]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^2]-[2tz(2(q+p)x-2(m+n))((c+b)((-q-p)((q+p)(m+n)-lx)+l(-m+(q+p)x-n))+l((q+p)(m+n)-lx)+(2bc+c+b)(-q-p)(-m+(q+p)x-n)+(ac+ab)(-q-p)(l-(q+p)^2)+al(l-(q+p)^2))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^3]+[6(l-(q+p)^2)tz(2(q+p)x-2(m+n))(a(((q+p)(m+n)-lx)^2+(l-(q+p)^2)(ly-(m+n)^2)+(l-(q+p)^2)((q+p)(m+n)-lx))+((q+p)(m+n)-lx)(ly-(m+n)^2)+(c+b)((-m+(q+p)x-n)(ly-(m+n)^2)+((q+p)(m+n)-lx)(x(m+n)-(q+p)y))+(ac+ab)((l-(q+p)^2)(x(m+n)-(q+p)y)+(-m+(q+p)x-n)((q+p)(m+n)-lx))+(2bc+c+b)(-m+(q+p)x-n)(x(m+n)-(q+p)y))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^4]-[2(l-(q+p)^2)tz((c+b)((m+n)^2+(q+p)(x(m+n)-(q+p)y)+x((q+p)(m+n)-lx)-2(-m+(q+p)x-n)(m+n)-ly)+(q+p)(ly-(m+n)^2)+a(2(q+p)((q+p)(m+n)-lx)-2(l-(q+p)^2)(m+n)+(q+p)(l-(q+p)^2))-(2bc+c+b)(x(m+n)-(q+p)y)-2(m+n)((q+p)(m+n)-lx)+(ac+ab)(-(q+p)(m+n)+(q+p)(-m+(q+p)x-n)+(l-(q+p)^2)x+lx)+(2bc+c+b)x(-m+(q+p)x-n))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^3]$

$h_{27}^{(U_4)}=[((c+b)((-q-p)(q+p)-l)+l(q+p)-(2bc+c+b)(-q-p))tz]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^2]-[2tz(2(q+p)x-2(n+m))((c+b)((-q-$

$$p)((q+p)(n+m)-lx)+l(-n+(q+p)x-m))+l((q+p)(n+m)-lx)+(2bc+c+b)(-q-$$

$$p)(-n+(q+p)x-m)+(ac+ab)(-q-p)(l-(q+p)^2)+al(l-(q+p)^2))]/[(-(n+m)^2+$$

$$2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]+[6(l-(q+p)^2)tz(2(q+p)x-2(n+m))(a(((q+$$

$$p)(n+m)-lx)^2+(l-(q+p)^2)(ly-(n+m)^2)+(l-(q+p)^2)((q+p)(n+m)-lx))+((q+$$

$$p)(n+m)-lx)(ly-(n+m)^2)+(c+b)((-n+(q+p)x-m)(ly-(n+m)^2)+((q+p)(n+$$

$$m)-lx)(x(n+m)-(q+p)y))+(ac+ab)((l-(q+p)^2)(x(n+m)-(q+p)y)+(-n+$$

$$(q+p)x-m)((q+p)(n+m)-lx))+(2bc+c+b)(-n+(q+p)x-m)(x(n+m)-(q+$$

$$p)y))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^4]-[2(l-(q+p)^2)tz((c+$$

$$b)((n+m)^2+(q+p)(x(n+m)-(q+p)y)+x((q+p)(n+m)-lx)-2(-n+(q+p)x-$$

$$m)(n+m)-ly)+(q+p)(ly-(n+m)^2)+a(2(q+p)((q+p)(n+m)-lx)-2(l-(q+$$

$$p)^2)(n+m)+(q+p)(l-(q+p)^2))-(2bc+c+b)(x(n+m)-(q+p)y)-2(n+m)((q+p)(n+$$

$$m)-lx)+(ac+ab)(-(q+p)(n+m)+(q+p)(-n+(q+p)x-m)+(l-(q+p)^2)x+lx)+$$

$$(2bc+c+b)x(-n+(q+p)x-m))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]$$

$$h_{28}^{(U_4)}=[tz(-(ac+ab)(l-(p+q)^2)-(2bc+c+b)(x(p+q)-n-m)+(c+b)(-(n+$$

$$m)(p+q)+(n+m)(-p-q)+2lx)-2(ac+ab)(-p-q)(p+q)-2al(p+q)+(2bc+$$

$$c+b)x(-p-q)+l(n+m))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^2]-$$

$$[2tz((ac+ab)(-y(l-(p+q)^2)-2(p+q)((n+m)x-y(p+q))+(n+m)(x(p+q)-n-$$

$$m)+x((n+m)(p+q)-lx))+a((n+m)(l-(p+q)^2)-2(p+q)((n+m)(p+q)-lx)+2(n+$$

$$m)((n+m)(p+q)-lx)-2(ly-(n+m)^2)(p+q))+(c+b)((n+m)((n+m)x-y(p+q))-$$

$$y((n+m)(p+q)-lx)+x(ly-(n+m)^2))+(2bc+c+b)x((n+m)x-y(p+q))-(2bc+c+$$

$$b)y(x(p+q)-n-m)+(n+m)(ly-(n+m)^2))(l-(p+q)^2)]/[(-y(p+q)^2+2(n+m)x(p+$$

$$q)+ly-lx^2-(n+m)^2)^3]-[2tz(2(n+m)x-2y(p+q))((ac+ab)(-p-q)(l-(p+q)^2)+$$

$$al(l-(p+q)^2)+(c+b)(l(x(p+q)-n-m)+(-p-q)((n+m)(p+q)-lx))+(2bc+c+$$

$$b)(-p-q)(x(p+q)-n-m)+l((n+m)(p+q)-lx))]/[(-y(p+q)^2+2(n+m)x(p+q)+$$

$$ly-lx^2-(n+m)^2)^3]+[4tz(p+q)(a(((n+m)(p+q)-lx)^2+((n+m)(p+q)-lx)(l-(p+$$

$$q)^2)+(ly-(n+m)^2)(l-(p+q)^2))+(ac+ab)(((n+m)x-y(p+q))(l-(p+q)^2)+((n+$$

$$m)(p+q)-lx)(x(p+q)-n-m))+(c+b)(((n+m)(p+q)-lx)((n+m)x-y(p+q))+(ly-$$

$$(n+m)^2)(x(p+q)-n-m))+(2bc+c+b)(x(p+q)-n-m)((n+m)x-y(p+q))+(ly-$$

$$(n+m)^2)((n+m)(p+q)-lx))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]+$$

$$[6tz(2(n+m)x-2y(p+q))(l-(p+q)^2)(a(((n+m)(p+q)-lx)^2+((n+m)(p+q)-lx)(l-$$

$(p+q)^2)+(ly-(n+m)^2)(l-(p+q)^2))+(ac+ab)(((n+m)x-y(p+q))(l-(p+q)^2)+((n+m)(p+q)-lx)(x(p+q)-n-m))+(c+b)(((n+m)(p+q)-lx)((n+m)x-y(p+q))+(ly-(n+m)^2)(x(p+q)-n-m))+(2bc+c+b)(x(p+q)-n-m)((n+m)x-y(p+q))+(ly-(n+m)^2)((n+m)(p+q)-lx))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^4]$

$h_{29}^{(U_4)} = [tz(-(ac+ab)(l-(q+p)^2)-(2bc+c+b)(x(q+p)-n-m)+(c+b)(-(n+m)(q+p)+(n+m)(-q-p)+2lx)-2(ac+ab)(-q-p)(q+p)-2al(q+p)+(2bc+c+b)x(-q-p)+l(n+m))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^2]-[2tz((ac+ab)(-y(l-(q+p)^2)-2(q+p)((n+m)x-y(q+p))+(n+m)(x(q+p)-n-m)+x((n+m)(q+p)-lx))+a((n+m)(l-(q+p)^2)-2(q+p)((n+m)(q+p)-lx)+2(n+m)((n+m)(q+p)-lx)-2(ly-(n+m)^2)(q+p))+(c+b)((n+m)((n+m)x-y(q+p))-y((n+m)(q+p)-lx)+x(ly-(n+m)^2))+(2bc+c+b)x((n+m)x-y(q+p))-(2bc+c+b)y(x(q+p)-n-m)+(n+m)(ly-(n+m)^2))(l-(q+p)^2)]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]-[2tz(2(n+m)x-2y(q+p))((ac+ab)(-q-p)(l-(q+p)^2)+al(l-(q+p)^2)+(c+b)(l(x(q+p)-n-m)+(-q-p)((n+m)(q+p)-lx))+(2bc+c+b)(-q-p)(x(q+p)-n-m)+l((n+m)(q+p)-lx))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]+[4tz(q+p)(a(((n+m)(q+p)-lx)^2+((n+m)(q+p)-lx)(l-(q+p)^2)+(ly-(n+m)^2)(l-(q+p)^2))+(ac+ab)(((n+m)x-y(q+p))(l-(q+p)^2)+((n+m)(q+p)-lx)(x(q+p)-n-m))+(c+b)(((n+m)(q+p)-lx)((n+m)x-y(q+p))+(ly-(n+m)^2)(x(q+p)-n-m))+(2bc+c+b)(x(q+p)-n-m)((n+m)x-y(q+p))+(ly-(n+m)^2)((n+m)(q+p)-lx))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]+[6tz(2(n+m)x-2y(q+p))(l-(q+p)^2)(a(((n+m)(q+p)-lx)^2+((n+m)(q+p)-lx)(l-(q+p)^2)+(ly-(n+m)^2)(l-(q+p)^2))+(ac+ab)(((n+m)x-y(q+p))(l-(q+p)^2)+((n+m)(q+p)-lx)(x(q+p)-n-m))+(c+b)(((n+m)(q+p)-lx)((n+m)x-y(q+p))+(ly-(n+m)^2)(x(q+p)-n-m))+(2bc+c+b)(x(q+p)-n-m)((n+m)x-y(q+p))+(ly-(n+m)^2)((n+m)(q+p)-lx))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^4]$

$h_{2\ 10}^{(U_4)} = 0$

$h_{33}^{(U_4)} = 0$

$h_{34}^{(U_4)} = [(c+b)(((n+m)(q+p)-lx)((n+m)x-(q+p)y)+((q+p)x-n-m)(ly-(n+m)^2))+(ac+ab)((l-(q+p)^2)((n+m)x-(q+p)y)+((n+m)(q+p)-lx)((q+p)x-n-m))+a((l-(q+p)^2)(ly-(n+m)^2)+((n+m)(q+p)-lx)^2+(l-(q+$

$p)^2)((n+m)(q+p)-lx))+(2bc+c+b)((q+p)x-n-m)((n+m)x-(q+p)y)+((n+m)(q+p)-lx)(ly-(n+m)^2)]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^2]$

$h_{35}^{(U_4)} = [t(a(-2x((n+m)(q+p)-xl)+y(l-(q+p)^2)-x(l-(q+p)^2)+yl-xl+(n+m)(q+p)-(n+m)^2)-x(yl-(n+m)^2)+y((n+m)(q+p)-xl)+(c+b)((((q+p)x-n-m)y-x((n+m)x-(q+p)y))+(ac+ab)(-(q+p)y-x((q+p)x-n-m)+(n+m)x))]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^2]-[2t(y-x^2)(a(((n+m)(q+p)-xl)^2+(l-(q+p)^2)(yl-(n+m)^2)+(l-(q+p)^2)((n+m)(q+p)-xl))+(c+b)((((q+p)x-n-m)(yl-(n+m)^2)+((n+m)x-(q+p)y)((n+m)(q+p)-xl))+(ac+ab)((((q+p)x-n-m)((n+m)(q+p)-xl)+((n+m)x-(q+p)y)(l-(q+p)^2))+((n+m)(q+p)-xl)(yl-(n+m)^2)+(2bc+c+b)((q+p)x-n-m)((n+m)x-(q+p)y))]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^3]$

$h_{36}^{(U_4)} = [t((c+b)((m+n)^2+(q+p)(x(m+n)-(q+p)y)+x((q+p)(m+n)-lx)-2(-m+(q+p)x-n)(m+n)-ly)+(q+p)(ly-(m+n)^2)+a(2(q+p)((q+p)(m+n)-lx)-2(l-(q+p)^2)(m+n)+(q+p)(l-(q+p)^2))-(2bc+c+b)(x(m+n)-(q+p)y)-2(m+n)((q+p)(m+n)-lx)+(ac+ab)(-(q+p)(m+n)+(q+p)(-m+(q+p)x-n)+(l-(q+p)^2)x+lx)+(2bc+c+b)x(-m+(q+p)x-n))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^2]-[2t(2(q+p)x-2(m+n))(a(((q+p)(m+n)-lx)^2+(l-(q+p)^2)(ly-(m+n)^2)+(l-(q+p)^2)((q+p)(m+n)-lx))+((q+p)(m+n)-lx)(ly-(m+n)^2)+(c+b)((-m+(q+p)x-n)(ly-(m+n)^2)+((q+p)(m+n)-lx)(x(m+n)-(q+p)y))+(ac+ab)((l-(q+p)^2)(x(m+n)-(q+p)y)+(-m+(q+p)x-n)((q+p)(m+n)-lx))+(2bc+c+b)(-m+(q+p)x-n)(x(m+n)-(q+p)y))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^3]$

$h_{37}^{(U_4)} = [t((c+b)((n+m)^2+(q+p)(x(n+m)-(q+p)y)+x((q+p)(n+m)-lx)-2(-n+(q+p)x-m)(n+m)-ly)+(q+p)(ly-(n+m)^2)+a(2(q+p)((q+p)(n+m)-lx)-2(l-(q+p)^2)(n+m)+(q+p)(l-(q+p)^2))-(2bc+c+b)(x(n+m)-(q+p)y)-2(n+m)((q+p)(n+m)-lx)+(ac+ab)(-(q+p)(n+m)+(q+p)(-n+(q+p)x-m)+(l-(q+p)^2)x+lx)+(2bc+c+b)x(-n+(q+p)x-m))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^2]-[2t(2(q+p)x-2(n+m))(a(((q+p)(n+m)-lx)^2+(l-(q+p)^2)(ly-(n+m)^2)+(l-(q+p)^2)((q+p)(n+m)-lx))+((q+p)(n+m)-lx)(ly-(n+m)^2)+(c+b)((-n+(q+p)x-m)(ly-(n+m)^2)+((q+p)(n+m)-lx)(x(n+m)-(q+p)y))+(ac+ab)((l-(q+p)^2)(x(n+m)-(q+p)y)+(-n+(q+p)x-m)((q+p)(n+m)-lx))+(2bc+c+b)(-n+$

$(q+p)x-m)(x(n+m)-(q+p)y))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]$

$h_{38}^{(U_4)} = [t((ac+ab)(-y(l-(p+q)^2)-2(p+q)((n+m)x-y(p+q))+(n+m)(x(p+q)-n-m)+x((n+m)(p+q)-lx))+a((n+m)(l-(p+q)^2)-2(p+q)((n+m)(p+q)-lx)+2(n+m)((n+m)(p+q)-lx)-2(ly-(n+m)^2)(p+q))+(c+b)((n+m)((n+m)x-y(p+q))-y((n+m)(p+q)-lx)+x(ly-(n+m)^2))+(2bc+c+b)x((n+m)x-y(p+q))-(2bc+c+b)y(x(p+q)-n-m)+(n+m)(ly-(n+m)^2))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^2]-[2t(2(n+m)x-2y(p+q))(a(((n+m)(p+q)-lx)^2+((n+m)(p+q)-lx)(l-(p+q)^2)+(ly-(n+m)^2)(l-(p+q)^2))+(ac+ab)(((n+m)x-y(p+q))(l-(p+q)^2)+((n+m)(p+q)-lx)(x(p+q)-n-m))+(c+b)(((n+m)(p+q)-lx)((n+m)x-y(p+q))+(ly-(n+m)^2)(x(p+q)-n-m))+(2bc+c+b)(x(p+q)-n-m)((n+m)x-y(p+q))+(ly-(n+m)^2)((n+m)(p+q)-lx))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]$

$h_{39}^{(U_4)} = [t((ac+ab)(-y(l-(q+p)^2)-2(q+p)((n+m)x-y(q+p))+(n+m)(x(q+p)-n-m)+x((n+m)(q+p)-lx))+a((n+m)(l-(q+p)^2)-2(q+p)((n+m)(q+p)-lx)+2(n+m)((n+m)(q+p)-lx)-2(ly-(n+m)^2)(q+p))+(c+b)((n+m)((n+m)x-y(q+p))-y((n+m)(q+p)-lx)+x(ly-(n+m)^2))+(2bc+c+b)x((n+m)x-y(q+p))-(2bc+c+b)y(x(q+p)-n-m)+(n+m)(ly-(n+m)^2))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^2]-[2t(2(n+m)x-2y(q+p))(a(((n+m)(q+p)-lx)^2+((n+m)(q+p)-lx)(l-(q+p)^2)+(ly-(n+m)^2)(l-(q+p)^2))+(ac+ab)(((n+m)x-y(q+p))(l-(q+p)^2)+((n+m)(q+p)-lx)(x(q+p)-n-m))+(c+b)(((n+m)(q+p)-lx)((n+m)x-y(q+p))+(ly-(n+m)^2)(x(q+p)-n-m))+(2bc+c+b)(x(q+p)-n-m)((n+m)x-y(q+p))+(ly-(n+m)^2)((n+m)(q+p)-lx))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]$

$h_{3\ 10}^{(U_4)} = 0$

$h_{44}^{(U_4)} = 0$

$h_{45}^{(U_4)} = [z(a(-2x((n+m)(q+p)-xl)+y(l-(q+p)^2)-x(l-(q+p)^2)+yl-xl+(n+m)(q+p)-(n+m)^2)-x(yl-(n+m)^2)+y((n+m)(q+p)-xl)+(c+b)((((q+p)x-n-m)y-x((n+m)x-(q+p)y))+(ac+ab)(-(q+p)y-x((q+p)x-n-m)+(n+m)x))]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^2]-[2(y-x^2)z(a(((n+m)(q+p)-xl)^2+(l-(q+p)^2)(yl-(n+m)^2)+(l-(q+p)^2)((n+m)(q+p)-xl))+(c+b)((((q+p)x-n-m)(yl-(n+m)^2)+((n+m)x-(q+p)y)((n+m)(q+p)-xl))+(ac+ab)((((q+p)x-n-m)((n+m)(q+p)-xl)+((n+m)x-$

$(q+p)y)(l-(q+p)^2))+((n+m)(q+p)-xl)(yl-(n+m)^2)+(2bc+c+b)((q+p)x-n-m)((n+m)x-(q+p)y))]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^3]$

$h_{46}^{(U_4)}=[z((c+b)((m+n)^2+(q+p)(x(m+n)-(q+p)y)+x((q+p)(m+n)-lx)-2(-m+(q+p)x-n)(m+n)-ly)+(q+p)(ly-(m+n)^2)+a(2(q+p)((q+p)(m+n)-lx)-2(l-(q+p)^2)(m+n)+(q+p)(l-(q+p)^2))-(2bc+c+b)(x(m+n)-(q+p)y)-2(m+n)((q+p)(m+n)-lx)+(ac+ab)(-(q+p)(m+n)+(q+p)(-m+(q+p)x-n)+(l-(q+p)^2)x+lx)+(2bc+c+b)x(-m+(q+p)x-n))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^2]-[2z(2(q+p)x-2(m+n))(a(((q+p)(m+n)-lx)^2+(l-(q+p)^2)(ly-(m+n)^2)+(l-(q+p)^2)((q+p)(m+n)-lx))+((q+p)(m+n)-lx)(ly-(m+n)^2)+(c+b)((-m+(q+p)x-n)(ly-(m+n)^2)+((q+p)(m+n)-lx)(x(m+n)-(q+p)y))+(ac+ab)((l-(q+p)^2)(x(m+n)-(q+p)y)+(-m+(q+p)x-n)((q+p)(m+n)-lx))+(2bc+c+b)(-m+(q+p)x-n)(x(m+n)-(q+p)y))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^3]$

$h_{47}^{(U_4)}=[z((c+b)((n+m)^2+(q+p)(x(n+m)-(q+p)y)+x((q+p)(n+m)-lx)-2(-n+(q+p)x-m)(n+m)-ly)+(q+p)(ly-(n+m)^2)+a(2(q+p)((q+p)(n+m)-lx)-2(l-(q+p)^2)(n+m)+(q+p)(l-(q+p)^2))-(2bc+c+b)(x(n+m)-(q+p)y)-2(n+m)((q+p)(n+m)-lx)+(ac+ab)(-(q+p)(n+m)+(q+p)(-n+(q+p)x-m)+(l-(q+p)^2)x+lx)+(2bc+c+b)x(-n+(q+p)x-m))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^2]-[2z(2(q+p)x-2(n+m))(a(((q+p)(n+m)-lx)^2+(l-(q+p)^2)(ly-(n+m)^2)+(l-(q+p)^2)((q+p)(n+m)-lx))+((q+p)(n+m)-lx)(ly-(n+m)^2)+(c+b)((-n+(q+p)x-m)(ly-(n+m)^2)+((q+p)(n+m)-lx)(x(n+m)-(q+p)y))+(ac+ab)((l-(q+p)^2)(x(n+m)-(q+p)y)+(-n+(q+p)x-m)((q+p)(n+m)-lx))+(2bc+c+b)(-n+(q+p)x-m)(x(n+m)-(q+p)y))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]$

$h_{48}^{(U_4)}=[z((ac+ab)(-y(l-(p+q)^2)-2(p+q)((n+m)x-y(p+q))+(n+m)(x(p+q)-n-m)+x((n+m)(p+q)-lx))+a((n+m)(l-(p+q)^2)-2(p+q)((n+m)(p+q)-lx)+2(n+m)((n+m)(p+q)-lx)-2(ly-(n+m)^2)(p+q))+(c+b)((n+m)((n+m)x-y(p+q))-y((n+m)(p+q)-lx)+x(ly-(n+m)^2))+(2bc+c+b)x((n+m)x-y(p+q))-(2bc+c+b)y(x(p+q)-n-m)+(n+m)(ly-(n+m)^2))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^2]-[2z(2(n+m)x-2y(p+q))(a(((n+m)(p+q)-lx)^2+((n+m)(p+q)-lx)(l-(p+q)^2)+(ly-(n+m)^2)(l-(p+q)^2))+(ac+ab)(((n+m)x-y(p+q))(l-(p+q)^2)+((n+m)(p+q)-lx)(x(p+q)-n-m))+(c+b)(((n+m)(p+q)-lx)((n+m)x-y(p+q))+$

$(ly-(n+m)^2)(x(p+q)-n-m))+(2bc+c+b)(x(p+q)-n-m)((n+m)x-y(p+q))+$
$(ly-(n+m)^2)((n+m)(p+q)-lx))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]$

$h_{49}^{(U_4)} = [z((ac+ab)(-y(l-(q+p)^2)-2(q+p)((n+m)x-y(q+p))+(n+m)(x(q+p)-$
$n-m)+x((n+m)(q+p)-lx))+a((n+m)(l-(q+p)^2)-2(q+p)((n+m)(q+p)-lx)+$
$2(n+m)((n+m)(q+p)-lx)-2(ly-(n+m)^2)(q+p))+(c+b)((n+m)((n+m)x-y(q+$
$p))-y((n+m)(q+p)-lx)+x(ly-(n+m)^2))+(2bc+c+b)x((n+m)x-y(q+p))-(2bc+$
$c+b)y(x(q+p)-n-m)+(n+m)(ly-(n+m)^2))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-$
$lx^2-(n+m)^2)^2]-[2z(2(n+m)x-2y(q+p))(a(((n+m)(q+p)-lx)^2+((n+m)(q+p)-$
$lx)(l-(q+p)^2)+(ly-(n+m)^2)(l-(q+p)^2))+(ac+ab)(((n+m)x-y(q+p))(l-(q+p)^2)+$
$((n+m)(q+p)-lx)(x(q+p)-n-m))+(c+b)(((n+m)(q+p)-lx)((n+m)x-y(q+p))+$
$(ly-(n+m)^2)(x(q+p)-n-m))+(2bc+c+b)(x(q+p)-n-m)((n+m)x-y(q+p))+$
$(ly-(n+m)^2)((n+m)(q+p)-lx))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]$

$h_{4\ 10}^{(U_4)} = 0$

$h_{55}^{(U_4)} = [t(a(2y+2x^2-2x)-2xy)z]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+$
$m)^2)^2]+[6t(y-x^2)^2z(a(((n+m)(q+p)-xl)^2+(l-(q+p)^2)(yl-(n+m)^2)+(l-(q+$
$p)^2)((n+m)(q+p)-xl))+(c+b)(((q+p)x-n-m)(yl-(n+m)^2)+((n+m)x-$
$(q+p)y)((n+m)(q+p)-xl))+(ac+ab)(((q+p)x-n-m)((n+m)(q+p)-xl)+$
$((n+m)x-(q+p)y)(l-(q+p)^2))+((n+m)(q+p)-xl)(yl-(n+m)^2)+(2bc+c+$
$b)((q+p)x-n-m)((n+m)x-(q+p)y))]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-$
$(n+m)^2)^4]-[4t(y-x^2)z(a(-2x((n+m)(q+p)-xl)+y(l-(q+p)^2)-x(l-(q+$
$p)^2)+yl-xl+(n+m)(q+p)-(n+m)^2)-x(yl-(n+m)^2)+y((n+m)(q+p)-xl)+$
$(c+b)((((q+p)x-n-m)y-x((n+m)x-(q+p)y))+(ac+ab)(-(q+p)y-x((q+$
$p)x-n-m)+(n+m)x))]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^3]$

$h_{56}^{(U_4)} = [tz(a(-2(m+n)-2(q+p)x+q+p)+2x(m+n)+(q+p)y+(c+b)(-y-$
$x^2)+2(ac+ab)x)]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^2]-[2tz(2(q+$
$p)x-2(m+n))(a(-(m+n)^2-2x((q+p)(m+n)-lx)+(q+p)(m+n)+(l-(q+p)^2)y+$
$ly-(l-(q+p)^2)x-lx)-x(ly-(m+n)^2)+(c+b)(y(-m+(q+p)x-n)-x(x(m+n)-$
$(q+p)y))+(ac+ab)(x(m+n)-x(-m+(q+p)x-n)+(-q-p)y)+y((q+p)(m+n)-$
$lx))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^3]+[6t(y-x^2)z(2(q+p)x-$
$2(m+n))(a(((q+p)(m+n)-lx)^2+(l-(q+p)^2)(ly-(m+n)^2)+(l-(q+p)^2)((q+$

$p)(m+n)-lx))+((q+p)(m+n)-lx)(ly-(m+n)^2)+(c+b)((-m+(q+p)x-n)(ly-$
$(m+n)^2)+((q+p)(m+n)-lx)(x(m+n)-(q+p)y))+(ac+ab)((l-(q+p)^2)(x(m+$
$n)-(q+p)y)+(-m+(q+p)x-n)((q+p)(m+n)-lx))+(2bc+c+b)(-m+(q+p)x-$
$n)(x(m+n)-(q+p)y))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^4]-[2t(y-$
$x^2)z((c+b)((m+n)^2+(q+p)(x(m+n)-(q+p)y)+x((q+p)(m+n)-lx)-2(-m+(q+$
$p)x-n)(m+n)-ly)+(q+p)(ly-(m+n)^2)+a(2(q+p)((q+p)(m+n)-lx)-2(l-(q+$
$p)^2)(m+n)+(q+p)(l-(q+p)^2))-(2bc+c+b)(x(m+n)-(q+p)y)-2(m+n)((q+p)(m+$
$n)-lx)+(ac+ab)(-(q+p)(m+n)+(q+p)(-m+(q+p)x-n)+(l-(q+p)^2)x+lx)+$
$(2bc+c+b)x(-m+(q+p)x-n))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^3]$

$h_{57}^{(U_4)} = [tz(a(-2(n+m)-2(q+p)x+q+p)+2x(n+m)+(q+p)y+(c+b)(-y-$
$x^2)+2(ac+ab)x)]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^2]-[2tz(2(q+$
$p)x-2(n+m))(a(-(n+m)^2-2x((q+p)(n+m)-lx)+(q+p)(n+m)+(l-(q+p)^2)y+$
$ly-(l-(q+p)^2)x-lx)-x(ly-(n+m)^2)+(c+b)(y(-n+(q+p)x-m)-x(x(n+m)-$
$(q+p)y))+(ac+ab)(x(n+m)-x(-n+(q+p)x-m)+(-q-p)y)+y((q+p)(n+m)-$
$lx))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]+[6t(y-x^2)z(2(q+p)x-$
$2(n+m))(a(((q+p)(n+m)-lx)^2+(l-(q+p)^2)(ly-(n+m)^2)+(l-(q+p)^2)((q+$
$p)(n+m)-lx))+((q+p)(n+m)-lx)(ly-(n+m)^2)+(c+b)((-n+(q+p)x-m)(ly-$
$(n+m)^2)+((q+p)(n+m)-lx)(x(n+m)-(q+p)y))+(ac+ab)((l-(q+p)^2)(x(n+$
$m)-(q+p)y)+(-n+(q+p)x-m)((q+p)(n+m)-lx))+(2bc+c+b)(-n+(q+p)x-$
$m)(x(n+m)-(q+p)y))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^4]-[2t(y-$
$x^2)z((c+b)((n+m)^2+(q+p)(x(n+m)-(q+p)y)+x((q+p)(n+m)-lx)-2(-n+(q+$
$p)x-m)(n+m)-ly)+(q+p)(ly-(n+m)^2)+a(2(q+p)((q+p)(n+m)-lx)-2(l-(q+$
$p)^2)(n+m)+(q+p)(l-(q+p)^2))-(2bc+c+b)(x(n+m)-(q+p)y)-2(n+m)((q+p)(n+$
$m)-lx)+(ac+ab)(-(q+p)(n+m)+(q+p)(-n+(q+p)x-m)+(l-(q+p)^2)x+lx)+$
$(2bc+c+b)x(-n+(q+p)x-m))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]$

$h_{58}^{(U_4)} = [tz(a(-2y(p+q)+2x(p+q)-2(n+m)x+n+m)+2(c+b)xy+(n+m)y+(ac+$
$ab)(-y-x^2))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^2]-[2tz(2(n+m)x-$
$2y(p+q))(a(y(l-(p+q)^2)-x(l-(p+q)^2)-2x((n+m)(p+q)-lx)+(n+m)(p+q)+ly-$
$lx-(n+m)^2)+(c+b)(y(x(p+q)-n-m)-x((n+m)x-y(p+q)))+(ac+ab)(-x(x(p+$
$q)-n-m)+y(-p-q)+(n+m)x)+y((n+m)(p+q)-lx)-x(ly-(n+m)^2))]/[(-y(p+$

$q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]-[2t(y-x^2)z((ac+ab)(-y(l-(p+q)^2)-2(p+q)((n+m)x-y(p+q))+(n+m)(x(p+q)-n-m)+x((n+m)(p+q)-lx))+a((n+m)(l-(p+q)^2)-2(p+q)((n+m)(p+q)-lx)+2(n+m)((n+m)(p+q)-lx)-2(ly-(n+m)^2)(p+q))+(c+b)((n+m)((n+m)x-y(p+q))-y((n+m)(p+q)-lx)+x(ly-(n+m)^2))+(2bc+c+b)x((n+m)x-y(p+q))-(2bc+c+b)y(x(p+q)-n-m)+(n+m)(ly-(n+m)^2))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]+[6t(y-x^2)z(2(n+m)x-2y(p+q))(a(((n+m)(p+q)-lx)^2+((n+m)(p+q)-lx)(l-(p+q)^2)+(ly-(n+m)^2)(l-(p+q)^2))+(ac+ab)(((n+m)x-y(p+q))(l-(p+q)^2)+((n+m)(p+q)-lx)(x(p+q)-n-m))+(c+b)(((n+m)(p+q)-lx)((n+m)x-y(p+q))+(ly-(n+m)^2)(x(p+q)-n-m))+(2bc+c+b)(x(p+q)-n-m)((n+m)x-y(p+q))+(ly-(n+m)^2)((n+m)(p+q)-lx))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^4]$

$h_{59}^{(U_4)}=[tz(a(-2y(q+p)+2x(q+p)-2(n+m)x+n+m)+2(c+b)xy+(n+m)y+(ac+ab)(-y-x^2))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^2]-[2tz(2(n+m)x-2y(q+p))(a(y(l-(q+p)^2)-x(l-(q+p)^2)-2x((n+m)(q+p)-lx)+(n+m)(q+p)+ly-lx-(n+m)^2)+(c+b)(y(x(q+p)-n-m)-x((n+m)x-y(q+p)))+(ac+ab)(-x(x(q+p)-n-m)+y(-q-p)+(n+m)x)+y((n+m)(q+p)-lx)-x(ly-(n+m)^2))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]-[2t(y-x^2)z((ac+ab)(-y(l-(q+p)^2)-2(q+p)((n+m)x-y(q+p))+(n+m)(x(q+p)-n-m)+x((n+m)(q+p)-lx))+a((n+m)(l-(q+p)^2)-2(q+p)((n+m)(q+p)-lx)+2(n+m)((n+m)(q+p)-lx)-2(ly-(n+m)^2)(q+p))+(c+b)((n+m)((n+m)x-y(q+p))-y((n+m)(q+p)-lx)+x(ly-(n+m)^2))+(2bc+c+b)x((n+m)x-y(q+p))-(2bc+c+b)y(x(q+p)-n-m)+(n+m)(ly-(n+m)^2))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]+[6t(y-x^2)z(2(n+m)x-2y(q+p))(a(((n+m)(q+p)-lx)^2+((n+m)(q+p)-lx)(l-(q+p)^2)+(ly-(n+m)^2)(l-(q+p)^2))+(ac+ab)(((n+m)x-y(q+p))(l-(q+p)^2)+((n+m)(q+p)-lx)(x(q+p)-n-m))+(c+b)(((n+m)(q+p)-lx)((n+m)x-y(q+p))+(ly-(n+m)^2)(x(q+p)-n-m))+(2bc+c+b)(x(q+p)-n-m)((n+m)x-y(q+p))+(ly-(n+m)^2)((n+m)(q+p)-lx))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^4]$

$h_{5\ 10}^{(U_4)}=0$

$h_{66}^{(U_4)}=[tz(-2((q+p)(m+n)-lx)+(c+b)(4(m+n)-2(-m+(q+p)x-n)+2(q+p)x)-4(q+p)(m+n)-2(2bc+c+b)x+a(2(q+p)^2-2(l-(q+p)^2))+(ac+ab)(-2q-$

$2p))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^2]+[4tz(a(((q+p)(m+n)-lx)^2+(l-(q+p)^2)(ly-(m+n)^2)+(l-(q+p)^2)((q+p)(m+n)-lx))+((q+p)(m+n)-lx)(ly-(m+n)^2)+(c+b)((-m+(q+p)x-n)(ly-(m+n)^2)+((q+p)(m+n)-lx)(x(m+n)-(q+p)y))+(ac+ab)((l-(q+p)^2)(x(m+n)-(q+p)y)+(-m+(q+p)x-n)((q+p)(m+n)-lx))+(2bc+c+b)(-m+(q+p)x-n)(x(m+n)-(q+p)y))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^3]+[6tz(2(q+p)x-2(m+n))^2(a(((q+p)(m+n)-lx)^2+(l-(q+p)^2)(ly-(m+n)^2)+(l-(q+p)^2)((q+p)(m+n)-lx))+((q+p)(m+n)-lx)(ly-(m+n)^2)+(c+b)((-m+(q+p)x-n)(ly-(m+n)^2)+((q+p)(m+n)-lx)(x(m+n)-(q+p)y))+(ac+ab)((l-(q+p)^2)(x(m+n)-(q+p)y)+(-m+(q+p)x-n)((q+p)(m+n)-lx))+(2bc+c+b)(-m+(q+p)x-n)(x(m+n)-(q+p)y))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^4]-[4tz(2(q+p)x-2(m+n))((c+b)((m+n)^2+(q+p)(x(m+n)-(q+p)y)+x((q+p)(m+n)-lx)-2(-m+(q+p)x-n)(m+n)-ly)+(q+p)(ly-(m+n)^2)+a(2(q+p)((q+p)(m+n)-lx)-2(l-(q+p)^2)(m+n)+(q+p)(l-(q+p)^2))-(2bc+c+b)(x(m+n)-(q+p)y)-2(m+n)((q+p)(m+n)-lx)+(ac+ab)(-(q+p)(m+n)+(q+p)(-m+(q+p)x-n)+(l-(q+p)^2)x+lx)+(2bc+c+b)x(-m+(q+p)x-n))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^3]$

$h_{67}^{(U_4)} = [tz(-2((q+p)(n+m)-lx)+(c+b)(4(n+m)-2(-n+(q+p)x-m)+2(q+p)x)-4(q+p)(n+m)-(2bc+c+b)x+(-2bc-c-b)x+a(2(q+p)^2-2(l-(q+p)^2))+(ac+ab)(-2q-2p))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^2]+[4tz(a(((q+p)(n+m)-lx)^2+(l-(q+p)^2)(ly-(n+m)^2)+(l-(q+p)^2)((q+p)(n+m)-lx))+((q+p)(n+m)-lx)(ly-(n+m)^2)+(c+b)((-n+(q+p)x-m)(ly-(n+m)^2)+((q+p)(n+m)-lx)(x(n+m)-(q+p)y))+(ac+ab)((l-(q+p)^2)(x(n+m)-(q+p)y)+(-n+(q+p)x-m)((q+p)(n+m)-lx))+(2bc+c+b)(-n+(q+p)x-m)(x(n+m)-(q+p)y))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]+[6tz(2(q+p)x-2(n+m))^2(a(((q+p)(n+m)-lx)^2+(l-(q+p)^2)(ly-(n+m)^2)+(l-(q+p)^2)((q+p)(n+m)-lx))+((q+p)(n+m)-lx)(ly-(n+m)^2)+(c+b)((-n+(q+p)x-m)(ly-(n+m)^2)+((q+p)(n+m)-lx)(x(n+m)-(q+p)y))+(ac+ab)((l-(q+p)^2)(x(n+m)-(q+p)y)+(-n+(q+p)x-m)((q+p)(n+m)-lx))+(2bc+c+b)(-n+(q+p)x-m)(x(n+m)-(q+p)y))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^4]-[2tz(2(q+p)x-2(n+m))((c+b)((n+m)^2+(q+p)(x(n+m)-(q+p)y)+x((q+p)(n+m)-lx)-2(-n+(q+p)x-m)(n+m)-$

$ly)+(q+p)(ly-(n+m)^2)+a(2(q+p)((q+p)(n+m)-lx)-2(l-(q+p)^2)(n+m)+(q+$
$p)(l-(q+p)^2))-(2bc+c+b)(x(n+m)-(q+p)y)-2(n+m)((q+p)(n+m)-lx)+(ac+$
$ab)(-(q+p)(n+m)+(q+p)(-n+(q+p)x-m)+(l-(q+p)^2)x+lx)+(2bc+c+b)x(-n+$
$(q+p)x-m))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]-[2tz(2(q+p)x-$
$2(n+m))((c+b)((n+m)^2+(q+p)(x(n+m)-(q+p)y)+x((q+p)(n+m)-lx)-2(-n+$
$(q+p)x-m)(n+m)-ly)+(q+p)(ly-(n+m)^2)+a(2(q+p)((q+p)(n+m)-lx)-2(l-(q+$
$p)^2)(n+m)+(q+p)(l-(q+p)^2))+(-2bc-c-b)(x(n+m)-(q+p)y)-2(n+m)((q+p)(n+$
$m)-lx)+(ac+ab)((-q-p)(n+m)+(q+p)(-n+(q+p)x-m)+(l-(q+p)^2)x+lx)+$
$(2bc+c+b)x(-n+(q+p)x-m))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]$

$h_{68}^{(U_4)} = [tz(a(-3(p+q)^2+2((n+m)(p+q)-lx)+6(n+m)(p+q)+l)-2(c+b)y(p+$
$q)+ly-(-2bc-c-b)y+(2bc+c+b)x^2-3(n+m)^2+(ac+ab)(-2n-2m))]/[(-y(p+$
$q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^2]-[2tz(2(n+m)x-2y(p+q))(a((p+q)(l-$
$(p+q)^2)-2(n+m)(l-(p+q)^2)+2(p+q)((n+m)(p+q)-lx))+(ac+ab)(x(l-(p+q)^2)+$
$(p+q)(x(p+q)-n-m)+(n+m)(-p-q)+lx)+(c+b)((p+q)((n+m)x-y(p+q))-$
$2(n+m)(x(p+q)-n-m)+x((n+m)(p+q)-lx)-ly+(n+m)^2)+(-2bc-c-b)((n+$
$m)x-y(p+q))+(2bc+c+b)x(x(p+q)-n-m)-2(n+m)((n+m)(p+q)-lx)+(ly-(n+$
$m)^2)(p+q))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]-[2tz(2x(p+q)-$
$2(n+m))((ac+ab)(-y(l-(p+q)^2)-2(p+q)((n+m)x-y(p+q))+(n+m)(x(p+q)-n-$
$m)+x((n+m)(p+q)-lx))+a((n+m)(l-(p+q)^2)-2(p+q)((n+m)(p+q)-lx)+2(n+$
$m)((n+m)(p+q)-lx)-2(ly-(n+m)^2)(p+q))+(c+b)((n+m)((n+m)x-y(p+q))-$
$y((n+m)(p+q)-lx)+x(ly-(n+m)^2))+(2bc+c+b)x((n+m)x-y(p+q))-(2bc+c+$
$b)y(x(p+q)-n-m)+(n+m)(ly-(n+m)^2))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-$
$(n+m)^2)^3]-[4txz(a(((n+m)(p+q)-lx)^2+((n+m)(p+q)-lx)(l-(p+q)^2)+(ly-(n+$
$m)^2)(l-(p+q)^2))+(ac+ab)(((n+m)x-y(p+q))(l-(p+q)^2)+((n+m)(p+q)-lx)(x(p+$
$q)-n-m))+(c+b)(((n+m)(p+q)-lx)((n+m)x-y(p+q))+(ly-(n+m)^2)(x(p+q)-$
$n-m))+(2bc+c+b)(x(p+q)-n-m)((n+m)x-y(p+q))+(ly-(n+m)^2)((n+m)(p+$
$q)-lx))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]+[6tz(2x(p+q)-2(n+$
$m))(2(n+m)x-2y(p+q))(a(((n+m)(p+q)-lx)^2+((n+m)(p+q)-lx)(l-(p+q)^2)+$
$(ly-(n+m)^2)(l-(p+q)^2))+(ac+ab)(((n+m)x-y(p+q))(l-(p+q)^2)+((n+m)(p+$
$q)-lx)(x(p+q)-n-m))+(c+b)(((n+m)(p+q)-lx)((n+m)x-y(p+q))+(ly-$

$(n+m)^2)(x(p+q)-n-m))+(2bc+c+b)(x(p+q)-n-m)((n+m)x-y(p+q))+(ly-(n+m)^2)((n+m)(p+q)-lx))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^4]$

$h_{69}^{(U_4)} = [tz(a(-3(q+p)^2+2((n+m)(q+p)-lx)+6(n+m)(q+p)+l)-2(c+b)y(q+p)+ly-(-2bc-c-b)y+(2bc+c+b)x^2-3(n+m)^2+(ac+ab)(-2n-2m))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^2]-[2tz(2(n+m)x-2y(q+p))(a((q+p)(l-(q+p)^2)-2(n+m)(l-(q+p)^2)+2(q+p)((n+m)(q+p)-lx))+(ac+ab)(x(l-(q+p)^2)+(q+p)(x(q+p)-n-m)+(n+m)(-q-p)+lx)+(c+b)((q+p)((n+m)x-y(q+p))-2(n+m)(x(q+p)-n-m)+x((n+m)(q+p)-lx)-ly+(n+m)^2)+(-2bc-c-b)((n+m)x-y(q+p))+(2bc+c+b)x(x(q+p)-n-m)-2(n+m)((n+m)(q+p)-lx)+(ly-(n+m)^2)(q+p))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]-[2tz(2x(q+p)-2(n+m))((ac+ab)(-y(l-(q+p)^2)-2(q+p)((n+m)x-y(q+p))+(n+m)(x(q+p)-n-m)+x((n+m)(q+p)-lx))+a((n+m)(l-(q+p)^2)-2(q+p)((n+m)(q+p)-lx)+2(n+m)((n+m)(q+p)-lx)-2(ly-(n+m)^2)(q+p))+(c+b)((n+m)((n+m)x-y(q+p))-y((n+m)(q+p)-lx)+x(ly-(n+m)^2))+(2bc+c+b)x((n+m)x-y(q+p))-(2bc+c+b)y(x(q+p)-n-m)+(n+m)(ly-(n+m)^2))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]-[4txz(a(((n+m)(q+p)-lx)^2+((n+m)(q+p)-lx)(l-(q+p)^2)+(ly-(n+m)^2)(l-(q+p)^2))+(ac+ab)(((n+m)x-y(q+p))(l-(q+p)^2)+((n+m)(q+p)-lx)(x(q+p)-n-m))+(c+b)(((n+m)(q+p)-lx)((n+m)x-y(q+p))+(ly-(n+m)^2)(x(q+p)-n-m))+(2bc+c+b)(x(q+p)-n-m)((n+m)x-y(q+p))+(ly-(n+m)^2)((n+m)(q+p)-lx))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]+[6tz(2x(q+p)-2(n+m))(2(n+m)x-2y(q+p))(a(((n+m)(q+p)-lx)^2+((n+m)(q+p)-lx)(l-(q+p)^2)+(ly-(n+m)^2)(l-(q+p)^2))+(ac+ab)(((n+m)x-y(q+p))(l-(q+p)^2)+((n+m)(q+p)-lx)(x(q+p)-n-m))+(c+b)(((n+m)(q+p)-lx)((n+m)x-y(q+p))+(ly-(n+m)^2)(x(q+p)-n-m))+(2bc+c+b)(x(q+p)-n-m)((n+m)x-y(q+p))+(ly-(n+m)^2)((n+m)(q+p)-lx))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^4]$

$h_{6\ 10}^{(U_4)} = 0$

$h_{77}^{(U_4)} = [tz(-2((q+p)(n+m)-lx)+(c+b)(4(n+m)-2(-n+(q+p)x-m)+2(q+p)x)-4(q+p)(n+m)-2(2bc+c+b)x+a(2(q+p)^2-2(l-(q+p)^2))+(ac+ab)(-2q-2p))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^2]+[4tz(a((((q+p)(n+m)-lx)^2+(l-(q+p)^2)(ly-(n+m)^2)+(l-(q+p)^2)((q+p)(n+m)-lx))+((q+p)(n+m)-$

$lx)(ly-(n+m)^2)+(c+b)((-n+(q+p)x-m)(ly-(n+m)^2)+((q+p)(n+m)-lx)(x(n+m)-(q+p)y))+(ac+ab)((l-(q+p)^2)(x(n+m)-(q+p)y)+(-n+(q+p)x-m)((q+p)(n+m)-lx))+(2bc+c+b)(-n+(q+p)x-m)(x(n+m)-(q+p)y))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]+[6tz(2(q+p)x-2(n+m))^2(a(((q+p)(n+m)-lx)^2+(l-(q+p)^2)(ly-(n+m)^2)+(l-(q+p)^2)((q+p)(n+m)-lx))+((q+p)(n+m)-lx)(ly-(n+m)^2)+(c+b)((-n+(q+p)x-m)(ly-(n+m)^2)+((q+p)(n+m)-lx)(x(n+m)-(q+p)y))+(ac+ab)((l-(q+p)^2)(x(n+m)-(q+p)y)+(-n+(q+p)x-m)((q+p)(n+m)-lx))+(2bc+c+b)(-n+(q+p)x-m)(x(n+m)-(q+p)y))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^4]-[4tz(2(q+p)x-2(n+m))((c+b)((n+m)^2+(q+p)(x(n+m)-(q+p)y)+x((q+p)(n+m)-lx)-2(-n+(q+p)x-m)(n+m)-ly)+(q+p)(ly-(n+m)^2)+a(2(q+p)((q+p)(n+m)-lx)-2(l-(q+p)^2)(n+m)+(q+p)(l-(q+p)^2))-(2bc+c+b)(x(n+m)-(q+p)y)-2(n+m)((q+p)(n+m)-lx)+(ac+ab)(-(q+p)(n+m)+(q+p)(-n+(q+p)x-m)+(l-(q+p)^2)x+lx)+(2bc+c+b)x(-n+(q+p)x-m))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]$

$h_{78}^{(U_4)} = [tz(a(-3(p+q)^2+2((n+m)(p+q)-lx)+6(n+m)(p+q)+l)-2(c+b)y(p+q)+ly-(-2bc-c-b)y+(2bc+c+b)x^2-3(n+m)^2+(ac+ab)(-2n-2m))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^2]-[2tz(2(n+m)x-2y(p+q))(a((p+q)(l-(p+q)^2)-2(n+m)(l-(p+q)^2)+2(p+q)((n+m)(p+q)-lx))+(ac+ab)(x(l-(p+q)^2)+(p+q)(x(p+q)-n-m)+(n+m)(-p-q)+lx)+(c+b)((p+q)((n+m)x-y(p+q))-2(n+m)(x(p+q)-n-m)+x((n+m)(p+q)-lx)-ly+(n+m)^2)+(-2bc-c-b)((n+m)x-y(p+q))+(2bc+c+b)x(x(p+q)-n-m)-2(n+m)((n+m)(p+q)-lx)+(ly-(n+m)^2)(p+q))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]-[2tz(2x(p+q)-2(n+m))((ac+ab)(-y(l-(p+q)^2)-2(p+q)((n+m)x-y(p+q))+(n+m)(x(p+q)-n-m)+x((n+m)(p+q)-lx))+a((n+m)(l-(p+q)^2)-2(p+q)((n+m)(p+q)-lx)+2(n+m)((n+m)(p+q)-lx)-2(ly-(n+m)^2)(p+q))+(c+b)((n+m)((n+m)x-y(p+q))-y((n+m)(p+q)-lx)+x(ly-(n+m)^2))+(2bc+c+b)x((n+m)x-y(p+q))-(2bc+c+b)y(x(p+q)-n-m)+(n+m)(ly-(n+m)^2))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]-[4txz(a(((n+m)(p+q)-lx)^2+((n+m)(p+q)-lx)(l-(p+q)^2)+(ly-(n+m)^2)(l-(p+q)^2))+(ac+ab)(((n+m)x-y(p+q))(l-(p+q)^2)+((n+m)(p+q)-lx)(x(p+q)-n-m))+(c+b)(((n+m)(p+q)-lx)((n+m)x-y(p+q))+(ly-(n+m)^2)(x(p+q)-$

$n-m))+(2bc+c+b)(x(p+q)-n-m)((n+m)x-y(p+q))+(ly-(n+m)^2)((n+m)(p+q)-lx))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]+[6tz(2x(p+q)-2(n+m))(2(n+m)x-2y(p+q))(a(((n+m)(p+q)-lx)^2+((n+m)(p+q)-lx)(l-(p+q)^2)+(ly-(n+m)^2)(l-(p+q)^2))+(ac+ab)(((n+m)x-y(p+q))(l-(p+q)^2)+((n+m)(p+q)-lx)(x(p+q)-n-m))+(c+b)(((n+m)(p+q)-lx)((n+m)x-y(p+q))+(ly-(n+m)^2)(x(p+q)-n-m))+(2bc+c+b)(x(p+q)-n-m)((n+m)x-y(p+q))+(ly-(n+m)^2)((n+m)(p+q)-lx))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^4]$

$h_{79}^{(U_4)} = [tz(a(-3(q+p)^2+2((n+m)(q+p)-lx)+6(n+m)(q+p)+l)-2(c+b)y(q+p)+ly-(-2bc-c-b)y+(2bc+c+b)x^2-3(n+m)^2+(ac+ab)(-2n-2m))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^2]-[2tz(2(n+m)x-2y(q+p))(a((q+p)(l-(q+p)^2)-2(n+m)(l-(q+p)^2)+2(q+p)((n+m)(q+p)-lx))+(ac+ab)(x(l-(q+p)^2)+(q+p)(x(q+p)-n-m)+(n+m)(-q-p)+lx)+(c+b)((q+p)((n+m)x-y(q+p))-2(n+m)(x(q+p)-n-m)+x((n+m)(q+p)-lx)-ly+(n+m)^2)+(-2bc-c-b)((n+m)x-y(q+p))+(2bc+c+b)x(x(q+p)-n-m)-2(n+m)((n+m)(q+p)-lx)+(ly-(n+m)^2)(q+p))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]-[2tz(2x(q+p)-2(n+m))((ac+ab)(-y(l-(q+p)^2)-2(q+p)((n+m)x-y(q+p))+(n+m)(x(q+p)-n-m)+x((n+m)(q+p)-lx))+a((n+m)(l-(q+p)^2)-2(q+p)((n+m)(q+p)-lx)+2(n+m)((n+m)(q+p)-lx)-2(ly-(n+m)^2)(q+p))+(c+b)((n+m)((n+m)x-y(q+p))-y((n+m)(q+p)-lx)+x(ly-(n+m)^2))+(2bc+c+b)x((n+m)x-y(q+p))-(2bc+c+b)y(x(q+p)-n-m)+(n+m)(ly-(n+m)^2))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]-[4txz(a(((n+m)(q+p)-lx)^2+((n+m)(q+p)-lx)(l-(q+p)^2)+(ly-(n+m)^2)(l-(q+p)^2))+(ac+ab)(((n+m)x-y(q+p))(l-(q+p)^2)+((n+m)(q+p)-lx)(x(q+p)-n-m))+(c+b)(((n+m)(q+p)-lx)((n+m)x-y(q+p))+(ly-(n+m)^2)(x(q+p)-n-m))+(2bc+c+b)(x(q+p)-n-m)((n+m)x-y(q+p))+(ly-(n+m)^2)((n+m)(q+p)-lx))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]+[6tz(2x(q+p)-2(n+m))(2(n+m)x-2y(q+p))(a(((n+m)(q+p)-lx)^2+((n+m)(q+p)-lx)(l-(q+p)^2)+(ly-(n+m)^2)(l-(q+p)^2))+(ac+ab)(((n+m)x-y(q+p))(l-(q+p)^2)+((n+m)(q+p)-lx)(x(q+p)-n-m))+(c+b)(((n+m)(q+p)-lx)((n+m)x-y(q+p))+(ly-(n+m)^2)(x(q+p)-n-m))+(2bc+c+b)(x(q+p)-n-m)((n+m)x-y(q+p))+(ly-(n+m)^2)((n+m)(q+p)-lx))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^4]$

$h_{7\ 10}^{(U_4)} = 0$

$h_{88}^{(U_4)} = [tz((ac+ab)(-2((n+m)x-y(p+q))+4y(p+q)+2(n+m)x)+a(-2((n+m)(p+q)-lx)-4(n+m)(p+q)-2(ly-(n+m)^2)+2(n+m)^2)-2(2bc+c+b)xy-2(c+b)(n+m)y)]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^2]-[4tz(2(n+m)x-2y(p+q))((ac+ab)(-y(l-(p+q)^2)-2(p+q)((n+m)x-y(p+q))+(n+m)(x(p+q)-n-m)+x((n+m)(p+q)-lx))+a((n+m)(l-(p+q)^2)-2(p+q)((n+m)(p+q)-lx)+2(n+m)((n+m)(p+q)-lx)-2(ly-(n+m)^2)(p+q))+(c+b)((n+m)((n+m)x-y(p+q))-y((n+m)(p+q)-lx)+x(ly-(n+m)^2))+(2bc+c+b)x((n+m)x-y(p+q))-(2bc+c+b)y(x(p+q)-n-m)+(n+m)(ly-(n+m)^2)))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]+[4tyz(a(((n+m)(p+q)-lx)^2+((n+m)(p+q)-lx)(l-(p+q)^2)+(ly-(n+m)^2)(l-(p+q)^2))+(ac+ab)(((n+m)x-y(p+q))(l-(p+q)^2)+((n+m)(p+q)-lx)(x(p+q)-n-m))+(c+b)(((n+m)(p+q)-lx)((n+m)x-y(p+q))+(ly-(n+m)^2)(x(p+q)-n-m))+(2bc+c+b)(x(p+q)-n-m)((n+m)x-y(p+q))+(ly-(n+m)^2)((n+m)(p+q)-lx))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]+[6tz(2(n+m)x-2y(p+q))^2(a(((n+m)(p+q)-lx)^2+((n+m)(p+q)-lx)(l-(p+q)^2)+(ly-(n+m)^2)(l-(p+q)^2))+(ac+ab)(((n+m)x-y(p+q))(l-(p+q)^2)+((n+m)(p+q)-lx)(x(p+q)-n-m))+(c+b)(((n+m)(p+q)-lx)((n+m)x-y(p+q))+(ly-(n+m)^2)(x(p+q)-n-m))+(2bc+c+b)(x(p+q)-n-m)((n+m)x-y(p+q))+(ly-(n+m)^2)((n+m)(p+q)-lx))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^4]$

$h_{89}^{(U_4)} = [tz((ac+ab)(-2((n+m)x-y(q+p))+4y(q+p)+2(n+m)x)+a(-2((n+m)(q+p)-lx)-4(n+m)(q+p)-2(ly-(n+m)^2)+2(n+m)^2)-(2bc+c+b)xy+(-2bc-c-b)xy-2(c+b)(n+m)y)]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^2]-[2tz(2(n+m)x-2y(q+p))((ac+ab)(-y(l-(q+p)^2)-2(q+p)((n+m)x-y(q+p))+(n+m)(x(q+p)-n-m)+x((n+m)(q+p)-lx))+a((n+m)(l-(q+p)^2)-2(q+p)((n+m)(q+p)-lx)+2(n+m)((n+m)(q+p)-lx)-2(ly-(n+m)^2)(q+p))+(c+b)((n+m)((n+m)x-y(q+p))-y((n+m)(q+p)-lx)+x(ly-(n+m)^2))+(2bc+c+b)x((n+m)x-y(q+p))-(2bc+c+b)y(x(q+p)-n-m)+(n+m)(ly-(n+m)^2)))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]-[2tz(2(n+m)x-2y(q+p))((ac+ab)(-y(l-(q+p)^2)-2(q+p)((n+m)x-y(q+p))+(n+m)(x(q+p)-n-m)+x((n+m)(q+p)-lx))+a((n+m)(l-(q+p)^2)-2(q+p)((n+m)(q+p)-lx)+2(n+m)((n+m)(q+p)-lx)-2(ly-(n+m)^2)(q+p))+(c+b)((n+$

$m)((n+m)x-y(q+p))-y((n+m)(q+p)-lx)+x(ly-(n+m)^2))+(2bc+c+b)x((n+$
$m)x-y(q+p))+(-2bc-c-b)y(x(q+p)-n-m)+(n+m)(ly-(n+m)^2))]/[(-y(q+p)^2+$
$2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]+[4tyz(a(((n+m)(q+p)-lx)^2+((n+m)(q+$
$p)-lx)(l-(q+p)^2)+(ly-(n+m)^2)(l-(q+p)^2))+(ac+ab)(((n+m)x-y(q+p))(l-(q+$
$p)^2)+((n+m)(q+p)-lx)(x(q+p)-n-m))+(c+b)(((n+m)(q+p)-lx)((n+m)x-y(q+$
$p))+(ly-(n+m)^2)(x(q+p)-n-m))+(2bc+c+b)(x(q+p)-n-m)((n+m)x-y(q+$
$p))+(ly-(n+m)^2)((n+m)(q+p)-lx))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+$
$m)^2)^3]+[6tz(2(n+m)x-2y(q+p))^2(a(((n+m)(q+p)-lx)^2+((n+m)(q+p)-lx)(l-$
$(q+p)^2)+(ly-(n+m)^2)(l-(q+p)^2))+(ac+ab)(((n+m)x-y(q+p))(l-(q+p)^2)+((n+$
$m)(q+p)-lx)(x(q+p)-n-m))+(c+b)(((n+m)(q+p)-lx)((n+m)x-y(q+p))+(ly-$
$(n+m)^2)(x(q+p)-n-m))+(2bc+c+b)(x(q+p)-n-m)((n+m)x-y(q+p))+(ly-$
$(n+m)^2)((n+m)(q+p)-lx))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^4]$

$h^{(U_4)}_{8\ 10} = 0$

$h^{(U_4)}_{99} = [tz((ac+ab)(-2((n+m)x-y(q+p))+4y(q+p)+2(n+m)x)+a(-2((n+$
$m)(q+p)-lx)-4(n+m)(q+p)-2(ly-(n+m)^2)+2(n+m)^2)-2(2bc+c+b)xy-2(c+$
$b)(n+m)y)]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^2]-[4tz(2(n+m)x-$
$2y(q+p))((ac+ab)(-y(l-(q+p)^2)-2(q+p)((n+m)x-y(q+p))+(n+m)(x(q+p)-$
$n-m)+x((n+m)(q+p)-lx))+a((n+m)(l-(q+p)^2)-2(q+p)((n+m)(q+p)-lx)+$
$2(n+m)((n+m)(q+p)-lx)-2(ly-(n+m)^2)(q+p))+(c+b)((n+m)((n+m)x-y(q+$
$p))-y((n+m)(q+p)-lx)+x(ly-(n+m)^2))+(2bc+c+b)x((n+m)x-y(q+p))-(2bc+$
$c+b)y(x(q+p)-n-m)+(n+m)(ly-(n+m)^2))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-$
$lx^2-(n+m)^2)^3]+[4tyz(a(((n+m)(q+p)-lx)^2+((n+m)(q+p)-lx)(l-(q+p)^2)+$
$(ly-(n+m)^2)(l-(q+p)^2))+(ac+ab)(((n+m)x-y(q+p))(l-(q+p)^2)+((n+m)(q+$
$p)-lx)(x(q+p)-n-m))+(c+b)(((n+m)(q+p)-lx)((n+m)x-y(q+p))+(ly-(n+$
$m)^2)(x(q+p)-n-m))+(2bc+c+b)(x(q+p)-n-m)((n+m)x-y(q+p))+(ly-(n+$
$m)^2)((n+m)(q+p)-lx))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]+$
$[6tz(2(n+m)x-2y(q+p))^2(a(((n+m)(q+p)-lx)^2+((n+m)(q+p)-lx)(l-(q+p)^2)+$
$(ly-(n+m)^2)(l-(q+p)^2))+(ac+ab)(((n+m)x-y(q+p))(l-(q+p)^2)+((n+m)(q+$
$p)-lx)(x(q+p)-n-m))+(c+b)(((n+m)(q+p)-lx)((n+m)x-y(q+p))+(ly-$
$(n+m)^2)(x(q+p)-n-m))+(2bc+c+b)(x(q+p)-n-m)((n+m)x-y(q+p))+(ly-$

$(n+m)^2)((n+m)(q+p)-lx))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^4]$

$h_{9\ 10}^{(U_4)}=0$

$h_{10\ 10}^{(U_4)}=0$

$h_{11}^{(U_5)}=[sz((ac+ab)(-2((n+m)(q+p)-lx)+4lx+2(n+m)(q+p))-2(2bc+c+b)((n+m)x-(q+p)y)-4(2bc+c+b)(n+m)x+(c+b)(2(n+m)^2-2(ly-(n+m)^2))+a(-2l(q+p)-2l(n+m)))]/[(-lx^2+2(n+m)(q+p)x-(q+p)^2y+ly-(n+m)^2)^2]-[4sz(2(n+m)(q+p)-2lx)((2bc+c+b)(n+m)(y-x^2)+(ac+ab)(-l(y-x^2)+(n+m)((q+p)x-n-m)+(q+p)((n+m)x-(q+p)y)-2x((n+m)(q+p)-lx))+a(-l((q+p)x-n-m)-l((n+m)x-(q+p)y)+(q+p)((n+m)(q+p)-lx)+(n+m)((n+m)(q+p)-lx)+(q+p)(ly-(n+m)^2))+(c+b)(2(n+m)((n+m)x-(q+p)y)-2(ly-(n+m)^2)x)-2(2bc+c+b)x((n+m)x-(q+p)y)+(n+m)(ly-(n+m)^2))]/[(-lx^2+2(n+m)(q+p)x-(q+p)^2y+ly-(n+m)^2)^3]+[4lsz((c+b)(((n+m)x-(q+p)y)^2+(ly-(n+m)^2)(y-x^2))+(2bc+c+b)((n+m)x-(q+p)y)(y-x^2)+(ac+ab)(((n+m)(q+p)-lx)(y-x^2)+((n+m)x-(q+p)y)((q+p)x-n-m))+a(((n+m)(q+p)-lx)((q+p)x-n-m)+(ly-(n+m)^2)((q+p)x-n-m)+((n+m)(q+p)-lx)((n+m)x-(q+p)y))+(ly-(n+m)^2)((n+m)x-(q+p)y))]/[(-lx^2+2(n+m)(q+p)x-(q+p)^2y+ly-(n+m)^2)^3]+[6sz(2(n+m)(q+p)-2lx)^2((c+b)(((n+m)x-(q+p)y)^2+(ly-(n+m)^2)(y-x^2))+(2bc+c+b)((n+m)x-(q+p)y)(y-x^2)+(ac+ab)(((n+m)(q+p)-lx)(y-x^2)+((n+m)x-(q+p)y)((q+p)x-n-m))+a(((n+m)(q+p)-lx)((q+p)x-n-m)+(ly-(n+m)^2)((q+p)x-n-m)+((n+m)(q+p)-lx)((n+m)x-(q+p)y))+(ly-(n+m)^2)((n+m)x-(q+p)y))]/[(-lx^2+2(n+m)(q+p)x-(q+p)^2y+ly-(n+m)^2)^4]$

$h_{12}^{(U_5)}=[s((c+b)(2(n+m)(-q-p)-2lx)-2(2bc+c+b)(-q-p)x+(ac+ab)((-q-p)(q+p)-l)+a(l(q+p)-l(-q-p))+l(n+m)+(2bc+c+b)(n+m))z]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^2]+[6(l-(q+p)^2)s(2(n+m)(q+p)-2lx)z((c+b)(((n+m)x-(q+p)y)^2+(y-x^2)(ly-(n+m)^2))+(ac+ab)((((q+p)x-n-m)((n+m)x-(q+p)y)+((n+m)(q+p)-lx)(y-x^2))+a(((n+m)(q+p)-lx)((n+m)x-(q+p)y)+((q+p)x-n-m)(ly-(n+m)^2)+((n+m)(q+p)-lx)((q+p)x-n-m))+(ly-(n+m)^2)((n+m)x-(q+p)y)+(2bc+c+b)(y-x^2)((n+m)x-(q+p)y))]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^4]-[2(l-(q+p)^2)sz((ac+ab)((q+p)((n+m)x-(q+p)y)-l(y-x^2)+(n+m)((q+p)x-n-m)-2x((n+m)(q+p)-lx))+(c+b)(2(n+$

$m)((n+m)x-(q+p)y)-2x(ly-(n+m)^2))+a(-l((n+m)x-(q+p)y)+(q+p)(ly-(n+m)^2)-l((q+p)x-n-m)+(q+p)((n+m)(q+p)-lx)+(n+m)((n+m)(q+p)-lx))-2(2bc+c+b)x((n+m)x-(q+p)y)+(n+m)(ly-(n+m)^2)+(2bc+c+b)(n+m)(y-x^2))]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^3]-[2s(2(n+m)(q+p)-2lx)z((c+b)(2(-q-p)((n+m)x-(q+p)y)+l(y-x^2)+ly-(n+m)^2)+l((n+m)x-(q+p)y)+(2bc+c+b)((n+m)x-(q+p)y)+(-q-p)(ly-(n+m)^2)+(2bc+c+b)(-q-p)(y-x^2)+(ac+ab)((-q-p)((q+p)x-n-m)-lx+(n+m)(q+p))+a(l((q+p)x-n-m)+(-q-p)((n+m)(q+p)-lx)))]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^3]$

$h_{13}^{(U_5)} = [s((ac+ab)((q+p)((n+m)x-(q+p)y)-l(y-x^2)+(n+m)((q+p)x-n-m)-2x((n+m)(q+p)-lx))+(c+b)(2(n+m)((n+m)x-(q+p)y)-2x(ly-(n+m)^2))+a(-l((n+m)x-(q+p)y)+(q+p)(ly-(n+m)^2)-l((q+p)x-n-m)+(q+p)((n+m)(q+p)-lx)+(n+m)((n+m)(q+p)-lx))-2(2bc+c+b)x((n+m)x-(q+p)y)+(n+m)(ly-(n+m)^2)+(2bc+c+b)(n+m)(y-x^2))]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^2]-[2s(2(n+m)(q+p)-2lx)((c+b)(((n+m)x-(q+p)y)^2+(y-x^2)(ly-(n+m)^2))+(ac+ab)(((q+p)x-n-m)((n+m)x-(q+p)y)+((n+m)(q+p)-lx)(y-x^2))+a(((n+m)(q+p)-lx)((n+m)x-(q+p)y)+((q+p)x-n-m)(ly-(n+m)^2)+((n+m)(q+p)-lx)((q+p)x-n-m))+(ly-(n+m)^2)((n+m)x-(q+p)y)+(2bc+c+b)(y-x^2)((n+m)x-(q+p)y))]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^3]$

$h_{14}^{(U_5)} = 0$

$h_{15}^{(U_5)} = [s(a(2(q+p)y-2(q+p)x-2(n+m)x+n+m)-2(c+b)xy+(n+m)y+(ac+ab)(3x^2-y))z]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^2]-[2s(a(-x((n+m)x-(q+p)y)+((q+p)x-n-m)y-x((q+p)x-n-m))+y((n+m)x-(q+p)y)+(c+b)y(y-x^2)-(ac+ab)x(y-x^2))z(2(n+m)(q+p)-2xl)]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^3]+[4sxz((c+b)((y-x^2)(yl-(n+m)^2)+((n+m)x-(q+p)y)^2)+a(((q+p)x-n-m)(yl-(n+m)^2)+((n+m)x-(q+p)y)((n+m)(q+p)-xl)+((q+p)x-n-m)((n+m)(q+p)-xl))+(ac+ab)((y-x^2)((n+m)(q+p)-xl)+((q+p)x-n-m)((n+m)x-(q+p)y))+((n+m)x-(q+p)y)(yl-(n+m)^2)+(2bc+c+b)(y-x^2)((n+m)x-(q+p)y))]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^3]+[6s(y-x^2)z(2(n+m)(q+p)-2xl)((c+b)((y-x^2)(yl-(n+m)^2)+((n+m)x-(q+p)y)^2)+a(((q+p)x-n-m)(yl-(n+m)^2)+((n+m)x-(q+p)y)((n+m)(q+p)-xl)+((q+$

$p)x-n-m)((n+m)(q+p)-xl))+(ac+ab)((y-x^2)((n+m)(q+p)-xl)+((q+p)x-n-m)((n+m)x-(q+p)y))+((n+m)x-(q+p)y)(yl-(n+m)^2)+(2bc+c+b)(y-x^2)((n+m)x-(q+p)y))]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^4]-[2s(y-x^2)z((c+b)(2(n+m)((n+m)x-(q+p)y)-2x(yl-(n+m)^2))+a((q+p)(yl-(n+m)^2)+(q+p)((n+m)(q+p)-xl)+(n+m)((n+m)(q+p)-xl)-((n+m)x-(q+p)y)l-((q+p)x-n-m)l)+(ac+ab)(-2x((n+m)(q+p)-xl)-(y-x^2)l+(q+p)((n+m)x-(q+p)y)+(n+m)((q+p)x-n-m))+(n+m)(yl-(n+m)^2)-2(2bc+c+b)x((n+m)x-(q+p)y)+(2bc+c+b)(n+m)(y-x^2))]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^3]$

$h_{16}^{(U_5)}=[sz(-3(m+n)^2+(c+b)(2(x(m+n)-(q+p)y)+6x(m+n))+(ac+ab)(-2m-2n)+(2bc+c+b)(y-x^2)+ly-2(2bc+c+b)x^2+a(-2lx+(q+p)^2+l))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^2]-[2sz(2(q+p)x-2(m+n))((m+n)(ly-(m+n)^2)+(c+b)(2(m+n)(x(m+n)-(q+p)y)-2x(ly-(m+n)^2))+a((q+p)(ly-(m+n)^2)-l(x(m+n)-(q+p)y)+(m+n)((q+p)(m+n)-lx)+(q+p)((q+p)(m+n)-lx)-l(-m+(q+p)x-n))+(ac+ab)((q+p)(x(m+n)-(q+p)y)-2x((q+p)(m+n)-lx)+(-m+(q+p)x-n)(m+n)-l(y-x^2))-2(2bc+c+b)x(x(m+n)-(q+p)y)+(2bc+c+b)(y-x^2)(m+n))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^3]-[4(q+p)sz((c+b)((x(m+n)-(q+p)y)^2+(y-x^2)(ly-(m+n)^2))+(x(m+n)-(q+p)y)(ly-(m+n)^2)+a((-m+(q+p)x-n)(ly-(m+n)^2)+((q+p)(m+n)-lx)(x(m+n)-(q+p)y)+(-m+(q+p)x-n)((q+p)(m+n)-lx))+(ac+ab)((-m+(q+p)x-n)(x(m+n)-(q+p)y)+(y-x^2)((q+p)(m+n)-lx))+(2bc+c+b)(y-x^2)(x(m+n)-(q+p)y))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^3]+[6sz(2(q+p)x-2(m+n))(2(q+p)(m+n)-2lx)((c+b)((x(m+n)-(q+p)y)^2+(y-x^2)(ly-(m+n)^2))+(x(m+n)-(q+p)y)(ly-(m+n)^2)+a((-m+(q+p)x-n)(ly-(m+n)^2)+((q+p)(m+n)-lx)(x(m+n)-(q+p)y)+(-m+(q+p)x-n)((q+p)(m+n)-lx))+(ac+ab)((-m+(q+p)x-n)(x(m+n)-(q+p)y)+(y-x^2)((q+p)(m+n)-lx))+(2bc+c+b)(y-x^2)(x(m+n)-(q+p)y))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^4]-[2sz(2(q+p)(m+n)-2lx)(a((m+n)^2+(q+p)(x(m+n)-(q+p)y)+x((q+p)(m+n)-lx)-2(-m+(q+p)x-n)(m+n)-(q+p)(m+n)+(q+p)(-m+(q+p)x-n)-ly+lx)+x(ly-(m+n)^2)+(c+b)(2x(x(m+n)-(q+p)y)-2(y-x^2)(m+n))-2(m+n)(x(m+n)-(q+p)y)+(ac+ab)(-x(m+n)+x(-m+(q+p)x-n)+(q+p)(y-x^2)+$

$(q+p)y)+(2bc+c+b)x(y-x^2))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^3]$

$h_{17}^{(U_5)} = [sz(-3(n+m)^2+(c+b)(2(x(n+m)-(q+p)y)+6x(n+m))+(ac+ab)(-2n-$
$2m)+(2bc+c+b)(y-x^2)+ly-2(2bc+c+b)x^2+a(-2lx+(q+p)^2+l))]/[(-(n+m)^2+$
$2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^2]-[2sz(2(q+p)x-2(n+m))((n+m)(ly-(n+$
$m)^2)+(c+b)(2(n+m)(x(n+m)-(q+p)y)-2x(ly-(n+m)^2))+a((q+p)(ly-(n+$
$m)^2)-l(x(n+m)-(q+p)y)+(n+m)((q+p)(n+m)-lx)+(q+p)((q+p)(n+m)-$
$lx)-l(-n+(q+p)x-m))+(ac+ab)((q+p)(x(n+m)-(q+p)y)-2x((q+p)(n+m)-$
$lx)+(-n+(q+p)x-m)(n+m)-l(y-x^2))-2(2bc+c+b)x(x(n+m)-(q+p)y)+$
$(2bc+c+b)(y-x^2)(n+m))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]-$
$[4(q+p)sz((c+b)((x(n+m)-(q+p)y)^2+(y-x^2)(ly-(n+m)^2))+(x(n+m)-(q+$
$p)y)(ly-(n+m)^2)+a((-n+(q+p)x-m)(ly-(n+m)^2)+((q+p)(n+m)-lx)(x(n+$
$m)-(q+p)y)+(-n+(q+p)x-m)((q+p)(n+m)-lx))+(ac+ab)((-n+(q+p)x-$
$m)(x(n+m)-(q+p)y)+(y-x^2)((q+p)(n+m)-lx))+(2bc+c+b)(y-x^2)(x(n+$
$m)-(q+p)y))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]+[6sz(2(q+$
$p)x-2(n+m))(2(q+p)(n+m)-2lx)((c+b)((x(n+m)-(q+p)y)^2+(y-x^2)(ly-(n+$
$m)^2))+(x(n+m)-(q+p)y)(ly-(n+m)^2)+a((-n+(q+p)x-m)(ly-(n+m)^2)+((q+$
$p)(n+m)-lx)(x(n+m)-(q+p)y)+(-n+(q+p)x-m)((q+p)(n+m)-lx))+(ac+$
$ab)((-n+(q+p)x-m)(x(n+m)-(q+p)y)+(y-x^2)((q+p)(n+m)-lx))+(2bc+c+$
$b)(y-x^2)(x(n+m)-(q+p)y))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^4]-$
$[2sz(2(q+p)(n+m)-2lx)(a((n+m)^2+(q+p)(x(n+m)-(q+p)y)+x((q+p)(n+$
$m)-lx)-2(-n+(q+p)x-m)(n+m)-(q+p)(n+m)+(q+p)(-n+(q+p)x-m)-$
$ly+lx)+x(ly-(n+m)^2)+(c+b)(2x(x(n+m)-(q+p)y)-2(y-x^2)(n+m))-2(n+$
$m)(x(n+m)-(q+p)y)+(ac+ab)(-x(n+m)+x(-n+(q+p)x-m)+(q+p)(y-x^2)+$
$(q+p)y)+(2bc+c+b)x(y-x^2))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]$

$h_{18}^{(U_5)} = [sz(a(2(n+m)(p+q)+2ly-2lx)-2(ac+ab)y(p+q)+2(2bc+c+b)xy-2(c+$
$b)(n+m)y)]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^2]-[2sz(2(n+m)x-$
$2y(p+q))((ac+ab)((p+q)((n+m)x-y(p+q))+(n+m)(x(p+q)-n-m)-2x((n+$
$m)(p+q)-lx)-l(y-x^2))+(c+b)(2(n+m)((n+m)x-y(p+q))-2x(ly-(n+m)^2))+$
$a(-l((n+m)x-y(p+q))-l(x(p+q)-n-m)+(p+q)((n+m)(p+q)-lx)+(n+m)((n+$
$m)(p+q)-lx)+(ly-(n+m)^2)(p+q))-2(2bc+c+b)x((n+m)x-y(p+q))+(n+m)(ly-$

$$(n+m)^2)+(2bc+c+b)(n+m)(y-x^2))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]-[2sz(2(n+m)(p+q)-2lx)((ac+ab)(x((n+m)x-y(p+q))-y(x(p+q)-n-m)+(n+m)(y-x^2))+a((n+m)((n+m)x-y(p+q))+(n+m)(x(p+q)-n-m)-y((n+m)(p+q)-lx)+x((n+m)(p+q)-lx)+x(ly-(n+m)^2))-2(c+b)y((n+m)x-y(p+q))-y(ly-(n+m)^2)-(2bc+c+b)y(y-x^2))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]-[4(n+m)sz((c+b)(((n+m)x-y(p+q))^2+(y-x^2)(ly-(n+m)^2))+(ac+ab)((x(p+q)-n-m)((n+m)x-y(p+q))+(y-x^2)((n+m)(p+q)-lx))+a(((n+m)(p+q)-lx)((n+m)x-y(p+q))+((n+m)(p+q)-lx)(x(p+q)-n-m)+(ly-(n+m)^2)(x(p+q)-n-m))+(ly-(n+m)^2)((n+m)x-y(p+q))+(2bc+c+b)(y-x^2)((n+m)x-y(p+q)))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]+[6sz(2(n+m)(p+q)-2lx)(2(n+m)x-2y(p+q))((c+b)(((n+m)x-y(p+q))^2+(y-x^2)(ly-(n+m)^2))+(ac+ab)((x(p+q)-n-m)((n+m)x-y(p+q))+(y-x^2)((n+m)(p+q)-lx))+a(((n+m)(p+q)-lx)((n+m)x-y(p+q))+((n+m)(p+q)-lx)(x(p+q)-n-m)+(ly-(n+m)^2)(x(p+q)-n-m))+(ly-(n+m)^2)((n+m)x-y(p+q))+(2bc+c+b)(y-x^2)((n+m)x-y(p+q)))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^4]$$

$$h_{19}^{(U_5)}=[sz(a(2(n+m)(q+p)+2ly-2lx)-2(ac+ab)y(q+p)+2(2bc+c+b)xy-2(c+b)(n+m)y)]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^2]-[2sz(2(n+m)x-2y(q+p))((ac+ab)((q+p)((n+m)x-y(q+p))+(n+m)(x(q+p)-n-m)-2x((n+m)(q+p)-lx)-l(y-x^2))+(c+b)(2(n+m)((n+m)x-y(q+p))-2x(ly-(n+m)^2))+a(-l((n+m)x-y(q+p))-l(x(q+p)-n-m)+(q+p)((n+m)(q+p)-lx)+(n+m)((n+m)(q+p)-lx)+(ly-(n+m)^2)(q+p))-2(2bc+c+b)x((n+m)x-y(q+p))+(n+m)(ly-(n+m)^2)+(2bc+c+b)(n+m)(y-x^2))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]-[2sz(2(n+m)(q+p)-2lx)((ac+ab)(x((n+m)x-y(q+p))-y(x(q+p)-n-m)+(n+m)(y-x^2))+a((n+m)((n+m)x-y(q+p))+(n+m)(x(q+p)-n-m)-y((n+m)(q+p)-lx)+x((n+m)(q+p)-lx)+x(ly-(n+m)^2))-2(c+b)y((n+m)x-y(q+p))-y(ly-(n+m)^2)-(2bc+c+b)y(y-x^2))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]-[4(n+m)sz((c+b)(((n+m)x-y(q+p))^2+(y-x^2)(ly-(n+m)^2))+(ac+ab)((x(q+p)-n-m)((n+m)x-y(q+p))+(y-x^2)((n+m)(q+p)-lx))+a(((n+m)(q+p)-lx)((n+m)x-y(q+p))+((n+m)(q+p)-lx)(x(q+p)-n-m)+(ly-(n+m)^2)(x(q+p)-n-m))+(ly-(n+m)^2)((n+m)x-y(q+p))+(2bc+c+b)(y-x^2)((n+$$

$m)x-y(q+p)))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]+[6sz(2(n+m)(q+p)-2lx)(2(n+m)x-2y(q+p))((c+b)(((n+m)x-y(q+p))^2+(y-x^2)(ly-(n+m)^2))+(ac+ab)((x(q+p)-n-m)((n+m)x-y(q+p))+(y-x^2)((n+m)(q+p)-lx))+a(((n+m)(q+p)-lx)((n+m)x-y(q+p))+((n+m)(q+p)-lx)(x(q+p)-n-m)+(ly-(n+m)^2)(x(q+p)-n-m))+(ly-(n+m)^2)((n+m)x-y(q+p))+(2bc+c+b)(y-x^2)((n+m)x-y(q+p))))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^4]$

$h_{1\ 10}^{(U_5)}=[((ac+ab)((q+p)((n+m)x-(q+p)y)-l(y-x^2)+(n+m)((q+p)x-n-m)-2x((n+m)(q+p)-lx))+(c+b)(2(n+m)((n+m)x-(q+p)y)-2x(ly-(n+m)^2))+a(-l((n+m)x-(q+p)y)+(q+p)(ly-(n+m)^2)-l((q+p)x-n-m)+(q+p)((n+m)(q+p)-lx)+(n+m)((n+m)(q+p)-lx))-2(2bc+c+b)x((n+m)x-(q+p)y)+(n+m)(ly-(n+m)^2)+(2bc+c+b)(n+m)(y-x^2))z]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^2]-[2(2(n+m)(q+p)-2lx)((c+b)(((n+m)x-(q+p)y)^2+(y-x^2)(ly-(n+m)^2))+(ac+ab)((((q+p)x-n-m)((n+m)x-(q+p)y)+((n+m)(q+p)-lx)(y-x^2))+a(((n+m)(q+p)-lx)((n+m)x-(q+p)y)+((q+p)x-n-m)(ly-(n+m)^2)+((n+m)(q+p)-lx)((q+p)x-n-m))+(ly-(n+m)^2)((n+m)x-(q+p)y)+(2bc+c+b)(y-x^2)((n+m)x-(q+p)y))z]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^3]$

$h_{22}^{(U_5)}=[(2l(-q-p)+2(2bc+c+b)(-q-p)+(c+b)(2(-q-p)^2+2l))sz]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^2]+[6(l-(q+p)^2)^2sz((c+b)(((n+m)x-(q+p)y)^2+(y-x^2)(ly-(n+m)^2))+(ac+ab)((((q+p)x-n-m)((n+m)x-(q+p)y)+((n+m)(q+p)-lx)(y-x^2))+a(((n+m)(q+p)-lx)((n+m)x-(q+p)y)+((q+p)x-n-m)(ly-(n+m)^2)+((n+m)(q+p)-lx)((q+p)x-n-m))+(ly-(n+m)^2)((n+m)x-(q+p)y)+(2bc+c+b)(y-x^2)((n+m)x-(q+p)y))]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^4]-[4(l-(q+p)^2)sz((c+b)(2(-q-p)((n+m)x-(q+p)y)+l(y-x^2)+ly-(n+m)^2)+l((n+m)x-(q+p)y)+(2bc+c+b)((n+m)x-(q+p)y)+(-q-p)(ly-(n+m)^2)+(2bc+c+b)(-q-p)(y-x^2)+(ac+ab)((-q-p)((q+p)x-n-m)-lx+(n+m)(q+p))+a(l((q+p)x-n-m)+(-q-p)((n+m)(q+p)-lx)))]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^3]$

$h_{23}^{(U_5)}=[s((c+b)(2(-q-p)((n+m)x-(q+p)y)+l(y-x^2)+ly-(n+m)^2)+l((n+m)x-(q+p)y)+(2bc+c+b)((n+m)x-(q+p)y)+(-q-p)(ly-(n+m)^2)+(2bc+c+b)(-q-p)(y-x^2)+(ac+ab)((-q-p)((q+p)x-n-m)-lx+(n+m)(q+p))+$

$a(l(((q+p)x-n-m)+(-q-p)((n+m)(q+p)-lx)))]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^2]-[2(l-(q+p)^2)s((c+b)(((n+m)x-(q+p)y)^2+(y-x^2)(ly-(n+m)^2))+(ac+ab)((((q+p)x-n-m)((n+m)x-(q+p)y)+((n+m)(q+p)-lx)(y-x^2))+a((((n+m)(q+p)-lx)((n+m)x-(q+p)y)+(((q+p)x-n-m)(ly-(n+m)^2)+((n+m)(q+p)-lx)(((q+p)x-n-m))+(ly-(n+m)^2)((n+m)x-(q+p)y)+(2bc+c+b)(y-x^2)((n+m)x-(q+p)y))]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^3]$

$h_{24}^{(U_5)}=0$

$h_{25}^{(U_5)}=[s((c+b)(2y-x^2)-(q+p)y+(-q-p)y+a((q+p)x-(-q-p)x-n-m)+(n+m)x-(ac+ab)x)z]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^2]-[2s(a(-x((n+m)x-(q+p)y)+((q+p)x-n-m)y-x((q+p)x-n-m))+y((n+m)x-(q+p)y)+(c+b)y(y-x^2)-(ac+ab)x(y-x^2))z(l-(q+p)^2)]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^3]-[2sz((c+b)((y-x^2)(yl-(n+m)^2)+((n+m)x-(q+p)y)^2)+a((((q+p)x-n-m)(yl-(n+m)^2)+((n+m)x-(q+p)y)((n+m)(q+p)-xl)+(((q+p)x-n-m)((n+m)(q+p)-xl))+(ac+ab)((y-x^2)((n+m)(q+p)-xl)+(((q+p)x-n-m)((n+m)x-(q+p)y))+((n+m)x-(q+p)y)(yl-(n+m)^2)+(2bc+c+b)(y-x^2)((n+m)x-(q+p)y))]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^3]+[6s(y-x^2)z(l-(q+p)^2)((c+b)((y-x^2)(yl-(n+m)^2)+((n+m)x-(q+p)y)^2)+a((((q+p)x-n-m)(yl-(n+m)^2)+((n+m)x-(q+p)y)((n+m)(q+p)-xl)+(((q+p)x-n-m)((n+m)(q+p)-xl))+(ac+ab)((y-x^2)((n+m)(q+p)-xl)+(((q+p)x-n-m)((n+m)x-(q+p)y))+((n+m)x-(q+p)y)(yl-(n+m)^2)+(2bc+c+b)(y-x^2)((n+m)x-(q+p)y))]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^4]-[2s(y-x^2)z(a((-q-p)((n+m)(q+p)-xl)+(((q+p)x-n-m)l)+(c+b)((y-x^2)l+yl+2(-q-p)((n+m)x-(q+p)y)-(n+m)^2)+(-q-p)(yl-(n+m)^2)+(ac+ab)(-xl+(-q-p)(((q+p)x-n-m)+(n+m)(q+p))+((n+m)x-(q+p)y)l+(2bc+c+b)((n+m)x-(q+p)y)+(2bc+c+b)(-q-p)(y-x^2))]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^3]$

$h_{26}^{(U_5)}=[sz((c+b)(2(-q-p)x-2(m+n))-2(-q-p)(m+n)+lx+(2bc+c+b)x+a((-q-p)(q+p)-l)+(ac+ab)(2q+2p))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^2]-[2sz(2(q+p)x-2(m+n))((c+b)(-(m+n)^2+2(-q-p)(x(m+n)-(q+p)y)+l(y-x^2)+ly)+(-q-p)(ly-(m+n)^2)+a((-q-p)((q+p)(m+n)-lx)+l(-m+(q+p)x-n))+l(x(m+n)-(q+p)y)+(2bc+c+b)(x(m+n)-(q+p)y)+(ac+ab)((q+p)(m+n)+(-q-$

$p)(-m+(q+p)x-n)-lx)+(2bc+c+b)(-q-p)(y-x^2))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^3]+[6(l-(q+p)^2)sz(2(q+p)x-2(m+n))((c+b)((x(m+n)-(q+p)y)^2+(y-x^2)(ly-(m+n)^2))+(x(m+n)-(q+p)y)(ly-(m+n)^2)+a((-m+(q+p)x-n)(ly-(m+n)^2)+((q+p)(m+n)-lx)(x(m+n)-(q+p)y)+(-m+(q+p)x-n)((q+p)(m+n)-lx))+(ac+ab)((-m+(q+p)x-n)(x(m+n)-(q+p)y)+(y-x^2)((q+p)(m+n)-lx))+(2bc+c+b)(y-x^2)(x(m+n)-(q+p)y))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^4]-[2(l-(q+p)^2)sz(a((m+n)^2+(q+p)(x(m+n)-(q+p)y)+x((q+p)(m+n)-lx)-2(-m+(q+p)x-n)(m+n)-(q+p)(m+n)+(q+p)(-m+(q+p)x-n)-ly+lx)+x(ly-(m+n)^2)+(c+b)(2x(x(m+n)-(q+p)y)-2(y-x^2)(m+n))-2(m+n)(x(m+n)-(q+p)y)+(ac+ab)(-x(m+n)+x(-m+(q+p)x-n)+(q+p)(y-x^2)+(q+p)y)+(2bc+c+b)x(y-x^2))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^3]$

$h_{27}^{(U_5)} = [sz((c+b)(2(-q-p)x-2(n+m))-2(-q-p)(n+m)+lx+(2bc+c+b)x+a((-q-p)(q+p)-l)+(ac+ab)(2q+2p))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^2]-[2sz(2(q+p)x-2(n+m))((c+b)(-(n+m)^2+2(-q-p)(x(n+m)-(q+p)y)+l(y-x^2)+ly)+(-q-p)(ly-(n+m)^2)+a((-q-p)((q+p)(n+m)-lx)+l(-n+(q+p)x-m))+l(x(n+m)-(q+p)y)+(2bc+c+b)(x(n+m)-(q+p)y)+(ac+ab)((q+p)(n+m)+(-q-p)(-n+(q+p)x-m)-lx)+(2bc+c+b)(-q-p)(y-x^2))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]+[6(l-(q+p)^2)sz(2(q+p)x-2(n+m))((c+b)((x(n+m)-(q+p)y)^2+(y-x^2)(ly-(n+m)^2))+(x(n+m)-(q+p)y)(ly-(n+m)^2)+a((-n+(q+p)x-m)(ly-(n+m)^2)+((q+p)(n+m)-lx)(x(n+m)-(q+p)y)+(-n+(q+p)x-m)((q+p)(n+m)-lx))+(ac+ab)((-n+(q+p)x-m)(x(n+m)-(q+p)y)+(y-x^2)((q+p)(n+m)-lx))+(2bc+c+b)(y-x^2)(x(n+m)-(q+p)y))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^4]-[2(l-(q+p)^2)sz(a((n+m)^2+(q+p)(x(n+m)-(q+p)y)+x((q+p)(n+m)-lx)-2(-n+(q+p)x-m)(n+m)-(q+p)(n+m)+(q+p)(-n+(q+p)x-m)-ly+lx)+x(ly-(n+m)^2)+(c+b)(2x(x(n+m)-(q+p)y)-2(y-x^2)(n+m))-2(n+m)(x(n+m)-(q+p)y)+(ac+ab)(-x(n+m)+x(-n+(q+p)x-m)+(q+p)(y-x^2)+(q+p)y)+(2bc+c+b)x(y-x^2))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]$

$h_{28}^{(U_5)} = [sz((c+b)(-2((n+m)x-y(p+q))-2y(-p-q))+(ac+ab)(-x(p+q)+x(-p-q)+2n+2m)+a(-(n+m)(p+q)+(n+m)(-p-q)+2lx)-(2bc+c+b)(y-x^2)-2ly-(2bc+c+b)y+(n+m)^2)]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^2]-$

$[2sz((ac+ab)(x((n+m)x-y(p+q))-y(x(p+q)-n-m)+(n+m)(y-x^2))+a((n+$
$m)((n+m)x-y(p+q))+(n+m)(x(p+q)-n-m)-y((n+m)(p+q)-lx)+x((n+$
$m)(p+q)-lx)+x(ly-(n+m)^2))-2(c+b)y((n+m)x-y(p+q))-y(ly-(n+m)^2)-$
$(2bc+c+b)y(y-x^2))(l-(p+q)^2)]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+$
$m)^2)^3]-[2sz(2(n+m)x-2y(p+q))((c+b)(2(-p-q)((n+m)x-y(p+q))+l(y-x^2)+$
$ly-(n+m)^2)+(ac+ab)((-p-q)(x(p+q)-n-m)+(n+m)(p+q)-lx)+a(l(x(p+q)-$
$n-m)+(-p-q)((n+m)(p+q)-lx))+l((n+m)x-y(p+q))+(2bc+c+b)((n+m)x-$
$y(p+q))+(ly-(n+m)^2)(-p-q)+(2bc+c+b)(y-x^2)(-p-q))]/[(-y(p+q)^2+2(n+$
$m)x(p+q)+ly-lx^2-(n+m)^2)^3]+[4sz(p+q)((c+b)(((n+m)x-y(p+q))^2+(y-x^2)(ly-$
$(n+m)^2))+(ac+ab)((x(p+q)-n-m)((n+m)x-y(p+q))+(y-x^2)((n+m)(p+q)-$
$lx))+a(((n+m)(p+q)-lx)((n+m)x-y(p+q))+((n+m)(p+q)-lx)(x(p+q)-n-$
$m)+(ly-(n+m)^2)(x(p+q)-n-m))+(ly-(n+m)^2)((n+m)x-y(p+q))+(2bc+c+$
$b)(y-x^2)((n+m)x-y(p+q))))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]+$
$[6sz(2(n+m)x-2y(p+q))(l-(p+q)^2)((c+b)(((n+m)x-y(p+q))^2+(y-x^2)(ly-$
$(n+m)^2))+(ac+ab)((x(p+q)-n-m)((n+m)x-y(p+q))+(y-x^2)((n+m)(p+q)-$
$lx))+a(((n+m)(p+q)-lx)((n+m)x-y(p+q))+((n+m)(p+q)-lx)(x(p+q)-n-$
$m)+(ly-(n+m)^2)(x(p+q)-n-m))+(ly-(n+m)^2)((n+m)x-y(p+q))+(2bc+c+$
$b)(y-x^2)((n+m)x-y(p+q))))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^4]$

$h_{29}^{(U_5)} = [sz((c+b)(-2((n+m)x-y(q+p))-2y(-q-p))+(ac+ab)(-x(q+p)+$
$x(-q-p)+2n+2m)+a(-(n+m)(q+p)+(n+m)(-q-p)+2lx)-(2bc+c+b)(y-x^2)-$
$2ly-(2bc+c+b)y+(n+m)^2)]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^2]-$
$[2sz((ac+ab)(x((n+m)x-y(q+p))-y(x(q+p)-n-m)+(n+m)(y-x^2))+a((n+$
$m)((n+m)x-y(q+p))+(n+m)(x(q+p)-n-m)-y((n+m)(q+p)-lx)+x((n+$
$m)(q+p)-lx)+x(ly-(n+m)^2))-2(c+b)y((n+m)x-y(q+p))-y(ly-(n+m)^2)-$
$(2bc+c+b)y(y-x^2))(l-(q+p)^2)]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+$
$m)^2)^3]-[2sz(2(n+m)x-2y(q+p))((c+b)(2(-q-p)((n+m)x-y(q+p))+l(y-x^2)+$
$ly-(n+m)^2)+(ac+ab)((-q-p)(x(q+p)-n-m)+(n+m)(q+p)-lx)+a(l(x(q+p)-$
$n-m)+(-q-p)((n+m)(q+p)-lx))+l((n+m)x-y(q+p))+(2bc+c+b)((n+m)x-$
$y(q+p))+(ly-(n+m)^2)(-q-p)+(2bc+c+b)(y-x^2)(-q-p))]/[(-y(q+p)^2+2(n+$
$m)x(q+p)+ly-lx^2-(n+m)^2)^3]+[4sz(q+p)((c+b)(((n+m)x-y(q+p))^2+(y-x^2)(ly-$

$(n+m)^2))+(ac+ab)((x(q+p)-n-m)((n+m)x-y(q+p))+(y-x^2)((n+m)(q+p)-lx))+a(((n+m)(q+p)-lx)((n+m)x-y(q+p))+((n+m)(q+p)-lx)(x(q+p)-n-m)+(ly-(n+m)^2)(x(q+p)-n-m))+(ly-(n+m)^2)((n+m)x-y(q+p))+(2bc+c+b)(y-x^2)((n+m)x-y(q+p)))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]+[6sz(2(n+m)x-2y(q+p))(l-(q+p)^2)((c+b)(((n+m)x-y(q+p))^2+(y-x^2)(ly-(n+m)^2))+(ac+ab)((x(q+p)-n-m)((n+m)x-y(q+p))+(y-x^2)((n+m)(q+p)-lx))+a(((n+m)(q+p)-lx)((n+m)x-y(q+p))+((n+m)(q+p)-lx)(x(q+p)-n-m)+(ly-(n+m)^2)(x(q+p)-n-m))+(ly-(n+m)^2)((n+m)x-y(q+p))+(2bc+c+b)(y-x^2)((n+m)x-y(q+p)))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^4]$

$h_{2\ 10}^{(U_5)} = [((c+b)(2(-q-p)((n+m)x-(q+p)y)+l(y-x^2)+ly-(n+m)^2)+l((n+m)x-(q+p)y)+(2bc+c+b)((n+m)x-(q+p)y)+(-q-p)(ly-(n+m)^2)+(2bc+c+b)(-q-p)(y-x^2)+(ac+ab)((-q-p)((q+p)x-n-m)-lx+(n+m)(q+p))+a(l((q+p)x-n-m)+(-q-p)((n+m)(q+p)-lx)))z]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^2]-[2(l-(q+p)^2)((c+b)(((n+m)x-(q+p)y)^2+(y-x^2)(ly-(n+m)^2))+(ac+ab)(((q+p)x-n-m)((n+m)x-(q+p)y)+((n+m)(q+p)-lx)(y-x^2))+a(((n+m)(q+p)-lx)((n+m)x-(q+p)y)+((q+p)x-n-m)(ly-(n+m)^2)+((n+m)(q+p)-lx)((q+p)x-n-m))+(ly-(n+m)^2)((n+m)x-(q+p)y)+(2bc+c+b)(y-x^2)((n+m)x-(q+p)y))z]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^3]$

$h_{33}^{(U_5)} = 0$

$h_{34}^{(U_5)} = 0$

$h_{35}^{(U_5)} = [s(a(-x((n+m)x-(q+p)y)+((q+p)x-n-m)y-x((q+p)x-n-m))+y((n+m)x-(q+p)y)+(c+b)y(y-x^2)-(ac+ab)x(y-x^2))]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^2]-[2s(y-x^2)((c+b)((y-x^2)(yl-(n+m)^2)+((n+m)x-(q+p)y)^2)+a(((q+p)x-n-m)(yl-(n+m)^2)+((n+m)x-(q+p)y)((n+m)(q+p)-xl)+((q+p)x-n-m)((n+m)(q+p)-xl))+(ac+ab)((y-x^2)((n+m)(q+p)-xl)+((q+p)x-n-m)((n+m)x-(q+p)y))+((n+m)x-(q+p)y)(yl-(n+m)^2)+(2bc+c+b)(y-x^2)((n+m)x-(q+p)y))]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^3]$

$h_{36}^{(U_5)} = [s(a((m+n)^2+(q+p)(x(m+n)-(q+p)y)+x((q+p)(m+n)-lx)-2(-m+(q+p)x-n)(m+n)-(q+p)(m+n)+(q+p)(-m+(q+p)x-n)-ly+lx)+x(ly-(m+n)^2)+(c+b)(2x(x(m+n)-(q+p)y)-2(y-x^2)(m+n))-2(m+n)(x(m+$

$n)-(q+p)y)+(ac+ab)(-x(m+n)+x(-m+(q+p)x-n)+(q+p)(y-x^2)+(q+$
$p)y)+(2bc+c+b)x(y-x^2))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^2]-$
$[2s(2(q+p)x-2(m+n))((c+b)((x(m+n)-(q+p)y)^2+(y-x^2)(ly-(m+n)^2))+$
$(x(m+n)-(q+p)y)(ly-(m+n)^2)+a((-m+(q+p)x-n)(ly-(m+n)^2)+((q+$
$p)(m+n)-lx)(x(m+n)-(q+p)y)+(-m+(q+p)x-n)((q+p)(m+n)-lx))+(ac+$
$ab)((-m+(q+p)x-n)(x(m+n)-(q+p)y)+(y-x^2)((q+p)(m+n)-lx))+(2bc+c+$
$b)(y-x^2)(x(m+n)-(q+p)y))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^3]$

$h_{37}^{(U_5)}=[s(a((n+m)^2+(q+p)(x(n+m)-(q+p)y)+x((q+p)(n+m)-lx)-$
$2(-n+(q+p)x-m)(n+m)-(q+p)(n+m)+(q+p)(-n+(q+p)x-m)-ly+lx)+$
$x(ly-(n+m)^2)+(c+b)(2x(x(n+m)-(q+p)y)-2(y-x^2)(n+m))-2(n+m)(x(n+$
$m)-(q+p)y)+(ac+ab)(-x(n+m)+x(-n+(q+p)x-m)+(q+p)(y-x^2)+(q+$
$p)y)+(2bc+c+b)x(y-x^2))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^2]-$
$[2s(2(q+p)x-2(n+m))((c+b)((x(n+m)-(q+p)y)^2+(y-x^2)(ly-(n+m)^2))+$
$(x(n+m)-(q+p)y)(ly-(n+m)^2)+a((-n+(q+p)x-m)(ly-(n+m)^2)+((q+$
$p)(n+m)-lx)(x(n+m)-(q+p)y)+(-n+(q+p)x-m)((q+p)(n+m)-lx))+(ac+$
$ab)((-n+(q+p)x-m)(x(n+m)-(q+p)y)+(y-x^2)((q+p)(n+m)-lx))+(2bc+c+$
$b)(y-x^2)(x(n+m)-(q+p)y))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]$

$h_{38}^{(U_5)}=[s((ac+ab)(x((n+m)x-y(p+q))-y(x(p+q)-n-m)+(n+m)(y-x^2))+$
$a((n+m)((n+m)x-y(p+q))+(n+m)(x(p+q)-n-m)-y((n+m)(p+q)-lx)+$
$x((n+m)(p+q)-lx)+x(ly-(n+m)^2))-2(c+b)y((n+m)x-y(p+q))-y(ly-(n+$
$m)^2)-(2bc+c+b)y(y-x^2))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^2]-$
$[2s(2(n+m)x-2y(p+q))((c+b)(((n+m)x-y(p+q))^2+(y-x^2)(ly-(n+m)^2))+$
$(ac+ab)((x(p+q)-n-m)((n+m)x-y(p+q))+(y-x^2)((n+m)(p+q)-lx))+$
$a(((n+m)(p+q)-lx)((n+m)x-y(p+q))+((n+m)(p+q)-lx)(x(p+q)-n-m)+$
$(ly-(n+m)^2)(x(p+q)-n-m))+(ly-(n+m)^2)((n+m)x-y(p+q))+(2bc+c+$
$b)(y-x^2)((n+m)x-y(p+q)))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]$

$h_{39}^{(U_5)}=[s((ac+ab)(x((n+m)x-y(q+p))-y(x(q+p)-n-m)+(n+m)(y-x^2))+$
$a((n+m)((n+m)x-y(q+p))+(n+m)(x(q+p)-n-m)-y((n+m)(q+p)-lx)+$
$x((n+m)(q+p)-lx)+x(ly-(n+m)^2))-2(c+b)y((n+m)x-y(q+p))-y(ly-(n+$
$m)^2)-(2bc+c+b)y(y-x^2))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^2]-$

$[2s(2(n+m)x-2y(q+p))((c+b)(((n+m)x-y(q+p))^2+(y-x^2)(ly-(n+m)^2))+(ac+ab)((x(q+p)-n-m)((n+m)x-y(q+p))+(y-x^2)((n+m)(q+p)-lx))+a(((n+m)(q+p)-lx)((n+m)x-y(q+p))+((n+m)(q+p)-lx)(x(q+p)-n-m)+(ly-(n+m)^2)(x(q+p)-n-m))+(ly-(n+m)^2)((n+m)x-y(q+p))+(2bc+c+b)(y-x^2)((n+m)x-y(q+p)))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]$

$h_{3\ 10}^{(U_5)} = [(c+b)(((n+m)x-(q+p)y)^2+(y-x^2)(ly-(n+m)^2))+(ac+ab)(((q+p)x-n-m)((n+m)x-(q+p)y)+((n+m)(q+p)-lx)(y-x^2))+a(((n+m)(q+p)-lx)((n+m)x-(q+p)y)+((q+p)x-n-m)(ly-(n+m)^2)+((n+m)(q+p)-lx)((q+p)x-n-m))+(ly-(n+m)^2)((n+m)x-(q+p)y)+(2bc+c+b)(y-x^2)((n+m)x-(q+p)y)]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^2]$

$h_{44}^{(U_5)} = 0$

$h_{45}^{(U_5)} = 0$

$h_{46}^{(U_5)} = 0$

$h_{47}^{(U_5)} = 0$

$h_{48}^{(U_5)} = 0$

$h_{49}^{(U_5)} = 0$

$h_{4\ 10}^{(U_5)} = 0$

$h_{55}^{(U_5)} = [6s(y-x^2)^2z((c+b)((y-x^2)(yl-(n+m)^2)+((n+m)x-(q+p)y)^2)+a((((q+p)x-n-m)(yl-(n+m)^2)+((n+m)x-(q+p)y)((n+m)(q+p)-xl)+((q+p)x-n-m)((n+m)(q+p)-xl))+(ac+ab)((y-x^2)((n+m)(q+p)-xl)+((q+p)x-n-m)((n+m)x-(q+p)y))+((n+m)x-(q+p)y)(yl-(n+m)^2)+(2bc+c+b)(y-x^2)((n+m)x-(q+p)y))]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^4]-[4s(y-x^2)(a(-x((n+m)x-(q+p)y)+((q+p)x-n-m)y-x((q+p)x-n-m))+y((n+m)x-(q+p)y)+(c+b)y(y-x^2)-(ac+ab)x(y-x^2))z]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^3]$

$h_{56}^{(U_5)} = [s(xy+a(-y-x^2+x))z]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^2]-[2sz(2(q+p)x-2(m+n))(a(-x(x(m+n)-(q+p)y)+y(-m+(q+p)x-n)-x(-m+(q+p)x-n))+y(x(m+n)-(q+p)y)+(c+b)y(y-x^2)+(-ac-ab)x(y-x^2))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^3]+[6s(y-x^2)z(2(q+p)x-2(m+n))((c+b)((x(m+n)-(q+p)y)^2+(y-x^2)(ly-(m+n)^2))+(x(m+n)-(q+p)y)(ly-(m+n)^2)+a((-m+(q+p)x-n)(ly-(m+n)^2)+((q+p)(m+n)-$

$lx)(x(m+n)-(q+p)y)+(-m+(q+p)x-n)((q+p)(m+n)-lx))+(ac+ab)((-m+(q+p)x-n)(x(m+n)-(q+p)y)+(y-x^2)((q+p)(m+n)-lx))+(2bc+c+b)(y-x^2)(x(m+n)-(q+p)y))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^4]-[2s(y-x^2)z(a((m+n)^2+(q+p)(x(m+n)-(q+p)y)+x((q+p)(m+n)-lx)-2(-m+(q+p)x-n)(m+n)-(q+p)(m+n)+(q+p)(-m+(q+p)x-n)-ly+lx)+x(ly-(m+n)^2)+(c+b)(2x(x(m+n)-(q+p)y)-2(y-x^2)(m+n))-2(m+n)(x(m+n)-(q+p)y)+(ac+ab)(-x(m+n)+x(-m+(q+p)x-n)+(q+p)(y-x^2)+(q+p)y)+(2bc+c+b)x(y-x^2))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^3]$

$h_{57}^{(U_5)}=[s(xy+a(-y-x^2+x))z]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^2]-[2sz(2(q+p)x-2(n+m))(a(-x(x(n+m)-(q+p)y)+y(-n+(q+p)x-m)-x(-n+(q+p)x-m))+y(x(n+m)-(q+p)y)+(c+b)y(y-x^2)+(-ac-ab)x(y-x^2))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]+[6s(y-x^2)z(2(q+p)x-2(n+m))((c+b)((x(n+m)-(q+p)y)^2+(y-x^2)(ly-(n+m)^2))+(x(n+m)-(q+p)y)(ly-(n+m)^2)+a((-n+(q+p)x-m)(ly-(n+m)^2)+((q+p)(n+m)-lx)(x(n+m)-(q+p)y)+(-n+(q+p)x-m)((q+p)(n+m)-lx))+(ac+ab)((-n+(q+p)x-m)(x(n+m)-(q+p)y)+(y-x^2)((q+p)(n+m)-lx))+(2bc+c+b)(y-x^2)(x(n+m)-(q+p)y))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^4]-[2s(y-x^2)z(a((n+m)^2+(q+p)(x(n+m)-(q+p)y)+x((q+p)(n+m)-lx)-2(-n+(q+p)x-m)(n+m)-(q+p)(n+m)+(q+p)(-n+(q+p)x-m)-ly+lx)+x(ly-(n+m)^2)+(c+b)(2x(x(n+m)-(q+p)y)-2(y-x^2)(n+m))-2(n+m)(x(n+m)-(q+p)y)+(ac+ab)(-x(n+m)+x(-n+(q+p)x-m)+(q+p)(y-x^2)+(q+p)y)+(2bc+c+b)x(y-x^2))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]$

$h_{58}^{(U_5)}=[s(a(2xy-x^2)-y^2)z]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^2]-[2s(y-x^2)z((ac+ab)(x((n+m)x-y(p+q))-y(x(p+q)-n-m)+(n+m)(y-x^2))+a((n+m)((n+m)x-y(p+q))+(n+m)(x(p+q)-n-m)-y((n+m)(p+q)-lx)+x((n+m)(p+q)-lx)+x(ly-(n+m)^2))-2(c+b)y((n+m)x-y(p+q))-y(ly-(n+m)^2)-(2bc+c+b)y(y-x^2))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]-[2sz(2(n+m)x-2y(p+q))(a(-x((n+m)x-y(p+q))+y(x(p+q)-n-m)-x(x(p+q)-n-m))+y((n+m)x-y(p+q))+(c+b)y(y-x^2)+(-ac-ab)x(y-x^2))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]+[6s(y-x^2)z(2(n+m)x-2y(p+q))((c+b)(((n+m)x-y(p+q))^2+$

$(y-x^2)(ly-(n+m)^2))+(ac+ab)((x(p+q)-n-m)((n+m)x-y(p+q))+(y-x^2)((n+m)(p+q)-lx))+a(((n+m)(p+q)-lx)((n+m)x-y(p+q))+((n+m)(p+q)-lx)(x(p+q)-n-m)+(ly-(n+m)^2)(x(p+q)-n-m))+(ly-(n+m)^2)((n+m)x-y(p+q))+(2bc+c+b)(y-x^2)((n+m)x-y(p+q)))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^4]$

$h_{59}^{(U_5)} = [s(a(2xy-x^2)-y^2)z]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^2]-[2s(y-x^2)z((ac+ab)(x((n+m)x-y(q+p))-y(x(q+p)-n-m)+(n+m)(y-x^2))+a((n+m)((n+m)x-y(q+p))+(n+m)(x(q+p)-n-m)-y((n+m)(q+p)-lx)+x((n+m)(q+p)-lx)+x(ly-(n+m)^2))-2(c+b)y((n+m)x-y(q+p))-y(ly-(n+m)^2)-(2bc+c+b)y(y-x^2))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]-[2sz(2(n+m)x-2y(q+p))(a(-x((n+m)x-y(q+p))+y(x(q+p)-n-m)-x(x(q+p)-n-m))+y((n+m)x-y(q+p))+(c+b)y(y-x^2)+(-ac-ab)x(y-x^2))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]+[6s(y-x^2)z(2(n+m)x-2y(q+p))((c+b)(((n+m)x-y(q+p))^2+(y-x^2)(ly-(n+m)^2))+(ac+ab)((x(q+p)-n-m)((n+m)x-y(q+p))+(y-x^2)((n+m)(q+p)-lx))+a(((n+m)(q+p)-lx)((n+m)x-y(q+p))+((n+m)(q+p)-lx)(x(q+p)-n-m)+(ly-(n+m)^2)(x(q+p)-n-m))+(ly-(n+m)^2)((n+m)x-y(q+p))+(2bc+c+b)(y-x^2)((n+m)x-y(q+p)))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^4]$

$h_{5\ 10}^{(U_5)} = [(a(-x((n+m)x-(q+p)y)+((q+p)x-n-m)y-x((q+p)x-n-m))+y((n+m)x-(q+p)y)+(c+b)y(y-x^2)+(-ac-ab)x(y-x^2))z]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^2]-[2(y-x^2)((c+b)(((n+m)x-(q+p)y)^2+(y-x^2)(ly-(n+m)^2))+(ac+ab)(((q+p)x-n-m)((n+m)x-(q+p)y)+((n+m)(q+p)-lx)(y-x^2))+a(((n+m)(q+p)-lx)((n+m)x-(q+p)y)+((q+p)x-n-m)(ly-(n+m)^2)+((n+m)(q+p)-lx)((q+p)x-n-m))+(ly-(n+m)^2)((n+m)x-(q+p)y)+(2bc+c+b)(y-x^2)((n+m)x-(q+p)y))z]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^3]$

$h_{66}^{(U_5)} = [sz(-2(x(m+n)-(q+p)y)+a(4(m+n)-2(-m+(q+p)x-n)+2(q+p)x-2q-2p)-4x(m+n)+(c+b)(2x^2-2(y-x^2))-2(ac+ab)x)]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^2]+[4sz((c+b)((x(m+n)-(q+p)y)^2+(y-x^2)(ly-(m+n)^2))+(x(m+n)-(q+p)y)(ly-(m+n)^2)+a((-m+(q+p)x-n)(ly-(m+n)^2)+((q+p)(m+n)-lx)(x(m+n)-(q+p)y)+(-m+(q+p)x-n)((q+p)(m+n)-lx))+(ac+ab)((-m+(q+p)x-n)(x(m+n)-(q+p)y)+(y-x^2)((q+p)(m+n)-lx))+(2bc+c+b)(y-x^2)(x(m+n)-(q+p)y))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-$

$lx^2)^3] + [6sz(2(q+p)x-2(m+n))^2((c+b)((x(m+n)-(q+p)y)^2+(y-x^2)(ly-(m+n)^2))+(x(m+n)-(q+p)y)(ly-(m+n)^2)+a((-m+(q+p)x-n)(ly-(m+n)^2)+((q+p)(m+n)-lx)(x(m+n)-(q+p)y)+(-m+(q+p)x-n)((q+p)(m+n)-lx))+(ac+ab)((-m+(q+p)x-n)(x(m+n)-(q+p)y)+(y-x^2)((q+p)(m+n)-lx))+(2bc+c+b)(y-x^2)(x(m+n)-(q+p)y))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^4]-[4sz(2(q+p)x-2(m+n))(a((m+n)^2+(q+p)(x(m+n)-(q+p)y)+x((q+p)(m+n)-lx)-2(-m+(q+p)x-n)(m+n)-(q+p)(m+n)+(q+p)(-m+(q+p)x-n)-ly+lx)+x(ly-(m+n)^2)+(c+b)(2x(x(m+n)-(q+p)y)-2(y-x^2)(m+n))-2(m+n)(x(m+n)-(q+p)y)+(ac+ab)(-x(m+n)+x(-m+(q+p)x-n)+(q+p)(y-x^2)+(q+p)y)+(2bc+c+b)x(y-x^2))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^3]$

$h_{67}^{(U_5)} = [sz(-2(x(n+m)-(q+p)y)+a(4(n+m)-2(-n+(q+p)x-m)+2(q+p)x-2q-2p)-4x(n+m)+(c+b)(2x^2-2(y-x^2))-2(ac+ab)x)]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^2]+[4sz((c+b)((x(n+m)-(q+p)y)^2+(y-x^2)(ly-(n+m)^2))+(x(n+m)-(q+p)y)(ly-(n+m)^2)+a((-n+(q+p)x-m)(ly-(n+m)^2)+((q+p)(n+m)-lx)(x(n+m)-(q+p)y)+(-n+(q+p)x-m)((q+p)(n+m)-lx))+(ac+ab)((-n+(q+p)x-m)(x(n+m)-(q+p)y)+(y-x^2)((q+p)(n+m)-lx))+(2bc+c+b)(y-x^2)(x(n+m)-(q+p)y))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]+[6sz(2(q+p)x-2(n+m))^2((c+b)((x(n+m)-(q+p)y)^2+(y-x^2)(ly-(n+m)^2))+(x(n+m)-(q+p)y)(ly-(n+m)^2)+a((-n+(q+p)x-m)(ly-(n+m)^2)+((q+p)(n+m)-lx)(x(n+m)-(q+p)y)+(-n+(q+p)x-m)((q+p)(n+m)-lx))+(ac+ab)((-n+(q+p)x-m)(x(n+m)-(q+p)y)+(y-x^2)((q+p)(n+m)-lx))+(2bc+c+b)(y-x^2)(x(n+m)-(q+p)y))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^4]-[2sz(2(q+p)x-2(n+m))(a((n+m)^2+(q+p)(x(n+m)-(q+p)y)+x((q+p)(n+m)-lx)-2(-n+(q+p)x-m)(n+m)-(q+p)(n+m)+(q+p)(-n+(q+p)x-m)-ly+lx)+x(ly-(n+m)^2)+(c+b)(2x(x(n+m)-(q+p)y)-2(y-x^2)(n+m))-2(n+m)(x(n+m)-(q+p)y)+(ac+ab)(-x(n+m)+x(-n+(q+p)x-m)+(q+p)(y-x^2)+(q+p)y)+(2bc+c+b)x(y-x^2))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]-[2sz(2(q+p)x-2(n+m))(a((n+m)^2+(q+p)(x(n+m)-(q+p)y)+x((q+p)(n+m)-lx)-2(-n+(q+p)x-m)(n+m)+(-q-p)(n+m)+(q+p)(-n+(q+p)x-m)-ly+lx)+x(ly-(n+m)^2)+(c+b)(2x(x(n+m)-(q+p)y)-2(y-x^2)(n+m))-2(n+$

$m)(x(n+m)-(q+p)y)+(ac+ab)(-x(n+m)+x(-n+(q+p)x-m)+(q+p)(y-x^2)+(q+p)y)+(2bc+c+b)x(y-x^2))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]$

$h_{68}^{(U_5)} = [sz(a(-2y(p+q)+2x(p+q)-2n-2m)-2(c+b)xy+2(n+m)y+2(ac+ab)y)]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^2]-[2sz(2(n+m)x-2y(p+q))(a((p+q)((n+m)x-y(p+q))+(p+q)(x(p+q)-n-m)-2(n+m)(x(p+q)-n-m)+x((n+m)(p+q)-lx)+(n+m)(-p-q)-ly+lx+(n+m)^2)+(c+b)(2x((n+m)x-y(p+q))-2(n+m)(y-x^2))+(ac+ab)(x(x(p+q)-n-m)+(y-x^2)(p+q)+y(p+q)-(n+m)x)-2(n+m)((n+m)x-y(p+q))+x(ly-(n+m)^2)+(2bc+c+b)x(y-x^2))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]-[2sz(2x(p+q)-2(n+m))((ac+ab)(x((n+m)x-y(p+q))-y(x(p+q)-n-m)+(n+m)(y-x^2))+a((n+m)((n+m)x-y(p+q))+(n+m)(x(p+q)-n-m)-y((n+m)(p+q)-lx)+x((n+m)(p+q)-lx)+x(ly-(n+m)^2))-2(c+b)y((n+m)x-y(p+q))-y(ly-(n+m)^2)-(2bc+c+b)y(y-x^2))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]-[4sxz((c+b)(((n+m)x-y(p+q))^2+(y-x^2)(ly-(n+m)^2))+(ac+ab)((x(p+q)-n-m)((n+m)x-y(p+q))+(y-x^2)((n+m)(p+q)-lx))+a(((n+m)(p+q)-lx)((n+m)x-y(p+q))+((n+m)(p+q)-lx)(x(p+q)-n-m)+(ly-(n+m)^2)(x(p+q)-n-m))+(ly-(n+m)^2)((n+m)x-y(p+q))+(2bc+c+b)(y-x^2)((n+m)x-y(p+q)))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]+[6sz(2x(p+q)-2(n+m))(2(n+m)x-2y(p+q))((c+b)(((n+m)x-y(p+q))^2+(y-x^2)(ly-(n+m)^2))+(ac+ab)((x(p+q)-n-m)((n+m)x-y(p+q))+(y-x^2)((n+m)(p+q)-lx))+a(((n+m)(p+q)-lx)((n+m)x-y(p+q))+((n+m)(p+q)-lx)(x(p+q)-n-m)+(ly-(n+m)^2)(x(p+q)-n-m))+(ly-(n+m)^2)((n+m)x-y(p+q))+(2bc+c+b)(y-x^2)((n+m)x-y(p+q)))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^4]$

$h_{69}^{(U_5)} = [sz(a(-2y(q+p)+2x(q+p)-2n-2m)-2(c+b)xy+2(n+m)y+2(ac+ab)y)]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^2]-[2sz(2(n+m)x-2y(q+p))(a((q+p)((n+m)x-y(q+p))+(q+p)(x(q+p)-n-m)-2(n+m)(x(q+p)-n-m)+x((n+m)(q+p)-lx)+(n+m)(-q-p)-ly+lx+(n+m)^2)+(c+b)(2x((n+m)x-y(q+p))-2(n+m)(y-x^2))+(ac+ab)(x(x(q+p)-n-m)+(y-x^2)(q+p)+y(q+p)-(n+m)x)-2(n+m)((n+m)x-y(q+p))+x(ly-(n+m)^2)+(2bc+c+b)x(y-x^2))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]-[2sz(2x(q+p)-2(n+m))((ac+ab)(x((n+m)x-y(q+p))-y(x(q+p)-n-m)+(n+m)(y-x^2))+a((n+m)((n+m)x-y(q+p))+$

$(n+m)(x(q+p)-n-m)-y((n+m)(q+p)-lx)+x((n+m)(q+p)-lx)+x(ly-(n+m)^2))-2(c+b)y((n+m)x-y(q+p))-y(ly-(n+m)^2)-(2bc+c+b)y(y-x^2))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]-[4sxz((c+b)(((n+m)x-y(q+p))^2+(y-x^2)(ly-(n+m)^2))+(ac+ab)((x(q+p)-n-m)((n+m)x-y(q+p))+(y-x^2)((n+m)(q+p)-lx))+a(((n+m)(q+p)-lx)((n+m)x-y(q+p))+((n+m)(q+p)-lx)(x(q+p)-n-m)+(ly-(n+m)^2)(x(q+p)-n-m))+(ly-(n+m)^2)((n+m)x-y(q+p))+(2bc+c+b)(y-x^2)((n+m)x-y(q+p)))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]+[6sz(2x(q+p)-2(n+m))(2(n+m)x-2y(q+p))((c+b)(((n+m)x-y(q+p))^2+(y-x^2)(ly-(n+m)^2))+(ac+ab)((x(q+p)-n-m)((n+m)x-y(q+p))+(y-x^2)((n+m)(q+p)-lx))+a(((n+m)(q+p)-lx)((n+m)x-y(q+p))+((n+m)(q+p)-lx)(x(q+p)-n-m)+(ly-(n+m)^2)(x(q+p)-n-m))+(ly-(n+m)^2)((n+m)x-y(q+p))+(2bc+c+b)(y-x^2)((n+m)x-y(q+p)))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^4]$

$h_{6\ 10}^{(U_5)} = [((c+b)(2x((n+m)x-(q+p)y)-2(n+m)(y-x^2))+a((q+p)((n+m)x-(q+p)y)-ly+(q+p)((q+p)x-n-m)-2(n+m)((q+p)x-n-m)+x((n+m)(q+p)-lx)+lx+(n+m)(-q-p)+(n+m)^2)+(ac+ab)((q+p)(y-x^2)+(q+p)y+x((q+p)x-n-m)-(n+m)x)-2(n+m)((n+m)x-(q+p)y)+x(ly-(n+m)^2)+(2bc+c+b)x(y-x^2))z]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^2]-[2(2(q+p)x-2(n+m))((c+b)(((n+m)x-(q+p)y)^2+(y-x^2)(ly-(n+m)^2))+(ac+ab)((((q+p)x-n-m)((n+m)x-(q+p)y)+((n+m)(q+p)-lx)(y-x^2))+a((((n+m)(q+p)-lx)((n+m)x-(q+p)y)+(((q+p)x-n-m)(ly-(n+m)^2)+((n+m)(q+p)-lx)(((q+p)x-n-m))+(ly-(n+m)^2)((n+m)x-(q+p)y)+(2bc+c+b)(y-x^2)((n+m)x-(q+p)y))z]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^3]$

$h_{77}^{(U_5)} = [sz(-2(x(n+m)-(q+p)y)+a(4(n+m)-2(-n+(q+p)x-m)+2(q+p)x-2q-2p)-4x(n+m)+(c+b)(2x^2-2(y-x^2))-2(ac+ab)x)]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^2]+[4sz((c+b)((x(n+m)-(q+p)y)^2+(y-x^2)(ly-(n+m)^2))+(x(n+m)-(q+p)y)(ly-(n+m)^2)+a((-n+(q+p)x-m)(ly-(n+m)^2)+((q+p)(n+m)-lx)(x(n+m)-(q+p)y)+(-n+(q+p)x-m)((q+p)(n+m)-lx))+(ac+ab)((-n+(q+p)x-m)(x(n+m)-(q+p)y)+(y-x^2)((q+p)(n+m)-lx))+(2bc+c+b)(y-x^2)(x(n+m)-(q+p)y))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]+[6sz(2(q+p)x-2(n+m))^2((c+b)((x(n+m)-(q+p)y)^2+(y-x^2)(ly-(n+$

$m)^2))+(x(n+m)-(q+p)y)(ly-(n+m)^2)+a((-n+(q+p)x-m)(ly-(n+m)^2)+((q+$

$p)(n+m)-lx)(x(n+m)-(q+p)y)+(-n+(q+p)x-m)((q+p)(n+m)-lx))+(ac+$

$ab)((-n+(q+p)x-m)(x(n+m)-(q+p)y)+(y-x^2)((q+p)(n+m)-lx))+(2bc+c+$

$b)(y-x^2)(x(n+m)-(q+p)y))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^4]-$

$[4sz(2(q+p)x-2(n+m))(a((n+m)^2+(q+p)(x(n+m)-(q+p)y)+x((q+p)(n+$

$m)-lx)-2(-n+(q+p)x-m)(n+m)-(q+p)(n+m)+(q+p)(-n+(q+p)x-m)-$

$ly+lx)+x(ly-(n+m)^2)+(c+b)(2x(x(n+m)-(q+p)y)-2(y-x^2)(n+m))-2(n+$

$m)(x(n+m)-(q+p)y)+(ac+ab)(-x(n+m)+x(-n+(q+p)x-m)+(q+p)(y-x^2)+$

$(q+p)y)+(2bc+c+b)x(y-x^2))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]$

$h_{78}^{(U_5)} = [sz(a(-2y(p+q)+2x(p+q)-2n-2m)-2(c+b)xy+2(n+m)y+2(ac+$

$ab)y)]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^2]-[2sz(2(n+m)x-2y(p+$

$q))(a((p+q)((n+m)x-y(p+q))+(p+q)(x(p+q)-n-m)-2(n+m)(x(p+q)-n-m)+$

$x((n+m)(p+q)-lx)+(n+m)(-p-q)-ly+lx+(n+m)^2)+(c+b)(2x((n+m)x-y(p+$

$q))-2(n+m)(y-x^2))+(ac+ab)(x(x(p+q)-n-m)+(y-x^2)(p+q)+y(p+q)-(n+$

$m)x)-2(n+m)((n+m)x-y(p+q))+x(ly-(n+m)^2)+(2bc+c+b)x(y-x^2))]/[(-y(p+$

$q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]-[2sz(2x(p+q)-2(n+m))((ac+ab)(x((n+$

$m)x-y(p+q))-y(x(p+q)-n-m)+(n+m)(y-x^2))+a((n+m)((n+m)x-y(p+q))+$

$(n+m)(x(p+q)-n-m)-y((n+m)(p+q)-lx)+x((n+m)(p+q)-lx)+x(ly-(n+$

$m)^2))-2(c+b)y((n+m)x-y(p+q))-y(ly-(n+m)^2)-(2bc+c+b)y(y-x^2))]/[(-y(p+$

$q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]-[4sxz((c+b)(((n+m)x-y(p+q))^2+(y-$

$x^2)(ly-(n+m)^2))+(ac+ab)((x(p+q)-n-m)((n+m)x-y(p+q))+(y-x^2)((n+m)(p+$

$q)-lx))+a(((n+m)(p+q)-lx)((n+m)x-y(p+q))+((n+m)(p+q)-lx)(x(p+q)-n-$

$m)+(ly-(n+m)^2)(x(p+q)-n-m))+(ly-(n+m)^2)((n+m)x-y(p+q))+(2bc+c+$

$b)(y-x^2)((n+m)x-y(p+q)))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]+$

$[6sz(2x(p+q)-2(n+m))(2(n+m)x-2y(p+q))((c+b)(((n+m)x-y(p+q))^2+(y-$

$x^2)(ly-(n+m)^2))+(ac+ab)((x(p+q)-n-m)((n+m)x-y(p+q))+(y-x^2)((n+m)(p+$

$q)-lx))+a(((n+m)(p+q)-lx)((n+m)x-y(p+q))+((n+m)(p+q)-lx)(x(p+q)-n-$

$m)+(ly-(n+m)^2)(x(p+q)-n-m))+(ly-(n+m)^2)((n+m)x-y(p+q))+(2bc+c+$

$b)(y-x^2)((n+m)x-y(p+q)))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^4]$

$h_{79}^{(U_5)} = [sz(a(-2y(q+p)+2x(q+p)-2n-2m)-2(c+b)xy+2(n+m)y+2(ac+$

$ab)y)]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^2]-[2sz(2(n+m)x-2y(q+p))(a((q+p)((n+m)x-y(q+p))+(q+p)(x(q+p)-n-m)-2(n+m)(x(q+p)-n-m)+x((n+m)(q+p)-lx)+(n+m)(-q-p)-ly+lx+(n+m)^2)+(c+b)(2x((n+m)x-y(q+p))-2(n+m)(y-x^2))+(ac+ab)(x(x(q+p)-n-m)+(y-x^2)(q+p)+y(q+p)-(n+m)x)-2(n+m)((n+m)x-y(q+p))+x(ly-(n+m)^2)+(2bc+c+b)x(y-x^2))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]-[2sz(2x(q+p)-2(n+m))((ac+ab)(x((n+m)x-y(q+p))-y(x(q+p)-n-m)+(n+m)(y-x^2))+a((n+m)((n+m)x-y(q+p))+(n+m)(x(q+p)-n-m)-y((n+m)(q+p)-lx)+x((n+m)(q+p)-lx)+x(ly-(n+m)^2))-2(c+b)y((n+m)x-y(q+p))-y(ly-(n+m)^2)-(2bc+c+b)y(y-x^2))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]-[4sxz((c+b)(((n+m)x-y(q+p))^2+(y-x^2)(ly-(n+m)^2))+(ac+ab)((x(q+p)-n-m)((n+m)x-y(q+p))+(y-x^2)((n+m)(q+p)-lx))+a(((n+m)(q+p)-lx)((n+m)x-y(q+p))+((n+m)(q+p)-lx)(x(q+p)-n-m)+(ly-(n+m)^2)(x(q+p)-n-m))+(ly-(n+m)^2)((n+m)x-y(q+p))+(2bc+c+b)(y-x^2)((n+m)x-y(q+p)))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]+[6sz(2x(q+p)-2(n+m))(2(n+m)x-2y(q+p))((c+b)(((n+m)x-y(q+p))^2+(y-x^2)(ly-(n+m)^2))+(ac+ab)((x(q+p)-n-m)((n+m)x-y(q+p))+(y-x^2)((n+m)(q+p)-lx))+a(((n+m)(q+p)-lx)((n+m)x-y(q+p))+((n+m)(q+p)-lx)(x(q+p)-n-m)+(ly-(n+m)^2)(x(q+p)-n-m))+(ly-(n+m)^2)((n+m)x-y(q+p))+(2bc+c+b)(y-x^2)((n+m)x-y(q+p)))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^4]$

$h_{7\ 10}^{(U_5)}=[((c+b)(2x((n+m)x-(q+p)y)-2(n+m)(y-x^2))+a((q+p)((n+m)x-(q+p)y)-ly+(q+p)((q+p)x-n-m)-2(n+m)((q+p)x-n-m)+x((n+m)(q+p)-lx)+lx+(n+m)(-q-p)+(n+m)^2)+(ac+ab)((q+p)(y-x^2)+(q+p)y+x((q+p)x-n-m)-(n+m)x)-2(n+m)((n+m)x-(q+p)y)+x(ly-(n+m)^2)+(2bc+c+b)x(y-x^2))z]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^2]-[2(2(q+p)x-2(n+m))((c+b)(((n+m)x-(q+p)y)^2+(y-x^2)(ly-(n+m)^2))+(ac+ab)(((q+p)x-n-m)((n+m)x-(q+p)y)+((n+m)(q+p)-lx)(y-x^2))+a(((n+m)(q+p)-lx)((n+m)x-(q+p)y)+((q+p)x-n-m)(ly-(n+m)^2)+((n+m)(q+p)-lx)((q+p)x-n-m))+(ly-(n+m)^2)((n+m)x-(q+p)y)+(2bc+c+b)(y-x^2)((n+m)x-(q+p)y))z]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^3]$

$h_{88}^{(U_5)}=[s(2(c+b)y^2+a(2(n+m)x-2(n+m)y)-2(ac+ab)xy)z]/[(-y(p+q)^2+$

$2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^2]-[4sz(2(n+m)x-2y(p+q))((ac+ab)(x((n+$
$m)x-y(p+q))-y(x(p+q)-n-m)+(n+m)(y-x^2))+a((n+m)((n+m)x-y(p+q))+$
$(n+m)(x(p+q)-n-m)-y((n+m)(p+q)-lx)+x((n+m)(p+q)-lx)+x(ly-(n+$
$m)^2))-2(c+b)y((n+m)x-y(p+q))-y(ly-(n+m)^2)-(2bc+c+b)y(y-x^2))]/[(-y(p+$
$q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]+[4syz((c+b)(((n+m)x-y(p+q))^2+(y-$
$x^2)(ly-(n+m)^2))+(ac+ab)((x(p+q)-n-m)((n+m)x-y(p+q))+(y-x^2)((n+$
$m)(p+q)-lx))+a(((n+m)(p+q)-lx)((n+m)x-y(p+q))+((n+m)(p+q)-lx)(x(p+$
$q)-n-m)+(ly-(n+m)^2)(x(p+q)-n-m))+(ly-(n+m)^2)((n+m)x-y(p+q))+$
$(2bc+c+b)(y-x^2)((n+m)x-y(p+q)))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-$
$(n+m)^2)^3]+[6sz(2(n+m)x-2y(p+q))^2((c+b)(((n+m)x-y(p+q))^2+(y-x^2)(ly-$
$(n+m)^2))+(ac+ab)((x(p+q)-n-m)((n+m)x-y(p+q))+(y-x^2)((n+m)(p+q)-$
$lx))+a(((n+m)(p+q)-lx)((n+m)x-y(p+q))+((n+m)(p+q)-lx)(x(p+q)-n-$
$m)+(ly-(n+m)^2)(x(p+q)-n-m))+(ly-(n+m)^2)((n+m)x-y(p+q))+(2bc+c+$
$b)(y-x^2)((n+m)x-y(p+q)))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^4]$

$h_{89}^{(U_5)} = [s(2(c+b)y^2+a(2(n+m)x-2(n+m)y)-2(ac+ab)xy)z]/[(-y(q+p)^2+$
$2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^2]-[2sz(2(n+m)x-2y(q+p))((ac+ab)(x((n+$
$m)x-y(q+p))-y(x(q+p)-n-m)+(n+m)(y-x^2))+a((n+m)((n+m)x-y(q+p))+$
$(n+m)(x(q+p)-n-m)-y((n+m)(q+p)-lx)+x((n+m)(q+p)-lx)+x(ly-(n+$
$m)^2))-2(c+b)y((n+m)x-y(q+p))-y(ly-(n+m)^2)-(2bc+c+b)y(y-x^2))]/[(-y(q+$
$p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]-[2sz(2(n+m)x-2y(q+p))((ac+$
$ab)(x((n+m)x-y(q+p))-y(x(q+p)-n-m)+(n+m)(y-x^2))+a((n+m)((n+m)x-$
$y(q+p))+(n+m)(x(q+p)-n-m)-y((n+m)(q+p)-lx)+x((n+m)(q+p)-lx)+$
$x(ly-(n+m)^2))-2(c+b)y((n+m)x-y(q+p))-y(ly-(n+m)^2)+(-2bc-c-b)y(y-$
$x^2))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]+[4syz((c+b)(((n+m)x-$
$y(q+p))^2+(y-x^2)(ly-(n+m)^2))+(ac+ab)((x(q+p)-n-m)((n+m)x-y(q+p))+$
$(y-x^2)((n+m)(q+p)-lx))+a(((n+m)(q+p)-lx)((n+m)x-y(q+p))+((n+m)(q+$
$p)-lx)(x(q+p)-n-m)+(ly-(n+m)^2)(x(q+p)-n-m))+(ly-(n+m)^2)((n+m)x-$
$y(q+p))+(2bc+c+b)(y-x^2)((n+m)x-y(q+p)))]/[(-y(q+p)^2+2(n+m)x(q+p)+$
$ly-lx^2-(n+m)^2)^3]+[6sz(2(n+m)x-2y(q+p))^2((c+b)(((n+m)x-y(q+p))^2+(y-$
$x^2)(ly-(n+m)^2))+(ac+ab)((x(q+p)-n-m)((n+m)x-y(q+p))+(y-x^2)((n+m)(q+$

$p)-lx))+a(((n+m)(q+p)-lx)((n+m)x-y(q+p))+((n+m)(q+p)-lx)(x(q+p)-n-m)+(ly-(n+m)^2)(x(q+p)-n-m))+(ly-(n+m)^2)((n+m)x-y(q+p))+(2bc+c+b)(y-x^2)((n+m)x-y(q+p))))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^4]$

$h_{8\ 10}^{(U_5)} = [((ac+ab)(x((n+m)x-(q+p)y)+(n+m)(y-x^2)-((q+p)x-n-m)y)+a((n+m)((n+m)x-(q+p)y)+x(ly-(n+m)^2)-((n+m)(q+p)-lx)y+(n+m)((q+p)x-n-m)+x((n+m)(q+p)-lx))-2(c+b)y((n+m)x-(q+p)y)-y(ly-(n+m)^2)+(-2bc-c-b)y(y-x^2))z]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^2]-[2(2(n+m)x-2(q+p)y)((c+b)(((n+m)x-(q+p)y)^2+(y-x^2)(ly-(n+m)^2))+(ac+ab)(((q+p)x-n-m)((n+m)x-(q+p)y)+((n+m)(q+p)-lx)(y-x^2))+a(((n+m)(q+p)-lx)((n+m)x-(q+p)y)+((q+p)x-n-m)(ly-(n+m)^2)+((n+m)(q+p)-lx)((q+p)x-n-m))+(ly-(n+m)^2)((n+m)x-(q+p)y)+(2bc+c+b)(y-x^2)((n+m)x-(q+p)y))z]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^3]$

$h_{99}^{(U_5)} = [s(2(c+b)y^2+a(2(n+m)x-2(n+m)y)-2(ac+ab)xy)z]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^2]-[4sz(2(n+m)x-2y(q+p))((ac+ab)(x((n+m)x-y(q+p))-y(x(q+p)-n-m)+(n+m)(y-x^2))+a((n+m)((n+m)x-y(q+p))+(n+m)(x(q+p)-n-m)-y((n+m)(q+p)-lx)+x((n+m)(q+p)-lx)+x(ly-(n+m)^2))-2(c+b)y((n+m)x-y(q+p))-y(ly-(n+m)^2)-(2bc+c+b)y(y-x^2))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]+[4syz((c+b)(((n+m)x-y(q+p))^2+(y-x^2)(ly-(n+m)^2))+(ac+ab)((x(q+p)-n-m)((n+m)x-y(q+p))+(y-x^2)((n+m)(q+p)-lx))+a(((n+m)(q+p)-lx)((n+m)x-y(q+p))+((n+m)(q+p)-lx)(x(q+p)-n-m)+(ly-(n+m)^2)(x(q+p)-n-m))+(ly-(n+m)^2)((n+m)x-y(q+p))+(2bc+c+b)(y-x^2)((n+m)x-y(q+p))))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]+[6sz(2(n+m)x-2y(q+p))^2((c+b)(((n+m)x-y(q+p))^2+(y-x^2)(ly-(n+m)^2))+(ac+ab)((x(q+p)-n-m)((n+m)x-y(q+p))+(y-x^2)((n+m)(q+p)-lx))+a(((n+m)(q+p)-lx)((n+m)x-y(q+p))+((n+m)(q+p)-lx)(x(q+p)-n-m)+(ly-(n+m)^2)(x(q+p)-n-m))+(ly-(n+m)^2)((n+m)x-y(q+p))+(2bc+c+b)(y-x^2)((n+m)x-y(q+p))))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^4]$

$h_{9\ 10}^{(U_5)} = [((ac+ab)(x((n+m)x-(q+p)y)+(n+m)(y-x^2)-((q+p)x-n-m)y)+a((n+m)((n+m)x-(q+p)y)+x(ly-(n+m)^2)-((n+m)(q+p)-lx)y+(n+m)((q+p)x-n-m)+x((n+m)(q+p)-lx))-2(c+b)y((n+m)x-(q+p)y)-y(ly-$

$(n+m)^2)+(-2bc-c-b)y(y-x^2))z]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^2]-[2(2(n+m)x-2(q+p)y)((c+b)(((n+m)x-(q+p)y)^2+(y-x^2)(ly-(n+m)^2))+(ac+ab)((((q+p)x-n-m)((n+m)x-(q+p)y)+((n+m)(q+p)-lx)(y-x^2))+a((((n+m)(q+p)-lx)((n+m)x-(q+p)y)+(((q+p)x-n-m)(ly-(n+m)^2)+((n+m)(q+p)-lx)(((q+p)x-n-m))+(ly-(n+m)^2)((n+m)x-(q+p)y)+(2bc+c+b)(y-x^2)((n+m)x-(q+p)y))z]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^3]$

$h^{(U_5)}_{10\ 10} = 0$

$h^{(U_6)}_{11} = [st((c+b)(-2((n+m)(q+p)-lx)+4lx+2(n+m)(q+p))-2(2bc+c+b)((q+p)x-n-m)-4(2bc+c+b)(q+p)x+(ac+ab)(2(q+p)^2-2(l-(q+p)^2))-2al(q+p)-2l(n+m))]/[(-lx^2+2(n+m)(q+p)x-(q+p)^2y+ly-(n+m)^2)^2]-[4st(2(n+m)(q+p)-2lx)((2bc+c+b)(q+p)(y-x^2)+(c+b)(-l(y-x^2)+(n+m)((q+p)x-n-m)+(q+p)((n+m)x-(q+p)y)-2x((n+m)(q+p)-lx))+(ac+ab)(2(q+p)((q+p)x-n-m)-2(l-(q+p)^2)x)+a(-l((q+p)x-n-m)+(q+p)((n+m)(q+p)-lx)+(q+p)(l-(q+p)^2)+(n+m)(l-(q+p)^2))-2(2bc+c+b)x((q+p)x-n-m)-l((n+m)x-(q+p)y)+(n+m)((n+m)(q+p)-lx))]/[(-lx^2+2(n+m)(q+p)x-(q+p)^2y+ly-(n+m)^2)^3]+[4lst((ac+ab)(((q+p)x-n-m)^2+(l-(q+p)^2)(y-x^2))+(2bc+c+b)((q+p)x-n-m)(y-x^2)+(c+b)(((n+m)(q+p)-lx)(y-x^2)+((n+m)x-(q+p)y)((q+p)x-n-m))+a((((n+m)(q+p)-lx)((q+p)x-n-m)+(l-(q+p)^2)((q+p)x-n-m)+(l-(q+p)^2)((n+m)x-(q+p)y))+((n+m)(q+p)-lx)((n+m)x-(q+p)y)))]/[(-lx^2+2(n+m)(q+p)x-(q+p)^2y+ly-(n+m)^2)^3]+[6st(2(n+m)(q+p)-2lx)^2((ac+ab)(((q+p)x-n-m)^2+(l-(q+p)^2)(y-x^2))+(2bc+c+b)((q+p)x-n-m)(y-x^2)+(c+b)(((n+m)(q+p)-lx)(y-x^2)+((n+m)x-(q+p)y)((q+p)x-n-m))+a((((n+m)(q+p)-lx)((q+p)x-n-m)+(l-(q+p)^2)((q+p)x-n-m)+(l-(q+p)^2)((n+m)x-(q+p)y))+((n+m)(q+p)-lx)((n+m)x-(q+p)y)))]/[(-lx^2+2(n+m)(q+p)x-(q+p)^2y+ly-(n+m)^2)^4]$

$h^{(U_6)}_{12} = [((c+b)((-q-p)(q+p)-l)+(2bc+c+b)(q+p)-l(-q-p))st]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^2]-[2st(2(n+m)(q+p)-2lx)((c+b)((-q-p)(((q+p)x-n-m)-lx+(n+m)(q+p))+(2bc+c+b)((q+p)x-n-m)+(-q-p)((n+m)(q+p)-lx)+a(-q-p)(l-(q+p)^2)+(ac+ab)(l-(q+p)^2))]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^3]+[6(l-(q+p)^2)st(2(n+m)(q+p)-2lx)((c+b)((((q+p)x-n-m)((n+m)x-(q+p)y)+((n+m)(q+p)-lx)(y-x^2))+a((l-(q+p)^2)((n+m)x-(q+$

$p)y)+((n+m)(q+p)-lx)((q+p)x-n-m)+(l-(q+p)^2)((q+p)x-n-m))+(ac+ab)((l-(q+p)^2)(y-x^2)+((q+p)x-n-m)^2)+((n+m)(q+p)-lx)((n+m)x-(q+p)y)+(2bc+c+b)((q+p)x-n-m)(y-x^2))]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^4]-[2(l-(q+p)^2)st((c+b)((q+p)((n+m)x-(q+p)y)-l(y-x^2)+(n+m)((q+p)x-n-m)-2x((n+m)(q+p)-lx))-l((n+m)x-(q+p)y)+(2bc+c+b)(q+p)(y-x^2)+(ac+ab)(2(q+p)((q+p)x-n-m)-2(l-(q+p)^2)x)+a(-l((q+p)x-n-m)+(q+p)((n+m)(q+p)-lx)+(q+p)(l-(q+p)^2)+(n+m)(l-(q+p)^2))-2(2bc+c+b)x((q+p)x-n-m)+(n+m)((n+m)(q+p)-lx))]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^3]$

$h_{13}^{(U_6)} = 0$

$h_{14}^{(U_6)} = [s((c+b)((q+p)((n+m)x-(q+p)y)-l(y-x^2)+(n+m)((q+p)x-n-m)-2x((n+m)(q+p)-lx))-l((n+m)x-(q+p)y)+(2bc+c+b)(q+p)(y-x^2)+(ac+ab)(2(q+p)((q+p)x-n-m)-2(l-(q+p)^2)x)+a(-l((q+p)x-n-m)+(q+p)((n+m)(q+p)-lx)+(q+p)(l-(q+p)^2)+(n+m)(l-(q+p)^2))-2(2bc+c+b)x((q+p)x-n-m)+(n+m)((n+m)(q+p)-lx))]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^2]-[2s(2(n+m)(q+p)-2lx)((c+b)(((q+p)x-n-m)((n+m)x-(q+p)y)+((n+m)(q+p)-lx)(y-x^2))+a((l-(q+p)^2)((n+m)x-(q+p)y)+((n+m)(q+p)-lx)((q+p)x-n-m)+(l-(q+p)^2)((q+p)x-n-m))+(ac+ab)((l-(q+p)^2)(y-x^2)+((q+p)x-n-m)^2)+((n+m)(q+p)-lx)((n+m)x-(q+p)y)+(2bc+c+b)((q+p)x-n-m)(y-x^2))]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^3]$

$h_{15}^{(U_6)} = [st((q+p)y+(c+b)(3x^2-y)+a(-2(q+p)x+q+p+2n+2m)-2(n+m)x-2(ac+ab)x)]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^2]-[2st(a(-(q+p)y-x((q+p)x-n-m)+(q+p)x+(n+m)x-n-m)-x((n+m)x-(q+p)y)-(c+b)x(y-x^2)+(ac+ab)(y-x^2))(2(n+m)(q+p)-2xl)]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^3]+[4stx((c+b)((y-x^2)((n+m)(q+p)-xl)+((q+p)x-n-m)((n+m)x-(q+p)y))+a(((q+p)x-n-m)((n+m)(q+p)-xl)+((n+m)x-(q+p)y)(l-(q+p)^2)+((q+p)x-n-m)(l-(q+p)^2))+(ac+ab)((y-x^2)(l-(q+p)^2)+((q+p)x-n-m)^2)+((n+m)x-(q+p)y)((n+m)(q+p)-xl)+(2bc+c+b)((q+p)x-n-m)(y-x^2))]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^3]+[6st(y-x^2)(2(n+m)(q+p)-2xl)((c+b)((y-x^2)((n+m)(q+p)-xl)+((q+p)x-n-m)((n+m)x-(q+p)y))+a(((q+p)x-n-m)((n+m)(q+p)-xl)+((n+m)x-(q+p)y)(l-(q+p)^2)+((q+p)x-n-m)(l-(q+p)^2))+(ac+$

$ab)((y-x^2)(l-(q+p)^2)+((q+p)x-n-m)^2)+((n+m)x-(q+p)y)((n+m)(q+p)-xl)+(2bc+c+b)((q+p)x-n-m)(y-x^2))]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^4]-[2st(y-x^2)((c+b)(-2x((n+m)(q+p)-xl)-(y-x^2)l+(q+p)((n+m)x-(q+p)y)+(n+m)((q+p)x-n-m))+a((q+p)((n+m)(q+p)-xl)+(q+p)(l-(q+p)^2)+(n+m)(l-(q+p)^2)-((q+p)x-n-m)l)+(ac+ab)(2(q+p)((q+p)x-n-m)-2x(l-(q+p)^2))+(n+m)((n+m)(q+p)-xl)-((n+m)x-(q+p)y)l+(2bc+c+b)(q+p)(y-x^2)-2(2bc+c+b)x((q+p)x-n-m))]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^3]$

$h_{16}^{(U_6)}=[st(2(q+p)(m+n)+(c+b)(-2m-2n)-2lx+2(2bc+c+b)x-2(ac+ab)(q+p)+2al)]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^2]-[4(q+p)st((c+b)((-m+(q+p)x-n)(x(m+n)-(q+p)y)+(y-x^2)((q+p)(m+n)-lx))+a((l-(q+p)^2)(x(m+n)-(q+p)y)+(-m+(q+p)x-n)((q+p)(m+n)-lx)+(l-(q+p)^2)(-m+(q+p)x-n))+((q+p)(m+n)-lx)(x(m+n)-(q+p)y)+(2bc+c+b)(y-x^2)(-m+(q+p)x-n)+(ac+ab)((-m+(q+p)x-n)^2+(l-(q+p)^2)(y-x^2))))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^3]-[2st(2(q+p)x-2(m+n))((c+b)((q+p)(x(m+n)-(q+p)y)-2x((q+p)(m+n)-lx)+(-m+(q+p)x-n)(m+n)-l(y-x^2))+a((q+p)((q+p)(m+n)-lx)+(l-(q+p)^2)(m+n)-l(-m+(q+p)x-n)+(q+p)(l-(q+p)^2))-l(x(m+n)-(q+p)y)+(m+n)((q+p)(m+n)-lx)-2(2bc+c+b)x(-m+(q+p)x-n)+(ac+ab)(2(q+p)(-m+(q+p)x-n)-2(l-(q+p)^2)x)+(2bc+c+b)(q+p)(y-x^2))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^3]-[2st(2(q+p)(m+n)-2lx)((q+p)(x(m+n)-(q+p)y)+(c+b)(-x(m+n)+x(-m+(q+p)x-n)+(q+p)(y-x^2)+(q+p)y)+x((q+p)(m+n)-lx)+a(-(q+p)(m+n)+(q+p)(-m+(q+p)x-n)+(l-(q+p)^2)x+lx+(q+p)^2-l)-2(ac+ab)(-m+(q+p)x-n)-(2bc+c+b)(y-x^2))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^3]+[6st(2(q+p)x-2(m+n))(2(q+p)(m+n)-2lx)((c+b)((-m+(q+p)x-n)(x(m+n)-(q+p)y)+(y-x^2)((q+p)(m+n)-lx))+a((l-(q+p)^2)(x(m+n)-(q+p)y)+(-m+(q+p)x-n)((q+p)(m+n)-lx)+(l-(q+p)^2)(-m+(q+p)x-n))+((q+p)(m+n)-lx)(x(m+n)-(q+p)y)+(2bc+c+b)(y-x^2)(-m+(q+p)x-n)+(ac+ab)((-m+(q+p)x-n)^2+(l-(q+p)^2)(y-x^2))))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^4]$

$h_{17}^{(U_6)}=[st(2(q+p)(n+m)+(c+b)(-2n-2m)-2lx+2(2bc+c+b)x-2(ac+ab)(q+p)+2al)]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^2]-[4(q+p)st((c+$

$b)((-n+(q+p)x-m)(x(n+m)-(q+p)y)+(y-x^2)((q+p)(n+m)-lx))+a((l-(q+p)^2)(x(n+m)-(q+p)y)+(-n+(q+p)x-m)((q+p)(n+m)-lx)+(l-(q+p)^2)(-n+(q+p)x-m))+((q+p)(n+m)-lx)(x(n+m)-(q+p)y)+(2bc+c+b)(y-x^2)(-n+(q+p)x-m)+(ac+ab)((-n+(q+p)x-m)^2+(l-(q+p)^2)(y-x^2)))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]-[2st(2(q+p)x-2(n+m))((c+b)((q+p)(x(n+m)-(q+p)y)-2x((q+p)(n+m)-lx)+(-n+(q+p)x-m)(n+m)-l(y-x^2))+a((q+p)((q+p)(n+m)-lx)+(l-(q+p)^2)(n+m)-l(-n+(q+p)x-m)+(q+p)(l-(q+p)^2))-l(x(n+m)-(q+p)y)+(n+m)((q+p)(n+m)-lx)-2(2bc+c+b)x(-n+(q+p)x-m)+(ac+ab)(2(q+p)(-n+(q+p)x-m)-2(l-(q+p)^2)x)+(2bc+c+b)(q+p)(y-x^2))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]-[2st(2(q+p)(n+m)-2lx)((q+p)(x(n+m)-(q+p)y)+(c+b)(-x(n+m)+x(-n+(q+p)x-m)+(q+p)(y-x^2)+(q+p)y)+x((q+p)(n+m)-lx)+a(-(q+p)(n+m)+(q+p)(-n+(q+p)x-m)+(l-(q+p)^2)x+lx+(q+p)^2-l)-2(ac+ab)(-n+(q+p)x-m)-(2bc+c+b)(y-x^2))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]+[6st(2(q+p)x-2(n+m))(2(q+p)(n+m)-2lx)((c+b)((-n+(q+p)x-m)(x(n+m)-(q+p)y)+(y-x^2)((q+p)(n+m)-lx))+a((l-(q+p)^2)(x(n+m)-(q+p)y)+(-n+(q+p)x-m)((q+p)(n+m)-lx)+(l-(q+p)^2)(-n+(q+p)x-m))+((q+p)(n+m)-lx)(x(n+m)-(q+p)y)+(2bc+c+b)(y-x^2)(-n+(q+p)x-m)+(ac+ab)((-n+(q+p)x-m)^2+(l-(q+p)^2)(y-x^2)))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^4]$

$h_{18}^{(U_6)}=[st(a(-3(p+q)^2-2lx+l)+(ac+ab)(2(x(p+q)-n-m)+6x(p+q))-2(c+b)y(p+q)+(2bc+c+b)(y-x^2)+ly-2(2bc+c+b)x^2+(n+m)^2)]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^2]-[2st(2(n+m)x-2y(p+q))(a((p+q)(l-(p+q)^2)+(n+m)(l-(p+q)^2)-l(x(p+q)-n-m)+(p+q)((n+m)(p+q)-lx))+(ac+ab)(2(p+q)(x(p+q)-n-m)-2x(l-(p+q)^2))+(c+b)((p+q)((n+m)x-y(p+q))+(n+m)(x(p+q)-n-m)-2x((n+m)(p+q)-lx)-l(y-x^2))-l((n+m)x-y(p+q))-2(2bc+c+b)x(x(p+q)-n-m)+(n+m)((n+m)(p+q)-lx)+(2bc+c+b)(y-x^2)(p+q))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]-[2st(2(n+m)(p+q)-2lx)(a(-y(l-(p+q)^2)+x(l-(p+q)^2)-2(p+q)((n+m)x-y(p+q))-2(p+q)(x(p+q)-n-m)+(n+m)(x(p+q)-n-m)+x((n+m)(p+q)-lx))+(c+b)(x((n+m)x-y(p+q))-y(x(p+q)-n-m)+(n+m)(y-x^2))+(ac+ab)(2x(x(p+q)-n-m)-2(y-x^2)(p+q))+(n+$

$m)((n+m)x-y(p+q))-y((n+m)(p+q)-lx)+(2bc+c+b)x(y-x^2))]/[(-y(p+q)^2+$
$2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]-[4(n+m)st((ac+ab)((x(p+q)-n-m)^2+(y-$
$x^2)(l-(p+q)^2))+a(((n+m)x-y(p+q))(l-(p+q)^2)+(x(p+q)-n-m)(l-(p+q)^2)+$
$((n+m)(p+q)-lx)(x(p+q)-n-m))+(c+b)((x(p+q)-n-m)((n+m)x-y(p+q))+$
$(y-x^2)((n+m)(p+q)-lx))+((n+m)(p+q)-lx)((n+m)x-y(p+q))+(2bc+c+$
$b)(y-x^2)(x(p+q)-n-m))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]+$
$[6st(2(n+m)(p+q)-2lx)(2(n+m)x-2y(p+q))((ac+ab)((x(p+q)-n-m)^2+(y-$
$x^2)(l-(p+q)^2))+a(((n+m)x-y(p+q))(l-(p+q)^2)+(x(p+q)-n-m)(l-(p+q)^2)+$
$((n+m)(p+q)-lx)(x(p+q)-n-m))+(c+b)((x(p+q)-n-m)((n+m)x-y(p+$
$q))+(y-x^2)((n+m)(p+q)-lx))+((n+m)(p+q)-lx)((n+m)x-y(p+q))+(2bc+$
$c+b)(y-x^2)(x(p+q)-n-m))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^4]$

$h_{19}^{(U_6)} = [st(a(-3(q+p)^2-2lx+l)+(ac+ab)(2(x(q+p)-n-m)+6x(q+p))-2(c+$
$b)y(q+p)+(2bc+c+b)(y-x^2)+ly-2(2bc+c+b)x^2+(n+m)^2)]/[(-y(q+p)^2+2(n+$
$m)x(q+p)+ly-lx^2-(n+m)^2)^2]-[2st(2(n+m)x-2y(q+p))(a((q+p)(l-(q+p)^2)+$
$(n+m)(l-(q+p)^2)-l(x(q+p)-n-m)+(q+p)((n+m)(q+p)-lx))+(ac+ab)(2(q+$
$p)(x(q+p)-n-m)-2x(l-(q+p)^2))+(c+b)((q+p)((n+m)x-y(q+p))+(n+m)(x(q+$
$p)-n-m)-2x((n+m)(q+p)-lx)-l(y-x^2))-l((n+m)x-y(q+p))-2(2bc+c+$
$b)x(x(q+p)-n-m)+(n+m)((n+m)(q+p)-lx)+(2bc+c+b)(y-x^2)(q+p))]/[(-y(q+$
$p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]-[2st(2(n+m)(q+p)-2lx)(a(-y(l-(q+$
$p)^2)+x(l-(q+p)^2)-2(q+p)((n+m)x-y(q+p))-2(q+p)(x(q+p)-n-m)+(n+$
$m)(x(q+p)-n-m)+x((n+m)(q+p)-lx))+(c+b)(x((n+m)x-y(q+p))-y(x(q+$
$p)-n-m)+(n+m)(y-x^2))+(ac+ab)(2x(x(q+p)-n-m)-2(y-x^2)(q+p))+(n+$
$m)((n+m)x-y(q+p))-y((n+m)(q+p)-lx)+(2bc+c+b)x(y-x^2))]/[(-y(q+p)^2+$
$2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]-[4(n+m)st((ac+ab)((x(q+p)-n-m)^2+(y-$
$x^2)(l-(q+p)^2))+a(((n+m)x-y(q+p))(l-(q+p)^2)+(x(q+p)-n-m)(l-(q+p)^2)+$
$((n+m)(q+p)-lx)(x(q+p)-n-m))+(c+b)((x(q+p)-n-m)((n+m)x-y(q+p))+$
$(y-x^2)((n+m)(q+p)-lx))+((n+m)(q+p)-lx)((n+m)x-y(q+p))+(2bc+c+$
$b)(y-x^2)(x(q+p)-n-m))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]+$
$[6st(2(n+m)(q+p)-2lx)(2(n+m)x-2y(q+p))((ac+ab)((x(q+p)-n-m)^2+(y-$
$x^2)(l-(q+p)^2))+a(((n+m)x-y(q+p))(l-(q+p)^2)+(x(q+p)-n-m)(l-(q+p)^2)+$

$((n+m)(q+p)-lx)(x(q+p)-n-m))+(c+b)((x(q+p)-n-m)((n+m)x-y(q+p))+(y-x^2)((n+m)(q+p)-lx))+((n+m)(q+p)-lx)((n+m)x-y(q+p))+(2bc+c+b)(y-x^2)(x(q+p)-n-m))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^4]$

$h_{1\ 10}^{(U_6)} = [t((c+b)((q+p)((n+m)x-(q+p)y)-l(y-x^2)+(n+m)((q+p)x-n-m)-2x((n+m)(q+p)-lx))-l((n+m)x-(q+p)y)+(2bc+c+b)(q+p)(y-x^2)+(ac+ab)(2(q+p)((q+p)x-n-m)-2(l-(q+p)^2)x)+a(-l((q+p)x-n-m)+(q+p)((n+m)(q+p)-lx)+(q+p)(l-(q+p)^2)+(n+m)(l-(q+p)^2))-2(2bc+c+b)x((q+p)x-n-m)+(n+m)((n+m)(q+p)-lx))]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^2]-[2t(2(n+m)(q+p)-2lx)((c+b)(((q+p)x-n-m)((n+m)x-(q+p)y)+((n+m)(q+p)-lx)(y-x^2))+a((l-(q+p)^2)((n+m)x-(q+p)y)+((n+m)(q+p)-lx)((q+p)x-n-m)+(l-(q+p)^2)((q+p)x-n-m))+(ac+ab)((l-(q+p)^2)(y-x^2)+((q+p)x-n-m)^2)+((n+m)(q+p)-lx)((n+m)x-(q+p)y)+(2bc+c+b)((q+p)x-n-m)(y-x^2))]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^3]$

$h_{22}^{(U_6)} = [6(l-(q+p)^2)^2st((c+b)(((q+p)x-n-m)((n+m)x-(q+p)y)+((n+m)(q+p)-lx)(y-x^2))+a((l-(q+p)^2)((n+m)x-(q+p)y)+((n+m)(q+p)-lx)((q+p)x-n-m)+(l-(q+p)^2)((q+p)x-n-m))+(ac+ab)((l-(q+p)^2)(y-x^2)+((q+p)x-n-m)^2)+((n+m)(q+p)-lx)((n+m)x-(q+p)y)+(2bc+c+b)((q+p)x-n-m)(y-x^2))]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^4]-[4(l-(q+p)^2)st((c+b)((-q-p)((q+p)x-n-m)-lx+(n+m)(q+p))+(2bc+c+b)((q+p)x-n-m)+(-q-p)((n+m)(q+p)-lx)+a(-q-p)(l-(q+p)^2)+(ac+ab)(l-(q+p)^2))]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^3]$

$h_{23}^{(U_6)} = 0$

$h_{24}^{(U_6)} = [s((c+b)((-q-p)((q+p)x-n-m)-lx+(n+m)(q+p))+(2bc+c+b)((q+p)x-n-m)+(-q-p)((n+m)(q+p)-lx)+a(-q-p)(l-(q+p)^2)+(ac+ab)(l-(q+p)^2))]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^2]-[2(l-(q+p)^2)s((c+b)(((q+p)x-n-m)((n+m)x-(q+p)y)+((n+m)(q+p)-lx)(y-x^2))+a((l-(q+p)^2)((n+m)x-(q+p)y)+((n+m)(q+p)-lx)((q+p)x-n-m)+(l-(q+p)^2)((q+p)x-n-m))+(ac+ab)((l-(q+p)^2)(y-x^2)+((q+p)x-n-m)^2)+((n+m)(q+p)-lx)((n+m)x-(q+p)y)+(2bc+c+b)((q+p)x-n-m)(y-x^2))]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^3]$

$h_{25}^{(U_6)} = [st(-(-q-p)x-(c+b)x+a(-q-p)+ac+ab)]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^2]-[2st(a(-(q+p)y-x((q+p)x-n-m)+(q+p)x+(n+m)x-n-m)-x((n+m)x-(q+p)y)-(c+b)x(y-x^2)+(ac+ab)(y-x^2))(l-(q+p)^2)]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^3]-[2st((c+b)((y-x^2)((n+m)(q+p)-xl)+((q+p)x-n-m)((n+m)x-(q+p)y))+a(((q+p)x-n-m)((n+m)(q+p)-xl)+((n+m)x-(q+p)y)(l-(q+p)^2)+((q+p)x-n-m)(l-(q+p)^2))+(ac+ab)((y-x^2)(l-(q+p)^2)+((q+p)x-n-m)^2)+((n+m)x-(q+p)y)((n+m)(q+p)-xl)+(2bc+c+b)((q+p)x-n-m)(y-x^2))]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^3]+[6st(y-x^2)(l-(q+p)^2)((c+b)((y-x^2)((n+m)(q+p)-xl)+((q+p)x-n-m)((n+m)x-(q+p)y))+a(((q+p)x-n-m)((n+m)(q+p)-xl)+((n+m)x-(q+p)y)(l-(q+p)^2)+((q+p)x-n-m)(l-(q+p)^2))+(ac+ab)((y-x^2)(l-(q+p)^2)+((q+p)x-n-m)^2)+((n+m)x-(q+p)y)((n+m)(q+p)-xl)+(2bc+c+b)((q+p)x-n-m)(y-x^2))]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^4]-[2st(y-x^2)((c+b)(-xl+(-q-p)((q+p)x-n-m)+(n+m)(q+p))+(-q-p)((n+m)(q+p)-xl)+a(-q-p)(l-(q+p)^2)+(ac+ab)(l-(q+p)^2)+(2bc+c+b)((q+p)x-n-m))]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^3]$

$h_{26}^{(U_6)} = [((c+b)(2q+2p)+(-q-p)(q+p)-2bc-c-b)st]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^2]-[2(l-(q+p)^2)st((q+p)(x(m+n)-(q+p)y)+(c+b)(-x(m+n)+x(-m+(q+p)x-n)+(q+p)(y-x^2)+(q+p)y)+x((q+p)(m+n)-lx)+a(-(q+p)(m+n)+(q+p)(-m+(q+p)x-n)+(l-(q+p)^2)x+lx+(q+p)^2-l)-2(ac+ab)(-m+(q+p)x-n)-(2bc+c+b)(y-x^2))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^3]-[2st(2(q+p)x-2(m+n))((c+b)((q+p)(m+n)+(-q-p)(-m+(q+p)x-n)-lx)+(-q-p)((q+p)(m+n)-lx)+(2bc+c+b)(-m+(q+p)x-n)+a(-q-p)(l-(q+p)^2)+(ac+ab)(l-(q+p)^2))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^3]+[6(l-(q+p)^2)st(2(q+p)x-2(m+n))((c+b)((-m+(q+p)x-n)(x(m+n)-(q+p)y)+(y-x^2)((q+p)(m+n)-lx))+a((l-(q+p)^2)(x(m+n)-(q+p)y)+(-m+(q+p)x-n)((q+p)(m+n)-lx)+(l-(q+p)^2)(-m+(q+p)x-n))+((q+p)(m+n)-lx)(x(m+n)-(q+p)y)+(2bc+c+b)(y-x^2)(-m+(q+p)x-n)+(ac+ab)((-m+(q+p)x-n)^2+(l-(q+p)^2)(y-x^2)))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^4]$

$h_{27}^{(U_6)} = [((c+b)(2q+2p)+(-q-p)(q+p)-2bc-c-b)st]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^2]-[2(l-(q+p)^2)st((q+p)(x(n+m)-(q+p)y)+(c+$

$b)(-x(n+m)+x(-n+(q+p)x-m)+(q+p)(y-x^2)+(q+p)y)+x((q+p)(n+m)-lx)+a(-(q+p)(n+m)+(q+p)(-n+(q+p)x-m)+(l-(q+p)^2)x+lx+(q+p)^2-l)-2(ac+ab)(-n+(q+p)x-m)-(2bc+c+b)(y-x^2))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]-[2st(2(q+p)x-2(n+m))((c+b)((q+p)(n+m)+(-q-p)(-n+(q+p)x-m)-lx)+(-q-p)((q+p)(n+m)-lx)+(2bc+c+b)(-n+(q+p)x-m)+a(-q-p)(l-(q+p)^2)+(ac+ab)(l-(q+p)^2))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]+[6(l-(q+p)^2)st(2(q+p)x-2(n+m))((c+b)((-n+(q+p)x-m)(x(n+m)-(q+p)y)+(y-x^2)((q+p)(n+m)-lx))+a((l-(q+p)^2)(x(n+m)-(q+p)y)+(-n+(q+p)x-m)((q+p)(n+m)-lx)+(l-(q+p)^2)(-n+(q+p)x-m))+((q+p)(n+m)-lx)(x(n+m)-(q+p)y)+(2bc+c+b)(y-x^2)(-n+(q+p)x-m)+(ac+ab)((-n+(q+p)x-m)^2+(l-(q+p)^2)(y-x^2)))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^4]$

$h_{28}^{(U_6)}=[st(-a(l-(p+q)^2)+(c+b)(-x(p+q)+x(-p-q)+2n+2m)-2a(-p-q)(p+q)-(n+m)(p+q)-2(ac+ab)(p+q)+(n+m)(-p-q)+lx+(2bc+c+b)x)]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^2]-[2st(a(-y(l-(p+q)^2)+x(l-(p+q)^2)-2(p+q)((n+m)x-y(p+q))-2(p+q)(x(p+q)-n-m)+(n+m)(x(p+q)-n-m)+x((n+m)(p+q)-lx))+(c+b)(x((n+m)x-y(p+q))-y(x(p+q)-n-m)+(n+m)(y-x^2))+(ac+ab)(2x(x(p+q)-n-m)-2(y-x^2)(p+q))+(n+m)((n+m)x-y(p+q))-y((n+m)(p+q)-lx)+(2bc+c+b)x(y-x^2))(l-(p+q)^2)]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]-[2st(2(n+m)x-2y(p+q))(a(-p-q)(l-(p+q)^2)+(ac+ab)(l-(p+q)^2)+(c+b)((-p-q)(x(p+q)-n-m)+(n+m)(p+q)-lx)+(2bc+c+b)(x(p+q)-n-m)+(-p-q)((n+m)(p+q)-lx))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]+[4st(p+q)((ac+ab)((x(p+q)-n-m)^2+(y-x^2)(l-(p+q)^2))+a(((n+m)x-y(p+q))(l-(p+q)^2)+(x(p+q)-n-m)(l-(p+q)^2)+((n+m)(p+q)-lx)(x(p+q)-n-m))+(c+b)((x(p+q)-n-m)((n+m)x-y(p+q))+(y-x^2)((n+m)(p+q)-lx))+((n+m)(p+q)-lx)((n+m)x-y(p+q))+(2bc+c+b)(y-x^2)(x(p+q)-n-m))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]+[6st(2(n+m)x-2y(p+q))(l-(p+q)^2)((ac+ab)((x(p+q)-n-m)^2+(y-x^2)(l-(p+q)^2))+a(((n+m)x-y(p+q))(l-(p+q)^2)+(x(p+q)-n-m)(l-(p+q)^2)+((n+m)(p+q)-lx)(x(p+q)-n-m))+(c+b)((x(p+q)-n-m)((n+m)x-y(p+q))+(y-x^2)((n+m)(p+q)-lx))+((n+m)(p+q)-lx)((n+m)x-y(p+q))+(2bc+$

$c+b)(y-x^2)(x(p+q)-n-m))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^4]$

$h_{29}^{(U_6)} = [st(-a(l-(q+p)^2)+(c+b)(-x(q+p)+x(-q-p)+2n+2m)-2a(-q-p)(q+p)-(n+m)(q+p)-2(ac+ab)(q+p)+(n+m)(-q-p)+lx+(2bc+c+b)x)]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^2]-[2st(a(-y(l-(q+p)^2)+x(l-(q+p)^2)-2(q+p)((n+m)x-y(q+p))-2(q+p)(x(q+p)-n-m)+(n+m)(x(q+p)-n-m)+x((n+m)(q+p)-lx))+(c+b)(x((n+m)x-y(q+p))-y(x(q+p)-n-m)+(n+m)(y-x^2))+(ac+ab)(2x(x(q+p)-n-m)-2(y-x^2)(q+p))+(n+m)((n+m)x-y(q+p))-y((n+m)(q+p)-lx)+(2bc+c+b)x(y-x^2))(l-(q+p)^2)]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]-[2st(2(n+m)x-2y(q+p))(a(-q-p)(l-(q+p)^2)+(ac+ab)(l-(q+p)^2)+(c+b)((-q-p)(x(q+p)-n-m)+(n+m)(q+p)-lx)+(2bc+c+b)(x(q+p)-n-m)+(-q-p)((n+m)(q+p)-lx))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]+[4st(q+p)((ac+ab)((x(q+p)-n-m)^2+(y-x^2)(l-(q+p)^2))+a(((n+m)x-y(q+p))(l-(q+p)^2)+(x(q+p)-n-m)(l-(q+p)^2)+((n+m)(q+p)-lx)(x(q+p)-n-m))+(c+b)((x(q+p)-n-m)((n+m)x-y(q+p))+(y-x^2)((n+m)(q+p)-lx))+((n+m)(q+p)-lx)((n+m)x-y(q+p))+(2bc+c+b)(y-x^2)(x(q+p)-n-m))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]+[6st(2(n+m)x-2y(q+p))(l-(q+p)^2)((ac+ab)((x(q+p)-n-m)^2+(y-x^2)(l-(q+p)^2))+a(((n+m)x-y(q+p))(l-(q+p)^2)+(x(q+p)-n-m)(l-(q+p)^2)+((n+m)(q+p)-lx)(x(q+p)-n-m))+(c+b)((x(q+p)-n-m)((n+m)x-y(q+p))+(y-x^2)((n+m)(q+p)-lx))+((n+m)(q+p)-lx)((n+m)x-y(q+p))+(2bc+c+b)(y-x^2)(x(q+p)-n-m))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^4]$

$h_{2\ 10}^{(U_6)} = [t((c+b)((-q-p)((q+p)x-n-m)-lx+(n+m)(q+p))+(2bc+c+b)((q+p)x-n-m)+(-q-p)((n+m)(q+p)-lx)+a(-q-p)(l-(q+p)^2)+(ac+ab)(l-(q+p)^2))]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^2]-[2(l-(q+p)^2)t((c+b)(((q+p)x-n-m)((n+m)x-(q+p)y)+((n+m)(q+p)-lx)(y-x^2))+a((l-(q+p)^2)((n+m)x-(q+p)y)+((n+m)(q+p)-lx)((q+p)x-n-m)+(l-(q+p)^2)((q+p)x-n-m))+(ac+ab)((l-(q+p)^2)(y-x^2)+((q+p)x-n-m)^2)+((n+m)(q+p)-lx)((n+m)x-(q+p)y)+(2bc+c+b)((q+p)x-n-m)(y-x^2))]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^3]$

$h_{33}^{(U_6)} = 0$

$h_{34}^{(U_6)} = 0$

$h_{35}^{(U_6)} = 0$

$h_{36}^{(U_6)} = 0$

$h_{37}^{(U_6)} = 0$

$h_{38}^{(U_6)} = 0$

$h_{39}^{(U_6)} = 0$

$h_{3\ 10}^{(U_6)} = 0$

$h_{44}^{(U_6)} = 0$

$h_{45}^{(U_6)} = [s(a(-(q+p)y - x((q+p)x - n - m) + (q+p)x + (n+m)x - n - m) - x((n+m)x - (q+p)y) - (c+b)x(y - x^2) + (ac+ab)(y - x^2))]/[(yl - x^2l - (q+p)^2y + 2(n+m)(q+p)x - (n+m)^2)^2] - [2s(y - x^2)((c+b)((y - x^2)((n+m)(q+p) - xl) + ((q+p)x - n - m)((n+m)x - (q+p)y)) + a(((q+p)x - n - m)((n+m)(q+p) - xl) + ((n+m)x - (q+p)y)(l - (q+p)^2) + ((q+p)x - n - m)(l - (q+p)^2)) + (ac+ab)((y - x^2)(l - (q+p)^2) + ((q+p)x - n - m)^2) + ((n+m)x - (q+p)y)((n+m)(q+p) - xl) + (2bc + c + b)(((q+p)x - n - m)(y - x^2))]/[(yl - x^2l - (q+p)^2y + 2(n+m)(q+p)x - (n+m)^2)^3]$

$h_{46}^{(U_6)} = [s((q+p)(x(m+n) - (q+p)y) + (c+b)(-x(m+n) + x(-m + (q+p)x - n) + (q+p)(y - x^2) + (q+p)y) + x((q+p)(m+n) - lx) + a(-(q+p)(m+n) + (q+p)(-m + (q+p)x - n) + (l - (q+p)^2)x + lx + (q+p)^2 - l) - 2(ac+ab)(-m + (q+p)x - n) - (2bc + c + b)(y - x^2))]/[(-(m+n)^2 + 2(q+p)x(m+n) - (q+p)^2y + ly - lx^2)^2] - [2s(2(q+p)x - 2(m+n))((c+b)((-m + (q+p)x - n)(x(m+n) - (q+p)y) + (y - x^2)((q+p)(m+n) - lx)) + a((l - (q+p)^2)(x(m+n) - (q+p)y) + (-m + (q+p)x - n)((q+p)(m+n) - lx) + (l - (q+p)^2)(-m + (q+p)x - n)) + ((q+p)(m+n) - lx)(x(m+n) - (q+p)y) + (2bc + c + b)(y - x^2)(-m + (q+p)x - n) + (ac+ab)((-m + (q+p)x - n)^2 + (l - (q+p)^2)(y - x^2)))]/[(-(m+n)^2 + 2(q+p)x(m+n) - (q+p)^2y + ly - lx^2)^3]$

$h_{47}^{(U_6)} = [s((q+p)(x(n+m) - (q+p)y) + (c+b)(-x(n+m) + x(-n + (q+p)x - m) + (q+p)(y - x^2) + (q+p)y) + x((q+p)(n+m) - lx) + a(-(q+p)(n+m) + (q+p)(-n + (q+p)x - m) + (l - (q+p)^2)x + lx + (q+p)^2 - l) - 2(ac+ab)(-n + (q+p)x - m) - (2bc + c + b)(y - x^2))]/[(-(n+m)^2 + 2(q+p)x(n+m) - (q+p)^2y + ly - lx^2)^2] - [2s(2(q+p)x - 2(n+m))((c+b)((-n + (q+p)x - m)(x(n+m) - (q+p)y) + (y - x^2)((q+p)(n+m) - lx)) + a((l - (q+p)^2)(x(n+m) - (q+p)y) + (-n + (q+p)x -$

$m)((q+p)(n+m)-lx)+(l-(q+p)^2)(-n+(q+p)x-m))+((q+p)(n+m)-lx)(x(n+m)-(q+p)y)+(2bc+c+b)(y-x^2)(-n+(q+p)x-m)+(ac+ab)((-n+(q+p)x-m)^2+(l-(q+p)^2)(y-x^2)))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]$

$h_{48}^{(U_6)}=[s(a(-y(l-(p+q)^2)+x(l-(p+q)^2)-2(p+q)((n+m)x-y(p+q))-2(p+q)(x(p+q)-n-m)+(n+m)(x(p+q)-n-m)+x((n+m)(p+q)-lx))+(c+b)(x((n+m)x-y(p+q))-y(x(p+q)-n-m)+(n+m)(y-x^2))+(ac+ab)(2x(x(p+q)-n-m)-2(y-x^2)(p+q))+(n+m)((n+m)x-y(p+q))-y((n+m)(p+q)-lx)+(2bc+c+b)x(y-x^2))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^2]-[2s(2(n+m)x-2y(p+q))((ac+ab)((x(p+q)-n-m)^2+(y-x^2)(l-(p+q)^2))+a(((n+m)x-y(p+q))(l-(p+q)^2)+(x(p+q)-n-m)(l-(p+q)^2)+((n+m)(p+q)-lx)(x(p+q)-n-m))+(c+b)((x(p+q)-n-m)((n+m)x-y(p+q))+(y-x^2)((n+m)(p+q)-lx))+((n+m)(p+q)-lx)((n+m)x-y(p+q))+(2bc+c+b)(y-x^2)(x(p+q)-n-m))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]$

$h_{49}^{(U_6)}=[s(a(-y(l-(q+p)^2)+x(l-(q+p)^2)-2(q+p)((n+m)x-y(q+p))-2(q+p)(x(q+p)-n-m)+(n+m)(x(q+p)-n-m)+x((n+m)(q+p)-lx))+(c+b)(x((n+m)x-y(q+p))-y(x(q+p)-n-m)+(n+m)(y-x^2))+(ac+ab)(2x(x(q+p)-n-m)-2(y-x^2)(q+p))+(n+m)((n+m)x-y(q+p))-y((n+m)(q+p)-lx)+(2bc+c+b)x(y-x^2))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^2]-[2s(2(n+m)x-2y(q+p))((ac+ab)((x(q+p)-n-m)^2+(y-x^2)(l-(q+p)^2))+a(((n+m)x-y(q+p))(l-(q+p)^2)+(x(q+p)-n-m)(l-(q+p)^2)+((n+m)(q+p)-lx)(x(q+p)-n-m))+(c+b)((x(q+p)-n-m)((n+m)x-y(q+p))+(y-x^2)((n+m)(q+p)-lx))+((n+m)(q+p)-lx)((n+m)x-y(q+p))+(2bc+c+b)(y-x^2)(x(q+p)-n-m))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]$

$h_{4\ 10}^{(U_6)}=[(c+b)(((q+p)x-n-m)((n+m)x-(q+p)y)+((n+m)(q+p)-lx)(y-x^2))+a((l-(q+p)^2)((n+m)x-(q+p)y)+((n+m)(q+p)-lx)((q+p)x-n-m)+(l-(q+p)^2)((q+p)x-n-m))+(ac+ab)((l-(q+p)^2)(y-x^2)+((q+p)x-n-m)^2)+((n+m)(q+p)-lx)((n+m)x-(q+p)y)+(2bc+c+b)((q+p)x-n-m)(y-x^2)]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^2]$

$h_{55}^{(U_6)}=[6st(y-x^2)^2((c+b)((y-x^2)((n+m)(q+p)-xl)+((q+p)x-n-m)((n+m)x-(q+p)y))+a(((q+p)x-n-m)((n+m)(q+p)-xl)+((n+m)x-(q+p)y)(l-$

$(q+p)^2)+((q+p)x-n-m)(l-(q+p)^2))+(ac+ab)((y-x^2)(l-(q+p)^2)+((q+p)x-n-m)^2)+((n+m)x-(q+p)y)((n+m)(q+p)-xl)+(2bc+c+b)((q+p)x-n-m)(y-x^2))]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^4]-[4st(y-x^2)(a(-(q+p)y-x((q+p)x-n-m)+(q+p)x+(n+m)x-n-m)-x((n+m)x-(q+p)y)-(c+b)x(y-x^2)+(ac+ab)(y-x^2))]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^3]$

$h_{56}^{(U_6)}=[st(a(2x-1)-x^2)]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^2]-[2st(2(q+p)x-2(m+n))(a(x(m+n)-m-x(-m+(q+p)x-n)+(-q-p)y+(q+p)x-n)-x(x(m+n)-(q+p)y)+(-c-b)x(y-x^2)+(ac+ab)(y-x^2))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^3]-[2st(y-x^2)((q+p)(x(m+n)-(q+p)y)+(c+b)(-x(m+n)+x(-m+(q+p)x-n)+(q+p)(y-x^2)+(q+p)y)+x((q+p)(m+n)-lx)+a(-(q+p)(m+n)+(q+p)(-m+(q+p)x-n)+(l-(q+p)^2)x+lx+(q+p)^2-l)-2(ac+ab)(-m+(q+p)x-n)-(2bc+c+b)(y-x^2))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^3]+[6st(y-x^2)(2(q+p)x-2(m+n))((c+b)((-m+(q+p)x-n)(x(m+n)-(q+p)y)+(y-x^2)((q+p)(m+n)-lx))+a((l-(q+p)^2)(x(m+n)-(q+p)y)+(-m+(q+p)x-n)((q+p)(m+n)-lx)+(l-(q+p)^2)(-m+(q+p)x-n))+((q+p)(m+n)-lx)(x(m+n)-(q+p)y)+(2bc+c+b)(y-x^2)(-m+(q+p)x-n)+(ac+ab)((-m+(q+p)x-n)^2+(l-(q+p)^2)(y-x^2)))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^4]$

$h_{57}^{(U_6)}=[st(a(2x-1)-x^2)]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^2]-[2st(2(q+p)x-2(n+m))(a(x(n+m)-n-x(-n+(q+p)x-m)+(-q-p)y+(q+p)x-m)-x(x(n+m)-(q+p)y)+(-c-b)x(y-x^2)+(ac+ab)(y-x^2))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]-[2st(y-x^2)((q+p)(x(n+m)-(q+p)y)+(c+b)(-x(n+m)+x(-n+(q+p)x-m)+(q+p)(y-x^2)+(q+p)y)+x((q+p)(n+m)-lx)+a(-(q+p)(n+m)+(q+p)(-n+(q+p)x-m)+(l-(q+p)^2)x+lx+(q+p)^2-l)-2(ac+ab)(-n+(q+p)x-m)-(2bc+c+b)(y-x^2))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]+[6st(y-x^2)(2(q+p)x-2(n+m))((c+b)((-n+(q+p)x-m)(x(n+m)-(q+p)y)+(y-x^2)((q+p)(n+m)-lx))+a((l-(q+p)^2)(x(n+m)-(q+p)y)+(-n+(q+p)x-m)((q+p)(n+m)-lx)+(l-(q+p)^2)(-n+(q+p)x-m))+((q+p)(n+m)-lx)(x(n+m)-(q+p)y)+(2bc+c+b)(y-x^2)(-n+(q+p)x-m)+(ac+ab)((-n+(q+p)x-m)^2+(l-(q+p)^2)(y-x^2)))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^4]$

$h_{58}^{(U_6)}=[st(xy+a(-y-x^2+x))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+$

$m)^2)^2] - [2st(y-x^2)(a(-y(l-(p+q)^2)+x(l-(p+q)^2)-2(p+q)((n+m)x-y(p+q))-2(p+q)(x(p+q)-n-m)+(n+m)(x(p+q)-n-m)+x((n+m)(p+q)-lx))+(c+b)(x((n+m)x-y(p+q))-y(x(p+q)-n-m)+(n+m)(y-x^2))+(ac+ab)(2x(x(p+q)-n-m)-2(y-x^2)(p+q))+(n+m)((n+m)x-y(p+q))-y((n+m)(p+q)-lx)+(2bc+c+b)x(y-x^2))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3] - [2st(2(n+m)x-2y(p+q))(a(-x(x(p+q)-n-m)+x(p+q)+y(-p-q)+(n+m)x-n-m)-x((n+m)x-y(p+q))+(-c-b)x(y-x^2)+(ac+ab)(y-x^2))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3] + [6st(y-x^2)(2(n+m)x-2y(p+q))((ac+ab)((x(p+q)-n-m)^2+(y-x^2)(l-(p+q)^2))+a(((n+m)x-y(p+q))(l-(p+q)^2)+(x(p+q)-n-m)(l-(p+q)^2)+((n+m)(p+q)-lx)(x(p+q)-n-m))+(c+b)((x(p+q)-n-m)((n+m)x-y(p+q))+(y-x^2)((n+m)(p+q)-lx))+((n+m)(p+q)-lx)((n+m)x-y(p+q))+(2bc+c+b)(y-x^2)(x(p+q)-n-m))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^4]$

$h_{59}^{(U_6)} = [st(xy+a(-y-x^2+x))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^2] - [2st(y-x^2)(a(-y(l-(q+p)^2)+x(l-(q+p)^2)-2(q+p)((n+m)x-y(q+p))-2(q+p)(x(q+p)-n-m)+(n+m)(x(q+p)-n-m)+x((n+m)(q+p)-lx))+(c+b)(x((n+m)x-y(q+p))-y(x(q+p)-n-m)+(n+m)(y-x^2))+(ac+ab)(2x(x(q+p)-n-m)-2(y-x^2)(q+p))+(n+m)((n+m)x-y(q+p))-y((n+m)(q+p)-lx)+(2bc+c+b)x(y-x^2))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3] - [2st(2(n+m)x-2y(q+p))(a(-x(x(q+p)-n-m)+x(q+p)+y(-q-p)+(n+m)x-n-m)-x((n+m)x-y(q+p))+(-c-b)x(y-x^2)+(ac+ab)(y-x^2))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3] + [6st(y-x^2)(2(n+m)x-2y(q+p))((ac+ab)((x(q+p)-n-m)^2+(y-x^2)(l-(q+p)^2))+a(((n+m)x-y(q+p))(l-(q+p)^2)+(x(q+p)-n-m)(l-(q+p)^2)+((n+m)(q+p)-lx)(x(q+p)-n-m))+(c+b)((x(q+p)-n-m)((n+m)x-y(q+p))+(y-x^2)((n+m)(q+p)-lx))+((n+m)(q+p)-lx)((n+m)x-y(q+p))+(2bc+c+b)(y-x^2)(x(q+p)-n-m))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^4]$

$h_{5\ 10}^{(U_6)} = [t(-x((n+m)x-(q+p)y)+a((-q-p)y-x((q+p)x-n-m)+(q+p)x+(n+m)x-n-m)+(-c-b)x(y-x^2)+(ac+ab)(y-x^2))]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^2] - [2t(y-x^2)((c+b)(((q+p)x-n-m)((n+m)x-(q+p)y)+((n+m)(q+p)-lx)(y-x^2))+a((l-(q+p)^2)((n+m)x-(q+p)y)+((n+m)(q+p)-lx)((q+p)x-n-m)+(l-(q+p)^2)((q+p)x-n-m))+(ac+ab)((l-(q+$

$p)^2)(y-x^2)+((q+p)x-n-m)^2)+((n+m)(q+p)-lx)((n+m)x-(q+p)y)+(2bc+c+b)((q+p)x-n-m)(y-x^2))]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^3]$

$h_{66}^{(U_6)} = [st(2(q+p)x-2(c+b)x+a(-2q-2p)+2(ac+ab))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^2]+[4st((c+b)((-m+(q+p)x-n)(x(m+n)-(q+p)y)+(y-x^2)((q+p)(m+n)-lx))+a((l-(q+p)^2)(x(m+n)-(q+p)y)+(-m+(q+p)x-n)((q+p)(m+n)-lx)+(l-(q+p)^2)(-m+(q+p)x-n))+((q+p)(m+n)-lx)(x(m+n)-(q+p)y)+(2bc+c+b)(y-x^2)(-m+(q+p)x-n)+(ac+ab)((-m+(q+p)x-n)^2+(l-(q+p)^2)(y-x^2)))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^3]-[4st(2(q+p)x-2(m+n))((q+p)(x(m+n)-(q+p)y)+(c+b)(-x(m+n)+x(-m+(q+p)x-n)+(q+p)(y-x^2)+(q+p)y)+x((q+p)(m+n)-lx)+a(-(q+p)(m+n)+(q+p)(-m+(q+p)x-n)+(l-(q+p)^2)x+lx+(q+p)^2-l)-2(ac+ab)(-m+(q+p)x-n)-(2bc+c+b)(y-x^2))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^3]+[6st(2(q+p)x-2(m+n))^2((c+b)((-m+(q+p)x-n)(x(m+n)-(q+p)y)+(y-x^2)((q+p)(m+n)-lx))+a((l-(q+p)^2)(x(m+n)-(q+p)y)+(-m+(q+p)x-n)((q+p)(m+n)-lx)+(l-(q+p)^2)(-m+(q+p)x-n))+((q+p)(m+n)-lx)(x(m+n)-(q+p)y)+(2bc+c+b)(y-x^2)(-m+(q+p)x-n)+(ac+ab)((-m+(q+p)x-n)^2+(l-(q+p)^2)(y-x^2)))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^4]$

$h_{67}^{(U_6)} = [st(2(q+p)x-2(c+b)x+a(-2q-2p)+2(ac+ab))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^2]+[4st((c+b)((-n+(q+p)x-m)(x(n+m)-(q+p)y)+(y-x^2)((q+p)(n+m)-lx))+a((l-(q+p)^2)(x(n+m)-(q+p)y)+(-n+(q+p)x-m)((q+p)(n+m)-lx)+(l-(q+p)^2)(-n+(q+p)x-m))+((q+p)(n+m)-lx)(x(n+m)-(q+p)y)+(2bc+c+b)(y-x^2)(-n+(q+p)x-m)+(ac+ab)((-n+(q+p)x-m)^2+(l-(q+p)^2)(y-x^2)))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]-[2st(2(q+p)x-2(n+m))((q+p)(x(n+m)-(q+p)y)+(c+b)(-x(n+m)+x(-n+(q+p)x-m)+(q+p)(y-x^2)+(q+p)y)+x((q+p)(n+m)-lx)+a(-(q+p)(n+m)+(q+p)(-n+(q+p)x-m)+(l-(q+p)^2)x+lx+(q+p)^2-l)-2(ac+ab)(-n+(q+p)x-m)-(2bc+c+b)(y-x^2))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]-[2st(2(q+p)x-2(n+m))((q+p)(x(n+m)-(q+p)y)+(c+b)(-x(n+m)+x(-n+(q+p)x-m)+(q+p)(y-x^2)+(q+p)y)+x((q+p)(n+m)-lx)+a((-q-p)(n+m)+(q+p)(-n+(q+p)x-m)+(l-(q+p)^2)x+lx+(q+p)^2-l)-2(ac+ab)(-n+$

$(q+p)x-m)+(-2bc-c-b)(y-x^2))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]+[6st(2(q+p)x-2(n+m))^2((c+b)((-n+(q+p)x-m)(x(n+m)-(q+p)y)+(y-x^2)((q+p)(n+m)-lx))+a((l-(q+p)^2)(x(n+m)-(q+p)y)+(-n+(q+p)x-m)((q+p)(n+m)-lx)+(l-(q+p)^2)(-n+(q+p)x-m))+((q+p)(n+m)-lx)(x(n+m)-(q+p)y)+(2bc+c+b)(y-x^2)(-n+(q+p)x-m)+(ac+ab)((-n+(q+p)x-m)^2+(l-(q+p)^2)(y-x^2)))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^4]$

$h_{68}^{(U_6)} = [st(a(2(p+q)-2n-2m)-2y(p+q)+2(c+b)y+2(n+m)x-2(ac+ab)x)]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^2]-[2st(2x(p+q)-2(n+m))(a(-y(l-(p+q)^2)+x(l-(p+q)^2)-2(p+q)((n+m)x-y(p+q))-2(p+q)(x(p+q)-n-m)+(n+m)(x(p+q)-n-m)+x((n+m)(p+q)-lx))+(c+b)(x((n+m)x-y(p+q))-y(x(p+q)-n-m)+(n+m)(y-x^2))+(ac+ab)(2x(x(p+q)-n-m)-2(y-x^2)(p+q))+(n+m)((n+m)x-y(p+q))-y((n+m)(p+q)-lx)+(2bc+c+b)x(y-x^2))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]-[4stx((ac+ab)((x(p+q)-n-m)^2+(y-x^2)(l-(p+q)^2))+a(((n+m)x-y(p+q))(l-(p+q)^2)+(x(p+q)-n-m)(l-(p+q)^2)+((n+m)(p+q)-lx)(x(p+q)-n-m))+(c+b)((x(p+q)-n-m)((n+m)x-y(p+q))+(y-x^2)((n+m)(p+q)-lx))+((n+m)(p+q)-lx)((n+m)x-y(p+q))+(2bc+c+b)(y-x^2)(x(p+q)-n-m))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]+[6st(2x(p+q)-2(n+m))(2(n+m)x-2y(p+q))((ac+ab)((x(p+q)-n-m)^2+(y-x^2)(l-(p+q)^2))+a(((n+m)x-y(p+q))(l-(p+q)^2)+(x(p+q)-n-m)(l-(p+q)^2)+((n+m)(p+q)-lx)(x(p+q)-n-m))+(c+b)((x(p+q)-n-m)((n+m)x-y(p+q))+(y-x^2)((n+m)(p+q)-lx))+((n+m)(p+q)-lx)((n+m)x-y(p+q))+(2bc+c+b)(y-x^2)(x(p+q)-n-m))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^4]-[2st(2(n+m)x-2y(p+q))(a((p+q)^2+x(l-(p+q)^2)+(p+q)(x(p+q)-n-m)+(n+m)(-p-q)+lx-l)+(c+b)(x(x(p+q)-n-m)+(y-x^2)(p+q)+y(p+q)-(n+m)x)+(p+q)((n+m)x-y(p+q))-2(ac+ab)(x(p+q)-n-m)+x((n+m)(p+q)-lx)+(-2bc-c-b)(y-x^2))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]$

$h_{69}^{(U_6)} = [st(a(2(q+p)-2n-2m)-2y(q+p)+2(c+b)y+2(n+m)x-2(ac+ab)x)]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^2]-[2st(2x(q+p)-2(n+m))(a(-y(l-(q+p)^2)+x(l-(q+p)^2)-2(q+p)((n+m)x-y(q+p))-2(q+p)(x(q+p)-n-m)+(n+m)(x(q+p)-n-m)+x((n+m)(q+p)-lx))+(c+b)(x((n+m)x-$

$y(q+p))-y(x(q+p)-n-m)+(n+m)(y-x^2))+(ac+ab)(2x(x(q+p)-n-m)-2(y-x^2)(q+p))+(n+m)((n+m)x-y(q+p))-y((n+m)(q+p)-lx)+(2bc+c+b)x(y-x^2))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]-[4stx((ac+ab)((x(q+p)-n-m)^2+(y-x^2)(l-(q+p)^2))+a(((n+m)x-y(q+p))(l-(q+p)^2)+(x(q+p)-n-m)(l-(q+p)^2)+((n+m)(q+p)-lx)(x(q+p)-n-m))+(c+b)((x(q+p)-n-m)((n+m)x-y(q+p))+(y-x^2)((n+m)(q+p)-lx))+((n+m)(q+p)-lx)((n+m)x-y(q+p))+(2bc+c+b)(y-x^2)(x(q+p)-n-m))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]+[6st(2x(q+p)-2(n+m))(2(n+m)x-2y(q+p))((ac+ab)((x(q+p)-n-m)^2+(y-x^2)(l-(q+p)^2))+a(((n+m)x-y(q+p))(l-(q+p)^2)+(x(q+p)-n-m)(l-(q+p)^2)+((n+m)(q+p)-lx)(x(q+p)-n-m))+(c+b)((x(q+p)-n-m)((n+m)x-y(q+p))+(y-x^2)((n+m)(q+p)-lx))+((n+m)(q+p)-lx)((n+m)x-y(q+p))+(2bc+c+b)(y-x^2)(x(q+p)-n-m))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^4]-[2st(2(n+m)x-2y(q+p))(a((q+p)^2+x(l-(q+p)^2)+(q+p)(x(q+p)-n-m)+(n+m)(-q-p)+lx-l)+(c+b)(x(x(q+p)-n-m)+(y-x^2)(q+p)+y(q+p)-(n+m)x)+(q+p)((n+m)x-y(q+p))-2(ac+ab)(x(q+p)-n-m)+x((n+m)(q+p)-lx)+(-2bc-c-b)(y-x^2))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]$

$h_{6\ 10}^{(U_6)} = [t((c+b)((q+p)(y-x^2)+(q+p)y+x((q+p)x-n-m)-(n+m)x)+(q+p)((n+m)x-(q+p)y)+(-2bc-c-b)(y-x^2)+a((q+p)((q+p)x-n-m)+(l-(q+p)^2)x+lx+(q+p)^2+(n+m)(-q-p)-l)-2(ac+ab)((q+p)x-n-m)+x((n+m)(q+p)-lx))]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^2]-[2t(2(q+p)x-2(n+m))((c+b)(((q+p)x-n-m)((n+m)x-(q+p)y)+((n+m)(q+p)-lx)(y-x^2))+a((l-(q+p)^2)((n+m)x-(q+p)y)+((n+m)(q+p)-lx)((q+p)x-n-m)+(l-(q+p)^2)((q+p)x-n-m))+(ac+ab)((l-(q+p)^2)(y-x^2)+((q+p)x-n-m)^2)+((n+m)(q+p)-lx)((n+m)x-(q+p)y)+(2bc+c+b)((q+p)x-n-m)(y-x^2))]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^3]$

$h_{77}^{(U_6)} = [st(2(q+p)x-2(c+b)x+a(-2q-2p)+2(ac+ab))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^2]+[4st((c+b)((-n+(q+p)x-m)(x(n+m)-(q+p)y)+(y-x^2)((q+p)(n+m)-lx))+a((l-(q+p)^2)(x(n+m)-(q+p)y)+(-n+(q+p)x-m)((q+p)(n+m)-lx)+(l-(q+p)^2)(-n+(q+p)x-m))+((q+p)(n+m)-lx)(x(n+m)-(q+p)y)+(2bc+c+b)(y-x^2)(-n+(q+p)x-m)+(ac+ab)((-n+(q+p)x-m)^2+(l-(q+p)^2)(y-x^2))))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-$

$lx^2)^3] - [4st(2(q+p)x - 2(n+m))((q+p)(x(n+m) - (q+p)y) + (c+b)(-x(n+m) + x(-n+(q+p)x-m) + (q+p)(y-x^2) + (q+p)y) + x((q+p)(n+m)-lx) + a(-(q+p)(n+m) + (q+p)(-n+(q+p)x-m) + (l-(q+p)^2)x + lx + (q+p)^2 - l) - 2(ac+ab)(-n+(q+p)x-m) - (2bc+c+b)(y-x^2))]/[(-(n+m)^2 + 2(q+p)x(n+m) - (q+p)^2y + ly - lx^2)^3] + [6st(2(q+p)x - 2(n+m))^2((c+b)((-n+(q+p)x-m)(x(n+m) - (q+p)y) + (y-x^2)((q+p)(n+m)-lx)) + a((l-(q+p)^2)(x(n+m)-(q+p)y) + (-n+(q+p)x-m)((q+p)(n+m)-lx) + (l-(q+p)^2)(-n+(q+p)x-m)) + ((q+p)(n+m)-lx)(x(n+m) - (q+p)y) + (2bc+c+b)(y-x^2)(-n+(q+p)x-m) + (ac+ab)((-n+(q+p)x-m)^2 + (l-(q+p)^2)(y-x^2))))]/[(-(n+m)^2 + 2(q+p)x(n+m) - (q+p)^2y + ly - lx^2)^4]$

$h_{78}^{(U_6)} = [st(a(2(p+q) - 2n - 2m) - 2y(p+q) + 2(c+b)y + 2(n+m)x - 2(ac+ab)x)]/[(-y(p+q)^2 + 2(n+m)x(p+q) + ly - lx^2 - (n+m)^2)^2] - [2st(2x(p+q) - 2(n+m))(a(-y(l-(p+q)^2) + x(l-(p+q)^2) - 2(p+q)((n+m)x - y(p+q)) - 2(p+q)(x(p+q)-n-m) + (n+m)(x(p+q)-n-m) + x((n+m)(p+q)-lx)) + (c+b)(x((n+m)x - y(p+q)) - y(x(p+q)-n-m) + (n+m)(y-x^2)) + (ac+ab)(2x(x(p+q)-n-m) - 2(y-x^2)(p+q)) + (n+m)((n+m)x - y(p+q)) - y((n+m)(p+q)-lx) + (2bc+c+b)x(y-x^2))]/[(-y(p+q)^2 + 2(n+m)x(p+q) + ly - lx^2 - (n+m)^2)^3] - [4stx((ac+ab)((x(p+q)-n-m)^2 + (y-x^2)(l-(p+q)^2)) + a(((n+m)x - y(p+q))(l-(p+q)^2) + (x(p+q)-n-m)(l-(p+q)^2) + ((n+m)(p+q)-lx)(x(p+q)-n-m)) + (c+b)((x(p+q)-n-m)((n+m)x - y(p+q)) + (y-x^2)((n+m)(p+q)-lx)) + ((n+m)(p+q)-lx)((n+m)x - y(p+q)) + (2bc+c+b)(y-x^2)(x(p+q)-n-m))]/[(-y(p+q)^2 + 2(n+m)x(p+q) + ly - lx^2 - (n+m)^2)^3] + [6st(2x(p+q) - 2(n+m))(2(n+m)x - 2y(p+q))((ac+ab)((x(p+q)-n-m)^2 + (y-x^2)(l-(p+q)^2)) + a(((n+m)x - y(p+q))(l-(p+q)^2) + (x(p+q)-n-m)(l-(p+q)^2) + ((n+m)(p+q)-lx)(x(p+q)-n-m)) + (c+b)((x(p+q)-n-m)((n+m)x - y(p+q)) + (y-x^2)((n+m)(p+q)-lx)) + ((n+m)(p+q)-lx)((n+m)x - y(p+q)) + (2bc+c+b)(y-x^2)(x(p+q)-n-m))]/[(-y(p+q)^2 + 2(n+m)x(p+q) + ly - lx^2 - (n+m)^2)^4] - [2st(2(n+m)x - 2y(p+q))(a((p+q)^2 + x(l-(p+q)^2) + (p+q)(x(p+q)-n-m) + (n+m)(-p-q) + lx - l) + (c+b)(x(x(p+q)-n-m) + (y-x^2)(p+q) + y(p+q) - (n+m)x) + (p+q)((n+m)x - y(p+q)) - 2(ac+ab)(x(p+q)-n-m) + x((n+m)(p+q)-lx) + (-2bc-c-b)(y-x^2))]/[(-y(p+q)^2 + 2(n+m)x(p+q) + ly - lx^2 - (n+m)^2)^3]$

$h_{79}^{(U_6)} = [st(a(2(q+p) - 2n - 2m) - 2y(q+p) + 2(c+b)y + 2(n+m)x - 2(ac+$

$ab)x)]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^2]-[2st(2x(q+p)-2(n+m))(a(-y(l-(q+p)^2)+x(l-(q+p)^2)-2(q+p)((n+m)x-y(q+p))-2(q+p)(x(q+p)-n-m)+(n+m)(x(q+p)-n-m)+x((n+m)(q+p)-lx))+(c+b)(x((n+m)x-y(q+p))-y(x(q+p)-n-m)+(n+m)(y-x^2))+(ac+ab)(2x(x(q+p)-n-m)-2(y-x^2)(q+p))+(n+m)((n+m)x-y(q+p))-y((n+m)(q+p)-lx)+(2bc+c+b)x(y-x^2))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]-[4stx((ac+ab)((x(q+p)-n-m)^2+(y-x^2)(l-(q+p)^2))+a(((n+m)x-y(q+p))(l-(q+p)^2)+(x(q+p)-n-m)(l-(q+p)^2)+((n+m)(q+p)-lx)(x(q+p)-n-m))+(c+b)((x(q+p)-n-m)((n+m)x-y(q+p))+(y-x^2)((n+m)(q+p)-lx))+((n+m)(q+p)-lx)((n+m)x-y(q+p))+(2bc+c+b)(y-x^2)(x(q+p)-n-m))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]+[6st(2x(q+p)-2(n+m))(2(n+m)x-2y(q+p))((ac+ab)((x(q+p)-n-m)^2+(y-x^2)(l-(q+p)^2))+a(((n+m)x-y(q+p))(l-(q+p)^2)+(x(q+p)-n-m)(l-(q+p)^2)+((n+m)(q+p)-lx)(x(q+p)-n-m))+(c+b)((x(q+p)-n-m)((n+m)x-y(q+p))+(y-x^2)((n+m)(q+p)-lx))+((n+m)(q+p)-lx)((n+m)x-y(q+p))+(2bc+c+b)(y-x^2)(x(q+p)-n-m))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^4]-[2st(2(n+m)x-2y(q+p))(a((q+p)^2+x(l-(q+p)^2)+(q+p)(x(q+p)-n-m)+(n+m)(-q-p)+lx-l)+(c+b)(x(x(q+p)-n-m)+(y-x^2)(q+p)+y(q+p)-(n+m)x)+(q+p)((n+m)x-y(q+p))-2(ac+ab)(x(q+p)-n-m)+x((n+m)(q+p)-lx)+(-2bc-c-b)(y-x^2))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]$

$h_{7\ 10}^{(U_6)}=[t((c+b)((q+p)(y-x^2)+(q+p)y+x((q+p)x-n-m)-(n+m)x)+(q+p)((n+m)x-(q+p)y)+(-2bc-c-b)(y-x^2)+a((q+p)((q+p)x-n-m)+(l-(q+p)^2)x+lx+(q+p)^2+(n+m)(-q-p)-l)-2(ac+ab)((q+p)x-n-m)+x((n+m)(q+p)-lx))]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^2]-[2t(2(q+p)x-2(n+m))((c+b)(((q+p)x-n-m)((n+m)x-(q+p)y)+((n+m)(q+p)-lx)(y-x^2))+a((l-(q+p)^2)((n+m)x-(q+p)y)+((n+m)(q+p)-lx)((q+p)x-n-m)+(l-(q+p)^2)((q+p)x-n-m))+(ac+ab)((l-(q+p)^2)(y-x^2)+((q+p)x-n-m)^2)+((n+m)(q+p)-lx)((n+m)x-(q+p)y)+(2bc+c+b)((q+p)x-n-m)(y-x^2))]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^3]$

$h_{88}^{(U_6)}=[st(a(-2((n+m)x-y(p+q))-2(x(p+q)-n-m)+4y(p+q)-4x(p+q)+2(n+m)x)+(ac+ab)(2x^2-2(y-x^2))-2(c+b)xy-2(n+m)y)]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^2]-[4st(2(n+m)x-2y(p+q))(a(-y(l-(p+q)^2)+x(l-(p+q)^2)-$

$2(p{+}q)((n{+}m)x{-}y(p{+}q)){-}2(p{+}q)(x(p{+}q){-}n{-}m){+}(n{+}m)(x(p{+}q){-}n{-}m){+}x((n{+}m)(p{+}q){-}lx)){+}(c{+}b)(x((n{+}m)x{-}y(p{+}q)){-}y(x(p{+}q){-}n{-}m){+}(n{+}m)(y{-}x^2)){+}(ac{+}ab)(2x(x(p{+}q){-}n{-}m){-}2(y{-}x^2)(p{+}q)){+}(n{+}m)((n{+}m)x{-}y(p{+}q)){-}y((n{+}m)(p{+}q){-}lx){+}(2bc{+}c{+}b)x(y{-}x^2))]/[({-}y(p{+}q)^2{+}2(n{+}m)x(p{+}q){+}ly{-}lx^2{-}(n{+}m)^2)^3]{+}[4sty((ac{+}ab)((x(p{+}q){-}n{-}m)^2{+}(y{-}x^2)(l{-}(p{+}q)^2)){+}a(((n{+}m)x{-}y(p{+}q))(l{-}(p{+}q)^2){+}(x(p{+}q){-}n{-}m)(l{-}(p{+}q)^2){+}((n{+}m)(p{+}q){-}lx)(x(p{+}q){-}n{-}m)){+}(c{+}b)((x(p{+}q){-}n{-}m)((n{+}m)x{-}y(p{+}q)){+}(y{-}x^2)((n{+}m)(p{+}q){-}lx)){+}((n{+}m)(p{+}q){-}lx)((n{+}m)x{-}y(p{+}q)){+}(2bc{+}c{+}b)(y{-}x^2)(x(p{+}q){-}n{-}m))]/[({-}y(p{+}q)^2{+}2(n{+}m)x(p{+}q){+}ly{-}lx^2{-}(n{+}m)^2)^3]{+}[6st(2(n{+}m)x{-}2y(p{+}q))^2((ac{+}ab)((x(p{+}q){-}n{-}m)^2{+}(y{-}x^2)(l{-}(p{+}q)^2)){+}a(((n{+}m)x{-}y(p{+}q))(l{-}(p{+}q)^2){+}(x(p{+}q){-}n{-}m)(l{-}(p{+}q)^2){+}((n{+}m)(p{+}q){-}lx)(x(p{+}q){-}n{-}m)){+}(c{+}b)((x(p{+}q){-}n{-}m)((n{+}m)x{-}y(p{+}q)){+}(y{-}x^2)((n{+}m)(p{+}q){-}lx)){+}((n{+}m)(p{+}q){-}lx)((n{+}m)x{-}y(p{+}q)){+}(2bc{+}c{+}b)(y{-}x^2)(x(p{+}q){-}n{-}m))]/[({-}y(p{+}q)^2{+}2(n{+}m)x(p{+}q){+}ly{-}lx^2{-}(n{+}m)^2)^4]$

$h_{89}^{(U_6)} = [st(a({-}2((n{+}m)x{-}y(q{+}p)){-}2(x(q{+}p){-}n{-}m){+}4y(q{+}p){-}4x(q{+}p){+}2(n{+}m)x){+}(ac{+}ab)(2x^2{-}2(y{-}x^2)){-}2(c{+}b)xy{-}2(n{+}m)y)]/[({-}y(q{+}p)^2{+}2(n{+}m)x(q{+}p){+}ly{-}lx^2{-}(n{+}m)^2)^2]{-}[4st(2(n{+}m)x{-}2y(q{+}p))(a({-}y(l{-}(q{+}p)^2){+}x(l{-}(q{+}p)^2){-}2(q{+}p)((n{+}m)x{-}y(q{+}p)){-}2(q{+}p)(x(q{+}p){-}n{-}m){+}(n{+}m)(x(q{+}p){-}n{-}m){+}x((n{+}m)(q{+}p){-}lx)){+}(c{+}b)(x((n{+}m)x{-}y(q{+}p)){-}y(x(q{+}p){-}n{-}m){+}(n{+}m)(y{-}x^2)){+}(ac{+}ab)(2x(x(q{+}p){-}n{-}m){-}2(y{-}x^2)(q{+}p)){+}(n{+}m)((n{+}m)x{-}y(q{+}p)){-}y((n{+}m)(q{+}p){-}lx){+}(2bc{+}c{+}b)x(y{-}x^2))]/[({-}y(q{+}p)^2{+}2(n{+}m)x(q{+}p){+}ly{-}lx^2{-}(n{+}m)^2)^3]{+}[4sty((ac{+}ab)((x(q{+}p){-}n{-}m)^2{+}(y{-}x^2)(l{-}(q{+}p)^2)){+}a(((n{+}m)x{-}y(q{+}p))(l{-}(q{+}p)^2){+}(x(q{+}p){-}n{-}m)(l{-}(q{+}p)^2){+}((n{+}m)(q{+}p){-}lx)(x(q{+}p){-}n{-}m)){+}(c{+}b)((x(q{+}p){-}n{-}m)((n{+}m)x{-}y(q{+}p)){+}(y{-}x^2)((n{+}m)(q{+}p){-}lx)){+}((n{+}m)(q{+}p){-}lx)((n{+}m)x{-}y(q{+}p)){+}(2bc{+}c{+}b)(y{-}x^2)(x(q{+}p){-}n{-}m))]/[({-}y(q{+}p)^2{+}2(n{+}m)x(q{+}p){+}ly{-}lx^2{-}(n{+}m)^2)^3]{+}[6st(2(n{+}m)x{-}2y(q{+}p))^2((ac{+}ab)((x(q{+}p){-}n{-}m)^2{+}(y{-}x^2)(l{-}(q{+}p)^2)){+}a(((n{+}m)x{-}y(q{+}p))(l{-}(q{+}p)^2){+}(x(q{+}p){-}n{-}m)(l{-}(q{+}p)^2){+}((n{+}m)(q{+}p){-}lx)(x(q{+}p){-}n{-}m)){+}(c{+}b)((x(q{+}p){-}n{-}m)((n{+}m)x{-}y(q{+}p)){+}(y{-}x^2)((n{+}m)(q{+}p){-}lx)){+}((n{+}m)(q{+}p){-}lx)((n{+}m)x{-}y(q{+}p)){+}(2bc{+}c{+}b)(y{-}x^2)(x(q{+}p){-}n{-}m))]/[({-}y(q{+}p)^2{+}2(n{+}m)x(q{+}p){+}ly{-}lx^2{-}(n{+}m)^2)^4]$

$h_{8\ 10}^{(U_6)} = [t((c+b)(x((n+m)x-(q+p)y)+(n+m)(y-x^2)-((q+p)x-n-m)y)+$

$a(-2(q+p)((n+m)x-(q+p)y)-(l-(q+p)^2)y-2(q+p)((q+p)x-n-m)+(n+m)((q+p)x-n-m)+x((n+m)(q+p)-lx)+(l-(q+p)^2)x)+(ac+ab)(2x((q+p)x-n-m)-2(q+p)(y-x^2))+(n+m)((n+m)x-(q+p)y)+(2bc+c+b)x(y-x^2)-((n+m)(q+p)-lx)y)]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^2]-[2t(2(n+m)x-2(q+p)y)((c+b)(((q+p)x-n-m)((n+m)x-(q+p)y)+((n+m)(q+p)-lx)(y-x^2))+a((l-(q+p)^2)((n+m)x-(q+p)y)+((n+m)(q+p)-lx)((q+p)x-n-m)+(l-(q+p)^2)((q+p)x-n-m))+(ac+ab)((l-(q+p)^2)(y-x^2)+((q+p)x-n-m)^2)+((n+m)(q+p)-lx)((n+m)x-(q+p)y)+(2bc+c+b)((q+p)x-n-m)(y-x^2))]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^3]$

$h_{99}^{(U_6)}=[st(a(-2((n+m)x-y(q+p))-2(x(q+p)-n-m)+4y(q+p)-4x(q+p)+2(n+m)x)+(ac+ab)(2x^2-2(y-x^2))-2(c+b)xy-2(n+m)y)]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^2]-[4st(2(n+m)x-2y(q+p))(a(-y(l-(q+p)^2)+x(l-(q+p)^2)-2(q+p)((n+m)x-y(q+p))-2(q+p)(x(q+p)-n-m)+(n+m)(x(q+p)-n-m)+x((n+m)(q+p)-lx))+(c+b)(x((n+m)x-y(q+p))-y(x(q+p)-n-m)+(n+m)(y-x^2))+(ac+ab)(2x(x(q+p)-n-m)-2(y-x^2)(q+p))+(n+m)((n+m)x-y(q+p))-y((n+m)(q+p)-lx)+(2bc+c+b)x(y-x^2))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]+[4sty((ac+ab)((x(q+p)-n-m)^2+(y-x^2)(l-(q+p)^2))+a(((n+m)x-y(q+p))(l-(q+p)^2)+(x(q+p)-n-m)(l-(q+p)^2)+((n+m)(q+p)-lx)(x(q+p)-n-m))+(c+b)((x(q+p)-n-m)((n+m)x-y(q+p))+(y-x^2)((n+m)(q+p)-lx))+((n+m)(q+p)-lx)((n+m)x-y(q+p))+(2bc+c+b)(y-x^2)(x(q+p)-n-m))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]+[6st(2(n+m)x-2y(q+p))^2((ac+ab)((x(q+p)-n-m)^2+(y-x^2)(l-(q+p)^2))+a(((n+m)x-y(q+p))(l-(q+p)^2)+(x(q+p)-n-m)(l-(q+p)^2)+((n+m)(q+p)-lx)(x(q+p)-n-m))+(c+b)((x(q+p)-n-m)((n+m)x-y(q+p))+(y-x^2)((n+m)(q+p)-lx))+((n+m)(q+p)-lx)((n+m)x-y(q+p))+(2bc+c+b)(y-x^2)(x(q+p)-n-m))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^4]$

$h_{9\ 10}^{(U_6)}=[t((c+b)(x((n+m)x-(q+p)y)+(n+m)(y-x^2)-((q+p)x-n-m)y)+a(-2(q+p)((n+m)x-(q+p)y)-(l-(q+p)^2)y-2(q+p)((q+p)x-n-m)+(n+m)((q+p)x-n-m)+x((n+m)(q+p)-lx)+(l-(q+p)^2)x)+(ac+ab)(2x((q+p)x-n-m)-2(q+p)(y-x^2))+(n+m)((n+m)x-(q+p)y)+(2bc+c+b)x(y-x^2)-((n+m)(q+p)-lx)y)]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^2]-$

$[2t(2(n+m)x-2(q+p)y)((c+b)(((q+p)x-n-m)((n+m)x-(q+p)y)+((n+m)(q+p)-lx)(y-x^2))+a((l-(q+p)^2)((n+m)x-(q+p)y)+((n+m)(q+p)-lx)((q+p)x-n-m)+(l-(q+p)^2)((q+p)x-n-m))+(ac+ab)((l-(q+p)^2)(y-x^2)+((q+p)x-n-m)^2)+((n+m)(q+p)-lx)((n+m)x-(q+p)y)+(2bc+c+b)((q+p)x-n-m)(y-x^2))]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^3]$

$h^{(U_6)}_{10\,10}=0$

$g_4=-\dfrac{1}{yl+2x(m+n)(p+q)-(m+n)^2-(p+q)^2y-x^2l}[a((yl-(m+n)^2)z+((p+q)(m+n)-xl)t+(x(m+n)-(p+q)y)s)+b(((p+q)(m+n)-xl)z+(l-(p+q)^2)t+((p+q)x-(m+n))s)+c((x(m+n)-(p+q)y)z+((p+q)x-(m+n))t+(y-x^2)s)]$

$h^{(W)}_{11}=[2cs]/[-lx^2+2(n+m)(q+p)x-(q+p)^2y+ly-(n+m)^2]-[2l(c(s(y-x^2)+t((q+p)x-n-m)+z((n+m)x-(q+p)y))+b(s((q+p)x-n-m)+z((n+m)(q+p)-lx)+(l-(q+p)^2)t)+a(s((n+m)x-(q+p)y)+t((n+m)(q+p)-lx)+(ly-(n+m)^2)z))]/[(-lx^2+2(n+m)(q+p)x-(q+p)^2y+ly-(n+m)^2)^2]+[2(2(n+m)(q+p)-2lx)(c(-2sx+(n+m)z+(q+p)t)+b((q+p)s-lz)+a((n+m)s-lt))]/[(-lx^2+2(n+m)(q+p)x-(q+p)^2y+ly-(n+m)^2)^2]-[2(2(n+m)(q+p)-2lx)^2(c(s(y-x^2)+t((q+p)x-n-m)+z((n+m)x-(q+p)y))+b(s((q+p)x-n-m)+z((n+m)(q+p)-lx)+(l-(q+p)^2)t)+a(s((n+m)x-(q+p)y)+t((n+m)(q+p)-lx)+(ly-(n+m)^2)z))]/[(-lx^2+2(n+m)(q+p)x-(q+p)^2y+ly-(n+m)^2)^3]$

$h^{(W)}_{12}=[(2(n+m)(q+p)-2lx)(c((-q-p)z+s)+a(lz+(-q-p)s))]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^2]+[(l-(q+p)^2)(c((n+m)z-2sx+(q+p)t)+b((q+p)s-lz)+a((n+m)s-lt))]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^2]-[2(l-(q+p)^2)(2(n+m)(q+p)-2lx)(c(z((n+m)x-(q+p)y)+s(y-x^2)+t((q+p)x-n-m))+a(s((n+m)x-(q+p)y)+z(ly-(n+m)^2)+t((n+m)(q+p)-lx))+b(((n+m)(q+p)-lx)z+s((q+p)x-n-m)+(l-(q+p)^2)t))]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^3]$

$h^{(W)}_{13}=[(2(n+m)(q+p)-2lx)(c((n+m)x-(q+p)y)+a(ly-(n+m)^2)+b((n+m)(q+p)-lx))]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^2]-[c(n+m)-bl]/[-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2]$

$h^{(W)}_{14}=[(2(n+m)(q+p)-2lx)(c((q+p)x-n-m)+a((n+m)(q+p)-lx)+b(l-(q+p)^2))]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^2]-[c(q+p)-$

$al]/[-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2]$

$h_{15}^{(W)} = -[-bz-at]/[yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2]+[(a(yz-tx)+b(t-xz))(2(n+m)(q+p)-2xl)]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^2]-[2x(a(z(yl-(n+m)^2)+t((n+m)(q+p)-xl)+s((n+m)x-(q+p)y))+b(z((n+m)(q+p)-xl)+t(l-(q+p)^2)+s((q+p)x-n-m))+c(((n+m)x-(q+p)y)z+s(y-x^2)+t((q+p)x-n-m)))]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^2]-[2(y-x^2)(2(n+m)(q+p)-2xl)(a(z(yl-(n+m)^2)+t((n+m)(q+p)-xl)+s((n+m)x-(q+p)y))+b(z((n+m)(q+p)-xl)+t(l-(q+p)^2)+s((q+p)x-n-m))+c(((n+m)x-(q+p)y)z+s(y-x^2)+t((q+p)x-n-m)))]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^3]+[(y-x^2)(b((q+p)s-zl)+a((n+m)s-tl)+c((n+m)z-2sx+(q+p)t))]/[(yl-x^2l-(q+p)^2y+2(n+m)(q+p)x-(n+m)^2)^2]$

$h_{16}^{(W)} = -[cz+as]/[-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2]+[2(q+p)(a(z(ly-(m+n)^2)+s(x(m+n)-(q+p)y)+t((q+p)(m+n)-lx))+c(z(x(m+n)-(q+p)y)+t(-m+(q+p)x-n)+s(y-x^2))+b(z((q+p)(m+n)-lx)+s(-m+(q+p)x-n)+(l-(q+p)^2)t))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^2]+[(2(q+p)x-2(m+n))(c(z(m+n)-2sx+(q+p)t)+a(s(m+n)-lt)+b((q+p)s-lz))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^2]+[(2(q+p)(m+n)-2lx)(a(-2z(m+n)+sx+(q+p)t)+c(xz-t)+b((q+p)z-s))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^2]-[2(2(q+p)x-2(m+n))(2(q+p)(m+n)-2lx)(a(z(ly-(m+n)^2)+s(x(m+n)-(q+p)y)+t((q+p)(m+n)-lx))+c(z(x(m+n)-(q+p)y)+t(-m+(q+p)x-n)+s(y-x^2))+b(z((q+p)(m+n)-lx)+s(-m+(q+p)x-n)+(l-(q+p)^2)t))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^3]$

$h_{17}^{(W)} = -[cz+as]/[-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2]+[2(q+p)(a(z(ly-(n+m)^2)+s(x(n+m)-(q+p)y)+t((q+p)(n+m)-lx))+c(z(x(n+m)-(q+p)y)+t(-n+(q+p)x-m)+s(y-x^2))+b(z((q+p)(n+m)-lx)+s(-n+(q+p)x-m)+(l-(q+p)^2)t))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^2]+[(2(q+p)x-2(n+m))(c(z(n+m)-2sx+(q+p)t)+a(s(n+m)-lt)+b((q+p)s-lz))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^2]+[(2(q+p)(n+m)-2lx)(a(-2z(n+m)+sx+(q+p)t)+c(xz-t)+b((q+p)z-s))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^2]-[2(2(q+p)x-2(n+m))(2(q+p)(n+m)-$

$2lx)(a(z(ly-(n+m)^2)+s(x(n+m)-(q+p)y)+t((q+p)(n+m)-lx))+c(z(x(n+m)-(q+p)y)+t(-n+(q+p)x-m)+s(y-x^2))+b(z((q+p)(n+m)-lx)+s(-n+(q+p)x-m)+(l-(q+p)^2)t))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]$

$h_{18}^{(W)} = -[ct+bs]/[-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2]+[2(n+m)(b(t(l-(p+q)^2)+s(x(p+q)-n-m)+z((n+m)(p+q)-lx))+c(z((n+m)x-y(p+q))+t(x(p+q)-n-m)+s(y-x^2))+a(s((n+m)x-y(p+q))+t((n+m)(p+q)-lx)+(ly-(n+m)^2)z))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^2]+[(2(n+m)x-2y(p+q))(c(t(p+q)+(n+m)z-2sx)+b(s(p+q)-lz)+a((n+m)s-lt))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^2]+[(2(n+m)(p+q)-2lx)(b(-2t(p+q)+(n+m)z+sx)+c(tx-yz)+a((n+m)t-sy))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^2]-[2(2(n+m)(p+q)-2lx)(2(n+m)x-2y(p+q))(b(t(l-(p+q)^2)+s(x(p+q)-n-m)+z((n+m)(p+q)-lx))+c(z((n+m)x-y(p+q))+t(x(p+q)-n-m)+s(y-x^2))+a(s((n+m)x-y(p+q))+t((n+m)(p+q)-lx)+(ly-(n+m)^2)z))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]$

$h_{19}^{(W)} = -[ct+bs]/[-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2]+[2(n+m)(b(t(l-(q+p)^2)+s(x(q+p)-n-m)+z((n+m)(q+p)-lx))+c(z((n+m)x-y(q+p))+t(x(q+p)-n-m)+s(y-x^2))+a(s((n+m)x-y(q+p))+t((n+m)(q+p)-lx)+(ly-(n+m)^2)z))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^2]+[(2(n+m)x-2y(q+p))(c(t(q+p)+(n+m)z-2sx)+b(s(q+p)-lz)+a((n+m)s-lt))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^2]+[(2(n+m)(q+p)-2lx)(b(-2t(q+p)+(n+m)z+sx)+c(tx-yz)+a((n+m)t-sy))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^2]-[2(2(n+m)(q+p)-2lx)(2(n+m)x-2y(q+p))(b(t(l-(q+p)^2)+s(x(q+p)-n-m)+z((n+m)(q+p)-lx))+c(z((n+m)x-y(q+p))+t(x(q+p)-n-m)+s(y-x^2))+a(s((n+m)x-y(q+p))+t((n+m)(q+p)-lx)+(ly-(n+m)^2)z))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]$

$h_{1\ 10}^{(W)} = [(2(n+m)(q+p)-2lx)(a((n+m)x-(q+p)y)+c(y-x^2)+b((q+p)x-n-m))]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^2]-[-2cx+b(q+p)+a(n+m)]/[-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2]$

$h_{22}^{(W)} = [2(q^2+2pq+p^2-l)((cq+cp-al)x-bq^2+(-2bp+an+am)q-bp^2+(an+am)p-cn-cm+bl)((lx+(-n-m)q+(-n-m)p)z+(-q-p)sx+(q^2+2pq+p^2-l)t+$

$(n+m)s)]/[((q^2+2pq+p^2-l)y+lx^2+((-2n-2m)q+(-2n-2m)p)x+n^2+2mn+m^2)^3]$

$h_{23}^{(W)} = -[(lx+(-n-m)q+(-n-m)p)((cq+cp-al)x-bq^2+(-2bp+an+am)q-bp^2+(an+am)p-cn-cm+bl)]/[((q^2+2pq+p^2-l)y+lx^2+((-2n-2m)q+(-2n-2m)p)x+n^2+2mn+m^2)^2]$

$h_{24}^{(W)} = -[(q^2+2pq+p^2-l)((cq+cp-al)x-bq^2+(-2bp+an+am)q-bp^2+(an+am)p-cn-cm+bl)]/[((q^2+2pq+p^2-l)y+lx^2+((-2n-2m)q+(-2n-2m)p)x+n^2+2mn+m^2)^2]$

$h_{25}^{(W)} = -[(b-ax)(z(xl+(-n-m)q+(-n-m)p)+t(-l+q^2+2pq+p^2)+(-q-p)sx+(n+m)s)]/[(x^2l+y(-l+q^2+2pq+p^2)+((-2n-2m)q+(-2n-2m)p)x+n^2+2mn+m^2)^2]+[2(x^2-y)(x(-al+cq+cp)+bl-bq^2+(-2bp+an+am)q-bp^2+(an+am)p-cn-cm)(z(xl+(-n-m)q+(-n-m)p)+t(-l+q^2+2pq+p^2)+(-q-p)sx+(n+m)s)]/[(x^2l+y(-l+q^2+2pq+p^2)+((-2n-2m)q+(-2n-2m)p)x+n^2+2mn+m^2)^3]-[(xz-t)(x(-al+cq+cp)+bl-bq^2+(-2bp+an+am)q-bp^2+(an+am)p-cn-cm)]/[(x^2l+y(-l+q^2+2pq+p^2)+((-2n-2m)q+(-2n-2m)p)x+n^2+2mn+m^2)^2]$

$h_{26}^{(W)} = -[((-q-p)z+s)(q(am-2bp+an)+p(am+an)-cm+(cq+cp-al)x-bq^2-bp^2-cn+bl)]/[(m^2+2nm+x(q(-2m-2n)+p(-2m-2n))+(q^2+2pq+p^2-l)y+lx^2+n^2)^2]-[(aq+ap-c)(s(m+n)+z(q(-m-n)+p(-m-n)+lx)+(-q-p)sx+(q^2+2pq+p^2-l)t)]/[(m^2+2nm+x(q(-2m-2n)+p(-2m-2n))+(q^2+2pq+p^2-l)y+lx^2+n^2)^2]+[2(2m+(-2q-2p)x+2n)(s(m+n)+z(q(-m-n)+p(-m-n)+lx)+(-q-p)sx+(q^2+2pq+p^2-l)t)(q(am-2bp+an)+p(am+an)-cm+(cq+cp-al)x-bq^2-bp^2-cn+bl)]/[(m^2+2nm+x(q(-2m-2n)+p(-2m-2n))+(q^2+2pq+p^2-l)y+lx^2+n^2)^3]$

$h_{27}^{(W)} = -[((-q-p)z+s)(q(an-2bp+am)+p(an+am)-cn+(cq+cp-al)x-bq^2-bp^2-cm+bl)]/[(n^2+2mn+x(q(-2n-2m)+p(-2n-2m))+(q^2+2pq+p^2-l)y+lx^2+m^2)^2]-[(aq+ap-c)(s(n+m)+z(q(-n-m)+p(-n-m)+lx)+(-q-p)sx+(q^2+2pq+p^2-l)t)]/[(n^2+2mn+x(q(-2n-2m)+p(-2n-2m))+(q^2+2pq+p^2-l)y+lx^2+m^2)^2]+[2(2n+(-2q-2p)x+2m)(s(n+m)+z(q(-n-m)+p(-n-m)+lx)+(-q-p)sx+(q^2+2pq+p^2-l)t)(q(an-2bp+am)+p(an+am)-cn+(cq+cp-al)x-bq^2-bp^2-cm+bl)]/[(n^2+2mn+x(q(-2n-2m)+p(-2n-2m))+(q^2+2pq+p^2-l)y+lx^2+m^2)^3]$

$h_{28}^{(W)} = -[(-2bp+cx-2bq+an+am)(t(p^2+2qp+q^2-l)+z((-n-m)p+lx+(-n-m)q)+sx(-p-q)+(n+m)s)]/[(y(p^2+2qp+q^2-l)+x((-2n-2m)p+$

$(-2n-2m)q) + lx^2 + n^2 + 2mn + m^2)^2] - [(t(2p+2q) + (-n-m)z - sx)(-bp^2 + x(cp+cq-al) + q(-2bp+an+am) + (an+am)p - bq^2 - cn - cm + bl)]/[(y(p^2+2qp+q^2-l) + x((-2n-2m)p + (-2n-2m)q) + lx^2 + n^2 + 2mn + m^2)^2] + [2(y(2p+2q) + (-2n-2m)x)(-bp^2 + x(cp+cq-al) + q(-2bp+an+am) + (an+am)p - bq^2 - cn - cm + bl)(t(p^2+2qp+q^2-l) + z((-n-m)p + lx + (-n-m)q) + sx(-p-q) + (n+m)s)]/[(y(p^2+2qp+q^2-l) + x((-2n-2m)p + (-2n-2m)q) + lx^2 + n^2 + 2mn + m^2)^3]$

$h_{29}^{(W)} = -[(-2bq + cx - 2bp + an + am)(t(q^2+2pq+p^2-l) + z((-n-m)q + lx + (-n-m)p) + sx(-q-p) + (n+m)s)]/[(y(q^2+2pq+p^2-l) + x((-2n-2m)q + (-2n-2m)p) + lx^2 + n^2 + 2mn + m^2)^2] - [(t(2q+2p) + (-n-m)z - sx)(-bq^2 + x(cq+cp-al) + (-2bp+an+am)q - bp^2 + (an+am)p - cn - cm + bl)]/[(y(q^2+2pq+p^2-l) + x((-2n-2m)q + (-2n-2m)p) + lx^2 + n^2 + 2mn + m^2)^2] + [2(y(2q+2p) + (-2n-2m)x)(-bq^2 + x(cq+cp-al) + (-2bp+an+am)q - bp^2 + (an+am)p - cn - cm + bl)(t(q^2+2pq+p^2-l) + z((-n-m)q + lx + (-n-m)p) + sx(-q-p) + (n+m)s)]/[(y(q^2+2pq+p^2-l) + x((-2n-2m)q + (-2n-2m)p) + lx^2 + n^2 + 2mn + m^2)^3]$

$h_{2\ 10}^{(W)} = -[((-q-p)x + n + m)((cq+cp-al)x - bq^2 + (-2bp+an+am)q - bp^2 + (an+am)p - cn - cm + bl)]/[((q^2+2pq+p^2-l)y + lx^2 + ((-2n-2m)q + (-2n-2m)p)x + n^2 + 2mn + m^2)^2]$

$h_{33}^{(W)} = 0$

$h_{34}^{(W)} = 0$

$h_{35}^{(W)} = [bx - ay]/[yl - x^2l - (q+p)^2y + 2(n+m)(q+p)x - (n+m)^2] - [(y - x^2)(-a(yl - (n+m)^2) - b((n+m)(q+p) - xl) - c((n+m)x - (q+p)y))]/[(yl - x^2l - (q+p)^2y + 2(n+m)(q+p)x - (n+m)^2)^2]$

$h_{36}^{(W)} = [2a(m+n) - cx - b(q+p)]/[-(m+n)^2 + 2(q+p)x(m+n) - (q+p)^2y + ly - lx^2] - [(2(q+p)x - 2(m+n))(-a(ly - (m+n)^2) - c(x(m+n) - (q+p)y) - b((q+p)(m+n) - lx))]/[(-(m+n)^2 + 2(q+p)x(m+n) - (q+p)^2y + ly - lx^2)^2]$

$h_{37}^{(W)} = [2a(n+m) - cx - b(q+p)]/[-(n+m)^2 + 2(q+p)x(n+m) - (q+p)^2y + ly - lx^2] - [(2(q+p)x - 2(n+m))(-a(ly - (n+m)^2) - c(x(n+m) - (q+p)y) - b((q+p)(n+m) - lx))]/[(-(n+m)^2 + 2(q+p)x(n+m) - (q+p)^2y + ly - lx^2)^2]$

$h_{38}^{(W)} = [cy - b(n+m)]/[-y(p+q)^2 + 2(n+m)x(p+q) + ly - lx^2 - (n+m)^2] - [(2(n+m)x - 2y(p+q))(-c((n+m)x - y(p+q)) - b((n+m)(p+q) - lx) - a(ly -$

$(n+m)^2))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^2]$

$h_{39}^{(W)} = [cy-b(n+m)]/[-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2] - [(2(n+m)x-2y(q+p))(-c((n+m)x-y(q+p))-b((n+m)(q+p)-lx)-a(ly-(n+m)^2))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^2]$

$h_{3\ 10}^{(W)} = 0$

$h_{44}^{(W)} = 0$

$h_{45}^{(W)} = -[((q+p)x-n-m)((aq+ap-c)y+cx^2+(-bq-bp-an-am)x+bn+bm)]/[((y-x^2)l+(-q^2-2pq-p^2)y+((2n+2m)q+(2n+2m)p)x-n^2-2mn-m^2)^2]$

$h_{46}^{(W)} = [c-a(q+p)]/[-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2] - [(2(q+p)x-2(m+n))(-a((q+p)(m+n)-lx)-c(-m+(q+p)x-n)-b(l-(q+p)^2))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^2]$

$h_{47}^{(W)} = [c-a(q+p)]/[-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2] - [(2(q+p)x-2(n+m))(-a((q+p)(n+m)-lx)-c(-n+(q+p)x-m)-b(l-(q+p)^2))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^2]$

$h_{48}^{(W)} = [2b(p+q)-cx-a(n+m)]/[-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2] - [(2(n+m)x-2y(p+q))(-b(l-(p+q)^2)-c(x(p+q)-n-m)-a((n+m)(p+q)-lx))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^2]$

$h_{49}^{(W)} = [2b(q+p)-cx-a(n+m)]/[-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2] - [(2(n+m)x-2y(q+p))(-b(l-(q+p)^2)-c(x(q+p)-n-m)-a((n+m)(q+p)-lx))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^2]$

$h_{4\ 10}^{(W)} = 0$

$h_{55}^{(W)} = -[2(y-x^2)((aq+ap-c)y+cx^2+(-bq-bp-an-am)x+bn+bm)(((q+p)y+(-n-m)x)z-sy+sx^2+(-q-p)tx+(n+m)t)]/[((y-x^2)l+(-q^2-2pq-p^2)y+((2n+2m)q+(2n+2m)p)x-n^2-2mn-m^2)^3]$

$h_{56}^{(W)} = [(t-xz)(x(-am-bq-bp-an)+bm+(aq+ap-c)y+cx^2+bn)]/[(-m^2+x(q(2m+2n)+p(2m+2n))-2nm+l(y-x^2)+(-q^2-2pq-p^2)y-n^2)^2] + [(b-ax)(t(m+n)+z(x(-m-n)+(q+p)y)-sy+sx^2+(-q-p)tx)]/[(-m^2+x(q(2m+2n)+p(2m+2n))-2nm+l(y-x^2)+(-q^2-2pq-p^2)y-n^2)^2] - [2(-2m+(2q+2p)x-2n)(t(m+n)+z(x(-m-n)+(q+p)y)-sy+sx^2+(-q-p)tx)(x(-am-bq-bp-an)+bm+(aq+ap-c)y+cx^2+bn)]/[(-m^2+x(q(2m+2n)+p(2m+$

$2n)) - 2nm + l(y - x^2) + (-q^2 - 2pq - p^2)y - n^2)^3]$

$h_{57}^{(W)} = [(t - xz)(x(-an - bq - bp - am) + bn + (aq + ap - c)y + cx^2 + bm)]/[(-n^2 + x(q(2n + 2m) + p(2n + 2m)) - 2mn + l(y - x^2) + (-q^2 - 2pq - p^2)y - m^2)^2] + [(b - ax)(t(n + m) + z(x(-n - m) + (q + p)y) - sy + sx^2 + (-q - p)tx)]/[(-n^2 + x(q(2n + 2m) + p(2n + 2m)) - 2mn + l(y - x^2) + (-q^2 - 2pq - p^2)y - m^2)^2] - [2(-2n + (2q + 2p)x - 2m)(t(n + m) + z(x(-n - m) + (q + p)y) - sy + sx^2 + (-q - p)tx)(x(-an - bq - bp - am) + bn + (aq + ap - c)y + cx^2 + bm)]/[(-n^2 + x(q(2n + 2m) + p(2n + 2m)) - 2mn + l(y - x^2) + (-q^2 - 2pq - p^2)y - m^2)^3]$

$h_{58}^{(W)} = [(ay - bx)(z(y(p + q) + (-n - m)x) + tx(-p - q) - sy + sx^2 + (n + m)t)]/[(y(-p^2 - 2qp - q^2) + x((2n + 2m)p + (2n + 2m)q) + l(y - x^2) - n^2 - 2mn - m^2)^2] + [(yz - tx)(x(-bp - bq - an - am) + y(ap + aq - c) + cx^2 + bn + bm)]/[(y(-p^2 - 2qp - q^2) + x((2n + 2m)p + (2n + 2m)q) + l(y - x^2) - n^2 - 2mn - m^2)^2] - [2(y(-2p - 2q) + (2n + 2m)x)(x(-bp - bq - an - am) + y(ap + aq - c) + cx^2 + bn + bm)(z(y(p + q) + (-n - m)x) + tx(-p - q) - sy + sx^2 + (n + m)t)]/[(y(-p^2 - 2qp - q^2) + x((2n + 2m)p + (2n + 2m)q) + l(y - x^2) - n^2 - 2mn - m^2)^3]$

$h_{59}^{(W)} = [(ay - bx)(z(y(q + p) + (-n - m)x) + tx(-q - p) - sy + sx^2 + (n + m)t)]/[(y(-q^2 - 2pq - p^2) + x((2n + 2m)q + (2n + 2m)p) + l(y - x^2) - n^2 - 2mn - m^2)^2] + [(yz - tx)(x(-bq - bp - an - am) + y(aq + ap - c) + cx^2 + bn + bm)]/[(y(-q^2 - 2pq - p^2) + x((2n + 2m)q + (2n + 2m)p) + l(y - x^2) - n^2 - 2mn - m^2)^2] - [2(y(-2q - 2p) + (2n + 2m)x)(x(-bq - bp - an - am) + y(aq + ap - c) + cx^2 + bn + bm)(z(y(q + p) + (-n - m)x) + tx(-q - p) - sy + sx^2 + (n + m)t)]/[(y(-q^2 - 2pq - p^2) + x((2n + 2m)q + (2n + 2m)p) + l(y - x^2) - n^2 - 2mn - m^2)^3]$

$h_{5\ 10}^{(W)} = [(x^2 - y)((aq + ap - c)y + cx^2 + (-bq - bp - an - am)x + bn + bm)]/[(l(y - x^2) + (-q^2 - 2pq - p^2)y + ((2n + 2m)q + (2n + 2m)p)x - n^2 - 2mn - m^2)^2]$

$h_{66}^{(W)} = [2az]/[-(m + n)^2 + 2(q + p)x(m + n) - (q + p)^2y + ly - lx^2] - [2(a(z(ly - (m + n)^2) + s(x(m + n) - (q + p)y) + t((q + p)(m + n) - lx)) + c(z(x(m + n) - (q + p)y) + t(-m + (q + p)x - n) + s(y - x^2)) + b(z((q + p)(m + n) - lx) + s(-m + (q + p)x - n) + (l - (q + p)^2)t))]/[(-(m + n)^2 + 2(q + p)x(m + n) - (q + p)^2y + ly - lx^2)^2] + [2(2(q + p)x - 2(m + n))(a(-2z(m + n) + sx + (q + p)t) + c(xz - t) + b((q + p)z - s))]/[(-(m + n)^2 + 2(q + p)x(m + n) - (q + p)^2y + ly - lx^2)^2] - [2(2(q + p)x - 2(m +$

$n))^2(a(z(ly-(m+n)^2)+s(x(m+n)-(q+p)y)+t((q+p)(m+n)-lx))+c(z(x(m+n)-(q+p)y)+t(-m+(q+p)x-n)+s(y-x^2))+b(z((q+p)(m+n)-lx)+s(-m+(q+p)x-n)+(l-(q+p)^2)t))]/[(-(m+n)^2+2(q+p)x(m+n)-(q+p)^2y+ly-lx^2)^3]$

$h_{67}^{(W)}=[2az]/[-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2]-[2(a(z(ly-(n+m)^2)+s(x(n+m)-(q+p)y)+t((q+p)(n+m)-lx))+c(z(x(n+m)-(q+p)y)+t(-n+(q+p)x-m)+s(y-x^2))+b(z((q+p)(n+m)-lx)+s(-n+(q+p)x-m)+(l-(q+p)^2)t))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^2]+[2(2(q+p)x-2(n+m))(a(-2z(n+m)+sx+(q+p)t)+c(xz-t)+b((q+p)z-s))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^2]-[2(2(q+p)x-2(n+m))^2(a(z(ly-(n+m)^2)+s(x(n+m)-(q+p)y)+t((q+p)(n+m)-lx))+c(z(x(n+m)-(q+p)y)+t(-n+(q+p)x-m)+s(y-x^2))+b(z((q+p)(n+m)-lx)+s(-n+(q+p)x-m)+(l-(q+p)^2)t))]/[(-(n+m)^2+2(q+p)x(n+m)-(q+p)^2y+ly-lx^2)^3]$

$h_{68}^{(W)}=-[bz+at]/[-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2]+[2x(b(t(l-(p+q)^2)+s(x(p+q)-n-m)+z((n+m)(p+q)-lx))+c(z((n+m)x-y(p+q))+t(x(p+q)-n-m)+s(y-x^2))+a(s((n+m)x-y(p+q))+t((n+m)(p+q)-lx)+(ly-(n+m)^2)z))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^2]+[(2(n+m)x-2y(p+q))(b(z(p+q)-s)+a(t(p+q)-2(n+m)z+sx)+c(xz-t))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^2]+[(2x(p+q)-2(n+m))(b(-2t(p+q)+(n+m)z+sx)+c(tx-yz)+a((n+m)t-sy))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^2]-[2(2x(p+q)-2(n+m))(2(n+m)x-2y(p+q))(b(t(l-(p+q)^2)+s(x(p+q)-n-m)+z((n+m)(p+q)-lx))+c(z((n+m)x-y(p+q))+t(x(p+q)-n-m)+s(y-x^2))+a(s((n+m)x-y(p+q))+t((n+m)(p+q)-lx)+(ly-(n+m)^2)z))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]$

$h_{69}^{(W)}=-[bz+at]/[-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2]+[2x(b(t(l-(q+p)^2)+s(x(q+p)-n-m)+z((n+m)(q+p)-lx))+c(z((n+m)x-y(q+p))+t(x(q+p)-n-m)+s(y-x^2))+a(s((n+m)x-y(q+p))+t((n+m)(q+p)-lx)+(ly-(n+m)^2)z))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^2]+[(2(n+m)x-2y(q+p))(b(z(q+p)-s)+a(t(q+p)-2(n+m)z+sx)+c(xz-t))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^2]+[(2x(q+p)-2(n+m))(b(-2t(q+p)+(n+m)z+sx)+c(tx-yz)+a((n+m)t-sy))]/[(-y(q+p)^2+2(n+m)x(q+$

$p) + ly - lx^2 - (n+m)^2)^2] - [2(2x(q+p) - 2(n+m))(2(n+m)x - 2y(q+p))(b(t(l - (q+p)^2) + s(x(q+p) - n - m) + z((n+m)(q+p) - lx)) + c(z((n+m)x - y(q+p)) + t(x(q+p) - n - m) + s(y - x^2)) + a(s((n+m)x - y(q+p)) + t((n+m)(q+p) - lx) + (ly - (n+m)^2)z))]/[(-y(q+p)^2 + 2(n+m)x(q+p) + ly - lx^2 - (n+m)^2)^3]$

$h^{(W)}_{6\ 10} = [(2(q+p)x - 2(n+m))(a((n+m)x - (q+p)y) + c(y - x^2) + b((q+p)x - n - m))]/[(-(q+p)^2y + ly - lx^2 + 2(n+m)(q+p)x - (n+m)^2)^2] - [ax - b]/[-(q+p)^2y + ly - lx^2 + 2(n+m)(q+p)x - (n+m)^2]$

$h^{(W)}_{77} = [2az]/[-(n+m)^2 + 2(q+p)x(n+m) - (q+p)^2y + ly - lx^2] - [2(a(z(ly - (n+m)^2) + s(x(n+m) - (q+p)y) + t((q+p)(n+m) - lx)) + c(z(x(n+m) - (q+p)y) + t(-n + (q+p)x - m) + s(y - x^2)) + b(z((q+p)(n+m) - lx) + s(-n + (q+p)x - m) + (l - (q+p)^2)t))]/[(-(n+m)^2 + 2(q+p)x(n+m) - (q+p)^2y + ly - lx^2)^2] + [2(2(q+p)x - 2(n+m))(a(-2z(n+m) + sx + (q+p)t) + c(xz - t) + b((q+p)z - s))]/[(-(n+m)^2 + 2(q+p)x(n+m) - (q+p)^2y + ly - lx^2)^2] - [2(2(q+p)x - 2(n+m))^2(a(z(ly - (n+m)^2) + s(x(n+m) - (q+p)y) + t((q+p)(n+m) - lx)) + c(z(x(n+m) - (q+p)y) + t(-n + (q+p)x - m) + s(y - x^2)) + b(z((q+p)(n+m) - lx) + s(-n + (q+p)x - m) + (l - (q+p)^2)t))]/[(-(n+m)^2 + 2(q+p)x(n+m) - (q+p)^2y + ly - lx^2)^3]$

$h^{(W)}_{78} = -[bz + at]/[-y(p+q)^2 + 2(n+m)x(p+q) + ly - lx^2 - (n+m)^2] + [2x(b(t(l - (p+q)^2) + s(x(p+q) - n - m) + z((n+m)(p+q) - lx)) + c(z((n+m)x - y(p+q)) + t(x(p+q) - n - m) + s(y - x^2)) + a(s((n+m)x - y(p+q)) + t((n+m)(p+q) - lx) + (ly - (n+m)^2)z))]/[(-y(p+q)^2 + 2(n+m)x(p+q) + ly - lx^2 - (n+m)^2)^2] + [(2(n+m)x - 2y(p+q))(b(z(p+q) - s) + a(t(p+q) - 2(n+m)z + sx) + c(xz - t))]/[(-y(p+q)^2 + 2(n+m)x(p+q) + ly - lx^2 - (n+m)^2)^2] + [(2x(p+q) - 2(n+m))(b(-2t(p+q) + (n+m)z + sx) + c(tx - yz) + a((n+m)t - sy))]/[(-y(p+q)^2 + 2(n+m)x(p+q) + ly - lx^2 - (n+m)^2)^2] - [2(2x(p+q) - 2(n+m))(2(n+m)x - 2y(p+q))(b(t(l - (p+q)^2) + s(x(p+q) - n - m) + z((n+m)(p+q) - lx)) + c(z((n+m)x - y(p+q)) + t(x(p+q) - n - m) + s(y - x^2)) + a(s((n+m)x - y(p+q)) + t((n+m)(p+q) - lx) + (ly - (n+m)^2)z))]/[(-y(p+q)^2 + 2(n+m)x(p+q) + ly - lx^2 - (n+m)^2)^3]$

$h^{(W)}_{79} = -[bz + at]/[-y(q+p)^2 + 2(n+m)x(q+p) + ly - lx^2 - (n+m)^2] + [2x(b(t(l - (q+p)^2) + s(x(q+p) - n - m) + z((n+m)(q+p) - lx)) + c(z((n+m)x - y(q+p)) + t(x(q+p) - n - m) + s(y - x^2)) + a(s((n+m)x - y(q+p)) + t((n+m)(q+p) - lx) +$

$(ly-(n+m)^2)z))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^2]+[(2(n+m)x-2y(q+p))(b(z(q+p)-s)+a(t(q+p)-2(n+m)z+sx)+c(xz-t))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^2]+[(2x(q+p)-2(n+m))(b(-2t(q+p)+(n+m)z+sx)+c(tx-yz)+a((n+m)t-sy))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^2]-[2(2x(q+p)-2(n+m))(2(n+m)x-2y(q+p))(b(t(l-(q+p)^2)+s(x(q+p)-n-m)+z((n+m)(q+p)-lx))+c(z((n+m)x-y(q+p))+t(x(q+p)-n-m)+s(y-x^2))+a(s((n+m)x-y(q+p))+t((n+m)(q+p)-lx)+(ly-(n+m)^2)z))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]$

$h_{7\ 10}^{(W)}=[(2(q+p)x-2(n+m))(a((n+m)x-(q+p)y)+c(y-x^2)+b((q+p)x-n-m))]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^2]-[ax-b]/[-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2]$

$h_{88}^{(W)}=[2bt]/[-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2]-[2y(b(t(l-(p+q)^2)+s(x(p+q)-n-m)+z((n+m)(p+q)-lx))+c(z((n+m)x-y(p+q))+t(x(p+q)-n-m)+s(y-x^2))+a(s((n+m)x-y(p+q))+t((n+m)(p+q)-lx)+(ly-(n+m)^2)z))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^2]+[2(2(n+m)x-2y(p+q))(b(-2t(p+q)+(n+m)z+sx)+c(tx-yz)+a((n+m)t-sy))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^2]-[2(2(n+m)x-2y(p+q))^2(b(t(l-(p+q)^2)+s(x(p+q)-n-m)+z((n+m)(p+q)-lx))+c(z((n+m)x-y(p+q))+t(x(p+q)-n-m)+s(y-x^2))+a(s((n+m)x-y(p+q))+t((n+m)(p+q)-lx)+(ly-(n+m)^2)z))]/[(-y(p+q)^2+2(n+m)x(p+q)+ly-lx^2-(n+m)^2)^3]$

$h_{89}^{(W)}=[2bt]/[-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2]-[2y(b(t(l-(q+p)^2)+s(x(q+p)-n-m)+z((n+m)(q+p)-lx))+c(z((n+m)x-y(q+p))+t(x(q+p)-n-m)+s(y-x^2))+a(s((n+m)x-y(q+p))+t((n+m)(q+p)-lx)+(ly-(n+m)^2)z))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^2]+[2(2(n+m)x-2y(q+p))(b(-2t(q+p)+(n+m)z+sx)+c(tx-yz)+a((n+m)t-sy))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^2]-[2(2(n+m)x-2y(q+p))^2(b(t(l-(q+p)^2)+s(x(q+p)-n-m)+z((n+m)(q+p)-lx))+c(z((n+m)x-y(q+p))+t(x(q+p)-n-m)+s(y-x^2))+a(s((n+m)x-y(q+p))+t((n+m)(q+p)-lx)+(ly-(n+m)^2)z))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]$

$h_{8\ 10}^{(W)}=[(2(n+m)x-2(q+p)y)(a((n+m)x-(q+p)y)+c(y-x^2)+b((q+p)x-$

$n-m))]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^2]-[bx-ay]/[-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2]$

$h^{(W)}_{99} = [2bt]/[-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2]-[2y(b(t(l-(q+p)^2)+s(x(q+p)-n-m)+z((n+m)(q+p)-lx))+c(z((n+m)x-y(q+p))+t(x(q+p)-n-m)+s(y-x^2))+a(s((n+m)x-y(q+p))+t((n+m)(q+p)-lx)+(ly-(n+m)^2)z))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^2]+[2(2(n+m)x-2y(q+p))(b(-2t(q+p)+(n+m)z+sx)+c(tx-yz)+a((n+m)t-sy))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^2]-[2(2(n+m)x-2y(q+p))^2(b(t(l-(q+p)^2)+s(x(q+p)-n-m)+z((n+m)(q+p)-lx))+c(z((n+m)x-y(q+p))+t(x(q+p)-n-m)+s(y-x^2))+a(s((n+m)x-y(q+p))+t((n+m)(q+p)-lx)+(ly-(n+m)^2)z))]/[(-y(q+p)^2+2(n+m)x(q+p)+ly-lx^2-(n+m)^2)^3]$

$h^{(W)}_{9\ 10} = [(2(n+m)x-2(q+p)y)(a((n+m)x-(q+p)y)+c(y-x^2)+b((q+p)x-n-m))]/[(-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2)^2]-[bx-ay]/[-(q+p)^2y+ly-lx^2+2(n+m)(q+p)x-(n+m)^2]$

$h^{(W)}_{10\ 10} = 0$

Bibliography

[1] Joel N. Hirschhorn and Mark J. Daly. Genome-wide association studies for common diseases and complex traits. *Nature Reviews Genetics*, 6(2):95–108, 2005.

[2] Jason B. Wolf, Edmund D. Brodie, and Michael John Wade. *Epistasis and the evolutionary process*. Oxford University Press, 2000.

[3] Valdar W Willis-Owen SA. Deciphering gene-environment interactions through mouse models of allergic asthma. *Journal of Allergy and Clinical Immunology*, 123(1):24–5, 2009.

[4] Carl D. Schlichting and Massimo Pigiucci. *Phenotypic Evolution: A Reaction Norm Perspective*. Sinauer Associates Inc, 1998.

[5] Yukio Horik awa, Naohisa Oda, Nancy J. Cox, Xiangquan Li, Marju Orho-Melander, Manami Hara, Yoshinori Hinokio, Tom H. Lindner, Hirosato Mashima, Peter E.h. Schwarz, and et al. Genetic variation in the gene encoding calpain-10 is associated with type 2 diabetes mellitus. *Nature Genetics*, 26(2):163–75, 2000.

[6] P. St George-Hyslop, R. Tanzi, R. Polinsky, J. Haines, L Nee, P. Watkins, R. Myers, R. Feldman, D Pollen, D Drachman, and et al. The genetic defect causing familial alzheimers disease maps on chromosome 21. *Science*, 235(4791):885–90, 1987.

[7] Nazneen Rahman, Sheila Seal, Deborah Thompson, Patrick Kelly, Anthony Renwick, Anna Elliott, Sarah Reid, Katarina Spanova, Rita Barfoot, Tasnim Chagtai, and et al. Palb2, which encodes a brca2-interacting protein, is a breast cancer susceptibility gene. *Nature Genetics*, 39(2):165–167, 2006.

[8] Stanley Lemeshow David W. Hosmer. *Applied Logistic Regression.* John Wiley & Sons, 2000.

[9] M D Ritchie, L W Hahn, N Roodi, L R Bailey, W D Dupont, F F Parl, and J H Moore. Multifactor-dimensionality reduction reveals high-order interactions among estrogen-metabolism genes in sporadic breast cancer. *American journal of human genetics*, 69(1):138–47, Jul 2001.

[10] Hongying Dai, Richard J Charnigo, Mara L Becker, J Steven Leeder, and Alison A Motsinger-Reif. Risk score modeling of multiple gene to gene interactions using aggregated-multifactor dimensionality reduction. *BioData Mining*, 6(1), 2013.

[11] Damian Gola, Jestinah M. Mahachie John, Kristel Van Steen, and Inke R. König. A roadmap to multifactor dimensionality reduction methods. *Briefings in Bioinformatics*, 17(2):293–308, 2015.

[12] Jiang Gui, Angeline S. Andrew, Peter Andrews, Heather M. Nelson, Karl T. Kelsey, Margaret R. Karagas, and Jason H. Moore. A robust multifactor dimensionality reduction method for detecting gene-gene interactions with application to the genetic analysis of bladder cancer susceptibility. *Annals of Human Genetics*, 75(1):20–28, 2010.

[13] Hao Mei, Deqiong Ma, Allison Ashley-Koch, and Eden R Martin. Extension of multifactor dimensionality reduction for identifying multilocus effects in the gaw14 simulated data. *BMC Genetics*, 6(Suppl 1), 2005.

[14] Y. Chung, S. Y. Lee, R. C. Elston, and T. Park. Odds ratio based multifactor-dimensionality reduction method for detecting gene-gene interactions. *Bioinformatics*, 23(1):71–76, Aug 2006.

[15] Adan Niu, Shuanglin Zhang, and Qiuying Sha. A novel method to detect gene-gene interactions in structured populations: Mdr-sp. *Annals of Human Genetics*, 75(6):742–754, Apr 2011.

[16] Xing Hua, Han Zhang, Hong Zhang, Yaning Yang, and Anthony Y.c. Kuk. Testing multiple gene interactions by the ordered combinatorial partitioning method in case–control studies. *Bioinformatics*, 26(15):1871–1878, Oct 2010.

[17] Jiang Gui, Jason H. Moore, Scott M. Williams, Peter Andrews, Hans L. Hillege, Pim Van Der Harst, Gerjan Navis, Wiek H. Van Gilst, Folkert W. Asselbergs, Diane Gilbert-Diamond, and et al. A simple and computationally efficient approach to multifactor dimensionality reduction analysis of gene-gene interactions for quantitative traits. *PLoS ONE*, 8(6), 2013.

[18] Folkert W. Asselbergs, Scott M. Williams, Patricia R. Hebert, Christopher S. Coffey, Hans L. Hillege, Gerjan Navis, Douglas E. Vaughan, Wiek H. Van Gilst, and Jason H. Moore. Epistatic effects of polymorphisms in genes from the renin-angiotensin, bradykinin, and fibrinolytic systems on plasma t-pa and pai-1 levels. *Genomics*, 89(3):362–369, 2007.

[19] Xiang-Yang Lou, Guo-Bo Chen, Lei Yan, Jennie Z. Ma, Jun Zhu, Robert C. Elston, and Ming D. Li. A generalized combinatorial approach for detecting gene-by-gene and gene-by-environment interactions with application to nicotine dependence. *The American Journal of Human Genetics*, 80(6):1125–1137, 2007.

[20] H. He, W.s. Oetting, M.j. Brott, and S. Basu. Pair-wise multifactor dimensionality reduction method to detect gene-gene interactions in a case-control study. *Human Heredity*, 69(1):60–70, 2010.

[21] Rebecca E. Crouch. Aggregated quantitative multifactor dimensionality reduction. http://uknowledge.uky.edu/statistics_etds/25, 2016. Theses and Dissertations–Statistics. 25.

[22] Wikipedia contributors. Minimum mean square error. https://en.wikipedia.org/wiki/Minimum_mean_square_error, Oct 2018.

[23] Richard Lowry. Concepts and applications of inferential statistics. http://vassarstats.net/textbook/ch4apx.html, 2015.

[24] I.A. Lee and K.J. Preacher. Calculation for the test of the difference between two dependent correlations with one variable in common [computer software]. http://www.quantpsy.org/corrtest/corrtest2.htm, Sep 2013.

[25] THOMAS S. FERGUSON. *A COURSE IN LARGE SAMPLE THEORY*. Chapman & Hall, 1996.

[26] Derivative calculator. https://www.derivative-calculator.net. Derivative Calculator • With Steps!

[27] Markus MacGill. Alzheimer's disease: Symptoms, stages, causes, and treatment. *Medical News Today*, Feb 2018.

[28] What are the signs of alzheimer's disease? https://www.nia.nih.gov/health/what-are-signs-alzheimers-disease. National Institute on Aging, U.S. Department of Health and Human Services.

[29] Latest alzheimer's facts and figures. https://www.alz.org/facts/. Latest Facts & Figures Report — Alzheimer's Association.

[30] What causes alzheimer's disease? https://www.nia.nih.gov/health/what-causes-alzheimers-disease. National Institute on Aging, U.S. Department of Health and Human Services.

[31] Kaj Blennow, Bruno Dubois, Anne M. Fagan, Piotr Lewczuk, Mony J. de Leon, and Harald Hampel. *Alzheimer's & dementia : the journal of the Alzheimer's Association*, Jan 2015.

[32] J C Lambert, C A Ibrahim-Verbaas, D Harold, A C Naj, R Sims, C Bellenguez, A L DeStafano, J C Bis, G W Beecham, B Grenier-Boley, and et al. Meta-analysis of 74,046 individuals identifies 11 new susceptibility loci for alzheimer's disease. *Nature genetics.*, Dec 2013.

[33] A Griciuc, A SerranoPozo, A R Parrado, A N Lesinski, C N Asselin, K Mullin, B Hooli, S H Choi, B T Hyman, R E Tanzi, and et al. Alzheimer's disease risk gene CD33 inhibits microglial uptake of amyloid beta. *Neuron*, May 2013.

[34] M Allen, M Kachadoorian, Z Quicksall, F Zou, H S Chai, C Younkin, J E Crook, V S Pankratz, M M Carrasquillo, S Krishnan, and et al. Association of MAPT haplotypes with Alzheimer's disease risk and MAPT brain gene expression levels. *Alzheimer's research & therapy.*, Jul 2014.

[35] Ariel A. Hippen, Mark T. W. Ebbert, Maria C. Norton, Joann T. Tschanz, Ronald G. Munger, Christopher D. Corcoran, and John S. K. Kauwe. Presenilin E318G variant and Alzheimer's disease risk: the cache county study. *BMC Genomics*, 17(S3), 2016.

[36] Jonsson T et al. Variant of TREM2 associated with the risk of Alzheimer's disease. *N. Engl. J. Med*, 368:117–127, 2013.

[37] Adni — Alzheimer's disease neuroimaging initiative. http://adni.loni.usc.edu/.

[38] Griffiths AJF, Miller JH JH, Suzuki DT, and et al. *An Introduction to Genetic Analysis.* W. H. Freeman, 2000. Quantifying heritability. Available from: https://www.ncbi.nlm.nih.gov/books/NBK21866/.

[39] Jian Yang, S. Hong Lee, Michael E. Goddard, and Peter M. Visscher. GCTA: A tool for genomewide complex trait analysis. *American Journal of Human Genetics*, 88:76–82, 2011.

[40] Nielsen R, Paul JS, Albrechtsen A, and Song YS. Genotype and SNP calling from next-generation sequencing data. *Nat. Rev. Genet.*, page 443–451, Dec 2011.

[41] Michael H. Kutner. *Applied regression analysis.* McGraw-Hill Create, 2013.

[42] Sklar JJ, OH VM, and Li SC. Factors affecting therapeutic compliance: a review from the patient's perspective. *Ther Clin Risk Manag.*, page 269–286, Dec 2008.

[43] What factors change demand? https://www.khanacademy.org/economics-finance-domain/ap-microeconomics/ap-supply-demand-equilibrium/ap-demand-curve-tutorial/a/what-factors-change-demand.

Vita

Ye Li was born in Qiqihar, China.She received a bachelor degree in mathematics and applied mathematics at Shandong Normal University, China in 2011, a master degree in applied mathematics at Northwest Normal University, China in 2014, and a master degree in statistics at University of Kentucky in May 2016. Her research adviser is Dr. Richard Charnigo(UKy) in dimension reduction in identifying genetic interaction.