2019

Boundary Layers in Periodic Homogenization

Jinping Zhuge
University of Kentucky, jpzhuge@gmail.com
Digital Object Identifier: https://doi.org/10.13023/etd.2019.263

STUDENT AGREEMENT:

I represent that my thesis or dissertation and abstract are my original work. Proper attribution has been given to all outside sources. I understand that I am solely responsible for obtaining any needed copyright permissions. I have obtained needed written permission statement(s) from the owner(s) of each third-party copyrighted matter to be included in my work, allowing electronic distribution (if such use is not permitted by the fair use doctrine) which will be submitted to UKnowledge as Additional File.

I hereby grant to The University of Kentucky and its agents the irrevocable, non-exclusive, and royalty-free license to archive and make accessible my work in whole or in part in all forms of media, now or hereafter known. I agree that the document mentioned above may be made available immediately for worldwide access unless an embargo applies.

I retain all other ownership rights to the copyright of my work. I also retain the right to use in future works (such as articles or books) all or part of my work. I understand that I am free to register the copyright to my work.

REVIEW, APPROVAL AND ACCEPTANCE

The document mentioned above has been reviewed and accepted by the student's advisor, on behalf of the advisory committee, and by the Director of Graduate Studies (DGS), on behalf of the program; we verify that this is the final, approved version of the student's thesis including all changes required by the advisory committee. The undersigned agree to abide by the statements above.

Jinping Zhuge, Student

Dr. Zhongwei Shen, Major Professor

Dr. Peter Hislop, Director of Graduate Studies

Boundary Layers in Periodic Homogenization

DISSERTATION

A dissertation submitted in partial
fulfillment of the requirements for
the degree of Doctor of Philosophy
in the College of Arts and Sciences
at the University of Kentucky

By
Jinping Zhuge
Lexington, Kentucky

Director: Dr. Zhongwei Shen, Professor of Mathematics
Lexington, Kentucky
2019

ABSTRACT OF DISSERTATION

Boundary Layers in Periodic Homogenization

The boundary layer problems in periodic homogenization arise naturally from the quantitative analysis of convergence rates. Formally they are second-order linear elliptic systems with periodically oscillating coefficient matrix, subject to periodically oscillating Dirichelt or Neumann boundary data. In this dissertation, for either Dirichlet problem or Neumann problem, we establish the homogenization results and obtain the nearly sharp convergence rates, provided the domain is strictly convex. Also, we show that the homogenized boundary data is in $W^{1,p}$ for any $p \in (1, \infty)$, which implies the C^α-Hölder continuity for any $\alpha \in (0, 1)$.

KEYWORDS: Boundary layers, Periodic homogenization, Convergence rates

Author's signature: Jinping Zhuge

Date: June 29, 2019

Boundary Layers in Periodic Homogenization

By
Jinping Zhuge

Director of Dissertation: Zhongwei Shen

Director of Graduate Studies: Peter Hislop

Date: June 29, 2019

Dedicated to my wife Rongmei,
my daughter Jasmine and my son Grayson

ACKNOWLEDGMENTS

Most work contained in this dissertation was joint with my advisor Zhongwei Shen. I am deeply indebted to Dr. Shen for his consistent academic mentorship, support and encouragement during the past 6 years. His broad knowledge, extreme patience, and kind personality are and will alway be greatly admired and appreciated.

I would like to thank professors, Peter Hislop, John Lewis, Ke-Fei Liu and Nathan Jacobs for serving as my dissertation committee and outside examiner. I would also like to thank professor Russell Brown for his helpful suggestions from time to time on teaching, job search and more.

Then, many thanks go to the faculty, visiting scholars and my fellow graduate students (especially PDErs) in the department of mathematics in the University of Kentucky, for all kinds of assistance (such as language proofreading and teaching substitute), discussions and conversations.

The special thank goes to professor Zhengqiu Zhang, who devoted his time to organizing excellent seminars and helping me build the mathematical foundation of harmonic analysis and PDEs when I was in Nankai University.

Finally, and from the bottom of my heart, I would like to thank my wife, Rongmei Xiao, for patience, care and love.

The work in this dissertation was supported in part by National Science Foundation Grants DMS-1161154 and DMS-1600520.

TABLE OF CONTENTS

Chapter 1 Introduction

1.1 Motivation in homogenization

During the last four decades, the theory of homogenization, or averaging of partial differential equations with rapidly oscillating coefficients, has been studied extensively. This theory has many important applications in various physical problems in composite or heterogeneous materials. Mathematically, the characteristics of a microscopically self-similar heterogeneous material are usually described by rescaled functions in the form of $A(x/\varepsilon)$, where x is the spatial variable and $\varepsilon > 0$ is a small scalar parameter that represents the scale of the microstructure in the material. The unrescaled function $A(y)$, with the typical microscopically self-similar structures in practice, may be periodic, almost-periodic or a realization of a stationary ergodic random field. For example, in the heat conductivity problem, we use a matrix $A(\varepsilon^{-1}x)$ to describe the thermal conductivity tensor of a material. Then at equilibrium, the temperature distribution in a material body Ω satisfies the following elliptic partial differential equation with a Dirichlet boundary condition

$$\begin{cases} -\mathrm{div}(A(x/\varepsilon)\nabla u_\varepsilon) = F & \text{in } \Omega, \\ u_\varepsilon = f & \text{on } \partial\Omega, \end{cases} \tag{1.1.1}$$

where u_ε, depending implicitly in ε, represents the temperature field in Ω. In practice, computing the solution of the equation (1.1.1) numerically with rapidly oscillating coefficients $A(x/\varepsilon)$ is a difficult task if ε is tiny. However, if we view the problem from a macroscopic (or mesoscopic) scale, the heterogeneous microstructure will be invisible and the material, as well as the solution of the involved PDE, will exhibit some sort of averaging or homogeneous properties. Of course, the self-similar structure, such as periodicity or stationary randomness, will play an essential role in the averaging process. This is exactly the core principle behind the homogenization theory, whose goal is to represent or approximate a complex, heterogeneous material by a simple, homogeneous one.

In this dissertation, we study the periodic homogenization of linear elliptic equations and systems, which means we assume that the coefficients involved in the PDEs are periodic and can be measured precisely in a single microscopic periodic cell (at a one-time cost). To explain the classical theory of homogenization, we take the modeling equation (1.1.1) for example. Let ε vary in $(0, 1)$. The elliptic equation (1.1.1) generates a sequence of weak solutions $\{u_\varepsilon : 0 < \varepsilon < 1\}$ which lie in the Sobolev space $H^1(\Omega)$. The H^1 norms of these solutions are uniformly bounded, independent of ε. The first question in homogenization is the asymptotic behavior of the solutions u_ε as ε approaching zero. The answer to this classical question composes of two parts: (1) as $\varepsilon \to 0$, the entire sequence of solutions $\{u_\varepsilon\}$ converges weakly to a function u_0 in $H^1(\Omega)$ and strongly in $L^2(\Omega)$; (2) The limit function u_0 satisfies the so-called homogenized equation

$$\begin{cases} -\mathrm{div}(\widehat{A}\nabla u_0) = F & \text{in } \Omega, \\ u_0 = f & \text{on } \partial\Omega, \end{cases} \tag{1.1.2}$$

where $\widehat{A}$ is a constant matrix called homogenized or effective coefficient matrix. In terms of the above property, we will say the equation (1.1.1) homogenizes to (1.1.2). Theoretically, $\widehat{A}$ depends only on the original coefficient matrix $A(y)$ and can be computed by solving a periodic cell problem at a one-time cost (the explicit formula of $\widehat{A}$ may be found in Chapter 2). The above classical homogenization result provides an effective way to find a good approximation of u_ε if the microscopic scale ε is relatively small compared to the scale of material body Ω. In other words, to solve (1.1.1), we do not compute u_ε directly as the computation could be very costly. Instead, we compute u_0, the solution of (1.1.2), which is supposed to be much easier to solve numerically since the coefficients are constant; while the classical (qualitative) homogenization theory assures that the error $|u_\varepsilon - u_0|$ is small in the sense of L^2.

Recently, people are more interested in quantitative estimates in homogenization. One of the central questions in quantitative homogenization is the convergence rate or the quantitative two-scale asymptotic expansion. It has been well-known that the solution u_ε of (1.1.1) has a formal two-scale expansion as follows

$$u_\varepsilon(x) = u_0(x) + \varepsilon\chi(x/\varepsilon)\cdot\nabla u_0(x) + \varepsilon^2\Upsilon(x/\varepsilon)\cdot\nabla^2 u_0(x) + \cdots \tag{1.1.3}$$

where u_0 is the homogenized solution in (1.1.1), and $\chi(y)$ and $\Upsilon(y)$ are the (first-order) corrector and second-order corrector. We point out that these correctors are also periodic matrix-valued functions that depends only on the coefficient matrix A and may be computed by solving certain periodic cell problems. Now a natural question is that in what sense the asymptotic expansion (1.1.3) may hold rigorously. For example, in view of (1.1.3), one may expect to have the following

$$u_\varepsilon = u_0 + O(\varepsilon) \qquad \text{in } L^2(\Omega). \tag{1.1.4}$$

The precise meaning of (1.1.4) is that there exists a positive constant C, independent of ε, so that $\|u_\varepsilon - u_0\|_{L^2(\Omega)} \le C\varepsilon$. In fact, this sharp estimate has been established in many literatures in various settings. On the other hand, to derive an expansion in $H^1(\Omega)$, one has to take the next term in (1.1.3) into consideration. Actually, we have the following sharp estimate

$$u_\varepsilon(x) = u_0(x) + \varepsilon\chi(x/\varepsilon)\cdot\nabla u_0(x) + O(\sqrt{\varepsilon}) \qquad \text{in } H^1(\Omega). \tag{1.1.5}$$

This result is unexpected since, intuitively, (1.1.3) suggests that we should have $O(\varepsilon)$ error in (1.1.5), instead of $O(\sqrt{\varepsilon})$. This phenomenon, caused by the boundary layer effect, can be fixed by subtracting an additional term that corrects the boundary discrepancy. Indeed, if v_ε^D is the solution of

$$\begin{cases} -\mathrm{div}(A(x/\varepsilon)\nabla v_\varepsilon^D) = 0 & \text{in } \Omega, \\ v_\varepsilon^D = -\chi(x/\varepsilon)\cdot\nabla u_0(x) & \text{on } \partial\Omega, \end{cases} \tag{1.1.6}$$

where the supscript D indicates this is a boundary layer in the Dirichlet problem, then we can recover the $O(\varepsilon)$ rate in $H^1(\Omega)$

$$u_\varepsilon(x) = u_0(x) + \varepsilon\chi(x/\varepsilon)\cdot\nabla u_0(x) + +\varepsilon v_\varepsilon^D + O(\varepsilon) \qquad \text{in } H^1(\Omega). \tag{1.1.7}$$

See Theorem 2.2 for more details.

Similar phenomenon also takes place in the Neumann problem. To this end, let us consider the heat conductivity problem with a Neumann boundary condition

$$\begin{cases} -\operatorname{div}(A(x/\varepsilon)\nabla u_\varepsilon) = F & \text{in } \Omega, \\ \dfrac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = g & \text{on } \partial\Omega, \end{cases} \tag{1.1.8}$$

where $\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = n\cdot A(x/\varepsilon)\nabla u_\varepsilon$ is the conormal derivative, and n denotes the unit outward normal. For the solvability of Neumann problems, we require the so-called compatibility condition, namely, $\int_\Omega F + \int_{\partial\Omega} g = 0$. Moreover, for the uniqueness of the solution u_ε of (1.1.8), we will always assume $\int_\Omega u_\varepsilon = 0$. Now, in the same sense as the Dirichlet problem, (1.1.8) homogenizes to

$$\begin{cases} -\operatorname{div}(\widehat{A}\nabla u_0) = F & \text{in } \Omega, \\ \dfrac{\partial u_0}{\partial \nu_0} = g & \text{on } \partial\Omega, \end{cases} \tag{1.1.9}$$

where $\frac{\partial u_0}{\partial \nu_0} = n\cdot \widehat{A}\nabla u_0$ is the conormal derivative associated with $\widehat{A}$. Again, here we assume $\int_\Omega u_0 = 0$.

For the quantitative estimate of the Neumann problem, we still have the sharp estimates (1.1.4) and (1.1.5) as expected. Furthermore, we also have the recovered $O(\varepsilon)$ rate in $H^1(\Omega)$

$$u_\varepsilon(x) = u_0(x) + \varepsilon\chi(x/\varepsilon)\cdot\nabla u_0(x) + +\varepsilon v_\varepsilon^N + O(\varepsilon) \qquad \text{in } H^1(\Omega), \tag{1.1.10}$$

where v_ε^N is a boundary layer term in the Neumann problem given by the following equation

$$\begin{cases} -\operatorname{div}(A(x/\varepsilon)\nabla v_\varepsilon^N) = 0 & \text{in } \Omega, \\ \dfrac{\partial v_\varepsilon^N}{\partial \nu_\varepsilon} = \dfrac{1}{2}\Big(n_k\dfrac{\partial}{\partial x_i} - n_i\dfrac{\partial}{\partial x_k}\Big)\Big(\phi_{kij}(x/\varepsilon)\dfrac{\partial u_0}{\partial x_j}\Big) & \text{on } \partial\Omega, \end{cases} \tag{1.1.11}$$

where $n = (n_1, n_2, \cdots, n_d)$ is the unit outward normal. We would like to say a few words about the equation (1.1.11). In the boundary condition of (1.1.11), the Einstein's summation convention is used (and will be used throughout this dissertation), i.e., all the repeated indices are summed (here i, j, k are all summed from 1 to d with d being the dimension). The functions $\phi_{kij}(y)$ are periodic functions determined only by A. Most importantly, for each i and k, $n_k\frac{\partial}{\partial x_i} - n_i\frac{\partial}{\partial x_k}$ is a tangential derivative on $\partial\Omega$ that allows the integration by parts on $\partial\Omega$. This special structure is critical in our analysis for Neumann problem.

Finally, we mention briefly the higher-order convergence rates in H^1. For either Dirichlet or Neumann problem, one may show that in $H^1(\Omega)$

$$u_\varepsilon(x) = u_0(x) + \varepsilon\chi(x/\varepsilon)\cdot\nabla u_0(x) + +\varepsilon\tilde{v}_\varepsilon^X + \varepsilon^2\Upsilon(x/\varepsilon)\cdot\nabla^2 u_0(x) + O(\varepsilon^2), \quad (1.1.12)$$

where $\tilde{v}_\varepsilon^X$ is the boundary layer term for either Dirichlet or Neumann problem with similar structure as (1.1.6) or (1.1.11). However, the equation for $\tilde{v}_\varepsilon^X$ is much more complicated and the details will be carried out in Chapter 3.

Now, we are in a position to explain the motivation of this dissertation. First of all, we note that the function v_ε^X (or $\tilde{v}_\varepsilon^X$), where $X = D$ or N, depends implicitly on ε through both the oscillating coefficient matrix and the oscillating boundary condition. And it is not hard to see $\|v_\varepsilon^X\|_{H^1(\Omega)} \simeq O(\varepsilon^{-\frac{1}{2}})$ which blows up as $\varepsilon \to 0$. Now a natural and fundamental question in homogenization is what happens to v_ε^X as ε approaching zero. Precisely, we would like to ask: does v_ε^X converge in $L^2(\Omega)$? With what hypothesis? If so, what is the (sharp) rate and what can we say about the homogenized equation? The purpose of this dissertation is to give a comprehensive study on these questions and eventually provide a better understanding of the boundary layer phenomenon in periodic homogenization.

1.2 Statement of main results

This dissertation reorganize and present our recent work contained mainly in [38, 37, 42] where we studied the homogenization and boundary layers for elliptic systems with oscillating Dirichlet or Neumann boundary data. We start by introducing a family of elliptic operators in divergence form with a small scale parameter $\varepsilon > 0$

$$\mathcal{L}_\varepsilon = -\mathrm{div}(A(x/\varepsilon)\nabla) = -\frac{\partial}{\partial x_i}\left(a_{ij}^{\alpha\beta}(x/\varepsilon)\frac{\partial}{\partial x_j}\right). \quad (1.2.1)$$

We assume that the coefficient matrix $A = A(y) = \left(a_{ij}^{\alpha\beta}\right)$, with $1 \le i, j \le d$ and $1 \le \alpha, \beta \le m$, satisfies the following standard assumptions

- Ellipticity: there exists $\mu > 0$ such that
$$\mu|\xi|^2 \le a_{ij}^{\alpha\beta}\xi_i^\alpha\xi_j^\beta \le \mu^{-1}|\xi|^2 \quad \text{for any } \xi = (\xi_i^\alpha) \in \mathbb{R}^{m\times d}; \quad (1.2.2)$$
- Periodicity: A is 1-periodic, that is
$$A(y+z) = A(y) \quad \text{for any } y \in \mathbb{R}^d \text{ and } z \in \mathbb{Z}^d; \quad (1.2.3)$$
- Smoothness:
$$a_{ij}^{\alpha\beta} \in C^\infty(\mathbb{T}^d) \quad \text{for } 1 \le \alpha, \beta \le m \text{ and } 1 \le i, j \le d. \quad (1.2.4)$$

Now, we consider the Neumann problem with both the zero-order and the first-order oscillating data

$$\begin{cases} \mathcal{L}_\varepsilon(u_\varepsilon) = 0 & \text{in } \Omega, \\ \dfrac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = T_{ij}\cdot\nabla_x\big\{g_{ij}(x, x/\varepsilon)\big\} + g_0(x, x/\varepsilon) - \gamma_\varepsilon & \text{on } \partial\Omega, \end{cases} \quad (1.2.5)$$

where $T_{ij} = (n_i e_j - n_j e_i)$ is a tangential vector field on $\partial\Omega$, and $\gamma_\varepsilon = \frac{1}{|\partial\Omega|}\int_{\partial\Omega} g_0(x, x/\varepsilon)\, d\sigma$ is a constant so that the compatibility condition for (1.2.5) is satisfied. This system arises when we construct the boundary layer term $\tilde{v}_\varepsilon^N$ in (1.1.12), and treats (1.1.11) as a special scalar case with $g_0 = 0$.

Throughout this dissertation, unless otherwise stated, we assume that $\Omega \subset \mathbb{R}^d$ is a bounded, smooth, and strictly convex domain in the sense that all the principle curvatures are strictly positive, and that $g(x, y) = \{g_0(x, y),\, g_{ij}(x, y)\}$ are smooth in $(x, y) \in \partial\Omega \times \mathbb{R}^d$ and 1-periodic in y, namely

$$g(x, y + z) = g(x, y) \quad \text{for any } x \in \partial\Omega, y \in \mathbb{R}^d \text{ and } z \in \mathbb{Z}^d. \tag{1.2.6}$$

The key reason that we require the strict convexity on the domain is because any periodic functions in $\mathbb{R}^d$ are somehow equidistributed on the strictly convex boundary (a $(d-1)$-dimensional surface), regardless of translations, rotations and scales. Although the geometry and regularity assumption on the domain might be weakened, the above equidistribution property seems to be a natural prerequisite for homogenization to take place, even in the case with constant coefficients. Precisely, under the above conditions, we are able to show that as $\varepsilon \to 0$, the unique solution of (1.2.5) with $\int_\Omega u_\varepsilon = 0$ converges strongly in $L^2(\Omega)$ to u_0, where u_0 is a solution of

$$\begin{cases} \mathcal{L}_0(u_0) = 0 & \text{in } \Omega, \\ \dfrac{\partial u_0}{\partial \nu_0} = T_{ij} \cdot \nabla_x \overline{g}_{ij} + \langle g_0 \rangle - \gamma_0 & \text{on } \partial\Omega. \end{cases} \tag{1.2.7}$$

The operator $\mathcal{L}_0$ is given by $\mathcal{L}_0 = -\mathrm{div}(\widehat{A}\nabla)$, with $\widehat{A}$ being the usual homogenized matrix of A, and

$$\langle g_0 \rangle(x) = \fint_{\mathbb{T}^d} g_0(x, y)\, dy \quad \text{and} \quad \gamma_0 = \fint_{\partial\Omega} \langle g_0 \rangle\, d\sigma. \tag{1.2.8}$$

The formulation for function $\{\overline{g}_{ij}\}$ in (1.2.7) on $\partial\Omega$ is much more involved and will be given explicitly in Chapter 3. Nevertheless, it is good to point out here that, unlike (1.2.8), $\overline{g}_{ij}(x)$ is not simply the trivial average of $g_{ij}(x, \cdot)$, but a complicated combination relying on A, $\{g_{ij}(x, \cdot) : 1 \le i, j \le d\}$, and the outward normal $n(x)$ to $\partial\Omega$.

In the following, we state our main results for the Neumann problem (1.2.5), including a convergence rate in L^2, which is optimal (up to an arbitrarily small exponent) for $d \ge 3$, and the $W^{1,p}$ regularity estimate of the homogenized boundary data $\overline{g}_{ij}$ for any $p \in (1, \infty)$.

Theorem 1.1. *Let Ω be a bounded smooth, strictly convex domain in $\mathbb{R}^d$, $d \ge 3$. Assume that $A(y)$ satisfies (1.2.2)-(1.2.4), and that $g_0(x, y)$ and $g_{ij}(x, y)$ are smooth and satisfy conditions (1.2.6). Let u_ε and u_0 be the solutions of (1.2.5) and (1.2.7), respectively, with $\int_\Omega u_\varepsilon = \int_\Omega u_0 = 0$. Then for any $\sigma \in (0, 1/2)$ and $\varepsilon \in (0, 1)$,*

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \le C_\sigma\, \varepsilon^{\frac{1}{2} - \sigma}, \tag{1.2.9}$$

where C_σ depends only on d, m, σ, A, Ω, and $g = \{g_0, g_{ij}\}$. Furthermore, the function $\overline{g} = \{\overline{g}_{ij}\}$ in (1.2.7) satisfies

$$\|\overline{g}\|_{W^{1,q}(\partial\Omega)} \le C_q \sup_{y\in\mathbb{T}^d} \|g(\cdot, y)\|_{C^1(\partial\Omega)} \quad \text{for any } q < \infty, \tag{1.2.10}$$

where C_q depends only on d, m, μ, q and $\|A\|_{C^k(\mathbb{T}^d)}$ for some $k = k(d) \ge 1$.

In recent years, there has been considerable interest in the homogenization of boundary value problems with oscillating boundary data [20, 21, 29, 3, 25, 4, 15, 13, 17, 5, 7] (also see related earlier work in [31, 32, 27, 28, 6]. In the case of Dirichlet problem (a general form of (1.1.6)),

$$\begin{cases} \mathcal{L}_\varepsilon(u_\varepsilon) = 0 & \text{in } \Omega, \\ \quad u_\varepsilon = f(x, x/\varepsilon) & \text{on } \partial\Omega, \end{cases} \tag{1.2.11}$$

where

$$f(x, y+z) = f(x, y) \quad \text{for any } x \in \partial\Omega, y \in \mathbb{R}^d \text{ and } z \in \mathbb{Z}^d, \tag{1.2.12}$$

major progress was made in [21] and later in [7]. Let u_ε be the solution of (1.2.11). Under the assumption that Ω is smooth and strictly convex in $\mathbb{R}^d$, $d \ge 2$, it was proved in [21] that

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \le C\, \varepsilon^{\frac{(d-1)}{3d+5} - \sigma}$$

for any $\sigma \in (0,1)$, where u_0 is the solution of the homogenized problem,

$$\begin{cases} \mathcal{L}_0(u_0) = 0 & \text{in } \Omega, \\ \quad u_0 = \overline{f} & \text{on } \partial\Omega, \end{cases} \tag{1.2.13}$$

and the homogenized data $\overline{f}$ at x depends on $f(x,\cdot)$, A, and $n(x)$. A sharper rate of convergence in L^2 was obtained recently in [7] for the Dirichlet problem (1.2.11), with $O(\varepsilon^{\frac{1}{2}-})$ for $d \ge 4$, $O(\varepsilon^{\frac{1}{3}-})$ for $d = 3$, and $O(\varepsilon^{\frac{1}{6}-})$ for $d = 2$. As demonstrated in [5] in the case of elliptic equations with constant coefficients, the optimal rate would be $O(\varepsilon^{\frac{1}{2}})$ for $d \ge 3$ (up to a factor of $\ln\varepsilon$ in the case of $d = 3$), and $O(\varepsilon^{\frac{1}{4}})$ for $d = 2$. Thus the convergence rates obtained in [7] for the Dirichlet problem are optimal for $d \ge 4$, up to an arbitrarily small exponent. In [38], we established the optimal convergence rates for both the Neumann and Dirichlet problems in any dimensions.

Regarding the regularity of the homogenized boundary data, under the same assumptions, it was proved in [21] that $\nabla_{\tan}\overline{f} \in L^{p,\infty}(\partial\Omega)$ with $p = \frac{d-1}{2}$. The result was improved in [7] to $\nabla_{\tan}\overline{f} \in L^{p,\infty}(\partial\Omega)$ with $p = \frac{2(d-1)}{3}$ if $d \ge 3$, and to $\overline{f} \in W^{1,p}(\partial\Omega)$ for any $p < \frac{2}{3}$ if $d = 2$. Further improvement was made in [38], where we proved that $\overline{f}, \overline{g} \in W^{1,p}(\partial\Omega)$ for any $p < d-1$. The regularity estimates were finally improved to $\overline{f}, \overline{g} \in W^{1,p}(\partial\Omega)$ for any $p < \infty$ in our recent paper [37]. In particular, this implies that $\overline{f}$ and $\overline{g}$ are C^σ-Hölder continuous for any $\sigma \in (0,1)$. However, whether these regularity estimates are optimal remains an interesting and challenging problem. We summarize the results for Dirichlet problems as follows.

Theorem 1.2. *Let Ω be a bounded smooth, strictly convex domain in $\mathbb{R}^d$, $d \geq 2$. Assume that $A(y)$ satisfies (1.2.2)-(1.2.4), and that $f(x,y)$ is smooth and satisfies (1.2.12). Let u_ε and u_0 are solutions of Dirichlet problems (1.2.11) and (1.2.13), respectively. Then for any $\sigma \in (0,1/2)$ and $\varepsilon \in (0,1)$,*

$$\|u_\varepsilon - u_0\|_{L^2(\Omega)} \leq C_\sigma \begin{cases} \varepsilon^{\frac{1}{2}-\sigma} & \text{if } d \geq 3, \\ \varepsilon^{\frac{1}{4}-\sigma} & \text{if } d = 2, \end{cases} \tag{1.2.14}$$

where C_σ depends only on d, m, σ, A, Ω and f. Furthermore, for any $d \geq 2$,

$$\|\overline{f}\|_{W^{1,q}(\partial\Omega)} \leq C_q \sup_{y\in\mathbb{T}^d} \|f(\cdot,y)\|_{C^1(\partial\Omega)} \quad \text{for any } q < \infty, \tag{1.2.15}$$

where $\overline{f}$ be the homogenized data in (1.2.13) and C_q depends only on d, m, μ, q and $\|A\|_{C^k(\mathbb{T}^d)}$ for some $k = k(d) \geq 1$.

We remark that Theorem 1.1 and 1.2 may be applied to establish the higher-order convergence rates. Indeed, one can prove, for either Neumann or Dirichlet problems,

$$u_\varepsilon = u_0 + \varepsilon\chi(x/\varepsilon)\nabla u_0 + \varepsilon v^{bl} + O(\varepsilon^{\frac{3}{2}-}), \tag{1.2.16}$$

where v^{bl} is the solution of some homogenized system independent of ε; see §3.8. The estimate (1.2.16) can be further used to study the first-order expansions of eigenvalues or eigenfunctions (eigenspaces). The exploitation in this direction may be found in [41, 30] and will not be included in this dissertation.

The organization of the dissertation is as follows: The preliminaries, including correctors, uniform Lipschitz estimates and the Diophantine condition, are given in Chapter 2. The proofs for Theorem 1.1 and 1.2 are very long and will be carried out across Chapter 3, 4 and 5. Particularly, in Chapter 3 and 4, we prove the convergence rates in Theorem 1.1 for Neumann problems and in Theorem 1.2 for Dirichlet problems, respectively. In Chapter 5, we establish the $W^{1,p}$ estimates of the homogenized boundary data for both theorems.

1.3 Notations

Most of the notations in this dissertation are standard. Some symbols are used with different meanings in the context. For example, we use $\delta(x)$ to denote the distance from x to the underlying boundary, use $\delta_y(x)$ to denote the Dirac function, use $\delta^{\alpha\beta}$ or δ_{ij} to denote the Kronecker delta function (identity matrix), and so on. Fortunately, these symbols are used locally and could be interpreted without ambiguity in the context. In the following, we list some frequently used global notations.

d	spatial dimension
m	dimension of the solution vector, or the number of equations in the system
$\mathcal{L}_\varepsilon$	$-\mathrm{div}(A(x/\varepsilon)\nabla)$, oscillating elliptic operator with $\varepsilon > 0$
$\mathcal{L}_0$	$-\mathrm{div}(\widehat{A}\nabla)$, homogenized elliptic operator

$\widehat{A}$	homogenized (effective) coefficient matrix
$\partial/\partial\nu_\varepsilon, \partial/\partial\nu_0$	conormal derivatives
μ	ellipticity constant
$i, j, k, \cdots$	subscripts, $1 \le i, j, k, \cdots \le d$
$\alpha, \beta, \gamma, \cdots$	superscripts, $1 \le \alpha, \beta, \gamma, \cdots \le m$
1-periodic	a function f is 1-periodic if $f(x+z) = f(x)$ for all $x \in \mathbb{R}^d$ and $z \in \mathbb{Z}^d$
$\mathbb{T}^d$	d-dimensional torus; we will identity a 1-periodic function in $\mathbb{R}^d$ as a function defined on $\mathbb{T}^d$; see (1.2.4) for example
$\mathbb{S}^{d-1}$	unit sphere in $\mathbb{R}^d$ with the usual topology
χ	(first-order) corrector
Υ	second-order corrector
Φ_ε	Dirichlet corrector
Ψ_ε	Neumann corrector
$P_{\Omega,\varepsilon}, P_\Omega$	Poison kernels in Ω
N_ε, N_0	Neumann functions
e^β, e_j	standard Cartesian coordinate vectors in $\mathbb{R}^m$ and $\mathbb{R}^d$
$P_j^\beta(x)$	an affine function $x_j e^\beta$
T_{ij}	a tangential vector in the form of $n_i e_j - n_j e_i$ on $\partial\Omega$, where $n = (n_1, n_2, \cdots, n_d)$ is the unit outward normal vector
$\mathbb{R}_+$	$[0, \infty)$
$\mathbb{H}_n^d(s)$	a half-space $\{x \in \mathbb{R}^d : x \cdot n < -s\}$
$I - n \otimes n$	the projection operator onto the orthogonal space of n
$\kappa = \kappa(n)$	the Diophantine constant of $n \in \mathbb{S}^{d-1}$
$L^{p,\infty}$	weak L^p space
$H^k, W^{k,p}$	Sobolev spaces
$\fint_E$	$\lvert E\rvert^{-1} \int_E$, i.e., average integral over E
$\langle f \rangle$	$\fint_{\mathbb{T}^d} f$, i.e., the average of a 1-periodic function f
$O(\varepsilon^{t-})$	of order $\varepsilon^{t-\sigma}$ for any $\sigma > 0$
$\overline{f}, \overline{g}$	homogenized boundary data for Dirichlet and Neumann problems
$C, c, \cdots$	generic constants independent of ε or κ

Chapter 2 Preliminaries

In this chapter, we introduce the definitions of correctors, flux correctors and the homogenized operators. To demonstrate the general approach for the quantitative periodic homogenization, we prove a sharp $O(\varepsilon)$ convergence rate in H^1 involving the boundary layers, as claimed in the introduction. We also introduce the (full-scale) uniform Lipschitz estimates in half-spaces which will be used in an essential way in the following chapters. Finally, we introduce the Diophantine condition which is a key ingredient that quantifies the geometry (strict convexity) of the boundary.

2.1 Correctors and homogenized operators

The correctors, arising from the two-scale asymptotic expansion, play a crucial role in homogenization theory [11, 22, 36]. The precise definition is given as follows. For $1 \le j \le d$ and $1 \le \beta \le m$, let $\chi = (\chi_j^\beta) = (\chi_j^{1\beta}, \chi_j^{2\beta}, \cdots, \chi_j^{m\beta})$ denote the correctors for $\mathcal{L}_\varepsilon$, which are 1-periodic functions satisfying the system

$$\begin{cases} \mathcal{L}_1\big(\chi_j^\beta + P_j^\beta\big) = 0 \quad \text{ in } \mathbb{R}^d, \\ \chi_j^\beta \text{ is 1-periodic and } \displaystyle\int_{\mathbb{T}^d} \chi_j^\beta = 0, \end{cases} \tag{2.1.1}$$

where $P_j^\beta(x) = x_j e^\beta$ and $e^\beta = (0, \cdots, 1, \cdots, 0)$ is the βth coordinate vector. Intuitively, the periodic corrector χ_j^β is the correction to a linear function P_j^β so that $\chi_j^\beta + P_j^\beta$ is an "almost linear" solution in the entire space $\mathbb{R}^d$.

With correctors, the homogenized operator may be given by $\mathcal{L}_0 = -\mathrm{div}(\widehat{A}\nabla)$, where the homogenized coefficient matrix $\widehat{A} = (\widehat{a}_{ij}^{\alpha\beta})$ is defined by

$$\widehat{A} = \int_{\mathbb{T}^d} A(I + \nabla\chi), \quad \text{or precisely} \quad \widehat{a}_{ij}^{\alpha\beta} = \int_{\mathbb{T}^d} \Big\{ a_{ij}^{\alpha\beta} + a_{ik}^{\alpha\gamma} \frac{\partial}{\partial y_k}(\chi_j^{\gamma\beta}) \Big\}.$$

It can be shown that $\widehat{A}$ also satisfies the ellipticity condition (1.2.2), possibly with a different ellipticity constant.

We also introduce the adjoint operator $\mathcal{L}_\varepsilon^* = -\mathrm{div}(A^*(x/\varepsilon)\nabla)$, where $A^* = (a_{ij}^{*\alpha\beta})$ with $a_{ij}^{*\alpha\beta} = a_{ji}^{\beta\alpha}$. Note that A^* also satisfies our standard assumptions (1.2.2) - (1.2.4). Then, we may similarly define the adjoint correctors $\chi^* = (\chi_j^{*\beta})$ and the adjoint homogenized operator $\mathcal{L}_0^* = -\mathrm{div}(\widehat{A}^*\nabla)$. Observe that the correctors and the homogenized operators defined above depend only on the original coefficient matrix A.

Another concept we need to use in studying the convergence rate is the flux corrector. The flux corrector is a matrix $B(y) = (b_{ij}^{\alpha\beta})$ defined by

$$b_{ij}^{\alpha\beta}(y) = a_{ij}^{\alpha\beta}(y) + a_{ik}^{\alpha\gamma}(y)\frac{\partial}{\partial y_k}\Big(\chi_j^{\gamma\beta}(y)\Big) - \widehat{a}_{ij}^{\alpha\beta}, \tag{2.1.2}$$

where the repeated index k is summed from 1 to d and γ from 1 to m. Observe that $B(y)$ is 1-periodic and smooth under our setting. Moreover, it follows from the definition of χ_j^β and $\widehat{a}_{ij}^{\alpha\beta}$ that

$$\frac{\partial}{\partial y_i}\left(b_{ij}^{\alpha\beta}\right) = 0 \quad \text{and} \quad \int_{\mathbb{T}^d} b_{ij}^{\alpha\beta} = 0. \tag{2.1.3}$$

Lemma 2.1. *There exist $\phi_{kij}^{\alpha\beta} \in H^1(\mathbb{T}^d)$, where $1 \le i, j, k \le d$ and $1 \le \alpha, \beta \le m$, such that*

$$b_{ij}^{\alpha\beta} = \frac{\partial}{\partial y_k}\left(\phi_{kij}^{\alpha\beta}\right) \quad \textit{and} \quad \phi_{kij}^{\alpha\beta} = -\phi_{ikj}^{\alpha\beta}. \tag{2.1.4}$$

If $\chi = (\chi_j^\beta)$ is Hölder continuous, then $\phi_{kij}^{\alpha\beta} \in L^\infty(\mathbb{T}^d)$.

The proof of the above lemma may be found in, for example, [36, Proposition 3.1.1]. In our setting, since χ is smooth, we may even show that $\phi = (\phi_{kij}^{\alpha\beta})$ is smooth.

2.2 Convergence rates

In this section, we will prove the convergence results (1.1.7) and (1.1.10) claimed in the introduction. These results show that the asymptotic analysis of the boundary layer terms is a natural and crucial question in periodic homogenization. We will state and prove these results separately for Dirichlet problem and Neumann problem.

Theorem 2.2. *Let u_ε be the weak solution of*

$$\begin{cases} \mathcal{L}_\varepsilon(u_\varepsilon) = F & \text{in } \Omega, \\ \quad u_\varepsilon = f & \text{on } \partial\Omega, \end{cases} \tag{2.2.1}$$

and u_0 be the weak solution of the homogenized system

$$\begin{cases} \mathcal{L}_0(u_0) = F & \text{in } \Omega, \\ \quad u_0 = f & \text{on } \partial\Omega. \end{cases} \tag{2.2.2}$$

Then

$$\|u_\varepsilon - u_0 - \varepsilon\chi(\cdot/\varepsilon)\nabla u_0 - \varepsilon v_\varepsilon^D\|_{H^1(\Omega)} \le C\varepsilon\|\nabla^2 u_0\|_{L^2(\Omega)}, \tag{2.2.3}$$

where v_ε^D is the weak solution of

$$\begin{cases} \mathcal{L}_\varepsilon(v_\varepsilon^D) = 0 & \text{in } \Omega, \\ \quad v_\varepsilon^D = -\chi_j^{\cdot\beta}(x/\varepsilon)\dfrac{\partial u_0^\beta}{\partial x_j} & \text{on } \partial\Omega. \end{cases} \tag{2.2.4}$$

Proof. The proof is quite standard in periodic homogenization by considering the first-order approximation in the asymptotic expansion. Let

$$w_\varepsilon^\beta(x) = u_\varepsilon^\beta(x) - u_0^\beta(x) - \varepsilon\chi_k^{\beta\gamma}(x/\varepsilon)\frac{\partial u_0^\gamma(x)}{\partial x_k}. \tag{2.2.5}$$

Then, we derive the system for w_ε

$$\mathcal{L}_\varepsilon(w_\varepsilon) = \mathcal{L}_\varepsilon(u_\varepsilon) - \mathcal{L}_\varepsilon(u_0) - \mathcal{L}_\varepsilon(\varepsilon\chi(x/\varepsilon)\nabla u_0). \tag{2.2.6}$$

Using the system (2.2.16) and (2.2.17), we have

$$\mathcal{L}_\varepsilon(u_\varepsilon) = F = \mathcal{L}_0(u_0). \tag{2.2.7}$$

Also, by a direct calculation, we have

$$\begin{aligned}\Big(\mathcal{L}_\varepsilon(\varepsilon\chi(x/\varepsilon)\nabla u_0)\Big)^\alpha &= -\frac{\partial}{\partial x_i}\Big(a_{ik}^{\alpha\gamma}(x/\varepsilon)\frac{\partial \chi_j^{\gamma\beta}}{\partial x_k}(x/\varepsilon)\frac{\partial u_0^\beta(x)}{\partial x_j}\Big)\\ &\quad - \varepsilon\frac{\partial}{\partial x_i}\Big(a_{ik}^{\alpha\gamma}(x/\varepsilon)\chi_j^{\gamma\beta}(x/\varepsilon)\frac{\partial^2 u_0^\beta(x)}{\partial x_k\partial x_j}\Big).\end{aligned} \tag{2.2.8}$$

Substituting (2.2.7) and (2.2.8) into (2.2.6) and by a careful calculation, we obtain

$$\begin{aligned}\Big(\mathcal{L}_\varepsilon(w_\varepsilon)\Big)^\alpha &= \frac{\partial}{\partial x_i}\Big[\Big(a_{ij}^{\alpha\beta}(x/\varepsilon) + a_{ik}^{\alpha\gamma}(x/\varepsilon)\frac{\partial \chi_j^{\gamma\beta}}{\partial x_k}(x/\varepsilon) - \widehat{a}_{ij}^{\alpha\beta}\Big)\frac{\partial u_0^\beta(x)}{\partial x_j}\Big]\\ &\quad + \varepsilon\frac{\partial}{\partial x_i}\Big(a_{ik}^{\alpha\gamma}(x/\varepsilon)\chi_j^{\gamma\beta}(x/\varepsilon)\frac{\partial^2 u_0^\beta(x)}{\partial x_k\partial x_j}\Big)\\ &= \frac{\partial}{\partial x_i}\Big[\frac{\partial}{\partial x_k}\Big(\varepsilon\phi_{kij}^{\alpha\beta}(x/\varepsilon)\Big)\frac{\partial u_0^\beta(x)}{\partial x_j}\Big]\\ &\quad + \varepsilon\frac{\partial}{\partial x_i}\Big(a_{ik}^{\alpha\gamma}(x/\varepsilon)\chi_j^{\gamma\beta}(x/\varepsilon)\frac{\partial^2 u_0^\beta(x)}{\partial x_k\partial x_j}\Big),\end{aligned} \tag{2.2.9}$$

where we also used (2.1.4) in the last equality. Observe that

$$\begin{aligned}&\frac{\partial}{\partial x_i}\Big[\frac{\partial}{\partial x_k}\Big(\varepsilon\phi_{kij}^{\alpha\beta}(x/\varepsilon)\Big)\frac{\partial u_0^\beta(x)}{\partial x_j}\Big]\\ &\quad = \frac{\partial^2}{\partial x_i\partial x_k}\Big[\varepsilon\phi_{kij}^{\alpha\beta}(x/\varepsilon)\frac{\partial u_0^\beta(x)}{\partial x_j}\Big] + \frac{\partial}{\partial x_i}\Big[\varepsilon\phi_{kij}^{\alpha\beta}(x/\varepsilon)\frac{\partial^2 u_0^\beta(x)}{\partial x_k\partial x_j}\Big].\end{aligned} \tag{2.2.10}$$

Now, the key observation here is that the anti-symmetry of ϕ in (2.1.4) with respect to indices i and k implies that the first term on the right-hand side of (2.2.10) vanishes in the sense of distribution. As a consequence, we obtain

$$\Big(\mathcal{L}_\varepsilon(w_\varepsilon)\Big)^\alpha = \varepsilon\frac{\partial}{\partial x_i}\Big[\phi_{kij}^{\alpha\beta}(x/\varepsilon)\frac{\partial^2 u_0^\beta(x)}{\partial x_k\partial x_j}\Big] + \varepsilon\frac{\partial}{\partial x_i}\Big[a_{ik}^{\alpha\gamma}(x/\varepsilon)\chi_j^{\gamma\beta}(x/\varepsilon)\frac{\partial^2 u_0^\beta(x)}{\partial x_k\partial x_j}\Big]. \tag{2.2.11}$$

Set

$$F_{\varepsilon,j}^\alpha(x) = \phi_{kij}^{\alpha\beta}(x/\varepsilon)\frac{\partial^2 u_0^\beta(x)}{\partial x_k\partial x_j} + a_{ik}^{\alpha\gamma}(x/\varepsilon)\chi_j^{\gamma\beta}(x/\varepsilon)\frac{\partial^2 u_0^\beta(x)}{\partial x_k\partial x_j}. \tag{2.2.12}$$

Since χ and ϕ are both bounded, one sees that $\|F^\alpha_{\varepsilon,j}\|_{L^2(\Omega)} \le C\|\nabla^2 u_0\|_{L^2(\Omega)}$, where C depends only on A. It follows that w_ε satisfies

$$\begin{cases} \mathcal{L}_\varepsilon(w_\varepsilon) = \varepsilon \mathrm{div}(F_\varepsilon) & \text{in } \Omega, \\ w_\varepsilon = -\varepsilon \chi_j^{\cdot\beta}(x/\varepsilon)\dfrac{\partial u_0^\beta}{\partial x_j} & \text{on } \partial\Omega. \end{cases} \tag{2.2.13}$$

Finally, let v_ε^D be the solution of (2.2.4). Then

$$\begin{cases} \mathcal{L}_\varepsilon(w_\varepsilon - \varepsilon v_\varepsilon^D) = \varepsilon \mathrm{div}(F_\varepsilon) & \text{in } \Omega, \\ w_\varepsilon - \varepsilon v_\varepsilon^D = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2.14}$$

A standard energy estimate gives

$$\|w_\varepsilon - \varepsilon v_\varepsilon^D\|_{H^1(\Omega)} \le C\varepsilon \|\nabla^2 u_0\|_{L^2(\Omega)}, \tag{2.2.15}$$

which implies the desired estimate. □

Theorem 2.3. *Let u_ε and u_0 be the weak solution of*

$$\begin{cases} \mathcal{L}_\varepsilon(u_\varepsilon) = F & \textit{in } \Omega, \\ \dfrac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = g & \textit{on } \partial\Omega, \end{cases} \tag{2.2.16}$$

and u_0 be the weak solution of the homogenized equation

$$\begin{cases} \mathcal{L}_0(u_0) = F & \textit{in } \Omega, \\ \dfrac{\partial u_0}{\partial \nu_0} = g & \textit{on } \partial\Omega. \end{cases} \tag{2.2.17}$$

Then

$$\|u_\varepsilon - u_0 - \varepsilon\chi(\cdot/\varepsilon)\nabla u_0 - \varepsilon v_\varepsilon^N\|_{H^1(\Omega)} \le C\varepsilon\|\nabla^2 u_0\|_{L^\infty(\Omega)}, \tag{2.2.18}$$

where v_ε^N is the solution of

$$\begin{cases} \mathcal{L}_\varepsilon(v_\varepsilon^N) = 0 & \textit{in } \Omega, \\ \dfrac{\partial v_\varepsilon^N}{\partial \nu_\varepsilon} = \dfrac{1}{2}\Big(n_k \dfrac{\partial}{\partial x_i} - n_i \dfrac{\partial}{\partial x_k}\Big)\Big(\phi_{kij}^{\cdot\beta}(x/\varepsilon)\dfrac{\partial u_0^\beta}{\partial x_j}\Big) & \textit{on } \partial\Omega, \end{cases} \tag{2.2.19}$$

and $n = (n_1, n_2, \cdots, n_d)$ is the unit outward normal.

Proof. The proof for Neumann problem is similar as Dirichlet problem, while the Neumann boundary condition needs to be handled more carefully. Let w_ε be defined as (2.2.5). It follows from the same argument that w_ε satisfies

$$\begin{cases} \mathcal{L}_\varepsilon(w_\varepsilon) = \varepsilon \mathrm{div}(F_\varepsilon) & \text{in } \Omega, \\ \dfrac{\partial w_\varepsilon}{\partial \nu_\varepsilon} = \dfrac{\partial}{\partial \nu_\varepsilon}\big(u_\varepsilon - u_0 - \varepsilon\chi(x/\varepsilon)\nabla u_0\big) & \text{on } \partial\Omega, \end{cases} \tag{2.2.20}$$

where F_ε is the same as (2.2.12).

Now, we need to analyze the boundary condition of w_ε. Note that

$$\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = g = \frac{\partial u_0}{\partial \nu_0} \qquad \text{on } \partial\Omega. \tag{2.2.21}$$

Then, by a careful calculation as in (2.2.9),

$$\begin{aligned}\Big(\frac{\partial w_\varepsilon}{\partial \nu_\varepsilon}\Big)^\alpha &= n_i \widehat{a}_{ij}^{\alpha\beta} \frac{\partial u_0^\beta}{\partial x_j} - n_i \widehat{a}_{ij}^{\alpha\beta}(x/\varepsilon)\frac{\partial u_0^\beta}{\partial x_j} - n_i a_{ik}^{\alpha\gamma}(x/\varepsilon)\frac{\partial}{\partial x_k}\Big(\varepsilon \chi_j^{\gamma\beta}(x/\varepsilon)\frac{\partial u_0^\beta}{\partial x_j}\Big)\\ &= -n_i \frac{\partial}{\partial x_k}\Big(\varepsilon \phi_{kij}^{\alpha\beta}(x/\varepsilon)\frac{\partial u_0^\beta}{\partial x_j}\Big) - \varepsilon n_i a_{ik}^{\alpha\gamma}(x/\varepsilon)\chi_j^{\gamma\beta}(x/\varepsilon)\frac{\partial^2 u_0^\beta}{\partial x_k \partial x_j}.\end{aligned} \tag{2.2.22}$$

Using the anti-symmetry of ϕ in (2.1.4), we observe that

$$-n_i \frac{\partial}{\partial x_k}\Big(\varepsilon \phi_{kij}^{\alpha\beta}(x/\varepsilon)\frac{\partial u_0^\beta}{\partial x_j}\Big) = \frac{1}{2}\Big(n_k \frac{\partial}{\partial x_i} - n_i \frac{\partial}{\partial x_k}\Big)\Big(\varepsilon \phi_{kij}^{\alpha\beta}(x/\varepsilon)\frac{\partial u_0^\beta}{\partial x_j}\Big). \tag{2.2.23}$$

Hence,

$$\begin{aligned}\Big(\frac{\partial w_\varepsilon}{\partial \nu_\varepsilon}\Big)^\alpha &= \frac{\varepsilon}{2}\Big(n_k \frac{\partial}{\partial x_i} - n_i \frac{\partial}{\partial x_k}\Big)\Big(\phi_{kij}^{\alpha\beta}(x/\varepsilon)\frac{\partial u_0^\beta}{\partial x_j}\Big)\\ &\qquad - \varepsilon n_i a_{ik}^{\alpha\gamma}(x/\varepsilon)\chi_j^{\gamma\beta}(x/\varepsilon)\frac{\partial^2 u_0^\beta}{\partial x_k \partial x_j}.\end{aligned} \tag{2.2.24}$$

Now, let v_ε^N be the solution of (2.2.19). Note that the compatibility condition is satisfied automatically for the Neumann problem (2.2.19) due to an integration by parts on the boundary; see Lemma 3.1. As a result, $w_\varepsilon - \varepsilon v_\varepsilon^N$ satisfies

$$\begin{cases}\mathcal{L}_\varepsilon(w_\varepsilon - \varepsilon v_\varepsilon^N) = \varepsilon \operatorname{div}(F_\varepsilon) & \text{in } \Omega,\\ \dfrac{\partial}{\partial \nu_\varepsilon}(w_\varepsilon - \varepsilon v_\varepsilon^N) = -\varepsilon n_i a_{ik}^{\alpha\gamma}(x/\varepsilon)\chi_j^{\gamma\beta}(x/\varepsilon)\dfrac{\partial^2 u_0^\beta}{\partial x_k \partial x_j} & \text{on } \partial\Omega.\end{cases} \tag{2.2.25}$$

Finally, a standard energy estimate gives

$$\|w_\varepsilon - \varepsilon v_\varepsilon^N\|_{H^1(\Omega)} \le C\varepsilon \|\nabla^2 u_0\|_{L^\infty(\Omega)}, \tag{2.2.26}$$

which implies the desired estimate. $\square$

2.3 Uniform Lipschitz estimates

In periodic homogenization, Lipschitz estimates (uniform in ε) are the optimal regularity for the solutions of general elliptic equations in divergence form. Historically, the interior and boundary Lipschitz estimates with Dirichlet condition was first proved by M. Avellaneda and F. Lin in [9] by using the compactness method. The boundary Lipschitz estimate with Neumann condition was proved by C. Kenig, F. Lin, and Z. Shen in [24], under the additional symmetry condition $A^* = A$. The symmetry condition was later removed by S. Armstrong and Z. Shen in [8]. In this section we state these Lipschitz estimates (with flat boundaries) which will be crucial for us.

Theorem 2.4 (Interior Lipschitz estimate)**.** *Suppose that $A = A(y)$ satisfies the ellipticity and periodicity conditions (1.2.2)-(1.2.3). Also assume that A satisfies the Hölder continuity condition:*

$$|A(x) - A(y)| \le \tau |x - y|^\sigma, \tag{2.3.1}$$

for some $\sigma \in (0,1)$ and $\tau \ge 0$. Let $u_\varepsilon \in H^1(B(x_0, r); \mathbb{R}^m)$ be a weak solution of

$$\mathcal{L}_\varepsilon(u_\varepsilon) = F \text{ in } B(x_0, r), \text{ where } F \in L^p(B(x_0, r); \mathbb{R}^m) \tag{2.3.2}$$

for some $p > d$. Then

$$\|\nabla u_\varepsilon\|_{L^\infty(B(x_0, r/2))} \le C_p \left\{ \left(\fint_{B(x_0,r)} |\nabla u_\varepsilon|^2 \right)^{1/2} + r \left(\fint_{B(x_0,r)} |F|^p \right)^{1/p} \right\}, \tag{2.3.3}$$

where C_p depends only on d, m, p, λ, σ and τ.

Theorem 2.4 was proved by M. Avellaneda and F. Lin in [9].

Theorem 2.5 (Lipschitz estimate with Dirichlet condition)**.** *Suppose that A satisfies the same conditions as in Theorem 2.4. Let $\Omega = \mathbb{H}_n^d(s)$ for some $n \in \mathbb{S}^{d-1}$ and $s \in \mathbb{R}$. Given $x_0 \in \partial\Omega$ and $r > 0$, let u_ε be a weak solution to*

$$\begin{cases} \mathcal{L}_\varepsilon(u_\varepsilon) = F & \text{in } B(x_0, r) \cap \Omega, \\ u_\varepsilon = f & \text{on } B(x_0, r) \cap \partial\Omega. \end{cases}$$

Then

$$\begin{aligned} \|\nabla u\|_{L^\infty(\Omega \cap B(x_0, r/2))} \le C_p \Bigg\{ & \left(\fint_{B(x_0,r)\cap\Omega} |\nabla u_\varepsilon|^2 \right)^{1/2} + r \left(\fint_{B(x_0,r)\cap\Omega} |F|^p \right)^{1/p} \\ & + \|\nabla_{\tan} f\|_{L^\infty(B(x_0,r)\cap\partial\Omega)} + r^\sigma \|\nabla_{\tan} f\|_{C^{0,\sigma}(B(x_0,r)\cap\partial\Omega)} \Bigg\}, \end{aligned} \tag{2.3.4}$$

where C_p depends only on d, m, p, λ, σ and τ.

Proof. Notice that a half-space $\mathbb{H}_n^d(s)$ is invariant under rescaling (or translation, rotation). Thus, by rescaling, we may assume $r = 1$. In this case the estimate (2.3.4) follows from the boundary Lipschitz estimate with Dirichlet boundary condition, proved in [9] for a general $C^{1,\alpha}$ domain. The fact that Ω has a flat boundary is essential here. For otherwise the constant C_p in (2.3.4) will depend on r, if r is large. □

Theorem 2.6 (Lipschitz estimate with Neumann condition)**.** *Let A and Ω be the same as in Theorem 2.5. Given $x_0 \in \partial\Omega$ and $r > 0$, let u_ε be a weak solution to*

$$\begin{cases} \mathcal{L}_\varepsilon(u_\varepsilon) = F & \text{in } B(x_0, r) \cap \Omega, \\ \dfrac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = g & \text{on } B(x_0, r) \cap \partial\Omega. \end{cases}$$

Then

$$\|\nabla u\|_{L^\infty(\Omega\cap B(x_0,r/2))} \le C_p \Bigg\{ \left(\fint_{B(x_0,r)\cap\Omega} |\nabla u_\varepsilon|^2 \right)^{1/2} + r \left(\fint_{B(x_0,r)\cap\Omega} |F|^p \right)^{1/p} + \|g\|_{L^\infty(B(x_0,r)\cap\partial\Omega)} + r^\sigma \|g\|_{C^{0,\sigma}(B(x_0,r)\cap\partial\Omega)} \Bigg\}, \tag{2.3.5}$$

where C_p depends only on d, m, p, λ, σ and τ.

Proof. By rescaling we may assume that $r = 1$. In this case the estimate (2.3.5) follows from the boundary Lipschitz estimate with Neumann boundary condition, proved in [24, 8]. As in the case of the Dirichlet condition, the fact that $\Omega = \mathbb{H}^d_n(s)$ has a flat boundary is essential for $r > 1$. $\square$

Remark 2.7. As we have pointed out, the flatness of the boundary in the last two theorems is crucial for the estimates to hold for large $r > 0$. However, if we restrict ourself to $0 < r < 1$, then the above Lipschitz estimates hold as long as the boundary is $C^{1,\alpha}$.

2.4 Diophantine condition

The Diophantine condition was first introduced in [20] (also used in [21, 7]) to study the boundary layer problem in polygonal convex domains. We give a precise definition as follows.

Definition 2.8. We say a unit vector $n \in \mathbb{S}^{d-1}$ satisfies the Diophantine condition, if there exists some $\kappa = \kappa(n) > 0$ so that

$$|(I - n\otimes n)\xi| \ge \kappa|\xi|^{-2} \quad \text{for any } \xi \in \mathbb{Z}^d \setminus \{0\}, \tag{2.4.1}$$

where $n\otimes n = (n_i n_j)_{d\times d}$. The largest possible number κ will be called the Diophantine constant of n.

Observe that $(I - n\otimes n)\xi = \xi - (\xi\cdot n)n$ is the projection vector of ξ onto the orthogonal plane of n. Intuitively, the Diophantine constant κ, arising from the number theory, quantifies the irrationality of a unit vector. Clearly, if n is rational (i.e., $n \in \mathbb{R}\mathbb{Z}^d$), then $\kappa(n) = 0$. We may also construct irrational directions whose Diophantine constants are zero by using Liouville numbers which are supposed to be arbitrarily close to rational numbers. Nonetheless, in the following lemma, we show that almost all the unit vectors satisfy the Diophantine condition with $\kappa > 0$.

Lemma 2.9. *Let Ω be a strictly convex C^2 domain. Then*

$$\frac{1}{\kappa(n(x))} \in L^{d-1,\infty}(\partial\Omega, d\sigma). \tag{2.4.2}$$

Proof. A key observation for the strictly convex domains is that for any $\omega \in \mathbb{S}^{d-1}$,

$$\sigma(\{x \in \partial\Omega : |(I - \omega \otimes \omega)n(x)| \le t\}) \le Ct^{d-1}, \tag{2.4.3}$$

if $t < 1$. This geometric property can be easily seen if $\Omega = B_1$, while the general case may follows by writing the boundary as a local graph (see [42]).

Now, let $t \in (0,1)$ and note that

$$\{x \in \partial\Omega : \kappa(n(x))^{-1} > t^{-1}\} \subset S_t := \bigcup_{\xi \in \mathbb{Z}^d \setminus \{0\}} \{x \in \partial\Omega : |(I - n(x) \otimes n(x)\xi| < t|\xi|^{-2}\}.$$

Using (2.4.3) and the fact $|(I - \omega \otimes \omega)n(x)| = |(I - n(x) \otimes n(x))\omega|$, we have

$$\begin{aligned}
&\sigma(\{x \in \partial\Omega : |(I - n(x) \otimes n(x))\xi| < t|\xi|^{-2}\}) \\
&\quad = \sigma(\{x \in \partial\Omega : |(I - n(x) \otimes n(x))\omega| < t|\xi|^{-3}, \omega = |\xi|^{-1}\xi\}) \\
&\quad = \sigma(\{x \in \partial\Omega : |(I - \omega \otimes \omega)n(x)| < t|\xi|^{-3}, \omega = |\xi|^{-1}\xi\}) \\
&\quad \le Ct^{d-1}|\xi|^{-3(d-1)}.
\end{aligned}$$

Since $3(d-1) > d$ for $d \ge 2$, it follows

$$\sigma(\{x \in \partial\Omega : \kappa(n(x))^{-1} > t^{-1}\}) \le \sigma(S_t) \le \sum_{\xi \in \mathbb{Z}^d \setminus \{0\}} Ct^{d-1}|\xi|^{-3(d-1)} \le Ct^{d-1},$$

for any $0 < t < 1$. This implies our desire result (2.4.2). $\square$

The property (2.4.2) will be used in an essential way throughout this dissertation and this is exactly the only property that we need from the strict convexity of the domains. We also emphasize that all the constants C in this dissertation will be independent of κ. In other words, if a constant depends on κ, it will be specified explicitly.

Next, we will show a quantitative equidistribution property of a periodic function restricted on a hyperplane. Let $n \in \mathbb{S}^{d-1}$ with $\kappa = \kappa(n) > 0$. Let M be a $d \times d$ orthogonal matrix so that its last column is n, namely, $Me_d = n$. Write $M = (N, n)$ where N is a $d \times (d-1)$ matrix. Now, observe that

$$I = MM^T = NN^T + n \otimes n.$$

This yields $|(I - n \otimes n)\xi| = |NN^T\xi| = |N^T\xi|$. Thus, the Diophantine condition (2.4.1) is equivalent to

$$|N^T\xi| \ge \kappa|\xi|^{-2} \quad \text{for any } \xi \in \mathbb{Z}^d \setminus \{0\}. \tag{2.4.4}$$

The following lemma is an analog of [7, Proposition 2.1].

Lemma 2.10 (Quantitative equidistribution)**.** *Let $n \in \mathbb{S}^{d-1}$ with $\kappa = \kappa(n) > 0$ and $\partial\mathbb{H}^d_n(0) = \{x : x \cdot n = 0\}$. Assume $f \in C^\infty(\mathbb{T}^d)$ (i.e., f is a smooth 1-periodic function) and $\varphi \in C^\infty(\partial\mathbb{H}^d_n(0))$. Then, for any $\ell \ge 0$,*

$$\begin{aligned}
&\left| \int_{\partial\mathbb{H}^d_n(0)} f(x/\varepsilon)\varphi(x)d\sigma - \langle f \rangle \int_{\partial\mathbb{H}^d_n(0)} \varphi(x)d\sigma \right| \\
&\qquad \le \left(\frac{\varepsilon}{2\pi\kappa}\right)^\ell \int_{\partial\mathbb{H}^d_n(0)} |\nabla^\ell_{\tan}\varphi(x)|d\sigma \sum_{0 \ne \xi \in \mathbb{Z}^d} |\widehat{f}(\xi)||\xi|^{2\ell},
\end{aligned}$$

where $\nabla_{\tan}$ *is the full tangential gradient on* $\partial\mathbb{H}_n^d(0)$.

Proof. First of all, we may use a change of variables to convert the integral on $\partial\mathbb{H}_n^d(0)$ to an integral on $\mathbb{R}^{d-1}$. Precisely, let M the $d\times d$ orthogonal matrix given above and $x = My = Ny'$ with $y = (y', 0)$. Then

$$\begin{aligned}\int_{\partial\mathbb{H}_n^d(0)} f(x/\varepsilon)\varphi(x)d\sigma &= \int_{\mathbb{R}^{d-1}} f(Ny'/\varepsilon)\varphi(Ny')dy' \\ &= \langle f\rangle \int_{\mathbb{R}^{d-1}} \varphi(Ny')dy' + \sum_{0\neq\xi\in\mathbb{Z}^d}\int_{\mathbb{R}^{d-1}} \widehat{f}(\xi)e^{2\pi i\xi\cdot Ny'/\varepsilon}\varphi(Ny')dy',\end{aligned} \tag{2.4.5}$$

where we have used the Fourier series expansion of f in the second identity and $\widehat{f}(\xi)$ is the Fourier coefficient. Note that $\langle f\rangle = \widehat{f}(0)$.

Now, fix $\xi \neq 0$. Using (2.4.4) and the integration by parts, we have

$$\begin{aligned}\Big|\int_{\mathbb{R}^{d-1}} \widehat{f}(\xi)e^{2\pi i\xi\cdot Ny'/\varepsilon}\varphi(Ny')dy'\Big| &= \Big|\int_{\mathbb{R}^{d-1}} \widehat{f}(\xi)e^{2\pi i\varepsilon^{-1}N^T\xi\cdot y'}\varphi(Ny')dy'\Big| \\ &= \Big|\int_{\mathbb{R}^{d-1}} \frac{\widehat{f}(\xi)e^{2\pi i\varepsilon^{-1}N^T\xi\cdot y'}}{(2\pi i\varepsilon^{-1}|N^T\xi|)^\ell}\Big(\frac{N^T\xi}{|N^T\xi|}\cdot\nabla\Big)^\ell(\varphi(Ny'))dy'\Big| \\ &\le \int_{\mathbb{R}^{d-1}} |\widehat{f}(\xi)||\xi|^{2\ell}\Big(\frac{\varepsilon}{2\pi\kappa}\Big)^\ell |\nabla^\ell(\varphi(Ny'))|dy'.\end{aligned}$$

Combing this with (2.4.5), we obtain

$$\begin{aligned}&\Big|\int_{\partial\mathbb{H}_n^d(0)} f(x/\varepsilon)\varphi(x)d\sigma - \langle f\rangle\int_{\mathbb{R}^{d-1}}\varphi(Ny')dy'\Big| \\ &\qquad\le \Big(\frac{\varepsilon}{2\pi\kappa}\Big)^\ell \int_{\mathbb{R}^{d-1}} |(N^T\nabla)^\ell\varphi(Ny')|dy' \sum_{0\neq\xi\in\mathbb{Z}^d}|\widehat{f}(\xi)||\xi|^{2\ell}.\end{aligned}$$

This yields the desired estimate by changing variables back to x. $\square$

Chapter 3 Neumann problems

In this chapter, we study the Neumann problem (1.2.5) and obtain the $O(\varepsilon^{\frac{1}{2}})$ convergence rate for $d \geq 3$. In the case of the Neumann problem with only zero-order oscillating data $g_0(x, x/\varepsilon) - \gamma_\varepsilon$, i.e., $g_{ij}(x, y) = 0$, the homogenization of (1.2.5) is well understood, mostly due to the fact that the Neumann function $N_\varepsilon(x, y)$ for $\mathcal{L}_\varepsilon$ in Ω converges pointwise to $N_0(x, y)$, the Neumann function for the homogenized operator $\mathcal{L}_0$ in Ω. In fact, it was proved in [25] that if Ω is a bounded $C^{1,1}$ domain in $\mathbb{R}^d$ and $d \geq 3$, then

$$|N_\varepsilon(x, y) - N_0(x, y)| \leq \frac{C\,\varepsilon \ln\left[\varepsilon^{-1}|x - y| + 2\right]}{|x - y|^{d-1}}, \tag{3.0.1}$$

for any $x, y \in \Omega$. This effectively reduces the problem to the case of operators with constant coefficients, which may be handled by the method of oscillatory integrals [3, 5]. Thus the real challenge for the Neumann problem starts with the first-order oscillating boundary data that includes terms in the form of $\varepsilon^{-1} g(x, x/\varepsilon)$. As we will show in the last section of this chapter, in the study of the higher-order convergence of solutions to the Neumann problems for $\mathcal{L}_\varepsilon$ with non-oscillating boundary data, one is forced to deal with a Neumann problem in the form of (1.2.5).

3.1 Neumann functions and Neumann correctors

Under the conditions (1.2.2)-(1.2.3) and $A \in C^\sigma(\mathbb{T}^d)$ for some $\sigma \in (0, 1)$, one may construct an $m \times m$ matrix of Neumann functions $N_\varepsilon(x, y) = (N_\varepsilon^{\alpha\beta}(x, y))$ in a bounded $C^{1,\alpha}$ domain Ω, such that

$$\begin{cases} \mathcal{L}_\varepsilon\big\{N_\varepsilon(\cdot, y)\big\} = \delta_y(x) I & \text{in } \Omega, \\ \dfrac{\partial}{\partial \nu_\varepsilon}\big\{N_\varepsilon(\cdot, y)\big\} = -|\partial\Omega|^{-1} I & \text{on } \partial\Omega, \\ \displaystyle\int_{\partial\Omega} N_\varepsilon(x, y)\, d\sigma(x) = 0, \end{cases} \tag{3.1.1}$$

where $I = I_{m\times m}$ and the operator $\mathcal{L}_\varepsilon$ acts on each column of $N_\varepsilon(\cdot, y)$. Let $u_\varepsilon \in H^1(\Omega; \mathbb{R}^m)$ be a solution to $\mathcal{L}_\varepsilon(u_\varepsilon) = F$ in Ω with $\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = h$ on $\partial\Omega$, then

$$u_\varepsilon(x) - \fint_{\partial\Omega} u_\varepsilon = \int_\Omega N_\varepsilon(x, y) F(y)\, dy + \int_{\partial\Omega} N_\varepsilon(x, y) h(y)\, d\sigma(y) \tag{3.1.2}$$

for any $x \in \Omega$. If $d \geq 3$, the Neumann functions satisfy the following estimates,

$$\begin{aligned} |N_\varepsilon(x, y)| &\leq C\,|x - y|^{2-d} \\ |\nabla_x N_\varepsilon(x, y)| + |\nabla_y N_\varepsilon(x, y)| &\leq C|x - y|^{1-d}, \\ |\nabla_x \nabla_y N_\varepsilon(x, y)| &\leq C|x - y|^{-d}, \end{aligned} \tag{3.1.3}$$

for any $x, y \in \Omega$. This was proved in [24], using boundary Lipschitz estimates with Neumann conditions, which require the additional assumption $A^* = A$. This additional assumption for the boundary Lipschitz estimates was removed later in [8]. As a result, the estimates in (3.1.3) hold if A satisfies (1.2.2)-(1.2.3) and is Hölder continuous. Note that if $x, y, z \in \Omega$ and $|x - z| \le (1/2)|x - y|$, it follows from (3.1.3) that

$$
\begin{aligned}
|N_\varepsilon(x, y) - N_\varepsilon(z, y)| &\le \frac{C|x - z|}{|x - y|^{d-1}}, \\
|\nabla_y \{N_\varepsilon(x, y) - N_\varepsilon(z, y)\}| &\le \frac{C|x - z|}{|x - y|^d}.
\end{aligned}
\tag{3.1.4}
$$

To study the boundary regularity for solutions of Neumann problems, the matrix of Neumann correctors $\Psi^\beta_{\varepsilon,j} = \big(\Psi^{\alpha\beta}_{\varepsilon,j}\big)$ for $\mathcal{L}_\varepsilon$ in Ω, defined by

$$
\begin{cases}
\mathcal{L}_\varepsilon\big(\Psi^\beta_{\varepsilon,j}\big) = 0 & \text{in } \Omega, \\
\dfrac{\partial}{\partial \nu_\varepsilon}\big(\Psi^\beta_{\varepsilon,j}\big) = \dfrac{\partial}{\partial \nu_0}\big(P^\beta_j\big) & \text{on } \partial\Omega,
\end{cases}
\tag{3.1.5}
$$

was introduced in [24], where $\partial u/\partial \nu_0$ denotes the conormal derivative associated with $\mathcal{L}_0$. One of the main estimates in [24] is the following Lipschitz estimate for $\Psi^\beta_{\varepsilon,j}$,

$$
\|\nabla \Psi^\beta_{\varepsilon,j}\|_{L^\infty(\Omega)} \le C. \tag{3.1.6}
$$

Let $N_0(x, y)$ denote the matrix of Neumann functions for $\mathcal{L}_0$ in Ω. It was proved in [25] that if Ω is $C^{1,1}$,

$$
|N_\varepsilon(x, y) - N_0(x, y)| \le \frac{C\varepsilon \ln\big[\varepsilon^{-1}|x - y| + 2\big]}{|x - y|^{d-1}} \tag{3.1.7}
$$

for any $x, y \in \Omega$, and that if Ω is $C^{2,\alpha}$ for some $\alpha \in (0, 1)$,

$$
\Big|\frac{\partial}{\partial y_i}\big\{N^{\gamma\alpha}_\varepsilon(x, y)\big\} - \frac{\partial}{\partial y_i}\big\{\Psi^{*\alpha\beta}_{\varepsilon,j}(y)\big\} \cdot \frac{\partial}{\partial y_j}\big\{N^{\gamma\beta}_0(x, y)\big\}\Big| \le \frac{C_\sigma\, \varepsilon^{1-\sigma}}{|x - y|^{d-\sigma}} \tag{3.1.8}
$$

for any $x, y \in \Omega$ and $\sigma \in (0, 1)$. The functions $(\Psi^{*\alpha\beta}_{\varepsilon,j})$ in (3.1.8) are the Neumann correctors, defined as in (3.1.5), for the adjoint operator $\mathcal{L}^*_\varepsilon$ in Ω. We remark that these estimates as well as (3.1.6) were proved in [24] under the additional assumption $A^* = A$. As in the case of (3.1.3), with the results in [8], they continue to hold without this assumption.

The estimates (3.1.7) and (3.1.8) mark the starting point of our investigation of the Neumann problem (1.2.5) with oscillating data. Indeed, let u_ε be the solution of (1.2.5) with $\int_{\partial\Omega} u_\varepsilon = 0$. It follows by (3.1.2) that

$$
\begin{aligned}
u_\varepsilon(x) = &\int_{\partial\Omega} N_\varepsilon(x, y)(T_{ij}(y) \cdot \nabla_y)\big\{g_{ij}(y, y/\varepsilon)\big\}\, d\sigma(y) \\
&+ \int_{\partial\Omega} N_\varepsilon(x, y) g_0(y, y/\varepsilon)\, d\sigma(y),
\end{aligned}
\tag{3.1.9}
$$

where $T_{ij} = n_i e_j - n_j e_i$, $n = (n_1, \cdots, n_d)$ is the outward normal to $\partial\Omega$, and $e_i = (0, \ldots, 1, \ldots, 0)$ with 1 in the i^{th} position.

Lemma 3.1. *Let Ω be a bounded Lipschitz domain in $\mathbb{R}^d$. Then, for $u, v \in C^1(\partial\Omega)$,*

$$\int_{\partial\Omega} ((n_i e_j - n_j e_i) \cdot \nabla u)\, v \, d\sigma = -\int_{\partial\Omega} u\, ((n_i e_j - n_j e_i) \cdot \nabla v)\, d\sigma. \tag{3.1.10}$$

Proof. Let u, v be extended as functions in $\overline{\Omega}$. Then, by the divergence theorem, we have

$$\begin{aligned}\int_{\partial\Omega} ((n_i e_j - n_j e_i) \cdot \nabla u)\, v \, d\sigma &= \int_{\Omega} \frac{\partial}{\partial x_i}\Big(\frac{\partial}{\partial x_j} u\, v\Big) - \frac{\partial}{\partial x_j}\Big(\frac{\partial}{\partial x_i} u\, v\Big) dx \\ &= \int_{\Omega} \frac{\partial u}{\partial x_j}\frac{\partial v}{\partial x_i} - \frac{\partial u}{\partial x_i}\frac{\partial v}{\partial x_j} dx.\end{aligned} \tag{3.1.11}$$

Similarly, the RHS of (3.1.10) gives the same result which completes the proof. □

It follows from (3.1.9) and (3.1.10) that

$$\begin{aligned}u_\varepsilon(x) = &-\int_{\partial\Omega} \big(T_{ij}(y) \cdot \nabla_y\big) N_\varepsilon(x, y) \cdot g_{ij}(y, y/\varepsilon)\, d\sigma(y) \\ &+ \int_{\partial\Omega} N_\varepsilon(x, y) g_0(y, y/\varepsilon)\, d\sigma(y).\end{aligned} \tag{3.1.12}$$

In view of (3.1.3) this implies that

$$\begin{aligned}|u_\varepsilon(x)| &\le C\, \|g\|_\infty \int_{\partial\Omega} \frac{d\sigma(y)}{|x-y|^{d-1}} + C\, \|g\|_\infty \int_{\partial\Omega} \frac{d\sigma(y)}{|x-y|^{d-2}} \\ &\le C\, \|g\|_\infty \big\{1 + |\ln \delta(x)|\big\},\end{aligned} \tag{3.1.13}$$

where $g = \{g_{ij}, g_0\}$ and $\delta(x) = \operatorname{dist}(x, \partial\Omega)$.

Remark 3.2. It follows from (3.1.13) that for any $1 < q < \infty$,

$$\|u_\varepsilon\|_{L^q(\Omega)} \le C_q\, \|g\|_\infty,$$

where C_q depends on q, A and Ω. By interpolation, this, together with (1.2.9), implies that

$$\|u_\varepsilon - u_0\|_{L^q(\Omega)} \le C_{q,\sigma}\, \varepsilon^{\frac{1}{q} - \sigma} \tag{3.1.14}$$

for any $2 < q < \infty$ and $\sigma \in (0, \frac{1}{q})$. Moreover, if $A^* = A$, it follows from [23] that

$$\|u_\varepsilon\|_{H^{1/2}(\Omega)} + \left(\int_{\Omega} |\nabla u_\varepsilon(x)|^2\, \delta(x)\, dx\right)^{1/2} \le C\|g\|_{L^2(\partial\Omega)}. \tag{3.1.15}$$

Thus, by interpolation, we may deduce from (1.2.9) and (3.1.15) that

$$\|u_\varepsilon - u_0\|_{H^\alpha(\Omega)} \le C_{\alpha,\sigma} \varepsilon^{\frac{1}{2} - \alpha - \sigma} \tag{3.1.16}$$

for any $\alpha \in (0, 1/2)$ and $\sigma \in (0, (1/2) - \alpha)$.

Using (3.1.7) and (3.1.8), we obtain

$$
\begin{aligned}
u_\varepsilon^\gamma(x) = &-\int_{\partial\Omega} \big(T_{ij}(y)\cdot\nabla_y\big)\Psi_{\varepsilon,k}^{*\alpha\beta}(y)\cdot \frac{\partial}{\partial y_k}\big\{N_0^{\gamma\beta}(x,y)\big\}\cdot g_{ij}^\alpha(y,y/\varepsilon)\,d\sigma(y)\\
&+\int_{\partial\Omega} N_0^{\gamma\alpha}(x,y) g_0^\alpha(y,y/\varepsilon)\,d\sigma(y) + R_\varepsilon^\gamma(x),
\end{aligned}
\tag{3.1.17}
$$

where the remainder R_ε satisfies

$$
\begin{aligned}
|R_\varepsilon(x)| \le & C\varepsilon^{1-\sigma}\|g\|_\infty \int_{\partial\Omega}\frac{d\sigma(y)}{|x-y|^{d-\sigma}}\\
&+ C\varepsilon\|g\|_\infty\int_{\partial\Omega}\frac{\ln[\varepsilon^{-1}|x-y|+2]}{|x-y|^{d-1}}d\sigma(y).
\end{aligned}
\tag{3.1.18}
$$

Lemma 3.3. *Let Ω be a bounded $C^{2,\alpha}$ domain for some $\alpha\in(0,1)$. Then the function R_ε, given by (3.1.17), satisfies*

$$
\|R_\varepsilon\|_{L^q(\Omega)} \le C\,\varepsilon^{\frac{1}{q}}(1+|\ln\varepsilon|)\|g\|_\infty, \tag{3.1.19}
$$

for any $1<q<\infty$, where C depends only on q, A and Ω.

Proof. Let $x\in\Omega$. If $\delta(x)=\text{dist}(x,\partial\Omega)\ge\varepsilon$, we may use (3.1.18) to show that

$$
|R_\varepsilon(x)| \le C_\sigma\left(\frac{\varepsilon}{\delta(x)}\right)^{1-\sigma}\|g\|_\infty \tag{3.1.20}
$$

for any $\sigma\in(0,1)$. If $\delta(x)\le\varepsilon$, the estimates in (3.1.3), as in (3.1.13), lead to

$$
|R_\varepsilon(x)| \le C\,\|g\|_\infty(1+|\ln\delta(x)|). \tag{3.1.21}
$$

It is not hard to verify that (3.1.19) follows from (3.1.20) and (3.1.21). □

As $\varepsilon\to 0$, the second term in the RHS of (3.1.17) converges to

$$
w_0^\gamma(x) = \int_{\partial\Omega} N_0^{\gamma\alpha}(x,y)\langle g_0^\alpha\rangle(y)\,d\sigma(y), \tag{3.1.22}
$$

where

$$
\langle g_0\rangle(y) = \fint_{\mathbb{T}^d} g_0(y,z)\,dz. \tag{3.1.23}
$$

More precisely, the following results on the convergence rate were obtained in [5].

Lemma 3.4. *Let w_ε denote the second term in the RHS of (3.1.17). Assume that Ω is a bounded smooth, uniformly convex domain in $\mathbb{R}^d$. Then, for any $1\le q<\infty$,*

$$
\|w_\varepsilon - w_0\|_{L^q(\Omega)} \le C_q\begin{cases}\varepsilon^{\frac{1}{q}} & \text{if } d=3,\\ \varepsilon^{\frac{3}{2q}} & \text{if } d=4,\\ \varepsilon^{\frac{2}{q}}(1+|\ln\varepsilon|)^{\frac{1}{q}} & \text{if } d\ge 5,\end{cases} \tag{3.1.24}
$$

where w_0 is given by (3.1.22).

Much of the rest of paper is devoted to the study of the first term in the RHS of (3.1.17). To this end we first replace the function $\Psi^{*\alpha\beta}_{\varepsilon,k}$ by

$$\psi^{*\alpha\beta}_{\varepsilon,k}(x) = \Psi^{*\alpha\beta}_{\varepsilon,k}(x) - P^{\alpha\beta}_k(x) - \varepsilon\chi^{*\alpha\beta}_k(x/\varepsilon), \tag{3.1.25}$$

where $(\chi^{*\alpha\beta}_k(y))$ denotes the matrix of correctors for $\mathcal{L}^*_\varepsilon$ in $\mathbb{R}^d$. Note that

$$\mathcal{L}^*_\varepsilon(\psi^{*\beta}_{\varepsilon,k}) = 0 \quad \text{ in } \Omega, \tag{3.1.26}$$

where $\psi^{*\beta}_{\varepsilon,k} = (\psi^{*1\beta}_{\varepsilon,k}, \dots, \psi^{*m\beta}_{\varepsilon,k})$.

We end this section with some observations on its conormal derivatives.

Lemma 3.5. *Let $\psi^{*\alpha\beta}_{\varepsilon,k}$ be defined by (3.1.25). Then*

$$\left(\frac{\partial}{\partial\nu^*_\varepsilon}\{\psi^{*\beta}_{\varepsilon,k}\}\right)^\alpha(x) = -n_i(x)b^{*\alpha\beta}_{ik}(x/\varepsilon) \quad \text{ for } x \in \partial\Omega, \tag{3.1.27}$$

where

$$b^{*\alpha\beta}_{ik}(y) = a^{*\alpha\beta}_{ik}(y) + a^{*\alpha\gamma}_{ij}(y)\frac{\partial}{\partial y_j}\big(\chi^{*\gamma\beta}_k\big) - \widehat{a}^{*\alpha\beta}_{ik} \tag{3.1.28}$$

and $\widehat{A^*} = (\widehat{a}^{*\alpha\beta}_{ij}) = (\widehat{A})^*$ is the homogenized matrix of A^*.

Proof. By the definitions (3.1.25) and (3.1.5),

$$\begin{aligned}
&\left(\frac{\partial}{\partial\nu^*_\varepsilon}\{\psi^{*\beta}_{\varepsilon,k}\}\right)^\alpha(x)\\
&= n_i a^{*\alpha\gamma}_{ij}(x/\varepsilon)\frac{\partial}{\partial x_j}\psi^{*\gamma\beta}_{\varepsilon,k}\\
&= n_i a^{*\alpha\gamma}_{ij}(x/\varepsilon)\frac{\partial}{\partial x_j}\big(\Psi^{*\gamma\beta}_{\varepsilon,k}(x) - P^{\gamma\beta}_k(x) - \varepsilon\chi^{*\gamma\beta}_k(x/\varepsilon)\big)\\
&= n_i\widehat{a}^{*\alpha\gamma}_{ij}\frac{\partial}{\partial x_j}P^{\gamma\beta}_k(x) - n_i a^{*\alpha\gamma}_{ij}(x/\varepsilon)\frac{\partial}{\partial x_j}P^{\gamma\beta}_k(x) - n_i a^{*\alpha\gamma}_{ij}(x/\varepsilon)\frac{\partial\chi^{*\gamma\beta}_k}{\partial x_j}(x/\varepsilon)\\
&= n_i\Big(\widehat{a}^{*\alpha\beta}_{ik} - a^{*\alpha\beta}_{ik}(x/\varepsilon) - a^{*\alpha\gamma}_{ij}(x/\varepsilon)\frac{\partial\chi^{*\gamma\beta}_k}{\partial x_j}(x/\varepsilon)\Big).
\end{aligned}$$

This proves the lemma. □

Note that by the definitions of correctors $\chi^{*\alpha\beta}_k$ and of the homogenized matrix $\widehat{A^*}$,

$$\frac{\partial}{\partial y_i}\{b^{*\alpha\beta}_{ik}\} = 0 \quad \text{ and } \quad \int_{\mathbb{T}^d} b^{*\alpha\beta}_{ik} = 0. \tag{3.1.29}$$

Similar as Lemma 2.1, this implies that there are 1-periodic functions $f^{\alpha\beta}_{\ell ik}$ with mean value zero such that

$$b^{*\alpha\beta}_{ik} = \frac{\partial}{\partial y_\ell}\{f^{\alpha\beta}_{\ell ik}\} \quad \text{ and } \quad f^{\alpha\beta}_{\ell ik} = -f^{\alpha\beta}_{i\ell k}. \tag{3.1.30}$$

As a result (see the proof of Theorem 2.3), we obtain

$$n_i(x)b_{ik}^{*\alpha\beta}(x/\varepsilon) = \frac{1}{2}(n_ie_j - n_je_i)\cdot\nabla_x\big\{\varepsilon f_{jik}^{\alpha\beta}(x/\varepsilon)\big\}. \tag{3.1.31}$$

This shows that $\varepsilon^{-1}\psi_{\varepsilon,k}^{*\beta}$ is a solutions of the Neumann problem (1.2.5) with $g_{ij}(x,y) = (1/2)f_{jik}^{\beta}(y)$ and $g_0 = 0$.

3.2 Neumann problems in half-spaces

For $n \in \mathbb{S}^{d-1}$ and $a \in \mathbb{R}$, let

$$\mathbb{H}_n^d(a) = \big\{x \in \mathbb{R}^d : x\cdot n < -a\big\} \tag{3.2.1}$$

denote a half-space with outward unit normal n. Consider the Neumann problem

$$\begin{cases} \operatorname{div}(A\nabla u) = 0 & \text{in } \mathbb{H}_n^d(a), \\ n\cdot A\nabla u = T\cdot\nabla g & \text{on } \partial\mathbb{H}_n^d(a), \end{cases} \tag{3.2.2}$$

where $T \in \mathbb{R}^d$, $|T| \le 1$ and $T\cdot n = 0$. We will assume that $g \in C^\infty(\mathbb{T}^d)$ with mean value zero and n satisfies the Diophantine condition (2.4.1) with constant $\kappa = \kappa(n) > 0$. Let M be a $d\times d$ orthogonal matrix such that $Me_d = -n$. Note that the last column of M is $-n$. Let N denote the $d\times(d-1)$ matrix of the first $d-1$ columns of M. Since $MM^T = I$, we see that

$$NN^T + n\otimes n = I, \tag{3.2.3}$$

where M^T denotes the transpose of M.

To study the solvability of the half-space problem (3.2.2), one first notices the boundary data $T\cdot\nabla g(\theta)$ and the coefficient matrix A are both quasi-periodic on $\partial\mathbb{H}_n^d(a)$. Recall that a quasi-periodic function is defined by restricting a periodic function in a lower dimensional hyperplane. Then, it is natural to expect that the solution of (3.2.2) also possesses the same quasi-periodic structure along every hyperplane parallel to the boundary $\partial\mathbb{H}_n^d(a)$. While in the direction of n, the solution will decay in some sense. As a result, we may assume by intuition that the solution of (3.2.2) is given by

$$u(x) = V((I-n\otimes n)x, -x\cdot n) = V(x-(x\cdot n)n, -x\cdot n), \tag{3.2.4}$$

where $V = V(\theta,t)$ is a function of $(\theta,t)\in\mathbb{T}^d\times[a,\infty)$, 1-periodic in θ. Note that

$$\nabla_x u = \Big(I-n\otimes n, -n\Big)\begin{pmatrix}\nabla_\theta \\ \partial_t\end{pmatrix} = M\begin{pmatrix}N^T\nabla_\theta \\ \partial_t\end{pmatrix}V, \tag{3.2.5}$$

where we have used (3.2.3). It follows from (3.2.2) and (3.2.5) that V is a solution of

$$\begin{cases} \begin{pmatrix}N^T\nabla_\theta \\ \partial_t\end{pmatrix}\cdot B\begin{pmatrix}N^T\nabla_\theta \\ \partial_t\end{pmatrix}V = 0 & \text{in } \mathbb{T}^d\times(a,\infty), \\ -e_{d+1}\cdot B\begin{pmatrix}N^T\nabla_\theta \\ \partial_t\end{pmatrix}V = T\cdot\nabla_\theta\widetilde{g} & \text{on } \mathbb{T}^d\times\{a\}, \end{cases} \tag{3.2.6}$$

where

$$B = B(\theta, t) = M^T A(\theta - tn) M, \tag{3.2.7}$$

$\widetilde{g}(\theta, t) = g(\theta - tn)$, and we have used the assumption that $T \cdot n = 0$ to obtain $T \cdot \nabla_x g = T \cdot \nabla_\theta \widetilde{g}$. Observe that if V^0 is a solution of (3.2.6) with $a = 0$ and

$$V^a(\theta, t) = V^0(\theta - an, t - a) \quad \text{ for } a \in \mathbb{R},$$

then V^a is a solution of (3.2.6). This follows from the fact that

$$B(\theta - an, t - a) = B(\theta, t) \quad \text{ and } \quad \widetilde{g}(\theta - an, t - a) = \widetilde{g}(\theta, t).$$

As a result, it suffices to study the boundary value problem (3.2.6) for $a = 0$. To this end, we shall consider the Neumann problem

$$\begin{cases} -\begin{pmatrix} N^T \nabla_\theta \\ \partial_t \end{pmatrix} \cdot B \begin{pmatrix} N^T \nabla_\theta \\ \partial_t \end{pmatrix} V - \lambda \Delta_\theta V = \begin{pmatrix} N^T \nabla_\theta \\ \partial_t \end{pmatrix} G & \text{in } \mathbb{T}^d \times \mathbb{R}_+, \\ -e_{d+1} \cdot B \begin{pmatrix} N^T \nabla_\theta \\ \partial_t \end{pmatrix} V = T \cdot \nabla_\theta g + e_{d+1} \cdot G & \text{on } \mathbb{T}^d \times \{0\}, \end{cases} \tag{3.2.8}$$

where $\lambda > 0$ and the term $-\lambda \Delta_\theta V$ is added to regularize the system.

Let

$$\mathcal{H} = \left\{ f \in H^1_{\text{loc}}(\mathbb{T}^d \times \mathbb{R}_+) : \int_0^\infty \int_{\mathbb{T}^d} \left(|\nabla_\theta f|^2 + |\partial_t f|^2 \right) < \infty \right\}. \tag{3.2.9}$$

We call $V \in \mathcal{H}$ a weak solution of (3.2.8) with $g \in H^1(\mathbb{T}^d)$ and $G \in L^2(\mathbb{T}^d \times \mathbb{R}_+)$, if

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{T}^d} \left\{ B \begin{pmatrix} N^T \nabla_\theta \\ \partial_t \end{pmatrix} V \cdot \begin{pmatrix} N^T \nabla_\theta \\ \partial_t \end{pmatrix} W + \lambda \begin{pmatrix} \nabla_\theta \\ 0 \end{pmatrix} V \cdot \begin{pmatrix} \nabla_\theta \\ 0 \end{pmatrix} W \right\} d\theta dt \\ &= -\int_{\mathbb{T}^d} (T \cdot \nabla_\theta g) \cdot W(\theta, 0)\, d\theta - \int_0^\infty \int_{\mathbb{T}^d} G \cdot \begin{pmatrix} N^T \nabla_\theta \\ \partial_t \end{pmatrix} W \, d\theta dt \end{aligned} \tag{3.2.10}$$

for any $W \in \mathcal{H}$.

Proposition 3.6. *Let $g \in H^1(\mathbb{T}^d)$ and $G \in L^2(\mathbb{T}^d \times \mathbb{R}_+)$. Then the boundary value problem (3.2.8) has a solution, unique up to a constant, in $\mathcal{H}$. Moreover, the solution V satisfies*

$$\int_0^\infty \int_{\mathbb{T}^d} \left(|N^T \nabla_\theta V|^2 + |\partial_t V|^2 \right) \le C \left\{ \|g\|^2_{H^1(\mathbb{T}^d)} + \|G\|^2_{L^2(\mathbb{T}^d \times \mathbb{R}_+)} \right\}, \tag{3.2.11}$$

$$\lambda \int_0^\infty \int_{\mathbb{T}^d} |\nabla_\theta V|^2 \le C \left\{ \|g\|^2_{H^1(\mathbb{T}^d)} + \|G\|^2_{L^2(\mathbb{T}^d \times \mathbb{R}_+)} \right\}, \tag{3.2.12}$$

where C depends only on d, m and μ.

Proof. This follows readily from the Lax-Milgram theorem. One only needs to observe that

$$\begin{aligned}
&\Big| \int_{\mathbb{T}^d} (T \cdot \nabla_\theta g) \cdot W(\theta, 0)\, d\theta \Big| \\
&\le C\|g\|_{H^1(\mathbb{T}^d)} \left(\int_0^1 \int_{\mathbb{T}^d} \left(|N^T \nabla_\theta W|^2 + |\partial_t W|^2 \right) \right)^{1/2}
\end{aligned} \tag{3.2.13}$$

for any $W \in \mathcal{H}$. Indeed, write

$$\begin{aligned}
\int_{\mathbb{T}^d} (T \cdot \nabla_\theta g) \cdot W(\theta, 0)\, d\theta = &\int_0^1 \int_{\mathbb{T}^d} (T \cdot \nabla_\theta g) \cdot (W(\theta, 0) - W(\theta, t))\, d\theta dt \\
&+ \int_0^1 \int_{\mathbb{T}^d} (T \cdot \nabla_\theta g) \cdot W(\theta, t)\, d\theta dt.
\end{aligned} \tag{3.2.14}$$

It is easy to see that the first term in the RHS of (3.2.14) is bounded by

$$C\|\nabla_\theta g\|_{L^2(\mathbb{T}^d)} \left(\int_0^1 \|\partial_t W\|^2_{L^2(\mathbb{T}^d)}\, dt \right)^{1/2}.$$

To handle the second term in the RHS of (3.2.14), we use

$$\int_0^1 \int_{\mathbb{T}^d} (T \cdot \nabla_\theta g) \cdot W(\theta, t)\, d\theta dt = -\int_0^1 \int_{\mathbb{T}^d} g \cdot (T \cdot \nabla_\theta W)(\theta, t)\, d\theta dt$$

and

$$T \cdot \nabla_\theta W = T \cdot N N^T \nabla_\theta W$$

to bound it by

$$C\|g\|_{L^2(\mathbb{T}^d)} \left(\int_0^1 \|N^T \nabla_\theta W\|^2_{L^2(\mathbb{T}^d)}\, dt \right)^{1/2}.$$

The estimate (3.2.13) now follows. □

Proposition 3.7. *Let $g \in H^k(\mathbb{T}^d)$ and $G \in L^2(\mathbb{R}_+, H^{k-1}(\mathbb{T}^d))$ for some $k \ge 1$. Then the solution of (3.2.8), given by Proposition 3.6, satisfies*

$$\begin{aligned}
&\int_0^\infty \left(\|N^T \nabla_\theta V\|^2_{H^{k-1}(\mathbb{T}^d)} + \|\partial_t V\|^2_{H^{k-1}(\mathbb{T}^d)} + \lambda \|V\|^2_{H^k(\mathbb{T}^d)} \right) dt \\
&\qquad \le C_k \left\{ \|g\|^2_{H^k(\mathbb{T}^d)} + \int_0^\infty \|G\|^2_{H^{k-1}(\mathbb{T}^d)} \right\} dt,
\end{aligned} \tag{3.2.15}$$

where C_k depends on d, m, k, μ and $\|A\|_{C^{k-1}(\mathbb{T}^d)}$.

Proof. The proof is standard. The case $k = 1$ is given in Proposition 3.6. To prove the estimate for $k = 2$, one applies the estimate for $k = 1$ to the quotient of difference $\{V(\theta + se_j, t) - V(\theta, t)\}\, s^{-1}$ and lets $s \to 0$. The general case follows similarly by an induction argument on k. □

Proposition 3.8. *Let $g \in H^{k+\ell-1}(\mathbb{T}^d)$ for some $k, \ell \geq 1$. Suppose that*

$$\partial_t^\alpha G \in L^2(\mathbb{R}_+, H^{k+\ell-2-\alpha}(\mathbb{T}^d)) \quad \text{for } \ 0 \leq \alpha \leq \ell - 1.$$

Then the solution of (3.2.8), given by Proposition 3.6, satisfies

$$\begin{aligned} &\int_0^\infty \|\partial_t^\ell V\|^2_{H^{k-1}(\mathbb{T}^d)}\, dt \\ &\leq C \left\{ \|g\|^2_{H^{k+\ell-1}(\mathbb{T}^d)} + \sum_{0 \leq \alpha \leq \ell-1} \int_0^\infty \|\partial_t^\alpha G\|^2_{H^{k+\ell-2-\alpha}(\mathbb{T}^d)} \right\} dt, \end{aligned} \tag{3.2.16}$$

where C depends on d, m, k, ℓ, μ and $\|A\|_{C^{k+\ell-2}(\mathbb{T}^d)}$.

Proof. The case $\ell = 1$ is contained in Proposition 3.7. To see the case $\ell = 2$, we observe that the second-order equation in (3.2.8) allows us to obtain

$$\begin{aligned} \partial_t^2 V = \ &\text{a linear combination of} \\ &\nabla_\theta (N^T \nabla_\theta) V, N^T \nabla_\theta V, \partial_t \nabla_\theta V, \partial_t V, \nabla_\theta G, \lambda \Delta_\theta V, \partial_t G \end{aligned} \tag{3.2.17}$$

with smooth coefficients. It follows that

$$\begin{aligned} \|\partial_t^2 V\|_{H^{k-1}(\mathbb{T}^d)} \leq C \Big\{ &\|N^T \nabla_\theta V\|_{H^k(\mathbb{T}^d)} + \|\partial_t V\|_{H^k(\mathbb{T}^d)} + \|G\|_{H^k(\mathbb{T}^d)} \\ &+ \|\partial_t G\|_{H^{k-1}(\mathbb{T}^d)} + \lambda \|V\|_{H^{k+1}(\mathbb{T}^d)} \Big\}. \end{aligned}$$

This, together with the estimate (3.2.15), gives (3.2.16) for $\ell = 2$. The general case follows by differentiating (3.2.17) in t and using an induction argument on ℓ. $\square$

Proposition 3.9. *Suppose that n satisfies the Diophantine condition (2.4.1) with constant $\kappa > 0$. Let V be the solution of (3.2.8), given by Proposition 3.6. Let*

$$\widetilde{V}(\theta, t) = V(\theta, t) - \fint_{\mathbb{T}^d} V(\cdot, t).$$

Then

$$\int_0^\infty \kappa^2 \|\widetilde{V}\|^2_{H^k(\mathbb{T}^d)} dt \leq C \Big\{ \|g\|^2_{H^{k+3}(\mathbb{T}^d)} + \int_0^\infty \|G\|^2_{H^{k+2}(\mathbb{T}^d)}\, dt \Big\}, \tag{3.2.18}$$

where C depends on d and k .

Proof. Recall (2.4.4) gives $|N^T \xi| \geq \kappa |\xi|^{-2}$ for any $\xi \in \mathbb{Z}^d \setminus \{0\}$. This implies that

$$\|N^T \nabla_\theta V\|_{H^{k+2}(\mathbb{T}^d)} \geq C\kappa \|\widetilde{V}\|_{H^k(\mathbb{T}^d)}, \tag{3.2.19}$$

which, together with (3.2.15), gives the estimate (3.2.18). To see (3.2.19), we use the Parseval's identity to obtain

$$\begin{aligned}\|N^T\nabla_\theta V\|^2_{H^{k+2}(\mathbb{T}^d)} &= \sum_{\xi\in\mathbb{Z}^d}(1+|\xi^2|)^{k+2}|N^T\xi\widehat{V}(\xi)|^2 \\ &\geq \sum_{0\neq\xi\in\mathbb{Z}^d}\frac{\kappa^2(1+|\xi|^2)^{k+2}}{|\xi|^4}|\widehat{V}(\xi)|^2 \\ &\geq C\kappa^2\sum_{0\neq\xi\in\mathbb{Z}^d}(1+|\xi|^2)^k|\widehat{V}(\xi)|^2 \\ &= C\kappa^2\|\widetilde{V}\|^2_{H^k(\mathbb{T}^d)}.\end{aligned} \tag{3.2.20}$$

This completes the proof of (3.2.19). □

Remark 3.10. Suppose that $g\in C^\infty(\mathbb{T}^d)$, $G\in C^\infty(\mathbb{T}^d\times\mathbb{R}_+)$ and $\partial_t^k\partial_\theta^\alpha G\in L^2(\mathbb{T}^d\times\mathbb{R}_+)$ for any k and α. For $\lambda>0$, let V_λ be the solution of (3.2.8), given by Proposition 3.6. By subtracting a constant we may assume that $\int_{\mathbb{T}^d}V_\lambda(\theta,0)d\theta=0$ and thus

$$V_\lambda(\theta,t)=\widetilde{V_\lambda}(\theta,t)+\int_0^t\int_{\mathbb{T}^d}\partial_s V_\lambda(\theta,s)d\theta ds.$$

It follows from Propositions 3.8 and 3.9 that the $L^2(\mathbb{T}^d\times(0,L))$ norm of $\partial_t^k\partial_\theta^\alpha V_\lambda$ is uniformly bounded in λ, for any k, α and $L\geq 1$. Hence, by Sobolev imbedding, the $C^k(\mathbb{T}^d\times(0,L))$ norm of V_λ is uniformly bounded in λ, for any $k\geq 0$ and $L\geq 1$. By a simple limiting argument this allows us to show that the Neumann problem (3.2.8) with $\lambda=0$ has a solution V, unique up to a constant, in $C^\infty(\mathbb{T}^d\times[0,\infty))$. Furthermore, by passing to the limit, estimates (3.2.11), (3.2.15), (3.2.16) and (3.2.18) continue to hold for this solution.

Proposition 3.11. *Suppose that n satisfies the Diophantine condition (2.4.1) with constant $\kappa>0$. Let V be the solution of (3.2.8) with $\lambda=0$, $g\in C^\infty(\mathbb{T}^d)$ and $G=0$, given by Remark 3.10. Then there exists a constant V_∞ such that for any $\ell\geq 1$,*

$$|\partial_\theta^\alpha(V-V_\infty)(\theta,t)|\leq\frac{C_{\alpha,\ell}}{\kappa(1+\kappa t)^\ell}, \tag{3.2.21}$$

for any $\alpha=(\alpha_1,\ldots,\alpha_d)$. Moreover, we have

$$|N^T\nabla_\theta(\partial_\theta^\alpha V)(\theta,t)|+|\partial_t^k\partial_\theta^\alpha V(\theta,t)|\leq\frac{C_{\alpha,\ell,k}}{(1+\kappa t)^\ell}, \tag{3.2.22}$$

where $k\geq 1$.

Proof. It follows from Propositions 3.7 and 3.8 by Sobolev imbedding that

$$|N^T\nabla_\theta(\partial_\theta^\alpha V)(\theta,t)|+|\partial_t^k\partial_\theta^\alpha V(\theta,t)|\leq C_{\alpha,k}$$

for any $\alpha=(\alpha_1,\ldots,\alpha_d)$ and $k\geq 1$. Next we note that the decay estimate in (3.2.22) follows by the exact argument as in the case of Dirichlet boundary conditions, given in [21, Proposition 2.6] (the proof does not use the boundary condition at $t = 0$). For the readers' convenience, we will present their proof here. Let

$$F(s) = \int_s^\infty \int_{\mathbb{T}^d} \Big(|N^T \nabla_\theta (\partial_\theta^\alpha V)|^2 + |\partial_t^k \partial_\theta^\alpha V|^2 \Big) \, d\theta dt.$$

We would like to show

$$F(s) \leq \frac{C_\ell}{(\kappa s)^\ell} \quad \text{for any } \ell \geq 1.$$

To this end, let $s > 0$ and

$$W(\theta, t) = V(\theta, t) - \int_{\mathbb{T}^d} V(\theta, s) d\theta.$$

Note that for $t > s$, W satisfies

$$-\begin{pmatrix} N^T \nabla_\theta \\ \partial_t \end{pmatrix} \cdot B \begin{pmatrix} N^T \nabla_\theta \\ \partial_t \end{pmatrix} W = 0.$$

Multiplying the above system by W and integrating over $\mathbb{T}^d \times [s, \infty)$, we obtain

$$\begin{aligned} &\int_s^\infty \int_{\mathbb{T}^d} B \begin{pmatrix} N^T \nabla_\theta \\ \partial_t \end{pmatrix} W \cdot \begin{pmatrix} N^T \nabla_\theta \\ \partial_t \end{pmatrix} W \, d\theta dt \\ &\qquad = \int_{\mathbb{T}^d} \left[\begin{pmatrix} 0_{d-1} \\ -1 \end{pmatrix} \cdot B(\theta, s) \begin{pmatrix} N^T \nabla_\theta \\ \partial_t \end{pmatrix} W(\theta, s) \right] W(\theta, s) d\theta. \end{aligned} \tag{3.2.23}$$

Note that $\nabla_\theta W = \nabla_\theta V$ and $\partial_t W = \partial_t V$. Then (3.2.23) implies

$$F(s) \leq C(-F'(s))^{1/2} \left(\int_{\mathbb{T}^d} |W(\theta, s)|^2 d\theta \right)^{1/2}. \tag{3.2.24}$$

To estimate the integral in (3.2.24), we need to use the equivalent Diophantine condition (2.4.4). Precisely,

$$\begin{aligned} \int_{\mathbb{T}^d} |W(\theta, s)|^2 d\theta &= \sum_{0 \neq \xi \in \mathbb{Z}^d} |\widehat{W}(\xi, s)|^2 \\ &\leq \Big(\sum_{0 \neq \xi \in \mathbb{Z}^d} |N^T \xi|^2 |\widehat{W}(\xi, s)|^2 \Big)^{1/p} \Big(\sum_{0 \neq \xi \in \mathbb{Z}^d} \frac{|\widehat{W}(\xi, s)|^2}{|N^T \xi|^{2p'/p}} \Big)^{1/p'} \\ &\leq \kappa^{-2/p} \Big(\int_{\mathbb{T}^d} |N^T \nabla_\theta W(\theta, s)|^2 \Big)^{1/p} \Big(\sum_{0 \neq \xi \in \mathbb{Z}^d} |\xi|^{4p'/p} |\widehat{W}(\xi, s)|^2 \Big)^{1/p'} \\ &\leq \kappa^{-2/p} (-F'(s))^{1/p} \|W(\cdot, s)\|_{H^{2/(p-1)}(\mathbb{T}^d)}^{2/p'}, \end{aligned}$$

where $p > 1$ and $1/p + 1/p' = 1$. Now using a simple observation

$$\begin{aligned} |W(\theta, s)|^2 &= \int_s^\infty W(\theta, t) \cdot \partial_t W(\theta, t) dt \\ &\leq \Big(\int_s^\infty |W(\theta, t)|^2 dt \Big)^{1/2} \Big(\int_s^\infty |\partial_t W(\theta, t)|^2 dt \Big)^{1/2}, \end{aligned}$$

and (3.2.15), (3.2.18), we have

$$\|W(\cdot, s)\|_{H^\ell(\mathbb{T}^d)} \leq C\kappa^{1/2},$$

for any $\ell \geq 0$. It follows that

$$\int_{\mathbb{T}^d} |W(\theta, s)|^2 d\theta \leq C\kappa^{-2/p-1/p'}(-F'(s))^{1/p} = C\kappa^{-1-1/p}(-F'(s))^{1/p}.$$

Substituting this into (3.2.24), we obtain

$$F(s) \leq C\Big(\frac{-F'(s)}{\kappa}\Big)^{\frac{1}{2}+\frac{1}{2p}}, \tag{3.2.25}$$

for any $p \in (1, \infty)$. This gives

$$F(s) \leq C_p(\kappa s)^{-\frac{p+1}{p-1}},$$

which shows that $F(s)$ may decay faster than any power of s as $s \to \infty$. This proves (3.2) as desired.

Next, by differentiating (3.2.8) in θ and t, and using a similar argument, we may show by induction that

$$F_{\alpha,k}(s) = \int_s^\infty \int_{\mathbb{T}^d} \Big(|N^T\nabla_\theta(\partial_\theta^\alpha V)|^2 + |\partial_t^k\partial_\theta^\alpha V|^2\Big)\, d\theta dt.$$

decay faster than any power of s. More precisely, assuming the estimate holds for all $|\alpha| + k < N$. Then if $|\alpha| + k = N$, for any $\ell > \frac{p+1}{p-1}$,

$$F_{\alpha,k}(s) \leq C_p\Big[\Big(\frac{-F'(s)}{\kappa}\Big)^{\frac{1}{2}+\frac{1}{2p}} + (\kappa s)^{-\ell}\Big].$$

Hence, we get

$$F_{\alpha,k}(s) + (\kappa s)^{-\frac{p+1}{p-1}} \leq C_p\Big(\frac{-F'(s)}{\kappa} + (\kappa s)^{-\frac{2p}{p-1}}\Big)^{\frac{1}{2}+\frac{1}{2p}}. \tag{3.2.26}$$

Set $G_{\alpha,k}(s) = F_{\alpha,k}(s) + (\kappa s)^{-\frac{p+1}{p-1}}$. Then, (3.2.26) implies

$$G_{\alpha,k}(s) \leq C\Big(\frac{-G'_{\alpha,k}(s)}{\kappa}\Big)^{\frac{1}{2}+\frac{1}{2p}}, \tag{3.2.27}$$

which, as before, yields the desired decay estimate of $F_{\alpha,k}(s)$. Now, by the Sobolev imbedding theorem, we establish

$$|N^T\nabla_\theta(\partial_\theta^\alpha V)(\theta, t)| + |\partial_t^k\partial_\theta^\alpha V(\theta, t)| \leq \frac{C_{\alpha,\ell,k}}{(\kappa t)^\ell},$$

which implies (3.2.22).

Finally, to show the existence of the constant limit at infinity, we note that (3.2.22) implies that

$$\lim_{t\to\infty} |\nabla_\theta V(\theta, t)| = 0 \tag{3.2.28}$$

uniformly in $\theta \in \mathbb{T}^d$. On the other hand,

$$|V(\cdot, s+h) - V(\cdot, s)| = \int_s^{s+h} |\partial_t V(\cdot, t)| dt \le \int_s^{s+h} \frac{C}{(1+\kappa t)^2} dt \le \frac{C}{\kappa s}.$$

Thus, $V(\cdot, t)$ is a Cauchy function and admits a unique limit as $t \to \infty$. Moreover, (3.2.28) implies that the limit is independent of θ. This shows the existence of $V_\infty := \lim_{t\to\infty} V(\cdot, t)$. As a consequence,

$$\begin{aligned} |\partial_\theta^\alpha (V - V_\infty)(\theta, t)| &\le \int_t^\infty |\partial_t \partial_\theta^\alpha V(\theta, s)|\, ds \le C \int_t^\infty \frac{ds}{(1+\kappa s)^{\ell+2}} \\ &\le \frac{C}{(1+\kappa t)^\ell} \int_t^\infty \frac{ds}{(1+\kappa s)^2} \\ &\le \frac{C}{\kappa (1+\kappa t)^\ell}, \end{aligned} \tag{3.2.29}$$

where we have used (3.2.22) for the second inequality. □

We now state and prove the main result of this section.

Theorem 3.12. *Let $n \in \mathbb{S}^{d-1}$ and $a \in \mathbb{R}$, where $d \ge 2$. Let $T \in \mathbb{R}^d$ such that $|T| \le 1$ and $T \cdot n = 0$. Suppose that $n \in \mathbb{S}^{d-1}$ satisfies the Diophantine condition (2.4.1) with constant $\kappa > 0$. Then for any $g \in C^\infty(\mathbb{T}^d)$, the Neumann problem (3.2.2) has a smooth solution u satisfying*

$$\begin{aligned} |u(x)| &\le \frac{C}{\kappa(1+\kappa|x\cdot n + a|)^\ell}, \\ |\partial_x^\alpha u(x)| &\le \frac{C}{(1+\kappa|x\cdot n + a|)^\ell}, \end{aligned} \tag{3.2.30}$$

for any $|\alpha| \ge 1$ and $\ell \ge 1$. The constant C depends at most on d, m, μ, α, ℓ as well as the $C^k(\mathbb{T}^d)$ norms of A and g for some $k = k(d, \alpha, \ell)$.

Proof. Let V be the solution of (3.2.8) with $\lambda = 0$, $g \in C^\infty(\mathbb{T}^d)$ and $G = 0$, given by Remark 3.10. Let

$$u(x) = V(x - (x\cdot n + a)n, -(x\cdot n + a)) - V_\infty.$$

Then u is a solution of the Neumann problem (3.2.2). The first inequality in (3.2.30) follows directly from (3.2.21). To see the second inequality, one uses (3.2.5) and (3.2.22). □

3.3 Refined estimates in half-spaces

Throughout this section we fix $n \in \mathbb{S}^{d-1}$ and $a \in \mathbb{R}$. We assume that $n \in \mathbb{S}^{d-1}$ satisfies the Diophantine condition (2.4.1) with constant $\kappa > 0$. However, we will be only interested in estimates that are independent of κ.

Our first result plays the same role as the maximum principle in the case of Dirichlet problem.

Theorem 3.13. *Let $T \in \mathbb{R}^d$ such that $|T| \le 1$ and $T \cdot n = 0$. Then for any $g \in C^\infty(\mathbb{T}^d)$, the solution u of Neumann problem (3.2.2), given by Theorem 3.12, satisfies*

$$|\nabla u(x)| \le \frac{C\,\|g\|_\infty}{|x \cdot n + a|}, \tag{3.3.1}$$

for any $x \in \mathbb{H}^d_n(a)$, where C depends only on d, m and μ as well as some Hölder norm of A.

Proof. By translation we may assume that $a = 0$. We choose a bounded smooth domain D such that

$$\begin{aligned} &B(0,1) \cap \mathbb{H}^d_n(0) \subset D \subset B(0,2) \cap \mathbb{H}^d_n(0),\\ &\overline{B(0,1)} \cap \partial\mathbb{H}^d_n(0) = \partial D \cap \partial \mathbb{H}^d_n(0). \end{aligned}$$

Let $v_\varepsilon(x) = \varepsilon u(x/\varepsilon)$. Since $\mathcal{L}_\varepsilon(v_\varepsilon) = 0$ in D,

$$v_\varepsilon(x) - v_\varepsilon(z) = \int_{\partial D} \big\{N_\varepsilon(x,y) - N_\varepsilon(z,y)\big\} \frac{\partial v_\varepsilon}{\partial \nu_\varepsilon}(y)\, d\sigma(y)$$

for any $x, z \in D$, where $N_\varepsilon(x,y)$ denotes the matrix of Neumann functions for $\mathcal{L}_\varepsilon$ in D. By a change of variables it follows that

$$u(x) - u(z) = \varepsilon^{d-2} \int_{\partial D_{1/\varepsilon}} \big\{N_\varepsilon(\varepsilon x, \varepsilon y) - N_\varepsilon(\varepsilon z, \varepsilon y)\big\} n(\varepsilon y) \cdot A(y) \nabla u(y)\, d\sigma(y),$$

where $D_{1/\varepsilon} = \big\{\varepsilon^{-1} y : y \in D\big\}$.

Fix $x, z \in \mathbb{H}^d_n(0)$ such that $|x - z| < (1/2)|x \cdot n| = (1/2)\mathrm{dist}(x, \partial\mathbb{H}^d_n(0))$. Choose $\eta_\varepsilon \in C^1_0(B(0,\varepsilon^{-1}))$ such that $0 \le \eta_\varepsilon \le 1$, $\eta_\varepsilon = 1$ on $B(0, \varepsilon^{-1} - 1)$ and $|\nabla \eta_\varepsilon| \le 1$, where $\varepsilon < 1/10$. Let $u(x) - u(z) = I_1 + I_2$, where

$$\begin{aligned} I_1 &= \varepsilon^{d-2} \int_{\partial D_{1/\varepsilon}} \eta_\varepsilon(y) \big\{N_\varepsilon(\varepsilon x, \varepsilon y) - N_\varepsilon(\varepsilon z, \varepsilon y)\big\} n(\varepsilon y) \cdot A(y) \nabla u(y)\, d\sigma(y)\\ &= \varepsilon^{d-2} \int_{\partial D_{1/\varepsilon}} \eta_\varepsilon(y) \big\{N_\varepsilon(\varepsilon x, \varepsilon y) - N_\varepsilon(\varepsilon z, \varepsilon y)\big\} T \cdot \nabla g(y)\, d\sigma(y)\\ &= -\varepsilon^{d-2} \int_{B(0,\varepsilon^{-1}) \cap \partial \mathbb{H}^d_n(0)} T \cdot \nabla_y \Big\{\eta_\varepsilon(y)(N_\varepsilon(\varepsilon x, \varepsilon y) - N_\varepsilon(\varepsilon z, \varepsilon y))\Big\} g(y)\, d\sigma(y), \end{aligned}$$

where we have used the Neumann condition for u as well as an integration by parts on the boundary. We now apply the estimates in (3.1.4). This gives

$$
\begin{aligned}
|I_1| &\le C|x-z|\|g\|_\infty \int_{\partial\mathbb{H}^d_n(0)} \frac{d\sigma(y)}{|x-y|^d} + C|x-z|\|g\|_\infty \int_{\frac{1}{\varepsilon}-1\le|y|\le\frac{1}{\varepsilon}} \frac{d\sigma(y)}{|x-y|^{d-1}}\\
&\le C_0\|g\|_\infty + C\varepsilon\|g\|_\infty|x-z|,
\end{aligned}
$$

if ε, which may depend on $|x|$, is sufficiently small. We point out that the constant C_0 in the estimate above depends only on d, m, μ and some Hölder norm of A.

Next, to handle I_2, we use the estimate

$$
|\nabla u(y)| \le \frac{C}{(1+\kappa|y\cdot n|)^2}
$$

from (3.2.30). This, together with (3.1.4), leads to

$$
\begin{aligned}
|I_2| &= \varepsilon^{d-2}\Big|\int_{\partial D_{1/\varepsilon}} (1-\eta_\varepsilon(y))\big\{N_\varepsilon(\varepsilon x,\varepsilon y) - N_\varepsilon(\varepsilon z,\varepsilon y)\big\} n(\varepsilon y)\cdot A(y)\nabla u(y)\, d\sigma(y)\Big|\\
&\le C_{x,z}\int_{\partial D_{1/\varepsilon}\cap\mathbb{H}^d_n(0)} \frac{d\sigma(y)}{|x-y|^{d-1}(1+\kappa|y\cdot n|)^2} + C_{x,z}\int_{\frac{1}{\varepsilon}-1\le|y|\le\frac{1}{\varepsilon}} \frac{d\sigma(y)}{|x-y|^{d-1}}\\
&\le C_{x,z}\varepsilon^{d-1}\int_{\partial D_{1/\varepsilon}\cap\mathbb{H}^d_n(0)} \frac{d\sigma(y)}{(1+\kappa|y\cdot n|)^2} + C_{x,z}\varepsilon,
\end{aligned}
$$

which shows that $I_2 \to 0$, as $\varepsilon \to 0$. As a result, we have proved that for any $x, z \in \mathbb{H}^d_n(0)$ with $|x-z| \le \operatorname{dist}(x, \partial\mathbb{H}^d_n(0))$,

$$
|u(x)-u(z)| = \lim_{\varepsilon\to 0}|I_1+I_2| \le C_0\|g\|_\infty.
$$

Since $\mathcal{L}_1(u)=0$ in $\mathbb{H}^d_n(0)$, by the interior Lipschitz estimates [9] for $\mathcal{L}_1$, we obtain

$$
|\nabla u(x)| \le \frac{C_0\|g\|_\infty}{|x\cdot n|},
$$

which completes the proof. $\square$

Let $\Omega = \mathbb{H}^d_n(a)$ and $\mathcal{L} = -\operatorname{div}(A(x)\nabla)$. In the rest of this section we consider the Dirichlet problem,

$$
\begin{cases}
\mathcal{L}(u) = \operatorname{div}(f) + h & \text{in } \Omega,\\
u = 0 & \text{on } \partial\Omega,
\end{cases}
\tag{3.3.2}
$$

and the Neumann problem,

$$
\begin{cases}
\mathcal{L}(u) = \operatorname{div}(f) & \text{in } \Omega,\\
\dfrac{\partial u}{\partial\nu} = -n\cdot f & \text{on } \partial\Omega,
\end{cases}
\tag{3.3.3}
$$

where A is assumed to satisfy the ellipticity condition (1.2.2) and $A \in C^\sigma(\mathbb{T}^d)$ for some $\sigma \in (0,1)$. We shall be interested in the weighted L^2 estimate,

$$\begin{aligned}\int_\Omega &|\nabla u(x)|^2 \big[\delta(x)\big]^\alpha \, dx \\ &\le C \int_\Omega |f(x)|^2 \big[\delta(x)\big]^\alpha \, dx + C \int_\Omega |h(x)|^2 \big[\delta(x)\big]^{\alpha+2} \, dx,\end{aligned} \tag{3.3.4}$$

where $-1 < \alpha < 0$ and

$$\delta(x) = \operatorname{dist}(x, \partial\Omega) = |a + (x \cdot n)|. \tag{3.3.5}$$

We start with some observations on the weight $\omega(x) = \big[\delta(x)\big]^\alpha$.

Lemma 3.14. *Let $\omega(x) = \big[\delta(x)\big]^\alpha$, where $-1 < \alpha < 0$ and $\delta(x)$ is defined by (3.3.5). Then $\omega(x)$ is an A_1 weight, i.e., for any ball $B \subset \mathbb{R}^d$,*

$$⨍_B \omega \le C \inf_B \omega, \tag{3.3.6}$$

where C depends only on d and α. Moreover, w satisfies the reverse Hölder's inequality,

$$\left(⨍_B \omega^p \, dx \right)^{1/p} \le C ⨍_B \omega \, dx, \tag{3.3.7}$$

where $1 < p < \infty$ and $\alpha p > -1$.

Proof. This is more or less well known and may be verified directly by reducing the problem to the case $\Omega = \mathbb{R}^d_+$ and $\delta(x) = |x_d|$. $\square$

It follows from (3.3.7) by Hölder's inequality that if $E \subset B$, then

$$\frac{\omega(E)}{\omega(B)} \le C \left(\frac{|E|}{|B|} \right)^{1-\frac{1}{p}}, \tag{3.3.8}$$

where $\omega(E) = \int_E \omega \, dx$. Since (3.3.8) implies that ω satisfies the doubling condition, $\omega(2B) \le C\omega(B)$, it is easy to see that (3.3.8) also holds if one replaces ball B by cube Q. In fact, if ω is an A_p weight in $\mathbb{R}^d$, i.e.,

$$⨍_B \omega \cdot \left(⨍_B \omega^{-\frac{1}{p-1}} \right)^{p-1} \le C, \tag{3.3.9}$$

then there exist some $\sigma > 0$ and $C > 0$ such that

$$\frac{\omega(E)}{\omega(Q)} \le C \left(\frac{|E|}{|Q|} \right)^\sigma \quad \text{for any } E \subset Q. \tag{3.3.10}$$

Functions satisfying (3.3.10) are called A_∞ weights. In the following we will also need the well known fact that if ω is an A_p weight for some $1 < p < \infty$, then

$$\int_{\mathbb{R}^d} |\mathcal{M}(f)|^p \, \omega \, dx \le C \int_{\mathbb{R}^d} |f|^p \, \omega \, dx, \tag{3.3.11}$$

where $\mathcal{M}(f)$ denotes the Hardy-Littlewood maximal function of f. This is the defining property of the A_p weights. Note that if ω is A_1, then ω is A_p for any $p > 1$. We refer the reader to [40] for the theory of weights in harmonic analysis.

Theorem 3.15. *Let ω be an A_1 weight in $\mathbb{R}^d$. Let $u \in H^1_{loc}(\Omega)$ be a weak solution of Dirichlet problem (3.3.2) with $h = 0$. Assume that*

$$\omega(B(x_0, R) \cap \Omega) \fint_{B(x_0,R)\cap\Omega} |\nabla u|^2 \to 0 \quad \text{as } R \to \infty, \tag{3.3.12}$$

for some $x_0 \in \partial\Omega$. Then

$$\int_\Omega |\nabla u|^2 \, \omega \, dx \le C \int_\Omega |f|^2 \, \omega \, dx, \tag{3.3.13}$$

where C depends only on d, m, μ, $\|A\|_{C^\sigma(\mathbb{T}^d)}$, and the constant in (3.3.6).

Proof. This is essentially proved in [34] by a real-variable method, originated in [12]. We provide a proof here for the reader's convenience. By translation we may assume that $a = 0$. We will also assume that $n = -e_d$ and thus $\Omega = \mathbb{R}^d_+$ for simplicity of exposition. We point out that the periodicity of the coefficient matrix is not used particularly in the proof, only the estimates in Theorem 2.6.

Fix $1 < p < 2$. Let $\rho \in (0, 1)$ be a small constant to be determined and $A = \rho^{-\sigma/2}$, where σ is given in (3.3.10). Let

$$\Omega_R = (-R, R) \times \cdots \times (-R, R) \times (0, 2R) \subset \mathbb{R}^d_+.$$

We fix $R > 1$ and consider the set

$$E(\lambda) = \big\{ x \in \Omega_R : \mathcal{M}_R(|\nabla u|^p)(x) > \lambda \big\}, \tag{3.3.14}$$

where $\mathcal{M}_R(F)$ is a localized Hardy-Littlewood maximal function of F, defined by

$$\mathcal{M}_R(F)(x) = \sup_{x \in Q \subset \Omega_{2R}} \fint_Q |F|.$$

Let

$$\lambda_0 = \frac{C_0}{|\Omega_{2R}|} \int_{\Omega_{2R}} |\nabla u|^p, \tag{3.3.15}$$

where C_0 is a large constant depending on d. For each $\lambda > \lambda_0$, we perform a Calderón-Zygmund decomposition to $E(A\lambda) \subset \Omega_R$. This produces a sequence of disjoint dyadic subcubes $\{Q_k\}$ of Ω_R such that

$$\begin{aligned} &|E(A\lambda) \setminus \cup_k Q_k| = 0, \\ &|E(A\lambda) \cap Q_k| > \rho |Q_k|, \quad |E(A\lambda) \cap Q_k^*| \le \rho |Q_k^*|, \end{aligned} \tag{3.3.16}$$

where Q_k^* denotes the dyadic parent of Q_k, i.e., Q_k is obtained by bisecting Q_k^* once. We claim that it is possible to choose $\rho, \gamma \in (0, 1)$ so that

$$\text{if } \big\{ x \in Q_k^* : \mathcal{M}_R(|f|^p)(x) \le \gamma\lambda \big\} \ne \emptyset, \text{ then } Q_k^* \subset E(\lambda). \tag{3.3.17}$$

The claim is proved by contraction. Suppose that there exists some x_0 such that $x_0 \in Q_k^* \setminus E(\lambda)$. Then, if a cube Q contains Q_k^* and $Q \subset \Omega_{2R}$,

$$\fint_Q |f|^p \le \gamma\lambda \quad \text{and} \quad \fint_Q |\nabla u|^p \le \lambda. \tag{3.3.18}$$

This implies that for any $x \in Q_k$,

$$\mathcal{M}_R(|\nabla u|^p)(x) \le \max(\mathcal{M}_{2Q_k^*}(|\nabla u|^p)(x), 5^d\lambda), \tag{3.3.19}$$

where

$$\mathcal{M}_{2Q_k^*}(|F|)(x) = \sup_{x\in Q\subset 2Q_k^*\cap\Omega} \fint_Q |F|.$$

We now write $u = v + w$, where v is a function such that

$$\mathcal{L}(v) = \operatorname{div}(f) \quad \text{in } \Omega\cap 5Q_k^* \quad \text{and} \quad v = 0 \quad \text{on } \partial\Omega\cap 5Q_k^*, \tag{3.3.20}$$

$$\int_{\Omega\cap 4Q_k^*} |\nabla v|^p \le C \int_{\Omega\cap 5Q_k^*} |f|^p, \tag{3.3.21}$$

and C depends only on d, m, p, μ and $\|A\|_{C^\sigma(\mathbb{T}^d)}$. The existence of such v follows from the boundary $W^{1,p}$ estimates for $\mathcal{L}_\varepsilon$ [9, 10]. Note that if $A > 5^d$,

$$\begin{aligned} |Q_k \cap E(A\lambda)| &\le \left|\left\{x\in Q_k : \mathcal{M}_{2Q_k^*}(|\nabla u|^p)(x) > A\lambda\right\}\right| \\ &\le \left|\left\{x\in Q_k : \mathcal{M}_{2Q_k^*}(|\nabla v|^p)(x) > (1/4)A\lambda\right\}\right| \\ &\quad + \left|\left\{x\in Q_k : \mathcal{M}_{2Q_k^*}(|\nabla w|^p)(x) > (1/4)A\lambda\right\}\right|. \end{aligned} \tag{3.3.22}$$

For the first term in the RHS of (3.3.22), we use the fact that the operator $\mathcal{M}$ is bounded from L^1 to weak-L^1. This, together with (3.3.21) and (3.3.18), shows that the term is dominated by

$$\frac{C}{A\lambda}\int_{\Omega\cap 5Q_k^*} |f|^p \le C\gamma A^{-1}|Q_k|,$$

where C depends only on d, m, μ and $\|A\|_{C^\sigma(\mathbb{T}^d)}$. Since $\mathcal{L}(w) = 0$ in $\Omega\cap 5Q_k^*$ and $w = 0$ on $\partial\Omega\cap 5Q_k^*$, in view of Theorem 2.6, we obtain

$$\begin{aligned} \|\nabla w\|_{L^\infty(\Omega\cap 2Q_k^*)} &\le C \fint_{\Omega\cap 4Q_k^*} |\nabla w| \\ &\le C\left(\fint_{\Omega\cap 4Q_k^*} |\nabla u|^p\right)^{1/p} + C\left(\fint_{\Omega\cap 4Q_k^*} |\nabla v|^p\right)^{1/p} \\ &\le C\lambda^{1/p} + C\left(\fint_{\Omega\cap 5Q_k^*} |f|^p\right)^{1/p} \\ &\le C\lambda^{1/p}, \end{aligned}$$

where we have also used estimates (3.3.18) and (3.3.21). It follows that the second term in the RHS of (3.3.22) is zero, if A is large. As a result, we have proved that

$$|Q_k \cap E(A\lambda)| \le C\gamma A^{-1}|Q_k| = C\gamma\rho^\sigma |Q_k|,$$

if $\rho \in (0,1)$ is sufficiently small. By choosing $\gamma \in (0,1)$ so small that $C\gamma\rho^\sigma < \rho$, we obtain $|Q_k \cap E(A\lambda)| < \rho|Q_k|$, which is in contradiction with (3.3.16). This proves the claim (3.3.17). We should point out that the choices of ρ and γ are uniform for all $\lambda > \lambda_0$.

To proceed, we use (3.3.10) and (3.3.16) to obtain

$$\omega(E(A\lambda) \cap Q_k) \le C\rho^\sigma \omega(Q_k). \tag{3.3.23}$$

This, together with (3.3.17), leads to

$$\begin{aligned}
\omega(E(A\lambda)) &\le \omega\big(E(A\lambda) \cap \big\{x \in \Omega_R : \mathcal{M}_R(|f|^p)(x) \le \gamma\lambda\big\}\big) \\
&\qquad + \omega\big\{x \in \Omega_R : \mathcal{M}_R(|f|^p)(x) > \gamma\lambda\big\} \\
&\le \sum_k \omega\big\{x \in E(A\lambda) \ \cap Q_k : \ \mathcal{M}_R(|f|^p)(x) \le \gamma\lambda\big\} \\
&\qquad + \omega\big\{x \in \Omega_R : \mathcal{M}_R(|f|^p)(x) > \gamma\lambda\big\} \\
&\le C\rho^\sigma \sum_k \omega(Q_k) + \omega\big\{x \in \Omega_R : \mathcal{M}_R(|f|^p)(x) > \gamma\lambda\big\},
\end{aligned} \tag{3.3.24}$$

for any $\lambda > \lambda_0$, where the last sum is taken only over those Q_k's such that $\{x \in Q_k : \mathcal{M}_R(|f|^p)(x) \le \gamma\lambda\} \ne \emptyset$. By the claim (3.3.17) this gives

$$\omega(E(A\lambda)) \le C\rho^\sigma \omega(E(\lambda)) + \omega\big\{x \in \Omega_R : \mathcal{M}_R(|f|^p)(x) > \gamma\lambda\big\} \tag{3.3.25}$$

for any $\lambda > \lambda_0$.

Finally, we multiply both sides of (3.3.25) by λ^t with $t = \frac{2}{p} - 1 \in (0,1)$, and integrate the resulting inequality in λ over the interval (λ_0, Λ) to obtain

$$\begin{aligned}
&A^{-1-t} \int_{A\lambda_0}^{A\Lambda} \lambda^t \omega(E(\lambda))\, d\lambda \\
&\qquad \le C\rho^\sigma \int_{\lambda_0}^{\Lambda} \lambda^t \omega(E(\lambda))\, d\lambda + C_\gamma \int_{\Omega_R} \Big\{\mathcal{M}_{2R}(|f|^p)\Big\}^{\frac{2}{p}} \omega\, dx.
\end{aligned}$$

Since $CA^{1+t}\rho^\sigma = C\rho^{-\frac{\sigma}{2}(1+t)}\rho^\sigma < (1/2)$ if $\rho > 0$ is small, this gives

$$\int_0^\Lambda \lambda^t \omega(E(\lambda))\, d\lambda \le C \int_0^{A\lambda_0} \lambda^t \omega(E(\lambda))\, d\lambda + C \int_{\Omega_{2R}} |f|^2 \omega\, dx,$$

where we have used the weighted norm inequality (3.3.11) as well as the fact that $2/p > 1$. By letting $\Lambda \to \infty$ we obtain

$$\begin{aligned}
\int_{\Omega_R} |\nabla u|^2 \omega\, dx &\le C\lambda_0^{\frac{2}{p}} \omega(\Omega_R) + C \int_{\Omega_{2R}} |f|^2 \omega\, dx \\
&= \frac{C\,\omega(\Omega_R)}{|\Omega_{2R}|} \int_{\Omega_{2R}} |\nabla u|^2\, dx + C \int_{\Omega_{2R}} |f|^2 \omega\, dx,
\end{aligned} \tag{3.3.26}$$

where we have used the fact $|F| \le \mathcal{M}(F)$. We complete the proof by letting $R \to \infty$ in (3.3.26) and using the assumption (3.3.12). $\square$

The next theorem treats the Neumann problem (3.3.3).

Theorem 3.16. *Let ω be an A_1 weight in $\mathbb{R}^d$. Let $u \in H^1_{loc}(\Omega)$ be a weak solution of the Neumann problem(3.3.3). Assume that u satisfies the condition (3.3.12). Then the estimate (3.3.13) holds with constant C depending only on d, m, μ, $\|A\|_{C^\sigma(\mathbb{T}^d)}$, and the constant in (3.3.6).*

Proof. The proof is similar to that of Theorem 3.15. The only difference is that in the place of (3.3.20), we need to find a function v such that,

$$\begin{cases} \mathcal{L}(v) = \operatorname{div}(f\varphi) & \text{in } \Omega \cap 5Q_k^*, \\ \dfrac{\partial v}{\partial \nu} = n \cdot (\varphi f) & \text{on } \partial\Omega \cap 5Q_k^*, \end{cases} \tag{3.3.27}$$

where $\varphi \in C_0^\infty(5Q_k^*)$, $0 \le \varphi \le 1$, and $\varphi = 1$ on $4Q_k^*$. The existence of functions satisfying (3.3.27) and the estimate (3.3.21) follows from the boundary $W^{1,p}$ estimates for $\mathcal{L}_\varepsilon$ with Neumann conditions [24, 8]. We omit the details and refer the reader to [19] for related $W^{1,p}$ estimates for Neumann problems. $\square$

Finally, we go back to the case where $\omega(x) = \big[\delta(x)\big]^\alpha$.

Theorem 3.17. *Let $-1 < \alpha < 0$, $\Omega = \mathbb{H}^d_n(a)$ and $\delta(x)$ be given by (3.3.5). Let $u \in H^1_{loc}(\Omega)$ be a weak solution of (3.3.2) in Ω. Assume that*

$$R^\alpha \int_{B(x_0,R)\cap\Omega} |\nabla u|^2 \to 0 \qquad \text{as } R \to \infty, \tag{3.3.28}$$

for some $x_0 \in \partial\Omega$. Then estimate (3.3.4) holds with a constant C depending only on d, m, μ, α and $\|A\|_{C^\sigma(\mathbb{T}^d)}$.

Proof. By translation we may assume that $a = 0$. Recall that $\omega(x) = \big[\delta(x)\big]^\alpha$ is an A_1 weight for $-1 < \alpha < 0$. Also observe that in this case the assumption (3.3.12) is reduced to (3.3.28). We may assume that

$$\int_\Omega |h(x)|^2 \big[\delta(x)\big]^{\alpha+2}\, dx = \int_{\partial\mathbb{H}^d_n(0)} \left\{ \int_0^\infty |h(x' - tn)|^2 t^{\alpha+2} dt \right\} d\sigma(x') < \infty.$$

For otherwise there is nothing to prove. It follows that for a.e. $x' \in \partial\mathbb{H}^d_n(0)$,

$$\int_s^\infty |h(x' - tn)| dt \le \left(\int_s^\infty |h(x' - tn)|^2 t^{\alpha+2} dt \right)^{1/2} \left(\int_s^\infty t^{-\alpha-2} dt \right)^{1/2} < \infty,$$

if $s > 0$. This allows us to write

$$h = \operatorname{div}(H),$$

where $H = (H_1, \dots, H_d)$ and

$$H_i(x) = n_i \int_0^\infty h(x - tn)\, dt.$$

As a result, we obtain $\mathcal{L}(u) = \operatorname{div}(f + H)$ in Ω. It follows by Theorem 3.15 that

$$\int_\Omega |\nabla u|^2 \omega\, dx \le C \int_\Omega |f|^2 \omega\, dx + C \int_\Omega |H|^2 \omega\, dx. \tag{3.3.29}$$

Finally, we observe that for $x' \in \partial \mathbb{H}^d_n(0)$,

$$\begin{aligned}
\int_0^\infty |H(x' - tn)|^2 t^\alpha\, dt &\le \int_0^\infty \Big| \int_0^\infty |h(x' - tn - sn)|\, ds \Big|^2 t^\alpha\, dt \\
&\le \int_0^\infty \Big| \int_t^\infty |h(x' - sn)|\, ds \Big|^2 t^\alpha\, dt \\
&\le \frac{4}{(\alpha+1)^2} \int_0^\infty |h(x' - tn)|^2 t^{\alpha+2}\, dt,
\end{aligned}$$

where $\alpha > -1$ and a Hardy inequality was used for the last step [39, p.272]. By integrating above inequalities in x' over $\partial \mathbb{H}^d_n(0)$, we obtain

$$\int_\Omega |H(x)|^2 \big[\delta(x)\big]^\alpha\, dx \le C \int_\Omega |h(x)|^2 \big[\delta(x)\big]^{\alpha+2}\, dx.$$

This, together with (3.3.29), gives the weighted estimate (3.3.4). □

The next theorem establishes (3.3.4) for the Neumann problem (3.3.3).

Theorem 3.18. *Let $-1 < \alpha < 0$, $\Omega = \mathbb{H}^d_n(a)$ and $\delta(x)$ be given by (3.3.5). Let $u \in H^1_{loc}(\Omega)$ be a weak solution of Neumann problem (3.3.3) in Ω. Suppose that u satisfies the condition (3.3.28) for some $x_0 \in \partial\Omega$. Then*

$$\int_\Omega |\nabla u(x)|^2 \big[\delta(x)\big]^\alpha\, dx \le C \int_\Omega |f(x)|^2 \big[\delta(x)\big]^\alpha\, dx, \tag{3.3.30}$$

where C depending only on d, m, μ, α and $\|A\|_{C^\sigma(\mathbb{T}^d)}$.

Proof. This follows directly from Theorem 3.16. □

Remark 3.19. Let $\Omega = \mathbb{H}^d_n(a)$. Let $u \in H^1_{\text{loc}}(\Omega)$ be a solution of $-\operatorname{div}(A(x)\nabla u) = \operatorname{div}(f)$ in Ω, with either Dirichlet condition $u = 0$ or Neumann condition $\frac{\partial u}{\partial \nu} = -n \cdot f$ on $\partial\Omega$. Let

$$\Omega_R = \big\{ x \in \Omega : |x - (a + x \cdot n)n| \le R \text{ and } |a + x \cdot n| \le 2R \big\}. \tag{3.3.31}$$

It follows from (3.3.26) that for $R \ge 1$

$$\int_{\Omega_R} |\nabla u|^2 \big[\delta(x)\big]^\alpha\, dx \le C R^\alpha \int_{\Omega_{2R}} |\nabla u|^2\, dx + C \int_{\Omega_{2R}} |f|^2 \big[\delta(x)\big]^\alpha\, dx, \tag{3.3.32}$$

where $-1 < \alpha < 0$ and C depends only on d, m, μ, α, and some Hölder norm of A. This will be used in §3.5.

Remark 3.20. The weighted estimates in Theorems 3.17 and 3.18 also hold for the range $0 < \alpha < 1$, which is not used in the paper. This may be proved by a duality argument.

3.4 Approximation of Neumann correctors

Throughout this section we assume that Ω is a bounded smooth, strictly convex domain in $\mathbb{R}^d$, $d \geq 3$, and that A is smooth and satisfies (1.2.2)-(1.2.3). For $g \in C^\infty(\mathbb{T}^d; \mathbb{R}^m)$, consider the Neumann problem

$$\begin{cases} \mathcal{L}_\varepsilon(u_\varepsilon) = 0 & \text{in } \Omega, \\ \dfrac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = T(x) \cdot \nabla g(x/\varepsilon) & \text{on } \partial\Omega, \end{cases} \tag{3.4.1}$$

where $T(x) = T_{ij}(x) = n_i(x)e_j - n_j(x)e_i$ for some $1 \leq i, j \leq d$ is a tangential vector field on $\partial\Omega$, and $n(x) = (n_1(x), \ldots, n_d(x))$ denotes the outward normal to $\partial\Omega$ at $x \in \partial\Omega$. Fix $x_0 \in \partial\Omega$. Assume that $n = n(x_0)$ satisfies the Diophantine condition (2.4.1) with constant $\kappa > 0$ (all constants C will be independent of κ). To approximate u_ε in a neighborhood of x_0, we solve the Neumann problem in a half-space

$$\begin{cases} \mathcal{L}_\varepsilon(v_\varepsilon) = 0 & \text{in } \mathbb{H}^d_n(a), \\ \dfrac{\partial v_\varepsilon}{\partial \nu_\varepsilon} = T(x_0) \cdot \nabla g(x/\varepsilon) & \text{on } \partial\mathbb{H}^d_n(a), \end{cases} \tag{3.4.2}$$

where $a = -x_0 \cdot n$ and $\partial\mathbb{H}^d_n(a)$ is the tangent plane of $\partial\Omega$ at x_0. Note that if $v_\varepsilon(x) = \varepsilon w(x/\varepsilon)$, then w is a solution of

$$\begin{cases} \mathcal{L}_1(w) = 0 & \text{in } \mathbb{H}^d_n(a\varepsilon^{-1}), \\ \dfrac{\partial w}{\partial \nu_1} = T(x_0) \cdot \nabla g(x) & \text{on } \partial\mathbb{H}^d_n(a\varepsilon^{-1}). \end{cases} \tag{3.4.3}$$

It then follows by Theorem 3.12 that (3.4.2) has a bounded smooth solution v_ε satisfying

$$\|\partial^\alpha_x v_\varepsilon\|_\infty \leq C_\alpha\, \varepsilon^{1-|\alpha|} \quad \text{for any } |\alpha| \geq 1. \tag{3.4.4}$$

In particular, $\|\nabla v_\varepsilon\|_\infty \leq C$. In view of Theorem 3.13, we also obtain the estimate

$$|\nabla v_\varepsilon(x)| \leq \frac{C\,\varepsilon}{|x \cdot n + a|}. \tag{3.4.5}$$

The goal of this section is prove the following.

Theorem 3.21. *Let u_ε be a solution of (3.4.1) and v_ε a solution of (3.4.2), constructed above. Let $\varepsilon \leq r \leq \sqrt{\varepsilon}$. Then, for any $\sigma \in (0,1)$,*

$$\|\nabla(u_\varepsilon - v_\varepsilon)\|_{L^\infty(B(x_0,r)\cap\Omega)} \leq C\sqrt{\varepsilon}\big\{1 + |\ln\varepsilon|\big\} + C\varepsilon^{-1-\sigma} r^{2+\sigma}, \tag{3.4.6}$$

where C depends on d, m, μ, σ, Ω, $\|A\|_{C^k(\mathbb{T}^d)}$ and $\|g\|_{C^k(\mathbb{T}^d)}$ for some $k = k(d) > 1$.

Let $N_\varepsilon(x,y)$ denote the matrix of Neumann functions for the operator $\mathcal{L}_\varepsilon$ in Ω. Since $\Omega \subset \mathbb{H}^d_n(a)$ and $\mathcal{L}_\varepsilon(u_\varepsilon - v_\varepsilon) = 0$ in Ω, we obtain the representation,

$$\begin{aligned} &u_\varepsilon(x) - v_\varepsilon(x) - \big\{u_\varepsilon(z) - v_\varepsilon(z)\big\} \\ &= \int_{\partial\Omega} \Big\{N_\varepsilon(x,y) - N_\varepsilon(z,y)\Big\} \left\{\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} - \frac{\partial v_\varepsilon}{\partial \nu_\varepsilon}\right\} d\sigma(y) \end{aligned} \tag{3.4.7}$$

for any $x, z \in \Omega$. Fix a cut-off function $\eta = \eta_\varepsilon \in C_0^\infty(B(x_0, 5\sqrt{\varepsilon}))$ such that $0 \le \eta \le 1$, $\eta = 1$ in $B(x_0, 4\sqrt{\varepsilon})$ and $|\nabla \eta| \le C\varepsilon^{-1/2}$. Let

$$I(x,z) = \int_{\partial\Omega} \eta(y) \Big\{ N_\varepsilon(x,y) - N_\varepsilon(z,y) \Big\} \left\{ \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} - \frac{\partial v_\varepsilon}{\partial \nu_\varepsilon} \right\} d\sigma(y), \tag{3.4.8}$$

$$J(x,z) = \int_{\partial\Omega} (1-\eta(y)) \Big\{ N_\varepsilon(x,y) - N_\varepsilon(z,y) \Big\} \left\{ \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} - \frac{\partial v_\varepsilon}{\partial \nu_\varepsilon} \right\} d\sigma(y). \tag{3.4.9}$$

We begin with the estimate of $I(x,z)$ in (3.4.8). Here it is essential to take advantage of the fact that the Neumann data of v_ε agrees with the Neumann data of u_ε at x_0. Furthermore, to fully utilize the decay estimates for the derivatives of Neumann functions, we need to transfer the derivative from $\partial(u_\varepsilon - v_\varepsilon)/\partial\nu_\varepsilon$ to the Neumann functions.

For $x \in \mathbb{H}^d_n(a)$, we use

$$\widehat{x} = x - ((x - x_0)\cdot n))n \in \partial\mathbb{H}^d_n(a) \tag{3.4.10}$$

to denote its projection onto the tangent plane of $\partial\Omega$ at x_0. Observe that if $x \in \partial\Omega$, then $|x - \widehat{x}| \le C|x - x_0|^2$.

Lemma 3.22. *Suppose that $x \in \partial\Omega$ and $|x - x_0| \le c_0$. Then*

$$\begin{aligned} & n(x)\cdot A^\varepsilon(x)\nabla v_\varepsilon(x) - n(x_0)\cdot A^\varepsilon(\widehat{x})\nabla v_\varepsilon(\widehat{x}) \\ & = T_{i\ell}(x)\cdot \nabla_x \left\{ \int_0^1 a^\varepsilon_{ij}(x) \frac{\partial v_\varepsilon}{\partial x_j}\big(sx + (1-s)\widehat{x}\big)\, ds\, ((x-x_0)\cdot n) n_\ell \right\} + R(x), \end{aligned} \tag{3.4.11}$$

where $T_{i\ell}(x) = n_i(x)e_\ell - n_\ell(x)e_i$, $n = n(x_0) = (n_1, \dots, n_d)$, and

$$|R(x)| \le C\big\{ |x - x_0| + \varepsilon^{-1}|x - x_0|^3 \big\}. \tag{3.4.12}$$

Proof. In view of (3.4.4) we have $\|\nabla v_\varepsilon\|_\infty \le C$ and $\|\nabla^2 v_\varepsilon\|_\infty \le C\varepsilon^{-1}$. It follows that

$$\begin{aligned} & n(x)\cdot A^\varepsilon(x)\nabla v_\varepsilon(x) - n(x_0)\cdot A^\varepsilon(\widehat{x})\nabla v_\varepsilon(\widehat{x}) \\ & = n(x_0)\cdot A^\varepsilon(x)\nabla v_\varepsilon(x) - n(x_0)\cdot A^\varepsilon(\widehat{x})\nabla v_\varepsilon(\widehat{x}) + O(|x-x_0|) \\ & = n_i(x_0)\int_0^1 \frac{\partial}{\partial s}\left\{ a^\varepsilon_{ij}\frac{\partial v_\varepsilon}{\partial x_j}\big(sx + (1-s)\widehat{x}\big)\right\} ds + O(|x-x_0|) \\ & = n_i(x_0)\int_0^1 \frac{\partial}{\partial x_\ell}\left(a^\varepsilon_{ij}\frac{\partial v_\varepsilon}{\partial x_j}\right)\big(sx + (1-s)\widehat{x}\big)((x-x_0)\cdot n)n_\ell(x_0)\, ds + O(|x-x_0|) \\ & = \int_0^1 \left(n_i(x_0)\frac{\partial}{\partial x_\ell} - n_\ell(x_0)\frac{\partial}{\partial x_i}\right)\left(a^\varepsilon_{ij}\frac{\partial v_\varepsilon}{\partial x_j}\right)\big(sx + (1-s)\widehat{x}\big)((x-x_0)\cdot n)n_\ell(x_0)\, ds \\ & \qquad + O(|x-x_0|), \end{aligned}$$

where we have used the equation $\mathcal{L}_\varepsilon(v_\varepsilon) = 0$ in the last step. Using the observation that

$$\begin{aligned} & \left(n_i(x_0)\frac{\partial}{\partial x_\ell} - n_\ell(x_0)\frac{\partial}{\partial x_i}\right)\Big(F(sx + (1-s)\widehat{x})\Big) \\ & \qquad = \left(n_i(x_0)\frac{\partial}{\partial x_\ell} - n_\ell(x_0)\frac{\partial}{\partial x_i}\right) F(sx + (1-s)\widehat{x}), \end{aligned}$$

we then obtain

$$
\begin{aligned}
&n(x)\cdot A^\varepsilon(x)\nabla v_\varepsilon(x) - n(x_0)\cdot A^\varepsilon(\widehat{x})\nabla v_\varepsilon(\widehat{x}) = \\
&\left(n_i(x_0)\frac{\partial}{\partial x_\ell} - n_\ell(x_0)\frac{\partial}{\partial x_i}\right)\left(\int_0^1 \left(a_{ij}^\varepsilon \frac{\partial v_\varepsilon}{\partial x_j}\right)\big(sx+(1-s)\widehat{x}\big)((x-x_0)\cdot n)n_\ell(x_0)\,ds\right) \\
&\qquad\quad + O(|x-x_0|) \\
&= O(|x-x_0|) + O(\varepsilon^{-1}|x-x_0|^3) + \\
&\left(n_i(x)\frac{\partial}{\partial x_\ell} - n_\ell(x)\frac{\partial}{\partial x_i}\right)\left(\int_0^1 \left(a_{ij}^\varepsilon \frac{\partial v_\varepsilon}{\partial x_j}\right)\big(sx+(1-s)\widehat{x}\big)((x-x_0)\cdot n)n_\ell(x_0)\,ds\right),
\end{aligned}
$$

where we have used the fact that $|(x-x_0)\cdot n|\le C|x-x_0|^2$ as well as the estimate $|\nabla(A^\varepsilon\nabla v_\varepsilon)|\le C\varepsilon^{-1}$ for the last step. □

Lemma 3.22 allows us to carry out an integration by parts on the boundary for $I(x,z)$.

Lemma 3.23. *Let $I(x,z)$ be given by (3.4.8). Suppose that $x,z\in B(x_0,3r)\cap\Omega$ for some $\varepsilon\le r\le\sqrt{\varepsilon}$ and $|x-z|\le(1/2)\delta(x)$, where $\delta(x)=dist(x,\partial\Omega)$. Then*

$$|I(x,z)|\le C\,r\sqrt{\varepsilon}. \tag{3.4.13}$$

Proof. Let $y\in\partial\Omega$ and $|y-x_0|\le 5\sqrt{\varepsilon}$. Using the Neumann conditions for u_ε, v_ε and Lemma 3.22, we see that

$$
\begin{aligned}
&\frac{\partial u_\varepsilon}{\partial\nu_\varepsilon}(y) - \frac{\partial v_\varepsilon}{\partial\nu_\varepsilon}(y) \\
&= T(y)\cdot\nabla g(y/\varepsilon) - T(x_0)\cdot\nabla g(\widehat{y}/\varepsilon) + n(x_0)\cdot A^\varepsilon(\widehat{y})\nabla v_\varepsilon(\widehat{y}) - n(y)\cdot A^\varepsilon(y)\nabla v_\varepsilon(y) \\
&= T(y)\cdot\nabla_y\Big\{\varepsilon g(y/\varepsilon) - \varepsilon g(\widehat{y}/\varepsilon)\Big\} - T_{i\ell}(y)\cdot\nabla_y\big\{f_{i\ell}(y)\big\} + O(|y-x_0|),
\end{aligned}
$$

where

$$f_{i\ell}(y) = \int_0^1 a_{ij}^\varepsilon(y)\frac{\partial v_\varepsilon}{\partial y_j}\big(sy+(1-s)\widehat{y}\big)\,ds\,((y-x_0)\cdot n)n_\ell$$

is given by Lemma 3.22. We have also used the observation,

$$T(x_0)\cdot\nabla_y\big\{\varepsilon g(\widehat{y}/\varepsilon)\big\} = T(x_0)\cdot\nabla g(\widehat{y}/\varepsilon),$$

in the computation above. This, together with (3.1.10), gives

$$I(x,z) = I_1 + I_2 + I_3,$$

where

$$
\begin{aligned}
I_1 &= -\int_{\partial\Omega} T(y)\cdot\nabla_y\Big\{\eta(y)\Big(N_\varepsilon x,y) - N_\varepsilon(z,y)\Big)\Big\}\Big\{\varepsilon g(y/\varepsilon) - \varepsilon g(\widehat{y}/\varepsilon)\Big\}\,d\sigma(y), \\
I_2 &= \int_{\partial\Omega} T_{i\ell}(y)\cdot\nabla_y\Big\{\eta(y)\Big(N_\varepsilon x,y) - N_\varepsilon(z,y)\Big)\Big\}f_{i\ell}(y)\,d\sigma(y), \\
|I_3| &\le C\int_{\partial\Omega}|\eta(y)||N_\varepsilon(x,y) - N_\varepsilon(z,y)||y-x_0|\,d\sigma(y).
\end{aligned}
$$

Since $|x-z| \le (1/2)\delta(x) \le (1/2)|y-x|$ for any $y \in \partial\Omega$, by (3.1.4), we obtain

$$
\begin{aligned}
|I_1| &\le C \int_{\partial\Omega} |\nabla_y \{\eta(y)\big(N_\varepsilon(x,y) - N_\varepsilon(z,y)\big)\}||y-x_0|^2\, d\sigma(y) \\
&\le C\sqrt{\varepsilon}\delta(x) \int_{4\sqrt{\varepsilon}\le|y-x_0|\le 5\sqrt{\varepsilon}} \frac{d\sigma(y)}{|y-x|^{d-1}} + C\,\delta(x) \int_{|y-x_0|\le C\sqrt{\varepsilon}} \frac{|y-x_0|^2}{|y-x|^d}\, d\sigma(y),
\end{aligned}
\tag{3.4.14}
$$

where we have used the fact $|y-\widehat{y}| \le C|y-x_0|^2$ and $|\nabla\eta(y)| \le C\varepsilon^{-1/2}$. For the first term in the RHS of (3.4.14), we note that if $|y-x_0| \ge 4\sqrt{\varepsilon}$, then

$$
|y-x| \ge |y-x_0| - |x-x_0| \ge 4\sqrt{\varepsilon} - 3r \ge \sqrt{\varepsilon}.
$$

For the second term, we use $|y-x_0| \le |y-x| + r$. This leads to

$$
\begin{aligned}
|I_1| &\le C\sqrt{\varepsilon}\,\delta(x) + C\delta(x) \int_{|y-x_0|\le C\sqrt{\varepsilon}} \frac{d\sigma(y)}{|y-x|^{d-2}} + Cr^2\delta(x) \int_{\partial\Omega} \frac{d\sigma(y)}{|y-x|^d} \\
&\le C\sqrt{\varepsilon}\,\delta(x) + Cr^2 \\
&\le Cr\sqrt{\varepsilon}.
\end{aligned}
$$

Since $|f_{i\ell}| \le C|y-x_0|^2$, the estimate of I_2 is the exactly same as that of I_1.

Finally, to handle I_3, we use (3.1.4) as well as $|y-x_0| \le |y-x|+r$ again to obtain

$$
\begin{aligned}
|I_3| &\le C \int_{|y-x_0|\le C\sqrt{\varepsilon}} \frac{|x-z|}{|y-x|^{d-1}} \big\{|y-x| + r\big\}\, d\sigma(y) \\
&\le C\, r\sqrt{\varepsilon}.
\end{aligned}
$$

This completes the proof. $\square$

To estimate $J(x,z)$ in (3.4.9), we split it as $J(x,z) = J_1 - J_2$, where

$$
\begin{aligned}
J_1(x,z) &= \int_{\partial\Omega} (1-\eta(y))\Big\{N_\varepsilon(x,y) - N_\varepsilon(z,y)\Big\} \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon}\, d\sigma(y), \\
J_2(x,z) &= \int_{\partial\Omega} (1-\eta(y))\Big\{N_\varepsilon(x,y) - N_\varepsilon(z,y)\Big\} \frac{\partial v_\varepsilon}{\partial \nu_\varepsilon}\, d\sigma(y).
\end{aligned}
\tag{3.4.15}
$$

Lemma 3.24. *Let $J_1(x,z)$ be given by (3.4.15). Suppose that $x, z \in B(x_0, 3r)\cap\Omega$ for some $\varepsilon \le r \le \sqrt{\varepsilon}$ and $|x-z| \le (1/2)\delta(x)$, where $\delta(x) = dist(x,\partial\Omega)$. Then*

$$
|J_1(x,z)| \le C\, r\sqrt{\varepsilon}. \tag{3.4.16}
$$

Proof. Using the Neumann condition for u_ε and (3.1.10), we see that

$$
J_1(x,z) = -\int_{\partial\Omega} T\cdot\nabla_y \Big\{(1-\eta(y))\big(N_\varepsilon(x,y) - N_\varepsilon(z,y)\big)\Big\} \varepsilon g(y/\varepsilon)\, d\sigma(y).
$$

It follows that

$$
\begin{aligned}
|J_1(x,z)| &\le C\varepsilon\|g\|_\infty \int_{\partial\Omega} |\nabla_y \Big\{(1-\eta(y))\big(N_\varepsilon(x,y) - N_\varepsilon(z,y)\big)\Big\}|\, d\sigma(y) \\
&\le C\sqrt{\varepsilon}|x-z| + C\varepsilon|x-z| \int_{|y-x_0|\ge 5\sqrt{\varepsilon}} \frac{d\sigma(y)}{|x-y|^d} \\
&\le C\, r\sqrt{\varepsilon},
\end{aligned}
$$

where we have used (3.1.4) for the second inequality. $\square$

It remains to estimate $J_2(x,z)$.

Lemma 3.25. *Let $J_2(x,z)$ be given by (3.4.15). Suppose that $x,z \in B(x_0,3r)\cap\Omega$ for some $\varepsilon \le r \le \sqrt{\varepsilon}$ and $|x-z| \le (1/2)\delta(x)$, where $\delta(x) = dist(x,\partial\Omega)$. Then*

$$|J_2(x,z)| \le C\, r\sqrt{\varepsilon}\big\{1+|\ln\varepsilon|\big\}. \tag{3.4.17}$$

Proof. It follows by the divergence theorem that

$$\begin{aligned}
J_2(x,z) &= -\int_\Omega (1-\eta(y))\nabla_y\big\{N_\varepsilon(x,y)-N_\varepsilon(z,y)\big\}\cdot A(y/\varepsilon)\nabla v_\varepsilon(y)\,dy\\
&\qquad -\int_\Omega \big\{N_\varepsilon(x,y)-N_\varepsilon(z,y)\big\}\nabla_y(1-\eta(y))\cdot A(y/\varepsilon)\nabla v_\varepsilon(y)\,dy\\
&= \int_\Omega A^*(y/\varepsilon)\nabla_y\big\{N_\varepsilon(x,y)-N_\varepsilon(z,y)\big\}\cdot\nabla_y(1-\eta(y))\,(v_\varepsilon(y)-E)\,dy\\
&\qquad -\int_\Omega \big\{N_\varepsilon(x,y)-N_\varepsilon(z,y)\big\}\nabla_y(1-\eta(y))\cdot A(y/\varepsilon)\nabla v_\varepsilon(y)\,dy,
\end{aligned}$$

where $E\in\mathbb{R}^m$ is a constant to be chosen. Here we have used $\mathcal{L}_\varepsilon(v_\varepsilon)=0$ in Ω for the first equality and

$$\begin{aligned}
\mathcal{L}^*_\varepsilon\big\{N_\varepsilon(x,\cdot)-N_\varepsilon(z,\cdot)\big\} &= 0 \quad \text{in } \Omega\setminus B(x_0,3\sqrt{\varepsilon}),\\
\frac{\partial}{\partial\nu^*_\varepsilon}\big\{N_\varepsilon(x,\cdot)-N_\varepsilon(z,\cdot)\big\} &= 0 \quad \text{on } \partial\Omega
\end{aligned}$$

for the second. As before, we apply the estimates in (3.1.4) to obtain

$$\begin{aligned}
|J_2(x,z)| &\le \frac{C|x-z|}{(\sqrt{\varepsilon})^{d+1}}\int_{B(x_0,5\sqrt{\varepsilon})\cap\mathbb{H}^d_n(a)}|v_\varepsilon-E| + \frac{C|x-z|}{(\sqrt{\varepsilon})^{d}}\int_{B(x_0,5\sqrt{\varepsilon})\cap\mathbb{H}^d_n(a)}|\nabla v_\varepsilon|\\
&\le \frac{Cr}{(\sqrt{\varepsilon})^{d}}\int_{B(x_0,5\sqrt{\varepsilon})\cap\mathbb{H}^d_n(a)}|\nabla v_\varepsilon|,
\end{aligned} \tag{3.4.18}$$

where we have chosen E to be the average of v_ε over $B(x_0,5\sqrt{\varepsilon})\cap\mathbb{H}^d_n(a)$ and used a Poincaré type inequality for the last step.

Finally, to estimate the integral in the RHS of (3.4.18), we split the region $B(x_0,5\sqrt{\varepsilon})\cap\mathbb{H}^d_n(a)$ into two parts. If $|x\cdot n+a|\le\varepsilon$, we use the estimate $\|\nabla v_\varepsilon\|_\infty \le C$. If $|x\cdot n+a|\ge\varepsilon$, we apply the refined estimate (3.4.5). This yields that

$$|J_2(x,z)| \le Cr\sqrt{\varepsilon}\big\{1+|\ln\varepsilon|\big\},$$

which completes the proof. □

We are now ready to give the proof of Theorem 3.21.

Proof of Theorem 3.21. Let $\varepsilon\le r\le\sqrt{\varepsilon}$. In view of Lemmas 3.23, 3.24 and 3.25, we have proved that if $x,z\in\Omega\cap B(x_0,3r)$ and $|x-z|<(1/2)\delta(x)$, where $\delta(x) = \text{dist}(x,\partial\Omega)$, then

$$|u_\varepsilon(x)-v_\varepsilon(x)-\{u_\varepsilon(z)-v_\varepsilon(z)\}| \le Cr\sqrt{\varepsilon}\big\{1+|\ln\varepsilon|\big\}. \tag{3.4.19}$$

Since $\mathcal{L}_\varepsilon(u_\varepsilon - v_\varepsilon) = 0$ in Ω, by the interior Lipschitz estimate for $\mathcal{L}_\varepsilon$ [9], it follows that for any $x \in B(x_0, 2r)$,

$$|\nabla u_\varepsilon(x) - \nabla v_\varepsilon(x)| \le Cr\sqrt{\varepsilon}\big\{1 + |\ln \varepsilon|\big\}\big[\delta(x)\big]^{-1}. \tag{3.4.20}$$

Thus, if $0 < p < 1$,

$$\left(\fint_{B(x_0,2r)\cap\Omega} |\nabla(u_\varepsilon - v_\varepsilon)|^p\right)^{1/p} \le C\sqrt{\varepsilon}\big\{1 + |\ln \varepsilon|\big\}. \tag{3.4.21}$$

Next, we estimate the $C^\sigma(B(x_0, 2r) \cap \partial\Omega)$ norm of

$$F(y) = \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} - \frac{\partial v_\varepsilon}{\partial \nu_\varepsilon}.$$

As in the proof of Lemma 3.23, we write

$$\begin{aligned}
F(y) &= \Big\{T(y)\cdot\nabla g(y/\varepsilon) - T(x_0)\cdot\nabla g(\widehat{y}/\varepsilon)\Big\}\\
&\qquad + \Big\{n(x_0)\cdot A(\widehat{y}/\varepsilon)\nabla w(\widehat{y}/\varepsilon) - n(y)\cdot A(y/\varepsilon)\nabla w(y/\varepsilon)\Big\}\\
&= F_1(y) + F_2(y),
\end{aligned}$$

where we have used the fact $v_\varepsilon(x) = \varepsilon w(x/\varepsilon)$ and w is a solution of (3.4.3). Using $|y - \widehat{y}| \le C|y - x_0|^2$ and $\|\nabla w\|_\infty + \|\nabla^2 w\|_\infty \le C$, it is easy to see that if $y \in B(x_0, 2r) \cap \partial\Omega$,

$$|F_1(y)| + |F_2(y)| \le C|y - x_0| + C\varepsilon^{-1}|y - x_0|^2 \le C\,\varepsilon^{-1}r^2, \tag{3.4.22}$$

where we also used the assumption $\varepsilon \le r$ for the last step. By extending $n(y)$ smoothly to a neighborhood of $\partial\Omega$, we may assume that $F(y)$ is defined in $B(x_0, c_0) \cap \mathbb{H}^d_n$. A computation shows that

$$|\nabla_y F(y)| \le C\big\{1 + \varepsilon^{-1}|y - x_0| + \varepsilon^{-2}|y - x_0|^2\big\} \le C\varepsilon^{-2}r^2, \tag{3.4.23}$$

where we have used the estimate $\|\nabla^3 w\|_\infty \le C$. By interpolation it follows from (3.4.22) and (3.4.23) that

$$\|F\|_{C^{0,\sigma}(B(x_0,2r)\cap\partial\Omega)} \le C\,(\varepsilon^{-1}r^2)^{1-\sigma}(\varepsilon^{-2}r^2)^\sigma = C\,\varepsilon^{-1-\sigma}r^2 \tag{3.4.24}$$

for any $\sigma \in (0, 1)$.

Finally, since $\mathcal{L}_\varepsilon(u_\varepsilon - v_\varepsilon) = 0$ in $\Omega \cap B(x_0, 2r)$, we apply the boundary Lipschitz estimate for solutions with Neumann data [24, 8] to obtain

$$\begin{aligned}
&\|\nabla(u_\varepsilon - v_\varepsilon)\|_{L^\infty(\Omega\cap B(x_0,r))} \le\\
&C\left(\fint_{B(x_0,2r)\cap\Omega} |\nabla(u_\varepsilon - v_\varepsilon)|^p\right)^{1/p} + C\|F\|_{L^\infty(B(x_0,2r)\cap\partial\Omega)} + Cr^\sigma\|F\|_{C^{0,\sigma}(B(x_0,2r)\cap\partial\Omega)}\\
&\le C\sqrt{\varepsilon}\big\{1 + |\ln \varepsilon|\big\} + C\varepsilon^{-1}r^2 + C\varepsilon^{-1-\sigma}r^{2+\sigma}\\
&\le C\sqrt{\varepsilon}\big\{1 + |\ln \varepsilon|\big\} + C\varepsilon^{-1-\sigma}r^{2+\sigma}.
\end{aligned}$$

This completes the proof. □

Recall that the function $\psi^{*\beta}_{\varepsilon,k} = (\psi^{*1\beta}_{\varepsilon,k}(y), \dots, \psi^{*m\beta}_{\varepsilon,k}(y)$ in (3.1.25) is a solution of the Neumann problem

$$\begin{cases} \mathcal{L}^*_\varepsilon(\psi^{*\beta}_{\varepsilon,k}) = 0 & \text{in } \Omega, \\ \dfrac{\partial}{\partial \nu^*_\varepsilon}\{\psi^{*\beta}_{\varepsilon,k}\} = -\dfrac{1}{2}(n_i(x)e_\ell - n_\ell(x)e_i)\cdot \nabla f^\beta_{\ell ik}(x/\varepsilon) & \text{on } \partial\Omega, \end{cases} \tag{3.4.25}$$

where the 1-periodic functions $f^\beta_{\ell ik} \in C^\infty(\mathbb{T}^d;\mathbb{R}^m)$ are given by (3.1.30). For each $x_0 \in \partial\Omega$ fixed and satisfying (2.4.1), in view of Theorem 3.21, we may approximate this function in a small neighborhood of x_0 by a solution of

$$\begin{cases} \mathcal{L}^*_\varepsilon(\phi^{*\beta,x_0}_{\varepsilon,k}) = 0 & \text{in } \mathbb{H}^d_n(a), \\ \dfrac{\partial}{\partial \nu^*_\varepsilon}\{\phi^{*\beta,x_0}_{\varepsilon,k}\} = -\dfrac{1}{2}(n_ie_\ell - n_\ell e_i)\cdot \nabla f^\beta_{\ell ik}(x/\varepsilon) & \text{on } \partial\mathbb{H}^d_n(a), \end{cases} \tag{3.4.26}$$

where $n = n(x_0)$ and $\mathbb{H}^d_n(a)$ is the tangent plane of $\partial\Omega$ at x_0. Recall that by a change of variables, a solution of (3.4.26) is given by

$$\phi^{\alpha\beta,x_0}_{\varepsilon,k}(x) = \varepsilon V^{*\alpha\beta,n}_k\left(\frac{x-(x\cdot n+a)n}{\varepsilon}, -\frac{x\cdot n + a}{\varepsilon}\right), \tag{3.4.27}$$

where $V^* = V^{*\beta,n}_k(\theta,t) = (V^{*1\beta,n}_k(\theta,t),\dots,V^{*m\beta,n}_k(\theta,t))$ is the smooth solution of

$$\begin{cases} \begin{pmatrix} N^T\nabla_\theta \\ \partial_t \end{pmatrix} \cdot M^T A^*(\theta - tn) M \begin{pmatrix} N^T\nabla_\theta \\ \partial_t \end{pmatrix} V^* = 0 & \text{in } \mathbb{T}^d\times\mathbb{R}_+, \\ -e_{d+1}\cdot M^T A^*(\theta) M \begin{pmatrix} N^T\nabla_\theta \\ \partial_t \end{pmatrix} V^* = -\dfrac{1}{2}(n_ie_\ell - n_\ell e_i)\cdot\nabla_\theta f^\beta_{lik} & \text{on } \mathbb{T}^d\times\{0\}, \end{cases} \tag{3.4.28}$$

given by Remark 3.10. As a result, we may deduce the following from Theorem 3.21.

Theorem 3.26. *Let $\varepsilon \le r \le \sqrt{\varepsilon}$ and $\sigma\in(0,1/2)$. Then for any $x\in B(x_0,r)\cap\Omega$,*

$$\begin{aligned} &\left|\nabla\left(\Psi^{*\alpha\beta}_{\varepsilon,k}(x) - P^{\alpha\beta}_k(x) - \varepsilon\chi^{*\alpha\beta}_k\left(\frac{x}{\varepsilon}\right) - \varepsilon V^{*\alpha\beta,n}_k\left(\frac{x-(x\cdot n+a)n}{\varepsilon}, -\frac{x\cdot n+a}{\varepsilon}\right)\right)\right| \\ &\le C\sqrt{\varepsilon}\{1+|\ln\varepsilon|\} + C\varepsilon^{-1-\sigma}r^{2+\sigma}, \end{aligned} \tag{3.4.29}$$

where C depends only on d, m, σ, μ, Ω and $\|A\|_{C^k(\mathbb{T}^d)}$ for some $k = k(d) > 1$.

3.5 Estimates of the homogenized data

Observe that by (3.4.27),

$$\begin{aligned} T_{ij}(x_0)\cdot\nabla_x\phi^{*\alpha\beta,x_0}_{\varepsilon,k}(x) &= T(x_0)\cdot(I - n\otimes n, -n)\begin{pmatrix}\nabla_\theta\\ \partial_t\end{pmatrix} V^{*\alpha\beta,n}_k\left(\frac{x}{\varepsilon},0\right) \\ &= T_{ij}(x_0)\cdot\nabla_\theta V^{*\alpha\beta,n}_k\left(\frac{x}{\varepsilon},0\right), \end{aligned} \tag{3.5.1}$$

where $x \in \partial\mathbb{H}_n^d(a)$ and we have used the fact $T_{ij}(x_0) \cdot n(x_0) = 0$. For $x \in \partial\Omega$, define

$$
\begin{aligned}
\widetilde{g}_k^\beta(x) &= T_{ij}(x) \cdot \Big\langle \nabla_\theta (V_k^{*\alpha\beta,n} + \chi_k^{*\alpha\beta})(\cdot, 0) g_{ij}^\alpha(x, \cdot) \Big\rangle + (T_{ij} \cdot \nabla_x) x_k \langle g_{ij}^\beta(x, \cdot) \rangle \\
&= T_{ij}(x) \cdot \int_{\mathbb{T}^d} (I - n \otimes n) \nabla_\theta (V_k^{*\alpha\beta,n} + \chi_k^{*\alpha\beta})(\theta, 0) g_{ij}^\alpha(x, \theta)\, d\theta \\
&\qquad + (n_i \delta_{jk} - n_j \delta_{ik}) \int_{\mathbb{T}^d} g_{ij}^\beta(x, \theta) d\theta \\
&= T_{ij}(x) \cdot \int_{\mathbb{T}^d} \Big(e_k \delta^{\alpha\beta} + \nabla_\theta \chi_k^{*\alpha\beta}(\theta) + \nabla_\theta V_k^{*\alpha\beta}(\theta, 0) \Big) g_{ij}^\alpha(x, \theta)\, d\theta,
\end{aligned}
\tag{3.5.2}
$$

where $n = n(x)$. Using the estimate $\|(I - n \otimes n)\nabla_\theta V^*\| \le C$ in Proposition 3.11, we obtain $\|\widetilde{g}\|_\infty \le C\|g\|_\infty$.

Let

$$
v_\varepsilon^\gamma(x) = -\int_{\partial\Omega} \big(T_{ij}(y) \cdot \nabla_y\big) \Psi_{\varepsilon,k}^{*\alpha\beta}(y) \cdot \frac{\partial}{\partial y_k} \big\{ N_0^{\gamma\beta}(x, y) \big\} \cdot g_{ij}^\alpha(y, y/\varepsilon)\, d\sigma(y) \tag{3.5.3}
$$

be the first term in the RHS of (3.1.17). In the next section we will show that as $\varepsilon \to 0$,

$$
v_\varepsilon^\gamma(x) \to v_0^\gamma(x) = -\int_{\partial\Omega} \frac{\partial}{\partial y_k} \big\{ N_0^{\gamma\beta}(x, y) \big\} \widetilde{g}_k^\beta(y)\, d\sigma(y). \tag{3.5.4}
$$

Fix $1 \le \gamma \le m$ and $1 \le k \le d$. Using

$$
\begin{aligned}
&n_i n_j \widehat{a}_{ij}^{*\alpha\beta} \frac{\partial}{\partial y_k} \big\{ N_0^{\gamma\beta}(x, y) \big\} \\
&= n_i \widehat{a}_{ij}^{*\alpha\beta} \Big((n_j e_k - n_k e_j) \cdot \nabla_y \Big) N_0^{\gamma\beta}(x, y) + n_k \left(\frac{\partial}{\partial \nu_0^*(y)} \big\{ N_0^\gamma(x, y) \big\} \right)^\alpha,
\end{aligned}
\tag{3.5.5}
$$

where $N_0^\gamma(x, y) = (N_0^{\gamma 1}(x, y), \cdots, N_0^{\gamma m}(x, y))$, we may write

$$
\frac{\partial}{\partial y_k} \big\{ N_0^{\gamma\beta}(x, y) \big\} = h^{*\beta\alpha} n_i \widehat{a}_{ij}^{*\alpha t} \Big((n_j e_k - n_k e_j) \cdot \nabla_y \Big) N_0^{\gamma t}(x, y) - h^{*\beta\gamma} n_k |\partial\Omega|^{-1}, \tag{3.5.6}
$$

where $h^*(y) = (h^{*\alpha\beta}(y))$ is the inverse of the $m \times m$ matrix $(\widehat{a}_{ij}^{*\alpha\beta} n_i n_j)$ and we have used the fact that the conormal derivative of the matrix of Neumann functions is $-|\partial\Omega|^{-1} I_{m\times m}$. It follows that

$$
\begin{aligned}
v_0^\gamma(x) = &-\int_{\partial\Omega} \big[(n_j e_k - n_k e_j) \cdot \nabla_y\big] N_0^{\gamma t}(x, y) \cdot h^{*\beta\alpha} n_i \widehat{a}_{ij}^{*\alpha t} \widetilde{g}_k^\beta(y)\, d\sigma(y) \\
&\qquad + \fint_{\partial\Omega} h^{*\beta\gamma} n_k(y) \widetilde{g}_k^\beta(y)\, d\sigma(y) \\
&= \int_{\partial\Omega} N_0^{\gamma t}(x, y) \Big[(n_j e_k - n_k e_j) \cdot \nabla_y \Big] \Big(n_i \widehat{a}_{ji}^{t\alpha} h^{\alpha\beta} \widetilde{g}_k^\beta(y) \Big)\, d\sigma(y) + \text{constant},
\end{aligned}
\tag{3.5.7}
$$

where $h = (h^*)^* = (h^{\alpha\beta})$ is the inverse of the matrix $(\widehat{a}_{ij}^{\alpha\beta} n_i n_j)$. This shows that v_0 is a solution of the following Neumann problem,

$$\begin{cases} \mathcal{L}_0(v_0) = 0 & \text{in } \Omega, \\ \left(\dfrac{\partial v_0}{\partial \nu_0}\right)^\gamma = \big(T_{jk} \cdot \nabla\big)\Big(n_i \widehat{a}_{ji}^{\gamma\alpha} h^{\alpha\beta} \widetilde{g}_k^\beta(y)\Big) & \text{on } \partial\Omega. \end{cases} \tag{3.5.8}$$

Thus the homogenized data $\overline{g}_{jk}^\gamma$ in (1.2.7) is given by

$$\begin{aligned} \overline{g}_{jk}^\gamma &= n_i \widehat{a}_{ji}^{\gamma\alpha} h^{\alpha\beta} \widetilde{g}_k^\beta \\ &= n_i \widehat{a}_{ji}^{\gamma\alpha} h^{\alpha\beta} T_{\ell r} \cdot \int_{\mathbb{T}^d} \Big(e_k \delta^{\nu\beta} + \nabla_\theta \chi_k^{*\nu\beta}(\theta) + \nabla_\theta V_k^{*\nu\beta}(\theta, 0)\Big) g_{\ell r}^\nu(x, \theta)\, d\theta, \end{aligned} \tag{3.5.9}$$

where $(h^{\alpha\beta})$ is the inverse of the matrix $(\widehat{a}_{ij}^{\alpha\beta} n_i n_j)$.

The rest of this section is devoted to the proof of the following.

Theorem 3.27. *Let $x, y \in \partial\Omega$ and $|x - y| \le c_0$. Suppose that $n(x)$ and $n(y)$ satisfy the Diophantine condition (2.4.1) with constants $\kappa(x)$ and $\kappa(y)$, respectively. Let $\overline{g} = (\overline{g}_k^\beta)$ be defined by (3.5.9). Then, for any $\sigma \in (0, 1)$,*

$$|\overline{g}(x) - \overline{g}(y)| \le \frac{C_\sigma |x - y|}{\kappa^{1+\sigma}} \sup_{z \in \mathbb{T}^d} \|g(\cdot, z)\|_{C^1(\partial\Omega)}, \tag{3.5.10}$$

where $\kappa = \max(\kappa(x), \kappa(y))$ and C_σ depends only on d, m, σ, μ, and $\|A\|_{C^k(\mathbb{T}^d)}$ for some $k = k(d) > 1$.

Remark 3.28. The estimate in (3.5.10) is not optimal. In Chapter 4, we will use a more delicate argument to improve the estimate to

$$|\overline{g}(x) - \overline{g}(y)| \le \frac{C_\sigma |x - y|}{\kappa^{\sigma}} \sup_{z \in \mathbb{T}^d} \|g(\cdot, z)\|_{C^1(\partial\Omega)}.$$

Nevertheless, (3.5.10) is sufficient for us to establish the nearly optimal convergence rates for all $d \ge 3$.

Assume that $n, \widetilde{n} \in \mathbb{S}^{n-1}$ satisfy the condition (2.4.1). Choose two orthogonal matrices M_n, $M_{\widetilde{n}}$ such that $M_n(e_d) = -n$, $M_{\widetilde{n}} e_d = -\widetilde{n}$ and $|M_n - M_{\widetilde{n}}| \le C|n - \widetilde{n}|$. Let N_n and $N_{\widetilde{n}}$ denote the $d \times (d-1)$ matrices of the first $d-1$ columns of M_n and $M_{\widetilde{n}}$, respectively. Let $V_n^*(\theta, t)$ and $V_{\widetilde{n}}^*(\theta, t)$ be the corresponding solutions of (3.4.28). We will show that

$$\int_{\mathbb{T}^d} |N_n^T \nabla_\theta \big(V_n^*(\theta, 0) - V_{\widetilde{n}}^*(\theta, 0)\big)|\, d\theta \le \frac{C_\sigma |n - \widetilde{n}|}{\kappa^{1+\sigma}}, \tag{3.5.11}$$

where $\kappa > 0$ is the constant in the Diophantine condition (2.4.1) for $\widetilde{n}$. Using $N_n N_n^T = I - n \otimes n$, $N_{\widetilde{n}} N_{\widetilde{n}}^T = I - \widetilde{n} \otimes \widetilde{n}$, and the estimate $|\nabla_\theta V_{\widetilde{n}}^*| \le C\kappa^{-1}$ from Proposition 3.11, it is not hard to see that (3.5.10) follows from (3.5.11). Furthermore, let

$$W(\theta, t) = V_n^*(\theta, t) - V_{\widetilde{n}}^*(\theta, t). \tag{3.5.12}$$

Since

$$\int_{\mathbb{T}^d} |N_n^T \nabla_\theta W(\theta,0)|\, d\theta \le \int_0^1 \int_{\mathbb{T}^d} |N_n^T \nabla_\theta W|\, d\theta dt + \int_0^1 \int_{\mathbb{T}^d} |N_n^T \nabla_\theta \partial_t W|\, d\theta dt,$$

it suffices to show that

$$\int_0^1 \int_{\mathbb{T}^d} \Big\{ |N_n^T \nabla_\theta W|^2 + |\nabla_\theta \partial_t W|^2 \Big\} d\theta dt \le \frac{C_\sigma |n-\widetilde{n}|^2}{\kappa^{2+\sigma}}, \tag{3.5.13}$$

for any $\sigma \in (0,1)$.

Let

$$B_n^*(\theta,t) = M_n^T A^*(\theta - tn) M_n \quad \text{and} \quad B_{\widetilde{n}}^*(\theta,t) = M_{\widetilde{n}}^T A^*(\theta - t\widetilde{n}) M_{\widetilde{n}}.$$

To prove (3.5.13), as in the case of Dirichlet condition [21, 7], we first note that W is a solution of the Neumann problem,

$$\begin{cases} -\begin{pmatrix} N_n^T\nabla_\theta \\ \partial_t \end{pmatrix} \cdot B_n^* \begin{pmatrix} N_n^T\nabla_\theta \\ \partial_t \end{pmatrix} W = \begin{pmatrix} N_n^T\nabla_\theta \\ \partial_t \end{pmatrix} G + H & \text{in } \mathbb{T}^d \times \mathbb{R}_+, \\ -e_{d+1} \cdot B_n^* \begin{pmatrix} N_n^T\nabla_\theta \\ \partial_t \end{pmatrix} W = h + e_{d+1}\cdot G & \text{on } \mathbb{T}^d \times \{0\}, \end{cases} \tag{3.5.14}$$

where $G = G_1 + G_2$, H, and h are given by

$$\begin{aligned} G_1 &= B_n^* \begin{pmatrix} (N_n - N_{\widetilde{n}})^T \nabla_\theta \\ 0 \end{pmatrix} V_{\widetilde{n}}^*, \\ G_2 &= \big(B_n^* - B_{\widetilde{n}}^*\big) \begin{pmatrix} N_{\widetilde{n}}^T\nabla_\theta \\ \partial_t \end{pmatrix} V_{\widetilde{n}}^*, \\ H &= \begin{pmatrix} (N_n - N_{\widetilde{n}})^T \nabla_\theta \\ 0 \end{pmatrix} B_{\widetilde{n}}^* \begin{pmatrix} N_{\widetilde{n}}^T\nabla_\theta \\ \partial_t \end{pmatrix} V_{\widetilde{n}}^*, \\ h &= -\frac{1}{2}\big[(n_i - \widetilde{n}_i)e_\ell - (n_\ell - \widetilde{n}_\ell)e_i\big] \cdot \nabla_\theta f_{i\ell}. \end{aligned} \tag{3.5.15}$$

Note that $|\partial_t^k \partial_\theta^\alpha (B_n^* - B_{\widetilde{n}}^*)| \le C(1+t)|n-\widetilde{n}|$. This, together with Proposition 3.11, gives

$$|\partial_t^k \partial_\theta^\alpha G_1(\theta,t)| \le \frac{C|n-\widetilde{n}|}{\kappa(1+\kappa t)^\ell}, \tag{3.5.16}$$

$$|\partial_t^k \partial_\theta^\alpha G_2(\theta,t)| \le \frac{C(t+1)|n-\widetilde{n}|}{(1+\kappa t)^\ell}, \tag{3.5.17}$$

$$|\partial_t^k \partial_\theta^\alpha H(\theta,t)| \le \frac{C|n-\widetilde{n}|}{(1+\kappa t)^\ell}, \tag{3.5.18}$$

for any α, k, ℓ, where C depends on d, m, α, k, ℓ and A.

To deal with the growth factor $t+1$ in (3.5.17) as well as the term H, we rely on the following weighted estimates.

Lemma 3.29. *Suppose that $n \in \mathbb{S}^{n-1}$ satisfies the Diophantine condition (2.4.1). Let U be a smooth solution of*

$$\begin{cases} -\begin{pmatrix} N_n^T \nabla_\theta \\ \partial_t \end{pmatrix} \cdot B_n^* \begin{pmatrix} N_n^T \nabla_\theta \\ \partial_t \end{pmatrix} U = \begin{pmatrix} N_n^T \nabla_\theta \\ \partial_t \end{pmatrix} F & \text{in } \mathbb{T}^d \times \mathbb{R}_+, \\ -e_{d+1} \cdot B_n^* \begin{pmatrix} N_n^T \nabla_\theta \\ \partial_t \end{pmatrix} U = e_{d+1} \cdot F & \text{on } \mathbb{T}^d \times \{0\}. \end{cases} \tag{3.5.19}$$

Assume that

$$\sup_{t>0} \Big\{ (1+t) \|\nabla_{\theta,t} U(\cdot, t)\|_{L^\infty(\mathbb{T}^d)} + (1+t)\|F(\cdot,t)\|_{L^\infty(\mathbb{T}^d)} \Big\} < \infty. \tag{3.5.20}$$

Then, for any $-1 < \alpha < 0$,

$$\int_0^\infty \int_{\mathbb{T}^d} \Big\{ |N_n^T \nabla_\theta U|^2 + |\partial_t U|^2 \Big\} t^\alpha \, d\theta dt \le C_\alpha \int_0^\infty \int_{\mathbb{T}^d} |F|^2 \, t^\alpha \, d\theta dt, \tag{3.5.21}$$

where C_α depends only on d, m, μ, α as well as some Hölder norm of A.

Proof. We will reduce the weighted estimate (3.5.21) to the analogous estimates we proved in §3.3 in a half-space. Let

$$u(x) = U(x - (x\cdot n)n, -x\cdot n) \quad \text{and} \quad f(x) = F(x-(x\cdot n)n, -x\cdot n).$$

Then u is a solution of the Neumann problem,

$$\begin{cases} -\text{div}(A(x)\nabla u) = \text{div}(f) & \text{in } \mathbb{H}_n^d(0), \\ n \cdot A(x)\nabla u = -n \cdot f & \text{on } \partial \mathbb{H}_n^d(0). \end{cases} \tag{3.5.22}$$

It follows from the estimate (3.3.32) that

$$\begin{aligned} &\frac{1}{R^{d-1}} \int_{\Omega_R} |N_n^T \nabla_\theta U(x-(x\cdot n)n, -x\cdot n)|^2 \, |x\cdot n|^\alpha \, dx \\ &\qquad + \frac{1}{R^{d-1}} \int_{\Omega_R} |\partial_t U(x-(x\cdot n)n, -x\cdot n)|^2 \, |x\cdot n|^\alpha \, dx \\ &\le C R^{\alpha+1-d} \int_{\Omega_{2R}} |\nabla u|^2 \, dx + \frac{C}{R^{d-1}} \int_{\Omega_{2R}} |F(x-(x\cdot n)n, -x\cdot n)|^2 \, |x\cdot n|^\alpha \, dx, \end{aligned} \tag{3.5.23}$$

where

$$\Omega_R = \big\{ x \in \mathbb{H}_n^d(0) : |x - (x\cdot n)n| \le R \text{ and } |x\cdot n| \le 2R \big\}.$$

Next, we compute the limit for each term in (3.5.23), as $R \to \infty$. In view of (3.5.20) it is clear that the first term in the RHS of (3.5.23) goes to zero. For the second term in the RHS of (3.5.23), we write it as

$$C \int_0^{2R} t^\alpha \left\{ \frac{1}{R^{d-1}} \int_{\substack{x\cdot n = t \\ |x+tn|<R}} |F(x+tn, t)|^2 \, d\sigma(x) \right\} dt. \tag{3.5.24}$$

Since $F(\theta,t)$ is 1-periodic in θ and n satisfies the Diophantine condition,

$$\frac{1}{R^{d-1}} \int_{\substack{x\cdot n=t\\ |x+tn|<R}} |F(x+tn,t)|^2\, d\sigma(x) \to C_d \int_{\mathbb{T}^d} |F(\theta,t)|^2\, d\theta \tag{3.5.25}$$

for each $t>0$, as $R\to\infty$. With the assumption (3.5.20) at our disposal, we apply the Dominated Convergence Theorem to deduce that the last integral in (3.5.23) converges to

$$C_d \int_0^\infty \int_{\mathbb{T}^d} |F(\theta,t)|^2\, t^\alpha\, d\theta dt.$$

A similar argument also shows that the LHS of (3.5.23) converges to

$$C_d \int_0^\infty \int_{\mathbb{T}^d} \Big\{ |N_n^T \nabla_\theta U|^2 + |\partial_t U|^2 \Big\}\, t^\alpha\, d\theta dt.$$

As a result, we have proved the estimate (3.5.21). □

Remark 3.30. The same argument as in the proof of Lemma 3.29 also gives a weighted estimate for Dirichlet problem. More precisely, suppose that $n\in\mathbb{S}^{n-1}$ satisfies the Diophantine condition (2.4.1). Let U be a smooth solution of

$$\begin{cases} -\begin{pmatrix} N_n^T\nabla_\theta \\ \partial_t \end{pmatrix} \cdot B_n^* \begin{pmatrix} N_n^T\nabla_\theta \\ \partial_t \end{pmatrix} U = \begin{pmatrix} N_n^T\nabla_\theta \\ \partial_t \end{pmatrix} F + H & \text{in } \mathbb{T}^d\times\mathbb{R}_+, \\ U = 0 & \text{on } \mathbb{T}^d\times\{0\}. \end{cases} \tag{3.5.26}$$

Assume that

$$\sup_{t>0} (1+t) \Big\{ \|\nabla_{\theta,t} U(\cdot,t)\|_{L^\infty(\mathbb{T}^d)} + \|F(\cdot,t)\|_{L^\infty(\mathbb{T}^d)} + (1+t)\|H(\cdot,t)\|_{L^\infty(\mathbb{T}^d)} \Big\} < \infty.$$

Then, for any $-1<\alpha<0$,

$$\begin{aligned} &\int_0^\infty \int_{\mathbb{T}^d} \Big\{ |N_n^T\nabla_\theta U|^2 + |\partial_t U|^2 \Big\}\, t^\alpha\, d\theta dt \\ &\qquad \le C_\alpha \int_0^\infty \int_{\mathbb{T}^d} \Big\{ |F|^2 + |H|^2 t^2 \Big\}\, t^\alpha\, d\theta dt, \end{aligned} \tag{3.5.27}$$

where C_α depends only on d, m, μ, α as well as some Hölder norm of A. This weighted estimate will also be used in the next chapter to establish the regularity estimate of the homogenized boundary data for the Dirichlet problem.

Lemma 3.31. *Suppose that* $n\in\mathbb{S}^{n-1}$ *satisfies the Diophantine condition (2.4.1). Let* U *be a smooth solution of*

$$\begin{cases} -\begin{pmatrix} N_n^T\nabla_\theta \\ \partial_t \end{pmatrix} \cdot B_n^* \begin{pmatrix} N_n^T\nabla_\theta \\ \partial_t \end{pmatrix} U = 0 & \text{in } \mathbb{T}^d\times\mathbb{R}_+, \\ -e_{d+1}\cdot B_n^* \begin{pmatrix} N_n^T\nabla_\theta \\ \partial_t \end{pmatrix} U = h & \text{on } \mathbb{T}^d\times\{0\}. \end{cases} \tag{3.5.28}$$

Assume that

$$\sup_{t>0}(1+t)\|\nabla_{\theta,t}U(\cdot,t)\|_{L^\infty(\mathbb{T}^d)} < \infty. \tag{3.5.29}$$

Then

$$\int_0^1\int_{\mathbb{T}^d}\Big\{|N_n^T\nabla_\theta U|^2 + |\partial_t U|^2\Big\}d\theta dt \le C\,\|h\|^2_{L^2(\mathbb{T}^d)}. \tag{3.5.30}$$

Proof. Let $u(x) = U(x-(x\cdot n)n, -x\cdot n)$. Note that u is a solution of the Neumann problem, $-\text{div}(A(x)\nabla u) = 0$ in $\mathbb{H}_n^d(0)$ and $n\cdot A(x)\nabla u = h$ on $\partial\mathbb{H}_n^d(0)$. Let $u_\varepsilon(x) = \varepsilon u(x/\varepsilon)$ and D be the same bounded smooth domain as in the proof of Theorem 3.13. Since $\mathcal{L}_\varepsilon(u_\varepsilon) = 0$ in D, it follows that

$$\int_{D(\varepsilon)}|\nabla u_\varepsilon|^2\,dx \le C\,\varepsilon\int_{\partial D}\Big|\frac{\partial u_\varepsilon}{\partial \nu_\varepsilon}\Big|^2\,d\sigma, \tag{3.5.31}$$

where $D(\varepsilon) = \{x\in D:\ \text{dist}(x,\partial D) < \varepsilon\}$. We remark that if D is a Lipschitz domain and $A^* = A$, the large-scale Rellich estimate (3.5.31) was proved in [26] by Rellich identities. If D is smooth, the symmetry condition is not needed. This may be proved by using the $O(\sqrt{\varepsilon})$ convergence rate in $H^1(D)$, as in [35].

By a change of variables and using the assumption (3.5.29), one may deduce from (3.5.31) that

$$\int_{\substack{x\in B(0,R)\cap\mathbb{H}_n^d(0)\\ 0<|x\cdot n|<1}}|\nabla u|^2\,dx \le C\int_{B(0,2R)\cap\partial\mathbb{H}_n^d(0)}|h|^2\,d\sigma + o(R^{d-1}), \tag{3.5.32}$$

as $R\to\infty$. We now divide both sides of (3.5.32) by R^{d-1} and then let $R\to\infty$. As in the proof of Lemma 3.29, this leads to the desired inequality (3.5.30). □

We are now in a position to give the proof of Theorem 3.27.

Proof of Theorem 3.27. Recall that it suffices to prove (3.5.13) with W given by (3.5.12). To do this we split W as $W = W_1 + W_2 + W_3$, where W_1 is a solution of (3.5.14) with $G = 0$ and $H = 0$, W_2 a solution with $H = 0$ and $h = 0$, and W_3 a solution with $G = 0$ and $h = 0$. In view of Proposition 3.11, we may require that

$$\sup_{t>0}(1+t)^\ell\|\nabla_{\theta,t}W_i(\cdot,t)\|_{L^\infty(\mathbb{T}^d)} < \infty$$

for $i = 1,2,3$, and for any $\ell\ge 1$.

To estimate W_1, we use Lemma 3.31 to obtain

$$\int_0^1\int_{\mathbb{T}^d}\Big\{|N_n^T\nabla_\theta W_1|^2 + |\partial_t W_1|^2\Big\}d\theta dt \le C\,\|h\|^2_{L^2(\mathbb{T}^d)} \le C|n-\widetilde{n}|^2. \tag{3.5.33}$$

To handle W_2, we use the weighted estimates in Lemma 3.29 with $\alpha = \sigma - 1$. This, together with estimates (3.5.16)-(3.5.18), leads to

$$\begin{aligned}
&\int_0^1 \int_{\mathbb{T}^d} \left\{ |N_n^T \nabla_\theta W_2|^2 + |\partial_t W_2|^2 \right\} d\theta dt \\
&\leq C \int_0^\infty \int_{\mathbb{T}^d} |G|^2 \, t^{\sigma-1} \, d\theta dt \\
&\leq C \int_0^\infty \left\{ \frac{(t+1)^2 |n - \widetilde{n}|^2}{(1+\kappa t)^4} + \frac{|n-\widetilde{n}|^2}{\kappa^2 (1+\kappa t)^4} \right\} t^{\sigma-1} \, dt \\
&\leq \frac{C|n-\widetilde{n}|^2}{\kappa^{2+\sigma}}.
\end{aligned} \tag{3.5.34}$$

Next, we note that by writing $H(\theta,t) = \partial_t \widetilde{H}(\theta,t)$, where

$$\widetilde{H}(\theta,t) = -\int_t^\infty H(\theta,s) ds,$$

we may reduce the estimate of W_3 to the previous two cases. Indeed, we split W_3 as $W_{31} + W_{32}$, where W_{31} is a solution of (3.5.14) with $G = 0$, $H = 0$, $h = e_d \cdot \widetilde{H}(\theta, 0)$, and W_{32} a solution of (3.5.14) with $G = (0, \widetilde{H})$, $H = 0$, and $h = 0$. Observe that by (3.5.18),

$$\begin{aligned}
\|\widetilde{H}(\cdot,0)\|^2_{L^2(\mathbb{T}^d)} &\leq \frac{C|n-\widetilde{n}|^2}{\kappa^2} \\
|\partial_t^k \partial_\theta^\alpha \widetilde{H}(\theta,t)| &\leq \frac{C|n-\widetilde{n}|}{\kappa(1+\kappa t)^\ell},
\end{aligned}$$

for any α, k and ℓ. As in the cases of W_1 and W_2, by Lemmas 3.29 and 3.31, we obtain

$$\int_0^1 \int_{\mathbb{T}^d} \left\{ |N_n^T \nabla_\theta W_3|^2 + |\partial_t W_3|^2 \right\} d\theta dt \leq \frac{C|n-\widetilde{n}|^2}{\kappa^{2+\sigma}}.$$

Consequently, we have proved that

$$\int_0^1 \int_{\mathbb{T}^d} \left\{ |N_n^T \nabla_\theta W|^2 + |\partial_t W|^2 \right\} d\theta dt \leq \frac{C|n-\widetilde{n}|^2}{\kappa^{2+\sigma}}. \tag{3.5.35}$$

Finally, we note that by differentiating the system (3.5.14), the function $\partial^\alpha W$ (with $|\alpha| = 1$) is a smooth solution to a Neumann problem of same type as W. In particular, let $N_{n,k}$ denote the kth column of N_n, and define the kth component of

$N_n^T \nabla_\theta$ by $\nabla_k = N_{n,k}^T \cdot \nabla_\theta$, where $1 \le k \le d-1$. We apply ∇_k to (5.3.21) and obtain

$$\begin{cases} -\begin{pmatrix} N_n^T \nabla_\theta \\ \partial_t \end{pmatrix} \cdot B_n^* \begin{pmatrix} N_n^T \nabla_\theta \\ \partial_t \end{pmatrix} \nabla_k W = \begin{pmatrix} N_n^T \nabla_\theta \\ \partial_t \end{pmatrix} \cdot \nabla_k G + \nabla_k H \\ \qquad\qquad + \begin{pmatrix} N_n^T \nabla_\theta \\ \partial_t \end{pmatrix} \cdot \nabla_k B_n^* \begin{pmatrix} N_n^T \nabla_\theta \\ \partial_t \end{pmatrix} W \\ \qquad\qquad \text{in } \mathbb{T}^d \times \mathbb{R}_+, \\ -e_{d+1} \cdot B_n^* \begin{pmatrix} N_n^T \nabla_\theta \\ \partial_t \end{pmatrix} \nabla_k W = e_{d+1} \cdot \nabla_k G + \nabla_k h \\ \qquad\qquad + e_{d+1} \cdot \nabla_k B_n^* \begin{pmatrix} N_n^T \nabla_\theta \\ \partial_t \end{pmatrix} W \\ \qquad\qquad \text{on } \mathbb{T}^d \times \{0\}. \end{cases} \tag{3.5.36}$$

Let $\eta(t)$ be a cut-off function such that $\eta(t) = 1$ for $t \in [0,1]$, $\eta(t) = 0$ for $t \in [2, \infty)$, $0 \le \eta(t) \le 1$ and $|\nabla \eta| \le C$. Now by integrating (3.5.36) against $\nabla_k(W\eta^2)$, we derive from integration by parts that

$$\begin{aligned} &\int_0^1 \int_{\mathbb{T}^d} \left(|N_n^T \nabla_\theta \nabla_k W|^2 + |\partial_t \nabla_k W|^2 \right) d\theta dt \\ &\quad \le C \int_0^2 \int_{\mathbb{T}^d} \left(|\nabla_k G|^2 + |\nabla_k H|^2 + |N_n^T \nabla_\theta W|^2 + |\partial_t W|^2 \right) d\theta dt + C \|h\|_{H^1(\mathbb{T}^d)}^2 \\ &\quad \le \frac{C|n - \widetilde{n}|^2}{\kappa^{2+\sigma}}, \end{aligned}$$

where we have used the fact $|\nabla_k W| \le |\nabla_n^T \nabla_\theta W|$. Consequently,

$$\int_0^1 \int_{\mathbb{T}^d} \left(|N_n^T \nabla_\theta \otimes N_n^T \nabla_\theta W|^2 + |\partial_t N_n^T \nabla_\theta W|^2 \right) d\theta dt \le \frac{C|n - \widetilde{n}|^2}{\kappa^{2+\sigma}},$$

which finishes the proof of (3.5.13). $\square$

3.6 A partition of unity

For $x \in \partial\Omega$, recall that

$$\kappa(x) = \sup \Big\{ \kappa \in [0,1] : \text{the Diophantine condition (2.4.1) holds} \\ \text{for } n(x) \text{ with constant } \kappa \Big\}. \tag{3.6.1}$$

Lemma 2.9 shows that $1/\kappa(x)$ belongs to the space $L^{d-1,\infty}(\partial\Omega)$. This means that there exists $C > 0$ such that

$$|\{x \in \partial\Omega : [\kappa(x)]^{-1} > \lambda\}| \le C\,\lambda^{1-d} \quad \text{for any } \lambda > 0. \tag{3.6.2}$$

Proposition 3.32. *Let $0 < q < d-1$. Then for any $x \in \partial\Omega$ and $0 < r < diam(\Omega)$,*

$$\left(\fint_{B(x,r)\cap\partial\Omega} \big(\kappa(y)\big)^{-q} \, d\sigma(y) \right)^{1/q} \le \frac{C}{r}, \tag{3.6.3}$$

where C depends only on d, q and Ω.

Proof. Note that

$$\begin{aligned}
\int_{B(x,r)\cap\partial\Omega} \big(\kappa(y)\big)^{-q} \, d\sigma(y) &= q \int_0^\infty \lambda^{q-1} |\{x \in B(x,r) \cap \partial\Omega : \big[\kappa(x)\big]^{-1} > \lambda\}| \, d\lambda \\
&\le C \int_0^\Lambda \lambda^{q-1} \cdot r^{d-1} \, d\lambda + C \int_\Lambda^\infty \lambda^{q-d} \, d\sigma \\
&\le C r^{d-1} \cdot \Lambda^q + C \Lambda^{q-d+1},
\end{aligned}$$

where we have used (3.6.2) for the first inequality. The proof is finished by optimizing Λ with $\Lambda = r^{-1}$. ☐

In this section we construct a partition of unity for $\partial\Omega$, which is adapted to the function $\kappa(x)$. We mention that a similar partition of unity, which plays an important role in the analysis of the oscillating Dirichlet problem, was given in [7]. Here we provide a more direct L^p-based approach.

We first describe such construction in the flat space.

Lemma 3.33. *Let Q_0 be a cube in $\mathbb{R}^{d-1}$ and $F \in L^p(12Q_0)$ for some $p > d-1$. Let $\tau > 0$ be a number such that*

$$\left(\int_{12Q_0} |F|^p \right)^{1/p} > \frac{\tau}{\big[\ell(Q_0)\big]^{1-\frac{d-1}{p}}}, \tag{3.6.4}$$

where $\ell(Q_0)$ denotes the side length of Q_0. Then there exists a finite sequence $\{Q_j\}$ of dyadic sub-cubes of Q_0 such that the interiors of Q_j's are mutually disjoint,

$$Q_0 = \cup Q_j, \tag{3.6.5}$$

$$\left(\int_{12Q_j} |F|^p \right)^{1/p} \le \frac{\tau}{\big[\ell(Q_j)\big]^{1-\frac{d-1}{p}}}, \tag{3.6.6}$$

$$\left(\int_{12Q_j^+} |F|^p \right)^{1/p} > \frac{\tau}{\big[\ell(Q_j^+)\big]^{1-\frac{d-1}{p}}}, \tag{3.6.7}$$

where Q_j^+ denotes the dyadic parent of Q_j, i.e., Q_j is obtained by bisecting Q_j^+ once. Moreover, if $4Q_j \cap 4Q_k \neq \emptyset$, then

$$(1/2)\ell(Q_k) \le \ell(Q_j) \le 2\ell(Q_k). \tag{3.6.8}$$

Proof. The lemma is proved by using a stoping time argument (Calderón-Zygmund decomposition). We begin by bisecting the sides of Q_0 and obtaining 2^{d-1} dyadic sub-cubes $\{Q'\}$. If a cube Q' satisfies

$$\left(\int_{Q'} |F|^p\right)^{1/p} \leq \frac{\tau}{\left[\ell(Q')\right]^{1-\frac{d-1}{p}}} \tag{3.6.9}$$

we stop and collect the cube. Otherwise, we repeat the same procedure on Q'. Since the RHS of (3.6.9) goes to ∞ as $\ell(Q') \to 0$, the procedure is stopped in a finite time. As a result, we obtain a finite number of sub-cubes with mutually disjoint interiors satisfying (3.6.5)-(3.6.7). We point out this decomposition was performed in the whole space in [33] in the study of negative eigenvalues for the Pauli operator. The inequalities in (3.6.8) were proved in [33] by adapting an argument found in [14]. The same argument works equally well in the setting of a finite cube Q_0. We omit the details. □

Remark 3.34. Note that the condition for selecting cubes $\{Q_j\}$ in the above lemma is equivalent to

$$\left(\fint_{12Q_j} |F|^p\right)^{1/p} \simeq \frac{\tau}{\ell(Q_j)}. \tag{3.6.10}$$

In particular, we may let $p \to \infty$ in the above condition and replace the LHS by $\|F\|_{L^\infty(12Q_j)}$. So (3.6.10) can be roughly interpreted as follows: around a point in Q_0 with F being relatively large, the decomposition will be finer with relative small cubes; while if F is relatively small over a particular region, then we need to enlarge these cubes so that we may still expect relatively large F at some points in the enlarged cubes.

Fix $x_0 \in \partial\Omega$. Let $c_0 > 0$ be sufficiently small so that $B(x_0, 10c_0\sqrt{d}) \cap \partial\Omega$ is given by the graph of a smooth function in a coordinate system, obtained from the standard system through rotation and translation. Let $\mathbb{H}^d_n(a)$ denote the tangent plane for $\partial\Omega$ at x_0, where $n = n(x_0)$ and $a = x_0 \cdot n$. For $x \in B(x_0, 5c_0\sqrt{d}) \cap \partial\Omega$, let

$$P(x; x_0) = x - ((x - x_0) \cdot n)n \tag{3.6.11}$$

denote its projection to $\mathbb{H}^d_n(a)$. The projection P is one-to-one in $B(x_0, 10c_0\sqrt{d}) \cap \partial\Omega$. To construct a partition of unity for $B(x_0, c_0) \cap \partial\Omega$, adapted to the function $\kappa(x)$, we use the inverse map P^{-1} to lift a partition on the tangent plane, given in Lemma 3.33, to $\partial\Omega$. More precisely, fix a cube Q_0 on the tangent plane $H^d_n(a)$ such that

$$B(x_0, 5c_0\sqrt{d}) \cap \partial\Omega \subset P^{-1}(Q_0) \subset B(x, 10c_0\sqrt{d}) \cap \partial\Omega.$$

We apply Lemma 3.33 to Q_0 with the bounded function $F(x) = \kappa(P^{-1}(x))$ and some $p > d - 1$. For each $0 < \tau < c_1$, this generates a finite sequence of sub-cubes $\{Q_j\}$ with the properties (3.6.5)-(3.6.8).

Let x_j denote the center of Q_j and r_j the side length. Let $\widetilde{x}_j = P^{-1}(x_j)$ and $\widetilde{Q}_j = P^{-1}(Q_j)$. We will use the notation $t\widetilde{Q}_j = P^{-1}(tQ_j)$ for $t > 0$ and call $\widetilde{Q}_j$ a cube on $\partial\Omega$. Then

$$\widetilde{Q}_0 = P^{-1}(Q_0) = \cup_{j=1}^N \widetilde{Q}_j.$$

For each $\widetilde{Q}_j$ with $j \geq 1$, we choose $\eta_j \in C_0^\infty(\mathbb{R}^d)$ such that $0 \leq \eta_j \leq 1$, $\eta_j = 1$ on $\widetilde{Q}_j$, $\eta_j = 0$ on $\partial\Omega \setminus 2\widetilde{Q}_j$, and $|\nabla \eta_j| \leq C r_j^{-1}$. Note that by the property (3.6.8), $1 \leq \sum_j \eta_j \leq C_0$ on $\widetilde{Q}_0$, where C_0 is a constant depending only on d and Ω. Let

$$\varphi_j(x) = \frac{\eta_j(x)}{\sum_{k=1}^N \eta_k(x)}.$$

Then

$$\sum_{j=1}^N \varphi_j(x) = 1 \quad \text{ for any } x \in \widetilde{Q}_0.$$

Observe that $0 \leq \varphi_j \leq 1$, $\varphi_j \geq C_0^{-1}$ on $\widetilde{Q}_j$, $\varphi_j = 0$ on $\partial\Omega \setminus 2\widetilde{Q}_j$, and $|\nabla \varphi_j| \leq C r_j^{-1}$. Furthermore, by the property (3.6.9), there are positive constants c_1 and c_2, depending only on d, p and Ω, such that

$$\left(\fint_{12\widetilde{Q}_j} (\kappa(x))^p d\sigma(x) \right)^{1/p} \leq \frac{c_1 \tau}{r_j}, \tag{3.6.12}$$

$$\left(\fint_{36\widetilde{Q}_j} (\kappa(x))^p \, d\sigma(x) \right)^{1/p} \geq \frac{c_2 \tau}{r_j}. \tag{3.6.13}$$

In particular, since $\kappa \leq 1$, it follows from (3.6.13) that

$$r_j \geq c_2 \, \tau. \tag{3.6.14}$$

Also, by (3.6.13), there exists some $z_j \in 36\widetilde{Q}_j$ such that

$$\kappa(z_j) \geq \frac{c_2 \tau}{r_j}. \tag{3.6.15}$$

Proposition 3.35. *There exists a constant C, depending only on d, p and Ω, such that*

$$r_j \leq C \sqrt{\tau}. \tag{3.6.16}$$

Proof. By Hölder's inequality,

$$\begin{aligned} 1 &\leq \left(\fint_{12\widetilde{Q}_j} \kappa^{-p'} \, d\sigma \right)^{1/p'} \left(\fint_{12\widetilde{Q}_j} \kappa^p \, d\sigma \right)^{1/p} \\ &\leq C \, r_j^{-1} \left(\fint_{12\widetilde{Q}_j} \kappa^p \, d\sigma \right)^{1/p}, \end{aligned} \tag{3.6.17}$$

where we have used Proposition 3.32 for the last step. Note that the condition $p > d - 1$ is equivalent to $p' < \frac{d-1}{d-2}$, which is less or equal to $d - 1$ if $d \geq 3$. In view of (3.6.12) and (3.6.17) we obtain (3.6.16). $\square$

Proposition 3.36. *Let $0 \le \alpha < d-1$. Then*

$$\sum_j r_j^{\alpha+d-1} \le C_\alpha \, \tau^\alpha, \tag{3.6.18}$$

where C_α depends only on d, p, α and Ω.

Proof. It follows from the first inequality in (3.6.17) and (3.6.12) that

$$r_j \le C\tau \left(\fint_{12\widetilde{Q}_j} \kappa^{-p'} d\sigma \right)^{1/p'}. \tag{3.6.19}$$

Let $\mathcal{M}_{\partial\Omega}(f)$ denote the Hardy-Littlewood maximal function of f on $\partial\Omega$, defined by

$$\mathcal{M}(f)(x) = \sup \left\{ \fint_{B(x,r)\cap\partial\Omega} |f| \, d\sigma : 0 < r < \operatorname{diam}(\Omega) \right\} \tag{3.6.20}$$

for $x \in \partial\Omega$. By (3.6.19) we obtain

$$r_j^\alpha \le C\tau^\alpha \left(\inf_{x \in 12\widetilde{Q}_j} \mathcal{M}_{\partial\Omega}(\kappa^{-p'})(x) \right)^{\alpha/p'} \tag{3.6.21}$$

Hence,

$$\begin{aligned} \sum_j r_j^{\alpha+d-1} &\le C\tau^\alpha \sum_j \int_{\widetilde{Q}_j} \left[\mathcal{M}_{\partial\Omega}(\kappa^{-p'}) \right]^{\alpha/p'} \\ &\le C\tau^\alpha \int_{\partial\Omega} \left[\mathcal{M}_{\partial\Omega}(\kappa^{-p'}) \right]^{\alpha/p'}. \end{aligned}$$

Finally, recall that $p' < \frac{d-1}{d-2}$ and $0 \le \alpha < d-1$. Choose $t > 1$ so that $p' < t\alpha < d-1$. Then

$$\begin{aligned} \int_{\partial\Omega} \left[\mathcal{M}_{\partial\Omega}(\kappa^{-p'}) \right]^{\alpha/p'} d\sigma &\le C \left(\int_{\partial\Omega} \left[\mathcal{M}_{\partial\Omega}(\kappa^{-p'}) \right]^{\alpha t/p'} d\sigma \right)^{1/t} \\ &\le C \left(\int_{\partial\Omega} (\kappa^{-1})^{\alpha t} d\sigma \right)^{1/t} < \infty, \end{aligned}$$

where we have used the fact that the operator $\mathcal{M}_{\partial\Omega}$ is bounded on $L^q(\partial\Omega)$ for $q > 1$. This completes the proof. □

3.7 Proof of Theorem 1.1: convergence rate

With the estimates in §3.4,§3.5 and §3.6, the line of argument is similar to that in [7] for the oscillating Dirichlet problem. Recall that

$$v_\varepsilon^\gamma(x) = -\int_{\partial\Omega} \frac{\partial}{\partial y_k} \left\{ N_0^{\gamma\beta}(x,y) \right\} \cdot \left(T_{ij}(y) \cdot \nabla_y \right) \Psi_{\varepsilon,k}^{*\alpha\beta}(y) \cdot g_{ij}^\alpha(y, y/\varepsilon) \, d\sigma(y)$$

and

$$v_0^\gamma(x) = -\int_{\partial\Omega} \frac{\partial}{\partial y_k}\big\{N_0^{\gamma\beta}(x,y)\big\}\widetilde{g}_k^\beta(y)\, d\sigma(y), \tag{3.7.1}$$

where the function $\widetilde{g}_k^\beta$ is given by (3.5.2). We will show that for any $\sigma \in (0,1)$,

$$\int_\Omega |v_\varepsilon - v_0|^2\, dx \le C_\sigma\, \varepsilon^{1-\sigma}. \tag{3.7.2}$$

This would imply that if u_ε and u_0 are solutions of (1.2.5) and (1.2.7) respectively, then there exists some constant E such that

$$\|u_\varepsilon - u_0 - E\|_{L^2(\Omega)} \le C_\sigma\, \varepsilon^{\frac{1}{2}-\sigma}.$$

It then follows that

$$\|u_\varepsilon - u_0 - \fint_\Omega (u_\varepsilon - u_0)\|_{L^2(\Omega)} \le C_\sigma\, \varepsilon^{\frac{1}{2}-\sigma},$$

which gives (1.2.9) in the case $\int_\Omega u_\varepsilon = \int_\Omega u_0 = 0$.

To prove (3.7.2) we first note that by using a partition of unity for $\partial\Omega$, without loss of generality, we may assume that there exists some $x_0 \in \partial\Omega$ such that for any $y \in \mathbb{T}^d$, $\text{supp}(g(\cdot,y)) \subset B(x_0, c_0)$, where $c_0 > 0$ is sufficiently small so that $B(x_0, 10c_0\sqrt{d}) \cap \partial\Omega$ is given by the graph of a smooth function in a coordinate system, obtained from the standard system by rotation and translation. We construct another partition of unity for $B(x_0, 5c_0\sqrt{d}) \cap \partial\Omega$, as described in Section 7, with

$$\tau = \varepsilon^{1-\sigma}, \tag{3.7.3}$$

adapted to the function $\kappa(x)$. Thus there exist a finite sequence $\{\varphi_j\}$ of C_0^∞ functions in $\mathbb{R}^d$ and a finite sequence $\{\widetilde{Q}_j\}$ of "cubes" on $\partial\Omega$, such that $\sum_j \varphi_j = 1$ on $B(x_0, 5c_0\sqrt{d}) \cap \partial\Omega$.

Next, observe that by the estimate $|\nabla_y N_0(x,y)| + |\nabla_y N_\varepsilon(x,y)| \le C|x-y|^{1-d}$,

$$|v_\varepsilon(x)| + |v_0(x)| \le C\big\{1 + |\ln\delta(x)|\big\},$$

where $\delta(x) = \text{dist}(x, \partial\Omega)$. This implies that

$$\begin{aligned}
\sum_j \int_{B(\widetilde{x}_j, Cr_j)\cap\Omega} |v_\varepsilon - v_0|^2\, dx &\le C \sum_j \int_{B(\widetilde{x}_j, Cr_j)\cap\Omega} \big(1 + |\ln\delta(x)|\big)^2\, dx \\
&\le C \sum_j r_j^d (1 + |\ln r_j|)^2 \\
&\le C\varepsilon^{1-\sigma}(1 + |\ln\varepsilon|)^2,
\end{aligned} \tag{3.7.4}$$

where we have used Propositions 3.35 and 3.36 (see §3.6 for the definitions of $\widetilde{x}_j$ and r_j). Also note that

$$|\cup_j B(\widetilde{x}_j, Cr_j)| \le C \sum_j r_j^d \le C\tau, \tag{3.7.5}$$

where we have used Proposition 3.36. To estimate the L^2 norm of $v_\varepsilon - v_0$ on the set

$$D = D_\varepsilon = \Omega \setminus \cup_j B(\widetilde{x}_j, Cr_j), \tag{3.7.6}$$

we introduce a function

$$\Theta_t(x) = \sum_j \frac{r_j^{d-1+t}}{|x-\widetilde{x}_j|^{d-1}}, \tag{3.7.7}$$

where $0 \le t < d-1$.

Lemma 3.37. *Let $\Theta_t(x)$ be defined by (3.7.7). Then, if $q \ge 1$ and $0 \le qt < d-1$,*

$$\int_D (\Theta_t(x))^q \, dx \le C\, \tau^{qt}. \tag{3.7.8}$$

Proof. Observe that if $x \notin B(\widetilde{x}_j, Cr_j)$, then

$$\frac{r_j^{d-1}}{|x-\widetilde{x}_j|^{d-1}} \le C \int_{\widetilde{Q}_j} \frac{d\sigma(y)}{|x-y|^{d-1}}.$$

Hence, for $x \in D$,

$$\begin{aligned} \Theta_t(x) &\le C \int_{\partial\Omega} \frac{f_t(y)}{|x-y|^{d-1}}\, d\sigma(y) \\ &\le C \left(\int_{\partial\Omega} \frac{|f_t(y)|^q}{|x-y|^{d-1}}\, d\sigma(y) \right)^{1/q} (1+|\ln \delta(x)|)^{1/q'}, \end{aligned}$$

where $f_t(y) = \sum_j r_j^t \varphi_j(y)$, $\delta(x) = \text{dist}(x, \partial\Omega)$, and we have used Hölder's inequality for the last step. It follows that

$$\int_D |\Theta_t(x)|^q \, dx \le C \int_{\partial\Omega} |f_t(y)|^q \, d\sigma \le C \sum_j r_j^{qt} r_j^{d-1} \le C\tau^{qt},$$

where have used Proposition 3.36. □

As in the case of Dirichlet problem in [7], we split $v_\varepsilon - v_0$ into several parts,

$$\begin{aligned} &-\big(v_\varepsilon^\gamma(x) - v_0^\gamma(x)\big) \\ &= \sum_j \int_{\partial\Omega} \partial_{y_k} N_0^{\gamma\beta}(x,y) \Big\{ \big(T_{i\ell}(y)\cdot\nabla_y\big) \Psi_{\varepsilon,k}^{*\alpha\beta}(y) \cdot g_{i\ell}^\alpha(y, y/\varepsilon) - \widetilde{g}_k^\beta(y) \Big\} \varphi_j(y)\, d\sigma(y) \\ &= I_1 + I_2 + I_3 + I_4 + I_5, \end{aligned} \tag{3.7.9}$$

where $I_1, I_2, \dots, I_5$ are defined below and handled separately. We will show that for $k = 1, 2, \dots, 5$,

$$\int_D |I_k(x)|^2 \, dx \le C_\sigma\, \varepsilon^{1-4\sigma}, \tag{3.7.10}$$

which, together with (3.7.4), gives (3.7.2), as $\sigma \in (0, 1/4)$ is arbitrary.

Estimate of I_1, where

$$
\begin{aligned}
I_1 &= \sum_j \int_{\partial\Omega} \partial_{y_k} N_0^{\gamma\beta}(x,y)\big(T_{i\ell}(y)\cdot\nabla_y\big)\big(\Psi^{*\alpha\beta}_{\varepsilon,k} - \Phi^{*\alpha\beta,z_j}_{\varepsilon,k}\big)\cdot g^\alpha_{i\ell}(y,y/\varepsilon)\varphi_j(y)\,d\sigma(y),\\
\Phi^{*\alpha\beta,z_j}_{\varepsilon,k}(y) &= y_k\delta^{\alpha\beta} + \varepsilon\chi^{*\alpha\beta}_k(y/\varepsilon) + \phi^{*\alpha\beta,z_j}_{\varepsilon,k}(y),
\end{aligned}
\tag{3.7.11}
$$

and z_j is given in (3.6.15). Here we use Theorem 3.26 to obtain that for any $\rho \in (0,1/2)$,

$$
\Big|\nabla\Big(\Psi^{*\alpha\beta}_{\varepsilon,k} - \Phi^{*\alpha\beta,z_j}_{\varepsilon,k}\Big)\Big| \le C\sqrt{\varepsilon}\big\{1+|\ln\varepsilon|\big\} + C\varepsilon^{-1-\rho}r_j^{2+\rho}, \tag{3.7.12}
$$

for $y \in 2\widetilde{Q}_j$. It follows from (3.7.12) that for $x \in D$,

$$
|I_1(x)| \le C\sqrt{\varepsilon}(1+|\ln\varepsilon|)\sum_j \frac{r_j^{d-1}}{|x-\widetilde{x}_j|^{d-1}} + C\varepsilon^{-1-\rho}\sum_j \frac{r_j^{2+\rho+d-1}}{|x-\widetilde{x}_j|^{d-1}}. \tag{3.7.13}
$$

We now use Lemma 3.37 to estimate the L^2 norm of I_1 on D. The first term in the RHS of (3.7.13) is harmless. For the second term we use the fact $r_j \le C\sqrt{\tau}$ to bound it by

$$
C\varepsilon^{-1-\rho}\tau^{\frac{1}{2}+\rho}\sum_j \frac{r_j^{1-\rho+d-1}}{|x-\widetilde{x}_j|^{d-1}}.
$$

Since $2(1-\rho) < d-1$ for $d \ge 3$, we obtain

$$
\begin{aligned}
\int_D |I_1(x)|^2\,dx &\le C\varepsilon(1+|\ln\varepsilon|)^2 + C\varepsilon^{-2-2\rho}\tau^{1+2\rho}\tau^{2(1-\rho)}\\
&\le C\,\varepsilon^{1-4\sigma},
\end{aligned}
$$

if ρ is sufficiently small.

Estimate of I_2, where

$$
\begin{aligned}
I_2 = &\sum_j \int_{\partial\Omega} \partial_{y_k} N_0^{\gamma\beta}(x,y)\big(T_{ij}(y)\cdot\nabla_y\big)\big(\Phi^{*\alpha\beta,z_j}_{\varepsilon,k}\big)\cdot g^\alpha_{ij}(y,y/\varepsilon)\varphi_j(y)\,d\sigma(y)\\
&- \sum_j \int_{\partial\mathbb{H}^d_j} \partial_{y_k} N_0^{\gamma\beta}(x,P_j^{-1}(y))\big(T_{i\ell}(P_j^{-1}(y))\cdot\nabla_y\big)\big(\Phi^{*\alpha\beta,z_j}_{\varepsilon,k}(y)\big)\\
&\qquad\qquad\cdot g^\alpha_{i\ell}(y,y/\varepsilon)\varphi_j(y)\,d\sigma(y),
\end{aligned}
\tag{3.7.14}
$$

where $\partial\mathbb{H}^d_j$ denotes the tangent plane for $\partial\Omega$ at z_j and P_j^{-1} is the inverse of the projection map from $B(z_j,Cr_j)\cap\partial\Omega$ to $\partial\mathbb{H}^d_j$. Here we rely on the estimates

$$
\begin{aligned}
|\nabla_y^2 N_0(x,y)| &\le C|x-y|^{-d},\\
|\nabla^2\Phi^{*\alpha\beta,z_j}_{\varepsilon,k}| &\le C\varepsilon^{-1},
\end{aligned}
\tag{3.7.15}
$$

as well as the observation that $|y - P_j^{-1}(y)| \le C r_j^2$ for $y \in B(\widetilde{x}_j, C r_j) \cap \partial\Omega$. It is not hard to see that for $x \in D$,

$$|I_2(x)| \le C\,\varepsilon^{-1} \sum_j \frac{r_j^{2+d-1}}{|x - \widetilde{x}_j|^{d-1}} \le C\varepsilon^{-1}\tau^{\frac{1+\rho}{2}} \sum_j \frac{r_j^{1-\rho+d-1}}{|x - \widetilde{x}_j|^{d-1}}, \tag{3.7.16}$$

which, by Lemma 3.37, leads to (3.7.10) for $k = 2$.

Estimate of I_3, where

$$\begin{aligned} I_3 = \sum_j \int_{\partial\mathbb{H}_j^d} \partial_{y_k} N_0^{\gamma\beta}(x, P_j^{-1}(y)) \big(T_{i\ell}(y) \cdot \nabla_y\big) \big(\Phi_{\varepsilon,k}^{*\alpha\beta, z_j}\big) \cdot g_{i\ell}^{\alpha}(y, y/\varepsilon) \varphi_j(y)\, d\sigma(y) \\ - \sum_j \int_{\partial\mathbb{H}_j^d} \partial_{y_k} N_0^{\gamma\beta}(x, P_j^{-1}(y)) \big(T_{i\ell}(z_j) \cdot \nabla_y\big) \big(\Phi_{\varepsilon,k}^{*\alpha\beta, z_j}\big) \cdot g_{i\ell}^{\alpha}(z_j, y/\varepsilon) \varphi_j(y)\, d\sigma(y). \end{aligned} \tag{3.7.17}$$

It is easy to see that for $x \in D$,

$$|I_3(x)| \le C \sum_j \frac{r_j^{1+d-1}}{|x - \widetilde{x}_j|^{d-1}} \le C\,\tau^{\frac{\rho}{2}} \sum_j \frac{r_j^{1-\rho+d-1}}{|x - \widetilde{x}_j|^{d-1}},$$

which may be handled by Lemma 3.37.

Estimate of I_4, where

$$\begin{aligned} I_4 = \sum_j \int_{\partial\mathbb{H}_j^d} \partial_{y_k} N_0^{\gamma\beta}(x, P_j^{-1}(y)) \big(T_{i\ell}(z_j) \cdot \nabla_y\big) \big(\Phi_{\varepsilon,k}^{*\alpha\beta, z_j}\big) \cdot g_{i\ell}^{\alpha}(z_j, y/\varepsilon) \varphi_j(y)\, d\sigma(y) \\ - \sum_j \int_{\partial\mathbb{H}_j^d} \partial_{y_k} N_0^{\gamma\beta}(x, P_j^{-1}(y)) \widetilde{g}_k^{\beta}(z_j) \varphi_j(y)\, d\sigma(y). \end{aligned} \tag{3.7.18}$$

The estimate of I_4 uses the fact that for each j, the function

$$\big(T_{i\ell}(z_j) \cdot \nabla_y\big) \big(\Phi_{\varepsilon,k}^{*\alpha\beta, z_j}\big) \cdot g_{i\ell}^{\alpha}(z_j, y/\varepsilon)$$

is of form $U(y/\varepsilon)$, where $U(x)$ is a smooth 1-periodic function whose mean value is given by $\widetilde{g}_k^{\beta}(z_j)$. Furthermore, the normal to the hyperplane $\partial\mathbb{H}_j^d$ is $n(z_j)$, which satisfies the Diophantine condition (2.4.1) with constant

$$\kappa(z_j) \ge \frac{c\,\tau}{r_j}.$$

It then follows from Lemma 2.10 that if $x \in D$,

$$|I_4(x)| \le C_N (\tau^{-1}\varepsilon)^N \sum_j \frac{r_j^{d-1}}{|x - \widetilde{x}_j|^{d-1}},$$

for any $N \ge 1$. Since $\tau = \varepsilon^{1-\sigma}$, this implies that

$$\int_D |I_4(x)|^2\, dx \le C_N \varepsilon^{N\sigma}.$$

Estimate of I_5, where

$$
\begin{aligned}
I_5 = &\sum_j \int_{\partial\mathbb{H}_j^d} \partial_{y_k} N_0^{\gamma\beta}(x, P_j^{-1}(y))\overline{g}_k^\beta(z_j)\varphi_j(y)\, d\sigma(y) \\
&- \sum_j \int_{\partial\Omega} \partial_{y_k} N_0^{\gamma\beta}(x, y)\overline{g}_k^\beta(y)\varphi_j(y)\, d\sigma(y).
\end{aligned}
\tag{3.7.19}
$$

Finally, to estimate I_5, we use the regularity estimate for $\overline{g}$ in Theorem 3.27 to obtain

$$
|\overline{g}(y) - \overline{g}(z_j)| \le \frac{C r_j}{\big[\kappa(z_j)\big]^{1+\rho}} \le \frac{C r_j^{2+\rho}}{\tau^{1+\rho}},
$$

for any $y \in B(\widetilde{x}_j, C r_j) \cap \partial\Omega$, where $\rho \in (0, 1/2)$. It follows that for any $x \in D$,

$$
\begin{aligned}
|I_5(x)| &\le C \sum_j \frac{r_j^d}{|x - \widetilde{x}_j|^{d-1}} + C\tau^{-1-\rho} \sum_j \frac{r_j^{2+\rho+d-1}}{|x - \widetilde{x}_j|^{d-1}} \\
&\le C \sum_j \frac{r_j^d}{|x - \widetilde{x}_j|^{d-1}} + C\tau^{-\frac{1}{2}} \sum_j \frac{r_j^{1-\rho+d-1}}{|x - \widetilde{x}_j|^{d-1}}.
\end{aligned}
$$

As before, by applying Lemma 3.37 and choosing $\rho > 0$ sufficiently small, we obtain the desired estimate for I_5. This completes the proof of (3.7.2) and therefore (1.2.9).

Remark 3.38. Let $\Theta_t(x)$ be defined by (3.7.7). It follows from the proof of Proposition 3.36 that for $x \in D$,

$$
\Theta_t(x) \le C\tau^t \int_{\partial\Omega} \frac{\big[\mathcal{M}_{\partial\Omega}(\kappa^{-q})\big]^{t/q}(y)}{|x - y|^{d-1}} d\sigma(y),
$$

where $q = p' < \frac{d-1}{d-2}$ and $t \ge 0$. Let u_ε and u_0 be solutions of (1.2.5) and (1.2.7), respectively. An inspection of our proof of Theorem 1.1 shows that for any $\sigma \in (0, 1/2)$, there exists a neighborhood Ω_ε of $\partial\Omega$ in Ω such that

$$
|\Omega_\varepsilon| \le C\, \varepsilon^{1-\sigma}, \tag{3.7.20}
$$

and for $x \in \Omega \setminus \Omega_\varepsilon$,

$$
|u_\varepsilon(x) - u_0(x) - E| \le C\varepsilon^{\frac{1}{2} - 4\sigma} \int_{\partial\Omega} \frac{\big[\mathcal{M}_{\partial\Omega}(\kappa^{-q})\big]^{\frac{1-\rho}{q}}(y)}{|x - y|^{d-1}} d\sigma(y), \tag{3.7.21}
$$

where $1 < q < d - 1$, $\rho = \rho(\sigma) > 0$ is small, and E is a constant. The boundary layer Ω_ε, which is given locally by the union of $B(\widetilde{x}_j, C r_j) \cap \Omega$, depends only on the function κ and Ω. Furthermore, if $F(x)$ denotes the integral in (3.7.21), then

$$
|F(x)| \le C(1 + |\ln \delta(x)|)^{1/s'} \left(\int_{\partial\Omega} \frac{\big[\mathcal{M}_{\partial\Omega}(\kappa^{-q})\big]^{\frac{s(1-\rho)}{q}}(y)}{|x - y|^{d-1}} d\sigma(y) \right)^{1/s},
$$

where $q < s(1-\rho) < d-1$. Since $\kappa^{-1} \in L^s(\partial\Omega)$ for any $1 < s < d-1$, $\mathcal{M}_{\partial\Omega}(\kappa^{-q}) \in L^{s/q}(\partial\Omega)$ for any $q < s < d-1$. It follows that $F \in L^s(\Omega)$ for any $q < s \le d-1$. This, together with (3.7.21),

$$\|u_\varepsilon - u_0 - E\|_{L^s(\Omega\setminus\Omega_\varepsilon)} \le C\,\varepsilon^{\frac{1}{2}-4\sigma} \quad \text{for any } 1 < s \le d-1. \tag{3.7.22}$$

Finally, assume that $\int_\omega u_\varepsilon = \int_\Omega u_0 = 0$. Since $\|u_\varepsilon - u_0\|_{L^2(\Omega)} \le C_\sigma\,\varepsilon^{\frac{1}{2}-\sigma}$ by Theorem 1.1, it follows from (3.7.22) that $|E| \le C\,\varepsilon^{\frac{1}{2}-4\sigma}$. As a result, estimates (3.7.21) and (3.7.22) hold with $E = 0$.

3.8 Higher-order convergence

In this section we use Theorem 1.1 to establish a higher-order rate of convergence in the two-scale expansion for the Neumann problem,

$$\begin{cases} \mathcal{L}_\varepsilon(u_\varepsilon) = F & \text{in } \Omega, \\ \dfrac{\partial u_\varepsilon}{\partial \nu_\varepsilon} = g & \text{on } \partial\Omega, \end{cases} \tag{3.8.1}$$

where F and g are smooth functions. Our goal is to prove the following.

Theorem 3.39. *Suppose that A and Ω satisfy the same conditions as in Theorem 1.1. Let u_ε be the solution of (3.8.1) with $\int_\Omega u_\varepsilon = 0$, and u_0 the solution of the homogenized problem. Then there exists a function v^{bl}, independent of ε, such that*

$$\|u_\varepsilon - u_0 - \varepsilon\chi_k(x/\varepsilon)\frac{\partial u_0}{\partial x_k} - \varepsilon v^{bl}\|_{L^2(\Omega)} \le C_\sigma \varepsilon^{\frac{3}{2}-\sigma}\|u_0\|_{W^{3,\infty}(\Omega)}, \tag{3.8.2}$$

for any $\sigma \in (0, 1/2)$, where C_σ depends only on d, m, σ, A and Ω. Moreover, the function v^{bl} is a solution to the Neumann problem

$$\begin{cases} \mathcal{L}_0(v^{bl}) = F_* & \textit{in } \Omega, \\ \dfrac{\partial v^{bl}}{\partial \nu_0} = g_* & \textit{on } \partial\Omega, \end{cases} \tag{3.8.3}$$

where $F_* = -\overline{c}_{ki\ell}\frac{\partial^3 u_0}{\partial x_k \partial x_i \partial x_\ell}$ for some constants $\overline{c}_{ki\ell}$, and g_* satisfies

$$\|g_*\|_{L^q(\partial\Omega)} \le C_q \|u_0\|_{W^{2,\infty}(\Omega)}, \tag{3.8.4}$$

for any $1 < q < d-1$.

Proof. For simplicity of exposition we will drop the superscripts in this section. Let (χ_k) be the (first-order) correctors, $(b_{k\ell})$ be the flux correctors and (ϕ_{kij}) be the 1-periodic functions defined in Lemma 2.1. The second-order correctors $(\Upsilon_{k\ell})$ with $1 \le k, \ell \le d$ are defined by

$$\begin{cases} -\dfrac{\partial}{\partial y_i}\left\{a_{ij}\dfrac{\partial \Upsilon_{k\ell}}{\partial y_j}\right\} = b_{k\ell} + \dfrac{\partial}{\partial y_i}\big(a_{i\ell}\chi_k\big) & \text{in } \mathbb{R}^d, \\ \Upsilon_{k\ell} \text{ is 1-periodic and } \displaystyle\int_{\mathbb{T}^d} \Upsilon_{k\ell} = 0. \end{cases} \tag{3.8.5}$$

Let

$$w_\varepsilon = u_\varepsilon - u_0 - \varepsilon \chi_k^\varepsilon \frac{\partial u_0}{\partial x_k} - \varepsilon^2 \chi_{k\ell}^\varepsilon \frac{\partial^2 u_0}{\partial x_k \partial x_\ell}, \tag{3.8.6}$$

where we have used the notation $f^\varepsilon(x) = f(x/\varepsilon)$. A direct computation shows that

$$\begin{aligned} \mathcal{L}_\varepsilon(w_\varepsilon) = &- \varepsilon\Big(\phi_{kij}^\varepsilon \delta_{j\ell} - a_{ij}^\varepsilon \chi_k^\varepsilon \delta_{j\ell} - a_{ij}^\varepsilon \big(\frac{\partial \Upsilon_{k\ell}}{\partial x_j}\big)^\varepsilon\Big) \frac{\partial^3 u_0}{\partial x_i \partial x_k \partial x_\ell} \\ &+ \varepsilon^2 \frac{\partial}{\partial x_i} \left\{ a_{ij}^\varepsilon \Upsilon_{k\ell}^\varepsilon \frac{\partial^3 u_0}{\partial x_j \partial x_k \partial x_\ell} \right\}. \end{aligned} \tag{3.8.7}$$

Let

$$\begin{cases} c_{ki\ell} = \phi_{kij}\delta_{j\ell} - a_{ij}\chi_k \delta_{j\ell} - a_{ij}\dfrac{\partial \Upsilon_{k\ell}}{\partial x_j}, \\ \overline{c}_{ki\ell} = \displaystyle\fint_{\mathbb{T}^d} c_{ki\ell}. \end{cases} \tag{3.8.8}$$

Note that by the definition of $\Upsilon_{k\ell}$,

$$\frac{\partial}{\partial x_i}\big(c_{ki\ell}\big) = 0.$$

It follows that there exist 1-periodic functions $f_{mki\ell}$ with $1 \le m, k, i, \ell \le d$ such that

$$c_{ki\ell} - \overline{c}_{ki\ell} = \frac{\partial}{\partial y_m}\big(f_{mki\ell}\big) \quad \text{and} \quad f_{mki\ell} = -f_{ikm\ell}. \tag{3.8.9}$$

This allows us to rewrite (3.8.7) as

$$\begin{aligned} \mathcal{L}_\varepsilon(w_\varepsilon) = &-\varepsilon \overline{c}_{ki\ell} \frac{\partial^3 u_0}{\partial x_i \partial x_k \partial x_\ell} - \varepsilon \left(\frac{\partial f_{mki\ell}}{\partial x_m}\right)^\varepsilon \frac{\partial^3 u_0}{\partial x_i \partial x_k \partial x_\ell} \\ &+ \varepsilon^2 \frac{\partial}{\partial x_i} \left\{ a_{ij}^\varepsilon \Upsilon_{k\ell}^\varepsilon \frac{\partial^3 u_0}{\partial x_j \partial x_k \partial x_\ell} \right\}. \end{aligned} \tag{3.8.10}$$

Next we compute the conormal derivative of w_ε. Again, a direct computation gives

$$\begin{aligned} \frac{\partial w_\varepsilon}{\partial \nu_\varepsilon} = &-n_i b_{ij}^\varepsilon \frac{\partial u_0}{\partial x_j} - \varepsilon n_i a_{ij}^\varepsilon \chi_k^\varepsilon \frac{\partial^2 u_0}{\partial x_j \partial x_k} - \varepsilon n_i a_{ij}^\varepsilon \left(\frac{\partial \Upsilon_{k\ell}}{\partial x_j}\right)^\varepsilon \frac{\partial^2 u_0}{\partial x_k \partial x_\ell} \\ &- \varepsilon^2 n_i a_{ij}^\varepsilon \Upsilon_{k\ell}^\varepsilon \frac{\partial^3 u_0}{\partial x_j \partial x_k \partial x_\ell}. \end{aligned} \tag{3.8.11}$$

Using (2.1.4) and (3.8.8), we further obtain

$$\frac{\partial w_\varepsilon}{\partial \nu_\varepsilon} = -\varepsilon n_i \frac{\partial}{\partial x_k}\left(\phi_{kij}^\varepsilon \frac{\partial u_0}{\partial x_j}\right) + \varepsilon n_i c_{ki\ell}^\varepsilon \frac{\partial^2 u_0}{\partial x_k \partial x_\ell} - \varepsilon^2 n_i a_{ij}^\varepsilon \Upsilon_{k\ell}^\varepsilon \frac{\partial^3 u_0}{\partial x_j \partial x_k \partial x_\ell} \tag{3.8.12}$$

In view of (3.8.10) and (3.8.12), we split $w_\varepsilon - \fint_\Omega w_\varepsilon$ as $w_\varepsilon^{(1)} + w_\varepsilon^{(2)} + w_\varepsilon^{(3)} + w_\varepsilon^{(4)}$, where

$$\begin{cases} \mathcal{L}_\varepsilon(w_\varepsilon^{(1)}) = 0 & \text{in } \Omega, \\ \dfrac{\partial}{\partial \nu_\varepsilon}\big(w_\varepsilon^{(1)}\big) = -\varepsilon n_i \dfrac{\partial}{\partial x_k}\left(\phi_{kij}^\varepsilon \dfrac{\partial u_0}{\partial x_j}\right) & \text{on } \partial\Omega, \end{cases} \tag{3.8.13}$$

$$
\begin{cases}
\mathcal{L}_\varepsilon(w_\varepsilon^{(2)}) = -\varepsilon \overline{c}_{ki\ell} \dfrac{\partial^3 u_0}{\partial x_i \partial x_k \partial x_\ell} & \text{in } \Omega, \\
\dfrac{\partial}{\partial \nu_\varepsilon}\big(w_\varepsilon^{(2)}\big) = \varepsilon n_i \overline{c}_{ki\ell} \dfrac{\partial^2 u_0}{\partial x_k \partial x_\ell} & \text{on } \partial\Omega,
\end{cases} \tag{3.8.14}
$$

$$
\begin{cases}
\mathcal{L}_\varepsilon(w_\varepsilon^{(3)}) = -\varepsilon \left(\dfrac{\partial f_{mki\ell}}{\partial x_m}\right)^\varepsilon \dfrac{\partial^3 u_0}{\partial x_i \partial x_k \partial x_\ell} & \text{in } \Omega, \\
\dfrac{\partial}{\partial \nu_\varepsilon}\big(w_\varepsilon^{(3)}\big) = \varepsilon n_i \big(c_{ki\ell}^\varepsilon - \overline{c}_{ki\ell}\big) \dfrac{\partial^2 u_0}{\partial x_k \partial x_\ell} & \text{on } \partial\Omega,
\end{cases} \tag{3.8.15}
$$

and

$$
\begin{cases}
\mathcal{L}_\varepsilon(w_\varepsilon^{(4)}) = \varepsilon^2 \dfrac{\partial}{\partial x_i}\left\{ a_{ij}^\varepsilon \Upsilon_{k\ell}^\varepsilon \dfrac{\partial^3 u_0}{\partial x_j \partial x_k \partial x_\ell}\right\} & \text{in } \Omega, \\
\dfrac{\partial}{\partial \nu_\varepsilon}\big(w_\varepsilon^{(4)}\big) = -\varepsilon^2 n_i a_{ij}^\varepsilon \Upsilon_{k\ell}^\varepsilon \dfrac{\partial^3 u_0}{\partial x_j \partial x_k \partial x_\ell} & \text{on } \partial\Omega.
\end{cases} \tag{3.8.16}
$$

We further require that

$$
\int_\Omega w_\varepsilon^{(1)} = \int_\Omega w_\varepsilon^{(2)} = \int_\Omega w_\varepsilon^{(3)} = \int_\Omega w_\varepsilon^{(4)} = 0. \tag{3.8.17}
$$

To proceed, we first note that by Poincaré inequality, (3.8.17) and energy estimates,

$$
\|w_\varepsilon^{(4)}\|_{L^2(\Omega)} \le C\, \|\nabla w_\varepsilon^{(4)}\|_{L^2(\Omega)} \le C\, \varepsilon^2 \|\nabla^3 u_0\|_{L^2(\Omega)}. \tag{3.8.18}
$$

The solution $w_\varepsilon^{(3)}$ may be handled in a similar manner. To see this we use the skew-symmetry of $f_{mki\ell}$ in m and i to write the RHS of the equation as

$$
-\varepsilon^2 \frac{\partial}{\partial x_m}\left(f_{mki\ell}^\varepsilon \frac{\partial^3 u_0}{\partial x_i \partial x_k \partial x_\ell}\right),
$$

while the Neumann data for $w_\varepsilon^{(3)}$ may be written as

$$
\frac{\varepsilon^2}{2}\left(n_i \frac{\partial}{\partial x_m} - n_m \frac{\partial}{\partial x_i}\right)\left(f_{mki\ell}^\varepsilon \frac{\partial^2 u_0}{\partial x_k \partial x_\ell}\right) - \varepsilon^2 n_i f_{mki\ell}^\varepsilon \frac{\partial^3 u_0}{\partial x_k \partial x_\ell \partial x_m}.
$$

As a result, we obtain

$$
\begin{aligned}
\|w_\varepsilon^{(3)}\|_{L^2(\Omega)} \le C \|\nabla w_\varepsilon^{(3)}\|_{L^2(\Omega)} &\le C\, \varepsilon^2 \left\{ \|\nabla^3 u_0\|_{L^2(\Omega)} + \|f^\varepsilon \nabla^2 u_0\|_{H^{\frac{1}{2}}(\partial\Omega)} \right\} \\
&\le C \varepsilon^{\frac{3}{2}} \|u_0\|_{W^{3,\infty}(\Omega)}.
\end{aligned} \tag{3.8.19}
$$

Next, we observe that $w_\varepsilon^{(2)}$ may be dealt with by the classical homogenization results for $\mathcal{L}_\varepsilon$. Indeed, let $v_0^{(2)}$ be the solution of

$$
\begin{cases}
\mathcal{L}_0(v_0^{(2)}) = -\overline{c}_{ki\ell} \dfrac{\partial^3 u_0}{\partial x_i \partial x_k \partial x_\ell} & \text{in } \Omega, \\
\dfrac{\partial v_0^{(2)}}{\partial \nu_0} = n_i \overline{c}_{ki\ell} \dfrac{\partial^2 u_0}{\partial x_k \partial x_\ell} & \text{on } \partial\Omega,
\end{cases} \tag{3.8.20}
$$

with $\int_\Omega v_0^{(2)} = 0$. It is well known that

$$\|w_\varepsilon^{(2)} - \varepsilon v_0^{(2)}\|_{L^2(\Omega)} \le C\,\varepsilon^2 \|u_0\|_{W^{3,\infty}(\Omega)}. \tag{3.8.21}$$

It remains to estimate the solution $w_\varepsilon^{(1)}$, which will be handled by using Theorem 1.1. Observe that by the skew-symmetry of ϕ_{kij} in k and i, the Neumann data of $w_\varepsilon^{(1)}$ may be written as

$$-\frac{\varepsilon}{2}\Big(T_{ik}\cdot\nabla\Big)\left(\phi_{kij}^\varepsilon \frac{\partial u_0}{\partial x_j}\right), \tag{3.8.22}$$

where $T_{ik} = n_i e_k - n_k e_i$. This allows us to apply Theorem 1.1 to deduce that

$$\|w_\varepsilon - \varepsilon v_0^{(1)}\|_{L^2(\Omega)} \le C_\sigma\,\varepsilon^{\frac{3}{2}-\sigma}\|u_0\|_{W^{2,\infty}(\Omega)}, \tag{3.8.23}$$

for any $\sigma \in (0, 1/2)$, where $v_0^{(1)}$ is a solution of the Neumann problem

$$\begin{cases} \mathcal{L}_0(v_0^{(1)}) = 0 & \text{in } \Omega,\\ \dfrac{\partial}{\partial \nu_0}\big(v_0^{(1)}\big) = (T_{ij}\cdot\nabla)(\overline{g}_{ij}) & \text{in } \partial\Omega, \end{cases} \tag{3.8.24}$$

and $\overline{g}_{ij} \in W^{1,q}(\partial\Omega)$ for any $1 < q < d-1$. We remark that the explicit dependence on the $W^{2,\infty}(\Omega)$ norm of u_0 in the RHS of (3.8.23) follows from the proof of Theorem 1.1. The key observation is that the fast variable y in the Neumann data (3.8.22) is separated from the slow variable x.

Let $v^{bl} = v_0^{(1)} + v_0^{(2)}$. In view of (3.8.18), (3.8.19), (3.8.21) and (3.8.23), we have proved that

$$\|w_\varepsilon - \fint_\Omega w_\varepsilon - \varepsilon v^{bl}\|_{L^2(\Omega)} \le C_\sigma\,\varepsilon^{\frac{3}{2}-\sigma}\|u_0\|_{W^{3,\infty}(\Omega)}. \tag{3.8.25}$$

Finally, we note that since $\int_\Omega u_\varepsilon = \int_\Omega u_0 = 0$,

$$\begin{aligned}\Big|\fint_\Omega w_\varepsilon\Big| &\le C\,\varepsilon\Big|\int_\Omega \chi_k(x/\varepsilon)\frac{\partial u_0}{\partial x_k}\,dx\Big| + C\varepsilon^2\|\nabla^2 u_0\|_\infty\\ &\le C\,\varepsilon^2\|u_0\|_{W^{2,\infty}(\Omega)},\end{aligned}$$

where the last step follows from the fact that χ_k is periodic with mean value zero. This, together with (3.8.25), yields the estimate (3.8.2) and thus completes the proof of Theorem 3.39. □

Chapter 4 Dirichlet problems

In this chapter, we study the boundary layer problems with oscillating Dirichlet boundary conditions,

$$\begin{cases} \mathcal{L}_\varepsilon(u_\varepsilon) = 0 & \text{in } \Omega, \\ u_\varepsilon = f(x, x/\varepsilon) & \text{on } \partial\Omega. \end{cases}$$

This problem has been studied in [21, 7, 42]. For operators with constant coefficients, the optimal convergence rates were shown in [5]. For oscillating coefficients, the optimal convergence rates for lower dimensions were claimed in [38] without a concrete proof. In this chapter, we will give a complete proof for the Dirichlet problem for all dimensions in strictly convex domains, which is a reduced version of the proof in [42] where we studied the Dirichlet problem in more general (non-convex) domains of finite type.

4.1 Dirichlet correctors

We introduce the matrix of Dirichlet boundary correctors $\Phi_\varepsilon = \Phi^\beta_{\varepsilon,j} = (\Phi^{1\beta}_{\varepsilon,j}, \Phi^{2\beta}_{\varepsilon,j}, \ldots, \Phi^{m\beta}_{\varepsilon,j})$ associated with $\mathcal{L}_\varepsilon$ in a bounded domain Ω. Indeed, for each $1 \le j \le d, 1 \le \beta \le m$, $\Phi^\beta_{\varepsilon,j}$ is the solution of

$$\begin{cases} \mathcal{L}_\varepsilon \Phi^\beta_{\varepsilon,j}(x) = 0 & \text{in } \Omega, \\ \Phi^\beta_{\varepsilon,j}(x) = P^\beta_j(x) & \text{on } \partial\Omega. \end{cases}$$

Let Ω be a bounded $C^{2,\sigma}$ domain and $\sigma \in (0,1)$. The matrix of Poisson kernel $P_{\Omega,\varepsilon} : \Omega \times \partial\Omega \mapsto \mathbb{R}^{m\times m}$, associated with $\mathcal{L}_\varepsilon$ in Ω, is defined by

$$P^{\alpha\beta}_{\Omega,\varepsilon}(x,y) = -n(y) \cdot a^{\zeta\beta}(y/\varepsilon)\nabla_y G^{\alpha\zeta}_{\Omega,\varepsilon}(x,y),$$

where $n(y)$ is the unit outer normal and $G_{\Omega,\varepsilon}$ is the matrix of Green's function associated with $\mathcal{L}_\varepsilon$ in Ω. The following uniform estimates in [9] will be useful,

$$|P_{\Omega,\varepsilon}(x,y)| \le \frac{C}{|x-y|^{d-1}}, \tag{4.1.1}$$

and

$$|P_{\Omega,\varepsilon}(x,y)| \le \frac{C\mathrm{dist}(x,\partial\Omega)}{|x-y|^{d}}. \tag{4.1.2}$$

Let P_Ω be the Poisson kernel associated with the homogenized operator $\mathcal{L}_0$ in Ω. Clearly, P_Ω possesses the same estimates (4.1.1) and (4.1.2).

Recall that the two-scale expansion of the Poisson kernel of $\mathcal{L}_\varepsilon$ in Ω was established in [25],

$$P^{\alpha\beta}_{\Omega,\varepsilon}(x,y) = P^{\alpha\zeta}_{\Omega}(x,y)\omega^{\zeta\beta}_\varepsilon(y) + R^{\alpha\beta}_\varepsilon(x,y) \qquad \text{for } x \in \Omega, y \in \partial\Omega, \tag{4.1.3}$$

where R_ε is the remainder term satisfying

$$|R_\varepsilon(x,y)| \le \frac{C\varepsilon \ln(2+\varepsilon^{-1}|x-y|)}{|x-y|^d}.$$

The highly oscillating factor $\omega_\varepsilon(y)$ in (4.1.3) is given by

$$\omega_\varepsilon^{\zeta\beta}(y) = h^{\zeta\nu}(y)\cdot n_k(y)n_\ell(y)\frac{\partial}{\partial y_\ell}\Phi^{*\rho\nu}_{\varepsilon,k}(y)\cdot a_{ij}^{\rho\beta}(y/\varepsilon)n_i(y)n_j(y), \tag{4.1.4}$$

and $h(y)$ is the inverse matrix of $\widehat{a}_{ij}(y)n_i(y)n_j(y)$.

Let u_ε be the solution of (1.2.11). By Poisson integral formula, we have

$$u_\varepsilon(x) = \int_{\partial\Omega} P_{\Omega,\varepsilon}(x,y)f(y,y/\varepsilon)d\sigma(y).$$

Note that (4.1.2) implies the Agmon-type maximum principle $\|u_\varepsilon\|_{L^\infty(\Omega)} \le C\|f\|_{L^\infty(\partial\Omega\times\mathbb{T}^d)}$, which we will often refer to. Define

$$\widetilde{u}_\varepsilon(x) = \int_{\partial\Omega} P_\Omega(x,y)\omega_\varepsilon(y)f(y,y/\varepsilon)d\sigma(y).$$

Lemma 4.1. *Let Ω be a bounded $C^{2,\sigma}$ domain and let (1.2.2), (1.2.3) and (1.2.4) hold. Then*

$$\|u_\varepsilon - \widetilde{u}_\varepsilon\|_{L^q} \le C\varepsilon^{1/q}(1+|\ln\varepsilon|)\|f\|_{L^\infty(\partial\Omega\times\mathbb{T}^d)}.$$

for any $1\le q<\infty$.

This follows readily from (4.1.3) and a similar proof can be found in [38, Lemma 2.3]. Thanks to Lemma 4.1, the estimate for $\|u_\varepsilon - u_0\|_{L^2(\Omega)}$ is reduced to $\|\widetilde{u}_\varepsilon - u_0\|_{L^2(\Omega)}$.

4.2 Dirichlet problems in half-spaces

For $n\in\mathbb{S}^{d-1}$ and $a\in\mathbb{R}$, let $\mathbb{H}^d_n(a)$ denote the half-space $\{x\in\mathbb{R}^d : x\cdot n < -a\}$ (also see (3.2.1)) with n being the unit outer normal to its boundary $\partial\mathbb{H}^d_n(a) = \{x\in\mathbb{R}^d : x\cdot n = -a\}$. Consider the Dirichlet problem

$$\begin{cases} -\mathrm{div}(A\nabla u(x)) = 0 & \text{in } \mathbb{H}^d_n(a), \\ \quad u(x) = f(x) & \text{on } \partial\mathbb{H}^d_n(a), \end{cases} \tag{4.2.1}$$

where A satisfies (1.2.2), (1.2.3) and (1.2.4), and f is smooth and 1-periodic. Instead of solving (4.2.1) directly, we try to find a solution of (4.2.1) with a particular form, i.e.,

$$u(x) = V^a(x-(x\cdot n)n, -x\cdot n), \tag{4.2.2}$$

where $V^a = V^a(\theta,t)$ is a function of $(\theta,t)\in\mathbb{T}^d\times[a,\infty)$. To identify the system satisfied for V^a, let M be a $d\times d$ orthogonal matrix whose last column is $-n$. Let N denote the $d\times(d-1)$ matrix of the first $d-1$ columns of M. Since $MM^T = I$, we see that $NN^T + n \otimes n = I$. It follows from (4.2.1) and the previous settings that V^a must be a solution of

$$\begin{cases} -\begin{pmatrix} N^T\nabla_\theta \\ \partial_t \end{pmatrix} \cdot B \begin{pmatrix} N^T\nabla_\theta \\ \partial_t \end{pmatrix} V = 0 & \text{in } \mathbb{T}^d \times (a, \infty), \\ V = F & \text{on } \mathbb{T}^d \times \{a\}, \end{cases} \tag{4.2.3}$$

where $B(\theta, t) = M^T A(\theta - tn)M$ and $F(\theta) = f(\theta)$. Observe that if V^a is a solution of (4.2.3) with $a \in \mathbb{R}$, then $V^a(\theta, t) = V^0(\theta - an, t - a)$, which reduces the problem to the particular case $a = 0$.

Now we collect some important results concerning the lifted system (4.2.3) in the following theorem.

Theorem 4.2. *Let $n \in \mathbb{S}^{d-1}, a = 0$ and $F \in C^\infty(\mathbb{T}^d)$. Then*

(i) The system (4.2.3) has a smooth solution V such that for all $k, s \geq 0$,

$$\int_0^\infty \|N^T\nabla_\theta \partial_t^k V\|^2_{H^s(\mathbb{T}^d)} + \|\partial_t^{k+1} V\|^2_{H^s(\mathbb{T}^d)} dt \leq C,$$

where C depends only on d, m, k, s, A and F.

(ii) If n satisfies the Diophantine condition with constant $\kappa > 0$ and V is the solution of (4.2.3) given in (i), then there exists a constant V_∞ such that for all $\alpha \in \mathbb{N}^d, k \geq 0$ and $s \geq 0$,

$$|N^T\nabla_\theta \partial_\theta^\alpha \partial_t^k V| + |\partial_\theta^\alpha \partial_t^{k+1} V| + \kappa |\partial_\theta^\alpha \partial_t^k (V - V_\infty)| \leq \frac{C}{(1 + \kappa t)^s},$$

where C depends only on d, m, k, α, s, A and F.

(iii) Let n satisfy the Diophantine condition with constant $\kappa > 0$ and $\widetilde{n}$ be any other unit vector in $\mathbb{S}^{d-1}$. Let V and $\widetilde{V}$ be the solutions of (4.2.3) corresponding to n and $\widetilde{n}$, respectively. Define $W = V - \widetilde{V}$. Then for any $0 < \sigma < 1$,

$$\int_0^1 \int_{\mathbb{T}^d} |\widetilde{N}^T \nabla_\theta W|^2 + |\partial_t W|^2 d\theta dt \leq C \frac{|n - \widetilde{n}|^2}{\kappa^{2+\sigma}}.$$

where $(\widetilde{N}, -\widetilde{n})$ is an orthogonal matrix and C depends only on d, m, σ, A and F.

The proof is similar to the Neumann problem (3.2.8). Actually, (i) and (ii) are more or less known and can be found in [20, 7, 29]. Statement (iii) was established in [38] recently for Neumann problems by applying a weighted estimate. The proof for Dirichlet problems is similar without any real difficulty by using the weighted estimate in Theorem 3.17. But again, this estimate will be further improved in the next chapter.

4.3 Approximation of Dirichlet correctors

From now on, we will assume that Ω is a smooth and strictly convex domain. In view of (4.1.3), to study the oscillating behavior of ω_ε, the difficulty is to understand the behavior of $\nabla\Phi_\varepsilon^*$ near the boundary. This can be done by studying $u_{\varepsilon,j}^{*\beta} = \Phi_{\varepsilon,j}^{*\beta}(x) - P_j^\beta(x) - \varepsilon\chi_j^{*\beta}(x/\varepsilon)$ for each $1 \le j \le d, 1 \le \beta \le m$. Clearly, by the definitions of Φ_ε^* and χ^*, $u_{\varepsilon,j}^\beta$ satisfies

$$\begin{cases} \mathcal{L}_\varepsilon^* u_{\varepsilon,j}^{*\beta}(x) = 0 & \text{in } \Omega, \\ \quad u_{\varepsilon,j}^{*\beta}(x) = -\varepsilon\chi_j^{*\beta}(x/\varepsilon) & \text{on } \partial\Omega. \end{cases} \tag{4.3.1}$$

Let us consider the general case of (4.3.1)

$$\begin{cases} \mathcal{L}_\varepsilon u_\varepsilon(x) = 0 & \text{in } \Omega, \\ \quad u_\varepsilon(x) = f_\varepsilon(x) = \varepsilon f(x/\varepsilon) & \text{on } \partial\Omega, \end{cases} \tag{4.3.2}$$

where $f(y)$ is 1-periodic and smooth. Fix $x_0 \in \partial\Omega$. To find an approximation of u_ε in a neighborhood of x_0, we solve the Dirichlet problem in a half-space

$$\begin{cases} \mathcal{L}_\varepsilon v_\varepsilon(x) = 0 & \text{in } \mathbb{H}_{n_0}^d(a), \\ \quad v_\varepsilon(x) = f_\varepsilon(x) & \text{on } \partial\mathbb{H}_{n_0}^d(a), \end{cases} \tag{4.3.3}$$

where $a = -x_0 \cdot n_0$ and $\partial\mathbb{H}_{n_0}^d(a)$ is the tangent plane of $\partial\Omega$ at x_0. Note that v_ε has a form of $v_\varepsilon(x) = \varepsilon v_1(x/\varepsilon)$, and v_1 is the solution of

$$\begin{cases} \mathcal{L}_1 v_1(x) = 0 & \text{in } \mathbb{H}_{n_0}^d(a/\varepsilon), \\ \quad v_1(x) = f(x) & \text{on } \partial\mathbb{H}_{n_0}^d(a/\varepsilon), \end{cases} \tag{4.3.4}$$

The existence of the solution of (4.3.4) or (4.3.3) as well as its estimates have been established via the half-space problem in Theorem 4.2 (i) and formula (4.2.2). Define $w_\varepsilon(x) = u_\varepsilon(x) - v_\varepsilon(x)$. Observe that by the definition of v_ε, w_ε is defined and actually a solution of $\mathcal{L}_\varepsilon w_\varepsilon(x) = 0$ only in Ω. Now we prove the following.

Theorem 4.3. *Let w_ε be constructed as above. Let $\varepsilon \le r \le \sqrt{\varepsilon}$. Then for any $\sigma \in (0,1)$,*

$$\|\nabla w_\varepsilon\|_{L^\infty(B(x_0,r)\cap\Omega)} \le C\sqrt{\varepsilon} + C\frac{r^{2+\sigma}}{\varepsilon^{1+\sigma}}, \tag{4.3.5}$$

where C depends on $d, m, \mu, \sigma, \Omega, A$ and f.

To prove the theorem, we require the following lemmas.

Lemma 4.4. *Let u_ε be a solution of (4.3.2), then one has for any $k \ge 0$,*

$$\|\nabla^k u_\varepsilon\|_{L^\infty(\Omega)} \le C\varepsilon^{1-k}, \tag{4.3.6}$$

where C is independent of ε.

Proof. For $k = 0$, we use the Agmon-type maximal principle to obtain

$$\|u_\varepsilon\|_{L^\infty(\Omega)} \le C\|f_\varepsilon\|_{L^\infty(\partial\Omega)} \le C\varepsilon. \tag{4.3.7}$$

For $k > 0$, we apply a blow-up argument. Set $u_\varepsilon(x) = \varepsilon u_1(x/\varepsilon)$. Then u_1 is a solution of

$$\begin{cases} \mathcal{L}_1 u_1(x) = 0 & \text{in } \Omega^\varepsilon, \\ \quad u_1(x) = f(x) & \text{on } \partial\Omega^\varepsilon, \end{cases} \tag{4.3.8}$$

where $\Omega^\varepsilon = \{x : \varepsilon x \in \Omega\}$. Note that the C^k character of Ω^ε is controlled by that of Ω. It follows from the local Schauder's estimate that for any $x \in \overline{\Omega^\varepsilon}$,

$$\|\nabla^k u_1\|_{L^\infty(B(x,1)\cap\Omega^\varepsilon)} \le C\|u_1\|_{L^\infty(B(x,2)\cap\Omega^\varepsilon)} + \|f\|_{C^{k,\alpha}(B(x,2)\cap\Omega^\varepsilon)}.$$

Since f is 1-periodic, then $\|f\|_{C^{k,\alpha}(B(x,2)\cap\Omega^\varepsilon)} \le C\|f\|_{C^{k,\alpha}(\mathbb{T}^d)}$. And by (4.3.7), $\|u_1\|_{L^\infty(\Omega^\varepsilon)} \le C$. It follows that

$$\|\nabla^k u_1\|_{L^\infty(\Omega^\varepsilon)} \le C,$$

for any $k > 0$, where C depends also on k. Changing variables back to u_ε, we obtain the desired estimates (4.3.6). $\square$

Lemma 4.5. *Let v_ε be constructed as above, then one has for $k \in \{0,1,2\}$,*

$$\|\nabla^k v_\varepsilon\|_{L^\infty(\mathbb{H}^d_{n_0}(a))} \le C\varepsilon^{1-k}, \tag{4.3.9}$$

where C is independent of ε.

Proof. Let $v_\varepsilon(x) = \varepsilon v_1(x/\varepsilon)$. Then v_1 is the solution of (4.3.4), which can also be given by the Poisson integral formula

$$v_1(x) = \int_{\partial\mathbb{H}^d_{n_0}(a/\varepsilon)} P_{\mathbb{H}}(x,y) f(y) d\sigma(y), \tag{4.3.10}$$

where $P_{\mathbb{H}}$ is the Poisson kernel of $\mathcal{L}_1$ in the half-space $\mathbb{H}^d_{n_0}(a/\varepsilon)$. A similar estimate as (4.1.2) in half-spaces was established in [21], i.e.,

$$P_{\mathbb{H}}(x,y) \le \frac{C\text{dist}(x, \partial\mathbb{H}^d_{n_0}(a/\varepsilon))}{|x-y|^d}, \qquad \text{for all } x \in \mathbb{H}^d_{n_0}(a/\varepsilon).$$

Then it follows from (4.3.10) that $\|v_1\|_{L^\infty(\mathbb{H}^d_{n_0}(a/\varepsilon))} \le C\|f\|_{L^\infty(\partial\mathbb{H}^d_{n_0}(a/\varepsilon))}$ (Agmon-type maximal principle). Thus, $\|v_\varepsilon\|_{L^\infty(\mathbb{H}^d_{n_0}(a))} \le C\varepsilon$ as desired for $k = 0$. The estimates for $k > 0$ follow similarly as Lemma 4.4 by the local Schauder's estimates. $\square$

Proof of Theorem 4.3. The proof follows a line of [7]. Let $y \in \partial\Omega$ and $|y - x_0| \le r_0$ for some r_0 depending only on Ω. We will use the following conventions: let $\widehat{y}$ denote the projection of y on $\partial\mathbb{H}^d_{n_0}(a)$ such that $y - \widehat{y}$ is a multiple of $n(x_0)$. Since both Ω is smooth and strictly convex near x_0, it is easy to see that for all y satisfying $|y - x_0| \le r_0$,

$$C^{-1}|y-x_0|^2 \le |y - \widehat{y}| \le C|y-x_0|^2. \tag{4.3.11}$$

This also implies $|y - x_0| \approx |\widehat{y} - x_0|$. On the other hand, let $n(y)$ and $\widehat{n} = n(x_0)$ denote the unit outer normal of $\partial\Omega$ and $\partial\mathbb{H}^d_{n_0}(a)$, respectively. Then

$$|\widehat{n} - n(y)| \le C|y - x_0|. \tag{4.3.12}$$

To prove the estimate (4.3.5), note that w_ε is a solution of

$$\mathcal{L}_\varepsilon w_\varepsilon = 0 \quad \text{subject to certain Dirichlet boundary condition on } \partial\Omega.$$

Indeed, it follows from the uniform Lipschitz estimate in $C^{1,\alpha}$ domains that

$$\begin{aligned}\|\nabla w_\varepsilon\|_{L^\infty(B_r\cap\Omega)} &\leq Cr^{-1}\|w_\varepsilon\|_{L^\infty(B_{2r}\cap\Omega)} \\ &\quad + C\|\nabla_{\tan} w_\varepsilon\|_{L^\infty(B_{2r}\cap\partial\Omega)} + Cr^\sigma\|\nabla_{\tan} w_\varepsilon\|_{C^\sigma(B_{2r}\cap\partial\Omega)}.\end{aligned} \tag{4.3.13}$$

Note that $\nabla_{\tan}$ can be written as $(I - n\otimes n)\nabla$ (which can be viewed as the projection of ∇ onto the tangent planes $n^\perp$), where n is the unit outer normal of $\partial\Omega$.

Estimate of $\nabla_{\tan} w_\varepsilon$: Using the fact $u_\varepsilon = f_\varepsilon(y) = \varepsilon f(x/\varepsilon)$ on $\partial\Omega$, we know

$$(I - n\otimes n)\nabla(u_\varepsilon - f_\varepsilon)(y) = 0 \qquad \text{on } \partial\Omega. \tag{4.3.14}$$

Similarly, taking advantage of the fact $v_\varepsilon = f_\varepsilon$ on the hyperplane $\partial\mathbb{H}^d_{n_0}(a)$, we have

$$(I - \widehat{n}\otimes \widehat{n})\nabla(v_\varepsilon - f_\varepsilon)(\widehat{y}) = 0 \qquad \text{on } \partial\mathbb{H}^d_{n_0}(a). \tag{4.3.15}$$

Combining (4.3.14) and (4.3.15), we have

$$\begin{aligned}|\nabla_{\tan} w_\varepsilon(y)| &= |(I - n\otimes n)\nabla(u_\varepsilon - v_\varepsilon)(y)| \\ &\leq |(I - n\otimes n)\nabla(u_\varepsilon - v_\varepsilon)(y) - (I - \widehat{n}\otimes\widehat{n})\nabla(u_\varepsilon - v_\varepsilon)(\widehat{y})| \\ &\leq |n\otimes n - \widehat{n}\otimes\widehat{n}|\|\nabla(u_\varepsilon - v_\varepsilon)\|_{L^\infty(B_{2r}\cap\Omega)} + |y-\widehat{y}|\|\nabla^2(u_\varepsilon - v_\varepsilon)\|_{L^\infty(B_{2r}\cap\Omega)} \\ &\leq Cr + C\varepsilon^{-1}r^2 \leq C\varepsilon^{-1}r^2,\end{aligned}$$

for $r \geq \varepsilon$, where we have used the mean value theorem in the first inequality and used Lemma 4.4 and 4.5 as well as (4.3.11) and (4.3.12) in the last inequality.

A similar argument also shows that $\|\nabla^2_{\tan} w_\varepsilon\|_{L^\infty(B_{2r}\cap\partial\Omega)} \leq C\varepsilon^{-2}r^2$, which, by interpolation, implies $\|\nabla_{\tan} w_\varepsilon\|_{C^\sigma(B_{2r}\cap\partial\Omega)} \leq C\varepsilon^{-1-\sigma}r^2$ for any $0 < \sigma < 1$.

Estimate of $w_\varepsilon(x)$: We first claim that

$$|w_\varepsilon(y)| \leq C|y - x_0|^2 \qquad \text{for all } y \in \partial\Omega\cap B(x_0, r_0). \tag{4.3.16}$$

Actually, write agian $w_\varepsilon = (u_\varepsilon - f_\varepsilon) - (v_\varepsilon - f_\varepsilon)$. Using the cancellation $u_\varepsilon - f_\varepsilon = 0$ on $\partial\Omega$ and mean value theorem, we have that for any $y \in \partial\Omega \cap B_{r_0}(x_0)$

$$\begin{aligned}|w_\varepsilon(y)| &= |v_\varepsilon(y) - f_\varepsilon(y)| \\ &\leq C|y-\widehat{y}|\|\nabla(u_\varepsilon - f_\varepsilon)\|_{L^\infty(B_{2r}\cap\Omega)} \\ &\leq C|y - x_0|^2,\end{aligned}$$

where in the last inequality we have used Lemma 4.5 and (4.3.11).

Then we take advantage of the Poisson integral formula and split it into two parts,

$$\begin{aligned}w_\varepsilon(x) &= \int_{\partial\Omega} P_{\Omega,\varepsilon}(x,y)w_\varepsilon(y)d\sigma(y) \\ &= \int_{\partial\Omega\cap\{|y-x_0|\leq c\sqrt{\varepsilon}\}} P_{\Omega,\varepsilon}(x,y)w_\varepsilon(y)d\sigma(y) + \int_{\partial\Omega\cap\{|y-x_0|> c\sqrt{\varepsilon}\}} P_{\Omega,\varepsilon}(x,y)w_\varepsilon(y)d\sigma(y)\end{aligned} \tag{4.3.17}$$

where $x \in \Omega$ and $P_{\Omega,\varepsilon}$ is the Poisson kernel of $\mathcal{L}_\varepsilon$ in Ω. To estimate the first term on the right-hand side of (4.3.17), we apply (4.1.2) and (4.3.16),

$$
\begin{aligned}
&\left| \int_{\partial\Omega \cap \{|y-x_0| \le c\sqrt{\varepsilon}\}} P_{\Omega,\varepsilon}(x,y) w_\varepsilon(y) d\sigma(y) \right| \\
&\le C \int_{\partial\Omega \cap \{|y-x_0| \le c\sqrt{\varepsilon}\}} \operatorname{dist}(x,\partial\Omega) \frac{|y-x_0|^2}{|x-y|^d} d\sigma(y) \\
&\le C \int_{\partial\Omega \cap \{|y-x_0| \le c\sqrt{\varepsilon}\}} \operatorname{dist}(x,\partial\Omega) \frac{|x-x_0|^2}{|x-y|^d} d\sigma(y) + C \int_{\partial\Omega \cap \{|y-x_0| \le c\sqrt{\varepsilon}\}} \frac{\operatorname{dist}(x,\partial\Omega)}{|x-y|^{d-2}} d\sigma(y) \\
&\le C|x-x_0|^2 + C \operatorname{dist}(x,\partial\Omega)\sqrt{\varepsilon} \\
&\le Cr^2 + r\sqrt{\varepsilon},
\end{aligned}
$$

where we have used the observation $|y-x_0|^2 \le 2|y-x|^2 + 2|x-x_0|^2$.

To bound the second term on the right-hand side of (4.3.17), we note that (4.3.6) and (4.3.9) give $\|w_\varepsilon\|_{L^\infty(\Omega)} \le C\varepsilon$. Then

$$
\begin{aligned}
\left| \int_{\partial\Omega \cap \{|y-x_0| > c\sqrt{\varepsilon}\}} P_{\Omega,\varepsilon}(x,y) w_\varepsilon(y) d\sigma(y) \right| &\le C\varepsilon \int_{\partial\Omega \cap \{|y-x_0| > c\sqrt{\varepsilon}\}} \frac{\operatorname{dist}(x,\partial\Omega)}{|y-x|^d} d\sigma(y) \\
&\le C\varepsilon \operatorname{dist}(x,\partial\Omega)(\sqrt{\varepsilon})^{-1} \le Cr\sqrt{\varepsilon}.
\end{aligned}
$$

It follows

$$
|w_\varepsilon(x)| \le Cr^2 + Cr\sqrt{\varepsilon}, \qquad \text{for all } x \in B(x_0, 2r) \cap \Omega.
$$

This, together with (4.3.13) and the estimates for $\nabla_{\tan} w_\varepsilon$, proves (4.3.5). $\square$

For each fixed $x_0 \in \partial\Omega$, the system (4.3.3) associated with the adjoint operator $\mathcal{L}_\varepsilon^*$ and $f_\varepsilon = -\varepsilon\chi_j^{*\beta}(x/\varepsilon)$ has a solution $v_{\varepsilon,j}^{*\beta}$ of form

$$
v_{\varepsilon,j}^{*\beta}(x) = \varepsilon V_j^{*\beta}\left(\frac{x - (x\cdot n_0 + a)n_0}{\varepsilon}, -\frac{x\cdot n_0 + a}{\varepsilon} \right), \qquad \text{for } x \cdot n_0 \le -a, \tag{4.3.18}
$$

where $a = -x_0 \cdot n_0$ and $V_j^{*\beta} = V_j^{*\beta}(\theta, t)$ is a solution of

$$
\begin{cases}
-\begin{pmatrix} N^T\nabla_\theta \\ \partial_t \end{pmatrix} \cdot B^* \begin{pmatrix} N^T\nabla_\theta \\ \partial_t \end{pmatrix} V_j^{*\beta} = 0 & \text{in } \mathbb{T}^d \times (0,\infty), \\
V_j^{*\beta} = -\chi_j^{*\beta} & \text{on } \mathbb{T}^d \times \{0\},
\end{cases}
$$

given by Theorem 4.2. Note that $V_j^{*\beta}$ also depends on n_0. Finally, we apply Theorem 4.3 to obtain the main theorem of this section as follows.

Theorem 4.6. *Let $\varepsilon \le r \le \sqrt{\varepsilon}$ and $\sigma \in (0,1)$. Then for any $x \in B(x_0, r) \cap \Omega$,*

$$
\left| \nabla \left(\Phi_{\varepsilon,j}^{*\beta}(x) - P_j^\beta(x) - \varepsilon\chi_j^{*\beta}(x/\varepsilon) - v_{\varepsilon,j}^{*\beta}(x) \right) \right| \le C\sqrt{\varepsilon} + C\frac{r^{2+\sigma}}{\varepsilon^{1+\sigma}}, \tag{4.3.19}
$$

where C depends on $d, m, \mu, \sigma, \Omega$ and A.

4.4 Proof of Theorem 1.2: convergence rate

In this section, we will establish the sharp convergence rate for Dirichlet problem (1.2.11). Due to Lemma 4.1, it is sufficient to estimate $\|\widetilde{u}_\varepsilon - u_0\|_{L^2(\Omega)}$, where $\widetilde{u}_\varepsilon$ and u_0 are defined by

$$\widetilde{u}_\varepsilon^\alpha(x) = \int_{\partial\Omega} P_\Omega^{\alpha\zeta}(x,y)\omega_\varepsilon^{\zeta\beta}(y) f^\beta(y, y/\varepsilon) d\sigma(y) \tag{4.4.1}$$

and

$$u_0^\alpha(x) = \int_{\partial\Omega} P_\Omega^{\alpha\zeta}(x,y)\overline{f}^\zeta(y) d\sigma(y). \tag{4.4.2}$$

Now we need to find an explicit expression for the homogenized data $\overline{f}$. Roughly speaking, the homogenized data $\overline{f}$ in (4.4.2) should be the weak limit of $\omega_\varepsilon(y) f(y/\varepsilon)$ as $\varepsilon \to 0$. By (4.1.4) and (4.3.19), for $y \in B(x_0, r) \cap \partial\Omega$, one has

$$\begin{aligned} &\omega_\varepsilon^{\zeta\beta}(y) f^\beta(y/\varepsilon) \\ &\quad = h^{\zeta\nu}(y) \cdot n_\ell \frac{\partial}{\partial y_\ell}[P_k^{\rho\nu}(y) + \varepsilon\chi_k^{*\rho\nu}(y/\varepsilon) + v_{\varepsilon,k}^{*\rho\nu,x_0}(y)] n_k \cdot a_{ij}^{\rho\beta}(y/\varepsilon) n_i n_j f^\beta(y, y/\varepsilon) \\ &\qquad + \text{Error terms}. \end{aligned} \tag{4.4.3}$$

Note that $v_\varepsilon^{*,x_0}(y)$ is given in (4.3.18) which depends also on x_0, and $n = n(y)$ is the unit outward normal at y. For a fixed $y \in \partial\Omega$, in view of the quantitative ergodic theorem [7, Proposition 2.1], we know that $\omega_\varepsilon(y) f(y/\varepsilon)$ converges to its average on the tangent plane $\mathbb{H}_n^d(a)$ at y, where $n = n(y)$. The only unclear term in (4.4.3) is $n \cdot \nabla v_{\varepsilon,k}^{*\nu,x_0}$. Actually, in view of (4.3.18), for $z \in \mathbb{H}_n^d(a)$, one has

$$\begin{aligned} n \cdot \nabla v_{\varepsilon,k}^{*\nu,x_0}(z) &= n \cdot (1 - n \otimes n, -n) \begin{pmatrix} \nabla_\theta \\ \partial_t \end{pmatrix} V_k^{*\nu,x_0}\left(\frac{z}{\varepsilon}, 0\right) \\ &= -\partial_t V_k^{*\nu,x_0}\left(\frac{z}{\varepsilon}, 0\right). \end{aligned} \tag{4.4.4}$$

Note that $V_k^{*,x_0}(\theta, t)$ is 1-periodic in θ. As a consequence, we can define the homogenized boundary data as follows:

$$\begin{aligned} &\overline{f}^\zeta(y) \\ &= h^{\zeta\nu}(y) \int_{\mathbb{T}^d} [\delta^{\rho\nu} + n(y) \cdot \nabla\chi^{*\rho\nu}(\theta) \cdot n(y) - \partial_t V^{*\rho\nu,y}(\theta, 0) \cdot n(y)] n_i(y) n_j(y) a_{ij}^{\rho\beta}(\theta) f^\beta(y, \theta) d\theta \end{aligned} \tag{4.4.5}$$

Remark 4.7. If the coefficient matrix $A = (a_{ij}^{\alpha\beta})$ is constant (or divergence free), then $\chi^* = 0$ and hence $V^* = 0$ in (4.4.5). Also in this case, one has $\widehat{A} = A$. By the definition of h, this implies that $h^{\zeta\nu}\delta^{\rho\nu} n_i n_j a_{ij}^{\rho\beta} = \delta^{\zeta\beta}$. As a result, (4.4.5) is reduced to

$$\overline{f}(y) = \int_{\mathbb{T}^d} f(y, \theta) d\theta.$$

This exactly coincides with the homogenized boundary data for Dirichlet problems with constant coefficients.

Theorem 4.8. *Let $x, y \in \partial\Omega$ and $|x-y| < r_0$. Suppose that $n(x), n(y)$ satisfies the Diophantine condition with constant $\kappa(x)$ and $\kappa(y)$ respectively. Let $\overline{f}$ be defined by (4.4.5). Then for any $\sigma \in (0,1)$,*

$$|\overline{f}(x) - \overline{f}(y)| \le C \frac{|x-y|}{\kappa^{1+\sigma}} \sup_{z \in \mathbb{T}^d} \|f(\cdot, z)\|_{C^1(\partial\Omega)}.$$

where $\kappa = \max\{\kappa(x), \kappa(y)\}$ and C depends only on d, m, σ, Ω and A.

The above theorem may be proved by the similar argument as Theorem 3.27 by using Theorem 4.2. Moreover, an improve estimate will be proved by using a more delicate argument in the next chapter. At this point, however, the above estimate is sufficient for us to establish the optimal convergence rate.

The rest of the proof is devoted to estimating $\|u_\varepsilon - u_0\|_{L^2(\Omega)}$. To begin with, we perform a partition of unity on $\partial\Omega$ and restrict ourself on $B(x_0, r_0) \cap \partial\Omega$ for some x_0 and $r_0 > 0$ sufficiently small. So without any loss of generality, we may assume $\operatorname{supp}(f(\cdot, y)) \subset B(x_0, r_0)$ for any $y \in \mathbb{T}^d$. Then we construct the another partition of unity on $B(x_0, r_0) \cap \partial\Omega$ adapted to the Diophantine function $\kappa(x)$, by exactly the same method described in §3.7, with

$$\tau = \varepsilon^{1-\sigma},$$

for some small constant $\sigma > 0$. Recall that, as in §3.6, there exist a finite sequence of $\{\varphi_j\}$ of C_0^∞ positive functions in $\mathbb{R}^d$ and a finite of sequence of *surface cubes* $\{\widetilde{Q}_j\}$ on $\partial\Omega$, such that $\sum_j \varphi_j = 1$ on $B(x_0, 2r_0) \cap \partial\Omega$. Note that φ_j is supported in $2\widetilde{Q}_j$ and $|\nabla^k \varphi_j| \le C r_j^{-k}$, where r_j is the *side length* of $\widetilde{Q}_j$ as before. Also, for each j, there exists some $z_j \in 36\widetilde{Q}_j$ such that

$$\kappa(z_j) \ge \frac{c\tau}{r_j} = \frac{c\varepsilon^{1-\sigma}}{r_j}. \tag{4.4.6}$$

Note that $\widetilde{x}_j$ is the center of $\widetilde{Q}_j$. Let Γ_ε denote a boundary layer

$$\Gamma_\varepsilon = \Omega \cap \left(\bigcup_j B(\widetilde{x}_j, C r_j) \right)$$

and $D_\varepsilon = \Omega \setminus \Gamma_\varepsilon$. By Proposition 3.36,

$$|\Gamma_\varepsilon| \le \sum_j |B(\widetilde{x}_j, C r_j)| \le C \sum_j r_j^d \le C\tau = C\varepsilon^{1-\sigma}.$$

Thus for any $q > 0$,

$$\int_{\Gamma_\varepsilon} |u_\varepsilon - u_0|^q \le C\varepsilon^{1-\sigma}, \tag{4.4.7}$$

where we have used the boundedness of u_ε and u_0.

To deal with the L^q norm of $u_\varepsilon - u_0$ on D_ε, we introduce a function as (3.7.7)

$$\Theta_t(x) = \sum_j \frac{r_j^{d-1+t}}{|x-\widetilde{x}_j|^{d-1}}, \tag{4.4.8}$$

where $0 \le t < d-1$. We mention that Lemma 3.37 for $\Theta_t(x)$ will play a key role and be used repeatedly in the following context.

As in the Neumann problem, we split $\widetilde{u}_\varepsilon - u_0$ into five parts

$$\begin{aligned}\widetilde{u}_\varepsilon(x) - u_0(x) &= \int_{\partial\Omega} P_\Omega(x,y)\omega_\varepsilon(y) f(y,y/\varepsilon) d\sigma(y) - \int_{\partial\Omega} P_\Omega(x,y)\overline{f}(y) d\sigma(y)\\ &= I_1 + I_2 + I_3 + I_4 + I_5,\end{aligned}$$

where $I_k, 1 \le k \le 5$, will be defined below and handled separately. We point out in advance that estimates for I_3 and I_4 essentially distinguish from the case of strictly convex domains and need more careful calculations.

Let $\delta > 0$ be an arbitrarily small exponent that might differ in each occurrence.

Estimate of I_1: Let

$$\begin{aligned}I_1 = &\int_{\partial\Omega} P_\Omega^{\alpha\zeta}(x,y)\omega_\varepsilon^{\zeta\beta}(y) f^\beta(y,y/\varepsilon) d\sigma(y)\\ &\qquad - \sum_j \int_{\partial\Omega} \varphi_j(y) P_\Omega^{\alpha\zeta}(x,y)\widetilde{\omega}_\varepsilon^{\zeta\beta,z_j}(y) f^\beta(y,y/\varepsilon) d\sigma(y),\end{aligned}$$

where

$$\widetilde{\omega}_\varepsilon^{\zeta\beta,z_j}(y) = h^{\zeta\nu}(y) n_\ell(y) \frac{\partial}{\partial y_\ell}\Big[P_k^{\rho\nu}(y) + \varepsilon\chi_k^{*\rho\nu}(y/\varepsilon) + v_{\varepsilon,k}^{*\rho\nu,z_j}(y)\Big] n_k(y) a_{im}^{\rho\beta}(y/\varepsilon) n_i(y) n_m(y), \tag{4.4.9}$$

and z_j's are specially selected as in (4.4.6). Note that I_1 comes from the error terms in (4.4.3), which by (4.3.5) is bounded by

$$C\sum_j \int_{\partial\Omega} \varphi_j(y) |P_\Omega(x,y)| \bigg(\sqrt{\varepsilon} + \frac{r_j^{2+\sigma}}{\varepsilon^{1+\sigma}} \wedge 1\bigg) d\sigma(y) = R_1 + R_2,$$

for any $\sigma \in (0,1)$. Observe that

$$R_1 \le C\sqrt{\varepsilon} \int_{\partial\Omega} |P_\Omega(x,y)| \le C\sqrt{\varepsilon}. \tag{4.4.10}$$

For R_2, using $|P_\Omega(x,y)| \le C|x-y|^{1-d}$ and $|x-y| \approx |x-\widetilde{x}_j|$ for $x \in D_\varepsilon, y \in B(\widetilde{x}_j, Cr_j)$, we have

$$R_2 = C\sum_j \int_{\partial\Omega} \varphi_j(y)|P_\Omega(x,y)| \bigg(\frac{r_j^{2+\sigma}}{\varepsilon^{1+\sigma}} \wedge 1\bigg) d\sigma(y) \le C\varepsilon^{-1-\sigma} \sum_j \frac{r_j^{2+\sigma+d-1}}{|x-\widetilde{x}_j|^{d-1}} \tag{4.4.11}$$

Now we estimate R_2 by Lemma 3.37 in two separate cases. If $2(2+\sigma) < d-1$, then we apply Lemma 3.37 directly with $q = 2$ and obtain

$$\int_{D_\varepsilon} |R_2(x)|^2 dx \le C\varepsilon^{-2(1+\sigma)}\tau^{2(2+\sigma)} \le C\varepsilon^{2-\delta}, \tag{4.4.12}$$

where we have used $\tau = \varepsilon^{1-\sigma}$ and chosen σ sufficiently small. Otherwise, we choose suitable $q < 2$ such that $q(2+\sigma) = d-1-\sigma < d-1$ and then apply Lemma 3.37

$$\int_{D_\varepsilon} |R_2(x)|^q dx \le C\varepsilon^{-q(1+\sigma)}\tau^{q(2+\sigma)} \le C\varepsilon^{\frac{1}{2}(d-1)-\delta},$$

where again, σ is chosen sufficiently small. Clearly, (4.4.11) also implies $|R_2| \le C$. Thus, a simple interpolation leads to

$$\int_{D_\varepsilon} |R_2(x)|^2 dx \le C\varepsilon^{\frac{1}{2}(d-1)-\delta}. \tag{4.4.13}$$

Combining (4.4.10), (4.4.12) and (4.4.13), we obtain

$$\int_{D_\varepsilon} |I_1(x)|^2 dx \le C\varepsilon^{1\wedge\frac{1}{2}(d-1)-\delta}.$$

Estimate of I_2: Set

$$\begin{aligned} I_2 = &\sum_j \int_{\partial\Omega} \varphi_j(y) P_\Omega^{\alpha\zeta}(x,y)\widetilde{\omega}_\varepsilon^{\zeta\beta,z_j}(y) f^\beta(y, y/\varepsilon) d\sigma(y) \\ &- \sum_j \int_{\partial\mathbb{H}_j^d} \varphi_j(P_j^{-1}(y)) P_\Omega^{\alpha\zeta}(x, P_j^{-1}(y))\widetilde{\omega}_\varepsilon^{\zeta\beta,z_j}(y) f^\beta(z_j, y/\varepsilon) d\sigma(y) \end{aligned} \tag{4.4.14}$$

where $\partial\mathbb{H}_j^d$ denotes the tangent plane for $\partial\Omega$ at z_j and P_j^{-1} is the inverse of the projection map from $B(z_j, Cr_j) \cap \partial\Omega$ to $\partial\mathbb{H}_j^d$. We clarify that in (4.4.9), $n(y)$ is the outer normal of $y \in \partial\Omega$. But in the second term of (4.4.14), y needs to belong to $\partial\mathbb{H}_j^d$ and hence we need to update $n(y) = n(z_j)$ for all $y \in \partial\mathbb{H}_j^d$. This modification leads to some harmless errors bounded by $Cr_j \le Cr_j^2/\varepsilon$. Then, for the same reason as I_2 in §3.7, we are able to bound I_2 by

$$|I_2| \le C\varepsilon^{-1} \sum_j \frac{r_j^{2+d-1}}{|x - \widetilde{x}_j|^{d-1}}.$$

Similar as (4.4.11), we estimate this in two cases and obtain

$$\int_{D_\varepsilon} |I_2(x)|^2 dx \le C\varepsilon^{2\wedge\frac{1}{2}(d-1)-\delta}.$$

Estimate of I_3: Set

$$\begin{aligned} I_3 = &\sum_j \int_{\partial\mathbb{H}_j^d} \varphi_j(P_j^{-1}(y)) P_\Omega^{\alpha\zeta}(x, P_j^{-1}(y))\widetilde{\omega}_\varepsilon^{\zeta\beta,z_j}(y) f^\beta(z_j, y/\varepsilon) d\sigma(y) \\ &- \sum_j \int_{\partial\mathbb{H}_j^d} \varphi_j(P_j^{-1}(y)) P_\Omega^{\alpha\zeta}(x, P_j^{-1}(y))\overline{f}^\zeta(z_j) d\sigma(y), \end{aligned}$$

where $\overline{f}$ is defined in (4.4.5). To estimate I_3, we apply the quantitative ergodic theorem in [7]. As we have mention in the estimate of I_2, the outer normal in the definition of $\widetilde{\omega}_\varepsilon^{\zeta\beta,z_j}(y)$ is constant on $\partial\mathbb{H}_j^d$ with Diophantine constant $\kappa(z_j)$, and therefore $\widetilde{\omega}_\varepsilon^{\zeta\beta,z_j}(y)$ is nothing but a slice of some 1-periodic function in $\mathbb{R}^d$ (see (4.4.4)). Note that by (4.4.6), $\kappa(z_j) > c\varepsilon^{1-\sigma}/r_j$. Then it follows from Lemma 2.10 that for any $N > 0$,

$$
\begin{aligned}
|I_3| &\le C\sum_j \Big(\frac{\varepsilon r_j}{\tau}\Big)^N \int_{2\widetilde{Q}_j} |\nabla^N(\varphi_j(y)P_\Omega(x,y))|d\sigma(y)\\
&\le C\sum_j \Big(\frac{\varepsilon r_j}{\tau}\Big)^N \sum_{k=0}^N \frac{r_j^{d-1-N+k}}{|x-\widetilde{x}_j|^{d-1+k}}\\
&\le C\varepsilon^{N\sigma}\sum_j \frac{r_j^{d-1}}{|x-\widetilde{x}_j|^{d-1}},
\end{aligned}
$$

where we have used $|\nabla^k\varphi_j| \le Cr_j^{-k}$, $|\nabla^k P_\Omega(x,y)| \le C|x-y|^{1-d-k}$ and $r_j \le C|x-\widetilde{x}_j| \approx C|x-y|$ for all $x \in D_\varepsilon$ and $y \in 2\widetilde{Q}_j$. Now, applying Lemma 3.37 with $t = 0$, we have

$$
\int_{D_\varepsilon} |I_3|^2 \le C\varepsilon^{N\sigma}.
$$

This is a desired estimate if we choose $N \ge 1$ large enough.

Estimate of I_4: Set

$$
\begin{aligned}
I_4 = &\sum_j \int_{\partial\mathbb{H}_j^d} \varphi_j(P_j^{-1}(y))P_\Omega^{\alpha\zeta}(x,P_j^{-1}(y))\overline{f}^\zeta(z_j)d\sigma(y)\\
&- \sum_j \int_{\partial\mathbb{H}_j^d} \varphi_j(P_j^{-1}(y))P_\Omega^{\alpha\zeta}(x,P_j^{-1}(y))\overline{f}^\zeta(P_j^{-1}(y))d\sigma(y).
\end{aligned}
$$

The estimate for I_4 essentially relies on the regularity of homogenized data $\overline{f}$. Indeed, by Proposition 4.8

$$
|\overline{f}(z_j) - \overline{f}(P_j^{-1}(y))| \le C\left(\frac{r_j}{\kappa(z_j)^{1+\sigma}}\right) \le C\left(\frac{r_j^{1+(1+\sigma)}}{\tau^{\sigma(1+\sigma)}}\right),
$$

where we also used $|z_j - P_j^{-1}(y)| \le Cr_j$. This leads to a bound for I_4

$$
|I_4| \le C\tau^{-(1+\sigma)}\sum_j \frac{r_j^{1+(1+\sigma)+d-1}}{|x-x_j|^{d-1}}.
$$

Using Lemma 3.37 and a familiar argument as before, we are able to show

$$
\int_{D_\varepsilon} |I_4|^2 \le C\varepsilon^{2\wedge\frac{1}{2}(d-1)-\delta}.
$$

Estimate of I_5: Finally, let

$$I_5 = \sum_j \int_{\partial \mathbb{H}^d_j} \varphi_j(P_j^{-1}(y)) P_\Omega^{\alpha\zeta}(x, P_j^{-1}(y)) \overline{f}^\zeta (P_j^{-1}(y)) d\sigma(y) - \int_{\partial\Omega} P_\Omega(x,y)\overline{f}(y) d\sigma(y).$$

A change of variables gives

$$|I_5| \le C \sum_j \frac{r_j^{1+d-1}}{|x - \widetilde{x}_j|^{d-1}}.$$

Then by Lemma 3.37 and a familiar argument, we obtain

$$\int_{D_\varepsilon} |I_5|^2 \le C\varepsilon^{2\wedge(d-1)-\delta}.$$

Combining the estimates of I_k, we have shown that

$$\int_{D_\varepsilon} |\widetilde{u}_\varepsilon - u_0|^2 \le C\varepsilon^{1\wedge \frac{1}{2}(d-1)-\delta},$$

for arbitrarily small $\delta > 0$. This, together with Lemma 4.1 and (4.4.7), ends the proof of (1.2.14).

The main purpose of this chapter is to show the $W^{1,p}$ estimate, with any $p \in (1,\infty)$, of the homogenized boundary data $\overline{f}$ and $\overline{g}$. This implies the C^{1-}-Hölder continuity, due to the Sobolev embedding theorem. We mention several related work regarding the continuity of homogenized boundary data. In [1], under the additional assumption that A is independent of some rational direction ν_0, it was proved that the homogenized Dirichlet data has a unique continuous extension to the set $\{x \in \partial\Omega : n(x)\cdot\nu_0 \neq 0\}$. The problem of Hölder continuity was also studied in [13, 16] for second-order nonlinear elliptic equations of form $F(D^2 u_\varepsilon, x/\varepsilon) = 0$. In particular, it was shown in [16] that if the homogenized operator $\overline{F}$ is either rotational invariant or linear, then the homogenized Dirichlet data is $C^{1/d-}$-Hölder continuous, and that the homogenized data may be discontinuous in general. Note that the linear elliptic equations in non-divergence form may be written in a divergence form with $\mathrm{div}(A) = 0$. In this case, the first-order correctors are trivial and therefore the homogenized data is smooth if Ω is smooth and satisfies some geometric conditions. In the nonlinear setting of divergence form, the $C^{1/d-}$-Hölder continuity and the possible discontinuity of the homogenized boundary data at rational directions have been studied recently in [18]. The main result of this chapter, on the C^{1-}-Hölder continuity of the homogenized data for linear elliptic systems in divergence form, was first proved in [37].

We point out that, unlike the optimal convergence rates, the assumption that Ω is strictly convex is not essential for the regularity theory of the homogenized data. In fact, the proof in this chapter goes through as long as one has $[\varkappa(n(x))]^{-1} \in L^q(\partial\Omega)$ for some $q > 0$ (see (2.4.1) for the definition of $\varkappa$). Consequently, the regularity results of Theorems 1.1 and 1.2 continue to hold for the domains of finite type considered in [42].

5.1 An introduction to the proofs

We briefly describe our main idea to (1.2.15) and (1.2.10), as well as some of the key estimates in the proof. Our starting point for the proof of (1.2.15) for Dirichlet problem is the formula for the homogenized data $\overline{f}$ discovered in (4.4.5). This formula reduces the problem to the study of continuity of solutions $V_n = V_n(\theta, t)$ with respect to $n \in \mathbb{S}^{d-1}$ for the Dirichlet problem in a half-space,

$$\begin{cases} -\begin{pmatrix} N_n^T \nabla_\theta \\ \partial_t \end{pmatrix} \cdot B_n \begin{pmatrix} N_n^T \nabla_\theta \\ \partial_t \end{pmatrix} V_n = 0 & \text{in } \mathbb{T}^d \times \mathbb{R}_+, \\ V_n(\theta, 0) = \phi(\theta) & \text{on } \mathbb{T}^d \times \{0\}, \end{cases} \tag{5.1.1}$$

where $\phi \in C^\infty(\mathbb{T}^d; \mathbb{R}^m)$, $B_n = B_n(\theta,t) = M_n^T A^*(\theta - tn) M_n$, M_n is any $d\times d$ orthogonal matrix whose last column is $-n$, and N_n is defined by $M_n = (N_n, -n)$. Note that M_n and N_n are not unique. However, as we have seen before, the solution V_n of (5.1.1) is independent of the choice of N_n.

We use $\mathbb{S}^{d-1}_R$, $\mathbb{S}^{d-1}_I$ and $\mathbb{S}^{d-1}_D$ to represent the sets of rational, irrational and Diophantine unit vectors (i.e., unit vectors satisfying the Diophantine condition (2.4.1)), respectively. Note that $\mathbb{S}^{d-1}_D$ is a subset of $\mathbb{S}^{d-1}_I$ and has full surface measure of $\mathbb{S}^{d-1}$.

Let $n, \widetilde{n} \in \mathbb{S}^{d-1}_D$. The key step in our proof is to show that for any $\sigma \in (0,1)$,

$$\left(\int_{\mathbb{T}^d} \left| \partial_t V_n(\theta, 0) - \partial_t V_{\widetilde{n}}(\theta, 0) \right|^2 d\theta \right)^{1/2} \le C_\sigma \varkappa^{-\sigma} |n - \widetilde{n}|, \tag{5.1.2}$$

where $\varkappa = \max\left\{ \varkappa(n), \varkappa(\widetilde{n}) \right\}$ and C_σ depends only on d, m, σ, λ, $\|A\|_{C^k(\mathbb{T}^d)}$ and $\|\phi\|_{C^k(\mathbb{T}^d)}$ for some $k = k(d, \sigma) > 1$. Observe that (5.1.2) shows that V_n is locally Lipschitz in n near a point with a Diophantine normal. Then (1.2.15) follows from (5.1.2) by using the representation formula (4.4.5), the fact $[\varkappa(n(x))]^{-1} \in L^{d-1}(\partial\Omega)$ and an approximation argument.

To prove (5.1.2), besides the pointwise decay estimates (depending on $\kappa(n)$) established in Theorem 4.2, one needs to fully take advantage of the fact that if

$$u^s(x) = V_n(x - (x \cdot n)n - sn, -x \cdot n - s), \tag{5.1.3}$$

then u^s is a solution of the Dirichlet problem in a half-space,

$$\begin{cases} \mathcal{L}_1^*(u^s) = 0 & \text{in } \mathbb{H}^d_n(s), \\ u^s = \phi & \text{on } \partial\mathbb{H}^d_n(s), \end{cases} \tag{5.1.4}$$

where $\mathcal{L}_1^* = \mathcal{L}_\varepsilon^*$, the adjoint of $\mathcal{L}_\varepsilon$, with $\varepsilon = 1$. In (5.1.4), $\mathbb{H}^d_n(s) = \mathbb{H}^d_n - sn$ and $\mathbb{H}^d_n = \{x \in \mathbb{R}^d : x \cdot n < 0\}$ is the half-space whose boundary contains the origin and with the outward normal n. This allows us to apply the maximal principle and the large-scale boundary regularity estimates for the operator $\mathcal{L}_1^*$. The technique was already used in [21, 7] to establish the boundedness of V_n. Here, among other things, we apply the technique to establish the uniform boundedness of $\nabla_\theta V_n$ as well as some uniform pointwise decay estimates for $\partial_t V_n$ and $N_n^T \nabla_\theta V_n$ (independent of n). Then, combining the energy and pointwise decay estimates, the uniform boundedness of V_n or $\nabla_\theta V_n$, and a weighted estimate (see Remark 3.30), we adopt a delicate interpolation argument to conclude (5.1.2).

We remark that the asymptotic behavior of the solution u^s of (5.1.4) as $x \cdot n \to -\infty$ is well understood thanks to [27, 6, 20, 21, 29, 2]. In particular, if n is irrational, it was shown in [29] that there exists a constant vector $\mu^*(n, \phi) \in \mathbb{R}^m$ independent of s such that

$$\mu^*(n, \phi) = \lim_{x \cdot n \to -\infty} u^s(x), \tag{5.1.5}$$

though the rate of convergence could be arbitrarily slow in general. On the other hand, if n is rational [27, 6], the above limit depends on s and possesses an exponential rate of convergence. The mapping $\mu : \mathbb{S}^{d-1}_I \times C^\infty(\mathbb{T}^d; \mathbb{R}^m) \mapsto \mathbb{R}^m$ defined via (5.1.5), but with $\mathcal{L}_1^*$ replaced by $\mathcal{L}_1$, is called the boundary layer tail (BLT) for Dirichlet problems associated with $\mathcal{L}_1$. It follows from [21] that

$$\overline{f}(x) = \mu(n(x), f(x, \cdot)), \qquad \text{if } n(x) \in \mathbb{S}^{d-1}_D. \tag{5.1.6}$$

Thus, by (1.2.15), $\|\mu(\cdot,\phi)\|_{W^{1,p}(\mathbb{S}^{d-1})} \le C\|\phi\|_{L^2(\mathbb{T}^d)}$ for any $1 < p < \infty$. Consequently, for any $0 < \alpha < 1$, $\mu(\cdot,\phi)$ extends to a Hölder continuous function of order α on $\mathbb{S}^{d-1}$ and

$$|\mu(n,\phi) - \mu(\widetilde{n},\phi)| \le C_\alpha |n - \widetilde{n}|^\alpha \|\phi\|_{L^2(\mathbb{T}^d)} \quad \text{for any } n, \widetilde{n} \in \mathbb{S}^{d-1}, \tag{5.1.7}$$

where C_α depends only on d, m, α and A.

Our approach to (1.2.10) for Neumann problems is similar to that used for (1.2.15). The starting point is a formula for the homogenized data $\{\overline{g}_{ij}\}$ obtained in (3.5.9); also see Theorem 5.11 for details. As in the case of Dirichlet problems, this formula reduces the problem to the study of the continuity in $n \in \mathbb{S}^{d-1}$ of solutions $U_n = U_n(\theta,t)$ to the Neumann problem,

$$\begin{cases} -\begin{pmatrix} N_n^T \nabla_\theta \\ \partial_t \end{pmatrix} \cdot B_n \begin{pmatrix} N_n^T \nabla_\theta \\ \partial_t \end{pmatrix} U_n = 0 & \text{in } \mathbb{T}^d \times \mathbb{R}_+, \\ -e_{d+1} \cdot B_n \begin{pmatrix} N_n^T \nabla_\theta \\ \partial_t \end{pmatrix} U_n = T_n \cdot \nabla_\theta \phi & \text{on } \mathbb{T}^d \times \{0\}, \end{cases} \tag{5.1.8}$$

where $T_n \in \mathbb{R}^d$, $|T_n| \le 1$ and $T_n \cdot n = 0$. Let $n, \widetilde{n} \in \mathbb{S}^{d-1}_D$. We will show in §5.2 that for any $\sigma \in (0,1)$,

$$\left(\int_{\mathbb{T}^d} \left| \nabla_\theta U_n(\theta,0) - \nabla_\theta U_{\widetilde{n}}(\theta,0) \right|^2 d\theta \right)^{1/2} \le C_\sigma \varkappa^{-\sigma} |n - \widetilde{n}|, \tag{5.1.9}$$

where $\varkappa = \max\left\{\varkappa(n), \varkappa(\widetilde{n})\right\}$ and C_σ depends only on d, m, σ, λ, $\|A\|_{C^k(\mathbb{T}^d)}$ and $\|\phi\|_{C^k(\mathbb{T}^d)}$ for some $k = k(d,\sigma) > 1$. Now (1.2.10) follows from (5.1.9), the fact $[\varkappa(n(x))]^{-1} \in L^{d-1}(\partial\Omega)$, and the representation formula mentioned above. Finally, we point out that the key estimates in the proof of (5.1.9) rely on the observation that if $u^s(x) = U_n(x - (x\cdot n)n - sn, -x\cdot n - s)$, then u^s is a solution to the Neumann problem,

$$\begin{cases} \mathcal{L}_1^*(u^s) = 0 & \text{in } \mathbb{H}_n^d(s), \\ \dfrac{\partial u^s}{\partial \nu_1^*} = T_n \cdot \nabla_x \phi & \text{on } \partial\mathbb{H}_n^d(s), \end{cases} \tag{5.1.10}$$

We refer the reader to §5.2 for details.

5.2 Regularity for Dirichlet problems

As we have seen in the previous section, the central problem for the regularity of $\overline{f}$ is to study the regularity of (5.1.4) with respect to n. However, the solvability of the Dirichlet problem (5.1.4) is not obvious, since the domain $\mathbb{H}_n^d(s)$ is unbounded and the boundary data does not decay. Nevertheless, by using Lipschitz estimates in [9] and an approximation argument, one may establish the existence of the Poisson kernel in a half-space and hence the solvability of (5.1.4) via the Poisson integral formula.

Theorem 5.1. *Let $\Omega = \mathbb{H}^d_n(s)$ for some $n \in \mathbb{S}^{d-1}$ and $s \in \mathbb{R}$. Then, for any bounded continuous function ϕ in $\mathbb{R}^d$, there exists a unique bounded function u in $C^\infty(\Omega;\mathbb{R}^m) \cap C(\overline{\Omega};\mathbb{R}^m)$ such that*

$$\begin{cases} \mathcal{L}_1^*(u) = 0 & \text{in } \Omega, \\ u = \phi & \text{on } \partial\Omega. \end{cases} \tag{5.2.1}$$

Moreover, the solution may be represented by

$$u(x) = \int_{\partial\Omega} P^*(x,y)\phi(y)\, d\sigma(y), \tag{5.2.2}$$

where the Poisson kernel $P^* = P^*(x,y)$ satisfies

$$|P^*(x,y)| \le \frac{C \min\{\delta(x), |x-y|\}}{|x-y|^d}, \tag{5.2.3}$$

$$|\nabla_x P^*(x,y)| \le \frac{C}{|x-y|^d} \tag{5.2.4}$$

for any $x \in \Omega$ and $y \in \partial\Omega$, $\delta(x) = \mathrm{dist}(x,\partial\Omega) = |s + x\cdot n|$, and C depends only on d, m, λ, and some Hölder norm of A on $\mathbb{T}^d$.

Proof. The theorem was proved in [21, Proposition 2.5]. □

Remark 5.2. By the boundary Lipschitz estimates in Theorem 2.5 and the Cacciopoli inequality, the uniqueness holds under the sublinear growth condition: $|u(x)| \le C_0(1+\delta(x))^\alpha$ for some $C_0 > 0$ and $\alpha \in (0,1)$. Also, it follows readily from (5.2.3) that the Miranda-Agmon maximum principle,

$$\|u\|_{L^\infty(\Omega)} \le C\|\phi\|_{L^\infty(\partial\Omega)} \tag{5.2.5}$$

holds, where C depends only on d, m, λ, and some Hölder norm of A on $\mathbb{T}^d$.

An alternative way to establish the solvability of (5.1.4) for periodic data ϕ is to lift the problem to a $(d+1)$-dimensional problem in the upper half-space. Fix $n \in \mathbb{S}^{d-1}$. Let $M = (N, -n)$ be a $d \times d$ orthogonal matrix such that the last column is $-n$ and the first $d-1$ column is a $d \times (d-1)$ matrix N. Now we seek a solution u of (5.1.4) in a particular form

$$u^s(x) = V(x - (x\cdot n)n - sn, -x\cdot n - s). \tag{5.2.6}$$

It is not hard to see that $V = V(\theta,t)$ has to satisfy the following lifted degenerate system,

$$\begin{cases} -\begin{pmatrix} N^T\nabla_\theta \\ \partial_t \end{pmatrix} \cdot B \begin{pmatrix} N^T\nabla_\theta \\ \partial_t \end{pmatrix} V = 0 & \text{in } \mathbb{T}^d \times (0,\infty), \\ V(\theta,0) = \phi(\theta) & \text{on } \mathbb{T}^d \times \{0\}, \end{cases} \tag{5.2.7}$$

where $B(\theta, t) = M^T A^*(\theta - tn)M$. Note that $MM^T = I$ implies $I = NN^T + n \otimes n$. It follows that

$$M\begin{pmatrix} N^T\nabla_\theta \\ \partial_t \end{pmatrix} = (I - n \otimes n)\nabla_\theta - n\partial_t. \tag{5.2.8}$$

Thus, the solution V is independent of the choice of N.

The well-posedness of (5.2.7) was given by [21, Propositions 2.1 and 2.6].

Lemma 5.3. *Let $n \in \mathbb{S}^{d-1}$. Then, for any $\phi \in C^\infty(\mathbb{T}^d; \mathbb{R}^m)$, the system (5.2.7) has a smooth solution $V = V(\theta, t)$ satisfying*

$$\left(\int_0^\infty \int_{\mathbb{T}^d} \left(|N^T\nabla_\theta \partial_\theta^\alpha \partial_t^j V|^2 + |\partial_\theta^\alpha \partial_t^{1+j} V|^2 d\theta\right) dt\right)^{1/2} \le C\|\phi\|_{C^{|\alpha|+j+1}(\mathbb{T}^d)}, \tag{5.2.9}$$

where $|\alpha|$, $j \ge 0$, and C depends only on d, m, $|\alpha|$, j and A. Moreover, if $n \in \mathbb{S}_D^{d-1}$ with Diophantine constant $\kappa > 0$, then there exists a constant V_∞ such that for any $|\alpha|$, j, $\ell \ge 0$,

$$|N^T\nabla_\theta \partial_\theta^\alpha \partial_t^j V| + |\partial_\theta^\alpha \partial_t^{1+j} V| + \kappa|\partial_\theta^\alpha (V - V_\infty)| \le \frac{C_\ell \|\phi\|_{C^k(\mathbb{T}^d)}}{(1 + \kappa t)^\ell}, \tag{5.2.10}$$

where $k = k(|\alpha|, j, \ell, d)$ and C_ℓ depends only on d, m, $|\alpha|$, j, ℓ and A.

Remark 5.4. The solution of (5.1.4) given by Theorem 5.1 coincides with the solution of (5.1.4) given by Lemma 5.3 via (5.2.6) for any $n \in \mathbb{S}^{d-1}$. To see this, let $w(x) = u^s(x) - V(x - (x \cdot n)n - sn, -x \cdot n - s)$. Clearly, w satisfies

$$\begin{cases} \mathcal{L}_1^* w = 0 & \text{in } \mathbb{H}_n^d(s), \\ w = 0 & \text{on } \partial\mathbb{H}_n^d(s). \end{cases} \tag{5.2.11}$$

Since u^s is bounded and V satisfies

$$\begin{aligned} |V(\theta, t)| &= \left| \int_0^t \partial_\rho V(\theta, \rho) d\rho + \phi(\theta) \right| \\ &\le \|\phi\|_\infty + t^{1/2} \left(\int_0^\infty |\partial_\rho V(\theta, \rho)|^2 \, d\rho \right)^{1/2} \\ &\le \|\phi\|_\infty + Ct^{1/2} \left(\int_0^\infty \|\partial_\rho V(\cdot, \rho)\|_{H^k(\mathbb{T}^d)}^2 \, d\rho \right)^{1/2} \\ &\le \|\phi\|_\infty + Ct^{1/2} \|f\|_{H^{k+2}(\mathbb{T}^d)}, \end{aligned}$$

for some $k \ge 1$, we conclude that w is of sublinear growth as $|x \cdot n| \to \infty$. Thus, by Remark 5.2, we obtain $w \equiv 0$.

Now we give an explicit expression for $\overline{f}(x)$ if $n(x) \in \mathbb{S}_D^{d-1}$. For $1 \le k \le d$ and $1 \le \beta \le m$, let $V_{n,k}^\beta = V_{n,k}^\beta(\theta, t)$ denote the solution of the following Dirichlet problem,

$$\begin{cases} -\begin{pmatrix} N^T\nabla_\theta \\ \partial_t \end{pmatrix} \cdot B_n \begin{pmatrix} N^T\nabla_\theta \\ \partial_t \end{pmatrix} V_{n,k}^\beta = 0 & \text{in } \mathbb{T}^d \times (0, \infty), \\ \qquad\qquad\qquad\qquad\qquad V_{n,k}^\beta = -\chi_k^{*\beta} & \text{on } \mathbb{T}^d \times \{0\}, \end{cases} \tag{5.2.12}$$

where $\chi_k^{*\beta}$ are the correctors for $\mathcal{L}_\varepsilon^*$, $B_n = M^T A^*(\theta - tn)M$, and $M = (N, -n)$ is an orthogonal matrix.

Theorem 5.5. *Let $x \in \partial\Omega$. Suppose that $n = n(x) \in \mathbb{S}_D^{d-1}$. Let $V_n(\theta, t)$ be the solution of (5.2.12). Then*

$$\overline{f}^\alpha(x) = \int_{\mathbb{T}^d} h^{\alpha\beta}\left[\delta^{\gamma\beta} + \frac{\partial}{\partial\theta_\ell}\chi_k^{*\gamma\beta}(\theta)n_\ell n_k - \partial_t V_{n,k}^{\gamma\beta}(\theta,0)\cdot n_k\right] a_{ij}^{\gamma\nu}(\theta) n_i n_j f^\nu(x,\theta)\, d\theta \tag{5.2.13}$$

for $1 \le \alpha \le m$, where $h = (h^{\alpha\beta})$ denotes the inverse matrix of the $m \times m$ matrix $(\widehat{a}_{ij}^{*\alpha\beta} n_i n_j)$.

Proof. This was proved in [7] (also see [42]). □

We now turn to the proof of (1.2.15). The key step is to prove the following.

Theorem 5.6. *Fix $\sigma \in (0,1)$. Let $x, y \in \partial\Omega$ and $|x-y| \le c_0$. Suppose that $n(x)$, $n(y) \in \mathbb{S}_D^{d-1}$. Then*

$$|\overline{f}(x) - \overline{f}(y)| \le C_\sigma \kappa^{-\sigma}|x-y| \left(\int_{\mathbb{T}^d} \|f(\cdot,y)\|_{C^1(\partial\Omega)}^2\, dy\right)^{1/2}, \tag{5.2.14}$$

where $\kappa = \max\{\kappa(n(x)), \kappa(n(y))\}$ and C_σ depends only on d, m, σ, λ, and $\|A\|_{C^k(\mathbb{T}^d)}$ for some $k = k(d,\sigma) \ge 1$.

To prove Theorem 5.6, in view of the formula (5.2.13), we investigate the continuity in n of the solution to the Dirichlet problem (5.2.7).

Lemma 5.7. *For $\phi \in C^\infty(\mathbb{T}^d; \mathbb{R}^m)$, let V be the solution of (5.2.7), given by Lemma 5.3, with $n \in \mathbb{S}^{d-1}$. Then*

$$|N^T \nabla_\theta V| + |\partial_t V| \le \frac{C\|\phi\|_{C^2(\mathbb{T}^d)}}{1+t}, \tag{5.2.15}$$

where C depends only on d, m and A. Moreover, for any $|\alpha|$, $j \ge 0$ and $0 < \sigma < 1$,

$$|N^T \nabla_\theta \partial_\theta^\alpha \partial_t^j V| + |\partial_\theta^\alpha \partial_t^{1+j} V| \le \frac{C_\sigma \|\phi\|_{C^k(\mathbb{T}^d)}}{(1+t)^{1-\sigma}}, \tag{5.2.16}$$

where $k = k(|\alpha|, j, \sigma, d)$ and C_σ depends only on d, m, $|\alpha|$, j, σ and A.

Proof. Let u^s be given by (5.2.6). Then

$$\begin{cases} \mathcal{L}_1^* u^s = 0 & \text{in } \mathbb{H}_n^d(s), \\ u^s = \phi & \text{on } \partial\mathbb{H}_n^d(s). \end{cases} \tag{5.2.17}$$

It follows from (5.2.6) that

$$V(\theta, t) = u^{-\theta\cdot n}(\theta - tn) \quad \text{for all } (\theta, t) \in \mathbb{T}^d \times \mathbb{R}_+, \tag{5.2.18}$$

and that $u^s(x)$ is smooth in s and x for $-x\cdot n-s>0$. Thanks to the fact $N^T\nabla_\theta(\theta\cdot n)=0$, the last equality implies that

$$\begin{cases} N^T\nabla_\theta V(\theta,t)=N^T\nabla_x u^{-\theta\cdot n}(\theta-tn),\\ \quad\ \partial_t V(\theta,t)=-n\cdot\nabla_x u^{-\theta\cdot n}(\theta-tn). \end{cases} \tag{5.2.19}$$

As a result, estimates for $N^T\nabla_\theta V$ and $\partial_t V$ may be reduced to the corresponding estimates for u^s.

It follows from the representation of Poisson integral (5.2.2) and the pointwise estimate (5.2.4) that

$$|\nabla u^s(x)|\le \frac{C\|\phi\|_\infty}{|s+x\cdot n|}. \tag{5.2.20}$$

To deal with the case where $|s+x\cdot n|=\operatorname{dist}(x,\partial\mathbb{H}^d_n(s))<1$, we first note that $\|u^s\|_\infty\le C\|\phi\|_\infty$ by (5.2.5). Next, by the boundary Lipschitz estimate, we obtain $|\nabla u^s(x)|\le C\|\phi\|_{C^2(\mathbb{T}^d)}$ if $\operatorname{dist}(x,\partial\mathbb{H}^d_n(s))<1$. This, together with (5.2.20) and (5.2.19), proves (5.2.15).

Finally, we prove the inequality (5.2.16) by using interpolation and the Sobolev embedding. Precisely, for any $L>0$, it follows from (5.2.15), (5.2.9) and interpolation that

$$\begin{aligned}\|N^T\nabla_\theta V\|_{H^{r+\frac{d}{2}+1}(\mathbb{T}^d\times[L,L+1])}&\le C\|N^T\nabla_\theta V\|^{1-\sigma}_{L^2(\mathbb{T}^d\times[L,L+1])}\|N^T\nabla_\theta V\|^{\sigma}_{H^{k-1}(\mathbb{T}^d\times[L,L+1])}\\ &\le C(1+L)^{-(1-\sigma)}\|\phi\|_{C^k(\mathbb{T}^d)},\end{aligned}$$

where $k=k(d,r,\sigma)\ge 1$ is sufficiently large. It follows from the Sobolev embedding theorem that

$$\begin{aligned}\sup_{(\theta,t)\in\mathbb{T}^d\times[L,L+1]}|N^T\nabla_\theta\partial_\theta^\alpha\partial_t^j V(\theta,t)|&\le C\|N^T\nabla_\theta V\|_{H^{r+d/2+1}(\mathbb{T}^d\times[L,L+1])}\\ &\le \frac{C\|\phi\|_{C^k(\mathbb{T}^d)}}{(1+L)^{1-\sigma}},\end{aligned}$$

which readily implies

$$|N^T\nabla_\theta\partial_\theta^\alpha\partial_t^j V(\theta,t)|\le \frac{C\|\phi\|_{C^k(\mathbb{T}^d)}}{(1+t)^{1-\sigma}}\qquad\text{for any } (\theta,t)\in\mathbb{T}^d\times\mathbb{R}_+, \tag{5.2.21}$$

where $|\alpha|+j\le r$. A similar argument gives the pointwise estimate for $|\partial_\theta^\alpha\partial_t^{1+j}V|$. $\square$

Lemma 5.8. *Let V be the solution of (5.2.7) with $n\in\mathbb{S}^{d-1}$. Then*

$$|V|+|\nabla_\theta V|\le C\|\phi\|_{C^2(\mathbb{T}^d)}, \tag{5.2.22}$$

where C depends only on d, m and A. Moreover, if $n\in\mathbb{S}^{d-1}_D$ with Diophantine constant $\kappa=\kappa(n)>0$, then for any $|\alpha|\ge 2$ and $0<\sigma<1$,

$$|\partial_\theta^\alpha V|\le C\kappa^{-\sigma}\|\phi\|_{C^k(\mathbb{T}^d)}, \tag{5.2.23}$$

where $k=k(d,|\alpha|,\sigma)>1$ and C depends only on d, m, $|\alpha|$, σ and A.

Proof. Again, the desired estimates for V will be reduced to estimates for solutions u^s of (5.2.17), where V and u^s are related by (5.2.18). First, since $\|u^s\|_\infty \le C\|\phi\|_\infty$, we obtain $|V| \le C\|\phi\|_\infty$. Next, by comparing u^s and $u^{s'}$ in the common domain, we may deduce from the boundary Lipschitz estimate and the Miranda-Agmon maximal principle (5.2.5) that

$$|u^s(x) - u^{s'}(x)| \le C|s - s'|\|\phi\|_{C^2(\mathbb{T}^d)}, \tag{5.2.24}$$

if $x \cdot n < -\max\{s, s'\}$. Observe that, to prove the boundedness of $\nabla_\theta V$, it suffices to prove the boundedness of $n \cdot \nabla_\theta V$, as $N^T\nabla_\theta V$ is bounded due to (5.2.15). To this end, note that

$$\begin{aligned}
&|V(\theta + rn, t) - V(\theta, t)| = |u^{-\theta\cdot n - r}(\theta + rn - tn) - u^{-\theta\cdot n}(\theta - tn)| \\
&\le |u^{-\theta\cdot n - r}(\theta + rn - tn) - u^{-\theta\cdot n - r}(\theta - tn)| + |u^{-\theta\cdot n - r}(\theta - tn) - u^{-\theta\cdot n}(\theta - tn)| \\
&\le |r|\|\nabla u^{-\theta\cdot n - r}\|_\infty + \|u^{-\theta\cdot n - r} - u^{-\theta\cdot n}\|_\infty \\
&\le C|r|\|\phi\|_{C^2(\mathbb{T}^d)},
\end{aligned}$$

where we have used (5.2.24) for the last step. Dividing by r on both sides and taking the limit as $r \to 0$, we obtain $|n \cdot \nabla_\theta V| \le C\|\phi\|_{C^2(\mathbb{T}^d)}$. This finishes the proof of (5.2.22).

Finally, to show (5.2.23), we use (5.2.22), (5.2.10) and an interpolation argument. Precisely, let $L > 0$ and $t \in [L, L+1]$,

$$\begin{aligned}
\sup_{(\theta,t)\in\mathbb{T}^d\times[L,L+1]} |\partial_\theta^\alpha V(\theta, t)| &\le C\|V\|_{H^{d/2+|\alpha|+1}(\mathbb{T}^d\times[L,L+1])} \\
&\le C\|V\|^{1-\sigma}_{H^1(\mathbb{T}^d\times[L,L+1])}\|V\|^{\sigma}_{H^r(\mathbb{T}^d\times[L,L+1])} \\
&\le C\kappa^{-\sigma}\|\phi\|_{C^k(\mathbb{T}^d)},
\end{aligned}$$

where $|\alpha| \ge 2$ and $r = r(d, \alpha, \sigma)$, $k = k(d, |\alpha|, \sigma)$ are sufficiently large. The desired estimate follows. $\square$

Now we are ready to prove Theorem 5.6.

Proof of Theorem 5.6. **Step 1: Set-up and reduction.**

Fix $n_1, n_2 \in \mathbb{S}^{d-1}_D$. We may assume that $\delta = |n_1 - n_2| > 0$ is sufficiently small. Let N_1 and N_2 be the $d \times (d-1)$ matrices such that both $M_1 = (N_1, -n_1)$ and $M_2 = (N_2, -n_2)$ are orthogonal matrices. Recall that solution V_1 (resp. V_2) of (5.2.7), associated with n_1 (resp. n_2), is independent of the choices of N_1 (resp. N_2). So without loss of generality, we may assume $|N_1 - N_2| \le C\delta$. To be precise, we write down the systems for V_1 and V_2 as follows:

$$\begin{cases}
-\begin{pmatrix} N_1^T\nabla_\theta \\ \partial_t \end{pmatrix} \cdot B_1 \begin{pmatrix} N_1^T\nabla_\theta \\ \partial_t \end{pmatrix} V_1 = 0 & \text{in } \mathbb{T}^d \times (0, \infty), \\
V_1 = \phi & \text{on } \mathbb{T}^d \times \{0\},
\end{cases} \tag{5.2.25}$$

and

$$\begin{cases} -\begin{pmatrix} N_2^T\nabla_\theta \\ \partial_t \end{pmatrix} \cdot B_2 \begin{pmatrix} N_2^T\nabla_\theta \\ \partial_t \end{pmatrix} V_2 = 0 & \text{in } \mathbb{T}^d \times (0,\infty), \\ V_2 = \phi & \text{on } \mathbb{T}^d \times \{0\}, \end{cases} \tag{5.2.26}$$

where $B_\ell(\theta,t) = M_\ell^T A^*(\theta - tn_\ell)M_\ell$ for $\ell = 1,2$ and $\phi = -\chi_k^{*\beta}$. In view of Theorem 5.5, to show (5.2.14), it suffices to prove that

$$\int_{\mathbb{T}^d} |\partial_t V_1(\theta,0) - \partial_t V_2(\theta,0)|^2 \, d\theta \le C\kappa^{-2\sigma}|n_1 - n_2|^2. \tag{5.2.27}$$

Define $W = V_1 - V_2$. Observe that

$$\int_{\mathbb{T}^d} |\partial_t W(\theta,0)|^2 \, d\theta \le 2\int_0^1 \int_{\mathbb{T}^d} |\partial_t W(\theta,t)|^2 \, d\theta dt + 2\int_0^1 \int_{\mathbb{T}^d} |\partial_t^2 W(\theta,t)|^2 \, d\theta dt. \tag{5.2.28}$$

Thus, the estimate (5.2.27) is further reduced to that for the two integrals in the RHS of (5.2.28). We may assume that $\kappa(n_1) \ge \kappa(n_2)$ and thus $\kappa = \kappa(n_1)$.

Step 2: Estimate for $\partial_t W$.

Note that W satisfies $W(\theta,0) = 0$ and

$$\begin{aligned} &-\begin{pmatrix} N_2^T\nabla_\theta \\ \partial_t \end{pmatrix} \cdot B_2 \begin{pmatrix} N_2^T\nabla_\theta \\ \partial_t \end{pmatrix} W \\ &= -\begin{pmatrix} N_2^T\nabla_\theta \\ \partial_t \end{pmatrix} \cdot B_2 \begin{pmatrix} N_2^T\nabla_\theta \\ \partial_t \end{pmatrix} V_1 \\ &= \left[\begin{pmatrix} N_1^T\nabla_\theta \\ \partial_t \end{pmatrix} \cdot B_1 \begin{pmatrix} N_1^T\nabla_\theta \\ \partial_t \end{pmatrix} - \begin{pmatrix} N_2^T\nabla_\theta \\ \partial_t \end{pmatrix} \cdot B_2 \begin{pmatrix} N_2^T\nabla_\theta \\ \partial_t \end{pmatrix} \right] V_1. \end{aligned} \tag{5.2.29}$$

By using

$$\begin{aligned} &\begin{pmatrix} N_1^T\nabla_\theta \\ \partial_t \end{pmatrix} \cdot B_1 \begin{pmatrix} N_1^T\nabla_\theta \\ \partial_t \end{pmatrix} - \begin{pmatrix} N_2^T\nabla_\theta \\ \partial_t \end{pmatrix} \cdot B_2 \begin{pmatrix} N_2^T\nabla_\theta \\ \partial_t \end{pmatrix} \\ &= -\begin{pmatrix} N_2^T\nabla_\theta \\ \partial_t \end{pmatrix} \cdot B_2 \begin{pmatrix} (N_2^T - N_1^T)\nabla_\theta \\ 0 \end{pmatrix} - \begin{pmatrix} N_2^T\nabla_\theta \\ \partial_t \end{pmatrix} \cdot (B_2 - B_1) \begin{pmatrix} N_1^T\nabla_\theta \\ \partial_t \end{pmatrix} \\ &\qquad + \begin{pmatrix} (N_2^T - N_1^T)\nabla_\theta \\ 0 \end{pmatrix} \cdot B_1 \begin{pmatrix} N_1^T\nabla_\theta \\ \partial_t \end{pmatrix}, \end{aligned}$$

the RHS of (5.2.29) can be written as

$$\begin{pmatrix} N_2^T\nabla_\theta \\ \partial_t \end{pmatrix} \cdot (G_1 + G_2) + H, \tag{5.2.30}$$

where

$$\begin{aligned} G_1 &= -B_2 \begin{pmatrix} (N_2^T - N_1^T)\nabla_\theta \\ 0 \end{pmatrix} V_1, \\ G_2 &= -(B_2 - B_1) \begin{pmatrix} N_1^T\nabla_\theta \\ \partial_t \end{pmatrix} V_1, \\ H &= \begin{pmatrix} (N_2^T - N_1^T)\nabla_\theta \\ 0 \end{pmatrix} \cdot B_1 \begin{pmatrix} N_1^T\nabla_\theta \\ \partial_t \end{pmatrix} V_1. \end{aligned}$$

Therefore, the equation (5.2.29) is reduced to

$$-\begin{pmatrix} N_2^T\nabla_\theta \\ \partial_t \end{pmatrix} \cdot B_2 \begin{pmatrix} N_2^T\nabla_\theta \\ \partial_t \end{pmatrix} W = \begin{pmatrix} N_2^T\nabla_\theta \\ \partial_t \end{pmatrix} \cdot G + H, \tag{5.2.31}$$

where $G = G_1 + G_2$.

Now, we are going to employ the weighted estimate established in §3.3. Precisely, applying (3.5.27) in Remark 3.30 to the system (5.2.31), we obtain

$$\begin{aligned} \int_0^\infty \int_{\mathbb{T}^d} & \left(|N_2^T\nabla_\theta W|^2 + |\partial_t W|^2\right) t^{\sigma-1}\, d\theta dt \\ & \le C \int_0^\infty \int_{\mathbb{T}^d} \left(|G|^2 + t^2 |H|^2\right) t^{\sigma-1}\, d\theta dt. \end{aligned} \tag{5.2.32}$$

Hence, it suffices to estimate the integrals involving G and H in (5.2.32).

Estimate for the integral with G_1: By the estimates for $|\nabla V_1|$ in (5.2.22) and (5.2.10), we have

$$|G_1(\theta,t)| \le C\delta |\nabla_\theta V_1(\theta,t)| \le C\delta \cdot 1^{1-\sigma}[\kappa^{-1}(1+\kappa t)^{-\ell}]^\sigma \tag{5.2.33}$$

for any $0 < \sigma < 1$. It follows that

$$\begin{aligned} \int_0^\infty \int_{\mathbb{T}^d} |G_1|^2 t^{\sigma-1}\, d\theta dt & \le C\delta^2 \kappa^{-2\sigma} \int_0^\infty \frac{dt}{t^{1-\sigma}(1+\kappa t)^{2\ell\sigma}} \\ & \le C\delta^2 \kappa^{-3\sigma} \int_0^\infty \frac{dt}{t^{1-\sigma}(1+t)^{2\ell\sigma}} \\ & \le C\delta^2 \kappa^{-3\sigma}, \end{aligned}$$

where we can simply choose $\ell = 1$ to ensure the convergence of the integral in the right-hand side.

Estimate for the integral with G_2: Note that an interpolation between (5.2.15) and (5.2.10) implies

$$|N_1^T\nabla_\theta V_1(\theta,t)| + |\partial_t V_1(\theta,t)| \le C(1+t)^{\sigma-1}(1+\kappa t)^{-\ell\sigma}. \tag{5.2.34}$$

Also note that $|B_1(\theta,t) - B_2(\theta,t)| \le Ct\delta$. It follows that

$$\begin{aligned} \int_0^\infty \int_{\mathbb{T}^d} |G_2|^2 t^{\sigma-1}\, d\theta dt & \le C\delta^2 \int_0^\infty \frac{t^{1+\sigma} dt}{(1+t)^{2(1-\sigma)}(1+\kappa t)^{2\ell\sigma}} \\ & \le C\delta^2 \kappa^{-3\sigma}, \end{aligned}$$

where we need to choose $\ell = 2$.

Estimate for the integral with H: Observe that

$$\begin{aligned} \int_0^\infty \int_{\mathbb{T}^d} |H|^2 t^{1+\sigma}\, d\theta dt & \le C\delta^2 \int_0^\infty \int_{\mathbb{T}^d} \left(|N_1^T\nabla_\theta V_1|^2 + |\partial_t V_1|^2\right) t^{1+\sigma}\, d\theta dt \\ & \quad + C\delta^2 \int_0^\infty \int_{\mathbb{T}^d} \left(|N_1^T\nabla_\theta \nabla_\theta V_1|^2 + |\partial_t \nabla_\theta V_1|^2\right) t^{1+\sigma}\, d\theta dt. \end{aligned}$$

The first term in the RHS is bounded by $\delta^2\kappa^{-3\sigma}$ by using (5.2.34). To handle the second integral, we apply the interpolation theorem between (5.2.16) and (5.2.10) to obtain

$$|N_1^T\nabla_\theta\nabla_\theta V_1(\theta,t)| + |\partial_t\nabla_\theta V_1(\theta,t)| \le C(1+t)^{-(1-\sigma)^2}(1+\kappa t)^{-\ell\sigma}. \tag{5.2.35}$$

Thus, the second term is bounded by

$$C\delta^2\int_0^\infty \frac{t^{1+\sigma}\,dt}{(1+t)^{2(1-2\sigma)}(1+\kappa t)^{2\ell\sigma}} \le C\delta^2\kappa^{-5\sigma}, \tag{5.2.36}$$

where we have chosen $\ell = 3$.

By combining the estimates above with (5.2.32), we obtain

$$\int_0^1\int_{\mathbb{T}^d}\left(|N_2^T\nabla_\theta W|^2 + |\partial_t W|^2\right)d\theta dt \le C_\sigma\delta^2\kappa^{-5\sigma}. \tag{5.2.37}$$

Step 3: Estimate for $\partial_t^2 W$.

Let N_{2j} denote the jth column of N_2 and define $\nabla_{2j} = N_{2j}^T\cdot\nabla_\theta$ for $1\le j\le d-1$. Note that ∇_{2j} is the jth component of $N_2^T\nabla_\theta$. Then we apply ∇_{2j} to (5.2.31) and obtain

$$\begin{aligned}-\begin{pmatrix}N_2^T\nabla_\theta\\ \partial_t\end{pmatrix}\cdot B_2\begin{pmatrix}N_2^T\nabla_\theta\\ \partial_t\end{pmatrix}\nabla_{2j}W &= \begin{pmatrix}N_2^T\nabla_\theta\\ \partial_t\end{pmatrix}\cdot\nabla_{2j}G + \nabla_{2j}H\\ &\quad+\begin{pmatrix}N_2^T\nabla_\theta\\ \partial_t\end{pmatrix}\cdot\nabla_{2j}B_2\begin{pmatrix}N_2^T\nabla_\theta\\ \partial_t\end{pmatrix}W,\end{aligned} \tag{5.2.38}$$

on $\mathbb{T}^d\times\mathbb{R}_+$ and $\nabla_{2j}W = 0$ on $\mathbb{T}^d\times\{0\}$. Let $\eta(t)$ be a cut-off function such that $\eta(t) = 1$ for $t\in[0,1]$, $\eta(t) = 0$ for $t\in[2,\infty)$, $0\le\eta(t)\le 1$ and $|\nabla\eta|\le C$. Now integrating (5.2.38) against $\eta^2\nabla_{2j}W$, we derive from integration by parts that

$$\begin{aligned}&\int_0^1\int_{\mathbb{T}^d}\left(|N_2^T\nabla_\theta\nabla_{2j}W|^2 + |\partial_t\nabla_{2j}W|^2\right)d\theta dt\\ &\quad\le C\int_0^2\int_{\mathbb{T}^d}\left(|\nabla_{2j}G|^2 + |\nabla_{2j}H|^2 + |N_2^T\nabla_\theta W|^2 + |\partial_t W|^2\right)d\theta dt\\ &\quad\le C\kappa^{-5\sigma}\delta^2,\end{aligned}$$

where we have used the fact $|\nabla_{2j}W|\le|N_2^T\nabla_\theta W|$. Consequently,

$$\int_0^1\int_{\mathbb{T}^d}\left(|N_2^T\nabla_\theta\otimes N_2^T\nabla_\theta W|^2 + |\partial_t N_2^T\nabla_\theta W|^2\right)d\theta dt \le C\kappa^{-5\sigma}\delta^2. \tag{5.2.39}$$

Now observe that by applying the product rule of differentiation,

$$\begin{aligned}&\begin{pmatrix}N_2^T\nabla_\theta\\ \partial_t\end{pmatrix}\cdot B_2\begin{pmatrix}N_2^T\nabla_\theta\\ \partial_t\end{pmatrix}W\\ &=\left[\begin{pmatrix}N_2^T\nabla_\theta\\ \partial_t\end{pmatrix}B_2\right]\cdot\begin{pmatrix}N_2^T\nabla_\theta\\ \partial_t\end{pmatrix}W + B_2:\left[\begin{pmatrix}N_2^T\nabla_\theta\\ \partial_t\end{pmatrix}\otimes\begin{pmatrix}N_2^T\nabla_\theta\\ \partial_t\end{pmatrix}\right]W\\ &=\left[\begin{pmatrix}N_2^T\nabla_\theta\\ \partial_t\end{pmatrix}B_2\right]\cdot\begin{pmatrix}N_2^T\nabla_\theta\\ \partial_t\end{pmatrix}W + B_2:\begin{bmatrix}N_2^T\nabla_\theta\otimes N_2^T\nabla_\theta & N_2^T\nabla_\theta\partial_t\\ (N_2^T\nabla_\theta\partial_t)^T & 0\end{bmatrix}W + b_{2,dd}\partial_t^2 W,\end{aligned}$$

where $b_{2,dd} = (b_{2,dd}^{\alpha\beta})_{1\le\alpha,\beta\le m}$ is positive due to the strong ellipticity condition. This gives

$$\begin{aligned} b_{2,dd}\partial_t^2 W = &-\left[\begin{pmatrix} N_2^T\nabla_\theta \\ \partial_t \end{pmatrix} B_2\right]\cdot\begin{pmatrix} N_2^T\nabla_\theta \\ \partial_t \end{pmatrix} W - B_2 : \begin{bmatrix} N_2^T\nabla_\theta\otimes N_2^T\nabla_\theta & N_2^T\nabla_\theta\partial_t \\ (N_2^T\nabla_\theta\partial_t)^T & 0 \end{bmatrix} W \\ &- \begin{pmatrix} N_2^T\nabla_\theta \\ \partial_t \end{pmatrix}\cdot G - H. \end{aligned}$$

Note that $|(b_{2,dd})^{-1}| \le C$. Thus, it follows from (5.2.37), (5.2.39) and the pointwise estimates of G and H for $t\in[0,1]$ that

$$\int_0^1\int_{\mathbb{T}^d} |\partial_t^2 W|^2\, d\theta dt \le C\delta^2\kappa^{-5\sigma}. \tag{5.2.40}$$

This completes the proof of Theorem 5.6. $\square$

Proof of Theorem 1.2: Regularity estimate. Note that $\partial\Omega$ is locally differential homeomorphic to $\mathbb{R}^{d-1}$. Thus, in view of Theorem 5.6, it suffices to prove the following claim: Let $F\in L^1(\mathbb{R}^{d-1};\mathbb{R}^m)$ and $G\in L^p(\mathbb{R}^{d-1})$ for some $1<p<\infty$. Suppose that for a.e. $x\in\mathbb{R}^{d-1}$,

$$|F(x)-F(y)| \le |x-y||G(x)|, \qquad \text{for a.e. } y\in\mathbb{R}^{d-1}. \tag{5.2.41}$$

Then

$$\left(\int_{\mathbb{R}^{d-1}} |\nabla F|^p\right)^{1/p} \le C\left(\int_{\mathbb{R}^{d-1}} |G|^p\right)^{1/p}, \tag{5.2.42}$$

where C depends only on d and p. Indeed, if the claim holds, then it follows from Theorem 5.6 that

$$\left(\int_{\partial\Omega} |\nabla_{\tan}\overline{f}|^p\right)^{1/p} \le C\left(\int_{\mathbb{T}^d} \|f(\cdot,y)\|^2_{C^1(\partial\Omega)} dy\right)^{1/2}\left(\int_{\partial\Omega} [\kappa(n(x))]^{-\sigma p}\, dx\right)^{1/p} \tag{5.2.43}$$

for any $0<\sigma<1$. Recall that $[\kappa(n(x))]^{-1}\in L^q(\partial\Omega)$ for any $q<d-1$ (see [?]). Thus, for any $p<\infty$, we choose $\sigma\in(0,1)$ so small that $\sigma p<d-1$. As a result, we obtain

$$\left(\int_{\partial\Omega} |\nabla_{\tan}\overline{f}|^p\right)^{1/p} \le C\left(\int_{\mathbb{T}^d} \|f(\cdot,y)\|^2_{C^1(\partial\Omega)} dy\right)^{1/2} \tag{5.2.44}$$

for any $p<\infty$. Note that $\overline{f}$ is bounded. We may conclude that $\overline{f}\in W^{1,p}(\partial\Omega;\mathbb{R}^m)$ and (1.2.15) holds.

It remains to prove the claim. Let $\varphi\in C_0^\infty(B(0,1))$ and $\int_{\mathbb{R}^{d-1}}\varphi = 1$. Set $\varphi_\varepsilon(x) = \varepsilon^{1-d}\varphi(x/\varepsilon)$. Define for any $\varepsilon>0$,

$$F_\varepsilon(x) = \int_{\mathbb{R}^{d-1}} F(y)\varphi_\varepsilon(x-y)\, dy. \tag{5.2.45}$$

Clearly, F_ε is smooth and $F_\varepsilon \to F$ in $L^1(\mathbb{R}^{d-1};\mathbb{R}^m)$ as $\varepsilon \to 0$. Moreover, for any $z \in B(x,\varepsilon)$,

$$
\begin{aligned}
\nabla F_\varepsilon(x) &= \int_{\mathbb{R}^{d-1}} F(y)\nabla\varphi_\varepsilon(x-y)\,dy \\
&= \int_{\mathbb{R}^{d-1}} (F(y)-F(z))\nabla\varphi_\varepsilon(x-y)\,dy.
\end{aligned}
$$

Using the assumption (5.2.41),

$$
\begin{aligned}
|\nabla F_\varepsilon(x)| &\le \fint_{B(x,\varepsilon)} |G(z)| \int_{B(x,\varepsilon)} |y-z||\nabla\varphi_\varepsilon(x-y)|\,dydz \\
&\le C \fint_{B(x,\varepsilon)} |G(z)|\,dz \\
&\le C \left(\fint_{B(x,\varepsilon)} |G(z)|^p\,dz\right)^{1/p}.
\end{aligned}
$$

Thus, by Fubini's Theorem, for any $\varepsilon > 0$

$$
\left(\int_{\mathbb{R}^{d-1}} |\nabla F_\varepsilon(x)|^p\,dx\right)^{1/p} \le C\left(\int_{\mathbb{R}^{d-1}} |G(z)|^p\,dz\right)^{1/p}. \tag{5.2.46}
$$

Since $\nabla F_\varepsilon \to \nabla F$ in the sense of distribution as $\varepsilon \to 0$, (5.2.42) follows from (5.2.46). $\square$

5.3 Regularity for Neumann problems

The approach for the Neumann problem is similar to the Dirichlet problem. We recall the explicit formula for $\overline{g}_{ij}$ given in (3.5.9), which involves a family of Neumann problems in the half-spaces:

$$
\begin{cases}
\mathcal{L}_1^* u^s = 0 & \text{in } \mathbb{H}_n^d(s), \\
n\cdot A^*\nabla u^s = T\cdot\nabla\phi & \text{on } \partial\mathbb{H}_n^d(s),
\end{cases} \tag{5.3.1}
$$

where T is a constant tangential vector, i.e., $T\cdot n = 0$, with $|T| \le 1$. We assume that $\phi \in C^\infty(\mathbb{T}^d;\mathbb{R}^m)$.

As far as we know, for arbitrary $n \in \mathbb{S}^{d-1}$, the solvability of (5.3.1) is not clear. But for $n \in \mathbb{S}_D^{d-1}$, it was shown in §3.2 that (5.3.1) is solvable by lifting the problem to a $(d+1)$-dimensional system in the upper half-space, in a manner similar to the case of Dirichlet condition. More precisely, we seek a solution in the form of

$$
u^s(x) = U(x-(x\cdot n+s)n, -(x\cdot n+s)), \tag{5.3.2}
$$

where U is a solution of the Neumann problem:

$$
\begin{cases}
-\begin{pmatrix} N^T\nabla_\theta \\ \partial_t \end{pmatrix}\cdot B\begin{pmatrix} N^T\nabla_\theta \\ \partial_t \end{pmatrix} U = 0 & \text{in } \mathbb{T}^d\times\mathbb{R}_+, \\
-e_{d+1}\cdot B\begin{pmatrix} N^T\nabla_\theta \\ \partial_t \end{pmatrix} U = T\cdot\nabla_\theta\phi & \text{on } \mathbb{T}^d\times\{0\},
\end{cases} \tag{5.3.3}
$$

with $B(\theta,t) = M^T A^*(\theta - tn)M$ and $M = (N, -n)$ being an orthogonal matrix. The solvability of (5.3.3) and related estimates contained in Proposition 3.7, 3.8 and 3.11 are addressed below.

Lemma 5.9. *Suppose that n satisfies the Diophantine condition with constant $\kappa > 0$. Then the Neumann problem (5.3.3) has a smooth solution U, and the solution is unique, up to a constant under the condition that $U \in L^\infty(\mathbb{T}^d \times \mathbb{R}_+)$, $\nabla_\theta U \in L^2(\mathbb{T}^d \times \mathbb{R}_+)$ and $\partial_t U \in L^2(\mathbb{T}^d \times \mathbb{R}_+)$. Moreover, the solution satisfies*

$$\int_0^\infty \int_{\mathbb{T}^d} \Big\{ |N^T \nabla_\theta \partial_\theta^\alpha \partial_t^j U|^2 + |\partial_\theta^\alpha \partial_t^{1+j} U|^2 \Big\} \, d\theta dt \le C \|\phi\|^2_{C^{|\alpha|+j+1}(\mathbb{T}^d)}, \tag{5.3.4}$$

for any $|\alpha|$, $j \ge 0$, where C depends only on d, m, $|\alpha|$, j, and A. Furthermore, there exists a constant vector U_∞ such that for any $|\alpha|$, j, $\ell \ge 0$,

$$|N^T \nabla_\theta \partial_\theta^\alpha \partial_t^j U| + |\partial_\theta^\alpha \partial_t^{1+j} U| + \kappa |\partial_\theta^\alpha (U - U_\infty)| \le \frac{C_\ell \|\phi\|_{C^k(\mathbb{T}^d)}}{(1+\kappa t)^\ell}, \tag{5.3.5}$$

where $k = k(|\alpha|, j, \ell, d)$ and C_ℓ depends only on d, m, $|\alpha|$, j, ℓ, and A.

Remark 5.10. Lemma 5.9 gives the existence of solutions to (5.3.1) for $s \in \mathbb{R}$ and $n \in \mathbb{S}^{d-1}_D$ via (5.3.2). Moreover, by the (large-scale) uniform boundary Lispchitz estimates for Neumann conditions in Theorem 5.4, the solution satisfying the sublinear growth as $x \cdot n \to -\infty$ is unique up to a constant.

Recall that $\mathbb{S}^{d-1}_D$ has full surface measure of $\mathbb{S}^{d-1}$. In the following, we reformulate the expression for $\overline{g}_{ij}$ defined a.e. on $\mathbb{S}^{d-1}$, as shown in §3.5.

Theorem 5.11. *Let $g = \{g_{ij}\}$, where $g_{ij} \in C^\infty(\partial\Omega \times \mathbb{T}^d; \mathbb{R}^m)$. Then, for any $x \in \partial\Omega$ with $n = n(x) \in \mathbb{S}^{d-1}_D$,*

$$\overline{g}^\gamma_{jk}(x) = n_i \widehat{a}^{\alpha\gamma}_{ji} h^{\alpha\beta} T_{\ell r} \cdot \int_{\mathbb{T}^d} \Big[e_k \delta^{\nu\beta} + \nabla_\theta \chi^{*\nu\beta}_k(\theta) + \nabla_\theta U^{\nu\beta}_{n,k}(\theta, 0) \Big] g^\nu_{\ell r}(x,\theta) \, d\theta, \tag{5.3.6}$$

where $(h^{\alpha\beta})$ denotes the inverse of the $m \times m$ matrix $(\widehat{a}^{*\alpha\beta}_{ij} n_i n_j)$ and $U^\beta_{n,k}$ is the solution of

$$\begin{cases} -\begin{pmatrix} N^T \nabla_\theta \\ \partial_t \end{pmatrix} \cdot B_n \begin{pmatrix} N^T \nabla_\theta \\ \partial_t \end{pmatrix} U^\beta_{n,k} = 0 & \text{in } \mathbb{T}^d \times (0,\infty), \\ -e_{d+1} \cdot B_n \begin{pmatrix} N^T \nabla_\theta \\ \partial_t \end{pmatrix} U^\beta_{n,k} = \dfrac{1}{2} T_{ij} \cdot \nabla_\theta \phi^\beta_{ij,k} & \text{on } \mathbb{T}^d \times \{0\}, \end{cases} \tag{5.3.7}$$

where $T_{ij} = n_i e_j - n_j e_i$, $B_n(\theta,t) = M^T A^*(\theta - tn)M$, and $\phi^\beta_{ij,k} = (\phi^{1\beta}_{ij,k}, \phi^{2\beta}_{ij,k}, \cdots, \phi^{m\beta}_{ij,k})$ are the 1-periodic smooth functions satisfying

$$\frac{\partial}{\partial \theta_i}\{\phi^{\alpha\beta}_{ij,k}\} = a^{*\alpha\beta}_{jk} + a^{*\alpha\gamma}_{j\ell} \frac{\partial}{\partial \theta_\ell} \chi^{*\gamma\beta}_k - \widehat{a}^{*\alpha\beta}_{jk} \quad \textit{and} \quad \phi^{\alpha\beta}_{ij,k} = -\phi^{\alpha\beta}_{ji,k}. \tag{5.3.8}$$

We point out that the functions $\phi^\beta_{ij,k}$, which are completely determined by A, are smooth as long as A is. The equations (5.3.8) for $\phi^\beta_{ij,k}$ will not be used in this paper.

Theorem 5.12. *Fix $\sigma \in (0,1)$. Let $x, y \in \partial\Omega$ and $|x-y| \le c_0$. Suppose that $n(x)$, $n(y) \in \mathbb{S}_D^{d-1}$. Then*

$$|\overline{g}(x) - \overline{g}(y)| \le C_\sigma \kappa^{-\sigma} |x-y| \left(\int_{\mathbb{T}^d} \|g(\cdot, y)\|^2_{C^1(\partial\Omega)} \, dy \right)^{1/2}, \tag{5.3.9}$$

where $\kappa = \max\left\{\kappa(n(x)), \kappa(n(y))\right\}$ and C_σ depends only on d, m, σ, λ, and $\|A\|_{C^k(\mathbb{T}^d)}$ for some $k = k(d,\sigma) \ge 1$.

To prove Theorem 5.12, the following two lemmas will be crucial.

Lemma 5.13. *Let $n \in \mathbb{S}_D^{d-1}$ and U be a solution of (5.3.3) corresponding to n. Then*

$$|N^T \nabla_\theta U| + |\partial_t U| \le \frac{C\|\phi\|_{C^k(\mathbb{T}^d)}}{1+t}, \tag{5.3.10}$$

where $k > d/2 + 1$ and C depends only on d, m and A. Moreover, for any $0 < \sigma < 1$,

$$|N^T \nabla_\theta \partial_\theta^\alpha \partial_t^j U| + |\partial_\theta^\alpha \partial_t^{1+j} U| \le \frac{C_\sigma \|\phi\|_{C^k(\mathbb{T}^d)}}{(1+t)^{1-\sigma}}, \tag{5.3.11}$$

where $k = k(|\alpha|, j, \sigma, d)$ and C_σ depends only on d, m, $|\alpha|$, j, σ and A.

Proof. Let u^s be the solution of (5.3.1), given by (5.3.2). Then it follows from Theorem 3.13 that

$$|\nabla u^s(x)| \le \frac{C\|\phi\|_\infty}{|x \cdot n + s|} \quad \text{for } x \cdot n + s < 0. \tag{5.3.12}$$

Observe that (5.3.2) is equivalent to $U(\theta, t) = u^{-\theta \cdot n}(\theta - tn)$ for any $(\theta, t) \in \mathbb{T}^d \times \mathbb{R}_+$. It follows that

$$\begin{cases} N^T \nabla_\theta U(\theta, t) = N^T \nabla_x u^{-\theta \cdot n}(\theta - tn), \\ \quad \partial_t U(\theta, t) = -n \cdot \nabla_x u^{-\theta \cdot n}(\theta - tn). \end{cases} \tag{5.3.13}$$

In view of (5.3.12) and (5.3.13) we obtain

$$|N^T \nabla_\theta U(\theta, t)| + |\partial_t U(\theta, t)| \le \frac{C\|\phi\|_{L^\infty}}{t}. \tag{5.3.14}$$

This gives (5.3.10) for $t \ge 1/2$. The case $t \in [0, 1/2]$ follows from (5.3.4) and the Sobolev embedding theorem in $\mathbb{T}^d \times [0,1]$, which requires $k > d/2 + 1$.

Finally, the estimate (5.3.11) follows from (5.3.10), (5.3.4) and an interpolation argument, as in the proof of Lemma 5.7. □

Lemma 5.14. *Let $n \in \mathbb{S}_D^{d-1}$ with Diophantine constant $\kappa > 0$ and U be a solution of (5.3.3) corresponding to n. Then there exists a constant vector U_∞ such that for any $0 < \sigma < 1$ and $|\alpha| \ge 0$*

$$|\partial_\theta^\alpha (U - U_\infty)| \le C_\sigma \kappa^{-\sigma} \|f\|_{C^k(\mathbb{T}^d)}. \tag{5.3.15}$$

where $k = k(\alpha, \sigma, d)$ and C_σ depends only on d, m, α, σ, and A.

Proof. We first observe that it suffices to show $|U - U_\infty| \le C_\sigma \kappa^{-\sigma} \|f\|_{C^k(\mathbb{T}^d)}$ for any $0 < \sigma < 1$. Then the case $|\alpha| > 0$ follows from this and (5.3.5) by an interpolation argument.

Note that $|U - U_\infty| \to 0$ as $t \to \infty$. It follows from (5.3.5) and (5.3.10) that

$$|\partial_t U(\theta, t)| \le C \frac{\|f\|^{1-\sigma}_{C^k(\mathbb{T}^d)}}{(1+t)^{1-\sigma}} \cdot \frac{\|f\|^{\sigma}_{C^k(\mathbb{T}^d)}}{(1+\kappa t)^{\sigma \ell}}. \tag{5.3.16}$$

Hence,

$$\begin{aligned}
\sup_{t>0} |(U - U_\infty)(\theta, t)| &\le \int_0^\infty |\partial_t U(\theta, t)|\, dt \\
&\le C\|f\|_{C^k(\mathbb{T}^d)} \int_0^\infty \frac{dt}{(1+t)^{1-\sigma}(1+\kappa t)^{\sigma\ell}} \\
&\le C\kappa^{-\sigma} \|f\|_{C^k(\mathbb{T}^d)}.
\end{aligned}$$

This completes the proof. □

Proof of Theorem 5.12. **Step 1: Set-up and reduction.** Let $n_1 = (n_{1,1}, \cdots, n_{1,d})$, $n_2 = (n_{2,1}, \cdots, n_{2,d}) \in \mathbb{S}^{d-1}_D$ and $\delta = |n_1 - n_2| > 0$. Choose $d \times (d-1)$ matrices N_1, N_2 such that both $M_1 = (N_1, -n_1)$ and $M_2 = (N_2, -n_2)$ are orthogonal and $|N_1 - N_2| \le C\delta$. Let U_1, U_2 be solutions of the systems in the form of (5.3.7) associated with n_1, n_2, respectively, i.e.,

$$\begin{cases}
-\begin{pmatrix} N_1^T \nabla_\theta \\ \partial_t \end{pmatrix} \cdot B_1 \begin{pmatrix} N_1^T \nabla_\theta \\ \partial_t \end{pmatrix} U_1 = 0 & \text{in } \mathbb{T}^d \times (0, \infty), \\
-e_{d+1} \cdot B_1 \begin{pmatrix} N_1^T \nabla_\theta \\ \partial_t \end{pmatrix} U_1 = T_{1,ij} \cdot \nabla_\theta \phi_{ij} & \text{on } \mathbb{T}^d \times \{0\},
\end{cases} \tag{5.3.17}$$

and

$$\begin{cases}
-\begin{pmatrix} N_2^T \nabla_\theta \\ \partial_t \end{pmatrix} \cdot B_2 \begin{pmatrix} N_2^T \nabla_\theta \\ \partial_t \end{pmatrix} U_2 = 0 & \text{in } \mathbb{T}^d \times (0, \infty), \\
-e_{d+1} \cdot B_2 \begin{pmatrix} N_2^T \nabla_\theta \\ \partial_t \end{pmatrix} U_2 = T_{2,ij} \cdot \nabla_\theta \phi_{ij} & \text{on } \mathbb{T}^d \times \{0\},
\end{cases} \tag{5.3.18}$$

where $T_{\ell,ij} = n_{\ell,i} e_j - n_{\ell,j} e_i$ are vectors orthogonal to n_ℓ and $B_\ell(\theta, t) = M_\ell^T A^*(\theta - tn_\ell) M_\ell$ for $\ell = 1, 2$.

Without loss of generality, we may assume that $\kappa = \kappa(n_1) \ge \kappa(n_2)$. In view of the formula (5.3.6), we only need to show that

$$\int_{\mathbb{T}^d} |T_{1,ij} \cdot \nabla_\theta U_1(\theta, 0) - T_{2,ij} \cdot \nabla_\theta U_2(\theta, 0)|^2\, d\theta \le C_\sigma \kappa^{-2\sigma} |n_1 - n_2|^2 \tag{5.3.19}$$

for $1 \leq i, j \leq d$. By the triangle inequality,

$$
\begin{aligned}
&\int_{\mathbb{T}^d} |T_{1,ij} \cdot \nabla_\theta U_1(\theta,0) - T_{2,ij} \cdot \nabla_\theta U_2(\theta,0)|^2 \, d\theta \\
&\qquad \leq 2 \int_{\mathbb{T}^d} |(T_{1,ij} - T_{2,ij}) \cdot \nabla_\theta U_1(\theta,0)|^2 \, d\theta + 2 \int_{\mathbb{T}^d} |T_{2,ij} \cdot \nabla_\theta (U_1(\theta,0) - U_2(\theta,0))|^2 \, d\theta \\
&\qquad \leq C\kappa^{-2\sigma}\delta^2 + C \int_{\mathbb{T}^d} |N_2^T \nabla_\theta (U_1(\theta,0) - U_2(\theta,0))|^2 \, d\theta,
\end{aligned}
$$

where in the last inequality we have used (5.3.15) and the fact that the columns of N_2 span the subspace orthogonal to n_2. Furthermore, we let $W = U_1 - U_2$ and note that

$$
\begin{aligned}
&\int_{\mathbb{T}^d} |N_2^T \nabla_\theta W(\theta,0)|^2 \, d\theta \\
&\quad \leq 2 \int_0^1 \int_{\mathbb{T}^d} |N_2^T \nabla_\theta W(\theta,t)|^2 \, d\theta dt + 2 \int_0^1 \int_{\mathbb{T}^d} |N_2^T \nabla_\theta \partial_t W(\theta,t)|^2 \, d\theta dt.
\end{aligned} \tag{5.3.20}
$$

As a result, it suffices to estimate the two terms in the RHS of the above inequality.

Step 2: Estimate for $N_2^T \nabla_\theta W$.

The argument here is similar to that for Dirichlet problems, with Lemmas 5.9, 5.13 and 5.14 in our disposal. Note that W satisfies

$$
\begin{cases}
-\begin{pmatrix} N_2^T \nabla_\theta \\ \partial_t \end{pmatrix} \cdot B_2 \begin{pmatrix} N_2^T \nabla_\theta \\ \partial_t \end{pmatrix} W = \begin{pmatrix} N_2^T \nabla_\theta \\ \partial_t \end{pmatrix} \cdot G + H & \text{in } \mathbb{T}^d \times \mathbb{R}_+, \\
\qquad -e_{d+1} \cdot B_2 \begin{pmatrix} N_2^T \nabla_\theta \\ \partial_t \end{pmatrix} W = e_{d+1} \cdot G + (T_{1,ij} - T_{2,ij}) \cdot \nabla_\theta \phi_{ij} & \text{on } \mathbb{T}^d \times \{0\},
\end{cases} \tag{5.3.21}
$$

where $G = G_1 + G_2$ and H are exactly the same as in (5.2.30) for Dirichlet problems.

Now, we will make use of Lemma 3.29 and 3.31 in an essential way. First, we split W as $W = W_1 + W_2 + W_3$, where

$$
\begin{cases}
-\begin{pmatrix} N_2^T \nabla_\theta \\ \partial_t \end{pmatrix} \cdot B_2 \begin{pmatrix} N_2^T \nabla_\theta \\ \partial_t \end{pmatrix} W_1 = 0 & \text{in } \mathbb{T}^d \times \mathbb{R}_+, \\
\qquad -e_{d+1} \cdot B_2 \begin{pmatrix} N_2^T \nabla_\theta \\ \partial_t \end{pmatrix} W_1 = (T_{1,ij} - T_{2,ij}) \cdot \nabla_\theta \phi_{ij} & \text{on } \mathbb{T}^d \times \{0\},
\end{cases} \tag{5.3.22}
$$

$$
\begin{cases}
-\begin{pmatrix} N_2^T \nabla_\theta \\ \partial_t \end{pmatrix} \cdot B_2 \begin{pmatrix} N_2^T \nabla_\theta \\ \partial_t \end{pmatrix} W_2 = \begin{pmatrix} N_2^T \nabla_\theta \\ \partial_t \end{pmatrix} \cdot G & \text{in } \mathbb{T}^d \times \mathbb{R}_+, \\
\qquad -e_{d+1} \cdot B_2 \begin{pmatrix} N_2^T \nabla_\theta \\ \partial_t \end{pmatrix} W_2 = e_{d+1} \cdot G & \text{on } \mathbb{T}^d \times \{0\},
\end{cases} \tag{5.3.23}
$$

and

$$
\begin{cases}
-\begin{pmatrix} N_2^T \nabla_\theta \\ \partial_t \end{pmatrix} \cdot B_2 \begin{pmatrix} N_2^T \nabla_\theta \\ \partial_t \end{pmatrix} W_3 = H & \text{in } \mathbb{T}^d \times \mathbb{R}_+, \\
\qquad -e_{d+1} \cdot B_2 \begin{pmatrix} N_2^T \nabla_\theta \\ \partial_t \end{pmatrix} W_3 = 0 & \text{on } \mathbb{T}^d \times \{0\}.
\end{cases} \tag{5.3.24}
$$

Estimate for W_1. Since ϕ_{ij} is smooth, we can show that (5.3.22) is solvable and the solution W_1 satisfies (3.5.29). Thus, by Lemma 3.31,

$$\begin{aligned}\int_0^2 \int_{\mathbb{T}^d} \left(|N_2^T \nabla_\theta W_1|^2 + |\partial_t W_1|^2\right) d\theta dt &\le C \int_{\mathbb{T}^d} |T_{1,ij} - T_{2,ij}|^2 |\nabla_\theta \phi_{ij}|^2 \, d\theta dt \\ &\le C\delta^2.\end{aligned} \tag{5.3.25}$$

Estimate for W_2. By Lemma 3.29, we have

$$\begin{aligned}\int_0^\infty \int_{\mathbb{T}^d} \left(|N_2^T \nabla_\theta W_2|^2 + |\partial_t W_2|^2\right) t^{\sigma-1} \, d\theta dt &\le C \int_0^\infty \int_{\mathbb{T}^d} |G|^2 t^{\sigma-1} \, d\theta dt \\ &\le C \sum_{k=1,2} \int_0^\infty \int_{\mathbb{T}^d} |G_k|^2 t^{\sigma-1} \, d\theta dt.\end{aligned}$$

Using (5.3.15) and (5.3.5), we obtain

$$|\nabla_\theta U_1| \le C\kappa^{-\sigma(1-\sigma)}[\kappa^{-1}(1+\kappa t)^{-\ell}]^\sigma \le C\kappa^{-2\sigma}(1+\kappa t)^{-\sigma\ell}. \tag{5.3.26}$$

Hence,

$$\begin{aligned}\int_0^\infty \int_{\mathbb{T}^d} |G_1|^2 t^{\sigma-1} \, d\theta dt &\le C\kappa^{-4\sigma}\delta^2 \int_0^\infty (1+\kappa t)^{-2\sigma\ell} t^{\sigma-1} \, dt \\ &\le C\kappa^{-5\sigma}\delta^2.\end{aligned}$$

Similarly, by (5.3.10) and (5.3.5), we have

$$|N_1^T \nabla_\theta U_1| + |\partial_t U_1| \le C(1+t)^{1-\sigma}(1+\kappa t)^{-\sigma\ell}. \tag{5.3.27}$$

It follows that

$$\begin{aligned}\int_0^\infty \int_{\mathbb{T}^d} |G_2|^2 t^{\sigma-1} \, d\theta dt &\le C\delta^2 \int_0^\infty t^2(1+t)^{2\sigma-2}(1+\kappa t)^{-2\sigma\ell} t^{\sigma-1} \, dt \\ &\le C\kappa^{-3\sigma}\delta^2.\end{aligned}$$

As a result, we may conclude that

$$\int_0^\infty \int_{\mathbb{T}^d} \left(|N_2^T \nabla_\theta W_2|^2 + |\partial_t W_2|^2\right) t^{\sigma-1} \, d\theta dt \le C\kappa^{-5\sigma}\delta^2. \tag{5.3.28}$$

Estimate for W_3. The estimate for W_3 can be reduced to the first two cases. Let

$$\widetilde{H}(\theta, t) = -\int_t^\infty H(\theta, s) ds. \tag{5.3.29}$$

Note that $\widetilde{H}$ is bounded for all $(\theta, t) \in \mathbb{T}^d \times \mathbb{R}_+$. Write

$$H(\theta, t) = \partial_t \widetilde{H}(\theta, t) = \begin{pmatrix} N_2^T \nabla_\theta \\ \partial_t \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \widetilde{H}(\theta, t) \end{pmatrix}. \tag{5.3.30}$$

Then, we can further decompose W_3 into $W_3 = W_{31} + W_{32}$, where

$$\begin{cases} -\begin{pmatrix} N_2^T\nabla_\theta \\ \partial_t \end{pmatrix} \cdot B_2 \begin{pmatrix} N_2^T\nabla_\theta \\ \partial_t \end{pmatrix} W_{31} = \begin{pmatrix} N_2^T\nabla_\theta \\ \partial_t \end{pmatrix} \cdot \begin{pmatrix} 0 \\ \widetilde{H}(\theta,t) \end{pmatrix} & \text{in } \mathbb{T}^d \times \mathbb{R}_+, \\ -e_{d+1} \cdot B_2 \begin{pmatrix} N_2^T\nabla_\theta \\ \partial_t \end{pmatrix} W_{32} = e_{d+1} \cdot \begin{pmatrix} 0 \\ \widetilde{H}(\theta,t) \end{pmatrix} & \text{on } \mathbb{T}^d \times \{0\}, \end{cases} \tag{5.3.31}$$

and

$$\begin{cases} -\begin{pmatrix} N_2^T\nabla_\theta \\ \partial_t \end{pmatrix} \cdot B_2 \begin{pmatrix} N_2^T\nabla_\theta \\ \partial_t \end{pmatrix} W_{32} = 0 & \text{in } \mathbb{T}^d \times \mathbb{R}_+, \\ -e_{d+1} \cdot B_2 \begin{pmatrix} N_2^T\nabla_\theta \\ \partial_t \end{pmatrix} W_{32} = -e_{d+1} \cdot \begin{pmatrix} 0 \\ \widetilde{H}(\theta,t) \end{pmatrix} & \text{on } \mathbb{T}^d \times \{0\}. \end{cases} \tag{5.3.32}$$

Now by applying Lemma 3.29 for W_{31}, we obtain

$$\int_0^\infty \int_{\mathbb{T}^d} \left(|N_2^T\nabla_\theta W_{31}|^2 + |\partial_t W_{31}|^2\right) t^{\sigma-1}\, d\theta dt \le C \int_0^\infty \int_{\mathbb{T}^d} |\widetilde{H}|^2 t^{\sigma-1}\, d\theta dt. \tag{5.3.33}$$

It follows from Hardy's inequality (see [39, p.272]) that

$$\begin{aligned} \int_0^\infty \int_{\mathbb{T}^d} |\widetilde{H}|^2 t^{\sigma-1}\, d\theta dt &= \int_{\mathbb{T}^d} \int_0^\infty \left| \int_t^\infty H(\theta,s) ds \right|^2 t^{\sigma-1}\, dt d\theta \\ &\le \frac{4}{(1-\sigma)^2} \int_{\mathbb{T}^d} \int_0^\infty |H(\theta,t)|^2 t^{\sigma-1+2}\, dt d\theta. \end{aligned}$$

Consequently,

$$\int_0^\infty \int_{\mathbb{T}^d} \left(|N_2^T\nabla_\theta W_{31}|^2 + |\partial_t W_{31}|^2\right) t^{\sigma-1}\, d\theta dt \le C \int_0^\infty \int_{\mathbb{T}^d} |H|^2 t^{1+\sigma}\, d\theta dt.$$

For W_{32}, using Lemma 3.31 and Hölder's inequality, we have

$$\begin{aligned} &\int_0^2 \int_{\mathbb{T}^d} \left(|N_2^T\nabla_\theta W_{32}|^2 + |\partial_t W_{32}|^2\right) d\theta dt \\ &\quad \le C \int_{\mathbb{T}^d} |\widetilde{H}(\theta,0)|^2\, d\theta \\ &\quad \le C \int_{\mathbb{T}^d} \left| \int_0^\infty |H(\theta,t)| dt \right|^2 d\theta \\ &\quad \le C \int_{\mathbb{T}^d} \int_0^\infty |H(\theta,t)|^2 (1+t)^{2-\alpha} dt \int_0^\infty (1+t)^{\alpha-2}\, dt d\theta \\ &\quad \le C \int_0^\infty \int_{\mathbb{T}^d} (1+t)^2 |H(\theta,t)|^2 t^{-\alpha}\, d\theta dt. \end{aligned}$$

Therefore,

$$
\begin{aligned}
\int_0^2 \int_{\mathbb{T}^d} & \left(|N_2^T \nabla_\theta W_3|^2 + |\partial_t W_3|^2\right) d\theta dt \\
& \le C \int_0^\infty \int_{\mathbb{T}^d} (1+t)^2 |H|^2 t^{\sigma-1}\, d\theta dt \\
& \le C\delta^2 \int_0^\infty (1+t)^{2-2(\sigma-1)^2} (1+\kappa t)^{-2\sigma\ell} t^{\sigma-1}\, dt \\
& \le C\kappa^{-5\sigma}\delta^2,
\end{aligned}
$$

where in the last inequality we have chosen $\ell \ge 2$.

Summing up the estimates for W_k, we arrive at

$$
\int_0^2 \int_{\mathbb{T}^d} \left(|N_2^T \nabla_\theta W|^2 + |\partial_t W|^2\right) d\theta dt \le C\kappa^{-5\sigma}\delta^2, \tag{5.3.34}
$$

which proves the first part of (5.3.20), as $\sigma \in (0,1)$ can be arbitrarily small.

Step 3: Estimate for $N_2^T \nabla_\theta \partial_t W$.

The argument is similar to Step 3 in the proof of Theorem 5.6. Let N_{2k} denote the kth column of N_2, and define the kth component of $N_2^T \nabla_\theta$ by $\nabla_{2k} = N_{2k}^T \cdot \nabla_\theta$. We apply ∇_{2k} to (5.3.21) and obtain

$$
\begin{cases}
-\begin{pmatrix} N_2^T \nabla_\theta \\ \partial_t \end{pmatrix} \cdot B_2 \begin{pmatrix} N_2^T \nabla_\theta \\ \partial_t \end{pmatrix} \nabla_{2k} W = \begin{pmatrix} N_2^T \nabla_\theta \\ \partial_t \end{pmatrix} \cdot \nabla_{2k} G + \nabla_{2k} H \\
\qquad\qquad + \begin{pmatrix} N_2^T \nabla_\theta \\ \partial_t \end{pmatrix} \cdot \nabla_{2k} B_2 \begin{pmatrix} N_2^T \nabla_\theta \\ \partial_t \end{pmatrix} W \\
\qquad\qquad \text{in } \mathbb{T}^d \times \mathbb{R}_+, \\
-e_{d+1} \cdot B_2 \begin{pmatrix} N_2^T \nabla_\theta \\ \partial_t \end{pmatrix} \nabla_{2k} W = e_{d+1} \cdot \nabla_{2k} G + \nabla_{2k} h \\
\qquad\qquad + e_{d+1} \cdot \nabla_{2k} B_2 \begin{pmatrix} N_2^T \nabla_\theta \\ \partial_t \end{pmatrix} W \\
\qquad\qquad \text{on } \mathbb{T}^d \times \{0\},
\end{cases} \tag{5.3.35}
$$

where $h = (T_{1,ij} - T_{2,ij}) \cdot \nabla_\theta f_{ij}$. Let $\eta(t)$ be a cut-off function such that $\eta(t) = 1$ for $t \in [0,1]$, $\eta(t) = 0$ for $t \in [2,\infty)$, $0 \le \eta(t) \le 1$ and $|\nabla \eta| \le C$. Now by integrating (5.3.35) against $\nabla_{2k}(W\eta^2)$, we derive from integration by parts that

$$
\begin{aligned}
\int_0^1 \int_{\mathbb{T}^d} & \left(|N_2^T \nabla_\theta \nabla_{2k} W|^2 + |\partial_t \nabla_{2k} W|^2\right) d\theta dt \\
& \le C \int_0^2 \int_{\mathbb{T}^d} \left(|\nabla_{2k} G|^2 + |\nabla_{2k} H|^2 + |N_2^T \nabla_\theta W|^2 + |\partial_t W|^2\right) d\theta dt + C\|h\|^2_{H^1(\mathbb{T}^d)} \\
& \le C\kappa^{-5\sigma}\delta^2.
\end{aligned}
$$

Consequently,

$$\int_0^1 \int_{\mathbb{T}^d} \left(|N_2^T \nabla_\theta \otimes N_2^T \nabla_\theta W|^2 + |\partial_t N_2^T \nabla_\theta W|^2 \right) d\theta dt \le C\kappa^{-5\sigma}\delta^2, \tag{5.3.36}$$

which finishes the proof. □

Proof of Theorem 1.1: Regularity estimate. With Theorem 5.12 at our disposal, the proof of (1.2.10) is identical to that of Theorem 1.2.15. □

Bibliography

[1] H. Aleksanyan. Regularity of boundary data in periodic homogenization of elliptic systems in layered media. *Manuscripta Math.*, pages 1–32, 2016.

[2] H. Aleksanyan. Slow convergence in periodic homogenization problems for divergence-type elliptic operators. *SIAM J. Math. Anal.*, 48(5):3345–3382, 2016.

[3] H. Aleksanyan, H. Shahgholian, and P. Sjölin. Applications of Fourier analysis in homogenization of Dirichlet problem I. Pointwise estimates. *J. Differential Equations*, 254(6):2626–2637, 2013.

[4] H. Aleksanyan, H. Shahgholian, and P. Sjölin. Applications of Fourier analysis in homogenization of Dirichlet problem III: Polygonal domains. *J. Fourier Anal. Appl.*, 20(3):524–546, 2014.

[5] H. Aleksanyan, H. Shahgholian, and P. Sjölin. Applications of Fourier analysis in homogenization of the Dirichlet problem: L^p estimates. *Arch. Ration. Mech. Anal.*, 215(1):65–87, 2015.

[6] G. Allaire and M. Amar. Boundary layer tails in periodic homogenization. *ESAIM Control Optim. Calc. Var.*, 4:209–243 (electronic), 1999.

[7] S. Armstrong, T. Kuusi, J.-C. Mourrat, and C. Prange. Quantitative analysis of boundary layers in periodic homogenization. *Arch. Ration. Mech. Anal.*, 226(2):695–741, 2017.

[8] S. Armstrong and Z. Shen. Lipschitz estimates in almost-periodic homogenization. *Comm. Pure Appl. Math.*, 10(10):1882–1923, 2016.

[9] M. Avellaneda and F. Lin. Compactness methods in the theory of homogenization. *Comm. Pure Appl. Math.*, 40:803–847, 1987.

[10] M. Avellaneda and F. Lin. L^p bounds on singular integrals in homogenization. *Comm. Pure Appl. Math.*, 44:897–910, 1991.

[11] A. Bensoussan, J.-L. Lions, and G. Papanicolaou. *Asymptotic analysis for periodic structures*, volume 5 of *Studies in Mathematics and its Applications*. North-Holland Publishing Co., Amsterdam-New York, 1978.

[12] L.. Caffarelli and I. Peral. On $W^{1,p}$ estimates for elliptic equations in divergence form. *Comm. Pure Appl. Math.*, 51(1):1–21, 1998.

[13] S. Choi and I. Kim. Homogenization for nonlinear PDEs in general domains with oscillatory Neumann boundary data. *J. Math. Pures Appl. (9)*, 102(2):419–448, 2014.

[14] C. Fefferman. On electrons and nuclei in a magnetic field. *Adv. Math.*, 124(1):100–153, 1996.

[15] W. Feldman. Homogenization of the oscillating Dirichlet boundary condition in general domains. *J. Math. Pures Appl. (9)*, 101(5):599–622, 2014.

[16] W. Feldman and I. Kim. Continuity and discontinuity of the boundary layer tail. *Ann. Sci. Éc. Norm. Supér. (4)*, 50(4):1017–1064, 2017.

[17] W. Feldman, I. Kim, and P. Souganidis. Quantitative homogenization of elliptic partial differential equations with random oscillatory boundary data. *J. Math. Pures Appl. (9)*, 103(4):958–1002, 2015.

[18] W. Feldman and Y. Zhang. Continuity properties for divergence form boundary data homogenization problems. *To appear in Anal. PDE.*

[19] J. Geng. $W^{1,p}$ estimates for elliptic problems with Neumann boundary conditions in Lipschitz domains. *Adv. Math.*, 229(4):2427–2448, 2012.

[20] D. Gérard-Varet and N. Masmoudi. Homogenization in polygonal domains. *J. Eur. Math. Soc. (JEMS)*, 13(5):1477–1503, 2011.

[21] D. Gérard-Varet and N. Masmoudi. Homogenization and boundary layers. *Acta Math.*, 209(1):133–178, 2012.

[22] V. Jikov, S. Kozlov, and O. Oleinik. *Homogenization of Differential Operators and Integral Functionals.* Springer-Verlag, Berlin, 1994.

[23] C. Kenig, F. Lin, and Z. Shen. Convergence rates in L^2 for elliptic homogenization problems. *Arch. Ration. Mech. Anal.*, 203(3):1009–1036, 2012.

[24] C. Kenig, F. Lin, and Z. Shen. Homogenization of elliptic systems with Neumann boundary conditions. *J. Amer. Math. Soc.*, 26(4):901–937, 2013.

[25] C. Kenig, F. Lin, and Z. Shen. Periodic homogenization of Green and Neumann functions. *Comm. Pure Appl. Math.*, 67(8):1219–1262, 2014.

[26] C. Kenig and Z. Shen. Layer potential methods for elliptic homogenization problems. *Comm. Pure Appl. Math.*, 64:1–44, 2011.

[27] S. Moskow and M. Vogelius. First-order corrections to the homogenized eigenvalues of a periodic composite medium. A convergence proof. *Proc. Roy. Soc. Edinburgh Sect. A*, 127:1263–1299, 1997.

[28] S. Moskow and M. Vogelius. First order corrections to the homogenized eigenvalues of a periodic composite medium. The case of Neumann boundary conditions. *Preprint, Rutgers University*, 1997.

[29] C. Prange. Asymptotic analysis of boundary layer correctors in periodic homogenization. *SIAM J. Math. Anal.*, 45(1):345–387, 2013.

[30] C. Prange. First-order expansion for the Dirichlet eigenvalues of an elliptic system with oscillating coefficients. *Asymptot. Anal.*, 83(3):207–235, 2013.

[31] F. Santosa and M. Vogelius. First-order corrections to the homogenized eigenvalues of a periodic composite medium. *SIAM J. Appl. Math.*, 53:1636–1668, 1993.

[32] F. Santosa and M. Vogelius. Erratum to the paper: First-order corrections to the homogenized eigenvalues of a periodic composite medium (SIAM J. Appl. Math. 53 (1993), 1636-1668). *SIAM J. Appl. Math.*, 55:864, 1995.

[33] Z. Shen. On moments of negative eigenvalues for the Pauli operator. *J. Differential Equations*, 149(2):292–327, 1998.

[34] Z. Shen. Bounds of Riesz transforms on L^p spaces for second order elliptic operators. *Ann. Inst. Fourier (Grenoble)*, 55(1):173–197, 2005.

[35] Z. Shen. Boundary estimates in elliptic homogenization. *Anal. PDE*, 10(3):653–694, 2017.

[36] Z. Shen. *Periodic homogenization of elliptic systems*, volume 269 of *Operator Theory: Advances and Applications*. Birkhäuser/Springer, Cham, 2018. Advances in Partial Differential Equations (Basel).

[37] Z. Shen and J. Zhuge. Regularity of homogenized boundary data in periodic homogenization of elliptic systems. *To appear in J. Eur. Math. Soc. (JEMS).*

[38] Z. Shen and J. Zhuge. Boundary layers in periodic homogenization of Neumann problems. *Comm. Pure Appl. Math.*, 71(11):2163–2219, 2018.

[39] E. Stein. *Singular Integrals and Differentiability Properties of Functions*. Princeton University Press, Princeton, 1970.

[40] E. Stein. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, volume 43 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1993. With the assistance of Timothy S. Murphy, Monographs in Harmonic Analysis, III.

[41] J. Zhuge. First-order expansions for eigenvalues and eigenfunctions in periodic homogenization. *To appear in Proc. Roy. Soc. Edinburgh Sect. A.*

[42] J. Zhuge. Homogenization and boundary layers in domains of finite type. *Comm. Partial Differential Equations*, 43(4):549–584, 2018.

Vita

Jinping Zhuge

Education

- Ph.D. in Mathematics, University of Kentucky, Lexington, KY. June 2019 (expected).
- M.A. in Mathematics, Nankai University, Tianjin, China. June 2013.
- B.S. in Mathematics, Central South University, Changsha, China. June 2010.

Selected awards and honors

- 2017 - 2018 Mathematics Department Fellowship, University of Kentucky
- 2016 Summer Research Assistantship, University of Kentucky
- 2014 - 2015 Max Steckler Fellowship, University of Kentucky
- 2010 - 2013 Academic Scholarships for Master, Nankai University
- 2010 Outstanding Graduates, Central South University

Publications and preprints

My papers are accessible on my personal web page: http://www.ms.uky.edu/∼jzh257/

- Oscillatory integrals and periodic homogenization of Robin boundary value problems (with Jun Geng). arXiv:1902.10332, (2019).
- Periodic homogenization of Green's functions for Stokes systems (with Shu Gu). Calc. Var. Partial Differential Equations (to appear), arXiv: 1710.05383.
- First-order expansions for eigenvalues and eigenfunctions in periodic homogenization. Proc. Roy. Soc. Edinburgh Sect. A. (to appear), arXiv:1804.10739, (2018).
- Regularity of homogenized boundary data in periodic homogenization of elliptic systems (with Zhongwei Shen). J. Eur. Math. Soc. (JEMS) (to appear), arXiv:1707.03160v1.
- Homogenization and boundary layers in domains of finite type. Comm. Partial Differential Equations, 43 (2018), no. 4, 549-584.
- Boundary layers in periodic homogenization of Neumann problems (with Zhongwei Shen). Comm. Pure Appl. Math., 71 (2018), no. 11, 2163-2219.

- Approximate correctors and convergence rates in almost-periodic homogenization (with Zhongwei Shen). J. Math. Pures Appl., 110 (2018), 187-238.

- Uniform boundary regularity in almost-periodic homogenization. J. Differential Equations, 262 (2017), no. 1, 418-453.

- Convergence rates in periodic homogenization of systems of elasticity (with Zhongwei Shen). Proc. Amer. Math. Soc., 145 (2017), no. 3, 1187-1202.

- Green matrices and continuity of the weak solutions for the elliptic systems with lower order terms (with Zhenqiu Zhang). Internat. J. Math., 27 (2016), no. 2, 1650010, 34 pp.