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Polytopes Associated to Graph Laplacians

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Polytopes Associated to Graph Laplacians

DISSERTATION

A dissertation submitted in partial
fulfillment of the requirements for the
degree of Doctor of Philosophy in the
College of Arts and Sciences at the
University of Kentucky

By
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ABSTRACT OF DISSERTATION

Polytopes Associated to Graph Laplacians

Graphs provide interesting ways to generate families of lattice polytopes. In particular, one can use matrices encoding the information of a finite graph to define vertices of a polytope. This dissertation initiates the study of the *Laplacian simplex*, P_G, obtained from a finite graph G by taking the convex hull of the columns of the Laplacian matrix for G. The Laplacian simplex is extended through the use of a parallel construction with a finite digraph D to obtain the *Laplacian polytope*, P_D.

Basic properties of both families of simplices, P_G and P_D, are established using techniques from Ehrhart theory. Motivated by a well-known conjecture in the field, our investigation focuses on reflexivity, the integer decomposition property, and unimodality of Ehrhart h^*-vectors of these polytopes. A systematic investigation of P_G for trees, cycles, and complete graphs is provided, which is enhanced by an investigation of P_D for cyclic digraphs. We form intriguing connections with other families of simplices and produce G and D such that the h^*-vectors of P_G and P_D exhibit extremal behavior.

KEYWORDS: lattice polytope, Ehrhart theory, Laplacian matrix, h^*-vector, unimodal sequence, digraph

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Polytopes Associated to Graph Laplacians

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Chapter 1 Introduction

1.1 Lattice polytopes

The integer points $\mathbb{Z}^d$ form a lattice in $\mathbb{R}^d$. We call the integer points *lattice points*.

Definition 1.1.1. A *lattice polytope*, denoted $\mathcal{P}$, of dimension d is the convex hull of finitely many points in $\mathbb{Z}^n$, called *vertices* of $\mathcal{P}$, which together affinely span a d-dimensional hyperplane of $\mathbb{R}^n$. If $\mathbf{v_1}, \ldots, \mathbf{v_m} \in \mathbb{Z}^n$ are the vertices of $\mathcal{P}$, then

$$\mathcal{P} = \left\{ \sum_{i=1}^{m} \lambda_i \mathbf{v_i} \mid \text{ each } \lambda_i \geq 0 \text{ and } \sum_{i=1}^{m} \lambda_i = 1 \right\} \subseteq \mathbb{R}^n.$$

is the *vertex description* of $\mathcal{P}$, and we write $\mathcal{P} = \operatorname{conv}(\mathbf{v_1}, \ldots, \mathbf{v_m})$.

Equivalently, a lattice polytope can be defined as the finite intersection of half spaces. This is called the *hyperplane description* of a polytope. Each polytope has both a vertex and hyperplane description; however, it is nontrivial to algorithmically interchange descriptions. A proof of this fact can be found in Appendix A [6]. Generally we will use the vertex description described in Definition 1.1.1. The minimum number of vertices a d-dimensional polytope can have is $d+1$. These polytopes are called *simplices*.

Definition 1.1.2. The convex hull of $d+1$ affinely independent vertices is called a *d-simplex*. For example, every 1-dimensional polytope, that is, every line segment, is a 1-simplex. Also, the 2-dimensional simplices are triangles, and the 3-dimensional simplices are tetrahedra.

We do not want to distinguish between certain transformations of a polytope, such as translations or reflections; therefore, we consider a lattice polytope up to the following equivalence. Two lattice polytopes $P \subseteq \mathbb{R}^n$ and $P' \subseteq \mathbb{R}^{n'}$ are said to be *unimodularly equivalent* if there exists an affine map from the affine span of P to the affine span of P' that maps $\mathbb{Z}^n \cap \operatorname{aff}(P)$ bijectively onto $\mathbb{Z}^{n'} \cap \operatorname{aff}(P')$ and maps P to P'. Consequently we consider lattice polytopes up to affine automorphisms of the lattice. The lattice polytopes in a given equivalence class are unimodular transformations of a representative lattice polytope. Sometimes it is convenient to work with full-dimensional lattice polytopes, i.e. lattice polytopes embedded in a space of their same dimension. Affine maps make this possible. One invariant of lattice equivalent polytopes is volume. It is convenient for combinatorists to use normalized volume when measuring to polytopes instead of the usual Euclidean volume. The relation between these two notions is simple.

Definition 1.1.3. If a polytope $\mathcal{P}$ is d-dimensional, its *normalized volume* $\operatorname{Vol}(\mathcal{P})$ is defined to be $d!$ times the relative Euclidean volume of $\mathcal{P}$.

As with other geometric objects, there is a notion of polar duality among polytopes.

Definition 1.1.4. The *(polar) dual polytope* of a full dimensional polytope $\mathcal{P} \subseteq \mathbb{R}^n$ which contains the origin in its interior is

$$\mathcal{P}^* := \{\mathbf{x} \in \mathbb{R}^n \mid \mathbf{x} \cdot \mathbf{y} \leq 1 \text{ for all } \mathbf{y} \in \mathcal{P}\}.$$

in particular, $(\mathcal{P}^*)^* = \mathcal{P}$.

The concept of polar duality extends beyond polytopes to any subset of $\mathbb{R}^n$. Dual polytopes allow us to define a paramount property a polytope can possess. Reflexive polytopes are a particularly important class of polytopes first introduced in [4]. There are equivalent ways to define reflexive polytopes. First we provide a geometric interpretation using the polar dual.

Definition 1.1.5. A lattice polytope $\mathcal{P}$ is called *reflexive* if it contains the origin in its interior, and its dual $\mathcal{P}^*$ is a lattice polytope.

We extend the definition of reflexive to all the lattice polytopes which are unimodularly equivalent to $\mathcal{P}$. As an example, any lattice translate of a reflexive polytope is also called reflexive. An alternative definition for reflexivity relies on the hyperplane description of a polytope. In practice, we use the more convenient interpretation to prove a polytope is reflexive.

Definition 1.1.6. A lattice polytope $\mathcal{P}$ is called *reflexive* if it has the hyperplane description

$$\mathcal{P} = \{\mathbf{x} \in \mathbb{R}^n \mid H\mathbf{x} \leq \mathbb{1}\},$$

where H is an integral matrix.

In [4], Victor Batyrev created the term *reflexive polytope* when he found applications to mirror symmetry in physical string theory. Observe both $\mathcal{P}$ and $\mathcal{P}^*$ are reflexive polytopes if $\mathcal{P}$ satisfies Definition 1.1.5. Thus they always appear as dual pairs. In particular, reflexive polytopes describe mirror families of Calabi-Yau manifolds and can be used to compute invariants of these Calabi-Yau varieties.

There are many open questions about basic properties of reflexive polytopes; however, there are still interesting known results. Haase and Melnikov provided an algorithm to show that any lattice polytope is isomorphic to a face of some reflexive polytope [17]. The *reflexive dimension* of a lattice polytope is the smallest d such that the polytope is lattice equivalent to the face of a reflexive d-polytope. They provided bounds on the reflexive dimension of certain polytopes.

From Definition 1.1.6, we observe that the origin is the only lattice point in the interior of a reflexive polytope. A polytope with exactly one interior lattice point is called *Fano*. Lagarias and Ziegler proved that for a fixed n, there are finitely many n-dimensional lattice Fano polytopes [26]. Thus we conclude there are a finite number of reflexive polytopes for a given dimension. In fact, Kreuzer and Skarke used a computer program to classify all reflexive polytopes up to dimension four [25]. Their results are displayed in Figure 1.1. In dimensions 1 and 2, observe the Fano lattice polytopes are exactly the reflexive polytopes of 1 and 2 dimensions. This equality breaks in dimensions 3 and beyond as not all Fano polytopes are reflexive.

Reflexive Polytopes	
Dimension	Number of equivalence classes
1	1
2	16
3	4,319
4	473,800,776
5	???

Figure 1.1: Number of Reflexive Polytopes

A generalization of reflexive polytopes was introduced in [23]. To understand the generalization, we need the following notions. A lattice point is *primitive* if the line segment joining it and the origin contains no other lattice points. The *local index* ℓ_F is equal to the integral distance from the origin to the affine hyperplane spanned by F.

Definition 1.1.7. A lattice polytope $\mathcal{P}$ is *ℓ-reflexive* if, for some $\ell \in \mathbb{Z}_{>0}$, the following conditions hold:

(i) $\mathcal{P}$ contains the origin in its (strict) interior;

(ii) The vertices of $\mathcal{P}$ are primitive;

(iii) For any facet F of $\mathcal{P}$ the local index $\ell_F = \ell$.

We refer to $\mathcal{P}$ as a *reflexive polytope of index ℓ*. The reflexive polytopes of index 1 are precisely the reflexive polytopes in Definitions 1.1.5, 1.1.6. See Figure 1.2 for an illustration of a 2-reflexive polytope compared to a reflexive polytope.

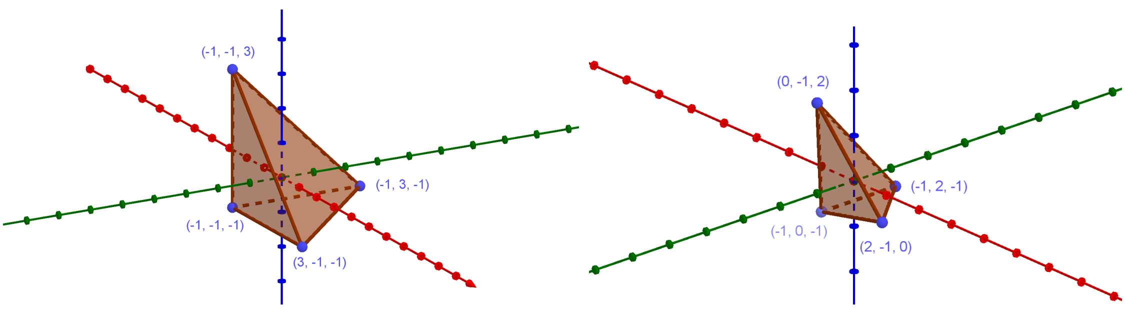


Figure 1.2: The simplex on the left, P_{K_4}, is reflexive while the simplex on the right, P_{C_4}, is 2-reflexive. These simplices are defined in 2.1.4 and specifically addressed in Sections 2.5 and 2.4, respectively.

1.2 Ehrhart theory

Ehrhart theory was developed to study discrete properties of polytopes including the lattice point enumeration of a polytope and its dilates. For $t \in \mathbb{Z}_{>0}$, the t^{th} *dilate* of P is given by $t\mathcal{P} := \{t\mathbf{p} \mid \mathbf{p} \in \mathcal{P}\}$. One technique used to recover dilates of polytopes is *coning over the polytope.*

Definition 1.2.1. Given $\mathcal{P} = \operatorname{conv}(\mathbf{v_1}, \ldots, \mathbf{v_m}) \subseteq \mathbb{R}^n$, we lift these vertices into $\mathbb{R}^{n+1}$ by appending 1 as the last coordinate to define $\mathbf{w_1} = (\mathbf{v_1}, 1), \ldots, \mathbf{w_m} = (\mathbf{v_m}, 1)$. The *cone over* $\mathcal{P}$ is

$$\operatorname{cone}(\mathcal{P}) = \{\lambda_1 \mathbf{w_1} + \lambda_2 \mathbf{w_2} + \cdots + \lambda_m \mathbf{w_m} \mid \lambda_1, \lambda_2, \ldots, \lambda_m \geq 0\} \subseteq \mathbb{R}^{n+1}.$$

For each $t \in \mathbb{Z}_{>0}$ we recover $t\mathcal{P}$ by considering $\operatorname{cone}(\mathcal{P}) \cap \{z_{n+1} = t\}$. To record the number of lattice points in these dilates, we let $L_{\mathcal{P}}(t) = |t\mathcal{P} \cap \mathbb{Z}^n|$.

In a classical result [16], Ehrhart proves that the number of lattice points in integer dilations of a d-dimensional lattice polytope behaves polynomially. He shows $L_{\mathcal{P}}(t)$, called the *Ehrhart polynomial of* $\mathcal{P}$, is a polynomial in degree $d = \dim(\mathcal{P})$ with generating function known as the *Ehrhart series* of $\mathcal{P}$. The series can be written

$$\operatorname{Ehr}_{\mathcal{P}}(z) = 1 + \sum_{t \geq 1} L_{\mathcal{P}}(t) z^t = \frac{h_d^* z^d + h_{d-1}^* z^{d-1} + \cdots + h_1^* z + h_0^*}{(1-z)^{d+1}}$$

where $h^*(z) := h_d^* z^d + \cdots + h_1^* z + h_0^*$ is a polynomial of degree at most d with non-negative integer coefficients and $h_0^* = 1$. We call this polynomial the *h^*-polynomial* of $\mathcal{P}$. This is an important invariant as it preserves much information about $\mathcal{P}$. For example, the following relations are well known [35]:

$$h_1^* = |\mathcal{P} \cap \mathbb{Z}^d| - n - 1, \qquad h_d^* = |\mathcal{P}^\circ \cap \mathbb{Z}^d|, \qquad 1 + \sum_{i=1}^{d} h_i^* = \operatorname{Vol}(\mathcal{P}),$$

where $\mathcal{P}^\circ$ denotes the relative interior of $\mathcal{P}$. Observe the third relation encodes the normalized volume of a polytope as the sum of the h^*-coefficients. The Euclidean volume can be recovered as $\operatorname{vol}(\mathcal{P}) = \frac{1}{d!} \sum_{i=0}^{d} h_i^*$. Note that if $\mathcal{P}$ and $\mathcal{Q}$ are lattice polytopes such that $\mathcal{Q}$ is the image of $\mathcal{P}$ under an affine unimodular transformation, then their Ehrhart series are equal.

Ehrhart theory is related to the study of a graded k-algebra associated with an integral convex polytope in commutative algebra. Indeed, the Hilbert and Ehrhart series are equal for the semi-group formed by the cone over a lattice polytope. This result (Corollary 33.5 [19]), among other interesting connections, is found in [19].

The h^*-polynomial of P is often identified with the vector of its coefficients $h^*(\mathcal{P}) = (h_0^*, h_1^*, \ldots, h_d^*)$, called the *$h^*$-vector* of $\mathcal{P}$. In the existing literature, *δ-polynomial* and *δ-vector* are sometimes used to denote the h^*-polynomial and h^*-vector. For the case of symmetric h^*-vectors, Hibi established the following connection to reflexive polytopes.

Theorem 1.2.2 (Hibi [20]). A d-dimensional lattice polytope $\mathcal{P} \subseteq \mathbb{R}^d$ containing the origin in its interior is reflexive if and only if $h^*(\mathcal{P})$ satisfies $h_i^* = h_{d-i}^*$ for $0 \leq i \leq \lfloor \frac{d}{2} \rfloor$.

Thus, when investigating symmetric h^*-vectors, reflexive polytopes form the correct class to work with. Notice the above classification provides a third way to identify reflexive polytopes. The following property is related to the h^*-vector.

Definition 1.2.3. A lattice polytope $\mathcal{P} \subseteq \mathbb{R}^d$ has the *integer decomposition property* if, for every integer $t \in \mathbb{Z}_{>0}$ and for all $\mathbf{p} \in t\mathcal{P} \cap \mathbb{Z}^d$, there exists $\mathbf{p_1}, \ldots, \mathbf{p_t} \in \mathcal{P} \cap \mathbb{Z}$ such that $\mathbf{p} = \mathbf{p_1} + \cdots + \mathbf{p_t}$. We frequently say $\mathcal{P}$ is IDP when $\mathcal{P}$ possesses this property.

It is well-known that if $\mathcal{P}$ admits a unimodular triangulation, then $\mathcal{P}$ is IDP; we will use this fact when analyzing complete graphs, see Corollary 2.5.5.

A vector $\mathbf{x} = (x_0, x_1, \ldots, x_d)$ is *unimodal* if there exists a $j \in [d]$ such that $x_i \leq x_{i+1}$ for all $0 \leq i < j$ and $x_k \geq x_{k+1}$ for all $j \leq k < d$. An important open problem in Ehrhart theory is to determine properties of $\mathcal{P}$ that imply unimodality of $h^*(\mathcal{P})$ [7].

The cause of unimodality for h^*-vectors in Ehrhart theory is mysterious. Many efforts have been made to find sufficient conditions for unimodality. It has been conjectured by Stanley [36] that a standard graded Cohen-Macaulay integral domain has a unimodal h-vector. In the context of lattice polytopes, Schepers and van Langenhoven [33] have raised the question of whether or not the integer decomposition property alone is sufficient to force unimodality of the h^*-vector for a lattice polytope. A weaker condition for unimodality is suggested in the following open conjecture.

Conjecture 1.2.4 (Hibi and Ohsugi [29])**.** If $\mathcal{P}$ is a lattice polytope that is reflexive and satisfies the integer decomposition property, then $\mathcal{P}$ has a unimodal Ehrhart h^*-vector.

In general, the interplay of the qualities of a lattice polytope being reflexive, satisfying the integer decomposition property, and having a unimodal h^*-vector is not well-understood. When new families of lattice polytopes are introduced, it is of interest to explore how these three properties behave for that family. Further, lattice simplices have been shown to be a rich source of examples and have been the subject of several recent investigations, especially in the context of Conjecture 1.2.4 [8, 10, 31, 34]. We will explore the interplay of these properties for Laplacian simplices in Chapter 2 as well as the extended Laplacian simplices in Chapter 3.

1.3 Edge polytopes and graph Laplacians

There are many fruitful ways to associate a polytope to a finite graph. These constructions require the use of matrices which encode all the information from a graph.

Definition 1.3.1. Let G be a finite simple graph. Then G has vertex set $V(G) = [n] := \{1, 2, \ldots, n\}$ and edge set $E(G)$ with no loops or multiple edges. The *degree* of vertex $i \in [n]$, denoted $\deg i$, is the number of edges incident to i in G.

(i) The *degree matrix*, denoted $D(G)$, is the $n \times n$ matrix with entries $a_{ii} = \deg i$ for each $i \in [n]$ and 0 otherwise.

(ii) The $\{0,1\}$-*adjacency matrix*, denoted $A(G)$, is the $n \times n$ matrix with entries $a_{ij} = 1$ if $\{i,j\} \in E(G)$ and 0 otherwise.

(iii) Assign a direction to each edge in G. The *signed vertex-edge incidence matrix*, denoted $N(G)$, is the $n \times |E(G)|$ matrix with columns c_i such that $(c_i)_j = -1$ and $(c_i)_k = 1$ where edge i is directed from vertex j to vertex k in G and 0 else.

(iv) The *Laplacian matrix*, denoted $L(G)$, is the difference $L(G) := D(G) - A(G)$. Equivalently, $L(G) := N(G) \cdot N(G)^T$. Consequently, L has rows and columns indexed by $[n]$ with entries $a_{ii} = \deg i$, $a_{ij} = -1$ if $\{i,j\} \in E(G)$, and 0 otherwise.

A *unimodular* matrix is a square integer matrix having determinant 1 or -1. The unimodular matrices of order n form a group, denoted $GL_n(\mathbb{Z})$. A *totally unimodular matrix* is a matrix for which every square non-singular submatrix is unimodular. Equivalently, every square submatrix of a totally unimodular matrix has determinant $0, 1$, or -1. The following Lemma gives an example of a family of totally unimodular matrices. Section 2.5 refers to this result.

Lemma 1.3.2. [3, Lemma 2.6] Let G be a graph with signed vertex-edge incidence matrix $N(G)$. Then $N(G)$ is totally unimodular.

The first well-known instance of polytopes arising from graphs is the *edge polytope* of G, obtained by taking the convex hull of the columns of the unsigned vertex-edge incidence matrix, $|N(G)|$. Many geometric, combinatorial, and algebraic properties of edge polytopes have been established over the past several decades. Some examples of such results are found in [27, 28, 38, 40].

We now lay the foundation for the study of the analogue of the edge polytope obtained by taking the convex hull of the columns of $L(G)$, see Definition 2.1.4.

Definition 1.3.3. A graph G is *connected* if there exists a path between every pair of vertices in G. A *spanning tree* of G is a connected subgraph of G which includes all the vertices of G with the minimum possible number of edges. The number of spanning tress of G is denoted $\kappa(G)$, or κ if G is understood.

Perhaps one of the most celebrated results in algebraic graph theory is *The Matrix-Tree Theorem*, also known as *Kirchhoff's Matrix-Tree Theorem.*

Theorem 1.3.4 (Kirchhoff [24])**.** For a connected graph G on n vertices, let $\lambda_1, \lambda_2, \ldots, \lambda_{n-1}$ be the non-zero eigenvalues of its Laplacian matrix. Then the number of spanning trees of G is

$$\kappa(G) = \frac{1}{n} \prod_{i=1}^{n-1} \lambda_i.$$

The following additional facts about the Laplacian matrix will be useful. Further interesting results are found in [3].

Proposition 1.3.5. The Laplacian matrix L of a connected graph G with vertex set $[n]$ satisfies the following:

(i) $L \in \mathbb{Z}^{n\times n}$ is symmetric.

(ii) Each row and column sum of L is 0.

(iii) $\mathrm{rk}\ L = \mathrm{rk}\ L(i) = n-1$ for each $i \in [n]$. Here $L(i)$ is the matrix L with the i^{th} column removed.

(iv) $\ker_{\mathbb{R}} L = \langle \mathbb{1} \rangle$ and $\mathrm{im}_{\mathbb{R}}\ L = \langle \mathbb{1} \rangle^{\perp}$.

(v) Any cofactor of L is equal to κ. (The Matrix-Tree Theorem [24]).

Proof. Items (i) and (ii) follow from the definition of L.

To show (iii), notice $\mathrm{rk}\ L \leq n-1$ since the rows of L are linearly dependent by item (ii). Recall $L = N(G) \cdot N(G)^T$, see Definition 1.3.1, which means $\mathrm{rk}\ L = \mathrm{rk}\ N(G)$. Suppose $\mathbf{x}$ is in the left nullspace of $N(G)$, i.e. $\mathbf{x}^T \cdot N(G) = \mathbf{0}$. Then $x_i - x_j = 0$ implies $x_i = x_j$ whenever vertex i and vertex j share an edge in G. Since G is connected, all coordinates of $\mathbf{x}$ must be equal. Thus $\mathrm{rk}\ N(G) \geq n-1$, which shows $\mathrm{rk}\ L = n-1$. To show $\mathrm{rk}\ L(i) = n-1$ for any $i \in [n]$, observe any choice of $n-1$ columns of L will form a basis of the column space of L because of the dependence $\mathbf{c_1} + \cdots + \mathbf{c_n} = \mathbf{0}$ where $\mathbf{c_i}$ is the i^{th} column of L.

For (iv), notice (ii) implies $\mathbb{1} \in \ker_{\mathbb{R}} L$ and by Rank-nullity theorem, the result follows.

Item (v) is a direct consequence of Theorem 1.3.4.

□

There has been recent research regarding graph Laplacians from the perspective of polyhedral combinatorics and integer-point enumeration. For example, M. Beck and the first author investigated hyperplane arrangements defined by graph Laplacians with connections to nowhere-harmonic colorings and inside-out polytopes [5]. A. Padrol and J. Pfeifle explored Laplacian Eigenpolytopes [30] with a focus on the effect of graph operations on the associated polytopes. B. Braun, R. Davis, J. Doering, A. Harrison, J. Noll, and C. Taylor studied integer-point enumeration for polyhedral cones constrained by graph Laplacian minors [9]. In a recent preprint [13], A. Dall and J. Pfeifle analyzed polyhedral decompositions of the zonotope defined as the Minkowski sum of the line segments from the origin to each column of $L(G)$ in order to give a polyhedral proof of the Matrix-Tree Theorem. The results in Chapters 2 and 3 extend this list.

Chapter 2 Laplacian Simplices Associated to Simple Graphs

This chapter initiates the study of Laplacian simplices and is based on joint work with Benjamin Braun. The results have been submitted for publication and can be found here [11].

In the entirety of this chapter, G is assumed to be a connected, simple graph with vertex set $[n]$ and Laplacian matrix L. We often refer to a submatrix of L defined by restricting to specified rows and columns. For $S, T \subseteq [n]$, define $L(S \mid T)$ to be the matrix with rows from L indexed by $[n] \setminus S$ and columns from L indexed by $[n] \setminus T$. Equivalently, $L(S \mid T)$ is obtained from L by the deletion of rows indexed by S and columns indexed by T. For simplicity, we define $L(i)$ to be the matrix obtained by deleting the i^{th} column of L, that is, $L(i) := L(\emptyset \mid i) \in \mathbb{Z}^{n\times(n-1)}$. We use the notation $[L \mid \mathbb{1}]$ to denote the matrix L with a column of all ones appended.

2.1 Laplacian simplices

We start by considering $L(i) \in \mathbb{Z}^{n\times(n-1)}$. We recognize the rows of $L(i)$ as integer points in $\mathbb{Z}^{n-1}$ and consider the convex hull of the rows, $\operatorname{conv}\left(L(i)^T\right)$. Using Proposition 1.3.5, notice the rows of $L(i)$ form a collection of n affinely independent lattice points, which makes $\operatorname{conv}\left(L(i)^T\right)$ an $n-1$ dimensional simplex. The next proposition shows that regardless of which column of L was deleted, the resulting lattice simplex is in the same equivalence class.

Proposition 2.1.1. The lattice simplices $\operatorname{conv}\left(L(i)^T\right)$ and $\operatorname{conv}\left(L(j)^T\right)$ are unimodularly equivalent for all $i, j \in [n]$.

Proof. Notice the matrices $L(i)$ and $L(j)$ differ by only one column when $i \neq j$. In particular, we can write $L(i) \cdot U = L(j)$ where $U \in \mathbb{Z}^{n-1\times n-1}$ has columns $\mathbf{c_k}$ for $1 \leq k \leq n-1$ defined to be

$$\mathbf{c_k} = \left\{ \begin{array}{ll} \mathbf{e}_\ell & \text{column } k \text{ in } L(j) \text{ is column } \ell \text{ in } L(i) \\ (-1,-1,\ldots,-1)^T & \text{column } k \text{ in } L(j) \text{ is not among columns of } L(i) \end{array} \right\}$$

where $\mathbf{e}_\ell$ is the standard basis vector with a 1 in the ℓ^{th} entry and 0 else.

Notice U has integer entries and $\det U = \pm 1$, as computed by expanding along the column with all entries equal to -1. This shows U is a unimodular matrix. Further, U maps the vertices of $\operatorname{conv}\left(L(i)^T\right)$ onto the vertices of $\operatorname{conv}\left(L(i)^T\right)$. Thus $\operatorname{conv}\left(L(i)^T\right)$ and $\operatorname{conv}\left(L(j)^T\right)$ are unimodularly equivalent lattice polytopes. □

Remark 2.1.2. The reason we delete a column from L before taking the convex hull of the rows is to ensure a full-dimensional simplex. Note $\operatorname{conv}\left(L^T\right)$ yields an $(n-1)$-simplex embedded in $\mathbb{R}^n$; however, the simplices $\operatorname{conv}\left(L^T\right)$ and $\operatorname{conv}\left(L(i)^T\right)$ are lattice equivalent. It is convenient to work with a simplex whose ambient space is of equal dimension.

Given a fixed graph G, we choose a representative for this equivalence class of lattice simplices to be used throughout this chapter, unless otherwise noted. Let $B = \{\mathbf{e_1} - \mathbf{e_2}, \mathbf{e_2} - \mathbf{e_3}, \ldots, \mathbf{e_{n-1}} - \mathbf{e_n}\}$ be the standard basis for the orthogonal complement of the vector $\mathbb{1} \in \mathbb{R}^n$, where $\mathbf{e_i} \in \mathbb{R}^n$ is the standard basis vector that contains a 1 in the i^{th} entry and 0 else. Then B is a basis of the column space of L. Define $L_B \in \mathbb{Z}^{n \times (n-1)}$ to be the representation of the matrix L with respect to the basis B. In practice, L_B can be computed using the matrix multiplication $L_B = L \cdot A$ where A is the upper triangular $n \times (n-1)$ matrix with entries

$$a_{ij} = \left\{ \begin{array}{ll} 1 & i \leq j \leq n-1 \\ 0 & \text{else} \end{array} \right\}. \tag{2.1}$$

Example 2.1.3. Given the cycle C_5 of length five, we have

$$L = \begin{bmatrix} 2 & -1 & 0 & 0 & -1 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & 0 & -1 & 2 \end{bmatrix} \qquad L_B = \begin{bmatrix} 2 & 1 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & -1 & -1 & -2 \end{bmatrix}.$$

This brings us to the object of study in this chapter.

Definition 2.1.4. The $n-1$ dimensional lattice simplex,

$$P_G := \operatorname{conv}\left((L_B)^T\right) \subseteq \mathbb{R}^{n-1}$$

is called the *Laplacian simplex* associated to the graph G.

Proposition 2.1.5. The Laplacian simplex satisfies the following properties.

(i) P_G is a representative of the equivalence class containing $\{\operatorname{conv}\left(L(i)^T\right)\}_{i \in [n]}$.

(ii) P_G has normalized volume equal to $n \cdot \kappa$.

(iii) P_G contains the origin in its strict interior.

Proof. (i) Notice we can write $L(n) \cdot A(n \mid \emptyset) = L_B$ where A is the matrix defined in equation (2.1). Let $U := A(n \mid \emptyset)$. Then U is the upper diagonal matrix of all ones so that $\det U = 1$. This implies P_G is unimodularly equivalent to $\operatorname{conv}\left(L(n)^T\right)$. By Proposition 2.1.1, the result follows.

(ii) Since P_G is a simplex, the normalized volume of P_G is equal to the determinant of the matrix $[L_B \mid \mathbb{1}]$, that is,

$$|\det [L_B \mid \mathbb{1}]| = \left| \sum_{i=1}^{n} (-1)^{i+n} M_{in} \right| = \left| \sum_{i=1}^{n} C_{in} \right|,$$

where $M_{i,n}$ is a minor of $[L_B \mid \mathbb{1}]$, $C_{i,n}$ is the corresponding cofactor, and the determinant is expanded along the appended column of ones. The relation

$L(n) \cdot U = L_B$ yields $L(i \mid n) \cdot U = L_B(i \mid \emptyset)$. Then for each cofactor,

$$\begin{aligned} C_{i,n} &= (-1)^{i+n} \det L_B(i \mid \emptyset) \\ &= (-1)^{i+n} \det(L(i \mid n) \cdot U) \\ &= (-1)^{i+n} \det L(i \mid n) \det U \\ &= (-1)^{i+n} \det L(i \mid n) \\ &= \bar{C}_{i,n} \\ &= \kappa \end{aligned}$$

where $\bar{C}_{i,n}$ is the cofactor of L, and the last equality is a result of the Matrix-Tree Theorem 1.3.4. Summing over all $i \in [n]$ yields the desired result.

(iii) Note the sum of all rows of L_B is $\mathbf{0}$, and L_B has no column with all entries equal to 0. It follows that $(0, \dots, 0) \in \mathbb{Z}^n$ is in the interior of P_G.

□

Example 2.1.6. The simplex P_{C_5} is obtained as the convex hull of the columns of the transpose of

$$L_B = \begin{bmatrix} 2 & 1 & 1 & 1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ -1 & -1 & -1 & -2 \end{bmatrix}.$$

The determinant of L_B with a column of ones appended is computed to be 25, which is indeed equal to $n \cdot \kappa$.

In the proof of (ii) in Proposition 2.1.5 above, we showed the minor obtained by deleting the i^{th} row of L_B is equal to the minor obtained by deleting the n^{th} column and the i^{th} row of L for some $i \in [n]$, i.e., $\det L_B(i \mid \emptyset) = \det L(i \mid n)$ for any $i \in [n]$. The second minors of L_B and L are related in the following manner, which we will need in subsequent sections.

Lemma 2.1.7. Let $i, k \in [n]$ and $j \in [n-1]$ such that $i \neq k$. Then

$$\det L_B(i, k \mid j) = \det L(i, k \mid j, n) + \det L(i, k \mid j+1, n).$$

In the case $j = n-1$, $\det L_B(i, k \mid n-1) = \det L(i, k \mid n-1, n)$.

Proof. Recall $L_B = L \cdot A$ where A is the $n \times (n-1)$ upper diagonal matrix defined in equation (2.1). It follows $L_B(i, k \mid j) = L(i, k \mid \emptyset) \cdot A(j)$. Apply the Cauchy-Binet formula to compute the determinant

$$\begin{aligned} \det L_B(i, k \mid j) &= \sum_{S \in \binom{[n]}{n-2}} \det L(i, k \mid \emptyset)_{[n-2], S} \det A(j)_{S, [n-2]} \\ &= \det L(i, k \mid \emptyset)_{[n-2], [n] \setminus \{j, n\}} \det A(j)_{[n] \setminus \{j, n\}, [n-2]} \\ &\qquad + \det L(i, k \mid \emptyset)_{[n-2], [n] \setminus \{(j+1), n\}} \det A(j)_{[n] \setminus \{(j+1), n\}, [n-2]} \\ &= \det L(i, k \mid j, n) + \det L(i, k \mid j+1, n). \end{aligned}$$

The only nonzero terms in the sum arise from choosing $(n-2)$ linearly independent rows in A. Based on the structure of A, there are only two ways to do this unless we are in the case $j = n-1$ in which there is exactly one way. $\square$

The following is a special case of a general characterization of reflexive simplices using cofactor expansions.

Theorem 2.1.8. The Laplacian simplex P_G is reflexive if and only if for each $i \in [n]$, κ divides

$$\sum_{k=1}^{n-1} C_{kj} = \sum_{k=1}^{n-1} (-1)^{k+j} M_{kj}$$

for each $1 \leq j \leq n-1$. Here C_{kj} is the cofactor and M_{kj} is the minor of the matrix $L_B(i \mid \emptyset) \in \mathbb{Z}^{(n-1)\times(n-1)}$.

Proof. We show P_G is reflexive by showing the vertices of its dual polytope are lattice points. By [41, Theorem 2.11], the hyperplane description of the dual polytope is given by $P_G^* = \{\mathbf{x} \in \mathbb{R}^{n-1} \mid L_B \cdot \mathbf{x} \leq \mathbb{1}\}$. Each intersection of $(n-1)$ hyperplanes will yield a unique vertex of P_G^* since any first minor of L_B is nonzero. Let $\{v_1, v_2, \ldots, v_n\}$ be the set of vertices of T_G^*. Each $\mathbf{v_i}$ satisfies

$$L_B(i \mid \emptyset)\mathbf{v_i} = \mathbb{1}$$

for $i \in [n]$. Reindex the rows of $L_B(i \mid \emptyset)$ in increasing order by $[n-1]$. We can write

$$\mathbf{v_i} = L_B(i \mid \emptyset)^{-1} \cdot \mathbb{1} = \frac{1}{\det L_B(i \mid \emptyset)} C^T \cdot \mathbb{1}$$

where C^T is the $(n-1)\times(n-1)$ matrix whose whose (j,k) entry is the (k,j) cofactor of $L_B(i \mid \emptyset)$, which we denote as C_{kj}. Since $\det L_B(i \mid \emptyset) = \det L(i \mid n) = \pm\kappa$, each vertex is of the form

$$\mathbf{v_i} = \frac{1}{\pm\kappa}\left(\sum_{k=1}^{n-1} C_{k1}, \sum_{k=1}^{n-1} C_{k2}, \ldots, \sum_{k=1}^{n-1} C_{k(n-1)}\right)^T,$$

which is a lattice point if and only if κ divides each coordinate. $\square$

Remark 2.1.9. Apply Lemma 2.1.7 to Theorem 2.1.8 to yield a condition on the second minors of L when determining if P_G is reflexive. Notice

$$\begin{aligned}(C^T)_{jk} &= C_{kj} \\ &= (-1)^{k+j} \det L_B(i,k \mid j) \\ &= (-1)^{k+j} \left(\det L(i,k \mid j,n) + \det L(i,k \mid j+1,n)\right),\end{aligned}$$

which shows for a given $\mathbf{v_i}$, its ℓ^{th} coordinate has the form

$$\frac{1}{\pm\kappa}\sum_{k=1}^{n-1} C_{k\ell} = \frac{1}{\pm\kappa}\sum_{k=1}^{n-1}(-1)^{k+\ell}\left(\det L(i,k \mid \ell,n) + \det L(i,k \mid \ell+1,n)\right).$$

Computing alternating sums of second minors of Laplacian matrices can be challenging. Thus, we often verify reflexivity by explicitly computing the vertices of P_G^* via ad hoc methods.

2.2 Interpretation of h^* vectors of Laplacian simplices

Simplices play a special role in Ehrhart theory, as there is a method for computing their h^*-vectors that is simple to state, although not always easy to apply.

Definition 2.2.1. Given a lattice simplex $\mathcal{P} \subset \mathbb{R}^{n-1}$ with vertices $\{\mathbf{v_i}\}_{i\in[n]}$, the *fundamental parallelepiped* of $\mathcal{P}$ is the subset of cone $(\mathcal{P})$, see Definition 1.2.1, defined by

$$\Pi_{\mathcal{P}} := \left\{ \sum_{i=1}^{n} \lambda_i(\mathbf{v_i}, 1) \mid 0 \leq \lambda_i < 1 \right\} .$$

Further, $|\Pi_{\mathcal{P}} \cap \mathbb{Z}^n|$ is equal to the determinant of the matrix whose i^{th} row is given by $(\mathbf{v_i}, 1)$.

Lemma 2.2.2 (see Chapter 3 of [6])**.** Given a lattice simplex $\mathcal{P}$,

$$h_i^*(\mathcal{P}) = |\Pi_{\mathcal{P}} \cap \{x \in \mathbb{Z}^n \mid x_n = i\}| .$$

Immediately we apply this lemma to state a further property of the Laplacian simplex.

Proposition 2.2.3. The Laplacian simplex satisfies $h_i^*(P_G) \geq 1$ for all $0 \leq i \leq n-1$.

Proof. Observe each column in L_B sums to 0. Consider lattice points of the form

$$\mathbf{p_i} = \left(\frac{i}{n}, \frac{i}{n}, \ldots, \frac{i}{n}\right) \cdot [L_B \mid \mathbb{1}] = (0, 0, \ldots, 0, i) \in \mathbb{Z}^{1\times n}$$

for each $0 \leq i < n$. Then $\mathbf{p_i} \in \Pi_{P_G} \cap \{\mathbf{x} \in \mathbb{Z}^n \mid x_n = i\}$ implies $h_i^*(P_G) \geq 1$ for each $0 \leq i \leq n-1$ by Lemma 2.2.2. □

Using the notation from Definition 2.2.1, let A be the $n \times n$ matrix whose i^{th} row is $(\mathbf{v_i}, 1)$. One approach to determine $h^*(\mathcal{P})$ in this case is to recognize that finding lattice points in $\Pi_{\mathcal{P}}$ is equivalent to finding integer vectors of the form $\lambda \cdot A$ with $0 \leq \lambda_i < 1$ for all i. Cramer's rule implies the $\lambda \in \mathbb{Q}^n$ that yield integer vectors will have entries of the form

$$\lambda_i = \frac{b_i}{\det A} < 1$$

for $b_i \in \mathbb{Z}_{\geq 0}$. In particular, if $\mathbf{x} = \frac{1}{\det(A)}\mathbf{b} \cdot A \in \mathbb{Z}^n$, then $b_i = \det A(i, \mathbf{x})$ where $A(i, \mathbf{x})$ is the matrix obtained by replacing the i^{th} row of A by $\mathbf{x}$. Since $A(i, \mathbf{x})$ is an integer matrix, $\det A(i, \mathbf{x}) \in \mathbb{Z}$. Notice that for any $\lambda \in \mathbb{Q}^n$, the last coordinate of λA is $\langle \lambda, \mathbb{1} \rangle = \sum_{i=1}^{n} \frac{b_i}{\det A}$. Thus, we have

$$\Pi_{\mathcal{P}} \cap \mathbb{Z}^n = \mathbb{Z}^n \cap \left\{ \frac{1}{\det A}\mathbf{b} \cdot A \mid 0 \leq b_i < \det(A), b_i \in \mathbb{Z}, \sum_{i=1}^{n} b_i \equiv 0 \bmod \det(A) \right\} .$$

One exhaustive method for determining the lattice points in $\Pi_{\mathcal{P}}$ is to find the $\det(A)$-many lattice points in the right-hand set above by first considering all the b-vectors that satisfy the given modular equation.

2.3 P_G associated to trees

This section contains results pertaining to P_G where G is a tree. A *tree* is a graph in which each pair of vertices is connected by a unique minimal path. We first consider the case where $G = P_k$, a path on k vertices. Label the vertices along the path with the elements of $[k]$ in increasing order. Then L and consequently L_B have the form

$$L = \begin{bmatrix} 1 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & 0 & & \vdots \\ 0 & -1 & 2 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 2 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 1 \end{bmatrix} \quad L_B = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ -1 & 1 & 0 & & \vdots \\ 0 & -1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & -1 & 1 \\ 0 & \cdots & \cdots & 0 & -1 \end{bmatrix}.$$

Observe that multiplication by the lower triangular matrix of all ones yields

$$L_B \cdot \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ \vdots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & 0 \\ 1 & \cdots & \cdots & \cdots & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & \ddots & & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \\ -1 & \cdots & \cdots & -1 & -1 \end{bmatrix}.$$

Since the lower triangular matrix is an element in $GL_{k-1}(\mathbb{Z})$, it follows that P_{P_k} is lattice equivalent to

$$S_{k-1}(1) := \mathrm{conv}\left(\mathbf{e_1}, \mathbf{e_2}, \ldots, \mathbf{e_{k-1}}, -\sum_{i=1}^{k-1} \mathbf{e_i}\right).$$

We leave it as an exercise for the reader to show that $S_{k-1}(1)$ is the unique reflexive $(k-1)$-polytope of minimal volume. This extends to all trees as follows.

Proposition 2.3.1. Let G be a tree on n vertices. Then P_G is unimodularly equivalent to $S_{n-1}(1)$.

Proof. Let G be a tree on n vertices. By Proposition 2.1.5, P_G is a simplex that contains the origin in its strict interior and has normalized volume equal to n, since G has only one spanning tree. Consider the triangulation of P_G that consists of creating a pyramid at the origin over each facet. Since G is a tree,

$$\mathrm{vol}(P_G) = \sum_{\text{facet}} \mathrm{vol}(F) = 1 \cdot n = n\,.$$

There are n facets, so each must have $\mathrm{vol}(F) = 1$. Applying a unimodular transformation to $n-1$ of the vertices of P_G, we can assume that the vertices of P_G are the n standard basis vectors and a single integer vector in the strictly negative orthant (so that the origin is in the interior of P_G). Because the normalized volume of the pyramid over each facet is equal to 1, it follows that the final vertex is $-\mathbb{1}$. $\square$

Corollary 2.3.2. The h^*-vector of the Laplacian simplex for any tree is $(1, 1, \ldots, 1)$ and hence is unimodal.

Corollary 2.3.3. The Laplacian simplex associated to a tree is IDP.

Corollary 2.3.4. Let G be a tree on n vertices with Laplacian matrix L_B. Then there exists $U \in \mathrm{G}L_{n-1}(\mathbb{Z})$ such that

$$L_B \cdot U = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \\ -1 & \cdots & \cdots & -1 \end{bmatrix}$$

The next proposition asserts that attaching an arbitrary tree with k vertices to a graph on n vertices yields a lattice isomorphism between the resulting Laplacian simplex and the Laplacian simplex associated to the graph obtained by attaching any other tree with k vertices at the same root.

Proposition 2.3.5. Let G be a connected graph on n vertices, and let v be a vertex of G. Let G' be the graph obtained from G by attaching k vertices such that G' restricted to the vertex set $\{v\} \cup [k]$ forms a tree, call it T. The edges of G' are the edges from G along with any edges among the vertices $\{v\} \cup [k]$. Let P be the graph obtained from G by attaching k vertices such that P restricted to the vertex set $\{v\} \cup [k]$ forms a path. Then $P_{G'} \cong P_P$.

Proof. The reduced Laplacian matrix associated to $P_{G'}$ is the following $(n+k) \times (n+k-1)$ matrix:

$$\left[\begin{array}{c|c} L_B(G) & 0 \\ \hline 0 & L_B(T) \end{array}\right]$$

Here $L_B(T) \in \mathbb{Z}^{(k+1)\times k}$ is the Laplacian matrix for T, the tree on $(k+1)$ vertices. Let $U \in \mathrm{G}L_k(\mathbb{Z})$ be the matrix such that $L_B(T) \cdot U$ gives the matrix with vertex set $S_k(1)$ as in Corollary 2.3.4. Then we have

$$\left[\begin{array}{c|c} L_B(G) & 0 \\ \hline 0 & L_B(T) \end{array}\right] \cdot \left[\begin{array}{c|c} I_{n-1} & 0 \\ \hline 0 & U \end{array}\right] = \left[\begin{array}{c|c} L_B(G) & 0 \\ \hline 0 & L_B(P) \end{array}\right].$$

For any set of k vertices we attach to a vertex $v \in V(G)$ to obtain a tree on the vertex set $\{v\} \cup [k]$, we get a corresponding unimodular matrix U such that the above multiplication holds. The determinant of the $(n-1+k) \times (n-1+k)$ transformation matrix is equal to the determinant of U, which is ± 1. Then $P_{G'}$ is lattice equivalent to P_P for any such G'. □

We will use this fact in Section 2.6 when we discuss graph constructions yielding reflexive Laplacian simplices.

2.4 P_G associated to cycles

A *cycle* graph on n vertices, denoted C_n, has vertex set $[n]$ and n edge set equal to $\{\{i, i+1\}, \{1, n\} \mid 1 \leq i \leq n-1\}$. For simplicity, we label the vertices of C_n cyclically.

Theorem 2.4.1. For $n \geq 3$, the simplex P_{C_n} is reflexive if and only if n is odd. For $k \geq 2$, the simplex $P_{C_{2k}}$ is 2-reflexive.

Proof. The matrices $L(C_n)$ and $L_B(C_n)$ have the form:

$$L = \begin{bmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & \ddots & & 0 \\ 0 & -1 & 2 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & & \ddots & -1 & 2 & -1 \\ -1 & 0 & \cdots & 0 & -1 & 2 \end{bmatrix} \qquad L_B = \begin{bmatrix} 2 & 1 & \cdots & \cdots & 1 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & 1 \\ -1 & -1 & \cdots & -1 & -2 \end{bmatrix}.$$

To show P_{C_n} is reflexive, we show $P^*_{C_n} = \{\mathbf{x} \mid L_B\mathbf{x} \leq \mathbb{1}\}$ is a lattice polytope. Each intersection of $(n-1)$ facet hyperplanes will yield a unique vertex of $P^*_{C_n}$, since the rank of L_B is $n-1$. For each $i \in [n]$, let $\mathbf{v_i} \in \mathbb{R}^{n-1}$ be the vertex that satisfies $L_B(i \mid \emptyset) \cdot \mathbf{v_i} = \mathbb{1}$. Solving the appropriate system of linear equations yields

$$\begin{aligned} \mathbf{v_1} &= \left(\frac{1-n}{2}, \frac{3-n}{2}, \frac{5-n}{2}, \ldots, \frac{n-5}{2}, \frac{n-3}{2}\right) = \left(\frac{(2j-1)-n}{2}\right)_{j=1}^{n-1} \\ \mathbf{v_i} &= \left(\left(\frac{(2j+1)+n-2i}{2}\right)_{j=1}^{i-1}, \left(\frac{(2j+1)-n-2i}{2}\right)_{j=i}^{n-1}\right), \text{ for } 2 \leq i \leq n-1 \\ \mathbf{v_n} &= \left(\frac{3-n}{2}, \frac{5-n}{2}, \frac{7-n}{2}, \ldots, \frac{n-3}{2}, \frac{n-1}{2}\right) = \left(\frac{(2j+1)-n}{2}\right)_{j=1}^{n-1} \end{aligned}$$

These are the vertices of $P^*_{C_n}$. Note $\mathbf{v_i} \in \mathbb{Z}^{n-1}$ only if n is odd. Then P_{C_n} is reflexive if and only if n is odd.

For the even case, observe that each vertex of $2P^*_{C_{2k}}$ has coordinates which are relatively prime. Then each of these vertices is primitive. Thus, for $n = 2k$ each vertex of $P^*_{C_{2k}}$ is a multiple of $\frac{1}{2}$, which allows us to write

$$P_{C_{2k}} = \left\{\mathbf{x} \mid \frac{1}{2}\tilde{A}\mathbf{x} \leq \mathbb{1}\right\} = \left\{\mathbf{x} \mid \tilde{A}\mathbf{x} \leq 2 \cdot \mathbb{1}\right\}$$

where $\tilde{A} \in \mathbb{Z}^{n\times(n-1)}$ is an integer matrix. The facets of $P_{C_{2k}}$ have supporting hyperplanes $\langle \mathbf{r_i}, \mathbf{x}\rangle = 2$ where $\mathbf{r_i}$ is the i^{th} row of $\tilde{A}$. Thus $P_{C_{2k}}$ is a 2-reflexive Laplacian simplex. $\square$

Example 2.4.2. Below are the dual polytopes to P_{C_n} for small n.

- $P^*_{C_3} = \operatorname{conv}((-1,0),(1,-1),(0,1))$
- $P^*_{C_4} = \operatorname{conv}\left((-\frac{3}{2},-\frac{1}{2},\frac{1}{2}),(\frac{3}{2},-\frac{3}{2},-\frac{1}{2}),(\frac{1}{2},\frac{3}{2},-\frac{3}{2}),(-\frac{1}{2},\frac{1}{2},\frac{3}{2})\right)$
- $P^*_{C_5} = \operatorname{conv}((-2,-1,0,1),(2,-2,-1,0),(1,2,-2,-1),(0,1,2,-2),(-1,0,1,2))$

Having found the vertices of the dual polytope, the hyperplane description of P_{C_n} is given by

$$P_{C_n} = \{\mathbf{x} \in \mathbb{R}^{n-1} \mid A\mathbf{x} \leq \mathbb{1}\}$$

where the rows of $A \subseteq \mathbb{R}^{n\times(n-1)}$ are the vertices of $P^*_{C_n}$.

Although $P_{C_{2k}}$ is not reflexive, we show next that whiskering C_{2k} results in a graph $W(C_{2k})$ such that $P_{W(C_{2k})}$ is reflexive. The technique of whiskering graphs has been studied previously in the context of Cohen-Macaulay edge ideals, see [14, Theorem 4.4] and [39].

Definition 2.4.3. To add a *whisker* at a vertex $x \in V(G)$, one adds a new vertex y and the edge connecting x and y. Let $W(G)$ denote the graph obtained by whiskering all vertices in G. We call $W(G)$ the *whiskered graph of* G. If $V(G) = \{x_1,\dots,x_n\}$ and $E(G) = E$, then $V(W(G)) = V(G) \cup \{y_1,\dots,y_n\}$ and $E(W(G)) = E \cup \{\{x_1,y_1\},\dots,\{x_n,y_n\}\}$.

Proposition 2.4.4. *$P_{W(C_n)}$ is reflexive for even integers $n \geq 2$.*

Proof. $W(C_n)$ is a graph with vertex set $[2n]$ and $2n$ edges. Label the vertices of C_n cyclically. For each vertex $i \in [n]$, label the vertex of the added whisker with $n+i$. The Laplacian matrix has the following form.

$$L = \left[\begin{array}{c|c} L + I_n & -I_n \\ \hline -I_n & I_n \end{array}\right]$$

Consequently, if A is the $n \times (n-1)$ matrix given by Equation (2.1), then

$$L_B = \left[\begin{array}{c|cccc} L_B(C_n) + A & & A^T & & \\ & 1 & \cdots & \cdots & 1 \\ \hline -A & & -A^T & & \\ & -1 & \cdots & \cdots & -1 \end{array}\right].$$

We show $P_{W(C_n)}$ is reflexive by showing $P^*_{W(C_n)}$ is a lattice polytope. Each vertex of the dual is a solution to $L_B(i \mid \emptyset)\mathbf{v_i} = \mathbb{1}$. We consider the following cases.

Case: $1 \leq i \leq n$. Multiply both sides of $L_B(i \mid \emptyset)\mathbf{v_i} = \mathbb{1}$ by the $(2n-1)\times(2n-1)$ upper diagonal matrix with the following entries.

$$x_{\ell k} = \begin{cases} 1, & \text{if } \ell = k \\ 1, & \text{if } \ell < k \text{ and } \{v_\ell, v_k\} \text{ is a whisker} \\ -1, & \text{if } n < \ell = k-1 \end{cases}$$

In this matrix, each of the first $n-1$ rows will have exactly two non-zero entries of value 1, which corresponds to adding the two rows of $L_B(i \mid \emptyset)$ that are indexed by the labels of a whisker in the graph. The last n rows will have an entry of 1 along the diagonal and an entry of -1 on the superdiagonal, which corresponds to subtracting consecutive rows in $L_B(i \mid \emptyset)$ to achieve cancellation. We obtain the following system of linear equations.

$$\left[\begin{array}{ccc|ccc} & L_B(C_n)(i \mid \emptyset) & & & 0 & \\ \hline & & & 0 & & \\ & -I_{n-1} & & \vdots & I_{n-1} & \\ & & & 0 & & \\ 0 & \cdots & 0 & -1 & \cdots \quad \cdots & -1 \end{array}\right] \mathbf{v_i} = \begin{bmatrix} 2 \\ \vdots \\ 2 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Let $(\mathbf{v_i^*})_j$ denote the j^{th} coordinate of the vertex $\mathbf{v_i} \in \mathbb{Q}^{n-1}$ of $P^*_{C_n}$ described in Proposition 2.4.1. Then the vertex $\mathbf{v_i}$ of $P^*_{W(C_n)}$ has the following form.

$$(\mathbf{v_i})_j = \begin{cases} 2(\mathbf{v_i^*})_j, & \text{if } 1 \leq j \leq n-1 \\ -1 - \sum_{k=1}^{n-1} 2(\mathbf{v_i^*})_k, & \text{if } j = n \\ 2(\mathbf{v_i^*})_{j-n}, & \text{if } n+1 \leq j \leq 2n-1 \end{cases}$$

Since $2(\mathbf{v_i^*})_j \in \mathbb{Z}$ by Proposition 2.4.1 for $1 \leq j \leq n-1$, then $\mathbf{v_i} \in \mathbb{Z}^{2n-1}$.

Case: $n+2 \leq i \leq 2n$. The strategy is to multiply the equality $L_B(i \mid \emptyset)\mathbf{v_i} = \mathbb{1}$ by the matrix that performs the following row operations. Let $\mathbf{r_m} \in \mathbb{Z}^{2n-1}$ denote the m^{th} row of $L_B(i \mid \emptyset)$. For each whisker with vertex labels $\{m, n+m\}$, replace $\mathbf{r_m}$ with $\mathbf{r_m} + \mathbf{r_{n+m}}$ for $m \in [n]$. Row $\mathbf{r_{i-n}}$ will not have a row to add because the index of its whisker is the index of the deleted row. Since the entries in each column of L_B sum to 0, the negative sum of all the rows of $L_B(i \mid \emptyset)$ is equal to the row removed. We recover the missing row by replacing $\mathbf{r_{i-n}}$ with $-\sum_{k=1}^{2n-1} \mathbf{r_k}$. Then as in the previous case, we want to replace $\mathbf{r_k}$ with $\mathbf{r_k} - \mathbf{r_{k+1}}$ for $n+1 \leq k \leq 2n-2$. Here $\mathbf{r_{i-n}}$ plays the role of the deleted $\mathbf{r_i}$. We obtain a similar system of linear equations found in the first case. The vertex $\mathbf{v_i}$ of $P^*_{W(C_n)}$ has the following form.

$$(\mathbf{v_i})_j = \begin{cases} 2(\mathbf{v_i^*})_j, & \text{if } 1 \le j \le n-1 \\ -1 - \sum_{k=1}^{n-1} 2(\mathbf{v_i^*})_k, & \text{if } j = n \\ 2(\mathbf{v_i^*})_{j-n}, & \text{if } n+1 \le j \le 2n-1 \text{ and } j \ne i-1, i \\ 2(\mathbf{v_i^*})_{j-n} + 2n, & \text{if } j = i-1 \\ 2(\mathbf{v_i^*})_{j-n} - 2n, & \text{if } j = i \end{cases}$$

Observe in the case $i = 2n$, the last equality is not applicable since $j \in [2n-1]$. Thus $\mathbf{v_i} \in \mathbb{Z}^{2n-1}$.

Case: $i = n+1$. Here $(\mathbf{v_i})_{i-1} = (\mathbf{v_i})_n = -(2n-1) - \sum_{k=1}^{n-1} 2(\mathbf{v_i^*})_k \in \mathbb{Z}$ and the other coordinates are as described above. Then $\mathbf{v_i} \in \mathbb{Z}^{2n-1}$. ☐

Example 2.4.5. The Laplacian simplex obtained from the graph in Figure 2.1 is reflexive by Proposition 2.4.4.

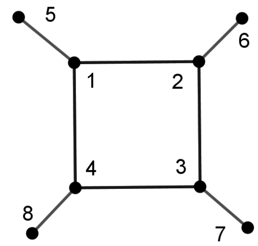


Figure 2.1: The whiskered 4-cycle, $W(C_4)$.

We extend Proposition 2.4.4 to a more general result, that whiskering a graph whose Laplacian simplex is 2-reflexive results in a graph whose Laplacian simplex is reflexive. Although even cycles are the only known graph type to result in 2-reflexive Laplacian simplices, Theorem 2.4.1, we include the following result.

Proposition 2.4.6. If G is a connected graph on n vertices such that P_G is 2-reflexive, then $P_{W(G)}$ is reflexive for all $n \ge 2$.

Proof. If P_G is 2-reflexive, then each vertex $\mathbf{v_i}$ of P^*_G satisfies $2\mathbf{v_i} \in \mathbb{Z}^{n-1}$ for each $1 \le i \le n$. As in the proof of Proposition 2.4.4, we can find descriptions of the vertices of $P^*_{W(G)}$ in terms of the coordinates from vertices of P^*_G to show they are lattice points. The result follows. ☐

The graph operation of whiskering not only behaves nicely with respect to 2-reflexive Laplacian simplices, it also preserves the reflexivity of Laplacian simplices as shown in the the following proposition.

Proposition 2.4.7. If G is a connected graph on n vertices such that P_G is reflexive, then $P_{W(G)}$ is reflexive for all $n \ge 1$.

Proof. If P_G is reflexive, then vertices of P_G^* are integer and satisfy $L_B(i \mid \emptyset)\mathbf{v_i} = \mathbb{1}$ for all $1 \leq i \leq n$. Observe $2\mathbf{v_i} \in \mathbb{Z}^{n-1}$ satisfies $L_B(i \mid \emptyset)2\mathbf{v_i} = 2 \cdot \mathbb{1}$. Following the proof technique in Proposition 2.4.4, we can find descriptions of the vertices of $P_{W(G)}^*$ in terms of the coordinates from vertices of P_G^* to show they are lattice points. □

The next topic of interest is the unimodality of the h^*-vectors of P_{C_n}. For odd n, our proof of the following theorem can be interpreted as establishing the existence of a weak Lefschetz element in the quotient of the semigroup algebra associated to cone (P_{C_n}) by the system of parameters corresponding to the ray generators of the cone. This proof approach is not universally applicable, as there are examples of reflexive IDP simplices with unimodal h^*-vectors for which this proof method fails [8].

Theorem 2.4.8. For odd n, $h^*(P_{C_n})$ is unimodal.

Proof. Recall from Lemma 2.2.2 that $h_i^*(P_{C_n})$ is the number of lattice points in $\Pi_{P_{C_n}}$ at height i. Theorem 2.4.1 shows $h_i^*(P_{C_n})$ is symmetric for odd n. Our goal is to prove that for $i \leq \lfloor n/2 \rfloor$ we have $h_i^* \leq h_{i+1}^*$. This will show that $h^*(P_{C_n})$ is unimodal.

While $\kappa = n$ for C_n, we will freely use both κ and n to denote this quantity, as it is often helpful to distinguish between the number of spanning trees and the number of vertices. Lattice points in the fundamental parallelepiped of P_{C_n} can be described as follows:

$$\mathbb{Z}^n \cap \left\{ \frac{1}{\kappa n}\mathbf{b} \cdot [L_B \mid \mathbb{1}] \mid 0 \leq b_i < \kappa n, b_i \in \mathbb{Z}_{\geq 0}, \sum_{i=1}^{n} b_i \equiv 0 \bmod \kappa n \right\}.$$

We will use the modular equation above extensively in our analysis. Denote the height of a lattice point in $\Pi_{P_{C_n}}$ by

$$h(\mathbf{b}) := \frac{\sum_{i=1}^n b_i}{n\kappa} \in \mathbb{Z}_{\geq 0}\,.$$

We first show that every lattice point in $\Pi_{P_{C_n}}$ arising from $\mathbf{b}$ satisfies

$$\frac{(k-j+1)(b_1 - b_n)}{\kappa n} + \frac{b_j - b_{k+1}}{\kappa n} \in \mathbb{Z}$$

for each $1 \leq j < k \leq n-1$. Since the lattice point lies in $\Pi_{P_{C_n}}$, we have the following constraint equations:

$$\frac{b_1 - b_n + b_i - b_{i+1}}{\kappa n} \in \mathbb{Z}$$

for each $1 \leq i \leq n-1$. Summing any consecutive set of these equations where $1 \leq j \leq k \leq n-1$ yields

$$\sum_{i=j}^{k} \left(\frac{b_1 - b_n}{\kappa n} + \frac{b_i - b_{i+1}}{\kappa n} \right) \in \mathbb{Z}\,.$$

The result follows.

Thus, each vector $\mathbf{b}$ corresponding to an integer point in $\Pi_{P_{C_n}}$ satisfies $\kappa \mid (b_1 - b_n)$, which follows from setting $j = 1$ and $k = n-1$. We next claim that every lattice point in $\Pi_{P_{C_n}}$ arises from $\mathbf{b} \in \mathbb{Z}^n$ such that $b_i \equiv b_j \mod (\kappa)$ for each $1 \leq i, j \leq n$. To prove this, set $\frac{b_1 - b_n}{\kappa} = B \in \mathbb{Z}$. Then for each $1 \leq i \leq n-1$, our constraint equation becomes $\frac{B}{n} + \frac{b_i - b_{i+1}}{\kappa n} = C$ for some $C \in \mathbb{Z}$. Then $\frac{b_i - b_{i+1}}{\kappa} = Cn - B \in \mathbb{Z}$ holds for each i. The result follows.

First Major Claim: For n odd, any lattice point in $\Pi_{P_{C_n}}$ arises from $\mathbf{b} \in \mathbb{Z}^n$ such that $b_i \equiv 0 \mod (\kappa)$ for each $1 \leq i \leq n$.

To prove this, let $b_i = m_i \kappa + \alpha$ such that $0 \leq m_i < \kappa$ and $0 \leq \alpha < \kappa$. Constraint equations yield

$$\frac{b_1 - b_n + b_i - b_{i+1}}{\kappa n} = \frac{m_1 - m_n + m_i - m_{i+1}}{n} \in \mathbb{Z}$$

using $\kappa = n$. Summing all $n-1$ integer expressions with linear coefficients yields

$$\sum_{i=1}^{k} i(m_1 - m_n + m_i - m_{i+1}) = \frac{n(n-1)}{2} m_1 + \sum_{i=1}^{n-1} m_i - (n-1)m_n - \frac{n(n-1)}{2} m_n,$$

which is divisible by n. Call the resulting sum An for some $A \in \mathbb{Z}$. Finally, notice the last constraint equation (corresponding to $h(\mathbf{b})$) can be written

$$\begin{aligned}\frac{\sum_{i=1}^{n} b_i}{\kappa n} &= \frac{\sum_{i=1}^{n} m_i + \alpha}{n} \\ &= \frac{m_n + An - \frac{n(n-1)}{2} m_1 + (n-1)m_n + \frac{n(n-1)}{2} m_n + \alpha}{n} \in \mathbb{Z}.\end{aligned}$$

Then n odd implies n divides $\frac{n(n-1)}{2}$ so that n divides α. Since $0 \leq \alpha < n$, then $\alpha = 0$ as desired.

Second Major Claim: Consider P_{C_n} for odd n. Suppose $h(\mathbf{b}) < \frac{n-1}{2}$. If $\mathbf{p} \in \Pi_{P_{C_n}} \cap \mathbb{Z}^n$, then $\mathbf{p} + (0, \cdots, 0, 1)^T \in \Pi_{P_{C_n}} \cap \mathbb{Z}^n$.

To establish this, it suffices to prove that for every $\mathbf{p} = \frac{1}{n^2} \mathbf{b} \cdot [L_B \mid \mathbb{1}] \in \Pi_{P_{C_n}} \cap \mathbb{Z}^n$ such that $h(\mathbf{b}) < \frac{n-1}{2}$, we have $b_i < n(n-1)$ for each i. This would imply

$$\mathbf{p} + (0, \cdots, 0, 1)^T = \frac{1}{n^2}(\mathbf{b} + n\mathbb{1}) \cdot [L_B \mid \mathbb{1}] \in \Pi_{P_{C_n}} \cap \mathbb{Z}^n\,,$$

providing an injection from the lattice points in $\Pi_{P_{C_n}}$ at height i to those at height $i+1$. Constraint equations yield, using the same notation as in the proof of our first major claim, that

$$-m_{j-1} + 2m_j - m_{j+1} \in n\mathbb{Z}$$

for each $1 \leq j \leq n$. Note that this comes from subtracting the two integers

$$\frac{m_1 + m_j - m_{j+1} - m_n}{n} - \frac{m_1 + m_{j-1} - m_j - m_n}{n} = \frac{2m_j - (m_{j-1} + m_{j+1})}{n} \in \mathbb{Z}$$

for each $2 \leq j \leq n-1$, as well as

$$\frac{2m_1 - m_2 - m_n}{n}, \frac{-(m_1 + m_{n-1} - 2m_n)}{n} \in \mathbb{Z}\,.$$

For a contradiction, suppose there exists a j such that $b_j = n(n-1)$. Then $m_j = n-1$. Constraints on the other variables m_i imply

$$0 \leq \frac{2(n-1) - (m_{j-1} + m_{j+1})}{n} \leq 1 \implies 2(n-1) - (m_{j-1} + m_{j+1}) = 0 \text{ or } n.$$

Case 1: If the above is 0, then

$$2(n-1) = m_{j-1} + m_{j+1} \implies m_{j-1} = m_{j+1} = n-1.$$

Apply these substitutions on other constraint equations to yield $m_i = n-1$ for all $1 \leq i \leq n$. Then

$$h(\mathbf{b}) = \frac{\sum_{i=1}^n m_i}{n} = \frac{n(n-1)}{n} = n - 1 > \frac{n-1}{2},$$

which is a contradiction.
Case 2: If the above is 1, then $n - 2 = m_{j-1} + m_{j+1}$. Adding subsequent constraint equations yields

$$\begin{aligned}(-m_j + 2m_{j-1} - m_{j-2}) + (-m_j + 2m_{j+1} - m_{j+2}) &= -2m_j + 2(m_{j-1} + m_{j+1}) - (m_{j-2} + m_{j+2}) \\ &= -2(n-1) + 2(n-2) - (m_{j-2} + m_{j+2}) \\ &= -2 - (m_{j-2} + m_{j+2})\end{aligned}$$

Since the above is in $n\mathbb{Z}$, it is equal to either $-2n$ or $-n$.
Case 2a: If the above is equal to $-2n$, then $m_{j-2} = m_{j+2} = n-1$. Then

$$-m_{j-3} + 2m_{j-2} - m_{j-1} = -m_{j-3} + m_{j+1} \in n\mathbb{Z} \implies m_{j-3} = m_{j+1}.$$

A similar argument shows $m_{j+3} = m_{j-1}$. Continuing in this way shows $m_{j\pm k} = m_{j\mp 1}$ for remaining m_i. Then for each of the $\frac{n-3}{2}$ pairs, $m_{j-k} + m_{j+k} = n - 2$ where $k \in \{1, \hat{2}, 3, \cdots, \frac{n-1}{2}\}$. But then

$$\begin{aligned}h(\mathbf{b}) &= \frac{\sum_{i=1}^n m_i}{n} \\ &= \frac{n - 1 + 2(n-1) + \frac{n-3}{2}(n-2)}{n} \\ &= \frac{n+1}{2},\end{aligned}$$

which is a contradiction.
Case 2b: If the above is equal to $-n$, then $m_{j-2} + m_{j+2} = n - 2$. Adding subsequent constraint equations as above yields $n - 2 - (m_{j-3} + m_{j+3})$. Since the above is in $n\mathbb{Z}$, it is equal to either $-2n$ or $-n$.
Case 2b(i): If the above is equal to $-n$, then $m_{j-3} = m_{j+3} = n - 1$. Following the same argument as Case 2a leads to the contradiction, $h(\mathbf{b}) = \dfrac{n+1}{2}$.

Case 2b(ii): If the above is equal to $-2n$, then $m_{j-3} + m_{j+3} = n - 2$. Continuing in this manner yields $m_{j-k} + m_{j+k} = n - 2$ for all $k \in \{1, 2, \cdots, \frac{n-1}{2}\}$. But then

$$h(\mathbf{b}) = \frac{n - 1 + \frac{(n-1)}{2}(n-2)}{n} = \frac{n-1}{2},$$

which is a contradiction. This concludes the proof of our second major claim.

The second claim implies that for $i \leq \lfloor n/2 \rfloor$, we have $h_i^* \leq h_{i+1}^*$. Thus, our proof is complete. $\square$

We next classify the lattice points in the fundamental parallelepiped for P_{C_n} by considering the matrix $[L_B \mid \mathbb{1}]$ over the ring $\mathbb{Z}/\kappa\mathbb{Z}$. Let

$$[\widetilde{L} \mid \mathbb{1}] := [L_B \mid \mathbb{1}] \bmod \kappa\,.$$

Recall that for a cycle we have $n = \kappa$.

Lemma 2.4.9. For C_n with odd n and corresponding $[L_B \mid \mathbb{1}]$, we have

$$\ker_{\mathbb{Z}/\kappa\mathbb{Z}}[\widetilde{L} \mid \mathbb{1}] = \{\mathbf{x} \in (\mathbb{Z}/\kappa\mathbb{Z})^n \mid \mathbf{x}[L_B \mid \mathbb{1}] \equiv \mathbf{0} \mod \kappa\} = \langle \mathbb{1}^n, (0, 1, \cdots, n-1)\rangle.$$

Proof. Consider the second principal minor of $[L_B \mid \mathbb{1}]$ with the first and n^{th} rows and columns deleted. The matrix $[L_B \mid \mathbb{1}](1, n \mid 1, n)$ is the lower diagonal matrix of the following form:

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & & \vdots \\ 0 & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & -1 & 1 \end{bmatrix}$$

Then $\det[L_B \mid \mathbb{1}](1, n \mid 1, n) = 1$ implies there are $n-2$ linearly independent columns, hence $\mathrm{rk}_{\mathbb{Z}/\kappa\mathbb{Z}}[L_B \mid \mathbb{1}] \geq n - 2$.

Since the entries in each column of $[L_B \mid \mathbb{1}]$ sum to 0, then

$$\mathbb{1} \cdot [L_B \mid \mathbb{1}] = (0, \ldots, 0, n) \equiv \mathbf{0} \mod \kappa$$

implies $\mathbb{1} \in \ker_{\mathbb{Z}/\kappa\mathbb{Z}}[L_B \mid \mathbb{1}]$. Consider

$$(0, 1, \ldots, n{-}1)\cdot \begin{bmatrix} 2 & 1 & 1 & \cdots & \cdots & 1 & 1 \\ -1 & 1 & 0 & \cdots & \cdots & 0 & 1 \\ 0 & -1 & 1 & \ddots & & \vdots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 & \vdots \\ 0 & \cdots & \cdots & 0 & -1 & 1 & 1 \\ -1 & -1 & -1 & \cdots & -1 & -2 & 1 \end{bmatrix} = \left(-n, \ldots, -n, \frac{n(n-1)}{2}\right) \equiv \mathbf{0} \mod \kappa.$$

This shows $(0, 1, \ldots, n-1) \in \ker_{\mathbb{Z}/\kappa\mathbb{Z}}[L_B \mid \mathbb{1}]$. Since these two vectors are linearly independent, we have $\mathrm{rk}_{\mathbb{Z}/\kappa\mathbb{Z}}[L_B \mid \mathbb{1}] \leq n - 2$.

Thus, the kernel is two-dimensional and we have found a basis. $\square$

Theorem 2.4.10. For odd $n \geq 3$, lattice points in $\Pi_{P_{C_n}}$ are of the form

$$\frac{(\alpha\mathbb{1} + \beta(0,1,\ldots,n-1)) \mod \kappa}{\kappa} \cdot [L_B \mid \mathbb{1}]$$

for all $\alpha, \beta \in \mathbb{Z}/\kappa\mathbb{Z}$. Thus, $h_i^*(P_G)$ is equal to the cardinality of

$$\left\{ \frac{(\alpha\mathbb{1} + \beta(0,1,\ldots,n-1)) \mod \kappa}{\kappa} \cdot [L_B \mid \mathbb{1}] \mid 0 \leq \alpha, \beta < \kappa - 1, \frac{1}{\kappa}\sum_{j=0}^{n-1}(\alpha + j\beta \mod \kappa) = i \right\}.$$

Proof. Since $|\Pi_{P_{C_n}} \cap \mathbb{Z}^n| = \sum_{i=0}^{n-1} h_i^*(P_{C_n}) = n\kappa = n^2$, there are n^2 lattice points in the fundamental parallelepiped. Similarly, there are n^2 possible linear combinations of $\mathbb{1}$ and $(0,1,2,\ldots,n-1)$ in $\mathbb{Z}/\kappa\mathbb{Z}$. We show that each such linear combination yields a lattice point. Recall the sum of the coordinates down each of the first $n-1$ columns of $[L_B \mid \mathbb{1}]$ is 0. Since

$$(\alpha\mathbb{1} + \beta(0,1,\ldots,n-1)) \cdot [L_B \mid \mathbb{1}] \equiv \mathbf{0} \mod \kappa$$

by Lemma 2.4.9, it follows that

$$(\alpha\mathbb{1} + \beta(0,1,\ldots,n-1) \mod \kappa) \cdot [L_B \mid \mathbb{1}] \equiv \mathbf{0} \mod \kappa.$$

Then $\dfrac{(\alpha\mathbb{1} + \beta(0,1,\ldots,n-1)) \mod \kappa}{\kappa} \cdot [L_B \mid \mathbb{1}]$ is a lattice point. Since we are reducing the numerators of the entries in the vector of coefficients modulo κ prior to dividing by κ, it follows that each entry in the coefficient vector is greater than or equal to 0 and strictly less than 1, and hence the resulting lattice point is an element of $\Pi_{P_{C_n}}$. $\square$

Theorem 2.4.11. Consider C_n where $n \geq 3$ is odd. Let $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ be the prime factorization of n where $p_1 > p_2 > \cdots > p_k$. Then

$$h^*(P_{C_n}) = (1,\ldots,1,h_m^*,h_{m+1}^*,\ldots,h_{\frac{n-1}{2}}^*,\ldots,h_{n-m-1}^*,h_{n-m}^*,1,\ldots,1)$$

where $m = \frac{1}{2}(n - p_1^{a_1}\cdots p_k^{a_k-1})$ and $h_m > 1$. Further, if $\mathbb{Z}_n^*$ denotes the group of units of $\mathbb{Z}_n$, we have that $h_{(n-1)/2}^* \geq n \cdot |\mathbb{Z}_n^*| + 1$. In particular, if n is prime, we have

$$h^*(P_{C_n}) = (1,\ldots,1,n^2-n+1,1,\ldots,1)$$

Proof. Keeping in mind that $n = \kappa$ for C_n, denote the height of the lattice point

$$\frac{(\alpha\mathbb{1} + \beta(0,1,\ldots,n-1)) \bmod n}{n} \cdot [L_B \mid \mathbb{1}]$$

in the fundamental parallelepiped by

$$h(\alpha,\beta) := \frac{1}{n}\sum_{j=0}^{n-1}((\alpha + j\beta) \bmod n).$$

Each $\alpha \in \mathbb{Z}/n\mathbb{Z}$ paired with $\beta = 0$ produces a lattice point at a unique height in $\Pi_{P_{C_n}}$, and thus each $h_i^* \geq 1$. Let $\mathbb{Z}_n^*$ denote the group of units of $\mathbb{Z}_n$. If $\beta \in \mathbb{Z}_n^*$, then $\beta(0, 1, \ldots, n-1) \bmod n$ yields a vector that is a permutation of $(0, 1, \ldots, n-1)$, and thus for any α we have the height of the resulting lattice point is $(n-1)/2$, proving that $h^*_{(n-1)/2} \geq n \cdot |\mathbb{Z}_n^*| + 1$. Thus, when n is an odd prime, it follows that

$$h^*(P_{C_n}) = (1, \ldots, 1, n^2 - n + 1, 1, \ldots, 1) .$$

Now, suppose that $\gcd(\beta, n) = \prod p_i^{b_i} \neq 1$. Then the order of β in $\mathbb{Z}_n$ is $\prod p_i^{a_i - b_i}$, and (after some reductions in summands modulo n)

$$h(\alpha, \beta) = \frac{1}{n} \cdot \prod p_i^{b_i} \cdot \left(\sum_{j=0}^{\prod p_i^{a_i - b_i} - 1} \left((\alpha + j \prod p_i^{b_i}) \bmod n \right) \right) .$$

Thus, we see that for a fixed β, the height is minimized (not uniquely) when $\alpha = 0$. In this case, we have

$$\begin{aligned} h(0, \beta) &= \frac{1}{n} \cdot \prod p_i^{b_i} \cdot \left(\sum_{j=0}^{\prod p_i^{a_i - b_i} - 1} \left(j \prod p_i^{b_i} \bmod n \right) \right) \\ &= \frac{1}{n} \cdot \prod p_i^{b_i} \cdot \prod p_i^{b_i} \cdot \left(\sum_{j=0}^{\prod p_i^{a_i - b_i} - 1} j \right) \\ &= \frac{n - \prod p_i^{b_i}}{2} . \end{aligned}$$

This value is minimized when $\prod p_i^{b_i} = p_1^{a_1} \cdots p_k^{a_k - 1}$, and this height is attained more than once by setting $\beta = p_1^{a_1} \cdots p_k^{a_k - 1}$ and $\alpha = 0, 1, 2, \ldots, p_1^{a_1} \cdots p_k^{a_k - 1} - 1$. $\square$

Corollary 2.4.12. P_{C_n} is not IDP for odd $n \geq 5$.

Proof. Theorem 2.4.11 yields $h_1^*(P_{C_n}) = 1$ for odd $n \geq 3$. It is known, see Section 1.2, that for an integral convex d-polytope $\mathcal{P}$, $h_1^*(\mathcal{P}) = |\mathcal{P} \cap \mathbb{Z}^n| - (d+1)$. In this case,

$$|P_{C_n} \cap \mathbb{Z}^n| = h_1^*(P_{C_n}) + (n-1) + 1 = n + 1$$

is the number of lattice points in P_{C_n}. In particular, the lattice points consist of the n vertices of P_{C_n} and the origin. Then $\Pi_{P_{C_n}} \cap \{\mathbf{x} \mid x_n = 1\} \cap \mathbb{Z}^n = (0, \ldots, 0, 1)$. If P_{C_n} is IDP, then every lattice point in $\Pi_{P_{C_n}}$ is of the form $(0, \ldots, 0, 1) + \cdots + (0, \ldots, 0, 1)$, which is not true by Proposition 2.4.10. The result follows. $\square$

2.5 P_G associated to complete graphs

The simplex P_{K_n} is a generalized permutohedron, where a *permutohedron* $P_n(x_1, \ldots, x_n)$ for $x_i \in \mathbb{R}$ is the convex hull of the $n!$ points obtained from $(x_1, \ldots, x_n)$ by permutations of the coordinates. For K_n, the Laplacian matrix has diagonal entries equal to $n-1$ and all other entries equal to -1. Thus $\operatorname{conv}\left(L(n)^T\right) = P_n(n-1, -1, \ldots, -1) \cong P_n(n, 0, \ldots, 0)$. Note that this implies that P_{K_n} is equivalent to the n^{th} dilate of an $(n-1)$-dimensional standard simplex.

Definition 2.5.1. The *standard simplex* Δ in dimension d is the convex hull of the $d+1$ points $\mathbf{e_1}, \ldots, \mathbf{e_d}$ and the origin.

Many properties of dilates of standard simplices and of generalized permutahedra [32] are known. While some of the findings in this section follow quickly from these general results, for the sake of completeness we will include proofs or proof outlines below.

Theorem 2.5.2. The simplices P_{K_n} are reflexive for $n \geq 1$.

Proof. Observe L_B is an $n \times (n-1)$ integer matrix of the form

$$L_B = \begin{bmatrix} (n-1) & (n-2) & (n-3) & \cdots & \cdots & 1 \\ -1 & (n-2) & (n-3) & \cdots & \cdots & 1 \\ -1 & -2 & (n-3) & \cdots & \cdots & \vdots \\ -1 & -2 & -3 & (n-4) & \cdots & \vdots \\ \vdots & \vdots & \vdots & -4 & \ddots & \vdots \\ \vdots & \vdots & \vdots & \vdots & & 1 \\ -1 & -2 & -3 & \cdots & \cdots & -(n-1) \end{bmatrix}.$$

To prove P_{K_n} is reflexive, we show $P_{K_n} = \{\mathbf{x} \in \mathbb{R}^{n-1} \mid A\mathbf{x} \leq \mathbb{1}\}$ for some $A \in \mathbb{Z}^{n \times (n-1)}$. We claim that A has the following form:

$$A = \begin{bmatrix} -1 & 0 & 0 & \cdots & 0 \\ 1 & -1 & 0 & & \vdots \\ 0 & 1 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & 1 & -1 \\ 0 & \cdots & \cdots & 0 & 1 \end{bmatrix} \in \{0, \pm 1\}^{n \times (n-1)}.$$

Let $\mathbf{r_i}$ be the i^{th} row of L_B. Observe that $A(i \mid \emptyset)\mathbf{r_i} = \mathbb{1}$ for each $1 \leq i \leq n$. Then $\{\mathbf{r_i}\}_{i=1}^n$ is a set of intersection points of defining hyperplanes of P_{K_n} taken $(n-1)$ at a time. Notice rk $A = n-1$, and further, each matrix $A(i \mid \emptyset)$ has full rank. This implies $\{\mathbf{r_i}\}_{i=1}^n$ is the set of unique intersection points. Thus $\{\mathbf{x} \mid A\mathbf{x} \leq \mathbb{1}\} = \operatorname{conv}(\mathbf{r_1}, \ldots, \mathbf{r_n}) = P_{K_n}$ shows that P_{K_n} is reflexive. □

One technique for proving a polytope is IDP is showing it has a regular unimodular triangulation. Here we uncover the meaning of these words. A *unimodular simplex* is a lattice polytope which is lattice equivalent to the standard simplex. Unimodular simplices have minimal possible normalized volume of 1. A *lattice subdivision* of a lattice polytope $\mathcal{P}$ of dimension d is a collection of lattice polytopes $\mathcal{T} = \{T_i\}_{i \in [n]}$ such that every face of T_i is in $\mathcal{T}$, $T_i \cap T_j \in \mathcal{T}$, and $\cup_{i=1}^n T_i = \mathcal{P}$. The maximal d-dimensional T_i are called *cells* of $\mathcal{T}$. A *unimodular triangulation* is a lattice subdivision for which each cell of the subdivision is a unimodular simplex. Finally, a subdivision is called *regular* if its cells are the domains of linearity of a convex piecewise linear function [18]. Informally, we create a regular subdivision of a lattice polytope by assigning different heights to each lattice point in the polytope and project their lower convex hull to obtain the desired subdivision.

Theorem 2.5.3. [18, Theorem 2.4] Suppose that $\mathcal{P} \subset \mathbb{R}^d$ is a *facet unimodular* lattice polytope, that is, the collection of primitive facet normals of $\mathcal{P}$ forms a unimodular matrix. Then the following are true.

(i) The canonical subdivision of $\mathcal{P}$ is regular, and all the cells are compressed.

(ii) $\mathcal{P}$ has a regular unimodular triangulation.

Proposition 2.5.4. The simplex P_{K_n} has a regular unimodular triangulation.

Proof. Since the matrix of the facet normals, see proof of Theorem 2.5.2, is a signed vertex-edge incidence matrix for a path, it is totally unimodular by Lemma 1.3.2. Thus, it follows from 2.5.3 that P_{K_n} has a regular unimodular triangulation. □

Corollary 2.5.5. The simplex P_{K_n} is IDP.

Proof. If P_{K_n} admits a unimodular triangulation, it follows that P_{K_n} is IDP because cone(P_{K_n}) is a union of unimodular cones with lattice-point generators of degree 1. □

Theorem 2.5.2 implies that $h^*(P_{K_n})$ is symmetric. The following theorem implies that if $\mathcal{P}$ is reflexive and admits a regular unimodular triangulation, then $h^*_{\mathcal{P}}$ is unimodal.

Theorem 2.5.6 (Athanasiadis [1])**.** Let $\mathcal{P}$ be a d-dimensional lattice polytope with $h^*(\mathcal{P}) = (h^*_0, h^*_1, \ldots, h^*_d)$. If $\mathcal{P}$ admits a regular unimodular triangulation, then $h^*_i \geq h^*_{d-i+1}$ for $1 \leq i \leq \lfloor (d+1)/2 \rfloor$,

$$h^*_{\lfloor (d+1)/2 \rfloor} \geq \cdots \geq h^*_{d-1} \geq h^*_d$$

and

$$h^*_i \leq \binom{h^*_1 + i - 1}{i}$$

for $0 \leq i \leq d$.

Corollary 2.5.7. For each $n \geq 2$, $h^*(P_{K_n})$ is unimodal.

As a consequence of computing lattice points in the fundamental parallelepiped, the following is an interesting classification of all lattice points in cone$\,(P_{K_n})$.

Theorem 2.5.8. The lattice points at height h in cone$\,(P_{K_n})$ are in bijection with weak compositions of hn of length n, where the height of the lattice point in the cone is given by the last coordinate of the lattice point.

Proof. This is a straightforward consequence of the fact that cone$\,(P_{K_n})$ is lattice equivalent to the cone over the n^{th} dilate of a standard simplex of dimension $n-1$. □

Corollary 2.5.9. The Ehrhart polynomial of P_{K_n} is $L_{P_{K_n}}(t) = \binom{tn+n-1}{n-1}$.

Proof. The number of weak compositions of tn of length n is $\binom{tn+n-1}{n-1}$. Then the result follows directly from Theorem 2.5.8. $\square$

Corollary 2.5.10. The lattice points of $\Pi_{P_{K_n}}$ are in bijection with weak compositions of hn of length n with each part of size strictly less than n.

Proof. Each $\mathbf{x} \in \Pi_{P_{K_n}} \cap \mathbb{Z}^n$ is of the form $\mathbf{x} = \dfrac{1}{\kappa n}\mathbf{b} \cdot [L_B \mid \mathbb{1}]$ such that $0 \leq \dfrac{b_i}{\kappa n} < 1$ for each $i \in [n]$, i.e., $0 \leq \dfrac{b_i}{\kappa} < n$. Each coordinate of the lattice point has the form $x_i = \left(\sum_{j=1}^{i} \dfrac{b_j}{\kappa}\right) - ih$, which is an integer. It follows by induction on j that κ divides b_j for each $1 \leq j \leq n$. Then it follows from $\dfrac{1}{\kappa}\sum_{i=1}^{n} b_i = hn$ that $\left(\dfrac{1}{\kappa}\mathbf{b}\right)$ is a weak composition of hn of length n with parts no greater than $n-1$.

With each $c \in \{\text{length } n \text{ weak compositions of } tn \text{ with parts of size less than } n\}$, associate κc with $\mathbf{b}$ such that the i^{th} coordinate of $\mathbf{b}$ is κ times the i^{th} part of the weak composition c. This $\mathbf{b}$ will generate a lattice point in the fundamental parallelepiped. The result follows. $\square$

Proposition 2.5.11. For each $n \geq 2$, the h^*-vector of P_{K_n} is given by

$$h^*(P_{K_n}) = (1, m_1, \ldots, m_{n-1})$$

where m_i is the number of weak compositions of in of length n with parts of size less than n.

Proof. From Lemma 2.2.2, h_i^* enumerates $|\{\Pi_{P_{K_n}} \cap \{x_n = i\} \cap \mathbb{Z}^n\}|$. By Corollary 2.5.10, the result follows. $\square$

2.6 Graph operations and Laplacian simplices

Connections across seemingly distinct fields is part of the natural beauty of mathematics. This section explores the correspondence between graph theoretic operations and polytopal properties. A preliminary topic of interest is to determine which graph structures yield Laplacian simplices in the same lattice equivalence class. The following operation is one such graph structure.

Proposition 2.6.1. Let G be a connected graph on n vertices such that the following cut is possible. Partition $V(G)$ into vertex sets A and B such that all edges between A and B are incident to a single vertex $x \in A$; label those edges $\{e_1, \ldots, e_k\}$. Additionally suppose x has a leaf with adjacent vertex $y \in A$. Form a new graph G' by moving the edges $\{e_1, \ldots, e_k\}$ previously incident to x to be incident to y. Then G' has vertex set $V(G)$, and edge set $(E(G) \setminus \{e_1, \ldots, e_k\}) \cup \{\{y, v_i\} : i = 1, \ldots, k\}$ where $e_i = \{x, v_i\} \in E(G)$. Then $P_G \cong P_{G'}$.

Proof. Label the vertices of G with $[n]$. Observe G' has the same labels since $V(G) = V(G')$. We refer to each vertex by its label for simplicity. Let $N_G(i)$ be the set of neighbors of vertex i in G, that is, $N_G(i) := \{j \in V(G) \mid \{i, j\} \in E(G)\}$. Let L be the Laplacian matrix of G and L' be the Laplacian matrix of G'. We describe row operations that take each row $\mathbf{r_i} \in L$ to row $\mathbf{r'_i} \in L'$. For each $i \in V(G)$, $1 \leq i \leq n$, we have the following cases.

Consider $i \in A$ such that $i \neq x, y$. Then $N_G(i) = N_{G'}(i)$, so we set $\mathbf{r'_i} = \mathbf{r_i}$ since the i^{th} row is the same in L and L'. Then $\mathbf{r'_i} \in L'$.

Consider $i \in B \setminus N_{G'}(x)$. Again, $N_G(i) = N_{G'}(i)$, so we set $\mathbf{r'_i} = \mathbf{r_i}$ and have $\mathbf{r'_i} \in L'$.

Consider $i \in B \cap N_{G'}(x)$. The degree of i is constant in G and G', but $\{i, x\} \in E(G)$ becomes $\{i, y\} \in E(G')$ in the described algorithm. Set $\mathbf{r'_i} = \mathbf{r_i} - \mathbf{r_y}$ to reflect the change in incident edges of i from G to G'. Since $y \in V(G)$ is a leaf, $\mathbf{r'_i}$ now has 0 in the x^{th} coordinate, -1 in the y^{th} coordinate, and all remaining coordinates are unchanged. Then $\mathbf{r'_i} \in L'$.

Consider $i = x$. Set $\mathbf{r'_x} = \mathbf{r_x} + \sum_{j \in B} \mathbf{r_j}$. Observe $N_G(x) \setminus N_{G'}(x) = \{v_1, \ldots, v_k\}$. Then adding $\sum_{\ell=1}^{k} \mathbf{r}_{\mathbf{v}_\ell}$ decreases the x^{th} coordinate of $\mathbf{r_x}$ by k, which is the new degree of vertex $x \in V(G')$. Adding the other rows does contribute to the x^{th} coordinate of $\mathbf{r'_x}$ since those vertices are not adjacent to $x \in V(G)$; however, we must add all rows corresponding to $j \in B$ to obtain a 0 in all coordinates indexed by $j \in B$. Notice the coordinates indexed by the vertices in A remain fixed. Then $\mathbf{r'_x} \in L'$.

Finally consider $i = y$. Set $\mathbf{r'_y} = (k+1)\mathbf{r_y} - \sum_{j \in B} \mathbf{r_j}$. The y^{th} coordinate of $\mathbf{r'_y}$ is $k + 1$, which is the degree of y in $V(G')$. Observe $N_{G'}(y) \setminus N_G(y) = \{v_1, \ldots, v_k\}$. Then subtracting $\sum_{\ell=1}^{k} \mathbf{r}_{\mathbf{v}_\ell}$ from $(k+1)\mathbf{r_y}$ ensures the x^{th} coordinate of $\mathbf{r'_y}$ is -1. We subtract all rows corresponding to $j \in B$ from $(k+1)\mathbf{r_y}$ to obtain a -1 in all coordinates of $\mathbf{r'_y}$ indexed by $\{v_\ell\}_{\ell=1}^{k}$. Then $\mathbf{r'_y} \in L'$.

It is straightforward to verify that the collection of row operations described above is a unimodular transformation of the Laplacian matrix and thus can be represented by the multiplication of unimodular matrix $U \in \mathbb{Z}^{n \times n}$ such that $U \cdot L = L'$. It follows that $U \cdot L(n) = L'(n)$. Thus $\operatorname{conv}\left(L(n)^T\right) = \operatorname{conv}\left(L'(n)^T\right)$, and we have shown $P_G \cong P_{G'}$. $\square$

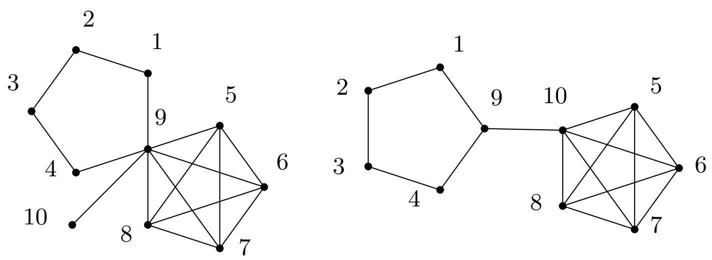


Figure 2.2: The wedge of K_5 and C_5 with a leaf and the bridge of K_5 and C_5.

Example 2.6.2. It is straightforward to verify that with the following assignment, the graphs in Figure 2.2 are related via Proposition 2.6.1, and thus their respective

Laplacian simplices are lattice equivalent. Consider $A = \{1, 2, 3, 4, 9, 10\}$, let $x = 9$, and let $y = 10$ in the graph on the left as the appropriate assignment.

It is not obvious which graph operations, aside from the transformations detailed in the proof of Proposition 2.6.1 and those found in Proposition 2.3.5, will result in unimodularly equivalent Laplacian simplices. It would be interesting to investigate this phenomenon further.

Given a graph G with reflexive P_G, we have already seen that whiskering a graph preserves the reflexivity of $P_{W(G)}$. We next assert in Theorem 2.6.5 that bridging two graphs under certain conditions can produce a graph with a reflexive Laplacian simplex. First we will require the following lemma.

Lemma 2.6.3. Let $A \in \mathbb{Z}^{k\times k}$. If $(\det A)$ divides mC_{ki} for each $i \in [k]$, where C_{ki} is the cofactor of A, and $A\mathbf{x} = \mathbb{1}$ has an integer solution $\mathbf{x} \in \mathbb{Z}^k$, then $A\mathbf{w} = [1, \ldots, 1, 1 + m]^T$ has an integer solution $\mathbf{w} \in \mathbb{Z}^k$.

Proof. Notice we can write

$$A\mathbf{w} = A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y} = \begin{bmatrix} 1 \\ \vdots \\ 1 \\ 1+m \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ \vdots \\ 0 \\ m \end{bmatrix}.$$

Solving the system $A\mathbf{y} = [0, \ldots, 0, m]^T$ yields

$$\mathbf{y} = A^{-1} \cdot \begin{bmatrix} 0 \\ \vdots \\ 0 \\ m \end{bmatrix} = \frac{1}{\det A} C^T \cdot \begin{bmatrix} 0 \\ \vdots \\ 0 \\ m \end{bmatrix} = \frac{m}{\det A} \begin{bmatrix} C_{k1} \\ C_{k2} \\ \vdots \\ C_{kk} \end{bmatrix}$$

in which C_{ki} is the cofactor of A. The above is an integer for each $i \in [k]$ by assumption. Set $w_j = x_j + y_j \in \mathbb{Z}$, and the result follows. $\square$

We apply Lemma 2.6.3 when considering a connected graph G on $m = n$ vertices with $A = L_B(i \mid \emptyset)$ for any $i \in [n]$. Here $\det L_B(i \mid \emptyset) = \pm\kappa$. Observe in this case the condition $A\mathbf{x} = \mathbb{1}$ for all $i \in [n]$ is equivalent to P_G being a reflexive Laplacian simplex.

Lemma 2.6.4. For all $n \geq 1$, $G = K_n$ satisfies the conditions of Lemma 2.6.3; that is, for each $i \in [n-1]$, κ divides nM_{nj} for each $1 \leq j \leq n-1$. Here $M_{nj} = \det L_B(i, n \mid j)$.

Proof. It is sufficient to show for each $1 \leq i, j \leq n-1$, κ divides nM_{ij} where $M_{ij} = \det L(i, n \mid j, n)$. By Lemma 2.1.7, this implies the result. For $G = K_n$, recall Cayley's formula yields $\kappa = n^{n-2}$. Then we must show n^{n-3} divides M_{ij}.

If $i = j$, using row operations on $L(i, n \mid i, n) \in \mathbb{Z}^{(n-2)\times(n-2)}$ which preserve the determinant, we have $M_{ii} = 2n^{n-3}$. If $i \neq j$, $L(i, n \mid j, n) \in \mathbb{Z}^{(n-2)\times(n-2)}$ contains exactly one row and one column with all entries of -1. A computation of the determinant using row reduction yields $M_{ij} = -n^{n-3}$. $\square$

It is clear that when $G = C_{2k+1}$ and G is a tree, $L(G)$ also satisfies the conditions of Lemma 2.6.3. In general, we show graphs which satisfy these conditions can be bridged in a way so that the resulting graph has a reflexive Laplacian simplex.

Theorem 2.6.5. Let G and G' be graphs with vertex set $[n]$ such that P_G and $P_{G'}$ are reflexive. Suppose κ_G divides nM_{ij} and $\kappa_{G'}$ divides nM'_{ij} for all $i, j \in [n-1]$, where $M_{ij} = \det L_B(i, n \mid j)$ with L as the Laplacian matrix of G, and M'_{ij} is defined similarly. Let H be the graph formed by G and G' with $V(H) = V(G) \uplus V(G')$ and $E(H) = E(G) \uplus E(G') \uplus \{i, i'\}$ where $i \in V(G)$ and $i' \in V(G')$. Then P_H is reflexive.

Proof. To show P_H is reflexive, we show P_H^* is a lattice simplex. Label the vertices of H such that $V(G) = [n]$, $V(G') = [2n] \setminus [n]$. Let L_B, $L_B(G)$, and $L_B(G')$ be the Laplacian matrices with basis B of the graphs H, G and G', respectively. Then L_B is of the form

$$\left[\begin{array}{c|c|c} & 0 & \\ L_B(G) & \vdots & 0 \\ & 0 & \\ & 1 & \\ \hline & -1 & \\ 0 & 0 & L_B(G') \\ & \vdots & \\ & 0 & \end{array}\right].$$

For $1 \le i \le 2n$, the vertex $\mathbf{v_i}$ of P_H^* is the solution to $L_B(i \mid \emptyset)\mathbf{v_i} = \mathbb{1}$. We consider two cases: $i \in [n-1]$ and $i = n$. The cases $i = n+1$ and $i \in [2n] \setminus [n+1]$ follow without loss of generality.

First suppose $i \in [n-1]$. Then $L_B(i \mid \emptyset)\mathbf{v_i} = \mathbb{1}$ can be solved the following way. Multiply each side of the equation on the left by the $(2n-1) \times (2n-1)$ unimodular matrix

$$\left[\begin{array}{c|cc|cccc} & 0 & 0 & & & & \\ I_{n-2} & \vdots & \vdots & & 0 & & \\ & 0 & 0 & & & & \\ \hline & 1 & 1 & 1 & \cdots & \cdots & 1 \\ & 0 & 1 & 1 & \cdots & \cdots & 1 \\ \cline{4-7} 0 & 0 & 0 & & & & \\ & \vdots & \vdots & & I_{n-1} & & \\ & 0 & 0 & & & & \end{array}\right]$$

to obtain

$$\left[\begin{array}{c|c|c} L_B(G)(i\mid\emptyset) & \begin{matrix}0\\ \vdots\\ \vdots\\ 0\end{matrix} & 0 \\ \hline & -1 & 0 \quad \cdots \quad 0 \\ \cline{3-3} 0 & \begin{matrix}0\\ \vdots \\ \vdots\\ 0\end{matrix} & L_B(G')(1\mid\emptyset)\end{array}\right] \mathbf{v_i} = \left[\begin{array}{c} 1\\ \vdots\\ 1\\ n+1\\ n\\ \hline 1\\ \vdots\\ \vdots\\ 1\end{array}\right].$$

We write $(\mathbf{v_i})_k$ to denote the k^{th} coordinate of $\mathbf{v_i}$. Then $(\mathbf{v_i})_k \in \mathbb{Z}$ for all $k \in [n-1]$ by Lemma 2.6.3. Observe from the above multiplication $(\mathbf{v_i})_n = -n$. Finally, $(\mathbf{v_i})_k \in \mathbb{Z}$ for all k, $n+1 \leq k \leq 2n-1$, as a consequence of $P_{G'}$ being reflexive, i.e., $P^*_{G'}$ is a lattice polytope.

Now suppose $i = n$. Replace I_{n-2} with I_{n-1} and follow the same argument as above. Then $(\mathbf{v_i})_n = -n$, and it follows all other coordinates of $\mathbf{v_i}$ are integers since P^*_G and $P^*_{G'}$ are lattice polytopes. $\square$

Remark 2.6.6. It follows from Theorem 2.6.5 that bridging a tree to a graph G with P_G reflexive and $L(G)$ satisfying the appropriate division condition on minors will result in a new reflexive Laplacian simplex. Further, Proposition 2.3.5 shows that the equivalence class of the resulting reflexive simplex is independent of the choice of tree used in the attachment. The following example demonstrates this construction.

Example 2.6.7. The two graphs in Figure 2.3 have the same Laplacian simplex, see Proposition 2.3.5. Additionally, the simplex will be reflexive by Theorem 2.6.5.

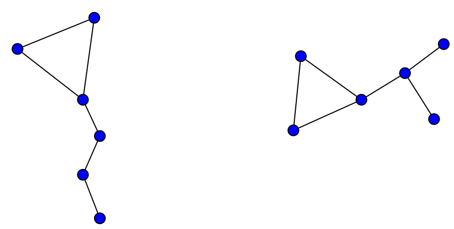


Figure 2.3: The bridge of K_3 with two different trees on 3 vertices.

Remark 2.6.8. One way to obtain the bridge graph construction described above is the application of Proposition 2.6.1 to the wedge of two graphs G and G' with a leaf attached to the wedge point. Thus, the wedge of G and G' with a leaf attached to the wedge point is reflexive if G and G' satisfy the conditions in Theorem 2.6.5.

Theorem 2.6.5 applies to graphs we have already studied, as seen in the following Corollaries.

Corollary 2.6.9. If $G, G' \in \{T_{2k+1}, C_{2k+1}, K_{2k+1}\}$ for a fixed $k \in \mathbb{Z}_{\geq 1}$, then the bridge graph between these is associated to a reflexive Laplacain simplex.

Proof. For cyclic graphs on n vertices, the number of spanning trees is n. This and Lemma 2.6.4 show that trees, cyclic graphs, and complete graphs satisfy the condition κ divides $|V(G)| \cdot M_{ij}$, as described in Lemma 2.6.3. Additionally, $P_{T_{2k+1}}, P_{K_{2k+1}}$ and $P_{C_{2k+1}}$ are reflexive Laplacian simplices as shown in Propositions 2.3.1, 2.4.1, and 2.5.2. $\square$

Example 2.6.10. The Laplacian simplex associated to the graphs in Figure 2.2 is reflexive by Corollary 2.6.9.

Corollary 2.6.11. If $G, G' \in \{T_{2k}, K_{2k}\}$ for a fixed $k \in \mathbb{Z}_{\geq 1}$, then the bridge graph between these is associated to a reflexive Laplacian simplex.

2.7 Further questions

This brief section highlights future questions and directions based on the results in the chapter.

Proposition 2.6.1 asserts the wedge with a leaf and the bridge graph yield Laplacian simplices of the same equivalence class. Perhaps there are other graph operations which preserve the equivalence class of the polytope. Additionally we can attempt to classify which simplices are Laplacian simplices.

Question 2.7.1. Which polytopes in the equivalence class of P_G can be recognized as the convex hull of a Laplacian matrix?

An important focus in the field is to understand reflexive polytopes, unimodal h^*-vectors, and IDP polytopes.

Question 2.7.2. Which graphs yield reflexive P_G? Is there a graph characteristic that implies reflexivity?

Theorems 2.3.1, 2.4.1, and 2.5.2, have addressed specific families of such graphs; however, there are many families still to be considered. We also have a characterization of reflexivity on the Laplacian matrix involving sums of second minors in Theorem 2.1.8, but this is cumbersome to check in practice. It would be nice to have a combinatorial property on the graph for this property.

Two graph operations which behave nicely with reflexivity are bridging and whiskering graphs.

Question 2.7.3. Which other graph operations preserve reflexivity of Laplacian simplices?

On the flip side, we can look at operations on polytopes to yield interesting connections with graph-theoretic operations.

Question 2.7.4. Are there polytopal operations on the Laplacian simplex which can be recognized as a graph operations on the underlying graph?

Of course we can also continue to investigate other properties.

Question 2.7.5. Which G have unimodal $h^*(P_G)$? Which P_G are IDP? Is there a graph characteristic which implies either of the above?

Chapter 3 Laplacian Simplices Associated to Digraphs

This chapter extends the study of Laplacian simplices by consider the construction from Chapter 2 on graphs with directed edges, known as digraphs. This chapter is based on joint work with coauthors Gabriele Balletti and Akiyoshi Tsuchiya. Most results will be published and can be found [2].

3.1 Digraphs

Definition 3.1.1. A *directed graph* or *digraph* consists of a vertex set $V(D) = [n]$ and a directed edge set $E(D)$. A *directed edge* $e = (i, j) \in E(D)$ points from a vertex i, called the *tail* of e, to another vertex j, called the *head* of e. In addition, we use the following notation and language for digraphs.

(i) The *outdegree* of i, $\text{outdeg}(i)$, is the number of edges with vertex i as a tail.

(ii) The *indegree* of i, $\text{indeg}(i)$, is the number of edges with vertex i as a head.

(iii) We call D *strongly connected* if it contains a directed path from i to j for every pair of distinct vertices $i, j \in [n]$.

(iv) We call D *weakly connected* if there exists a path (not necessarily directed) between i and j for every pair of distinct vertices $i, j \in [n]$.

(v) A *converging tree* is a weakly connected digraph having one vertex with outdegree 0, called the *root*, and all other vertices have outdegree 1.

(vi) We say that a subgraph D' of D is *spanning* if the vertex set of D' is $[n]$.

Multiple directed edges between vertices are allowed in D. Let $a_{i,j}$ be the number of directed edges having tail on the vertex i and head on the vertex j of D, with $i, j \in [n]$ and $i \neq j$. Since loops will not affect the Laplacian matrix, we assume D to be without loops, and thus $a_{i,i} = 0$ for all $i \in [n]$. We also assume D has no isolated vertices, i.e. vertices with indegree and outdegree equal to zero. The information from a digraph is encoded in matrices similar to those used in the undirected graph case, Definition 1.3.1.

Definition 3.1.2. Let D be a digraph with vertex set $[n]$ and directed edge set $E(D)$.

(i) The *outdegree matrix*, denoted $O(D)$, is the $n\times n$ matrix with entries $(d_{i,j})_{1\leq i,j\leq n}$, with $d_{i,j} = \text{outdeg}(i)$, if $i = j$, and $d_{i,j} = 0$ otherwise.

(ii) The *adjacency matrix*, denoted $A(D)$, is the $n\times n$ matrix with entries $(a_{i,j})_{1\leq i,j\leq n}$. As above, $a_{i,j}$ is the number of directed edges with tail i and head j, with $i, j \in [n]$ and $i \neq j$.

(iii) The *Laplacian matrix* is defined to be the $n \times n$ matrix $L(D) := O(D) - A(D)$.

Observe the sum of the entries in each row of $L(D)$ is zero. Thus the rank of the Laplacian matrix is never maximal, i.e.

$$\mathrm{rk}(L(D)) \leq n-1. \tag{3.1}$$

A combinatorial interpretation for having equality in (3.1) is given by the Matrix-Tree Theorem, which is presented here in its generalized version for digraphs. The interpretation is given in terms of *spanning converging trees* of D.

Definition 3.1.3. For any $i \in [n]$, we denote by c_i the number of spanning trees which converge to i, i.e. the converging trees of D with n vertices having i as the root. We denote by $c(D)$ the total number of spanning converging trees of D, i.e. $c(D) := \sum_{i=1}^{n} c_i$. The number $c(D)$ is usually referred to as the *complexity* of the digraph D.

Theorem 3.1.4 (Matrix-Tree Theorem [37, Theorem 5.6.4])**.** Let D be a digraph without loops on the vertex set $[n]$. Let $i, j \in [n]$, and $L(i \mid j)$ the matrix obtained from $L(D)$ by removing its i^{th} row and j^{th} column. Then the determinant of $L(i \mid j)$ equals, up to sign, the number of spanning trees of D converging to i, i.e.

$$(-1)^{i+j} \det L(i \mid j) = \det L(i \mid j) = c_i.$$

In particular, the complexity of D is

$$c(D) = \sum_{i=1}^{n} \det L(i \mid i).$$

3.2 Laplacian polytopes

Let D be a digraph on the vertex set $[n]$. To D we associate a convex polytope in $\mathbb{R}^n$ having vertices in the integer lattice $\mathbb{Z}^n$.

Definition 3.2.1. We call the *Laplacian polytope* associated to D the polytope

$$P_D := \mathrm{conv}\,(\mathbf{v}_1, \ldots, \mathbf{v}_n) \subseteq \mathbb{R}^n,$$

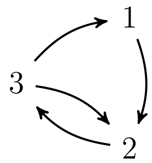
where $\mathbf{v}_i$ is the i^{th} row of the Laplacian matrix of D.

The polytope P_D is not full-dimensional. Since the sum of the entries in each row of $L(D)$ is zero, P_D is contained in the hyperplane $H := \{\mathbf{x} = (x_1, \ldots, x_n) \mid \sum_{i=1}^{n} x_i = 0\}$ of $\mathbb{R}^n$. In particular, the dimension of the Laplacian polytope, $\dim(P_D)$, equals the rank of the Laplacian matrix $L(D)$. When the rank of $L(D)$ is equal to $n-1$, then P_D is a simplex, called the *Laplacian simplex* associated to D.

Remark 3.2.2. The Laplacian simplex in this context is a generalization of the Laplacian simplex, see Definition 2.1.4, explored in Chapter 2. For a connected simple graph G with edges $E(G)$, define a digraph D_G with directed edge set $E(D_G) = \{\{i,j\},\{j,i\} \mid \{i,j\} \in E(G)\}$. The Laplacian matrix $L(D_G)$ is equal to the Laplacian matrix $L(G)$, and thus the resulting simplices are equal, that is, $P_G = P_{D_G}$.

Given a Laplacian simplex P_D, one can easily get a full-dimensional unimodularly equivalent copy of P_D by considering the lattice polytope defined as the convex hull of the rows of $L(D)$ with one column deleted. An example of this can be observed in Example 3.2.3.

Example 3.2.3. Let D be the following digraph with its Laplacian matrix $L(D)$.



$$L(D) = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

Note that $L(D)$ has rank two, which means P_D is a two dimensional simplex in $\mathbb{R}^3$. Full-dimensional unimodularly equivalent copies of P_D can be obtained by deleting any of the columns of $L(D)$ and considering the convex hull of the rows as in Figure 3.1.

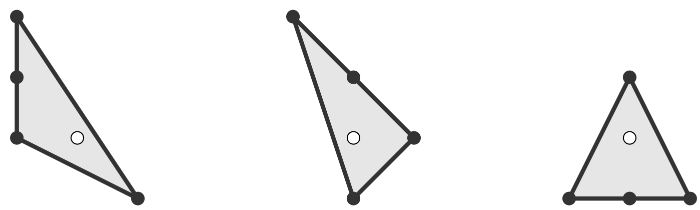


Figure 3.1: Three unimodularly equivalent full-dimensional copies of P_D obtained by deleting the first, second, and third columns of $L(D)$, respectively.

From the Matrix-Tree Theorem (Theorem 3.1.4), the following characterization can be immediately obtained.

Proposition 3.2.4. Let D be a digraph on n vertices. The following are equivalent:

(i) D has positive complexity $c(D)$.

(ii) $\operatorname{rk} L(D) = n - 1$.

(iii) P_D is an $(n-1)$-simplex.

We focus our attention to the case in which a digraph D on n vertices defines an $(n-1)$-simplex. Proposition 3.2.4 asserts we will always assume the digraph D has positive complexity. As another consequence of Theorem 3.1.4, we deduce that numbers of spanning converging trees encode the barycentric coordinates of $\mathbf{0}$, where $\mathbf{0}$ is the origin of the lattice.

Proposition 3.2.5. Let D be a digraph with positive complexity. Then the numbers of spanning converging trees $c_1, \ldots, c_n$ of D encode the unique linear dependence among the vertices $\mathbf{v}_1, \ldots, \mathbf{v}_n$ of P_D, i.e.

$$\sum_{i=1}^{n} c_i \mathbf{v}_i = \mathbf{0}.$$

Proof. Since the determinant of $L(D)$ is zero, the Laplace expansion along the j^{th} column of $L(D)$ yields $\sum_{i=1}^{n}(-1)^{i+j}\det L(i \mid j)v_{i,j} = 0$, where $L(i,j)$ is the matrix of $L(D)$ obtained by removing the i^{th} row and j^{th} column of $L(D)$, and $v_{i,j}$ is the j^{th} entry of $\mathbf{v}_i$. By Theorem 3.1.4, $\det L(i \mid i) = c_i$. □

Corollary 3.2.6. Let D be a digraph on n vertices having positive complexity. Then $\mathbf{0} \in P_D$. Moreover, $\mathbf{0}$ is an interior point of P_D if and only if D is strongly connected.

Proof. The first statement is a direct consequence of Proposition 3.2.5. For the second it is enough to note that D is strongly connected if and only if each vertex has at least one spanning converging tree. □

In this setting we prove a formula for the normalized volume of P_D.

Proposition 3.2.7. Let D be a digraph with positive complexity. Then its normalized volume equals the complexity of D, i.e.

$$\operatorname{Vol}(P_D) = c(D).$$

Proof. In this case, P_D is a $(n-1)$-simplex by Proposition 3.2.4. For $i = 1,\dots,n$, we denote by F_i the facet of P_D not containing the vertex i. Let $S_i := \operatorname{conv}(\mathbf{0} \cup F_i)$ and $I := \{i \in [n] \mid \mathbf{0} \notin F_i\}$. By Proposition 3.2.5, $\mathbf{0} \in P_D$, so the set $\{S_i \mid i \in I\}$ forms a triangulation of P_D. In particular

$$\operatorname{Vol}(P_D) = \sum_{i \in I} \operatorname{Vol}(S_i).$$

Let S_i' be the unimodularly equivalent copy of S_i obtained as the convex hull of the rows of $L(i \mid i)$.

$$\operatorname{Vol}(P_D) = \sum_{i\in I}\operatorname{Vol}(S_i) = \sum_{i\in I}\operatorname{Vol}(S_i') = \sum_{i\in I}\det L(D)_{i,i} = \sum_{i\in I} c_i = \sum_{i=1}^{n} c_i,$$

where the fourth equality follows from Theorem 3.1.4. □

Recall from Proposition 2.1.5 the normalized volume of the Laplacian simplex P_G is $|V(G)|$ times the number of spanning trees of G. If we realize G as a digraph using Remark 3.2.2, each vertex in D_G has the same number of converging spanning trees c_i. More concretely, for each spanning tree of G, we can generate $|V(G)|$ spanning converging trees in D_G by choosing each vertex to be the root. Thus we also have $\operatorname{Vol} P_G = c(D_G)$.

3.3 Connections with other families of simplices

Laplacian simplices associated to strongly connected digraphs have interesting intersections with the study of weighted projective space arising from algebraic geometry as well as the study of other families of simplices. We use these connections to describe properties of Laplacian simplices with particular attention to reflexivity, the integer decomposition property, and h^*-vectors of lattice polytopes. First we establish a connection with weighted projective space.

Definition 3.3.1. Given positive integers $\lambda_1, \dots, \lambda_n$ which are coprime, i.e. such that $\gcd\{\lambda_1, \dots, \lambda_n\} = 1$, we define the polynomial algebra $S(\lambda_1, \dots, \lambda_n) \coloneqq \mathbb{C}[x_1, \dots, x_n]$ graded by $\deg x_i \coloneqq \lambda_i$. A *weighted projective space* with weights $\lambda_1, \dots, \lambda_n$ is the projective variety $\mathbb{P}(\lambda_1, \dots, \lambda_n) \coloneqq \mathrm{Proj}(S(\lambda_1, \dots, \lambda_n))$.

Since $\mathbb{P}(\lambda_1, \dots, \lambda_n)$ is a toric variety, it corresponds to a fan Δ which can be characterized as follows.

Definition 3.3.2. Let $\mathbf{v}_1, \dots, \mathbf{v}_n$ be primitive lattice points which generate the lattice and satisfy $\sum_{i=1}^{n} \lambda_i \mathbf{v}_i = \mathbf{0}$, where $\gcd\{\lambda_1, \dots, \lambda_n\} = 1$. Then, up to isomorphism, the fan Δ is the fan whose rays are generated by the $\mathbf{v}_i$.

Note the fan Δ identifies uniquely the simplex $S_\Delta \coloneqq \mathrm{conv}\,(\mathbf{v}_1, \dots, \mathbf{v}_n)$. With an abuse of terminology, we say a simplex is the weighted projective space $\mathbb{P}(\lambda_1, \dots, \lambda_n)$ if it is unimodularly equivalent to the simplex S_Δ. For a detailed description, see [15, 21].

Definition 3.3.3. Given $(x_1, \dots, x_n) \in \mathbb{Z}^n_{>0}$, we say that the sequence $x_1, \dots, x_n$ is *well-formed* if, for any $i \in [n]$, $\gcd\{x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n\} = 1$.

Proposition 3.3.4. Let D be a strongly connected digraph such that the sequence $c_1, \dots, c_n$ is well-formed. Then P_D is equivalent to the weighted projective space $\mathbb{P}(c_1, \dots, c_n)$.

Proof. Let $M \coloneqq \mathbb{Z}^n \cap \mathrm{aff}\,(P_D)$ be the ambient lattice of P_D. We first prove that all the vertices of P_D are primitive in M. Suppose that there exists $j \in [n]$ such that $\mathbf{v}_j$ can be written as $k\mathbf{u}$ with $\mathbf{u} \in M$ primitive and $k \in \mathbb{Z}_{>0}$. Then $k | \det L(i \mid i)$ for any $i \in [n]$, $i \neq j$. Recall $\det L(i \mid i) = c_i$. Since $c_1, \dots, c_n$ is a well-formed sequence, we get $k = 1$.

Now we prove that the vertices of P_D span the lattice. Let L be the lattice spanned by all the vertices, and L_i the lattice spanned by all the vertices $\mathbf{v}_j$ such that $j \neq i$. Then we have the following inclusions of subgroups of M: $L_i \subseteq L \subseteq \mathbb{Z}^n$. In particular for all i, $|M : L||L : L_i| = |M : L_i| = \det L(i \mid i) = c_i$, which implies that $L = M$. □

In [12, 22], characterizations for properties of weighted projective spaces are given in terms of their weights and are used to perform classifications. We use these results to translate properties of D to properties of P_D. Motivated by the open questions mentioned in Section 1.2, we focus on reflexivity, the integer decomposition property, and a description of the h^*-polynomial.

We use the following result of Conrads, presented below in a slightly weaker form.

Proposition 3.3.5 ([12, Proposition 5.1])**.** Let $S = \mathrm{conv}\,(\mathbf{v}_1, \dots, \mathbf{v}_n)$ be an $(n-1)$-simplex such that $\sum_{i=1}^{n} q_i \mathbf{v}_i = \mathbf{0}$ for some positive integers $q_1, \dots, q_n$ satisfying $\gcd(q_1, \dots, q_n) = 1$. Then S is reflexive if and only if

$$q_i \quad \text{divides the total weight} \quad \sum_{j=1}^{n} q_j \quad \textit{for} \quad i = 1, \dots, n. \tag{3.2}$$

From this we can derive the following corollary.

Corollary 3.3.6. Let D be a strongly connected digraph such that $\gcd\{c_1, \dots, c_n\} = 1$. Then P_D is reflexive if and only if c_i divides $c(D)$ for all i.

Proposition 3.3.5 is also used by Braun–Davis–Solus [10] to define an interesting class of reflexive simplices. In particular they are interested in studying the integer decomposition property and unimodality of the h^*-vectors of such simplices constructed the following way. Let $\mathbf{q} = (q_1, \dots, q_n)$ be a nondecreasing sequence of positive integers satisfying the condition $q_j | (1 + \sum_{i \neq j} q_i)$ for all $j \in [n]$. For such a vector $\mathbf{q}$, the simplex $\Delta_{(1,\mathbf{q})}$ is defined as

$$\Delta_{(1,\mathbf{q})} := \operatorname{conv}\left(\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n, -\sum_{i=1}^{n} q_i \mathbf{e}_i\right),$$

where $\mathbf{e}_i \in \mathbb{R}^n$ is the i^{th} standard basis vector. By Proposition 3.3.5, $\Delta_{(1,\mathbf{q})}$ is a reflexive simplex. Note that the condition $q_j | (1 + \sum_{i \neq j} q_i)$ for all $j \in [n]$ implies that the sequence $1, q_1, \dots, q_n$ is well-formed, so $\Delta_{(1,\mathbf{q})}$ is equivalent to the weighted projective space with weights $(1, q_1, \dots, q_n)$.

The next proposition shows the simplices $\Delta_{(1,\mathbf{q})}$ are a subfamily of Laplacian simplices arising from special star-shaped, strongly connected digraphs.

Proposition 3.3.7. Let $\mathbf{q} = (q_1, \dots, q_n)$ be any nondecreasing sequence of positive integers such that $\gcd\{q_1, \dots, q_n\} = 1$. Then there is a strongly connected digraph D such that P_D is unimodularly equivalent to $\mathbb{P}(1, q_1, \dots, q_n)$. In particular, if $\mathbf{q}$ satisfies the condition $q_j | (1 + \sum_{i \neq j} q_i)$ for all $j = 1, \dots, n$, then P_D is unimodularly equivalent to $\Delta_{(1,\mathbf{q})}$.

Proof. As in Figure 3.2, we define D as the star-shaped digraph on the vertices $1, \dots, n+1$ such that

1. for $i = 1, \dots, n$ there are q_i many edges directed from 1 to $i+1$;
2. for $i = 1, \dots, n$ there is one edge directed from $i+1$ to 1.

It is easy to verify that $c_1 = 1$ and, for $i \geq 2$, $c_i = q_{i-1}$. Proposition 3.3.4 concludes the proof. □

In [10] an explicit formula for the h^*-polynomial of the simplices $\Delta_{(1,\mathbf{q})}$ is given. Such a formula can be also extracted from [22], where it is proved in the more general setting of weighted projective spaces; however, the formulation given in [10] perfectly fits our needs.

Theorem 3.3.8 ([10, Theorem 2.5])**.** The h^*-polynomial of $\Delta_{(1,\mathbf{q})}$ is

$$h^*(z) = \sum_{b=0}^{q_1 + \cdots + q_n} z^{w(b)}$$

where

$$w(b) := b - \sum_{i=1}^{n} \left\lfloor \frac{q_i b}{1 + q_1 + \cdots + q_n} \right\rfloor.$$

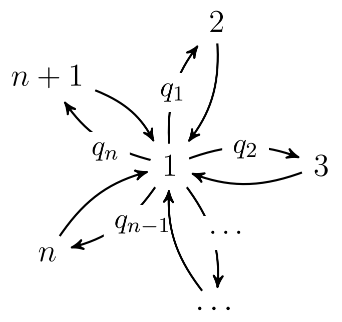


Figure 3.2: The star shaped digraph D such that $P_D = \mathbb{P}(1, q_1, \dots, q_n)$. The label on an edge from i to j represents the total number of edges from i to j.

Finally, in [10], necessary conditions for a $\Delta_{(1,\mathbf{q})}$ simplex to be IDP are given.

Lemma 3.3.9 ([10, Corollary 2.7])**.** If $\Delta_{(1,\mathbf{q})}$ is IDP, then for all $j = 1, 2, \dots, n$

$$\frac{1}{q_j} + \sum_{i \neq j} \left\{ \frac{q_i}{q_j} \right\} = 1,$$

where $\left\{ \frac{q_i}{q_j} \right\}$ denotes the fractional part of $\frac{q_i}{q_j}$.

3.4 Laplacian simplices associated to cycle digraphs

We now want to extend the study of Laplacian simplices associated to cycle graphs from Section 2.4. Before we saw the Laplacian simplex associated to a cycle is reflexive if and only if the cycle has odd length n; in that case it has a unimodal h^*-vector and fails to be IDP for $n \geq 5$ [Theorems 2.4.1, 2.4.8, 2.4.11]. We generalize this study by extending the notion of cycle graphs to cycle digraphs. A natural way to extend is to consider digraphs whose underlying simple graphs are cycle graphs.

Definition 3.4.1. The *underlying simple graph* G_D of a digraph D is the simple undirected graph on the vertex set $V(G_D) \coloneqq V(D)$ such that the edge $\{i, j\}$ is in $E(G_D)$ if and only if there is at least one directed edge between i and j in D (in either of the two directions).

Since we are interested in reflexivity, we know by Corollary 3.2.6 that D has to be strongly connected; therefore, D needs to contain a cycle entirely oriented in one of the two possible directions. This generalization of cycle graphs will be made clear later, see Definition 3.4.4. Moreover, in order to ensure the presence of no more than one interior point, we will assume for each couple of vertices i, j of D, there is at most one oriented edge from i to j. The cycle digraphs we will examine are more generally considered *simple* digraphs.

Definition 3.4.2. A *simple* digraphs contains at most one directed edge from i to j, for any pair of vertices $i, j \in [n]$, $i \neq j$. Note the presence of both a directed edge from i to j and one from j to i is allowed.

As in the previous section, we restrict our attention to those digraphs having positive complexity. This case still generalizes the work in Chapter 2 and defines polytopes with at most one interior point. We prove that all the Laplacian simplices of a simple digraph on n vertices are subpolytopes of P_{K_n}, the Laplacian simplex associated to the complete simple digraph. Observe P_{K_n} is equivalent to the n^{th} dilation of an $(n-1)$-dimensional unimodular simplex, and therefore it has exactly one interior lattice point.

Proposition 3.4.3. Let D be a simple digraph on n vertices. Then P_D is a subpolytope of P_{K_n}. In particular, if D is strongly connected, then P_D has exactly one interior lattice point.

Proof. Corollary 3.2.6 implies P_D has at least one interior lattice point, so the second statement follows directly from the first one. In order to prove the first part, we show that any vertex $\mathbf{u}$ of P_D is in P_{K_n}. Up to a relabeling of the vertices, we can assume that $\mathbf{u} = (a, -1, \ldots, -1, 0, \ldots, 0)$, where a equals the number entries of u which are equal to -1. We know that the Laplacian $L(K_n)$ is

$$L(K_n) = \begin{bmatrix} n-1 & -1 & \ldots & -1 \\ -1 & n-1 & \ldots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \ldots & n-1 \end{bmatrix}.$$

We denote by $\mathbf{v}_i$ the i^{th} row of $L(K_n)$, as well as the corresponding vertex of P_{K_n}. It is then enough to prove that $\mathbf{u}$ can be written as a convex combination of the vertices of K_n, i.e. that $\mathbf{u} = \sum_{i=0}^{n} \lambda_i \mathbf{v}_i$, with $0 \leq \lambda_i \leq 1$ and $\sum_{i=0}^{n} \lambda_i = 1$. This can be done with the following choice of barycentric coordinates:

$$\lambda_i = \begin{cases} \frac{a+1}{n}, & \text{if } i = 1 \\ 0, & \text{if } 2 \leq i \leq a+1 \\ \frac{1}{n}, & \text{if } a+2 \leq i \leq n \end{cases}.$$

This proves P_D is a subpolytope of P_{K_n}. $\square$

The rest of this section is aimed to generalize the results of P_{C_n} to the case of directed cycles. Note that in order to have reflexivity (or, in particular, to have one interior lattice point) we need the digraph to be strongly connected (Corollary 3.2.6). Therefore, all cycles we consider will always contain a cycle entirely oriented in one of the two possible directions and some additional edges directed in the opposite direction. Informally speaking, we define a cycle digraph to have all the edges pointing clockwise and some edges pointing counterclockwise.

Definition 3.4.4. Let $n \geq 3$. We say that a digraph D on the vertex set $[n]$ is a *cycle digraph* if, up to a relabeling of the vertices, $E(D) = \overrightarrow{E}(D) \cup \overleftarrow{E}(D)$, where

$$\begin{aligned} \overrightarrow{E}(D) &= \{(1,2), (2,3), \ldots, (n-1,n), (n,1)\}, \\ \overleftarrow{E}(D) &\subseteq \{(n,n-1), (n-1,n-2), \ldots, (2,1), (1,n)\}. \end{aligned}$$

If such a relabeling exists, D is completely determined by $\overleftarrow{E}(D)$, and we denote it by $D = C_n^S$, where $S \subseteq [n]$ is the set of the tails of the directed edges in $\overleftarrow{E}(D)$. As an example see Figure 3.3.

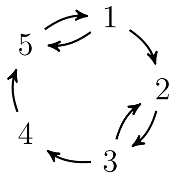
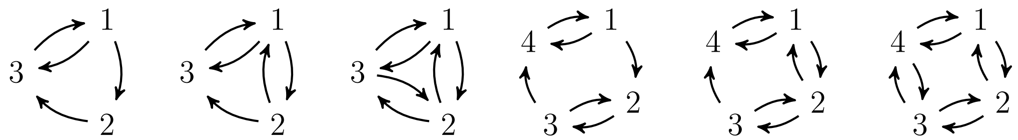


Figure 3.3: The cycle digraph $C_5^{\{1,3\}}$.

We first prove for most of the directed cycles, the associated Laplacian simplex has no lattice points other than its vertices and the origin. Borrowing some terminology from the algebraic geometers, we call a simplex with this property *terminal Fano*.

Theorem 3.4.5. Let D be any cycle digraph. P_D is terminal Fano if and only if D is not, up to a relabeling of the vertices, one of the following six exceptional directed cycles.



Proof. We prove that, for $n \geq 5$, $P_{C_n^S}$ is terminal Fano for all $S \subseteq [n]$. The lower dimensional cases are checked individually, leading to the six exceptional cases above. For each $i \in [n]$, we have $\mathbf{v}_i = a_{i-1}\mathbf{e}_{i-1} + b_i\mathbf{e}_i - \mathbf{e}_{i+1}$ where for each $j \in [n]$ $a_j \in \{-1, 0\}$ and $b_j = 1 - a_j \in \{1, 2\}$, and $a_0 = a_n$, $\mathbf{e}_0 = \mathbf{e}_n$ and $\mathbf{e}_{n+1} = \mathbf{e}_1$.

Assume that $P_{C_n^S}$ is not terminal Fano. Let $\mathbf{x} = (x_1, \ldots, x_n)$ be a lattice point in $P_{C_n^S} \setminus \{\mathbf{v}_1, \ldots, \mathbf{v}_n, \mathbf{0}\}$ and set $\mathbf{x} = \sum_{i=1}^n \lambda_i \mathbf{v}_i$ with $0 \leq \lambda_1, \ldots, \lambda_n < 1$ and $\lambda_1 + \cdots + \lambda_n = 1$. Then one has $x_i = -\lambda_{i-1} + b_i\lambda_i + a_i\lambda_{i+1} \in \{-1, 0, 1\}$ for each i, where $\lambda_0 = \lambda_n$ and $\lambda_{n+1} = \lambda_1$.

Suppose that there exists $i \in [n]$ such that $x_i = -1$. We can assume without loss of generality that $x_2 = -1$. Then we obtain $a_2 = -1$, $0 < \lambda_1, \lambda_3 < 1$ and $\lambda_j = 0$ for any $j \neq 1, 3$. This implies that $x_4 = -\lambda_3 + b_4\lambda_4 + a_4\lambda_5 = -\lambda_3 \notin \mathbb{Z}$, a contradiction. Hence we have $x_i \in \{0, 1\}$ for each i. Since $\mathbf{x} \neq \mathbf{0}$, we can assume without loss of generality that $x_2 = 1$. Then one has $b_2 = 2$ and $\lambda_2 \geq 1/2$.

If $b_3 = 1$, it follows that $x_3 = 0, \lambda_2 = \lambda_3 = 1/2$ and $\lambda_j = 0$ for any $j \neq 2, 3$. This implies that $x_4 = -\lambda_3 + b_4\lambda_4 + a_4\lambda_5 = -\lambda_3 \notin \mathbb{Z}$, a contradiction. Hence one has $b_3 = 2$.

If $x_3 = 1$, then $\lambda_3 \geq 1/2$, hence one has $\lambda_2 = \lambda_3 = 1/2$ and $\lambda_j = 0$ for any $j \neq 2, 3$. However, we obtain $x_4 = -\lambda_3 + b_4\lambda_4 + a_4\lambda_5 = -\lambda_3 \notin \mathbb{Z}$, a contradiction. Hence $x_3 = 0$.

If $b_1 = 1$, then one has $\lambda_0 = \lambda_1 = 0$. Since $2\lambda_2 - \lambda_3 = 1$ and $-\lambda_2 + 2\lambda_3 - \lambda_4 = 0$, it follows that $3\lambda_2 = \lambda_4 + 2 \geq 2$. Hence one has $\lambda_2 = 2/3, \lambda_3 = 1/3$ and $\lambda_j = 0$ for any $j \neq 2, 3$. However, we obtain $x_4 = -\lambda_3 + b_4\lambda_4 + a_4\lambda_5 = -\lambda_3 \notin \mathbb{Z}$, a contradiction. Thus, $b_1 = 2$.

Then it follows from $\lambda_2 \geq 1/2$ that $\lambda_2 = 1/2, \lambda_1 = \lambda_3 = 1/4$ and $\lambda_j = 0$ for $j \in [n] \setminus \{1, 2, 3\}$. This implies that $x_4 = -\lambda_3 + b_4\lambda_4 + a_4\lambda_5 = -\lambda_3 \notin \mathbb{Z}$, a contradiction. Therefore, $P_{C_n^S}$ is terminal Fano. □

Now we characterize reflexivity for Laplacian simplices $P_{C_n^S}$, extending Theorem 2.4.1.

Theorem 3.4.6. The Laplacian simplex $P_{C_n^S}$ associated to a cycle digraph C_n^S is reflexive if and only if one of the following conditions is satisfied:

(i) $S = \varnothing$, or

(ii) $S = [n]$ and $n = 2$, or

(iii) $S = [n]$ and n is odd, or

(iv) $\varnothing \subsetneq S \subsetneq [n]$, such that $k|c(D)$ for each integer $1 \leq k \leq K + 1$, where K is the longest chain of consecutive edges pointing counterclockwise, i.e.

$$K \coloneqq \max\{j \mid \{a+1, \ldots, a+j\} \subseteq S, \text{ for some } a \in [n]\},$$

where, since $S \subsetneq [n]$, we have assumed without loss of generality, that $1 \notin S$.

Proof. If S satisfies (1) or (2), then thanks to Corollary 3.3.6, it trivial to check that $P_{C_n^S}$ is reflexive. If S satisfies (3), then $P_{C_n^S}$ is reflexive by Theorem 2.4.1.

Suppose now that S satisfies (4). In particular we have assumed that $1 \notin S$. This implies vertex n has exactly one spanning converging tree, i.e. $c_n = 1$. As usual, c_i denotes the number of spanning trees which converge to vertex i. Then $\gcd(c_1, \ldots, c_n) = 1$, and $P_{C_n^S}$ is a weighted projective space by Proposition 3.3.4. For each vertex i we denote by K_i the length of the longest chain of consecutive edges pointing counterclockwise ending in i, i.e.

$$K_i \coloneqq \max\{j \mid \{i+1, \ldots, i+j\} \subseteq S\}, \qquad \text{for } 1 \leq i \leq n \;.$$

In particular, $K_n = 0$ and $K = \max\{K_i \mid i \in [n]\}$. Given $i \in [n]$, note there are exactly $K_i + 1$ spanning trees converging to i. There are K_i having edge set

$$\{(j, j-1), \ldots, (i+1, i), (j+1, j+2), \ldots, (n-1, n), (n, 1), \ldots, (i-1, i)\},$$

for all $j \in \{i+1, \ldots, i+K_i\}$, plus an additional "clockwise tree" with edges

$$\{(i+1, i+2), \ldots, (n-1, n), (n, 1), \ldots, (i-1, i)\}.$$

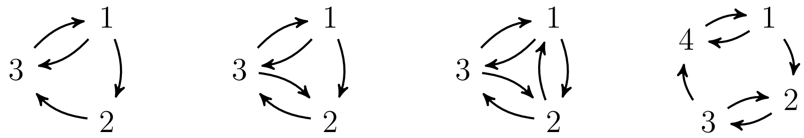
By Corollary 3.3.6, $P_{C_n^S}$ is reflexive if and only if $c_i|c(D)$, for all $i \in [n]$. We conclude by noting that if $c_i > 1$ for some $i \in [n]$, then $c_{i+1} = c_i - 1$, in particular $\{c_i \mid i \in [n]\} = \{1, \ldots, K+1\}$. □

We now have all the tools to completely characterize all reflexive IDP simplices arising from cycle digraphs and extend Corollary 2.4.12.

Theorem 3.4.7. Let C_n^S be a cycle digraph on n vertices such that $P_{C_n^S}$ is reflexive. Then $P_{C_n^S}$ possesses the integer decomposition property if and only if D satisfies one of the following conditions:

(i) $S = \varnothing$, or

(ii) D is, up to a relabeling of the vertices, one of the following directed cycles.



Proof. If $S = \varnothing$, then C_n^S is known to be a reflexive IDP simplex. If $S = [n]$, from Theorem 2.4.1 $P_{C_n^{[n]}}$ is reflexive if and only if n is odd. In this case it is known $P_{C_n^{[n]}}$ is IDP if and only if $n = 3$ [Theorem 2.4.11].

Now, assume that $\varnothing \neq S \neq [n]$ and $P_{C_n^S}$ is IDP. We use the same notation introduced in Theorem 3.4.6. Then we can assume $c_1 = 1, c_2 = K+1, c_3 = K, \ldots, c_{K+1} = 2$. Set $\mathbf{q} = (c_2, \ldots, c_n)$. It follows that $P_{C_n^S}$ is unimodularly equivalent to $\Delta_{(1,\mathbf{q})}$. By Lemma 3.3.9, we know that for each $2 \leq j \leq n$,

$$\frac{1}{c_j} + \sum_{i \neq j} \left\{ \frac{c_i}{c_j} \right\} = 1. \tag{3.3}$$

But, if $K \geq 3$, by (3.3) we get

$$\frac{1}{K+1} + \sum_{i=3}^{n} \left\{ \frac{c_i}{K+1} \right\} \geq \frac{1}{K+1} + \frac{K-1}{K+1} + \frac{K}{K+1} > 1,$$

so $K \in \{1, 2\}$. By applying (3.3) in these cases one gets $n \leq 4$. We conclude by checking all the cycle digraph having up to four vertices. $\square$

As an application of the tools developed in this section, we build a special family of cycle digraphs whose Laplacian simplices are reflexive and have non unimodal h^*-vectors.

Theorem 3.4.8. Let $\alpha, \beta, k \in \mathbb{Z}_{>0}$ such that $\alpha \leq \beta \leq k-1$ and $\alpha + \beta \leq k+1$. Let $D = C_n^S$ be the cycle digraph with $n \coloneqq 6(k+1) - 2\alpha - \beta$, and $S \coloneqq S_1 \cup S_2 \cup S_3$ where

$$\begin{aligned} S_1 &\coloneqq \{1 + 3h \mid 0 \leq h \leq \alpha - 1\}, \\ S_2 &\coloneqq \{2 + 3h \mid 0 \leq h \leq \alpha - 1\}, \\ S_3 &\coloneqq \{3\alpha + 1 + 2h \mid 0 \leq h \leq \beta - \alpha - 1\}. \end{aligned}$$

Then P_D is a reflexive simplex of dimension $6(k+1) - 2\alpha - \beta - 1$ with symmetric and nonunimodal h^*-vector

$$(\underbrace{1, \ldots, 1}_{2(k+1)-\alpha}, \underbrace{2, \ldots, 2}_{\alpha}, \underbrace{1, \ldots, 1}_{(k+1)-\alpha-\beta}, \underbrace{2, \ldots, 2}_{\beta}, \underbrace{1, \ldots, 1}_{(k+1)-\alpha-\beta}, \underbrace{2, \ldots, 2}_{\alpha}, \underbrace{1, \ldots, 1}_{2(k+1)-\alpha}).$$

Proof. An example of the digraph in the statement is demonstrated in Example 3.4.9. The digraph has no more than two consecutive vertices with outdegree two, so the number of spanning trees converging to each of the vertices of D is at most three. Specifically,

$$c_{i-1} = \begin{cases} 3, & \text{if } i \in S_1, \\ 2, & \text{if } i \in S_2 \cup S_3, \\ 1, & \text{if } i \in [n] \setminus S. \end{cases}$$

Above, we set c_0 to be c_n. Since each c_i divides $c(D) = \sum_{i=1}^{n} c_i = 6(k+1)$, then P_D is reflexive by Theorem 3.4.6. Now we use Theorem 3.3.8 to describe its h^*-polynomial. In particular,

$$h^*(z) = \sum_{b=0}^{c(D)-1} z^{w(b)}, \quad \text{with } w(b) = b - \sum_{i=1}^{n} \left\lfloor \frac{c_i b}{6(k+1)} \right\rfloor.$$

In our case this becomes

$$w(b) = b - \alpha \left\lfloor \frac{b}{2(k+1)} \right\rfloor - \beta \left\lfloor \frac{b}{3(k+1)} \right\rfloor,$$

which yields

$$w(b) = \begin{cases} b, & \text{if} \quad 0 \le b \le 2(k+1) - 1, \\ b - \alpha, & \text{if} \quad 2(k+1) \le b \le 3(k+1) - 1, \\ b - \alpha - \beta, & \text{if} \quad 3(k+1) \le b \le 4(k+1) - 1, \\ b - 2\alpha - \beta, & \text{if} \quad 4(k+1) \le b \le 6(k+1) - 1. \end{cases}$$

From this, using the condition $\alpha + \beta \le k + 1$, we deduce the i^{th} coefficient of the h^*-polynomial:

$$h_i^* = \begin{cases} 2, & \text{if} \quad \begin{cases} 2(k+1) - \alpha \le i \le 2(k+1) - 1, \text{ or} \\ 3(k+1) - \alpha - \beta \le i \le 3(k+1) - \alpha - 1, \text{ or} \\ 4(k+1) - 2\alpha - \beta \le i \le 4(k+1) - \alpha - \beta - 1; \end{cases} \\ 1, & \text{otherwise.} \end{cases}$$

□

Example 3.4.9. Figure 3.4 provides an example of the digraph constructed in Theorem 3.4.8. In this case, $\alpha = \beta = 1$ and $k = 2$. The Laplacian simplex associated to this digraph has h^*-vector $(1, 1, 1, 1, 1, 2, 1, 2, 1, 2, 1, 1, 1, 1, 1)$.

3.5 Digraph operations and Laplacian polytopes

To goal of this section is to consider constructions on D with P_D reflexive that yield new reflexive simplices of higher dimension. This is analogous to the problem explored in Section 2.6. First we comment on a couple of general constructions.

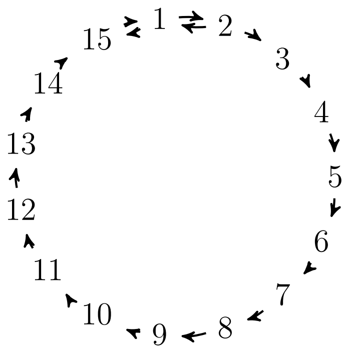


Figure 3.4: An example of the construction of Theorem 3.4.8.

Definition 3.5.1. For a polytope $\mathcal{P} \subset \mathbb{R}^{d-1}$ with vertices $\mathbf{v_1}, \ldots, \mathbf{v_m}$, define the *pyramid over* $\mathcal{P}$ as

$$\mathrm{Pyr}(\mathcal{P}) = \mathrm{conv}\left((\mathbf{v_1}, 0), \ldots, (\mathbf{v_m}, 0), (0, \ldots, 0, 1)\right) \subset \mathbb{R}^d.$$

Proposition 3.5.2. Given a digraph D with Laplacian polytope $P_D \subseteq \mathbb{R}^{n-1}$, the lattice pyramid over P_D can be obtained from the digraph formed by attaching a new vertex via one edge directed into D. Concisely, let H be the digraph with $V(H) = V(D) \uplus \{v\}$ and $E(H) = E(D) \uplus \{vv_i\}$ for some existing $v_i \in V(D)$. Then $\mathrm{Pyr}(P_D) = \mathrm{conv}\left((\mathbf{v_1}, 0), \ldots, (\mathbf{v_n}, 0), (0, \ldots, 0, 1)\right) = P_H \subseteq \mathbb{R}^n$ where $\mathbf{v_i} \in \pi_i(P_D)$.

Proof. Let $L(D) \in \mathbb{Z}^{n \times n}$ be the Laplacian matrix for D. The addition of an edge pointing from a new vertex v to an existing $v_i \in V(D)$ contributes a row with a 1 in the $(n+1)^{\text{th}}$ position, a -1 in the i^{th} position, and 0 else. Delete column i and take the convex hull of the rows to see $P_H = \mathrm{Pyr}(P_D)$. □

The polytopal operation of pyramiding over a Laplacian polytope can be easily generated using the above proposition to adjust the underlying digraph. It does not matter which existing vertex is the head of the added edge. It would be interesting to find other polytopal operations which can be stated as a graph-theoretic operation.

Any graph operation which creates the situation $\{c_i\}_{i \in V(D)} = \{c_j\}_{j \in V(D')}$ for two digraphs D and D' will result in the same Laplacian polytope, $P_D \cong P_{D'}$.

Remark 3.5.3. We have shown $P_D \cong P_{D'}$ if and only if $\{c_i\}_{i \in V(D)} = \{c_j\}_{j \in V(D')}$. As a consequence, attaching any directed tree (one directed edge in each direction for every pair of adjacent vertices) of a fixed number of vertices to a digraph D at a fixed vertex $v \in V(D)$ results in the same Laplacian polytope. More generally, attaching vertices and directed edges such that $\{c_i\}_{i \in V(D)}$ and $\{c_j\}_{j \in V(D')}$ remain equal will preserve the Laplacian polytope.

Given a reflexive simplex P_D, the following proposition provides a construction to generate a new digraph D' such that $P_{D'}$ is reflexive with $\mathrm{Vol}(P_{D'}) = m\,\mathrm{Vol}(P_D)$ for any $m \in \mathbb{Z}_{>1}$.

Proposition 3.5.4. Let P_D be a reflexive simplex with digraph D on n vertices such that $\gcd(c_1, \ldots, c_n) = 1$. Let T be any simple directed tree on k vertices which is strongly connected. Let D' be the digraph with $V(D') = V(D) \uplus V(T)$ and $E(D') = E(D) \uplus E(T) \uplus \{uv, vu\}$ for some $u \in V(D)$ and $v \in V(T)$. Then $P_{D'}$ is reflexive with $\mathrm{Vol}(P_{D'}) = c(D) + kc_u$ if and only if c_i divides kc_u for each $i \in V(D)$.

Proof. Observe that since T is a simple strongly connected directed tree, for any $j \in V(D') \cap V(T)$ the number of spanning trees of D' which converge to j is c_u. Then

$$\mathrm{Vol}(P_{D'}) = \sum_{i \in V(D') \cap V(D)} c_i + \sum_{j \in V(D') \cap V(T)} c_j = c(D) + kc_u.$$

The result follows. Additionally, if T and T' are two distinct simple directed trees on k vertices, the above construction results in the same Laplacian simplex. □

Corollary 3.5.5. Let $m \in \mathbb{Z}_{>1}$, and refer to the notation of the above proposition. If T is a simple strongly connected directed tree on $(m-1) \cdot \frac{c(D)}{c_u}$ vertices, then $P_{D'}$ is reflexive with $\mathrm{Vol}(P_{D'}) = m \cdot c(D)$.

We next provide another construction to generate a new digraph corresponding to a reflexive Laplacian simplex with volume equal to $m\mathrm{Vol}(P_D)$ for any $m \in \mathbb{Z}_{>1}$ given a reflexive simplex P_D. This uses the whisker operation from Definition 2.4.3.

Proposition 3.5.6. Let P_D be a reflexive simplex with digraph D on n vertices such that $\gcd(c_1, \ldots, c_n) = 1$. Then $P_{W(D)}$ is a reflexive simplex with $\mathrm{Vol}\, P_{W(D)} = 2 \cdot c(D)$.

Proof. Observe the number of spanning trees converging to $u_j \in V(D)$ in D is equal to the number of spanning trees converging to u_j in $W(D)$. Also for any whisker $v_j \in V(W(D)) \setminus V(D)$, the number of spanning converging trees in $W(D)$ is c_{u_j}, $u_j \in V(D)$ being the vertex whiskered with v_j. Then $\mathrm{Vol}(P_{W(D)}) = 2 \cdot c(D)$, and the result follows. □

To achieve our construction, we must now consider attaching whiskers to all vertices $v_i \in V(W(D)) \setminus V(D)$. Denote the resulting digraph as $W(D)_2$. Then $W(D)_k$ is the digraph obtained by whiskering all vertices $v_i \in V(W(D)_{k-1}) \setminus V(W(D)_{k-2})$ for $k \in \mathbb{Z}_{\geq 3}$ where $W(D) := W(D)_1$. The construction we desire is the following generalization of Proposition 3.5.6.

Proposition 3.5.7. Let P_D be a reflexive simplex with digraph D on n vertices such that $\gcd(c_1, \ldots, c_n) = 1$. For $m \in \mathbb{Z}_{\geq 1}$, $P_{W(D)_m}$ is a reflexive simplex with $\mathrm{Vol}(P_{W(D)_m}) = (m+1) \cdot c(D)$.

Proof. The proof is similar to that of Proposition . All vertices in D will have constant c_i. All vertices whiskered at $i \in V(D)$ or whiskered at whiskers of i will have c_i spanning converging trees. Then $\mathrm{Vol}(P_{W(D)_m}) = (m+1) \cdot c(D)$. □

Observe the two construtions defined above yield two distinct reflexive Laplacian simplices with equal volume.

3.6 Further questions

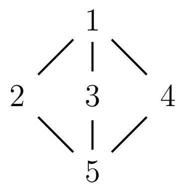
Observe that in the case of the undirected cycles in Section 2.4, the reflexivity is influenced by the number of vertices of the graph, (Theorem 2.4.1). On the other hand, when passing to the directed case we discussed in Section 3.4, it is clear (from Theorem 3.4.6) that one can build reflexive Laplacian simplices starting from cycles of any length. This can be done by orienting a cycle in one of the two directions.

It is natural to wonder how the structure of the *underlying* simple graph G_D, see Definition 3.4.1, of a digraph D plays a role in determining the reflexivity of P_D.

Definition 3.6.1. We define an *oriented graph* to be a simple digraph D such that if there is an edge pointing from i to j, then there is no edge pointing from j to i.

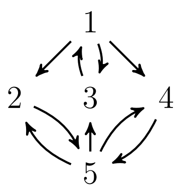
The following examples show that obtaining reflexive Laplacian simplices from digraphs with a fixed underlying simple graph is not an easy task. Example 3.6.2 shows there is a simple graph G_1 such that any of its orientations is a digraph whose Laplacian simplex is not a full-dimensional reflexive simplex. However, if we do not require the digraph to be an oriented graph, there is a simple digraph D_1 (Example 3.6.3) having G_1 as its underlying graph such that P_{D_1} is a full-dimensional reflexive simplex. On the other hand, in Example 3.6.4 we show there is a graph G_2 which is not the underlying graph of any simple digraph whose Laplacian simplex is reflexive. However, if we do not require the digraph to be simple, then there is a digraph D_2 (Example 3.6.5) having G_2 as its underlying graph such that P_{D_2} is reflexive.

Example 3.6.2. Let G be the following graph.

$G_1 =$ 

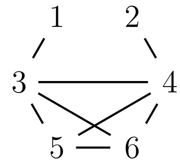
Assume D is an orientation of G_1 such that P_D is a reflexive 4-simplex. Since D must be strongly connected, we may assume, without loss of generality, that $(5,3),(3,1),(1,2),(2,5)$ are edges of D. It follows that either $(1,4),(4,5)$ or $(5,4),(4,1)$ are in $E(D_1)$. In both cases, P_D is not reflexive. So none of the orientations of G_1 lead to a reflexive simplex.

Example 3.6.3. Let D_1 be the following simple digraph.

$D_1 =$ 

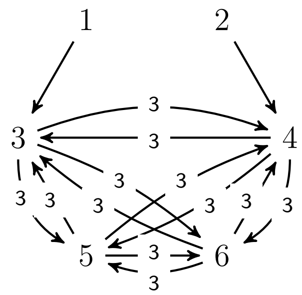
Note that its underlying simple graph is still G_1 of Example 3.6.2, but P_{D_1} is reflexive.

Example 3.6.4. Let G_2 be the following graph.

$G_2 =$ 

Note that there are finitely many possible directed simple graphs having G_2 as an underlying graph. A computer-assisted check shows none of them produces a reflexive Laplacian simplex.

Example 3.6.5. Let D_2 be the following digraph (the label on an edge from i to j, if present, represents the total number of edges from i to j).

$D_2 =$ 

Then P_{D_2} is a reflexive simplex.

In general it is still unclear how the underlying graph affects the reflexivity of the Laplacian simplex of a digraph. Examples 3.6.2 and 3.6.4 show that this is a nontrivial question. We conclude with the following three open questions.

Question 3.6.6. For which simple graphs G on $[n]$, do there exist an oriented graph D on $[n]$ such that $G_D = G$ and P_D is a reflexive $(n-1)$-simplex?

Question 3.6.7. For which simple graphs G on $[n]$, do there exist a simple digraph D on $[n]$ such that $G_D = G$ and P_D is a reflexive $(n-1)$-simplex?

Question 3.6.8. For any simple graph G on $[n]$, does there exist a digraph D on $[n]$ such that $G_D = G$ and P_D is a reflexive $(n-1)$-simplex?

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