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A Combinatorial Miscellany: Antipodes, Parking Cars, and Descent Set Powers

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A Combinatorial Miscellany: Antipodes, Parking Cars, and Descent Set Powers

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DISSERTATION

A dissertation submitted in partial
fulfillment of the requirements for
the degree of Doctor of Philosophy
in the College of Arts and Sciences
at the University of Kentucky

By
Alexander Thomas Happ
Lexington, Kentucky

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ABSTRACT OF DISSERTATION

A Combinatorial Miscellany: Antipodes, Parking Cars, and Descent Set Powers

In this dissertation we first introduce an extension of the notion of parking functions to cars of different sizes. We prove a product formula for the number of such sequences and provide a refinement using a multi-parameter extension of the Abel–Rothe polynomial. Next, we study the incidence Hopf algebra on the noncrossing partition lattice. We demonstrate a bijection between the terms in the canceled chain decomposition of its antipode and noncrossing hypertrees. Thirdly, we analyze the sum of the rth powers of the descent set statistic on permutations and how many small prime factors occur in these numbers. These results depend upon the base p expansion of both the dimension and the power of these statistics. Finally, we inspect the f-vector of the descent polytope $\mathrm{DP}_{\mathbf{v}}$, proving a maximization result using an analogue of the boustrophedon transform.

KEYWORDS: set partition, parking function, descent set, polytope, Hopf algebra, antipode

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A Combinatorial Miscellany: Antipodes, Parking Cars, and Descent Set Powers

By
Alexander Thomas Happ

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Date:	April 25, 2018

To my loving parents and my lovely wife.

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am eternally grateful to her for the essential part she has played in making this work possible.

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Chapter 1 Introduction

> A Miscellany is a collection without a natural ordering relation; I shall not attempt a spurious unity by imposing artificial ones.
>
> *J. E. Littlewood*
> *A Mathematician's Miscellany*

Before we commence an analysis of each of the topics of our combinatorial miscellany in turn, it will be useful to review certain preliminary concepts. This is the purpose of Chapter 1.

In Chapter 2, we provide a generalization on the classical notion of parking functions introduced by Konheim and Weiss in 1966. Our generalization of parking sequences departs from the theme of prior generalizations in allowing the size of the cars themselves to vary. We provide an enumeration of parking sequences and prove it using a generalization of the elegant proof for classical parking functions commonly attributed to Pollak. We also discuss a refinement of these parking sequences involving parking the cars after a trailer, whose multi-parameter enumeration we show to be related to the classical Abel–Rothe polynomials.

Chapter 3 concerns the reduced incidence Hopf algebra on posets. We focus on the noncrossing partition lattice in particular and use a map between chains in the noncrossing partition lattice and noncrossing hypertrees to prove a cancellation-free expression for its antipode. The fibers of this map are shown to collapse to a single element by both a geometric argument on the Euler characteristic of the permutahedron and a sign-reversing involution. The latter argument actually provides a Morse matching on the subposet of the order complex of the noncrossing partition lattice given by each fiber, which elicits questions concerning its topology.

Chapter 4 shifts the focus to a study of permutations and a commonly-studied statistic on them involving descents. Namely, we study sums of powers of these descent set statistics and how many small prime factors arise in them. The results, somewhat surprisingly, depend on the sum of the digits in the base p expansions of both the dimension n of the permutations and the power r of the sum. Tables 4.1 and 4.2 provide a summary of the results of this chapter.

Finally, Chapter 5 studies a polytope defined using these same descent set statistics on permutations. We extend the so-called boustrophedon transform used for computing descent set statistics to the face numbers of descent polytopes. We demonstrate the utility of this transform by proving a maximization result with relative ease.

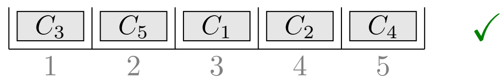


Figure 1.1: The final positions of the cars $C_1, \ldots, C_5$ after successfully parking according to the preferences $(3, 4, 1, 3, 1)$. Hence, this sequence of preferences is a parking function.

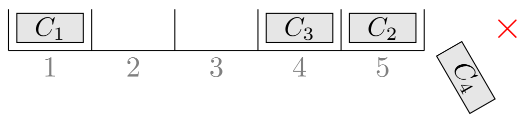


Figure 1.2: The final positions of the cars $C_1, \ldots, C_5$ after attempting to park according to the preferences $(1, 5, 4, 4, 2)$. This sequence of preferences is therefore *not* a parking function.

1.1 Parking functions

Parking functions were first introduced by Konheim and Weiss in 1966 [27]. The original definition was a colorful reformulation of a computer storage problem in terms of n cars parking in a linear parking lot of n labeled spaces. Each car would have a preferred spot in mind and would, in order, enter the parking lot and attempt to park in its preferred spot. If this spot was occupied, the car would move to the next available spot. A *parking function*, then, was a sequence of parking preferences that would allow all n cars to park according to this rule without leaving the parking lot. See Figures 1.1 and 1.2 for examples. It turns out this definition is equivalent to the following definition, which strips away these colorful roots while retaining the name.

Definition 1.1.1. *Let $\vec{a} = (a_1, a_2, \ldots, a_n)$ be a sequence of positive integers, and let $b_1 \leq b_2 \leq \cdots \leq b_n$ be the increasing rearrangement of $\vec{a}$. Then the sequence $\vec{a}$ is a parking function if and only if $b_i \leq i$ for all indices i.*

This immediately tells us that any permutation of the entries in a parking function is also a parking function, an observation that is not immediately clear using the original definition.

Theorem 1.1.2 (Foata and Riordan [19])**.** *The number of parking functions with n cars is $(n+1)^{n-1}$.*

It will be worth our while to review the following elegant proof of this fact, commonly attributed to Pollak; see [39].

Proof of Theorem 1.1.2. Add an additional space $n+1$ to the parking lot, and arrange the spaces in a circle. Allow $n+1$ also as a preferred space.

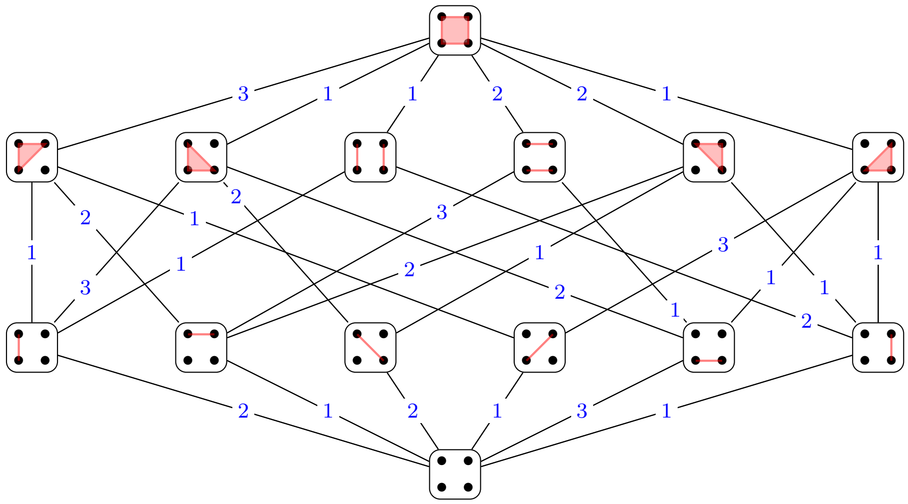


Figure 1.3: The Hasse diagram of the noncrossing partition lattice on 4 elements, NC_4. Reading the labels of the edges as one ascends any maximal chain in the diagram gives a parking function.

Now all cars will be able to park, and since there are still only n cars, there will be one empty space. The sequence $\vec{a} = (a_1, a_2, \ldots, a_n)$ is a parking function in the original sense if and only if the empty space in this arrangement is $n+1$.

If $\vec{a}$ leads to car C_i parking in space p_i, then $(a_1+j, a_2+j, \ldots, a_n+j) \pmod{n+1}$ will lead to car C_i parking in space $p_i + j$. Hence, exactly one of the vectors

$$(a_1+i, a_2+i, \ldots, a_n+i) \pmod{n+1}$$

for $0 \le i \le n$ is a parking function, and the total number of parking functions is given by

$$\frac{(n+1)^n}{n+1} = (n+1)^{n-1}. \qquad \square$$

The reader may notice that this is Cayley's formula for the number of labeled trees on $n+1$ nodes. Foata and Riordan found a bijection between parking functions on n cars and labeled trees on $n+1$ nodes using Prüfer codes [19]. Stanley discovered that parking functions could be used for an EL-labeling of the noncrossing partition lattice [46]; see Figure 1.3. For more on the noncrossing partition lattice, see Section 1.4. Further connections have been found to other structures, such as priority queues [21], Gončarov polynomials [32], and hyperplane arrangements [47].

The notion of a parking function has been generalized in myriad ways; see the sequence of papers [5, 30, 31, 32, 54]. Most generalizations use the increasing rearrangement formulation of Definition 1.1.1.

1.2 Permutations and descents

We let $\mathfrak{S}_n$ denote the symmetric group on n elements. That is, $\mathfrak{S}_n$ is the group of *permutations* on the elements $[n] = \{1, 2, \ldots, n\}$ under the operation of composition. We use one-line notation to refer to a permutation. For $\sigma \in \mathfrak{S}_n$, we write $\sigma = \sigma_1\sigma_2\cdots\sigma_n$, where $\sigma(i) = \sigma_i$. For example, the permutation $3142 \in \mathfrak{S}_4$ represents the map with the assignments $1 \mapsto 3$, $2 \mapsto 1$, $3 \mapsto 4$, and $4 \mapsto 2$.

A permutation statistic is a map from the symmetric group to the nonnegative integers. Permutation statistics were first studied at length by Major Percy MacMahon [34] in the early 20th century, and they have since earned quite a bit of attention. Some very commonly studied permutation statistics are descents, excedances, inversions, and the major index. For more on permutation statistics, see [34] or [49, Sections 1.3 through 1.6]. We will be largely concerned with descents.

For a permutation σ in the symmetric group $\mathfrak{S}_n$, we define the *descent set* of σ to be the subset of $[n-1] = \{1, 2, \ldots, n-1\}$ given by

$$\mathrm{Des}(\sigma) = \{i \in [n-1] \,:\, \sigma_i > \sigma_{i+1}\}.$$

For example, our permutation from above has descent set $\mathrm{Des}(3142) = \{1, 3\}$.

The *descent set statistics* $\beta_n(S)$ are defined for subsets S of $[n-1]$ by

$$\beta_n(S) = \left|\{\pi \in \mathfrak{S}_n \,:\, \mathrm{Des}(\pi) = S\}\right|.$$

Further, define $\alpha_n(S)$ by the sum

$$\alpha_n(S) = \sum_{T \subseteq S} \beta_n(T).$$

Observe that $\alpha_n(S)$ enumerates the number of permutations in $\mathfrak{S}_n$ with descent set contained in the set S. It is straightforward, then, using inclusion-exclusion to invert this relation to get

$$\beta_n(S) = \sum_{T \subseteq S} (-1)^{|S-T|} \cdot \alpha_n(T).$$

Let us define a bijection co from subsets of the set $[n-1]$ to compositions of n by sending the set $S = \{s_1 < s_2 < \cdots < s_{k-1}\}$ to the composition $\mathrm{co}(S) = (c_1, c_2, \ldots, c_k)$, where $c_i = s_i - s_{i-1}$ with $s_0 = 0$ and $s_k = n$. See, for instance, [4] or [48, Section 7.19]. It is now straightforward to observe that $\alpha_n(S)$ is given by the multinomial coefficient

$$\binom{n}{\mathrm{co}(S)} = \binom{n}{c_1, c_2, \ldots, c_k}.$$

These $\alpha_n(S)$ and $\beta_n(S)$ may equivalently be thought of as the flag f-vector and h-vector of the Boolean algebra B_n; see [49, Section 3.13].

Another commonly used method for computing the value of $\beta_n(S)$ is through the use of a triangular array, originally developed by de Bruijn [6] and later generalized first by Millar, Sloane, and Young [36] to the boustrophedon transform, then again by Ehrenborg and Mahajan [14]. The triangular array for some subset S of the positive integers and sequence of numbers $\mathbf{a} = (a_0, a_1, a_2 \ldots)$ has entries $t_{i,j}$ for $0 \leq i \leq n$ and $0 \leq j \leq i$. The first row of the triangle is initialized as $t_{0,0} = a_0$, and the first

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 0 & \rightarrow & 1 & & & & \\
 & & & 1 & \leftarrow & 1 & \leftarrow & 0 & & & \\
 & & 2 & \leftarrow & 1 & \leftarrow & 0 & \leftarrow & 0 & & \\
 & 0 & \rightarrow & 2 & \rightarrow & 3 & \rightarrow & 3 & \rightarrow & 3 & \\
11 & \leftarrow & 11 & \leftarrow & 9 & \leftarrow & 6 & \leftarrow & 3 & \leftarrow & 0
\end{array}$$

Figure 1.4: The triangular array of the boustrophedon transform for the set $S = \{2, 3, 5\} \subseteq [5]$ and sequence $\mathbf{a} = (1, 0, 0, \ldots)$. Hence, $\beta_6(\{2, 3, 5\}) = 11+11+9+6+3+0 = 40$.

element of each subsequent row is $t_{i,0} = a_i$. The remainder of the triangle is computed recursively via

$$t_{i,j} = \begin{cases} t_{i,j-1} + t_{i-1,j-1} & \text{if } i-1, i \in S \text{ or } i-1, i \notin S, \\ t_{i,j-1} + t_{i-1,i-j} & \text{otherwise.} \end{cases}$$

Taking the sequence $\mathbf{a} = (1, 0, 0, \ldots)$ and considering S a subset of $[n-1]$, it turns out that this process computes exactly the descent set statistic $\beta_n(S)$. Namely, it is exactly the sum of the entries in the last row of the array

$$\beta_n(S) = \sum_{k=0}^{n} t_{n,k}.$$

With some reversal of the order of entries in a row, we can picture the recursion for this transform so that the ith row is created by adding along its own and the previous row, where we add from left to right if $i \notin S$ and from right to left if $i \in S$. An example is given in Figure 1.2.

In fact, when computing the descent set statistics with the boustrophedon transform where $\vec{a} = (1, 0, 0, \ldots)$, the transform can be simplified to consist merely of the repeated application of two linear operators $\mathbb{N}^k \longrightarrow \mathbb{N}^{k+1}$ of consecutive partial sums defined by

$$\begin{aligned} (p_1, p_2, \ldots, p_k) &\longmapsto (0, p_1, p_1 + p_2, \ldots, p_1 + p_2 + \cdots + p_k) \qquad \text{and} \\ (p_1, p_2, \ldots, p_k) &\longmapsto (p_1 + p_2 + \cdots + p_k, p_2 + p_3 + \cdots + p_k, \ldots, p_k, 0), \end{aligned}$$

applied to the initial sequence (1). This lends itself readily to an efficient method of dynamic computation for the descent set statistic.

It is a classical result that the descent set statistic is maximized on the sets

$$\begin{aligned} S_{\text{odd}} &= \{1, 3, 5, \ldots\} \cap \{1, \ldots, n\} \qquad \text{and} \\ S_{\text{even}} &= \{2, 4, 6, \ldots\} \cap \{1, \ldots, n\}, \end{aligned}$$

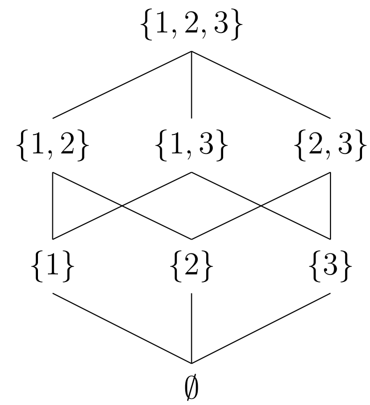


Figure 1.5: The Hasse diagram of the Boolean algebra B_3.

producing the so-called alternating permutations; see [37] or [6]. It is notable that $\beta_n(S_{\text{odd}}) = \beta_n(S_{\text{even}}) = E_n$ where E_n is the nth Euler number.

Ehrenborg and Mahajan [14] used this transform to prove a number of inequalities for descent set statistics according to the sets S.

1.3 Posets

A *partially ordered set* P, often called a *poset* for short, is a set along with a binary relation $\leq$ satisfying

Reflexivity: For each $p \in P$, we have $p \leq p$,

Antisymmetry: If $p \leq q$ and $q \leq p$, then $p = q$,

Transitivity: If $p \leq q$ and $q \leq r$, then $p \leq r$.

If $p \leq q$ and $p \neq q$, we write $p < q$. We say that q *covers* p, denoted $p \prec q$, if $p < q$ and there exists no element $x \in P$ such that $p < x < q$. The relation between p and q in this case is known as a cover relation. We can visualize a poset P as a graph where the vertices are the elements of P, and the edges are the cover relations. We typically draw the graph so that if $p < q$, then q appears above p in the picture. We call such a diagram the *Hasse diagram* of P.

The standard example of a poset is the Boolean algebra B_n whose elements are the subsets of $[n]$, ordered so that $A \leq B$ if A is a subset of B. See Figure 1.5 for the Hasse diagram of the Boolean algebra B_3.

It is worth noting that the relation $\leq$ need not be a total order. That is, for $p, q \in P$, it need not be the case that $p \leq q$ or $q \leq p$. For example, the elements $\{2\}$ and $\{3\}$ in the Boolean algebra B_3 can be seen in Figure 1.5 to have no relation. We call such elements *incomparable*.

We use $\hat{0}$ to denote the minimal element of P, that is, the unique element such that $\hat{0} \leq p$ for all $p \in P$, if such an element exists. Similarly, we let $\hat{1}$ denote the maximal element of P, if it exists. In the Boolean algebra B_n, the minimal element is $\hat{0} = \emptyset$, and the maximal element is $\hat{1} = [n]$.

The interval $[p, q]$ is defined to be the set $[p, q] = \{x \in P : p \leq x \leq q\}$, which can be viewed as a subposet of P with the order on the elements induced from P. A *chain* c in the poset P is a set of distinct, totally ordered elements in P, that is, $c = \{p_0 < p_1 < \cdots < p_k\}$. If each of these relations is a cover relation, and $c = \{p = p_0 \prec p_1 \prec \cdots \prec p_k = q\}$, then we say c is a *saturated chain* in the interval $[p, q]$. We say the *length* of c is $\ell(c) = k$. If every saturated chain of P from a minimal element to a maximal element is of the same length n, we say the poset P is of *rank* n. In such a poset, we may define a rank function $\rho : P \longrightarrow [n]$ so that, for $p \in P$, the rank of p, denoted $\rho(p)$, is the length of any saturated chain from a minimal element of P to p. We often extend this definition to $\rho(p, q) = \rho(p) - \rho(q)$, called the *rank difference* of p and q. A ranked poset that has a unique minimal and maximal element $\hat{0}$ and $\hat{1}$ is called a *graded* poset. A poset P with a finite number of elements is said to be *finite*, whereas a poset P whose intervals are all finite is said to be *locally finite*.

Given two posets P and Q, we define the *Cartesian product poset* $P \times Q$ to be the set of pairs (p, q) for $p \in P$ and $q \in Q$ ordered so that $(p_1, q_1) \leq (p_2, q_2)$ if and only if $p_1 \leq_P p_2$ and $q_1 \leq_Q q_2$. We define the *dual* of a poset P to be the poset P^* with all order relations reversed. That is, $p \leq_P q$ if and only if $q \leq_{P^*} p$.

For two elements $p, q \in P$, we define the *meet* of p and q, denoted $p \wedge q$, to be the greatest lower bound of both p and q. That is, $p \wedge q$ is the unique element x (if it exists) such that $x \leq p$ and $x \leq q$, but for any other element $y \leq p$ and $y \leq q$, we have $y \leq x$. We define the *join* of p and q (if it exists), denoted $p \vee q$, to be the least upper bound of both p and q in a similar way. A *lattice* is a poset in which the meet and join of each pair of elements exists. In Figure 1.5, one can see that the meet and join in a Boolean algebra correspond to intersections and unions of sets, respectively.

For more basic terminology and examples of posets, see [49, Chapter 3].

We will now introduce the *incidence algebra* on posets. Let P be a locally finite poset, and let I denote the set of all closed intervals of P. For a K a field, we will consider the collection $I(P)$ of all functions $f : I \longrightarrow K$, for which we write $f(p, q)$ to mean $f([p, q])$.

We define addition and subtraction in $I(P)$ pointwise and define the algebra multiplication to be given by convolution. That is, if $f, g \in I(P)$, then $f * g$ is defined by

$$f * g(p, q) = \sum_{p \leq x \leq q} f(p, x)g(x, q).$$

The identity in the incidence algebra, denoted δ, is defined by

$$\delta(p, q) = \begin{cases} 1 & \text{if } p = q, \\ 0 & \text{if } p \neq q. \end{cases}$$

A useful function in the incidence algebra is the zeta function ζ, which takes the value 1 on all intervals of P. While interesting in its own right, we will be concerned mostly with its convolutional inverse, called the *Möbius function* μ. It can be shown that the relation $\mu * \zeta = \delta$ is equivalent to

$$\begin{aligned} \mu(p,p) &= 1, \quad \text{for all } p \in P, \\ \mu(p,q) &= -\sum_{p \le x < q} \mu(p,x), \quad \text{for all } p < q \text{ in } P. \end{aligned}$$

We take this as our definition of μ, though it is not the most efficient means of computation. If P has extremal elements $\hat{0}$ and $\hat{1}$, we let $\mu(P)$ indicate the value of $\mu([\hat{0},\hat{1}])$ in P.

The Möbius function is closely related to the number theoretic Möbius function. Namely, consider the divisor lattice D_n, whose elements are the positive integer divisors of n ordered so that $i \le j$ if i divides j. Then $\mu(p,q)$ for $p,q \in D_n$ corresponds exactly to the number theoretic Möbius function $\mu(q/p)$.

An important method of computation for the Möbius function is due to Philip Hall's theorem. We will make reference to this theorem multiple times throughout the dissertation.

Theorem 1.3.1 (Philip Hall's theorem)**.** *Let P be a finite poset, and let $\widehat{P}$ denote P with a $\hat{0}$ and $\hat{1}$ adjoined. Let c_i be the number of chains $\hat{0} = p_0 < p_1 < \cdots < p_i = \hat{1}$ of length i between $\hat{0}$ and $\hat{1}$. (Thus, $c_0 = 0$ and $c_1 = 1$). Then*

$$\mu_{\widehat{P}}(\hat{0},\hat{1}) = c_0 - c_1 + c_2 - c_3 + \cdots .$$

For more on the Möbius function, see [49, Sections 3.7 through 3.10]

1.4 Partitions

We will consider multiple types of partitions. First, an *integer partition* λ of a positive integer n is a sequence $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ of positive integers such that $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_k \ge 1$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_k = n$. The *length* of the partition λ is given by $\ell(\lambda) = k$.

A *set partition* π of $[n]$ is a collection of subsets $B_1, B_2, \ldots, B_k$ of $[n]$, called *blocks*, such that

- For each index i, we have $B_i \neq \emptyset$,
- For all pairs of distinct indices i and j, we have $B_i \cap B_j = \emptyset$,
- The union $\bigcup_{i=1}^{k} B_i$ is the entire set $[n]$.

We write $\pi = B_1|B_2|\cdots|B_k$, and we often omit commas between elements in a block when this will not cause confusion. Let $|\pi|$ denote the number of blocks in the partition π.

The set of all set partitions of $[n]$ forms a graded lattice called the partition lattice, denoted Π_n, with the partial order that $\pi \le \tau$ if every block of π is contained in a block of τ. In this case, we say π is a *refinement* of τ. The unique minimal element $\hat{0}$ is the partition consisting of all singleton blocks $1|2|\cdots|n$, and the unique maximal

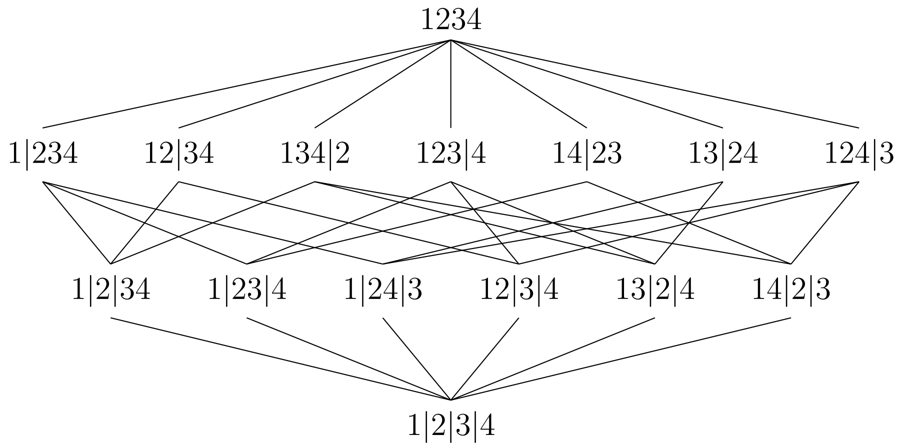


Figure 1.6: The partition lattice Π_4.

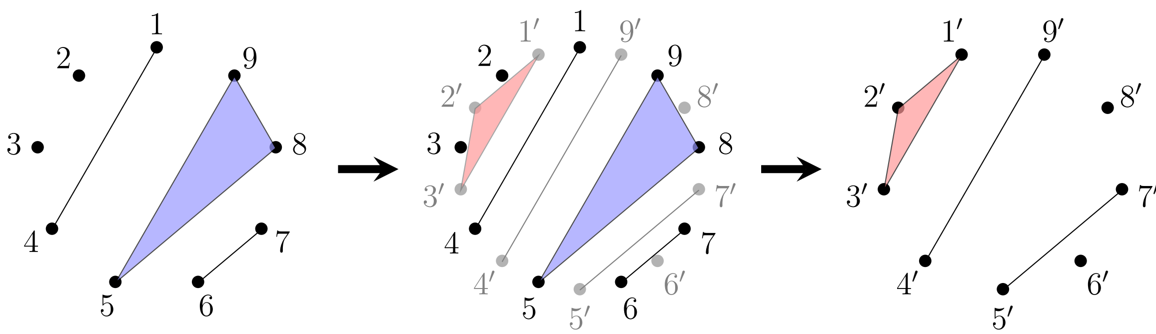


Figure 1.7: The noncrossing partition 14|2|3|589|67 and its complement, 123|49|57|6|8.

element $\hat{1}$ is the partition containing a single block $123\cdots n$. See Figure 1.6 for the Hasse diagram of the partition lattice Π_4 on four elements.

A partition σ of $[n]$ is *noncrossing* if it has the property that if $a < b < c < d$ and some block B of σ contains both a and c, while some block B' of σ contains both b and d, then B and B' are the same block, that is, $B = B'$. Geometrically, σ is a partition of the vertices of a regular n-gon (labeled by the set $[n]$) with the property that the convex hulls of its blocks are pairwise disjoint.

The set of noncrossing partitions on $[n]$ comprises a sublattice of Π_n, known as the noncrossing partition lattice, NC_n. See Figure 1.3 for an example of the noncrossing partition lattice NC_4, and notice that the partition 13|24 is the only partition from Π_4 that is not included. The noncrossing partition lattice was originally studied by Kreweras [28].

An interesting property of the noncrossing partition lattice NC_n is that it is self-dual. In fact, there is a classical complementation map due to Kreweras [28] that

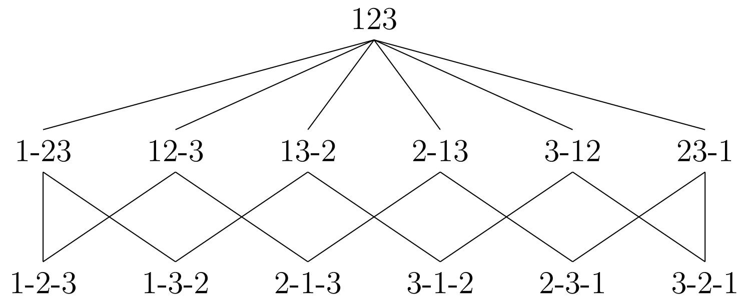


Figure 1.8: The poset Q_3 of ordered set partitions on 3 elements.

involves viewing the partition geometrically and swapping blocks with empty regions; see Figure 1.7 for an example. While this map is not itself an involution, there is a slight tweak of it due to Simion and Ullman [45] that is an order-reversing involution.

One final variant on set partitions we will consider are ordered set partitions. An *ordered set partition* is a partition where the blocks are given a linear order. That is, $(C_1, C_2, \ldots, C_r)$ is an ordered set partition if $\{C_1, C_2, \ldots, C_r\}$ is a set partition. We will often write C_1-C_2-$\cdots$-C_r for an ordered set partition as opposed to $C_1|C_2|\cdots|C_r$ for a set partition. Let Q_k denote the set of all ordered set partitions on the set $[k]$. We make Q_k into a poset by joining adjacent blocks, that is, the cover relation is

$$(C_1, \ldots, C_{i-1}, C_i, C_{i+1}, C_{i+2}, \ldots, C_r) \prec (C_1, \ldots, C_{i-1}, C_i \cup C_{i+1}, C_{i+2}, \ldots, C_r).$$

Note that Q_k has one maximal element ($[k]$) but $k!$ minimal elements. See Figure 1.8 for an example of the poset Q_3. In fact, if we join a minimal element $\hat{0}$ to Q_k, we obtain the face lattice of the permutahedron; see Section 1.7.

The dual of this polytope is simplicial, and it may be realized geometrically as a subdivision of the $(k-2)$-dimensional sphere, where we embed the sphere S^{k-2} in $\mathbb{R}^k$ by

$$S^{k-2} = \{(x_1, x_2, \ldots, x_k) \in \mathbb{R}^k \ : \ x_1^2 + x_2^2 + \cdots + x_k^2 = 1, \ x_1 + x_2 + \cdots + x_k = 0\}.$$

The result is a $(k-2)$-dimensional sphere cut by $\binom{k}{2}$ hyperplanes $x_i = x_j$ for $1 \le i < j \le k$. A picture for $k = 4$ is given in Figure 1.9.

1.5 Noncrossing hypertrees

A *graph* is a set V of *vertices* along with a set E of *edges*, which are 2-element subsets of V. We will consider only undirected simple graphs, meaning that the edges have no orientation, there can be at most edge between any two given vertices, and these two vertices must be distinct. If we allow the edges to consist of subsets of V of cardinality 2 or more, then we call this a *hypergraph*, and the edges become *hyperedges*. A hypergraph is *connected* if for all vertices x and y in the vertex set $[n]$ there exists a sequence of vertices $x = x_0, x_1, \ldots, x_k = y$ and a sequence of edges $E_1, E_2, \ldots, E_k$

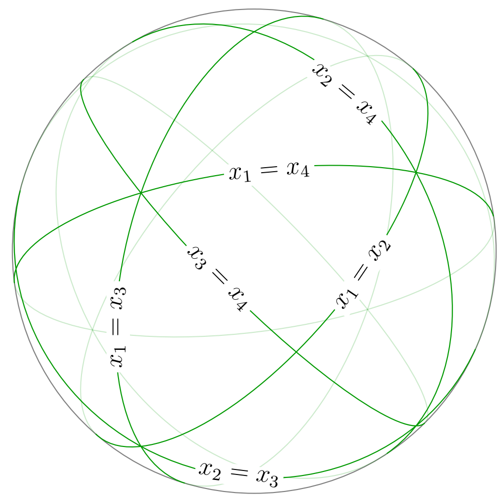


Figure 1.9: A geometric realization of the dual of the permutahedron of order 4 as a subdivision of the 2-sphere. Compare with Figure 1.10.

such that $x_{i-1}, x_i \in E_i$ for all $1 \leq i \leq k$. A *hypertree* H on $[n]$, then, is a connected hypergraph such that $\sum_{E \in H}(|E| - 1) = n - 1$.

We will primarily be considering a smaller class of hypergraphs satisfying a noncrossing property. Namely, take n points on a circle labeled counterclockwise from 1 to n, and consider graphs whose vertices are the given points and whose edges are rectilinear and do not cross. We call these *noncrossing graphs.* If we allow the edges to consist of more than just two vertices, then these become *noncrossing hypergraphs.* Then we may consider trees of this same type, called *noncrossing hypertrees.* We denote the set of noncrossing hypertrees on n vertices as NCHT_n. While hypertrees and the noncrossing property have received substantial attention individually, their union has received relatively little. For more on noncrossing hypertrees, see [35].

1.6 The reduced incidence Hopf algebra of posets

In this section, we will hearken back to Section 1.3 and our introduction of the incidence algebra of posets. Joni and Rota in [25] established that it is often more convenient and natural to consider incidence coalgebras than algebras, and, in fact, the reduced incidence coalgebra naturally extends to a bialgebra. Schmitt [42] later showed that the reduced incidence bialgebra of posets could be further extended to a Hopf algebra structure.

Let us be a bit more precise. Let P be a locally finite poset. Recall that $[p, q]$ for $p, q \in P$ denotes the subposet of P on all elements x such that $p \leq x$ and $x \leq q$. Then we can define the coalgebra $C(P)$ over the field K to be the vector space spanned by all intervals $[p, q]$ in P with a comultiplication $\Delta : C(P) \longrightarrow C(P) \otimes C(P)$ defined by

$$\Delta([p,q]) = \sum_{x\in[p,q]} [p,x]\otimes[x,q],$$

and the counit $\varepsilon : C(P) \longrightarrow K$ defined by

$$\varepsilon([p,q]) = \begin{cases} 1, & \text{if } p = q, \\ 0, & \text{otherwise.} \end{cases}$$

The necessary properties of coassociativity and cocommutativity follow nicely. We should notice now that the algebra dual to $C(P)$ is $\mathrm{Hom}(C(P), K)$, which is isomorphic to $I(P)$, the incidence algebra from Section 1.3.

At this point, if we want to extend to a bialgebra, we first need to restrict ourselves to a reduced incidence coalgebra by considering only isomorphism classes of intervals. In particular, we define the equivalence relation $\sim$ so that $[p,q] \sim [r,s]$ if and only if $[p,q]$ is isomorphic to $[r,s]$, and we form the quotient space $C_\sim(P) = C(P)/\sim$. We let $\overline{[p,q]}$ denote the isomorphism class of the interval $[p,q]$. If we take the Cartesian product as our multiplication of classes of intervals and the isomorphism class of the single-element interval as the unit 1, this extends to a bialgebra. Further, this bialgebra is endowed with a linear map $S : C_\sim(P) \longrightarrow C_\sim(P)$ called the antipode, satisfying the relation

$$\varepsilon(\overline{[p,q]})\cdot 1 = \sum_{x\in\overline{[p,q]}} S(\overline{[p,x]})\cdot\overline{[x,q]}.$$

The antipode can be thought of as a generalization of the Möbius function in the following way. If we define the linear function $\zeta : C_\sim(P) \longrightarrow K$ so that $\zeta(\overline{[p,q]}) = 1$ for any interval $[p,q]$ as in Section 1.3, then it satisfies $\mu([p,q]) = \zeta(S(\overline{[p,q]}))$.

Schmitt [42] provided an expression for the antipode $S(P)$ of a poset P with unique minimal element $\hat{0}$ and unique maximal element $\hat{1}$ that sums over all chains in P; see [42, Theorem 6.1]. See also [24, Theorem 1].

Theorem 1.6.1 (Schmitt)**.** *The antipode of a poset P in the reduced incidence Hopf algebra of posets is given by*

$$S(P) = \sum_{c} (-1)^k \cdot [x_0,x_1]\cdot[x_1,x_2]\cdots[x_{k-1},x_k], \tag{1.6.1}$$

where the sum is over all chains $c = \{\hat{0} = x_0 < x_1 < \cdots < x_k = \hat{1}\}$ in the poset P.

This can be thought of as an extension of Hall's formula for the Möbius function, and in general, there will be mass cancellation in this formula. Very few posets have had their antipode determined in an explicit, cancellation-free form. Haiman and Schmitt computed the antipode of the partition lattice [24, Corollary 1], and Einziger in her dissertation computed the antipode of the noncrossing partition lattice in terms of polygon dissections where each region has an even number of sides; see [17, Theorem 8.7]. For more on the reduced incidence Hopf algebra, see [25] or [42], and for other Hopf algebras appearing in combinatorics, see [2].

1.7 Polytopes

A *convex polytope* $P \subset \mathbb{R}^d$ may be defined in two equivalent ways: as the bounded intersection of finitely many closed halfspaces, or as the convex hull of finitely many points in $\mathbb{R}^d$. The equivalence of these definitions is a fundamental result of the study of polytopes. While we may gain intuition about these objects from those in $\mathbb{R}^3$, it is often the case that this intuition is proven false in higher dimensions, and polytope theory has provided a number of useful tools for visualizing higher-dimensional polytopes; see [22] or [55].

For any d-dimensional polytope $P \subset \mathbb{R}^d$, the empty set and P itself are the *improper faces* of P. Every subset F of P is a *proper face* of P if $F = P \cap H$ for some supporting hyperplane H of P. Faces of dimension 0 we call *vertices*, those of dimension 1 are *edges*, and those of dimension $d-1$ are *facets*. We let $f_i(P)$ denote the number of faces of dimension i in P, and the *f-polynomial* is the (finite) sum $f(P) = \sum_{i\geq 0} f_i(P) \cdot t^i$. Another way to express the f-polynomial is as the sum $\sum_F t^{\dim(F)}$, where the sum is over all nonempty faces F.

The *face lattice* of a convex polytope P is the poset $L(P)$ of all faces of P, partially ordered by inclusion. The *dual* of P, denoted P^*, is the dual of the face lattice of P. For more on posets, see Section 1.3.

The simplest family of polytopes are the simplices. We define a d-simplex as the convex hull of any $d+1$ affinely independent points in some $\mathbb{R}^n$ with $n \geq d$. The triangle and the tetrahedron are the familiar members of this family from 2 and 3 dimensions. We say a d-dimensional polytope P is *simplicial* when every one of its faces is a simplex. We call it *simple*, on the other hand, if every one of its vertices is adjacent to exactly d facets. An important relationship between these definitions is that they are dual to one another. That is, the dual of a simplicial polytope is simple, and the dual of a simple polytope is simplicial.

Another family of polytopes we will be interested in are the so-called *permutahedra*. First investigated by Schoute [43] in 1911, the permutahedron of order d is defined to be the convex hull of all permutations π in $\mathfrak{S}_d$ taken as points $\vec{x}$ in $\mathbb{R}^d$ via $\pi_i \mapsto x_i$. One should note that every such point $\vec{x}$ sits inside the hyperplane

$$x_1 + x_2 + \cdots + x_n = 1 + 2 + \cdots + n,$$

so the permutahedron of order d is itself $(d-1)$-dimensional. The permutahedron of order 3 is a hexagon, and the permutahedron of order 4 is a truncated octahedron; see Figure 1.10.

Another family of polytopes we will consider that have not been as widely studied are the descent polytopes. For a word $\mathbf{v} = \mathbf{v}_1\mathbf{v}_2\cdots\mathbf{v}_{n-1}$ of length $n-1$ in the letters $\mathbf{x}$ and $\mathbf{y}$, we define the *descent polytope* $\mathrm{DP}_{\mathbf{v}}$ to be the n-dimensional polytope

$$\mathrm{DP}_{\mathbf{v}} = \{(x_1, x_2, \ldots, x_n) \in [0,1]^n \;\;:\;\; x_i \leq x_{i+1} \text{ if } \mathbf{v}_i = \mathbf{x}, \text{ and } x_i \geq x_{i+1} \text{ if } \mathbf{v}_i = \mathbf{y}\}.$$

Descent polytopes briefly appeared in [13, Subsection 4.2], but it was in the paper [3] they were first studied for their own sake. See Figure 1.11 for an example of the descent polytopes in dimension 3.

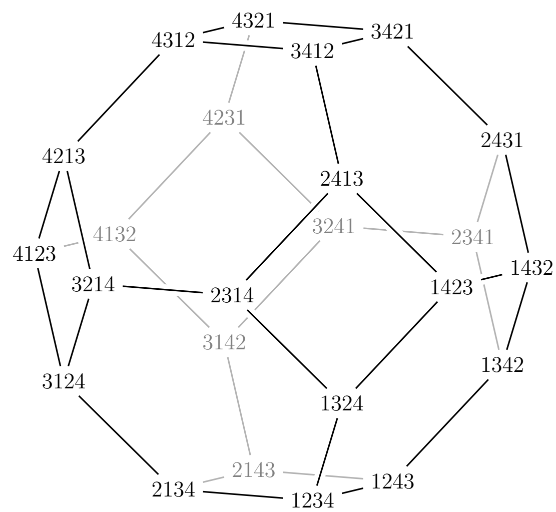


Figure 1.10: The permutahedron of order 4. Compare with its dual in Figure 1.9.

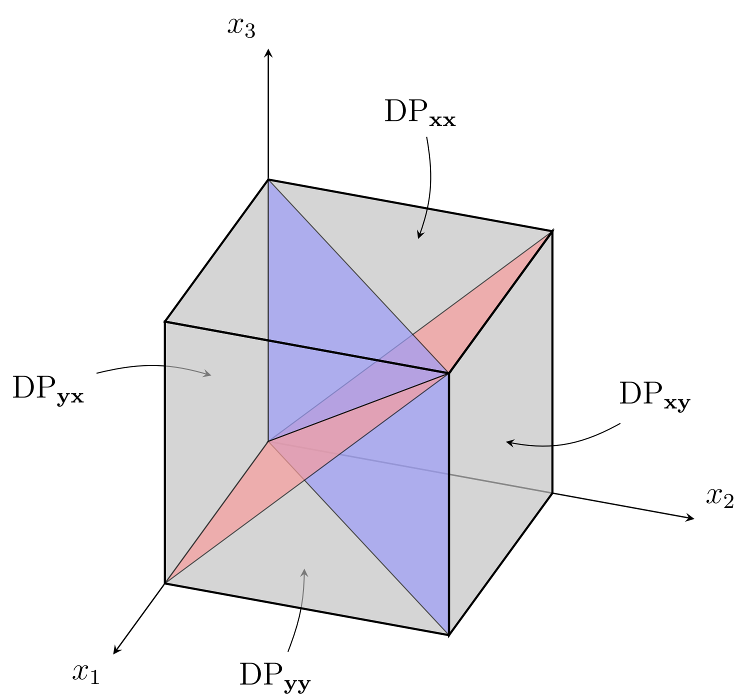


Figure 1.11: The four descent polytopes of dimension 3 sitting inside the unit cube. Note that $\mathrm{DP}_{\mathbf{xx}}$ and $\mathrm{DP}_{\mathbf{yy}}$ are tetrahedra, while $\mathrm{DP}_{\mathbf{xy}}$ and $\mathrm{DP}_{\mathbf{yx}}$ are square pyramids.

1.8 The order complex and discrete Morse theory

An *abstract simplicial complex* Δ is a finite collection of sets that is closed under subsets. That is, if $X \in \Delta$ and $Y \leq X$, then we must have $Y \in \Delta$. The elements in Δ are called *faces*, and maximal faces are called *facets*. The *dimension* of a face $X \in \Delta$ is given by $|X| - 1$. Faces of dimension d are referred to as d-faces.

To every abstract simplicial complex Δ there is a closely related topological space called the *geometric realization* of Δ. In this way, we are able to talk about various topological properties of a simplicial complex. To take this one step further, there is also a common technique for associating any poset with a simplicial complex. Namely, given a poset P, we define the *order complex* of P, denoted $\Delta(P)$, to be the simplicial complex whose i-faces are the chains of size $i + 1$ in P.

We may conversely encode the structure of an abstract simplicial complex Δ in a poset $P(\Delta)$, called the *face poset* of Δ and defined to be the poset of nonempty faces of Δ ordered by inclusion. This notion of face poset is compatible with the notion of the order complex in that for any simplicial complex Δ, we get the homeomorphism (of geometric realizations) $\Delta \cong \Delta(P(\Delta))$. For more details on the interplay between simplicial complexes and posets, see Wachs' overview article [53].

We now explain the rudiments of discrete Morse theory, developed by Forman [20]. First, a *partial matching* M of a poset P is a collection of edges from the Hasse diagram of P such that each element in P is in at most one edge of the matching M. One way we may define a matching on a poset is with a function u along with its inverse d so that if $u(a) = b$ or $d(b) = a$ for $a \prec b$, then the edge between a and b is included in the matching M. If we think of the Hasse diagram as a graph, then this coincides with the graph theoretic notion of matching.

Let us further consider the edges in the Hasse diagram of P to be initially oriented down so that $a \prec b$ means that the edge between a and b is directed from b to a. Flip the orientation of every edge contained in M so that it is now oriented up. If, at this point, there are no directed cycles in the Hasse diagram, we call this matching M of P a *discrete Morse matching*. Any unmatched elements of P are called *critical cells*. Then we have the following theorem from Forman [20].

Theorem 1.8.1 (Forman)**.** *Let Δ be an abstract simplicial complex. If the face poset $P(\Delta)$ has a discrete Morse matching, then the geometric realization of Δ is homotopy equivalent to a CW complex with exactly one cell of dimension p for each critical cell of dimension p.*

Hence, a discrete Morse matching on a face poset can dramatically simplify the computation of homotopy type for a simplicial complex.

1.9 Euler characteristics

The *Euler characteristic* χ was classically defined for the surfaces of polyhedra, which Euler found to satisfy $\chi(P) = f_0 - f_1 + f_2 = 2$, where f_0, f_1, and f_2 are the number of vertices, edges, and faces in a 2-dimensional convex polyhedron P. This idea can generalize to convex polytopes (or simplicial complexes) of any dimension so that

$$\chi(P) = f_0 - f_1 + f_2 - f_3 + \cdots .$$

Often, it is convenient to consider the *reduced Euler characteristic* $\widetilde{\chi}(P)$, which includes the empty set in the sum as $f_{-1} = 1$. That is, the reduced Euler characteristic is given by the sum

$$\widetilde{\chi}(P) = -f_{-1} + f_0 - f_1 + f_2 - f_3 + \cdots .$$

In general, the Euler-Poincaré formula gives the Euler characteristic in terms of an alternating sum of the *Betti numbers*, where the ith Betti number, denoted b_i, is the dimension of the ith homology group of the space.

Using the notions of the Euler characteristic and the order complex of a poset, we can rephrase Philip Hall's theorem on the Möbius function.

Theorem 1.9.1 (Philip Hall)**.** *Let P be a bounded poset, and let $\overline{P}$ denote the poset P with the elements $\hat{0}$ and $\hat{1}$ removed. Then the Möbius function of P is given by*

$$\mu(P) = \widetilde{\chi}(\Delta(\overline{P})).$$

The most useful feature of the Euler characteristic is its homotopy invariance, and it turns out to be one of the simplest methods for distinguishing topological spaces up to homotopy. However, this definition of the Euler characteristic is *not* additive. That is, for M and N two subspaces of a topological space X, the Euler characteristic does *not* necessarily satisfy that

$$\chi(M \cup N) = \chi(M) + \chi(N) - \chi(M \cap N).$$

Klain and Rota [26] developed the theory of the Euler characteristic as a valuation satisfying this additive property on various classes of objects, including polytopes. In general, for a topological space X, one may define the *Euler characteristic on compact support* to be the alternating sum of the dimensions of the cohomology groups on compact support of X. Using this alternative definition, one trades homotopy invariance for this additive property. One consequence of this definition that we will make use of in Chapter 3 is that the Euler characteristic on compact support of an open ball of dimension k is $(-1)^k$.

The results of Chapter 2 have appeared in the two papers [10, 12], and the results of Chapter 4 have appeared in [11].

Chapter 2 Parking cars of different sizes

2.1 Introduction

Recall from Section 1.1 that classical parking functions were first introduced by Konheim and Weiss [27]. The original concept was that of a linear parking lot with n available spaces, and n cars with a stated parking preference. Each car would, in order, attempt to park in its preferred spot. If the car found its preferred spot occupied, it would move to the next available slot. A parking function is a sequence of parking preferences that would allow all n cars to park according to this rule. This definition is equivalent to the following formal definition:

Definition 2.1.1. *Let $\vec{a} = (a_1, a_2, \ldots, a_n)$ be a sequence of positive integers, and let $b_1 \leq b_2 \leq \cdots \leq b_n$ be the increasing rearrangement of $\vec{a}$. Then the sequence $\vec{a}$ is a parking function if and only if $b_i \leq i$ for all indices i.*

It is well known that the number of such parking functions is $(n+1)^{n-1}$. This is Cayley's formula for the number of labeled trees on $n+1$ nodes and Foata and Riordan found a bijective proof [19] using Prüfer codes. Stanley discovered the relationship between parking functions and noncrossing partitions [46]. Further connections have been found to other structures, such as priority queues [21], Gončarov polynomials [32] and hyperplane arrangements [47].

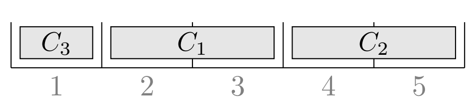
The notion of a parking function has been generalized in myriad ways; see the sequence of papers [5, 30, 31, 32, 54]. We present here a different generalization, returning to the original idea of parking cars. This time the cars have different sizes, and each takes up a number of adjacent parking spaces.

Definition 2.1.2. *Let there be n cars $C_1, \ldots, C_n$ of sizes $y_1, \ldots, y_n$, where $y_1, \ldots, y_n$ are positive integers. Assume there are $\sum_{i=1}^{n} y_i$ spaces in a row. Furthermore, let car C_i have the preferred spot c_i. Now let the cars in the order C_1 through C_n park according to the following rule:*

> *Starting at position c_i, car C_i looks for the first empty spot $j \geq c_i$. If the spaces j through $j + y_i - 1$ are empty, then car C_i parks in these spots. If any of the spots $j+1$ through $j + y_i - 1$ is already occupied, then there will be a collision, and the result is not a parking sequence.*

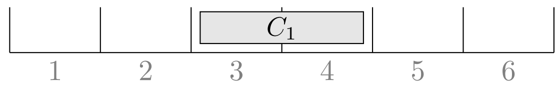
Iterate this rule for all the cars $C_1, C_2, \ldots, C_n$. We call $(c_1, \ldots, c_n)$ a parking sequence *for $\vec{y} = (y_1, \ldots, y_n)$ if all n cars can park without any collisions and without leaving the $\sum_{i=1}^{n} y_i$ parking spaces.*

As an example, consider three cars of sizes $\vec{y} = (2, 2, 1)$ with preferences $\vec{c} = (2, 3, 1)$. Then there are $2 + 2 + 1 = 5$ available parking spaces, and the final configuration of the cars is



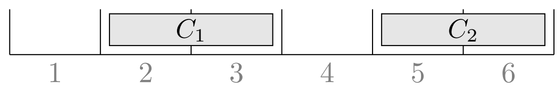
All cars are able to park, so this yields a parking sequence.

There are two ways in which a sequence can fail to be a parking sequence. Either a collision occurs, or a car passes the end of the parking lot. As an example, consider three cars with $\vec{y} = (2, 2, 2)$ and preferences $\vec{c} = (3, 2, 1)$. Then we have $2 + 2 + 2 = 6$ parking spots, and the first car parks in its desired spot:



However, the second car prefers spot 2, and since spot 2 is open, he tries to take spots 2 and 3, but collides with C_1 in the process. Hence, this is not a parking sequence.

If, instead, we had $\vec{y} = (2, 2, 2)$ and $\vec{c} = (2, 5, 5)$, then again the first two cars are able to park with no difficulty:



But car C_3 will pass by all the parking spots after his preferred spot without seeing an empty spot. Hence, this also fails to be a parking sequence.

The classical notion of parking function is obtained when all the cars have size 1, that is, $\vec{y} = (1, 1, \ldots, 1)$. Note in this case that there are no possible collisions.

In the classical case, any permutation of a parking function is again a parking function. This is not true for cars of larger size. As an example, note for $\vec{y} = (2, 2)$ that $\vec{c} = (1, 2)$ is a parking sequence. However, the rearrangement $\vec{c}' = (2, 1)$ is not a parking sequence. This shows that the notion of parking sequence differs from the notion of parking function in the papers [5, 30, 31, 32, 54].

The classical result is that the number of parking functions is given by $(n+1)^{n-1}$; see [27]. For cars of bigger sizes we have the following result:

Theorem 2.1.3. *The number of parking sequences $f(\vec{y})$ for car sizes $\vec{y} = (y_1, \ldots, y_n)$ is given by the product*

$$f(\vec{y}) = (y_1 + n) \cdot (y_1 + y_2 + n - 1) \cdots (y_1 + \cdots + y_{n-1} + 2). \tag{2.1.1}$$

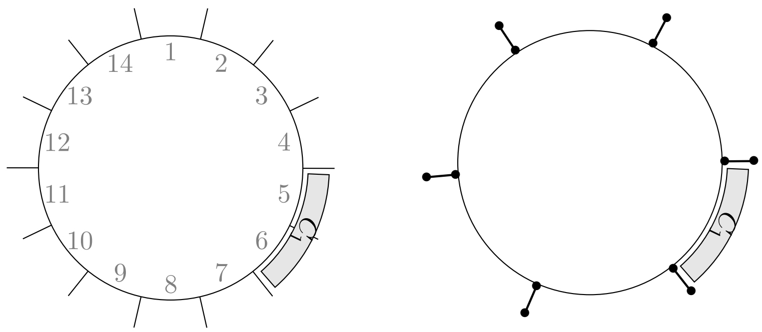
2.2 Circular parking arrangements

Consider $M = y_1 + y_2 + \cdots + y_n + 1$ parking spaces arranged in a circle. We will consider parking cars on this circular arrangement, without a cliff for cars to fall off. Observe that when all the cars have parked, there will be one empty spot left over. We claim that there are

$$M \cdot f(\vec{y}) = (y_1 + n) \cdot (y_1 + y_2 + n - 1) \cdots (y_1 + \cdots + y_n + 1). \tag{2.2.1}$$

such circular parking sequences. The first car C_1 has M ways to choose its parking spot.

The next step is counterintuitive. After car C_1 has parked, erase the markings for the remaining $y_2 + \cdots + y_n + 1$ spots and put in $n + 1$ dividers. These dividers create $n+1$ intervals on the circle, where one interval is taken up by C_1. Furthermore, these dividers are on wheels and can freely move along the circle. Each interval will accept one (and only one) car. For example, consider the case where $n = 5$ and $\vec{y} = (2, 5, 1, 3, 2)$ so that $M = 2 + 5 + 1 + 3 + 2 + 1 = 14$, and $c_1 = 5$.

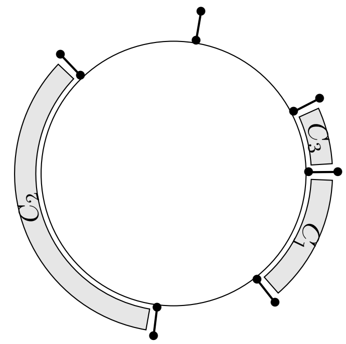


We will now create a circular parking sequence, but only at the end do we obtain the exact positions of cars C_2 through C_{n+1}. That is, instead of focusing on the number of specific spot preferences each car could have, we keep track of the order the cars park in, which will then determine the exact locations of the cars.

The second car has two options. The first is that it has a desired position already taken by C_1. In this case, it will cruise until the next empty spot. This can happen in y_1 ways, and then car C_2 obtains the next open interval after the interval C_1 is in. Otherwise, the car C_2 has a preferred spot not already taken. In this case C_2 has n open intervals to choose from. The total number of options for C_2 is $y_1 + n$.

The third car C_3 has the same options. First, it may desire a spot that is already taken, in which case it will have to cruise until the next open interval. This can happen in $y_1 + y_2$ ways. Note that this count applies to both the case when C_1 and C_2 are parked next to each other, and when C_1 and C_2 have open intervals between them. Otherwise, C_3 has $n - 1$ open intervals to pick from.

In general, car C_i has $y_1 + \cdots + y_{i-1} + n + 2 - i$ choices. This pattern continues up to C_n, which has $y_1 + \cdots + y_{n-1} + 2$ possibilities. For example, suppose C_2 and C_3 in our above example have parked as below:



Then C_4 may either cruise on C_1 and C_3 (in $y_1 + y_3$ ways), it may cruise on C_2 (in y_2 ways), or it can pick one of the three available intervals directly. In total, C_4 has $(y_1 + y_3) + y_2 + 3 = 11$ ways to park.

One can imagine that when we park a car, we do not set the parking brake, but put the car in neutral, so that the car and the dividers can move as necessary to make room for future cars.

Thus the total number of circular parking arrangements of this type is

$$M \cdot (y_1 + n) \cdot (y_1 + y_2 + n - 1) \cdots (y_1 + \cdots + y_{n-1} + 2),$$

where the ith factor is the number of options for the car C_i. This proves the claim about the number of circular parking sequences in (2.2.1).

Hence, to prove Theorem 2.1.3 we need only observe that the circular parking sequences with spot M empty are the same as our parking sequences. This follows from the observation that no car in the circular arrangement has preference M, since otherwise this spot would not be empty. Furthermore, no car would cruise by this empty spot.

Observe that the set of circular parking sequences is invariant under rotation. That is, if $(c_1, c_2, \ldots, c_n)$ is a parking sequence, then so is the sequence $(c_1 + a, c_2 + a, \ldots, c_n + a)$, where all the additions are modulo M. In particular, the number of circular parking sequences with spot M empty is given by $1/M \cdot M \cdot f(\vec{y}) = f(\vec{y})$.

2.3 Parking cars after a trailer

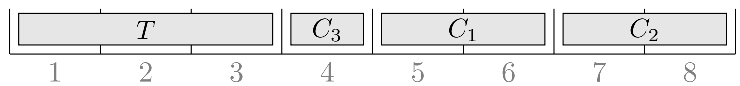
We now introduce a refinement of the result by adding a trailer.

Definition 2.3.1. *Let there be n cars $C_1, \ldots, C_n$ of sizes $y_1, \ldots, y_n$, where $y_1, \ldots, y_n$ are positive integers. Assume there are $z - 1 + \sum_{i=1}^{n} y_i$ spaces in a row, where the trailer occupies the $z - 1$ first spaces. Furthermore, let car C_i have the preferred spot c_i. Now let the cars in the order C_1 through C_n park according to the following rule:*

> *Starting at position c_i, car C_i looks for the first empty spot $j \geq c_i$. If the spaces j through $j + y_i - 1$ are empty, then car C_i parks in these spots. If any of the spots $j + 1$ through $j + y_i - 1$ is already occupied, then there will be a collision, and the result is not a parking sequence.*

Iterate this rule for all the cars $C_1, C_2, \ldots, C_n$. We call $(c_1, \ldots, c_n)$ a parking sequence *for $\vec{y} = (y_1, \ldots, y_n)$ if all n cars can park without any collisions and without leaving the $z - 1 + \sum_{i=1}^{n} y_i$ parking spaces.*

As an example, consider three cars of sizes $\vec{y} = (2, 2, 1)$, a trailer of size 3, that is $z = 4$, and the preferences $\vec{c} = (5, 6, 2)$. Then there are $2 + 2 + 1 = 5$ available parking spaces after the trailer, and the final configuration of the cars is



All cars are able to park, so this yields a parking sequence.

2.4 The result

We now have the main result. Observe that when setting $z = 1$, this expression reduces to equation (2.1.1).

Theorem 2.4.1. *The number of parking sequences $f(\vec{y}; z)$ for car sizes $\vec{y} = (y_1, \ldots, y_n)$ and a trailer of length $z - 1$ is given by the product*

$$f(\vec{y}; z) = z \cdot (z + y_1 + n - 1) \cdot (z + y_1 + y_2 + n - 2) \cdots (z + y_1 + \cdots + y_{n-1} + 1).$$

The first part of our proof comes from the following identity. Let $\dot{\cup}$ denote disjoint union of sets.

Lemma 2.4.2. *The number of parking sequences for car sizes $(y_1, \ldots, y_n, y_{n+1})$ and a trailer of length $z - 1$ satisfies the recurrence*

$$f(\vec{y}, y_{n+1}; z) = \sum_{L \dot{\cup} R = \{1,\ldots,n\}} \left(z + \sum_{l \in L} y_l \right) \cdot f(\vec{y}_L; z) \cdot f(\vec{y}_R; 1),$$

where $\vec{y}_S = (y_{s_1}, \ldots, y_{s_k})$ for $S = \{s_1 < s_2 < \cdots < s_k\} \subseteq \{1, \ldots, n\}$.

Proof. Consider the situation required for the last car C_{n+1} to park successfully:

– Car C_{n+1} must see, to the left of its vacant spot, the trailer along with a subset of the cars labeled with indices L occupying the first $z - 1 + \sum_{l \in L} y_l$ spots. Hence, the restriction $\vec{c}_L$ of $\vec{c} = (c_1, c_2, \ldots, c_{n+1})$ to the indices in L must be a parking sequence for $\vec{y}_L$ and trailer of length $z - 1$. This can be done in $f(\vec{y}_L; z)$ possible ways.

– Car C_{n+1} must have a preference c_{n+1} that lies in the range $[1, z + \sum_{l \in L} y_l]$.

– Car C_{n+1} must see, to the right of its vacant spot, the complementary subset of cars labeled with indices $R = \{1, 2, \ldots, n\} - L$ occupying the last $\sum_{r \in R} y_r$ spots. These cars must have parked successfully with preferences $\vec{c}_R$ and no trailer, that is, $z = 1$. This is enumerated by $f(\vec{y}_R; 1)$.

Now summing over all decompositions $L \dot\cup R = \{1, 2, \ldots, n\}$, the recursion follows. $\square$

The next piece of the proof of Theorem 2.4.1 utilizes a multi-parameter convolution identity due to Strehl [50]. Let $\mathbf{x} = (x_{i,j})_{1 \leq i < j}$ and $\mathbf{y} = (y_j)_{1 \leq j}$ be two infinite sets of parameters. For a finite subset A of the positive integers, define the two sums

$$\mathbf{x}^A_{>a} = \sum_{j \in A, j > a} x_{a,j} \quad \text{and} \quad \mathbf{y}^A_{\leq a} = \sum_{j \in A, j \leq a} y_j.$$

Define the polynomials $t_A(\mathbf{x}, \mathbf{y}; z)$ and $s_A(\mathbf{x}, \mathbf{y}; z)$ by

$$t_A(\mathbf{x}, \mathbf{y}; z) = z \cdot \prod_{a \in A - \max(A)} (z + \mathbf{y}^A_{\leq a} + \mathbf{x}^A_{>a}),$$
$$s_A(\mathbf{x}, \mathbf{y}; z) = \prod_{a \in A} (z + \mathbf{y}^A_{\leq a} + \mathbf{x}^A_{>a}).$$

Note that, when A is the empty set, we set $t_A(\mathbf{x}, \mathbf{y}; z)$ to be 1. We directly have that

$$(z + \mathbf{y}^A_{\leq \max(A)}) \cdot t_A(\mathbf{x}, \mathbf{y}; z) = z \cdot s_A(\mathbf{x}, \mathbf{y}; z). \tag{2.4.1}$$

Now Theorem 1, equation (6) in [50] states:

Theorem 2.4.3 (Strehl)**.** *The polynomials $s_L(\mathbf{x}, \mathbf{y}; z)$ and $t_R(\mathbf{x}, \mathbf{y}; w)$ satisfy the following convolution identity:*

$$s_A(\mathbf{x}, \mathbf{y}; z + w) = \sum_{L \dot\cup R = A} s_L(\mathbf{x}, \mathbf{y}; z) \cdot t_R(\mathbf{x}, \mathbf{y}; w). \tag{2.4.2}$$

Strehl first interprets $s_A(\mathbf{x}, \mathbf{y}; z)$ and $t_A(\mathbf{x}, \mathbf{y}; z)$ as sums of weights on functions, then translates these via a bijection to sums of weights on rooted, labeled trees where the $x_{i,j}$'s record ascents, and the y_j's record descents. The proof of (2.4.2) then follows from the structure inherent in splitting a tree into two. A similar result using the same bijection was discovered by Eğecioğlu and Remmel in [7].

Proof of Theorem 2.4.1. The proof follows from noticing that our proposed expression for $f(\vec{y}; z)$ is Strehl's polynomial $t_{\{1,2,\ldots,n\}}(\mathbf{1}, \mathbf{y}; z)$. By induction we obtain

$$\begin{aligned}
f(\vec{y}, y_{n+1}; z) &= \sum_{L \dot\cup R = \{1,2,\ldots,n\}} \left(z + \sum_{l \in L} y_l \right) \cdot f(\vec{y}_L; z) \cdot f(\vec{y}_R; 1) \\
&= \sum_{L \dot\cup R = \{1,2,\ldots,n\}} (z + \mathbf{y}^L_{\leq \max(L)}) \cdot t_L(\mathbf{1}, \mathbf{y}; z) \cdot t_R(\mathbf{1}, \mathbf{y}; 1) \\
&= \sum_{L \dot\cup R = \{1,2,\ldots,n\}} z \cdot s_L(\mathbf{1}, \mathbf{y}; z) \cdot t_R(\mathbf{1}, \mathbf{y}; 1) \\
&= z \cdot s_{\{1,2,\ldots,n\}}(\mathbf{1}, \mathbf{y}; z + 1) \\
&= t_{\{1,2,\ldots,n+1\}}(\mathbf{1}, \mathbf{y}; z),
\end{aligned}$$

where we used the recursion in Lemma 2.4.2, equation (2.4.1) and Theorem 2.4.3. □

2.5 Concluding remarks

The idea of considering a circular arrangement goes back to Pollak; see [39]. In fact, when all the cars have size 1, our argument without a trailer reduces to his argument that the number of classical parking functions is $(n+1)^{n-1}$.

The idea of not using fixed coordinates when placing cars in the circular arrangement is reminiscent of the argument Athanasiadis used to compute the characteristic polynomial of the Shi arrangement [1].

The polynomial $t_A(\mathbf{x}, \mathbf{y}; z)$ satisfies the following convolution identity; see [50, Equation (7)],

$$t_A(\mathbf{x}, \mathbf{y}; z + w) = \sum_{B \dot\cup C = A} t_B(\mathbf{x}, \mathbf{y}; z) \cdot t_C(\mathbf{x}, \mathbf{y}; w). \tag{2.5.1}$$

Hence it is suggestive to think of this polynomial as of binomial type and the polynomial $s_A(\mathbf{x}, \mathbf{y}; w)$ as an associated Sheffer sequence; see [40]. When setting all the parameters $\mathbf{x}$ to be constant and also the parameters $\mathbf{y}$ to be constant, we obtain the classical Abel–Rothe polynomials. Hence it is natural to ask if other sequences of binomial type and their associated Sheffer sequences have multi-parameter extensions. Since the Hopf algebra $\mathbf{k}[x]$ explains sequences of binomial type, one wonders if there is a Hopf algebra lurking in the background explaining equations (2.5.1) and (2.4.2).

Chapter 3 The antipode of the noncrossing partition lattice

3.1 Introduction

Recall from Section 1.6 that the antipode of the reduced incidence Hopf algebra of posets is an extension of the Möbius function of a poset. For instance, Hall's formula for the Möbius function as the alternating sum of chains was extended to the antipode by Schmitt [42]. However, very few posets have had their antipode determined in an explicit form. Haiman and Schmitt computed the antipode of the partition lattice [24, Corollary 1], and Einziger in her dissertation computed the antipode of the noncrossing partition lattice in terms of polygon dissections where each region has an even number of sides; see [17, Theorem 8.7]. These dissections are in bijective correspondence with noncrossing hypertrees; see [35, Remarks 3.2 and 3.3].

We present here a different approach to the antipode of the noncrossing partition lattice directly in terms of noncrossing hypertrees. Our approach is based on a map φ from chains in the noncrossing partition lattice to noncrossing hypertrees. Chains that belong to the same fiber of this map have the same poset product that appears in the Schmitt formula. The last step is to show that the alternating sum over each fiber cancels all elements but one. We do this by a geometric argument using the Euler characteristic with compact support. In this proof the permutahedron has a cameo appearance similar to that in the paper [15].

For more work on computing the antipode of posets, see [16, 17], and for other Hopf algebras appearing in combinatorics, see [2].

3.2 Preliminaries

Noncrossing partitions

Let $[n]$ denote the set $\{1, 2, \ldots, n\}$. A *(set) partition* $\pi = \{B_1, B_2, \ldots, B_k\}$ of the set $[n]$ is a collection of non-empty blocks which are disjoint and whose union is the set $[n]$. Let $|\pi|$ denote the number of blocks of the partition π. Let Π_n be the set of all partitions of $[n]$. Order Π_n such that for π and τ in Π_n, we have the inequality $\pi \leq \tau$ if each block of π is contained in a block of τ. Note that the minimal element of Π_n is the partition $1|2|\cdots|n$, and the maximal element is the singleton block $[n]$. In fact, the poset Π_n is a lattice, hence known as the partition lattice.

A partition σ of $[n]$ is *noncrossing* if it has the property that if $a < b < c < d$ and some block B of σ contains both a and c, while some block B' of σ contains both b and d, then B and B' are the same block, that is, $B = B'$. Geometrically, σ is a partition of the vertices of a regular n-gon (labeled by the set $[n]$) with the property that the convex hulls of its blocks are pairwise disjoint. For example, Figure 3.1 shows the noncrossing partition $14|2|3|589|67$. Then the set of noncrossing partitions on $[n]$ comprises a sublattice of Π_n, known as the noncrossing partition lattice, NC_n. Note

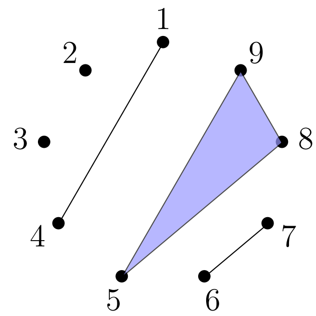


Figure 3.1: A noncrossing partition on 9 elements.

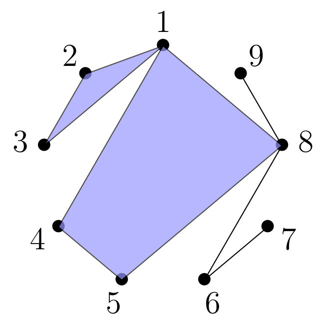


Figure 3.2: A noncrossing hypertree on 9 vertices.

that such a partition when pictured in this way divides the n-gon into regions. For instance, in our example in Figure 3.1, this partition has three regions: $\{1,2,3,4\}$, $\{1,4,5,9\}$, and $\{5,6,7,8\}$.

Recall that an *integer partition* λ is a sequence $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ of positive integers such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k \geq 1$. The length of a partition λ is given by $\ell(\lambda) = k$. Lastly, define the partition $\lambda + \mathbf{1}$ to be the partition $(\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_k + 1)$.

For a partition λ, we define the product of noncrossing lattices NC_λ to be the Cartesian product $\mathrm{NC}_\lambda = \prod_{i=1}^{\ell(\lambda)} \mathrm{NC}_{\lambda_i}$.

For more on the structure of the noncrossing partition lattice NC_n, see for instance [44] or [46].

Noncrossing hypertrees

A *hypergraph* H on a vertex set $[n]$ is a collection of subsets, called edges, of $[n]$ of cardinality at least 2. A hypergraph is *connected* if for all vertices x and y in the vertex set $[n]$ there exists a sequence of vertices $x = x_0, x_1, \ldots, x_k = y$ and a sequence of edges $E_1, E_2, \ldots, E_k$ such that $x_{i-1}, x_i \in E_i$ for all $1 \leq i \leq k$. A *hypertree* H on $[n]$, then, is a connected hypergraph such that $\sum_{E \in H}(|E| - 1) = n - 1$. Note that there are equivalent ways to define a hypertree, such as a connected hypergraph without any cycles, but we have picked the definition that best fits our purposes.

Take n points on a circle labeled counterclockwise from 1 to n, and consider graphs whose vertices are the given points and whose edges are rectilinear and do not cross. We call these *noncrossing graphs*. If we allow the edges to consist of more than just two vertices, then these become *noncrossing hypergraphs*. Then we may consider trees of this same type, called *noncrossing hypertrees*. We denote the set of noncrossing hypertrees on n vertices as NCHT_n. For example, Figure 3.2 shows a noncrossing hypertree on 9 vertices. These look quite similar to noncrossing partitions, and they again divide the n-gon into regions. The regions in the hypertree in Figure 3.2 are $\{1, 3, 4\}$, $\{1, 8, 9\}$, $\{5, 6, 8\}$, and $\{6, 7, 8\}$. For more on noncrossing hypertrees, see [35].

For a hypertree T on the vertex set $[n]$, let the type of the hypertree T be the partition $\mathrm{type}(T)$ consisting of the cardinalities of the edges of T in weakly decreasing order. Thus, the type of a tree on n vertices from classical graph theory is the partition $(2, 2, \ldots, 2)$.

Posets and the reduced incidence Hopf algebra

Given a finite poset P, let the order complex of P, denoted by $\Delta(P)$, be the simplicial complex consisting of all chains in the poset P. Observe that this includes the empty chain. For a poset P with a minimal element $\hat{0}$ and a maximal element $\hat{1}$, we let $\overline{P}$ denote the poset $P - \{\hat{0}, \hat{1}\}$, that is, with the minimal and maximal elements removed. For a poset P with minimal element $\hat{0}$ and maximal element $\hat{1}$, we will identify the complex $\Delta(\overline{P})$ with all chains in P containing the two extreme elements $\hat{0}$ and $\hat{1}$, that is,

$$\Delta(\overline{P}) = \{c \in \Delta(P) \ : \ c = \{\hat{0} = x_0 < x_1 < \cdots < x_k = \hat{1}\}\}.$$

The reduced incidence Hopf algebra on posets consists of the linear span of all isomorphism types of finite posets having a unique minimal element $\hat{0}$ and a unique maximal element $\hat{1}$. That is, we view two posets to be equal in this Hopf algebra if they are isomorphic. The product in this algebra is the Cartesian product of posets. The unit consists of the isomorphism type of the one-element poset. The coproduct (unfortunately also denoted by Δ) of a poset P is given by

$$\Delta(P) = \sum_{\hat{0} \leq x \leq \hat{1}} [\hat{0}, x] \otimes [x, \hat{1}].$$

Every Hopf algebra is endowed with an antipode. In the reduced incidence Hopf algebra, the antipode $S(P)$ of a poset P has an expression summing over all chains in P; see [42, Theorem 6.1]. See also [24, Theorem 1].

Theorem 3.2.1 (Schmitt). *The antipode of a poset P in the reduced incidence Hopf algebra of posets is given by*

$$S(P) = \sum_{c} (-1)^k \cdot [x_0, x_1] \cdot [x_1, x_2] \cdots [x_{k-1}, x_k], \tag{3.2.1}$$

where the sum is over all chains $c = \{\hat{0} = x_0 < x_1 < \cdots < x_k = \hat{1}\}$ in the poset P.

This formula for the antipode has the disadvantage that it is not cancellation-free in general.

3.3 The map from chains to noncrossing hypertrees

We now produce a map φ from chains in $\Delta(\overline{\mathrm{NC}_n})$ to noncrossing hypertrees. We build up the map step by step as we ascend the chain.

Definition 3.3.1. *For a block B of a noncrossing partition π and R a region of π adjacent to the block B, let $v_R(B)$ be the vertex of the region R in the most positive orientation (counterclockwise) that belongs to the block B.*

Definition 3.3.2. *For two noncrossing partitions π and σ in NC_n such that $\pi < \sigma$, define the hypergraph $\varphi(\pi, \sigma)$ as follows. For each region R of π and a maximal collection of blocks $B_1, B_2, \ldots, B_k$ of π, where $k \geq 2$, which are adjacent to the region R and are all contained in one block of the noncrossing partition σ, add the edge $\{v_R(B_1), v_R(B_2), \ldots, v_R(B_k)\}$ to the hypergraph $\varphi(\pi, \sigma)$.*

Note that the edge $\{v_R(B_1), v_R(B_2), \ldots, v_R(B_k)\}$ is contained in the region R of the partition π, and it is also contained in the associated block of the partition σ.

Lemma 3.3.3. *For two noncrossing partitions π and σ such that $\pi < \sigma$, the following poset isomorphism holds*

$$[\pi, \sigma] \cong \prod_{E \in \varphi(\pi,\sigma)} \mathrm{NC}_{|E|} .$$

Proof. Consider an edge E in $\varphi(\pi, \sigma)$ of size k. This edge must join together k blocks of π to form a single block (or subset of a block) of σ. All of the partial ways to join these blocks is isomorphic to the noncrossing partition lattice $\mathrm{NC}_k = \mathrm{NC}_{|E|}$. Since the edges of the hypergraph $\varphi(\pi, \sigma)$ are disjoint, these events are independent of each other, which on the poset level translates to the Cartesian product of posets. $\square$

Definition 3.3.4. *For a chain $c = \{\hat{0} = \pi_0 < \pi_1 < \cdots < \pi_k = \hat{1}\}$ in the noncrossing partition lattice NC_n, define the hypergraph $\varphi(c)$ to be the union of the hypergraphs $\bigcup_{i=1}^{k} \varphi(\pi_{i-1}, \pi_i)$.*

See Figure 3.3 for an example of this construction.

By combining Definition 3.3.4 and Lemma 3.3.3, we have the following corollary:

Corollary 3.3.5. *Let $c = \{\hat{0} = \pi_0 < \pi_1 < \cdots < \pi_k = \hat{1}\}$ be a chain in the noncrossing partition lattice NC_n. Then the following isomorphism holds*

$$\prod_{i=1}^{k} [\pi_{i-1}, \pi_i] \cong \prod_{E \in \varphi(c)} \mathrm{NC}_{|E|} . \qquad (3.3.1)$$

Especially, given another chain $d = \{\hat{0} = \sigma_0 < \sigma_1 < \cdots < \sigma_r = \hat{1}\}$ such that $\varphi(c) = \varphi(d)$, the two poset products $\prod_{i=1}^{k} [\pi_{i-1}, \pi_i]$ and $\prod_{i=1}^{r} [\sigma_{i-1}, \sigma_i]$ are isomorphic.

Lemma 3.3.6. *For a chain c in the noncrossing partition lattice NC_n, the hypergraph $\varphi(c)$ is a noncrossing hypertree.*

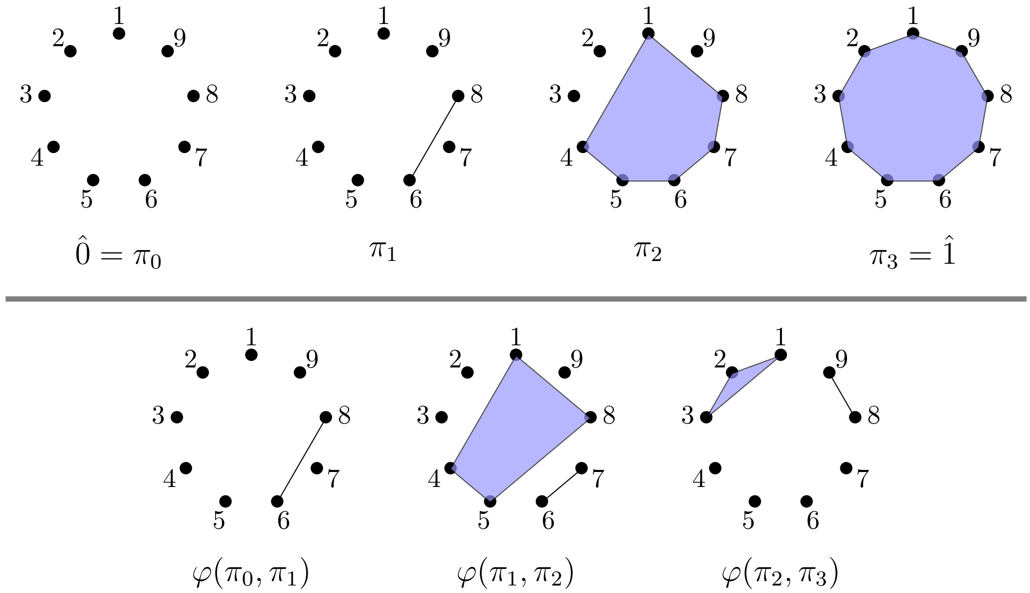


Figure 3.3: A chain in the noncrossing partition lattice NC_9, followed by the associated hypergraphs $\varphi(\pi_{i-1}, \pi_i)$. Note that the union of the hypergraphs is the hypertree displayed in Figure 3.2.

Proof. First, observe that $\varphi(c)$ is connected, since the chain c ends at $\pi_k = \hat{1}$, where all blocks have been merged. Now by considering the rank of the poset in equation (3.3.1), we have that

$$\sum_{E \in \varphi(c)} (|E| - 1) = \sum_{E \in \varphi(c)} \rho(\mathrm{NC}_{|E|}) = \sum_{i=1}^{k} \rho([\pi_{i-1}, \pi_i]) = \sum_{i=1}^{k} (\rho(\pi_i) - \rho(\pi_{i-1})) = n - 1.$$

Hence we conclude $\varphi(c)$ is a hypertree.

Finally, let $E \in \varphi(\pi_{i-1}, \pi_i)$, $F \in \varphi(\pi_{j-1}, \pi_j)$ be two edges of the hypertree $\varphi(c)$. If $i = j$, they were added at the same time and hence are disjoint. Without loss of generality we may assume $i < j$, that is, $i \leq j - 1$. Then E is contained in a block of π_i, which is contained in a block of π_{j-1}. Since F is contained in a region of π_{j-1}, we have that E and F intersect in at most a single point, and hence the hypertree is noncrossing. $\square$

3.4 A geometric approach to the fibers

We call two edges E and F in a noncrossing hypertree H *adjacent* if they share a vertex and they also border a common region of H. Note that the adjacency relation forms a tree where the vertices are the edges of the hypertree.

For a moment, consider the noncrossing hypertree with two adjacent edges $\{2, 3, 4\}$ and $\{1, 2, 5\}$. What do chains in the fiber of this hypertree under φ look like? Since

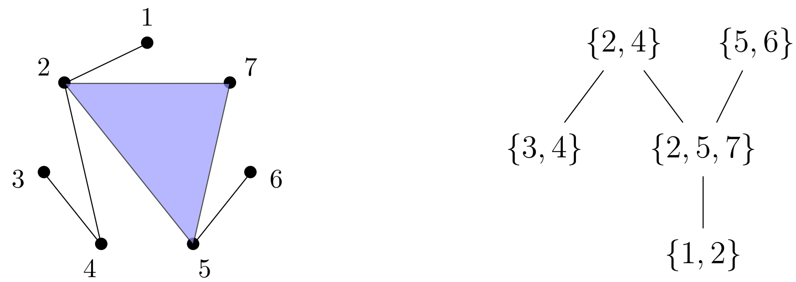


Figure 3.4: A noncrossing hypertree on 7 vertices along with the Hasse diagram of the induced relations on its hyperedges under the map φ.

they share the vertex 2, we know that they were not added at the same step of the chain in the noncrossing partition lattice. Furthermore, if the edge $\{2, 3, 4\}$ was added first, then the elements 2, 3, and 4 belong to the same block B in the partition at this moment of the chain. Now when the elements 1 and 5 are joined together with the block B, we would pick the element in the most positive orientation from B. However, this element would be 4, not the element 2. Hence we, conclude that the edge $\{1, 2, 5\}$ was added to the tree before the edge $\{2, 3, 4\}$. This motivates the following definition.

Definition 3.4.1. *Let E and F be two adjacent edges in a hypertree T, and assume that they share the vertex v. If the block E is on the left from the perspective of the vertex v looking into the hypertree, then we give the two blocks the order relation $E < F$.*

In other words, the order relation $E < F$ between two edges implies that the edge E was added before the edge F in every chain in the fiber of the hypertree. This gives us a tree with an order relation on each edge.

Example 3.4.2. Consider the hypertree H on 7 vertices with the edges $\{1, 2\}$, $\{2, 4\}$, $\{2, 5, 7\}$, $\{3, 4\}$, and $\{5, 6\}$, pictured in Figure 3.4. We extract the relations on these edges using Definition 3.4.1, yielding

$$\{1, 2\} < \{2, 5, 7\}, \qquad \{2, 5, 7\} < \{5, 6\}, \qquad \{2, 5, 7\} < \{2, 4\}, \qquad \{3, 4\} < \{2, 4\}.$$

Any chain c that induces this list of hyperedges without violating any of these relations is in the fiber $\varphi^{-1}(H)$.

An *ordered (set) partition* is a (set) partition where the blocks have a linear order. That is, $(C_1, C_2, \ldots, C_r)$ is an ordered set partition if $\{C_1, C_2, \ldots, C_r\}$ is a partition. Let Q_k denote the set of all ordered set partitions on the set $[k]$. We make Q_k into a poset by joining adjacent blocks, that is, the cover relation is

$$(C_1, \ldots, C_{i-1}, C_i, C_{i+1}, C_{i+2}, \ldots, C_r) \prec (C_1, \ldots, C_{i-1}, C_i \cup C_{i+1}, C_{i+2}, \ldots, C_r).$$

Note that Q_k has one maximal element ($[k]$) but $k!$ minimal elements. In fact, if we join a minimal element $\hat{0}$ to Q_k, we obtain the face lattice of the permutahedron.

This is the simple $(k-1)$-dimensional polytope that is the convex hull of the $k!$ permutations.

For our purposes, it is better to view the dual polytope which is simplicial. However, we will view the dual as a subdivision of the $(k-2)$-dimensional sphere, where we describe the sphere S^{k-2} as the set

$$S^{k-2} = \{(x_1, x_2, \dots, x_k) \in \mathbb{R}^k \ : \ x_1^2 + x_2^2 + \cdots + x_k^2 = 1,\ x_1 + x_2 + \cdots + x_k = 0\}.$$

An ordered set partition $\tau = (C_1, C_2, \dots, C_r)$ with at least two blocks corresponds to the subset $g(\tau)$ of the sphere given by the equalities and inequalities

(a) if i and j belong to the same block of the ordered set partition τ, then $x_i = x_j$,

(b) if i belongs to a block which is earlier than the block that j belongs to in the ordered set partition τ, then $x_i < x_j$.

Note that the set $g(\tau)$ is homeomorphic to an $(r-2)$-dimensional open ball.

Given the noncrossing hypertree H with the k edges $E_1, E_2, \dots, E_k$, define a map ψ from the fiber $\varphi^{-1}(H)$ to the ordered set partitions Q_k as follows. The chain $c = \{\hat{0} = \pi_0 < \pi_1 < \cdots < \pi_r = \hat{1}\}$ of length r in the noncrossing partition lattice is sent to the ordered set partition $\psi(c) = (C_1, C_2, \dots, C_r)$ into r blocks, where the ith block is described by $C_i = \{j \ : \ E_j \in \varphi(\pi_{i-1}, \pi_i)\}$. That is, the ordered set partition $\psi(c)$ encodes which edges of H are added in the ith step in the chain c. See Figure 3.5 for an example of this construction.

We can now prove our desired equality.

Proposition 3.4.3. *For all noncrossing hypertrees H on n elements with k edges, we have*

$$\sum_{c \in \varphi^{-1}(H)} (-1)^{\ell(c)} = (-1)^k.$$

Proof. Recall that the Euler characteristic with compact support of an r-dimensional open ball is $(-1)^r$; see [23]. Using this property, we reformulate the sum as computing the Euler characteristic of a union:

$$\sum_{c \in \varphi^{-1}(H)} (-1)^{\ell(c)} = \sum_{c \in \varphi^{-1}(H)} \chi_c(g(\psi(c))) = \chi_c\left(\bigcup_{c \in \varphi^{-1}(H)} g(\psi(c)) \right),$$

where we used that the Euler characteristic with compact support is an additive valuation. The union of the right-hand side is the subset of the sphere S^{k-2} intersected with the open halfspaces $x_i < x_j$ if $E_i < E_j$ in the hypertree H. This intersection yields an open ball of dimension $k-2$. The Euler characteristic with compact support of this ball is $(-1)^k$, completing the proof. $\square$

One can avoid the Euler characteristic with compact support by not working on the sphere. Instead, we can use the dual of the permutahedron, which is the barycentric subdivision of the boundary of the simplex, and the Euler characteristic as a valuation, as presented in [26].

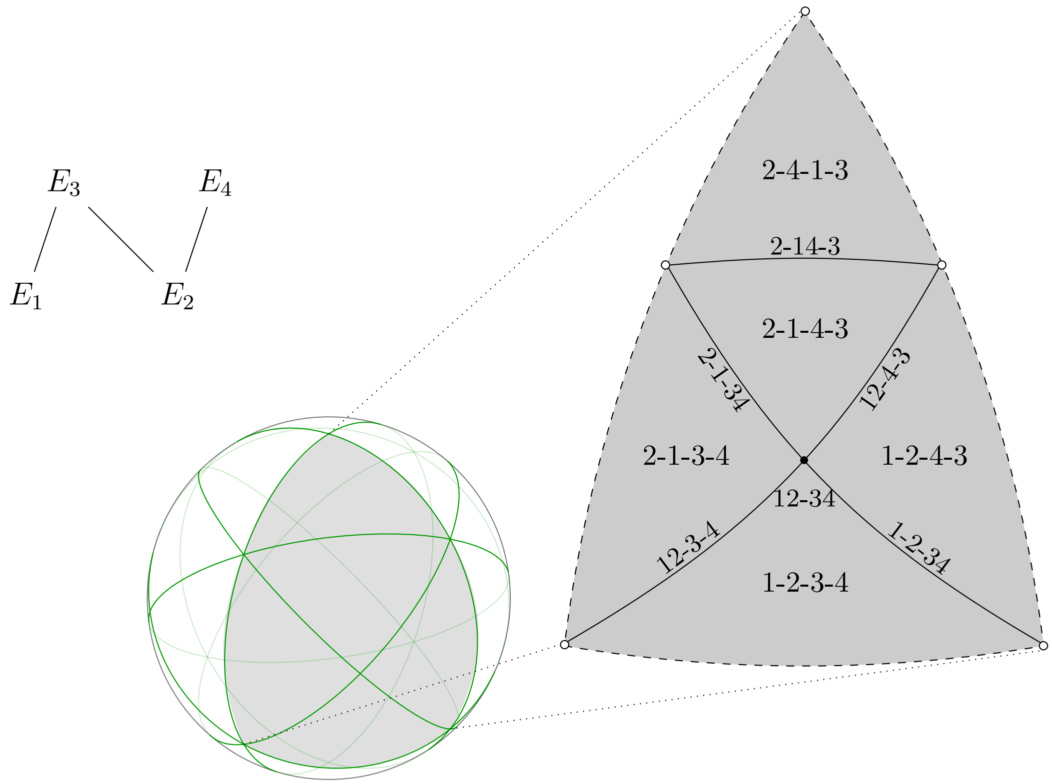


Figure 3.5: The poset order on the edges of some hypertree and the associated complex cut out from the sphere S^2 by the inequalities $x_1 < x_3$, $x_2 < x_3$, and $x_2 < x_4$, where we label the faces with the associated ordered partitions.

3.5 A sign-reversing involution approach to the fibers

In this section we present a Morse matching proving Proposition 3.4.3. Suppose the noncrossing hypertree H has k edges. Let Q_H be the collection of all ordered set partitions compatible with the hypertree H.

Our first step is to label the edges of the hypertree H with indices $1, 2, 3, \ldots, k$ so that if E_i is adjacent to E_j and E_j is closer to E_k than E_i, then $i < j$. Then we may think of the hypertree as a rooted tree with root E_k, and we could remove the edges in the order E_1, E_2, …, E_{k-1} until we are left with just the root E_k. We shall now refer to the edges $E_1, E_2, \ldots, E_k$ as nodes of this rooted tree, which are also endowed with a partial order via Definition 3.4.1.

Furthermore, for $1 \le i \le k-1$, let $p(i)$ be the index of the parent of the node E_i. That is, $E_{p(i)}$ is the first node on the path from E_i to the root E_k. We call a node E_i an *ascent node* if $E_i < E_{p(i)}$ (in the partial order of Definition 3.4.1), and otherwise we call E_i a *descent node*. Let $a(1)$ through $a(k_a)$ be the indices of all the ascent nodes in order, that is, $a(1) < a(2) < \cdots < a(k_a)$. Similarly, let $b(1) < b(2) < \cdots < b(k_b)$ be all the indices of the descent nodes in order. Note that $k_a + k_b = k - 1$, as E_k is neither an ascent nor a descent node.

Consider an ordered partition σ on the set $[k]$. We say that σ has *type* q if it is of the form

$$(\{a(1)\}, \{a(2)\}, \ldots, \{a(\alpha)\}, C_1, C_2, \ldots, C_r, \{b(\beta)\}, \{b(\beta-1)\}, \ldots, \{b(1)\}), \quad (3.5.1)$$

where $\alpha + \beta + 1 = q$ is maximal. That is, $C_1 \neq \{a(\alpha+1)\}$ and $C_r \neq \{b(\beta+1)\}$.

We now define a matching on the collection Q_H. Consider an ordered set partition σ of type $q \leq k-1$. That is, σ is given in equation (3.5.1) with $r \geq 1$. Assume that the block C_i of σ contains the element q, that is, $q \in C_i$. We have four cases to consider:

(1) Suppose E_q is an ascent node, that is, $E_q < E_{p(q)}$, and C_i is the singleton block $\{q\}$. Note that $i \neq 1$, or the type of σ would be at least $q+1$. Merge the block C_i with the previous block C_{i-1} to obtain the ordered partition $u(\sigma)$, that is,

$$u(\sigma) = (\ldots, C_{i-2}, C_{i-1} \cup C_i, C_{i+1}, \ldots).$$

Match σ with this new partition $u(\sigma)$. Note that $u(\sigma)$ is also contained in Q_H because the requirement that $E_q < E_{p(q)}$ is still satisfied, and no other relations have been disturbed.

(2) Suppose E_q is an ascent node as in (1), and q is contained in the block C_i, but C_i is not a singleton block, that is, $\{q\} \subsetneqq C_i$. Then we will split off the element q by itself to the right. In particular, we create the ordered partition $d(\sigma)$ by

$$d(\sigma) = (\ldots, C_{i-1}, C_i - \{q\}, \{q\}, C_{i+1}, \ldots).$$

Match σ with this new partition $d(\sigma)$. Note again that no relations can have been disturbed, so $d(\sigma) \in Q_H$.

(3) Suppose E_q is a descent node, that is, $E_q > E_{p(q)}$, and C_i is the singleton block $\{q\}$. Note that $i \neq r$, since this would imply the type of σ is greater than q. Merge the block C_i with the subsequent block C_{i+1} to obtain the ordered partition $u(\sigma)$, that is,

$$u(\sigma) = (\ldots, C_{i-1}, C_i \cup C_{i+1}, C_{i+2}, \ldots).$$

Match σ with this new partition $u(\sigma)$.

(4) Suppose E_q is a descent node as in (3), and q is contained in the block C_i, but C_i is not a singleton block, that is, $\{q\} \subsetneqq C_i$. Then we will split off the element q by itself to the left. In particular, we create the ordered partition $d(\sigma)$ by

$$d(\sigma) = (\ldots, C_{i-1}, \{q\}, C_i - \{q\}, C_{i+1}, \ldots).$$

Match σ with this new partition $d(\sigma)$.

Lemma 3.5.1. *The above rules (1) through (4) define a matching on Q_H. Furthermore, the only unmatched ordered set partition is given by*

$$(\{a(1)\}, \{a(2)\}, \ldots, \{a(k_a)\}, \{k\}, \{b(k_b)\}, \{b(k_b - 1)\}, \ldots, \{b(1)\}). \qquad (3.5.2)$$

Proof. For any set partition σ of type strictly less than k meeting the requirements of conditions (1) and (3), we have that $d(u(\sigma)) = \sigma$. Similarly, for an ordered set partition σ satisfying conditions (2) and (4), we have $u(d((\sigma)) = \sigma$. Hence these four conditions describe a matching. Finally, we note also that none of the four types of matchings changes the type of an ordered partition σ. Since the ordered set partition in (3.5.2) is the unique ordered set partition of type k, this implies that it does not get matched. □

Example 3.5.2. Consider the noncrossing hypertree H from Example 3.4.2 along with the partial order on its edges unduced by the map φ.

One possible indexing of the edges that satisfies the requirements of our matching would be

$$E_1 = \{5, 6\}, \quad E_2 = \{3, 4\}, \quad E_3 = \{1, 2\}, \quad E_4 = \{2, 4\}, \quad E_5 = \{2, 5, 7\}.$$

The unmatched ordered set partition for this labeling is 2-3-5-4-1. See Figure 3.6 for the entire poset Q_H with its matching under this labeling of the edges.

Lemma 3.5.3. *If σ and τ are two ordered set partitions such that $\sigma \leq \tau$, then the inequality* $\mathrm{type}(\sigma) \geq \mathrm{type}(\tau)$ *holds.*

Proof. Since we join blocks together of σ to obtain τ, we cannot increase the number of singleton blocks at either end of the ordered set partition, and this is the only way to increase the type of a partition. □

Proposition 3.5.4. *Assume σ and τ are two different ordered set partitions of the same type q and that they satisfy the cover relations $\sigma \prec u(\sigma) \succ \tau \prec u(\tau)$. Note this implies that $\{q\}$ is a singleton block in both σ and τ. Then the element q is closer to the parent $p(q)$ in the ordered set partition σ than in τ.*

Proof. Note that the block $\{q\}$ joined the block away from the parent $p(q)$ in order to form $u(\sigma)$. Then to obtain τ, the element q had to split from this block, continuing to move away from the parent $p(q)$. □

This proposition proves that no directed cycles are possible in the matching, suggesting the Morse property.

Theorem 3.5.5. *The above matching on Q_H is a discrete Morse matching.*

Proof. Assume we have a directed cycle. Then Lemma 3.5.3 implies that all the types are the same in the directed cycle. But Proposition 3.5.4 implies that the element q always moves away from its parent, which is impossible in a cycle. □

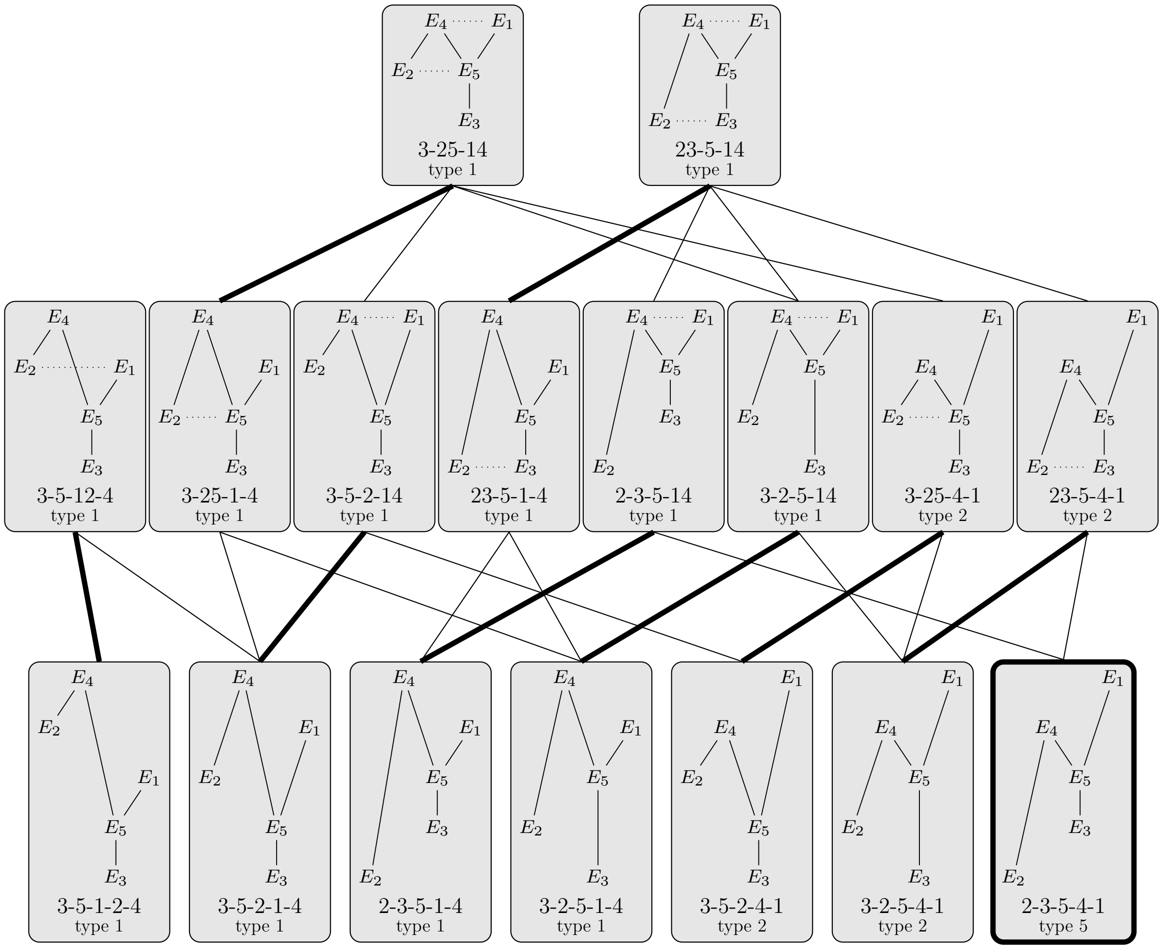


Figure 3.6: The poset Q_H for the hypertree from Example 3.4.2 with the indexing on the edges of H from Example 3.5.2. The matching is signified by the thicker lines in the Hasse diagram.

Second proof of Proposition 3.4.3. The matching yields a sign-reversing involution where the only fixed point is the critical cell. $\square$

While the Morse property was not necessary to prove Proposition 3.4.3, it does raise questions concerning the topological properties of these posets Q_H. Recall also that the elements in Q_H correspond to chains in the noncrossing partition lattice, so Q_H is the dual of a subposet of the order complex $\Delta(\overline{\mathrm{NC}_n})$.

3.6 Conclusion and closing remarks

Combining Proposition 3.4.3 with Schmitt's antipode formula in Theorem 3.2.1 yields the main results:

Theorem 3.6.1. *The antipode of the noncrossing partition lattice is given by*

$$S(\mathrm{NC}_n) = \sum_{H} (-1)^{|H|} \cdot \mathrm{NC}_{\mathrm{type}(H)},$$

where the sum is over all noncrossing hypertrees H on n vertices.

As a corollary we have

Corollary 3.6.2. *The antipode of the noncrossing partition lattice is given by*

$$S(\mathrm{NC}_n) = \sum_{\lambda \vdash n-1} (-1)^{\ell(\lambda)} \cdot \left\{ \textit{number of noncrossing hypertrees of type } \lambda + \mathbf{1} \right\} \cdot \mathrm{NC}_{\lambda+\mathbf{1}} .$$

Einziger combines her expression for the antipode using polygon dissections with the enumerative results in [18] to give the following expression

$$S(\mathrm{NC}_n) = \frac{1}{2n+1} \cdot \sum_{k=1}^{n} (-1)^k \cdot \binom{2n+k}{k} \cdot \sum_{i_1+i_2+\cdots+i_k=n} \mathrm{NC}_{i_1} \cdot \mathrm{NC}_{i_2} \cdots \mathrm{NC}_{i_k};$$

see Theorem 8.9 in [17].

Chapter 4 Sums of powers of the descent set statistic

4.1 Introduction

It has always been interesting to study divisibility properties of sequences defined combinatorially. Three classical examples are Fibonacci numbers, the partition function, and binomial coefficients. The Fibonacci numbers satisfy $\gcd(F_m, F_n) = F_{\gcd(m,n)}$. Ramanujan discovered that the partition function satisfies, among other relations, that 5 divides $p(5n+4)$. The binomial coefficients are well-studied modulo a prime; see the theorems of Lucas and Kummer in Section 4.2. In this chapter we consider divisibility properties of the sum of powers of the descent set statistic from permutation enumeration. The descent set statistic was first studied by MacMahon [34].

Recall from Section 1.2 that for a permutation π in the symmetric group $\mathfrak{S}_n$, the descent set of π is the subset of $[n-1] = \{1, 2, \ldots, n-1\}$ given by $\mathrm{Des}(\pi) = \{i \in [n-1] \ : \ \pi_i > \pi_{i+1}\}$. The descent set statistics $\beta_n(S)$ are defined for subsets S of $[n-1]$ by

$$\beta_n(S) = \left|\{\pi \in \mathfrak{S}_n \ : \ \mathrm{Des}(\pi) = S\}\right|.$$

Since there are $n!$ permutations, we directly have

$$n! = \sum_{S \subseteq [n-1]} \beta_n(S).$$

Define A_n^r to be the sum of the rth powers of the descent set statistics, that is,

$$A_n^r = \sum_{S \subseteq [n-1]} \beta_n(S)^r.$$

This quantity occurs naturally as moments of the random variable $\mathrm{Des}(S)$, where the set S is chosen with a uniform distribution from all subsets of the set $[n-1]$.

In Section 4.3 we give two expressions, depending on the parity of r for A_n^r; see Lemma 4.3.1. We continue by showing that for an odd prime p and an even positive integer r, if m and n contain the same non-zero digits in base p, then the prime p dividing A_m^r is equivalent to p dividing A_n^r. In Section 4.4 we give lower bounds for the number of prime factors in A_n^r. These bounds depend on the digit sum of n in base p. Unfortunately, we do not obtain any bound when p is an odd prime and r is even. In Section 4.5 we sharpen the results by collecting terms together occurring in the expansion of Lemma 4.3.1. The method of collection is by considering orbits of a group action. First we use the cyclic group $\mathbb{Z}_{p^k}$, and then we use a group defined by the action on the balanced p-ary tree of cyclically rotating the branches under any node. The lower bounds obtained in this section for the prime factors of p in A_n^r now also depend on the base p expansion of r.

We end in the concluding remarks by presenting two tables obtained by computation to compare our bounds with the actual number of factors of 2 and 3 occurring in A_n^r.

4.2 Preliminaries

Define $\alpha_n(S)$ by the sum

$$\alpha_n(S) = \sum_{T \subseteq S} \beta_n(T).$$

Observe that $\alpha_n(S)$ enumerates the number of permutations in $\mathfrak{S}_n$ with descent set contained in the set S. Especially, we know that $A_n^1 = \alpha_n([n-1]) = n!$. For more on descents; see [49, Section 1.4].

Define a bijection co from subsets of the set $[n-1]$ to compositions of n by sending the set $S = \{s_1 < s_2 < \cdots < s_{k-1}\}$ to the composition $\mathrm{co}(S) = (c_1, c_2, \ldots, c_k)$, where $c_i = s_i - s_{i-1}$ with $s_0 = 0$ and $s_k = n$. See, for instance, [4] or [48, Section 7.19]. It is now straightforward to observe that $\alpha_n(S)$ is given by the multinomial coefficient $\binom{n}{\mathrm{co}(S)}$.

Using elementary number theory we have three observations.

Proposition 4.2.1. *Let p be a prime. Assume that r and s are both greater than or equal to k and $r \equiv s \pmod{p^{k-1} \cdot (p-1)}$. Then the congruence $A_n^r \equiv A_n^s \pmod{p^k}$ holds. Especially, the statement p^k divides A_n^r is equivalent to p^k divides A_n^s.*

Proof. We may assume that $r < s$, that is, $s - r = p^{k-1} \cdot (p-1) \cdot j$ for a positive integer j. For an integer x which is relative prime to the prime p, Euler's theorem implies that $x^s \equiv x^r \cdot (x^{p^{k-1}(p-1)})^j \equiv x^r \pmod{p^k}$. For an integer x which is divisible by the prime p, we have $x^s \equiv 0 \equiv x^r \pmod{p^k}$ since $r, s \geq k$. Thus for all integers x we have $x^s \equiv x^r \pmod{p^k}$, and we conclude

$$A_n^s \equiv \sum_{S \subseteq [n-1]} \beta_n(S)^s \equiv \sum_{S \subseteq [n-1]} \beta_n(S)^r \equiv A_n^r \pmod{p^k}. \qquad \square$$

When the prime p is 2 and $k \geq 3$, we have an improvement of a factor of 2.

Proposition 4.2.2. *Assume that r and s are both greater than or equal to $k \geq 3$ and $r \equiv s \pmod{2^{k-2}}$. Then the congruence $A_n^r \equiv A_n^s \pmod{2^k}$ holds. Especially, the statement 2^k divides A_n^r is equivalent to 2^k divides A_n^s.*

Proof. For an odd integer x we know that $x^{2^{k-2}} \equiv 1 \pmod{2^k}$, which yields the better bound using the same argument as in the proof of Proposition 4.2.1. $\square$

Proposition 4.2.3. *Let p be a prime and r an integer such that $r \geq k \cdot p$. If p^k divides the k numbers $A_n^{r-(p-1)}$, $A_n^{r-2\cdot(p-1)}$, through $A_n^{r-k\cdot(p-1)}$, then p^k divides A_n^r.*

Proof. By Fermat's little theorem we know $x^{p-1} - 1 \equiv 0 \pmod{p}$ for x relative prime to p. Hence the kth power of this quantity is divisible by p^k, that is, $(x^{p-1} - 1)^k \equiv 0 \pmod{p^k}$. Note that $x^k \equiv 0 \pmod{p^k}$ for x not relative prime to p. Multiplying these two statements we obtain

$$x^{k \cdot p} - \binom{k}{1} \cdot x^{k \cdot p - (p-1)} + \cdots + (-1)^k \cdot x^k \equiv 0 \pmod{p^k}$$

for all x. Multiply this polynomial relation with $x^{r-k\cdot p}$, substitute x to be $\beta_n(S)$, and sum over all $S \subseteq [n-1]$ to obtain the linear recursion

$$A_n^r - \binom{k}{1} \cdot A_n^{r-(p-1)} + \cdots + (-1)^k \cdot A_n^{r-k\cdot(p-1)} \equiv 0 \pmod{p^k}.$$

This relation yields the result. □

Example 4.2.4. Note using Table 4.1 that for $8 \leq n \leq 20$, the power 2^5 divides A_n^r when $5 \leq r \leq 9$. Hence, Proposition 4.2.3 gives that 2^5 divides A_n^r for $r \geq 5$.

Example 4.2.5. Using Table 4.2 we know for $n = 6$ and $8 \leq n \leq 20$ that 3^2 divides A_n^3 and A_n^5. Hence, Proposition 4.2.3 implies for r odd and $r \geq 3$ that 3^2 divides A_n^r. Similarly, we know for $n \in \{9, 10, 12, 13, 15, 16, 18, 19, 20\}$ that 3^3 divides A_n^3, A_n^5 and A_n^7. Therefore, for these same values of n, 3^3 divides A_n^r for r odd and $r \geq 3$.

Remark 4.2.6. Note that Propositions 4.2.1 through 4.2.3 apply to any sequence of the form $\sum_{i=1}^{N} c_i \cdot d_i^r$ where c_i and d_i are integers.

We end this section by reviewing Lucas' theorem, see [33, Chapter XXIII, Section 228], and Kummer's theorem, see [29], for multinomial coefficients.

Theorem 4.2.7 (Lucas). *Let p be a prime and $\vec{c} = (c_1, c_2, \ldots, c_k)$ be a weak composition of n, that is, 0 is allowed as an entry. Expand n and each c_i in base p, that is, $n = \sum_{j\geq 0} n_j \cdot p^j$ and $c_i = \sum_{j\geq 0} c_{i,j} \cdot p^j$ where $0 \leq n_j, c_{i,j} \leq p-1$. Let $\vec{c}_j$ be the weak composition $\vec{c}_j = (c_{1,j}, c_{2,j}, \ldots, c_{k,j})$. Then the multinomial coefficient $\binom{n}{\vec{c}}$ modulo p is given by*

$$\binom{n}{\vec{c}} \equiv \prod_{j\geq 0} \binom{n_j}{\vec{c}_j} \pmod{p}.$$

The power of the prime p in the factorization of the binomial coefficient $\binom{n}{c}$ is given by the number of carries of the addition $c + (n-c)$ in base p. This result is due to Kummer. We are going to need an extension of this result for the multinomial coefficient $\binom{n}{\vec{c}}$. Hence we define $\text{carries}_p(\vec{c})$ to be the sum of the carries when adding $c_1 + c_2 + \cdots + c_k$ in base p. To give a better definition, for a positive integer n, let $u_p(n)$ be the sum of the digits when n is written in base p. More formally, for $n = \sum_{i\geq 0} n_i \cdot p^i$, where $0 \leq n_i \leq p-1$, the function $u_p(n)$ is given by the sum $\sum_{i\geq 0} n_i$. Furthermore, for a composition $\vec{c} = (c_1, c_2, \ldots, c_k)$, define $u_p(\vec{c})$ to be the sum of the digits when all the parts of $\vec{c}$ are written in base p, that is, $u_p(\vec{c}) = \sum_{i=1}^{k} u_p(c_i)$.

Definition 4.2.8. *For a composition $\vec{c}$ of n, define the sum of the carries of the digits in base p by*

$$\text{carries}_p(\vec{c}) = \frac{u_p(\vec{c}) - u_p(n)}{p-1}.$$

The motivation for this definition is as follows. If one lines up the parts $c_1, c_2, \ldots, c_k$ of $\vec{c}$ in base p, note that any one of the $u_p(\vec{c})$ units in any of these addends has only two options: It may either contribute to a carry along with another $p-1$ units in its column, or it can directly become one of the $u_p(n)$ units in n.

Theorem 4.2.9 (Kummer). *For a prime p and a composition $\vec{c} = (c_1, c_2, \ldots, c_k)$ of n, the largest power d such that p^d divides the multinomial coefficient $\binom{n}{\vec{c}}$ is given by* $\mathrm{carries}_p(\vec{c})$.

Corollary 4.2.10. *Let p be a prime. Then the number of factors of p in A_n^1 is $(n - u_p(n))/(p-1)$.*

Proof. Note that $A_n^1 = n! = \binom{n}{1,1,\ldots,1}$. Hence by Kummer's theorem the number of factors of p is $\mathrm{carries}_p(1,1,\ldots,1) = (u_p(1,1,\ldots,1) - u_p(n))/(p-1)$. $\square$

4.3 Divisibility by odd primes

First, we express the sum $A_n^r = \sum_{S \subseteq [n-1]} \beta_n(S)^r$ in terms of $\alpha_n(S)$.

Lemma 4.3.1. *When r is even, A_n^r is given by*

$$A_n^r = \sum_{T_1, T_2, \ldots, T_r \subseteq [n-1]} (-1)^{\sum_{i=1}^r |T_i|} \cdot 2^{n-1-|\bigcup_{i=1}^r T_i|} \cdot \prod_{i=1}^r \alpha_n(T_i). \tag{4.3.1}$$

When r is odd, we have

$$A_n^r = \sum_{\substack{T_1, T_2, \ldots, T_r \subseteq [n-1] \\ T_1 \cup T_2 \cup \cdots \cup T_r = [n-1]}} (-1)^{n-1+\sum_{i=1}^r |T_i|} \cdot \prod_{i=1}^r \alpha_n(T_i). \tag{4.3.2}$$

Proof. We begin by expanding $\beta_n(S)$ in terms of $\alpha_n(S)$:

$$\begin{aligned}
A_n^r &= \sum_{S \subseteq [n-1]} \beta_n(S)^r \\
&= \sum_{S \subseteq [n-1]} \prod_{i=1}^r \left(\sum_{T_i \subseteq S} (-1)^{|S - T_i|} \cdot \alpha_n(T_i) \right) \\
&= \sum_{T_1, T_2, \ldots, T_r \subseteq [n-1]} \; \sum_{T_1 \cup T_2 \cup \cdots \cup T_r \subseteq S \subseteq [n-1]} (-1)^{r \cdot |S|} \cdot (-1)^{\sum_{i=1}^r |T_i|} \cdot \prod_{i=1}^r \alpha_n(T_i).
\end{aligned}$$

When r is even, we have $(-1)^{r \cdot |S|} = 1$, and the inner sum has $2^{n-1-|\bigcup_{i=1}^r T_i|}$ terms. When r is odd, the inner sum is zero unless the union $\bigcup_{i=1}^r T_i$ is the whole set $[n-1]$. $\square$

Theorem 4.3.2. *Let p be an odd prime and r an even positive integer. Assume that m and n contain the same non-zero digits when written in base p. Then the congruence $2^{-m} \cdot A_m^r \equiv 2^{-n} \cdot A_n^r \pmod{p}$ holds. Especially, the prime p divides A_m^r if and only if p divides A_n^r.*

Proof. Let m and n have the base p expansions $m = \sum_{j \geq 0} m_j \cdot p^j$ and $n = \sum_{j \geq 0} n_j \cdot p^j$. Then there exists a permutation π on the non-negative integers such that $m_j = n_{\pi(j)}$ for all $j \geq 0$. Note that this permutation may not be unique, but we fix one such permutation π for the remainder of the argument. Essentially, π permutes the powers of the prime p. Define a bijection f on the non-negative integers by $f\left(\sum_{j \geq 0} a_j \cdot p^j\right) = \sum_{j \geq 0} a_j \cdot p^{\pi(j)}$, where $0 \leq a_j \leq p - 1$. Note that

$$f(m) = \sum_{j \geq 0} m_j \cdot p^{\pi(j)} = \sum_{j \geq 0} n_{\pi(j)} \cdot p^{\pi(j)} = \sum_{j \geq 0} n_j \cdot p^j = n.$$

Furthermore, when there are no carries adding x and y in base p, this function is additive, that is, $f(x + y) = f(x) + f(y)$. Also note that the inverse function f^{-1} is additive under the same condition. In terms of compositions, we have that if $\vec{c} = (c_1, c_2, \ldots, c_k)$ is a composition of m such that $\mathrm{carries}_p(\vec{c}) = 0$, then the composition $f(\vec{c}) = (f(c_1), f(c_2), \ldots, f(c_k))$ is a composition of $f(m) = n$.

Let the non-carry power set $\mathrm{NCP}(m)$ be the collection of all subsets of $[m-1]$ whose associated composition has no carries when added in base p, that is,

$$\mathrm{NCP}(m) = \{T \subseteq [m-1] \ : \ \mathrm{carries}_p(\mathrm{co}(T)) = 0\}.$$

Observe that $\mathrm{NCP}(m)$ is closed under inclusion. Note that we can define a bijection $f : \mathrm{NCP}(m) \longrightarrow \mathrm{NCP}(n)$ by composing the three maps

$$\begin{aligned} \mathrm{NCP}(m) &\xrightarrow{\mathrm{co}} \{\vec{c} \in \mathrm{Comp}(m) : \mathrm{carries}_p(\vec{c}) = 0\} \\ &\xrightarrow{f} \{\vec{d} \in \mathrm{Comp}(n) : \mathrm{carries}_p(\vec{d}) = 0\} \\ &\xrightarrow{\mathrm{co}^{-1}} \mathrm{NCP}(n). \end{aligned}$$

Since the compositions $\vec{c}$ and $f(\vec{c})$ have the same length, the function f preserves cardinality. But there is a more direct description of the last map f on sets. For $T = \{t_1 < t_2 < \cdots < t_k\} \in \mathrm{NCP}(m)$, we claim that $f(T) = \{f(t_1), f(t_2), \ldots, f(t_k)\}$. Let $\vec{c}$ be the composition $\mathrm{co}(T)$. By definition, the ith element of $f(T)$ is the initial partial sum of the i first elements of $f(\vec{c})$, that is, $f(c_1) + \cdots + f(c_i)$. Since the whole sum $c_1 + \cdots + c_k$ has no carries, the partial sum also has no carries. Hence, the ith element of $f(T)$ is given by $f(c_1) + \cdots + f(c_i) = f(c_1 + \cdots + c_i) = f(t_i)$, proving the claim.

Also note that for a composition $\vec{c}$ without any carries, we have by Lucas' Theorem that

$$\binom{m}{\vec{c}} \equiv \binom{f(m)}{f(\vec{c})} \pmod{p},$$

since the factors of the product in Lucas' Theorem are permuted by the permutation π. Hence, for a set T in $\mathrm{NCP}(m)$ we know that $\alpha_m(T) = \alpha_n(f(T))$.

We now use the expansion in equation (4.3.1). Let $\vec{c}^{\,i}$ be the composition associated with the subset T_i of $[m-1]$. Similarly, let U_i be the subset of $[n-1]$ associated with the composition $f(\vec{c}^{\,i}) = \vec{d}^{\,i}$. Next we study the two unions $\bigcup_{i=1}^{r} T_i$ and $\bigcup_{i=1}^{r} U_i$. However, they may not be in the collection $\mathrm{NCP}(m)$, respectively, $\mathrm{NCP}(n)$.

For I a non-empty subset of the index set $[r]$, let T_I be the intersection $\bigcap_{i \in I} T_i$. Note that T_I belongs to $\mathrm{NCP}(m)$ since this collection is closed under inclusion. Similarly, let U_I be the intersection $\bigcap_{i \in I} U_i$ which belongs to $\mathrm{NCP}(n)$. Note that $f(T_I) = U_I$, so the two sets T_I and U_I have the same cardinality. By inclusion-exclusion we have

$$\left| \bigcup_{i=1}^{r} T_i \right| = \sum_{\emptyset \subsetneq I \subseteq [r]} (-1)^{|I|-1} \cdot |T_I| = \sum_{\emptyset \subsetneq I \subseteq [r]} (-1)^{|I|-1} \cdot |U_I| = \left| \bigcup_{i=1}^{r} U_i \right|.$$

Now observe that the non-zero terms in equation (4.3.1) modulo p are the terms where T_i belongs to $\mathrm{NCP}(m)$. Hence, modulo p we have that

$$\begin{aligned} A_m^r &\equiv \sum_{T_1, T_2, \ldots, T_r \in \mathrm{NCP}(m)} (-1)^{\sum_{i=1}^{r} |T_i|} \cdot 2^{m-1-|\bigcup_{i=1}^{r} T_i|} \cdot \prod_{i=1}^{r} \alpha_m(T_i) \\ &\equiv 2^{m-n} \cdot \sum_{U_1, U_2, \ldots, U_r \in \mathrm{NCP}(n)} (-1)^{\sum_{i=1}^{r} |U_i|} \cdot 2^{n-1-|\bigcup_{i=1}^{r} U_i|} \cdot \prod_{i=1}^{r} \alpha_n(U_i) \\ &\equiv 2^{m-n} \cdot A_n^r \pmod{p}. \end{aligned}$$

This proves the identity. Finally, since 2 is invertible modulo p, we obtain that A_m^r and A_n^r either both have a factor of p or none of them have a factor of p. $\square$

Corollary 4.3.3. *When r is even and p is an odd prime, the congruence $A_{pn}^r \equiv A_n^r \pmod{p}$ holds.*

Proof. Since n and $p \cdot n$ have the same non-zero digits modulo p, Theorem 4.3.2 applies. Hence, it is enough to observe that $2^{pn} \equiv (2^n)^p \equiv 2^n \pmod{p}$ using Fermat's little theorem. $\square$

Corollary 4.3.4. *When r is even and p is an odd prime, $A_{p^k}^r$ is not divisible by p.*

Proof. It is enough to check that $A_1^r = 1$ is not divisible by p. $\square$

Example 4.3.5. We can compute A_{14}^2 to observe that this number has a factor of 3. Hence by Proposition 4.2.1 we know that for all even r, the prime 3 divides A_{14}^r. Furthermore, 14 in base 3 consists of two 1's and one 2. Hence Theorem 4.3.2 implies for $n = 16, 22, 32, 34, 38, 42, 46, 48, 58, 64, 66, 86, 88, \ldots$ that 3 divides A_n^r as well.

Example 4.3.6. Note that 5 divides $A_3^2 = 10$. Hence, we know that 5 divides $A_n^{4 \cdot i+2}$ for n of the form $3 \cdot 5^k$. One may compute that 5 also divides A_{12}^2 and A_{13}^2. This implies that 5 divides $A_n^{4 \cdot i+2}$ for n belonging to the following two sequences: 12, 52, 60, 252, 260, 300, 1252, 1260, 1300, ... and $13, 17, 53, 65, 77, 85, 253, 265, 325, 377, 385, \ldots$.

4.4 On the number of prime factors

Recall that $\text{carries}_p(\vec{c})$ is the sum of carries when adding $c_1 + c_2 + \cdots + c_k$ in base p. Also recall that for a positive integer n, that $u_p(n)$ is the sum of the digits when n is written in base p and that for a composition $\vec{c} = (c_1, c_2, \ldots, c_k)$, $u_p(\vec{c})$ is the sum of the digits when all the parts of $\vec{c}$ are written in base p. Similarly, define the depth of n to be $d_p(n) = u_p(n) - 1$, that is, the sum of the digits of n in base p beyond the requisite digit greater than zero in its first position. Further, define the depth of a composition $\vec{c}$ to be the sum of the depth of each of its parts, that is, $d_p(\vec{c}) = \sum_{i=1}^{k} d_p(c_i)$. The next lemma is direct.

Lemma 4.4.1. *For a composition $\vec{c}$ into k parts, $u_p(\vec{c}) = d_p(\vec{c}) + k$.*

Recall according to the map co from subsets $S \subseteq [n-1]$ to compositions $\vec{c}$ of n that the number of parts k of $\vec{c}$ is one more than the cardinality of S. Combining this observation with the previous two lemmas yields the next result.

Proposition 4.4.2. *For a set $S \subseteq [n-1]$ and its associated composition $\text{co}(S) = \vec{c}$ of n, the sum of the carries $\text{carries}_p(\vec{c})$ is given by $(d_p(\vec{c}) + |S| - d_p(n))/(p-1)$.*

This gives way to the main result in this section.

Theorem 4.4.3. *When r is odd and p is prime, the sum A_n^r has at least*

$$\frac{n - 1 - r \cdot d_p(n)}{p - 1}$$

factors of p.

Proof. Consider a term in equation (4.3.2), where we let $\text{co}(T_i) = \vec{c}^{\,i}$. The number of factors of p in this term is given by

$$\sum_{i=1}^{r} \text{carries}_p(\vec{c}^{\,i}) = \sum_{i=1}^{r} \frac{d_p(\vec{c}^{\,i}) + |T_i| - d_p(n)}{p-1} \geq \sum_{i=1}^{r} \frac{|T_i| - d_p(n)}{p-1} \geq \frac{n - 1 - r \cdot d_p(n)}{p-1},$$

since $d_p(\vec{c}^{\,i}) \geq 0$ for all i and $\sum_{i=1}^{r} |T_i| \geq |\bigcup_{i=1}^{r} T_i| = n - 1$. □

We can say something stronger when the prime p is 2.

Theorem 4.4.4. *The sum A_n^r is divisible by $2^{n-1-r \cdot d_2(n)}$.*

Proof. The case when r is odd follows from Theorem 4.4.3. Now suppose r is even, and consider a term in equation (4.3.1). The number of factors of 2 in this term is given by

$$\begin{aligned} n - 1 - \left|\bigcup_{i=1}^{r} T_i\right| + \sum_{i=1}^{r} \text{carries}_2(\vec{c}^{\,i}) &\geq n - 1 - \sum_{i=1}^{r} |T_i| + \sum_{i=1}^{r} \text{carries}_2(\vec{c}^{\,i}) \\ &= n - 1 - \sum_{i=1}^{r} |T_i| + \sum_{i=1}^{r} (d_2(\vec{c}^{\,i}) + |T_i| - d_2(n)) \\ &\geq n - 1 - r \cdot d_2(n). \end{aligned}$$

□

Since $d_2(2^k) = 0$, we have the following corollary.

Corollary 4.4.5. *When n is a power of 2, then A_n^r is divisible by 2^{n-1}.*

In this case, we actually have equality.

Proposition 4.4.6. *When n is a power of 2, then 2^{n-1} is the highest power dividing A_n^r.*

Proof. For x an r-tuple $(T_1, T_2, \dots, T_r)$, let $h(x)$ represent the associated term in Lemma 4.3.1. Note that the expression for $h(x)$ depends on the parity of r. Observe in the proofs of Theorems 4.4.3 and 4.4.4 that we have equality in the bound for those terms where the sets T_i are disjoint and $d_2(\vec{c}^{\,i}) = 0$ for all i. Note that the latter condition requires all the parts of $\vec{c}^{\,i}$ to be powers of 2. Let X be the collection of all such r-tuples.

Consider an r-tuple $x = (T_1, T_2, \dots, T_r) \in X$, and choose the smallest index $1 \le k \le \lfloor r/2 \rfloor$ such that $T_{2k-1} \neq T_{2k}$, if one exists. Let x' be obtained by switching the $(2k-1)$st and $2k$th subsets, that is, $x' = (T_1, \dots, T_{2k-2}, T_{2k}, T_{2k-1}, T_{2k+1}, \dots, T_r)$. Observe that $h(x') = h(x)$. Since the subsets T_i are all disjoint, the only case where such a k does not exist is when $T_1, T_2, \dots, T_{2\cdot\lfloor r/2 \rfloor}$ are all empty. This occurs in a single r-tuple x_0, where $x_0 = (\emptyset, \emptyset, \dots, \emptyset)$ if r is even, and $x_0 = (\emptyset, \emptyset, \dots, \emptyset, [n-1])$ if r is odd. When r is even, we directly observe that $h(x_0) \equiv 2^{n-1} \pmod{2^n}$. When r is odd we have $h(x_0) \equiv n! \equiv 2^{n-1} \pmod{2^n}$, using that n is a power of 2. Now, after pairing up all these terms except the term $h(x_0)$, the result follows by

$$A_n^r \equiv \sum_{x \in X} h(x) \equiv h(x_0) \equiv 2^{n-1} \pmod{2^n}. \qquad \square$$

4.5 Improving the bound

We now improve upon the bounds of Theorems 4.4.3 and 4.4.4.

Proposition 4.5.1. *When p is an odd prime and $n \ge 2$, the sum $A_n^{p^k}$ has at least*

$$\frac{n - 1 - p^k \cdot d_p(n)}{p-1} + k$$

factors of p.

Proof. Consider the term indexed by the p^k-tuple $(T_1, T_2, \dots, T_{p^k})$ in equation (4.3.2), and consider the action by the shift $(T_2, T_3, \dots, T_{p^k}, T_1)$. Note that the size of the orbit of this action is p^i, for some $0 \le i \le k$. Grouping these p^i identical terms gives i factors of p. However, this means that our tuple $(T_1, T_2, \dots, T_{p^k})$ can be written as $(T_1, T_2, \dots, T_{p^i}, T_1, T_2, \dots, T_{p^i}, \dots, T_1, T_2, \dots, T_{p^i})$ up to a cyclic shift, and further, $\bigcup_{j=1}^{p^i} T_j = [n-1]$. Hence, the number of factors of p in these terms is

$$\begin{aligned} i + \sum_{j=1}^{p^k} \text{carries}_p(\vec{c}^{\,j}) &= i + \sum_{j=1}^{p^k} \frac{d_p(\vec{c}^{\,j}) + |T_j| - d_p(n)}{p-1} \\ &\geq i + \frac{\left(\sum_{j=1}^{p^k} |T_j|\right) - p^k \cdot d_p(n)}{p-1} \\ &\geq i + \frac{p^{k-i} \cdot (n-1) - p^k \cdot d_p(n)}{p-1} \\ &= i + \frac{(p^{k-i}-1) \cdot (n-1)}{p-1} + \frac{n-1-p^k \cdot d_p(n)}{p-1} \\ &\geq k + \frac{n-1-p^k \cdot d_p(n)}{p-1}, \end{aligned}$$

where in the last step we used $(p^{k-i}-1)/(p-1) = 1 + p + \cdots + p^{k-i-1} \geq k-i$ and $n-1 \geq 1$. $\square$

The above proof uses the action of the cyclic group $\mathbb{Z}_{p^k}$ to collect terms together. We can improve the bound of Proposition 4.5.1 in some cases by using a larger group acting on the r-tuples.

Let q be the prime power p^k. We define the group G_q acting on the set $[q]$. The generators are indexed by pairs (a, b) where $1 \leq a \leq k$ and $0 \leq b \leq p^{k-a} - 1$. The generator $\sigma_{a,b}$ is given by the following product of p-cycles,

$$\sigma_{a,b} = \prod_{i=1}^{p^{a-1}} (i + bp^a, i + bp^a + p^{a-1}, \ldots, i + bp^a + (p-1)p^{a-1}).$$

To give a geometric picture of the action of this group, consider a balanced p-ary tree of depth k. This tree has q leaves, which we label 1 through q. Furthermore, the tree has $(q-1)/(p-1)$ internal nodes, which are indexed by the pairs (a, b). The a coordinate states that the internal node is at depth $k - a$. The b coordinate indicates which node at that depth, reading from left to right. The generator $\sigma_{a,b}$ then cyclically shifts the p children of this node. See Figure 4.1 for an example.

With this geometric picture, it is straightforward to observe that the group has order $p^{(q-1)/(p-1)}$. Given a q-tuple of sets $x = (T_1, T_2, \ldots, T_q)$, let the group G_q act on x by permuting the indices. Let Orb_x be the orbit of the q-tuple x, that is, $\text{Orb}_x = \{g \cdot x : g \in G_q\}$. Note that the cardinality of the orbit Orb_x is a power of p.

Additionally, for an r-tuple $x = (T_1, \ldots, T_r)$ let $f_n^r(x) = (-1)^{\sum_{i=1}^r |T_i|} \cdot \prod_{i=1}^r \alpha_n(T_i)$.

Proposition 4.5.2. *Let $q = p^k$ and $d_p(n) > 0$. For a q-tuple $x = (T_1, T_2, \ldots, T_q)$, the sum $\sum_{y \in \text{Orb}_x} f_n^q(y)$ has at least*

$$\frac{q - 1 + |\bigcup_{i=1}^q T_i| - q \cdot d_p(n)}{p-1}$$

factors of p.

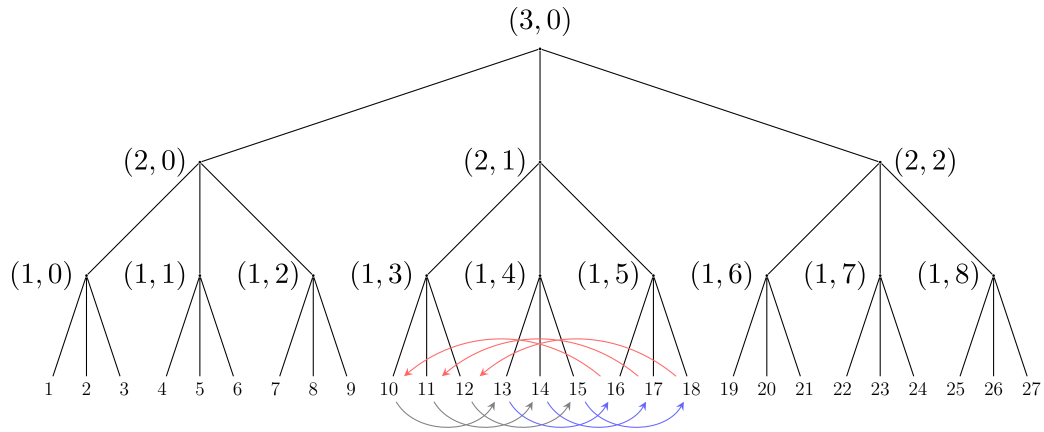


Figure 4.1: A balanced ternary tree of depth 3 with the action of $\sigma_{2,1}$ shown.

Proof. The proof is by induction on k. The induction basis is $k = 0$, that is, $q = 1$. Here Orb_x consists only of (T). The number of p-factors are

$$\mathrm{carries}_p(\vec{c}) = \frac{d_p(\vec{c}) + |T| - d_p(n)}{p-1} \geq \frac{|T| - d_p(n)}{p-1},$$

since $d_p(\vec{c}) \geq 0$, which completes the basis of the induction.

Now assume that the statement is true for all p-powers strictly less than q. Notice that $f_n(y) = f_n(x)$ for all $y \in \mathrm{Orb}_x$. Hence,

$$\sum_{y \in \mathrm{Orb}_x} f_n(y) = |\,\mathrm{Orb}_x\,| \cdot f_n^q(x) = |\,\mathrm{Orb}_x\,| \cdot (-1)^{\sum_{i=1}^q |T_i|} \cdot \prod_{i=1}^{q} \alpha_n(T_i).$$

Furthermore, the number of factors of p in the last expression is

$$\log_p(|\,\mathrm{Orb}_x\,|) + \sum_{i=1}^{q} \mathrm{carries}_p(\vec{c}^{\,i}) = \log_p(|\,\mathrm{Orb}_x\,|) + \sum_{i=1}^{q} \frac{d_p(\vec{c}^{\,i}) + |T_i| - d_p(n)}{p-1}.$$

For $0 \leq b \leq p-1$ let x_b denote the q/p-tuple $(T_{b \cdot q/p+1}, \ldots, T_{(b+1)\cdot q/p})$, that is, the q/p-tuple of sets below the node $(k-1, b)$ in the tree.

First, assume that the stabilizer of x contains an element involving the permutation $\sigma_{k,0}$. That is, the stabilizer contains a rotation centered at the root $(k, 0)$ of the tree. Then the leaves below the nodes $(k-1, 0)$ are the same as the leaves below $(k-1, b)$. Then the cardinality of the orbit Orb_x is the same as the size of the orbit Orb_{x_0}. Hence we can apply the induction hypotheses to the node $(k-1, 0)$ of the tree:

$$\begin{aligned}
&\log_p(|\operatorname{Orb}_x|) + \sum_{i=1}^{q} \operatorname{carries}_p(\vec{c}^{\,i}) \\
&= \log_p(|\operatorname{Orb}_{x_0}|) + \sum_{i=1}^{q/p} \operatorname{carries}_p(\vec{c}^{\,i}) + \sum_{i=q/p+1}^{q} \operatorname{carries}_p(\vec{c}^{\,i}) \\
&\geq \frac{q/p - 1 + \left|\bigcup_{i=1}^{q/p} T_i\right| - q/p \cdot d_p(n)}{p-1} + \sum_{i=q/p+1}^{q} \frac{d_p(\vec{c}^{\,i}) + |T_i| - d_p(n)}{p-1} \\
&= \frac{q/p - 1 + |\bigcup_{i=1}^{q} T_i| - q \cdot d_p(n)}{p-1} + \sum_{i=q/p+1}^{q} \frac{d_p(\vec{c}^{\,i}) + |T_i|}{p-1}.
\end{aligned}$$

If T_i is non-empty, then $|T_i| \geq 1$. If T_i is empty, then $\vec{c}^{\,i}$ is the composition n, so $d_p(\vec{c}^{\,i}) = d_p(n) \geq 1$ by our assumption. In both cases we have $d_p(\vec{c}^{\,i}) + |T_i| \geq 1$ for all $q/p + 1 \leq i \leq q$. Thus, we can apply this inequality

$$\log_p(|\operatorname{Orb}_x|) + \sum_{i=1}^{q} \operatorname{carries}_p(\vec{c}^{\,i}) \geq \frac{q/p - 1 + |\bigcup_{i=1}^{q} T_i| - q \cdot d_p(n)}{p-1} + \frac{q - q/p}{p-1},$$

which yields the bound.

It remains to consider the case when the stabilizer of x does not contain a rotation centered at the root $(k, 0)$. Now the cardinality of the orbit of x is given by the product

$$|\operatorname{Orb}_x| = \prod_{b=0}^{p-1} |\operatorname{Orb}_{x_b}|.$$

Hence we apply the induction hypotheses to each child of the root

$$\begin{aligned}
\log_p(|\operatorname{Orb}_x|) + \sum_{i=1}^{q} \operatorname{carries}_p(\vec{c}^{\,i}) &= \sum_{b=0}^{p-1} \left(\log_p(|\operatorname{Orb}_{x_b}|) + \sum_{i=1}^{q/p} \operatorname{carries}_p(\vec{c}^{\,b \cdot q/p + i}) \right) \\
&\geq \sum_{b=0}^{p-1} \frac{q/p - 1 + \left|\bigcup_{i=1}^{q/p} T_{b \cdot q/p + i}\right| - q/p \cdot d_p(n)}{p-1} \\
&\geq \frac{q - p + |\bigcup_{i=1}^{q} T_i| - q \cdot d_p(n)}{p-1},
\end{aligned}$$

which yields the bound. This completes the second case and the induction. □

Theorem 4.5.3. *For r odd and $d_p(n) > 0$, the sum A_n^r contains at least*

$$\left\lceil \frac{r - u_p(r) + n - 1 - r \cdot d_p(n)}{p-1} \right\rceil$$

factors of p.

Proof. Let $r = \sum_{i=1}^{u_p(r)} q_i$ where q_i is a power of p. Note that a power p^j occurs at most $p-1$ times in this sum. Now define the group G to be the Cartesian product $G = \prod_{i=1}^{u_p(r)} G_{q_i}$. Furthermore, let G act on the set $[r]$ by letting the G_{q_i} act on the interval $[q_1 + \cdots + q_{i-1} + 1, q_1 + \cdots + q_{i-1} + q_i]$. The action of the group G can be viewed as forest consisting of $u_p(r)$ trees. Finally, let G act on a r-tuple by acting on the indices of the tuple.

Note that the function f_n^r is multiplicative in the following meaning. For an r-tuple $x = (T_1, \ldots, T_r)$ define x_i to be the q_i-tuple $(T_{q_1+\cdots+q_{i-1}+1}, \ldots, T_{q_1+\cdots+q_{i-1}+q_i})$. Then we have

$$f_n^r(T_1, \ldots, T_r) = \prod_{i=1}^{u_p(r)} f_n^{q_i}(x_i).$$

Now the sum over an orbit of the r-tuple $x = (T_1, \ldots, T_r)$ factors as

$$\sum_{y \in \mathrm{Orb}_x} f_n^r(y) = \prod_{i=1}^{u_p(r)} \sum_{y_i \in \mathrm{Orb}_{x_i}} f_n^{q_i}(y_i).$$

Hence we can apply Proposition 4.5.2 to each factor, and the sum over the orbit has at least

$$\begin{aligned} &\sum_{i=1}^{u_p(r)} \frac{1}{p-1} \cdot \left(q_i - 1 + \left| \bigcup_{j=q_1+\cdots+q_{i-1}+1}^{q_1+\cdots+q_i} T_j \right| - q_i \cdot d_p(n) \right) \\ &\geq \frac{1}{p-1} \cdot \left(r - u_p(r) + \left| \bigcup_{j=1}^{r} T_j \right| - r \cdot d_p(n) \right) \qquad (4.5.1) \\ &= \frac{1}{p-1} \cdot \left(r - u_p(r) + n - 1 - r \cdot d_p(n) \right), \end{aligned}$$

where the last equality comes from the assumption $T_1 \cup T_2 \cup \cdots \cup T_r = [n-1]$ in equation (4.3.2). □

Again, we can make a stronger statement when $p = 2$.

Theorem 4.5.4. *For $d_2(n) > 0$, the sum A_n^r contains at least $r - u_2(r) + n - 1 - r \cdot d_2(n)$ factors of 2.*

Proof. The case where r is odd follows from Theorem 4.5.3. We retain the notation of the proof of Theorem 4.5.3. Note that in that proof, we did not use the parity of r until the very end. Now assume that r is even. For an r-tuple $x = (T_1, \ldots, T_r)$ define the function $g_n^r(x) = 2^{n-1-|\bigcup_{i=1}^r T_i|} \cdot f_n^r(x)$, which is the expression in equation (4.3.1). Hence the number of factors of 2 in the sum over the orbit

$$\sum_{y \in \mathrm{Orb}_x} g_n^r(y) = 2^{n-1-|\bigcup_{i=1}^r T_i|} \cdot \sum_{y \in \mathrm{Orb}_x} f_n^r(y)$$

is bounded from below by the sum of $n-1-|\bigcup_{i=1}^{r} T_i|$ and the expression (4.5.1). That is,

$$n-1-\left|\bigcup_{i=1}^{r} T_i\right|+r-u_2(r)+\left|\bigcup_{j=1}^{r} T_j\right|-r\cdot d_2(n)=n-1+r-u_2(r)-r\cdot d_2(n).\square$$

Theorem 4.5.4 improves upon Theorem 4.4.4 by at least 1 when n is not a 2-power.

Corollary 4.5.5. *When n is not a power of 2 and $r \geq 2$, then A_n^r has at least $n-r\cdot d_2(n)$ factors of 2.*

Proof. Note that $r \geq 2$ implies that $r > u_2(r)$, that is, $r-u_2(r)-1 \geq 0$. Hence by Theorem 4.5.4 we have $r-u_2(r)+n-1-r\cdot d_2(n) \geq n-r\cdot d_2(n)$. $\square$

Corollary 4.5.6. *Let n satisfy the inequality $2^k \leq n \leq 2^{k+1}-1$. Then A_n^2 is divisible by 2^{2^k-1}.*

Proof. Write n as the sum 2^k+a. When $a=0$ there is nothing to prove by Corollary 4.4.5. When $a \geq 1$ we have $d_2(n)=u_2(a)$. Furthermore, since for each 2-power 2^j, where $j \geq 1$, we have $2^j-2\cdot u_2(2^j) \geq 0$. But for $j=0$ we have $2^j-2\cdot u_2(2^j)=-1$. Hence for all non-negative a we have $a-2\cdot u_2(a) \geq -1$. Hence the bound by Corollary 4.5.5 yields $n-2\cdot d_2(n)=2^k+a-2\cdot u_2(a) \geq 2^k-1$. $\square$

4.6 Concluding remarks

Some of the results in this chapter are reminiscent of results in the papers [4, 8, 9], where there are results which depend on the binary expansion of the parameters. However, as the reader can see from Tables 4.1 and 4.2, where we present computational results for the numbers of factors of the primes 2 and 3 in A_n^r, a lot of work remains in order to understand these numbers.

In Section 4.5 there were two groups and their actions that helped us in improving the bounds. The first group is the cyclic group $\mathbb{Z}_{p^k}$ in Proposition 4.5.1 and the second group is G_{p^k}. Is there another group and associated group action that would provide a new bound?

Are there values of n such that A_n^2 is divisible by 3, other than those values listed in Example 4.3.5? Similarly, are there values of n other than those in Example 4.3.6 such that 5 divides A_n^2?

Corollary 4.5.6 yields a weakly increasing function $w(n)$ such that $2^{w(n)}$ divides A_n^2. Are there such lower bound functions $w(n)$ for other primes p and other powers r so that $p^{w(n)}$ divides A_n^r?

A final question is to understand the asymptotic behavior of A_n^r as n tends to infinity. How similar is this behavior to Stirling's formula? Using the classical inequality $(x_1+\cdots+x_m)/m \leq \sqrt[r]{(x_1^r+\cdots+x_m^r)/m} \leq \max(x_1,\ldots,x_m)$, we obtain the following bounds on A_n^r,

$$2^{-(r-1)\cdot(n-1)} \leq \frac{A_n^r}{(n!)^r} \leq 2^{n-1}\cdot\left(\frac{E_n}{n!}\right)^r,$$

Table 4.1: A comparison of our best prediction of the number of factors of 2 in A_n^r with the actual number. Predictions are given first, colored according to whether the result is given by Proposition 4.2.1, Proposition 4.2.2, Theorem 4.4.4, or Theorem 4.5.4, and the actual value is given second.

$\mathbf{n}$	$\mathbf{n}_2$	$d_2(\mathbf{n})$	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$	$r=6$	$r=7$	$r=8$
2	10	0	1, 1	1, 1	1, 1	1, 1	1, 1	1, 1	1, 1	1, 1
3	11	1	1, 1	1, 1	1, 1	1, 1	1, 1	1, 1	1, 1	1, 1
4	100	0	3, 3	3, 3	3, 3	3, 3	3, 3	3, 3	3, 3	3, 3
5	101	1	3, 3	3, 6	2, 2	3, 3	2, 2	3, 3	2, 2	3, 3
6	110	1	4, 4	4, 4	3, 3	4, 4	3, 3	3, 5	3, 3	4, 4
7	111	2	4, 4	3, 3	1, 2	2, 5	2, 2	3, 4	2, 2	4, 5
8	1000	0	7, 7	7, 7	7, 7	7, 7	7, 7	7, 7	7, 7	7, 7
9	1001	1	7, 7	7, 7	6, 7	7, 8	6, 6	6, 6	5, 7	7, 7
10	1010	1	8, 8	8, 8	7, 7	8, 8	7, 7	7, 8	6, 10	8, 8
11	1011	2	8, 8	7, 8	5, 5	5, 6	3, 6	3, 5	3, 5	4, 7
12	1100	1	10, 10	10, 10	9, 9	10, 10	9, 9	9, 11	8, 12	10, 10
13	1101	2	10, 10	9, 9	7, 7	7, 10	5, 7	4, 7	3, 6	4, 11
14	1110	2	11, 11	10, 11	8, 10	8, 13	6, 8	5, 9	3, 7	4, 11
15	1111	3	11, 11	9, 9	6, 7	5, 8	3, 7	3, 8	3, 6	4, 8
16	10000	0	15, 15	15, 15	15, 15	15, 15	15, 15	15, 15	15, 15	15, 15
17	10001	1	15, 15	15, 15	14, 15	15, 17	14, 14	14, 14	13, 14	15, 15
18	10010	1	16, 16	16, 16	15, 15	16, 16	15, 15	15, 17	14, 15	16, 16
19	10011	2	16, 16	15, 15	13, 14	13, 13	11, 12	10, 16	8, 11	9, 14
20	10100	1	18, 18	18, 18	17, 17	18, 18	17, 17	17, 19	16, 18	18, 18

Table 4.2: A comparison of our best prediction of the number of factors of 3 in A_n^r with the actual number. Predictions are given first, colored according to whether the result is given by Proposition 4.2.1, Example 4.2.5, Example 4.3.5, Theorem 4.4.3, Proposition 4.5.1, or Theorem 4.5.3, and the actual value is given second.

n	$\mathbf{n}_3$	$d_3(\mathbf{n})$	$r=1$	$r=2$	$r=3$	$r=4$	$r=5$	$r=6$	$r=7$	$r=8$	$r=9$
2	2	1	0, 0	0, 0	0, 0	0, 0	0, 0	0, 0	0, 0	0, 0	0, 0
3	10	0	1, 1	0, 0	2, 2	0, 0	1, 1	0, 0	1, 1	0, 0	3, 3
4	11	1	1, 1	0, 0	1, 2	0, 0	1, 1	0, 0	1, 1	0, 0	2, 3
5	12	2	1, 1	0, 0	1, 1	0, 0	1, 1	0, 0	1, 2	0, 0	1, 1
6	20	1	2, 2	0, 0	2, 4	0, 0	1, 2	0, 0	2, 2	0, 0	2, 5
7	21	2	2, 2	0, 0	1, 2	0, 0	1, 1	0, 0	1, 1	0, 0	2, 3
8	22	3	2, 2	0, 0	1, 2	0, 0	1, 2	0, 0	2, 2	0, 0	2, 2
9	100	0	4, 4	0, 0	5, 6	0, 0	4, 4	0, 0	4, 4	0, 0	6, 7
10	101	1	4, 4	0, 0	4, 6	0, 0	3, 4	0, 0	3, 4	0, 0	4, 6
11	102	2	4, 4	0, 0	3, 3	0, 0	1, 2	0, 0	2, 2	0, 0	2, 4
12	110	1	5, 5	0, 0	5, 6	0, 0	4, 5	0, 0	4, 5	0, 0	5, 7
13	111	2	5, 5	0, 0	4, 4	0, 0	2, 3	0, 0	2, 3	0, 0	3, 5
14	112	3	5, 5	0, 1	3, 4	1, 2	1, 2	1, 1	2, 2	1, 1	2, 5
15	120	2	6, 6	0, 0	5, 5	0, 0	3, 5	0, 0	2, 5	0, 0	3, 6
16	121	3	6, 6	1, 1	4, 5	1, 1	1, 3	1, 1	2, 3	1, 1	3, 7
17	122	4	6, 6	0, 0	3, 4	0, 0	1, 2	0, 0	2, 2	0, 0	2, 3
18	200	1	8, 8	0, 0	8, 11	0, 0	7, 8	0, 0	7, 8	0, 0	8, 12
19	201	2	8, 8	0, 0	7, 8	0, 0	5, 7	0, 0	4, 7	0, 0	4, 10
20	202	3	8, 8	0, 0	6, 8	0, 0	3, 5	0, 0	2, 6	0, 0	3, 7

where E_n denotes the nth Euler number. Note that the asymptotic behavior of the Euler number is described by $E_n/n! \sim 2 \cdot (2/\pi)^{n+1}$. Thus as n tends to infinity, the quantity $A_n^r/(n!)^r$ is bounded between two exponential functions. From numerical data we make the conjecture that there is a constant c_r such that a good approximation of $\log(A_n^r)$ is given by $n \cdot \log(n) + c_r \cdot n$.

Chapter 5 The boustrophedon transform for descent polytopes

5.1 Introduction

Recall from Section 1.7 that for a word $\mathbf{v} = \mathbf{v}_1\mathbf{v}_2\cdots\mathbf{v}_{n-1}$ of length $n-1$ in the letters $\mathbf{x}$ and $\mathbf{y}$, we define the *descent polytope* $\mathrm{DP}_{\mathbf{v}}$ to be the n-dimensional polytope

$$\mathrm{DP}_{\mathbf{v}} = \{(x_1, x_2, \ldots, x_n) \in [0,1]^n \ : \ x_i \leq x_{i+1} \text{ if } \mathbf{v}_i = \mathbf{x}, \text{ and } x_i \geq x_{i+1} \text{ if } \mathbf{v}_i = \mathbf{y}\}.$$

Descent polytopes briefly appeared in [13, Subsection 4.2], but it was in the paper [3] they were first studied for their own sake.

The volume of the descent polytope $\mathrm{DP}_{\mathbf{v}}$ is given by $1/n!$ times the descent set statistic. It is a classical result that the descent set statistic is maximized when the word $\mathbf{v}$ is alternating; see [6, 37, 38, 41, 51, 52]. Hence it is natural to ask if there are other statistics of the descent polytope which are maximized for the alternating word. The paper [3, Corollary 2.5] proves that the number of i-dimensional faces of the descent polytope $\mathrm{DP}_{\mathbf{v}}$ is also maximized for the alternating word.

We show here an alternative way to compute the f-polynomial of the descent polytope $\mathrm{DP}_{\mathbf{v}}$. Our method is reminiscent of the boustrophedon transform to compute the descent set statistics due to de Bruijn [6]; see also [51, 52]. The boustrophedon transform consists of two linear operators $\mathbb{N}^n \longrightarrow \mathbb{N}^{n+1}$ defined by $(p_1, p_2, \ldots, p_n) \longmapsto (0, p_1, p_1 + p_2, \ldots, p_1 + \cdots + p_n)$ and $(p_1, p_2, \ldots, p_n) \longmapsto (p_1 + \cdots + p_n, p_2 + \cdots + p_n, \ldots, p_n, 0)$. Applying $n-1$ of these two operators on the starting vector (1) yields a vector of length n whose entries sum to the corresponding descent set statistic; see [6]. The name boustrophedon was first used by Millar, Sloane, and Young [36] in a slightly different context, but the name extends naturally to this situation.

Our method to compute the f-polynomial of descent polytopes consists of two linear operators on the space $\mathbb{N}[t]^3$; see equations (5.2.1) and (5.2.2). We obtain an element in $\mathbb{N}[t]^3$ that we call the refined f-polynomial. Again, by summing the entries we obtain the desired f-polynomial. The advantage of this approach is that it is straightforward to obtain the necessary inequalities for the refined f-polynomial. These inequalities immediately yield the maximizing result for the f-polynomial of the descent polytope. This follows the same outline as the de Bruijn proof for the descent set statistic.

5.2 The boustrophedon transform

For a polytope P, let the face number $f_i(P)$ denote the number of i-dimensional faces of P. Define the f-polynomial to be the (finite) sum $f(P) = \sum_{i \geq 0} f_i(P) \cdot t^i$. Note that we do not include the empty face. That is, the f-polynomial encodes the f-vector as its coefficients. Another way to express the f-polynomial is as the sum $\sum_F t^{\dim(F)}$, where the sum is over all non-empty faces F.

We define the right action of an $\mathbf{xy}$-word on the space $\mathbb{N}[t]^3$ as follows. The empty word 1 is the identity action, so that $(p, q, r) \circ 1 = (p, q, r)$. For a non-empty $\mathbf{xy}$-word the action is defined recursively by the two cases

$$(p, q, r) \circ \mathbf{xv} = (p,\ \ t \cdot p + (1 + t) \cdot q,\ \ p + q + r) \circ \mathbf{v}, \tag{5.2.1}$$

$$(p, q, r) \circ \mathbf{yv} = (p + q + r,\ \ (1 + t) \cdot q + t \cdot r,\ \ r) \circ \mathbf{v}. \tag{5.2.2}$$

We introduce a refinement of the f-polynomial of the descent polytope. Let H^n_c be the hyperplane $x_n = c$ in $\mathbb{R}^n$. For $c = 0, 1$ let $f^c(\mathrm{DP}_\mathbf{v})$ denote the sum $\sum_F t^{\dim(F)}$, where F ranges over all non-empty faces of $\mathrm{DP}_\mathbf{v}$ contained in the hyperplane H^n_c. Finally, let $f^{1/2}(\mathrm{DP}_\mathbf{v})$ be the f-polynomial of the remaining faces, that is, the sum $\sum_F t^{\dim(F)}$ where the face F is not contained in either of the hyperplanes H^n_0 or H^n_1. We define the *refined f-polynomial* of the descent polytope $\mathrm{DP}_\mathbf{v}$ to be the triplet $\mathrm{rf}(\mathrm{DP}_\mathbf{v}) = (f^0(\mathrm{DP}_\mathbf{v}),\ f^{1/2}(\mathrm{DP}_\mathbf{v}),\ f^1(\mathrm{DP}_\mathbf{v}))$.

Another way to describe the refined f-polynomial is as follows

$$\mathrm{rf}(\mathrm{DP}_\mathbf{v}) = (f(\mathrm{DP}_\mathbf{v} \cap H^n_0),\ \ t \cdot f(\mathrm{DP}_\mathbf{v} \cap H^n_{1/2}),\ \ f(\mathrm{DP}_\mathbf{v} \cap H^n_1)).$$

The middle entry needs a quick explanation. Namely, a face F in the descent polytope which is not included in the two hyperplanes H^n_0 and H^n_1 does indeed intersect the hyperplane $H^n_{1/2}$. Finally, there is a dimension shift since $\dim(F \cap H^n_{1/2}) = \dim(F) - 1$, explaining the factor of t. In fact, the hyperplane $H^n_{1/2}$ can be replaced by any hyperplane H^n_c for $0 < c < 1$.

Theorem 5.2.1. *Let $\mathbf{v}$ be an $\mathbf{xy}$-word of length $n-1$. Then the refined f-polynomial of the descent polytope $\mathrm{DP}_\mathbf{v}$ is given by $(1, t, 1) \circ \mathbf{v}$, that is,*

$$\mathrm{rf}(\mathrm{DP}_\mathbf{v}) = (1, t, 1) \circ \mathbf{v}.$$

Proof. The proof is by induction on the word $\mathbf{v}$. When the word $\mathbf{v}$ is empty, the associated descent polytope DP_1 is the line segment $[0, 1]$ having the three faces $x_1 = 0$, $0 \le x_1 \le 1$ and $x_1 = 1$, yielding the triplet $(1, t, 1)$ and completing the induction base.

Assume now that the result holds for the word $\mathbf{v}$ of length $n - 1$, and consider a face F of the descent polytope $\mathrm{DP}_\mathbf{v}$. What faces in the one-dimension-higher descent polytope $\mathrm{DP}_{\mathbf{vx}}$ use F as a building block, and how do these faces contribute to the refined f-polynomial? We have three cases.

(0) The face F is contained in H^n_0. Then this face yields the following three faces of $\mathrm{DP}_{\mathbf{vx}}$: $F \times \{0\}$, $F \times [0, 1]$ and $F \times \{1\}$, which contribute $\left(t^{\dim(F)}, t^{\dim(F)+1}, t^{\dim(F)}\right)$ to the refined f-polynomial of $\mathrm{DP}_{\mathbf{vx}}$.

(1/2) The face F is between H^n_0 and H^n_1. Then we obtain the three faces

$$\{(x_1, \ldots, x_n, x_{n+1}) \ :\ (x_1, \ldots, x_n) \in F,\ x_n = x_{n+1}\},$$
$$\{(x_1, \ldots, x_n, x_{n+1}) \ :\ (x_1, \ldots, x_n) \in F,\ x_n \le x_{n+1}\},$$

and $F \times \{1\}$, which contribute $\left(0, t^{\dim(F)} + t^{\dim(F)+1}, t^{\dim(F)}\right)$.

(1) The face F is contained in H_1^n, and it only creates the face $F \times \{1\}$ with contribution $\big(0, 0, t^{\dim(F)}\big)$.

Note also that $\mathrm{DP}_{\mathbf{vx}}$ has no other faces. To summarize, we obtain the map $(p, q, r) \longmapsto (p, t \cdot p + (1 + t) \cdot q, p + q + r)$, which is the linear map described in equation (5.2.1). This completes the induction step in the case $\mathbf{vx}$. The second case is the word $\mathbf{vy}$, which is symmetrical to the first case. □

Let Σ denote the sum, that is, $\Sigma(p, q, r) = p + q + r$. Then the next result follows directly.

Corollary 5.2.2. *Let* $\mathbf{v}$ *be an* $\mathbf{xy}$*-word. The* f*-polynomial of the descent polytope* $\mathrm{DP}_{\mathbf{v}}$ *is given by*

$$f(\mathrm{DP}_{\mathbf{v}}) = \Sigma((1, t, 1) \circ \mathbf{v}).$$

Let star denote reversing the triplet, that is, $(p, q, r)^* = (r, q, p)$. Furthermore, let $\overline{\mathbf{v}}$ denote the result of exchanging the letters $\mathbf{x}$ and $\mathbf{y}$ in the word $\mathbf{v}$. From equations (5.2.1) and (5.2.2), we obtain that $((p, q, r) \circ \mathbf{v})^* = (p, q, r)^* \circ \overline{\mathbf{v}}$.

Define a partial order on $\mathbb{N}[t]$ by letting $p \leq p'$ if the polynomial $p' - p$ only has non-negative coefficients. Extend this partial order to $\mathbb{N}[t]^3$ by comparing entrywise. That is, $(p, q, r) \leq (p', q', r')$ if $p \leq p'$, $q \leq q'$, and $r \leq r'$. Note this inequality also implies that $(p, q, r) \circ \mathbf{v} \leq (p', q', r') \circ \mathbf{v}$ for any $\mathbf{xy}$-word $\mathbf{v}$.

Lemma 5.2.3. *For a triplet* $(p, q, r) \in \mathbb{N}[t]^3$*, the following two inequalities hold:* $(p, q, r) \circ \mathbf{xx} \leq ((p, q, r) \circ \mathbf{xy})^*$ *and* $(p, q, r) \circ \mathbf{yy} \leq ((p, q, r) \circ \mathbf{yx})^*$*.*

Proof. It is enough to verify the first inequality for the three unit cases $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. We have in each case $(1, 2t + t^2, 2 + t) = (2 + t, 2t + t^2, 1)^*$, $(0, 1 + 2t + t^2, 2 + t) \leq (2 + t, 1 + 3t + t^2, 1)^*$, and $(0, 0, 1) \leq (1, t, 1)^*$. The second inequality of the lemma follows by applying the involution $*$. □

Proposition 5.2.4. *Let* $\mathbf{v}$ *and* $\mathbf{w}$ *be two* $\mathbf{xy}$*-words such that the last letter of* $\mathbf{v}$ *is the same as the first letter of* $\mathbf{w}$*. For a triplet* $(p, q, r) \in \mathbb{N}[t]^3$*, we have* $(p, q, r) \circ \mathbf{vw} \leq ((p, q, r) \circ \mathbf{v}\overline{\mathbf{w}})^*$*.*

Proof. Factor $\mathbf{v}$ and $\mathbf{w}$ as $\mathbf{v}' \cdot \mathbf{u}$ and $\mathbf{u} \cdot \mathbf{w}'$, where $\mathbf{u}$ is either the letter $\mathbf{x}$ or $\mathbf{y}$. Then we have $(p, q, r) \circ \mathbf{vu} = ((p, q, r) \circ \mathbf{v}') \circ \mathbf{uu} \leq (((p, q, r) \circ \mathbf{v}') \circ \mathbf{u}\overline{\mathbf{u}})^* = ((p, q, r) \circ \mathbf{v}\overline{\mathbf{u}})^*$ by the previous lemma. Now applying $\mathbf{w}'$ yields the desired inequality $(p, q, r) \circ \mathbf{vw} \leq ((p, q, r) \circ \mathbf{v}\overline{\mathbf{u}})^* \circ \mathbf{w}' = ((p, q, r) \circ \mathbf{v}\overline{\mathbf{w}})^*$. □

Let $\mathbf{z}_n$ be the alternating word of length n starting with the letter $\mathbf{x}$, and let $\overline{\mathbf{z}_n}$ be the alternating word of length n starting with $\mathbf{y}$. It is now straightforward to obtain the maximization result of [3, Corollary 2.5].

Theorem 5.2.5. *The* f*-polynomial of the two descent polytopes* $\mathrm{DP}_{\mathbf{z}_{n-1}}$ *and* $\mathrm{DP}_{\overline{\mathbf{z}_{n-1}}}$ *is coefficientwise maximal among the* f*-vectors of all descent polytopes of dimension* n*. That is,* $f(\mathrm{DP}_{\mathbf{v}}) \leq f(\mathrm{DP}_{\mathbf{z}_{n-1}}) = f(\mathrm{DP}_{\overline{\mathbf{z}_{n-1}}})$ *for an* $\mathbf{xy}$*-word* $\mathbf{v}$ *of length* $n - 1$*.*

Proof. Proposition 5.2.4 shows that the refined f-polynomial $\mathrm{rf}(\mathrm{DP}_{\mathbf{v}}) = (1, t, 1) \circ \mathbf{v}$ is maximized for the two alternating words $\mathbf{z}_{n-1}$ and $\overline{\mathbf{z}_{n-1}}$. Applying the functional Σ yields the result. $\square$

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- Max Steckler Fellowship, University of Kentucky, 2016
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- Van Meter Graduate Fellowship, University of Kentucky, 2012–2015
- Haslam Scholar, University of Tennessee, 2008–2012

Publications

- "Parking cars of different sizes," The American Mathematical Monthly, Vol. 123, No. 10 (December 2016), pp. 1045–1048 (with Richard Ehrenborg), 2016.
- "Parking cars after a trailer," Australasian Journal of Combinatorics, Vol. 70, No. 3 (February 2018), pp. 402–406 (with Richard Ehrenborg), 2018.
- "On the powers of the descent set statistic," Advances in Applied Mathematics, Vol. 96 (May 2018), pp. 1–17 (with Richard Ehrenborg), 2018.
- "Box polynomials and the excedance matrix," submitted to Discrete Mathematics (with Richard Ehrenborg, Dustin Hedmark, and Cyrus Hettle), 24 pages, arXiv:1708.09804.
- "The boustrophedon transform for descent polytopes," submitted to European Journal of Combinatorics (with Richard Ehrenborg).
- "The antipode of the noncrossing partition lattice," submitted to Advances in Applied Mathematics (with Richard Ehrenborg).